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THE SADDLE-POINT METHOD IN $\mathbb{C}^N$ AND THE GENERALIZED AIRY FUNCTIONS

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Abstract. — We give a new version of the saddle-point method in N complex variables, for any $N \geq 2$. We apply our theorem to the asymptotic analysis of suitable multiple integrals of Airy's type.

Résumé (*La méthode du col dans $\mathbb{C}^N$ et les fonctions d'Airy généralisées*). — Nous donnons une nouvelle version de la méthode du col en N variables complexes, pour tout $N \geq 2$. Nous appliquons notre théorème à l'analyse asymptotique de certaines intégrales multiples du type d'Airy.

1. Introduction

1.1. The saddle-point method in $\mathbb{C}$, a generalization of Laplace's method for real integrals, yields asymptotic formulae for integrals

$$I(\tau) = \int_\gamma e^{\tau h(z)} g(z)\, \mathrm{d}z, \tag{1}$$

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where z is a complex variable, as the real parameter τ tends to $+\infty$. In (1), γ is a path contained in an open set $\varDelta \subset \mathbb{C}$ and not necessarily bounded, and $g(z)$ and $h(z)$ are holomorphic functions in $\varDelta$.

The origin of the saddle-point method can be traced back to a posthumous paper of Riemann [13]. Several authors, since the end of the nineteenth century (see, e.g., [8], [3], [2], [15]), applied the saddle-point method to integrals of type (1). The basic principle of the method, in its standard version, consists in replacing γ with a new integration path λ, equivalent to γ by Cauchy's theorem so that

$$I(\tau) = \int_{\lambda} e^{\tau h(z)} g(z)\,\mathrm{d}z, \tag{2}$$

where λ contains a 'nondegenerate' (or 'simple') saddle-point z_0 of $e^{h(z)}$, i.e., at which

$$h'(z_0) = 0, \quad h''(z_0) \neq 0, \tag{3}$$

and, along λ, $|e^{h(z)}| = \exp(\operatorname{Re} h(z))$ is maximal at z_0 and at no other point on λ. Under such conditions, and assuming $g(z_0) \neq 0$ and the integral (2) to be absolutely convergent, the main term in an asymptotic expansion of $I(\tau)$, as $\tau \to +\infty$, is determined by the values $g(z_0)$, $h(z_0)$ and $h''(z_0)$.

One of the earliest applications (in [2]) of the saddle-point method concerns the asymptotic study of the Airy function

$$\operatorname{Ai}(t) := \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \exp\left(t\zeta - \tfrac{1}{3}\zeta^3\right) \mathrm{d}\zeta \quad (t \in \mathbb{R}, t \to +\infty), \tag{4}$$

where the integration path is the union $\gamma_1 \cup \gamma_2$ of two of the three half-lines defined by

$$\gamma_k = \left\{ \varrho e^{2k\pi i/3} \mid 0 \leq \varrho < +\infty \right\} \quad (k = 0, 1, 2). \tag{5}$$

In (4), $\gamma_1 \cup \gamma_2$ is oriented from $e^{4\pi i/3}\infty$ to $e^{2\pi i/3}\infty$. The integral (4) was introduced by Airy [1] in connection with a problem in optics, and is transformed into an integral (1) by setting

$$\zeta = \tau^{1/3} z, \quad t = \tau^{2/3} \quad (\tau > 0). \tag{6}$$

This substitution yields

$$\operatorname{Ai}(\tau^{2/3}) = \frac{\tau^{1/3}}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \exp\left(\tau\left(z - \frac{1}{3}z^3\right)\right) \mathrm{d}z, \tag{7}$$

and this integral is of type (1) with $g(z) = 1$ and $h(z) = z - \frac{1}{3}z^3$. The solutions of $h'(z) = 0$ are $z = \pm 1$, and the relevant saddle-point for the integral (7) to apply the saddle-point method is seen to be $z_0 = -1$.

Similarly, let

$$\mathrm{Ai}_k(t) := \frac{1}{2\pi i} \int\limits_{\gamma_0 \cup \gamma_k} \exp\left(t\zeta - \frac{1}{3}\zeta^3\right) \mathrm{d}\zeta \quad (k = 1, 2), \tag{8}$$

with γ_0, γ_1 and γ_2 defined in (5), where the path $\gamma_0 \cup \gamma_k$ is oriented from $e^{2k\pi i/3}\infty$ to $+\infty$. With the substitution (6) we get

$$\mathrm{Ai}_k(\tau^{2/3}) = \frac{\tau^{1/3}}{2\pi i} \int\limits_{\gamma_0 \cup \gamma_k} \exp\left(\tau\left(z - \frac{1}{3}z^3\right)\right) \mathrm{d}z, \tag{9}$$

and for the integrals (9) with $k = 1, 2$ the relevant saddle-point is $z_0 = 1$.

Applying to the integrals (7) and (9) the asymptotic formula (23) below with $z_0 = -1$ and $z_0 = 1$ respectively, with $g(z) = 1$ and $f(z) = \exp(z - \frac{1}{3}z^3)$, and with $\tau = t^{3/2}$ in place of n, one easily gets, for $t \to +\infty$,

$$\mathrm{Ai}(t) \sim \frac{1}{2\sqrt{\pi}} t^{-1/4} \exp\left(-\frac{2}{3}t^{3/2}\right)$$

and

$$\mathrm{Ai}_k(t) \sim -\frac{i}{2\sqrt{\pi}} t^{-1/4} \exp\left(\frac{2}{3}t^{3/2}\right) \quad (k = 1, 2).$$

We refer to [4], pp. 279–289, or to [12], pp. 40–61, for a detailed treatment of the saddle-point method in $\mathbb{C}$ and its applications to the Airy integrals.

1.2. The problem of extending the saddle-point method to integrals

$$\int\limits_{\Gamma} e^{\tau h(z_1,\dots,z_N)} g(z_1, \dots, z_N)\, \mathrm{d}z_1 \cdots \mathrm{d}z_N \tag{10}$$

over suitable manifolds Γ in $\mathbb{C}^N$ with $N \geq 2$ was studied by Fedoryuk [6]. In [7], Chapter 1, Section 4.5, Fedoryuk gives a brief account of his method. As is well known, the complex Morse lemma ([5], Prop. 3.15, p. 142, or [7], p. 125) ensures that in a neighbourhood of a nondegenerate saddle-point $\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ of $\exp h(z_1, \dots, z_N)$ (see Definition 3.2 below) there exists a local change of variables transforming $h(z_1, \dots, z_N) - h\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ into a sum of squares. Similarly to [16], Theorem 1, pp. 480–482, using Morse's lemma one gets an expansion of the integral (10) into an asymptotic power series of τ^{-1} as $\tau \to +\infty$, provided the integration manifold Γ can be transformed into a manifold Λ equivalent to Γ by Cauchy–Poincaré's theorem, thus preserving the value of (10), containing the nondegenerate saddle-point $\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ of $\exp h(z_1, \dots, z_N)$ as an interior point, and such that

$$\max_{(z_1,\dots,z_N)\in\Lambda} \operatorname{Re} h(z_1, \dots, z_N) \tag{11}$$

is attained only at $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$. Moreover, the coefficients of such an asymptotic series can be computed using Fedoryuk's method (see [16], Theorem 2, p. 483 and [7], formula (1.61), p. 125). Thus the main difficulty to get the asymptotic expansion of (10) through Fedoryuk's method is to locate the relevant nondegenerate saddle-point $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ and prove the existence of a manifold Λ containing $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ and satisfying the properties above.

In order to find a constructive process to transform Γ into an equivalent manifold of 'steepest descent' for $\operatorname{Re} h(z_1,\dots,z_N)$ thus ensuring that, on such a manifold, (11) is attained only at $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$, Fedoryuk introduced techniques from algebraic topology based on homology groups, which, beside their theoretical interest, proved to be difficult to apply in concrete examples. In fact, in an example of dimension $N=2$ arising from catastrophe theory, Ursell [14] showed the non-uniqueness of steepest descent surfaces (see also the discussion in Kaminski [11]), with the result that in most cases there is no available method to transform the integration surface Γ into an equivalent surface Λ satisfying the required properties, and not even a criterion to find towards which nondegenerate saddle-point the surface Γ should be deformed.

The main example considered by Ursell [14] is an integral in $\mathbb{C}^2$ representing a natural two-dimensional generalization of the Airy integral (4)–(7). Ursell obtained results on the asymptotic behaviour of such an integral over a surface with four nearly coincident saddle-points. His final comment is: "For two complex variables little seems to be known ... More work is needed on a method of steepest descents for two complex variables, particularly on the deformation of the two-dimensional surfaces of integration".

The main purpose of the present paper is to circumvent the difficulties involved in Fedoryuk's topological deformation process by introducing a more flexible analytic method to find the relevant nondegenerate saddle-point $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ of $f(z_1,\dots,z_N)$ for an N-dimensional integral

$$\int_\Gamma f(z_1,\dots,z_N)^n g(z_1,\dots,z_N)\,\mathrm{d}z_1\cdots\mathrm{d}z_N \quad (n\in\mathbb{N},\ n\to+\infty), \tag{12}$$

for any fixed $N\geq 2$. For the treatment of (12) with $n\in\mathbb{N}$, we need not assume $f(z_1,\dots,z_N)\neq 0$. In Theorem 4.2 we obtain an asymptotic formula for the integral (12) under assumptions which permit us to avoid the search for an equivalent integration manifold of steepest descent for $|f(z_1,\dots,z_N)|$. In Section 5 we give a self-contained proof of Theorem 4.2. We treat (12) as an N-times iterated integral, and we apply the one-dimensional steepest descent method to each variable successively. This allows us to dispense with the global deformation process of the integration manifold. Our method, being independent of Morse's lemma, in principle could be extended, under suitable new assumptions, to the asymptotic analysis of the integral (12) in the neighbourhood of a degenerate saddle-point of $f(z_1, \ldots, z_N)$.

The applications we give in Section 6 show that in several interesting cases the assigned integration manifold Γ can rather easily be transformed into an equivalent manifold Λ satisfying the assumptions of Theorem 4.2. Our Theorem 4.2 generalizes to any dimension the result proved for $N = 2$ by Hata in [9], where the author applies his method to prove nonquadraticity measures for logarithms of suitable rational numbers and concludes the introduction with the words: "To establish the $\mathbb{C}^N$-saddle method may be an interesting problem itself".

Our result is based on the notion of 'admissible' saddle-point of f, which we introduce in Definition 3.3 below. In Remark 3.4 we show that such a notion is not essentially restrictive: up to applying a suitable invertible linear transformation of the variables $z_1, \ldots, z_N$, every nondegenerate saddle-point $\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)$ is transformed into an admissible saddle-point.

If $f(z_1, \ldots, z_N) \neq 0$ and

$$f(z_1, \ldots, z_N) = \exp h(z_1, \ldots, z_N) \tag{13}$$

with a given holomorphic function $h(z_1, \ldots, z_N)$, there is no ambiguity on the value of the logarithm of f, and hence on the power

$$f(z_1, \ldots, z_N)^\tau = \exp\left(\tau \log f(z_1, \ldots, z_N)\right)$$

for $\tau \notin \mathbb{Z}$, provided one takes $\log f(z_1, \ldots, z_N) = h(z_1, \ldots, z_N)$, whence

$$f(z_1, \ldots, z_N)^\tau = \exp(\tau\, h(z_1, \ldots, z_N)) \tag{14}$$

as in (10). In this case our Theorem 4.2 holds with $\tau \in \mathbb{R}$, $\tau \to +\infty$, in place of the integer exponent $n \to +\infty$ in (12).

In Section 6 we apply Theorem 4.2 to prove asymptotic formulae for N-fold Airy integrals of the type considered by Ursell [14] for $N = 2$, but without restrictions concerning the mutual distance of the saddle-points. We give a full treatment of such integrals for $N = 2$. For arbitrary N, we prove the required asymptotic formula for a suitable choice of the N integration paths.

2. The saddle-point method in $\mathbb{C}$

We briefly recall some well known aspects of the classical one-dimensional saddle-point method which will be used in the following sections. The aim of the method is to prove an asymptotic formula for an integral

$$I_n = \int_\lambda f(z)^n g(z)\, \mathrm{d}z \quad (n \in \mathbb{N},\, n \to +\infty), \tag{15}$$

where λ is a piecewise continuously differentiable path contained in an open set $\Delta \subset \mathbb{C}$ and not necessarily bounded, and $f(z)$ and $g(z)$ are holomorphic functions in Δ.

We assume that the path λ contains a nondegenerate saddle-point z_0 of $f(z)$, i.e., a point satisfying

$$f(z_0) \neq 0, \quad f'(z_0) = 0, \quad f''(z_0) \neq 0, \tag{16}$$

at which $g(z_0) \neq 0$. Moreover, we assume

$$|f(z)| < |f(z_0)| \tag{17}$$

for all z in the closure of λ in $\mathbb{C} \cup \{\infty\}$, $z \neq z_0$.

