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Bernard Host & Bryna Kra

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PARALLELEPIPEDS, NILPOTENT GROUPS AND GOWERS NORMS

by Bernard Host & Bryna Kra

Abstract. — In his proof of Szemerédi's Theorem, Gowers introduced certain norms that are defined on a parallelepiped structure. A natural question is on which sets a parallelepiped structure (and thus a Gowers norm) can be defined. We focus on dimensions 2 and 3 and show when this possible, and describe a correspondence between the parallelepiped structures and nilpotent groups.

Résumé (*Parallélépipèdes, groupes nilpotents et normes de Gowers*)
Dans sa preuve du théorème de Szemerédi, Gowers a introduit certaines normes définies par sommation sur des parallélépipèdes. Il est naturel de se demander sous quelles hypothèses on peut généraliser sa définition des parallélépipèdes et donc de ses normes. Nous nous restreignons aux dimensions 2 et 3 et décrivons une correspondance entre structures de parallélépipèdes et groupes nilpotents.

1. Introduction

In his proof of Szemerédi's Theorem [17], Gowers [4] introduced certain norms for functions on $\mathbb{Z}/N\mathbb{Z}$. Shortly thereafter, these norms were adapted

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Bernard Host, Université Paris-Est, Laboratoire Analyse et Mathématiques Appliquées, UMR CNRS 8050, 5 bd Descartes, 77454 Marne la Vallée Cedex 2, France • *E-mail :* bernard.host@univ-mlv.fr

Bryna Kra, Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA • *E-mail :* kra@math.northwestern.edu

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for a variety of other uses, including Green and Tao's major breakthrough showing that the primes contain arbitrarily long arithmetic progression [7] and their use in deriving finer asymptotics on structures in the primes ([5] and [6]). Similar seminorms were independently introduced by the authors and used to show convergence of some multiple ergodic averages [10]. Since then, the seminorms have been used for a variety of related problems in ergodic theory, including multiple averages along polynomial times ([9] and [15]), averages for certain commuting transformation [2] and averages along the primes [1].

Our goal here is to introduce and describe the most general context in which the first two Gowers norms can be defined. We call a 'parallelogram structure,' respectively a 'parallelepiped structure,' the weakest structure a set must have so that one can define a Gowers 2-norm, respectively a Gowers 3-norm, on the set.

The first Gowers norm is the absolute value of the sum of the values of the function, and in fact is only a seminorm. The second Gowers norm can be completely described using Fourier analysis (in terms of the ℓ^4 norm of the Fourier transform), and thus is closely linked to the abelian group structure of the circle. Analogously, in ergodic theory the second seminorm can be characterized completely by the Kronecker factor in a measure preserving system (see Furstenberg [3]), which is the largest abelian group rotation factor. The third Gowers norm is less well understood and can not be simply described in terms of Fourier analysis. In ergodic theory the third seminorm corresponds to a 2-step nilsystem, and more generally the k-th seminorm corresponds to a $(k-1)$-step nilsystem. (See [10] for the definition and precise statement; in the current context, the definition is given in Section 3.7.) In combinatorics, Green and Tao [5] have recently given a weak inverse theorem, but for the third Gowers norm the correspondence with a 2-step nilpotent group is not yet completely understood. We give conditions on a set that explain to what extent the correspondence with nilsystems can be made precise.

We start by defining the Gowers norms for $k \geq 2$. Let P denote the subset

$$\{(x_{00}, x_{01}, x_{10}, x_{11}) \in (\mathbb{Z}/N\mathbb{Z})^4 \colon x_{00} - x_{01} - x_{10} + x_{11} = 0\}$$

of $(\mathbb{Z}/N\mathbb{Z})^4$. For a function $f \colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, the *second Gowers norm* $\|f\|_{U_2}$ is defined [(1)] by

$$\|f\|_{U_2}^4 = \sum_{(x_{00},x_{01},x_{10},x_{11})\in\mathsf{P}} f(x_{00})\overline{f(x_{01})}\,\overline{f(x_{10})}f(x_{11}) \ .$$

(Although this agrees with Gowers's original definition, Green and Tao prefer to normalize the sum and define the norm as an average instead of a sum. In

(1) The notation $\|\cdot\|_{U_k}$ was introduced later in the work of Green and Tao [7].

our context we prefer to work with the sum.) To define the norms U_k for $k \geq 3$, we need some notation.

Notation. — The elements of $\{0,1\}^k$ are written without commas and parentheses. For $\epsilon = \epsilon_1 \dots \epsilon_k \in \{0,1\}^k$ we write

$$|\epsilon| = \epsilon_1 + \cdots + \epsilon_k \ .$$

For $x = (x_1, \dots, x_k) \in (\mathbb{Z}/N\mathbb{Z})^k$ and $\epsilon \in \{0,1\}^k$, we write

$$\epsilon \cdot x = \epsilon_1 x_1 + \cdots + \epsilon_k x_k \ .$$

Let $C\colon \mathbb{C} \to \mathbb{C}$ denote complex conjugation. Therefore, for $n \in \mathbb{N} \cup \{0\}$ and $\xi \in \mathbb{C}$,

$$C^n \xi = \begin{cases} \xi & \text{if } n \text{ is even} \\ \overline{\xi} & \text{if } n \text{ is odd} \ . \end{cases}$$

Definition of the Gowers norms. — For $k \geq 3$, the k-th Gowers norm $\|f\|_{U_k}$ for a function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined to be the sum over k-dimensional parallelepipeds:

$$\|f\|_{U_k}^{2^k} = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^k} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} \prod_{\epsilon \in \{0,1\}^k} C^{|\epsilon|} f(n + \epsilon \cdot x) \ .$$

For $k = 3$, $\|f\|_{U_3}^8$ can be written as the sum

$$(1) \qquad \sum_{x,m,n,p \in \mathbb{Z}/N\mathbb{Z}} f(x)\overline{f(x+m)}\,\overline{f(x+n)}f(x+m+n) \\ \overline{f(x+p)}f(x+m+p)f(x+n+p)\overline{f(x+m+n+p)} \ .$$

A natural question is on which sets a parallelepiped structure, and thus a Gowers norm, can be defined. More interesting is understanding to what extent the correspondence with a k-step nilpotent group can be made in this more general setting. We restrict ourselves to the cases $k = 2$ and $k = 3$ and characterize to what extent this correspondence can be made precise. As the precise definitions of parallelogram and parallelepiped structures are postponed until we have developed some machinery, we only give a loose overview of the results. Essentially, the properties included in the definition of a parallelepiped structure are exactly those needed in order to define a Gowers type norm.

For a two dimensional parallelogram, we completely characterize possible parallelogram structures by an abelian group (Corollary 1). This means that a parallelogram structure arises from a 1-step nilpotent group. For the corresponding three dimensional case, the situation becomes more complex. In Theorem 1, Theorem 2 and Corollary 2 we show under some additional hypotheses, a parallelepiped structure corresponds to a 2-step nilpotent group.

However, there are examples (Example 6) for which this hypothesis is not satisfied. On the other hand, we are able to show (Theorem 3) that in all cases, the parallelepiped structure can be embedded in a 2-step nilsystem.

In Section 7, we outline to what extent these results can be carried out in higher dimensions.

For all sets on which it is possible to define these structures, one can naturally define the corresponding Gowers norm U_k. We expect that these norms should have other applications, outside of those already developed by Gowers, Green and Tao, and the authors. The results of this paper already have an application in topological dynamics by Host and Maass [11], where they give a new characterization of 2-step nilsystems and 2-step nilsequences.

More notation. — Parallelogram structures and parallelepiped structures are defined as subsets of the Cartesian powers X^4 and X^8 of some sets or groups and so we introduce some notation.

When X is a set, we write $X^{[2]} = X \times X \times X \times X$ and let $X^{[3]}$ denote the analogous product with 8 terms.

A point in $X^{[2]}$ is written $\mathbf{x} = (x_0, x_1, x_2, x_3)$ or $\mathbf{x} = (x_{00}, x_{01}, x_{10}, x_{11})$ and a point in $X^{[3]}$ is written $\underline{\mathbf{x}} = (x_0, \dots, x_7)$ or $\underline{\mathbf{x}} = (x_{000}, x_{001}, \dots, x_{111})$. More succinctly, we denote $\mathbf{x} \in X^{[2]}$ by $\mathbf{x} = (x_i \colon 0 \le i \le 3)$ or $\mathbf{x} = (x_\epsilon \colon \epsilon \in \{0,1\}^2)$, and use similar notation for points in $X^{[3]}$.

It is convenient to identify $\{0,1\}^2$ with the set of vertices of the Euclidean unit square. Then the second type of notation allows us to view each coordinate of a point $\mathbf{x}$ of $X^{[2]}$ as lying at the corresponding vertex.

Each Euclidean isometry of the square permutes the vertices and thus the coordinates of $\mathbf{x}$. The permutations of $X^{[2]}$ defined in this way are called the *Euclidean permutations of* $X^{[2]}$. For example, the maps

$$\mathbf{x} \mapsto (x_{10}, x_{11}, x_{00}, x_{01}) \text{ and } \mathbf{x} \mapsto (x_{10}, x_{00}, x_{11}, x_{01})$$

are Euclidean permutations. We use the same vocabulary for $X^{[3]}$, with the Euclidean 3-dimensional unit cube replacing the square.

If $r \colon X \to Y$ is a map, by $r^{[2]} \colon X^{[2]} \to Y^{[2]}$ we mean

$$r^{[2]}(\mathbf{x}) = \big(r(x_{00}), r(x_{01}), r(x_{10}), r(x_{11})\big) \ .$$

Similarly, $r^{[3]}$ is defined as the corresponding map $r^{[3]} \colon X^{[3]} \to Y^{[3]}$.

2. Parallelograms

2.1. Definition and examples. — We start with a formal definition of a parallelogram structure on an arbitrary set:

DEFINITION 1. — Let X be a nonempty set. A *weak parallelogram structure* on X is a subset P of $X^{[2]}$ so that:

i) Equivalence relation: The relation $\sim$ on X^2 defined by $(x_{00}, x_{01}) \sim (x_{10}, x_{11})$ if and only if $(x_{00}, x_{01}, x_{10}, x_{11}) \in \mathsf{P}$ is an equivalence relation.
ii) Symmetry: If $(x_{00}, x_{01}, x_{10}, x_{11}) \in \mathsf{P}$, then $(x_{00}, x_{10}, x_{01}, x_{11}) \in \mathsf{P}$.
iii) Closing parallelogram property: For all $x_{00}, x_{01}, x_{10} \in X$, there exists $x_{11} \in X$ such that $(x_{00}, x_{01}, x_{10}, x_{11}) \in \mathsf{P}$.

We say that P is a *strong parallelogram structure* if the element x_{11} in iii) is unique.

DEFINITION 2. — We call the quotient space $X^2/\sim$ the *base* of the structure P and denote it by B. We write the equivalence class of an element (x, y) of X^2 as $\langle x, y\rangle$.

The only part of the definition of a parallelogram structure that does not appear to be completely natural is the transitivity in the equivalence relation $\sim$. We shall justify this assumption later (Proposition 2).

LEMMA 1. — *Let P be a weak parallelogram structure on a nonempty set X.*

i) *For all $x_0, x_1 \in X$, $(x_0, x_0, x_1, x_1) \in \mathsf{P}$.*
ii) *P is invariant under all Euclidean permutations of $X^{[2]}$.*
iii) *The relation $\sim$ can be rewritten as: for $x_{00}, x_{01}, x_{10}, x_{11} \in X$, $(x_{00}, x_{01}, x_{10}, x_{11}) \in \mathsf{P}$ if and only if $\langle x_{00}, x_{01}\rangle = \langle x_{10}, x_{11}\rangle$ if and only if $\langle x_{00}, x_{10}\rangle = \langle x_{01}, x_{11}\rangle$.*
iv) *All pairs (x, x) with $x \in X$ belong to the same $\sim$-equivalence class.*

Proof. — i) Reflexivity of $\sim$ implies that for all $x_0, x_1 \in X$ we have $(x_0, x_1, x_0, x_1) \in \mathsf{P}$. By property ii) of the definition, $(x_0, x_0, x_1, x_1) \in \mathsf{P}$. Part ii) follows from the symmetry of $\sim$ and property ii) of the definition. Part iii) follows from the definition of $\sim$ and ii), and part iv) follows immediately from i). □

NOTATION. — We denote the equivalence class of all pairs $(x, x), x \in B$ by 1_B.

2.2. Seminorm for parallelograms. — Given a weak parallelogram structure on a set X we can define a seminorm similar to the Gowers norm $\|\cdot\|_{U_2}$.

NOTATION. — If X is a set, $\mathcal{F}(X)$ denotes the space of complex valued functions on X with finite support.

PROPOSITION 1. — *Let X be a nonempty set and let P be a weak parallelogram structure on X. For any function $f \in \mathcal{F}(X)$ on X, we have*

$$\sum_{\mathbf{x}\in\mathsf{P}} f(x_{00})\overline{f(x_{01})}\,\overline{f(x_{10})}f(x_{11}) \geq 0\ . \tag{2}$$

Letting

$$\|f\|_{\mathsf{P}} := \Bigl(\sum_{\mathbf{x}\in\mathsf{P}} f(x_{00})\overline{f(x_{01})}\,\overline{f(x_{10})}f(x_{11})\Bigr)^{1/4}\ , \tag{3}$$

the map $f \mapsto \|f\|_{\mathsf{P}}$ is a seminorm on $\mathcal{F}(X)$ and it is a norm if and only if the structure P is strong.

