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FINITENESS OF TOTALLY GEODESIC EXCEPTIONAL DIVISORS IN HERMITIAN LOCALLY SYMMETRIC SPACES

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ABSTRACT. — We prove that on a smooth complex surface which is a compact quotient of the bidisc or of the 2-ball, there is at most a finite number of totally geodesic curves with negative self-intersection. More generally, there are only finitely many exceptional totally geodesic divisors in a compact Hermitian locally symmetric space of noncompact type of dimension at least 2. This is deduced from a convergence result for currents of integration along totally geodesic subvarieties in compact Hermitian locally symmetric spaces, which itself follows from an equidistribution theorem for totally geodesic submanifolds in a locally symmetric space of finite volume.

RÉSUMÉ (*Finitude du nombre de diviseurs totalement géodésiques exceptionnels dans les variétés localement symétriques hermitiennes*). — Nous prouvons que sur une surface complexe lisse qui est un quotient compact du bidisque ou de la boule de dimension 2, il n'y a qu'un nombre fini de courbes totalement géodésiques d'auto-intersection strictement négative. Plus généralement, il n'y a qu'un nombre fini de diviseurs totalement géodésiques exceptionnels dans une variété localement symétrique (de type non compact) hermitienne compacte de dimension au moins 2. Ces énoncés

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sont déduits d'un théorème de convergence de courants d'intégration le long de sous-variétés totalement géodésiques dans les variétés localement symétriques hermitiennes compactes, lui-même obtenu à partir d'un résultat d'équidistribution des sous-variétés totalement géodésiques dans les variétés localement symétriques de volume fini.

1. Introduction

Our motivation for writing this note comes from a question about totally geodesic curves in compact quotients of the 2-ball related to the so-called Bounded Negativity Conjecture. This conjecture states that if X is a smooth complex projective surface, there exists a number $b(X) \geq 0$ such that any negative curve on X has self-intersection at least $-b(X)$. On a Shimura surface X, i.e. an arithmetic compact quotient of the bidisc or of the 2-ball, one can ask whether such a conjecture holds for Shimura (totally geodesic) curves. In [2], using an inequality of Miyaoka [15], it was proved that on a quaternionic Hilbert modular surface, that is, a compact quotient of the bidisc, there are only a finite number of negative Shimura curves. The same question for Picard modular surfaces, i.e. quotients of the 2-ball, was open as we learned from discussions with participants of the MFO mini-workshops "Kähler Groups" (`http://www.mfo.de/occasion/1409a/www_view`) and "Negative Curves on Algebraic Surfaces" (`http://www.mfo.de/occasion/1409b/www_view`). See the report [9] and [2, Remarks 3.3 & 3.7]. There was a general feeling that this should follow from an equidistribution result about totally geodesic submanifolds in locally symmetric manifolds. Using such a result, we prove that this is indeed true (we include the already known case of the bidisc since this is also implied by the same method):

Theorem 1.1. — *Let X be a closed complex surface whose universal cover is biholomorphic to either the 2-ball or the bidisc. Then X only supports a finite number of totally geodesic curves with negative self-intersection.*

More generally, let X be a closed Hermitian locally symmetric space of non-compact type of complex dimension $n \geq 2$. Then X only supports a finite number of exceptional totally geodesic divisors.

It is known that the irreducible Hermitian symmetric spaces of noncompact type admitting totally geodesic divisors are those associated with the Lie groups $\mathrm{SU}(n,1)$, $n \geq 1$, and $\mathrm{SO}_0(p,2)$, $p \geq 3$, and then that the divisors are associated with the subgroups $\mathrm{SU}(n-1,1)$ and $\mathrm{SO}_0(p-1,2)$ respectively, see [18, 4]. Note, however, that Theorem 1.1 also applies in the case of reducible symmetric spaces.

The first assertion of this result has been obtained independently and at the same time by M. Möller and D. Toledo [16], who also participated in the aforementioned workshops. We refer to their paper for background on Shimura surfaces and Shimura curves, and in particular for a discussion of the arithmetic quotients of the 2-ball and of the bidisc, which admit infinite families of pairwise distinct totally geodesic curves. Their proof is based on an equidistribution theorem for curves in 2-dimensional Hermitian locally symmetric spaces [16, § 2]. Here we have chosen to present a more general result, see Theorem 1.2 below, in the hope that it can be useful in a wider setting (and indeed it implies the second assertion of the theorem).

Henceforth we will be interested in closed totally geodesic (possibly singular) submanifolds in non-positively curved locally symmetric manifolds of finite volume.

Let $\mathcal{X}$ be a symmetric space of noncompact type, $G = \mathrm{Isom}_0(\mathcal{X})$ the connected component of the isometry group of $\mathcal{X}$, Γ a torsion-free lattice of G and X the quotient locally symmetric manifold $\Gamma\backslash\mathcal{X}$.

Complete connected totally geodesic (smooth) submanifolds of $\mathcal{X}$ are naturally symmetric spaces themselves, and we will call such a subset $\mathcal{Y}$ a symmetric subspace of noncompact type of $\mathcal{X}$ if as a symmetric space it is of noncompact type, i.e. it has no Euclidean factor. Up to the action of G, there is only a finite number of symmetric subspaces of noncompact type in $\mathcal{X}$, see Fact 2.4. The orbit of $\mathcal{Y}$ under G will be called the *kind* of $\mathcal{Y}$.

A subset Y of X will be called a *closed totally geodesic submanifold of noncompact type* of X if it is of the form $\Gamma\backslash\Gamma\mathcal{Y}$, where $\mathcal{Y}$ is a symmetric subspace of noncompact type of $\mathcal{X}$ such that if $S_{\mathcal{Y}} < G$ is the stabilizer of $\mathcal{Y}$ in G, $\Gamma\cap S_{\mathcal{Y}}$ is a lattice in $S_{\mathcal{Y}}$. The *kind* of $Y = \Gamma\backslash\Gamma\mathcal{Y}$ is by definition the kind of $\mathcal{Y}$.

It will simplify the exposition to consider only symmetric subspaces of $\mathcal{X}$ passing through a fixed point $o \in \mathcal{X}$. Therefore, we define equivalently a closed totally geodesic submanifold of noncompact type Y of X to be a subset of the form $\Gamma\backslash\Gamma g\mathcal{Y}$, where $\mathcal{Y} \subset \mathcal{X}$ is a symmetric subspace of noncompact type passing through $o \in \mathcal{X}$, and $g \in G$ is such that if $S_{\mathcal{Y}} < G$ is the stabilizer of $\mathcal{Y}$ in G, $\Gamma \cap gS_{\mathcal{Y}}g^{-1}$ is a lattice in $gS_{\mathcal{Y}}g^{-1}$.

Such a Y is indeed a closed totally geodesic submanifold of X, which might be singular, and it supports a natural probability measure μ_Y which can be defined as follows. By assumption, the (right) $S_{\mathcal{Y}}$-orbit $\Gamma\backslash\Gamma gS_{\mathcal{Y}} \subset \Gamma\backslash G$ is closed and supports a unique $S_{\mathcal{Y}}$-invariant probability measure ([19, Chap. 1]). We will denote by μ_Y the probability measure on X whose support is Y and which is defined as the push forward of the previous measure by the projection $\pi : \Gamma\backslash G \longrightarrow X = \Gamma\backslash G/K$, where K is the isotropy subgroup of G at o. In the special case when $S_{\mathcal{Y}} = G$, we obtain the natural probability measure μ_X on X.

We will say that a closed totally geodesic submanifold of noncompact type $Y = \Gamma\backslash\Gamma g\mathcal{Y}$ as mentioned above is a *local factor* if $\mathcal{Y} \subset \mathcal{X}$ is a *factor*, meaning that there exists a totally geodesic isometric embedding $f : \mathcal{Y} \times \mathbb{R} \longrightarrow \mathcal{X}$ such that $f(y, 0) = y$ for all $y \in \mathcal{Y}$.

We may now state the following.

THEOREM 1.2. — *Let $\mathcal{X}$ be a symmetric space of noncompact type, Γ a torsion-free lattice of the connected component G of its isometry group and X the quotient manifold $\Gamma\backslash\mathcal{X}$. Let $(Y_j)_{j\in\mathbb{N}}$ be a sequence of closed totally geodesic submanifolds of noncompact type of X. Assume that no subsequence of $(Y_j)_{j\in\mathbb{N}}$ is either composed of local factors or contained in a closed totally geodesic proper submanifold of X.*

Then the sequence of probability measures $(\mu_{Y_j})_{j\in\mathbb{N}}$ converges to the probability measure μ_X.