By Cauchy's theorem we may plainly assume that, in a neighbourhood of z_0, λ coincides with the line tangent at z_0 to the path η of steepest descent for $|f(z)|$, i.e., of maximal slope for $|f(z)|$ satisfying (17). It is easily seen that this tangent line (the line of steepest descent for $|f(z)|$ at z_0) has the parametric equation

$$z = z_0 + re^{i\vartheta}, \quad r \in \mathbb{R}, \tag{18}$$

where

$$\vartheta = h\pi - \frac{1}{2} \arg\left(-\frac{f''(z_0)}{f(z_0)}\right), \quad h \in \mathbb{Z}. \tag{19}$$

In (19), the parity of the integer h must be chosen so that the orientation of the line (18) for increasing r agrees with the orientation of the path λ in (15).

We prove (19). Since η is the path of steepest descent for $|f|$, the gradient $\nabla|f|$ is tangent to η. By the Cauchy–Riemann equations, $\nabla \arg f$ is orthogonal to $\nabla|f|$. Thus $\arg f$ is constant along η, i.e.,

$$\arg \frac{f(z)}{f(z_0)} = 0.$$

Therefore, by (17),

$$\frac{f(z)}{f(z_0)} = \left|\frac{f(z)}{f(z_0)}\right| < 1 \quad \text{for all } z \in \eta,\ z \neq z_0. \tag{20}$$

By (16), Taylor's formula yields

$$\frac{f(z)}{f(z_0)} = 1 + \frac{f''(z_0)}{f(z_0)} \frac{(z-z_0)^2}{2!} + O\left(|z-z_0|^3\right),$$

whence, by (20),

$$-\frac{f''(z_0)}{f(z_0)}(z-z_0)^2 + O\left(|z-z_0|^3\right) > 0 \tag{21}$$

for $z \in \eta,\ z \neq z_0,\ z \to z_0$.

In a neighbourhood of z_0 we parametrize η with $r \in \mathbb{R}$ such that $|r| = |z-z_0|$. Hence, for $z \in \eta$ and for any sufficiently small $|r|$,

$$z = z_0 + re^{i\vartheta(r)}$$

where

$$\arg(z - z_0) = \begin{cases} \vartheta(r) & \text{for } r > 0 \\ \vartheta(r) + \pi & \text{for } r < 0. \end{cases}$$

Thus, dividing (21) by $|z - z_0|^2$,

$$-\frac{f''(z_0)}{f(z_0)} e^{2i\vartheta(r)} + O(|r|) > 0.$$

For $r \to 0$ we get

$$-\frac{f''(z_0)}{f(z_0)} e^{2i\vartheta} > 0, \tag{22}$$

where

$$\vartheta = \lim_{r\to 0} \vartheta(r)$$

is the argument of the tangent vector to η at z_0. By (22),

$$2\vartheta + \arg\left(-\frac{f''(z_0)}{f(z_0)}\right) = 2h\pi, \quad h \in \mathbb{Z},$$

and (19) follows.

As is well known (see, e.g., [4], pp. 279–285), under the above assumptions (16)–(17), and assuming $g(z_0) \neq 0$ and the integral (15) to be absolutely convergent for every sufficiently large n, the following asymptotic formula holds:

$$I_n = \sqrt{2\pi}\, e^{i\vartheta} g(z_0) \sqrt{\frac{|f(z_0)|}{|f''(z_0)|}}\, \frac{f(z_0)^n}{\sqrt{n}}\, (1 + o(1)) \qquad (n \to +\infty) \tag{23}$$

with ϑ given by (19).

3. Definitions and preliminary results

Let f be a function of N complex variables $z_1, \ldots, z_N$, holomorphic in an open set $\Delta \subset \mathbb{C}^N$ and such that, for each $j = 1, \ldots, N$, $\partial f / \partial z_j$ does not vanish identically.

Definition 3.1. — A point $\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) \in \Delta$ is a saddle-point of f if

$$\begin{cases} f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) \neq 0 \\ \frac{\partial f}{\partial z_j}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) = 0 \quad \text{for } j = 1, \ldots, N. \end{cases} \tag{24}$$

Let $H(z_1,\dots,z_N)$ denote the hessian determinant of f:

$$H(z_1,\dots,z_N)=\det\begin{pmatrix}\frac{\partial^2 f}{\partial z_1^2} & \cdots & \frac{\partial^2 f}{\partial z_1\partial z_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial z_N\partial z_1} & \cdots & \frac{\partial^2 f}{\partial z_N^2}\end{pmatrix}.$$

Definition 3.2. — A saddle-point $\left(z_1^{(0)},\dots,z_N^{(0)}\right)\in\mathit{\Delta}$ of f is nondegenerate if

$$H\left(z_1^{(0)},\dots,z_N^{(0)}\right)\neq 0.$$

Clearly the notion of nondegenerate saddle-point is independent of the ordering $z_1,\dots,z_N$ of the variables.

For a given ordering $z_1,\dots,z_N$, we also define the minors:

$$H_j=H_j(z_1,\dots,z_N):=\det\begin{pmatrix}\frac{\partial^2 f}{\partial z_1^2} & \cdots & \frac{\partial^2 f}{\partial z_1\partial z_j}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial z_j\partial z_1} & \cdots & \frac{\partial^2 f}{\partial z_j^2}\end{pmatrix}\quad (j=1,\dots,N). \tag{25}$$

Definition 3.3. — A saddle-point $\left(z_1^{(0)},\dots,z_N^{(0)}\right)\in\mathit{\Delta}$ of f is admissible with respect to the ordering $z_1,\dots,z_N$ of the variables if

$$H_j\left(z_1^{(0)},\dots,z_N^{(0)}\right)\neq 0\quad\text{for } j=1,\dots,N.$$

Remark 3.4. — Since $H_N(z_1,\dots,z_N)=H(z_1,\dots,z_N)$, an admissible saddle-point with respect to an ordering of the variables is also a nondegenerate saddle-point. Conversely, if $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ is a nondegenerate saddle-point of f, there exists a homogeneous linear transformation of the variables:

$$\begin{cases} z_1=a_{11}w_1+\cdots+a_{1N}w_N\\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ z_N=a_{N1}w_1+\cdots+a_{NN}w_N\end{cases} \tag{26}$$

with coefficients $a_{hk}\in\mathbb{C}$, depending on f and on $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ and satisfying

$$\det\begin{pmatrix}a_{11} & \dots & a_{1N}\\ \vdots & \ddots & \vdots\\ a_{N1} & \dots & a_{NN}\end{pmatrix}\neq 0,$$

such that the point $\left(w_1^{(0)},\dots,w_N^{(0)}\right)$, corresponding to $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ through (26), is an admissible saddle-point of the function

$$\widehat{f}(w_1,\dots,w_N) := f(a_{11}w_1+\cdots+a_{1N}w_N,\dots,a_{N1}w_1+\cdots+a_{NN}w_N)$$

with respect to the ordering $w_1,\dots,w_N$.

To prove this, denote

$$\mathcal{A}=\begin{pmatrix} a_{11} & \dots & a_{1N}\\ \vdots & \ddots & \vdots\\ a_{N1} & \dots & a_{NN}\end{pmatrix}$$

and

$$\mathcal{H}_0=\left.\begin{pmatrix} \frac{\partial^2 f}{\partial z_1^2} & \cdots & \frac{\partial^2 f}{\partial z_1\partial z_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial z_N\partial z_1} & \cdots & \frac{\partial^2 f}{\partial z_N^2}\end{pmatrix}\right|_{\left(z_1^{(0)},\dots,z_N^{(0)}\right)}, \tag{27}$$

whence $\det\mathcal{H}_0 = H\left(z_1^{(0)},\dots,z_N^{(0)}\right)\neq 0$. We have

$$\begin{pmatrix}\frac{\partial}{\partial w_1}\\ \vdots\\ \frac{\partial}{\partial w_N}\end{pmatrix}\widehat{f} = {}^t\!\mathcal{A}\cdot\begin{pmatrix}\frac{\partial}{\partial z_1}\\ \vdots\\ \frac{\partial}{\partial z_N}\end{pmatrix} f.$$

Here and in the sequel we denote by ${}^t\mathcal{M}$ the transpose of a matrix $\mathcal{M}$. It follows that

$$\widehat{\mathcal{H}_0} := \left.\begin{pmatrix} \frac{\partial^2 \widehat{f}}{\partial w_1^2} & \cdots & \frac{\partial^2 \widehat{f}}{\partial w_1\partial w_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2 \widehat{f}}{\partial w_N\partial w_1} & \cdots & \frac{\partial^2 \widehat{f}}{\partial w_N^2}\end{pmatrix}\right|_{\left(w_1^{(0)},\dots,w_N^{(0)}\right)} = {}^t\!\mathcal{A}\,\mathcal{H}_0\mathcal{A}.$$

Since the matrix $\mathcal{H}_0$ is symmetric, by a theorem of Autonne–Takagi (see [10], p. 153, Corollary 2.6.6 (a)) there exist a unitary matrix $\mathcal{U}$ and a diagonal matrix $\mathcal{D}$ such that

$$\mathcal{H}_0=\mathcal{U}\,\mathcal{D}\,{}^t\mathcal{U}.$$

Choosing $\mathcal{A}={}^t(\mathcal{U}^{-1})=({}^t\mathcal{U})^{-1}$, whence $\det\mathcal{A}\neq 0$, we get

$$\mathcal{D}=\mathcal{U}^{-1}\,\mathcal{H}_0\,{}^t\mathcal{U}^{-1} = {}^t\!\mathcal{A}\,\mathcal{H}_0\mathcal{A}=\widehat{\mathcal{H}_0}.$$

Hence $\widehat{\mathcal{H}_0}$ is diagonal and nonsingular because

$$\det\widehat{\mathcal{H}_0}=(\det\mathcal{A})^2\det\mathcal{H}_0\neq 0.$$

Therefore

$$\det \widehat{\mathcal{H}_0} = \left(\frac{\partial^2 \widehat{f}}{\partial w_1^2} \cdots \frac{\partial^2 \widehat{f}}{\partial w_N^2}\right)\left(w_1^{(0)}, \ldots, w_N^{(0)}\right) \neq 0,$$

whence, for $j = 1, \ldots, N$,

$$\widehat{H}_j\left(w_1^{(0)}, \ldots, w_N^{(0)}\right) = \det \left.\begin{pmatrix} \frac{\partial^2 \widehat{f}}{\partial w_1^2} & 0 & \ldots & 0 \\ 0 & \frac{\partial^2 \widehat{f}}{\partial w_2^2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\partial^2 \widehat{f}}{\partial w_j^2} \end{pmatrix}\right|_{\left(w_1^{(0)}, \ldots, w_N^{(0)}\right)} \neq 0.$$

Let $\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)$ be an admissible saddle-point of f with respect to the ordering $z_1, \ldots, z_N$. By the implicit function theorem, for each $j = 1, \ldots, N-1$ the system

$$\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_j} = 0 \tag{28}$$

is locally solvable with respect to $z_1, \ldots, z_j$. In other words, in an open neighbourhood ν_{j+1} of $\left(z_{j+1}^{(0)}, \ldots, z_N^{(0)}\right)$ in $\mathbb{C}^{N-j}$ there exist j holomorphic functions

$$Z_{1j}(z_{j+1}, \ldots, z_N), \ \ldots, \ Z_{jj}(z_{j+1}, \ldots, z_N)$$

satisfying the following j identities:

$$\begin{cases} \frac{\partial f}{\partial z_1}(Z_{1j}(z_{j+1}, \ldots, z_N), \ldots, Z_{jj}(z_{j+1}, \ldots, z_N), z_{j+1}, \ldots, z_N) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial f}{\partial z_j}(Z_{1j}(z_{j+1}, \ldots, z_N), \ldots, Z_{jj}(z_{j+1}, \ldots, z_N), z_{j+1}, \ldots, z_N) = 0 \end{cases} \tag{29}$$

for all $(z_{j+1}, \ldots, z_N) \in \nu_{j+1}$, with

$$z_1^{(0)} = Z_{1j}\left(z_{j+1}^{(0)}, \ldots, z_N^{(0)}\right), \ \ldots, \ z_j^{(0)} = Z_{jj}\left(z_{j+1}^{(0)}, \ldots, z_N^{(0)}\right). \tag{30}$$

Moreover, for each $1 \leq k < j < N$ and for each $(z_{j+1}, \ldots, z_N) \in \nu_{j+1}$, we can solve the system of the first k equations in (28)–(29) with respect to $z_1 = Z_{1j}$, $\ldots$, $z_k = Z_{kj}$. Thus, shrinking here and in what follows the neighbourhood ν_{j+1} if necessary, we get the k identities

$$\begin{cases} Z_{1j}(z_{j+1}, \ldots, z_N) = \\ \quad Z_{1k}(Z_{k+1,j}(z_{j+1}, \ldots, z_N), \ldots, Z_{jj}(z_{j+1}, \ldots, z_N), z_{j+1}, \ldots, z_N) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Z_{kj}(z_{j+1}, \ldots, z_N) = \\ \quad Z_{kk}(Z_{k+1,j}(z_{j+1}, \ldots, z_N), \ldots, Z_{jj}(z_{j+1}, \ldots, z_N), z_{j+1}, \ldots, z_N) \end{cases} \tag{31}$$

for all $(z_{j+1}, \ldots, z_N) \in \nu_{j+1}$.

Lemma 3.5. — *For all k,j such that $1 \le k \le j < N$ we have*

$$\frac{\partial Z_{kj}}{\partial z_{j+1}}(z_{j+1},\dots,z_N) = (-1)^{k+j+1}\,\frac{\delta_{j+1,k}}{H_j}\bigg|_{(Z_{1j}(z_{j+1},\dots,z_N),\dots,Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N)},$$

where H_j is defined in (25), and $\delta_{j+1,k}$ is the determinant obtained by removing the last row and the k-th column from the determinant H_{j+1}.