Proof. — We first note that if F, G are functions on X^2 with finite support, then

$$\sum_{\mathbf{x}\in\mathsf{P}} F(x_{00},x_{01})G(x_{10},x_{11}) = \sum_{z\in B}\Bigl(\sum_{\substack{(x,y)\in X^2\\ \langle x,y\rangle=z}} F(x,y)\Bigr)\Bigl(\sum_{\substack{(x,y)\in X^2\\ \langle x,y\rangle=z}} G(x,y)\Bigr)\ .$$

In particular, taking $G = \overline{F}$,

$$\sum_{\mathbf{x}\in\mathsf{P}} F(x_{00},x_{01})\overline{F(x_{10},x_{11})} \geq 0\ . \tag{4}$$

We deduce also that for $F, G \in \mathcal{F}(X^2)$,

$$\Bigl|\sum_{\mathbf{x}\in\mathsf{P}} F(x_{00},x_{01})G(x_{10},x_{11})\Bigr| \leq \Bigl(\sum_{\mathbf{x}\in\mathsf{P}} F(x_{00},x_{01})\overline{F(x_{10},x_{11})}\Bigr)^{1/2}\Bigl(\sum_{\mathbf{x}\in\mathsf{P}} G(x_{00},x_{01})\overline{G(x_{10},x_{11})}\Bigr)^{1/2} \tag{5}$$

Taking $F(x_{00},x_{01}) = f(x_{00})\overline{f(x_{01})}$ in (4) we obtain (3).

We now use this to show the *Cauchy-Schwarz-Gowers Inequality*: for four functions $f_{00}, f_{01}, f_{10}, f_{11} \in \mathcal{F}(X)$,

$$\Bigl|\sum_{\mathbf{x}\in\mathsf{P}} f_{00}(x_{00})\overline{f_{01}(x_{01})}\,\overline{f_{10}(x_{10})}f_{11}(x_{11})\Bigr| \leq \|f_{00}\|_{\mathsf{P}}\,\|f_{01}\|_{\mathsf{P}}\,\|f_{10}\|_{\mathsf{P}}\,\|f_{11}\|_{\mathsf{P}}\ . \tag{6}$$

Setting $F(x_{00}, x_{01}) = f_{00}(x_{00})\overline{f_{01}(x_{01})}$ and $G(x_{10}, x_{11}) = f_{10}(x_{10})\overline{f_{11}(x_{11})}$ in (5), we have that the square of the left hand side of (6) is bounded by

$$\sum_{\mathbf{x}\in\mathsf{P}} f_{00}(x_{00})\overline{f_{01}(x_{01})}\,\overline{f_{00}(x_{01})}f_{01}(x_{11}) \cdot \sum_{\mathbf{x}\in\mathsf{P}} f_{10}(x_{00})\overline{f_{11}(x_{01})}\,\overline{f_{10}(x_{10})}f_{11}(x_{11}) \ .$$

By symmetry, the first sum can be rewritten as

$$\sum_{\mathbf{x}\in\mathsf{P}} f_{00}(x_{00})\overline{f_{00}(x_{01})}\,\overline{f_{01}(x_{01})}f_{01}(x_{11}) \ .$$

After a second use of (5), we obtain that this sum is bounded by $\|f_{00}\|_{\mathsf{P}}^2 \cdot \|f_{01}\|_{\mathsf{P}}^2$. Using the same method for the second term we obtain the Cauchy-Schwarz-Gowers Inequality.

Subadditivity of $\|\cdot\|_{\mathsf{P}}$ follows easily and thus $\|\cdot\|_{\mathsf{P}}$ is a seminorm.

Assuming now that P is a strong parallelogram structure we show that $\|\cdot\|_{\mathsf{P}}$ is actually a norm. Let $f \in \mathcal{F}(X)$ be a function such that $\|f\|_{\mathsf{P}} = 0$. Let a be an arbitrary point of X and $g = \mathbf{1}_{\{a\}}$. By the Cauchy-Schwarz-Gowers Inequality,

$$0 = \sum_{\mathbf{x}\in\mathsf{P}} g(x_{00})g(x_{01})g(x_{10})f(x_{11}) = \sum_{x_{11}\in X,\ (a,a,a,x_{11})\in\mathsf{P}} f(x_{11}) = f(a)$$

and so f is identically zero.

Conversely, if the structure is not strong, we claim that $\|\cdot\|_{\mathsf{P}}$ is not a norm. This can be shown directly, but in the interest of brevity we postpone the proof until Section 2.5, after we have developed certain properties of parallelogram structures. □

Before further developing the theory of parallelogram structures, we justify the assumption of transitivity:

Proposition 2. — *Let X be a finite set and let $\mathsf{P} \subset X^4$ satisfy all assumptions of the definition of a strong parallelogram structure other than transitivity of $\sim$. Assume that the positivity relation (4) is satisfied. Then P is a strong parallelogram structure.*

Proof. — Assume that X has n elements. For (x_0, x_1) and (x_2, x_3) in X^2, we define

$$M_{(x_0,x_1),(x_2,x_3)} = \begin{cases} 1 & \text{if } (x_0, x_1, x_2, x_3) \in \mathsf{P} \\ 0 & \text{otherwise} \ . \end{cases}$$

This defines a $n^2 \times n^2$ matrix M.

This matrix has 1's on the diagonal and thus $\mathrm{Trace}(M) = n^2$. It is symmetric by the symmetry of P and is a positive matrix by hypothesis (4). Thus its eigenvalues $\lambda_1, \dots, \lambda_{n^2}$ are nonnegative. By the unique closing parallelogram property, the sum of the entries for each row is n. Therefore $\lambda_i \leq n$ for

$i = 1, \dots, n^2$. Furthermore, all the diagonal elements of M^2 are equal to n and $\operatorname{Trace}(M^2) = n^3$. We have

$$n^3 = \operatorname{Trace}(M^2) = \sum_{i=1}^{n^2} \lambda_i^2 \leq \sum_{i=1}^{n^2} n\lambda_i = n \operatorname{Trace}(M) = n^3$$

and thus all λ_i are either 0 or n, and $M^2 = n \cdot M$. Transitivity follows. □

2.3. Examples. — We give two examples that illustrate, in a sense to be explained, the general behavior of weak and strong parallelogram structures:

Example 1. — If G is an abelian group (written with multiplicative notation), then

$$\begin{aligned} \mathsf{P}_G &:= \big\{ \mathbf{g} = (g_{00}, g_{01}, g_{10}, g_{11}) \in G^{[2]} \colon g_{00} g_{01}^{-1} g_{10}^{-1} g_{11} = 1 \big\} \\ &= \big\{ (g, gs, gt, gst) \colon g, s, t \in G \big\} \end{aligned}$$

is a strong parallelogram structure on G.

Example 2. — Let X be a set, G an abelian group, P_G the strong parallelogram structure on G defined in Example 1 and $\pi \colon X \to G$ a surjection. Let P be the inverse image of P_G under the map $\pi^{[2]} \colon X^{[2]} \to G^{[2]}$:

$$\begin{aligned} \mathsf{P} &= \{ \mathbf{x} \in X^{[2]} \colon \pi^{[2]} \mathbf{x} \in \mathsf{P}_G \} \\ &= \big\{ \mathbf{x} \in X^{[2]} \colon \pi(x_{00})\, \pi(x_{01})^{-1} \pi(x_{10})^{-1} \pi(x_{11}) = 1 \big\} . \end{aligned}$$

Then P is a weak parallelogram structure on X; it is not strong unless π is a bijection.

2.4. Description of parallelogram structures. — We give a complete description of parallelogram structures.

Lemma 2. — *The set B can be endowed with a multiplication such that*

(7) $$\langle a, b \rangle \cdot \langle b, c \rangle = \langle a, c \rangle \text{ for all } a, b, c \in X .$$

With this multiplication, B is an abelian group.

Proof. — Let $s, t \in B$ and let $a \in X$. By part iii) of the definition of a parallelogram, there exists $b \in X$ with $\langle a, b \rangle = s$ and there exists $c \in X$ with $\langle b, c \rangle = t$. We check that $\langle a, c \rangle$ does not depend on the choices of a, b, c but only on s and t. Let $a', b', c' \in X$ satisfy $\langle a', b' \rangle = s$ and $\langle b', c' \rangle = t$. By part iii) of Lemma 1, $\langle a, a' \rangle = \langle b, b' \rangle = \langle c, c' \rangle$ and thus $\langle a, c \rangle = \langle a', c' \rangle$. It follows immediately that the multiplication in B satisfying (7) is uniquely defined.

By construction, this multiplication is associative and admits the class 1_B as unit element. The inverse of the class $\langle a, b \rangle$ is the class $\langle b, a \rangle$.

We are left with showing that the operation is commutative. Given s, t, a, b, c as above, we can choose $d \in X$ such that $\langle a, d\rangle = t$. Then $(a, b, d, c) \in \mathsf{P}$ and thus $\langle d, c\rangle = \langle a, b\rangle = s$ and so $t \cdot s = \langle a, d\rangle \cdot \langle d, c\rangle = \langle a, c\rangle = s \cdot t$. □

Notation. — Henceforth we implicitly chose a point $e \in X$ and define a map $\pi \colon X \to B$ by

$$\pi(x) = \langle e, x\rangle \text{ for every } x \in X\ .$$

Corollary 1. — *Any strong parallelogram structure is isomorphic (in the obvious sense) to a strong parallelogram structure of the type described in Example 1.*

Any weak parallelogram structure is isomorphic (in the obvious sense) to a weak parallelogram structure of the type described in Example 2.

Proof. — Assume that we have a nonempty set X with parallelogram structure P, quotient space B, a point $e \in X$, and a map $\pi \colon X \to B$ defined as above. Let P_B denote the parallelogram structure on B. For every $\mathbf{x} \in X^{[2]}$,

$$\begin{aligned}\pi(x_{00}) \cdot \pi(x_{01})^{-1} \cdot \pi(x_{10})^{-1} \cdot \pi(x_{11}) &= \langle e, x_{00}\rangle \cdot \langle e, x_{01}\rangle^{-1} \cdot \langle e, x_{10}\rangle^{-1} \cdot \langle e, x_{11}\rangle \\ &= \langle x_{00}, x_{01}\rangle \cdot \langle x_{10}, x_{11}\rangle^{-1}\end{aligned}$$

and this is equal to 1_B if and only if $\mathbf{x} \in \mathsf{P}$. Therefore the structure P is equal to the inverse image of P_B under $\pi^{[2]}$.

If P is strong, then π is a bijection. Identifying a point of X with its image under π we get that P is defined as in Example 1. □

The following proposition follows easily from the preceding discussion:

Lemma 3. — *Let P be a parallelogram structure on a nonempty set X. For $x, y \in X$, the following are equivalent :*

i) $(x, x, x, y) \in \mathsf{P}$.
ii) $\pi(x) = \pi(y)$.
iii) $\langle x, y\rangle = 1_B$.
iv) *For all $a, b, c \in X$ such that $(a, b, c, x) \in \mathsf{P}$, we have $(a, b, c, y) \in \mathsf{P}$.*
v) *There exist $a, b, c \in X$ with $(a, b, c, x) \in \mathsf{P}$ and $(a, b, c, y) \in \mathsf{P}$.*

Notation. — We denote the equivalence relation on X defined by these conditions by $\equiv$, or $\underset{\mathsf{P}}{\equiv}$ when we want to emphasize the underyling parallelogram structure.

This relation is equality if and only if the structure P is strong. Therefore the map $\pi \colon X \to B$ induces a bijection from the quotient space of $X/\underset{\mathsf{P}}{\equiv}$ onto B. We identify these two sets.

2.5. End of the proof of Proposition 1. — Assume that the structure P is not strong. We show that $\|\cdot\|_{\mathsf{P}}$ is not a norm. By Lemma 3, there exist distinct points $a, b \in X$ with the same image under π and furthermore, taking any quadruple all of whose entries are either a or b is a parallelogram in P. Setting $f = \mathbf{1}_{\{a\}} - \mathbf{1}_{\{b\}}$, the sum in the definition of the seminorm has 8 terms that are 1 and 8 that are -1, and so we have a nonzero function with $\|f\|_{\mathsf{P}} = 0$.

2.6. More examples. — We continue this section with another example that plays a significant role in the sequel. To do so, we introduce some notation that may seem a bit strange at the moment, but lends itself easily to the sort of generalization needed later.

DEFINITION 3. — Let G be a group. We write $G^{[2,2]}$ for the *diagonal subgroup* of $G^{[2]}$;

$$G^{[2,2]} := \big\{(g,g,g,g) \colon g \in G\big\} \ .$$

We write $G^{[2,1]}$ for the subgroup of $G^{[2]}$ spanned by the elements

$$(g,g,1,1) \ ; \ (g,1,g,1) \ ; (g,g,g,g) \text{ for } g \in G \ .$$

$G^{[2,1]}$ is called the *two dimensional edge group* of G.

