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A RIEMANN-ROCH-HIRZEBRUCH FORMULA FOR TRACES OF DIFFERENTIAL OPERATORS

BY MARKUS ENGELI AND GIOVANNI FELDER

ABSTRACT. – Let D be a holomorphic differential operator acting on sections of a holomorphic vector bundle on an n-dimensional compact complex manifold. We prove a formula, conjectured by Feigin and Shoikhet, giving the Lefschetz number of D as the integral over the manifold of a differential form. The class of this differential form is obtained via formal differential geometry from the canonical generator of the Hochschild cohomology $HH^{2n}(\mathcal{D}_n, \mathcal{D}_n^*)$ of the algebra of differential operators on a formal neighbourhood of a point. If D is the identity, the formula reduces to the Riemann-Roch-Hirzebruch formula.

RÉSUMÉ. – Soit D un opérateur différentiel holomorphe opérant sur les sections d'un fibré vectoriel holomorphe sur une variété complexe de dimension n. Nous démontrons une formule, conjecturée par Feigin et Shoikhet, donnant le nombre de Lefschetz de D comme intégrale d'une forme différentielle sur la variété. La classe de cette forme différentielle est obtenue, via la géométrie différentielle formelle du générateur canonique de la cohomologie de Hochschild $HH^{2n}(\mathcal{D}_n, \mathcal{D}_n^*)$ de l'algèbre des opérateurs différentiels sur un entourage formel d'un point. Si D est l'identité, la formule se réduit à la formule de Riemann-Roch-Hirzebruch.

1. Introduction

Let $E \to X$ be a holomorphic vector bundle of rank r on a compact connected complex manifold X of complex dimension n. Let $\mathcal{D}_E$ be the sheaf of holomorphic differential operators acting on sections of E.

Global differential operators $D \in \mathcal{D}_E(X) = \Gamma(X, \mathcal{D}_E)$ act on the sheaf cohomology groups $H^j(X, E)$ of E and thus we have algebra homomorphisms

$$H^j : \mathcal{D}_E(X) \to \mathrm{End}(H^j(X, E)).$$

Since the cohomology of E is finite dimensional, we can consider the *Lefschetz number* (or supertrace) $L\colon \mathcal{D}_E(X) \to \mathbb{C}$,

$$D \mapsto L(D) = \sum_{j=0}^{n} (-1)^j \operatorname{tr}(H^j(D)).$$

If $D = \mathrm{Id}$ is the identity then $L(\mathrm{Id})$ is the holomorphic Euler characteristic of E; it is given by the Riemann–Roch–Hirzebruch theorem as the integral over X of a characteristic class. Our aim is to generalize this formula to the case of a general differential operator D by writing the Lefschetz number as the integral over X of a differential form $\chi_0(D)$ whose value at a point $x \in X$ depends on finitely many derivatives of the coefficients of D at x.

The formula for the differential form χ_0 depends on the choice of a connection on the holomorphic vector bundles $T^{1,0}X$ and E and is similar to the formula written in [6] for the canonical trace of the quantum algebra of functions in deformation quantization of symplectic manifolds. Its ingredients are the Hochschild cocycle of [6] and formal differential geometry. Let $\mathcal{D}_{n,r} = M_r(\mathcal{D}_n)$ be the algebra of r by r matrices with coefficients in the algebra of formal differential operators $\mathcal{D}_n = \mathbb{C}[[y_1, \dots, y_n]][\partial_{y_1}, \dots, \partial_{y_n}]$. By [8], the continuous Hochschild cohomology $HH^\bullet(\mathcal{D}_{n,r}, \mathcal{D}^*_{n,r})$ is one-dimensional, concentrated in degree $2n$ and is generated by a $2n$-cocycle $\tau^r_{2n}\colon \mathcal{D}_{n,r}^{\otimes(2n+1)} \to \mathbb{C}$ given in [6] by an explicit integral formula. Formal differential geometry, see [3], gives a realization of $\mathcal{D}_E(X)$ as the algebra of horizontal sections for a flat connection ∇ on the bundle of algebras $\hat{\mathcal{D}}_E = J_1E \times_G \mathcal{D}_{n,r} \to X$ with fibre $\mathcal{D}_{n,r}$. Here $J_1E \to X$ denotes the extended frame bundle, whose fibre at $x \in X$ consists of pairs of bases, one of $T^{1,0}_xX$ and one of E_x; it is a principal bundle for the group $G = GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$. More generally, let J_pE be the complex manifold of p-jets at 0 of local bundle isomorphisms $\mathbb{C}^n \times \mathbb{C}^r \to E$. These manifolds come with holomorphic G-equivariant submersions $J_{p+1} \to J_p$ with contractible fibres. The flat connection depends on the choice (unique up to homotopy) of a G-equivariant section $\phi\colon J_1E \to J_\infty E = \varprojlim J_pE$. Such sections can be constructed out of connections on J_1E. Upon local trivialization of J_1E the flat connection has the form $\nabla(\hat{D}) = d\hat{D} + [\omega, \hat{D}]$ for some 1-form ω on X with values in the first order differential operators in $\mathcal{D}_{n,r}$ and the isomorphism $\mathcal{D}_E(X) \to Ker(\nabla)$ sends D to its Taylor expansion $\hat{D} = \phi_*D$ with respect to the local coordinates and trivialization of E given by ϕ.

With these notations the formula for $\chi_0(D)$ in terms of the horizontal section $\hat{D}$ associated with D is

$$\chi_0(D) = \tau^r_{2n}(\hat{D}, \omega, \dots, \omega).$$

The multilinear form τ^r_{2n} on $\mathcal{D}_{n,r}$ is extended to differential forms with values in $\mathcal{D}_{n,r}$ by linearity: if $\omega = \sum \omega_j dx_j$ in terms of local real coordinates x_j, $j = 1, \dots, 2n$,

$$\chi_0(D) = \sum \tau^r_{2n}(\hat{D}, \omega_{j_1}, \dots, \omega_{j_{2n}})\, dx_{j_1} \wedge \cdots \wedge dx_{j_{2n}}.$$

The local objects $\hat{D}$ and ω depend on a choice of a local trivialization of J_1E, but the differential form χ_0 is globally defined as a consequence of the fact that τ^r_{2n} is basic for the action of G. Our main result is

THEOREM 1.1. – *For any $D \in \mathcal{D}_E(X)$,*

$$L(D) = \frac{1}{(2\pi i)^n} \int_X \chi_0(D).$$

Moreover, for the identity differential operator, it is known [8, 17] that the class of $\chi_0(\mathrm{Id})$ is the component of degree $2n$ of the Hirzebruch class $\mathrm{td}(T_X)\mathrm{ch}(E)$ and thus we recover the Riemann–Roch–Hirzebruch theorem. Also, the direct calculation of [6] shows that $\chi_0(\mathrm{Id})$ is the representative of the Hirzebruch class given by the Chern–Weil map in terms of the curvature of the connection on $T^{1,0}X \oplus E$ canonically associated with ϕ.

The proof of the theorem is obtained by showing that the linear functions $T_1 = L$ and $T_2 = \int_X \chi_0$ on the Hochschild 0-th homology

$$HH_0(\mathcal{D}_E(X)) = \mathcal{D}_E(X)/[\mathcal{D}_E(X), \mathcal{D}_E(X)]$$

are proportional to a third linear function T_3 constructed essentially in [4, 21]: a global differential operator $D \in \mathcal{D}_E(X)$ defines a global 0-cycle in the complex of sheaves $\mathcal{C}_\bullet(\mathcal{D}_E)$ of Hochschild chains of $\mathcal{D}_E$, which is quasi-isomorphic to the complex of sheaves $\mathbb{C}_X[2n]$ of locally constant continuous functions concentrated in degree $-2n$. Thus there is a map $T_3 \colon HH_0(\mathcal{D}_E(X)) \to H^0(X, \mathbb{C}_X[2n]) = H^{2n}(X, \mathbb{C}) \simeq \mathbb{C}$.

The statement of Theorem 1.1 was conjectured around 2001 by B. Feigin and B. Shoikhet. In the case of curves a formula for $L(D)$ in terms of residues had been found by A. Beilinson and V. Schechtman (Lemma 2.2.3 in [1], see also [19]). A formula for the normalized trace in deformation quantization of a symplectic manifold, analogous to the one of Theorem 1.1 was proposed in [6]. The proof of that formula is simpler since the space of traces is one-dimensional in that situation, so one just has to check the normalization. The difficulty here is that $HH_0(\mathcal{D}_E(X))$ is not one-dimensional in general. An indirect approach to proving that $T_1 = T_3$, proposed in [7], is to embed $\mathcal{D}_E(X)$ in a suitable complex of algebras with one-dimensional cohomology and show that both T_1 and T_3 extend to chain maps on this complex. If the Euler characteristic of E does not vanish one can then deduce from the classical Riemann–Roch–Hirzebruch theorem that $T_1 = C \cdot T_3$ for some C. The rigorous completion of this programme presents some technical difficulties but it should lead to a proof of $T_1 = T_3$ if E has non-vanishing Euler characteristic. In a very recent preprint [18], A. Ramadoss shows that the approach of [7] could be extended to the much more general case where X admits a vector bundle with non-vanishing Euler characteristic.

Our result gives in particular a different direct proof of the fact that $T_1 = T_3$, without assumptions on X or E. It does not use the Riemann–Roch–Hirzebruch theorem.

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2. Hochschild homology of the algebra of differential operators

2.1. Hochschild homology

Let A be an algebra over $\mathbb{C}$ with unit 1 and set $\bar{A} = A/\mathbb{C}1$. We denote $\bar{a}$ the class in $\bar{A}$ of $a \in A$. The Hochschild homology $HH_\bullet(A)$ of A with coefficients in the bimodule A is the homology of the (normalized) Hochschild chain complex $\cdots \xrightarrow{b} C_q(A) \xrightarrow{b} C_{q-1}(A) \xrightarrow{b} \cdots$ with

$$C_q(A) = A \otimes \bar{A}^{\otimes q}, \qquad q \geq 0,$$

and differential

$$\begin{aligned} b(a_0, \dots, a_q) = {} & \sum_{j=0}^{q-1} (-1)^j (a_0, \dots, a_j a_{j+1}, \dots, a_q) \\ & + (-1)^q (a_q a_0, a_1, \dots, a_{q-1}). \end{aligned} \tag{1}$$

Here $a_0, \dots, a_q \in A$ and we write $(a_0, \dots, a_q)$ instead of $a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_q$. For topological algebras one has to take the projective tensor product, as explained in [5], Ch. II.

Let $\mathcal{O}_n = \mathbb{C}[[y_1, \dots, y_n]]$ be the algebra of formal powers series in n variables and $\mathcal{D}_n = \mathcal{O}_n[\partial_{y_1}, \dots, \partial_{y_n}]$ the algebra of formal differential operators. Let also $\mathcal{O}_n^{\mathrm{pol}} = \mathbb{C}[y_1, \dots, y_n]$, $\mathcal{D}_n^{\mathrm{pol}} = \mathcal{O}_n^{\mathrm{pol}}[\partial_{y_1}, \dots, \partial_{y_n}]$ be the subalgebras of polynomial functions and differential operators. As shown by Feigin and Tsygan [8], the Hochschild homology of $\mathcal{D}_n^{\mathrm{pol}}$ is one-dimensional and concentrated in degree $2n$. A representative of a generator of $HH_{2n}(\mathcal{D}_n)$ in the normalized Hochschild chain complex is

$$c_{2n} = \sum_{\pi \in S_{2n}} \mathrm{sgn}(\pi)\, 1 \otimes u_{\pi(1)} \otimes \cdots \otimes u_{\pi(2n)}, \quad u_{2j-1} = \partial_{y_j},\, u_{2j} = y_j.$$

Thus there is a unique linear form on Hochschild homology whose value on c_{2n} is one. This linear form is the class of a cocycle in the complex dual to the Hochschild complex. An explicit formula for such a cocycle τ_{2n} was found in [6]. It has the following properties.

(i) τ_{2n} extends to a linear form on $\mathcal{D}_n^{\otimes(2n+1)}$ obeying the cocycle condition $\tau_{2n} \circ b = 0$, where b is the Hochschild differential, and the normalization condition: $\tau_{2n}(D_0, \dots, D_{2n}) = 0$ if $D_j = 1$ for some $j \geq 1$.

(ii) τ_{2n} is invariant under the action of $GL_n(\mathbb{C})$ on $\mathcal{D}_n$ by linear coordinate transformations. Moreover, if $a = \sum a_{jk} y_k \partial_{y_j} + b$, $a_{jk}, b \in \mathbb{C}$, then $\sum_{j=1}^{2n} (-1)^j \tau_{2n}(D_0, \dots, D_{j-1}, a, D_j, \dots, D_{2n-1}) = 0$.

(iii) $\tau_{2n}(c_{2n}) = 1$.

More generally, let $M_r(A) \simeq M_r(\mathbb{C}) \otimes A$ denote the algebra of r by r matrices with entries in an associative algebra A. Since Hochschild homology is Morita invariant, $HH_\bullet(M_r(\mathcal{D}_n)) \simeq HH_\bullet(\mathcal{D}_n)$ is also one-dimensional and is spanned by c_{2n} where we view $\mathcal{D}_n$ as a subalgebra of $M_r(\mathcal{D}_n)$ via $D \to \mathrm{Id} \otimes D$. Define a cocycle τ_{2n}^r by

$$\tau_{2n}^r(A_0 \otimes D_0, \dots, A_{2n} \otimes D_{2n}) = \mathrm{tr}(A_0 \cdots A_{2n}) \tau_{2n}(D_0, \dots, D_{2n}),$$

$A_i \in M_r(\mathbb{C})$, $D_i \in \mathcal{D}_n$. As a consequence of the properties of τ_{2n}, τ_{2n}^r obeys:

(i) τ_{2n}^r is a linear form on $M_r(\mathcal{D}_n)^{\otimes(2n+1)}$ obeying the cocycle condition $\tau_{2n}^r \circ b = 0$ and $\tau_{2n}^r(D_0, \dots, D_{2n}) = 0$ if, for some $j \geq 1$, D_j is the multiplication by a constant matrix.

(ii) τ_{2n}^r is invariant under the action of $G = GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$ where $GL_r(\mathbb{C})$ acts on $M_r(\mathcal{D}_n)$ by conjugation. Moreover, if $a = \sum a_{jk} y_k \partial_{y_j} + b$, $a_{jk} \in \mathbb{C}$, $b \in M_r(\mathbb{C})$ then

$$\sum_{j=1}^{2n} (-1)^j \tau_{2n}^r(D_0, \ldots, D_{j-1}, a, D_j, \ldots, D_{2n-1}) = 0.$$

(iii) $\tau_{2n}^r(c_{2n}) = r$.