Proof. — Differentiating the identities (29) with respect to z_{j+1} we get

$$\begin{cases} \frac{\partial^2 f}{\partial z_1^2}\frac{\partial Z_{1j}}{\partial z_{j+1}} + \cdots + \frac{\partial^2 f}{\partial z_1 \partial z_j}\frac{\partial Z_{jj}}{\partial z_{j+1}} + \frac{\partial^2 f}{\partial z_1 \partial z_{j+1}} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\partial^2 f}{\partial z_j \partial z_1}\frac{\partial Z_{1j}}{\partial z_{j+1}} + \cdots + \frac{\partial^2 f}{\partial z_j^2}\frac{\partial Z_{jj}}{\partial z_{j+1}} + \frac{\partial^2 f}{\partial z_j \partial z_{j+1}} = 0, \end{cases}$$

and the lemma follows from Cramer's rule. □

Lemma 3.6. — *Let*

$$\widetilde{f}_j(z_{j+1},\dots,z_N) := f(Z_{1j}(z_{j+1},\dots,z_N),\dots,Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N)$$

for $j = 1,\dots,N-1$. Then

$$\frac{\partial^2 \widetilde{f}_j}{\partial z_{j+1}^2} = \frac{H_{j+1}}{H_j}\bigg|_{(Z_{1j}(z_{j+1},\dots,z_N),\dots,Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N)}.$$

Proof. — Owing to the identities (29), for $l = j+1,\dots,N$ we get

$$\frac{\partial \widetilde{f}_j}{\partial z_l} = \frac{\partial f}{\partial z_l}(Z_{1j}(z_{j+1},\dots,z_N),\dots,Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N). \tag{32}$$

Hence, by Lemma 3.5,

$$\begin{aligned} \frac{\partial^2 \widetilde{f}_j}{\partial z_{j+1}^2} &= \frac{\partial^2 f}{\partial z_{j+1}\partial z_1}\frac{\partial Z_{1j}}{\partial z_{j+1}} + \cdots + \frac{\partial^2 f}{\partial z_{j+1}\partial z_j}\frac{\partial Z_{jj}}{\partial z_{j+1}} + \frac{\partial^2 f}{\partial z_{j+1}^2} \\ &= \frac{1}{H_j}\left(\frac{\partial^2 f}{\partial z_{j+1}\partial z_1}(-1)^{j+2}\delta_{j+1,1} + \cdots + \frac{\partial^2 f}{\partial z_{j+1}\partial z_j}(-1)^{2j+1}\delta_{j+1,j} + \frac{\partial^2 f}{\partial z_{j+1}^2}H_j\right) \\ &= \frac{H_{j+1}}{H_j}, \end{aligned}$$

where in the last equality we have applied the Laplace expansion of the determinant H_{j+1} along the last row. □

4. Further lemmas and statement of the main theorem

4.1. Let $N \geq 2$, let $f(z_1, \dots, z_N)$ and $g(z_1, \dots, z_N)$ be holomorphic functions in an open set $\Delta \subset \mathbb{C}^N$, let $\left(z_1^{(0)}, \dots, z_N^{(0)}\right) \in \Delta$ be an admissible saddle-point of $f(z_1, \dots, z_N)$ with respect to the ordering $z_1, \dots, z_N$ (see Definition 3.3), and let $g\left(z_1^{(0)}, \dots, z_N^{(0)}\right) \neq 0$. For any integer $n \geq 1$, let

(33)
$$I_n = \int\limits_{\lambda_N} \mathrm{d}z_N \int\limits_{\lambda_{N-1}(z_N)} \mathrm{d}z_{N-1} \cdots \int\limits_{\lambda_2(z_3,\dots,z_N)} \mathrm{d}z_2 \int\limits_{\lambda_1(z_2,\dots,z_N)} f(z_1, \dots, z_N)^n g(z_1, \dots, z_N)\, \mathrm{d}z_1$$

be an N-fold integral, where for each $j = 1, \dots, N-1$ the path λ_j depends on $z_{j+1} \in \lambda_{j+1}, \dots, z_N \in \lambda_N$. We assume $\lambda_1, \dots, \lambda_N$ to be (not necessarily bounded) piecewise continuously differentiable paths such that $(z_1, \dots, z_N) \in \Delta$ for all $z_N \in \lambda_N$, $z_{N-1} \in \lambda_{N-1}(z_N)$, $\dots$, $z_1 \in \lambda_1(z_2, \dots, z_N)$, and the integral I_n to be absolutely convergent for every sufficiently large n.

Let

$$z_N^{(0)} \in \lambda_N \tag{34}$$

be an interior point of λ_N, and let the maximality condition

$$\begin{aligned} \left|\widetilde{f}_{N-1}(z_N)\right| &= |f(Z_{1,N-1}(z_N), \dots, Z_{N-1,N-1}(z_N), z_N)| \\ &< \left|f\left(z_1^{(0)}, \dots, z_{N-1}^{(0)}, z_N^{(0)}\right)\right| \end{aligned} \tag{35}$$

hold for all $z_N \in \nu_N \cap \lambda_N,\ z_N \neq z_N^{(0)}$, where the functions $Z_{1,N-1}, \dots, Z_{N-1,N-1}$ and the neighbourhood ν_N are defined by the identities (29) with $j = N-1$, and

$$\widetilde{f}_{N-1}(z_N) = f(Z_{1,N-1}(z_N), \dots, Z_{N-1,N-1}(z_N), z_N)$$

is defined as in Lemma 3.6. We have $\widetilde{f}_{N-1}\left(z_N^{(0)}\right) = f\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ by (30). By Definition 3.3, by (32) with $l = N$ and Lemma 3.6 with $j = N-1$, by (16) and by (35), $z_N^{(0)}$ is a nondegenerate saddle-point of the function $\widetilde{f}_{N-1}(z_N)$. Thus we may clearly assume, without loss of generality, that in a circular neighbourhood of centre $z_N^{(0)}$ and radius

$$\varrho_N > 0 \tag{36}$$

the path λ_N coincides with the line of steepest descent at $z_N^{(0)}$ for $\left|\widetilde{f}_{N-1}(z_N)\right|$.

Next we assume conditions similar to (34)–(35), successively for $j = N-1$, $N-2, \dots, 1$. For each j with $1 \leq j \leq N-1$ and for any fixed $(z_{j+1}, \dots, z_N) \in \nu_{j+1}$ such that $z_N \in \lambda_N$, $z_{N-1} \in \lambda_{N-1}(z_N)$, $\dots$, $z_{j+1} \in \lambda_{j+1}(z_{j+2}, \dots, z_N)$, let

$$Z_{jj}(z_{j+1}, \dots, z_N) \in \lambda_j(z_{j+1}, \dots, z_N) \tag{37}$$

be an interior point of $\lambda_j(z_{j+1},\dots,z_N)$, and let

$$\left|\widetilde{f}_{j-1}(z_j,z_{j+1},\dots,z_N)\right| < \left|\widetilde{f}_j(z_{j+1},\dots,z_N)\right| \tag{38}$$

for all $z_j \in \lambda_j(z_{j+1},\dots,z_N)$, $z_j \neq Z_{jj}(z_{j+1},\dots,z_N)$, $(z_j,z_{j+1},\dots,z_N) \in \nu_j$, where $\widetilde{f}_{j-1}$ and $\widetilde{f}_j$ are defined as in Lemma 3.6, and $\widetilde{f}_0 := f$. For $z_j = Z_{jj}(z_{j+1},\dots,z_N)$, the two sides of (38) are equal by the identities (31) with $k = j-1$.

For any fixed $(z_{j+1},\dots,z_N) \in \nu_{j+1}$, $z_j = Z_{jj}(z_{j+1},\dots,z_N)$ is a nondegenerate saddle-point of $\widetilde{f}_{j-1}(z_j,z_{j+1},\dots,z_N)$. For, by (38),

$$\widetilde{f}_{j-1}\left(Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N\right) \neq 0;$$

by Lemma 3.6 and by the identities (31)

$$\frac{\partial^2 \widetilde{f}_{j-1}}{\partial z_j^2}\left(Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N\right) = \left.\frac{H_j}{H_{j-1}}\right|_{(Z_{1j}(z_{j+1},\dots,z_N),\dots,Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N)} \neq 0;$$

and by (32), (31) and the last of (29)

$$\frac{\partial \widetilde{f}_{j-1}}{\partial z_j}\left(Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N\right) = \frac{\partial f}{\partial z_j}\left(Z_{1j}(z_{j+1},\dots,z_N),\dots,Z_{jj}(z_{j+1},\dots,z_N),z_{j+1},\dots,z_N\right) = 0.$$

Thus we may assume that in a circular neighbourhood of centre $Z_{jj}(z_{j+1},\dots,z_N)$ and radius

$$\varrho_j(z_{j+1},\dots,z_N) > 0 \tag{39}$$

the path $\lambda_j(z_{j+1},\dots,z_N)$ is the line of steepest descent for $\left|\widetilde{f}_{j-1}(z_j,z_{j+1},\dots,z_N)\right|$ at $z_j = Z_{jj}(z_{j+1},\dots,z_N)$.

By applying the maximality assumptions (38) successively for $j = 1,2,\dots,N-1$ and then (35) at the N-th step, we get the inequality

$$|f(z_1,\dots,z_N)| < \left|f\left(z_1^{(0)},\dots,z_N^{(0)}\right)\right| \tag{40}$$

for all $(z_1,\dots,z_N)$ in a suitable neighbourhood of $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ such that $z_N \in \lambda_N,\dots,z_1 \in \lambda_1(z_2,\dots,z_N)$ and $(z_1,\dots,z_N) \neq \left(z_1^{(0)},\dots,z_N^{(0)}\right)$. We require that (40) holds also away from $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$. We assume that for any neighbourhood σ of $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ there exists a real number $\mu=\mu(\sigma)$ with $0<\mu<1$ such that

$$|f(z_1,\dots,z_N)|\le\mu\left|f\left(z_1^{(0)},\dots,z_N^{(0)}\right)\right| \tag{41}$$

for all $z_N\in\lambda_N$, ..., $z_1\in\lambda_1(z_2,\dots,z_N)$ satisfying $(z_1,\dots,z_N)\notin\sigma$.

Remark 4.1. — For every $(z_{j+1},\dots,z_N)\in\nu_{j+1}$ with $z_N\in\lambda_N$, ..., $z_{j+1}\in\lambda_{j+1}(z_{j+2},\dots,z_N)$, the radius $\varrho_j(z_{j+1},\dots,z_N)$ is not uniquely defined. Since $z_j=Z_{jj}(z_{j+1},\dots,z_N)$ is a nondegenerate saddle-point of $\widetilde{f}_{j-1}(z_j,z_{j+1},\dots,z_N)$, we can plainly choose (39) to be a continuous function of $z_{j+1},\dots,z_N$. Thus for each $j=1,\dots,N-1$ there exists $\varrho_j>0$ such that $\varrho_j(z_{j+1},\dots,z_N)\ge\varrho_j$ for all $z_N\in\lambda_N$, ..., $z_{j+1}\in\lambda_{j+1}(z_{j+2},\dots,z_N)$ with $(z_{j+1},\dots,z_N)$ in a neighbourhood of $\left(z_{j+1}^{(0)},\dots,z_N^{(0)}\right)$. Defining

$$\varrho=\min\{\varrho_1,\dots,\varrho_N\}$$

with ϱ_N in (36), for each $j=1,\dots,N$ the path λ_j is the line of steepest descent for $\left|\widetilde{f}_{j-1}(z_j,\dots,z_N)\right|$ at $z_j=Z_{jj}(z_{j+1},\dots,z_N)$ $(j=1,\dots,N-1)$, $z_N=z_N^{(0)}$, in the neighbourhood of centre $\left(z_j^{(0)},\dots,z_N^{(0)}\right)$ and radius $\varrho>0$.

4.2. We can now state our main theorem.

Theorem 4.2. — *Let $N\ge2$, let $f(z_1,\dots,z_N)$ and $g(z_1,\dots,z_N)$ be holomorphic functions in an open set $\Delta\subset\mathbb{C}^N$, let $\left(z_1^{(0)},\dots,z_N^{(0)}\right)\in\Delta$ be an admissible saddle-point of $f(z_1,\dots,z_N)$ with respect to the ordering $z_1,\dots,z_N$, and let $g\left(z_1^{(0)},\dots,z_N^{(0)}\right)\neq0$. Let $H_j(z_1,\dots,z_N)(j=1,\dots,N-1)$ and $H_N(z_1,\dots,z_N)=H(z_1,\dots z_N)$ be the hessian determinants defined in* (25)*, and let $H_0(z_1,\dots,z_N):=1$. Under the assumptions in Section 4.1 (in particular* (34), (35)*,* (37)*,* (38) *and* (41)*), for the integral I_n defined in* (33) *the following asymptotic formula holds as $n\to+\infty$:*

$$I_n=(2\pi)^{N/2}e^{i(\vartheta_1+\cdots+\vartheta_N)}g\left(z_1^{(0)},\dots,z_N^{(0)}\right)\frac{\left|f\left(z_1^{(0)},\dots,z_N^{(0)}\right)\right|^{N/2}}{\sqrt{\left|H\left(z_1^{(0)},\dots,z_N^{(0)}\right)\right|}}\times\frac{f\left(z_1^{(0)},\dots,z_N^{(0)}\right)^n}{n^{N/2}}(1+o(1)), \tag{42}$$

where, for $j = 1, \ldots, N$,

$$\vartheta_j = h_j\pi - \frac{1}{2}\arg\left(-\frac{1}{f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)}\cdot\frac{H_j\left(z_1^{(0)},\ldots,z_N^{(0)}\right)}{H_{j-1}\left(z_1^{(0)},\ldots,z_N^{(0)}\right)}\right) \tag{43}$$

with $h_j \in \mathbb{Z}$.