REMARK 1.3. — Although we have not been able to find its exact statement in the literature, this theorem is certainly known to experts in homogeneous dynamics and follows from several equidistribution results originating in the work of M. Ratner on unipotent flows, see in particular the work of A. Eskin, S. Mozes and N. Shah [17] and [11]. In the case of special subvarieties of Shimura varieties, a very similar result has been obtained by L. Clozel and E. Ullmo [8, 22]. From the perspective of geodesic flows (which is in a sense orthogonal to unipotent flows), Theorem 1.2 can probably also be deduced from A. Zeghib's article [24] (at least for ball quotients it can be).

The proof we give here is based on a result of Y. Benoist and J.-F. Quint [3], see Section 3.1. As we just said, anterior results certainly imply Theorem 1.2 and moreover the scope of [3] is far larger than the problem at hand, but to our mind, the way the result of [3] is formulated makes it easier to apply to our situation.

REMARK 1.4. — A symmetric subspace $\mathcal{Y} \subset \mathcal{X}$ of noncompact type is the orbit of a point in $\mathcal{X}$ under a connected semisimple subgroup without compact factors $H_{\mathcal{Y}}$ of $G = \mathrm{Isom}_0(\mathcal{X})$. The assumption that $\mathcal{Y}$ is not a factor means that the centralizer $Z_G(H_{\mathcal{Y}})$ of $H_{\mathcal{Y}}$ in G is compact. Another equivalent formulation is that $\mathcal{Y}$ is the only totally geodesic orbit of $H_{\mathcal{Y}}$ in $\mathcal{X}$. See Fact 2.2 for a proof.

This assumption seems quite strong, but the conclusion of Theorem 1.2 is false in general without it, as the following simple example shows. Let $X = \Sigma_1 \times \Sigma_2$ be the product of two Riemann surfaces of genus at least 2 and let $(z_j)_{j\in\mathbb{N}}$ be a sequence of distinct points in Σ_1 such that no subsequence is contained in a proper geodesic of Σ_1. Set $Y_j = \{z_j\} \times \Sigma_2$. Then for any subsequence of (z_j) converging to some $z \in \Sigma_1$, the corresponding subsequence of measures μ_{Y_j} converges to $\mu_{\{z\}\times\Sigma_2}$.

We observe that for rank 1 symmetric spaces, and in the case of uniform *irreducible* lattices of the bidisc, the assumption is automatically satisfied (see the proof of Theorem 1.1 in Section 3.4).

It would be interesting to know whether it is still needed if one assumes e.g. that $\mathcal{X}$ or Γ is irreducible.

In the case of Hermitian locally symmetric spaces, Theorem 1.2 gives a convergence result for currents of integration along closed complex totally geodesic subvarieties (suitably renormalized) from which Theorem 1.1 will follow. Recall that on a complex manifold X of dimension n, a current T of bidegree $(n-p,n-p)$ is said to be *(weakly) positive* if for any choice of smooth $(1,0)$-forms $\alpha_1,\dots,\alpha_p$ on X, the distribution $T\wedge i\alpha_1\wedge\bar{\alpha}_1\wedge\cdots\wedge i\alpha_p\wedge\bar{\alpha}_p$ is a positive measure.

Corollary 1.5. — *Let X and $(Y_j)_{j\in\mathbb{N}}$ satisfy the assumptions of Theorem 1.2 and assume in addition that X is a compact Hermitian locally symmetric space of complex dimension n and that the Y_js are complex p-dimensional subvarieties of X of the same kind.*

Then there exists a closed positive $(n-p,n-p)$-form Ω on X (in the sense of currents), induced by a G-invariant $(n-p,n-p)$-form on $\mathcal{X}=G/K$, such that for any (p,p)-form η on X,

$$\lim_{j\to+\infty}\frac{1}{\mathrm{vol}(Y_j)}\int_{Y_j}\eta=\frac{1}{\mathrm{vol}(X)}\int_X\eta\wedge\Omega$$

Moreover, up to a positive constant, Ω depends only on the kind of the Y_js and if the Y_js are divisors, i.e. if $p=n-1$, then for any j, the $(1,1)$-form Ω restricted to Y_j does not vanish.

Since our initial interest was in 2-ball quotients, we underline that ball quotients X satisfying the assumptions of this corollary exist: the arithmetic manifolds whose fundamental groups are the so-called uniform lattices of type I in the automorphism group $\mathrm{PU}(n,1)$ of the n-ball are examples of manifolds supporting infinitely many complex totally geodesic subvarieties of dimension p for each $1\leq p<n$, not all contained in a proper totally geodesic subvariety. Moreover, any complex p-dimensional totally geodesic subvariety of X is itself a quotient of the p-ball and, as already mentioned, is not a local factor in X (because the n-ball is a rank 1 symmetric space).

In the case of 2-ball quotients, the form Ω of Corollary 1.5 is proportional to the Kähler form induced by the unique (up to a positive constant) $\mathrm{SU}(2,1)$-invariant Kähler form on the ball $\mathbb{H}^2_{\mathbb{C}}$. In the case of quotients of the bidisc, and if ω denotes the unique (up to a positive constant) $\mathrm{SU}(1,1)$-invariant Kähler form on $\mathbb{H}^1_{\mathbb{C}}$, Ω is proportional to the Kähler form induced by the $\mathrm{SU}(1,1)\times$

SU(1, 1)-invariant form $\omega_1 + \omega_2$ on the bidisc $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$, where ω_1, resp. ω_2, means ω on the first, resp. second, factor. See Section 3.3.

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2. Preliminary results

For the reader's convenience, we prove here some more or less well-known and/or easy facts that will be used in the rest of the paper. Good references for the material discussed in this section are [12] and [23].

As in the introduction, $\mathcal{X}$ is a symmetric space of noncompact type, G is the connected component of the isometry group of $\mathcal{X}$ (it is therefore a semisimple real Lie group without compact factors and the connected component of the real points of a semisimple algebraic group defined over $\mathbb{R}$), $o \in \mathcal{X}$ is a fixed origin, $K < G$ is the isotropy group of G at o, so that K is a maximal compact subgroup of G and $\mathcal{X} = G/K$.

We write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G given by the geodesic symmetry s_o around $o \in \mathcal{X}$.

We let $\mathcal{Y} \subset \mathcal{X}$ be a symmetric subspace of noncompact type containing the point o. Its tangent space at o can be identified with a Lie triple system $\mathfrak{q} \subset \mathfrak{p}$, so that setting $\mathfrak{l} := [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$, $\mathfrak{h} := \mathfrak{l} \oplus \mathfrak{q}$ is a semisimple Lie subalgebra of $\mathfrak{g}$. The corresponding connected Lie subgroup H of G is semisimple, without compact factors, has finite center, and its orbit through o is $\mathcal{Y}$. Let $Z_G(H)$ be the centralizer of H in G.

Fact 2.1. — *Let S_0 be the connected component of the stabilizer S of $\mathcal{Y}$ in G. Then $S_0 = HU$, where U is the connected component of $Z_G(H) \cap K$.*

Proof. — If $u \in U = (Z_G(H) \cap K)_0$, then certainly $u \in S_0$ since $u_{|\mathcal{Y}} = \mathrm{id}_{\mathcal{Y}}$. Hence, $HU < S_0$.