Remark 2.1. – For any associative algebra A, denote by A_{Lie} the Lie algebra A with bracket $[a, b] = ab - ba$. Then A_{Lie} acts on $C_p(A)$ via

$$L_a(a_0, \ldots, a_p) = \sum_{j=0}^{p} (a_0, \ldots, [a, a_j], \ldots, a_p), \qquad a \in A_{\mathrm{Lie}}$$

and we have a Cartan formula $L_a = b \circ \iota_a + \iota_a \circ b$ with

$$\iota_a(a_0, \ldots, a_p) = \sum_{j=1}^{p} (-1)^{j+1} (a_0, \ldots, a_{j-1}, a, a_j, \ldots, a_p).$$

It follows that A_{Lie} acts trivially on the cohomology. The property (ii) may be rephrased as saying that τ_{2n} is G-*basic*, namely G-invariant and obeying $\tau_{2n}^r \circ \iota_a = 0$, for a in the Lie algebra of G embedded in $\mathcal{D}_{n,r}$ as a Lie algebra of first order operators.

It also follows that the cohomology class of τ_{2n}^r is invariant under coordinate transformations.

2.2. Hochschild chain complex of the sheaf of differential operators

Let $\mathcal{D}_E$ be the sheaf of differential operators on E. In terms of holomorphic coordinates and a local holomorphic trivialization of E, a local section of $\mathcal{D}_E$ has the form

$$\sum_I a_I(z_1, \ldots, z_n) \partial_{z_1}^{i_1} \cdots \partial_{z_n}^{i_n}, \qquad I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n,$$

with holomorphic matrix-valued coefficients a_I, vanishing except for finitely many multi-indices I. The sheaf $\mathcal{D}_E$ is a sheaf of locally convex algebras: for any open set $U \subset X$, the locally convex subalgebra $\mathcal{D}_E(U)^{\leq k}$ of operators of order at most k is the space of sections of some vector bundle over U and has the topology of uniform convergence on compact subsets. Then the inductive limit $\mathcal{D}_E(U) = \cup_k \mathcal{D}_E(U)^{\leq k}$ with the inductive limit topology is a complete locally convex algebra. This is the topology considered in [4]. Then one has the following result:

Theorem 2.2 ([4, 21]). – *Every point of X has a coordinate neighbourhood U such that $HH_p(\mathcal{D}_E(U)) = 0$ for $p \neq 2n$ and $HH_{2n}(\mathcal{D}_E(U))$ is one-dimensional generated by the class of*

$$c_E(U) = \sum_{\pi \in S_{2n}} \mathrm{sgn}(\pi)(1, x_{\pi(1)}, \ldots, x_{\pi(2n)}),$$

where $x_{2j-1} = \partial_{z_j}, x_{2j} = z_j$. Here we identify $x \in \mathcal{D}(U)$ with the multiple of the identity $\mathrm{Id}_r \otimes x \in M_r \otimes \mathcal{D}(U) \simeq \mathcal{D}_E(U)$, with respect to some trivialization of E.

2.3. Formal differential geometry

We recall some notions of formal differential geometry [9, 10, 11], following [3].

Let $W_n = \oplus_i \mathcal{O}_n \partial_{y_i}$ be the Lie algebra of formal vector fields and $gl_r(\mathcal{O}_n)$ denote $M_r(\mathcal{O}_n)$ viewed as a Lie algebra, with commutator bracket. The Lie algebra W_n acts on $gl_r(\mathcal{O}_n)$ by derivations and we can thus define the semidirect product

$$W_{n,r} = W_n \ltimes gl_r(\mathcal{O}_n).$$

This Lie algebra is embedded in $M_r(\mathcal{D}_n)$ (viewed as Lie algebra with commutator bracket) as a Lie subalgebra of first order differential operators. It should be regarded as the Lie algebra of infinitesimal automorphisms of the trivial bundle of rank r over a formal neighbourhood of $0 \in \mathbb{C}^n$.

A *local parametrization* of E is a holomorphic bundle isomorphism $U \times \mathbb{C}^r \to E|_V$ from the trivial bundle over some neighbourhood $U \subset \mathbb{C}^n$ of 0 to the restriction of E to some open set V. Let J_pE be the complex manifold of p-jets at $0 \in \mathbb{C}^n$ of local parametrizations. In particular, J_1E is the extended frame bundle, whose fibre at $x \in X$ is the space of pairs of bases of the holomorphic tangent space at x and the fibre of E at x respectively. The group $G = GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$ acts freely on the right on each J_pE, $p = 1, 2, \ldots$ by linear transformations of $\mathbb{C}^n \times \mathbb{C}^r$ and J_1E is a principal G-bundle over X. The complex manifolds J_pE form a projective system with surjective G-equivariant submersions $J_pE \to J_qE$, $p > q$. The projective limit $J_\infty E$ is, in the language of [3], a holomorphic *principal* $W_{n,r}$*-space*. Namely, there is a Lie algebra homomorphism $W_{n,r} \to \mathcal{V}(J_\infty E)$ from $W_{n,r}$ to the Lie algebra of holomorphic vector fields on $J_\infty E$, which is an isomorphism $W_{n,r} \to T^{1,0}_\phi J_\infty E$ at each point $\phi \in J_\infty E$. The inverse map defines a holomorphic one-form $\Omega_{\mathrm{MC}} \in \Omega^{1,0}(J_\infty E, W_{n,r})$ with values in $W_{n,r}$ and the homomorphism property is equivalent to the Maurer–Cartan equation

$$d\Omega_{\mathrm{MC}} + \frac{1}{2}[\Omega_{\mathrm{MC}}, \Omega_{\mathrm{MC}}] = 0.$$

Moreover, the fibres of the bundle $J_\infty E/G \to J_1E/G = X$ are contractible and therefore there exists a smooth section (unique up to homotopy) $\phi : X \to J_\infty E/G$ or, equivalently, a smooth G-equivariant section $\tilde{\phi} : J_1E \to J_\infty E$. The Maurer–Cartan form Ω_{MC} pulls back to a G-equivariant 1-form $\tilde{\phi}^*\Omega_{\mathrm{MC}}$ on J_1E obeying the Maurer–Cartan equation. This induces a flat connection on the associated bundle

$$\hat{\mathcal{D}}_E = J_1E \times_G M_r(\mathcal{D}_n) \to X.$$

The horizontal sections are in one-to-one correspondence with global differential operators: to $D \in \mathcal{D}_E(X)$ there corresponds the horizontal section $\hat{D}$. Its value at $x \in X$ is the Taylor expansion at 0 of D with respect to the coordinates and the trivialization defined by ϕ at the point x. Conversely, every horizontal section comes from a differential operator. In explicit terms, let us choose a local trivialization of $J_1E = U \times G$ over $U \subset X$. Then the restriction of ϕ to U is given by a map $\phi^U : U \to J_\infty E|_U$ and $\omega = \phi^*\Omega_{\mathrm{MC}}$ is a $W_{n,r}$-valued 1-form on U. The Taylor expansion $\hat{D}$ is given on U by a map $U \to M_r(\mathcal{D}_n)$, $x \mapsto \hat{D}_x$ obeying

$$d\hat{D} + [\omega, \hat{D}] = 0.$$

A change of trivialization is given by a gauge transformation $g \colon U \to G$. The section changes as $\hat{D}_x \mapsto g_x \cdot \hat{D}_x$ and ω as $\omega \mapsto g \cdot \omega - dgg^{-1}$ and dgg^{-1} is a 1-form with values in the Lie algebra of G, embedded in $M_r(\mathcal{D}_n)$ as the Lie algebra of first order operators of the form $\sum a_{jk} y_k \frac{\partial}{\partial y_j} + b$, $a_{jk} \in \mathbb{C}$, $b \in M_r(\mathbb{C})$.

Proposition 2.3. – *Let $\Omega^\bullet$ be the complex of sheaves of complex-valued smooth differential forms on X with de Rham differential and let $\mathcal{C}(\mathcal{D}_E)$ be the complex of sheaves of Hochschild chains of $\mathcal{D}_E$. There is a homomorphism of complexes of sheaves*

$$\chi_\bullet \colon \mathcal{C}_\bullet(\mathcal{D}_E) \to \Omega^{2n-\bullet},$$

depending on a choice of section of $J_\infty E/G \to X$, inducing an isomorphism of the cohomology sheaves. The map $\chi_0 \colon \mathcal{D}_E(X) \to \Omega^{2n}(X)$ on global differential operators is the map appearing in Theorem 1.1 and χ_{2n} maps $(D_0, \ldots, D_{2n}) \in \mathcal{C}_{2n}(U)$ to the function $\tau^r_{2n}(\hat{D}_0, \ldots, \hat{D}_{2n})$ on the open set U.

The rest of this section is dedicated to the construction of $\chi_\bullet$ and the proof of Proposition 2.3.

2.4. Shift by a Maurer–Cartan element

We start by a general construction on the chain complex of an arbitrary differential graded algebra. Let $A = \oplus_{j\in\mathbb{Z}} A^j$ be a differential graded algebra with unit and with differential $d \colon A^j \to A^{j+1}$. We denote by $|a| = j$ the degree of a homogeneous element $a \in A^j$. The Hochschild chain complex of A is $C^\bullet(A) = \oplus_{p\in\mathbb{Z}} C^p(A)$ with

$$C^p(A) = \Pi_{\sum j_r - q = p} A^{j_0} \otimes \bar{A}^{j_1} \otimes \cdots \otimes \bar{A}^{j_q}.$$

The differential of the Hochschild complex is defined as the total differential $\delta = b + (-1)^p d$ on $C^p(A)$.[1] The Hochschild differential b is defined as in (1) except that the last term has an additional sign $(-1)^{|a_p|(|a_0|+\cdots+|a_{p-1}|)}$ and the differential d is extended as a derivation of degree 1 for the tensor product:

$$d(a_0, \ldots, a_p) = \sum_{j=0}^{p} (-1)^{|a_0|+\cdots+|a_{j-1}|} (a_0, \ldots, da_j, \ldots, a_p).$$

A *Maurer–Cartan element* of A is an element $\omega \in A^1$ of degree 1 obeying the Maurer–Cartan equation

$$d\omega + \omega^2 = 0.$$

The Maurer–Cartan equation implies that the linear endomorphism d_ω of A given by $d_\omega a = da + \omega a - (-1)^{|a|} a\omega$ is a differential. Moreover d_ω is a derivation of degree 1 of the algebra A and therefore the algebra A with differential d_ω is a differential graded algebra. We call this differential graded algebra the twist of A by ω and denote it A_ω.

The symmetric group S_p acts on $A \otimes \bar{A}^p$ by permutations of the last p factors with signs: the transposition of neighbouring factors a and b is accompanied by the sign $(-1)^{|a|\cdot|b|}$. Recall that the shuffle product $C_p(A) \otimes C_q(A) \to C_{p+q}(A)$ is defined by

$$(a_0, \ldots, a_p) \times (b_0, \ldots, b_q) = (-1)^{|b_0| \sum |a_j|} \mathrm{sh}_{p,q}(a_0 b_0, a_1, \ldots, a_p, b_1, \ldots, b_q),$$

[1] We introduce the upper index notation C^q to have a differential of degree one as d is. Thus in the ungraded case we have $C^q(A) = C_{-q}(A)$, concentrated in negative degrees.

where $\mathrm{sh}_{p,q}x = \sum_{\pi\in S_{p,q}} \mathrm{sgn}(\pi)\pi\cdot x$, with sum over (p,q)-shuffles in S_{p+q}, namely over the permutations that preserve the ordering of the first p and of the last q letters. The shuffle product is associative and if A is Abelian (which we do not assume) it is a homomorphism of complexes, see [16, 14].

PROPOSITION 2.4. – *Let A_ω be the twist of A by a Maurer–Cartan element $\omega \in A^1$. Let $(\omega)_k = (1,\omega,\dots,\omega)$ with k factors of ω. Then the map*

$$(a_0,\dots,a_p) \to \sum_{k\geq 0}(-1)^k(a_0,\dots,a_p)\times(\omega)_k$$

is an isomorphism of complexes $C(A_\omega)\to C(A)$.

We split the proof into a few steps.

LEMMA 2.5. – *$b(\omega)_0 = 0$ and, for $k\geq 1$, $b(\omega)_k = d(\omega)_{k-1}$.*

Proof. – The first statement is obvious. Let $k\geq 1$. Then

$$\begin{aligned}
b(\omega)_k &= b(1,\omega,\dots,\omega)\\
&= (\omega,\dots,\omega) + \sum_{j=1}^{k-1}(-1)^j(1,\omega,\dots,\omega^2,\dots,\omega) + (-1)^k(-1)^{k-1}(\omega,\dots,\omega)\\
&= -\sum_{j=1}^{k-1}(-1)^j(1,\omega,\dots,d\omega,\dots,\omega) = d(\omega)_{k-1}.
\end{aligned}$$ □

LEMMA 2.6. – *Let $a\in C^p(A)$. Then*

$$\begin{aligned}
b\,(a\times(\omega)_k) &= b\,a\times(\omega)_k + (-1)^p a\times b(\omega)_k\\
&\quad -(-1)^p\sum_{k=0}^{p}(-1)^{|a_0|+\cdots+|a_{k-1}|}(a_0,\dots,[\omega,a_k],\dots,a_p)\times(\omega)_{k-1},
\end{aligned}$$

where $[a,a'] = aa' - (-1)^{|a|\cdot|a'|}a'a$ is the graded commutator.

Proof. – For simplicity, we give the proof in the case where all a_j are of degree 0, which is the case appearing in our application. The additional signs appearing in the general case can be treated easily.

If we write out the sum over shuffles we see that there are four types of terms appearing on the left-hand side: those containing the products a_ja_{j+1}, ω^2, ωa_j and $a_j\omega$. The terms of the first and of the second type combine to give the first two terms on the right-hand side. The proof that the signs match is the same as in the proof of the homomorphism property for commutative algebras, see [14], Proposition 4.2.2, so we consider only the last two types. Consider a shuffle π appearing on the left-hand side such that l out of the k factors ω have been shuffled to the left of a_j. Then the term containing the product ωa_j comes with a sign $\mathrm{sgn}(\pi)(-1)^{j-1+l}$. The same term occurs for a shuffle π' in $(a_0,\dots,[\omega,a_j],\dots,a_p)\times(\omega)_{k-1}$ with a sign equal to $\mathrm{sgn}(\pi')(-1)^{l-1}$, where $(-1)^{l-1}$ is the Koszul sign coming from the fact that $l-1$ factors ω are permuted by π' to the left of the odd element $[\omega,a_j]$. The signs of the shuffles are related by $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi')(-1)^{p-j+1}$. The ratio of signs is thus $(-1)^{p-1}$, as claimed. The same reasoning can be applied to $a_j\omega$. □

LEMMA 2.7. – *Let $a \in C^p(A)$ and set $\delta_\omega a = b\,a + (-1)^p d_\omega a$. Then*

$$\delta \sum_{k\geq 0}(-1)^k a \times (\omega)_k = \sum_{k\geq 0}(-1)^k \delta_\omega a \times (\omega)_k.$$

Proof. – This follows from the previous lemma by inserting the definitions and summing over k. □

LEMMA 2.8. – *The map $C(A_\omega) \to C(A)$ of Proposition 2.4 is an isomorphism.*

Proof. – The map is the shuffle multiplication by $\psi = \sum(-1)^k(\omega)_k$. We claim that the inverse map is the shuffle multiplication by $\bar{\psi} = \sum(\omega)_k$. To prove this, we use that the shuffle product is associative, so that it suffices to show that $\psi \times \bar{\psi} = 1$. This follows from

$$(\omega)_k \times (\omega)_l = \sum_{\pi \in S_{k,l}} (\omega)_{k+l} = \binom{k+l}{k} (\omega)_{k+l}. \qquad \square$$

Proposition 2.4 follows from the last two lemmata.