REMARK 4.3. — In formula (43) one can choose any value of the argument (e.g., the principal argument). Accordingly, as in (18)–(19), the parity of the integer h_j must be taken so that the orientation of the line of steepest descent for $|f|$ at $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$ with respect to the variable z_j agrees with the orientation of the path λ_j in the integral I_n.

REMARK 4.4. — For $N = 1$, the asymptotic formula (42) becomes (23). For $N = 2$, (42) was proved by Hata [9]. In the proof of Theorem 4.2, for the $o(1)$ in (42) we shall obtain $O\left((\log n)^{\frac{3}{2}+\varepsilon}/\sqrt{n}\right)$. However this form of the error term is immaterial, since by Fedoryuk's theorem ([7], formula (1.61), p. 125) the term $o(1)$ in (42) can be expanded into an asymptotic series of the form $\sum_{k=1}^{\infty} c_k n^{-k}$, and therefore is $O(1/n)$.

REMARK 4.5. — If $f(z_1,\ldots,z_N)$ is written in the exponential form

$$f(z_1,\ldots,z_N) = \exp h(z_1,\ldots,z_N)$$

with a given function $h(z_1,\ldots,z_N)$ holomorphic in Δ, according to our discussion in Section 1 regarding (13)–(14), the proof of Theorem 4.2 can be modified in an obvious way to yield the asymptotic formula (42) with a real parameter $\tau \to +\infty$ in place of the integer $n \to +\infty$.

We postpone the proof of Theorem 4.2 to Section 5. In this section we prove some lemmas.

First of all, we parametrize the whole integration paths λ_N, $\lambda_{N-1}(z_N),\ldots,$ $\lambda_1(z_2,\ldots,z_N)$ respectively by parameters $r_N, r_{N-1},\ldots,r_1$ varying from -1 to 1, so that $z_N^{(0)}$ corresponds to $r_N = 0$, and similarly $Z_{jj}(z_{j+1},\ldots,z_N)$ corresponds to $r_j = 0$ for $j = 1,\ldots,N-1$.

By notation abuse, for $z_N \in \lambda_N, r_N \mapsto z_N$, we write

$$\begin{cases} z_N = \lambda_N(r_N), & -1 \le r_N \le 1 \\ z_N^{(0)} = \lambda_N(0), \end{cases} \tag{44}$$

and subsequently, for $j = N-1, N-2, \ldots, 1$,

$$\begin{cases} z_j = \lambda_j(r_j; r_{j+1},\ldots,r_N) := (\lambda_j(z_{j+1},\ldots,z_N))(r_j), & -1 \le r_j \le 1 \\ Z_{jj}(z_{j+1},\ldots,z_N) = \lambda_j(0; r_{j+1},\ldots,r_N). \end{cases} \tag{45}$$

Thus, defining

$$F(r_1,\dots,r_N) := f(\lambda_1(r_1;r_2,\dots,r_N),\dots,\lambda_{N-1}(r_{N-1};r_N),\lambda_N(r_N)),\\ G(r_1,\dots,r_N) := g(\lambda_1(r_1;r_2,\dots,r_N),\dots,\lambda_{N-1}(r_{N-1};r_N),\lambda_N(r_N)),\tag{46}$$

for the integral (33) we get

$$I_n = \int_{-1}^{1}\cdots\int_{-1}^{1} F(r_1,\dots,r_N)^n G(r_1,\dots,r_N) \\ \times \frac{\partial\lambda_1}{\partial r_1}\cdots\frac{\partial\lambda_{N-1}}{\partial r_{N-1}}\frac{\mathrm{d}\lambda_N}{\mathrm{d}r_N}\mathrm{d}r_1\cdots\mathrm{d}r_N.\tag{47}$$

By Remark 4.1, there exists $\widehat{r}$ with $0<\widehat{r}<1$ such that, for

$$r_1,\dots,r_N\in[-\widehat{r},\widehat{r}],\tag{48}$$

the paths $\lambda_N(r_N)$, …, $\lambda_1(r_1;r_2,\dots,r_N)$ are the lines of steepest descent for

$$|f(Z_{1,N-1}(z_N),\dots,Z_{N-1,N-1}(z_N),z_N)| \ \text{ at } \ z_N^{(0)},\\ \cdots\cdots\cdots\\ |f(z_1,\dots,z_N)| \ \text{ at } \ Z_{11}(z_2,\dots,z_N)$$

respectively, whence, by (18),

$$\begin{cases}\lambda_N(r_N)=z_N^{(0)}+r_N\,e^{i\vartheta_N},\\ \lambda_{N-1}(r_{N-1};r_N)=Z_{N-1,N-1}(\lambda_N(r_N))+r_{N-1}e^{i\vartheta_{N-1}(r_N)},\\ \cdots\cdots\cdots\\ \lambda_1(r_1;r_2,\dots,r_N)=Z_{11}(\lambda_2(r_2;r_3,\dots,r_N),\dots,\lambda_N(r_N))\\ \qquad\qquad +r_1e^{i\vartheta_1(r_2,\dots,r_N)}.\end{cases}\tag{49}$$

By (19), (30) and Lemma 3.6,

$$\vartheta_N = h_N\pi-\frac{1}{2}\arg\left(-\frac{1}{f}\,\frac{H_N}{H_{N-1}}\right)\bigg|_{(z_1^{(0)},\dots,z_N^{(0)})},\quad h_N\in\mathbb{Z},\tag{50}$$

as in (43) for $j=N$, and, for $j=1,\dots,N-1$,

$$\vartheta_j(r_{j+1},\dots,r_N)=\\ h_j\pi-\frac{1}{2}\arg\left(-\frac{1}{f}\,\frac{H_j}{H_{j-1}}\right)\bigg|_{(Z_{1,j-1}(\lambda_j,\dots,\lambda_N),\dots,Z_{j-1,j-1}(\lambda_j,\dots,\lambda_N),\lambda_j,\dots,\lambda_N)},\\ h_j\in\mathbb{Z}.\tag{51}$$

From (30) and (49) we obtain

$$\begin{cases} \lambda_N(0) = z_N^{(0)}, \\ \lambda_{N-1}(0;0) = Z_{N-1,N-1}\left(z_N^{(0)}\right) = z_{N-1}^{(0)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_1(0;0,\ldots,0) = Z_{11}\left(z_2^{(0)},\ldots,z_N^{(0)}\right) = z_1^{(0)}. \end{cases} \tag{52}$$

Hence, by (51), (52) and (30),

$$\vartheta_j(0,\ldots,0) = \vartheta_j$$

defined in (43).

Lemma 4.6. — *For each $k, j = 1, \ldots, N$,*

$$\left.\frac{\partial \lambda_k}{\partial r_j}\right|_{r_k=\cdots=r_N=0} = \begin{cases} 0, & \text{if } 1 \le j < k \le N \\ e^{i\vartheta_j}, & \text{if } j = k \\ e^{i\vartheta_j}\frac{\partial Z_{k,j-1}}{\partial z_j}\left(z_j^{(0)},\ldots,z_N^{(0)}\right), & \text{if } 1 \le k < j \le N. \end{cases} \tag{53}$$

Proof. — For $j \le k$, (53) is an immediate consequence of (49). If $k = j-1$, from (49) and (52) we get

$$\begin{aligned} &\left.\frac{\partial \lambda_{j-1}}{\partial r_j}\right|_{r_{j-1}=\cdots=r_N=0} \\ &= \left.\frac{\partial}{\partial r_j} Z_{j-1,j-1}(\lambda_j(r_j; r_{j+1},\ldots,r_N),\ldots,\lambda_N(r_N))\right|_{r_j=\cdots=r_N=0} \\ &= \left.\left(\frac{\partial Z_{j-1,j-1}}{\partial z_j}\frac{\partial \lambda_j}{\partial r_j}\right)\right|_{r_j=\cdots=r_N=0} = e^{i\vartheta_j}\,\frac{\partial Z_{j-1,j-1}}{\partial z_j}(\lambda_j(0;0,\ldots,0),\ldots,\lambda_N(0)) \\ &= e^{i\vartheta_j}\,\frac{\partial Z_{j-1,j-1}}{\partial z_j}\left(z_j^{(0)},\ldots,z_N^{(0)}\right), \end{aligned}$$

i.e., (53) for $k = j-1$.

For any $k < j$ we use descending induction. We assume (53) for $k = j-1, j-2, \ldots, l+1$ with $1 \le l \le j-2$, and prove (53) for $k = l$. By (49) we have

$$\begin{aligned} \left.\frac{\partial \lambda_l}{\partial r_j}\right|_{r_l=\cdots=r_N=0} &= \left.\frac{\partial}{\partial r_j} Z_{ll}(\lambda_{l+1}(r_{l+1}; r_{l+2},\ldots,r_N),\ldots,\lambda_N(r_N))\right|_{r_{l+1}=\cdots=r_N=0} \\ &= \left.\left(\frac{\partial Z_{ll}}{\partial z_{l+1}}\frac{\partial \lambda_{l+1}}{\partial r_j} + \cdots + \frac{\partial Z_{ll}}{\partial z_j}\frac{\partial \lambda_j}{\partial r_j}\right)\right|_{r_{l+1}=\cdots=r_N=0}, \end{aligned}$$

whence, by the inductive assumption and by (52),

$$\frac{\partial\lambda_l}{\partial r_j}\bigg|_{r_l=\cdots=r_N=0} = e^{i\vartheta_j}\left(\frac{\partial Z_{ll}}{\partial z_{l+1}}\frac{\partial Z_{l+1,j-1}}{\partial z_j}+\cdots+\frac{\partial Z_{ll}}{\partial z_{j-1}}\frac{\partial Z_{j-1,j-1}}{\partial z_j}+\frac{\partial Z_{ll}}{\partial z_j}\right)\bigg|_{\left(z_{l+1}^{(0)},\ldots,z_N^{(0)}\right)}.$$

From the last identity (31), with k, j replaced by l, $j-1$ respectively, we get

$$\begin{aligned}\frac{\partial}{\partial z_j}Z_{l,j-1}(z_j,\ldots,z_N) &= \frac{\partial}{\partial z_j}Z_{ll}(Z_{l+1,j-1}(z_j,\ldots,z_N),\ldots,Z_{j-1,j-1}(z_j,\ldots,z_N),z_j,\ldots,z_N)\\ &= \frac{\partial Z_{ll}}{\partial z_{l+1}}\,\frac{\partial Z_{l+1,j-1}}{\partial z_j}+\cdots+\frac{\partial Z_{ll}}{\partial z_{j-1}}\,\frac{\partial Z_{j-1,j-1}}{\partial z_j}+\frac{\partial Z_{ll}}{\partial z_j}\,.\end{aligned}$$

Therefore

$$\frac{\partial\lambda_l}{\partial r_j}\bigg|_{r_l=\cdots=r_N=0} = e^{i\vartheta_j}\,\frac{\partial Z_{l,j-1}}{\partial z_j}\left(z_j^{(0)},\ldots,z_N^{(0)}\right). \qquad \square$$

LEMMA 4.7. — *Let*

$$\mathcal{S}_N=\begin{pmatrix}\alpha_{11} & \ldots & \alpha_{1N}\\ \vdots & \ddots & \vdots\\ \alpha_{N1} & \ldots & \alpha_{NN}\end{pmatrix}$$

be a symmetric matrix ($\alpha_{hk}=\alpha_{kh}$) *such that, for each* $j=1,\ldots,N$, *the submatrix*

$$\mathcal{S}_j=\begin{pmatrix}\alpha_{11} & \ldots & \alpha_{1j}\\ \vdots & \ddots & \vdots\\ \alpha_{j1} & \ldots & \alpha_{jj}\end{pmatrix}$$

is nonsingular, i.e., $\det\mathcal{S}_j\neq 0$. *Let*

$$\mathcal{T}_N=\begin{pmatrix}\beta_{11} & \ldots & \beta_{1N}\\ \vdots & \ddots & \vdots\\ \beta_{N1} & \ldots & \beta_{NN}\end{pmatrix}$$

be the upper triangular matrix defined by

$$\beta_{kj}=\begin{cases}0, & \textit{if } 1\le j<k\le N\\ 1, & \textit{if } j=k\\ (-1)^{k+j}\frac{d_{jk}}{\det\mathcal{S}_{j-1}}, & \textit{if } 1\le k<j\le N,\end{cases} \tag{54}$$

where d_{jk} is the determinant of the matrix obtained by removing the j-th row and the k-th column in $\mathcal{S}_j$. Then ${}^t\mathcal{T}_N\mathcal{S}_N\mathcal{T}_N$ is the diagonal matrix given by

$$
{}^t\mathcal{T}_N\,\mathcal{S}_N\,\mathcal{T}_N = \begin{pmatrix} \det\mathcal{S}_1 & 0 & \dots & 0 \\ 0 & \frac{\det\mathcal{S}_2}{\det\mathcal{S}_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\det\mathcal{S}_N}{\det\mathcal{S}_{N-1}} \end{pmatrix}.
$$