The group S_0 is stable by the Cartan involution of G defined by conjugacy by the geodesic symmetry s_o w.r.t. the point $o \in \mathcal{Y}$, because $\mathcal{Y}$ being totally geodesic, it is preserved by the symmetries w.r.t. its points. Therefore, the Lie algebra $\mathfrak{s}$ of S is stable under the corresponding Cartan involution of $\mathfrak{g}$. Hence, we have $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{q}$, where $\mathfrak{m} = \mathfrak{s} \cap \mathfrak{k}$ is a subalgebra of $\mathfrak{k}$ containing $\mathfrak{l}$ and $\mathfrak{q} = \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s} \cap \mathfrak{p}$ because $\mathfrak{q}$ can be identified with the tangent space at o of the orbit $\mathcal{Y}$ of o under H which is also the orbit of o under S. The fact that $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{l}$ and $[\mathfrak{m}, \mathfrak{q}] \subset \mathfrak{q}$ implies that $\mathfrak{l}$ is an ideal in $\mathfrak{m}$ (hence, $\mathfrak{h}$ is an ideal in $\mathfrak{s}$), which in turn implies that the orthogonal $\mathfrak{u}$ of $\mathfrak{l}$ in $\mathfrak{m}$ for the Killing form of $\mathfrak{g}$ is an ideal in $\mathfrak{m}$ hence in $\mathfrak{s}$. Therefore, $\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{h}$ is a direct sum of ideals. Now $\mathfrak{u}$ is included in the Lie algebra of $Z_G(H) \cap K$, hence the result. □

Fact 2.2. — *The following assertions are equivalent:*

- *the symmetric subspace $\mathcal{Y} = Ho \subset \mathcal{X}$ is not a factor;*
- *the centralizer $Z_G(H)$ of H in G is a subgroup of K;*
- *the subgroup H of G has only one totally geodesic orbit in $\mathcal{X}$.*

Proof. — Suppose first that $\mathcal{Y}$ is a factor. This means that there exists a totally geodesic isometric embedding $f : \mathcal{Y} \times \mathbb{R} \longrightarrow \mathcal{X}$ such that $f(y, 0) = y$ for all $y \in \mathcal{Y}$. Hence, there exists v of unit norm in the orthogonal complement $\mathfrak{q}^\perp$ of $\mathfrak{q}$ in $\mathfrak{p}$ such that $[v, \mathfrak{q}] = 0$ (geometrically, and if we identify $\mathfrak{p}$ with $T_o\mathcal{X}$ and $\mathfrak{q}$ with $T_o\mathcal{Y}$, $-\|[v, x]\|^2$ is the sectional curvature of the 2-plane generated by two orthonormal vectors x and v. If $x \in \mathfrak{q}$, this is zero because x and v belong to different factors of a Riemannian product). Since $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$, v commutes with $\mathfrak{h}$. Hence, H commutes with the noncompact 1-parameter subgroup of transvections along the geodesic defined by v.

Assume now that the connected component $Z_G(H)$ of the centralizer of H in G is not included in K and let us prove that H has (at least) two distinct, hence disjoint, totally geodesic orbits $\mathcal{Y}$ and $z\mathcal{Y} = zHo = Hzo$ for some $z \in Z_G(H)$. Let indeed $z \in Z_G(H)$ and suppose by contradiction that $z\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$, i.e. there exists $y_0 \in \mathcal{Y}$ such that $zy_0 \in \mathcal{Y}$. Then z stabilizes $\mathcal{Y} = Hy_0$ and, if d is the distance in $\mathcal{X}$, for all $h \in H$, we have $d(hy_0, zhy_0) = d(hy_0, hzy_0) = d(y_0, zy_0)$, which means that $y \mapsto d(y, zy)$ is constant on $\mathcal{Y}$, equal to t_z say. If $t_z > 0$, z acts on $\mathcal{Y}$ as a non trivial Clifford translation. This implies that $\mathcal{Y}$ splits a line, that is $\mathcal{Y}$ is isometric to a product $\mathcal{Z} \times \mathbb{R}$, see e.g. [7, p. 235]. This is not possible, since H is semisimple. Hence, $t_z = 0$, so that z fixes $\mathcal{Y}$ pointwise and belongs to K.

Finally, assume that H has two distinct totally geodesic orbits $\mathcal{Y} = Ho$ and $\mathcal{Y}' = Hgo$ for some $g \in G$. Then, d being the distance in $\mathcal{X}$, the function $x \mapsto d(x, \mathcal{Y}')$ is convex on $\mathcal{X}$ because $\mathcal{Y}'$ is totally geodesic. Its restriction to $\mathcal{Y}$ is bounded by $d(o, go)$; hence it is constant, equal to a, say, because $\mathcal{Y}$ is also totally geodesic. Therefore, the convex hull of these two orbits is isometric to $\mathcal{Y} \times [0, a]$ and $\mathcal{Y}$ is a factor. See e.g. [7, Chap. II.2]. □

Fact 2.3. — *Assume that $Z_G(H)$ is compact and that L is a connected Lie subgroup of G containing H. Then L has a totally geodesic orbit in $\mathcal{X}$.*

Proof. — It is enough to show that the Lie algebra $\mathfrak{l}$ of L is stable by a Cartan involution of $\mathfrak{g}$. By [5, Lemma 1.5], since G is a connected linear semisimple Lie group and is therefore the connected component of the real points $\mathbf{G}(\mathbb{R})$ of an algebraic group $\mathbf{G}$ defined over $\mathbb{R}$, it suffices to prove that $\mathfrak{l}$ is reductive and algebraic in $\mathfrak{g}$. We are going to show that $\mathfrak{l}$ admits a Levi decomposition of the form $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ and $\mathfrak{a}$ are ideals of $\mathfrak{l}$, $\mathfrak{s}$ is semisimple, $\mathfrak{a}$ is abelian and all the elements of $\mathfrak{a}$ are semisimple for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. This will imply that $\mathfrak{l}$ is reductive in $\mathfrak{g}$, i.e. that the adjoint action of $\mathfrak{l}$ on $\mathfrak{g}$ is semisimple. Since, moreover, in this case $\mathfrak{a} \subset Z_{\mathfrak{g}}(\mathfrak{l}) \subset Z_{\mathfrak{g}}(\mathfrak{h})$ is a compact subalgebra, $\mathfrak{l}$ is indeed algebraic.

The desired decomposition $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{a}$ will be established if we prove that L does not normalize any Lie subalgebra of $\mathfrak{g}$ containing only nilpotent elements.

Indeed, assuming the latter, let $\mathfrak{r}$ be the radical of $\mathfrak{l}$. First, $[\mathfrak{r}, \mathfrak{r}]$ must be trivial since it only contains nilpotent elements and it is normalized by L. Therefore, $\mathfrak{r}$ is abelian. Now, we note that if $r = r_s + r_n$ and $r' = r'_s + r'_n$ are elements of $\mathfrak{r}$ written in terms of their Jordan-Chevalley decomposition, then we also have $[r_n, r'_n] = 0$. Indeed, $[r_s, r'] + [r_n, r'] = [r, r'] = 0$. As $\mathrm{ad}(r_s)$ and $\mathrm{ad}(r_n)$ are polynomials in $\mathrm{ad}(r)$, and since $\mathrm{ad}(r_n)$ is nilpotent, we must have $[r_s, r'] = [r_n, r'] = 0$. Reasoning in the same way with $[r_n, r']$, we see in particular that $[r_n, r'_n] = 0$.

Then $\mathfrak{r}_n := \{r_n | r \in \mathfrak{r}\}$ is a subalgebra of $\mathfrak{g}$ containing only nilpotent elements, and it is normalized by L, since for any $g \in G$ and any $r \in \mathfrak{r}$, $(\mathrm{Ad}(g)(r))_n = \mathrm{Ad}(g)(r_n)$. As a consequence, $\mathfrak{r}_n$ is trivial, i.e. $\mathfrak{r}$ is abelian and only contains semisimple elements. Finally, as $\mathfrak{r}$ is an ideal in $\mathfrak{l}$, its adjoint action on $\mathfrak{l}$ is nilpotent hence trivial, i.e. $\mathfrak{r}$ is central in $\mathfrak{l}$. The desired decomposition $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{a}$ is then a Levi decomposition with $\mathfrak{a} = \mathfrak{r}$.

To conclude, assume for the sake of contradiction that L normalizes a non-trivial Lie subalgebra $\mathfrak{u}$ which contains only nilpotent elements, and let U be the unipotent subgroup of G whose Lie algebra is $\mathfrak{u}$. Then U is the set of real points of a connected algebraic unipotent subgroup $\mathbf{U}$ of $\mathbf{G}$ defined over $\mathbb{R}$. By [6, Corollaire 3.9], there exists a proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over $\mathbb{R}$, whose unipotent radical we denote by $\mathbf{R}$ such that $\mathbf{U} \subset \mathbf{R}$ and $N_{\mathbf{G}}(\mathbf{U}) \subset \mathbf{P}$, where $N_{\mathbf{G}}(\mathbf{U})$ is the normalizer of $\mathbf{U}$ in $\mathbf{G}$. In particular, we have $H \subset L \subset N_G(U) \subset N_{\mathbf{G}}(\mathbf{U})(\mathbb{R}) \subset \mathbf{P}(\mathbb{R})$.