2.5. Hochschild and de Rham cohomology

We construct a homomorphism of complexes of sheaves

$$\chi_\bullet \colon \mathcal{C}_\bullet(\mathcal{D}_E) \to \Omega^{2n-\bullet}$$

from the Hochschild chain complex of the sheaf $\mathcal{D}_E$ to the sheaf of smooth de Rham forms. It is based on formal geometry and thus depends on a choice of section of $J_\infty E/G$ (but the map induced on homology is canonical). The map $\chi_0 \colon \mathcal{D}_E(X) \to \Omega^{2n}(X)$ on global differential operators is the one appearing in Theorem 1.1.

To do this we apply the previous constructions to the smooth de Rham complex $A = \Omega(U, \hat{\mathcal{D}}_E)$ with values in the vector bundle $\hat{\mathcal{D}}_E$ on some open subset $U \subset X$. Let $\hat{D} \in A^0$ denote the horizontal section corresponding to a differential operator $D \in \mathcal{D}_E(U)$. Locally, upon trivialization of $T^{1,0}X$ and E, the condition of horizontality is $d\hat{D} + [\omega, \hat{D}] = 0$ for some Maurer–Cartan element ω.

PROPOSITION 2.9. – *Let U be a sufficiently small open neighbourhood of any point in X. Let $D_0, \ldots, D_p \in \mathcal{D}_E(U)$ be differential operators on U and $\hat{D}_0, \ldots, \hat{D}_p \in A^0$ be the corresponding horizontal sections of $\hat{\mathcal{D}}_E$ on U. Then the differential $(2n-p)$-forms on U*

$$\chi_p(D_0, \ldots, D_p) = \tau_{2n}^r \left(\mathrm{sh}_{p,2n-p}(\hat{D}_0, \hat{D}_1, \ldots, \hat{D}_p, \omega, \ldots, \omega)\right), \tag{2}$$

are well-defined (i.e., independent of the trivialization of J_1E), continuous, and obey the relations

$$d \circ \chi_p = (-1)^{p-1} \chi_{p-1} \circ b.$$

Proof. – If we change trivialization of the extended frame bundle J_1E, then $\hat{D}, \omega$ change by the action of an element of G, under which τ_{2n}^r is invariant, and the shift of ω by a one-form with values in the Lie algebra of G embedded in A. By property (ii) of τ_{2n}^r, see 2.1, the right-hand side of (2) is unaffected by such a shift. The continuity is clear: since τ_{2n}^r depends non-trivially only on finitely many Taylor coefficients of its arguments, the C^ℓ-norms on compact subsets of $\chi_p(D_0, \ldots, D_p)$ are estimated by $C^{\ell'}$-norms of the coefficients of $D_0, \ldots, D_p$ which by analyticity are in turn controlled by sup norms on (slightly larger) compact subsets.

In the notation of Proposition 2.4,

$$\chi_p(D_0, \dots, D_p) = \tau_{2n}^r \left((\hat{D}_0, \dots, \hat{D}_p) \times (\omega)_{2n-p} \right).$$

The cochain $a = (\hat{D}_0, \dots, \hat{D}_p)$ obeys $d_\omega(a) = 0$, thus the homomorphism property of Proposition 2.4 reduces to

$$\delta \sum_{k \geq 0} (-1)^k a \times (\omega)_k = \sum_{k \geq 0} (-1)^k b\, a \times (\omega)_k, \qquad \delta = b \pm d.$$

The component of Hochschild degree $2n$ is

$$(-1)^{p-1} b(a \times (\omega)_{2n-p+1}) + (-1)^p (-1)^p d(a \times (\omega)_{2n-p}) = (-1)^{p-1} b\, a \times (\omega)_{2n-p+1}.$$

If we apply τ_{2n}^r the first term vanishes and we obtain the claim. □

Since the expressions on the right-hand side of (2) are local, it is clear that χ_p are compatible with the inclusion of open sets and thus define maps of sheaves. Moreover, by the normalization of τ_{2n}^r, we see that $\chi_{2n}(c_E(U)) = r$, where $c_E(U)$ is the generator of Theorem 2.2. Thus $\chi_\bullet$ induces a non-trivial map on homology. By Theorem 2.2 this map is an isomorphism. This concludes the proof of Proposition 2.3.

3. The third trace

The idea of the proof of Theorem 1.1 is to show that the two traces $T_1(D) = L(D) = \sum (-1)^j \operatorname{tr}(H^j(D))$ and $T_2(D) = \int_X \chi_0(D)$ are both proportional to a third linear form $T_3 \colon \mathcal{D}_E(X) \to \mathbb{C}$ constructed via Theorem 2.2 with the help of a finite open cover $\mathcal{U} = (U_\alpha)$.

Consider the Hochschild complex of sheaves $\mathcal{C}_\bullet(\mathcal{D}_E)$:

$$\cdots \to \mathcal{D}_E \otimes \bar{\mathcal{D}}_E \otimes \bar{\mathcal{D}}_E \to \mathcal{D}_E \otimes \bar{\mathcal{D}}_E \to \mathcal{D}_E \to 0.$$

Let $\mathcal{U} = (U_\alpha)$ be a sufficiently fine open cover of X. Let $C^{p,q} = \check{C}^q(\mathcal{U}, \mathcal{C}_{-p}(\mathcal{D}_E))$, ($q \geq 0, p \leq 0$), where $\check{C}^q(\mathcal{U}, \mathcal{F}) = \oplus_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$ be the Hochschild–Čech double complex. Global differential operators $D \in \mathcal{D}_E(X)$ define cocycles in $C^{0,0}$. The restriction $D|_{U_\alpha}$ of D to a sufficiently small open set is a Hochschild boundary by Theorem 2.2. Thus $D|_{U_\alpha} = bD^{(1)}_\alpha$ for some $D^{(1)} \in C^{-1,0}$. Since b and the Čech differential commute, $(\check{\delta} D^{(1)})_{\alpha\beta} = D^{(1)}_\beta - D^{(1)}_\alpha$ is a Hochschild cycle for the algebra $\mathcal{D}_E(U_\alpha \cap U_\beta)$ and is thus exact. Continuing in this way we can "climb the staircase", see Fig. 1, and find $D^{(j)} \in C^{-j,j-1}$, $j = 1, \dots, 2n$, such that

$$bD^{(1)} = D, \qquad \check{\delta} D^{(j)} = bD^{(j+1)}, \qquad j = 1, \dots, 2n-1. \tag{3}$$

Finally we get to the point where the Hochschild homology is nontrivial and we obtain

$$\check{\delta} D^{(2n)} = s + bD^{(2n+1)}, \tag{4}$$

where $s \in C^{2n,-2n}$ has the form

$$s_{\alpha_0, \dots, \alpha_{2n}} = \lambda_{\alpha_0, \dots, \alpha_{2n}}(D) c_E(U_{\alpha_0} \cap \cdots \cap U_{\alpha_{2n}}), \tag{5}$$

for some Čech cocycle $\lambda(D) \in \check{C}^{2n}(\mathcal{U}, \mathbb{C})$ with values in the locally constant sheaf $\mathbb{C}_X$. Its class $[\lambda(D)] \in H^{2n}(X, \mathbb{C}) \simeq \mathbb{C}$ is (up to sign) $T_3(D)$.

$$
\begin{array}{cccccc|l}
s & & & & & & 2n \\
\uparrow & & & & & & \\
D^{(2n)} \to & & & & & & 2n-1 \\
& \ddots & & & & & \\
& & \uparrow & & & & \\
& & D^{(2)} \to & & & & 1 \\
& & & \uparrow & & & \\
& & & D^{(1)} \to & D & & 0 \\
\hline
-2n & & & -1 & 0 & &
\end{array}
$$

FIGURE 1. The Hochschild–Čech double complex

4. The first trace is proportional to the third…

Here we study the first trace $T_1 = L$ and describe it in terms of local Čech data. Let $(\Omega^{(0,\bullet)}(X,E),\bar\partial)$ be the Dolbeault complex with values in the holomorphic vector bundle E. We fix hermitian metrics on T_X and on E. These metrics induce an L^2 inner product $\langle\,,\,\rangle$ on the Dolbeault complex and a self-adjoint positive semidefinite Laplace operator $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, with discrete spectrum. By Hodge theory, the cohomology group $H^j(X,E)$ is isomorphic to the space of harmonic forms $\mathrm{Ker}(\Delta_{\bar\partial})$. Moreover we have the following standard fact.

PROPOSITION 4.1. – *For any $D \in \mathcal{D}_E(X)$ and $t > 0$, $De^{-t\Delta_{\bar\partial}}$ is a trace class operator on the Hilbert space of square integrable Dolbeault forms. The expression*

$$\sum_{j=0}^{n}(-1)^j \mathrm{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar\partial}})$$

is independent of t and is equal to $T_1(D) = L(D)$.

Proof. – We refer, e.g., to [2] for the trace class property. The independence of t is checked by differentiation:

$$
\begin{aligned}
\frac{d}{dt}\,\mathrm{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar\partial}}) &= -\mathrm{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar\partial}}(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial))\\
&= -\mathrm{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar\partial}}\bar\partial\bar\partial^*) - \mathrm{tr}_{\Omega^{(0,j-1)}(X,E)}(\bar\partial De^{-t\Delta_{\bar\partial}}\bar\partial^*)\\
&= -\mathrm{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar\partial}}\bar\partial\bar\partial^*) - \mathrm{tr}_{\Omega^{(0,j-1)}(X,E)}(De^{-t\Delta_{\bar\partial}}\bar\partial\bar\partial^*).
\end{aligned}
$$

Here we use the fact that $\bar\partial$ commutes both with D (since D is holomorphic) and with the Laplacian. Taking the sum with alternating signs yields the claim.

Thus we can evaluate the sum in the limit $t \to \infty$. Since 0 is an isolated eigenvalue of the positive semidefinite operator $\Delta_{\bar\partial}$ we obtain the alternating sum of traces on harmonic forms, namely $L(D)$. □

4.1. The σ-cocycle

We introduce our main technical tool, a cocycle in a double complex associated to an open set U. Here we describe its properties and postpone its construction by heat kernel methods to Section 6.

Let U be a sufficiently small open neighbourhood of an arbitrary point in X. Let $A = \mathcal{D}_E(U)$ and let $B = C^\infty(U)$ be the algebra of smooth complex valued functions on U. Consider further $C_p(A) = A \otimes \bar{A}^{\otimes p}$ with Hochschild differential b of degree -1 and $C_p(B) = B \otimes \bar{B}^{\otimes p}$ with differential s of degree $+1$ given by

$$s(\rho_0 \otimes \cdots \otimes \rho_p) = 1 \otimes \rho_0 \otimes \cdots \otimes \rho_p.$$

Let $C_p^c(B)$ be the subcomplex spanned by $(\rho_0, \ldots, \rho_p)$ with compact common support $\cap_i \mathrm{supp}(\rho_i)$. Let us denote by $[f(t)]_- = a_{-N}t^{-N} + \cdots + a_{-1}t^{-1} + a_0$ the non-positive part of an asymptotic Laurent series $f(t) \sim \sum_{j \geq -N} a_j t^j$ in t.

PROPOSITION 4.2. – *Let $U \subset X$ be an open set. Let $A = \mathcal{D}_E(U)$, $B = C^\infty(U)$. For each choice of hermitian metrics on T_X and E, there exist linear maps*

$$\sigma_p \colon C_p(A) \otimes C_p^c(B) \to \mathbb{C}[t^{-1}],$$

such that the coefficients of $\sigma_p(D_0, \ldots D_p; \rho_0, \ldots, \rho_p)$ are continuous in $(D_0, \ldots, D_p)$ and satisfy the following

(i) *Let $C_p^{\varnothing}(B)$ be the subcomplex spanned by $(\rho_0, \ldots, \rho_p)$ with empty common support $\cap_{i=0}^p \mathrm{supp}(\rho_i)$. Then σ_p vanishes on $C_p(A) \otimes C_p^{\varnothing}(B)$.*

(ii) *For any $D \in C_{p+1}(A)$ and $\rho \in C_p^c(B)$,*

$$\sigma_p(bD \otimes \rho) = \sigma_{p+1}(D \otimes s\rho), \qquad p \geq 0,$$

(iii) $\sigma_0(D, \rho) = \left[\sum_{j=0}^n (-1)^j \mathrm{tr}_{\Omega^{(0,j)}(U,E)}(\rho D e^{-t\Delta_{\bar\partial}})\right]_-$, $(n = \dim_{\mathbb{C}}(X))$.

(iv) *Suppose that U is some coordinate neighbourhood of a point and let $c_E(U)$ be the cocycle appearing in Theorem 2.2. Assume further that $\rho_0, \ldots, \rho_{2n} \in C_c^\infty(U)$ are functions such that the metrics on T_X and E are flat on some neighbourhood of $\cap_{i=0}^{2n} \mathrm{supp}(\rho_i)$. Then*

$$\sigma_{2n}(c_E(U); \rho_0, \ldots, \rho_{2n}) = \frac{r}{(2\pi i)^n} \int_U \rho_0 d\rho_1 \cdots d\rho_{2n},$$

where r is the rank of E.

The proof is contained in Section 6. We first show how to use it to prove that T_1 is proportional to T_3.

4.2. A local formula for the Lefschetz number

Here it is useful to replace the open cover considered in Section 3 by a refinement obtained from a triangulation of X. Then the Hochschild–Čech cochains $(D^{(j)})$, constructed in Section 3 out of a global differential operator D, define cochains, still denoted by $(D^{(j)})$ for the refinement. Choose a smooth finite triangulation $|K| \to X$ of X, with underlying simplicial complex K, with fixed total ordering of its vertices. The *open star* of the triangulation is the open cover $\mathcal{U} = (U_\alpha)_{\alpha \in K_0}$ of X labeled by the set of vertices of the triangulation, such that U_α is the complement of the simplices not containing α. By construction, for all $\alpha_0 < \cdots < \alpha_p$,

(a) $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ is empty or contractible.
(b) If $p>2n$, then $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ is empty.