While the set of generators given for the edge group is a minimal one, it is not the most natural for understanding the name we give the group. Using the analogy with the Euclidean square $\{0,1\}^2$, the set of generators for the edge group consists of all elements of $G^{[2]}$ where we place g's in entries corresponding to an edge of $\{0,1\}^2$ and 1's elsewhere. This point of view becomes more natural and useful in the generalization to three dimensions.

NOTATION. — Let G be a group. We write G_2 for its commutator subgroup. Recall that G_2 is the subgroup of G spanned by the elements $[g,h]$, $g,h \in G$, where $[g,h] = ghg^{-1}h^{-1}$. G_3 denotes the second commutator subgroup of G, that is, the subgroup of G spanned by the elements $[g,u]$ for $g \in G$ and $u \in G_2$.

By a short computation, we have:

LEMMA 4. — *Let G be a group. Then*

$$\begin{aligned}(8)\quad G^{[2,1]} &= \big\{\mathbf{g} \in G^{[2]} : g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} \in G_2\big\} \\ &= \big\{(g, gh, gk, ghku) \colon g,h,k \in G,\ u \in G_2\big\} \ .\end{aligned}$$

In particular, if G is abelian then $G^{[2,1]}$ is equal to the set P_G of Example 1:

$$(9)\quad G^{[2,1]} := \big\{\mathbf{g} \in G^{[2]} : g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} = 1\big\} = \big\{(g, gs, gt, gst) \colon g,s,t \in G\big\} \ .$$

EXAMPLE 3. — Let G be a group and F a subgroup of G containing G_2. Thus F is normal in G and $B = G/F$ is abelian. Let $\pi\colon G \to B$ be the natural homomorphism and let P be the weak parallelogram structure on G defined in Example 2:

$$\mathsf{P} = \left\{\mathbf{g} \in G^{[2]} \colon \pi^{[2]}(\mathbf{g}) \in (G/F)^{[2,1]}\right\} = \{\mathbf{g} \in G^{[2]} \colon g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} \in F\} \ .$$

It is easy to check that P is a subgroup of $G^{[2]}$ and that $\mathsf{P} = G^{[2,1]}F^{[2]}$.

3. Parallelepipeds

3.1. Notation. — Parallelepipeds are the three dimensional generalization of parallelograms, and so naturally arise as subsets of $X^{[3]}$. Recall that we identify $\{0,1\}^3$ with the set of vertices of the unit Euclidean cube. Under this identification, we can naturally associate appropriate subsets of $\{0,1\}^3$ with vertices, edges, or faces of the unit cube.

Thus if $\underline{\mathbf{x}} \in X^{[3]}$ and η is an edge of the unit cube, the element $\mathbf{x}_\eta :=\{x_\epsilon \colon \epsilon \in \eta\}$ of $X \times X$ is called an *edge* of $\underline{\mathbf{x}}$. Similarly, if σ is a face of the unit cube, the element $\underline{\mathbf{x}}_\sigma := \{x_\epsilon \colon \epsilon \in \sigma\}$ of $X^{[2]}$ is called a *face* of $\underline{\mathbf{x}}$. By mapping each vertex of σ to a vertex of $\{0,1\}^2$ in increasing lexicographic order we can consider $\underline{\mathbf{x}}_\sigma$ as an element of $X^{[2]}$.

In particular, $\mathbf{x}' = (x_{000}, x_{001}, x_{010}, x_{011})$ and $\mathbf{x}'' = (x_{100}, x_{101}, x_{110}, x_{111})$ are opposite faces of $\underline{\mathbf{x}}$ and we often write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, naturally identifying $X^{[3]}$ with $X^{[2]} \times X^{[2]}$.

3.2. Definition of a parallelepiped. —

DEFINITION 4. — Assume that X is a nonempty set with a weak parallelogram structure P. A *weak parallelepiped structure* Q is a subset of $X^{[3]}$ satisfying:

i) Parallelograms: For every $\mathbf{x} \in \mathsf{Q}$ and every face σ of $\{0,1\}^3$, $\mathbf{x}_\sigma \in \mathsf{P}$.
ii) Symmetries: Q is invariant under all Euclidean permutations of $\{0,1\}^3$.
iii) Equivalence relation: The relation $\approx$ (written $\underset{\mathsf{Q}}{\approx}$ in case of ambiguity) on P defined by $\mathbf{x}' \approx \mathbf{x}''$ if and only if $(\mathbf{x}', \mathbf{x}'') \in \mathsf{Q}$ is an equivalence relation.
iv) Closing parallelepiped property: If $x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}$ are seven points of X satisfying

$$(x_{000}, x_{001}, x_{010}, x_{011}), (x_{000}, x_{010}, x_{100}, x_{110}), \text{ and } (x_{000}, x_{001}, x_{100}, x_{101}) \in \mathsf{P} \ ,$$

then there exists $x_{111} \in X$ such that

$$(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}) \in \mathsf{Q} \ .$$

We say that Q is a *strong parallelepiped structure* if the element x_{111} in iv) is unique.

A parallelepiped structure on X with parallelograms P and parallelepipeds Q is denoted by (P, Q).

NOTATION. — We denote the quotient space $\mathsf{P}/\approx$ by P and denote the equivalence class of a parallelogram $\mathbf{x} \in \mathsf{P}$ by $[\mathbf{x}]$ or by $[\mathbf{x}]_{\mathsf{Q}}$.

We start with some properties that follow immediately from the definition. Let $x_{000}, \dots, x_{110}$ be seven points in X satisfying the hypotheses of condition iv). Then there exists $x_{111} \in X$ such that $(x_{010}, x_{110}, x_{011}, x_{111}) \in \mathsf{P}$ and one can easily check that all faces of $\underline{\mathbf{x}} = (x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111})$ belong to P. Therefore condition iv) can be rewritten as :

v) Let $\underline{\mathbf{x}} \in X^{[3]}$ be such that each of its faces belongs to P. Then there exists $x'_{111} \in X$ such that $(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x'_{111}) \in \mathsf{Q}$.

We also note that some of the conditions in the definition of a parallelepiped are redundant: if Q is invariant under the permutations given in condition ii) and if the relation $\approx$ is transitive, then $\approx$ is an equivalence relation.

LEMMA 5. — *Let X be a nonempty set with parallelepiped structure (P, Q). Then*

i) *For $x, y \in X$, $[x, y, x, y]$ depends only on $\langle x, y\rangle$.*
ii) *All parallelograms of the form (a, a, a, a) for some $a \in X$ belong to the same $\approx$ equivalence class.*

Proof. — If $\langle x, y\rangle = \langle x', y'\rangle$, then $\mathbf{x} = (x, y, x', y') \in \mathsf{P}$. By reflexivity of the relation $\approx$, we have $(\mathbf{x}, \mathbf{x}) \in \mathsf{Q}$. By part ii) of the definition of a parallelepiped, $(x, y, x, y, x', y', x', y') \in \mathsf{Q}$ and $[x, y, x, y] = [x', y', x', y']$. Part ii) is an immediate corollary. □

NOTATION. — The common equivalence class in part ii) is called the *trivial class* and is denoted by 1_P.

We note that we did not make use of the closing parallelepiped property, part iv) of the definition of a parallelepiped property in this lemma.

3.3. Reduction to strong parallelepiped structures. — For the moment, we assume that Q satisfies the first three properties of the definition, but do not assume the closing parallelepiped property, part iv) of the definition.

PROPOSITION 3. — *Let X be a nonempty set with parallelogram structure P and a subset Q of $X^{[3]}$ only satisfying properties i), ii) and iii) of definition 4 of a parallelepiped structure. For $x, x' \in X$ the following are equivalent:*

i) $(x, x, x, x') \in \mathsf{P}$ *and* $[x, x, x, x'] = 1_P$.
ii) *For all* $a, b, c \in X$ *such that* $(a, b, c, x) \in \mathsf{P}$*, we have* $(a, b, c, x') \in \mathsf{P}$ *and* $[a, b, c, x] = [a, b, c, x']$.
iii) *There exist* $a, b, c \in X$ *with* $(a, b, c, x) \in \mathsf{P}$*,* $(a, b, c, x') \in \mathsf{P}$ *and* $[a, b, c, x] = [a, b, c, x']$.

Proof. — Each of the three properties implies that $\langle x, x' \rangle = 1_B$ and thus that $(x, x, x, x') \in \mathsf{P}$. Assume first that $[x, x, x, x'] = 1_P$. Let $a, b, c \in X$ be such that $(a, b, c, x) \in \mathsf{P}$. Since $[c, c, c, c] = 1_P = [x, x, x, x']$, by symmetry we have that $[c, x, c, x] = [c, x, c, x']$. By Lemma 5, part i) we have $[a, b, a, b] = [c, x, c, x] = [c, x, c, x']$ and thus $[a, b, c, x] = [a, b, c, x']$ and x, x' satisfy the second property.

The second condition trivially implies the third.

Assume now that there exist $a, b, c \in X$ with $(a, b, c, x) \in \mathsf{P}$ and $[a, b, c, x] = [a, b, c, x']$. The same argument implies that $[x, x, x, x'] = [x, x, x, x] = 1_P$, and our claim is proved. □

NOTATION. — For $x, x' \in X$, we write $x \underset{\mathsf{Q}}{\equiv} x'$ if x, x' satisfy any of the three equivalent properties in Proposition 3.

The second property implies that $\underset{\mathsf{Q}}{\equiv}$ is an equivalence relation. Moreover, Q is saturated for this relation meaning that for $\underline{\mathbf{x}}, \underline{\mathbf{x}}' \in X^{[3]}$ we have if $\underline{\mathbf{x}} \in \mathsf{Q}$ and $x_\epsilon \underset{\mathsf{Q}}{\equiv} x'_\epsilon$ for every $\epsilon \in \{0,1\}^3$, then $\underline{\mathbf{x}}' \in \mathsf{Q}$. In particular, the structure (P, Q) is strong if and only if the relation $\underset{\mathsf{Q}}{\equiv}$ is equality.

Let Y be the quotient space $X/\underset{\mathsf{Q}}{\equiv}$ and $r \colon X \to Y$ the quotient map. Let P_Y and Q_Y be the images of P and Q under $r^{[2]}$ and $r^{[3]}$, respectively. Then P and Q are the inverse images of P_Y and Q_Y under these maps.

If Q satisfies the closing parallelepiped property, then Q_Y also satisfies this property and $(\mathsf{P}_Y, \mathsf{Q}_Y)$ is a parallelepiped structure on Y. This structure is strong because clearly the relation $\underset{\mathsf{Q}_Y}{\equiv}$ is the identity.

We have thus shown:

PROPOSITION 4. — *Every parallelepiped structure on a nonempty set is the inverse image of a strong parallelepiped structure.*

Thus the study of parallelepiped structures reduces to the study of strong ones and so in the sequel, we generally consider strong parallelepiped structures.

3.4. The seminorm associated to a parallelepiped structure. — Given a parallelepiped structure, one can define a third (Gowers) norm, giving us a three dimensional version of Proposition 1.

Proposition 5. — *Let* (P,Q) *be a parallelepiped structure on the set* X*. For every* $f \in \mathcal{F}(X)$,

$$\sum_{\underline{\mathbf{x}}\in\mathsf{Q}} \prod_{\epsilon\in\{0,1\}^3} C^{|\epsilon|} f(x_\epsilon) \geq 0 \tag{10}$$

and thus we can define

$$\|f\|_{\mathsf{Q}} := \Big(\sum_{\underline{\mathbf{x}}\in\mathsf{Q}} \prod_{\epsilon\in\{0,1\}^3} C^{|\epsilon|} f(x_\epsilon)\Big)^{1/8} . \tag{11}$$

The map $f \mapsto \|f\|_{\mathsf{Q}}$ *is a seminorm on* $\mathcal{F}(X)$ *and it is a norm if and only if the structure* (P,Q) *is strong.*

We omit the proof, as it is exactly the same as the proof of the two dimensional version, Proposition 1. For the converse implication to show that if we have a norm the structure is strong, we use Proposition 3 instead of Lemma 3. Note that we have included in the definition of a parallelepiped structure all the properties needed to parallel the steps in Gowers's original proof. (The notation is given in the introduction.)

The assumption of transitivity of the relation $\approx$ in the definition of parallelepipeds is related to positivity, as for parallelograms in Proposition 2:

Proposition 6. — *Let* X *be a finite set, let* P *be a parallelogram structure on* X *and let* $\mathsf{Q} \subset X^{[3]}$ *satisfy all the assumptions of the definition of a strong parallelepiped structure other than transitivity of* $\approx$*. Assume that the following positivity relation holds: for every function* F *on* X^4 *with finite support,*

$$\sum_{\underline{\mathbf{x}}\in\mathsf{Q}} F(\mathbf{x}')\overline{F(\mathbf{x}'')} \geq 0 .$$

Then Q *is a strong parallelogram structure.*

We omit the proof, as it is more intricate but similar to the proof of Proposition 2.

3.5. First examples: Abelian parallelepiped structures. — We begin with some examples of parallelepiped structures on abelian groups. We need some notation and definitions similar to those introduced for Definition 3 in the two-dimensional case.

