Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

GEOMETRIC INSTABILITY FOR NLS ON SURFACES

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Tome 136
Fascicule 1

2008

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 167-193

Bull. Soc. math. France
136 (1), 2008, p. 167–193

THE WKB METHOD AND GEOMETRIC INSTABILITY FOR NONLINEAR SCHRÖDINGER EQUATIONS ON SURFACES

BY LAURENT THOMANN

ABSTRACT. — In this paper we are interested in constructing WKB approximations for the nonlinear cubic Schrödinger equation on a Riemannian surface which has a stable geodesic. These approximate solutions will lead to some instability properties of the equation.

RÉSUMÉ (*Méthode WKB et instabilité géométrique pour les équations de Schrödinger non linéaires sur des surfaces*)

À l'aide de la méthode WKB nous construisons des solutions approchées à l'équation de Schrödinger cubique sur une variété qui possède une géodésique stable. Cette construction permet d'obtenir des résultats d'instabilités dans des espaces de Sobolev.

1. Introduction

Let (M, g) be a Riemannian surface (i.e., a Riemannian manifold of dimension 2), orientable or not. We assume that M is either compact or a compact perturbation of the euclidian space, so that the Sobolev embeddings are true. Consider $\Delta = \Delta_g$ the Laplace-Beltrami operator. In this paper we are interested in constructing WKB approximations for the nonlinear cubic Schrödinger equation

Texte reçu le 20 octobre 2006, révisé le 23 avril 2007 et le 22 janvier 2008

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2000 Mathematics Subject Classification. — 35Q55; 35B35; 35R25.

Key words and phrases. — nonlinear Schrödinger equation, instability, quasimode.

$$(1)\qquad \begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = \varepsilon |u|^2 u(t,x), \quad \varepsilon = \pm 1, \\ u(0,x) = u_0(x) \in H^\sigma(M), \end{cases}$$

that is, given a small parameter $0 < h < 1$ and an integer N, functions $u_N(h)$ satisfying

$$(2)\qquad i\partial_t u_N(h) + \Delta u_N(h) = \varepsilon |u_N(h)|^2 u_N(h) + R_N(h),$$

with $\|u_N(h)\|_{H^\sigma} \sim 1$ and $\|R_N(h)\|_{H^\sigma} \leq C_N h^N$.
Here h is introduced so that $u_N(h)$ oscillates with frequency $\sim \frac{1}{h}$.
These approximate solutions to (1) will lead to some instability properties in the following sense (where h^{-1} will play the role of n):

Definition 1.1. — We say that the Cauchy problem (1) is unstable near 0 in $H^\sigma(M)$, if for all $C > 0$ there exist times $t_n \longrightarrow 0$ and $u_{1,n}, u_{2,n} \in H^\sigma(M)$ solutions of (1) so that

$$\begin{aligned} \|u_{1,n}(0)\|_{H^\sigma(M)},\ \|u_{2,n}(0)\|_{H^\sigma(M)} &\leq C, \\ \|u_{1,n}(0) - u_{2,n}(0)\|_{H^\sigma(M)} &\longrightarrow 0, \\ \limsup \|u_{1,n}(t_n) - u_{2,n}(t_n)\|_{H^\sigma(M)} &\geq \frac{1}{2}C, \end{aligned}$$

when $n \longrightarrow +\infty$.

This means that the problem is not uniformly well-posed, if we refer to the following definition:

Definition 1.2. — Let $\sigma \in \mathbb{R}$. Denote by $B_{R,\sigma}$ the ball of radius R in H^σ. We say that the Cauchy problem (1) is uniformly well-posed in H^σ if the flow map

$$u_0 \in B_{R,\sigma} \cap H^1(M) \longmapsto \Phi_t(u_0) \in H^\sigma(M),$$

is uniformly continuous for any t.

We now state our instability result:

Proposition 1.3. — *Let $0 < \sigma < \frac{1}{4}$, and assume that M has a stable and non degenerated periodic geodesic (see Assumptions 1 and 2), then the Cauchy problem* (1) *is not uniformly well-posed.*

This problem is motivated by the following results: Let (M, g) be a riemannian compact surface, then in [5], N. Burq, P. Gérard and N. Tzvetkov prove that (1) is uniformly well-posed in $H^\sigma(M)$ for $\sigma > \frac{1}{2}$. Whereas, in [4], they show that (1) is unstable on the sphere $\mathbb{S}^2$ for $0 < \sigma < \frac{1}{4}$. In fact they construct solutions of (1) of the form

$$(3)\qquad u_n^\kappa(t,x) = \kappa e^{i\lambda_n^\kappa t}(n^{\frac{1}{4}-\sigma}\psi_n(x) + r_n(t,x)),$$

where $0<\kappa<1$, $\psi_n=(x_1+ix_2)^n$ is a spherical harmonic which concentrates on the equator of the sphere when $n\longrightarrow+\infty$ and where r_n is an error term which is small. To obtain instability, they consider $\kappa_n\longrightarrow\kappa$, then

$$\|u_n^{\kappa}(0)-u_n^{\kappa_n}(0)\|_{H^\sigma(\mathbb{S}^2)}\lesssim|\kappa-\kappa_n|\longrightarrow 0,$$

but

$$\|u_n^{\kappa}(t_n)-u_n^{\kappa_n}(t_n)\|_{H^\sigma(\mathbb{S}^2)}\gtrsim\kappa|\mathrm{e}^{i\lambda_n^{\kappa}t_n}-\mathrm{e}^{i\lambda_n^{\kappa_n}t_n}|\longrightarrow 2\kappa,$$

with a suitable choice of $t_n\longrightarrow 0$.
We follow this strategy but as the surface is not rotation invariant, the ansatz will be more complicated than (3).
This result is sharp, because in [6] they show that (1) is uniformly well-posed on $\mathbb{S}^2$ when $\sigma>\frac{1}{4}$.
On the other hand, in [3] J. Bourgain shows that (1) is uniformly well-posed on the rational torus $\mathbb{T}^2$ when $\sigma>0$.
These results show how the geometry of M can lead to instability for the equation (1). Therefore it seems reasonable to obtain a result like Proposition 1.3 with purely geometric assumptions.

We first make the following assumption on M:

Assumption 1. — *The manifold M has a periodic geodesic.*

Denote by γ such a geodesic, then there exists a system of coordinates (s,r) near γ, say for $(s,r)\in\mathbb{S}^1\times]-r_0,r_0[$, called Fermi coordinates such that (see [13], p. 80)

1. The curve $r=0$ is the geodesic γ parametrized by arclength and
2. The curves $s=$ constant are geodesics parametrized by arclength. The curves $r=$ constant meet these curves perpendicularly.
3. In this system the metric writes

$$g=\begin{pmatrix}1&0\\0&a^2(s,r)\end{pmatrix}.$$

We set the length of γ equal to 2π. Denote by $R(s,r)$ the Gauss curvature at (s,r), then a is the unique solution of

$$\begin{cases}\dfrac{\partial^2 a}{\partial r^2}+R(s,r)a=0,\\ a(s,0)=1,\ \dfrac{\partial a}{\partial r}(s,0)=0.\end{cases}\tag{4}$$

The initial conditions traduce the fact that the curve $r=0$ is a unit-speed geodesic. In these coordinates the Laplace-Beltrami operator is

$$\Delta := \frac{1}{\sqrt{\det g}}\operatorname{div}(\sqrt{\det g}\, g^{-1}\nabla) = \frac{1}{a}\partial_s(\frac{1}{a}\partial_s) + \frac{1}{a}\partial_r(a\partial_r).$$

A function on M, defined locally near γ, can be identified with a function of $[0,2\pi]\times]-r_0,r_0[$ such that

$$\forall (s,r)\in[0,2\pi]\times]-r_0,r_0[\quad f(s+2\pi,r)=f(s,\omega r)$$

where $\omega=1$ if M is orientable and $\omega=-1$ if M is not. Define

$$\omega_1=\frac{1}{2}(\omega-1)\in\{-1,0\}. \tag{5}$$

From (4) we deduce that a admits the Taylor expansion

$$a = 1-\frac{1}{2}R(s)r^2+R_3(s)r^3+\cdots+R_p(s)r^p+o(r^p), \tag{6}$$

with $R(s)=R(s,0)$ and

$$R_k(s)=\frac{1}{k!}\frac{\partial^k a}{\partial r^k}(s,0), \tag{7}$$

for $k\geq 3$.
As $a(s+2\pi,r)=a(s,\omega r)$, we deduce $R(s+2\pi)=R(s)$ and for all $j\geq 3$, $R_j(s+2\pi)=\omega^j R_j(s)$.
Let $p_2=\frac{1}{a^2}\sigma^2+\rho^2$ be the principal symbol of Δ, and