LEMMA 4.3. – *Let (ρ_α) be a partition of unity subordinate to the open covering (U_α). Let $D\in\mathcal{D}_E(X)$ and $s\in\check{C}^{2n}(\mathcal{U},\mathcal{C}_{2n}(\mathcal{D}_E))$ be the cocycle (5). Then*

$$\sum_{p=0}^{2n}(-1)^p\mathrm{tr}(H^p(D))=\sum_{\alpha_0<\cdots<\alpha_{2n}}\sigma_{2n}(s_{\alpha_0,\dots,\alpha_{2n}};\rho_{\alpha_0,\dots,\alpha_{2n}}).$$

Here we use the abbreviation

$$\rho_{\alpha_0,\dots,\alpha_q}=\sum_{\pi\in S_{q+1}}\mathrm{sgn}(\pi)\rho_{\alpha_{\pi(0)}}\otimes\cdots\otimes\rho_{\alpha_{\pi(q)}}.$$

Proof. – Out of D we construct the cochains $D^{(j)}$ obeying (3).

$$\begin{aligned}T_1(D)&=\sum_{j=0}^{n}(-1)^j\left[\mathrm{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar\partial}})\right]_-\\&=\sum_\alpha\sum_{j=0}^{n}(-1)^j\left[\mathrm{tr}_{\Omega^{(0,j)}(X,E)}(\rho_\alpha De^{-t\Delta_{\bar\partial}})\right]_-\\&=\sum_\alpha\sigma_0(D_\alpha;\rho_\alpha),\qquad D_\alpha=D|_{U_\alpha}\in\mathcal{D}_E(U_\alpha).\end{aligned}$$

Now $D_\alpha=bD^{(1)}_\alpha$ and Proposition 4.2 (ii) implies

$$\begin{aligned}T_1(D)&=\sum_\alpha\sigma_1(D^{(1)}_\alpha;1,\rho_\alpha)\\&=\sum_{\alpha,\beta}\sigma_1(D^{(1)}_\alpha;\rho_\beta,\rho_\alpha)\\&=\sum_{\alpha\neq\beta}\sigma_1(D^{(1)}_\alpha;\rho_\beta,\rho_\alpha)+\sum_\beta\sigma_1(D^{(1)}_\beta;\rho_\beta,\rho_\beta)\\&=\sum_{\alpha\neq\beta}\sigma_1(D^{(1)}_\alpha-D^{(1)}_\beta;\rho_\beta,\rho_\alpha).\end{aligned}$$

In the last step we have replaced the last occurrence of ρ_β by $-\sum_{\alpha\neq\beta}\rho_\alpha \mod \mathbb{C}1$. We see that $(\check\delta D^{(1)})_{\beta,\alpha}$ appears. Thus we can iterate the procedure. At the q-th step we obtain similarly for $q<2n$,

$$\begin{aligned}\sum_{\alpha_0<\cdots<\alpha_q}\sigma_q(\check\delta D^{(q)}_{\alpha_0,\dots,\alpha_q};\rho_{\alpha_0,\dots,\alpha_q})&=\sum_{\alpha_0<\cdots<\alpha_q}\sigma_q(bD^{(q+1)}_{\alpha_0,\dots,\alpha_q};\rho_{\alpha_0,\dots,\alpha_q})\\&=\sum_{\alpha_0<\cdots<\alpha_q}\sigma_{q+1}(D^{(q+1)}_{\alpha_0,\dots,\alpha_q};1\otimes\rho_{\alpha_0,\dots,\alpha_q})\\&=\sum_{\alpha_0<\cdots<\alpha_{q+1}}\sigma_{q+1}(\check\delta D^{(q+1)}_{\alpha_0,\dots,\alpha_{q+1}};\rho_{\alpha_0,\dots,\alpha_{q+1}}).\end{aligned}$$

If $q = 2n$ we have an additional term containing s and we obtain

$$T_1(D) = \sum_{\alpha_0<\cdots<\alpha_{2n}} \sigma_{2n}(s_{\alpha_0,\ldots,\alpha_{2n}}; \rho_{\alpha_0,\ldots,\alpha_{2n}}) + \sum_{\alpha_0<\cdots<\alpha_{2n+1}} \sigma_{2n+1}(\check{\delta} D^{(2n+1)}_{\alpha_0,\ldots,\alpha_{2n+1}}; \rho_{\alpha_0,\ldots,\alpha_{2n+1}}).$$

Since there are no non-empty $(2n+2)$-fold intersections, $(\rho_{\alpha_0}, \ldots, \rho_{2n+1})$ belongs to $C^{\varnothing}(B)$ and therefore, by Proposition 4.2, (i), the second term vanishes. □

Let us now choose the hermitian metrics so that they are flat on the disjoint closed sets $\cap_{j=0}^{2n}\mathrm{supp}(\rho_{\alpha_i})$, $\alpha_0 < \cdots < \alpha_{2n}$. By Proposition 4.2, (iv), we then obtain

$$\sum_{p=0}^{2n}(-1)^p \mathrm{tr}(H^p(D)) = (2n+1)!\frac{r}{(2\pi i)^n} \sum_{\alpha_0<\cdots<\alpha_{2n}} \lambda_{\alpha_0,\ldots,\alpha_{2n}}(D) \int_X \rho_{\alpha_0} d\rho_{\alpha_1} \cdots d\rho_{\alpha_{2n}}.$$

Now the common support of the functions ρ_{α_i} in each summand is contained in a simplex $\sigma_{\alpha_0,\ldots,\alpha_{2n}}$. Moreover each of the functions vanishes on the corresponding face and $\sum_{i=0}^{2n} \rho_{\alpha_i} = 1$ on some neighbourhood of the simplex. Therefore the integral may be evaluated as follows.

LEMMA 4.4. – *Let $H_p \in \mathbb{R}^{p+1}$ be the hyperplane $\sum_{i=0}^{p} t_i = 1$ and $\Delta_p = H_p \cap [0,1]^{p+1}$ the standard simplex, with (standard) orientation given by the ordered basis $\partial_{t_1}, \ldots, \partial_{t_p}$. Let $\rho_0, \ldots, \rho_p$ be smooth functions defined on some open neighbourhood $U \subset H_p$ of Δ_p such that $\rho_0 + \cdots + \rho_p = 1$ and $\rho_i(t) = 0$ if $t_i \leq 0$. Then*

$$\int_{\Delta_p} \rho_0 d\rho_1 \cdots d\rho_p = \frac{1}{(p+1)!}.$$

Proof. – We prove by induction in p the more general formula

$$\int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_p = \frac{k!}{(p+k)!}, \qquad k = 0, 1, 2, \ldots$$

This formula trivially holds for $p = 0$. By the Stokes theorem,

$$\begin{aligned}\int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_p &= -\int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_{p-1} d\rho_0 \\ &= (-1)^p \frac{1}{k+1} \int_{\Delta_p} d(\rho_0^{k+1} d\rho_1 \cdots d\rho_{p-1}) \\ &= (-1)^p \frac{1}{k+1} \int_{\partial\Delta_p} \rho_0^{k+1} d\rho_1 \cdots d\rho_{p-1}\end{aligned}$$

Since ρ_j vanishes on the j-th face of Δ_p, only the p-th face (where $t_p = 0$) contributes. This face is Δ_{p-1} and the restriction of $\rho_0, \ldots, \rho_{p-1}$ obeys the assumptions of the lemma. Taking into account the sign $(-1)^p$ relating the orientation of Δ_{p-1} to the induced orientation, we obtain

$$\int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_p = \frac{1}{k+1} \int_{\Delta_{p-1}} \rho_0^{k+1} d\rho_1 \cdots d\rho_{p-1},$$

proving the induction step. □

COROLLARY 4.5. – *Let* $\epsilon(\alpha_0,\dots,\alpha_{2n}) \in \{-1,1\}$ *be the orientation of the simplex* $\sigma_{\alpha_0,\dots,\alpha_{2n}}$ *relative to the canonical orientation of* X. *Then*

$$T_1(D) = \frac{r}{(2\pi i)^n} \sum_{\alpha_0<\cdots<\alpha_{2n}} \lambda_{\alpha_0,\dots,\alpha_{2n}}(D)\epsilon(\alpha_0,\dots,\alpha_{2n}).$$

5. …and so is the second

Let $T_2(D) = \int_X \chi_0(D)$ be the second trace. Let C be the cell decomposition of X dual to the triangulation of subsection 4.2. Its cells are in one-to-one correspondence with the simplices of the triangulation. We denote by $C_{\alpha_0,\dots,\alpha_p}$ the $(2n-p)$-cell corresponding to the simplex $\sigma_{\alpha_0,\dots,\alpha_p}$ with vertices $\alpha_0,\dots,\alpha_p$. We orient the dual cells by the condition that $C_{\alpha_0,\dots,\alpha_p}\cdot\sigma_{\alpha_0,\dots,\alpha_p} = 1$ on the intersection index (see Appendix A).

PROPOSITION 5.1. – *Let* $s = s(D)$ *be the cocycle* (5). *Then*

$$T_2(D) = \sum_{\alpha_0<\cdots<\alpha_{2n}} \int_{C_{\alpha_0,\dots,\alpha_{2n}}} \chi_{2n}(s_{\alpha_0,\dots,\alpha_{2n}}),$$

where χ_{2n} *is defined in Proposition* 2.9 *for the open set* $U_{\alpha_0}\cap\cdots\cap U_{\alpha_{2n}}$.

Proof. – We first prove by induction that for all $p = 0,\dots,2n-1$,

$$T_2(D) = \sum_{\alpha_0<\cdots<\alpha_p} \int_{C_{\alpha_0,\dots,\alpha_p}} \chi_p(bD^{(p+1)}_{\alpha_0,\dots,\alpha_p}), \tag{6}$$

and then deduce the claim by doing a further induction step. For $p = 0$, Equation (6) follows from

$$T_2(D) = \sum_{\alpha} \int_{C(\alpha)} \chi_0(D|_{U_\alpha}),$$

and $D|_{U_\alpha} = bD^{(1)}_\alpha$. Assume that the claim is proved up to some $p < 2n-1$. Then, by Proposition 2.3 and the Stokes theorem (the signs are discussed in the appendix, see (15)), we get

$$\begin{aligned} T_2(D) &= \sum_{\alpha_0<\cdots<\alpha_p} \int_{C_{\alpha_0,\dots,\alpha_p}} \chi_p(bD^{(p+1)}_{\alpha_0,\dots,\alpha_p}) \\ &= (-1)^p \sum_{\alpha_0<\cdots<\alpha_p} \int_{C_{\alpha_0,\dots,\alpha_p}} d\chi_{p+1}(D^{(p+1)}_{\alpha_0,\dots,\alpha_p}) \\ &= (-1)^p(-1)^p \sum_{\beta,\alpha_0<\cdots<\alpha_p} \int_{C_{\beta,\alpha_0,\dots,\alpha_p}} \chi_{p+1}(D^{(p+1)}_{\alpha_0,\dots,\alpha_p}) \\ &= \sum_{\alpha_0<\cdots<\alpha_{p+1}} \int_{C_{\alpha_0,\dots,\alpha_{p+1}}} \chi_{p+1}((\check{\delta}D^{(p+1)})_{\alpha_0,\dots,\alpha_{p+1}}). \end{aligned}$$

Since $\check{\delta}D^{(p+1)} = bD^{(p+2)}$ if $p < 2n-1$ the induction step is complete.

Now we do this step once more for $p = 2n - 1$. The calculation is the same but the conclusion is different since $\check{\delta} D^{(2n)} = s + bD^{(2n+1)}$. We obtain

$$T_2(D) = \sum_{\alpha_0 < \cdots < \alpha_{2n}} \int_{C(\alpha_0, \ldots, \alpha_{2n})} \chi_{2n}((s + bD^{(2n+1)})_{\alpha_0, \ldots, \alpha_{2n}}).$$

Moreover, χ_{2n} coincides with τ_{2n} composed with the Taylor expansion and thus is a cocycle, i.e., it vanishes on exact chains such as $bD^{(2n+1)}$.

The integral over the 0-dimensional cycle $C_{\alpha_0, \ldots, \alpha_{2n}}$ is the evaluation of the integrand times the sign of the orientation, that is the sign $\epsilon(\alpha_0, \ldots, \alpha_{2n})$ of the orientation of $\sigma_{\alpha_0, \ldots, \alpha_{2n}}$ relative to the orientation of X. □

Corollary 5.2. – *We have*

$$T_2(D) = r \sum_{\alpha_0 < \cdots < \alpha_{2n}} \lambda_{\alpha_0, \ldots, \alpha_{2n}}(D) \epsilon(\alpha_0, \ldots, \alpha_{2n}).$$

5.1. Proof of Theorem 1.1

Recall that $T_1(D) = L(D)$ and that $T_2(D) = \int_X \chi_0(D)$. Theorem 1.1 follows from Corollary 4.5 and Corollary 5.2. The missing step is the proof of Proposition 4.2, which appears in the next section.

6. Asymptotic topological quantum mechanics

In this section we prove Proposition 4.2 and give in particular the construction of σ_p. Roughly speaking, σ_p is the cup product of a cochain Ψ, constructed using topological quantum mechanics and a cochain Z taking care of the partition of unity. The formula for Ψ is a version of the JLO cocycle [12] and is a regularized version of a cocycle appearing in "topological quantum mechanics" [15, 7]. It is constructed with heat kernel methods. Here we need only the asymptotic behaviour of these objects as time (or inverse temperature [12]) tends to zero, which allows us to replace the heat kernel by a better behaved parametrix with support in a neighbourhood of the diagonal.

We work in the context of Section 4 and fix in particular hermitian metrics on the holomorphic vector bundles $T^{1,0}X$ and E.

6.1. A parametrix for the heat equation

We summarize here what we need about the heat kernel and refer to [2] for more details and proofs. The heat operator $e^{-t\Delta_{\bar\partial}}$ is an integral operator with kernel $k_t \in \oplus_p \Gamma(X \times X, E^{0,p} \boxtimes (E^{0,p})^*)$, where $E^{0,p} = \wedge^p (T^{0,1}X)^* \otimes E$: for any smooth section $\phi \in \Omega^{0,\bullet}(X, E)$,

$$e^{-t\Delta_{\bar\partial}} \phi(z) = \int_X k_t(z, z') \cdot \phi(z') |dz'|, \qquad t > 0,$$

is the solution of the heat equation $\partial_t u + \Delta_{\bar\partial} u = 0$ with initial data ϕ. Here $|dz'|$ denotes the Riemannian volume form. Let $d(z, z')$ denote the geodesic distance between $z, z' \in X$. Then the heat kernel has an asymptotic expansion as $t \to 0$,

$$k_t(z, z') \sim \frac{1}{(\pi t)^n} e^{-\frac{d(z,z')^2}{t}} \left(\Phi_0(z, z') + t\Phi_1(z, z') + t^2 \Phi_2(z, z') + \cdots \right). \tag{7}$$

The smooth kernels $\Phi_j(z,z')$ can be chosen to vanish except on an arbitrary small neighbourhood $d(z,z') < \varepsilon$ of the diagonal. The precise meaning of the expansion is that if k_t^N is the truncation of the series at the N-th term and $\|\cdot\|_\ell$ denotes the C^ℓ-norm on sections of the hermitian bundle $E^{0,p} \boxtimes (E^{0,p})^*$ on $X \times X$; then for all $\ell, j, \alpha \geq 0$ and N sufficiently large, depending on ℓ, j, α,

$$\|\partial_t^j(k_t - k_t^N)\|_\ell = O(t^\alpha), \qquad \|(\partial_t + \Delta_{\bar\partial})k_t^N\|_\ell = O(t^\alpha). \tag{8}$$

Also, with the same hypotheses, for any smooth section ϕ,

$$\lim_{t\to 0^+} \|K_t^N\phi - \phi\|_\ell = 0, \tag{9}$$

where K_t^N denotes the integral operator with kernel k_t^N.