NOTATION. — Let G be a group. For a nonempty subset α of $\{0,1\}^3$ and $g \in G$, we write $g^{[3,\alpha]}$ for the element of $G^{[3]}$ given by

$$\left(g^{[3,\alpha]}\right)_\epsilon = \begin{cases} g & \text{if } \epsilon \in \alpha \\ 1 & \text{otherwise} . \end{cases}$$

For $\eta \in \{0,1\}^3$ we write $g^{[3,\eta]}$ instead of $g^{[3,\{\eta\}]}$.

In particular $g^{[3,\varnothing]} = \underline{\mathbf{1}} := (1,\dots,1)$.

Note that the elements $g^{[3,\eta]}$, $g \in G$ and $\eta \in \{0,1\}^3$, generate $G^{[3]}$. If $\alpha = \{0,1\}^3$ then $g^{[3,\alpha]}$ is the diagonal element $(g,g,\dots,g)$.

DEFINITION 5. — Let G be a group.

The *diagonal group* $G^{[3,3]}$ is the subgroup of $G^{[3]}$ consisting in elements of the form $(g,g,\dots,g)$ for $g \in G$.

The *edge group* $G^{[3,1]}$ is the subgroup of $G^{[3]}$ spanned by the elements of the form $g^{[3,e]}$ where $g \in G$ and e is an edge of the cube $\{0,1\}^3$.

The *face group* $G^{[3,2]}$ is the subgroup of $G^{[3]}$ spanned by the elements of the form $g^{[3,f]}$ where $g \in G$ and f is a face of the cube $\{0,1\}^3$.

The following Proposition follows immediately:.

PROPOSITION 7. — *If G is an abelian group, then*

$$G^{[3,1]} = \Big\{\underline{\mathbf{g}} \in G^{[3]} \colon \prod_{\epsilon\in\{0,1\}^3} g_\epsilon^{(-1)^{|\epsilon|}} = 1\Big\} \tag{12}$$

and

$$G^{[3,2]} = \big\{\underline{\mathbf{g}} \in G^{[3]} \colon \textit{ every face of } \underline{\mathbf{g}} \textit{ belongs to } G^{[2,1]}\big\} .$$

EXAMPLE 4. — Let G be an abelian group, let $\mathsf{P} = G^{[2,1]}$ be the strong parallelogram structure defined as in Example 1 and let $\mathsf{Q} = G^{[3,2]}$. Then (P,Q) is a (strong) parallelepiped structure on G.

EXAMPLE 5. — Let G be an abelian group and F a subgroup of G. Define

$$\mathsf{P} = G^{[2,1]}F^{[2]} \text{ and } \mathsf{Q} = G^{[3,2]}F^{[3,1]} .$$

Then (P,Q) is a (strong) parallelepiped structure on X.

This assertion is a particular case of a more general statement (Proposition 9) and so we omit the proof.

3.6. Some nonabelian examples. — More interesting are parallelepiped structures on nonabelian groups. It is here that the structures take on nontrivial properties.

We begin with an elementary remark:

REMARK 1. — Let G be a group and let α,β be two subsets of the cube $\{0,1\}^3$. Then, for every $g,h\in G$, the commutator of the two elements $g^{[3,\alpha]}$ and $h^{[3,\beta]}$ of $G^{[3]}$ is

$$\big[g^{[3,\alpha]},h^{[3,\beta]}\big]=[g,h]^{[3,\alpha\cap\beta]}\ .$$

LEMMA 6. — *Let G be a group. Then:*

i) $G^{[3,2]}\supset G_2^{[3,1]}\supset G_3^{[3]}$.

ii) *Let $g\in G$ and $\eta\in\{0,1\}^3$. Then $g^{[3,\eta]}\in G^{[3,2]}$ if and only if $g\in G_3$.*

iii) $G^{[2,2]}G_2^{[2,1]}G_3^{[2]}$ *is a normal subgroup of* $G^{[2,1]}$.

Moreover, under the identification $G^{[3]}=G^{[2]}\times G^{[2]}$,

$$G^{[3,2]}=\big\{\underline{\mathbf{g}}=(\mathbf{g}',\mathbf{g}'')\in G^{[2,1]}\times G^{[2,1]}\colon g'g''^{-1}\in G^{[2,2]}G_2^{[2,1]}G_3^{[2]}\big\}\ . \tag{13}$$

In a more general context, the proof is contained in sections 5 and 11 of [10]. The idea is to find the natural setting in which these cubic structures form a group, much as Hall [8], Petresco [16], Lazard [12], and later Leibman [13] & [14], did for arithmetic progressions. These groups also arise in [6], and Green and Tao refer to $G^{[3,2]}$ as the Hall-Petresco cube group. For completeness, we summarize the argument.

Proof. — *Part i)* Each face of $\{0,1\}^3$ is the union of two edges. If σ and τ are two faces of $\{0,1\}^3$, then $\sigma\cap\tau$ is a face, an edge, or the empty set. Conversely, each edge can be written as the intersection of two faces. By Remark 1, the commutator subgroup of $G^{[3,2]}$ is therefore equal to $G_2^{[3,1]}$. By a similar argument, the second commutator subgroup of $G^{[3,2]}$ is $G_3^{[3]}$, and part i) follows.

Part iii) We write $\mathbf{1}=(1,1,1,1)\in G^{[2]}$ and $K=G^{[2,2]}G_2^{[2,1]}G_3^{[2]}$. One can check directly that K is a normal subgroup of $G^{[2,1]}$.

It follows immediately from the definition of $G^{[3,2]}$ that for $\underline{\mathbf{g}}=(\mathbf{g}',\mathbf{g}'')\in G^{[3,2]}$ we have that $\mathbf{g}'$ and $\mathbf{g}''$ belong to $G^{[2,1]}$ and that for $\mathbf{g}\in G^{[2,1]}$ we have $(\mathbf{g},\mathbf{g})\in G^{[3,2]}$. Therefore

$$G^{[3,2]}=\big\{(\mathbf{g}',\mathbf{g}'')\in G^{[2,1]}\times G^{[2,1]}\colon g'g''^{-1}\in L\big\}\ ,$$

where

$$L=\big\{\mathbf{g}\in G^{[2,1]}\colon(\mathbf{g},\mathbf{1})\in G^{[3,2]}\big\}\ .$$

We are left with checking that $L=K$. The inclusion $K\subset L$ follows immediately from the definition of $G^{[3,2]}$ and part i). Moreover, the subset of $G^{[3]}$ on the right hand side of (13) is a subgroup of $G^{[3]}$ containing the generators of $G^{[3,2]}$ and thus containing this group. This implies that $K \supset L$.

Part ii) If $g \in G_3$ and $\eta \in \{0,1\}^3$, then $g^{[3,\eta]} \in G^{[3,2]}$ by part i). Conversely, let $g \in G$ and assume that $g^{[3,\eta]} \in G^{[3,2]}$ for some vertex η. We have to show that $g \in G_3$. By the symmetries of $G^{[3,2]}$, we can restrict to the case that $\eta = 111$.

By part iii), $(1,1,1,g)$ belongs to $G^{[2,2]}G_2^{[2,1]}G_3^{[2]}$ and we can write

$$(1,1,1,g) = (h,h,h,h).\mathbf{u}.\mathbf{v} \text{ with } h \in G,\ \mathbf{u} \in G_2^{[2,1]} \text{ and } \mathbf{v} \in G_3^{[3]}\ .$$

Looking at the congruences modulo G_2 of the first coordinate, we have that $h \in G_2$ and thus $(h,h,h,h) \in G_2^{[2,1]}$. Substituting $(h,h,h,h).\mathbf{u}$ for $\mathbf{u}$, we reduce to the case that $h = 1$.

Recall that G_3 is a normal subgroup of G_2 and that G_2/G_3 is abelian. Let $\bar{\mathbf{u}}$ be the element of $(G_2/G_3)^{[2]}$ obtained by reducing each coordinate of $\mathbf{u}$ modulo G_3. Then $\bar{\mathbf{u}}$ belongs to $(G_2/G_3)^{[2,1]}$, its first three coordinates are equal to 1 and by Lemma 4 its last coordinate is also equal to 1. This means that $u_{11} \in G_3$ and it follows that $g \in G_3$. □

We now turn to several nonabelian generalizations of the previous examples. The first one generalizes Example 4.

Proposition 8. — *Let G be a group, $\mathsf{P} = G^{[2,1]}$, and $\mathsf{Q} = G^{[3,2]}$. Then (P,Q) is a parallelepiped structure on G and this structure is strong if and only if G is 2-step nilpotent.*

We postpone the proof until after a second example of a group parallelepiped structure, which generalizes Example 5 to the nonabelian setting.

Proposition 9. — *Let G be a group and F a subgroup of G with*

$$G_2 \subset F \subset Z(G)\ . \tag{14}$$

We define (as in Example 5) $\mathsf{P} = G^{[2,1]}F^{[2]}$ and $\mathsf{Q} = G^{[3,2]}F^{[3,1]}$. Then (P,Q) is a strong parallelepiped structure on G.

Note that condition (14) implies in particular that G is a 2-step nilpotent group.

Proof. — The symmetries of Q are obvious. Q is clearly a subgroup of $G^{[3]}$ and it is not difficult to deduce form part iii) of Lemma 6 that $G^{[2,2]}F^{[2,1]}$ is normal in $G^{[2,1]}F^{[2]}$ and that

$$\mathsf{Q} = \{(\mathbf{g}',\mathbf{g}'') \in G^{[2]} \times G^{[2]} : \mathbf{g}' \in G^{[2,1]}F^{[2]},\quad \mathbf{g}'\mathbf{g}''^{-1} \in G^{[2,2]}F^{[2,1]}\}\ . \tag{15}$$

It follows that for every element $\underline{\mathbf{g}} = (\mathbf{g}',\mathbf{g}'') \in \mathsf{Q}$, the "first" face $\mathbf{g}'$ of $\underline{\mathbf{g}}$ belongs to P. Thus by symmetry, all faces belong to P. It also follows from (15) that $\underset{\mathsf{Q}}{\approx}$ is an equivalence relation on P.

Let Q' be the subgroup of $G^{[3]}$ consisting of elements $\underline{\mathbf{g}}$ such that each face of $\underline{\mathbf{g}}$ belongs to P. Then Q' contains $F^{[3]}$ and the quotient group $\mathsf{Q}'/F^{[3]}$ is the abelian parallelepiped structure $(G/F)^{[3,2]}$ on G/F defined in Example 4 and thus $\mathsf{Q}' = G^{[3,2]}F^{[3]}$.

Let $\underline{\mathbf{g}} \in \mathsf{Q}'$. We write $\underline{\mathbf{g}} = \underline{\mathbf{j}}\underline{\mathbf{h}}$ with $\underline{\mathbf{j}} \in G^{[3,1]}$ and $\underline{\mathbf{h}}$ of $F^{[3]}$. Since F is abelian, by (12) there exists an element $\underline{\mathbf{h}}' \in F^{[3,1]}$ that is equal to $\underline{\mathbf{h}}$ other than in the last coordinate. Thus $\underline{\mathbf{g}}' := \underline{\mathbf{j}}\underline{\mathbf{h}}' \in \mathsf{Q}$ and coincides with $\underline{\mathbf{g}}$ other than in the last coordinate. Therefore, condition v) is satisfied.

Finally we show that the structure is strong. Assume not. Since Q is a group, there exists $\underline{\mathbf{g}} \in \mathsf{Q}$ of the form $(1,1,\dots,1,g)$ for some $g \in G$ not equal to 1. By (15), $(1,1,1,g) \in G^{[2,2]}F^{[2,1]}$. Applying formula (9) for $F^{[2,1]}$, we have that $g = 1$, a contradiction. □

We now return to Proposition 8, and show that (P,Q) as defined in this example is a parallelepiped structure on G.

Proof of Proposition 8. — First consider the case that G is 2-step nilpotent. Then the hypotheses of Proposition 9 are satisfied with $F = G_2$. In order to show that (P,Q) is a strong parallelepiped structure we check that P and Q are equal to the sets defined in this Proposition. By Lemma 4, we have $G_2^{[2]} \subset G^{[2,1]}$ and thus $G^{[2,1]}G_2^{[2]} = G^{[2,1]} = \mathsf{P}$; by part i) of Lemma 6 we have that $G^{[3,2]}G_2^{[3,1]} = G^{[3,2]} = \mathsf{Q}$.

We now turn to the general case. Recall that G/G_3 is a 2-step nilpotent group. Since the group P contains $G_3^{[2]}$ and Q contains $G_3^{[3]}$, P is the inverse image of $(G/G_3)^{[2,1]}$ in $G^{[2]}$ and Q is the inverse image of $(G/G_3)^{[3,2]}$ in $G^{[3]}$. The assertion follows immediately. □

3.7. Nilparallelepiped structures. — Let G be a group and F a subgroup of G satisfying $G_2 \subset F \subset Z(G)$ (hypotheses (14) of Proposition 9). Let Γ be a subgroup, not necessarily normal, of G. We build a structure on the coset space $X = G/\Gamma$. By substituting $G/(\Gamma\cap F)$ for G, $F/(\Gamma\cap F)$ for F, and $\Gamma/(\Gamma\cap F)$ for Γ we reduce to the case that $\Gamma\cap F = \{1\}$.