Proof. — Let

$$
{}^t\mathcal{T}_N\,\mathcal{S}_N = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{pmatrix},
$$

let $\det\mathcal{S}_0 := 1$, and $d_{jj} := \det\mathcal{S}_{j-1}$ for $j = 1, \dots, N$. Since $\alpha_{hk} = \alpha_{kh}$, from (54) we get

$$
\gamma_{jk} = \sum_{h=1}^{N} \beta_{hj}\alpha_{kh} = \frac{1}{\det\mathcal{S}_{j-1}} \sum_{h=1}^{j} (-1)^{h+j} d_{jh}\alpha_{kh}.
$$

If $k \leq j$, the last sum is the Laplace expansion along the last row of the determinant of the matrix obtained from $\mathcal{S}_j$ by replacing the j-th row with the k-th row. Therefore

$$
\gamma_{jj} = \frac{\det\mathcal{S}_j}{\det\mathcal{S}_{j-1}}\,, \tag{55}
$$

and

$$
\gamma_{jk} = 0 \quad \text{for } k < j, \tag{56}
$$

whence ${}^t\mathcal{T}_N\,\mathcal{S}_N$ is an upper triangular matrix. Thus $({}^t\mathcal{T}_N\mathcal{S}_N)\mathcal{T}_N$ is the product of two upper triangular matrices, and hence is upper triangular. Moreover

$$
{}^t({}^t\mathcal{T}_N\,\mathcal{S}_N\mathcal{T}_N) = {}^t\mathcal{T}_N{}^t\mathcal{S}_N\mathcal{T}_N = {}^t\mathcal{T}_N\mathcal{S}_N\mathcal{T}_N,
$$

because $\mathcal{S}_N$ is symmetric. Thus ${}^t\mathcal{T}_N\,\mathcal{S}_N\,\mathcal{T}_N$ is upper triangular and symmetric, and hence is diagonal.

The j-th entry on the diagonal of $({}^t\mathcal{T}_N\,\mathcal{S}_N)\mathcal{T}_N$ is

$$
\gamma_{j1}\beta_{1j} + \cdots + \gamma_{jj}\beta_{jj} + \cdots + \gamma_{jN}\beta_{Nj}, \tag{57}
$$

with $\beta_{j+1,j} = \cdots = \beta_{Nj} = 0$ by (54), and $\gamma_{j1} = \cdots = \gamma_{j,j-1} = 0$ by (56). Also, by (54), $\beta_{jj} = 1$ for $j = 1, \dots, N$. Thus (57) equals

$$
\gamma_{jj} = \frac{\det\mathcal{S}_j}{\det\mathcal{S}_{j-1}}
$$

by (55). □

LEMMA 4.8. — *For $r > 0$, let $E_N(r)$ and $R_N(r)$ be the functions defined by*

$$E_N(r) = \int\limits_{x_1^2+\cdots+x_N^2 \le r^2} \cdots \int e^{-(x_1^2+\cdots+x_N^2)} \mathrm{d}x_1 \cdots \mathrm{d}x_N = \pi^{N/2}\left(1 - R_N(r)\right). \tag{58}$$

Then, as $r \to \infty$,

$$R_N(r) = O\left(\frac{e^{-r^2/N}}{r}\right). \tag{59}$$

Proof. — We consider the error function

$$\operatorname{erf} t := \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} \mathrm{d}x \qquad (t > 0)$$

and the complementary error function

$$\operatorname{erfc} t := 1 - \operatorname{erf} t = \frac{2}{\sqrt{\pi}} \int_t^{+\infty} e^{-x^2} \mathrm{d}x.$$

Using L'Hôpital's rule we get

$$\lim_{t \to +\infty} \frac{\int_t^{+\infty} e^{-x^2} \mathrm{d}x}{\left(t\, e^{t^2}\right)^{-1}} = \frac{1}{2},$$

whence

$$\operatorname{erfc} t \sim \frac{e^{-t^2}}{\sqrt{\pi} t} \qquad (t \to +\infty). \tag{60}$$

We remark that (60) is the first term in the well known asymptotic expansion

$$\operatorname{erfc} t = \frac{e^{-t^2}}{\sqrt{\pi} t}\left(1 + \sum_{l=1}^{L-1} (-1)^l \frac{(2l-1)!!}{(2t^2)^l} + O\left(\frac{1}{t^{2L}}\right)\right) \qquad (t \to +\infty)$$

for any integer $L \ge 1$, although (60) suffices for our purposes.

The hypercube defined by the inequalities

$$-\frac{r}{\sqrt{N}} \le x_j \le \frac{r}{\sqrt{N}} \quad (j = 1, \ldots, N)$$

is plainly contained in the sphere $x_1^2 + \cdots + x_N^2 \le r^2$. Therefore

$$E_N(r) > \int_{-r/\sqrt{N}}^{r/\sqrt{N}} \cdots \int_{-r/\sqrt{N}}^{r/\sqrt{N}} e^{-(x_1^2+\cdots+x_N^2)}\,\mathrm{d}x_1\cdots\mathrm{d}x_N \tag{61}$$

$$= \left(\int_{-r/\sqrt{N}}^{r/\sqrt{N}} e^{-x^2}\,\mathrm{d}x\right)^N = \left(\sqrt{\pi}\,\mathrm{erf}\,\frac{r}{\sqrt{N}}\right)^N = \pi^{N/2}\left(1-\mathrm{erfc}\,\frac{r}{\sqrt{N}}\right)^N.$$

Since $(1-X)^N \ge 1 - NX$ for $0 < X < 1$, from (58) and (61) we get

$$\pi^{N/2}\left(1 - R_N(r)\right) = E_N(r) > \pi^{N/2}\left(1 - N\mathrm{erfc}\,\frac{r}{\sqrt{N}}\right),$$

whence

$$R_N(r) < N\mathrm{erfc}\,\frac{r}{\sqrt{N}} = O\left(\frac{e^{-r^2/N}}{r}\right) \quad (r \to +\infty)$$

by (60). □

Remark 4.9. — A standard but tedious calculation yields the exact value of $R_N(r)$, namely

$$R_N(r) = \begin{cases} e^{-r^2}\sum\limits_{l=0}^{m-1}\frac{r^{2l}}{l!}, & \text{for } N = 2m \\ \mathrm{erfc}\; r + \frac{e^{-r^2}}{\sqrt{\pi}\,r}\sum\limits_{l=1}^{m}\frac{(2r^2)^l}{(2l-1)!!}, & \text{for } N = 2m+1. \end{cases} \tag{62}$$

Thus, by (60) and (62), the asymptotic formula (59) can be improved to the exact order of magnitude:

$$R_N(r) = O(e^{-r^2} r^{N-2}) \quad (r \to \infty), \tag{63}$$

but we need not use (63) for the proof of Theorem 4.2.

5. Proof of Theorem 4.2

By (46),

$$\frac{\partial F}{\partial r_j} = \sum_{l=1}^{j} \frac{\partial f}{\partial z_l}\frac{\partial \lambda_l}{\partial r_j} \quad (j = 1, \ldots, N),$$

with

$$\left.\frac{\partial f}{\partial z_l}\right|_{r_1=\cdots=r_N=0} = \frac{\partial f}{\partial z_l}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) = 0 \qquad (l = 1, \ldots, N) \tag{64}$$

by (24), whence

$$\frac{\partial F}{\partial r_j}(0,\ldots,0)=0 \quad (j=1,\ldots,N).$$

Also, by (64),

$$\begin{aligned}\frac{\partial^2 F}{\partial r_k \partial r_j}(0,\ldots,0) &= \sum_{l=1}^{j}\left(\frac{\partial}{\partial r_k}\frac{\partial f}{\partial z_l}\right)\frac{\partial \lambda_l}{\partial r_j}\bigg|_{r_1=\cdots=r_N=0}\\ &= \sum_{h=1}^{k}\sum_{l=1}^{j}\frac{\partial^2 f}{\partial z_h \partial z_l}\frac{\partial \lambda_h}{\partial r_k}\frac{\partial \lambda_l}{\partial r_j}\bigg|_{r_1=\cdots=r_N=0}.\end{aligned}$$

Hence, with the notation (27),

$$(65)\quad \frac{\partial^2 F}{\partial r_k \partial r_j}(0,\ldots,0)= \left(\left(\frac{\partial \lambda_1}{\partial r_k},\ldots,\frac{\partial \lambda_k}{\partial r_k},0,\ldots,0\right)\mathcal{H}_0 \, {}^t\left(\frac{\partial \lambda_1}{\partial r_j},\ldots,\frac{\partial \lambda_j}{\partial r_j},0,\ldots,0\right)\right)\bigg|_{r_1=\cdots=r_N=0}.$$

Here and in what follows, we identify a 1×1 matrix with its entry.

Let

$$\mathcal{T}_0=\begin{pmatrix} D_{11} & \ldots & D_{1N}\\ \vdots & \ddots & \vdots\\ D_{N1} & \ldots & D_{NN}\end{pmatrix}$$

be the upper triangular matrix defined by

$$(66)\qquad D_{kj}=\begin{cases}0, & \text{if } 1\le j<k\le N\\ 1, & \text{if } j=k\\ (-1)^{k+j}\ \frac{\delta_{jk}}{H_{j-1}}\Big|_{(z_1^{(0)},\ldots,z_N^{(0)})}, & \text{if } 1\le k<j\le N,\end{cases}$$

where H_{j-1} is defined by (25), and, as in Lemma 3.5, δ_{jk} is the determinant obtained by removing the last row and the k-th column from the determinant H_j. Let $\mathcal{E}_0$ be the diagonal matrix

$$\mathcal{E}_0=\begin{pmatrix} e^{i\vartheta_1} & 0 & \ldots & 0\\ 0 & e^{i\vartheta_2} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & e^{i\vartheta_N}\end{pmatrix},$$

with ϑ_j defined in (43). By Lemmas 4.6 and 3.5, and by (66), the vectors on the left and on the right of $\mathcal{H}_0$ in (65) are, respectively, the transpose of the

k-th column and the j-th column in $\mathcal{T}_0\mathcal{E}_0$. Therefore, by (65),

$$\left.\begin{pmatrix} \frac{\partial^2 F}{\partial r_1^2} & \cdots & \frac{\partial^2 F}{\partial r_1 \partial r_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial r_N \partial r_1} & \cdots & \frac{\partial^2 F}{\partial r_N^2} \end{pmatrix}\right|_{r_1=\cdots=r_N=0} = {}^t(\mathcal{T}_0\mathcal{E}_0)\mathcal{H}_0\mathcal{T}_0\mathcal{E}_0. \tag{67}$$

By Lemma 4.7 with $\mathcal{S}_N = \mathcal{H}_0$ and $\mathcal{T}_N = \mathcal{T}_0$, the matrix ${}^t\mathcal{T}_0\,\mathcal{H}_0\,\mathcal{T}_0$ is diagonal, and its entries on the diagonal are

$$\frac{H_j\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}{H_{j-1}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)} \qquad (j = 1, \ldots, N).$$

Since the diagonal matrices commute and $\mathcal{E}_0$ is diagonal, we get

$${}^t(\mathcal{T}_0\mathcal{E}_0)\mathcal{H}_0\mathcal{T}_0\mathcal{E}_0 = \mathcal{E}_0({}^t\mathcal{T}_0\mathcal{H}_0\mathcal{T}_0)\mathcal{E}_0 = \mathcal{E}_0^2({}^t\mathcal{T}_0\mathcal{H}_0\mathcal{T}_0),$$

whence

$${}^t(\mathcal{T}_0\mathcal{E}_0)\mathcal{H}_0\mathcal{T}_0\mathcal{E}_0 = \left.\begin{pmatrix} e^{2i\vartheta_1}\frac{H_1}{H_0} & 0 & \cdots & 0 \\ 0 & e^{2i\vartheta_2}\frac{H_2}{H_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2i\vartheta_N}\frac{H_N}{H_{N-1}} \end{pmatrix}\right|_{\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}. \tag{68}$$

Thus, by (67) and (68), in a neighbourhood of $r_1 = \cdots = r_N = 0$ we obtain by Taylor's formula

$$\begin{aligned} F(r_1, \ldots, r_N) &= f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) \\ &+ \frac{1}{2!}(r_1, \ldots, r_N) \left.\begin{pmatrix} e^{2i\vartheta_1}H_1/H_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2i\vartheta_N}H_N/H_{N-1} \end{pmatrix}\right|_{\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)} {}^t(r_1, \ldots, r_N) \\ &+ O\left(|r_1|^3 + \cdots + |r_N|^3\right) \\ &= f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) + \frac{1}{2}\sum_{j=1}^N e^{2i\vartheta_j}\frac{H_j\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}{H_{j-1}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)} r_j^2 \\ &+ O\left(|r_1|^3 + \cdots + |r_N|^3\right). \end{aligned}$$

Therefore

$$F(r_1, \ldots, r_N) = f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)\left(1 - \sum_{j=1}^N A_j r_j^2 + O\left(|r_1|^3 + \cdots + |r_N|^3\right)\right) \tag{69}$$

where

$$A_j = -\frac{1}{2} e^{2i\vartheta_j} \frac{H_j\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}{f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) H_{j-1}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)} \quad (j = 1, \ldots, N).$$

By (43),

$$\arg(2A_j) = 2\vartheta_j + \arg\left(-\frac{H_j\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}{f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) H_{j-1}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}\right) = 2h_j\pi,$$

whence A_j is real and positive. It follows that

$$(70) \quad A_j = \frac{1}{2}\left|\frac{H_j\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}{f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) H_{j-1}\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)}\right| > 0 \quad (j = 1, \ldots, N).$$

Let μ_1 be a constant satisfying

$$0 < \mu_1 < \min\{A_1, \ldots, A_N\}.$$

By (69), there exists a constant h_0 with $0 < h_0 \leq \widehat{r}$, where $\widehat{r}$ is the constant in (48), such that

$$(71) \qquad |F(r_1, \ldots, r_N)| \leq \left|f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)\right| \left(1 - \mu_1(r_1^2 + \cdots + r_N^2)\right)$$

for all $(r_1, \ldots, r_N) \in [-h_0, h_0]^N$.