Now, by [23, Corollary 3.14.3] there exists a Levi factor $\mathfrak{m}$ of the Lie algebra of $P := \mathbf{P}(\mathbb{R})$, which contains the Lie algebra $\mathfrak{h}$ of H. Since P is a proper parabolic subgroup of G, $Z_G(\mathfrak{m})$ is noncompact, which is impossible since it is contained in $Z_G(H)$. □

FACT 2.4. — *Up to the action of the stabilizer K of the point $o \in \mathcal{X}$, there are only finitely many symmetric subspaces of noncompact type in $\mathcal{X}$ passing through o. In particular, up to the action of its isometry group, a symmetric space of noncompact type admits only finitely many symmetric subspaces of noncompact type. As we said in the introduction, the orbit of a symmetric subspace $\mathcal{Y}$ of $\mathcal{X}$ under G is called the kind of $\mathcal{Y}$.*

Proof. — This follows from the fact that there are only finitely many G-conjugacy classes of semisimple subalgebras in the Lie algebra of a connected real Lie group G, see e.g. [21, Prop. 12.1]. Let us indeed consider a totally geodesic subspace of noncompact type $\mathcal{Y}'$ in the symmetric space of noncompact type $\mathcal{X}$. Up to the action of the isometry group, we may assume that $\mathcal{Y}'$ contains the point o. Its tangent space at o then identifies with a Lie triple system $\mathfrak{q}'$ of $\mathfrak{p}$ such that $\mathfrak{h}' := [\mathfrak{q}', \mathfrak{q}'] \oplus \mathfrak{q}'$ is a semisimple subalgebra of the Lie algebra $\mathfrak{g}$. Let H' be the corresponding connected subgroup of isometries of $\mathcal{X}$. We have $\mathcal{Y}' = H'o$. Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_m$ be a system of representatives of the conjugacy classes of semisimple subalgebras of $\mathfrak{g}$. We may assume that the $\mathfrak{h}_i$ are stable under the Cartan involution given by the geodesic symmetry around the point o so that if H_i is the connected subgroup of isometries corresponding to $\mathfrak{h}_i$, the orbit $\mathcal{Y}_i = H_i o$ is a totally geodesic subspace of $\mathcal{X}$.

We have $\mathfrak{h}' = g^{-1}\mathfrak{h}_j g$ for some isometry g of $\mathcal{X}$ and some $1 \leq j \leq m$, so that $H' = g^{-1}H_j g$. Then the semisimple subgroup H_j has two totally geodesic orbits $H_j o$ and $H_j g o$ in $\mathcal{X}$. It follows from the proof of Fact 2.2 that there exist a transvection z in the centralizer of H_j, $h \in H_j$ and $k \in K$ such that $g = hzk$. Hence $H' = k^{-1}H_j k$ and $\mathcal{Y}' = k^{-1}\mathcal{Y}_j$. □

Fact 2.5. — *Assume that $\mathcal{X}$ is a Hermitian symmetric space and that $\mathcal{Y}$ is a totally geodesic divisor of $\mathcal{X}$. Then either $\mathcal{Y}$ is not a factor or $\mathcal{X} = \mathcal{Y} \times \mathbb{H}^1_{\mathbb{C}}$, where $\mathbb{H}^1_{\mathbb{C}}$ is the hyperbolic disc. Moreover, if $\mathcal{Y}$ is not a factor and $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_\ell$ is the decomposition of $\mathcal{X}$ in a product of irreducible Hermitian symmetric spaces, then either*

1. *there exist $i \in \{1, \ldots, \ell\}$ and a totally geodesic subspace $\mathcal{Y}_i$ of $\mathcal{X}_i$ such that $\mathcal{Y} = \mathcal{Y}_i \times \prod_{j \neq i} \mathcal{X}_j$, or*
2. *there exist $i \neq j$ in $\{1, \ldots, \ell\}$ and a holomorphic isometry (up to scaling) $\varphi : \mathcal{X}_i \to \mathcal{X}_j$ such that $\mathcal{Y} = \{(x, \varphi(x)) \mid x \in \mathcal{X}_i\} \times \prod_{k \neq i,j} \mathcal{X}_k$.*

For dimensional reasons, in the first case $\mathcal{Y}_i$ is a divisor in $\mathcal{X}_i$ and in the second case $\mathcal{X}_i$ and $\mathcal{X}_j$ are both isometric (up to scaling) to the hyperbolic disc $\mathbb{H}^1_{\mathbb{C}}$.

Proof. — If $\mathcal{Y}$ is a factor then as we saw there exists $v \in \mathfrak{q}^\perp \subset \mathfrak{p}$, which commutes with every element of $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$. The complex structure on $\mathfrak{p}$ is given by $\mathrm{ad}(z)$ for an element z in the center of $\mathfrak{k}$. Since $\mathcal{Y}$ is complex, $\mathfrak{q}$ is invariant by $\mathrm{ad}(z)$, and hence $\mathrm{ad}(z)v$ also belongs to $\mathfrak{q}^\perp$. Hence, $\mathfrak{p} = \mathfrak{q} \oplus \mathbb{R}v \oplus \mathbb{R}\mathrm{ad}(z)v$ because $\dim_{\mathbb{C}} \mathfrak{q} = \dim_{\mathbb{C}} \mathfrak{p} - 1$. It is easily checked that $\mathbb{R}v \oplus \mathbb{R}\mathrm{ad}(z)v$ is a Lie triple system of $\mathfrak{p}$ and that $\mathbb{R}[\mathrm{ad}(z)v, v] \oplus \mathbb{R}v \oplus \mathbb{R}\mathrm{ad}(z)v$ is a Lie subalgebra of $\mathfrak{g}$ which commutes with $\mathfrak{h}$. This subalgebra is either isomorphic to $\mathbb{C}$ or to $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{su}(1, 1)$. Since $\mathfrak{g}$ is semisimple, it is $\mathfrak{sl}(2, \mathbb{R})$.

If $\mathcal{Y}$ is not a factor, then it is a *maximal* totally geodesic subspace of $\mathcal{X}$, in the terminology of [13], meaning that if $\mathcal{Z}$ is a totally geodesic subspace of $\mathcal{X}$ containing $\mathcal{Y}$ then either $\mathcal{Z} = \mathcal{Y}$ or $\mathcal{Z} = \mathcal{X}$. Indeed, if $\mathcal{Z} \neq \mathcal{Y}$ then $\mathcal{Z}$ is a (real) hypersurface of $\mathcal{X}$. By [13, Corollary 3.5], totally geodesic hypersurfaces of $\mathcal{X}$

must be of the form $\mathcal{Z}_i \times \prod_{j\neq i} \mathcal{X}_j$ where $\mathcal{Z}_i$ is a totally geodesic hypersurface of $\mathcal{X}_i$. Then necessarily $\mathcal{X}_i$ has constant sectional curvature (see e.g. [4]) and since it is a Hermitian symmetric space, it must be isometric to the disc $\mathbb{H}^1_{\mathbb{C}}$. Therefore, $\mathcal{Z} \simeq \mathbb{R} \times \prod_{j\neq i} \mathcal{X}_j$ and by the same argument either $\mathcal{Y} \simeq \{\mathrm{pt}\} \times \prod_{j\neq i} \mathcal{X}_j$, or there is a second factor $\mathcal{X}_j$, $j \neq i$, isometric to a disc $\mathbb{H}^1_{\mathbb{C}}$ and $\mathcal{Y} \simeq \mathbb{R}^2 \times \prod_{k\neq i,j} \mathcal{X}_k$. This is a contradiction, since in the former case $\mathcal{Y}$ is a factor, while in the latter it is not a complex submanifold of $\mathcal{X}$. Hence, $\mathcal{Y}$ is indeed maximal, and we may apply [13, Theorem 3.4], which exactly gives the alternative in our statement. □

3. Proofs

3.1. The main ingredient. — Theorem 1.2 follows from considerations originating in the celebrated results of M. Ratner on unipotent flows, see e.g. [20] for a survey. The key result we are going to use is Y. Benoist and J.-F. Quint's Theorem 1.5 in [3].