Proposition 10. — *Let G be a group and F a subgroup of G satisfying hypothesis (14) of Proposition 9,*

$$G_2 \subset F \subset Z(G)\ , \tag{14}$$

and let Γ be a subgroup of G with

$$\Gamma\cap F = \{1\}\ . \tag{16}$$

Set $X = G/\Gamma$*. Let* $\mathsf{P} = G^{[2,1]}F^{[2]}$, $\mathsf{Q} = G^{[3,2]}F^{[3,1]}$, $p\colon G \to G/\Gamma$ *be projection, and let*

$$\mathsf{P}_X = p^{[2]}(\mathsf{P}) \quad \textit{and} \quad \mathsf{Q}_X = p^{[3]}(\mathsf{Q})\ .$$

Then $(\mathsf{P}_X, \mathsf{Q}_X)$ *is a strong parallelepiped structure on* X*.*

A parallelepiped structure defined by $G, F, \Gamma, X, \mathsf{P}_X, \mathsf{Q}_X$ as in Proposition 10 is called a *nilmanifold parallelepiped structure*, and more succinctly we refer to it as a *nilparallelepiped structure.*

Before the proof of Proposition 10, we have a lemma.

Lemma 7. — *Maintaining notation as in Proposition 10, if* $\mathbf{g} = (g_{00}, g_{01}, g_{10}, g_{11}) \in G^{[2]}$ *is such that* $p^{[2]}(\mathbf{g}) \in \mathsf{P}_X$*, then there exists* $g'_{11} \in G$ *such that* $p(g'_{11}) = p(g_{11})$ *and* $(g_{00}, g_{01}, g_{10}, g'_{11}) \in \mathsf{P}$.

Proof. — By definition there exists $\mathbf{h} \in \mathsf{P}$ with $p^{[2]}(\mathbf{h}) = p^{[2]}(\mathbf{g})$ and thus there exists $\boldsymbol{\gamma} \in \Gamma^{[2]}$ with $\mathbf{g} = \mathbf{h}\boldsymbol{\gamma}$. Since Γ is abelian, it follows from Lemma 4 that there exists $\theta \in \Gamma$ with $(\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}\theta) \in \Gamma^{[2,1]}$ and the point $g'_{11} = g_{11}\theta$ satisfies the announced properties. □

Proof of Proposition 10. — P_X is clearly the weak parallelogram structure defined as in Example 2 by the natural projection of X on the base group $B = G/F\Gamma$. If $\underline{\mathbf{x}} \in \mathsf{Q}_X$ then every face of $\underline{\mathbf{x}}$ belongs to P_X by definition. The symmetries of Q are obvious.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be three parallelograms in P_X, such that $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{y}, \mathbf{z})$ belong to Q_X. This means that there exist two parallelepipeds $\underline{\mathbf{g}} = (\mathbf{g}', \mathbf{g}'')$ and $\underline{\mathbf{h}} = (\mathbf{h}', \mathbf{h}'')$ in Q such that

$$p^{[2]}(\mathbf{g}') = \mathbf{x}\ ,\ p^{[2]}(\mathbf{g}'') = p^{[2]}(\mathbf{h}') = \mathbf{y} \text{ and } p^{[2]}(\mathbf{h}'') = \mathbf{z}\ .$$

Let $\boldsymbol{\gamma} = \mathbf{g}''^{-1}\mathbf{h}'$. By (15), we have $\boldsymbol{\gamma} \in \Gamma^{[2]} \cap G^{[2,1]}F^{[2]}$ and thus $\gamma_{00}\gamma_{01}^{-1}\gamma_{10}^{-1}\gamma_{11} \in \Gamma \cap F = \{1\}$ and so $\boldsymbol{\gamma} \in \Gamma^{[2,1]}$. By part iii) of Lemma 6 again, $(\mathbf{g}'\boldsymbol{\gamma}, \mathbf{h}') \in \mathsf{Q}$ and thus $(\mathbf{g}'\boldsymbol{\gamma}, \mathbf{h}'') \in \mathsf{Q}$ by transitivity. The projection of this parallelepiped on X is $(\mathbf{x}, \mathbf{z})$ and thus $(\mathbf{x}, \mathbf{z}) \in \mathsf{Q}$. This shows that the relation $\approx_{\mathsf{Q}_X}$ on P_X is transitive and we deduce that it is an equivalence relation.

Consider now a point $\underline{\mathbf{x}} \in X^{[3]}$ such that each of its faces belongs to P_X. By Lemma 7, there exists $\underline{\mathbf{g}} \in G^{[3]}$ with $p^{[3]}(\underline{\mathbf{g}}) = \underline{\mathbf{x}}$ and such that $\underline{\mathbf{g}}_f \in \mathsf{P}$ when f is any of the three faces of $\{0,1\}^3$ containing 000 and also for the face $\{100, 101, 110, 111\}$. It follows that the two remaining faces of $\underline{\mathbf{g}}$ also are parallelograms of G. Since Q is a parallelepiped structure on G, we can modify (again using Lemma 7) g_{111} in order to get a parallelepiped in G. Projecting this parallelepiped on X, we get a parallelepiped in Q_X coinciding with $\underline{\mathbf{x}}$ other than in the last coordinate.

Thus we have that Q_X is a parallelepiped structure on X and we are left with showing it is strong. If not, there exist two parallelepipeds $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in \mathsf{Q}_X$

with $x_\epsilon = y_\epsilon$ for every $\epsilon \neq 111$ and $x_{111} \neq y_{111}$. Let $\underline{\mathbf{g}}, \underline{\mathbf{h}}$ be two parallelepipeds in G with $p^{[3]}(\underline{\mathbf{g}}) = \underline{\mathbf{x}}$ and $p^{[3]}(\underline{\mathbf{h}}) = \underline{\mathbf{y}}$. Writing $\underline{\mathbf{u}} = \underline{\mathbf{g}}^{-1}\underline{\mathbf{h}}$, we have $\underline{\mathbf{u}} \in \mathsf{Q}$ with $u_\epsilon \in \Gamma$ for every $\epsilon \neq 111$ and $u_{111} \notin \Gamma$. By part iii) of Lemma 6, there exists $\mathbf{v} \in G^{[2,2]}G_2^{[2,1]}$ with $v_\epsilon \in \Gamma$ for $\epsilon \neq 11$ and $v_{11} \notin \Gamma$. By Lemma 4, $\mathbf{v}$ can be written as $\mathbf{v} = (v, vs, vt, vst)$ with $v \in G$ and $s, t \in G_2$ and we get that v, vs, vt and thus vst belong to Γ, a contradiction. □

3.8. Nilpotent groups appear. — All the examples of strong parallelepiped structures considered thus far have a striking feature in common. There exists a 2 step nilpotent group G acting transitively on X. We explore this further.

In all examples given so far, the group $G^{[2]}$ acts on $X^{[2]}$ in a natural way and we write $(\mathbf{g}, \mathbf{x}) \mapsto \mathbf{g} \cdot \mathbf{x}$ for this action. It can be checked that for every parallelogram $\mathbf{x} \in \mathsf{P}$ and every $g \in G$, we have that $g^{[2]} \cdot \mathbf{x}$ is a parallelogram, equivalent to P under the relation $\approx$.

This is not merely coincidence; it reflects an underlying structure, so long as certain algebraic obstructions are avoided. We denote the action of G on X by $(g, x) \mapsto g \cdot x$. This motivates the following definition and notation.

Let $g\colon x \mapsto g \cdot x$ be a transformation of X. We recall that $g^{[2]}$ and $g^{[3]}$ are the diagonal transformations of $X^{[2]}$ and $X^{[3]}$, respectively. More generally, if α is a subset of the cube $\{0,1\}^3$, $g^{[3,\alpha]}$ is the transformation of $X^{[3]}$ given by

$$\left(g^{[3,\alpha]} \cdot \underline{\mathbf{x}}\right)_\epsilon = \begin{cases} g \cdot x_\epsilon & \text{if } \epsilon \in \alpha \\ x_\epsilon & \text{otherwise.} \end{cases}$$

This notation is coherent with the notation introduced in Section 3.5.

Definition 6. — Let (P, Q) be a strong parallelepiped structure on a nonempty set X. The *structure group* of (P, Q), written G or G_Q, is the group of bijections $x \mapsto g \cdot x$ of X such that for every parallelogram $\mathbf{x} \in \mathsf{P}$, $g^{[2]} \cdot \mathbf{x}$ is a parallelogram and $g^{[2]} \cdot \mathbf{x} \approx \mathbf{x}$.

We can rephrase this condition:

Proposition 11. — *If (P, Q) is a strong parallelepiped structure on a nonempty set X, then the structure group G is the group of bijections $x \mapsto g \cdot x$ of X such that for every parallelepiped $\underline{\mathbf{x}} \in \mathsf{Q}$ and every face f of $\{0,1\}^3$, we have $g^{[3,f]} \cdot \underline{\mathbf{x}} \in \mathsf{Q}$.*

Proof. — Let ϕ be the face $\{\epsilon \in \{0,1\}^3 \colon \epsilon_3 = 1\}$.

Assume first that $g \in \mathsf{G}$ and let $\underline{\mathbf{x}} \in \mathsf{Q}$. As usual, we write $\underline{\mathbf{x}} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in \mathsf{P}$ and we have $g^{[3,\phi]} \cdot \underline{\mathbf{x}} = (\mathbf{x}', g^{[2]} \cdot \mathbf{x}'')$. By hypothesis, $g^{[2]} \cdot \mathbf{x}'' \in \mathsf{P}$ and $g^{[2]} \cdot \mathbf{x}'' \approx \mathbf{x}''$, meaning that $(\mathbf{x}'', g^{[2]} \cdot \mathbf{x}'') \in \mathsf{Q}$. By transitivity, $g^{[3,\phi]} \cdot \underline{\mathbf{x}} \in \mathsf{Q}$. By symmetry, the same result also holds for the other faces of the cube.

Assume now that $x \mapsto g \cdot x$ is a bijection of X such that for every $\underline{\mathbf{x}} \in \mathsf{Q}$ and every face f we have $g^{[3,f]} \cdot \underline{\mathbf{x}} \in \mathsf{Q}$. Let $\mathbf{x} \in \mathsf{P}$. Then $\underline{\mathbf{x}} := (\mathbf{x}, \mathbf{x}) \in \mathsf{Q}$ and so $(\mathbf{x}, g^{[2]} \cdot \mathbf{x}) = g^{[3,\phi]} \cdot \underline{\mathbf{x}} \in \mathsf{Q}$. Thus $g^{[2]} \cdot \mathbf{x} \in \mathsf{P}$ and $g^{[2]} \cdot \mathbf{x} \approx \mathbf{x}$. □

PROPOSITION 12. — *Let* (P, Q) *be a strong parallelepiped structure on a nonempty set* X. *Then the structure group* G *is* 2*-step nilpotent.*

Proof. — The group $G^{[3]}$ acts on $X^{[3]}$ in the natural way. Let $g \in \mathsf{G}$ and $\underline{\mathbf{x}} \in \mathsf{Q}$. Write $\underline{\mathbf{x}} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in \mathsf{P}$. Then $g^{[2]} \cdot \mathbf{x}'$ is a parallelogram equivalent to $\mathbf{x}'$. Thus $(g^{[2]} \cdot \mathbf{x}', \mathbf{x}') \in \mathsf{Q}$ and so $(g^{[2]} \cdot \mathbf{x}', \mathbf{x}'') \in \mathsf{Q}$. By symmetries of Q, for every face f of $\{0,1\}^3$ we have that $g^{[3,f]} \cdot \mathbf{x} \in \mathsf{Q}$. Therefore, Q is invariant under the group $G^{[3,2]}$.

Let $g \in \mathsf{G}_3$. By Lemma 6, part iii) $(1, \dots, 1, g) \in G^{[3,2]}$. For every $x \in X$, we have $(x, \dots, x, x) \in \mathsf{Q}$ and thus $(x, \dots, x, g \cdot x) \in \mathsf{Q}$. Since Q is a strong structure, $g \cdot x = x$ and $g = 1$. Thus G is 2-step nilpotent. □

NOTATION. — Let $g \in \mathsf{G}$. For arbitrary $x, y \in X$, $g^{[2]}$ maps the parallelogram (x, x, y, y) to an equivalent one and this implies that

$$\pi(g \cdot x)\, \pi(x)^{-1} = \pi(g \cdot y)\, \pi(y)^{-1} \ .$$

Therefore there exists an element $p(g) \in B$ such that

$$\pi(g \cdot x) = p(g)\, \pi(x) \text{ for every } x \in X \ .$$

The map $p\colon G \to B$ defined in this way is clearly a group homomorphism.

In all examples considered thus far, the group G is included in $\mathsf{G}(X)$. But we note that $\mathsf{G}(X)$ may be substantially larger than G. Consider the situation of Example 5. Let $\phi\colon B \to F$ be a group homomorphism and define a transformation h on X by

$$h \cdot x = \phi(\pi(x)) \cdot x \ .$$

Then h belongs to $\mathsf{G}(X)$ and is not translation by an element of G.