6.2. Hochschild cohomology

Let A be an associative algebra with unit and let $(M = \oplus M^j, d_M)$ be a complex of A-bimodules such that $M^j = 0$ for all but finitely many j. Recall that the Hochschild cochain complex $C^\bullet(A,M)$ with values in M is the total complex of the double complex

$$C^{p,q}(A,M) = \mathrm{Hom}(A^{\otimes p}, M^q)$$

and differential $\delta = d_H + (-1)^p d_M \colon C^{p,q} \to C^{p+1,q} \oplus C^{p,q+1}$ with

$$\begin{aligned} d_H\varphi(a_1,\dots,a_{p+1}) = {} & a_1\varphi(a_2,\dots,a_{p+1}) \\ & + \sum_{l=1}^{p}(-1)^l\varphi(a_1,\dots,a_la_{l+1},\dots,a_{p+1}) \\ & +(-1)^{p+1}\varphi(a_1,\dots,a_p)a_{p+1}. \end{aligned}$$

The complex of A-bimodules dual to M is $(M^* = \oplus(M^*)^j, d_{M^*})$ with $(M^*)^j = (M^{-j})^*$, $d_{M^*}\varphi = (-1)^j\varphi\circ d_M$ for $\varphi \in (M^j)^*$ and action of A defined by $a\cdot\varphi(x) = \varphi(xa), \varphi\cdot a(x) = \varphi(ax), a \in A, x \in M$. With these definitions, $C^\bullet(A,A^*)$ is the complex dual to the Hochschild chain complex $C_\bullet(A)$.

With any homomorphism $\bullet\colon M_1 \otimes_A M_2 \to M_3$ of complexes of A-bimodules is associated a chain map, the *cup product* $\cup\colon C^{p,q}(A,M_1)\otimes C^{p',q'}(A,M_2) \to C^{p+q,p'+q'}(A,M_3)$,

$$\varphi\cup\psi(a_1,\dots,a_{p+q}) = (-1)^{qp'}\varphi(a_1,\dots,a_p)\bullet\psi(a_{p+1},\dots,a_{p+q}).$$

We will use this construction in two special cases: (a) $M_1 = M_2 = M_3 = M$ is a differential graded algebra whose product factors through $M\otimes_A M$ defining thus a map $\bullet\colon M\otimes_A M \to M$. (b) $M_1 = M$ is a complex of A-bimodules, $M_2 = M^*$, $M_3 = A^*$ with zero differential and $\bullet\colon M^*\otimes_A M \to A^*$ is the map $(\varphi, x)\mapsto (y\mapsto \varphi(xy))$.

6.3. A JLO-type cocycle in the Hochschild–Dolbeault double complex

Let U be an open subset of X and $A = \mathcal{D}_E(U)$ be the algebra of differential operators on the restriction of E to U. The Dolbeault complex $(M_c(U) = \Omega_c^{0,\bullet}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{D}_E(U), \bar\partial\otimes\mathrm{id})$ with compact support and values in $\mathcal{D}_E$ is a locally convex differential graded algebra and an A-bimodule. In local coordinates it is the graded algebra generated by $M_r(C_c^\infty(U))$ of degree 0, $d\bar z_i$ of degree 1 and ∂_{z_i} of degree zero. The algebra $M_c(U)$ is the inductive limit over j and K of the locally convex subalgebras $M_{K,j}$ of operators of order at most j and with support on a compact subset $K \subset U$. The space $M_{K,j}$ is the space of sections $x \to D_x$ of some vector bundle on U with support in K, and has the topology defined by the system of seminorms given by the C^ℓ-norms, for all ℓ.

PROPOSITION 6.1. – *Let $U \subset X$ be an open subset, $A = \mathcal{D}_E(U)$ and $M_c = M_c(U)$ be the Dolbeault complex with values in A and compact support. Let k_s^N be a parametrix, with support in some small neighbourhood of the diagonal, obtained by truncating the formal series (7) at the N-th term. Suppose that $D_0 \in M_c^p$, $D_1, \dots, D_p \in A$. Then, for any sufficiently large N,*

$$\Psi_p(D_0, \dots, D_p) \\ = (-1)^{\frac{p(p+1)}{2}} \left[\int_{t\Delta_p} \mathrm{Str}(D_0 K_{s_0}^N [\bar\partial^*, D_1] K_{s_1}^N \cdots [\bar\partial^*, D_p] K_{s_p}^N) ds_1 \cdots ds_p \right]_- ,$$

where Str *denotes the alternating sum of traces over the Hilbert space of square integrable sections of $\wedge (T^{0,1}U)^* \otimes E|_U$, is independent of N for large N and defines a continuous cocycle*

$$\Psi = \sum_p \Psi_p \in \oplus_{p=0}^n \mathrm{Hom}(M_c^p \otimes \bar A^{\otimes p}, \mathbb{C})[t^{-1}] \simeq C^0(A, M_c^*)[t^{-1}].$$

Proof. – The alternating trace $\mathrm{Str}(D_0 K_{s_0}^N [\bar\partial^*, D_1] K_{s_1}^N \cdots [\bar\partial^*, D_p] K_{s_p}^N)$ is the integral $\int_X \alpha_p |dx|$ of some function $\alpha_p \in C^\infty(X \times \Delta_p)$ with support in some neighbourhood of the support of D_0. This function has the form

$$\alpha_p(x, s) = \int_{X^p} \mathrm{str} \left(D_0 \prod_{j=0}^p [\bar\partial^*, D_j] k_{s_j}^N (x_j, x_{j+1}) \right) \prod_{j=1}^p |dx_j|, \quad x_0 = x_{p+1} = x,$$

where the differential operators $[\bar\partial^*, D_j]$ act with derivatives with respect to x_j (the product is the composition of linear maps in the conventional order). The supertrace str is the alternating sum of traces over the fibres $\wedge^j T_x^{0,1} X^* \otimes E_x$ at $x \in U$. The integral is actually over a small neighbourhood of $(x, \dots, x) \in X^p$. Since $k_s^N(z, z')$ is a smooth kernel, $\alpha_p(x, s)$ is smooth for s in the interior of the simplex $t\Delta_p$. It is also continuous on its boundary for any fixed t, uniformly in x, as can be seen using (9). By rescaling $s = ts'$ we see that $\int_{t\Delta_p} \alpha_p(x, s) \prod ds_i$ has an asymptotic expansion as a Laurent series in t whose singular part is not affected by corrections of order s^{N+1} to k_s^N for large enough N. Thus the expression for Ψ_p is independent of N for N large enough. For further details see appendix B.

The proof of the cocycle relation is similar to the proof in [12]. The Hochschild differential $d_H \Psi_p$ can be written as the alternating sum of integrals of a differential form on $X \times t\partial_i \Delta_{p+1}$, where $\partial_i \Delta_{p+1}$ is the i-th face $s_i = 0$ of the simplex Δ_{p+1}. Using the Stokes theorem and heat equation for $k_{s_i}^N$ (which holds up to terms we can neglect by (8)) to compute the differential with respect to s we obtain

$$\Psi_p(D_0 D_1, \dots, D_{p+1}) - \Psi_p(D_0, D_1 D_2, \dots, D_{p+1}) + \cdots \\ + (-1)^{p+1} \Psi_p(D_{p+1} D_0, \dots, D_p) = \Psi_{p+1}([\bar\partial, D_0], \dots, D_{p+1}),$$

which is the claim. □

6.4. Construction of σ_p

Let now $\rho_0, \ldots, \rho_p \in C^\infty(U)$. View $C^\infty(U)$ as a subalgebra of $M = \Omega^{0,\bullet}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{D}_E(U)$ embedded as $C^\infty(U) \otimes \mathrm{id}$. Since $C^0(A, M) = M$, we may consider ρ_i as a 0-cochain and define

$$Z^p(\rho_0, \ldots, \rho_p) = \rho_0 \cup \delta\rho_1 \cup \cdots \cup \delta\rho_p \in C^p(A, M),$$

where the cup product is defined using the product $M \otimes_A M \to M$. Clearly

$$\delta Z^p(\rho_0, \ldots, \rho_p) = Z^{p+1}(1, \rho_0, \ldots, \rho_p). \tag{10}$$

If $\cap_i \mathrm{supp}(\rho_i)$ is compact, then $Z^p(\rho_0, \ldots, \rho_p)$ takes values in differential operators with compact support and therefore is a cochain in $C^p(A, M_c)$.

Let $\cup\colon C^\bullet(A, M_c^*) \otimes C^\bullet(A, M_c) \to C^\bullet(A, A^*)$ be the cup product associated with the map $M_c^* \otimes_A M_c \to A^*$ sending $\varphi \otimes x$ to the linear form $a \mapsto \varphi(xa)$. We set

$$\sigma_p(\rho_0, \ldots, \rho_p) = \Psi \cup Z^p(\rho_0, \ldots, \rho_p) \in C^p(A, A^*)[t^{-1}].$$

6.5. Proof of Proposition 4.2

Claim (ii) follows from the fact that Ψ is a cocycle and Equation (10). To prove the remaining claims let us write σ_p more explicitly:

$$\begin{aligned}\sigma_p(D_0, \ldots, D_p; \rho_0, \ldots, \rho_p) \\ = \sum_{j=0}^{p} (-1)^{j(p-j)} \Psi_j(Z^p_{p-j}(D_{j+1}, \ldots, D_p; \rho_0, \ldots, \rho_p) D_0, D_1, \ldots, D_j).\end{aligned} \tag{11}$$

The component Z^p_{p-j} in $\mathrm{Hom}(\bar{A}^{\otimes p-j}, M_c^j)$ of Z^p is given by

$$Z^p_{p-j}(D_{j+1}, \ldots, D_p; \rho_0, \ldots, \rho_p) = \sum_{\pi \in S_{p-j,j}} \mathrm{sgn}(\pi) \rho_0 B_{\pi(1)}(\rho_1) \cdots B_{\pi(p)}(\rho_p),$$

where $B_i(\rho) = [D_{j+i}, \rho]$ for $i = 1, \ldots, p-j$, and $B_i(\rho) = [\bar\partial, \rho]$ for $i = p-j+1, \ldots, p$. From these expressions it is clear that (i) and (iii) hold. For (iii) see also Appendix B, Remark B.2. Let us turn to (iv). We need to evaluate $\sigma_{2n}(c_E(U); \rho_0, \ldots, \rho_{2n})$. By multiplying ρ_0 by a partition of unity we may assume that the support of ρ_0 is contained in a small coordinate neighbourhood of a point. We have to compute a sum of $(2n)!$ terms of the form (11) where $D_0 = 1$ and the remaining D_k are partial derivatives ∂_{z_i} or operators of multiplication by z_i. The arguments D_k occurring in Z^{2n}_{2n-j} appear in the combination $[D_k, \rho_l]$ which vanishes if $D_k = z_i$. Therefore the only non-vanishing terms in the sum (11) have $j \geq n$ and $D_{j+1}, \ldots, D_{2n}$ are all derivatives ∂_{z_i}. On the other hand, if $j > n$ then Z^{2n}_{2n-j} vanishes since a product of more than n $(0,1)$-forms is zero. Thus only the term with $j = n$ survives and we have (setting $\partial_i = \partial_{z_i}$)

$$Z^{2n}_n(\partial_{i_1}, \ldots, \partial_{i_n}; \rho_0, \ldots, \rho_{2n}) = \rho_0 \frac{\partial \rho_1}{\partial z_{i_1}} \cdots \frac{\partial \rho_n}{\partial z_{i_n}} \bar\partial\rho_{n+1} \cdots \bar\partial\rho_{2n} + \cdots$$

where the dots denote the remaining shuffles. Therefore

$$\sigma_{2n}(c_E(U); \rho_0, \ldots, \rho_{2n}) = (-1)^{n(n+1)/2}(-1)^n \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \Psi_n(B, z_{\pi(1)}, \ldots, z_{\pi(n)}), \tag{12}$$

where B is the multiplication operator

$$B = \sum_{\pi \in S_{2n}} \mathrm{sgn}(\pi)\rho_0 \frac{\partial \rho_{\pi(1)}}{\partial z_1} \cdots \frac{\partial \rho_{\pi(n)}}{\partial z_n} \frac{\partial \rho_{\pi(n+1)}}{\partial \bar{z}_1} \cdots \frac{\partial \rho_{\pi(2n)}}{\partial \bar{z}_n} \, d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

The sign $(-1)^{n(n+1)/2}$ is the sign of the permutation mapping $(\partial_1, z_1, \ldots, \partial_n, z_n)$ to $(z_1, \ldots, z_n, \partial_1, \ldots, \partial_n)$; the sign $(-1)^n$ is the sign appearing in (11) for $j = n$, $p = 2n$. Note that since B is the operator of multiplication by a $(0, n)$-form, the only trace appearing in the alternating sum defining Ψ_n is the trace over $\Omega^{0,n}$ and it comes with a sign $(-1)^n$. Let us calculate $\Psi_n(B, z_1, \ldots, z_n)$. The calculation for all other permutations is similar and gives the same contribution to the sum over S_n.

$$\begin{aligned}
&\Psi_n(B, z_1, \ldots, z_n) \\
&\qquad = (-1)^n (-1)^{n(n+1)/2} \int_{t\Delta_n} \mathrm{tr}_{\Omega^{0,n}} (B K_{s_0} [\bar{\partial}^*, z_1] K_{s_1} \cdots [\bar{\partial}^*, z_n] K_{s_n}) \, ds_1 \cdots ds_n.
\end{aligned}$$

With our assumption on the metrics, the heat kernel is the standard heat kernel on $\mathbb{C}^n$. In this case

$$\bar{\partial} = \sum d\bar{z}_i \frac{\partial}{\partial \bar{z}_i}, \qquad \bar{\partial}^* = -\sum \frac{\partial}{\partial z_i} \iota_{\frac{\partial}{\partial \bar{z}_i}}, \quad \Delta_{\bar{\partial}} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j},$$

where ι denotes interior multiplication. Thus $\Delta_{\bar{\partial}}$ is -4 times the standard Laplacian and the kernel of K_t is

$$k_t(z, z') = \frac{1}{(\pi t)^n} e^{-\frac{|z - z'|^2}{t}}.$$

Now $[\bar{\partial}^*, z_i] = -\iota_{\partial/\partial \bar{z}_i}$, which commutes with K_t. The heat operators combine to $K_{s_0} \cdots K_{s_n} = K_t$, since $\sum s_i = t$ on $t\Delta_n$. The product $(-\iota_{\partial/\partial \bar{z}_1}) \cdots (-\iota_{\partial/\partial \bar{z}_n})$ acting on the basis $d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ gives $(-1)^n (-1)^{n(n-1)/2}$. Let us write $B = b(z) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$. Then we obtain

$$\begin{aligned}
\Psi_n(B, z_1, \ldots, z_n) &= (-1)^n \int_U b(z) \mathrm{tr}_{\mathbb{C}^r} k_t(z, z) |dz| \int_{t\Delta_n} ds_1 \cdots ds_n \\
&= \frac{(-1)^n r}{n! \pi^n} \int_U b(z) |dz|.
\end{aligned}$$

The standard volume form $|dz|$ is

$$\begin{aligned}
|dz| &= (-2i)^{-n} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\
&= (-2i)^{-n} (-1)^{n(n-1)/2} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\end{aligned}$$

Thus $b(z)|dz| = (2i)^{-n} (-1)^{n(n+1)/2} \rho_0 d\rho_1 \cdots d\rho_{2n}$. Inserting this in the formula (12) gives the formula that had to be proved.