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}s(t)=\frac{\partial p_2}{\partial\sigma}=\frac{2\sigma}{a^2},\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t)=-\frac{\partial p_2}{\partial s}=-\partial_s(\frac{1}{a^2})\sigma^2,\\ \frac{\mathrm{d}}{\mathrm{d}t}r(t)=\frac{\partial p_2}{\partial\rho}=2\rho,\ \frac{\mathrm{d}}{\mathrm{d}t}\rho(t)=-\frac{\partial p_2}{\partial r}=-\partial_r(\frac{1}{a^2})\sigma^2,\\ s(0)=s_0,\ \sigma(0)=\sigma_0, r(0)=r_0,\ \rho(0)=\rho_0, \end{cases} \tag{8}$$

its associated hamiltonian system, where $p_2=p_2(s(t),r(t),\sigma(t),\rho(t))$. The system (8) admits a unique solution and defines the hamiltonian flow

$$\Phi_t:(s_0,\sigma_0,r_0,\rho_0)\longmapsto(s(t),\sigma(t),r(t),\rho(t)).$$

The curve $\Gamma=\{(s(t)=t,\sigma(t)=1/2,r(t)=0,\rho(t)=0),t\in[0,2\pi]\}$ is solution of (8) and its projection in the (s,r) space is the curve γ. Now denote by ϕ the Poincaré map associated to the trajectory Γ and to the hyperplane $\Sigma=\{s=0\}$. There exists a neighborhood $\mathcal{N}$ of $(\sigma=1/2,r=0,\rho=0)$ such that the following makes sense: solve the system (8) with the initial conditions $(0,\sigma_0,r_0,\rho_0)\in\{0\}\times\mathcal{N}$ and let T be such that $s(T)=2\pi$, then ϕ is the application

$$\phi:(r_0,\rho_0)\longmapsto(r(T),\rho(T)).$$

Moreover, the Poincaré map is continuously differentiable (see [14] p. 193). To obtain its differential $\mathrm{d}\phi(0,0)$ at $(0,0)$, we linearize the system (8) about the orbit Γ, i.e.,

$$\left\{\begin{array}{l}\frac{\mathrm{d}}{\mathrm{d}t}s(t)=2\sigma,\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t)=0,\\ \frac{\mathrm{d}}{\mathrm{d}t}r(t)=2\rho,\ \frac{\mathrm{d}}{\mathrm{d}t}\rho(t)=-\frac{1}{2}R(s(t))r,\end{array}\right. \tag{9}$$

then $\sigma=\frac{1}{2}$, $s(t)=t$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r\\ \rho\end{pmatrix}=\begin{pmatrix}0 & 2\\ -R/2 & 0\end{pmatrix}\begin{pmatrix} r\\ \rho\end{pmatrix}. \tag{10}$$

Hence the application $\mathrm{d}\phi(0,0)$ is

$$\mathrm{d}\phi(0,0):(r_0,\rho_0)\longmapsto(r(2\pi),\rho(2\pi)), \tag{11}$$

where (r,ρ) solves (10). As $\mathrm{d}\phi(0,0)$ is symplectic, it admits two eigenvalues Λ and Λ^{-1} that are called the characteristic multipliers of the system (10). We add the following assumption on γ, which can be formulated in terms of the eigenvalues of $\mathrm{d}\phi(0,0)$:

Assumption 2. — *The geodesic γ is stable, i.e., $\mathrm{d}\phi(0,0)$ is a rotation. Then the multipliers take the form $\Lambda=e^{i\lambda}$ and $\Lambda^{-1}=e^{-i\lambda}$ with $\lambda\in\mathbb{R}$. We assume moreover that there exist $\tau,\mu>0$ such that*

$$\forall(p,q)\in\mathbb{Z}\times\mathbb{N}\quad|p-q\frac{\lambda}{\pi}|\geq\frac{\mu}{|(p,q)|^{\tau}}, \tag{12}$$

where $|(p,q)|=|p|+|q|$. When this condition is fulfilled, we say that γ is non degenerated.

Remark 1.4. — Almost every $\lambda\in\mathbb{R}$ satisfies (12) with $\tau>1$. This is an easy consequence of [1] p. 159, e.g.

Examples 1. — *Let M be a surface which has a periodic geodesic γ. In the general case, the eigenvalues of $\mathrm{d}\phi(0,0)$ defined by (11) are $\Lambda=\rho e^{i\lambda}$ and $\Lambda^{-1}=\rho^{-1}e^{-i\lambda}$, with $\Lambda+\Lambda^{-1}\in\mathbb{R}_+$, i.e.,*

$$(\rho-\rho^{-1})\sin\lambda=0. \tag{13}$$

Assume that M is a surface of revolution and that $R>0$ on γ. Then the characteristic multipliers are

$$\Lambda=\rho e^{2\pi i\sqrt{R}}\quad\textit{and}\quad\Lambda^{-1}=\rho^{-1}e^{-2\pi i\sqrt{R}}.$$

i) *If $\lambda=2\pi\sqrt{R}$ satisfies (12) then $\rho=1$ and M satisfies the assumptions.*

ii) *Let* $2\sqrt{R} \notin \mathbb{N}$. *Let* $\tilde{M}$ *be a perturbation of* M, *and denote by*

$$\tilde{\Lambda} = \tilde{\rho} e^{i\tilde{\lambda}} \quad \text{and} \quad \tilde{\Lambda}^{-1} = \tilde{\rho}^{-1} e^{-i\tilde{\lambda}},$$

the new characteristic multipliers.
By (13), $\tilde{\rho} = 1$, *and Assumption 2 is satisfied almost surely.*

iii) *Let* $a > 0$, *then the torus* $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/a\mathbb{Z}$ *is not under the hypotheses: in this case* $\mathrm{d}\phi(0,0)$ *is not diagonalizable.*

Notice that the function r which satisfies (10) is solution of

$$\ddot{y}(s) + R(s)y(s) = 0. \tag{14}$$

Consider a_0 the solution of (14) with initial conditions $a_0(0) = 1$ and $\dot{a}_0(0) = i$. Then, from the Floquet theory, there exists a 2π-periodic function P so that

$$a_0(s) = \mathrm{e}^{i\frac{\lambda}{2\pi}s} P(s)$$

(or $a_0(s) = \exp\left(-i\frac{\lambda}{2\pi}s\right)P(s)$, but λ can be replaced with $-\lambda$).
Here, and in all the paper we denote by $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}s}f$ if f is differentiable. This notation is motived by the fact that s will play the role of a time variable (see section 2).

In order to prove Proposition 1.3, we construct stationnary approximate solutions of (1), as stated in the following theorem

Theorem 1.5. — *Assume 1 and 2. Let* $h \in]0,1]$ *such that* $\frac{1}{h} \in \mathbb{N}$, *let* $\kappa, \sigma > 0$ *and* $k \in \mathbb{N}$. *Let* λ *be given by Assumption 2 and* ω_1 *by* (5).
Define $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$.
Then for all $N \in \mathbb{N}$, *there exist* $\lambda_N(k) \in \mathbb{R}$ *and a family* $u_N(h)$ *such that* $C_1 h^\sigma \leq \|u_N(h)\|_{L^2(M)} \leq C_2 h^\sigma$ *with* $C_1, C_2 > 0$ *independent of* N *and* h, *and*

$$-\Delta u_N(h) = \lambda_N(k) u_N(h) - \varepsilon |u_N(h)|^2 u_N(h) + h^N g_N(h) \tag{15}$$

with for all $N \in \mathbb{N}$

$$\|h^N g_N(h)\|_{H^n(M)} \lesssim h^{N-n}.$$

Moreover

$$\lambda_N(k) = \frac{1}{h^2} - \frac{2}{h}E_0(k) + \frac{1}{\sqrt{h}}\varepsilon\kappa^2 h^{2\sigma} C_0 + \mathcal{O}(1),$$

where $C_0 > 0$ *is independent of* ε, κ *and* σ.

Remark 1.6. — The analog of Theorem 1.5 was proved by J. Ralston in [15] for the linear case ($\varepsilon = 0$), with the same type of assumptions.

REMARK 1.7. — Consider the more general equations

$$i\partial_t u + \Delta u = F(u), \tag{16}$$

where $F : \mathbb{C} \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function. The result of Theorem 1.5 is likely to hold with other nonlinearities $F(u)$, for example for $F(z) = z^3$, $F(z) = z^4$ or $F(z) = (1 + |z|^2)^\alpha z$ with $\alpha < 1$. However, the instability phenomenon is strongly related to the gauge invariance of the equation (16).

The scheme of the paper is the following: Thanks to a scaling, we reduce the problem (15) to the resolution of linear Schrödinger equations with a harmonic time dependent potential, and we will see, using Assumption 2, that these equations have periodic solutions. To prove Proposition 1.3 we show that the family $u_N(h)$ provides good approximations of (1) in times where instability occurs.

NOTATIONS 1.8. — *In this paper c, C denote constants the value of which may change from line to line. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \leq a \leq Cb$, $a \leq Cb$ respectively. By $\delta_{i,j}$ we mean the Kronecker symbol, i.e., $\delta_{i,j} = 0$ for $i \neq j$ and $\delta_{i,i} = 1$.*

REMARK 1.9. — In the sequel we do not always mention the dependence on h of the functions: we will write u, f, r_i, ... instead of u_h, f_h, $r_{i,h}$, ...

ACKNOWLEDGEMENTS. — *The author would like to thank his adviser N. Burq for his constant guidance in this work, P. Pansu for his help in the frame of geometry, and B. Helffer for having pointed out the reference* [11].

2. The WKB construction

Consider the equation

$$-\Delta u = \lambda u - \varepsilon |u|^2 u. \tag{17}$$

Given $h > 0$, we are looking for a solution of the form

$$u = \delta h^{-\frac{1}{4}} \mathrm{e}^{i\frac{s}{h}} f(s, r, h), \tag{18}$$

where $\delta = \kappa h^\sigma$, with $\kappa > 0$ and $0 \leq \sigma \leq \frac{1}{4}$. In all this section, δ will play the role of a parameter.

We try to find a solution (u, λ) of (17) of the form

$$u \sim \sum_{j\geq 0} h^{j/2} u_j, \quad \lambda \sim h^{-2} \sum_{j\geq 0} h^{j/2} \lambda_j.$$

As we will see, identifying each power of h will lead to a linear equation which can be solved with a suitable choice of λ_j.

Choose h such that $h^{-1} \in \mathbb{N}$, this ensures that $\exp i\frac{s}{h}$ is $2\pi-$periodic. Such a condition on h is natural and is known as a Bohr-Sommerfeld quantification condition.

With the ansatz (18), equation (17) becomes

$$-\frac{1}{a^2}\left(\frac{2i}{h}\partial_s f+\partial_s^2 f-\frac{1}{h^2}f\right)-\frac{1}{a}\partial_s\left(\frac{1}{a}\right)\left(\frac{i}{h}f+\partial_s f\right)$$
$$-\partial_r^2 f-\frac{\partial_r a}{a}\partial_r f=\lambda f-\varepsilon\delta^2 h^{-\frac{1}{2}}|f|^2 f. \tag{19}$$

We make the change of variables $x=\frac{r}{\sqrt{h}}$ and set $v(s,x,h)=f(s,\sqrt{h}x,h)$. Thus $\partial_r f=\frac{1}{\sqrt{h}}\partial_x v$ and $\partial_r^2 f=\frac{1}{h}\partial_x^2 v$.

Therefore we now have to find $v\sim\sum_{j\geq 0}h^{j/2}v_j$.

Using (6) we obtain the following Taylor expansions in h

$$\frac{1}{a^2}=1+hRx^2-2h^{\frac{3}{2}}R_3x^3+\mathcal{O}(h^2),$$

$$a^{-1}\partial_s(a^{-1})=\mathcal{O}(h)\quad\text{and}\quad a^{-1}\partial_r a=\mathcal{O}(h^{\frac{1}{2}}).$$

Equation (19) can therefore be written, after multiplication by $\frac{1}{2}h$

$$\begin{aligned}&i\partial_s v+\frac{1}{2}\partial_x^2 v-\frac{1}{2}Rx^2 v\\&\quad=\frac{1-\lambda h^2}{2h}v+h^{\frac{1}{2}}R_3x^3v+\frac{1}{2}\varepsilon\delta^2h^{\frac{1}{2}}|v|^2v+hPv,\end{aligned}\tag{20}$$

where

$$P=A_1\partial_s^2+A_2\partial_s+A_3\partial_x+A_4 \tag{21}$$

is a second order differential operator with coefficients $A_j=A_j(s,x,h)$ satisfying $A_j(s+2\pi,x,h)=A_j(s,\omega x,h)$ for $0\leq j\leq 4$.

Denote by $E=\frac{1-\lambda h^2}{2h}=E_0+h^{\frac{1}{2}}E_1+\cdots+h^{\frac{p}{2}}E_p+o(h^{\frac{p}{2}})$ and write $v=v_0+h^{\frac{1}{2}}v_1+\cdots+h^{\frac{p}{2}}v_p+o(h^{\frac{p}{2}})$ and by identifying the powers of h we obtain the system of equations:

$$\left(i\partial_s+\frac{1}{2}\partial_x^2-\frac{1}{2}Rx^2-E_0\right)v_0=0, \tag{22}$$

$$\left(i\partial_s+\frac{1}{2}\partial_x^2-\frac{1}{2}Rx^2-E_0\right)v_1=E_1v_0+R_3x^3v_0+\frac{1}{2}\varepsilon\delta^2|v_0|^2v_0, \tag{23}$$

$$\cdots=\cdots$$

$$\left(i\partial_s+\frac{1}{2}\partial_x^2-\frac{1}{2}Rx^2-E_0\right)v_p=E_pv_0+Q_p. \tag{24}$$

$$\cdots=\cdots$$

so that the $(j+1)$th equation of unknown (v_j,E_j) corresponds to the annihilation of the coefficient of $h^{\frac{j}{2}}$ in (20).

Here Q_p is a function which only depends on x,s, $(v_j)_{j\leq p-1}$ and $(E_j)_{j\leq p-1}$.

REMARK 2.1. — Notice that thanks to the scaling, we have reduced the problem (17) to the resolution of linear equations. However we have to solve them exactly; no smallness assumption on x is possible, as x can be of size $\sim \frac{1}{\sqrt{h}}$.

In this section we will show

PROPOSITION 2.2. — *For all $p \in \mathbb{N}$, there exist $(E_0, \cdots, E_p) \in \mathbb{R}^{p+1}$ and $(v_0, \cdots, v_p) \in (\mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R})))^{p+1}$ with $v_0 \neq 0$, which solve the system (22)-(24).*

This permits us to construct approximate solutions of (17); more precisely, we will obtain the following proposition, which is the main result of this section.

PROPOSITION 2.3. — *Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be such that $0 \leq \chi \leq 1$, $\chi = 1$ on $[-r_0/2, r_0/2]$ and suppose moreover that χ is an even function. Let $\delta > 0$. Denote by*

$$u_p(s, r) = \delta h^{-\frac{1}{4}} \chi(r) e^{i\frac{s}{h}}(v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p)(s, \frac{r}{\sqrt{h}}) \tag{25}$$

and by

$$\lambda_p = \frac{1}{h^2} - \frac{2}{h}(E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p). \tag{26}$$

Then u_p satisfies $\|u_p\|_{L^2(M)} \sim \delta$ and

$$-\Delta u_p = \lambda_p u_p - \varepsilon |u_p|^2 u_p + h^{\frac{p-1}{2}} g_p(h) \tag{27}$$

with

$$\forall h \in]0, 1], \ \forall n \in \mathbb{N}, \quad \|h^{\frac{p-1}{2}} g_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \lesssim \delta h^{\frac{p-1}{2} - n}.$$

2.1. Preliminaries: the analysis of the linear equations. — We will solve the system (22)-(24) for $x \in \mathbb{R}$. Notice that the Fermi coordinates are only defined for $|r| \leq r_0$ i.e., for $x \leq \frac{r_0}{\sqrt{h}}$. That's the reason why we need the cutoff which appears in the Proposition 2.3.

We first give an expansion of the operator P defined by (21).

LEMMA 2.4. — *Let*

$$P(s, x, h) = A_1(s, x, h)\partial_s^2 + A_2(s, x, h)\partial_s + A_3(s, x, h)\partial_x + A_4(s, x, h),$$

be the differential operator defined by (21). Then for all $p \geq 2$, P can be written

$$P(s, x, h) = \sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k(s, x) + h^{\frac{p}{2}} \tilde{P}_p(s, x, h), \tag{28}$$

so that
i) *For all* $0 \leq k \leq p-1$,

$$P_k(s,x) = A_1^k(s,x)\partial_s^2 + A_2^k(s,x)\partial_s + A_3^k(s,x)\partial_x + A_4^k(s,x),$$

where $A_j^k \in \mathcal{C}^\infty([0,2\pi]\times\mathbb{R})$, *for all* $s\in[0,2\pi]$ *the function* $x\mapsto A_j^k(s,x)$ *is a polynomial and* $A_j^k(s+2\pi,x) = A_j^k(s,\omega x)$.
ii) *Let* $\chi \in \mathcal{C}_0^\infty(]-r_0,r_0[)$ *and* $v\in\mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$, *then for all* $n\in\mathbb{N}$, *there exists* $C = C(p,n)$ *independent of* $h\in]0,1]$ *so that*

$$\|\chi(h^{\frac{1}{2}}x)\tilde{P}_p v(s,x)\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C. \tag{29}$$