4. Description of parallelepipeds structures

Henceforth, (P, Q) is a (strong) parallelepiped structure on a set X.

Our goal is to characterize when a parallelepiped structure is a nilparallelepiped structure, and we do so in Theorem 1. Moreover, in Theorem 2 and Corollary 2, we give sufficient conditions for these conditions to hold. We also give an example of a parallelepiped structure without this property (Example 6) and show that in the general case a parallelepiped structure can be imbedded in a nilstructure (Proposition 16) in a sense explained below.

For parallelograms we use the notation introduced in Section 2.6. B is the base group of the parallelogram structure P and $\pi\colon X \to B$ the surjection defined in Section 2.4. We recall that for $x, y \in X$, $\langle x, y\rangle$ is the equivalence class of the pair (x, y) under the relation $\underset{\mathsf{P}}{\sim}$, that is

$$\langle x, y\rangle = \pi(y)\,\pi(x)^{-1}\ .$$

For parallelepipeds we use the notation of Section 3.2. The equivalence class (under the relation $\approx$) of a parallelogram $\mathbf{x}$ is denoted by $[\mathbf{x}]$. We denote the quotient space $\mathsf{P}/\approx$ by P.

4.1. The groups P_s. — For every $s \in B$, we define

$$X_s := \big\{(x_{01}, x_{10}) \in X^2 \colon \langle x_{01}, x_{10}\rangle = s\big\} \subset X^2\ .$$
$$\mathsf{P}_s := \big\{\mathbf{x} \in \mathsf{P} \colon \langle x_{00}, x_{01}\rangle = s\big\} \subset X^{[2]}\ .$$
$$\mathsf{Q}_s := \big\{\underline{\mathbf{y}} \in \mathsf{Q} \colon \langle y_{000}, y_{001}\rangle = s\big\} \subset X^{[3]}\ .$$

If two parallelograms $\mathbf{x}, \mathbf{y} \in \mathsf{P}$ are equivalent under the relation $\approx$, then $\langle x_{00}, x_{01}\rangle = \langle y_{00}, y_{01}\rangle$ and thus they belong to the same P_s. Therefore each set P_s is a union of equivalence classes under the relation $\approx$. Writing P_s for the set of equivalence classes of parallelepipeds belonging to P_s, we have a partition of P:

$$P = \bigcup_{s\in B} P_s\ .$$

We identify $X^{[2]}$ with $(X \times X)^2$ in the natural way:

$$\big(x_{00}, x_{01}, x_{10}, x_{11}\big) = \big((x_{00}, x_{01}), (x_{10}, x_{11})\big)$$

and $X^{[3]}$ with $(X^2)^{[2]}$:

$$\begin{aligned}&\big(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}\big)\\ &\qquad\qquad = \big((x_{000}, x_{001}), (x_{010}, x_{011}), (x_{100}, x_{101}), (x_{110}, x_{111})\big)\ .\end{aligned}$$

Therefore we view $X_s \times X_s$ as a subset of $X^{[2]}$ and $X_s^{[2]}$ as a subset of $X^{[3]}$. For $s \in B$, we have:

$$\mathsf{P}_s = \mathsf{P} \cap (X_s \times X_s) \text{ and } \mathsf{Q}_s = \mathsf{Q} \cap X_s^{[2]}\ .$$

We reformulate this for clarity. A pair of elements of X_s represents four points in X forming a parallelogram of X and this parallelogram belongs to P_s. An element $\underline{\mathbf{x}} \in X_s^{[2]}$ consists in four points of X_s, that is, in eight points of X, and these eight points form a parallelepiped in X if and only if $\underline{\mathbf{x}} \in \mathsf{Q}_s$.

Lemma 8. — *Maintaining the above notation, Q_s is a parallelogram structure on X_s.*

Proof. — All the properties are immediate other than the closing parallelepiped property. Let (x_0, x_1), (x_2, x_3), (x_4, x_5) be three points in X_s. Choose $x_6 \in X$ with $\langle x_4, x_6\rangle = \langle x_0, x_2\rangle$. The seven points $x_0, \dots, x_6$ satisfy the closing parallelepiped property and so there exists $x_7 \in X$ such that $(x_0, \dots, x_6, x_7) \in \mathsf{Q}$. This parallelepiped actually belongs to Q_s and thus so does $\big((x_0, x_1), (x_2, x_3), (x_4, x_5), (x_6, x_7)\big)$ and our claim is proved. $\square$

We note that this parallelogram structure is not strong, as there is freedom in the choice of x_6 in the above construction.

Recall that $\underset{\mathsf{P}_s}{\sim}$ denotes the equivalence relation on X_s^2 associated to the parallelogram structure Q_s on X_s: two pairs of points in X_s are equivalent under this relation if they form a parallelogram in Q_s. If we consider these two pairs as parallelograms in X, then these parallelograms belong to P_s and Lemma 8 implies that they are equivalent under the relation $\approx$. Therefore we can identify the two quotient spaces

$$(X_s)^2/\underset{\mathsf{P}_s}{\sim} \quad = \quad \mathsf{P}_s/\approx \quad = \quad P_s\ .$$

By Lemma 2, the quotient space $(X_s)^2/\underset{\mathsf{P}_s}{\sim} = P_s$ can be endowed with a multiplication that gives it the structure of an abelian group. For clarity, we rewrite this multiplication in the present notation, viewing P_s as $\mathsf{P}_s/\approx$.

Let u, v be two classes in P_s. Let (x_0, x_1, x_2, x_3) be a parallelogram in the class u. As $\langle x_2, x_3\rangle = s$, there exist two points x_4 and x_5 in X such that (x_2, x_3, x_4, x_5) is a parallelogram in the class v. Then uv is the class of the parallelogram (x_0, x_1, x_4, x_5).

4.2. A homomorphism. — Let $s \in B$. If two parallelograms $\mathbf{x}$ and $\mathbf{y}$ of X belonging to P_s are equivalent under the relation $\approx$, then $\langle x_{00}, x_{10}\rangle = \langle y_{00}, y_{10}\rangle$. Therefore there exists a map $q_s \colon P_s \to B$ such that

$$q_s([\mathbf{x}]) = \langle x_{00}, x_{10}\rangle \text{ for every parallelogram } \mathbf{x} \in \mathsf{P}_s\ .$$

This map is clearly a group homomorphism from P_s onto B. The kernel of this homomorphism consists in the set of equivalence classes of parallelograms $\mathbf{x} \in \mathsf{P}$ with $\langle x_{00}, x_{01}\rangle = s$ and $\pi(x_{10}) = \pi(x_{00})$.

5. The fiber group

In this Section, (P, Q) is a strong parallelepiped structure on a nonempty set X and we maintain the notation of the preceding Section.

5.1. Vertical parallelograms. — We begin with some simple observations and some more vocabulary. For $b \in B$, the *fiber* F_b of b is defined by $F_b := \pi^{-1}(\{b\})$.

LEMMA 9. —

i) *If* $\mathbf{x} \in \mathsf{P}$, *then* $[x_{00}, x_{00}, x_{01}, x_{01}] = [x_{10}, x_{10}, x_{11}, x_{11}]$ *and* $[x_{00}, x_{01}, x_{00}, x_{01}] = [x_{10}, x_{11}, x_{10}, x_{11}]$.

ii) *If* x *and* y *belong to the same fiber, then* $[x, x, y, y] = [x, y, x, y] = 1$.

Proof. — For part i), since $(\mathbf{x}, \mathbf{x}) \in \mathsf{Q}$, both properties follow from the symmetries of Q. By assumption, $(x, x, x, y) \in \mathsf{P}$ and part ii) follows from part i). □

If $x_{00}, x_{01}, x_{10}, x_{11}$ are four points in the same fiber, then $(x_{00}, x_{01}, x_{10}, x_{11}) \in \mathsf{P}$. Thus it makes sense to define:

DEFINITION 7. — A parallelogram with its 4 vertices in the same fiber is called a *vertical parallelogram.*

A parallelogram equivalent to a vertical one is also vertical, and thus the family of vertical parallelograms is a union of equivalence classes.

LEMMA 10. — *If* $x_{00}, x_{01}, x_{10}, x_{11}$ *are four points in the same fiber, then*

$$[x_{00}, x_{01}, x_{10}, x_{11}] = [x_{00}, x_{10}, x_{01}, x_{11}] \ .$$

Proof. — There exists a unique $y \in X$ such that $[x_{00}, x_{01}, x_{10}, x_{11}] = [x_{00}, x_{01}, x_{01}, y]$. Thus $(x_{00}, x_{01}, x_{10}, x_{11}, x_{00}, x_{01}, x_{01}, y) \in \mathsf{Q}$. By symmetry, $(x_{00}, x_{10}, x_{01}, x_{11}, x_{00}, x_{01}, x_{01}, y) \in \mathsf{Q}$ and thus $[x_{00}, x_{10}, x_{01}, x_{11}] = [x_{00}, x_{01}, x_{01}, y] = [x_{00}, x_{01}, x_{10}, x_{11}]$. □

5.2. The fiber group F and its action on X

NOTATION. — To avoid cumbersome notation, we henceforth write 1 for the unit element 1_B of B. We let F denote the kernel of the homomorphism $q_1 : P_1 \to B$.

In other words, F is the set of equivalence classes (under the relation $\approx$) of vertical parallelograms. Recall that the multiplication in F satisfies:

(17) if $x_0, x_1, x_2, x_3, x_4, x_5$ belong to the same fiber, then

$$[x_0, x_1, x_2, x_3]\,[x_2, x_3, x_4, x_5] = [x_0, x_1, x_4, x_5] \ .$$

We now define an action of F on X, mapping each fiber to itself and use this to describe the vertical parallelograms.

Let $x \in X$ and $u \in F$. Recall that u is the class of some vertical parallelogram. It thus follows from the closing parallelepiped property that there exists $y \in X$ such that $[x, x, x, y] = u$. As the structure is strong, this point y is unique.

Notation. — For $x \in X$ and $u \in F$, we write $u \cdot x$ for the point of X defined by $[x, x, x, y] = u$.

Lemma 11. — *The map $(u, x) \mapsto u \cdot x$ is an action of the group F on the set X. This action preserves each fiber and acts transitively and freely on each fiber.*

Proof. — Note that $1_P \cdot x = x$ for every $x \in X$. We are left with showing that for $u, v \in F$ and $x \in X$, we have $(vu) \cdot x = v \cdot (u \cdot x)$. Let $y = u \cdot x$ and $z = v \cdot y = v \cdot (u \cdot x)$. Then $v = [u \cdot x, u \cdot x, u \cdot x, v \cdot (u \cdot x)] = [x, u \cdot x, x, u \cdot x][x, u \cdot x, x, v \cdot (u \cdot x)] = [x, u \cdot x, x, v \cdot (u \cdot x)]$ and $[x, x, x, v \cdot (u \cdot x)] = [x, x, x, u \cdot x]\,[x, u \cdot x, x, v \cdot (u \cdot x)] = uv = vu$ because F is abelian. Thus we have an action.

By construction, F preserves each fiber. For every x, y in the same fiber, there exists a unique $u \in F$ such that $u \cdot x = y$, namely $u = [x, x, x, y]$. This means that the action of F on each fiber is free and transitive. □

Proposition 13. — i) *Let $(x, u \cdot x, v \cdot x, w \cdot x)$ be a vertical parallelogram. Then the equivalence class in F of this parallelogram is the element $wu^{-1}v^{-1}$ of F.*

ii) *For every parallelogram $\mathbf{x} \in \mathsf{P}$ and $u \in F$, we have $(u \cdot x_{00}, u \cdot x_{01}, x_{10}, x_{11}) \in \mathsf{P}$ and $[u \cdot x_{00}, u \cdot x_{01}, x_{10}, x_{11}] = [\mathbf{x}]$.*

iii) *Every transformation in $F^{[2,1]}$ maps every parallelogram to an equivalent one.*

Proof. — We have

$$\begin{aligned}
w =&[x, x, x, w \cdot x] = [x, x, x, u \cdot x]\,[x, u \cdot x, v \cdot x, w \cdot x]\,[v \cdot x, w \cdot x, x, w \cdot x]\\
=&u\,[x, u \cdot x, v \cdot x, w \cdot x]\,[v \cdot x, x, w \cdot x, w \cdot x]\\
=&u\,[x, u \cdot x, v \cdot x, w \cdot x]\,[v \cdot x, x, x, x]\,[x, x, w \cdot x, w \cdot x]\\
=&u\,[x, u \cdot x, v \cdot x, w \cdot x]\,[v \cdot x, x, x, x] = u\,[x, u \cdot x, v \cdot x, w \cdot x]\,[x, x, x, v \cdot x]\\
=&u\,[x, u \cdot x, v \cdot x, w \cdot x]\,v\ .
\end{aligned}$$

This proves part i).

We now prove part ii). Let $\mathbf{x} \in \mathsf{P}$ and let $u \in F$. By Lemma 9, part i) we have $[x_{00}, x_{01}, x_{00}, x_{01}] = [x_{10}, x_{11}, x_{10}, x_{11}]$. By part i), $[x_{00}, x_{00}, u \cdot x_{00}, x_{00}] = u^{-1} = [x_{01}, x_{01}, u \cdot x_{01}, x_{01}]$ and thus $[x_{00}, x_{01}, u \cdot x_{00}, u \cdot x_{01}] = [x_{00}, x_{01}, x_{00}, x_{01}] = [x_{10}, x_{11}, x_{10}, x_{11}]$ by Lemma 9, part i). The claim follows by symmetry.