Let us begin by giving the definitions needed to quote a downgraded version of [3, Theorem 1.5]. Our notations are a bit different from those of [3]. Let G be a real Lie group, Γ a lattice in G, and H a Lie subgroup of G such that $\mathrm{Ad}(H)$ is a semisimple subgroup of $\mathrm{GL}(\mathfrak{g})$ with no compact factors.

A closed subset Z of $\Gamma\backslash G$ is called a finite volume homogeneous subspace if the stabilizer G_Z of Z in G acts transitively on Z and preserves a Borel probability measure μ_Z on Z. If moreover G_Z contains H, Z is said H-ergodic if H acts ergodically on (Z, μ_Z).

Let $C \subset \Gamma\backslash G$ be a compact subset of $\Gamma\backslash G$ and $E_C(H)$ be the set of H-invariant and H-ergodic finite volume homogeneous subspaces Z of $\Gamma\backslash G$ such that $Z \cap C \neq \emptyset$. We may identify $E_C(H)$ with a set of Borel probability measures on $\Gamma\backslash G$ through the map $Z \mapsto \mu_Z$ (note that μ_Z is unique by ergodicity). In particular, $E_C(H)$ is endowed with the topology of weak convergence, so that a sequence (Z_n) in $E_C(H)$ converges toward $Z \in E_C(H)$ if and only if μ_{Z_n} converges toward μ_Z.

Then [3, Theorem 1.5] implies the following:

Theorem 3.1. — *Let G be a real Lie group, Γ a lattice in G, and H a Lie subgroup of G such that* $\mathrm{Ad}(H)$ *is a semisimple subgroup of* $\mathrm{GL}(\mathfrak{g})$ *without compact factors. Let $C \subset \Gamma\backslash G$ be a compact subset. Then*

1. *the space $E_C(H)$ is compact;*
2. *if (Z_n) is a sequence of $E_C(H)$ converging to $Z \in E_C(H)$, there exists a sequence (ℓ_n) of elements of the centralizer of H in G such that $Z_n \cdot \ell_n \subset Z$ for n large.*

We will apply this result to semisimple groups G and to semisimple subgroups H of G whose centralizer in G is compact. In this case we will also use the following consequence of [10, Theorem 2.1]:

Proposition 3.2. — *Let G be a semisimple real Lie group, H a semisimple Lie subgroup of G without compact factors and assume that $Z_G(H)$ is compact. Let Γ be a lattice in G. Then there exists a compact subset $C \subset \Gamma\backslash G$ such that for any $g \in G$, $\Gamma gh \in C$ for some $h \in H$ (depending on g).*

Proof. — We saw in the proof of Fact 2.3 that since $Z_G(H)$ is compact, H is not contained in any proper parabolic subgroup of G. A fortiori, for any $g \in G$, gHg^{-1} is not contained in any proper Γ-rational parabolic subgroup of G, if we use the terminology of [10]. Let us choose a probability measure on G, supported on a bounded open subset of H. We may apply [10, Theorem 2.1] and taking $\epsilon = 1/2$ in property (R2) we get the existence of a compact subset C of $\Gamma\backslash G$ with the property that for all $\Gamma g \in \Gamma\backslash G$, there exists some $m \geq 1$ (depending on g) such that the set $\{(h_1, \ldots, h_m) \in H^m | \Gamma g h_1 \ldots h_m \in C\} \subset G^m$ has measure at least $1/2$ for the product measure. The proposition follows immediately. □

3.2. Proof of Theorem 1.2. — We recall the notation from the introduction. We have a symmetric space of noncompact type $\mathcal{X}$ and G is the connected component of the isometry group of $\mathcal{X}$. In particular, G is a connected semisimple real Lie group with trivial center and without compact factors. We fix an origin $o \in \mathcal{X}$ and let K be the isotropy group of G at o. Moreover, Γ is a torsion-free lattice of G and X is the finite volume locally symmetric space $\Gamma\backslash\mathcal{X}$.

Let (Y_j) be a sequence of compact totally geodesic submanifolds of noncompact type of X as in the statement of the theorem.

The first thing to note is that by the finiteness result of Fact 2.4, up to extracting subsequences, we may assume that the submanifolds Y_j are all of the same kind, meaning that they are of the form $\Gamma\backslash\Gamma g_j \mathcal{Y}$, where $\mathcal{Y}$ is a fixed symmetric subspace of $\mathcal{X}$ passing through the origin o, and $g_j \in G$ are such that $\Gamma \cap g_j S g_j^{-1}$ is a lattice in $g_j S g_j^{-1}$, S being the stabilizer of $\mathcal{Y}$ in G. We call H the connected semisimple subgroup without compact factors and with finite center such that $\mathcal{Y} = Ho$.

Moreover, by Fact 2.2, the hypothesis that the totally geodesic submanifolds Y_j are not local factors implies that the centralizer $Z_G(H)$ of H in G is included in K and that $\mathcal{Y}$ is the only totally geodesic orbit of H in $\mathcal{X}$.

Consider the (right) S-invariant subsets $\Gamma\backslash\Gamma g_j S$ in $\Gamma\backslash G$. They support natural S-invariant probability measures, but these measures might be non-ergodic with respect to the action of $H \subset S$. To get rid of this problem, we need to consider the action of H on the orbit of a smaller subgroup than S. Let S_0 be the connected component of the stabilizer S of $\mathcal{Y}$. By Fact 2.1, there exists a subgroup $U < K$ centralizing H such that $S_0 = HU$. The intersection $H \cap U$ is the center of H and hence is finite. The intersection $g_j^{-1}\Gamma g_j \cap S$ is by assumption a lattice in S, and therefore the intersection $g_j^{-1}\Gamma g_j \cap S_0$ is a lattice in S_0, since S_0 has finite index in S. Let M_j be the "projection" of $g_j^{-1}\Gamma g_j \cap S_0$ to U, namely the group $\{u \in U$ such that $u = \gamma h$ for some $\gamma \in g_j^{-1}\Gamma g_j \cap S_0$ and $h \in H\}$, and let $\bar{M}_j$ be the closure of M_j. Then $\bar{M}_j$ is a compact subgroup of U and we let $S_j := \bar{M}_j H$. This time, the right action of H on the S_j-invariant probability measure μ_j supported on $Z_j := \Gamma\backslash\Gamma g_j S_j$ is ergodic. Indeed, a H-orbit in Z_j is the same as a (left) $M_j H$-orbit in S_j (because H is obviously normal in S_j) and the group $M_j H$ is dense in S_j by construction (see [14, Prop. I.(4.5.1)]). Note, however, that the push forward of the measure μ_j by the projection $\pi : \Gamma\backslash G \longrightarrow X = \Gamma\backslash G/K$ is the same as the push forward of the S-invariant probability measure on $\Gamma\backslash\Gamma g_j S$ by π. Indeed, $g_j S g_j^{-1}$ (resp. $g_j S_j g_j^{-1}$) is unimodular because it contains its intersection with Γ as a lattice, and its Haar measure suitably normalized induces the S-invariant (resp. S_j-invariant) probability measure on $\Gamma\backslash\Gamma g_j S$ (resp. $\Gamma\backslash\Gamma g_j S_j$). The push forward of these Haar measures define $g_j H g_j^{-1}$-invariant measures supported on $g_j\mathcal{Y}$, and hence they must be proportional. Finally, as they both induce probability measures on Y_j, they are equal.

As $Z_G(H)$ is compact, there exists by Proposition 3.2 a compact subset C of $\Gamma\backslash G$ with the property that for all g_j, there exists $h_j \in H$ with $\Gamma g_j h_j \in C$. Therefore, since obviously $\Gamma g_j h_j \in \operatorname{supp}\mu_j$, the measures μ_j belong to the set $E_C(H)$ of H-invariant and H-ergodic finite volume homogeneous subspaces of $\Gamma\backslash G$ intersecting C, which is compact by Theorem 3.1.

Hence, after extraction of a subsequence, the sequence (μ_j) converges weakly to a H-ergodic probability measure μ whose support $\operatorname{supp}\mu$ is a closed G^μ-homogeneous subset of $\Gamma\backslash G$, where $G^\mu := \{g \in G : \mu g = \mu\}$ is a (closed) Lie subgroup of G containing H. Moreover, by the second part of the theorem, there exists a sequence (ℓ_j) of elements of the centralizer of H in G such that for j large enough, $\operatorname{supp}\mu_j \subset (\operatorname{supp}\mu)\cdot\ell_j$.