Appendix A

Triangulations and signs

Let T be a smooth finite triangulation of the oriented d-dimensional manifold X. Let $\sigma_{\alpha_0, \ldots, \alpha_p} \subset X$ denote the simplex with vertices $\alpha_0, \ldots, \alpha_p$. It is the image of the standard oriented simplex $\Delta_p = \{t \in [0,1]^{p+1} \mid \sum t_i = 1\}$ sending the i-th vertex with $t_i = 1$ to α_i and thus it comes with an orientation, for which

$$\partial\sigma_{\alpha_0,\dots,\alpha_p} = \sum_{j=0}^{p}(-1)^j\sigma_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_p}. \tag{13}$$

The cells of the dual cell decomposition T^* of X (see [13]) are in one-to-one correspondence with the simplices of the triangulation. The $(d-p)$-cell $C_{\alpha_0,\dots,\alpha_p}$ intersects only the p-simplex $\sigma_{\alpha_0,\dots,\alpha_p}$ and meets it transversally in exactly one interior point. Let us orient the cells by the condition that the intersection index is one:

$$C_{\alpha_0,\dots,\alpha_p}\cdot\sigma_{\alpha_0,\dots,\alpha_p} = 1. \tag{14}$$

This means in particular that the top-dimensional cells C_α have the same orientation as X. With this convention both $C_{\alpha_0,\dots,\alpha_p}$ and $\sigma_{\alpha_0,\dots,\alpha_p}$ change their orientation under permutation of the indices according to the sign of the permutation.

If c_p is a p-cell of T^* and c'_{d-p+1} is a $(d-p+1)$-cell of T, we have

$$\partial c_p\cdot c'_{d-p+1} = (-1)^p c_p\cdot\partial c'_{d-p+1}.$$

By combining this equation with (13) and (14) we obtain the formula for the boundary of dual cells:

$$\partial C_{\alpha_0,\dots,\alpha_p} = (-1)^{d+p}\sum_{\beta} C_{\beta,\alpha_0,\dots,\alpha_p}, \tag{15}$$

with summation over all β such that $\beta,\alpha_0,\dots,\alpha_p$ are the vertices of a simplex of the triangulation.

Appendix B

Heat kernel estimates and asymptotic expansion

In this section, we show the existence of the asymptotic expansion in the definition of the JLO-cocycle (see Proposition 6.1). In the first subsection, it is shown that the integrand in the definition of the JLO-cocyle is smooth for $s\in[0,1]^{p+1}\setminus\{0\}$. In the second subsection, we apply this result to compute the asymptotic expansion. In particular it will follow from this computation that Ψ_p is well defined and continuous in the operators $D_0,\dots,D_p$.

B.1. Heat kernel approximation

In order to show that the integrand $f(s)$ in the formula for Ψ_p is smooth for $s\in[0,1]^{p+1}\setminus\{0\}$, we need some estimates for the approximated heat kernel. We recall from [2] the notions of a generalized Laplacian and the corresponding heat kernel. A *generalized Laplacian* H acting on sections of a vector bundle $\mathcal{E}$ over a d-dimensional Riemannian manifold (X,g) is a second-order differential operator, which in local coordinates can be written as $H = -\sum_{i,j=1}^d g^{ij}\partial_i\partial_j +$ first order terms. It is easy to verify that $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is 4 times such a Laplacian if we set $\mathcal{E} = E\otimes\Lambda^\bullet T^{*(0,1)}X$. Therefore we may directly use the results about the heat kernel from [2] considering X as a smooth $2n$-dimensional Riemannian manifold.

We write $\mathfrak{D}_{\mathcal{E}}(X)$ for the space of smooth differential operators acting on smooth sections $\Gamma(X,\mathcal{E}) = \oplus\Omega^{0,j}(X)$. $\Gamma(X,\mathcal{E})$ is a locally convex space where the norms are the C^k-norms. These norms can be constructed by choosing a finite open cover of coordinate neighbourhoods of X. We then consider a cover of X of compact sets that are slightly smaller than the previous open sets. The C^k-norms are then defined by the sum of the C^k-norms on the compact sets and with respect to the corresponding coordinates. Furthermore we can assume that the C^k-norms on $\Gamma(X,\mathcal{E})$ are increasing, i.e. $\|\phi\|_k \leq \|\phi\|_\ell$ for $k \leq \ell$.

The spaces $\mathfrak{D}^j_E(X) \subset \mathfrak{D}_E(X)$ of differential operators of order j are spaces of sections of a certain hermitian vector bundle over X, and so one can define increasing C^k-norms on them in a similar way as above. Then $\mathfrak{D}_{\mathcal{E}}(X)$ is an LF-space which is the strict inductive limit of $\mathfrak{D}^j_{\mathcal{E}}(X)$, see, e.g., [20].

For two vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over the manifold X, we denote by $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ the external tensor product which is a vector bundle over $X \times X$. The *heat kernel* $k_t(x,y)$ is a family of sections $k_t \in \Gamma(X \times X, \mathcal{E} \boxtimes \mathcal{E}^*)$ defined for $t > 0$ which is C^1 with respect to t and C^2 with respect to x and solves the equation

$$\partial_t k_t(x,y) + \Delta_{\bar\partial} k_t(x,y) = 0$$

with initial condition $\lim_{t\to 0} k_t(x,y) = \delta(x-y)$ where δ is the Dirac distribution with respect to the Riemannian density on X. The heat kernel exists and is unique. There is an approximation to the heat kernel $k^N_t(x,y)$ of the form

$$k^N_t(x,y) = (\pi t)^{-n} e^{-d(x,y)^2/t} \sum_{i=0}^{N} t^i \Psi_i(x,y)$$

where $d(x,y)$ is the geodesic distance and $\Psi_i(x,y)$ are linear maps $\mathcal{E}_y \to \mathcal{E}_x$ depending smoothly on (x,y) and with support in the set where $d(x,y) \leq \varepsilon$ for some fixed ε which can be chosen arbitrarily small. Furthermore $\Psi_0(x,x)$ is the identity and the maps Ψ_i can be chosen so that the following theorem holds:

THEOREM B.1. – *Let here $\|\ .\ \|_\ell$ be C^ℓ-norms for sections in the bundle $\mathcal{E} \boxtimes \mathcal{E}^*$.*

(i) *k^N_t approximates the heat kernel k_t in the sense that*

$$\|\partial^m_t (k_t - k^N_t)\|_\ell = \mathcal{O}(t^{N-n-\ell/2-m}) \quad \textit{for } t \to 0\,.$$

(ii) *k^N_t is an approximate solution of the heat equation such that the remainder $r^N_t(x,y) := (\partial_t + \Delta_{\bar\partial}) k^N_t(x,y)$ satisfies the estimates*

$$\|\partial^k_t r^N_t\|_\ell < C t^{N-n/2-k-\ell/2}$$

for some constant C depending on ℓ and k.

Proof. – See [2], theorem 2.23 or 2.30 for part (i) and theorem 2.20 for part (ii). $\square$

REMARK B.2. – For any $D_0 \in M^0_c$ we have

$$\Psi_0(D_0) := [\mathrm{Str}(D_0 K^N_t)]_- = [\mathrm{Str}(D_0 K_t)]_-\ .$$

This follows directly from the estimates about the approximated heat kernel in part i) of the above theorem. This remark will be generalized for Ψ_p, $p = 0,\ldots,2n$ in remark B.12.

We define the operator K_t on smooth sections $\varphi \in \Gamma(X, \mathcal{E})$ by

$$(K_t\varphi)(x) = \int_X k_t(x,y)\varphi(y)|dy|_g \tag{16}$$

where $|dy|_g$ is the Riemannian density on X. $\varphi_t := K_t\varphi$ is a solution of the heat equation $\partial_t\varphi_t + \Delta_{\bar\partial}\varphi_t = 0$ with initial condition $\lim_{t\to 0}\varphi_t = \varphi$. In the same way, we also define the operators K_t^N that correspond to the approximated heat kernel k_t^N. We also set $K_0 = K_0^N = \mathrm{Id}$.

The operator K_t satisfies the following estimates:

Lemma B.3. – *We write $\|.\|_\ell$, $\ell \geq 0$ for the C^ℓ-norms on $\Gamma(X, \mathcal{E})$ or $\mathfrak{D}_{\mathcal{E}}(X)$. Fix a $\delta > 0$ small enough. Then for each ℓ and each of the following inequalities there is a constant C so that for all $s, s' \in [\delta, 1]$ and $t \in [0, 1]$,*

$$\begin{aligned}
&(i) && \|K_t^N\varphi - \varphi\|_\ell \leq C\,\|\varphi\|_{\ell+1}\sqrt{t}\\
&(ii) && \|K_s^N\varphi - K_{s'}^N\varphi\|_\ell \leq C\,\|\varphi\|_0\,|s-s'|\\
&(iii) && \|K_s^N\varphi\|_\ell \leq C\,\|\varphi\|_0\\
&(iv) && \|DK_0^N\varphi\|_\ell = \|D\varphi\|_\ell \leq C\,\|D\|_\ell\,\|\varphi\|_{\ell+d}
\end{aligned}$$

for every differential operator $D \in \mathfrak{D}_{\mathcal{E}}(X)$ of degree d.

Proof. – (i) The proof is essentially the same as for the first part of Theorem 2.29 in [2]. We consider the formula (16) for $K_t^N\varphi$, change to exponential coordinates for y ($y \mapsto \exp_x y$) and write $\varphi(x,y) := \varphi(\exp_x y)$ and $\Psi_j(x,y) = \Psi_j(x, \exp_x y)$, in the latter case with a slight abuse of the notation. We may assume that ε in the definition of k_t^N is smaller than the injectivity radius of the exponential map, so that the previous change to exponential coordinates in well defined. The substitution $y = \sqrt{t}v$ leads to

$$(K_t^N\varphi - \varphi)(x) = \frac{1}{\pi^n}\int\limits_{T_xX} e^{-\|v\|^2}\left(\sum_{j=0}^N t^j\Psi_j(x,\sqrt{t}v)\varphi(x,\sqrt{t}v)\rho(x,\sqrt{t}v) - \varphi(x,0)\right)dv$$

where we used $\frac{1}{\pi^n}\int_{T_xX} e^{-\|v\|^2} = 1$, and $\rho(x,y) := \sqrt{\det(\exp_x^* g(y))}$ is the factor coming from the Riemannian density. As $\Psi_j(x,y) = 0$ for $\|y\| > \varepsilon$, it is a compactly supported function on TX. For $j > 0$, it is therefore clear—by taking the supremum over y—that $t^j\Psi_j(x,y)\varphi(x,y)\rho(x,y)$ is bounded by a constant times $\sqrt{t}\|\varphi\|_0$. For $j = 0$, we write $f(x,y) = \Psi_0(x,y)\varphi(x,y)\rho(x,y)$. As $f(x,0) = \varphi(x,0)$, we get by the mean value theorem for the t^0-term

$$\frac{1}{\pi^n}\int_{T_xX} ve^{-\|v\|^2}\,\partial_y f(x,\sqrt{t'}v)\sqrt{t}dv$$

for some $t' \in [0,t]$. This expression is bounded by a constant times $\sqrt{t}\|\varphi\|_1$ and the claim follows in the case $\ell = 0$. For $\ell > 0$, we use the same arguments, but the function $f(x,y)$ is replaced by $\partial_x^\alpha f(x,y)$ where $|\alpha| \leq \ell$.

(ii) K_s^N is an integral operator with kernel with C^1-dependence on s for $s > 0$. Therefore the mean value theorem tells us that

$$|\partial_x^\alpha k_s^N(x,y) - \partial_x^\alpha k_{s'}^N(x,y)| \leq \sup_{s\in[\delta,1]}|\partial_s\partial_x^\alpha k_s^N(x,y)|\,|s-s'|$$

from which the claim follows.

(iii) Is obvious as the kernel is smooth in x for all $s \in [\delta, 1]$.

(iv) Also obvious. □

By iterating the above lemma, we find the following estimate:

LEMMA B.4. – *Let $D_i \in \mathfrak{D}_{\mathcal{E}}(X)$ be differential operators of degree d_i, $i = 1, \dots, m$. Fix a $\delta > 0$ and a set $I \subset \{1, \dots, m\}$. Let $s_i \in [0,1]$, $s'_i = 0$ for $i \in I$ and $s_i, s'_i \in [\delta, 1]$ for $i \notin I$. Then there is a constant C and an $L \leq \ell + m + \sum_{i=1}^m d_i$ so that*

$$\|D_1 K^N_{s_1} D_2 K^N_{s_2} \cdots D_m K^N_{s_m} \varphi - D_1 K^N_{s'_1} D_2 K^N_{s'_2} \cdots D_m K^N_{s'_m} \varphi\|_\ell \leq C \|\varphi\|_L \left(\sum_{i \in I} \sqrt{s_i} + \sum_{i \notin I} |s_i - s'_i| \right) \prod_{j=1}^k \|D_j\|_L .$$

Proof. – Using the triangle inequality and Lemma B.3, we find

$$\|DK^N_s \varphi_1 - DK^N_{s'} \varphi_2\|_\ell \leq \|(DK^N_s - DK^N_{s'})\varphi_1\|_\ell + \|DK^N_{s'}(\varphi_1 - \varphi_2)\|_\ell$$
$$\begin{cases} \overset{s'=0}{\leq} C\sqrt{s}\|\varphi_1\|_{\ell+d+1}\|D\|_\ell + C\|\varphi_1 - \varphi_2\|_{\ell+d}\|D\|_\ell \\ \overset{s'\geq\delta}{\leq} C|s - s'|\,\|D\|_\ell\|\varphi_1\|_0 + C\|\varphi_1 - \varphi_2\|_0\|D\|_\ell\,. \end{cases}$$

The proof is straightforward by induction on m. □

LEMMA B.5. – *The function $f(s)$, which is the integrand in the definition of Ψ_p, is continuous for $s \in [0,1]^{p+1} \setminus \{0\}$. In particular the integral over $t\Delta_p$ in the definition of Ψ_p (see Proposition 6.1) is well defined for $t \in (0,1]$.*

Proof. – An operator D on $\Gamma(X, \mathcal{E})$ with continuous kernel $D(x,y) \in \Gamma(X \times X, \mathcal{E} \boxtimes \mathcal{E}^*)$ is of trace class, and the supertrace can be written as

$$\begin{aligned} \mathrm{Str}(D) &= \sum_{k=0}^n (-1)^k \,\mathrm{Tr}_{\Omega^{0,k}(X,E)}(D) \\ &= \sum_{k=0}^n (-1)^k \int_X \mathrm{tr}_{E_x \otimes \Lambda^{0,k}(T_x X)}\, D(x,x) |dx|_g\,. \end{aligned}$$