The group $F^{[2]}$ acts on $X^{[2]}$ coordinate-wise. By part ii) and the symmetries of Q, we have the statement in part iii). □

From this lemma, we deduce that Q is invariant under the subgroup $F^{[3,1]}$ of $F^{[3]}$. Conversely, we have:

Proposition 14. — *Let $\mathbf{x} \in \mathsf{P}$ and $\mathbf{u} \in F^{[2]}$. If the parallelograms $\mathbf{x}$ and $\mathbf{u}\cdot\mathbf{x}$ are equivalent, then $\mathbf{u} \in F^{[2,1]}$.*

Proof. — There exists u'_{11} such that $(u_{00}, u_{01}, u_{10}, u'_{11}) \in F^{[2,1]}$ and $[u_{00} \cdot x_{00}, u_{01} \cdot x_{01}, u_{10} \cdot x_{10}, u'_{11} \cdot x_{11}] = [\mathbf{x}] = [u_{00} \cdot x_{00}, u_{01} \cdot x_{01}, u_{10} \cdot x_{10}, u_{11} \cdot x_{11}]$. Since the structure is strong, $u'_{11} \cdot x_{11} = u_{11} \cdot x_{11}$. But since F acts freely on each fiber, $u'_{11} = u_{11}$. □

Remark 2. — Returning to the parallelogram structure Q_s on X_s, we have already noticed that it is not strong. As in Section 2.4, we have a strong parallelogram structure by taking a quotient of X_s by some equivalence relation. Using the preceding proposition, we have that two elements (x, y) and (x', y') of X_s are equivalent under this relation if and only if there exists $u \in F$ with $x' = u \cdot x$ and $y' = u \cdot y$.

Proposition 15. — *The group F is included in the center of G.*

Proof. — For every $u \in F$, we have $u^{[2]} \in F^{[2,1]}$ and so $F \subset \mathsf{G}$.

Let $g \in \mathsf{G}$ and $u \in F$. Let f be a face of the cube and e an edge of the cube with $f \cap e = \{111\}$. Then the transformations $g^{[3,f]}$ and $u^{[3,e]}$ map Q to itself, and thus so does the commutator of these transformations. It is immediate to check that this commutator is equal to $[g; u]^{[3,e]}$. This means that for $\mathbf{x} \in \mathsf{Q}$, we also have $(x_{000}, x_{001}, \ldots, x_{110}, [g; u] \cdot x_{111}) \in \mathsf{Q}$. As the structure is strong, $[g; u] \cdot x_{111} =_{111}$. Thus $[g; u]$ is the identity transformation. □

5.3. Structure Theorem. — We are now ready to characterize parallelepiped structures that are nilstructures. Recall (see Section 2.4) that e is a point in X and that the map $\pi \colon X \to B$ is defined by $\pi(x) = \langle e, x\rangle$ for every $x \in X$.

Theorem 1. — *Let G be a subgroup of G containing F and assume that G acts transitively on X. Let Γ be the stabilizer of some $e \in X$ in G and identify X with G/Γ in the natural way. Then the parallelogram structure (P, Q) on X coincides with the structure $(\mathsf{P}_X, \mathsf{Q}_X)$ associated to G, F and Γ as in Proposition 10.*

Thus the parallelepiped structure (P, Q) is isomorphic to a nilmanifold parallelepiped structure as in Proposition 10 if and only if its structure group G acts transitively on X.

Proof. — *Step 1.* We first show that G, F and Γ satisfy the hypotheses of Proposition 10.

We have already shown that F is included in the center of G. Since F acts freely on X, $\Gamma \cap F = \{1\}$. We are left with showing:

Claim 1. — F contains the commutator subgroup G_2 of G.

Recall that $\mathsf{G}^{[3,2]} \supset \mathsf{G}_2^{[3,1]}$. Therefore every element of $\mathsf{G}_2^{[3,1]}$ leave Q invariant and thus every element of $\mathsf{G}_2^{[2,1]}$ maps each parallelogram to an equivalent one. On the other hand, since B is abelian, g belongs to the kernel of p and thus maps every point of X to a point in the same fiber.

Let $g \in \mathsf{G}_2$ and $x, y \in X$. There exist $u, v \in F$ with $g \cdot x = u \cdot x$ and $g \cdot y = v \cdot y$. As $(x, y, x, y) \in \mathsf{P}$ and $(g, g, 1, \tau) \in \mathsf{G}_2^{[2,1]}$ we have that $[x, y, x, y] = [g \cdot x, g \cdot y, x, y] = [u \cdot x, v \cdot y, x, y]$. By Proposition 14, we have $u = v$.

Therefore there exists $u \in F$ with $g \cdot x = u \cdot x$ for every x, meaning that $g = u$. This proves the claim. Therefore we can define the structure $(\mathsf{P}_X, \mathsf{Q}_X)$ as in Proposition 10.

Step 2. Recall the map $p \colon G \to B$ defined by $\pi(g \cdot x) = p(g)\,\pi(x)$ for every $x \in X$. We show that

Claim 2. — $\ker(p) = F\Gamma$.

The kernel of p contains clearly contains F. For $\gamma \in \Gamma$ we have $\pi(e) = \pi(\gamma \cdot e) = p(\gamma)\,\pi(e)$ and thus $\gamma \in \ker(p)$. We get that $\ker(p) \supset F\Gamma$. Conversely, let $g \in \ker(p)$. Then $g \cdot e$ belongs to the same fiber as e and there exists $u \in F$ with $g \cdot e = u \cdot e$ and we have $u^{-1} g \in \Gamma$. This proves the claim.

As the parallelogram structure P is associated to the projection $X \to B$ and P_X is associated to the projection $X \to \mathsf{G}/F\Gamma$ we have that $\mathsf{P} = \mathsf{P}_X$.

Step 3. We are left with showing that $\mathsf{Q} = \mathsf{Q}_X$.

Since$(e, e, \cdots, e) \in \mathsf{Q}$, for every $\mathbf{g} \in G^{[3,2]}$ we have $(g_{000} \cdot e, g_{001} \cdot e, \cdots, g_{111} \cdot e) \in \mathsf{Q}$ by definition of the group G. Under our identification, this means that $\mathsf{Q}_X \subset \mathsf{Q}$.

Let $\underline{\mathbf{x}} \in \mathsf{Q}$. As $\mathsf{P} = \mathsf{P}_X$, each face of $\underline{\mathbf{x}}$ belongs to P_X and by condition v) (applied to the structure $(\mathsf{P}_X, \mathsf{Q}_X)$) there exists x'_{111} such that $(x_{000}, x_{001}, \ldots, x_{110}, x'_{111}) \in \mathsf{Q}_X$. As $\mathsf{Q}_X \subset \mathsf{Q}$, these 8 points form a parallelepiped in Q. Since this last structure is strong, $x'_{111} = x_{111}$ and thus $\underline{\mathbf{x}} \in \mathsf{Q}_X$. ☐

5.4. An example. — Let G be a 2-step nilpotent group and let $F, \mathsf{P}, \mathsf{Q}$, and Q be as in Proposition 9. We compute the groups P_s and F in this case. Recall that $B = G/F$ and that $\pi\colon G \to B$ is the natural projection.

Let $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathsf{P}$. As $\mathsf{P} = G^{[2,1]}F^{[2]}$, $\mathbf{x}$ can be written as $\mathbf{x} = (g, ga, gb, gabu)$ with $g, a, b \in G$ and $u \in F$. We remark that $\mathbf{x} \in \mathsf{P}_s$ with $s = \pi(a)$ and that $q_s([\mathbf{x}]) = \pi(b)$.

Let $\mathbf{x}'$ be another parallelogram in P_s. Write it as $\mathbf{x}' = (g', g'a', g'b', g'a'b'u')$ with $g', a', b' \in G$ and $u' \in F$. We have that $\mathbf{x}$ and $\mathbf{x}'$ are equivalent if and only if $(\mathbf{x}, \mathbf{x}')$ belongs to $\mathsf{Q} = G^{[3,2]}F^{[3,1]}$. By a short computation using Lemma 6, part iii), we have that this is equivalent to the condition that there exist $h \in G$ and $v, w \in F$ with $g' = gh$, $a' = av$ and $b' = aw$.

Therefore the equivalence class $[\mathbf{x}]$ of $\mathbf{x} \in \mathsf{P}$ is characterized by the elements $s = \pi(a)$ and $\pi(b)$ of B and the element u of F.

For each $s \in B$, this defines a bijection of P_s onto $B \times F$. The homomorphism $q_s\colon P_s \to B$ corresponds to the natural projection $B \times F \to B$. Using the definition of the multiplication in P_s, we can easily check that it corresponds to multiplication coordinate by coordinate on $B \times F$. Applying this for $s = 1$, we have that the fiber group of (P, Q) is F.

Assume furthermore that Γ is a subgroup of G satisfying hypothesis (16) of Proposition 10 and let P_X, Q_X be as in this Proposition. We now have that $B = G/F\Gamma$ and that $\pi\colon X = G/\Gamma \to B$ is the natural projection. By elementary algebraic (but relatively long) computations, it is possible to show that the fiber group of $(\mathsf{P}_X, \mathsf{Q}_X)$ is F and we can determine the groups P_s. In this case, the group G is a subgroup of $\mathsf{G}_{\mathsf{Q}_X}$, acts transitively on X, and the general result (Theorem 2) stated in the following section gives that for every s the group P_s is the direct sum of B and F.

6. Conditions for transitivity

For $s \in B$, let F_s be the kernel of the group homomorphism $q_s\colon P_s \to B$ defined in Section 4.2. Thus F_s is the family of $\approx$-equivalence classes of parallelograms $\mathbf{x}$ with $\langle x_{00}, x_{01}\rangle = s$ and $\langle x_{00}, x_{10}\rangle = 1$. This means that x_{00} and x_{10} lie in the same fiber and therefore x_{01} and x_{11} also lie the same fiber.

Therefore, if $\mathbf{x}$ is a parallelogram its equivalence class $[\mathbf{x}]$ belongs to F_s if and only if $\mathbf{x}$ can be written in the form:

$$(a, b, u \cdot a, v \cdot b) \text{ with } a, b \in X,\ \langle a, b\rangle = s \text{ and } u, v \in F\ .$$

Let $a', b' \in X$ with $\langle a', b'\rangle = s$ and $u', v' \in F$. Then $[a, b, u \cdot a, v \cdot b] = [a', b', u' \cdot a', v' \cdot b']$ if and only if $[a, b, a', b'] = [u \cdot a, v \cdot b, u' \cdot a', v' \cdot b']$. By Proposition 13, part iii) this last class is equal to $[a, b, a', uv^{-1}u'^{-1}v' \cdot b']$. This is equivalent to $[a, b, a', b']$ if and only if $uv^{-1}u'^{-1}v' = 1$.

Therefore by passing to the quotient, the map

$$(a, b, u \cdot a, v \cdot b) \mapsto vu^{-1}$$

induces a one to one map $j_s \colon F_s \to F$.

The map j_s is clearly onto. Moreover, it follows immediately from the definition of the multiplication in F that this map is a group homomorphism. We conclude that $j_s \colon F_s \to F$ is a group isomorphism.

For every $s \in B$, we have an exact sequence

$$0 \longrightarrow F \xrightarrow{i_s} P_s \xrightarrow{q_s} B \longrightarrow 0 \ , \tag{18}$$

where $i_s \colon F \to P_s$ is the reciprocal map of the isomorphism $j_s \colon F_s \to F$.

THEOREM 2. — *The group* G *acts transitively on* X *if and only if for every* $s \in B$*, the exact sequence* (18) *splits.*

Proof. — Since the subgroup F of G acts transitively on each fiber, the group G acts transitively on X if and only if the map $p \colon \mathsf{G} \to B$ is onto.

First step. Let $s \in B$. First we assume that the exact sequence (18) splits. This means that there exists a group homomorphism $\phi \colon B \to P_s$ with $q_s \circ \phi = \mathbf{Id}_B$. We build a transformation h of X, belonging to G, with $p(g) = s$.

We choose $a, b \in X$ with $\langle a, b \rangle = s$. Let $x \in X$. Then we define $g \cdot x$ to be the unique point $g \cdot x$ in X such that $(a, b, x, g \cdot x)$ is a parallelogram in the class $\phi(\langle a, x \rangle)$.