We are going to prove that $G^\mu = G$, and the compactness of the centralizer of H will come into play in order to neutralize the effect of the ℓ_js.

By Fact 2.3, the connected component G_0^μ of G^μ has a totally geodesic orbit $G_0^\mu x^\mu$ in $\mathcal{X} = G/K$ for some $x^\mu \in \mathcal{X}$. Moreover, we know that $\operatorname{supp}\mu = \Gamma\backslash\Gamma g G^\mu$ for some $g \in G$, and we have seen above that if j is large, then $Z_j\ell_j = \Gamma\backslash\Gamma g_j \bar{M}_j H \ell_j \subset \Gamma\backslash\Gamma g G^\mu$ for some ℓ_j in the centralizer of H. Hence $\Gamma\backslash\Gamma g_j HK/K \subset \Gamma\backslash\Gamma g G_0^\mu K/K$, because $\ell_j \in K$ by assumption. Therefore, there exists $\gamma_j \in \Gamma$ such that $\gamma_j g_j \mathcal{Y} \subset g G_0^\mu o \subset \mathcal{X}$.

This actually implies that, for j large, the totally geodesic submanifolds $\gamma_j g_j\mathcal{Y}$ are all included in the totally geodesic submanifold $gG_0^\mu x^\mu$ and thus that $G_0^\mu o = G_0^\mu x^\mu$.

Indeed, let d be the distance in the symmetric space $\mathcal{X}$. The function

$$\begin{aligned}\mathcal{X} &\longrightarrow \mathbb{R}\\ x &\longmapsto d(x, gG_0^\mu x^\mu)\end{aligned}$$

is convex because $gG_0^\mu x^\mu$ is totally geodesic. Its restriction to $gG_0^\mu o$ is bounded because for all $a \in G_0^\mu$, $d(gao, gax^\mu) = d(o, x^\mu)$. Therefore, its restriction to $\gamma_j g_j \mathcal{Y}$, which is also totally geodesic, is both convex and bounded, hence constant equal to some $d_j \in \mathbb{R}$. Then, if $d_j \neq 0$, there exists an isometric embedding from the product $(\gamma_j g_j \mathcal{Y}) \times \mathbb{R}$ to $\mathcal{X}$ which maps $(\gamma_j g_j \mathcal{Y}) \times \{0\}$ to $\gamma_j g_j \mathcal{Y}$ and $(\gamma_j g_j \mathcal{Y}) \times \{d_j\}$ to a totally geodesic subspace of $gG_0^\mu x^\mu$; see e.g. [7, Chap. II.2]. This is a contradiction to the fact that $\mathcal{Y}$ is not a factor; hence $d_j = 0$ for all j large, as claimed.

As a consequence, $\Gamma\backslash\Gamma gG_0^\mu o$ is a closed totally geodesic submanifold in X which contains the submanifolds Y_j (for j large enough). Since no subsequence of (Y_j) is contained in a closed totally geodesic proper submanifold of X, $\Gamma\backslash\Gamma gG_0^\mu o = \Gamma\backslash G/K$, which implies $G_0^\mu = G$ since G_0^μ is reductive.

In conclusion, (μ_j) is a sequence of elements of $E_C(H)$ which is compact, and its sole limit point is the unique G-invariant probability measure on $\Gamma\backslash G$, so that it converges to this unique measure. Hence, the sequence (μ_{Y_j}) converges to μ_X.

Remark 3.3. — It is straightforward that for any compact subgroup $L \subset K$, the pushforward of the measures μ_j to $\Gamma\backslash G/L$ converges towards the pushforward to $\Gamma\backslash G/L$ of the G-invariant probability measure on $\Gamma\backslash G$.

3.3. Proof of Corollary 1.5. — The kind of the Y_js is fixed so that they are of the form $\Gamma\backslash\Gamma g_j\mathcal{Y}$, where $\mathcal{Y}$ is a fixed Hermitian symmetric subspace of $\mathcal{X}$ of dimension p passing through the origin o, and $g_j \in G$ are such that $\Gamma \cap g_j S g_j^{-1}$ is a lattice in $g_j S g_j^{-1}$, S being the stabilizer of $\mathcal{Y}$ in G.

We call $K_\mathcal{Y} \subset K$ the compact subgroup $K_\mathcal{Y} = K \cap S$. The homogeneous space $G/K_\mathcal{Y}$ is the G-orbit of $T_o\mathcal{Y}$ in the Grassmann manifold of complex p-planes in the tangent bundle $T\mathcal{X}$ of $\mathcal{X}$. It is a bundle over $\mathcal{X}$ and we let $\hat{X} := \Gamma\backslash G/K_\mathcal{Y}$ be the corresponding Grassmann bundle over X.

Every submanifold Y_j has a natural lift $\hat{Y}_j$ to $\hat{X}$: a smooth point y of Y_j defines the point $\hat{y} = T_yY_j$ in $\hat{X}$. In fact, $\hat{Y}_j$ is smooth and isomorphic to $(\Gamma \cap g_j S g_j^{-1})\backslash g_j\mathcal{Y}$ and the natural morphism $\nu_j : \hat{Y}_j \longrightarrow Y_j$ is an immersion which is generically one-to-one. In particular, $\hat{Y}_j$ is the normalization of Y_j.

For each j, we denote by $\mu_{\hat{Y}_j}$ the probability measure on $\hat{X}$ which is obtained by taking the direct image of the measure μ_j on $\Gamma\backslash G$ defined in the proof of Theorem 1.2. We emphasize that the support of the measure $\mu_{\hat{Y}_j}$ is indeed $\hat{Y}_j$. Let ω_X be the Kähler form on X induced by a G-invariant Kähler form ω on $\mathcal{X}$. Define $\mathrm{vol}(Y_j) = \frac{1}{p!}\int_{Y_j} \omega_X^p$ where $\int_{Y_j}$ means integration over the smooth part of Y_j, i.e. $\mathrm{vol}(Y_j) = \frac{1}{p!}\int_{\hat{Y}_j} \nu_j^*\omega_X^p$. Then the probability measure with support $\hat{Y}_j$ and density $\frac{1}{p!\mathrm{vol}(Y_j)}\nu_j^*\omega_X^p$ is equal to $\mu_{\hat{Y}_j}$. This is again due to the fact that they are both induced by $g_jHg_j^{-1}$-invariant measures on the orbit $g_jHK_{\mathcal{Y}} \subset G/K_{\mathcal{Y}}$.

For any (p,p)-form η on $\mathcal{X}$, we define a function φ_η on G by

$$\varphi_\eta(g) := p!\frac{\eta(e_1,\dots,e_{2p})}{\omega^p(e_1,\dots,e_{2p})}$$

where $(e_1,\dots,e_{2p})$ is any basis of $T_{gK}g\mathcal{Y}$. Said another way, in restriction to $T_{gK}g\mathcal{Y}$ the two (p,p)-forms η and $\frac{1}{p!}\omega^p$ are proportional, and $\varphi_\eta(g)$ is the coefficient of proportionality. Note that the action of $K_{\mathcal{Y}}$ on G by right multiplication induces the trivial action on φ_η by construction so that we will see it as a function on $G/K_{\mathcal{Y}}$ as well. Moreover, if η is the lift of a form on X then φ_η is well defined on $\Gamma\backslash G$ (and $\hat{X} = \Gamma\backslash G/K_{\mathcal{Y}}$). Then by the proof of Theorem 1.2 and Remark 3.3 we have

$$\begin{aligned}\frac{1}{\mathrm{vol}(Y_j)}\int_{Y_j}\eta = \frac{1}{\mathrm{vol}(Y_j)}\int_{\hat{Y}_j}\nu_j^*\eta = \frac{1}{p!\mathrm{vol}(Y_j)}\int_{\hat{Y}_j}\varphi_\eta\nu_j^*\omega_X^p\\ = \int_{\hat{X}}\varphi_\eta d\mu_{\hat{Y}_j} \underset{j\to+\infty}{\longrightarrow} \int_{\hat{X}}\varphi_\eta\, d\mu_{\hat{X}}\end{aligned}$$

where $d\mu_{\hat{X}}$ is the probability measure on $\hat{X}$ induced by the Haar measure dg on G normalized in such a way that $\int_{\Gamma\backslash G} dg = 1$.