For the integral over $t\Delta_p$ in the definition of Ψ_p to be convergent, it is sufficient to show that the function $f(s) := \mathrm{Str}(D_0 K^N_{s_0}[\bar\partial^*, D_1] K^N_{s_1} \cdots [\bar\partial^*, D_p] K^N_{s_p})$ is continuous in s for $s \in t\Delta_p$. As the heat kernel $k^N_{s_i}$ is C^1 with respect to s_i for $s_i > 0$, this is clear except for points on the boundary of $t\Delta_p$. For a point $s' \in t\partial\Delta_p$, let I be the subset of $\{1, \dots, n\}$ so that $s'_i = 0 \Leftrightarrow i \in I$ and take a $\delta > 0$ so that $s'_i > \delta\ \forall i \notin I$. As there is at least one $i \notin I$ and as the trace is cyclic, we can w.l.o.g. assume that $p \notin I$. To simplify the notation, we set $A_{s_0 \dots s_{p-1}} = D_0 K^N_{s_0}[\bar\partial^*, D_1] K^N_{s_1} \cdots [\bar\partial^*, D_{p-1}] K^N_{s_{p-1}}$ and $B_{s_p} = [\bar\partial^*, D_p] K^N_{s_p}$. We write the supertrace as

$$\mathrm{Str}(D_0 K^N_{s_0} \cdots [\bar\partial^*, D_p] K^N_{s_p}) = \sum_{\substack{k=0\dots n \\ i=1\dots i_k}} (-1)^k \int_{X \times X} \langle v^k_i | A_{s_0 \dots s_{p-1}}(x,y) B_{s_p}(y,x) | v^k_i \rangle \, dx\, dy$$

where $\{v_i^k\}$ for fixed k and $i = 1, \ldots, i_k$ is a basis for $E \otimes \Lambda^{0,k}(T_x X)$. Now we consider $A_{s_0 \ldots s_{p-1}}$ as operator acting on $\varphi_{s_p} := B_{s_p}(\,\cdot\,, x)v$ where $x \in X$ and $v \in \mathcal{E}_x$ are considered as parameters. Then we get by the triangle inequality and Lemma B.4 that

$$\begin{aligned}
&\|A_{s_0 \ldots s_{p-1}} \varphi_{s_p} - A_{s'_0 \ldots s'_{p-1}} \varphi_{s'_p}\|_0 \\
&\quad \leq \|(A_{s_0 \ldots s_{p-1}} - A_{s'_0 \ldots s'_{p-1}}) \varphi_{s_p}\|_0 + \|A_{s'_0 \ldots s'_{p-1}}(\varphi_{s_p} - \varphi_{s'_p})\|_0 \\
&\quad \leq \widetilde{C} \left(\sum_{i \in I \setminus \{p\}} \sqrt{s_i} + \sum_{i \notin I \cup \{p\}} |s'_i - s_i| \right) \|\varphi_{s_p}\|_L + \widetilde{C} \|\varphi_{s_p} - \varphi_{s'_p}
\end{aligned}$$

where $\widetilde{C} = C \|D_0\|_L \prod_{j=1}^{p-1} \|[\bar\partial^*, D_j]\|_L$. We use the mean value theorem and find

$$\begin{aligned}
\|\varphi_s - \varphi_{s'}\|_L &\leq |s - s'| \sup_{\substack{(x,v) \in \mathcal{E}, \|v\| \leq 1 \\ u \in [\delta, 1]}} \|B_u(\,\cdot\,, x)v\|_{L+1} \\
&\leq C |s - s'| \, \|[\bar\partial^*, D_p]\|_{L+1} \,.
\end{aligned}$$

As the integral of the trace is over a compact set, we have shown that f is continuous in s for $s \in t\Delta$ and

$$|f(s) - f(s')| \leq C \left(\sum_{i \in I} \sqrt{s_i} + \sum_{i \notin I} |s'_i - s_i| \right) \prod_{j=0}^{p} \|D_j\|_{L+2}. \tag{17}$$ □

Proposition B.6. – *The function $f(s)$ (see Lemma B.5) is k-times continuously differentiable for $s \in [0,1]^{p+1} \setminus \{0\}$ and $N = N_k$ large enough.*

Proof. – The proof works in exactly the same way as in the previous lemmata (B.3, B.4, B.5). We generalize the estimates in Lemma B.3 by adding time derivatives: Fix a $\delta > 0$ and assume $s, s' \in [\delta, 1]$ and $t \in [0,1]$. Then for each ℓ, m and each of the following inequalities there is a constant C so that

$$\begin{aligned}
&(i) && \|\partial_t^m K_t^N \varphi - (-\Delta_{\bar\partial})^m \varphi\|_\ell \leq C \, \|\varphi\|_{2m+\ell+1} \sqrt{t} \\
&(ii) && \|\partial_s^m K_s^N \varphi - \partial_{s'}^m K_{s'}^N \varphi\|_\ell \leq C \, \|\varphi\|_0 \, |s - s'| \\
&(iii) && \|\partial_s^m K_s^N \varphi\|_\ell \leq C \, \|\varphi\|_0 \\
&(iv) && \|(-\Delta_{\bar\partial})^m D K_0^N \varphi\|_\ell = \|(-\Delta_{\bar\partial})^m D \varphi\|_\ell \leq C \, \|\varphi\|_{\ell+d+2m}
\end{aligned}$$

where $\|.\|_\ell$, $\ell \geq 0$ are C^ℓ-norms on $\Gamma(X, \mathcal{E})$, $\mathfrak{D}_{\mathcal{E}}(X)$ respectively. We only prove the first estimate as the others are easy to show (see Lemma B.3). Recall from Theorem B.1 that the remainder $r_t^N = (\partial_t + \Delta_{\bar\partial}) k_t^N$ satisfies $\|\partial_t^k r_t^N\|_\ell < C t^{N-k-(n+\ell)/2}$. By the iterated application of $\partial_t k_t^N = -\Delta_{\bar\partial} k_t^N + r_t^N$, we find

$$\partial_t^m k_t^N = (-\Delta_{\bar\partial})^m k_t^N + \sum_{j=0}^{m-1} (-\Delta_{\bar\partial})^{m-1-j} \partial_t^j r_t^N$$

and hence the estimate

$$\|\partial_t^m K_t^N\varphi - (-\Delta_{\bar\partial})^m K_t^N\varphi\|_\ell \le \sum_{j=0}^{m-1} \|\Delta_{\bar\partial}^{m-1-j}\partial_t^j r_t^N\varphi\|_\ell$$

$$\le \sum_{j=0}^{m-1} \|\Delta_{\bar\partial}^{m-1-j}\|_\ell \|\partial_t^j r_t^N\|_{\ell+2(m-1-j)}\|\varphi\|_0 \le C\|\varphi\|_0 t^{N-m+1-(n+\ell)/2}.$$

We require N to be large enough, namely $N \ge \frac{n+\ell-1}{2} + m$. On the other hand we have the estimate

$$\|(-\Delta_{\bar\partial})^m K_t^N\varphi - (-\Delta_{\bar\partial})^m\varphi\|_\ell \le C\|\Delta_{\bar\partial}^m\|_\ell\|K_t^N\varphi - \varphi\|_{\ell+2m} \le C\|\varphi\|_{2m+\ell+1}\sqrt{t}.$$

The estimate (i) then follows by the triangle inequality.

Using the above estimates, it is now straightforward to generalize Lemma B.4 to

$$\|D_1\partial_{s_1}^{m_1}K_{s_1}^N D_2\partial_{s_2}^{m_2}K_{s_2}^N\cdots D_p\partial_{s_p}^{m_p}K_{s_p}^N\varphi - D_1\partial_{s_1'}^{m_1}K_{s_1'}^N D_2\partial_{s_2'}^{m_1}K_{s_2'}^N\cdots D_m\partial_{s_p'}^{m_p}K_{s_p'}^N\varphi\|_\ell$$

$$\le C\|\varphi\|_L\left(\sum_{i\in I}\sqrt{s_i} + \sum_{i\notin I}|s_i - s_i'|\right)$$

which is true for some $L \le \ell + \sum_i(d_i + 2m_i) + 1$. Then we see as in Lemma $B.5$ that the partial derivatives of $f(s)$ up to degree k are continuous. $\square$

B.2. Computation of Ψ_p and power counting

In this subsection, we explain an algorithm to compute Ψ_p which will lead to the result summarized in the following proposition:

PROPOSITION B.7. – *Let n be the complex dimension of X. Take the operators D_0, $D_1,\dots,D_p$ as in Proposition 6.1. We write d for the sum of the degrees of the differential operators $D_0, [\bar\partial^*, D_1],\dots,[\bar\partial^*, D_p]$ which are defined on a small (see remark below) open set $U \subset X$. Recall that the approximated heat kernel depends on the constants N and ε. Then for N big enough and ε small enough, $\Psi_p(D_0,\dots,D_p)$ is well defined and a polynomial in t^{-1} of degree $n-p+\lfloor\frac{d}{2}\rfloor$. More precisely, for $N \ge n-p+\lfloor\frac{d}{2}\rfloor$ and $\varepsilon < \frac{1}{p+1}\,\mathrm{dist}(X\setminus U, \mathrm{supp}(D_0))$, where dist means the geodesic distance, it is independent of N and $\varepsilon > 0$. Furthermore, $\Psi_p(D_0,\dots,D_p)$ depends continuously on $D_0, D_1,\dots,D_p$.*

REMARK B.8. – The set U in the above proposition has to be small in the sense that Lemma B.10 holds for any compact subset $K \subset U$. For larger U the above proposition would still be true with the exception that the upper bound for ε would need a more careful definition.

The main idea of the computation is to "move" the operators $[\bar\partial^*, D_i]$ in the formula for Ψ_p (see Proposition 6.1) to the left and to use a saddle point approximation for the heat kernel integrals. As a preparation for this computation, we formulate the following three lemmata:

LEMMA B.9. – *Let $U \subset X$ be an open subset of X so that the exponential map w.r.t. any point in U and restricted to the preimage of U is a diffeomorphism. Assume that $x_1, x_2 \in U$; then (in local coordinates) there is a smooth matrix valued function $a(x_1, x_2)$ so that*

$$\frac{\partial}{\partial x_2} d(x_1, x_2)^2 = a(x_1, x_2) \frac{\partial}{\partial x_1} d(x_1, x_2)^2 .$$

Proof. – We construct such a map explicitly: We introduce the coordinates $(x, \xi) = (x_1, \log_{x_1} x_2)$ and $(y, \eta) = (x_2, \log_{x_2} x_1)$. Obviously $|\xi| = d(x_1, x_2) = |\eta|$. Therefore we find

$$\frac{\partial}{\partial x_2} d(x_1, x_2)^2 = \frac{\partial \xi}{\partial x_2} \frac{\partial}{\partial \xi} |\xi|^2 = \frac{\partial \xi}{\partial x_2} \frac{\partial \eta}{\partial \xi} \frac{\partial}{\partial \eta} |\eta|^2 = \frac{\partial \xi}{\partial x_2} \frac{\partial \eta}{\partial \xi} \frac{\partial x_1}{\partial \eta} \frac{\partial}{\partial x_1} d(x_1, x_2)^2 .$$

As the exponential coordinates are smooth coordinates, the lemma follows. □

LEMMA B.10. – *Let $K \subset X$ be a sufficiently small compact neighbourhood of any point, so that the exponential map w.r.t. any point in K and restricted to the preimage of K is injective. Take $x_1, x_2, x_3 \in K$ and $s_1, s_2 \in (0, 1]$; then for fixed x_1, x_3, s_1, s_2 the function*

$$f(x_2) = \frac{d(x_1, x_2)^2}{s_1} + \frac{d(x_2, x_3)^2}{s_2}$$

has a unique minimum in the point $\bar{x}$ that lies on the geodesic through x_1 and x_3 and satisfies $d(x_1, \bar{x})/s_1 = d(x_3, \bar{x})/s_2$. We choose exponential coordinates $\xi = \log_{\bar{x}} x_2$ and expand f in the point $\bar{x}$. This leads to the following expressions for f:

$$\begin{aligned} f(x_2) &= \frac{d(x_1, x_3)^2}{s_1 + s_2} + \left(\frac{1}{s_1} + \frac{1}{s_2} \right) G_{ij}(s_1, s_2, x_1, x_2(\xi), x_3) \xi^i \xi^j \\ &= \frac{d(x_1, x_3)^2}{s_1 + s_2} + \left(\frac{1}{s_1} + \frac{1}{s_2} \right) G_{ij}(s_1, s_2, x_1, \bar{x}, x_3) \xi^i \xi^j \\ &\quad + G_{ijk}(s_1, s_2, x_1, x_2(\xi), x_3) \xi^i \xi^j \xi^k \end{aligned}$$

for smooth functions G_{ij} and G_{ijk}. The matrix $G_{ij}(s_1, s_2, x_1, x_2, x_3)$ defined and bounded on $([0, 1]^2 \setminus \{0\}) \times K^3$ is positive definite and there is a constant $C > 0$ so that the smallest eigenvalue of the matrix is greater than C for all $x_1, x_2, x_3 \in K$ and $s_1, s_2 \in [0, 1]$. Furthermore G_{ij} is homogeneous of degree 0 in s_1, s_2 and we have $G_{ij}(s_1, s_2, x_1, \bar{x}, x_3) \to \delta_{ij}$ for $|x_1 - x_3| \to 0$.

Proof. – If x_2 is not on the geodesic between x_1 and x_3, it is easy to see that there is always a point on the geodesic for which one term of f has the same value and the other one is smaller. For x_2 on the geodesic we have $d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3)$ from which $d(x_1, \bar{x})/s_1 = d(x_3, \bar{x})/s_2$ follows. The critical point $\bar{x}$ of the smooth function $s_1 s_2 f(x_2)$ is a smooth function of x_1, x_3, s_1, s_2, homogeneous of degree zero in s_1, s_2, as long as the Hessian is nondegenerate, which is the case if K is small enough and $s_1/(s_1 + s_2) \in (-\varepsilon_1, 1 + \varepsilon_1)$ for some $\varepsilon_1 > 0$. The expansion of f is just the Taylor expansion (with remainder) in the point $x_2 = \bar{x}$. This gives

$$\begin{aligned} &G_{ij}(s_1, s_2, x_1, x_2(\xi), x_3) \\ &\quad = \frac{1}{s_1 + s_2} \int_0^1 (1 - u) \frac{\partial^2}{\partial \eta^i \partial \eta^j} (s_2 d(x_1, \exp_{\bar{x}}(\eta))^2 + s_1 d(\exp_{\bar{x}}(\eta), x_3)^2)|_{\eta = u\xi} du. \end{aligned}$$

From this expression we see that G_{ij} is homogeneous in s and is smooth for $s_1/(s_1+s_2) \in (-\varepsilon_1, 1+\varepsilon_1)$, $x-1, x_3 \in K$. In particular it is a bounded continuous function on $([0,1]^2 \smallsetminus \{0\}) \times K^2$.