By the assumption (41) there exists a constant μ_2 with $0 < \mu_2 < 1$ such that

$$(72) \qquad |F(r_1, \ldots, r_N)| \leq \mu_2 \left|f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)\right|$$

for all $(r_1, \ldots, r_N) \in [-1, 1]^N \setminus [-h_0, h_0]^N$.

Let $K > 0$ be a constant to be chosen later, and let $n_1 > e^{2K}$ be an integer such that

$$(73) \qquad \frac{(\log n_1)^K}{\sqrt{n_1}} \leq \min\left\{h_0, \sqrt{\frac{1-\mu_2}{\mu_1}}\right\}.$$

Let $\Omega_{n_1} \supset \Omega_{n_1+1} \supset \Omega_{n_1+2} \supset \ldots$ be the sequence of spheres defined by

$$\Omega_n := \left\{(r_1, \ldots, r_N) \;\middle|\; r_1^2 + \cdots + r_N^2 \leq \frac{(\log n)^{2K}}{n}\right\} \qquad (n \geq n_1).$$

For $(r_1, \ldots, r_N) \in [-h_0, h_0]^N \setminus \Omega_n$ we get by (71)

$$(74) \qquad |F(r_1, \ldots, r_N)| \leq \left(1 - \mu_1 \frac{(\log n)^{2K}}{n}\right) \left|f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right)\right|.$$

By (73),

$$\mu_2 \leq 1 - \mu_1 \frac{(\log n_1)^{2K}}{n_1} \leq 1 - \mu_1 \frac{(\log n)^{2K}}{n} \quad (n \geq n_1).$$

Therefore, by (72), the inequality (74) holds for all $(r_1, \ldots, r_N) \in [-1,1]^N \setminus \Omega_n (n \geq n_1)$. Using (74), we show that the contribution given by $[-1,1]^N \setminus \Omega_n$ to the integral I_n in (47) is negligible. For this purpose we use the asymptotic formulae

$$\left(1 - \mu_1 \frac{(\log n)^{2K}}{n}\right)^n = O\left(e^{-\mu_1 (\log n)^{2K}}\right) \quad (n \to +\infty), \tag{75}$$

and, for $(r_1, \ldots, r_N) \in \Omega_n,\ n \to +\infty$,

$$\begin{aligned} G(r_1, \ldots, r_N) &\frac{\partial \lambda_1}{\partial r_1} \cdots \frac{\mathrm{d}\lambda_N}{\mathrm{d}r_N} \\ &= e^{i(\vartheta_1 + \cdots + \vartheta_N)} g\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) (1 + O(|r_1| + \cdots + |r_N|)) \\ &= e^{i(\vartheta_1 + \cdots + \vartheta_N)} g\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) \left(1 + O\left(\frac{(\log n)^K}{\sqrt{n}}\right)\right) \end{aligned} \tag{76}$$

and

$$\begin{aligned} &\left(1 + \frac{O\left(|r_1|^3 + \cdots + |r_N|^3\right)}{1 - \sum_{j=1}^N A_j r_j^2}\right)^n \\ &= 1 + O\left(n \left(\frac{(\log n)^K}{\sqrt{n}}\right)^3\right) = 1 + O\left(\frac{(\log n)^{3K}}{\sqrt{n}}\right). \end{aligned} \tag{77}$$

For (76) we have used (53).

Let

$$J_n := \int_{[-1,1]^N \setminus \Omega_n} \!\!\cdots \int F(r_1, \ldots, r_N)^n G(r_1, \ldots, r_N) \frac{\partial \lambda_1}{\partial r_1} \cdots \frac{\mathrm{d}\lambda_N}{\mathrm{d}r_N} \, \mathrm{d}r_1 \cdots \mathrm{d}r_N.$$

By the absolute convergence of I_n for every sufficiently large n (say, for $n \geq n_0$), and by (74) and (75), we get

$$\begin{aligned} |J_n| \leq &\left(1 - \mu_1 \frac{(\log n)^{2K}}{n}\right)^{n-n_0} \left| f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) \right|^{n-n_0} \\ &\times \int_{\lambda_N} |\mathrm{d}z_N| \cdots \int_{\lambda_1(z_2, \ldots, z_N)} |f(z_1, \ldots, z_N)|^{n_0} |g(z_1, \ldots, z_N)| \, |\mathrm{d}z_1| \\ = \ &O\left(e^{-\mu_1 (\log n)^{2K}} \left| f\left(z_1^{(0)}, \ldots, z_N^{(0)}\right) \right|^n \right). \end{aligned}$$

Hence, by (69), (76) and (77),

(78)

$$\begin{aligned} I_n &= \int_{\Omega_n}\!\!\cdots\!\int F(r_1,\ldots,r_N)^n\, G(r_1,\ldots,r_N)\frac{\partial\lambda_1}{\partial r_1}\cdots\frac{\mathrm{d}\lambda_N}{\mathrm{d}r_N}\mathrm{d}r_1\cdots\mathrm{d}r_N \;+\; J_n \\ &= e^{i(\vartheta_1+\cdots+\vartheta_N)} g\left(z_1^{(0)},\ldots,z_N^{(0)}\right) f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)^n \left(1+O\left(\frac{(\log n)^{3K}}{\sqrt{n}}\right)\right) \\ &\quad\times \int_{\Omega_n}\!\!\cdots\!\int \left(1-\sum_{j=1}^N A_j r_j^2\right)^n \mathrm{d}r_1\cdots\mathrm{d}r_N \\ &\quad + O\left(e^{-\mu_1(\log n)^{2K}}\left|f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|^n\right). \end{aligned}$$

We shall now prove an asymptotic formula ((84) below) for the integral

$$\int_{\Omega_n}\!\!\cdots\!\int \left(1-\sum_{j=1}^N A_j r_j^2\right)^n \mathrm{d}r_1\cdots\mathrm{d}r_N. \tag{79}$$

Substituting $\sqrt{A_j n}\, r_j = u_j (j=1,\ldots,N)$, (79) becomes

$$\frac{1}{\sqrt{A_1\ldots A_N}\; n^{N/2}} \int_{\Omega_n^*}\!\!\cdots\!\int \left(1-\frac{1}{n}\sum_{j=1}^N u_j^2\right)^n \mathrm{d}u_1\cdots\mathrm{d}u_N \tag{80}$$

with

$$\Omega_n^* = \left\{(u_1,\ldots,u_N)\;\middle|\; \frac{u_1^2}{A_1}+\cdots+\frac{u_N^2}{A_N} \le (\log n)^{2K}\right\}. \tag{81}$$

If $(u_1,\ldots,u_N)\in\Omega_n^*$ then

$$\sum_{j=1}^N u_j^2 \;\le\; A(\log n)^{2K}$$

where $A=\max\{A_1,\ldots,A_N\}$. Hence

$$\begin{aligned} \left(1-\frac{1}{n}\sum_{j=1}^N u_j^2\right)^n &= \exp\left(-n\left(\frac{1}{n}\sum_{j=1}^N u_j^2 + O\left(\frac{1}{n^2}\left(\sum_{j=1}^N u_j^2\right)^2\right)\right)\right) \\ &= \exp\left(-\sum_{j=1}^N u_j^2\right)\left(1+O\left(\frac{(\log n)^{4K}}{n}\right)\right). \end{aligned}$$

Therefore

$$\int_{\Omega_n^*}\!\!\cdots\!\int \left(1-\frac{1}{n}\sum_{j=1}^{N} u_j^2\right)^n \mathrm{d}u_1\cdots\mathrm{d}u_N = \left(1+O\left(\frac{(\log n)^{4K}}{n}\right)\right)\int_{\Omega_n^*}\!\!\cdots\!\int \exp\left(-\sum_{j=1}^{N} u_j^2\right) \mathrm{d}u_1\cdots\mathrm{d}u_N. \tag{82}$$

Let $\mu_0=\min\{A_1,\ldots,A_N\}$. From (81) and Lemma 4.8 we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus\Omega_n^*}\!\!\cdots\!\int \exp\left(-\sum_{j=1}^{N} u_j^2\right) \mathrm{d}u_1\cdots\mathrm{d}u_N\\ &< \int_{\sum_{j=1}^N u_j^2>\mu_0(\log n)^{2K}}\!\!\cdots\!\int \exp\left(-\sum_{j=1}^{N} u_j^2\right) \mathrm{d}u_1\cdots\mathrm{d}u_N = \pi^{N/2} R_N\left(\sqrt{\mu_0}\,(\log n)^K\right)\\ &= O\left((\log n)^{-K}e^{-(\mu_0/N)(\log n)^{2K}}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\Omega_n^*}\!\!\cdots\!\int \exp\left(-\sum_{j=1}^{N} u_j^2\right) \mathrm{d}u_1\cdots\mathrm{d}u_N\\ &= \pi^{N/2}-\int_{\mathbb{R}^N\setminus\Omega_n^*}\!\!\cdots\!\int \exp\left(-\sum_{j=1}^{N} u_j^2\right) \mathrm{d}u_1\cdots\mathrm{d}u_N\\ &= \pi^{N/2}\left(1+O\left((\log n)^{-K}e^{-(\mu_0/N)(\log n)^{2K}}\right)\right).\end{aligned} \tag{83}$$

From (80), (82) and (83) we get, for $n\to+\infty$,

$$\begin{aligned}&\int_{\Omega_n}\!\!\cdots\!\int \left(1-\sum_{j=1}^{N} A_j r_j^2\right)^n \mathrm{d}r_1\cdots\mathrm{d}r_N = \frac{\pi^{N/2}}{\sqrt{A_1\cdots A_N}\,n^{N/2}}\\ &\times\left(1+O\left(\frac{(\log n)^{4K}}{n}\right)\right)\left(1+O\left((\log n)^{-K}e^{-(\mu_0/N)(\log n)^{2K}}\right)\right).\end{aligned} \tag{84}$$

Since $(\log n)^{4K}/n = o\left((\log n)^{3K}/\sqrt{n}\right)$, by (78) and (84) we obtain

$$\begin{aligned} I_n = {} & e^{i(\vartheta_1+\cdots+\vartheta_N)} g\left(z_1^{(0)},\ldots,z_N^{(0)}\right) \frac{\pi^{N/2}}{\sqrt{A_1\cdots A_N}} \frac{f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)^n}{n^{N/2}} \\ & + O\left(\frac{(\log n)^{3K}}{n^{(N+1)/2}} \left| f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|^n\right) \\ & + O\left(n^{-N/2}(\log n)^{-K} e^{-(\mu_0/N)(\log n)^{2K}} \left| f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|^n\right) \\ & + O\left(e^{-\mu_1(\log n)^{2K}} \left| f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|^n\right). \end{aligned}$$

If $K > 1/2$, the last two error terms are negligible in comparison with

$$O\left(\frac{(\log n)^{3K}}{n^{(N+1)/2}} \left| f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|^n\right).$$

Choosing $K = \frac{1}{2} + \frac{\varepsilon}{3}$ with an arbitrarily small $\varepsilon > 0$ we conclude that

$$I_n = C\,\frac{f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)^n}{n^{N/2}} \left(1 + O\left(\frac{(\log n)^{\frac{3}{2}+\varepsilon}}{\sqrt{n}}\right)\right)$$

where, by (70),

$$\begin{aligned} C & = e^{i(\vartheta_1+\cdots+\vartheta_N)} g\left(z_1^{(0)},\ldots,z_N^{(0)}\right) \frac{\pi^{N/2}}{\sqrt{A_1\cdots A_N}} \\ & = (2\pi)^{N/2} e^{i(\vartheta_1+\cdots+\vartheta_N)} g\left(z_1^{(0)},\ldots,z_N^{(0)}\right) \frac{\left| f\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|^{N/2}}{\sqrt{\left| H\left(z_1^{(0)},\ldots,z_N^{(0)}\right)\right|}}. \qquad \square \end{aligned}$$

6. Generalized Airy functions

6.1. In [14] Ursell studies the asymptotic behaviour of certain double integrals depending on a large parameter. Special cases of the integrals considered by Ursell can be written in the following form, generalizing to the two-dimensional case the Airy integrals (7)–(9):

(85)

$$\left(\frac{\tau^{1/3}}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \exp\left(\tau(-\frac{1}{3}z_1^3 - \frac{1}{3}z_2^3 + az_1 + bz_2 + cz_1z_2)\right) \mathrm{d}z_1\,\mathrm{d}z_2 \quad (\tau \to +\infty),$$

where each of Γ_1 and Γ_2 is the union of two of the three half-lines (5). Combining Ursell's Theorems 1 and 2 ([14], pp. 254–255) with Lemma 2 ([14], p. 262) one obtains an asymptotic expansion for the integrals (85) under the restriction that $|a|$, $|b|$ and $|c|$ are small enough (this restriction implies that the four saddle-points in (85) are all close to $(0,0)$).