For any (p,p)-form η, we also define a function ψ_η on G by

$$\psi_\eta(g) := \int_K \varphi_\eta(gk)dk$$

where dk is the Haar probability measure on K. Actually, ψ_η is well defined on $\mathcal{X} = G/K$ and in the same way as φ_η, it can be seen as a function on $X = \Gamma\backslash G/K$ if η comes from X.

Now, as

$$\int_{\Gamma\backslash G}\varphi_\eta(gk)dg = \int_{\Gamma\backslash G}\varphi_\eta(g)dg$$

for any $k \in K$ (just because dg is right invariant), we get for any (p,p)-form η on X

$$\begin{aligned}\int_{\hat{X}}\varphi_\eta d\mu_{\hat{X}} = \int_{\Gamma\backslash G}\varphi_\eta(g)dg = \int_K\int_{\Gamma\backslash G}\varphi_\eta(gk)dgdk\\ = \int_{\Gamma\backslash G}\psi_\eta(g)dg = \frac{1}{n!\mathrm{vol}(X)}\int_X \psi_\eta\omega_X^n.\end{aligned}$$

Let us consider the linear form $\eta \mapsto \psi_\eta(e)$ on the space of (p,p)-forms on $\mathcal{X}$. It only depends on $\eta(o)$; hence there exists a (p,p)-form Ψ on $T_o\mathcal{X}$ such that $\psi_\eta(e) = \langle\eta,\Psi\rangle_o$. Moreover, for any $k \in K$, $\psi_\eta(k) = \psi_{k^*\eta}(e) = \langle k^*\eta,\Psi\rangle_o =$

$\langle \eta, (k^{-1})^*\Psi\rangle_o$. As ψ_η is K-invariant, we conclude that Ψ is also K-invariant. Therefore, Ψ is the restriction of a (unique) G-invariant form on $\mathcal{X}$ that we still denote by Ψ. Then, for any $g \in G$, $\psi_\eta(g) = \psi_{g^*\eta}(e) = \langle g^*\eta, \Psi\rangle_o = \langle \eta, (g^{-1})^*\Psi\rangle_{go} = \langle \eta, \Psi\rangle_{go}$ i.e. $\psi_\eta = \langle \eta, \Psi\rangle$ on $\mathcal{X}$ for any η. Finally,

$$\frac{1}{n!\mathrm{vol}(X)}\int_X \psi_\eta \omega_X^n = \frac{1}{n!\mathrm{vol}(X)}\int_X \langle \eta, \Psi\rangle \omega_X^n = \frac{1}{\mathrm{vol}(X)}\int_X \eta \wedge \star\Psi$$

where $\star$ is the Hodge star operator. Setting $\Omega := \star\Psi$, we get the desired result. The form Ω is closed and positive since this is the case for each current of integration over Y_j (note that, actually, a G-invariant form of even degree is automatically closed as its differential is a G-invariant form of odd degree and G contains an involution admitting o as an isolated fixed point, i.e. its differential at o is $-\mathrm{id}_{T_o\mathcal{X}}$).

By construction, Ω only depends on $\mathcal{Y}$, i.e. on the kind of the Y_js, and on the choice of a G-invariant Kähler form ω on $\mathcal{X}$. More precisely, from these arguments, we see that Ω is recovered from $\mathcal{Y}$ in the following way: let $(e_1, \dots, e_{2p})$ be a direct basis of $T_o\mathcal{Y}$, and Ψ be the real (p,p)-form on $\mathcal{X}$ obtained by averaging $e_1^* \wedge \dots \wedge e_{2p}^*$ by K in order to get a K-invariant form on $T_o\mathcal{X}$ and transporting it on $\mathcal{X}$ by G; then, up to a positive constant, we have $\Omega = \star\Psi$ (the choice of ω plays its role in the definition of $\star$).

If $\mathcal{X}$ is irreducible, ω is unique up to a positive constant. If $\mathcal{X}$ is not irreducible, then this is not the case. However, if ω' is another G-invariant Kähler form, the restriction of ω^p and ω'^p to $\mathcal{Y}$ only differ by a multiplicative positive constant c. Indeed, this is clear at o and the two forms are S-invariant. As a consequence, the corresponding functions φ_η and ψ_η differ by the constant $1/c$ and so the resulting forms Ω only differ by a positive constant.

In general, if $(\eta_1, \dots, \eta_m)$ is an orthonormal basis of G-invariant (p,p)-forms on $\mathcal{X}$, then it is straightforward from the construction above that $\Omega = \sum_{i=1}^m \varphi_{\eta_i}(\star\eta_i)$, the φ_{η_i} being constant since η_i is G-invariant.

Assume now that the Y_js are divisors, that is $p = n-1$.

Let $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_\ell$ be the decomposition of $\mathcal{X}$ in a product of irreducible Hermitian symmetric spaces $\mathcal{X}_i$ of dimension n_i and isometry group G_i, and ω_i the unique (up to a positive constant) G_i-invariant Kähler form on $\mathcal{X}_i$. If the Y_js are divisors, then Ω is induced by $\sum_{i=1}^\ell a_i\omega_i$ for some non-negative real numbers a_i. The positive $(n-1,n-1)$-forms $\eta_i := \omega_i^{n_i-1} \wedge \bigwedge_{j\neq i}\omega_j^{n_j}$ make up an orthogonal basis of G-invariant $(n-1,n-1)$-forms on X.

By Fact 2.5, one can assume that $\mathcal{Y} = \mathcal{D} \times \mathcal{X}_{k+1} \times \dots \times \mathcal{X}_\ell$, where either $k = 1$ and $\mathcal{D}$ is a divisor in $\mathcal{X}_1$, or $k = 2$, $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{H}^1_{\mathbb{C}}$ and $\mathcal{D} \simeq \mathbb{H}^1_{\mathbb{C}}$ which is diagonally embedded in $\mathcal{X}_1 \times \mathcal{X}_2$.

In the first case, the restriction of all the η_i's to $\mathcal{Y}$ vanishes, except when $i = 1$ and this implies that only $a_1 > 0$. Similarly, in the second case, only η_1

and η_2 do not vanish on $\mathcal{Y}$, and hence only a_1 and a_2 are positive (and equal if ω_1 and ω_2 are chosen in such a way that $\mathcal{X}_1$ and $\mathcal{X}_2$ are isometric).

As a consequence, Ω is positive (as a (1,1)-form) only if $\mathcal{X}$ is irreducible or if $\mathcal{X} = \mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$ and $\mathcal{Y} \simeq \mathbb{H}^1_{\mathbb{C}}$ is diagonally embedded. However, in all cases, the restriction of Ω to Y_j never vanishes.

Remark 3.4. — In the case of the n-ball, it is well known that for any $1 \leq p \leq n$, the space of G-invariant (p,p)-forms is 1-dimensional and generated by ω^p; hence, in this case we always have $\Omega = \frac{p!}{n!}\omega_X^p$.

3.4. Proof of Theorem 1.1. — We only prove the second assertion of the theorem, since the first is just a particular case by Grauert's criterion, see for example [1, p. 91].

Recall that a compact divisor D in a complex manifold X is *exceptional* if there exists a neighborhood U of D in X, a proper bimeromorphic map $\phi : U \longrightarrow U'$ onto a (possibly singular) analytic space U' and a point $x' \in U'$ such that $\phi(D) = \{x'\}$, and ϕ induces a biholomorphism between $U \backslash D$ and $U' \backslash \{x'\}$.

Let $\mathcal{X}$ be the universal cover of the Hermitian locally symmetric space X, G the connected component of the isometry group of $\mathcal{X}$ and Γ the torsion-free lattice of G such that $X = \Gamma \backslash \mathcal{X}$.

Assume that there exist infinitely many totally geodesic (irreducible) exceptional divisors $(D_j)_{j\in\mathbb{N}}$ in X. As before, because of Fact 2.4, we may assume that the totally geodesic divisors D_j are of the form $\Gamma\backslash\Gamma g_j\mathcal{D}$, where $\mathcal{D}$ is a totally geodesic divisor of $\mathcal{X}$ containing a fixed point $o \in \mathcal{X}$.

By Fact 2.5, either D_j is not a local factor or $\mathcal{X} = g_j\mathcal{D} \times \mathbb{H}^1_{\mathbb{C}}$. In the latter case, $G \simeq H_j \times \mathrm{PU}(1,1)$, where H_j is the connected component of the isometry group of $g_j\mathcal{D}$, and this implies that the lattice Γ is not irreducible. Indeed, if S_j is the stabilizer of $g_j\mathcal{D}$ then by assumption $\Gamma \cap S_j$ is a lattice in $S_j = H_j \times U$ where U is a compact subgroup of $\mathrm{PU}(1,1)$. By [19, Thm 1.13], $\Gamma \cdot (H_j \times U)$ is closed in G. This is not possible if Γ is irreducible because then the projection of Γ onto $\mathrm{PU}(1,1)$ is dense, so that $\Gamma \cdot (H_j \times U)$ is dense in $G = H_j \times \mathrm{PU}(1,1)$. Therefore, up to a finite covering, we have $X = D_j \times \Sigma_j$ (where Σ_j is a curve) and in particular, D_j is not exceptional.

Hence, we may assume that the D_js are of the same kind and are not local factors, so that Corollary 1.5 applies.

Let ω_X be a Kähler form on X induced by a G-invariant Kähler form of $\mathcal{X}$. We may choose ω_X so that it represents the first Chern class $c_1(K_X)$ of the canonical bundle K_X. All volumes will be computed w.r.t. ω_X.

Each of the divisors D_j defines an integral class $[D_j] \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{R})$. We write $[D_j] \cdot [\eta] = \int_{D_j} \eta$ for any class $[\eta] \in H^{n-1,n-1}(X,\mathbb{R})$ and we set $A_j := \frac{\mathrm{vol}(D_j)}{n\mathrm{vol}(X)} = \frac{[D_j]\cdot[\omega_X]^{n-1}}{n!\mathrm{vol}(X)}$. Since the pairing between $H^{1,1}(X,\mathbb{R})$ and

$H^{n-1,n-1}(X,\mathbb{R})$ is non degenerate, Corollary 1.5 implies that

$$(\star)\qquad \frac{1}{A_j}[D_j] \underset{j\to+\infty}{\longrightarrow} [\Omega]$$

for some closed non-negative $(1,1)$-form Ω, which does not vanish in restriction to the D_js. The canonical class $[K_X]$ of X is ample and is equal to $[\omega_X]$, so let $m \in \mathbb{N}^\star$ be large enough and $H_1,\dots,H_{n-2} \in |mK_X|$ be irreducible divisors in the linear system $|mK_X|$ such that $N := H_1 \cap \dots \cap H_{n-2} \subset X$ is a smooth surface intersecting all the divisors D_j transversally.

Consider now the sequence of curves $(C_j)_{j\in\mathbb{N}}$ of N defined by $C_j = N \cap D_j$. By definition, for any j, there exists a bimeromorphic map $\phi_j : X \longrightarrow X'_j$ which contracts the divisor D_j, so that the curve C_j is contracted by the morphism $\phi_{j|N} : N \longrightarrow \phi_j(N)$, hence has negative self-intersection by Grauert's criterion (see [1, p. 91] for instance).

Let Ω_N be the restriction of Ω to N and consider now the intersection numbers on N

$$I_j := \left([\Omega_N] - \frac{1}{A_j}[C_j]\right)\cdot\frac{1}{A_j}[C_j] = \left([\Omega] - \frac{1}{A_j}[D_j]\right)\cdot\frac{1}{A_j}[D_j]\cdot m^{n-2}[\omega_X]^{n-2}$$

(here we used $[N] = m^{n-2}[\omega_X]^{n-2}$).

On the one hand, $I_j \underset{j\to+\infty}{\longrightarrow} 0$ by $(\star)$, and on the other hand, since $C_j^2 < 0$, we have $I_j \geq [\Omega]\cdot\frac{1}{A_j}[C_j] = m^{n-2}\frac{1}{A_j}[D_j]\cdot[\Omega]\cdot[\omega_X]^{n-2} = cm^{n-2}n!\,\mathrm{vol}(X)$ for any j and for some positive constant c, a contradiction. We used the fact that since Ω does not vanish in restriction to the D_js, the restriction of $\Omega\wedge\omega_X^{n-2}$ to D_j is equal to c times the restriction of ω_X^{n-1} to D_j for some positive constant c, because both are invariant forms of bidegree $(n-1,n-1)$.

BIBLIOGRAPHY

[1] W. Barth, K. Hulek, C. Peters & A. Van de Ven, "Compact complex surfaces", *Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4*, vol. 2, Springer, 2004.

[2] T. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, S. Müller-Stach, X. Roulleau & T. Szemberg, "Negative curves on algebraic surfaces", *Duke Math. J.* **162** (2013), p. 1877–1894.

[3] Y. Benoist & J.-F. Quint "Stationary measures and invariant subsets of homogeneous spaces (III)", *Ann. of Math.* **178** (2013), p. 1017–1059.

[4] J. Berndt & C. Olmos, "On the index of symmetric spaces", *J. reine angew. Math.*, Ahead of Print DOI 10.1515/crelle-2015-0060, 2015.

[5] A. Borel & Harish-Chandra, "Arithmetic subgroups of algebraic groups", *Ann. of Math.* **75** (1962), p. 485–535.

[6] A. Borel & J. Tits, "Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I", *Invent. Math.* **12** (1971), p. 95–104.
[7] M. Bridson & A. Haefliger, "Metric spaces of non-positive curvature", *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 319, Springer, 1999.
[8] L. Clozel & E. Ullmo, "Equidistribution de sous-variétés spéciales", *Ann. of Math.* **161** (2005), p. 1571–1588.
[9] S. Di Rocco, A. Küronya, S. Müller-Stach & T. Szemberg, "Mini-Workshop: Negative Curves on Algebraic Surfaces", *Mathematisches Forschungsinstitut Oberwolfach, Report No. 10/2014*, available at http://www.mfo.de/document/1409b/OWR_2014_10.pdf.
[10] A. Eskin & G. Margulis, "Recurrence properties of random walks on finite volume homogeneous manifolds", *Random walks and geometry*, Walter de Gruyter, 2004, p. 431–444.
[11] A. Eskin, S. Mozes & N. Shah, "Unipotent flows and counting lattice points on homogeneous spaces", *Ann. of Math.* **143** (1996), p. 253–299.
[12] S. Helgason, "Differential geometry, Lie groups, and symmetric spaces", *Corrected reprint of the 1978 original, Graduate Studies in Mathematics, 34*, American Mathematical Society, 2001.
[13] A. Kollross, "Polar actions on symmetric spaces", *J. Differential Geom.* **77** (2007), p. 425–482.
[14] G. A. Margulis, "Discrete subgroups of semisimple Lie groups", *Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17*, Springer, 1991.
[15] Y. Miyaoka, "The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem", *Publ. Res. Inst. Math. Sci.* **44** (2008), p. 403–417.
[16] M. Möller & D. Toledo, "Bounded negativity of self-intersection numbers of Shimura curves in Shimura surfaces", *Algebra Number Theory* **9** (2015), p. 897–912.
[17] S. Mozes & N. Shah, "On the space of ergodic invariant measures of unipotent flows", *Ergodic Theory Dynam. Systems* **15** (1995), p. 149–159.
[18] A. L. Onishchik, "Totally geodesic submanifolds of symmetric spaces" (Russian), in Geometric methods in problems of algebra and analysis no. 2 (Russian), *Yaroslav. Gos. Univ., Yaroslavl'*, **161** (1980), p. 64–85.
[19] M. S. Raghunathan, "Discrete subgroups of Lie groups", *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 68. Springer, 1972.
[20] M. Ratner, "Invariant measures and orbit closures for unipotent actions on homogeneous spaces", *Geom. Funct. Anal.* **4** (1994), p. 236–257.
[21] R. W. Richardson Jr., "A rigidity theorem for subalgebras of Lie and associative algebras", *Illinois J. Math.* **11** (1967), p. 92–110.
[22] E. Ullmo, "Equidistribution de sous-variétés spéciales II", *J. Reine Angew. Math.* **606** (2007), p. 193–216.

[23] V. S. Varadarajan, "Lie groups, Lie algebras, and their representations", *Reprint of the 1974 edition. Graduate Texts in Mathematics*, vol. 102. Springer, 1984.

[24] A. Zeghib, "Ensembles invariants des flots géodésiques des variétés localement symétriques", *Ergodic Theory Dynam. Systems* **15** (1995), p. 379–412.