For a Euclidean metric it is an application of the law of cosines to show that $G_{ij} = \delta_{ij}$. By rescaling $x_i \mapsto \lambda x_i$ and taking into account that $d(\lambda x_i, \lambda x_j)/\lambda \to |x_i - x_j|$ for $\lambda \to 0$, we see that $G_{ij}(s_1, s_2, x_1, x_2, x_3) \to \delta_{ij}$ if $|x_1 - x_2| + |x_2 - x_3| \to 0$. Therefore also $G_{ij}(s_1, s_2, x_1, \bar{x}, x_3) \to \delta_{ij}$ for $|x_1 - x_3| \to 0$. As K is small, we are still close to the Euclidean case and therefore $G_{ij} - \delta_{ij}$ is small, from which the existence of C follows. □

LEMMA B.11 (Asymptotic expansion under the integral). – *We write $[f(t)]_t$ for the asymptotic expansion of the function f in the variable t in $t=0$. In the following cases we are allowed to interchange the asymptotic expansion and the integration:*

(i) *Let $f : [0,1]^{p+1} \setminus \{0\} \to \mathbb{C}$ be a smooth function and assume that there is an $n \in \mathbb{N}$ so that $F(s,t) := t^n f(st)$ can be continued to a function in $C^\infty(\Delta_p \times [0,1])$* [(2)]. *Then*

$$\Big[\int_{\Delta_p} f(st)ds\Big]_t = \int_{\Delta_p} [f(st)]_t \, ds .$$

(ii) *Let $K, G_{ij}, G_{ijk}, \bar{x}, x_2(\xi)$ be as in Lemma B.10. Let $H : \mathbb{R}^{2n} \to \mathbb{C}$ be a smooth function with support in a small neighbourhood of the origin. We abbreviate $G_{ij} := G_{ij}(s_1, s_2, x_1, \bar{x}, x_3)$, $G_{ij}(\xi) := G_{ij}(s_1, s_2, x_1, x_2(\xi), x_3)$ and $G_{ijk}(\xi) := G_{ijk}(s_1, s_2, x_1, x_2(\xi), x_3)$. Then*

$$\Big[\int_{\mathbb{R}^{2n}} H(\sqrt{t}\xi) e^{-aG_{ij}(\sqrt{t}\xi)\xi^i\xi^j} \, d\xi\Big]_{\sqrt{t}} = \int_{\mathbb{R}^{2n}} [H(\sqrt{t}\xi) e^{-aG_{ijk}(\sqrt{t}\xi)\xi^i\xi^j\xi^k\sqrt{t}}]_{\sqrt{t}} e^{-aG_{ij}\xi^i\xi^j} \, d\xi .$$

where a is any positive constant.

Proof. – (i) As $[f(st)]_t = t^{-n}[F(st)]_t$, it suffices to show that we can interchange the integral and the asymptotic expansion for F. Because F is smooth, its asymptotic expansion is given by the Taylor series and we have to show that in the following expression the limit and the integral are interchangeable:

$$\lim_{t \to 0} \int_{\Delta_p} \frac{F(s,t) - \sum_{k=0}^{\ell} \partial_t^k F(s,t) t^k / k!}{t^{\ell+1}} ds .$$

This is true because the integrand is dominated by $\sup\limits_{t \in [0,1]} |\partial_t^{\ell+1} F(s,t)|/(\ell+1)!$.

(ii) As in part (i), we consider the remainder of the Taylor expansion:

$$\frac{(\partial/\partial\sqrt{t})^m}{m!} H(\sqrt{t}\xi) e^{-aG_{ij}(\sqrt{t}\xi)\xi^i\xi^j} = \sum_{|\alpha|=m} \xi^\alpha \frac{\partial_\eta^\alpha}{\alpha!} H(\eta) e^{-aG_{ijk}(\eta)\eta^i\xi^j\xi^k}\Big|_{\eta=\sqrt{t}\xi} e^{-aG_{ij}\xi^i\xi^j} .$$

(2) By "C^∞ on a closed set" we mean that every derivative exists in the interior and extends continuously to the boundary.

As H has compact support, we can estimate this by

$$\|H(\eta)G_{ijk}(\eta)\eta^i\|_m(1+\|\xi\|^2)^m\xi^\alpha e^{-aG_{ij}\xi^i\xi^j-aG_{ijk}\xi^i\xi^j\xi^k\sqrt{t}}\,.$$

According to Lemma B.10, there is a constant C, so that

$$G_{ij}\xi^i\xi^j+G_{ijk}(\sqrt{t}\xi)\xi^i\xi^j\xi^k\sqrt{t}=G_{ij}(\sqrt{t}\xi)\xi^i\xi^j>C\|\xi\|^2\,,$$

for all ξ such that $\sqrt{t}\xi$ is in the support of H. Thus it follows again by the dominated convergence theorem that the asymptotic expansion and the integral commute. □

Proof of Proposition B.7. – We write Latin letters for indices in $\mathbb{N}_0$ and Greek letters for multiindices in $\mathbb{N}_0^{2n}$.

We consider again the function $f(s):=\operatorname{Str}(D_0K^N_{s_0}[\bar\partial^*,D_1]K^N_{s_1}\cdots[\bar\partial^*,D_p]K^N_{s_p})$. As we are going to show, the asymptotic expansion of $f(st)$ w.r.t. t in $t=0$ exists, has lowest order $-n-\lfloor\frac{d}{2}\rfloor$ and the coefficients are smooth functions of $s\in\Delta_p$. Therefore the function $F(s,t):=t^{n+\lfloor\frac{d}{2}\rfloor}f(st)$ is smooth [(3)] and we can apply Lemma B.11 (i):

$$\Psi_p(D_0,\ldots,D_p)=(-1)^{\frac{p(p+1)}{2}}\int_{\Delta_p}[t^pf(st)]_-ds\,.$$

To compute the asymptotic expansion of $f(st)$, we consider the kernel

$$(D_0K^N_{s_0}[\bar\partial^*,D_1]K^N_{s_1}\cdots[\bar\partial^*,D_p]K^N_{s_p})(x_0,x_{p+1})\,.$$

Recall that D_0 has compact support $K\subset U\subset X$ where U is an open set (see also Proposition 6.1). As $K^N_{s_i}(x_i,x_{i+1})$ vanishes for $d(x_i,x_{i+1})>\varepsilon$, there is a $\varepsilon>0$ so that $(p+1)\varepsilon$ is smaller than the geodesic distance between K and $X\setminus U$. Then in the above kernel only the values of terms inside a compact subset K_ε of U play a role and therefore it is well defined. We assume that K_ε is small enough to apply Lemma B.10.

We want to "move" the operators $[\bar\partial^*,D_i]$ to the left. First just consider a term $K^N_{s_1}DK^N_{s_2}$. We may assume that D in local coordinates has the form $\rho(x)\partial^\alpha$ where $\operatorname{supp}\rho\subset K_\varepsilon$. Explicitly, the above term is given by the integral

$$\int_X\sum_{0\le i,j\le N}s_1^is_2^j\Psi_i(x_1,x_2)\frac{e^{-d(x_1,x_2)^2/s_1}}{(\pi s_1)^n}\rho(x_2)\partial^\alpha_{x_2}\left(\Psi_j(x_2,x_3)\frac{e^{-d(x_2,x_3)^2/s_2}}{(\pi s_2)^n}\right)|dx_2|_g\,.$$

We write $|dx_2|_g=\sigma(x_2)dx_2$ and integrate by parts to bring the $\partial^\alpha_{x_2}$-operator to the left. Then we make repeatedly use of Lemma B.9 to "replace" the x_2-derivatives by x_1-derivatives, i.e. we use an identity of the form

$$\partial^\alpha_{x_2}e^{-d(x_1,x_2)^2/s_1}=\sum_{\beta+\gamma=\alpha}h_{\beta,\gamma}(x_1,x_2)\partial^\gamma_{x_1}e^{-d(x_1,x_2)^2/s_1}\,,$$

which holds for some smooth functions $h_{\beta,\gamma}$. Writing down again the integral, we find an expression of the form

$$\int_X\sum_{0\le i,j\le N}\sum_{\alpha'\le\alpha}s_1^is_2^jH_{i,j,\alpha'}(x_1,x_2,x_3)\partial^{\alpha'}_{x_1}\frac{e^{-d(x_1,x_2)^2/s_1-d(x_2,x_3)^2/s_2}}{(\pi s_1)^n(\pi s_2)^n}dx_2$$

(3) From Proposition B.6 follows that F is smooth for $(s,t)\in\Delta_p\times(0,1]$. The existence of the asymptotic expansion shows that its derivatives can be continued to $t=0$. Hence $F\in C^\infty(\Delta_p\times[0,1])$

where $H_{i,j,\alpha'}$ are smooth functions. If we apply the above procedure to shift all derivatives in the expression $D_0 K^N_{s_0} \dots [\bar\partial^*, D_p] K^N_{s_p}$ to the left, we get

$$\int_{X^p} \sum_{|\gamma|\le N} \sum_{|\alpha|\le d} s^\gamma H_{\gamma,\alpha}(x_0,\dots,x_p)\partial^\alpha_{x_0} \frac{e^{-\sum_{j=0}^p d(x_j,x_{j+1})^2/s_j}}{(\pi s_0)^n \dots (\pi s_p)^n} dx_1 \dots dx_p\,. \tag{18}$$

We omitted the terms for which $|\gamma| := \sum_{j=1}^{p-1} \gamma_j > N$, but we will see later that they would only produce (irrelevant) terms of higher order in t. We rewrite the exponent in the above expression using Lemma B.10 repeatedly:

$$\sum_{j=0}^p \frac{d(x_j,x_{j+1})^2}{s_j} = \frac{d(x_0,x_{p+1})^2}{s_0+\dots+s_p}$$
$$+ \sum_{\ell=1}^p \left(\frac{1}{s_0+\dots+s_{\ell-1}} + \frac{1}{s_\ell} \right) G_{ij}(s_0+\dots+s_{\ell-1}, s_\ell, x_0, x_\ell, x_{\ell+1}) \xi^i_\ell \xi^j_\ell,$$

where $\xi_\ell = \ln_{\bar x_\ell} x_\ell$, $\bar x_\ell = \bar x_\ell(x_{\ell-1}, x_{\ell+1})$. Now we change to the variables ξ^i in the integral and rescale $\xi^i \mapsto \sqrt{t}\xi^i$ as well as $s_i \mapsto ts_i$ so that $(s_0,\dots,s_p) \in \Delta_p$. We temporarily forget the last term in the exponent and suppress the arguments of G_{ij}:

$$t^{p-n} \int_{(T_{\bar x}X)^p} \sum_{\substack{|\gamma|\le N\\ |\alpha|\le d}} s^\gamma H_{\gamma,\alpha}(x_0,\dots,x_{p+1}) \partial^\alpha_{x_0} \frac{e^{-\sum_{\ell=1}^p \left(\frac{1}{s_0+\dots+s_{\ell-1}} + \frac{1}{s_\ell}\right) G_{ij}\xi^i_\ell \xi^j_\ell}}{(\pi s_0)^n \dots (\pi s_p)^n} d\xi_1 \dots d\xi_p \tag{19}$$

where the Jacobi determinant has been absorbed in $H_{\gamma,\alpha}$. Due to Lemma B.11 ii) we are allowed to expand asymptotically w.r.t. $\sqrt{t}$ under the integral. Keep in mind that $x_\ell = \exp_{\bar x_\ell}(\sqrt{t}\xi_\ell)$ so that the arguments of $H_{\gamma,\alpha}$ as well as of G_{ij} depend on $\sqrt{t}$. In the expansion of the exponent, there will be singular terms in s, namely powers of the factor

$$\frac{1}{s_0+\dots+s_{\ell-1}} + \frac{1}{s_\ell} = \frac{s_0+\dots+s_\ell}{(s_0+\dots+s_{\ell-1})s_\ell}\,,$$

but as these factors only appear paired with $\xi^i_\ell \xi^j_\ell$, the singularities cancel as we see in the following computation. After the expansion we have to compute integrals of the form

$$\int_{T_{\bar x}X} \xi^\beta_\ell e^{-\frac{s_0+\dots+s_\ell}{(s_0+\dots+s_{\ell-1})s_\ell} G_{ij}\xi^i_\ell \xi^j_\ell} d\xi_\ell = C_\beta(s,x) \left(\frac{(s_0+\dots+s_{\ell-1})s_\ell}{(s_0+\dots+s_\ell)} \right)^{\frac{|\beta|}{2}+n}$$

where $C_\beta(s,x)$ is a smooth function, homogeneous of degree 0 in s, vanishing unless $|\beta| = \sum \beta_i$ is even. Terms with $|\beta|$ even correspond to even terms in the asymptotic expansion in powers of $\sqrt{t}$. Therefore we actually have an asymptotic series in t.

We repeat the above steps for $\xi_2,\dots,\xi_p$. As

$$\prod_{\ell=1}^p \frac{(s_0+s_1+\dots+s_{\ell-1})s_\ell}{s_0+s_1+\dots+s_\ell} = \frac{s_0 s_1 \dots s_p}{s_0+s_1+\dots+s_p}\,,$$

the singularities from the denominator in equation (19) disappear, and we get

$$(D_0 K^N_{ts_0} \cdots D_p K^N_{ts_p})(x_0, x_{p+1}) = t^{p-n} \sum_{|\alpha| \leq d} \sum_{k=0}^{N} t^k f_k(s, x_0, x_{p+1}) \partial^\alpha_{x_0} e^{-\frac{d(x_0, x_{p+1})^2}{t}} + \mathcal{O}(t^{p-n+N+1})$$

for smooth functions $f_k : \Delta_p \times K \times K_\varepsilon \to \mathbb{C}$. Remember that

$$f(s,t) = \int_K (D_0 K^N_{ts_0} \cdots D_p K^N_{ts_p})(x_0, x_0) dx_0 \,.$$

The integral over $x_0 \in K$ and the asymptotic expansion commute for the same reason as in Lemma B.11. We see in the above formula that the negative powers in t are only produced by the derivative $\partial^\alpha_{x_0}$. As $\lim_{x_p \to x_0} \partial^\alpha_{x_0} d(x_0, x_p)^2 = 0$ for $|\alpha| = 1$, we need at least two derivatives to get a negative power in t. Thus the negative power is at most $\lfloor \frac{|\alpha|}{2} \rfloor$.

In formula (18), the coefficient functions of the operators $D_0, D_1, \ldots, D_p$ have been absorbed in the function $H_{\gamma,\alpha}$. It is easy to check that they enter linearly and with derivatives of order at most d, which is the sum of the degrees of the differential operators, in this function. After formula (19) when we do the expansion, we get an additional derivative for every order of $\sqrt{t}$. Therefore the coefficients of Ψ_p only depend on finitely many derivatives of the operators $D_0, D_1, \ldots, D_p$ restricted to the compact set K_ε that was mentioned in the beginning of the proof. This means that we can estimate Ψ_p by a product of C^k-norms over the compact set K_ε of the operators D_i. As the operators $D_1, \ldots, D_p$ are holomorphic and actually defined on an open set containing K_ε, we can use the Cauchy integral formula to estimate their C^k-norms by the sup-norms over a compact set that is slightly bigger than K_ε. This shows that Ψ_p is continuous in the operators $D_0, \ldots, D_p$. $\square$

REMARK B.12. – If we replace in the formula for Ψ_p one of the approximated heat kernels k^N by the exact heat kernel k, the dx-integral still is over a compact set. Thus we can choose ε small enough so that the formula for Ψ_p is still well defined. From the above proof it is also clear that this procedure does not change the value of Ψ_p.

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