In the spirit of [14], we apply our Theorem 4.2 to study in some cases the asymptotic behaviour of N-dimensional integrals of Airy's type:

$$\left(\frac{\tau^{1/3}}{2\pi i}\right)^N \int_{\Gamma_1} \cdots \int_{\Gamma_N} \exp\left(\tau\, h(z_1,\ldots,z_N)\right) \mathrm{d}z_1 \cdots \mathrm{d}z_N \quad (\tau \to +\infty), \tag{86}$$

where, as in (85), each of $\Gamma_1,\ldots,\Gamma_N$ is the union of two of the three half-lines (5), and where

$$h(z_1,\ldots,z_N) = \sum_{j=1}^{N} z_j + \sum_{1\le k<l\le N} z_k z_l - \sum_{j=1}^{N} \tfrac{1}{3} z_j^3 \tag{87}$$

is a cubic polynomial with no condition about the vicinity of the saddle-points.

6.2. In this section we study the asymptotic behaviour of the integrals (86) for $N=2$, i.e.,

$$\left(\frac{\tau^{1/3}}{2\pi i}\right)^2 \int_{\Gamma_2} \mathrm{d}z_2 \int_{\Gamma_1} \exp\left(\tau\left(z_1+z_2+z_1z_2-\frac{1}{3}z_1^3-\frac{1}{3}z_2^3\right)\right) \mathrm{d}z_1, \tag{88}$$

where each of Γ_1 and Γ_2 is the union of two of the three half-lines γ_0, γ_1, γ_2 in (5).

Up to complex conjugation or to the interchange of z_1 and z_2, we have four distinct cases:

(i) $\Gamma_1 = \Gamma_2 = \gamma_0 \cup \gamma_1$,
(ii) $\Gamma_1 = \Gamma_2 = \gamma_1 \cup \gamma_2$,
(iii) $\Gamma_1 = \gamma_1 \cup \gamma_2,\ \Gamma_2 = \gamma_0 \cup \gamma_1$,
(iv) $\Gamma_1 = \gamma_0 \cup \gamma_2,\ \Gamma_2 = \gamma_0 \cup \gamma_1$.

In the above cases (i)–(iv) we denote the function (88) by ${}_2\mathrm{Ai}_1(\tau^{2/3})$, ${}_2\mathrm{Ai}_2(\tau^{2/3})$, ${}_2\mathrm{Ai}_3(\tau^{2/3})$, ${}_2\mathrm{Ai}_4(\tau^{2/3})$, respectively.

Let

$$h(z_1,z_2) = z_1 + z_2 + z_1z_2 - \frac{1}{3}z_1^3 - \frac{1}{3}z_2^3. \tag{89}$$

The saddle-points of $f(z_1,z_2) = \exp h(z_1,z_2)$ are the solutions of the system

$$\begin{cases} \frac{\partial h}{\partial z_1} = 1 + z_2 - z_1^2 = 0 \\ \frac{\partial h}{\partial z_2} = 1 + z_1 - z_2^2 = 0. \end{cases} \tag{90}$$

Eliminating z_2 we get

$$z_1(z_1+1)(z_1-\varphi^+)(z_1-\varphi^-) = 0$$

with

$$\varphi^\pm = \frac{1 \pm \sqrt{5}}{2}\,.$$

Thus the saddle-points are

$$(z_1, z_2) = (0, -1), (-1, 0), (\varphi^+, \varphi^+), (\varphi^-, \varphi^-).$$

Moreover

$$\frac{\partial^2 f}{\partial z_1^2} = f(z_1, z_2) \left(\frac{\partial^2 h}{\partial z_1^2} + \left(\frac{\partial h}{\partial z_1} \right)^2 \right),$$

$$\frac{\partial^2 f}{\partial z_1 \partial z_2} = f(z_1, z_2) \left(\frac{\partial^2 h}{\partial z_1 \partial z_2} + \frac{\partial h}{\partial z_1} \frac{\partial h}{\partial z_2} \right),$$

$$\frac{\partial^2 f}{\partial z_2^2} = f(z_1, z_2) \left(\frac{\partial^2 h}{\partial z_2^2} + \left(\frac{\partial h}{\partial z_2} \right)^2 \right).$$

Hence at each saddle-point we get

$$H_1(z_1, z_2) = \frac{\partial^2 f}{\partial z_1^2} = f(z_1, z_2) \frac{\partial^2 h}{\partial z_1^2} = f(z_1, z_2)(-2z_1),$$

$$H(z_1, z_2) = \frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - \left(\frac{\partial^2 f}{\partial z_1 \partial z_2} \right)^2 = f(z_1, z_2)^2 \left(\frac{\partial^2 h}{\partial z_1^2} \frac{\partial^2 h}{\partial z_2^2} - \left(\frac{\partial^2 h}{\partial z_1 \partial z_2} \right)^2 \right)$$

$$= f(z_1, z_2)^2 (4z_1 z_2 - 1).$$

Therefore the admissible saddle-points with respect to the ordering z_1, z_2 of the variables are

$$(z_1, z_2) = (-1, 0), (\varphi^+, \varphi^+), (\varphi^-, \varphi^-). \tag{91}$$

With notation as in (29), from the first equation (90) we get

$$Z_{11}(z_2) = \pm\sqrt{1 + z_2} \tag{92}$$

in the cut plane $z_2 \in \mathbb{C} \setminus (-\infty, -1]$, where $\sqrt{1 + z_2} > 0$ for $z_2 > -1$. In (92) we must take the $-$ sign for $z_2 = 0$ or $z_2 = \varphi^-$, and the $+$ sign for $z_2 = \varphi^+$.

We now apply Theorem 4.2 to obtain asymptotic formulae for (88) in the above cases (i)–(iv).

(i) $${}_2\mathrm{Ai}_1\,.$$

This is the special case, for $N = 2$, of the function ${}_N\mathrm{Ai}_1$ defined in (98) below. From the discussion in Section 6.3 we see that the relevant saddle-point for ${}_2\mathrm{Ai}_1(\tau^{2/3})$ is (φ^+, φ^+). Setting $N = 2$ in the asymptotic formula (108) below, we obtain

$${}_2\mathrm{Ai}_1(t) \sim -\frac{t^{-1/2}}{2\pi\sqrt{5 + 2\sqrt{5}}} \exp\left(\frac{7 + 5\sqrt{5}}{6} t^{3/2} \right) \tag{93}$$

for $t \to +\infty$.

(ii) $${}_2\mathrm{Ai}_2$$

In this case we show that, according to (91)–(92), the relevant saddle-point is $(z_1, z_2) = (\varphi^-, \varphi^-)$, with

$$z_1 = Z_{11}(z_2) = -\sqrt{1+z_2}. \tag{94}$$

This function is a one-to-one mapping of

$$\Delta_2 := \{z_2 \in \mathbb{C} \setminus (-\infty, -1]\}$$

onto

$$\Delta_1 := \left\{z_1 \in \mathbb{C} \;\middle|\; \operatorname{Re} z_1 < 0\right\},$$

with fixed point φ^-. In order to apply Theorem 4.2 we must change the integration path $\gamma_1 \cup \gamma_2$ for z_2 to an equivalent path $\lambda_2 \subset \Delta_2$ passing through φ^- and such that

$$\operatorname{Re} h(-\sqrt{1+z_2}, z_2) < \operatorname{Re} h(\varphi^-, \varphi^-) = h(\varphi^-, \varphi^-)$$

for all $z_2 \in \lambda_2,\ z_2 \neq \varphi^-$, where h is the polynomial (89).

It is convenient to seek the image $\widetilde{\lambda}_1 = Z_{11}(\lambda_2) \subset \Delta_1$ of λ_2 through (94), so that

$$\operatorname{Re} h(z_1, z_1^2 - 1) \ <\ h(\varphi^-, \varphi^-)$$

for all $z_1 \in \widetilde{\lambda}_1,\ z_1 \neq \varphi^-$, since $z_2 = z_1^2 - 1$ is the inverse of (94). From (89) we get

$$h\left(z_1, z_1^2 - 1\right) = -\frac{1}{3} z_1^6 + z_1^4 + \frac{2}{3} z_1^3 - \frac{2}{3}\,. \tag{95}$$

We choose $\widetilde{\lambda}_1$ to be the path of steepest descent for $|\exp h(z_1, z_1^2 - 1)|$ containing $z_1 = \varphi^-$. Arguing as in Section 2 we see that $\widetilde{\lambda}_1$ is defined by

$$\arg \exp h\left(z_1, z_1^2 - 1\right) = \operatorname{Im} h\left(z_1, z_1^2 - 1\right) = \operatorname{Im} h(\varphi^-, \varphi^-) = 0,$$

i.e., by (95),

$$\operatorname{Im}(z_1^6 - 3z_1^4 - 2z_1^3) = 0.$$

Writing $z_1 = x_1 + iy_1$ we easily get the equation

$$y_1\left(3x_1y_1^4 + \left(1 + 6x_1 - 10x_1^3\right) y_1^2 - 3x_1^2 - 6x_1^3 + 3x_1^5\right) = 0.$$

Hence $\widetilde{\lambda}_1$ is the connected component in Δ_1 of the quintic in $\mathbb{R}^2$

$$3x_1y_1^4 + \left(1 + 6x_1 - 10x_1^3\right) y_1^2 - 3x_1^2 - 6x_1^3 + 3x_1^5 = 0$$

containing the point $x_1 = \varphi^-$, $y_1 = 0$ and having asymptotes $y_1 = \pm\sqrt{3}\,x_1$, i.e., γ_1 and γ_2. Thus, writing $z_2 = x_2 + iy_2$, we see that the path λ_2 contains the point $x_2 = \varphi^-$, $y_2 = 0$ and has asymptotes $y_2 = \pm\sqrt{3}\,(x_2 + 1)$. Hence λ_2 is equivalent to $\gamma_1 \cup \gamma_2$ by Cauchy's theorem.

For any fixed $z_2 \in \lambda_2$ we must change the integration path $\gamma_1 \cup \gamma_2$ for z_1 to an equivalent path $\lambda_1(z_2)$ through $Z_{11}(z_2) = -\sqrt{1+z_2}$ so that

$$\operatorname{Re} h(z_1, z_2) < \operatorname{Re} h(-\sqrt{1+z_2},\, z_2) \tag{96}$$

for all $z_1 \in \lambda_1(z_2),\ z_1 \neq -\sqrt{1+z_2}$.

If $z_2 = \varphi^-$ we choose $\lambda_1(\varphi^-) = \widetilde{\lambda}_1$. If $z_2 \neq \varphi^-$, let $V_1(z_2)$ and $V_2(z_2)$ be the 'valley-sets' of $\operatorname{Re} h(z_1, z_2)$, i.e., the two connected components of the open set $V(z_2)$, with vertex at the saddle-point $z_1 = -\sqrt{1+z_2}$, such that (96) holds for all $z_1 \in V(z_2)$.

It is easily seen that $\operatorname{Re} h(z_1, z_2)$ is strictly monotonic for

$$z_1 = iy_1, \quad -\infty < y_1 < +\infty.$$

Therefore $W_1(z_2) := V_1(z_2) \cap \Delta_1$ and $W_2(z_2) := V_2(z_2) \cap \Delta_1$ are both unbounded. Hence we may choose $\lambda_1(z_2) \subset W_1(z_2) \cup W_2(z_2) \cup \{-\sqrt{1+z_2}\,\}$, with asymptotes γ_1 and γ_2.

We recall that $\tau^{2/3} = t$. We apply Theorem 4.2 with $\tau \in \mathbb{R}$ in place of n (see Remark 4.5). From (43) we get $\vartheta_1 = \vartheta_2 = \pi/2$. Then the asymptotic formula (42) yields

$${}_2\mathrm{Ai}_2(t) \sim \frac{t^{-1/2}}{2\pi\sqrt{5-2\sqrt{5}}}\, \exp\left(\frac{7-5\sqrt{5}}{6}\, t^{3/2}\right).$$

(iii) ${}_2\mathrm{Ai}_3$

In this case the relevant saddle-point is $(z_1, z_2) = (-1, 0)$. Let λ_2 be the curve defined in the previous case (ii), let λ_2' be the part of λ_2 lying in the halfplane $\operatorname{Im} z_2 \geq 0$, and let λ_2'' be the half-line

$$\lambda_2'' = \left\{ z_2 \in \mathbb{R} \;\middle|\; \varphi^- \leq z_2 < +\infty \right\}.$$

We replace the integration path $\gamma_0 \cup \gamma_1$ for z_2 with

$$\mu_2 := \lambda_2' \cup \lambda_2''.$$

We easily find

$$h(-\sqrt{1+z_2}, z_2) < h(-1, 0) = \max_{z_2 \in \mu_2} \operatorname{Re} h(-\sqrt{1+z_2}, z_2)$$

for all $z_2 \in \lambda_2''$, $z_2 \neq 0$. Similarly to case (ii), for any fixed $z_2 \in \lambda_2''$ we replace the integration path $\gamma_1 \cup \gamma_2$ for z_1 with the steepest descent path $\mu_1(z_2)$ for $|\exp h(z_1, z_2)|$ through $-\sqrt{1+z_2}$, which clearly has asymptotes γ_1 and γ_2. For any fixed $z_2 \in \lambda_2'$ we argue as in case (ii).

We now have $\vartheta_1 = \pi/2$ and $\vartheta_2 = 0$. Thus (42) yields

$${}_2\mathrm{Ai}_3(t) \sim -\frac{it^{-1/2}}{2\pi}\, \exp\left(-\frac{2}{3}\, t^{3/2}\right). \tag{97}$$

(iv) ${}_2\mathrm{Ai}_4$

Plainly

$$ {}_2\mathrm{Ai}_4(t) = {}_2\mathrm{Ai}_1(t) + {}_2\mathrm{Ai}_3(t). $$

Hence from (93) and (97) we obtain

$$ {}_2\mathrm{Ai}_4(t) \sim -\frac{t^{-1/2}}{2\pi\sqrt{5+2\sqrt{5}}} \exp\left(\frac{7+5\sqrt{5}}{6} t^{3/2}\right). $$

6.3. In this section we apply Theorem 4.2 to the N-dimensional Airy integral

$$ {}_N\mathrm{Ai}_1(\tau^{2/3}) := \left(\frac{\tau^{1/3}}{2\pi i}\right)^N \int_{\Gamma_1} \cdots \int_{\Gamma_N} \exp\left(\tau h(z_1,\ldots,z_N)\right) \mathrm{d}z_1 \cdots \mathrm{d}z_N \tag{98} $$

with $h(z_1,\ldots,z_N)$ given by (87), and with

$$ \Gamma_1 = \cdots = \Gamma_N = \gamma_0 \cup \gamma_1 \tag{99} $$

oriented from $e^{2\pi i/3}\infty$ to $+\infty$, where γ_0 and γ_1 are defined in (5).

The saddle-points of $f(z_1,\ldots,z_N) = \exp h(z_1,\ldots,z_N)$ are the solutions of the system

$$ \begin{cases} \frac{\partial h}{\partial z_1} = 1 + z_2 + z_3 + \cdots + z_N - z_1^2 = 0 \\ \frac{\partial h}{\partial z_2} = 1 + z_1 + z_3 + \cdots + z_N - z_2^2 = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{\partial h}{\partial z_N} = 1 + z_1 + z_2 + \cdots + z_{N-1} - z_N^2 = 0. \end{cases} \tag{100} $$

Using the symmetry of the system (100) we seek solutions satisfying $z_1 = \cdots = z_N$. We set $z_1 = \cdots = z_N = \varphi_N$, say. Then (100) yields

$$ \varphi_N^2 - (N-1)\varphi_N - 1 = 0, $$

whence the saddle-points

$$ \left(z_1^{(0)},\ldots,z_N^{(0)}\right) = \begin{cases} (\varphi_N^+,\ldots,\varphi_N^+) \\ (\varphi_N^-,\ldots,\varphi_N^-) \end{cases} $$

where

$$ \varphi_N^{\pm} = \frac{N-1 \pm \sqrt{(N-1)^2+4}}{2}. \tag{101} $$

The relevant saddle-point for (98)–(99) turns out to be

$$ \left(z_1^{(0)},\ldots,z_N^{(0)}\right) = \left(\varphi_N^+,\ldots,\varphi_N^+\right). \tag{102} $$

It is easy to see that (102) is an admissible saddle-point. By (25) and (100) we have, for $j = 1, \dots, N$,

$$H_j\left(\varphi_N^+, \dots, \varphi_N^+\right) = \exp\left(j\ h\left(\varphi_N^+, \dots, \varphi_N^+\right)\right) \cdot \det \begin{pmatrix} -2\varphi_N^+ & 1 & \dots & 1 \\ 1 & -2\varphi_N^+ & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -2\varphi_N^+ \end{pmatrix}.$$

This $j \times j$ determinant equals $(-2\varphi_N^+ - 1)^{j-1}(-2\varphi_N^+ + j - 1)$, as is easy to prove by induction on j. Therefore

(103) $$H_j\left(\varphi_N^+, \dots, \varphi_N^+\right) = (-1)^j \left(2\varphi_N^+ + 1\right)^{j-1} \left(2\varphi_N^+ + 1 - j\right) \exp\left(j\ h\left(\varphi_N^+, \dots, \varphi_N^+\right)\right) \quad (j = 1, \dots, N).$$

By (101) we get $H_j(\varphi_N^+, \dots, \varphi_N^+) \neq 0$. Hence (102) is admissible.

By the system (100) and by (29), for each $j = 1, \dots, N-1$ the functions

$$Z_{1j}(z_{j+1}, \dots, z_N), \dots, Z_{jj}(z_{j+1}, \dots, z_N)$$

are defined by

$$1 + (j-1)Z_{kj} + z_{j+1} + \dots + z_N - Z_{kj}^2 = 0,$$

whence

(104) $$Z_{1j}(z_{j+1}, \dots, z_N) = \dots = Z_{jj}(z_{j+1}, \dots, z_N) = \frac{j - 1 + \sqrt{(j-1)^2 + 4(1 + z_{j+1} + \dots + z_N)}}{2},$$

where the square root is positive for $z_{j+1} > 0, \dots, z_N > 0$. The $+$ sign preceding the square root is justified by the condition

$$Z_{kj}\left(\varphi_N^+, \dots, \varphi_N^+\right) = \varphi_N^+.$$

With the notation of Theorem 4.2 we choose

(105) $$\lambda_N = \Gamma_N = \gamma_0 \cup \gamma_1.$$

By (87) and (104) we easily get

$$h(Z_{1,N-1}(z_N), \dots, Z_{N-1,N-1}(z_N), z_N) = \frac{(N-1)(N-2)}{2}\left(1 + \frac{(N-2)^2}{6}\right) + \left(1 + \frac{(N-1)(N-2)}{2}\right) z_N + \frac{N-1}{12}\left((N-2)^2 + 4(1+z_N)\right)^{3/2} - \frac{z_N^3}{3}.$$

By elementary arguments we see that

$$\operatorname{Re} h(Z_{1,N-1}(z_N), \dots, Z_{N-1,N-1}(z_N), z_N)$$

along γ_0 is maximal at $z_N = \varphi_N^+$, and along γ_1 is maximal at $z_N = 0$. Hence the assumptions (34) and (35) are satisfied for $z_N^{(0)} = \varphi_N^+$ and for λ_N given by (105).

For each $j = 1, \ldots, N-1$, the existence of a path $\lambda_j(z_{j+1}, \ldots, z_N)$ equivalent to (99) by Cauchy's theorem, containing the point $Z_{jj}(z_{j+1}, \ldots, z_N)$ given by (104), and satisfying (38), can be proved similarly to the case of ${}_2\mathrm{Ai}_2(\tau^{2/3})$, as follows.

Let $\Theta \subset \mathbb{C}$ be the angular region defined by

$$\Theta := \left\{ z \in \mathbb{C} \;\middle|\; -\pi/6 \le \arg z \le 2\pi/3 \right\},$$

and let

$$\delta = \delta' \cup \gamma_1$$

be the border of Θ, with

$$\delta' = \left\{ \varrho e^{-\pi i/6} \;\middle|\; 0 \le \varrho < +\infty \right\}.$$

For any fixed $z_{j+1}, \ldots, z_N \in \Theta$, from (104) we get

$$-\pi/12 \le \arg Z_{kj}(z_{j+1}, \ldots, z_N) \le \pi/3,$$

and

$$\begin{aligned} & h(Z_{1,j-1}(z_j, \ldots, z_N), \ldots, Z_{j-1,j-1}(z_j, \ldots, z_N), z_j, \ldots, z_N) \\ & = s_1 + s_2 + s_3 + \frac{1}{2}(j-1)(j-2)\left(s_1 + 1 + \frac{1}{6}(j-2)^2\right) \\ & \quad + \frac{1}{12}(j-1)\left((j-2)^2 + 4(1 + z_j + s_1)\right)^{3/2} \\ & \quad + \left(1 + \frac{(j-1)(j-2)}{2} + s_1\right) z_j - \frac{1}{3} z_j^3 \end{aligned} \tag{106}$$

with

$$s_1 = \sum_{m=j+1}^{N} z_m, \quad s_2 = \sum_{j+1 \le k < l \le N} z_k z_l, \quad s_3 = -\frac{1}{3} \sum_{m=j+1}^{N} z_m^3.$$

For $z_j \in \delta$ moving from $e^{2\pi i/3}\infty$ to $e^{-\pi i/6}\infty$, a straightforward computation shows that

$$\mathrm{Re}\left(\left((j-2)^2 + 4(1 + z_j + s_1)\right)^{3/2}\right) \text{ and } \mathrm{Re}\left(\left(1 + \frac{(j-1)(j-2)}{2} + s_1\right) z_j\right)$$

are both increasing, whereas $-\frac{1}{3}\,\mathrm{Re}(z_j^3)$ increases for $z_j \in \gamma_1$ and vanishes identically for $z_j \in \delta'$. Thus from (106) we see that

$$\mathrm{Re}\, h(Z_{1,j-1}(z_j, \ldots, z_N), \ldots, Z_{j-1,j-1}(z_j, \ldots, z_N), z_j, \ldots, z_N)$$

is increasing. Hence the intersections with Θ of the valley-sets in the plane of the variable z_j with vertex at the saddle-point $z_j = Z_{jj}(z_{j+1}, \ldots, z_N)$ are both unbounded. Moreover, for $z_j \to +\infty$ we have $-z_j^3/3 \to -\infty$ whence

$$\exp h(Z_{1,j-1}(z_j, \ldots, z_N), \ldots, Z_{j-1,j-1}(z_j, \ldots, z_N), z_j, \ldots, z_N) \to 0.$$

Thus there exists $z_j^* > 0$ such that the halfline $\{z_j^* \le z_j < +\infty\}$ is contained in one of the valley-sets above. This proves the existence in Θ of a path $\lambda_j(z_{j+1}, \ldots, z_N)$ for z_j equivalent to (99) by Cauchy's theorem and satisfying (37)–(38).

In order to get the asymptotic formula (42) for ${}_N\mathrm{Ai}_1(\tau^{2/3})$, we compute $f(\varphi_N^+, \ldots, \varphi_N^+) = \exp h(\varphi_N^+, \ldots, \varphi_N^+)$ and the determinants $H_j(\varphi_N^+, \ldots, \varphi_N^+)$ given by (103). From (87) and (101) we obtain

$$\begin{aligned} h\left(\varphi_N^+, \ldots, \varphi_N^+\right) &= \frac{N(N-1)}{6} + \left(\frac{2}{3}N + \frac{N(N-1)^2}{6}\right)\varphi_N^+ \\ &= \frac{N(N-1)}{2}\left(1 + \frac{(N-1)^2}{6}\right) + \frac{N}{3}\left(1 + \frac{(N-1)^2}{4}\right)\sqrt{(N-1)^2+4}. \end{aligned} \tag{107}$$

From (103) we get, for $j = 0, 1, \ldots, N$,

$$\begin{aligned} H_j\left(\varphi_N^+, \ldots, \varphi_N^+\right) &= (-1)^j \left(N + \sqrt{(N-1)^2+4}\right)^{j-1} \\ &\quad \times \left(N - j + \sqrt{(N-1)^2+4}\right) \exp\left(jh(\varphi_N^+, \ldots, \varphi_N^+)\right). \end{aligned}$$

Therefore

$$-\frac{1}{f(\varphi_N^+, \ldots, \varphi_N^+)} \, \frac{H_j(\varphi_N^+, \ldots, \varphi_N^+)}{H_{j-1}(\varphi_N^+, \ldots, \varphi_N^+)} > 0.$$

Thus, by (43) and Remark 4.3,

$$\vartheta_1 = \cdots = \vartheta_N = 0.$$

From (98), (103) and (42) we obtain

$$\begin{aligned} {}_N\mathrm{Ai}_1(\tau^{2/3}) \sim (-i)^N \frac{\tau^{-N/6}}{(2\pi)^{N/2}} \left(2\varphi_N^+ + 1\right)^{-(N-1)/2} \left(2\varphi_N^+ + 1 - N\right)^{-1/2} \\ \times \exp\left(\tau h\left(\varphi_N^+, \ldots, \varphi_N^+\right)\right). \end{aligned}$$

Substituting $\tau^{2/3} = t$, by (101) and (107) we finally get

$$\begin{aligned} {}_N\mathrm{Ai}_1(t) \sim (-i)^N \frac{t^{-N/4}}{(2\pi)^{N/2}} \left(N + \sqrt{(N-1)^2+4}\right)^{-(N-1)/2} \left((N-1)^2+4\right)^{-1/4} \\ \times \exp\left(\left(\frac{N(N-1)}{2}\left(1 + \frac{(N-1)^2}{6}\right)\right.\right. \\ \left.\left. + \frac{N}{3}\left(1 + \frac{(N-1)^2}{4}\right)\sqrt{(N-1)^2+4}\right) t^{3/2}\right). \end{aligned} \tag{108}$$

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