Proof. — We first compute the coefficients of P.
By the Taylor formula near $r=0$ we have

$$\begin{aligned}\frac{1}{a^2}(s,r) = 1 + R(s)r^2 - 2R_3(s)r^3 + \sum_{k=4}^{p+3} r^k R_k(s)\\ + \frac{r^{p+4}}{(p+3)!}\int_0^1 (1-t)^{p+3}\frac{\partial^{p+4}}{\partial r^{p+4}}\Big(\frac{1}{a^2}\Big)(s,tr)\,\mathrm{d}t,\end{aligned}$$

where R_k is given by (7).
Now write $r=\sqrt{h}x$ and obtain

$$\frac{1}{a^2}(s,\sqrt{h}x) = 1 + hR(s)x^2 - 2h^{\frac{3}{2}}R_3(s)x^3 + h^2 I_1(s,x,h), \tag{30}$$

where

$$\begin{aligned}I_1(s,x,h) = \sum_{k=4}^{p+3} h^{\frac{k-4}{2}} x^k R_k(s)+\\ + h^{\frac{p}{2}}\frac{x^{p+4}}{(p+3)!}\int_0^1 (1-t)^{p+3}\frac{\partial^{p+4}}{\partial r^{p+4}}\Big(\frac{1}{a^2}\Big)(s,\sqrt{h}xt)\,\mathrm{d}t.\end{aligned} \tag{31}$$

Similarly

$$\frac{1}{a}\partial_s(\frac{1}{a})(s,\sqrt{h}x) = hI_2(s,x,h), \tag{32}$$

with

$$\begin{aligned}I_2(s,x,h) = \sum_{k=2}^{p+1} h^{\frac{k-2}{2}}\frac{x^k}{k!}\frac{1}{a}\partial_s(\frac{1}{a})(s,0)+\\ + h^{\frac{p}{2}}\frac{x^{p+2}}{(p+1)!}\int_0^1 (1-t)^{p+1}\frac{\partial^{p+2}}{\partial r^{p+2}}\Big(\frac{1}{a}\partial_s(\frac{1}{a})\Big)(s,\sqrt{h}xt)\,\mathrm{d}t,\end{aligned} \tag{33}$$

and

$$\frac{\partial_r a}{a}(s,\sqrt{h}x) = h^{\frac{1}{2}} I_3(s,x,h), \tag{34}$$

where

$$I_3(s,x,h) = \sum_{k=1}^{p} h^{\frac{k-1}{2}} \frac{x^k}{k!} \frac{\partial^k}{\partial r^k}\left(\frac{\partial_r a}{a}\right)(s,0) + \\ + h^{\frac{p}{2}} \frac{x^{p+1}}{p!} \int_0^1 (1-t)^p \frac{\partial^{p+1}}{\partial r^{p+1}}\left(\frac{\partial_r a}{a}\right)(s,\sqrt{h}xt)\,\mathrm{d}t. \tag{35}$$

Plug the expressions (30), (32) and (34) in equation (20), and deduce that coefficients A_j are

$$\begin{aligned}
A_1 &= \frac{1}{2}(-1 - hRx^2 + 2h^{\frac{3}{2}} R_3 x^3 - h^2 I_1),\\
A_2 &= -iRx^2 + 2ih^{\frac{1}{2}} R_3 x^3 - ihI_1 - \frac{1}{2} I_2,\\
A_3 &= -\frac{1}{2} I_3,\\
A_4 &= \frac{1}{2}(I_1 - iI_2).
\end{aligned}$$

Then with the developments (31), (33) and (35), we see that for all $1 \le j \le 4$ and $0 \le k \le p-1$, $x \mapsto A_j^k(s,x)$ is a polynomial. Moreover as $a(s+2\pi,x) = a(s,\omega x)$, we also have $A_j^k(s+2\pi,x) = A_j^k(s,\omega x)$.

To obtain the bound (29), we now have to control the integral rests which appear in (31),(33) and (35).

Let $q \in \mathbb{N}^*$ and let $(s,r) \mapsto f(s,r)$ be one of the functions a^{-2}, $a^{-1}\partial_s(a^{-1})$ or $a^{-1}\partial_r$. Let $\chi \in \mathcal{C}^\infty\,(]-r_0,r_0[)$ and define F_q by

$$F_q(s,x) = \chi(\sqrt{h}x)\int_0^1 (1-t)^{q-1} \frac{\partial^q}{\partial r^q} f(s,\sqrt{h}xt)\,\mathrm{d}t.$$

As $f \in \mathcal{C}^\infty\,([0,2\pi]\times]-r_0,r_0[)$, we deduce that for all $n_1,n_2 \in \mathbb{N}$ there exists $C = C(q,n_1,n_2)$, independent of $h \in]0,1]$ so that

$$\forall (s,x) \in [0,2\pi]\times\mathbb{R}, \qquad |\partial_s^{n_1}\partial_x^{n_2} F_q(s,x)| \le C. \tag{36}$$

Now let $v \in \mathcal{C}^\infty\,([0,2\pi],\mathcal{S}(\mathbb{R}))$ and $n \in \mathbb{N}$. We can assume that $n \ge 2$, so that H^n is an algebra. Then by (36)

$$\|x^q\, F_q\, v\|_{H^n([0,2\pi]\times\mathbb{R})} \le C\|F_q\|_{H^n([0,2\pi]\times\mathbb{R})}\|x^q\, v\|_{H^n([0,2\pi]\times\mathbb{R})} \le C, \tag{37}$$

and this yields ii). □

Consider the Hilbertian basis of $L^2(\mathbb{R})$ composed of the Hermite functions $(\varphi_k)_{k\geq 0}$ which are the eigenfunctions of the harmonic oscillator $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$, i.e., $H\varphi_k = (k+\frac{1}{2})\varphi_k$. Moreover $\varphi_k(x) = P_k(x)\mathrm{e}^{-x^2/2}$ where P_k is a polynomial of degree k with $P_k(-x) = (-1)^k P_k(x)$. The link between the s-dependent operator $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$ and H is given by the following result proved by M. Combescure in [11].

Theorem 2.5. — *Let $a_0 : \mathbb{R} \longrightarrow \mathbb{C}$ be the solution of (14) with $a_0(0) = 1$, $\dot{a}_0(0) = i$. Define*

$$\alpha = \log|a_0|,\ \beta = \frac{1}{2i}\log\frac{a_0}{\overline{a_0}},$$

let the unitary transform $T(s)$ be defined by

$$T(s) = e^{i\dot{\alpha}(s)x^2/2}e^{-i\alpha(s)D},\ \textit{where}\ D = -\frac{i}{2}(x\cdot\nabla + \nabla\cdot x),$$

and let $U(s,\tau)$ be the unitary evolution operator for $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$, i.e., $U(s,\tau)\varphi$ is the unique solution of the problem

$$\begin{cases} \left(i\partial_s + \dfrac{1}{2}\partial_x^2 - \dfrac{1}{2}R(s)x^2\right)u = 0, \\ u(\tau,x) = \varphi(x) \in L^2(\mathbb{R}). \end{cases}$$

Then we have for any $s,\tau \in \mathbb{R}$

$$U(s,\tau) = T(s)e^{-i(\beta(s)-(\beta(\tau))H}T(\tau)^{-1}.$$

Remark 2.6. — The functions α and β are well defined: suppose that there exists s_0 such that $a_0(s_0) = 0$, then $\operatorname{Re} a_0$ and $\operatorname{Im} a_0$ are linearly dependent, which is impossible with this choice of the initial conditions.

Remark 2.7. — Define $\theta(s) = \beta(s) - \frac{\lambda}{2\pi}s$ where λ is given by Assumption 2. Then α and θ are 2π-periodic real functions. Moreover $\alpha(0) = \dot{\alpha}(0) = \beta(0) = \theta(0) = 0$.

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space, i.e., the space of smooth functions which are fast decreasing and their derivatives too.

Proposition 2.8. — *Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ and $E \in \mathbb{C}$. Let $f \in \mathcal{C}^\infty([0,2\pi]\times\mathbb{R},\mathbb{R})$ be such that*

$$\forall n\in\mathbb{N},\ \forall s\in[0,2\pi],\quad \partial_s^n f(s,\cdot)\in\mathcal{S}(\mathbb{R}),$$

in other words $f \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.
Let $\psi \in \mathcal{C}^1\big([0,2\pi],L^2(\mathbb{R})\big)\cap\mathcal{C}^0\big([0,2\pi],H^2(\mathbb{R})\big)$ be the solution of

$$\begin{cases} i\partial_s\psi + \frac{1}{2}\partial_x^2\psi - \frac{1}{2}R(s)x^2\psi - E\psi = f, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{38}$$

Then $\psi \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$.

Proof. — By replacing ψ with $\mathrm{e}^{iEt}\psi$, we can assume that $E = 0$. The solution of equation (38) is given by

$$\begin{aligned} \psi(s,\cdot) &= U(s,0)\psi_0 - i\int_0^s U(s,\tau)f(\tau,\cdot)\,\mathrm{d}\tau \\ &= T(s)\mathrm{e}^{-i\beta(s)H}\left(\psi_0 - i\int_0^s e^{i\beta(\tau)H}T(\tau)^{-1}f(\tau,\cdot)\,\mathrm{d}\tau\right). \end{aligned} \tag{39}$$

As D is a transport operator, we have

$$T, T^{-1} : \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right) \longrightarrow \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right),$$

we only have to show that

$$\mathrm{e}^{i\beta H} : \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right) \longrightarrow \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right).$$

This follows from the fact that β is regular and $\mathrm{e}^{iH} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$. □

The description of U given in Theorem 2.5 yields the following representation of $U(s,0)\varphi_k$:

PROPOSITION 2.9. — *For all $k \in \mathbb{N}$ and $s, x \in \mathbb{R}$ we have*

$$U(s,0)\varphi_k(x) = e^{i\dot\alpha(s)x^2/2}e^{-i(\frac{1}{2}+k)\beta(s)}e^{-\frac{1}{2}\alpha(s)}\varphi_k\left(xe^{-\alpha(s)}\right). \tag{40}$$

Proof. — According to Theorem 2.5, and as $H\varphi_k = (k+\frac{1}{2})\varphi_k$,

$$U(s,0)\varphi_k = \mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(k+\frac{1}{2})\beta(s)}\mathrm{e}^{-i\alpha(s)D}\varphi_k.$$

Denote by $f(s) = \mathrm{e}^{-i\alpha(s)D}\varphi_k$. Then f is solution of the transport equation

$$\partial_s f = -\frac{1}{2}\dot\alpha(s)\left(x\partial_x f + \partial_x(xf)\right) = -\frac{1}{2}\dot\alpha(s)\left(f + 2x\partial_x f\right)$$

with Cauchy data $f(0,x) = \varphi_k(x)$. Make the change of variables $\sigma = \alpha(s)$ and set $g(\sigma) = f(s)$. Therefore g satisfies $\partial_\sigma g = -\frac{1}{2}(g + 2x\partial_x g)$. The equation $x = \dot x$, $x(0) = x_0$ admits the solution $x(\tau) = x_0\mathrm{e}^\tau$ and the characteristics method gives $g(\tau, x(\tau)) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x_0) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x(\tau)\mathrm{e}^{-\tau})$, hence

$$f(s) = \mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_k(x\mathrm{e}^{-\alpha(s)}).$$

□

Corollary 2.10. — *Let $k \in \mathbb{N}$, define $\omega_1 = \frac{1}{2}(\omega - 1)$ and $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$. Then*

$$\begin{aligned} w_k &= e^{-isE_0(k)}U(s,0)\varphi_k \\ &= e^{-isE_0(k)}\, e^{i\dot\alpha(s)x^2/2}\, e^{-i(\frac{1}{2}+k)\beta(s)}\, e^{-\frac{1}{2}\alpha(s)}\varphi_k\left(x e^{-\alpha(s)}\right) \end{aligned} \tag{41}$$

is solution of the equation

$$\left(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}R(s)x^2 - E_0(k)\right) w_k(s,x) = 0.$$

Proof. — On the one hand, from Proposition 2.9 we deduce

$$\begin{aligned} w_k(s+2\pi, x) &= \mathrm{e}^{-2i\pi E_0(k)}\mathrm{e}^{-i\lambda(\frac{1}{2}+k)}w_k(s,x) = \mathrm{e}^{-ik\omega_1\pi}w_k(s,x) \\ &= (-1)^{k\omega_1}w_k(s,x) = w_k(s,\omega x). \end{aligned}$$

On the other hand, w_k satisfies (22) because of the definition of $U(s,0)$. □

Fix $k_0 \in \mathbb{N}$ and take $v_0 = w_{k_0}$ with the previous choice of $E_0(k_0)$. This choice corresponds to the k_0th level of energy for the harmonic oscillator.

Remark 2.11. — Until now we did not use the restriction (12), but it will be crucial in the following.

Proposition 2.12. — *For all $p \geq 0$, there exist $E_p \in \mathbb{C}$ and $v_p \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ which solve (24).*

Remark 2.13. — As stated in Theorem 1.5, the E_j's are in fact real numbers. This will be proved in Lemma 2.17.

Proof. — We proceed by induction on $p \in \mathbb{N}$.
For $p = 0$ the result was proved in Corollary 2.10.
Let $p \geq 1$, and suppose that for all $j \leq p-1$ there exist $E_j \in \mathbb{C}$ and $v_j \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ which solve the $(j+1)$th equation of (22). When $p \geq 2$, set

$$\begin{aligned} \tilde v_{p-1} &= h^{\frac{1}{2}}v_1 + \cdots + h^{\frac{p-1}{2}}v_{p-1}, \\ \tilde E_{p-1} &= h^{\frac{1}{2}}E_1 + \cdots + h^{\frac{p-1}{2}}E_{p-1} \end{aligned}$$

and $\tilde v_0 = \tilde E_0 = 0$. By (28), the function Q_p given by (24) is the coefficient of $h^{\frac{p}{2}}$ in the expansion in h of

$$\tilde E_{p-1}\tilde v_{p-1} + \frac{1}{2}\varepsilon\delta^2|v_0 + \tilde v_{p-1}|^2(v_0 + \tilde v_{p-1}) + h\left(\sum_{k=0}^{p-1} h^{\frac{k}{2}}P_k\right)(v_0 + \tilde v_{p-1}).$$

Now using the regularity of the v_j's and the fact that for all $0 \leq k \leq p-1$, P_k is an operator

$$P_k : \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R})),$$

we obtain $Q_p \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.
Moreover Q_p satisfies, $\forall (s,x)\in[0,2\pi]\times\mathbb{R}$

$$Q_p(s+2\pi,x)=Q_p(s,\omega x)$$

because this property holds for the v_j's, and a.
Define $F_p(s,x)=\mathrm{e}^{-i\dot\alpha(s)\mathrm{e}^{2\alpha(s)}x^2/2}Q_p(s,xe^{\alpha(s)})$, then $F_p\in\mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$ and satisfies $Q_p(s,x)=\mathrm{e}^{i\dot\alpha(s)x^2/2}F_p(s,x\mathrm{e}^{-\alpha(s)})$ and $F_p(s+2\pi,x)=F_p(s,\omega x)$. Let us decompose F_p on the basis $(\varphi_j)_{j\geq 0}$: there exists a unique family of smooth functions $(g_j^p(s))_{j\geq0}\in l^2(\mathbb{N})$ so that

$$F_p(s,y)=\sum_{j\geq0}g_j^p(s)\varphi_j(y). \tag{42}$$

Then

$$Q_p(s,x)=\sum_{j\geq0}g_j^p(s)\mathrm{e}^{i\dot\alpha(s)x^2/2}\varphi_j(x\mathrm{e}^{-\alpha(s)})=\sum_{j\geq0}h_j^p(s)w_j(s,x), \tag{43}$$

where according to (41)

$$h_j^p(s)=\mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac12+j)\beta(s)}\mathrm{e}^{\frac12\alpha(s)}g_j^p(s). \tag{44}$$

We have

$$Q_p(s,\omega x)=\sum_{j\geq0}h_j^p(s)w_j(s,\omega x),$$

but also

$$\begin{aligned}Q_p(s,\omega x)=Q_p(s+2\pi,x)&=\sum_{j\geq0}h_j^p(s+2\pi)w_j(s+2\pi,x)\\&=\sum_{j\geq0}h_j^p(s+2\pi)w_j(s,\omega x),\end{aligned}$$

and from the uniqueness of the h_j^p's we deduce $h_j^p(s+2\pi)=h_j^p(s)$.
We are now looking for a solution of (24) of the form

$$v_p(s,x)=\sum_{j\geq0}e_j^p(s)w_j(s,x) \tag{45}$$

where the e_j^p's are 2π-periodic functions. For all $j\geq0$, by Corollary 2.10 we have

$$\left(i\partial_s+\frac12\partial_x^2-\frac12Rx^2\right)\left(e_j^pw_j\right)=i\dot e_j^pw_j+\left(E_0(k_0)-E_0(j)\right)e_j^pw_j,$$

hence we have to solve the equations

$$i\dot e_j^p+\left(E_0(k_0)-E_0(j)\right)e_j^p=h_j^p+\delta_{j,k_0}E_p. \tag{46}$$

As $E_0(k_0) - E_0(j) = \frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})$, the solutions of (46) take the form

$$e_j^p(s) = \mathrm{e}^{\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\Big(C_j^p - i\int_0^s h_j^p(\tau)\mathrm{e}^{-\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}\,\mathrm{d}\tau\Big) \tag{47}$$

for $j \neq k_0$, and

$$e_{k_0}^p(s) = C_{k_0}^p - i\int_0^s h_{k_0}^p(\tau)\,\mathrm{d}\tau - iE_p s.$$

The constants $C_j^p \in \mathbb{C}$ and $E_p \in \mathbb{C}$ have to be determined such that $e_j^p(s+2\pi) = e_j^p(s)$.

• Case $j = k_0$:

$$e_{k_0}^p(s+2\pi) = -i\int_0^{2\pi} h_{k_0}^p(\tau)\,\mathrm{d}\tau - 2\pi i E_p + e_{k_0}^p(s),$$

thus $e_{k_0}^p$ is 2π-periodic iff

$$E_p = -\frac{1}{2\pi}\int_0^{2\pi} h_{k_0}^p(\tau)\,\mathrm{d}\tau. \tag{48}$$

• Case $j \neq k_0$:

Denote by $\tilde{h}_j^p : \tau \longmapsto h_j^p(\tau)\mathrm{e}^{-i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}$ and by $K = \mathrm{e}^{i(k_0-j)(\pi\omega_1-\lambda)}$. Then

$$\begin{aligned}\int_0^{s+2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau &= \int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau + \int_{2\pi}^{s+2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau\\ &= \int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau + K^{-1}\int_0^{s} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau,\end{aligned}$$

and by (47)

$$\begin{aligned}e_j^p(s+2\pi) &= K\mathrm{e}^{i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\Big(C_j^p - i\int_0^{s+2\pi}\tilde{h}_j^p(\tau)\,\mathrm{d}\tau\Big)\\ &= \mathrm{e}^{i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\Big(KC_j^p - iK\int_0^{2\pi}\tilde{h}_j^p(\tau)\,\mathrm{d}\tau - i\int_0^{s}\tilde{h}_j^p(\tau)\,\mathrm{d}\tau\Big).\end{aligned} \tag{49}$$

Notice that $K \neq 1$, as $\lambda \notin \pi\mathbb{Q}$ and choose

$$C_j^p = \frac{iK}{K-1} \int_0^{2\pi} \tilde{h}_j^p(\tau)\,\mathrm{d}\tau,$$

then according to (47)and (49), the function e_j^p is $2\pi-$periodic.

Now, we show that the constants C_j^p are uniformly bounded in $j \geq 0$, so that the function v_p given by (45) is well defined. We first need the

Lemma 2.14. — *Let $(h_j^p)_{j\geq 0} \in l^2(\mathbb{N})$ be the family of $2\pi-$periodic functions defined by (44) and $h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$ its Fourier decomposition. Then for all $n_1, n_2 \in \mathbb{N}$ there exists $C^p > 0$ such that for all $j \in \mathbb{N}$*

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c_{l,j}^p|^2 \leq C^p.$$

Proof. — Consider the function $F_p \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$ which defines the family $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ with (42). Denote by $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$. Let $n_1, n_2 \in \mathbb{N}$ and decompose the function $\partial_s^{n_2} H^{n_1} F_p$ on the basis $(\varphi_j)_{j\geq 0}$

$$\partial_s^{n_2} H^{n_1} F_p(s,y) = \sum_{j\geq 0} \tilde{g}_j^p(s)\varphi_j(y)$$

where $(\tilde{g}_j^p)_{j\geq 0}$ is a smooth family of functions in $l^2(\mathbb{N})$.
Using that $H\varphi_j = (j+\frac{1}{2})\varphi_j$ and that $F_p \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$, we have for all $n_1, n_2 \in \mathbb{N}$

$$\partial_s^{n_2} H^{n_1} F_p(s,y) = \sum_{j\geq 0} (j+\frac{1}{2})^{n_1} (g_j^p)^{(n_2)}(s)\varphi_j(y).$$

By uniqueness of such a decomposition,

$$\left((j+\frac{1}{2})^{n_1}(g_j^p)^{(n_2)}\right)_{j\geq 0} = (\tilde{g}_j^p)_{j\geq 0} \in l^2(\mathbb{N}).$$

Then by the definition (44) of h_j^p, an easy induction on $n_1, n_2 \in \mathbb{N}$ shows that $\left(j^{n_1}(h_j^p)^{(n_2)}\right)_{j\geq 0} \in l^2(\mathbb{N})$. Write the Fourier decomposition of h_j^p

$$h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$$

and by Parseval

$$\sum_{j\geq 0}\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}(s)|c_{l,j}^p|^2 = \sum_{j\geq 0} j^{2n_1} \int_0^{2\pi} |(h_j^p)^{(n_2)}(s)|^2\,\mathrm{d}s \leq C^p.$$

In particular, for all $j \in \mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c^p_{l,j}|^2 \leq C^p,$$

hence the result. ☐

End of the proof of Proposition 2.12: Using the Fourier decomposition of h_j we obtain

$$\begin{aligned} C^p_j &= \frac{iK}{K-1}\int_0^{2\pi} \tilde{h}^p_j(\tau)\,\mathrm{d}\tau \\ &= \frac{iK}{K-1}\sum_{l\in\mathbb{Z}} c^p_{l,j}\int_0^{2\pi} \mathrm{e}^{i(l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi}))\tau} \\ &= -i\sum_{l\in\mathbb{Z}} \frac{c^p_{l,j}}{l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})}. \end{aligned} \tag{50}$$

With Assumption 2 we have

$$\begin{aligned} \big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| &= \frac{1}{2}|(2l-(k_0-j)\omega_1)+(k_0-j)\frac{\lambda}{\pi}| \\ &\geq \frac{1}{2}\frac{\mu}{|(2l-(k_0-j)\omega_1, k_0-j)|^\tau}, \end{aligned}$$

and for $j\geq k_0$, $|2l-(k_0-j)\omega_1|+|k_0-j|\leq 2(|l|+|j|)$, then

$$\big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| \geq \frac{c\mu}{(|l|+|j|)^\tau}. \tag{51}$$

Hence, from (50) and (51) we deduce

$$|C^p_j| \lesssim \sum_{l\in\mathbb{Z}} |c^p_{l,j}|(|j|+|l|)^\tau \lesssim \sum_{l\in\mathbb{Z}} |c^p_{l,j}|(|j|^\tau+|l|^\tau). \tag{52}$$

By Cauchy-Schwarz and Lemma 2.14, from (52) we obtain

$$\begin{aligned} |C^p_j| &\lesssim \sum_{l\in\mathbb{Z}} \frac{1+|l|}{1+|l|}|c^p_{l,j}|(|j|^\tau+|l|^\tau) \\ &\lesssim \Big(\sum_{l\in\mathbb{Z}}\frac{1}{(1+|l|)^2}\Big)^{\frac{1}{2}}\Big(\sum_{l\in\mathbb{Z}}|c^p_{l,j}|^2(1+|l|)^2(|j|^{2\tau}+|l|^{2\tau})\Big)^{\frac{1}{2}} \\ &\leq C^p. \end{aligned} \tag{53}$$

Set

$$v_p(s,x) = \sum_{j\geq 0} e^p_j(s) w_j(s,x).$$

For all $j \in \mathbb{N}$, $s \longmapsto e_j^p(s)w_j(s,x)$ is continuous and there exists $c>0$ such that for all $j>k_0$, and for all $s \in [0,2\pi]$

$$|e_j^p(s)w_j(s,x)| \lesssim |g_j^p(s)||\varphi_j(cx)|$$

and this shows that $v_p \in C\left([0,2\pi], L^2(\mathbb{R})\right)$. Now using Proposition 2.8 we conclude, by uniqueness of such a solution, that $v_p \in \mathcal{C}^\infty\left([0,2\pi], \mathcal{S}(\mathbb{R})\right)$. □

2.2. The nonlinear analysis and proof of Proposition 2.3

Lemma 2.15. — *The constant E_1 given by Proposition 2.12 writes $E_1 = -\varepsilon\delta^2 C_0$ where $C_0>0$ is independent of ε and δ.*

Proof. — Consider the equation

$$\left(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\right)v_1 = E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0,$$

with

$$v_0(s,x) = \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right).$$

By the definition of Q_p (see (24)),

$$Q_1(s,x) = R_3(s)x^3 v_0(s,x) + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0(s,x),$$

and by (43), Q_1 can be written

$$Q_1(s,x) = \sum_{j\geq 0} h_j^1(s)w_j(s,x).$$

According to formula (48), we only have to compute the term $h_{k_0}^1$ in the previous expansion.
Write the expansion of $|\varphi_{k_0}|^2\varphi_{k_0}$ on the basis $(\varphi_j)_{j\geq 0}$:

$$|\varphi_{k_0}|^2\varphi_{k_0} = \sum_{j\geq 0} p_j\varphi_j, \tag{54}$$

with $p_j \in \mathbb{R}$ and $p_j = 0$ for $j-k_0 = 1$ *mod* 2 as $\varphi_k(-x) = (-1)^k\varphi_k(x)$.
Then by (54) and the expression (41) of w_j

$$\begin{aligned}
|v_0|^2 v_0(s,x) &= \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}|\varphi_{k_0}|^2\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right)\\
&= \sum_{j\geq 0} p_j \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}\varphi_j\left(x\mathrm{e}^{-\alpha(s)}\right)\\
&= \sum_{j\geq 0} f_j(s)w_j(s,x)
\end{aligned}$$

where

$$f_j(s) = p_j \mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{-\alpha(s)}$$
$$= p_j \mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{-\alpha(s)}.$$

Therefore $f_{k_0}(s) = p_{k_0}\mathrm{e}^{-\alpha(s)}$ with, using (54), $p_{k_0} = \int_{\mathbb{R}} |\phi_{k_0}|^4 > 0$. In the same manner we write

$$x^3\varphi_{k_0}(x) = \sum_{j\geq 0} q_j\varphi_j(x), \tag{55}$$

with $q_j = 0$ when $j - k_0 = 0 \ mod \ 2$ and by (55) we have

$$\begin{aligned} &R_3(s)x^3 v_0(s,x) \\ &\quad = R_3(s)\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\mathrm{e}^{-i(\frac12+k_0)\beta(s)}\mathrm{e}^{\frac52\alpha(s)}(x\mathrm{e}^{-\alpha(s)})^3\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right) \\ &\quad = \sum_{j\geq 0} q_j R_3(s)\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{-i(\frac12+k_0)\beta(s)}\mathrm{e}^{\frac52\alpha(s)}\mathrm{e}^{i\dot\alpha(s)x^2/2}\varphi_j\left(x\mathrm{e}^{-\alpha(s)}\right). \end{aligned}$$

By (41) we have

$$\mathrm{e}^{i\dot\alpha(s)x^2/2}\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right) = \mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac12+j)\beta(s)}\mathrm{e}^{\frac12\alpha(s)}.$$

Then

$$R_3(s)x^3 v_0(s,x) = \sum_{j\geq 0}\tilde f_j(s) w_j(s,x),$$

where

$$\tilde f_j(s) = q_j R_3(s)\mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{3\alpha(s)}$$
$$= q_j R_3(s)\mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{3\alpha(s)}.$$

Then $\tilde f_{k_0} = 0$ as $q_j = 0$ when $j - k_0 = 0 \ mod \ 2$. Thus

$$h^1_{k_0}(s) = \frac12\varepsilon\delta^2 f_{k_0}(s) = \frac12\varepsilon\delta^2 p_{k_0}\mathrm{e}^{-\alpha(s)}.$$

Finally, from (48) we deduce

$$E_1 = -\frac{1}{4\pi}\varepsilon\delta^2 p_{k_0}\int_0^{2\pi}\mathrm{e}^{-\alpha(\tau)}\,\mathrm{d}\tau = -\varepsilon\delta^2 C_0,$$

where $C_0 > 0$ as $p_{k_0} > 0$. □

LEMMA 2.16. — *Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\psi = 0$ near 0, and let $f \in \mathcal{S}(\mathbb{R})$. Then for all $n, N \in \mathbb{N}$, there exists $C = C(n,N)$ so that*

$$\|\psi(h^{\frac12}\cdot)f\|_{H^n(\mathbb{R})} \leq Ch^N. \tag{56}$$

Proof. — We only show (56) for $n = 0$, the general case follows from the Leibniz rule. We can assume that $\operatorname{supp}\psi \subset [a, b]$ with $a > 0$. Then as $f \in \mathcal{S}(\mathbb{R})$, for all $N \in \mathbb{N}$, there exists $C_N > 0$ so that

$$|f(x)| \leq C_N \frac{1}{1 + |x|^N}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} |\psi(h^{\frac{1}{2}}x)|^2 |f(x)|^2 \,\mathrm{d}x &= h^{-\frac{1}{2}} \int_a^b |\psi(x)|^2 |f(h^{-\frac{1}{2}}x)|^2 \,\mathrm{d}x \\ &\leq C_N h^{N-\frac{1}{2}} \int_a^b |\psi(x)|^2 \frac{1}{h^N + x^{2N}} \,\mathrm{d}x \\ &\leq C_N h^{N-\frac{1}{2}},\end{aligned}$$

hence the result. □

Proof of Proposition 2.3. — Let $p \geq 1$, and consider

$$V_p(s, x) = \left(v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p\right)(s, x),$$

and

$$\tilde{E}_p = E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p,$$

where the v_j's and the E_j's are given by Proposition 2.12.
Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be an even function such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $[-r_0/2, r_0/2]$.
We claim that there exists $G_p(h) \in \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R}))$, so that

$$\forall n \in \mathbb{N}, \quad \|G_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C_{n,p}, \tag{57}$$

where $C_{n,p}$ is independent of $h \in]0, 1]$, and such that $G_p(h)$ satisfies

$$\begin{aligned}\chi(h^{\frac{1}{2}}x)\Big((i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - \tilde{E}_p)V_p \\ -h^{\frac{1}{2}}R_3x^3V_p - \frac{1}{2}\varepsilon\delta^2 h^{\frac{1}{2}}|V_p|^2V_p - hPV_p\Big) = h^{\frac{p+1}{2}}G_p(h).\end{aligned} \tag{58}$$

By construction of the v_j's and the E_j's, in the l.h.s. of (58), the coefficient of h^j cancels for $0 \leq j \leq p$.
Then write the expansion in powers of h

$$\frac{1}{2}\varepsilon\delta^2|V_p|^2V_p = \sum_{k=0}^{3p+1} h^{\frac{k}{2}} V_p^k,$$

and use (28) to obtain

$$hPV_p = h\left(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k + h^{\frac{p}{2}}\tilde{P}_p\right)\left(\sum_{k=0}^{p} h^{\frac{k}{2}} v_k\right) := \sum_{k=0}^{2p+2} h^{\frac{k}{2}} W_p^k$$

We therefore obtain the explicit formula of $G_p(h)$

$$\begin{aligned} h^{\frac{p+1}{2}} G_p(h) &:= -\chi(h^{\frac12}x)\sum_{k=p+1}^{2p+2} h^{\frac{k}{2}} W_p^k - \chi(h^{\frac12}x)\sum_{k=p+1}^{3p+1} h^{\frac{k}{2}} V_p^k - \chi(h^{\frac12}x)h^{\frac{p+1}{2}} R_3x^3v_p \\ &= -h^{\frac{p+1}{2}}\chi(h^{\frac12}x)\left(\sum_{l=0}^{p+1} h^{\frac{l}{2}} W_p^{l+p+1}\sum_{l=0}^{2p} h^{\frac{l}{2}} V_p^{l+p+1} + R_3x^3v_p\right). \end{aligned}$$

The bound (57) then follows from an application of Lemma 2.4.

Denote by $\tilde{V}_p = \chi(h^{\frac12}x)V_p$, and write

$$\begin{aligned} P\tilde{V}_p &= (A_1\partial_s^2 + A_2\partial_s + A_3\partial_x + A_4)(\chi(h^{\frac12}x)V_p) \\ &= \chi(h^{\frac12}x)PV_p + h^{\frac12}\chi'(h^{\frac12}x)A_3V_p. \end{aligned}$$

By (58) we deduce that

$$\begin{aligned} &(i\partial_s + \frac12\partial_x^2 - \frac12 Rx^2 - \tilde{E}_p)\tilde{V}_p - h^{\frac12}R_3x^3\tilde{V}_p - \frac12\varepsilon\delta^2h^{\frac12}|\tilde{V}_p|^2\tilde{V}_p - hP\tilde{V}_p \\ &= h^{\frac{p+1}{2}}G_h^p + h^{\frac12}\chi'(h^{\frac12}x)\partial_xV_p + \frac12 h\chi''(h^{\frac12}x)V_p \\ &\quad + \frac12\varepsilon\delta^2h^{\frac12}\chi(1-\chi^2)(h^{\frac12}x)|V_p|^2V_p - h^{\frac32}\chi'(h^{\frac12}x)A_3V_p \\ &:= h^{\frac{p+1}{2}}\tilde{G}_p(h). \end{aligned}$$

Each of the functions χ', χ'' and $\chi(1-\chi^2)$ vanishes near 0, hence by Lemma 2.16 and (57)

$$\forall n \in \mathbb{N}, \quad \|\tilde{G}_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \le C_{n,p}. \tag{59}$$

Finally, set

$$u_p = \delta h^{-\frac14} \mathrm{e}^{i\frac{s}{h}} V_p(s, \frac{r}{\sqrt{h}}),$$

then

$$-\Delta u_p - \lambda_p u_p + \varepsilon|u_p|^2u_p = \frac{2}{h}\mathrm{e}^{i\frac{s}{h}} h^{\frac{p+1}{2}}\tilde{G}_p(h),$$

and $g_p(h) = 2\mathrm{e}^{i\frac{s}{h}}\tilde{G}_p(h)$ satisfies the conclusion of Proposition 2.3 by (59).

□

Lemma 2.17. — *Let $p \ge 1$ and E_p given by Proposition (2.12). Then $E_p \in \mathbb{R}$.*

Proof. — We already know that $E_0, E_1 \in \mathbb{R}$. Let $p \ge 3$. Multiply (27) by $\overline{u}_p$, integrate on M and take the imaginary part

$$0 = \|u_p\|_{L^2}^2 \operatorname{Im}\lambda_p + h^{\frac{p-1}{2}} \operatorname{Im}\int g_p(h)\overline{u}_p.$$

As $\|u_p\|_{L^2} \sim 1$ and $\|g_p\|_{L^2} \lesssim 1$, we obtain the estimate

$$|\operatorname{Im} \lambda_p| \lesssim h^{\frac{p-1}{2}} \|g_p\|_{L^2} \|u_p\|_{L^2} \lesssim h^{\frac{p-1}{2}}$$

and as

$$\operatorname{Im} \lambda_p = -2(\operatorname{Im} E_2 + h^{\frac{1}{2}} \operatorname{Im} E_3 + \cdots + h^{\frac{p-1}{2}} \operatorname{Im} E_p)$$

it follows that for all $0 \leq j \leq p-1$, $\operatorname{Im} E_j = 0$, i.e., $E_j \in \mathbb{R}$. □

3. The instability for the nonlinear Schrödinger equation

3.1. The error estimate

PROPOSITION 3.1. — *Let $\alpha > 0$, $\sigma \in]0, \frac{1}{4}]$ and let $v \in H^2(M)$ be such that*

$$\|v\|_{L^2} \lesssim 1, \quad \|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}, \quad \|\Delta v\|_{L^\infty} \lesssim h^{-\frac{9}{4}+\sigma},$$

and suppose that v satisfies

$$i\partial_t v + \Delta v = \varepsilon |v|^2 v + h^\alpha R(h),$$

with for all $\beta \in [0, 2]$, $\|R(h)\|_{H^\beta} \lesssim h^{-\beta}$. Let u be solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u, \\ u(0, x) = v(0, x). \end{cases}$$

Then, if $\alpha > \frac{1}{4} + 3\sigma$ we have

$$\|(u - v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \text{when} \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$.

Proof. — Define $w = u - v$ and

$$E(t) = \|w\|_{L^2}^2 + \|h^2 \Delta w\|_{L^2}^2.$$

We have $E(0) = 0$ and the following estimates:

$$\|w\|_{L^2} \leq E^{\frac{1}{2}}, \quad \|\Delta w\|_{L^2} \leq h^{-2} E^{\frac{1}{2}}, \quad \|\nabla w\|_{L^2} \leq h^{-1} E^{\frac{1}{2}}. \tag{60}$$

The function w satisfies the equation

$$i\partial_t w + \Delta w = \varepsilon(|w + v|^2 (w + v) - |v|^2 v) - h^\alpha R(h). \tag{61}$$

The energy method gives

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|w\|_{L^2}^2 &= \operatorname{Im} \int \overline{w} \left(\varepsilon(|w + v|^2 (w + v) - |v|^2 v) - h^\alpha R(h) \right) \\ &\lesssim h^\alpha \|w\|_{L^2} + \|w\|_{L^4}^4 + \|w\|_{L^2}^2 \|v\|_{L^\infty}^2. \end{aligned}$$

The Gagliardo-Nirenberg inequality gives

$$\|w\|_{L^4}^4 \lesssim \|w\|_{L^2}^2 \|\nabla w\|_{L^2}^2 \lesssim h^{-2} E^2,$$

and as $\|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma} E + h^{-2} E^2. \tag{62}$$

Now, apply Δ to (61)

$$i\partial_t \Delta w + \Delta^2 w = \varepsilon \Delta A - h^\alpha \Delta R(h), \tag{63}$$

with

$$\begin{aligned} A &= |w+v|^2(w+v) - |v|^2 v \\ &= 2w|v|^2 + \overline{w}v^2 + w^2\overline{v} + 2|w|^2 v + |w|^2 w, \end{aligned}$$

then

$$\begin{aligned} |\Delta A| \lesssim\ & |v|^2|\Delta w| + |v||\nabla v||\nabla w| + |\nabla v|^2|w| + |v||\Delta v||w| \\ & +|\Delta v||w|^2 + |w|^2|\Delta w| + |w||\nabla w|^2, \end{aligned}$$

hence

$$\begin{aligned} \|\Delta A\|_{L^2} \lesssim\ & \|v\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^\infty}\|\nabla w\|_{L^2} + \|\nabla v\|_{L^\infty}^2\|w\|_{L^2} \\ & +\|v\|_{L^\infty}\|\Delta v\|_{L^\infty}\|w\|_{L^2} + \|\Delta v\|_{L^\infty}\|w\|_{L^4}^2 \\ & +\|w\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|w\|_{L^2}\|\nabla w\|_{L^4}^2. \end{aligned} \tag{64}$$

The following inequality holds in dimension 2

$$\|w\|_{L^\infty} \lesssim \|w\|_{L^2}^{\frac{1}{2}}\|\Delta w\|_{L^2}^{\frac{1}{2}} \lesssim h^{-1}E^{\frac{1}{2}},$$

and with (60) and (64) we deduce

$$\|\Delta A\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-\frac{13}{4}+\sigma}E + h^{-4}E^{\frac{3}{2}}.$$

But

$$h^{-\frac{13}{4}+\sigma}E = h^{-\frac{5}{4}+\sigma}E^{\frac{1}{4}}h^{-2}E^{\frac{3}{4}} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}},$$

and we obtain

$$\|\Delta(A)\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}}. \tag{65}$$

Now, using (65) and $\|\Delta(R(h))\|_{L^2} \lesssim h^{-2}$, the energy method and the Cauchy-Schwarz inequality gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w\|_{L^2}^2 &= \operatorname{Im}\int \Delta\overline{w}\,(\Delta A - h^\alpha \Delta R(h)) \\ &\lesssim h^{-2}E^{\frac{1}{2}}(h^{\alpha-2} + h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}}), \end{aligned} \tag{66}$$

therefore from (62) and (66) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma}E + h^{-2}E^2.$$

Interpolation gives

$$\|w\|_{H^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{\dot{H}^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{L^2}^{1-\frac{\sigma}{2}} \|\Delta w\|_{L^2}^{\frac{\sigma}{2}} \lesssim h^{-\sigma} E^{\frac{1}{2}} := F.$$

The function F satisfies $F(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma} F + h^{-2+2\sigma} F^3. \tag{67}$$

As long as $h^{-2+2\sigma} F^3 \lesssim h^{-\frac{1}{2}+2\sigma} F$, i.e., $F \lesssim h^{\frac{3}{4}}$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma} F,$$

and the Gronwall inequality yields

$$F \lesssim h^{\alpha+\frac{1}{2}-3\sigma} \mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma} t}.$$

The nonlinear term in (67) can be removed with the continuity argument for times such that

$$h^{\alpha+\frac{1}{2}-3\sigma} \mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma} t} \lesssim h^{\frac{3}{4}+\eta},$$

with $\eta > 0$ i.e., for $t \lesssim (\alpha - \frac{1}{4} - 3\sigma - \eta) h^{\frac{1}{2}-2\sigma} \log \frac{1}{h}$, which is possible with η small enough as we assume $\alpha > \frac{1}{4} + 3\sigma$. □

Corollary 3.2. — *Let $\kappa > 0$, $0 < \sigma < \frac{1}{4}$ and set $\delta = \kappa h^\sigma$. Denote by $v = e^{-i\lambda_3 t} u_3$ where u_3 and λ_3 are defined by (25) and (26) respectively.*
Let u be solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u, \\ u(0,x) = v(0,x). \end{cases}$$

Then $\|v\|_{H^\sigma} \sim 1$ and

$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \textit{when} \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$.

Proof. — The result directly follows from Propositions 2.3 and 3.1, as for all $0 < \sigma < \frac{1}{4}$, we have $\sigma + 1 > \frac{1}{4} + 3\sigma$. □

3.2. The instability argument. — Let $\kappa, \kappa_h > 0$ and consider $v = v^1$ defined in Corollary 3.2 associated with κ and v^2 associated with κ_h. Let u be a solution of

$$\begin{cases} i\partial_t u^j + \Delta u^j = \varepsilon |u^j|^2 u^j, \\ u^j(0,x) = v^j(0,x), \end{cases}$$

and $t_h \sim h^{\frac{1}{2}-2\sigma} \log \frac{1}{h}$. Then

$$\begin{aligned} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \|(v^2 - v^1)(t_h)\|_{H^\sigma} - \|(u^2 - v^2)(t_h)\|_{H^\sigma} \\ - \|(u^1 - v^1)(t_h)\|_{H^\sigma}. \end{aligned} \tag{68}$$

From Corollary 3.2 we deduce that for $j = 1, 2$

$$\|(u^j - v^j)(t_h)\|_{H^\sigma} \longrightarrow 0. \tag{69}$$

Observe that

$$\|(v^2 - v^1)(t_h)\|_{H^\sigma} \sim \left| \mathrm{e}^{-i\lambda_3^2 t_h} - \mathrm{e}^{-i\lambda_3^1 t_h} \right| = \left| \mathrm{e}^{i(\lambda_3^2 - \lambda_3^1)t_h} - 1 \right|,$$

from Lemma 2.15 we have

$$(\lambda_3^2 - \lambda_3^1)t_h \sim h^{2\sigma-1}(\kappa - \kappa_h)t_h \sim (\kappa - \kappa_h) \log \frac{1}{h}.$$

It is possible to choose κ_h such that $\kappa_h \longrightarrow \kappa$ and $(\kappa - \kappa_h) \log \frac{1}{h} \longrightarrow \infty$. Then using (68) and (69)

$$\limsup_{h \longrightarrow 0} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \limsup_{h \longrightarrow 0} \|(v^2 - v^1)(t_h)\|_{H^\sigma} \geq 2,$$

even though

$$\|(u^2 - u^1)(0)\|_{H^\sigma} = \|(v^2 - v^1)(0)\|_{H^\sigma} \sim |\kappa - \kappa_h|,$$

which tends to 0 with h. According to Definition 1.1, we have proved Proposition 1.3.

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