Let $\mathbf{x}$ be a parallelogram in X. We have $\langle a, x_{00} \rangle \, \langle a, x_{01} \rangle^{-1} \, a\vec{x}_{10}^{\,-1} \, \langle a, x_{11} \rangle = 1$. Since ϕ is a group homomorphism, we have

$$\phi(\langle a, x_{00} \rangle) \, \phi(\langle a, x_{01} \rangle^{-1}) \, \phi(a\vec{x}_{10}^{\,-1}) \, \phi(\langle a, x_{11} \rangle) = 1 \ ,$$

This means that

$$[a, b, x_{00}, g \cdot x_{00}] \, [a, b, x_{01}, g \cdot x_{01}]^{-1} \, [a, b, x_{10}, g \cdot x_{10}]^{-1} \, [a, b, x_{11}, g \cdot x_{11}] = 1 \ .$$

By definition of the multiplication in P_s, we have that $[x_{00}, g \cdot x_{00}, x_{01}, , g \cdot x_{01}] = [x_{10}, g \cdot x_{10}, x_{11}, , g \cdot x_{11}]$ and thus $[\mathbf{x}] = [g \cdot x_{00}, g \cdot x_{01}, g \cdot x_{10}, g \cdot x_{11}]$.

Therefore the transformation g of X maps every parallelogram to an equivalent one and thus it belongs to G. By construction, $p(g) = s$.

Second step. Conversely, let $s \in B$ and assume that there exists $g \in \mathsf{G}$ with $p(g) = s$.

We choose $a \in X$ and define $b = g \cdot a$. For every $x \in X$, define $\psi(x) \in P_s$ to be the $\approx$-equivalence class of the parallelogram $(a, b, x, g \cdot x)$. Then for every $x \in X$, we have $q_s(\psi(x)) = \langle a, x \rangle$. Moreover, $\psi(a) = [a, b, a, b] = 1$ and for every $x \in X$ we have $q_s(\psi(x)) = \langle a, x \rangle$.

By the definition of G and the same computation as in the first step, we have that:

(19) $$\text{if } \mathbf{x} \text{ is a parallelogram, } \psi(x_{00})\psi(x_{01})^{-1}\psi(x_{10})^{-1}\psi(x_{11}) = 1\ .$$

We deduce that for $x \in X$, $\psi(x)$ depends only in $\pi(x)$ and thus on $\langle a, x\rangle$. Thus there exists a map $\phi\colon B \to P_s$ with $\psi(x) = \phi(\langle a, x\rangle)$ for every $x \in X$.

Thus the relation (19) implies that, when $b_{00}, b_{01}, b_{10}, b_{11}$ are four points in B with $b_{00}b_{01}^{-1}b_{10}^{-1}b_{11} = 1$ then $\phi(b_{00})\phi(b_{01})^{-1}\phi(x_{10})^{-1}\phi(x_{11}) = 1$. As $\phi(1) = \psi(a) = 1$, it follows that ϕ is a group homomorphism.

We have $q_s \circ \phi = \mathbf{Id}_B$ and so the exact sequence (18) splits. □

It follows immediately that:

Corollary 2. — *If either the fiber group is injective or if the base group is projective, then the parallelepiped structure is a nilparallelepiped structure, as in Proposition 10.*

6.1. Imbeddings. — In this Section we first show that there exist parallelepiped structures such that the structure group does not act transitively on the space, and so the parallelepiped structure is not a nilstructure. After the example, we show that every parallelepiped structure can be imbedded in a nilstructure. We start with a proposition that defines this imbedding.

Proposition 16. — *Let (P, Q) be a parallelepiped structure on a set X, assume that B is the base is the base of X, $\pi\colon X \to B$ the natural projection, and let Y be a subset of X satisfying the following two conditions:*

i) *The restriction to Y of π is onto.*
ii) *If $\underline{\mathbf{x}} \in \mathsf{Q}$ is a parallelepiped in X such that the seven points $x_{000}, \dots, x_{110}$ belong to Y, then $x_{111} \in Y$.*

Then $(\mathsf{P} \cap Y^{[2]}, \mathsf{Q} \cap Y^{[3]})$ is a parallelepiped structure on Y.

Definition 8. — If the conditions in the proposition are satisfied, we say that the parallelepiped structure $(Y, \mathsf{P} \cap Y^{[2]}, \mathsf{Q} \cap Y^{[3]})$ is *embedded* in $(X, \mathsf{P}, \mathsf{Q})$.

The proof of the proposition follows immediately from the definitions.

Example 6 (A parallelepiped structure which is not a nilstructure)

Let $p > 2$ be a prime number and let $B = \mathbb{Z}/p\mathbb{Z}$ and $F = \mathbb{Z}/p^2\mathbb{Z}$.

In this example we use additive notation for B and F. Let (P, Q) the (abelian) parallelepiped structure on $X = B \times F$ defined as in Example 5 (but with additive notation):

$$\mathsf{P} = B^{[2,1]} \times F^{[2]}\ ,\ \mathsf{Q} = B^{[3,2]} \times F^{[3,1]}\ .$$

The base group is B and $\pi\colon X \to B$ is projection onto the first coordinate. In this case the groups P_s, $s \in B$, introduced in Section 4.1 can easily be defined explicitly and we do so now.

Let $s \in B$. Then $P_s = B \times F$ (see Section 5.4). The element $\big((b_{00}, f_{00}), ((b_{01}, f_{01}), ((b_{10}, f_{10}), ((b_{11}, f_{11})\big)$ of $X^{[2]} = B^{[2]} \times F^{[2]}$ is a parallelogram belonging to P_s if and only if $b_{01} - b_{00} = b_{11} - b_{10} = s$. In this case, the class of this parallelogram is the element $(b_{10} - b_{00}, f_{00} - f_{01} - f_{10} + f_{11})$ of P_s. The homomorphism $q_s\colon P_s \to B$ is the first coordinate and the imbedding $F \to P_s$ is the map $f \mapsto (0, f)$.

Let X' be the subset of X given by

$$X' = \left\{(b, f)\colon f = \frac{b(b-1)}{2} \bmod p\mathbb{Z}/p^2\mathbb{Z}\right\} \ .$$

(Here we consider $p\mathbb{Z}/p^2\mathbb{Z}$ as a subgroup of $\mathbb{Z}/p^2\mathbb{Z} = F$ and identify $(\mathbb{Z}/p^2\mathbb{Z})/(p\mathbb{Z}/p^2\mathbb{Z})$ with $\mathbb{Z}/p\mathbb{Z} = B$ in the obvious way.) It is easy to check that if $\mathbf{x}$ is a parallelepiped of X such that seven of its edges belong to X', then the remaining edge also belongs to X'. Therefore, $\mathsf{P}' := \mathsf{P} \cap {X'}^{[2]}$ and $\mathsf{Q}' := \mathsf{Q} \cap {X'}^{[3]}$ is a parallelepiped structure on X', imbedded in the parallelepiped structure on X.

The fiber group is $F' = p\mathbb{Z}/p^2\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. For $s \in B$, the family of parallelograms of this structure is written P'_s, meaning that $\mathsf{P}'_s = \mathsf{P}_s \cap \mathsf{P}'$. Let P'_s be the abelian group of classes of parallelograms of this family. An immediate computation shows that P'_s is the subgroup

$$P'_s = \big\{(b, f) \in B \times F\colon f = sb \bmod p\mathbb{Z}/p^2\mathbb{Z}\big\}$$

of P_s. For $s \neq 0$ this group is isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ and thus it is not isomorphic to the direct sum $B' \oplus F' \approx \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Therefore the exact sequence (18) does not split and so there is no $g \in \mathsf{G}(X)$ projecting on s.

THEOREM 3. — *Every parallelepiped structure can be imbedded in a nilparallelepiped structure.*

Proof. — Let (P, Q) be a parallelepiped structure on the set X, let B be the base group and let F be the fiber group.

The abelian group F can be imbedded as a subgroup of a divisible group E. We write $X \otimes E$ for the set $X \times E$, quotiented by the equivalence relation given by

$$(x, e) \cong (x', e') \text{ if there exist } u \in F \text{ with } x' = u \cdot x \text{ and } e' = u^{-1}e \ .$$

We write $j\colon X \times E \to X \otimes E$ for the quotient map.

We now define a parallelepiped structure $(\mathsf{P}_E, \mathsf{Q}_E)$ on $X \otimes E$.

Let $(x, e), (x', e') \in X \times E$. If $(x, e) \cong (x', e')$, then $\pi(x) = \pi(x')$. Therefore we can define a map $\pi_E\colon X \otimes E \to B$ by mapping the equivalence class $j(x, e)$

to $\pi(x)$. We define P_E to be the parallelogram structure on $X \otimes E$ associated with this projection.

Let $(\mathsf{P}_E, \mathsf{Q}_E)$ be the parallelepiped structure on the abelian group E defined as in Example 4 and let $\mathsf{Q}_{X\otimes E}$ be the image of $\mathsf{Q} \times \mathsf{Q}_E$ under the natural projection $X^{[3]} \times E^{[3]} = (X \times E)^{[3]} \to (X \otimes E)^{[3]}$. We claim that $(\mathsf{P}_E, \mathsf{Q}_E)$ is a parallelepiped structure on $X \otimes E$.

For every $\mathbf{x} \in \mathsf{Q}_{X\otimes E}$, every face of $\mathbf{x}$ obviously belongs to $\mathsf{P}_{X\otimes E}$. The symmetries of $\mathsf{Q}_{X\otimes E}$ are obvious. Transitivity follows from Proposition 13, part iii) and Proposition 14. The closing parallelepiped property follows in the same way that it does in Lemma 7.

The base group of $\mathsf{P}_{X\otimes E}$ is B and the fiber group of $\mathsf{Q}_{X\otimes E}$ is F. Since F is a divisible group, the structure $(\mathsf{P}_{X\otimes E}, \mathsf{Q}_{X\otimes E})$ on $X \otimes E$ is a nilstructure.

On the other hand, the map $X \to X \otimes E$ associating $x \in X$ to the equivalence class of $j(x, 1)$ is one to one, and we can consider X as a subset of $X \otimes E$. Thus $\mathsf{P} = \mathsf{P}_{X\otimes E} \cap X^{[2]}$ and $\mathsf{Q} = \mathsf{Q}_{X\otimes E} \cap X^{[3]}$. □

7. Higher levels?

Gowers norms (for all $k \geq 2$) have already been used in several contexts and it is natural to ask to what extent the results here can be generalized for $k \geq 4$.

In the setting of ergodic theory, the authors define seminorms for all $k \geq 1$ and measures on Cartesian products of the space playing the same role played by the structures of this paper. The "strong" structures of this paper correspond to the "systems of order k" of [10]. These systems are completely characterized in terms of nilmanifolds. But the descriptions we give in the context of the present paper are substantially weaker.

Our definitions of parallelogram and parallelepiped structures extend immediately to structures of any dimension k, for which basic models are given by $(k-1)$-step nilmanifolds.

The main constructions and results of Section 3 extend directly, for example the reduction to strong structures (Proposition 4), the definition of the seminorm (Proposition 5), and the definition of the structure group (Definition 6). In particular, this structure group is always $(k-1)$-step nilpotent.

The results of Sections 4-6 are more difficult to extend. The main difficulty is that a good description of a structure of dimension k is only possible when the underlying structure of dimension $k-1$ arises from a nilmanifold.

BIBLIOGRAPHY

[1] N. Frantzikinakis, B. Host & B. Kra – "Multiple recurrence and convergence for sequences related to the prime numbers", *J. reine angew. Math.* **611** (2007), p. 131–144.

[2] N. Frantzikinakis & B. Kra – "Convergence of multiple ergodic averages for some commuting transformations", *Ergodic Theory Dynam. Systems* **25** (2005), p. 799–809.

[3] H. Furstenberg – "Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions", *J. Analyse Math.* **31** (1977), p. 204–256.

[4] W. T. Gowers – "A new proof of Szemerédi's theorem", *Geom. Funct. Anal.* **11** (2001), p. 465–588.

[5] B. Green & T. Tao – "An inverse theorem for the Gowers U^3 norm", to appear in *Proc. Edin. Math. Soc.*

[6] ______, "Linear equations in the primes", to appear in *Ann. of Math.*

[7] ______, "The primes contain arbitrarily long arithmetic progressions", *Ann. of Math.* **167** (2008).

[8] P. Hall – "A contribution to the theory of groups of prime-power order", *Proc. London Math. Soc. (2)* **36** (1933), p. 29–95.

[9] B. Host & B. Kra – "Convergence of polynomial ergodic averages", *Israel J. Math.* **149** (2005), p. 1–19, Probability in mathematics.

[10] ______, "Nonconventional ergodic averages and nilmanifolds", *Ann. of Math. (2)* **161** (2005), p. 397–488.

[11] B. Host & A. Maass – "Nilsystèmes d'ordre deux et parallélépipèdes", to appear in *Bull. Soc. Math. France.*

[12] M. Lazard – "Sur les groupes nilpotents et les anneaux de Lie", *Ann. Sci. Ecole Norm. Sup. (3)* **71** (1954), p. 101–190.

[13] A. Leibman – "Polynomial sequences in groups", *J. Algebra* **201** (1998), p. 189–206.

[14] ______, "Polynomial mappings of groups", *Israel J. Math.* **129** (2002), p. 29–60.

[15] ______, "Convergence of multiple ergodic averages along polynomials of several variables", *Israel J. Math.* **146** (2005), p. 303–315.

[16] J. Petresco – "Sur les commutateurs", *Math. Z.* **61** (1954), p. 348–356.

[17] E. Szemerédi – "On sets of integers containing no k elements in arithmetic progression", *Acta Arith.* **27** (1975), p. 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik.