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LIFTING THE CARTIER TRANSFORM OF OGUS-VOLOGODSKY MODULO p^n

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LIFTING THE CARTIER TRANSFORM OF OGUS-VOLOGODSKY MODULO p^n

Daxin Xu

Abstract. – Let W be the ring of the Witt vectors of a perfect field of characteristic p, $\mathfrak{X}$ a smooth formal scheme over W, $\mathfrak{X}'$ the base change of $\mathfrak{X}$ by the Frobenius morphism of W, $\mathfrak{X}'_2$ the reduction modulo p^2 of $\mathfrak{X}'$ and X the special fiber of $\mathfrak{X}$. We lift the Cartier transform of Ogus-Vologodsky defined by $\mathfrak{X}'_2$ modulo p^n. More precisely, we construct a functor from the category of p^n-torsion $\mathscr{O}_{\mathfrak{X}'}$-modules with integrable p-connection to the category of p^n-torsion $\mathscr{O}_{\mathfrak{X}}$-modules with integrable connection, each subject to suitable nilpotence conditions. Our construction is based on Oyama's reformulation of the Cartier transform of Ogus-Vologodsky in characteristic p. If there exists a lifting $F:\mathfrak{X}\to\mathfrak{X}'$ of the relative Frobenius morphism of X, our functor is compatible with a functor constructed by Shiho from F. As an application, we give a new interpretation of Faltings' relative Fontaine modules and of the computation of their cohomology.

Résumé **(Relèvement de la transformée de Cartier d'Ogus-Vologodsky modulo p^n)**
Soient W l'anneau des vecteurs de Witt d'un corps parfait de caractéristique $p>0$, $\mathfrak{X}$ un schéma formel lisse sur W, $\mathfrak{X}'$ le changement de base de $\mathfrak{X}$ par l'endomorphisme de Frobenius de W, $\mathfrak{X}'_2$ la réduction modulo p^2 de $\mathfrak{X}'$ et X la fibre spéciale de $\mathfrak{X}$. On relève la transformée de Cartier d'Ogus-Vologodsky définie par $\mathfrak{X}'_2$. Plus précisément, on construit un foncteur de la catégorie des $\mathscr{O}_{\mathfrak{X}'}$-modules de p^n-torsion à p-connexion intégrable dans la catégorie des $\mathscr{O}_{\mathfrak{X}}$-modules de p^n-torsion à connexion intégrable, chacune étant soumise à des conditions de nilpotence appropriées. S'il existe un relèvement $F:\mathfrak{X}\to\mathfrak{X}'$ du morphisme de Frobenius relatif de X, notre foncteur est compatible avec une construction « locale » de Shiho définie par F. Comme application de la transformée de Cartier modulo p^n, on donne une nouvelle interprétation des modules de Fontaine relatifs introduits par Faltings et du calcul de leur cohomologie.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. – In his seminal work [**34**], Simpson established a deep relation between complex representations of the fundamental group of a projective complex manifold X and Higgs modules on X, leading to a theory called *nonabelian Hodge theory*. Recall that a Higgs module on X is a coherent sheaf M together with an $\mathscr{O}_X$-linear morphism $\theta : M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/\mathbb{C}}$ such that $\theta \wedge \theta = 0$. (Simpson's result uses, but is much deeper than, the Riemann-Hilbert correspondence relating representations of the fundamental group and modules with integrable connection.) In [**14**], Faltings developed a partial p-adic analog of Simpson correspondence for p-adic local systems on varieties over p-adic fields.

On the other hand, in [**31**], Ogus and Vologodsky constructed a version of nonabelian Hodge theory in characteristic p. If X is a smooth scheme over a perfect field k of characteristic $p > 0$, they established an equivalence, called *Cartier transform*, between certain modules with integrable connection on X/k and certain Higgs modules on X/k, depending on a lifting of X' (the base change of X by the Frobenius morphism of k) to $\mathrm{W}_2(k)$. They also constructed a canonical quasi-isomorphism betweeen certain truncations of the de Rham complex of a module with integrable connection and of the Higgs complex of its Cartier transform. This result generalizes the Cartier isomorphism and the decomposition of the de Rham complex given by Deligne-Illusie [**11**]; it is also an analog of a corresponding result in Simpson's theory.

The relation between Faltings' p-adic Simpson correspondence and the Cartier transform is not yet understood. The first difficulty is to lift the Cartier transform modulo p^n. This is our main goal in the present article. Shiho [**33**] constructed a "local" lifting of the Cartier transform modulo p^n under the assumption of a lifting of the relative Frobenius morphism modulo p^{n+1}. In [**32**], Oyama gave a new construction of the Cartier transform of Ogus-Vologodsky as the inverse image by a morphism of topoi. His work is inspired by Tsuji's approach to the p-adic Simpson correspondence ([**2**] IV). In this article, we use Oyama topoi to "glue" Shiho's functor and obtain a lifting of the Cartier transform modulo p^n under the (only) assumption that X lifts to a smooth formal scheme over W.

1.2. – Shiho's construction applies to modules with λ-*connection*, a notion of introduced by Deligne. Let $f : X \to S$ be a smooth morphism of schemes, M an $\mathscr{O}_X$-module and $\lambda \in \Gamma(S, \mathscr{O}_S)$. A λ-*connection on* M *relative to* S is an $f^{-1}(\mathscr{O}_S)$-linear morphism $\nabla : M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/S}$ such that $\nabla(xm) = x\nabla(m) + \lambda m \otimes d(x)$ for every local sections x of $\mathscr{O}_X$ and m of M. 1-connections correspond to the classical notion of connections, and 0-connections to Higgs fields. The integrability of λ-connections is defined in the same way as for connections. We denote by $\mathrm{MIC}(X/S)$ (resp. λ-$\mathrm{MIC}(X/S)$) the category of $\mathscr{O}_X$-modules with integrable connection (resp. λ-connection) relative to S.

1.3. – In the following, if we use a gothic letter $\mathfrak{T}$ to denote an adic formal W-scheme, the corresponding roman letter T will denote its special fiber. Let $\mathfrak{X}$ be a smooth formal scheme over W and n an integer $\geqslant 1$. We denote by $\sigma : \mathrm{W} \to \mathrm{W}$ the Frobenius automorphism of W, by $\mathfrak{X}'$ the base change of $\mathfrak{X}$ by σ and by $\mathfrak{X}_n$ the reduction of $\mathfrak{X}$ modulo p^n. In [**33**], Shiho constructed a "local" lifting modulo p^n of the Cartier transform of Ogus-Vologodsky defined by $\mathfrak{X}'_2$, using a lifting $F_{n+1} : \mathfrak{X}_{n+1} \to \mathfrak{X}'_{n+1}$ of the relative Frobenius morphism $F_{X/k} : X \to X'$ of X.

The image of the differential morphism $dF_{n+1} : F^*_{n+1}(\Omega^1_{\mathfrak{X}'_{n+1}/\mathrm{W}_{n+1}}) \to \Omega^1_{\mathfrak{X}_{n+1}/\mathrm{W}_{n+1}}$ of F_{n+1} is contained in $p\Omega^1_{\mathfrak{X}_{n+1}/\mathrm{W}_{n+1}}$. Dividing by p, it induces an $\mathscr{O}_{\mathfrak{X}_n}$-linear morphism

$$dF_{n+1}/p : F^*_n(\Omega^1_{\mathfrak{X}'_n/\mathrm{W}_n}) \to \Omega^1_{\mathfrak{X}_n/\mathrm{W}_n}.$$

Shiho defined a functor (depending on F_{n+1}) ([**33**] 2.5)

$$\begin{aligned} \Phi_n : p\text{-}\mathrm{MIC}(\mathfrak{X}'_n/\mathrm{W}_n) &\to \mathrm{MIC}(\mathfrak{X}_n/\mathrm{W}_n) \\ (M', \nabla') &\mapsto (F^*_n(M'), \nabla), \end{aligned} \tag{1.3.1}$$

where $\nabla : F^*_n(M') \to \Omega^1_{\mathfrak{X}_n/\mathrm{W}_n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} F^*_n(M')$ is the integrable connection defined for every local section e of M' by

$$\nabla(F^*_n(e)) = (\mathrm{id} \otimes \frac{dF_{n+1}}{p})(F^*_n(\nabla'(e))). \tag{1.3.2}$$

Shiho showed that the functor Φ_n induces an equivalence of categories between the full subcategories of p-$\mathrm{MIC}(\mathfrak{X}'_n/\mathrm{W}_n)$ and of $\mathrm{MIC}(\mathfrak{X}_n/\mathrm{W}_n)$ consisting of quasi-nilpotent objects ([**33**] Thm. 3.1). When $n = 1$, Ogus and Vologodsky proved that the functor Φ_1 is compatible with the Cartier transform defined by $\mathfrak{X}'_2$ ([**31**] Thm. 2.11; [**33**] 1.12).

1.4. – The categories of connections and their analogs we will be studying can be understood geometrically using the language of groupoids. Our groupoids will be relatively affine and hence correspond to Hopf algebras. If $(\mathscr{T}, A)$ is a ringed topos, *a Hopf A-algebra* is the data of a ring B of $\mathscr{T}$ together with five homomorphisms

$$A \overset{d_2}{\underset{d_1}{\rightrightarrows}} B, \qquad \delta : B \to B \otimes_A B \text{ (comultiplication)},$$

$$\pi : B \to A \text{ (counit)}, \qquad \sigma : B \to B \text{ (antipode)},$$

where the tensor product $B \otimes_A B$ is taken on the left (resp. right) for the A-algebra structure of B defined by d_2 (resp. d_1), satisfying the compatibility conditions for coalgebras (cf. 4.2, **[4]** II 1.1.2).

A B-stratification on an A-module M is a B-linear isomorphism

$$\varepsilon : B \otimes_A M \xrightarrow{\sim} M \otimes_A B, \tag{1.4.1}$$

where the tensor product is taken on the left (resp. right) for the A-algebra structure defined by d_2 (resp. d_1), satisfying $\pi^*(\varepsilon) = \mathrm{id}_M$ and a cocycle condition (cf. 5.4).

1.5. – A classical example of a Hopf algebra is given by the PD-envelope of the diagonal immersion. Let $\mathfrak{X}$ be a smooth formal W-scheme, $\mathfrak{X}^2$ the product of two copies of $\mathfrak{X}$ over W. For any $n \geqslant 1$, we denote by $P_{\mathfrak{X}_n}$ the PD-envelope of the diagonal immersion $\mathfrak{X}_n \to \mathfrak{X}_n^2$ compatible with the canonical PD-structure on $(\mathrm{W}_n, p\,\mathrm{W}_n)$ and by $P_{\mathfrak{X}}$ the associated adic formal W-scheme. The $\mathscr{O}_{\mathfrak{X}}$-bialgebra $\mathscr{O}_{P_{\mathfrak{X}}}$ of $\mathfrak{X}_{\mathrm{zar}}$ is naturally equipped with a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure (i.e., for every $n \geqslant 1$, a Hopf $\mathscr{O}_{\mathfrak{X}_n}$-algebra structure on $\mathscr{O}_{P_{\mathfrak{X}_n}}$, which is compatible) (cf. 4.7, 5.10).

A quasi-nilpotent integrable connection relative to W_n on an $\mathscr{O}_{\mathfrak{X}_n}$-module M (cf. 5.3) is equivalent to an $\mathscr{O}_{P_{\mathfrak{X}}}$-stratification on M (**[5]** 4.12). Following Shiho **[33]**, we give below an analogous description of p-connections; the relevant Hopf algebra is constructed by dilatation (certain distinguished open subset of admissible blow-up) in formal geometry.

1.6. – We define by dilatation an adic formal $\mathfrak{X}^2$-scheme $R_{\mathfrak{X}}$ satisfying the following conditions (3.5).

(i) The canonical morphism $R_{\mathfrak{X},1} \to X^2$ factors through the diagonal immersion $X \to X^2$.

(ii) Let $X \to \mathfrak{X}^2$ be the morphism induced by the diagonal immersion. For any flat formal W-scheme $\mathfrak{Y}$ and any W-morphisms $f : \mathfrak{Y} \to \mathfrak{X}^2$ and $g : Y \to X$ which fit into the following commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathfrak{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & \mathfrak{X}^2, \end{array}$$

there exists a unique W-morphism $f' : \mathfrak{Y} \to R_{\mathfrak{X}}$ lifting f.

We denote abusively by $\mathscr{O}_{R_{\mathfrak{X}}}$ the direct image of $\mathscr{O}_{R_{\mathfrak{X}}}$ via the morphism $R_{\mathfrak{X},\mathrm{zar}} \to \mathfrak{X}_{\mathrm{zar}}$ (i). Using the universal property of $R_{\mathfrak{X}}$, we show that $\mathscr{O}_{R_{\mathfrak{X}}}$ is equipped with a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure (4.11).

The diagonal immersion $\mathfrak{X} \to \mathfrak{X}^2$ induces a closed immersion $\iota : \mathfrak{X} \to R_{\mathfrak{X}}$ (3.5). For any $n \geqslant 1$, we denote by $T_{\mathfrak{X},n}$ the PD-envelope of $\iota_n : \mathfrak{X}_n \to R_{\mathfrak{X},n}$ compatible with the canonical PD-structure on $(\mathrm{W}_n, p\,\mathrm{W}_n)$. The schemes $\{T_{\mathfrak{X},n}\}_{n\geqslant 1}$ form an adic inductive system and we denote by $T_{\mathfrak{X}}$ the associated adic formal W-scheme. By the universal property of PD-envelope, the formal Hopf algebra structure on $\mathscr{O}_{R_{\mathfrak{X}}}$ extends to a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure on the $\mathscr{O}_{\mathfrak{X}}$-bialgebra $\mathscr{O}_{T_{\mathfrak{X}}}$ of $\mathfrak{X}_{\mathrm{zar}}$ (5.15).

In ([**33**] Prop. 2.9), Shiho showed that for any $n \geqslant 1$ and any $\mathscr{O}_{\mathfrak{X}_n}$-module M, an $\mathscr{O}_{T_{\mathfrak{X}}}$-stratification on M is equivalent to a quasi-nilpotent integrable p-connection on M (cf. 5.17).

1.7. – Shiho's local construction deals with modules with p-connection and connection, which is different to the (global) Cartier transform of Ogus-Vologodsky. We need a fourth Hopf algebra, introduced by Oyama [**32**], and we will use it to define a notion of stratification that will enable us to globalize Shiho's construction.

For any k-scheme Y, we denote by $\underline{Y}$ the closed subscheme of Y defined by the ideal sheaf of $\mathscr{O}_Y$ consisting of the sections of $\mathscr{O}_Y$ whose pth power is zero. In (3.5), 4.9, we construct an adic formal $\mathfrak{X}^2$-scheme $Q_{\mathfrak{X}}$ satisfying the following conditions.

(i) The canonical morphism $\underline{Q_{\mathfrak{X},1}} \to X^2$ factors through the diagonal immersion $X \to X^2$.

(ii) For any flat formal W-scheme $\mathfrak{Y}$ and any W-morphisms $f : \mathfrak{Y} \to \mathfrak{X}^2$ and $g : \underline{Y} \to X$ which fit into the following commutative diagram

$$\begin{tikzcd} \underline{Y} \arrow[r] \arrow[d, "g"'] & \mathfrak{Y} \arrow[d, "f"] \\ X \arrow[r] & \mathfrak{X}^2, \end{tikzcd}$$

there exists a unique W-morphism $f' : \mathfrak{Y} \to Q_{\mathfrak{X}}$ lifting f.

We denote abusively by $\mathscr{O}_{Q_{\mathfrak{X}}}$ the direct image of $\mathscr{O}_{Q_{\mathfrak{X}}}$ via the morphism $Q_{\mathfrak{X},\mathrm{zar}} \to \mathfrak{X}_{\mathrm{zar}}$ (i). It is also equipped with a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure (4.11).

Let $P_{\mathfrak{X}}$ be the formal $\mathfrak{X}^2$-scheme defined in 1.5, $\iota : \mathfrak{X} \to P_{\mathfrak{X}}$ the canonical morphism lifting the diagonal immersion $\mathfrak{X} \to \mathfrak{X}^2$ and $\mathscr{J}$ the PD-ideal of $\mathscr{O}_{P_X}$ associated to ι_1. For any local section of $\mathscr{J}$, we have $x^p = p!x^{[p]} = 0$. Then we deduce a closed immersion $\underline{P_X} \to X$ over $\mathfrak{X}^2$. By the universal property of $Q_{\mathfrak{X}}$, we obtain an $\mathfrak{X}^2$-morphism $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$.

1.8. – The groupoids and Hopf algebras constructed above give a geometric interpretation of Shiho's functor Φ and of a variation of Φ which can be globalized. Let $F : \mathfrak{X} \to \mathfrak{X}'$ be a lifting of the relative Frobenius morphism $F_{X/k}$ of X. By the universal properties of $R_{\mathfrak{X}'}$ and of PD-envelopes, the morphism $F^2 : \mathfrak{X}^2 \to \mathfrak{X}'^2$ induce morphisms $\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$ (6.6) and $\varphi : P_{\mathfrak{X}} \to T_{\mathfrak{X}'}$ (6.8) above F^2 which fit into a commutative diagram (6.9.1)

(1.8.1)
$$\begin{tikzcd} P_{\mathfrak{X}} \arrow[r, "\varphi"] \arrow[d, "\lambda"'] & T_{\mathfrak{X}'} \arrow[d, "\varpi"] \\ Q_{\mathfrak{X}} \arrow[r, "\psi"] & R_{\mathfrak{X}'}, \end{tikzcd}$$

where $\varpi : T_{\mathfrak{X}'} \to R_{\mathfrak{X}'}$ (1.6) and $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$ (1.7) are independent of F. Moreover, ψ and φ induce homomorphisms of formal Hopf algebras $\mathscr{O}_{R_{\mathfrak{X}'}} \to F_*(\mathscr{O}_{Q_{\mathfrak{X}}})$ and $\mathscr{O}_{T_{\mathfrak{X}'}} \to F_*(\mathscr{O}_{P_{\mathfrak{X}}})$. The above diagram induces a commutative diagram (6.9.2)

$$
\begin{tikzcd}
\left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{O}_{R_{\mathfrak{X}'}}\text{-stratification}\end{array}\right\} \arrow[r, "\psi_n^*"] \arrow[d, "\varpi_n^*"'] & \left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{O}_{Q_{\mathfrak{X}}}\text{-stratification}\end{array}\right\} \arrow[d, "\lambda_n^*"] \\
\left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{O}_{T_{\mathfrak{X}'}}\text{-stratification}\end{array}\right\} \arrow[r, "\varphi_n^*"] & \left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{O}_{P_{\mathfrak{X}}}\text{-stratification}\end{array}\right\}.
\end{tikzcd}
\tag{1.8.2}
$$

In ([**33**] 2.17), Shiho showed that the functor φ_n^* is compatible with the functor Φ_n defined by F (1.3.1), via the equivalence between the category of modules with quasi-nilpotent integrable connection (resp. p-connection) and the category of modules with $\mathscr{O}_{P_{\mathfrak{X}}}$-stratification (resp. $\mathscr{O}_{T_{\mathfrak{X}}}$-stratification).

1.9. – Let us explain the Oyama sites $\mathscr{E}$ and $\underline{\mathscr{E}}$ whose crystals corresponding to $\mathscr{O}_{Q_{\mathfrak{X}}}$ and $\mathscr{O}_{R_{\mathfrak{X}}}$ stratification, and a morphism of topoi which will be used to lift the Cartier transform and to globalize the funtor ψ_n^*.

Let X be a scheme over k. An object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is a triple $(U, \mathfrak{T}, u)$ consisting of an open subscheme U of X, a flat formal W-scheme $\mathfrak{T}$ and an affine k-morphism $u : T \to U$ (resp. $u : \underline{T} \to U$ (1.7)). Morphisms are defined in a natural way (cf. 7.1). We denote by $\mathscr{E}'$ Oyama's category associated to the k-scheme X'. We denote by $\widetilde{\mathscr{E}}$ (resp. $\underline{\widetilde{\mathscr{E}}}$) the topos of sheaves of sets on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) with respect to the Zarisiki topology (7.8).

Let $(U, \mathfrak{T}, u)$ be an object of $\underline{\mathscr{E}}$. The relative Frobenius morphism $F_{T/k} : T \to T'$ factors through a k-morphism $f_{T/k} : T \to \underline{T}'$. We have a commutative diagram

$$
\begin{tikzcd}
U \arrow[d, "F_{U/k}"'] & \underline{T} \arrow[l, "u"'] \arrow[r, hook] \arrow[d, "F_{\underline{T}/k}"'] & T \arrow[d, "F_{T/k}"] \arrow[dl, "f_{T/k}"'] \\
U' & \underline{T}' \arrow[l, "u'"'] \arrow[r, hook] & T',
\end{tikzcd}
\tag{1.9.1}
$$

where the vertical arrows denote the relative Frobenius morphisms. Then $(U', \mathfrak{T}, u' \circ f_{T/k})$ is an object of $\mathscr{E}'$. We obtain a functor (9.1.2)

$$\rho : \underline{\mathscr{E}} \to \mathscr{E}', \qquad (U, \mathfrak{T}, u) \mapsto (U', \mathfrak{T}, u' \circ f_{T/k}). \tag{1.9.2}$$

The functor ρ is continuous and cocontinuous (9.3) and induces a morphism of topoi (9.1.3)

$$\mathrm{C}_{X/\mathrm{W}} : \underline{\widetilde{\mathscr{E}}} \to \widetilde{\mathscr{E}}' \tag{1.9.3}$$

such that its inverse image functor is induced by the composition with ρ.

1.10. – Let n be an integer $\geqslant 1$. The contravariant functor $(U, \mathfrak{T}, u) \mapsto \Gamma(\mathfrak{T}, \mathscr{O}_{\mathfrak{T}_n})$ defines a sheaf of rings on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) that we denote by $\mathscr{O}_{\mathscr{E},n}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$). By definition, we have $\mathrm{C}^*_{X/\mathrm{W}}(\mathscr{O}_{\mathscr{E}',n}) = \mathscr{O}_{\underline{\mathscr{E}},n}$. To give an $\mathscr{O}_{\mathscr{E},n}$-module (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module) $\mathscr{F}$ amounts to give the following data (8.2):

(i) For every object $(U, \mathfrak{T}, u)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $u_*(\mathscr{O}_{\mathfrak{T}_n})$-module $\mathscr{F}_{(U,\mathfrak{T})}$ of U_{zar}.
(ii) For every morphism $f : (U_1, \mathfrak{T}_1, u_1) \to (U_2, \mathfrak{T}_2, u_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $u_{1*}(\mathscr{O}_{\mathfrak{T}_{1,n}})$-linear morphism

$$c_f : u_{1*}(\mathscr{O}_{\mathfrak{T}_{1,n}}) \otimes_{(u_{2*}(\mathscr{O}_{\mathfrak{T}_{2,n}}))|_{U_1}} (\mathscr{F}_{(U_2,\mathfrak{T}_2)})|_{U_1} \to \mathscr{F}_{(U_1,\mathfrak{T}_1)},$$

satisfying a cocycle condition for the composition of morphisms as in ([**5**] 5.1).

Following ([**5**] 6.1), we say that $\mathscr{F}$ is a *crystal* if c_f is an isomorphism for every morphism f and that $\mathscr{F}$ is *quasi-coherent* if $\mathscr{F}_{(U,\mathfrak{T})}$ is a quasi-coherent $u_*(\mathscr{O}_{\mathfrak{T}_n})$-module of U_{zar} for every object $(U, \mathfrak{T}, u)$. We denote by $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E},n})$ (resp. $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n})$) the category of quasi-coherent crystals of $\mathscr{O}_{\mathscr{E},n}$-modules (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules).

The following are the main results of this article.

Proposition 1.11 (8.10). – *Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme and X its special fiber. There exists a canonical equivalence of categories between the category $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E},n})$ (resp. $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n})$) and the category of quasi-coherent $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathscr{O}_{R_{\mathfrak{X}}}$-stratification (resp. $\mathscr{O}_{Q_{\mathfrak{X}}}$-stratification)* (1.4), *1.6, 1.7.*

Theorem 1.12 (9.12). – *Let X be a smooth k-scheme. Then, for any $n \geqslant 1$, the inverse image and the direct image functors of the morphism* $\mathrm{C}_{X/\mathrm{W}}$ (1.9.3) *induce equivalences of categories quasi-inverse to each other*

$$\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E}',n}) \rightleftarrows \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n}). \tag{1.12.1}$$

The theorem is proved by fppf descent for quasi-coherent modules.

We call *Cartier equivalence modulo p^n* the equivalence of categories $\mathrm{C}^*_{X/\mathrm{W}}$ (1.12.1). Indeed, given a smooth formal W-scheme $\mathfrak{X}$ with special fiber X, Oyama proved 1.12 in the case $n = 1$ and showed that $\mathrm{C}^*_{X/\mathrm{W}}$ is compatible with the Cartier transform of Ogus-Vologodsky defined by the lifting $\mathfrak{X}'_2$ of X' (cf. [**32**] Section 1.5). In Section 12, we reprove the later result in a different way (12.22).

The following result explains the relation between the Cartier equivalence $\mathrm{C}^*_{X/\mathrm{W}}$ and Shiho's construction, in the presence of a lifting of Frobenius.

Proposition 1.13 (9.17). – *Let $\mathfrak{X}$ be a smooth formal* W*-scheme, X its special fiber, $F : \mathfrak{X} \to \mathfrak{X}'$ a lifting of the relative Frobenius morphism $F_{X/k}$ of X and ψ_n^* the functor* *defined by F in* (1.8.2)*. Then, the following diagram* (1.11)

$$\begin{tikzcd} \mathscr{C}(\mathscr{O}_{\mathscr{E}',n}) \arrow[rr, "\mathrm{C}^*_{X/\mathrm{W}}"] \arrow[d, "\mu"', "\wr"] & & \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n}) \arrow[d, "\nu", "\wr"'] \\ \left\{\begin{array}{c}\mathscr{O}_{\mathfrak{X}'_n}\text{-modules}\\ \text{with } \mathscr{O}_{R_{\mathfrak{X}'}}\text{-stratification}\end{array}\right\} \arrow[rr, "\psi_n^*"] & & \left\{\begin{array}{c}\mathscr{O}_{\mathfrak{X}_n}\text{-modules}\\ \text{with } \mathscr{O}_{Q_{\mathfrak{X}}}\text{-stratification}\end{array}\right\} \end{tikzcd} \tag{1.13.1}$$

is commutative up to a functorial isomorphism. That is, for every crystal $\mathscr{M}$ of $\mathscr{O}_{\mathscr{E}',n}$-modules of $\widetilde{\mathscr{E}}'$, we have a canonical functorial isomorphism

$$\eta_F : \psi_n^*(\mu(\mathscr{M})) \xrightarrow{\sim} \nu(\mathrm{C}^*_{X/\mathrm{W}}(\mathscr{M})). \tag{1.13.2}$$

In the diagram (1.13.1), while $\mathrm{C}^*_{X/\mathrm{W}}$ does not depend on models of X, ψ_n^* depends on the lifting F of the relative Frobenius morphism and the vertical functors μ, and ν depend on the formal model $\mathfrak{X}$ of X. The isomorphism η_F depends also on F. For different choice of liftings of Frobenius, η_F can be related by an explicit formula encorded in Oyama topos (9.22).

By 1.8, the equivalence of categories $\mathrm{C}^*_{X/\mathrm{W}}$ (1.12.1) is compatible with Shiho's functor Φ_n defined by F (1.3.1). In the case $n=1$, an analogous relation between the Cartier transform and Φ_1 is shown in ([**31**] 2.11).

1.14. – In [**15**], Fontaine and Laffaille introduced the notion of *Fontaine module* to study p-adic Galois representations. It is inspired by the work of Mazur ([**27**], [**28**]) and Ogus ([**5**], § 8) on the Katz conjecture. Let $\sigma : \mathrm{W} \to \mathrm{W}$ be the Frobenius endomorphism and $K_0 = \mathrm{W}[\frac{1}{p}]$. A Fontaine module is a triple $(M, M^\bullet, \varphi^\bullet)$ made of a W-module of finite length M, a decreasing filtration $\{M^i\}_{i\in\mathbb{Z}}$ such that $M^0 = M$, $M^p = 0$ and W-linear morphisms

$$\varphi^i : \mathrm{W}\otimes_{\sigma,\mathrm{W}} M^i \to M, \quad 0 \leqslant i \leqslant p-1, \tag{1.14.1}$$

such that $\varphi^i|_{M^{i+1}} = p\varphi^{i+1}$ and $\sum_{i=0}^{p-1} \varphi^i(M^i) = M$. The φ^i's are called *divided Frobenius morphisms.*

The main result of Fontaine-Laffaille is the construction of a fully faithful and exact functor from the category of Fontaine modules of length $\leqslant p-2$ to the category of torsion $\mathbb{Z}_p$-representations of the Galois group G_{K_0} of K_0 ([**36**] Thm. 2). Its essential image consists of torsion crystalline representations of G_{K_0} with weights $\leqslant p-2$ (cf. [**8**] 3.1.3.3).

Fontaine and Messing showed that there exists a natural Fontaine module structure on the crystalline cohomology of a smooth proper scheme $\mathscr{X}$ over W of relative dimension $\leqslant p-1$ ([**16**] II.2.7). Then they deduced the degeneration of the Hodge to de Rham spectral sequence of $\mathscr{X}_n/\mathrm{W}_n$.

1.15. – A generalization of Fontaine modules in a relative situation was proposed by Faltings in [**13**]. Relative Fontaine modules can be viewed as an analog of variation of Hodge structures on smooth formal schemes over W. Let $\mathfrak{X} = \mathrm{Spf}(R)$ be an affine smooth formal scheme over W, X its special fiber and $F : \mathfrak{X} \to \mathfrak{X}$ a σ-lifting of the absolute Frobenius morphism F_X of X. *A Fontaine module over* $\mathfrak{X}$ *with respect to* F is a quadruple $(M, \nabla, M^{\bullet}, \varphi_{\mathrm{F}}^{\bullet})$ made of a coherent, torsion $\mathscr{O}_{\mathfrak{X}}$-module M, an integrable connection ∇ on M, a decreasing exhaustive filtration $M^{\bullet}$ on M of length at most $p-1$ satisfying Griffiths' transversality, and a family of divided Frobenius morphisms $\{\varphi_F^{\bullet}\}$ as in (1.14.1) satisfying a compatibility condition between $\{\varphi_F^{\bullet}\}$ and ∇ (cf. [**13**] 2.c, 2.d).

Using the connection, Faltings glued the categories of Fontaine modules with respect to different Frobenius liftings by a *Taylor formula* (cf. [**13**] Thm. 2.3). By gluing local data, he defined Fontaine modules over a general smooth formal W-scheme $\mathfrak{X}$, even if there is no lifting of F_X.

If $\mathfrak{X}$ is the p-adic completion of a smooth, proper W-scheme $\mathscr{X}$, Faltings associated to each Fontaine module of length $\leqslant p-2$ over $\mathfrak{X}$ a representation of the étale fundamental group of $\mathscr{X}_{K_0}$ on a torsion $\mathbb{Z}_p$-module. Moreover, Faltings generalized Fontaine-Messing's result for the crystalline cohomology of a relative Fontaine module.

1.16. – Let $\mathfrak{X}$ be a smooth formal W-scheme. As an application of their Cartier transform [**31**], Ogus and Vologodsky proposed an interpretation of p-torsion Fontaine modules over $\mathfrak{X}$ ([**31**], 4.16). *A p-torsion Fontaine module over* $\mathfrak{X}$ is a triple $(M, \nabla, M^{\bullet})$ as in 1.15 such that $pM = 0$ and equipped with a horizontal isomorphism

$$\phi : \mathrm{C}_{\mathfrak{X}_2'}^{-1}(\pi^*(\mathrm{Gr}(M), \kappa)) \xrightarrow{\sim} (M, \nabla), \tag{1.16.1}$$

where κ is the Higgs field on $\mathrm{Gr}(M)$ induced by ∇ and Griffiths' transversality, and $\pi : X' \to X$ is the base change of the Frobenius morphism of k to X. Given a σ-lifting $F : \mathfrak{X} \to \mathfrak{X}$ of F_X, such a morphism ϕ is equivalent to a family of divided Frobenius morphisms $\{\varphi_F^{\bullet}\}$ with respect to F (1.15) (cf. 13.20).

By Griffiths' transversality, the de Rham complex $M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k}$ is equipped with a decreasing filtration

$$\mathrm{F}^i(M \otimes_{\mathscr{O}_X} \Omega^q_{X/k}) = M^{i-q} \otimes_{\mathscr{O}_X} \Omega^q_{X/k}. \tag{1.16.2}$$

Let ℓ be the length of the filtration $M^{\bullet}$ (i.e., $M^0 = M, M^{\ell+1} = 0$) and d the relative dimension of $\mathfrak{X}$ over W.

(i) For any i, m, the canonical morphism $\mathbb{H}^m(\mathrm{F}^{i+1}(M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k})) \to \mathbb{H}^m(\mathrm{F}^i(M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k}))$ is injective. The morphism ϕ (1.16.1) induces a family of divided Frobenius morphisms on $(\mathbb{H}^m(M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k}), \{\mathbb{H}^m(\mathrm{F}^i)\}_{i \leqslant p-1})$ which make it into a Fontaine module over W (1.14).

(ii) The hypercohomology spectral sequence of the filtered de Rham complex $(M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k}, \mathrm{F}^i)$ degenerates at E_1.

1.17. – For any $n \geqslant 1$, using the Cartier transform modulo p^n, we reformulate Faltings' definition of p^n-torsion Fontaine modules over $\mathfrak{X}$ following Ogus-Vologodsky (13.7). The Taylor formula used by Faltings to glue the data relative to different liftings of F_X is naturally encoded in Oyama topos (13.22). Following Faltings' strategy, we prove the analog of the previous results (1.16(i-ii)) on the crystalline cohomology of a p^n-torsion Fontaine module over $\mathfrak{X}$ (14.1).

1.18. – Section 2 contains the main notation and general conventions. In Section 3, we recall the notion of dilatation in formal geometry. In Section 4, after recalling the notions of Hopf algebras and groupoids, we present the constructions of groupoids $R_{\mathfrak{X}}$ and $Q_{\mathfrak{X}}$ (1.6), 1.7. In Section 5, we recall the notions of modules with integrable λ-connection (1.2) and of modules with stratification (1.4) and we discuss the relation between them. Following [**33**], we present the construction of Shiho's functor Φ_n (1.3.1) in Section 6. In Section 7, we explain the Oyama topoi $\widetilde{\mathscr{E}}$ and $\underline{\widetilde{\mathscr{E}}}$ (1.9) and their fppf variants. Section 8 is devoted to the study of crystals in the Oyama topoi (1.10). In Section 9, we study the morphism of topoi $\mathrm{C}_{X/\mathrm{W}}$ (1.9.3) and prove our main Results 1.12 and 1.13. We recall the construction of the Cartier transform of Ogus-Vologodsky [**31**] in Section 10. Section 11 is devoted to several rings of differential operators after Oyama and serves as a preparation for next section. In Section 12, we compare the Cartier equivalence $\mathrm{C}^*_{X/\mathrm{W}}$ (1.12.1) and the Cartier transform of Ogus-Vologodsky. In Section 13, we introduce a notion of relative Fontaine modules using Oyama topoi (1.17). We compare it with Faltings' definition [**13**] and Tsuji's definition [**35**]. In Section 14, we construct a Fontaine module structure on the crystalline cohomology of a relative Fontaine module.

After finishing this article, I learned from Arthur Ogus that Vadim Vologodsky has skecthed a similar approach for lifting the Cartier transform in a short note [1] without providing any detail.

A different approach to the formulation of a Cartier transform modulo p^n and its relationship to Fontaine modules was taken in [**26**].

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1. Available at `http://pages.uoregon.edu/vvologod/papers/p-adiccartier.pdf`.

CHAPTER 2

NOTATIONS AND CONVENTIONS

2.1. – In this article, p denotes a prime number, k a perfect field of characteristic p, W the ring of Witt vectors of k and $\sigma : \mathrm{W} \to \mathrm{W}$ the Frobenius automorphism of W. For any integer $n \geqslant 1$, we set $\mathrm{W}_n = \mathrm{W}/p^n\,\mathrm{W}$ and $\mathscr{S} = \mathrm{Spf}(\mathrm{W})$.

2.2. – Let X be a scheme over k. We denote by F_X the absolute Frobenius morphism of X and by $F_{X/k} : X \to X' = X \otimes_{k,F_k} k$ the relative Frobenius morphism. Then we have a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "F_{X/k}"] \arrow[dr] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
 & \operatorname{Spec} k \arrow[r, "F_k"] & \operatorname{Spec} k.
\end{tikzcd}
\tag{2.2.1}
$$

2.3. – Let X be a scheme over k. We denote by $\underline{X}$ the scheme theoretic image of $F_X : X \to X$ ([**22**] 6.10.1 and 6.10.5). By ([**22**] 6.10.4), $\underline{X}$ is the closed subscheme of X defined by the ideal sheaf of $\mathscr{O}_X$ consisting of the sections of $\mathscr{O}_X$ whose pth power is zero. It is clear that the correspondence $X \mapsto \underline{X}$ is functorial. Note that the canonical morphism $\underline{X} \to X$ induces an isomorphism of the underlying topological spaces.

The relative Frobenius morphism $F_{X/k} : X \to X'$ factors through $\underline{X}'$. We denote the induced morphism by $f_{X/k} : X \to \underline{X}'$. By definition, the homomorphism $\mathscr{O}_{\underline{X}'} \to f_{X/k*}(\mathscr{O}_X)$ is injective, i.e., $f_{X/k}$ is scheme theoretically dominant ([**22**] 5.4.2).

If $Y \to X$ and $Z \to X$ are two morphisms of k-schemes, by functoriality, we have a canonical morphism

$$\underline{Y \times_X Z} \to \underline{Y} \times_{\underline{X}} \underline{Z}. \tag{2.3.1}$$

Since X is affine if and only if $\underline{X}$ is affine (cf. [**22**] 2.3.5), we verify that (2.3.1) is an affine morphism.

2.4. – In this article, we follow the conventions of [**1**] for adic rings ([**1**] 1.8.4) and adic formal schemes ([**1**] 2.1.24). Note that these notions are stronger than those introduced by Grothendieck in ([**22**] 0.7.1.9 and 10.4.2).

If $\mathfrak{X}$ is an adic formal scheme such that $p\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$, for any integer $n \geqslant 1$, we denote the usual scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/p^n\mathscr{O}_{\mathfrak{X}})$ by $\mathfrak{X}_n$.

2.5. – We say that an adic formal $\mathscr{S}$-scheme $\mathfrak{X}$ ([**1**] 2.2.7) is *flat over* $\mathscr{S}$ (or that $\mathfrak{X}$ is a flat formal $\mathscr{S}$-scheme) if the morphism $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}$ induced by multiplication by p is injective (i.e., if $\mathscr{O}_{\mathfrak{X}}$ is rig-pur in the sense of ([**1**] 2.10.1.4.2)). It is clear that the above condition is equivalent to the fact that, for every affine open formal subscheme U of $\mathfrak{X}$, the algebra $\Gamma(U, \mathscr{O}_{\mathfrak{X}})$ is flat over W.

Let A be an adic W-algebra ([**1**] 1.8.4.5). Then A is flat over W if and only if $A_n = A/p^nA$ is flat over W_n for all integers $n \geqslant 1$. Indeed, we only need to prove that the condition is sufficient. Let a be an element of A such that $pa = 0$. For any integer $n \geqslant 1$, by the flatness of A_n over W_n, the image of a in A_n is contained in $p^{n-1}A/p^nA$. Since A is separated, we see that $a = 0$, i.e., A is flat over W. We deduce that an adic formal $\mathscr{S}$-scheme $\mathfrak{X}$ is flat over $\mathscr{S}$ if and only if $\mathfrak{X}_n$ is flat over $\mathscr{S}_n$ (2.4) for all integers $n \geqslant 1$.

CHAPTER 3

BLOW-UPS AND DILATATIONS

3.1. – Let A be an adic ring (2.4), J an ideal of definition of finite type of A. We put $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A/J)$ and $\mathfrak{X} = \mathrm{Spf}(A)$. The formal scheme $\mathfrak{X}$ is the completion of X along X'. For any A-module M, we denote by $\widetilde{M}$ the associated $\mathscr{O}_X$-module and by M^{Δ} the completion of $\widetilde{M}$ along X' ([**1**] 2.7.1), which is an $\mathscr{O}_{\mathfrak{X}}$-module.

Let $\mathfrak{a}$ be an open ideal of finite type of A. We denote by $\mathfrak{a}\mathscr{O}_{\mathfrak{X}}$ the ideal sheaf of $\mathscr{O}_{\mathfrak{X}}$ associated to the presheaf defined by $U \mapsto \mathfrak{a}\Gamma(U, \mathscr{O}_{\mathfrak{X}})$. By ([**1**] 2.1.13), we have $\mathfrak{a}^{\Delta} = \mathfrak{a}\mathscr{O}_{\mathfrak{X}}$.

Let B be an adic ring, $u : A \to B$ an adic homomorphism ([**1**] 1.8.4.5) and $f : \mathfrak{Y} = \mathrm{Spf}(B) \to \mathfrak{X} = \mathrm{Spf}(A)$ the associated morphism. In view of ([**1**] 2.5.11), we have a canonical functorial isomorphism

$$f^*(\mathfrak{a}^{\Delta}) \xrightarrow{\sim} (\mathfrak{a} \otimes_A B)^{\Delta}. \tag{3.1.1}$$

Then, we deduce a canonical isomorphism

$$f^*(\mathfrak{a}^{\Delta})\mathscr{O}_{\mathfrak{Y}} \xrightarrow{\sim} (\mathfrak{a}B)^{\Delta}. \tag{3.1.2}$$

Indeed, by definition, $f^*(\mathfrak{a}^{\Delta})\mathscr{O}_{\mathfrak{Y}}$ is the image of the morphism $f^*(\mathfrak{a}^{\Delta}) \to \mathscr{O}_{\mathfrak{Y}} = f^*(\mathscr{O}_{\mathfrak{X}})$, which clearly factors through $(\mathfrak{a}B)^{\Delta}$. The isomorphism (3.1.2) follows from the fact that $\mathfrak{a}^{\Delta} = \mathfrak{a}\mathscr{O}_{\mathfrak{X}}$ and $(\mathfrak{a}B)^{\Delta} = (\mathfrak{a}B)\mathscr{O}_{\mathfrak{Y}}$.

3.2. – Let $\mathfrak{X}$ be an adic formal scheme, $\mathscr{J}$ an ideal of definition of finite type of $\mathfrak{X}$ and $\mathscr{A}$ an open ideal of finite type of $\mathfrak{X}$ ([**1**] 2.1.19). For any $n \geqslant 1$, we denote by $\mathfrak{X}_n$ the usual scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^n)$ and we set

$$\mathfrak{X}'_n = \mathrm{Proj}(\bigoplus_{m \geqslant 0} \mathscr{A}^m \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}_n}). \tag{3.2.1}$$

The sequence $(\mathfrak{X}'_n)$ forms an adic inductive $(\mathfrak{X}_n)$-system ([**1**] 2.2.13). We call its inductive limit $\mathfrak{X}'$ the *admissible blow-up of $\mathscr{A}$ in $\mathfrak{X}$* ([**1**] 3.1.2). By ([**1**] 2.2.14, 2.3.13), $\mathfrak{X}'$ is an adic formal $\mathfrak{X}$-scheme of finite type. Note that $\mathfrak{X}'_n$ is different from the blow-up of $\mathfrak{X}_n$ along $(\mathscr{A} + \mathscr{J}^n)/\mathscr{J}^n$.

3.3. – Let $\mathfrak{X}$ be a flat formal $\mathscr{S}$-scheme locally of finite type (2.5, [**1**] 2.3.13) and $\mathscr{A}$ an open ideal of finite type of $\mathfrak{X}$ containing p. Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be the admissible blow-up of $\mathscr{A}$ in $\mathfrak{X}$. Then the ideal $\mathscr{A}\mathscr{O}_{\mathfrak{X}'}$ is invertible ([**1**] 3.1.4(i)), and $\mathfrak{X}'$ is flat over $\mathscr{S}$ ([**1**] 3.1.4(ii)). We denote by $\mathfrak{X}_{(\mathscr{A}/p)}$ the maximal open formal subscheme of $\mathfrak{X}'$ on which

$$(\mathscr{A}\mathscr{O}_{\mathfrak{X}'})|\mathfrak{X}_{(\mathscr{A}/p)} = (p\mathscr{O}_{\mathfrak{X}'})|\mathfrak{X}_{(\mathscr{A}/p)} \tag{3.3.1}$$

and we call it *the dilatation of $\mathscr{A}$ with respect to p.*

Note that $\mathfrak{X}_{(\mathscr{A}/p)}$ is the complement of $\mathrm{Supp}(\mathscr{A}\mathscr{O}_{\mathfrak{X}'}/p\mathscr{O}_{\mathfrak{X}'})$ in $\mathfrak{X}'$ ([**22**] 0.5.2.2). In view of ([**1**] 3.1.5 and 3.2.7), the above definition coincides with the notion of dilatation of $\mathscr{A}$ with respect to p introduced in ([**1**] 3.2.3.4). We denote the restriction of $\varphi : \mathfrak{X}' \to \mathfrak{X}$ to $\mathfrak{X}_{(\mathscr{A}/p)}$ by

$$\psi : \mathfrak{X}_{(\mathscr{A}/p)} \to \mathfrak{X}. \tag{3.3.2}$$

We set $\mathscr{A}^\sharp = \mathscr{A}^p + p\mathscr{O}_{\mathfrak{X}}$ the open ideal of $\mathfrak{X}$. If $\mathscr{A}$ is locally generated by $\{a_1, \dots, a_n\}$ then $\mathscr{A}^\sharp$ is locally generated by $\{p, a_1^p, \dots, a_n^p\}$.

3.4. – Keep the notation of 3.3 and assume moreover that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. There exists an open ideal of finite type $\mathfrak{a}$ of A containing p such that $\mathfrak{a}^\Delta = \mathscr{A}$ ([**1**] 2.1.10 and 2.1.13). Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(A/pA)$, $\phi : X' \to X$ be the blow-up of $\widetilde{\mathfrak{a}}$ in X and $Y' = \phi^{-1}(Y)$; so $\mathfrak{X}$ is the completion of X along Y. Then $\mathfrak{X}'$ is canonically isomorphic to the completion of X' along Y' and φ is the extension of ϕ to the completions ([**1**] 3.1.3).

Let $(a_i)_{0 \leqslant i \leqslant n}$ be a finite set of generators of $\mathfrak{a}$ such that $a_0 = p$. For any $0 \leqslant i \leqslant n$, let U_i be the maximal open subset of X' where a_i generates $\widetilde{\mathfrak{a}}\mathscr{O}_{X'}$. Since $\widetilde{\mathfrak{a}}\mathscr{O}_{X'}$ is invertible, $(U_i)_{0 \leqslant i \leqslant n}$ form an open covering of X'. It is well known that U_i is the affine scheme associated to the A-algebra A_i defined as follows:

$$A_i' = A\Big[\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\Big] = \frac{A[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]}{(a_j - a_i x_j)_{j \neq i}}, \quad A_i = \frac{A_i'}{(A_i')_{a_i\text{-tor}}},$$

where $(A_i')_{a_i\text{-tor}}$ denotes the ideal of a_i-torsion elements of A_i'. Let $\widehat{A}_i$ be the separated completion of A_i for the p-adic topology and $\widehat{U}_i = \mathrm{Spf}(\widehat{A}_i)$. Then $(\widehat{U}_i)_{0 \leqslant i \leqslant n}$ form a covering of $\mathfrak{X}'$; for any $0 \leqslant i \leqslant n$, $\widehat{U}_i$ is the maximal open of $\mathfrak{X}'$ where a_i generates the invertible ideal $\mathfrak{a}^\Delta\mathscr{O}_{\mathfrak{X}'}$ ([**1**] 3.1.7(ii)). The open formal subscheme $\mathfrak{X}_{(\mathscr{A}/p)}$ of $\mathfrak{X}'$ is equal to $\widehat{U}_0 = \mathrm{Spf}(\widehat{A}_0)$. In particular, we see that, in the general setting of 3.3, $\psi : \mathfrak{X}_{(\mathscr{A}/p)} \to \mathfrak{X}$ is affine ([**1**] 2.3.4).

Let $A\{x_1, \dots, x_n\}$ denote the p-adic completion of the polynomial ring in n variables $A[x_1, \dots, x_n]$, which is flat over $A[x_1, \dots, x_n]$ by ([**1**] 1.12.12). If A_0' is flat over W, then we have $A_0 = A_0'$ and deduce a canonical isomorphism ([**1**] 1.12.16(iv))

$$A\Big\{\frac{a_1}{p}, \dots, \frac{a_n}{p}\Big\} = \frac{A\{x_1, \dots, x_n\}}{(a_i - px_i)_{1 \leqslant i \leqslant n}} \xrightarrow{\sim} \widehat{A}_0. \tag{3.4.1}$$

Proposition 3.5. – *Let $\mathfrak{X}$ be a flat formal $\mathscr{S}$-scheme locally of finite type, $i : T \to \mathfrak{X}_1$ an immersion (not necessary closed). There exists a formal $\mathfrak{X}$-scheme $\psi : \mathfrak{X}_{(T/p)} \to \mathfrak{X}$ (resp. $\psi : \mathfrak{X}^{\sharp}_{(T/p)} \to \mathfrak{X}$) unique up to canonical isomorphisms satisfying the following conditions:*

(i) *The canonical morphism $(\mathfrak{X}_{(T/p)})_1 \to \mathfrak{X}_1$ (resp. $\underline{(\mathfrak{X}^{\sharp}_{(T/p)})_1} \to (\mathfrak{X}^{\sharp}_{(T/p)})_1 \to \mathfrak{X}_1$) factors through the immersion $T \to \mathfrak{X}_1$.*

(ii) *Let $\mathfrak{Y}$ be a flat formal $\mathscr{S}$-scheme, $Y = \mathfrak{Y}_1$ and $f : \mathfrak{Y} \to \mathfrak{X}$ an $\mathscr{S}$-morphism. Suppose that there exists a k-morphism $g : Y \to T$ (resp. $g : \underline{Y} \to T$) which fits into the following diagram:*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & \mathfrak{Y} \arrow[d, "f"] \\ T \arrow[r] & \mathfrak{X} \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} \underline{Y} \arrow[r] \arrow[d, "g"'] & \mathfrak{Y} \arrow[d, "f"] \\ T \arrow[r] & \mathfrak{X}. \end{tikzcd}\right) \tag{3.5.1}$$

Then there exists a unique $\mathscr{S}$-morphism $f' : \mathfrak{Y} \to \mathfrak{X}_{(T/p)}$ (resp. $f' : \mathfrak{Y} \to \mathfrak{X}^{\sharp}_{(T/p)}$) such that $f = \psi \circ f'$. If $T \to \mathfrak{X}$ and f are moreover closed immersion, then so is $f' : \mathfrak{Y} \to \mathfrak{X}_{(T/p)}$.

Proof. – It suffices to prove the existence. The uniqueness follows from (i) and the universal property (ii). We first prove the case where $T \to \mathfrak{X}$ is a closed immersion and denote the associated ideal sheaf by $\mathscr{A}$.

In the first situation, we take $\mathfrak{X}_{(T/p)}$ to be the dilatation $\mathfrak{X}_{(\mathscr{A}/p)}$. Since $p \in \mathscr{A}$, the commutativity of the first diagram of (3.5.1) is equivalent to $\mathscr{A}\mathscr{O}_{\mathfrak{Y}} = p\mathscr{O}_{\mathfrak{Y}}$. To verify condition (ii), we can reduce to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ is affine and then to the case where $\mathfrak{Y} = \mathrm{Spf}(B)$ is affine and the morphism f is associated to an adic homomorphism $u : A \to B$. We take again the notation of 3.4. By (3.1.2), we have $(pB)^{\Delta} = (\mathfrak{a}B)^{\Delta}$. The open ideals of finite type pB and $\mathfrak{a}B$ are complete by ([7] III § 2.12 Cor. 1 of Prop. 16) and separated as submodules of B. We deduce that $pB = \mathfrak{a}B$ by taking $\Gamma(\mathfrak{Y}, -)$. Since B is flat over W, the homomorphism u extends uniquely to an A-homomorphism $A_0 \to B$ and then to an adic A-homomorphism $w : \widehat{A}_0 \to B$ by p-adic completion. In the first situation, we take for f' the morphism induced by w which is uniquely determined by f. If $u : A \to B$ is moreover surjective, then so is w.

In the second situation, the commutativity of the second diagram of (3.5.1) means the p-th power of every local section of $\mathscr{A}\mathscr{O}_{\mathfrak{Y}}/p\mathscr{O}_{\mathfrak{Y}}$ is zero in $\mathscr{O}_Y$, which is equivalent to the fact that $\mathscr{A}^{\sharp}\mathscr{O}_{\mathfrak{Y}} = p\mathscr{O}_{\mathfrak{Y}}$ (3.3). We take $\mathfrak{X}^{\sharp}_{(T/p)}$ to be the dilatation $\mathfrak{X}_{(\mathscr{A}^{\sharp}/p)}$. Then condition (ii) in this situation follows from the first situation.

In general, let $\mathfrak{U}$ be an open formal subscheme of $\mathfrak{X}$ such that $i(T) \subset \mathfrak{U}$ and that $T \to \mathfrak{U}$ is a closed immersion and, let $\mathscr{A}$ be the open ideal of finite type associated to $T \to \mathfrak{U}$. We take $\mathfrak{X}_{(T/p)}$ (resp. $\mathfrak{X}^{\sharp}_{(T/p)}$) to be the dilatation $\mathfrak{U}_{(\mathscr{A}/p)}$ (resp. $\mathfrak{U}_{(\mathscr{A}^{\sharp}/p)}$). For any morphism $f : \mathfrak{Y} \to \mathfrak{X}$ as in (ii), by (3.5.1), the morphism f factors through the open subscheme $\mathfrak{U}$ of $\mathfrak{X}$. Then the assertion follows from the case where $T \to \mathfrak{X}$ is a closed immersion. □

Remark 3.6. – The formal $\mathfrak{X}$-scheme $\mathfrak{X}_{(T/p)}$ (resp. $\mathfrak{X}^{\sharp}_{(T/p)}$ is the same as the Higgs envelope $\widetilde{R}_T(\mathfrak{X})$ (resp. $\widetilde{Q}_T(\mathfrak{X})$) introduced in ([**32**] page 6).

The next result shows that the constructions of 3.5 are compatible with étale localization.

Proposition 3.7 ([**32**] 1.1.3). – *Let $\mathfrak{X}$, $\mathfrak{Y}$ be two flat formal $\mathscr{S}$-schemes locally of finite type, $f : \mathfrak{X} \to \mathfrak{Y}$ an étale $\mathscr{S}$-morphism* ([**1**] 2.4.5). *Suppose that there exist a k-scheme T and two immersions $i : T \to \mathfrak{X}_1$, $j : T \to \mathfrak{Y}_1$ such that $j = f \circ i$. Then f induces canonical isomorphisms of the formal schemes*

$$\mathfrak{X}_{(T/p)} \xrightarrow{\sim} \mathfrak{Y}_{(T/p)}, \qquad \mathfrak{X}^{\sharp}_{(T/p)} \xrightarrow{\sim} \mathfrak{Y}^{\sharp}_{(T/p)}. \tag{3.7.1}$$

Proof. – By the universal property (3.5), the composition $\mathfrak{X}_{(T/p)} \to \mathfrak{X} \to \mathfrak{Y}$ induces a canonical $\mathfrak{Y}$-morphism $u : \mathfrak{X}_{(T/p)} \to \mathfrak{Y}_{(T/p)}$. We consider the commutative diagram

$$\begin{tikzcd}
(\mathfrak{Y}_{(T/p)})_1 \arrow[r] \arrow[d] & T \arrow[r] & \mathfrak{X}_1 \arrow[d] \\
\mathfrak{Y}_{(T/p)} \arrow[rr] & & \mathfrak{Y}.
\end{tikzcd} \tag{3.7.2}$$

Since $f : \mathfrak{X} \to \mathfrak{Y}$ is étale, the composition of the top horizontal morphisms and $\mathfrak{X}_1 \to \mathfrak{X}$ lifts uniquely to a $\mathfrak{Y}$-morphism $g : \mathfrak{Y}_{(T/p)} \to \mathfrak{X}$. By (3.7.2) and the universal property, g induces an $\mathfrak{X}$-morphism $v : \mathfrak{Y}_{(T/p)} \to \mathfrak{X}_{(T/p)}$. The following diagram

$$\begin{tikzcd}
\mathfrak{Y}_{(T/p)} \arrow[r, "v"] \arrow[dr, "g"'] & \mathfrak{X}_{(T/p)} \arrow[r, "u"] \arrow[d] & \mathfrak{Y}_{(T/p)} \arrow[d] \\
& \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y}
\end{tikzcd} \tag{3.7.3}$$

is commutative. By the universal property, we deduce that $u \circ v = \mathrm{id}$.

We consider the diagram

$$\begin{tikzcd}
\mathfrak{X}_{(T/p)} \arrow[r, "u"] \arrow[d, "\psi"'] & \mathfrak{Y}_{(T/p)} \arrow[d] \arrow[dl, "g"'] \\
\mathfrak{X} \arrow[r, "f"] & \mathfrak{Y},
\end{tikzcd} \tag{3.7.4}$$

where the lower triangle is commutative. Since ψ_1 and g_1 factor through T, we have $\psi_1 = (g \circ u)_1$. Since f is étale and the square of (3.7.4) commutes, there exists one and only one lifting of $(\mathfrak{X}_{(T/p)})_1 \to \mathfrak{X}_1$ to a $\mathfrak{Y}$-morphism $\mathfrak{X}_{(T/p)} \to \mathfrak{X}$. We deduce that $\psi = g \circ u$, i.e., the diagram (3.7.4) commutes. Then we have $\psi \circ v \circ u = \psi$. By the universal property, we deduce that $v \circ u = \mathrm{id}$. The first isomorphism follows.

The second isomorphism can be verified in the same way. $\square$

The next result shows that the constructions of 3.5 are compatible with flat base change.

PROPOSITION 3.8 ([**32**] 1.1.4). – *Let $\mathfrak{X}$, $\mathfrak{Y}$ be two flat formal $\mathscr{S}$-schemes locally of finite type, $f : \mathfrak{X} \to \mathfrak{Y}$ a flat $\mathscr{S}$-morphism, $T \to \mathfrak{Y}_1$ an immersion and $S = T \times_{\mathfrak{Y}} \mathfrak{X}$. Then f induces canonical isomorphisms of formal schemes*

$$\mathfrak{X}_{(S/p)} \xrightarrow{\sim} \mathfrak{Y}_{(T/p)} \times_{\mathfrak{Y}} \mathfrak{X}, \qquad \mathfrak{X}^{\sharp}_{(S/p)} \xrightarrow{\sim} \mathfrak{Y}^{\sharp}_{(T/p)} \times_{\mathfrak{Y}} \mathfrak{X}. \tag{3.8.1}$$

Proof. – By 3.5, we can reduce to the case where $T \to \mathfrak{Y}$ is a closed immersion. Let $\mathscr{A}$ (resp. $\mathscr{B}$) be the open ideal of finite type associated to the closed immersion $T \to \mathfrak{Y}$ (resp. $S \to \mathfrak{X}$). Put $\mathfrak{Z} = \mathfrak{Y}_{(\mathscr{B}/p)} \times_{\mathfrak{Y}} \mathfrak{X}$. By the universal property, the morphism f induces a $\mathfrak{Y}$-morphism $\mathfrak{X}_{(\mathscr{A}/p)} \to \mathfrak{Y}_{(\mathscr{B}/p)}$ and then an $\mathfrak{X}$-morphism $u : \mathfrak{X}_{(\mathscr{A}/p)} \to \mathfrak{Z}$.

Since f is flat, $\mathfrak{Z}$ is flat over $\mathscr{S}$. We have $\mathscr{A}\mathscr{O}_{\mathfrak{Z}} = \mathscr{B}\mathscr{O}_{\mathfrak{Z}} = p\mathscr{O}_{\mathfrak{Z}}$. By the universal property, we deduce an $\mathfrak{X}$-morphism $v : \mathfrak{Z} \to \mathfrak{X}_{(\mathscr{A}/p)}$. Since u, v are $\mathfrak{X}$-morphisms, we deduce that $u \circ v = \mathrm{id}$ and $v \circ u = \mathrm{id}$ by the universal property as in the proof of 3.7.

The second isomorphism can be verified in the same way. $\square$

CHAPTER 4

HOPF ALGEBRAS AND GROUPOIDS

4.1. – In this section, we review the notion of Hopf algebras and groupoids following [**4**] and the construction of certain Hopf algebras used in [**32**].

Let $(\mathscr{T}, A)$ be a ringed topos. For any A-bimodules (resp. A-bialgebras) M and N of $\mathscr{T}$, $M \otimes_A N$ denotes the tensor product of the right A-module M and the left A-module N, and we regard $M \otimes_A N$ as an A-bimodule (resp. A-bialgebra) through the left A-action on M and the right A-action on N.

Definition 4.2 ([**4**] II 1.1.2 and [**32**] 1.2.1). – Let $(\mathscr{T}, A)$ be a ringed topos. *A Hopf A-algebra* is the data of an A-bialgebra B and three ring homomorphisms

(4.2.1)
$$\delta : B \to B \otimes_A B \text{ (comultiplication)}, \quad \pi : B \to A \text{ (counit)}, \quad \sigma : B \to B \text{ (antipode)}$$

satisfying the following conditions.

(a) δ and π are A-bilinear and the following diagrams are commutative:

(4.2.2)
$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[d, "\delta"'] & B \otimes_A B \arrow[d, "\delta \otimes \mathrm{id}_B"] \\ B \otimes_A B \arrow[r, "\mathrm{id}_B \otimes \delta"] & B \otimes_A B \otimes_A B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "\mathrm{id}_B"] & B \otimes_A B \arrow[d, "\pi \cdot \mathrm{id}_B"] \\ B \otimes_A B \arrow[r, "\mathrm{id}_B \cdot \pi"] & B. \end{tikzcd}$$

(b) σ is a homomorphism of A-algebras for the left (resp. right) A-action on the source and the right (resp. left) A-action on the target, and satisfies $\sigma^2 = \mathrm{id}_B$, $\pi \circ \sigma = \pi$.

(c) The following diagrams are commutative:

(4.2.3)
$$\begin{tikzcd} B \arrow[r, "\pi"] \arrow[d, "\delta"'] & A \arrow[d, "d_1"] \\ B \otimes_A B \arrow[r, "\mathrm{id}_B \cdot \sigma"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\pi"] \arrow[d, "\delta"'] & A \arrow[d, "d_2"] \\ B \otimes_A B \arrow[r, "\sigma \cdot \mathrm{id}_B"] & B. \end{tikzcd}$$

where d_1 (resp. d_2) is the structural homomorphism of the left (resp. right) A-algebra B.

Such a data is also called an affine groupoid of $(\mathscr{T}, A)$ by Berthelot ([**4**] II 1.1.2).

DEFINITION 4.3 ([**4**] II 1.1.6). – Let $f : (\mathscr{T}', A') \to (\mathscr{T}, A)$ be a morphism of ringed topoi, B a Hopf A-algebra and B' a Hopf A'-algebra. A *homomorphism of Hopf algebras* is an A-bilinear homomorphism $B \to f_*(B')$ compatible with comultiplications, counits and antipodes.

4.4. – Let $(\mathscr{T}, A)$ be a ringed topos, M and N two A-bimodules. We denote by $\mathscr{H}om_{ll}(M, N)$ (resp. $\mathscr{H}om_{lr}(M, N)$) the sheaf of A-linear homomorphisms from the left A-module M to the left (resp. right) A-module N. We put $M^\vee = \mathscr{H}om_{ll}(M, A)$. The actions of A on M induces a natural A-bimodule structure on $M^\vee$. There exists a canonical A-bilinear morphism

$$M^\vee \otimes_A N^\vee \to (M \otimes_A N)^\vee \tag{4.4.1}$$

which sends $\varphi \otimes \psi$ to θ defined by $\theta(m \otimes n) = \varphi(m\psi(n))$ for all local sections m of M and n of N.

Let B be a Hopf A-algebra. By (4.4.1), we obtain a morphism

$$B^\vee \times B^\vee \to (B \otimes_A B)^\vee \xrightarrow{\delta^\vee} B^\vee. \tag{4.4.2}$$

Letting $\pi : B \to A$ be the unit element, the above morphism induces a non-commutative ring structure on $B^\vee$. The homomorphism $\pi : B \to A$ induces a ring homomorphism $i : A = A^\vee \to B^\vee$. In this way, we regard $B^\vee$ as a non commutative A-algebra.

A homomorphism of Hopf A-algebras $\varphi : B \to C$ induces a homomorphism of A-algebras $\varphi^\vee : C^\vee \to B^\vee$.

4.5. – Let $f : (\mathscr{T}', A') \to (\mathscr{T}, A)$ be a morphism of ringed topoi, $(B, \delta_B, \sigma_B, \pi_B)$ a Hopf A-algebra. Suppose that the left and the right A-algebra structures on B are equal. Then $(f^*(B), f^*(\delta_B), f^*(\pi_B)\ , f^*(\sigma_B))$ form a Hopf A'-algebra.

Let $(B', \delta_{B'}, \sigma_{B'}, \pi_{B'})$ be a Hopf A'-algebra and $u : B \to f_*(B')$ a homomorphism of Hopf algebras (4.3). By adjunction, we obtain homomorphisms of A'-algebras

$$u^\sharp : f^*(B) \to B', \qquad \widetilde{u} : f^*(B) \otimes_{A'} f^*(B) \to B' \otimes_{A'} B' \tag{4.5.1}$$

for the left A'-algebra structures on the targets. Then the diagrams

$$\begin{tikzcd} f^*(B) \arrow[r, "f^*(\delta_B)"] \arrow[d, "u^\sharp"'] & f^*(B) \otimes_{A'} f^*(B) \arrow[d, "\widetilde{u}"] \\ B' \arrow[r, "\delta_{B'}"] & B' \otimes_{A'} B' \end{tikzcd} \qquad \begin{tikzcd} f^*(B) \arrow[r, "f^*(\pi_B)"] \arrow[d, "u^\sharp"'] & A' \arrow[d, equal] \\ B' \arrow[r, "\pi_{B'}"] & A' \end{tikzcd} \tag{4.5.2}$$

are commutative. The restriction of $\widetilde{u}$ on $f^{-1}(B \otimes_A B)$ is given by $u^\sharp|_{f^{-1}(B)}$. In view of (4.4) and (4.5.2), the homomorphism $u^\sharp$ induces a homomorphism of A'-algebras $(u^\sharp)^\vee : (B')^\vee \to (f^*(B))^\vee$.

4.6. – In the following, we review the notion of *formal groupoid*. Let $\mathfrak{X}$ be an adic formal $\mathscr{S}$-scheme. For any integer $r \geqslant 1$, let $\mathfrak{X}^{r+1}$ be the fiber product of $(r+1)$-copies of $\mathfrak{X}$ over $\mathscr{S}$. We consider $\mathfrak{X}$ as an adic formal $(\mathfrak{X}^{r+1})$-scheme by the diagonal immersion $\Delta(r) : \mathfrak{X} \to \mathfrak{X}^{r+1}$.

We denote by $\tau : \mathfrak{X}^2 \to \mathfrak{X}^2$ the morphism which exchanges the factors of $\mathfrak{X}^2$, by $p_i : \mathfrak{X}^2 \to \mathfrak{X}$ $(i = 1, 2)$ the canonical projections and by $p_{ij} : \mathfrak{X}^3 \to \mathfrak{X}^2$ $(1 \leqslant i < j \leqslant 3)$ be the projection whose composition with p_1 (resp. p_2) is the projection $\mathfrak{X}^3 \to \mathfrak{X}$ on the i-th (resp. j-th) factor.

For any formal $\mathfrak{X}^2$-schemes $\mathfrak{Y}$ and $\mathfrak{Z}$, $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$ denotes the product of $\mathfrak{Y} \to \mathfrak{X}^2 \xrightarrow{p_2} \mathfrak{X}$ and $\mathfrak{Z} \to \mathfrak{X}^2 \xrightarrow{p_1} \mathfrak{X}$, and we regard $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$ as a formal $\mathfrak{X}^2$-scheme by the projection $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{X}^2 \times_{\mathfrak{X}} \mathfrak{X}^2 = \mathfrak{X}^3 \xrightarrow{p_{13}} \mathfrak{X}^2$.

Definition 4.7. – Let $\mathfrak{X}$ be an adic formal $\mathscr{S}$-scheme. *A formal $\mathfrak{X}$-groupoid over $\mathscr{S}$* is the data of an adic formal $\mathfrak{X}^2$-scheme $\mathfrak{G}$ and three adic $\mathscr{S}$-morphisms (4.6)

$$\alpha : \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \to \mathfrak{G}, \quad \iota : \mathfrak{X} \to \mathfrak{G}, \quad \eta : \mathfrak{G} \to \mathfrak{G} \tag{4.7.1}$$

satisfying the following conditions.

(i) α and ι are $\mathfrak{X}^2$-morphisms and the following diagrams are commutative:

$$\begin{tikzcd} \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[r, "\mathrm{id} \times \alpha"] \arrow[d, "\alpha \times \mathrm{id}"'] & \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[d, "\alpha"] \\ \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[r, "\alpha"] & \mathfrak{G} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{G} \arrow[r, "\iota \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \iota"'] \arrow[dr, "\mathrm{id}"] & \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[d, "\alpha"] \\ \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[r, "\alpha"] & \mathfrak{G}. \end{tikzcd} \tag{4.7.2}$$

(ii) The morphism η is compatible with $\tau : \mathfrak{X}^2 \to \mathfrak{X}^2$ and we have $\eta^2 = \mathrm{id}_{\mathfrak{G}}$, $\eta \circ \iota = \iota$.

(iii) Let q_1 (resp. q_2) be the projection $\mathfrak{G} \to \mathfrak{X}$ induced by p_1 (resp. p_2). The following diagrams are commutative:

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "q_1"] \arrow[d, "\mathrm{id} \times \eta"'] & \mathfrak{X} \arrow[d, "\iota"] \\ \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[r, "\alpha"] & \mathfrak{G} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{G} \arrow[r, "q_2"] \arrow[d, "\eta \times \mathrm{id}"'] & \mathfrak{X} \arrow[d, "\iota"] \\ \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[r, "\alpha"] & \mathfrak{G}. \end{tikzcd} \tag{4.7.3}$$

(iv) The morphism of underlying topological spaces $|\mathfrak{G}| \to |\mathfrak{X}^2|$ factors through $\Delta : |\mathfrak{X}| \to |\mathfrak{X}^2|$.

Let $\varpi : \mathfrak{G}_{\mathrm{zar}} \to \mathfrak{X}_{\mathrm{zar}}$ be the morphism of topoi induced by $\mathfrak{G}_{\mathrm{zar}} \to \mathfrak{X}^2_{\mathrm{zar}}$ and its factorization through Δ. Then we have $\varpi_*(\mathscr{O}_{\mathfrak{G}}) = q_{1*}(\mathscr{O}_{\mathfrak{G}}) = q_{2*}(\mathscr{O}_{\mathfrak{G}})$. In this way, we regard $\mathscr{O}_{\mathfrak{G}}$ as an $\mathscr{O}_{\mathfrak{X}}$-bialgebra of $\mathfrak{X}_{\mathrm{zar}}$. Then, the formal $\mathfrak{X}$-groupoid structure on $\mathfrak{G}$ induces a *formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure* on $\mathscr{O}_{\mathfrak{G}}$, that is for every $n \geqslant 1$, a Hopf $\mathscr{O}_{\mathfrak{X}_n}$-algebra structure on $\mathscr{O}_{\mathfrak{G}_n}$ which are compatible.

DEFINITION 4.8. – Let $\mathfrak{X}, \mathfrak{Y}$ be two adic formal $\mathscr{S}$-schemes, $f : \mathfrak{Y} \to \mathfrak{X}$ an $\mathscr{S}$-morphism, $(\mathfrak{G}, \alpha_{\mathfrak{G}}, \iota_{\mathfrak{G}}, \eta_{\mathfrak{G}})$ a formal $\mathfrak{X}$-groupoid and $(\mathfrak{H}, \alpha_{\mathfrak{H}}, \iota_{\mathfrak{H}}, \eta_{\mathfrak{H}})$ a formal $\mathfrak{Y}$-groupoid. *A morphism of formal groupoids above* f is an $\mathfrak{X}^2$-morphism $\varphi : \mathfrak{H} \to \mathfrak{G}$ compatible with α's, ι's and η's.

A morphism of formal groupoids $\varphi : \mathfrak{H} \to \mathfrak{G}$ induces a homomorphism of formal Hopf algebras $\mathscr{O}_{\mathfrak{G}} \to f_*(\mathscr{O}_{\mathfrak{H}})$, that is for every $n \geqslant 1$, $\mathscr{O}_{\mathfrak{G}_n} \to f_*(\mathscr{O}_{\mathfrak{H}_n})$ is a homomorphism of Hopf algebras.

4.9. – In the remainder of this section, $\mathfrak{X}$ denotes a smooth formal $\mathscr{S}$-scheme. We put $X = \mathfrak{X}_1$. We denote by $R_{\mathfrak{X}}(r)$ (resp. $Q_{\mathfrak{X}}(r)$) the dilatation $(\mathfrak{X}^{r+1})_{(X/p)}$ (resp. $(\mathfrak{X}^{r+1})^{\sharp}_{(X/p)}$) with respect to the diagonal immersion $X \to \mathfrak{X}^{r+1}$ (3.5). By 3.5(i), the canonical morphisms $(R_{\mathfrak{X}}(r))_1 \to X^{r+1}$ and $\underline{(Q_{\mathfrak{X}}(r))_1} \to (Q_{\mathfrak{X}}(r))_1 \to X^{r+1}$ factor through the diagonal immersion $X \to X^{r+1}$.

To simplify the notation, we put $R_{\mathfrak{X}} = R_{\mathfrak{X}}(1)$, $Q_{\mathfrak{X}} = Q_{\mathfrak{X}}(1)$, $\mathscr{R}_{\mathfrak{X}} = \mathscr{O}_{R_{\mathfrak{X}}}$ and $\mathscr{Q}_{\mathfrak{X}} = \mathscr{O}_{Q_{\mathfrak{X}}}$. Following Oyama [**32**], we will present the formal $\mathfrak{X}$-groupoid structure on $R_{\mathfrak{X}}$ and $Q_{\mathfrak{X}}$.

Our notations are different to those of [**32**]. In ([**32**] 1.2), $\widetilde{R}_{\mathfrak{X}}$ (resp. $\widetilde{Q}_{\mathfrak{X}}$) denotes the formal $\mathfrak{X}^2$-scheme constructed by dilatation and $R_{\mathfrak{X}}$ (resp. $Q_{\mathfrak{X}}$) denotes its reduction modulo p.

PROPOSITION 4.10 ([**32**] 1.2.5 and 1.2.6). – *Let r, r' be two integers $\geqslant 1$. There exists canonical isomorphisms*

$$R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r') \xrightarrow{\sim} R_{\mathfrak{X}}(r + r'), \qquad Q_{\mathfrak{X}}(r) \times_{\mathfrak{X}} Q_{\mathfrak{X}}(r') \xrightarrow{\sim} Q_{\mathfrak{X}}(r + r'), \tag{4.10.1}$$

where the projections $R_{\mathfrak{X}}(r) \to \mathfrak{X}$ and $Q_{\mathfrak{X}}(r) \to \mathfrak{X}$ (resp. $R_{\mathfrak{X}}(r') \to \mathfrak{X}$ and $Q_{\mathfrak{X}}(r') \to \mathfrak{X}$) are induced by the projection $\mathfrak{X}^{r+1} \to \mathfrak{X}$ on the last factor (resp. $\mathfrak{X}^{r'+1} \to \mathfrak{X}$ on the first factor).

Proof. – By 4.9, we have a commutative diagram

$$\begin{tikzcd}
(R_{\mathfrak{X}}(r))_1 \times_X (R_{\mathfrak{X}}(r'))_1 \arrow[r] \arrow[d] & R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r') \arrow[d] \\
X \arrow[r] & \mathfrak{X}^{r+1} \times_{\mathfrak{X}} \mathfrak{X}^{r'+1}.
\end{tikzcd}$$

By the universality of $R_{\mathfrak{X}}(r + r')$ (3.5), we obtain an $(\mathfrak{X}^{r+r'+1})$-morphism

$$\varphi : R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r') \to R_{\mathfrak{X}}(r + r'). \tag{4.10.2}$$

On the other hand, by the universal property of $R_{\mathfrak{X}}(r)$ and $R_{\mathfrak{X}}(r')$, the projection $\mathfrak{X}^{r+r'+1} \to \mathfrak{X}^{r+1}$ on the first $(r+1)$-factors (resp. $\mathfrak{X}^{r+r'+1} \to \mathfrak{X}^{r'+1}$ on the last $(r'+1)$-factors) induces a morphism $R_{\mathfrak{X}}(r + r') \to R_{\mathfrak{X}}(r)$ (resp. $R_{\mathfrak{X}}(r + r') \to R_{\mathfrak{X}}(r')$) and hence an $(\mathfrak{X}^{r+r'+1})$-morphism

$$\psi : R_{\mathfrak{X}}(r + r') \to R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r'). \tag{4.10.3}$$

The composition $R_{\mathfrak{X}}(r+r') \xrightarrow{\psi} R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r') \xrightarrow{\varphi} R_{\mathfrak{X}}(r+r') \to \mathfrak{X}^{r+r'+1}$ is the canonical morphism $R_{\mathfrak{X}}(r+r') \to \mathfrak{X}^{r+r'+1}$. By the universal property of $R_{\mathfrak{X}}(r+r')$, we have $\varphi \circ \psi = \mathrm{id}$. Let q_1 (resp. q_2) denote the projection on the first (resp. second) factor of $R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r')$. We consider the commutative diagram

(4.10.4)
$$\begin{tikzcd}
R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r') \arrow[r, "\varphi"] \arrow[dr] & R_{\mathfrak{X}}(r+r') \arrow[r, "\psi"] \arrow[d] \arrow[dr] & R_{\mathfrak{X}}(r) \times_{\mathfrak{X}} R_{\mathfrak{X}}(r') \arrow[d, "q_1"] \\
& \mathfrak{X}^{r+r'+1} \arrow[dr] & R_{\mathfrak{X}}(r) \arrow[d] \\
& & \mathfrak{X}^{r+1}.
\end{tikzcd}$$

By the universal property of $R_{\mathfrak{X}}(r)$, we see that $q_1 \circ \psi \circ \varphi$ is equal to q_1. Similarly, we verify that $q_2 \circ \psi \circ \varphi$ is equal to q_2. Hence, we have $\psi \circ \varphi = \mathrm{id}$.

Since X is reduced, we have $\underline{X} = X$. We have a canonical morphism

$$\underline{(Q_{\mathfrak{X}}(r))_1 \times_X (Q_{\mathfrak{X}}(r'))_1} \to \underline{(Q_{\mathfrak{X}}(r))_1} \times_X \underline{(Q_{\mathfrak{X}}(r'))_1}$$

(2.3.1) and the following commutative diagram (4.9)

$$\begin{tikzcd}
\underline{(Q_{\mathfrak{X}}(r))_1 \times_X (Q_{\mathfrak{X}}(r'))_1} \arrow[r] & \underline{(Q_{\mathfrak{X}}(r))_1} \times_X \underline{(Q_{\mathfrak{X}}(r'))_1} \arrow[r] \arrow[d] & Q_{\mathfrak{X}}(r) \times_{\mathfrak{X}} Q_{\mathfrak{X}}(r') \arrow[d] \\
& X \arrow[r] & \mathfrak{X}^{r+1} \times_{\mathfrak{X}} \mathfrak{X}^{r'+1}.
\end{tikzcd}$$

By the universal property of $Q_{\mathfrak{X}}(r+r')$ (3.5), we obtain an $(\mathfrak{X}^{r+r'+1})$-morphism

(4.10.5)
$$Q_{\mathfrak{X}}(r) \times_{\mathfrak{X}} Q_{\mathfrak{X}}(r') \to Q_{\mathfrak{X}}(r+r').$$

By repeating the proof for $R_{\mathfrak{X}}(r+r')$, we verify that the above morphism is an isomorphism. □

PROPOSITION 4.11. – *The formal $\mathfrak{X}^2$-scheme $R_{\mathfrak{X}}$ (resp. $Q_{\mathfrak{X}}$) has a natural formal $\mathfrak{X}$-groupoid structure.*

Proof. – We follow the proof of ([**32**] 1.2.7) where the author proves the analogous results for the X^2-schemes $R_{\mathfrak{X},1}$ and $Q_{\mathfrak{X},1}$. By 4.9, the morphism of the underlying topological spaces $|R_{\mathfrak{X}}| \to |\mathfrak{X}^2|$ (resp. $|Q_{\mathfrak{X}}| \to |\mathfrak{X}^2|$) factors through the diagonal immersion $|\mathfrak{X}| \to |\mathfrak{X}^2|$. We consider the following commutative diagram (4.9):

(4.11.1)
$$\begin{tikzcd} (R_{\mathfrak{X}}(2))_1 \arrow[r, hook] \arrow[dd] & R_{\mathfrak{X}}(2) \arrow[d] \\ & \mathfrak{X}^3 \arrow[d, "p_{13}"] \\ X \arrow[r] & \mathfrak{X}^2. \end{tikzcd}$$

By the universal property of $R_{\mathfrak{X}}$ (3.5), we deduce an adic $\mathfrak{X}^2$-morphism

(4.11.2)
$$\alpha_R : R_{\mathfrak{X}}(2) \to R_{\mathfrak{X}}.$$

We identify $R_{\mathfrak{X}}(2)$ (resp. $R_{\mathfrak{X}}(3)$) and $R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}}$ (resp. $R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}}$) by 4.10. The diagrams

$$\begin{tikzcd} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\mathrm{id} \times \alpha_R"] \arrow[d] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^4 \arrow[r, "p_{124}"] & \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \end{tikzcd}$$

$$\begin{tikzcd} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R \times \mathrm{id}"] \arrow[d] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^4 \arrow[r, "p_{134}"] & \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \end{tikzcd}$$

are commutative and the compositions of the lower horizontal arrows coincide. By the universal property of $R_{\mathfrak{X}}$, we deduce that $\alpha_R \circ (\mathrm{id} \times \alpha_R) = \alpha_R \circ (\alpha_R \times \mathrm{id})$.

For any integer $r \geqslant 1$, we consider the following commutative diagram:

(4.11.3)
$$\begin{tikzcd} & \mathfrak{X} \arrow[d, "\Delta(r)"] \\ X \arrow[ur, hook] \arrow[r] & \mathfrak{X}^{r+1}. \end{tikzcd}$$

By the universal property of $R_{\mathfrak{X}}(r)$, we deduce a $\mathfrak{X}^{r+1}$-morphism

(4.11.4)
$$\iota_R(r) : \mathfrak{X} \to R_{\mathfrak{X}}(r).$$

In view of 3.5, since $\mathfrak{X} \to \mathfrak{X}^{r+1}$ is an immersion, $\iota_R(r)$ is a closed immersion. When $r = 1$, the diagrams

$$\begin{tikzcd} R_{\mathfrak{X}} \arrow[r, "\mathrm{id} \times \iota_R"] \arrow[d] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^2 \arrow[r, "(p_1, p_2, p_2)"] & \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \end{tikzcd} \qquad \begin{tikzcd} R_{\mathfrak{X}} \arrow[r, "\iota_R \times \mathrm{id}"] \arrow[d] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^2 \arrow[r, "(p_1, p_1, p_2)"] & \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \end{tikzcd}$$

are commutative and the compositions of the lower horizontal arrows are equal to $\mathrm{id}_{\mathfrak{X}^2}$. By the universal property of $R_{\mathfrak{X}}$, we deduce that $\alpha_R \circ (\mathrm{id} \times \iota_R) = \alpha_R \circ (\iota_R \times \mathrm{id}) = \mathrm{id}$.

We consider the following commutative diagram (4.6):

(4.11.5)
$$\begin{tikzcd} R_{\mathfrak{X},1} \arrow[r, hook] \arrow[dd] & R_{\mathfrak{X}} \arrow[d] \\ & \mathfrak{X}^2 \arrow[d, "\tau"] \\ X \arrow[r] & \mathfrak{X}^2. \end{tikzcd}$$

By the universal property of $R_{\mathfrak{X}}$, we deduce an adic morphism

(4.11.6)
$$\eta_R : R_{\mathfrak{X}} \to R_{\mathfrak{X}}.$$

Since $\tau \circ \Delta = \Delta$, we deduce that $\eta_R \circ \iota_R = \iota_R$ by the universal property of $R_{\mathfrak{X}}$. By construction, η_R satisfies the condition 4.7(ii). The diagrams

(4.11.7)
$$\begin{tikzcd} R_{\mathfrak{X}} \arrow[r, "\mathrm{id} \times \eta_R"] \arrow[d] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^2 \arrow[r, "(p_1,p_2,p_1)"] & \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \end{tikzcd} \qquad \begin{tikzcd} R_{\mathfrak{X}} \arrow[r, "q_1"] \arrow[d] & \mathfrak{X} \arrow[r, "\iota_R"] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^2 \arrow[r, "p_1"] & \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X}^2 \end{tikzcd}$$

are commutative and the compositions of the lower horizontal arrows coincide. By the universal property of $R_{\mathfrak{X}}$, we deduce that $\alpha_R \circ (\mathrm{id} \times \eta_R) = \iota_R \circ q_1$. We prove the equality $\alpha_R \circ (\eta_R \times \mathrm{id}) = \iota_R \circ q_2$ in the same way.

The proposition for $\mathscr{Q}_{\mathfrak{X}}$ can be verified in exactly the same way using 4.9 and 4.10. □

4.12. – We put $\mathfrak{Y} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$ and we present local descriptions for $\mathscr{R}_{\mathfrak{Y}}$ and $\mathscr{Q}_{\mathfrak{Y}}$ (4.9). Put $\xi_i = 1 \otimes T_i - T_i \otimes 1 \in \mathscr{O}_{\mathfrak{Y}^2}$. The ideal $\mathscr{A}$, associated to the diagonal closed immersion $\mathfrak{Y}_1 \to \mathfrak{Y}^2$, is generated by $p, \xi_1, \xi_2, \dots, \xi_d$. The algebra $W[1 \otimes T_1, \dots, 1 \otimes T_d, T_1 \otimes 1, \dots, T_d \otimes 1, x_1, \dots, x_d]/(\xi_i - px_i)_{1 \leqslant i \leqslant d}$ is free over W. Hence, we have an isomorphism of $\mathscr{O}_{\mathfrak{Y}^2}$-algebras (3.4.1), 4.9

(4.12.1)
$$\mathscr{O}_{\mathfrak{Y}^2}\Big\{\frac{\xi_1}{p}, \dots, \frac{\xi_d}{p}\Big\} = \frac{\mathscr{O}_{\mathfrak{Y}^2}\{x_1, \dots, x_d\}}{(\xi_i - px_i)_{1 \leqslant i \leqslant d}} \xrightarrow{\sim} \mathscr{R}_{\mathfrak{Y}}.$$

By 3.3, the ideal $\mathscr{A}^\sharp$ is generated by $p, \xi_1^p, \dots, \xi_d^p$. The algebra $W[1 \otimes T_1, \dots, 1 \otimes T_d, T_1 \otimes 1, \dots, T_d \otimes 1, x_1, \dots, x_d]/(\xi_i^p - px_i)_{1 \leqslant i \leqslant d}$ is free over W. Hence, we have an isomorphism of $\mathscr{O}_{\mathfrak{Y}^2}$-algebras (3.4.1)

(4.12.2)
$$\mathscr{O}_{\mathfrak{Y}^2}\Big\{\frac{\xi_1^p}{p}, \dots, \frac{\xi_d^p}{p}\Big\} = \frac{\mathscr{O}_{\mathfrak{Y}^2}\{x_1, \dots, x_d\}}{(\xi_i^p - px_i)_{1 \leqslant i \leqslant d}} \xrightarrow{\sim} \mathscr{Q}_{\mathfrak{Y}}.$$

Lemma 4.13. – *Let d be the relative dimension of $\mathfrak{X}$ over $\mathscr{S}$ and $\widehat{\mathbb{A}}^d_{\mathfrak{X}}$ the d-dimensional affine space. Assume that there exists an étale $\mathscr{S}$-morphism $f:\mathfrak{X}\to\mathfrak{Y}=\mathrm{Spf}(\mathrm{W}\{T_1,\dots,T_d\})$. Considering $R_{\mathfrak{X}}$ as a formal $\mathfrak{X}$-scheme via the morphism $q_1: R_{\mathfrak{X}}\to\mathfrak{X}$ (resp. $q_2: R_{\mathfrak{X}}\to\mathfrak{X}$), then f induces an isomorphism over $\mathfrak{X}$:*

$$\lambda: R_{\mathfrak{X}} \xrightarrow{\sim} \widehat{\mathbb{A}}^d_{\mathfrak{X}}, \tag{4.13.1}$$

such that $\lambda\circ\iota_R:\mathfrak{X}\to\widehat{\mathbb{A}}^d_{\mathfrak{X}}$ is the closed immersion associated to the zero section of $\widehat{\mathbb{A}}^d_{\mathfrak{X}}$.

Proof. – We follow the proof of ([**32**] 1.1.8) and we first prove the assertions for $\mathfrak{Y}$. Observe that q_1 and q_2 are affine. For any $1\leqslant i\leqslant d$, we have $1\otimes T_i = p(\frac{\xi_i}{p})+T_i\otimes 1$ in $\mathscr{R}_{\mathfrak{Y}}$. By (4.12.1), we deduce the isomorphisms

$$\mathscr{O}_{\mathfrak{Y}}\{x_1,\dots,x_d\}\xrightarrow{\sim} q_{1*}(\mathscr{R}_{\mathfrak{Y}}),\qquad \mathscr{O}_{\mathfrak{Y}}\{x_1,\dots,x_d\}\xrightarrow{\sim} q_{2*}(\mathscr{R}_{\mathfrak{Y}}) \tag{4.13.2}$$

where x_i is sent to $\frac{\xi_i}{p}$ in both cases. The isomorphisms (4.13.1) for $\mathfrak{Y}$ follows. We put $Y=\mathfrak{Y}_1$ and we consider the following commutative diagram

$$\begin{tikzcd} & \mathfrak{X}^2 \arrow[d, "\mathrm{id}\times f"] \\ X \arrow[ur, "\Delta"] \arrow[r] \arrow[d, "f_1"'] & \mathfrak{X}\times_{\mathscr{S}}\mathfrak{Y} \arrow[d, "f\times\mathrm{id}"] \\ Y \arrow[r, "\Delta"] & \mathfrak{Y}^2, \end{tikzcd} \tag{4.13.3}$$

where the square is Cartesian. By 3.7 and 3.8, we deduce an isomorphism

$$R_{\mathfrak{X}}\xrightarrow{\sim} R_{\mathfrak{Y}}\times_{\mathfrak{Y}^2}(\mathfrak{X}\times_{\mathscr{S}}\mathfrak{Y}) = R_{\mathfrak{Y}}\times_{q_1,\mathfrak{Y}}\mathfrak{X}. \tag{4.13.4}$$

Considering $R_{\mathfrak{X}}$ as a formal $\mathfrak{X}$-scheme via q_1, the isomorphism (4.13.1) follows from that of $R_{\mathfrak{Y}}$. The another isomorphism can be verified in the same way. In view of the construction of ι_R (4.11.4), the composition $\lambda\circ\iota_R$ corresponds to the zero section. □

Corollary 4.14. – *Keep assumptions of 4.13 and consider $\frac{\xi_i}{p}$'s as sections of $\mathscr{R}_{\mathfrak{X}}$. We have the following isomorphisms of $\mathscr{O}_{\mathfrak{X}}$-algebras*

$$\mathscr{O}_{\mathfrak{X}}\{x_1,\dots,x_d\}\xrightarrow{\sim} q_{1*}(\mathscr{R}_{\mathfrak{X}}),\qquad \mathscr{O}_{\mathfrak{X}}\{x_1,\dots,x_d\}\xrightarrow{\sim} q_{2*}(\mathscr{R}_{\mathfrak{X}}) \tag{4.14.1}$$

where x_i is sent to $\frac{\xi_i}{p}$ in both cases.

4.15. – We put $\mathfrak{Y}=\mathrm{Spf}(\mathrm{W}\{T_1,\dots,T_d\})$. By (4.12.2), we have following isomorphisms (4.15.1)

$$\frac{\mathscr{O}_{\mathfrak{Y}}\{x_1,\dots,x_d,y_1,\dots,y_d\}}{(y_i^p-px_i)_{1\leqslant i\leqslant d}}\xrightarrow{\sim} q_{1*}(\mathscr{Q}_{\mathfrak{Y}}),\qquad \frac{\mathscr{O}_{\mathfrak{Y}}\{x_1,\dots,x_d,y_1,\dots,y_d\}}{(y_i^p-px_i)_{1\leqslant i\leqslant d}}\xrightarrow{\sim} q_{2*}(\mathscr{Q}_{\mathfrak{Y}})$$

where x_i is sent to $\frac{\xi_i^p}{p}$ and y_i is sent to ξ_i in both cases.

Assume that there exists an étale $\mathscr{S}$-morphism $f:\mathfrak{X}\to\mathfrak{Y}=\mathrm{Spf}(\mathrm{W}\{T_1,\dots,T_d\})$. We consider the ξ_i's and $\frac{\xi_i^p}{p}$'s as sections of $\mathscr{Q}_{\mathfrak{X}}$. By 3.7, 3.8 and (4.13.3), we deduce the following isomorphisms

(4.15.2)

$$\frac{\mathscr{O}_{\mathfrak{X}}\{x_1,\dots,x_d,y_1,\dots,y_d\}}{(y_i^p-px_i)_{1\leqslant i\leqslant d}}\xrightarrow{\sim}q_{1*}(\mathscr{Q}_{\mathfrak{X}}),\qquad \frac{\mathscr{O}_{\mathfrak{X}}\{x_1,\dots,x_d,y_1,\dots,y_d\}}{(y_i^p-px_i)_{1\leqslant i\leqslant d}}\xrightarrow{\sim}q_{2*}(\mathscr{Q}_{\mathfrak{X}})$$

where x_i is sent to $\frac{\xi_i^p}{p}$ and y_i is sent to ξ_i in both cases.

4.16. – Let n be an integer $\geqslant 1$. We describe the Hopf algebra structure of $\mathscr{R}_{\mathfrak{X},n}$ and $\mathscr{Q}_{\mathfrak{X},n}$ in terms of a system of local coordinates. Keep the assumption and notation of 4.13. The homomorphism $\mathscr{O}_{\mathfrak{X}_n^2}\to\mathscr{O}_{\mathfrak{X}_n^2}\otimes_{\mathscr{O}_{\mathfrak{X}_n}}\mathscr{O}_{\mathfrak{X}_n^2}$ induced by $p_{13}:\mathfrak{X}^3\to\mathfrak{X}^2$ sends ξ_i to $1\otimes\xi_i+\xi_i\otimes 1$. The homomorphism $\mathscr{O}_{\mathfrak{X}_n^2}\to\mathscr{O}_{\mathfrak{X}_n^2}$ induced by $\tau:\mathfrak{X}^2\to\mathfrak{X}^2$, sends ξ_i to $-\xi_i$. In view of the proof of 4.11, we have following descriptions:

(4.16.1)

$$\begin{cases}\delta:\mathscr{R}_{\mathfrak{X},n}\to\mathscr{R}_{\mathfrak{X},n}\otimes_{\mathscr{O}_{\mathfrak{X}_n}}\mathscr{R}_{\mathfrak{X},n} & \frac{\xi_i}{p}\mapsto 1\otimes\frac{\xi_i}{p}+\frac{\xi_i}{p}\otimes 1\\ \sigma:\mathscr{R}_{\mathfrak{X},n}\to\mathscr{R}_{\mathfrak{X},n} & \frac{\xi_i}{p}\mapsto -\frac{\xi_i}{p}\\ \pi:\mathscr{R}_{\mathfrak{X},n}\to\mathscr{O}_{\mathfrak{X}_n} & \frac{\xi_i}{p}\mapsto 0\end{cases}$$

(4.16.2)

$$\begin{cases}\delta:\mathscr{Q}_{\mathfrak{X},n}\to\mathscr{Q}_{\mathfrak{X},n}\otimes_{\mathscr{O}_{\mathfrak{X}_n}}\mathscr{Q}_{\mathfrak{X},n} & \xi_i\mapsto 1\otimes\xi_i+\xi_i\otimes 1\\ & \frac{\xi_i^p}{p}\mapsto 1\otimes\frac{\xi_i^p}{p}+\sum_{j=1}^{p-1}\frac{(p-1)!}{j!(p-j)!}\xi_i^j\otimes\xi_i^{p-j}+\frac{\xi_i^p}{p}\otimes 1\\ \sigma:\mathscr{Q}_{\mathfrak{X},n}\to\mathscr{Q}_{\mathfrak{X},n} & \xi_i\mapsto -\xi_i,\quad \frac{\xi_i^p}{p}\mapsto\frac{(-\xi_i)^p}{p}\\ \pi:\mathscr{Q}_{\mathfrak{X},n}\to\mathscr{O}_{\mathfrak{X}_n} & \xi_i\mapsto 0,\quad \frac{\xi_i^p}{p}\mapsto 0\end{cases}$$

CHAPTER 5

CONNECTIONS AND STRATIFICATIONS

5.1. – Let S be a scheme, $f : X \to S$ a smooth morphism, M an $\mathscr{O}_X$-module and $\lambda \in \Gamma(S, \mathscr{O}_S)$. We say (abusively) that a morphism of $\mathscr{O}_X$-modules $u : M \to N$ is *$\mathscr{O}_S$-linear* if it is $f^{-1}(\mathscr{O}_S)$-linear. *A λ-connection on M* relative to S is an $\mathscr{O}_S$-linear morphism

$$\nabla : M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \tag{5.1.1}$$

such that for every local sections f of $\mathscr{O}_X$ and e of M, we have $\nabla(fe) = \lambda e \otimes d(f) + f\nabla(e)$. We will simply call ∇ a λ-connection on M when there is no risk of confusion. For any $q \geqslant 0$, the morphism ∇ extends to a unique $\mathscr{O}_S$-linear morphism

$$\nabla_q : M \otimes_{\mathscr{O}_X} \Omega^q_{X/S} \to M \otimes_{\mathscr{O}_X} \Omega^{q+1}_{X/S}, \tag{5.1.2}$$

such that for every local sections ω of $\Omega^q_{X/S}$ and e of M, we have $\nabla_q(e \otimes \omega) = \lambda e \otimes d(\omega) + \nabla(e) \wedge \omega$. The composition $\nabla_1 \circ \nabla$ is $\mathscr{O}_X$-linear. We say that ∇ is *integrable* if $\nabla_1 \circ \nabla = 0$.

Let (M, ∇) and (M', ∇') be two $\mathscr{O}_X$-modules with λ-connection. A morphism from (M, ∇) to (M', ∇') is an $\mathscr{O}_X$-linear morphism $u : M \to M'$ such that $(\mathrm{id} \otimes u) \circ \nabla = \nabla' \circ u$.

Classically, 1-connections are called *connections* and integrable 0-connections are called *Higgs fields. A Higgs module* is an $\mathscr{O}_X$-module equipped with a Higgs field.

We denote by $\mathrm{MIC}(X/S)$ (resp. λ-$\mathrm{MIC}(X/S)$, resp. $\mathrm{HIG}(X/S)$) the category of $\mathscr{O}_X$-modules with integrable connection (resp. λ-connection, resp. Higgs field) relative to S.

Let (M, ∇) be an object of λ-$\mathrm{MIC}(X/S)$. We deduce that $\nabla_{q+1} \circ \nabla_q = 0$ for all $q \geqslant 0$. Then we can associate to (M, ∇) a *λ-de Rham complex*:

$$M \xrightarrow{\nabla} M \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \xrightarrow{\nabla_1} M \otimes_{\mathscr{O}_X} \Omega^2_{X/S} \xrightarrow{\nabla_2} \cdots . \tag{5.1.3}$$

Classically, 0-de Rham complexes are called *Dolbeault complexes* in ([**34**] p. 24, [**2**] I.2.3) or *Higgs complexes* in ([**31**] p. 2).

5.2. – Let d, n be integers $\geqslant 1$, S a $(\mathbb{Z}/p^n\mathbb{Z})$-scheme, $f : X \to \mathbb{A}^d_S = \mathrm{Spec}(\mathscr{O}_S[T_1, \dots, T_d])$ an étale S-morphism. For any $1 \leqslant i \leqslant d$, we denote by t_i the image of T_i in $\mathscr{O}_X$. Let m be an integer $\geqslant 0$ and (M, ∇) an $\mathscr{O}_X$-module with integrable p^m-connection relative to S. There are $\mathscr{O}_S$-linear endomorphisms $\nabla_{\partial_1}, \dots, \nabla_{\partial_d}$ of M such that for every local section e of M, we have

$$\nabla(e) = \sum_{i=1}^{d} \nabla_{\partial_i}(e) \otimes dt_i. \tag{5.2.1}$$

Since ∇ is integrable, we have $\nabla_{\partial_i} \circ \nabla_{\partial_j} = \nabla_{\partial_j} \circ \nabla_{\partial_i}$ for all $1 \leqslant i, j \leqslant d$. Therefore, for every multi-index $I = (i_1, \dots, i_d) \in \mathbb{N}^d$, the endomorphism $\nabla_{\partial^I} = \prod_{j=1}^{d} (\nabla_{\partial_j})^{i_j}$ is well-defined.

Following ([**5**] 4.10, [**33**] Definition 1.5), we say that (M, ∇) *is quasi-nilpotent with respect to* f if, for any open subscheme U of X and any section $e \in M(U)$, there exists a Zariski covering $\{U_j \to U\}_{j \in J}$ and a family of integers $\{N_j\}_{j \in J}$ such that $\nabla_{\partial^I}(e|_{U_j}) = 0$ for all $j \in J$ and $I \in \mathbb{N}^d$ with $|I| \geqslant N_j$.

If $f' : X \to \mathbb{A}^d_S$ is another étale S-morphism, (M, ∇) is quasi-nilpotent with respect to f if and only if it is quasi-nilpotent with respect to f' ([**5**] 4.13, [**33**] Lemma 1.6). Note that this result requires that $p^n \mathscr{O}_S = 0$ for some $n > 0$.

Definition 5.3 ([**5**] 4.13; [**33**] Definition 1.8). – Let n be an integer $\geqslant 1$, S a $(\mathbb{Z}/p^n\mathbb{Z})$-scheme, X a smooth S-scheme and (M, ∇) an $\mathscr{O}_X$-module with integrable p^m-connection relative to S. We say that (M, ∇) is *quasi-nilpotent* if for any point x of X, there exists a Zariski neighborhood U of x in X and an étale S-morphism $f : U \to \mathbb{A}^d_S$ such that $(M, \nabla)|_U$ is quasi-nilpotent with respect to f (5.2).

We denote by $\mathrm{MIC}^{\mathrm{qn}}(X/S)$ (resp. λ-$\mathrm{MIC}^{\mathrm{qn}}(X/S)$) the full subcategory of $\mathrm{MIC}(X/S)$ (resp. λ-$\mathrm{MIC}(X/S)$) consisting of the quasi-nilpotent objects.

Definition 5.4. – Let $(\mathscr{T}, A)$ be a ringed topos, (B, δ, π, σ) a Hopf A-algebra (4.2) and M an A-module. *A B-stratification on M* is a B-linear isomorphism $\varepsilon : B \otimes_A M \xrightarrow{\sim} M \otimes_A B$ (4.1) such that:

(i) $\pi^*(\varepsilon) = \mathrm{id}_M$.

(ii) (cocycle condition) The following diagram is commutative:

$$\begin{tikzcd} B \otimes_A B \otimes_A M \arrow[rr, "\delta^*(\varepsilon)"] \arrow[dr, "\mathrm{id}_B \otimes \varepsilon"'] & & M \otimes_A B \otimes_A B \\ & B \otimes_A M \otimes_A B. \arrow[ur, "\varepsilon \otimes \mathrm{id}_B"'] & \end{tikzcd} \tag{5.4.1}$$

Given two A-modules with B-stratification (M_1, ε_1) and (M_2, ε_2), a morphism from (M_1, ε_1) to (M_2, ε_2) is an A-linear morphism $f : M_1 \to M_2$ compatible with ε_1 and ε_2. The A-module $M_1 \otimes_A M_2$ has a canonical B-stratification

$$B \otimes_A M_1 \otimes_A M_2 \xrightarrow{\varepsilon_1 \otimes \mathrm{id}_{M_2}} M_1 \otimes_A B \otimes_A M_2 \xrightarrow{\mathrm{id}_{M_1} \otimes \varepsilon_2} M_1 \otimes_A M_2 \otimes_A B. \tag{5.4.2}$$

The above stratification on the tensor product makes the category of A-modules with B-stratification into a tensor category.

Let n be an integer $\geqslant 1$, $\mathfrak{X}$ an adic formal $\mathscr{S}$-scheme, M an $\mathscr{O}_{\mathfrak{X}_n}$-module and $\mathfrak{G}$ a formal $\mathfrak{X}$-groupoid over $\mathscr{S}$ (4.7). We call abusively *$\mathscr{O}_{\mathfrak{G}}$-stratification on M* instead of $\mathscr{O}_{\mathfrak{G}_n}$-stratification on M.

We have a simpler description of a stratification.

LEMMA 5.5 ([**32**] 1.2.4). – *Let $(\mathscr{T}, A)$ be a ringed topos, B a Hopf A-algebra and M an A-module. A B-stratification on M is equivalent to an A-linear morphism $\theta : M \to M \otimes_A B$ for the right A-action on the target satisfying the following conditions:*

(i) *The composition* $(\mathrm{id}_M \otimes \pi) \circ \theta : M \to M \otimes_A B \to M$ *is the identity morphism.*

(ii) *The following diagram is commutative*

(5.5.1)
$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d, "\theta"'] & M \otimes_A B \arrow[d, "\theta \otimes \mathrm{id}_B"] \\ M \otimes_A B \arrow[r, "\mathrm{id}_M \otimes \delta"] & M \otimes_A B \otimes_A B. \end{tikzcd}$$

Let $\theta : M \to M \otimes_A B$ be an A-linear morphism satisfying the conditions of 5.5 and

$$\alpha(\theta) : B^\vee \otimes_A M \to M, \quad \varphi \otimes m = (\mathrm{id} \otimes \varphi)\big(\theta(m)\big)$$

the associated A-linear morphism. In view of conditions (i-ii) of 5.5, the morphism $\alpha(\theta)$ makes M into a left $B^\vee$-module (cf. [**32**] Proof of 1.2.9 page 18 for details).

5.6. – Let $f : (\mathscr{T}', A') \to (\mathscr{T}, A)$ be a morphism of ringed topoi, B a Hopf A-algebra and B' a Hopf A'-algebra. A homomorphism of Hopf algebras $B \to f_*(B')$ (4.3) induces a functor (5.4) (cf. [**4**] II 1.2.5):

(5.6.1)
$$\begin{aligned} \left\{ \begin{array}{c} A\text{-modules} \\ \text{with } B\text{-stratification} \end{array} \right\} &\to \left\{ \begin{array}{c} A'\text{-modules} \\ \text{with } B'\text{-stratification} \end{array} \right\} \\ (M, \varepsilon) &\mapsto (f^*(M), f^{-1}(\varepsilon) \otimes_{f^{-1}(B)} B'). \end{aligned}$$

5.7. – In the remainder of this section, $\mathfrak{X}$ denotes a smooth formal $\mathscr{S}$-scheme. For any integer $n \geqslant 1$, we equip $(\mathrm{W}_n, p\,\mathrm{W}_n)$ with the canonical PD-structure γ_n. We briefly review the formal groupoid structure on the PD-envelope of the diagonal immersion following [**4**].

Let r, n be integers $\geqslant 1$. We denote by $\mathfrak{X}_n^{r+1}$ the product of $(r+1)$-copies of $\mathfrak{X}_n$ over $\mathscr{S}_n$ (2.4) and by $P_{\mathfrak{X}_n}(r)$ the PD-envelope of the diagonal immersion $\mathfrak{X}_n \to \mathfrak{X}_n^{r+1}$ compatible with the PD-structure γ_n ([**4**] I 4.3.1). By extension of scalars ([**5**] 3.20.8, [**4**] I 2.8.2), we have a canonical PD-isomorphism $P_{\mathfrak{X}_n}(r) \times_{\mathscr{S}_n} \mathscr{S}_m \xrightarrow{\sim} P_{\mathfrak{X}_m}(r)$ for all integers $1 \leqslant m < n$. The inductive limit $P_{\mathfrak{X}}(r)$ of the inductive system $(P_{\mathfrak{X}_n}(r))_{n \geqslant 1}$ is an adic affine formal $(\mathfrak{X}^{r+1})$-scheme ([**1**] 2.3.10). We drop (r) from the notation when $r = 1$.

5.8. – For a commutative ring A, we denote by $A\langle x_1,\dots,x_d\rangle$ the PD polynomial ring in d variables ([**4**] I 1.5). If A is an adic ring such that pA is an ideal of the definition, we denote by $A\langle\!\langle x_1,\dots,x_d\rangle\!\rangle$ the p-adic completion of the PD polynomial algebra $A\langle x_1,\dots,x_d\rangle$.

5.9. – Assume that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X}\to\widehat{\mathbb{A}}^d_{\mathscr{S}}=\mathrm{Spf}(\mathrm{W}\{T_1,\dots,T_d\})$ and we set t_i the image of T_i in $\mathscr{O}_{\mathfrak{X}}$ for all $1\leqslant i\leqslant d$. We note ξ_i the section $1\otimes t_i-t_i\otimes 1$ of $\mathscr{O}_{\mathfrak{X}^2}$ and also its image in $\mathscr{O}_{P_{\mathfrak{X}}}$. By ([**4**] I 4.4.1 and 4.5.3), we deduce the following PD-isomorphisms (5.8)

$$\mathscr{O}_{\mathfrak{X}}\langle\!\langle x_1,\dots,x_d\rangle\!\rangle\xrightarrow{\sim}q_{1*}(\mathscr{O}_{P_{\mathfrak{X}}}),\qquad \mathscr{O}_{\mathfrak{X}}\langle\!\langle x_1,\dots,x_d\rangle\!\rangle\xrightarrow{\sim}q_{2*}(\mathscr{O}_{P_{\mathfrak{X}}}),\tag{5.9.1}$$

where $q_1,q_2:P_{\mathfrak{X}}\to\mathfrak{X}$ are the canonical morphisms and x_i is sent to ξ_i. In general, we deduce that $P_{\mathfrak{X}}$ is flat over $\mathscr{S}$ (2.5).

For any integers $r,r'\geqslant 1$, by ([**4**] II 1.3.4 and 1.3.5), we deduce a canonical isomorphism of formal $(\mathfrak{X}^{r+r'+1})$-schemes

$$P_{\mathfrak{X}}(r)\times_{\mathfrak{X}}P_{\mathfrak{X}}(r')\xrightarrow{\sim}P_{\mathfrak{X}}(r+r').\tag{5.9.2}$$

Proposition 5.10. – *The formal $\mathfrak{X}^2$-scheme $P_{\mathfrak{X}}$ has a natural formal $\mathfrak{X}$-groupoid structure.*

Proof. – For any $r\geqslant 1$, the diagonal immersion $\Delta(r):\mathfrak{X}\to\mathfrak{X}^{r+1}$ induces a canonical $(\mathfrak{X}^{r+1})$-morphism $\iota_P(r):\mathfrak{X}\to P_{\mathfrak{X}}(r)$. Set $X=\mathfrak{X}_1$ and let J be the PD-ideal of $\mathscr{O}_{P_X}$ associated to the closed immersion $X\to P_X$. For any local section x of J, we have $x^p=p!x^{[p]}=0$. Hence, we have a closed immersion $\underline{P_X}\hookrightarrow X$ (2.3). Since X is reduced, the composition $\underline{X}\to\underline{P_X}\to X$ is an isomorphism. We deduce an isomorphism:

$$\underline{P_X}\xrightarrow{\sim}X,\tag{5.10.1}$$

Hence the morphism of the underlying topological spaces $|P_{\mathfrak{X}}|\to|\mathfrak{X}^2|$ factors through $\Delta:|\mathfrak{X}|\to|\mathfrak{X}^2|$. The canonical morphism $P_{\mathfrak{X}}(2)\to\mathfrak{X}^3\xrightarrow{p_{13}}\mathfrak{X}^2$ is compatible with $\iota_P(2)$ and Δ. By the universal property of $(P_{\mathfrak{X}_n})_{n\geqslant 1}$, we deduce an $\mathfrak{X}^2$-morphism $\alpha_P:P_{\mathfrak{X}}(2)\to P_{\mathfrak{X}}$. Similary, by the universal property of $(P_{\mathfrak{X}_n})_{n\geqslant 1}$, the composition $P_{\mathfrak{X}}\to\mathfrak{X}^2\xrightarrow{\tau}\mathfrak{X}^2$ (4.6) induces a morphism $\eta_P:P_{\mathfrak{X}}\to P_{\mathfrak{X}}$. By the universal property of $(P_{\mathfrak{X}_n})_{n\geqslant 1}$, we verify that $(\alpha_P,\iota_P,\eta_P)$ defines a formal $\mathfrak{X}$-groupoid structure on $P_{\mathfrak{X}}$ (cf. the proof of 4.11). □

To simplify the notations, we put $\mathscr{P}_{\mathfrak{X}}=\mathscr{O}_{P_{\mathfrak{X}}}$ considered as a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra by 5.10.

Proposition 5.11 ([**5**] 4.12). – *Let n be an integer $\geqslant 1$. There is a canonical equivalence of categories between the category of $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathscr{P}_{\mathfrak{X}}$-stratification and the category* $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ *(5.3).*

We now explain the relationships among the various groupoids and stratifications we have constructed.

PROPOSITION 5.12. – *Let $Q_{\mathfrak{X}}$ be the formal $\mathfrak{X}$-groupoid defined in 4.11. We have a canonical morphism of formal $\mathfrak{X}$-groupoids* (5.10)

$$\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}. \tag{5.12.1}$$

Proof. – The isomorphism $\underline{P_X} \simeq X$ (5.10.1) fits into a commutative diagram

$$\begin{tikzcd} \underline{P_X} \arrow[r] \arrow[d] & P_{\mathfrak{X}} \arrow[d] \\ X \arrow[r, "\Delta"] & \mathfrak{X}^2. \end{tikzcd} \tag{5.12.2}$$

By the universal property of $Q_{\mathfrak{X}}$ (3.5), we deduce a canonical $\mathfrak{X}^2$-morphism $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$. We denote by $(\alpha_P, \iota_P, \eta_P)$ (resp. $(\alpha_Q, \iota_Q, \eta_Q)$) the formal groupoid structure on $P_{\mathfrak{X}}$ (resp. $Q_{\mathfrak{X}}$). The diagrams
(5.12.3)

$$\begin{tikzcd} P_{\mathfrak{X}} \times_{\mathfrak{X}} P_{\mathfrak{X}} \arrow[r, "\lambda^2"] \arrow[d] & Q_{\mathfrak{X}} \times_{\mathfrak{X}} Q_{\mathfrak{X}} \arrow[r, "\alpha_Q"] \arrow[d] & Q_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^3 \arrow[r, equal] & \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \end{tikzcd} \qquad \begin{tikzcd} P_{\mathfrak{X}} \times_{\mathfrak{X}} P_{\mathfrak{X}} \arrow[r, "\alpha_P"] \arrow[d] & P_{\mathfrak{X}} \arrow[r, "\lambda"] \arrow[d] & Q_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^3 \arrow[r, "p_{13}"] & \mathfrak{X}^2 \arrow[r, equal] & \mathfrak{X}^2 \end{tikzcd}$$

are commutative and the compositions of the lower horizontal arrows coincide. By the universal property of $Q_{\mathfrak{X}}$, we deduce that $\alpha_Q \circ \lambda^2 = \lambda \circ \alpha_P$. The composition $\mathfrak{X} \xrightarrow{\iota_P} P_{\mathfrak{X}} \to \mathfrak{X}^2$ is the diagonal immersion. By the universal property of $Q_{\mathfrak{X}}$, we deduce that $\lambda \circ \iota_P = \iota_Q$. The following diagrams

$$\begin{tikzcd} P_{\mathfrak{X}} \arrow[r, "\eta_P"] \arrow[d] & P_{\mathfrak{X}} \arrow[r, "\lambda"] \arrow[d] & Q_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^2 \arrow[r, "\tau"] & \mathfrak{X}^2 \arrow[r, equal] & \mathfrak{X}^2 \end{tikzcd} \qquad \begin{tikzcd} P_{\mathfrak{X}} \arrow[r, "\lambda"] \arrow[d] & Q_{\mathfrak{X}} \arrow[r, "\eta_Q"] \arrow[d] & Q_{\mathfrak{X}} \arrow[d] \\ \mathfrak{X}^2 \arrow[r, equal] & \mathfrak{X}^2 \arrow[r, "\tau"] & \mathfrak{X}^2 \end{tikzcd} \tag{5.12.4}$$

are commutative and the compositions of the lower arrows coincide. By the universal property of $Q_{\mathfrak{X}}$, we deduce that $\lambda \circ \eta_P = \eta_Q \circ \lambda$. The proposition follows. □

5.13. – We take again the notation of 4.9 for $\mathfrak{X}$. For any integers $n, r \geqslant 1$, we denote by $T_{\mathfrak{X},n}(r)$ the PD-envelope of the closed immersion $\mathfrak{X}_n \hookrightarrow (R_{\mathfrak{X}}(r))_n$ (4.11.4) compatible with the PD-structure γ_n. By extension of scalars ([**4**] I 2.8.2), we have a canonical isomorphism of PD-schemes $T_{\mathfrak{X},n}(r) \times_{\mathscr{S}_n} \mathscr{S}_m \xrightarrow{\sim} T_{\mathfrak{X},m}(r)$ for all integers $1 \leqslant m < n$. The inductive limit $T_{\mathfrak{X}}(r)$ of the inductive system $(T_{\mathfrak{X},n}(r))_{n \geqslant 1}$ is an adic affine formal $R_{\mathfrak{X}}(r)$-scheme. We denote by

$$\varpi(r) : T_{\mathfrak{X}}(r) \to R_{\mathfrak{X}}(r) \tag{5.13.1}$$

the canonical morphism. We set $T_{\mathfrak{X}} = T_{\mathfrak{X}}(1)$ and $\varpi = \varpi(1)$.

5.14. – Let n, r, r' be integers $\geqslant 1$. We denote by $J_{(R_{\mathfrak{X}}(r))_n}$ the ideal of $\mathscr{O}_{(R_{\mathfrak{X}}(r))_n}$ associated to the closed immersion $\mathfrak{X}_n \to (R_{\mathfrak{X}}(r))_n$, which induces an isomorphism on the underlying topological spaces. Via the $(\mathfrak{X}_n^{r+r'+1})$-isomorphism (4.10.1)

$$(R_{\mathfrak{X}}(r))_n \times_{\mathfrak{X}_n} (R_{\mathfrak{X}}(r'))_n \xrightarrow{\sim} (R_{\mathfrak{X}}(r+r'))_n,$$

the ideal $J_{(R_{\mathfrak{X}}(r))_n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{O}_{(R_{\mathfrak{X}}(r'))_n} + \mathscr{O}_{(R_{\mathfrak{X}}(r))_n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_{(R_{\mathfrak{X}}(r'))_n}$ corresponds to $J_{(R_{\mathfrak{X}}(r+r'))_n}$. In view of ([**4**] II 1.3.5), we deduce a canonical isomorphism of PD-schemes

$$\lambda_n : T_{\mathfrak{X},n}(r) \times_{\mathfrak{X}_n} T_{\mathfrak{X},n}(r') \xrightarrow{\sim} T_{\mathfrak{X},n}(r+r'), \tag{5.14.1}$$

where the projections $T_{\mathfrak{X},n}(r) \to \mathfrak{X}_n$ (resp. $T_{\mathfrak{X},n}(r') \to \mathfrak{X}_n$) is induced by the projection $\mathfrak{X}_n^{r+1} \to \mathfrak{X}_n$ on the last factor (resp. $\mathfrak{X}_n^{r'+1} \to \mathfrak{X}_n$ on the first factor). In view of the construction of (5.14.1), the isomorphisms λ_m and λ_n are compatibles for all integers $1 \leqslant m < n$. We deduce a canonical isomorphism of formal $(\mathfrak{X}^{r+r'+1})$-schemes

$$T_{\mathfrak{X}}(r) \times_{\mathfrak{X}} T_{\mathfrak{X}}(r') \xrightarrow{\sim} T_{\mathfrak{X}}(r+r'). \tag{5.14.2}$$

Proposition 5.15. – *The formal $\mathfrak{X}^2$-scheme $T_{\mathfrak{X}}$ has a natural formal $\mathfrak{X}$-groupoid structure such that the morphism $\varpi : T_{\mathfrak{X}} \to R_{\mathfrak{X}}$* (5.13.1) *is a morphism of formal $\mathfrak{X}$-groupoids* (4.8, 4.11).

Proof. – Since the morphism of underlying topological spaces $|R_{\mathfrak{X}}| \to |\mathfrak{X}^2|$ factors through $\Delta : |\mathfrak{X}| \to |\mathfrak{X}^2|$, the same holds for $|T_{\mathfrak{X}}|$. The $\mathfrak{X}^2$-morphism $\iota_R(r) : \mathfrak{X} \to R_{\mathfrak{X}}(r)$ (4.11.4) induces a canonical $\mathfrak{X}^2$-morphism

$$\iota_T(r) : \mathfrak{X} \to T_{\mathfrak{X}}. \tag{5.15.1}$$

The composition of $\varpi(2)$ and α_R (4.11.2)

$$T_{\mathfrak{X}}(2) \to R_{\mathfrak{X}}(2) \to R_{\mathfrak{X}} \tag{5.15.2}$$

is compatible with $\iota_T(2)$ and ι_R. By the universal property of $(T_{\mathfrak{X},n})_{n\geqslant 1}$, we deduce an $\mathfrak{X}^2$-morphism

$$\alpha_T : T_{\mathfrak{X}}(2) \to T_{\mathfrak{X}} \tag{5.15.3}$$

compatible with α_R. We identify $T_{\mathfrak{X}}(2)$ and $T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}}$ (resp. $T_{\mathfrak{X}}(3)$ and $T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}}$) by 5.14. The diagrams

$$\begin{tikzcd}
T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \arrow[r, "\mathrm{id}\times\alpha_T"] \arrow[d] & T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \arrow[r, "\alpha_T"] \arrow[d] & T_{\mathfrak{X}} \arrow[d] \\
R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\mathrm{id}\times\alpha_R"] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] & R_{\mathfrak{X}}
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \arrow[r, "\alpha_T\times\mathrm{id}"] \arrow[d] & T_{\mathfrak{X}} \times_{\mathfrak{X}} T_{\mathfrak{X}} \arrow[r, "\alpha_T"] \arrow[d] & T_{\mathfrak{X}} \arrow[d] \\
R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R\times\mathrm{id}"] & R_{\mathfrak{X}} \times_{\mathfrak{X}} R_{\mathfrak{X}} \arrow[r, "\alpha_R"] & R_{\mathfrak{X}}
\end{tikzcd}$$

are commutative and the compositions of the lower horizontal arrows coincide. By the universal property of $(T_{\mathfrak{X},n})_{n\geqslant 1}$, we deduce that $\alpha_T \circ (\mathrm{id} \times \alpha_T) = \alpha_T \circ (\alpha_T \times \mathrm{id})$. Since $\eta_R \circ \iota_R = \iota_R$ (4.7)(ii)), by the universal property of $(T_{\mathfrak{X},n n\geqslant 1}$, the composition $T_{\mathfrak{X}} \to R_{\mathfrak{X}} \xrightarrow{\eta_R} R_{\mathfrak{X}}$ (4.11.6) induces a morphism

$$\eta_T : T_{\mathfrak{X}} \to T_{\mathfrak{X}}. \tag{5.15.4}$$

By the universal property of $(T_{\mathfrak{X},n})_{n\geqslant 1}$, we verify that $(\alpha_T, \iota_T, \eta_T)$ is a formal $\mathfrak{X}$-groupoid structure on $T_{\mathfrak{X}}$ (cf. the proof of 4.11). In view of the proof, the morphism ϖ is clearly a morphism of formal groupoids. □

5.16. – In the following, we recall Shiho's intepretation of integrable p-connections in terms of $\mathscr{T}_{\mathfrak{X}}$-stratifications [**33**].

To simplify the notation, we set $\mathscr{T}_{\mathfrak{X}} = \mathscr{O}_{T_{\mathfrak{X}}}$ considered as a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra (4.7) and we present a local description for it. Assume that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We put $\xi_i = 1 \otimes T_i - T_i - 1$ and we consider $\frac{\xi_i}{p}$ as the section of $\mathscr{R}_{\mathfrak{X}}$ (4.12) and of $\mathscr{T}_{\mathfrak{X}}$.

Let n be an integer $\geqslant 1$. By 4.13 and 4.14, the closed immersion $\mathfrak{X}_n \to R_{\mathfrak{X},n}$ of smooth $\mathscr{S}_n$-schemes is regular ([**20**] 17.12.1) and $(\frac{\xi_1}{p}, \dots, \frac{\xi_d}{p})$ is a regular sequence which generates $J_{R_{\mathfrak{X}},n}$ (5.14). In view of ([**4**] I 4.5.1 and 4.5.2), we deduce the following isomorphisms:

$$\mathscr{O}_{\mathfrak{X}_n}\langle x_1, \dots, x_d\rangle \xrightarrow{\sim} q_{1*}(\mathscr{O}_{T_{\mathfrak{X},n}}) \qquad \mathscr{O}_{\mathfrak{X}_n}\langle x_1, \dots, x_d\rangle \xrightarrow{\sim} q_{2*}(\mathscr{O}_{T_{\mathfrak{X},n}}), \tag{5.16.1}$$

where $q_1, q_2 : T_{\mathfrak{X},n} \to \mathfrak{X}_n$ are the canonical projections and x_i is sent to $\frac{\xi_i}{p}$ in both cases.

Then we deduce the following isomorphisms (5.8)

$$\mathscr{O}_{\mathfrak{X}}\langle\langle x_1, \dots, x_d\rangle\rangle \xrightarrow{\sim} q_{1*}(\mathscr{T}_{\mathfrak{X}}) \qquad \mathscr{O}_{\mathfrak{X}}\langle\langle x_1, \dots, x_d\rangle\rangle \xrightarrow{\sim} q_{2*}(\mathscr{T}_{\mathfrak{X}}), \tag{5.16.2}$$

where $q_1, q_2 : T_{\mathfrak{X}} \to \mathfrak{X}$ are the canonical projections and x_i is send to $\frac{\xi_i}{p}$ in both cases. For any multi-index $I = (i_1, i_2, \dots, i_d) \in \mathbb{N}^d$, we put $(\frac{\xi}{p})^{[I]} = \prod_{j=1}^d (\frac{\xi_j}{p})^{[i_j]} \in \mathscr{T}_{\mathfrak{X}}$.

Proposition 5.17 ([**33**] Prop. 2.9). – *There is a canonical equivalence of categories between the category of $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathscr{T}_{\mathfrak{X}}$-stratification and the category* p-$\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ (5.3).

We recall the description of this equivalence in the local case (cf. [**33**] Prop. 2.9). Suppose that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We take again the notation of 5.2 and 5.16. Let (M, ∇) be an $\mathscr{O}_{\mathfrak{X}_n}$-modules with quasi-nilpotent integrable p-connection. The associated stratification $\varepsilon : \mathscr{T}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} M \xrightarrow{\sim} M \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{T}_{\mathfrak{X}}$ is defined, for every local section m of M by

$$\varepsilon(1 \otimes m) = \sum_{I \in \mathbb{N}^d} \nabla_{\partial^I}(m) \otimes \left(\frac{\xi}{p}\right)^{[I]}, \tag{5.17.1}$$

where the right hand side is a locally finite sum since ∇ is quasi-nilpotent.

LEMMA 5.18. – *There exists a canonical morphism of formal $\mathfrak{X}$-groupoids*

$$\varsigma : R_{\mathfrak{X}} \to P_{\mathfrak{X}}. \tag{5.18.1}$$

Proof. – Recall (3.5) that we have a commutative diagram

$$\begin{tikzcd} R_{\mathfrak{X},1} \arrow[r] \arrow[d] & R_{\mathfrak{X}} \arrow[d] \\ X \arrow[r] & \mathfrak{X}^2. \end{tikzcd}$$

Since $R_{\mathfrak{X}}$ is flat over W, the ideal (p) of $\mathscr{O}_{R_{\mathfrak{X}}}$ has a canonical PD-structure. By the universal property of PD-envelope, we deduce a canonical $\mathfrak{X}^2$-morphism $\varsigma : R_{\mathfrak{X}} \to P_{\mathfrak{X}}$. We denote by $(\alpha_R, \iota_R, \eta_R)$ (resp. $(\alpha_P, \iota_P, \eta_P)$) the formal groupoid structure on $R_{\mathfrak{X}}$ (resp. $P_{\mathfrak{X}}$). By the universal property of $(P_{\mathfrak{X},n})_{n\geqslant 1}$, we verify that ς is a morphism of formal $\mathfrak{X}$-groupoids (cf. the proof of 5.12). □

5.19. – Let $s : \mathscr{P}_{\mathfrak{X}} \to \mathscr{R}_{\mathfrak{X}}$ be the homomorphism of formal Hopf algebras induced by ς. We present a local description of s. Suppose that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We take again the notation of 4.14 and 5.9. In view of the construction of ς, for any multi-index $I \in \mathbb{N}^d$, we have

$$s(\xi^{[I]}) = \frac{p^{|I|}}{I!}\left(\frac{\xi}{p}\right)^I. \tag{5.19.1}$$

We denote by J_R (resp. J_P) the ideal sheaf of $\mathscr{R}_{\mathfrak{X}}$ (resp. $\mathscr{P}_{\mathfrak{X}}$) associated to the closed immersion ι_R (resp. ι_P). Note that the p-adic valuation of $I!$ is $\leqslant \sum_{k\geqslant 1}\lfloor \frac{|I|}{p^k}\rfloor < |I|$. By (5.19.1), we deduce that in general, s is injective and $s(J_P) \subset pJ_R$. Then we have $s(J_P^{[i]}) \subset p^i J_R^i$ for any $1 \leqslant i \leqslant p-1$. By dividing by p^i, we obtain $\mathscr{O}_{\mathfrak{X}}$-bilinear morphisms

$$s^i : J_P^{[i]} \to J_R^i \qquad \forall\ 0 \leqslant i \leqslant p-1. \tag{5.19.2}$$

CHAPTER 6

LOCAL CONSTRUCTIONS OF SHIHO

In the section, we review Shiho's local Cartier transform [**33**] (which depends on a lifting of Frobenius) and explain how it can be understood in terms of the groupoids $P_{\mathfrak{X}}, T_{\mathfrak{X}}, R_{\mathfrak{X}}$ and $Q_{\mathfrak{X}}$.

We denote by $\mathfrak{X}$ a smooth formal $\mathscr{S}$-scheme, by X the special fiber of $\mathfrak{X}$, by $\mathfrak{X}' = \mathfrak{X} \times_{\mathscr{S},\sigma} \mathscr{S}$ the base change of $\mathfrak{X}$ by σ (2.1) and by $\pi : \mathfrak{X}' \to \mathfrak{X}$ the canonical projection.

6.1. – Let n be an integer $\geqslant 1$. We assume that there exists an $\mathscr{S}_{n+1}$-morphism $F_{n+1} : \mathfrak{X}_{n+1} \to \mathfrak{X}'_{n+1}$ (2.4) whose reduction modulo p is the relative Frobenius morphism $F_{X/k}$ of X (2.2) and we denote by F_n the reduction modulo p^n of F_{n+1}. The morphism F_{n+1} induces an $(\mathscr{O}_{\mathfrak{X}_{n+1}})$-linear morphism $dF_{n+1} : F_{n+1}^*(\Omega^1_{\mathfrak{X}'_{n+1}/\mathscr{S}_{n+1}}) \to \Omega^1_{\mathfrak{X}_{n+1}/\mathscr{S}_{n+1}}$ whose image is contained in $p\Omega^1_{\mathfrak{X}_{n+1}/\mathscr{S}_{n+1}}$. By dividing by p, it induces an $\mathscr{O}_{\mathfrak{X}_n}$-linear morphism

$$\frac{dF_{n+1}}{p} : F_n^*(\Omega^1_{\mathfrak{X}'_n/\mathscr{S}_n}) \to \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}. \tag{6.1.1}$$

Let (M', ∇') be an $\mathscr{O}_{\mathfrak{X}'_n}$-module with an integrable p-connection relative to $\mathscr{S}_n$ (5.1). We denote by ζ_n the composition

(6.1.2)

$$\zeta_n : F_n^*(\Omega^1_{\mathfrak{X}'_n/\mathscr{S}_n} \otimes_{\mathscr{O}_{\mathfrak{X}'_n}} M') \xrightarrow{\sim} F_n^*(\Omega^1_{\mathfrak{X}'_n/\mathscr{S}_n}) \otimes_{\mathscr{O}_{\mathfrak{X}_n}} F_n^*(M') \to \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} F_n^*(M'),$$

i.e., the composition of $\frac{dF_{n+1}}{p} \otimes \mathrm{id}$ and the canonical isomorphism.

Shiho constructs a W_n-linear morphism $\nabla : F_n^*(M') \to \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} F_n^*(M')$ as follows. For any local sections f of $\mathscr{O}_{\mathfrak{X}_n}$ and e of M', we put

$$\nabla(fF_n^*(e)) = f\zeta_n(F_n^*(\nabla'(e))) + df \otimes (F_n^*(e)). \tag{6.1.3}$$

The morphism ∇ is well-defined and is an integrable connection on $F_n^*(M')$ relative to $\mathscr{S}_n$ (cf. [**33**] page 805-806). Shiho defines a functor (cf. [**33**] 2.5)

$$\begin{aligned} \Phi_n : p\text{-MIC}(\mathfrak{X}'_n/\mathscr{S}_n) &\to \mathrm{MIC}(\mathfrak{X}_n/\mathscr{S}_n), \\ (M', \nabla') &\mapsto (F_n^*(M'), \nabla). \end{aligned} \tag{6.1.4}$$

The functor Φ_1 sends quasi-nilpotent objects to quasi-nilpotent objects (cf. 6.2 below). By dévissage ([**33**] 1.13), the same holds for Φ_n. It induces an equivalence of the categories ([**33**] 3.1):

$$\Phi_n : p\text{-MIC}^{\text{qn}}(\mathfrak{X}'_n/\mathscr{S}_n) \xrightarrow{\sim} \text{MIC}^{\text{qn}}(\mathfrak{X}_n/\mathscr{S}_n). \tag{6.1.5}$$

Let (M',∇') be an object of p-MIC$(\mathfrak{X}'_n/\mathscr{S}_n)$ and $(M,\nabla) = \Phi_n(M',\nabla')$. In view of (6.1.3), the adjunction morphism of $\mathrm{id}_M \otimes \wedge^\bullet\left(\frac{dF_{n+1}}{p}\right)$ (6.1.1) induce a W-linear morphism of complexes (5.1.3)

$$\lambda : M' \otimes_{\mathscr{O}_{\mathfrak{X}'_n}} \Omega^\bullet_{\mathfrak{X}'_n/\mathscr{S}_n} \to F_{X/k*}(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n}). \tag{6.1.6}$$

Indeed, for local coordinates $t_1,\dots,t_d$ of $\mathfrak{X}'_n$ over $\mathscr{S}_n$, any local section of $M' \otimes_{\mathscr{O}_{\mathfrak{X}'_n}} \Omega^q_{\mathfrak{X}'_n/\mathscr{S}_n}$ can be written as a sum of sections of the form $m \otimes dt_{i_1} \wedge \cdots \wedge dt_{i_q}$. Using (6.1.3), one verifies that (6.1.6) is a morphism of complexes.

Lemma 6.2. – *Let (M',θ) be a Higgs module on X'/k (5.1) and ∇ the integrable connection on $M = F^*_{X/k}(M')$ constructed in 6.1. If (M',θ) is quasi-nilpotent (5.3), then so is (M,∇).*

Proof. – The question being local, we can reduce to the case where there exists an étale $\mathscr{S}_2$-morphism $\mathfrak{X}_2 \to \mathbb{A}^d_{\mathscr{S}_2} = \mathrm{Spec}(\mathrm{W}_2[T_1,\dots,T_d])$. For any $1 \leqslant i \leqslant d$, let t_i be the image of T_i in $\mathscr{O}_X$ and $t'_i = \pi^*(t_i) \in \mathscr{O}_{X'}$. There exists a section a_i of $\mathscr{O}_X$ such that $\frac{dF_2}{p}(dt'_i) = t_i^{p-1}dt_i + da_i$ and an $\mathscr{O}_{X'}$-linear morphism $\theta_i : M' \to M'$ such that for every local section e of M', we have $\theta(e) = \sum_{i=1}^d dt'_i \otimes \theta_i(e)$. We set ∂_i the dual of dt_i. Then we have

$$\nabla_{\partial_i}(F^*_{X/k}(e)) = t_i^{p-1}F^*_{X/k}(\theta_i(e)) + \sum_{j=1}^d \frac{\partial a_j}{\partial t_i} F^*_{X/k}(\theta_j(e)). \tag{6.2.1}$$

We denote by $\psi : M \to M \otimes_{\mathscr{O}_X} F^*_X(\Omega^1_{X/k})$ the p-curvature associated to ∇ ([**25**] 5.0). There exists $\mathscr{O}_X$-linear endomorphisms $\psi_i : M \to M$ for $1 \leqslant i \leqslant d$ such that

$$\psi = \sum_{i=1}^d \psi_i \otimes F^*_X(dt_i). \tag{6.2.2}$$

Recall ([**25**] 5.2) that ψ_i and ψ_j commutes for $1 \leqslant i,j \leqslant d$. For any $I = (i_1,\dots,i_d) \in \mathbb{N}^d$, we put $\psi_I = \prod_{j=1}^d \psi_j^{i_j}$ and $\theta_I = \prod_{j=1}^d \theta_j^{i_j}$. The pth iterate $\partial_i^{(p)}$ of ∂_i is zero ([**25**] 5.0). Then we have

$$\psi_i = (\nabla_{\partial_i})^p. \tag{6.2.3}$$

By (6.2.1) and induction, one verifies that for any integer $l \geqslant 1$, there exist elements $\{a_{l,I} \in \mathscr{O}_X\}_{I\in\mathbb{N}^d, 1\leqslant|I|\leqslant l}$ such that for every local section e of M', we have

$$(\nabla_{\partial_i})^l(F^*_{X/k}(e)) = \sum_{1\leqslant|I|\leqslant l} a_{l,I} F^*_{X/k}(\theta_I(e)). \tag{6.2.4}$$

Since the ψ_i's are $\mathscr{O}_X$-linear morphisms, if there exists an integer N such that $\theta_I(e) = 0$ for all $|I| \geqslant N$, then $\psi_I(F^*_{X/k}(e)) = 0$ for all $|I| \geqslant N$ by (6.2.3) and (6.2.4). We deduce that ∇ is quasi-nilpotent. □

REMARK 6.3. – Given an $\mathscr{O}_{X'}$-module M', the *Frobenius descent connection* ∇_{can} on $F^*_{X/k}(M')$ is defined for local sections m of M' and f of $\mathscr{O}_X$, by

$$\nabla_{\mathrm{can}}(fF^*_{X/k}(m)) = m \otimes df. \tag{6.3.1}$$

It is integrable and of p-curvature zero. Cartier descent states that the functor $M' \mapsto (F^*_{X/k}(M'), \nabla_{\mathrm{can}})$ induces an equivalence of categories between the category of quasi-coherent $\mathscr{O}_{X'}$-modules and the full subcategory of $\mathrm{MIC}^{\mathrm{qn}}(X/k)$ consisting of quasi-coherent objects whose p-curvature is zero (**[25]** 5.1). Considering $\mathscr{O}_{X'}$-modules as Higgs modules with the zero Higgs field, by (6.1.3), we see that Φ_1 is compatible with Cartier descent.

6.4. – Let (M', θ) be a Higgs module on X'/k and ℓ an integer $\geqslant 0$. We suppose that (M', θ) is nilpotent of level $\leqslant \ell$, i.e., there exists an increasing filtration of M':

$$0 = N'_0 \subset N'_1 \subset \cdots \subset N'_\ell \subset N'_{\ell+1} = M', \tag{6.4.1}$$

such that $\theta(N'_i) \subset N'_{i-1} \otimes_{\mathscr{O}_{X'}} \Omega^1_{X'/k}$ for $1 \leqslant i \leqslant \ell+1$. Then the induced Higgs field on $\mathrm{gr}^i_{N'}(M')$ is trivial.

We set $(M, \nabla) = \Phi_1(M', \theta)$ and $N_i = \Phi_1(N'_i, \theta|N'_i)$ for $0 \leqslant i \leqslant \ell+1$. By (6.1.3), we see that ∇ induces an integrable connection on each graded piece $\mathrm{gr}^i_N(M)$, with zero p-curvature.

We have a filtration on the de Rham complex $M \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/k}$ (resp. the Dolbeault complex $M' \otimes_{\mathscr{O}_{X'}} \Omega^\bullet_{X'/k}$) defined by:

$$N_i \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/k} \qquad (\text{resp. } N'_i \otimes_{\mathscr{O}_{X'}} \Omega^\bullet_{X'/k}).$$

PROPOSITION 6.5. – *The morphism of complexes* (6.1.6) *induces for every* $i \in [1, \ell+1]$ *a quasi-isomorphism*

$$N'_i \otimes_{\mathscr{O}_{X'}} \Omega^\bullet_{X'/k} \to F_{X/k*}(N_i \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/k}). \tag{6.5.1}$$

Proof. – We first consider the case where $\ell = 0$, i.e., θ is the zero Higgs field. We follow a similar argument of (**[30]** 1.2) where Ogus shows an analogous result in the level of cohomology of complexes. Then ∇ is the Frobenius descent connection on $M = F^*_{X/k}(M')$. When $M' = \mathscr{O}_{X'}$, the morphism on the cohomology induced by λ (6.1.6) is the Cartier isomorphism (**[25]** 7.2)

$$\mathrm{C}^{-1}_{X/k} : \Omega^i_{X'/k} \xrightarrow{\sim} \mathscr{H}^i(F_{X/k*}(\Omega^\bullet_{X/k})). \tag{6.5.2}$$

Since $F_{X/k}$ induces an isomorphism on the underlying topological spaces, we have an isomorphism of complexes

$$M' \otimes_{\mathscr{O}_{X'}} F_{X/k*}(\Omega^\bullet_{X/k}) \xrightarrow{\sim} F_{X/k*}(M \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/k}). \tag{6.5.3}$$

Since $F_{X/k*}(\Omega^\bullet_{X/k})$ is a complex of flat $\mathscr{O}_{X'}$-modules whose cohomology sheaves are also flat (6.5.2), the canonical morphism

$$M'\otimes_{\mathscr{O}_{X'}}\Omega^i_{X'/k} \xrightarrow{\sim} M'\otimes_{\mathscr{O}_{X'}}\mathscr{H}^i(F_{X/k*}(\Omega^\bullet_{X/k})) \to \mathscr{H}^i(M'\otimes_{\mathscr{O}_{X'}}F_{X/k*}(\Omega^\bullet_{X/k})) \tag{6.5.4}$$

is an isomorphism. The assertion in the case $\ell = 0$ follows.

We prove the general case by induction on i. The assertion for $i = 1$ is already proved. If the assertion is true for $i-1$, then the assertion for i follows by dévissage from the induction hypothesis. $\square$

In the remainder of this section, we suppose that there exists an $\mathscr{S}$-morphism $F : \mathfrak{X} \to \mathfrak{X}'$ which lifts the relative Frobenius morphism $F_{X/k}$ of X. We take again the notation of $R_{\mathfrak{X}}$, $Q_{\mathfrak{X}}$, $T_{\mathfrak{X}}$ and $P_{\mathfrak{X}}$ (4.11, 5.15, 5.10).

Proposition 6.6. – *The morphism F induces a morphism of formal groupoids above F (4.8)*

$$\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}. \tag{6.6.1}$$

Proof. – First, we show that there exists a unique morphism $g : Q_{\mathfrak{X},1} \to X'$ which fits into a commutative diagram

$$\begin{tikzcd} Q_{\mathfrak{X},1} \arrow[r] \arrow[dd, "g"'] & Q_{\mathfrak{X}} \arrow[d] \\ & \mathfrak{X}^2 \arrow[d, "F^2"] \\ X' \arrow[r, "\Delta"] & \mathfrak{X}'^2, \end{tikzcd} \tag{6.6.2}$$

where the bottom map is induced by the diagonal immersion. The problem being local on $\mathfrak{X}$, we can assume that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1,\dots,T_d\})$. For any $1 \leqslant i \leqslant d$, we put t_i the image of T_i in $\mathscr{O}_{\mathfrak{X}}$, $t'_i = \pi^*(t_i) \in \mathscr{O}_{\mathfrak{X}'}$, $\xi_i = 1\otimes t_i - t_i\otimes 1 \in \mathscr{O}_{\mathfrak{X}^2}$ and $\xi'_i = 1\otimes t'_i - t'_i\otimes 1 \in \mathscr{O}_{\mathfrak{X}'^2}$. Locally, there is a section a_i of $\mathscr{O}_{\mathfrak{X}}$ such that $F^*(t'_i) = t_i^p + pa_i$. Then we have

$$\begin{aligned} F^{2*}(\xi'_i) &= 1\otimes t_i^p - t_i^p\otimes 1 + p(1\otimes a_i - a_i\otimes 1) \\ &= (\xi_i + t_i\otimes 1)^p - t_i^p\otimes 1 + p(1\otimes a_i - a_i\otimes 1) \\ &= \xi_i^p + \sum_{j=1}^{p-1}\binom{p}{j}\xi_i^j(t_i\otimes 1)^{p-j} + p(1\otimes a_i - a_i\otimes 1). \end{aligned} \tag{6.6.3}$$

Since $\xi_i^p = p\cdot\left(\frac{\xi_i^p}{p}\right)$ in $Q_{\mathfrak{X}}$, the image of $F^{2*}(\xi'_i)$ in $Q_{\mathfrak{X}}$ is contained in $pQ_{\mathfrak{X}}$. Then the existence and the uniqueness of g follow. By the universal property of $R_{\mathfrak{X}'}$ (3.5), we deduce an $\mathfrak{X}'^2$-morphism $\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$. Using the universal property of $R_{\mathfrak{X}'}$, we verify that ψ is a morphism of formal groupoids above F (cf. the proof of 5.12). $\square$

6.7. – We denote the composition of $\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$ and $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$ (5.12.1) by

$$\phi : P_{\mathfrak{X}} \to R_{\mathfrak{X}'}. \tag{6.7.1}$$

The morphism ϕ is a morphism of formal groupoids above F (4.8).

Lemma 6.8 ([**33**] 2.14). – *The morphism ϕ induces a morphism of formal groupoids above F*

$$\varphi : P_{\mathfrak{X}} \to T_{\mathfrak{X}'}. \tag{6.8.1}$$

Proof. – For any $n \geqslant 1$, by $\phi \circ \iota_P = F \circ \iota_{R'}$ (4.7)(ii)) and the universal property of $T_{\mathfrak{X}',n}$ (5.13), $\phi_n : P_{\mathfrak{X}_n} \to R_{\mathfrak{X}',n}$ induces a PD-$R_{\mathfrak{X}',n}$-morphism $\varphi_n : P_{\mathfrak{X}_n} \to T_{\mathfrak{X}',n}$. For any $1 \leqslant m < n$, since ϕ_m and ϕ_n are compatible, we see that φ_m and φ_n are compatible. Hence we obtain a $\mathfrak{X}'^2$-morphism $\varphi : P_{\mathfrak{X}} \to T_{\mathfrak{X}'}$. It follows from 6.7 and the universal property of $(T_{\mathfrak{X}',n n\geqslant 1}$ that φ is a morphism of formal groupoids above F. □

6.9. – We have a commutative diagram of formal groupoids

$$\begin{tikzcd} P_{\mathfrak{X}} \arrow[r, "\varphi"] \arrow[d, "\lambda"'] \arrow[dr, "\phi"] & T_{\mathfrak{X}'} \arrow[d, "\varpi"] \\ Q_{\mathfrak{X}} \arrow[r, "\psi"] & R_{\mathfrak{X}'}, \end{tikzcd} \tag{6.9.1}$$

where φ, ϕ, ψ are induced by F. By 5.6, we deduce a commutative diagram:

$$\begin{tikzcd} \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{R}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \arrow[r, "\psi_n^*"] \arrow[d, "\varpi_n^*"'] & \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{Q}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\} \arrow[d, "\lambda_n^*"] \\ \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{T}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \arrow[r, "\varphi_n^*"] & \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{P}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\}. \end{tikzcd} \tag{6.9.2}$$

6.10. – Let n be an integer $\geqslant 1$ and F_n (resp. F_{n+1}) the reduction modulo p^n (resp. p^{n+1}) of F. In ([**33**] Prop. 2.17), Shiho shows that, through the equivalences of categories 5.11 and 5.17, the functor

$$\begin{aligned} \varphi_n^* : \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{T}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} &\to \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{P}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\} \\ (M,\varepsilon) &\mapsto (F_n^*(M), \varphi^{-1}(\varepsilon) \otimes_{\varphi^{-1}(\mathscr{T}_{\mathfrak{X}'})} \mathscr{P}_{\mathfrak{X}}) \end{aligned} \tag{6.10.1}$$

coincides with the functor induced by F_{n+1} (6.1.5)

$$\Phi_n : p\text{-MIC}^{\mathrm{qn}}(\mathfrak{X}'_n/\mathscr{S}_n) \xrightarrow{\sim} \mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n).$$

Indeed, if we write down a p-connection and the associated connection in terms of local coordinates (as in (6.2.1)), then we can verify the compatibility between φ_n and Φ_n using Formula (6.6.3) (cf. [**33**] 2.17 for more details).

CHAPTER 7

OYAMA TOPOI

In this section, X denotes a k-scheme. We explain two "crystalline like" topoi introduced by Oyama [**32**] associated to X. When X admits a smooth lifting $\mathfrak{X}$ to W, crystals on these topoi (introduced in § 8) are equivalent to modules with $\mathscr{R}_{\mathfrak{X}}$-stratification and (resp. $\mathscr{Q}_{\mathfrak{X}}$-stratification) discussed in 5, and are independent of the choice of any lifting of X.

If we use a gothic letter $\mathfrak{T}$ to denote an adic formal $\mathscr{S}$-scheme, the corresponding roman letter T will denote its special fiber $\mathfrak{T} \otimes_{\mathrm{W}} k$.

Definition 7.1 ([**32**] 1.3.1). – We define the category $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$) [1] as follows.

(i) An object of $\mathscr{E}(X/\mathscr{S})$ is a triple $(U, \mathfrak{T}, u)$ consisting of an open subscheme U of X, a flat formal $\mathscr{S}$-scheme $\mathfrak{T}$ (2.5) and an affine k-morphism $u : T \to U$.

(ii) An object of $\underline{\mathscr{E}}(X/\mathscr{S})$ is a triple $(U, \mathfrak{T}, u)$ consisting of an open subscheme U of X, a flat formal $\mathscr{S}$-scheme $\mathfrak{T}$ and an affine k-morphism $u : \underline{T} \to U$, where $\underline{T}$ is the closed subscheme of T defined in 2.3.

(iii) Let $(U_1, \mathfrak{T}_1, u_1)$ and $(U_2, \mathfrak{T}_2, u_2)$ be two objects of $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$). A morphism from $(U_1, \mathfrak{T}_1, u_1)$ to $(U_2, \mathfrak{T}_2, u_2)$ consists of an $\mathscr{S}$-morphism $f : \mathfrak{T}_1 \to \mathfrak{T}_2$ and an X-morphism $g : U_1 \to U_2$ such that $g \circ u_1 = u_2 \circ f_s$ (resp. $g \circ u_1 = u_2 \circ \underline{f_s}$), where f_s is the reduction modulo p of f.

We denote an object $(U, \mathfrak{T}, u)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) simply by $(U, \mathfrak{T})$, if there is no risk of confusion.

To simplify the notation, we drop $(X/\mathscr{S})$ in the notation $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$) and we write simply $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), if there is no risk of confusion. We put $\mathscr{E}' = \mathscr{E}(X'/\mathscr{S})$ (2.2).

1. In [**32**], the category $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$) is denoted by $\mathrm{HIG}^{\gamma}(X/\mathscr{S})$ (resp. $\mathrm{CRIS}^{\gamma}(X/\mathscr{S})$).

Lemma 7.2. – *Let $f : \mathfrak{X}_1 \to \mathfrak{X}_2$ be an $\mathscr{S}$-morphism of flat formal $\mathscr{S}$-schemes and $f_s : T_1 \to T_2$ its special fiber.*

(i) *If f_s is an isomorphism, then f is an isomorphism.*

(ii) *If f_s is flat, then the morphism $\mathfrak{X}_{1,n} \to \mathfrak{X}_{2,n}$ induced by f (2.4) is flat for all integers $n \geqslant 1$.*

Proof. – We can reduce to the case where $\mathfrak{X}_1 = \mathrm{Spf}(B)$, $\mathfrak{X}_2 = \mathrm{Spf}(A)$ are affine ([**1**] 2.1.37) and f is induced by an adic W-homomorphism $u : A \to B$. For any integers $n \geqslant 1$, we put $A_n = A/p^nA$, $B_n = B/p^nB$, $\mathrm{gr}^n(A) = p^nA/p^{n+1}A$ and $\mathrm{gr}^n(B) = p^nB/p^{n+1}B$. Since A and B are flat over W, the canonical morphism of B_1-modules

$$B_1 \otimes_{A_1} \mathrm{gr}^n(A) \to \mathrm{gr}^n(B) \tag{7.2.1}$$

is an isomorphism.

(i) If u induces an isomorphism $A_1 \xrightarrow{\sim} B_1$, we deduce that u is an isomorphism by (7.2.1) and ([**7**] III § 2.8 Cor. 3 to Théo. 1).

(ii) If B_1 is flat over A_1, we deduce that B_n is flat over A_n for all integers $n \geqslant 1$ by (7.2.1) and the local criterion of flatness ([**7**] III § 5.2 Théo. 1). □

7.3. – We say that a morphism $(U_1, \mathfrak{X}_1, u_1) \to (U_2, \mathfrak{X}_2, u_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is *flat* if the special fiber $T_1 \to T_2$ of the morphism $\mathfrak{X}_1 \to \mathfrak{X}_2$ is flat.

Let $(U_1, \mathfrak{X}_1, u_1) \to (U, \mathfrak{X}, u)$ be a flat morphism and $(U_2, \mathfrak{X}_2, u_2) \to (U, \mathfrak{X}, u)$ a morphism of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). Their fiber product is represented by the fiber products $\mathfrak{X}_{12} = \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{X}_2$ (which is flat over $\mathscr{S}$ in view of 7.2(ii) and 2.5) and $U_{12} = U_1 \times_U U_2$ endowed with the affine morphism $T_1 \times_T T_2 \to U_1 \times_U U_2$ (resp. composite morphism $\underline{T_1 \times_T T_2} \to \underline{T_1} \times_{\underline{T}} \underline{T_2} \to U_1 \times_U U_2$ (2.3.1)) induced by u_1, u_2 and u.

Definition 7.4. – (i) We say that a morphism $f : (U_1, \mathfrak{X}_1) \to (U_2, \mathfrak{X}_2)$ of $\mathscr{E}$ is *Cartesian* if the canonical morphism $T_1 \to T_2 \times_{U_2} U_1$ is an isomorphism.

(ii) We say that a morphism $f : (U_1, \mathfrak{X}_1, u_1) \to (U_2, \mathfrak{X}_2, u_2)$ of $\underline{\mathscr{E}}$ is *Cartesian* if $T_1 \to T_2$ is an open immersion and the canonical morphism $\underline{T}_1 \to \underline{T}_2 \times_{U_2} U_1$ is an isomorphism. (2)

If $f : (U_1, \mathfrak{X}_1) \to (U_2, \mathfrak{X}_2)$ is a Cartesian morphism of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), T_1 is an open subscheme of T_2 and f identifies $\mathfrak{X}_1$ with the open formal subscheme induced by $\mathfrak{X}_2$ on T_1 by 7.2(i).

A Cartesian morphism is clearly flat (7.3). By 7.3, the base change of a Cartesian morphism of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is Cartesian. The composition of Cartesian morphisms of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is clearly Cartesian.

2. The above definition of Cartesian morphism in $\underline{\mathscr{E}}$ is equivalent to the original definition in ([**32**] 1.3.1), where Oyama requires the canonical morphism $T_2 \to U_2' \times_{U_1', u_1' \circ f_{T_1/k}} T_1$ is an isomorphism (2.3).

7.5. – Let $(U, \mathfrak{T}, u)$ be an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) and V an open subscheme of U. Note that the canonical morphism $\underline{T} \to T$ induces an isomorphism on the underlying topological spaces. We denote by $\mathfrak{T}_V$ the restriction of $\mathfrak{T}$ to the open subset $u^{-1}(V)$ of the topological space $|T|$. Then, we obtain an object $(V, \mathfrak{T}_V)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) and a Cartesian morphism $(V, \mathfrak{T}_V) \to (U, \mathfrak{T})$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$).

Any morphism $(U_1, \mathfrak{T}_1) \to (U_2, \mathfrak{T}_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) factors uniquely through the Cartesian morphism $(U_1, (\mathfrak{T}_2)_{U_1}) \to (U_2, \mathfrak{T}_2)$. The category $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is fibered over the category $\mathbf{Zar}_{/X}$ of open subschemes of X by the functor

$$\pi : \mathscr{E} \to \mathbf{Zar}_{/X} \quad (\text{resp. } \underline{\mathscr{E}} \to \mathbf{Zar}_{/X}) \quad (U, \mathfrak{T}) \mapsto U, \tag{7.5.1}$$

7.6. – Let $(U, \mathfrak{T})$ be an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). By 7.5, we have a functor

$$\alpha_{(U,\mathfrak{T})} : \mathbf{Zar}_{/U} \to \mathscr{E} \quad (\text{resp. } \underline{\mathscr{E}}) \qquad V \mapsto (V, \mathfrak{T}_V). \tag{7.6.1}$$

Let $f : (U_1, \mathfrak{T}_1) \to (U_2, \mathfrak{T}_2)$ be a morphism of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), $\jmath_f : \mathbf{Zar}_{/U_1} \to \mathbf{Zar}_{/U_2}$ the functor induced by composing with $U_1 \to U_2$. Then, the morphism f induces a morphism of functors:

$$\beta_f : \alpha_{(U_1,\mathfrak{T}_1)} \to \alpha_{(U_2,\mathfrak{T}_2)} \circ \jmath_f.$$

Let $\mathscr{F}$ be a presheaf on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). We denote by $\mathscr{F}_{(U,\mathfrak{T})}$ the presheaf $\mathscr{F} \circ \alpha_{(U,\mathfrak{T})}$ on $\mathbf{Zar}_{/U}$. The morphism β_f induces a morphism of presheaves:

$$\gamma_f : \mathscr{F}_{(U_2,\mathfrak{T}_2)}|_{U_1} \to \mathscr{F}_{(U_1,\mathfrak{T}_1)}. \tag{7.6.2}$$

It is clear that $\gamma_{\mathrm{id}} = \mathrm{id}$. By construction, if f is a Cartesian morphism, then γ_f is an isomorphism. If $g : (U_2, \mathfrak{T}_2) \to (U_3, \mathfrak{T}_3)$ is another morphism of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), we verify that $\gamma_{g\circ f} = \gamma_f \circ \jmath_f^*(\gamma_g)$, where $\jmath_f^*$ denotes the localisation functor from the category of presheaves on $\mathbf{Zar}_{/U_2}$ to the category of presheaves on $\mathbf{Zar}_{/U_1}$.

Proposition 7.7. – *A presheaf $\mathscr{F}$ on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is equivalent to the following data:*

(i) *For every object $(U, \mathfrak{T})$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) a presheaf $\mathscr{F}_{(U,\mathfrak{T})}$ on $\mathbf{Zar}_{/U}$,*
(ii) *For every morphism $f : (U_1, \mathfrak{T}_1) \to (U_2, \mathfrak{T}_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) a morphism $\gamma_f : \mathscr{F}_{(U_2,\mathfrak{T}_2)}|_{U_1} \to \mathscr{F}_{(U_1,\mathfrak{T}_1)}$,*

subject to the following conditions

(a) *If f is the identity morphism of $(U, \mathfrak{T})$, then γ_f is the identity of $\mathscr{F}_{(U,\mathfrak{T})}$.*
(b) *If f is a Cartesian morphism, then γ_f is an isomorphism.*
(c) *If f and g are two composable morphisms of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), then we have $\gamma_{g\circ f} = \gamma_f \circ \jmath_f^*(\gamma_g)$.*

Proof. – Let $\{\mathscr{F}_{(U,\mathfrak{T})}, \gamma_f\}$ be a data as in the proposition. We associate to it a presheaf on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) as follows. Let $(U, \mathfrak{T})$ be an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). We define $\mathscr{F}(U, \mathfrak{T}) = \mathscr{F}_{(U,\mathfrak{T})}(U)$. For any morphism $f : (V, \mathfrak{Z}) \to (U, \mathfrak{T})$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), we deduce a morphism from γ_f

$$\mathscr{F}_{(U,\mathfrak{T})}(U) \to \mathscr{F}_{(U,\mathfrak{T})}(V) \to \mathscr{F}_{(V,\mathfrak{Z})}(V). \tag{7.7.1}$$

In view of conditions (a) and (c), the correspondence

$$(U,\mathfrak{T}) \mapsto \mathscr{F}_{(U,\mathfrak{T})}(U) \tag{7.7.2}$$

is a presheaf. In view of condition (b), the above construction is quasi-inverse to the construction of 7.6. The assertion follows. □

We call *descent data associated to* $\mathscr{F}$ the data $\{\mathscr{F}_{(U,\mathfrak{T})}, \gamma_f\}$ as in the proposition.

Definition 7.8 ([**32**] 1.3.1). – Let $(U,\mathfrak{T},u)$ be an object of $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$). We denote by $\mathrm{Cov}(U,\mathfrak{T},u)$ the set of families of Cartesian morphisms $\{(U_i,\mathfrak{T}_i,u_i) \to (U,\mathfrak{T},u)\}_{i\in I}$ such that $\{U_i \to U\}_{i\in I}$ is a Zariski covering.

7.9. – By 7.4, we see that the sets $\mathrm{Cov}(U,\mathfrak{T})$ for $(U,\mathfrak{T}) \in \mathbf{Ob}(\mathscr{E}(X/\mathscr{S}))$ (resp. $(U,\mathfrak{T}) \in \mathbf{Ob}(\underline{\mathscr{E}}(X/\mathscr{S}))$) form a pretopology ([**3**] II 1.3). We call the topology on $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$) associated to the pretopology defined by the $\mathrm{Cov}(U,\mathfrak{T})$'s *Zariski topology.* We denote by $\widetilde{\mathscr{E}}(X/\mathscr{S})$ (resp. $\widetilde{\underline{\mathscr{E}}}(X/\mathscr{S})$) the topos of sheaves of sets on $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$), equipped with the Zariski topology.

Proposition 7.10. – *Let $\mathscr{F}$ be a presheaf on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) and $\{\mathscr{F}_{(U,\mathfrak{T})}, \gamma_f\}$ the associated descent data* (7.7). *Then $\mathscr{F}$ is a sheaf for the Zariski topology, if and only if for each object $(U,\mathfrak{T})$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), the presheaf $\mathscr{F}_{(U,\mathfrak{T})}$* (7.6) *is a sheaf of the Zariski topos U_{zar}.*

Proof. – Let $(U,\mathfrak{T})$ be an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). The functor $\alpha_{(U,\mathfrak{T})}$ (7.6.1) sends morphisms of $\mathbf{Zar}_{/U}$ to Cartesian morphisms of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) and commutes with fibered products. It is clearly continuous for the Zariski topologies. Hence, if $\mathscr{F}$ is a sheaf, then $\mathscr{F}_{(U,\mathfrak{T})}$ is a sheaf of U_{zar} ([**3**] III 1.2).

Conversely, suppose that each presheaf $\mathscr{F}_{(U,\mathfrak{T})}$ is a sheaf of U_{zar}. Let $\{(U_i,\mathfrak{T}_i) \to (U,\mathfrak{T})\}_{i\in I}$ be an element of $\mathrm{Cov}(U,\mathfrak{T})$. In view of condition (b) of 7.7, we deduce that the sequence

$$\mathscr{F}(U,\mathfrak{T}) \to \prod_{i\in I} \mathscr{F}(U_i,\mathfrak{T}_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_{ij},\mathfrak{T}_{ij})$$

is exact, where $(U_{ij},\mathfrak{T}_{ij}) = (U_i,\mathfrak{T}_i) \times_{(U,\mathfrak{T})} (U_j,\mathfrak{T}_j)$. Hence, $\mathscr{F}$ is a sheaf. □

7.11. – Recall that a family of morphisms of schemes $\{f_i : T_i \to T\}_{i\in I}$ is called an *fppf covering* if each morphism f_i is flat and locally of finite presentation and if $|T| = \bigcup_{i\in I} f_i(|T_i|)$ (cf. [**12**] IV 6.3.2).

We say that a family of $\mathscr{S}$-morphisms of flat formal $\mathscr{S}$-schemes $\{f_i : \mathfrak{T}_i \to \mathfrak{T}\}_{i\in I}$ is an *fppf covering* if each morphism f_i is locally of finite presentation ([**1**] 2.3.15) and if its special fiber $\{T_i \to T\}_{i\in I}$ is an fppf covering of schemes. By ([**1**] 2.3.16) and 7.2(ii), a family of $\mathscr{S}$-morphisms of flat formal $\mathscr{S}$-schemes $\{\mathfrak{T}_i \to \mathfrak{T}\}_{i\in I}$ is an fppf covering if and only if the family $\{\mathfrak{T}_{i,n} \to \mathfrak{T}_n\}_{i\in I}$ is an fppf covering of schemes for all integers $n \geqslant 1$.

Recall that an adic formal $\mathscr{S}$-scheme $\mathfrak{T}$ is flat over $\mathscr{S}$ if and only if $\mathfrak{T}_n$ is flat over $\mathscr{S}_n$ for all integers $n \geqslant 1$ (2.5). Since fppf coverings of schemes are stable by base change and by composition, the same holds for fppf coverings of flat formal $\mathscr{S}$-schemes.

DEFINITION 7.12. – Let $(U, \mathfrak{T}, u)$ be an object of $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$). We denote by $\mathrm{Cov}_{\mathrm{fppf}}(U, \mathfrak{T}, u)$ the set of families of flat morphisms $\{(U_i, \mathfrak{T}_i, u_i) \to (U, \mathfrak{T}, u)\}_{i\in I}$ (7.3) such that $\{U_i \to U\}_{i\in I}$ is a Zariski covering and that $\{\mathfrak{T}_i \to \mathfrak{T}\}_{i\in I}$ is an fppf covering (7.11).

It is clear that $\mathrm{Cov}_{\mathrm{fppf}}(U, \mathfrak{T}, u)$ contains $\mathrm{Cov}(U, \mathfrak{T}, u)$ (7.8).

7.13. – By 7.3 and 7.11, we see that the sets $\mathrm{Cov}_{\mathrm{fppf}}(U, \mathfrak{T})$ for $(U, \mathfrak{T}) \in \mathbf{Ob}(\mathscr{E}(X/\mathscr{S}))$ (resp. $(U, \mathfrak{T}) \in \mathbf{Ob}(\underline{\mathscr{E}}(X/\mathscr{S}))$) form a pretopology ([**3**] II 1.3). We call *fppf topology* the topology on $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$) associated to the pretopology defined by the sets $\mathrm{Cov}_{\mathrm{fppf}}(U, \mathfrak{T})$. We denote by $\widetilde{\mathscr{E}}(X/\mathscr{S})_{\mathrm{fppf}}$ (resp. $\widetilde{\underline{\mathscr{E}}}(X/\mathscr{S})_{\mathrm{fppf}}$) the topos of sheaves of sets on $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$), equipped with the fppf topology.

7.14. – Let $\mathscr{F}$ be a sheaf of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$ (resp. $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$) and $(U, \mathfrak{T})$ an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). Since $\mathscr{F}$ is also a sheaf for the Zariski topology, the presheaf $\mathscr{F}_{(U,\mathfrak{T})}$ (7.7) is a sheaf of U_{zar}.

Let $\{f : (U, \mathfrak{Z}) \to (U, \mathfrak{T})\}$ be an element of $\mathrm{Cov}_{\mathrm{fppf}}(U, \mathfrak{T})$ and put $(U, \mathfrak{Z} \times_{\mathfrak{T}} \mathfrak{Z}) = (U, \mathfrak{Z}) \times_{(U,\mathfrak{T})} (U, \mathfrak{Z})$. Since $\mathscr{F}$ is a sheaf for the fppf topology, we deduce an exact sequence of U_{zar} (7.6.2)

$$\mathscr{F}_{(U,\mathfrak{T})} \xrightarrow{\gamma_f} \mathscr{F}_{(U,\mathfrak{Z})} \rightrightarrows \mathscr{F}_{(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}. \tag{7.14.1}$$

7.15. – Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, $\widehat{\mathscr{C}}$ (resp. $\widehat{\mathscr{D}}$) the category of presheaves of sets on $\mathscr{C}$ (resp. $\mathscr{D}$) and $u : \mathscr{C} \to \mathscr{D}$ a functor. We have a functor

$$\widehat{u}^* : \widehat{\mathscr{D}} \to \widehat{\mathscr{C}} \qquad \mathscr{G} \mapsto \widehat{u}^*(\mathscr{G}) = \mathscr{G} \circ u. \tag{7.15.1}$$

It admits a right adjoint ([**3**] I 5.1)

$$\widehat{u}_* : \widehat{\mathscr{C}} \to \widehat{\mathscr{D}}. \tag{7.15.2}$$

Let $\mathscr{C}$ and $\mathscr{D}$ be two sites [3]. If $u : \mathscr{C} \to \mathscr{D}$ is a cocontinuous (resp. continuous) functor and $\mathscr{F}$ (resp. $\mathscr{G}$) is a sheaf on $\mathscr{C}$ (resp. $\mathscr{D}$), then $\widehat{u}_*(\mathscr{F})$ (resp. $\widehat{u}^*(\mathscr{G})$) is a sheaf on $\mathscr{D}$ (resp. $\mathscr{C}$) ([**3**] III 1.2, 2.2).

Let $\widetilde{\mathscr{C}}$ (resp. $\widetilde{\mathscr{D}}$) be the topos of the sheaves of sets on $\mathscr{C}$ (resp. $\mathscr{D}$) and $u : \mathscr{C} \to \mathscr{D}$ a cocontinuous functor. Then u induces a morphism of topoi $g : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{D}}$ defined by $g_* = \widehat{u}_*$ and $g^* = a \circ \widehat{u}^*$, where a is the sheafification functor (cf. [**3**] III 2.3).

3. We suppose that the site $\mathscr{C}$ is small.

7.16. – Note that the fppf topology on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is finer than the Zariski topology. Equipped with the fppf topology on the source and the Zariski topology on the target, the identical functors $\mathscr{E} \to \mathscr{E}$ and $\underline{\mathscr{E}} \to \underline{\mathscr{E}}$ are cocontinuous. By 7.15, they induce morphisms of topoi

$$\sigma : \widetilde{\mathscr{E}}_{\mathrm{fppf}} \to \widetilde{\mathscr{E}}, \qquad \sigma : \widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}} \to \widetilde{\underline{\mathscr{E}}}. \tag{7.16.1}$$

If $\mathscr{F}$ is a sheaf of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$ (resp. $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$), $\sigma_*(\mathscr{F})$ is equal to $\mathscr{F}$. If $\mathscr{G}$ is a sheaf of $\widetilde{\mathscr{E}}$ (resp. $\widetilde{\underline{\mathscr{E}}}$), then $\sigma^*(\mathscr{G})$ is the sheafification of $\mathscr{G}$ with respect to the fppf topology.

CHAPTER 8

CRYSTALS IN OYAMA TOPOI

In this section, we keep the notation in § 7 and we study crystals in Oyama topoi and explain their interpretations in terms of modules with stratification (8.10). Let n an integer $\geqslant 1$.

8.1. – We define a presheaf of rings $\mathscr{O}_{\mathscr{E}(X/\mathscr{S}),n}$ on $\mathscr{E}(X/\mathscr{S})$ (resp. $\mathscr{O}_{\underline{\mathscr{E}}(X/\mathscr{S}),n}$ on $\underline{\mathscr{E}}(X/\mathscr{S})$) (7.1) by

$$(U,\mathfrak{X}) \mapsto \Gamma(\mathfrak{X},\mathscr{O}_{\mathfrak{X}_n}). \tag{8.1.1}$$

For any object $(U,\mathfrak{X})$ of $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$) and any element $\{(U_i,\mathfrak{X}_i) \to (U,\mathfrak{X})\}_{i\in I}$ of $\mathrm{Cov}_{\mathrm{fppf}}(U,\mathfrak{X})$ (7.12), $\{\mathfrak{X}_{i,n} \to \mathfrak{X}_n\}_{i\in I}$ is an fppf covering of schemes (7.11). By fppf descent for quasi-coherent modules (**[21]** VIII 1.2), $\mathscr{O}_{\mathscr{E}(X/\mathscr{S}),n}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}}(X/\mathscr{S}),n}$) is a sheaf for the fppf topology (7.13). Since the fppf topology is finer than the Zariski topology, it is also a sheaf for the Zariski topology (7.9).

8.2. – For any object $(U,\mathfrak{X},u)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), we consider $\mathscr{O}_{\mathfrak{X}_n}$ as a sheaf of T_{zar} (resp. $\underline{T}_{\mathrm{zar}}$). We have (7.10)

$$(\mathscr{O}_{\mathscr{E},n})_{(U,\mathfrak{X},u)} = u_*(\mathscr{O}_{\mathfrak{X}_n}) \quad (\text{resp. } (\mathscr{O}_{\underline{\mathscr{E}},n})_{(U,\mathfrak{X},u)} = u_*(\mathscr{O}_{\mathfrak{X}_n})).$$

A morphism $f:(U_1,\mathfrak{X}_1,u_1) \to (U_2,\mathfrak{X}_2,u_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) induces a morphism of ringed topoi

$$\widetilde{f}_n : (U_{1,\mathrm{zar}}, u_{1*}(\mathscr{O}_{\mathfrak{X}_{1,n}})) \to (U_{2,\mathrm{zar}}, u_{2*}(\mathscr{O}_{\mathfrak{X}_{2,n}})). \tag{8.2.1}$$

By 7.7 and 7.10 and a standard argument, an $\mathscr{O}_{\mathscr{E},n}$-module of $\widetilde{\mathscr{E}}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module of $\widetilde{\underline{\mathscr{E}}}$) is equivalent to the following data:

(i) For every object $(U,\mathfrak{X},u)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $u_*(\mathscr{O}_{\mathfrak{X}_n})$-module $\mathscr{F}_{(U,\mathfrak{X})}$ of U_{zar}.

(ii) For every morphism $f:(U_1,\mathfrak{X}_1,u_1) \to (U_2,\mathfrak{X}_2,u_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $u_{1*}(\mathscr{O}_{\mathfrak{X}_{1,n}})$-linear morphism $c_f : \widetilde{f}_n^*(\mathscr{F}_{(U_2,\mathfrak{X}_2)}) \to \mathscr{F}_{(U_1,\mathfrak{X}_1)}$,

which is subject to the following conditions:

(a) If f is the identity morphism, then c_f is the identity.

(b) If f is a Cartesian morphism, then c_f is an isomorphism.

(c) If f and g are two composable morphisms of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), then we have $c_{g\circ f} = c_f \circ \widetilde{f}_n^*(c_g)$.

We call the data $\{\mathscr{F}_{(U,\mathfrak{T})}, c_f\}$ *the linearized descent data associated to the $\mathscr{O}_{\mathscr{E},n}$-module (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module) $\mathscr{F}$* of $\widetilde{\mathscr{E}}$ (resp. of $\widetilde{\underline{\mathscr{E}}}$).

An $\mathscr{O}_{\mathscr{E},n}$-module (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module) $\mathscr{F}$ of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$ (resp. $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$) gives rise to an $\mathscr{O}_{\mathscr{E},n}$-module (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module) $\sigma_*(\mathscr{F})$ of $\widetilde{\mathscr{E}}$ (resp. $\widetilde{\underline{\mathscr{E}}}$) (7.16.1). We can associate to $\mathscr{F}$ a linearized descent data $\{\mathscr{F}_{(U,\mathfrak{T})}, c_f\}$ by that of $\sigma_*(\mathscr{F})$.

Definition 8.3 ([**32**] 1.3.3). – Let $\mathscr{F}$ be an $\mathscr{O}_{\mathscr{E},n}$-module of $\widetilde{\mathscr{E}}$ (resp. an $\mathscr{O}_{\mathscr{E},n}$-module of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$, resp. an $\mathscr{O}_{\underline{\mathscr{E}},n}$-module of $\widetilde{\underline{\mathscr{E}}}$, resp. an $\mathscr{O}_{\underline{\mathscr{E}},n}$-module of $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$).

(i) We say that $\mathscr{F}$ is *quasi-coherent* if for every object $(U,\mathfrak{T},u)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), the $u_*(\mathscr{O}_{\mathfrak{T}_n})$-module $\mathscr{F}_{(U,\mathfrak{T})}$ of U_{zar} (8.2) is quasi-coherent ([**22**] 0.5.1.3).

(ii) We say that $\mathscr{F}$ is a *crystal* or a *crystal of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}$* (resp. *$\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$*, resp. *$\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\widetilde{\underline{\mathscr{E}}}$*, resp. *$\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$*) if for every morphism $f:(U_1,\mathfrak{T}_1)\to(U_2,\mathfrak{T}_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), the canonical morphism c_f (8.2) is an isomorphism.

We denote by $\mathscr{C}(\mathscr{O}_{\mathscr{E},n})$ (resp. $\mathscr{C}_{\mathrm{fppf}}(\mathscr{O}_{\mathscr{E},n})$, resp. $\mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n})$, resp. $\mathscr{C}_{\mathrm{fppf}}(\mathscr{O}_{\underline{\mathscr{E}},n})$) the category of crystals of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}$ (resp. $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$, resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\widetilde{\underline{\mathscr{E}}}$, resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$) and we use the notation $\mathscr{C}^{\mathrm{qcoh}}(-)$ or $\mathscr{C}^{\mathrm{qcoh}}_{\mathrm{fppf}}(-)$ to denote the full subcategory consisting of quasi-coherent crystals.

Proposition 8.4. – *A quasi-coherent $\mathscr{O}_{\mathscr{E},n}$-module $\mathscr{F}$ of $\widetilde{\mathscr{E}}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module of $\widetilde{\underline{\mathscr{E}}}$) is equivalent to*

(i) *For every object $(U,\mathfrak{T})$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), a quasi-coherent $\mathscr{O}_{\mathfrak{T}_n}$-module $\mathscr{F}_{(U,\mathfrak{T})}$ of $\mathfrak{T}_{n,\mathrm{zar}}$;*

(ii) *For every morphism $f:(U_1,\mathfrak{T}_1)\to(U_2,\mathfrak{T}_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $(\mathscr{O}_{\mathfrak{T}_{1,n}})$-linear morphism of $\mathfrak{T}_{1,\mathrm{zar}}$:*

$$c_f : f_n^*(\mathscr{F}_{(U_2,\mathfrak{T}_2)}) \to \mathscr{F}_{(U_1,\mathfrak{T}_1)}, \tag{8.4.1}$$

where f_n denotes the morphism $\mathfrak{T}_{1,n}\to\mathfrak{T}_{2,n}$; which are subject to similar conditions (a-c) in 8.2.

The assertion follows from the following proposition.

Proposition 8.5. – *Let $u:T\to U$ be an affine morphism of schemes, $i:T\to\mathscr{T}$ a nilpotent closed immersion. We denote by $v:(\mathscr{T},\mathscr{O}_{\mathscr{T}})\to(U,u_*(\mathscr{O}_{\mathscr{T}}))$ the morphism of ringed topoi induced by u. Then, the inverse image and direct image functors of v induce equivalences of categories quasi-inverse to each other between the category of quasi-coherent $\mathscr{O}_{\mathscr{T}}$-modules of $\mathscr{T}_{\mathrm{zar}}$ and the category of quasi-coherent $u_*(\mathscr{O}_{\mathscr{T}})$-modules of U_{zar}.*

The proposition follows from 8.7 and 8.8.

LEMMA 8.6. – *We keep the assumption of 8.5. The restriction of the functor v_* to the category of quasi-coherent $\mathscr{O}_{\mathscr{T}}$-modules is exact.*

Proof. – The functor v_* is left exact. Let $\mathscr{M} \to \mathscr{N}$ be a surjection of quasi-coherent $\mathscr{O}_{\mathscr{T}}$-modules. To show that $v_*(\mathscr{M}) \to v_*(\mathscr{N})$ is surjective, it suffices to show that for any affine open subscheme V of U, the morphism $v_*(\mathscr{M})(V) \to v_*(\mathscr{N})(V)$ is surjective. Since u is affine, the open subscheme $\mathscr{T}_V$ of $\mathscr{T}$ associated to the open subset $u^{-1}(V)$ of T is affine. Since $\mathscr{M}$, $\mathscr{N}$ are quasi-coherent and $\mathscr{T}_V$ is affine, the morphism $\mathscr{M}(\mathscr{T}_V) \to \mathscr{N}(\mathscr{T}_V)$ is surjective. Then the assertion follows. □

LEMMA 8.7. – *We keep the assumption of 8.5. For any quasi-coherent $\mathscr{O}_{\mathscr{T}}$-module $\mathscr{M}$, $v_*(\mathscr{M})$ is quasi-coherent and the adjunction morphism $v^*(v_*(\mathscr{M})) \to \mathscr{M}$ is an isomorphism.*

Proof. – The question being local, we may assume that U is affine and so is $\mathscr{T}$. Then the quasi-coherent $\mathscr{O}_{\mathscr{T}}$ module $\mathscr{M}$ admits a presentation

$$\mathscr{O}_{\mathscr{T}}^{\oplus J} \to \mathscr{O}_{\mathscr{T}}^{\oplus I} \to \mathscr{M} \to 0. \tag{8.7.1}$$

The canonical morphism $(v_*(\mathscr{O}_{\mathscr{T}}))^{\oplus I} \to v_*(\mathscr{O}_{\mathscr{T}}^{\oplus I})$ is clearly an isomorphism. Since v_* is exact (8.6), we obtain a presentation of $v_*(\mathscr{M})$

$$(v_*(\mathscr{O}_{\mathscr{T}}))^{\oplus J} \to (v_*(\mathscr{O}_{\mathscr{T}}))^{\oplus I} \to v_*(\mathscr{M}) \to 0.$$

The first assertion follows.

The exact sequence (8.7.1) induces a commutative diagram

$$\begin{tikzcd}
(v^*(v_*(\mathscr{O}_{\mathscr{T}})))^{\oplus J} \arrow[r] \arrow[d] & (v^*(v_*(\mathscr{O}_{\mathscr{T}})))^{\oplus I} \arrow[r] \arrow[d] & v^*(v_*(\mathscr{M})) \arrow[r] \arrow[d] & 0 \\
\mathscr{O}_{\mathscr{T}}^{\oplus J} \arrow[r] & \mathscr{O}_{\mathscr{T}}^{\oplus I} \arrow[r] & \mathscr{M} \arrow[r] & 0.
\end{tikzcd}$$

Since v_* is exact and v^* is right exact, horizontal arrows are exact. The first two vertical arrows are isomorphisms. Then the second assertion follows. □

LEMMA 8.8. – *We keep the assumption of 8.5. For any quasi-coherent $u_*(\mathscr{O}_{\mathscr{T}})$-module $\mathscr{N}$, $v^*(\mathscr{N})$ is quasi-coherent and the adjunction morphism $\mathscr{N} \to v_*(v^*(\mathscr{N}))$ is an isomorphism.*

Proof. – The pull-back functor sends quasi-coherent objects to quasi-coherent objects ([**22**] 5.1.4).

The second assertion being local, we may assume that U is affine and that $\mathscr{N}$ admits a presentation:

$$(u_*(\mathscr{O}_{\mathscr{T}}))^{\oplus J} \to (u_*(\mathscr{O}_{\mathscr{T}}))^{\oplus I} \to \mathscr{N} \to 0$$

Then we deduce a commutative diagram

$$\begin{tikzcd}
(u_*(\mathscr{O}_{\mathscr{T}}))^{\oplus J} \arrow[r] \arrow[d] & (u_*(\mathscr{O}_{\mathscr{T}}))^{\oplus I} \arrow[r] \arrow[d] & \mathscr{N} \arrow[r] \arrow[d] & 0 \\
(v_*(v^*(u_*(\mathscr{O}_{\mathscr{T}}))))^{\oplus J} \arrow[r] & (v_*(v^*(u_*(\mathscr{O}_{\mathscr{T}}))))^{\oplus I} \arrow[r] & v_*(v^*(\mathscr{N})) \arrow[r] & 0.
\end{tikzcd}$$

Since v_* is exact (8.6) and v^* is right exact, the horizontal arrows are exact. The first two vertical arrows are isomorphisms. Then the second assertion follows. □

PROPOSITION 8.9. – *Let X be a k-scheme. The direct image functors $\sigma_* : \widetilde{\mathscr{E}}_{\mathrm{fppf}} \to \widetilde{\mathscr{E}}$ and $\sigma_* : \widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}} \to \widetilde{\underline{\mathscr{E}}}$* (7.16.1) *induce equivalences of categories*

$$\mathscr{C}^{\mathrm{qcoh}}_{\mathrm{fppf}}(\mathscr{O}_{\mathscr{E},n}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E},n}), \qquad \mathscr{C}^{\mathrm{qcoh}}_{\mathrm{fppf}}(\mathscr{O}_{\underline{\mathscr{E}},n}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n}). \tag{8.9.1}$$

Proof. – The functor σ_* sends quasi-coherent crystals of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}_{\mathrm{fppf}}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}}$) to quasi-coherent crystals of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\widetilde{\underline{\mathscr{E}}}$). The functors (8.9.1) are clearly fully faithful. It suffices to show the essential surjectivity.

Let $\mathscr{F}$ be a quasi-coherent crystal of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}$ and $\{\mathscr{F}_{(U,\mathfrak{T})}, c_f\}$ the associated linearized descent data. We consider $\mathscr{F}_{(U,\mathfrak{T})}$ as an $\mathscr{O}_{\mathfrak{T}_n}$-module of T_{zar} (8.4). To show $\mathscr{F}$ is a sheaf for fppf topology, we prove that, for any element $\{(U_i,\mathfrak{T}_i) \to (U,\mathfrak{T})\}_{i\in I}$ of $\mathrm{Cov}_{\mathrm{fppf}}(U,\mathfrak{T})$ (7.12), the sequence

$$0 \to \Gamma(T, \mathscr{F}_{(U,\mathfrak{T})}) \to \prod_{i\in I} \Gamma(T_i, \mathscr{F}_{(U_i,\mathfrak{T}_i)}) \to \prod_{i,j\in I} \Gamma(T_{ij}, \mathscr{F}_{(U_{ij},\mathfrak{T}_{ij})}) \tag{8.9.2}$$

is exact, where $(U_{ij},\mathfrak{T}_{ij}) = (U_i,\mathfrak{T}_i) \times_{(U,\mathfrak{T})} (U_j,\mathfrak{T}_j)$ (7.3). Then, we have

$$\mathscr{F}_{(U_i,\mathfrak{T}_i)} \simeq f_i^*(\mathscr{F}_{(U,\mathfrak{T})}), \quad \mathscr{F}_{(U_{ij},\mathfrak{T}_{ij})} \simeq f_{ij}^*(\mathscr{F}_{(U,\mathfrak{T})}), \tag{8.9.3}$$

where f_i (resp. f_{ij}) denotes the morphism $\mathfrak{T}_i \to \mathfrak{T}$ (resp. $\mathfrak{T}_{ij} \to \mathfrak{T}$). It suffices to show that the sequence

$$0 \to \mathscr{F}_{(U,\mathfrak{T})} \to \prod_{i\in I} f_{i*}(\mathscr{F}_{(U_i,\mathfrak{T}_i)}) \to \prod_{i,j\in I} f_{ij*}(\mathscr{F}_{(U_{ij},\mathfrak{T}_{ij})}) \tag{8.9.4}$$

is exact. The question being local, we can suppose that U is quasi-compact and quasi-separated and so is $\mathfrak{T}$. Hence we can reduce to the case where the set I is finite. Note that the schemes $\mathfrak{T}_{i,n}$ and $\mathfrak{T}_n$ are quasi-compact and quasi-separated and that the morphisms f_i and f_{ij} are quasi-compact. Then the exactness of (8.9.4) follows from the fppf descent for quasi-coherent modules.

The assertion for quasi-coherent crystals of $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules can be verified in the same way. □

PROPOSITION 8.10. – *Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme and X its special fiber. There exists a canonical equivalence of tensor categories between the category $\mathscr{C}(\mathscr{O}_{\mathscr{E},n})$ (resp. $\mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n})$) and the category of $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathscr{R}_{\mathfrak{X}}$-stratification (resp. $\mathscr{Q}_{\mathfrak{X}}$-stratification)* (4.11), 5.4.

Proof. – The proof is standard. We briefly explain the construction following ([**32**] 1.3.4) where the author deals with the case $n = 1$.

The triple $(X, \mathfrak{X}, \mathrm{id})$ (resp. $(X, \mathfrak{X}, \underline{X} \to X)$) is an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$). By 4.9, $(X, R_{\mathfrak{X}}(r))$ (resp. $(X, Q_{\mathfrak{X}}(r))$) is an object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) for all integers $r \geqslant 1$. We take again the notation of the proof of 4.11. We denote by $q_1, q_2 : (X, R_{\mathfrak{X}}) \to (X, \mathfrak{X})$ the canonical morphisms. Let $\mathscr{F}$ be a crystal of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}$. We set $\mathscr{J}$ to be the $\mathscr{O}_{\mathfrak{X}_n}$-module $\mathscr{F}_{(X,\mathfrak{X})}$. Then we deduce isomorphisms of $\mathscr{R}_{\mathfrak{X}}$-modules of X_{zar} (8.2)

$$c_{q_1} : \widetilde{q}^*_{1,n}(\mathscr{J}) \xrightarrow{\sim} \mathscr{F}_{(X,R_{\mathfrak{X}})} \qquad c_{q_2} : \widetilde{q}^*_{2,n}(\mathscr{J}) \xrightarrow{\sim} \mathscr{F}_{(X,R_{\mathfrak{X}})}. \tag{8.10.1}$$

We denote by ε the composition of c_{q_2} and the inverse of c_{q_1}. By a standard argument, we can show that ε defines a $\mathscr{R}_{\mathfrak{X}}$-stratification on $\mathscr{J}$. It is clear that the correspondence $\mathscr{F} \to (\mathscr{J}, \varepsilon)$ is functorial and is compatible with tensor products (5.4).

Conversely, let $\mathscr{J}$ be an $\mathscr{O}_{\mathfrak{X}_n}$-module with an $\mathscr{R}_{\mathfrak{X}}$-stratification $\varepsilon : \widetilde{q}^*_{2,n}(\mathscr{J}) \xrightarrow{\sim} \widetilde{q}^*_{1,n}(\mathscr{J})$. Let $(U, \mathfrak{T}, u)$ be an object of $\mathscr{E}$ such that U is affine. Since $\mathfrak{X}$ is smooth over $\mathscr{S}$ and $\mathfrak{T}$ is affine, the k-morphism $u : T \to U$ extends to a morphism $\varphi : (U, \mathfrak{T}) \to (X, \mathfrak{X})$ of $\mathscr{E}$. We define $\mathscr{F}_{(U,\mathfrak{T})}$ to be the $\mathscr{O}_{\mathfrak{T}_n}$-module $\widetilde{\varphi}^*_n(\mathscr{J})$ of U_{zar}. By a standard arguement, we can show that this definition of $\mathscr{F}_{(U,\mathfrak{T})}$ is independent of the choice of the deformation $\mathfrak{T} \to \mathfrak{X}$ of $u : T \to U$ up to a canonical isomorphism which comes from the stratification.

Let $g : (U_1, \mathfrak{T}_1) \to (U_2, \mathfrak{T}_2)$ be a morphism of $\mathscr{E}$ such that U_1 and U_2 are affine. There exists morphisms $\varphi_1 : (U_1, \mathfrak{T}_1) \to (X, \mathfrak{X})$ and $\varphi_2 : (U_2, \mathfrak{T}_2) \to (X, \mathfrak{X})$ of $\mathscr{E}$. Then there exists a unique morphism $h : (U_1, \mathfrak{T}_1) \to (X, R_{\mathfrak{X}})$ such that $q_1 \circ h = \varphi_1$ and $q_2 \circ h = \varphi_2 \circ g$. We deduce a canonical $\mathscr{O}_{\mathfrak{T}_{1,n}}$-linear isomorphism of $U_{1,\mathrm{zar}}$

$$c_g : \widetilde{g}^*_n(\mathscr{F}_{(U_2,\mathfrak{T}_2)}) = \widetilde{h}^*_n(\widetilde{q}^*_{2,n}(\mathscr{J})) \xrightarrow[\sim]{\widetilde{h}^*_n(\varepsilon)} \widetilde{h}^*_n(\widetilde{q}^*_{1,n}(\mathscr{J})) = \mathscr{F}_{(U_1,\mathfrak{T}_1)}. \tag{8.10.2}$$

By gluing the constructions for affine objects, we obtain an isomorphism c_g for a general morphism g of $\mathscr{E}$. We deduce the cocycle properties for c_g by the cocycle condtion of ε.

Hence we obtain a linearized descent data $\{\mathscr{F}_{(U,\mathfrak{T},u)}, c_f\}$ such that each morphism c_f is an isomorphism. Then, we get a crystal of $\mathscr{O}_{\mathscr{E},n}$-modules $\mathscr{F}$ of $\widetilde{\mathscr{E}}$ by 8.2. The correspondence $(\mathscr{J}, \varepsilon) \mapsto \mathscr{F}$ is clearly functorial and quasi-inverse to the previous construction.

The assertion for crystals of $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules can be verified in the same way. $\square$

CHAPTER 9

CARTIER EQUIVALENCE

In this section, we show that the relative Frobenius morphism of X induces an equivalence of topoi between $\underline{\widetilde{\mathscr{E}}}_{\mathrm{fppf}}$ and $\widetilde{\mathscr{E}}'_{\mathrm{fppf}}$. Then we prove that this equivalence globalizes Shiho's local Cartier transform modulo p^n explained in § 6.

9.1. – Let $(U,\mathfrak{T},u)$ be an object of $\underline{\mathscr{E}}$. We have a commutative diagram

$$\begin{tikzcd} U \arrow[d, "F_{U/k}"'] & \underline{T} \arrow[l, "u"'] \arrow[r, hook] \arrow[d, "F_{\underline{T}/k}"'] & T \arrow[d, "F_{T/k}"] \arrow[dl, "f_{T/k}"'] \\ U' & \underline{T}' \arrow[l, "u'"'] \arrow[r, hook] & T', \end{tikzcd} \tag{9.1.1}$$

where the vertical arrows denote the relative Frobenius morphisms (2.2, 2.3). It is clear that $(U',\mathfrak{T},u'\circ f_{T/k})$ is an object of $\mathscr{E}'$ (7.1). Moreover, the correspondence $(U,\mathfrak{T},u)\mapsto(U',\mathfrak{T},u'\circ f_{T/k})$ is functorial. We denote by ρ the functor defined as above:

$$\rho:\underline{\mathscr{E}}\to\mathscr{E}',\qquad (U,\mathfrak{T},u)\mapsto(U',\mathfrak{T},u'\circ f_{T/k}). \tag{9.1.2}$$

We will show in 9.3(ii) that ρ is continuous and cocontinuous with respect to either Zariski or fppf topology. By 7.15, the functor ρ (9.1.2) induces morphisms of topoi

$$\mathrm{C}_{X/\mathscr{S}}:\underline{\widetilde{\mathscr{E}}}\to\widetilde{\mathscr{E}}', \tag{9.1.3}$$

$$\mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}:\underline{\widetilde{\mathscr{E}}}_{\mathrm{fppf}}\to\widetilde{\mathscr{E}}'_{\mathrm{fppf}}, \tag{9.1.4}$$

such that the pullback functor is induced by the composition with ρ. They fit into a commutative diagram (7.16.1)

$$\begin{tikzcd} \underline{\widetilde{\mathscr{E}}}_{\mathrm{fppf}} \arrow[r, "\mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}"] \arrow[d, "\sigma"'] & \widetilde{\mathscr{E}}'_{\mathrm{fppf}} \arrow[d, "\sigma'"] \\ \underline{\widetilde{\mathscr{E}}} \arrow[r, "\mathrm{C}_{X/\mathscr{S}}"] & \widetilde{\mathscr{E}}'. \end{tikzcd} \tag{9.1.5}$$

One of the main results in this section is the following.

THEOREM 9.2 (cf. [**32**] 1.4.6). – *For any smooth k-scheme X, the morphism $\mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}$: $\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}} \to \widetilde{\mathscr{E}}'_{\mathrm{fppf}}$ is an equivalence of topoi.*

The theorem follows from 9.3, 9.8 and 9.10.

PROPOSITION 9.3 (cf. [**32**] 1.4.1). – (i) *The functor ρ (9.1.2) is fully faithful.*

(ii) *Equipped with the Zariski topology (7.9) (resp. fppf topology (7.13)) on both sides, the functor ρ is continuous and cocontinuous* ([**3**] III 1.1, 2.1).

LEMMA 9.4. – *Let Y, Z be two k-schemes and $g_1, g_2 : \underline{Y} \to Z$ two k-morphisms. We put $h_i = g_i' \circ f_{Y/k} : Y \to \underline{Y}' \to Z'$ (9.1.1) for $i = 1, 2$. If $h_1 = h_2$, then $g_1 = g_2$.*

Proof. – Let U be an affine open subscheme of Z. Since $f_{Y/k}$ is a homeomorphism and $h_1 = h_2$, we have $g_1^{-1}(U) = g_2^{-1}(U)$. Hence we can reduce to case where Z is affine.

Since the morphism $f_{Y/k}$ is scheme theoretically dominant (2.3), we deduce that $g_1' = g_2'$ by ([**22**] 5.4.1). The functor $X \mapsto X'$ from the category of k-schemes to itself is clearly faithful. Then the lemma follows. □

LEMMA 9.5. – *Let $(U, \mathfrak{T}, u)$ be an object of $\underline{\mathscr{E}}$ and $g : (V', \mathfrak{Z}, w) \to \rho(U, \mathfrak{T}, u)$ a morphism of $\mathscr{E}'$. Then there exist an object $(V, \mathfrak{Z}, v)$ of $\underline{\mathscr{E}}$ and a morphism $f : (V, \mathfrak{Z}, v) \to (U, \mathfrak{T}, u)$ of $\underline{\mathscr{E}}$ such that $g = \rho(f)$. If g is Cartesian (resp. flat), so is f.*

Proof. – Put $V = F_{U/k}^{-1}(V')$. Since the composition $Z \to T \xrightarrow{u' \circ f_{T/k}} U'$ factors through $V' \subset U'$, the composition $\underline{Z} \to \underline{T} \xrightarrow{u} U$ factors through V. We obtain a k-morphism $v : \underline{Z} \to V$ such that $w = v' \circ f_{Z/k}$. Since $f_{Z/k}$ is separated, v' is affine ([**18**] 1.6.2(v)) and so is v. Hence, we get an object $(V, \mathfrak{Z}, v)$ of $\underline{\mathscr{E}}$ and a morphism $f : (V, \mathfrak{Z}, v) \to (U, \mathfrak{T}, u)$ of $\underline{\mathscr{E}}$ such that $g = \rho(f)$. □

9.6. – *Proof of 9.3.* (i) The functor ρ is clearly faithful. We prove its fullness. Let $(U_1, \mathfrak{T}_1, u_1)$, $(U_2, \mathfrak{T}_2, u_2)$ be two objects of $\underline{\mathscr{E}}$ and $g : \rho(U_1, \mathfrak{T}_1, u_1) \to \rho(U_2, \mathfrak{T}_2, u_2)$ a morphism of $\mathscr{E}'$. Since $U_1' \subset U_2' \subset U'$, we have $U_1 \subset U_2 \subset U$. It suffices to show that the diagram

$$\begin{tikzcd} \underline{T}_1 \arrow[r, "\underline{g}_s"] \arrow[d, "u_1"'] & \underline{T}_2 \arrow[d, "u_2"] \\ U_1 \arrow[r] & U_2 \end{tikzcd}$$

is commutative. It follows from 9.4 applied to the compositions $\underline{T_1} \xrightarrow{u_1} U_1 \to U_2$ and $\underline{T_1} \to \underline{T_2} \xrightarrow{u_2} U_2$.

(ii) A family of morphisms $\{(U_i, \mathfrak{T}_i) \to (U, \mathfrak{T})\}_{i \in I}$ of $\underline{\mathscr{E}}$ belongs to $\mathrm{Cov}(U, \mathfrak{T})$ (resp. $\mathrm{Cov}_{\mathrm{fppf}}(U, \mathfrak{T})$) if and only if, its image by ρ belongs to $\mathrm{Cov}(\rho(U, \mathfrak{T}))$ (resp. $\mathrm{Cov}_{\mathrm{fppf}}(\rho(U, \mathfrak{T}))$). The functor ρ sends flat morphisms to flat morphisms and it commutes with the fibered product of a flat morphism and a morphism of $\underline{\mathscr{E}}$. Indeed, by functoriality, if $T_1 \to T$ and $T_2 \to T$ are two morphisms of k-schemes, the canonical morphism $f_{T_1/k} \times f_{T_2/k} : T_1 \times_T T_2 \to \underline{T_1}' \times_{\underline{T}'} \underline{T_2}'$ is equal to the composition $T_1 \times_T T_2 \to (\underline{T_1 \times_T T_2})' \to (\underline{T_1} \times_{\underline{T}} \underline{T}_2)' = \underline{T_1}' \times_{\underline{T}'} \underline{T_2}'$. Then the continuity of ρ follows from ([**3**] III 1.6).

Let $\{(U_i', \mathfrak{T}_i) \to \rho(U, \mathfrak{T})\}_{i\in I}$ be an element of $\mathrm{Cov}(\rho(U,\mathfrak{T}))$ (resp. $\mathrm{Cov}_{\mathrm{fppf}}(\rho(U,\mathfrak{T}))$). By 9.5, there exists an element $\{(U_i, \mathfrak{T}_i) \to (U, \mathfrak{T})\}_{i\in I}$ of $\mathrm{Cov}(U,\mathfrak{T})$ (resp. $\mathrm{Cov}_{\mathrm{fppf}}(U,\mathfrak{T})$) mapping by ρ to the given element. Then, ρ is cocontinuous by ([**3**] III 2.1). □

9.7. – To prove 9.2, we use (local) liftings of the Frobenius morphism to construct fppf converings. Let X be an affine scheme and $h : X \to Y = \mathrm{Spec}(k[T_1, \dots, T_d])$ an étale morphism. By ([**24**] 3.2), the following diagram is Cartesian (2.2)

(9.7.1)
$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "h"] \\ Y \arrow[r, "F_Y"] & Y. \end{tikzcd}$$

We put $\mathfrak{Y} = \mathrm{Spf}(\mathrm{W}\{\mathrm{T}_1, \dots, T_d\})$ and we denote by $F_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{Y}$ the affine morphism defined by $\sigma : \mathrm{W} \to \mathrm{W}$ and $T_i \mapsto T_i^p$. Since X is étale over Y, there exists a unique deformation $\mathfrak{X}$ of X over $\mathfrak{Y}$ up to a unique isomorphism. The formal scheme $\mathfrak{X} \times_{\mathfrak{Y}, F_{\mathfrak{Y}}} \mathfrak{Y}$ is also a deformation of $X = X \times_{Y, F_Y} Y$ over $\mathfrak{Y}$. Then we deduce a Cartesian diagram

(9.7.2)
$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ \mathfrak{Y} \arrow[r, "F_{\mathfrak{Y}}"] & \mathfrak{Y}. \end{tikzcd}$$

In particular, we obtain a morphism of finite type $F_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}$ above σ, which lifts the absolute Frobenius morphism F_X of X. We put $\mathfrak{X}' = \mathfrak{X} \times_{\mathscr{S}, \sigma} \mathscr{S}$. Then we obtain an $\mathscr{S}$-morphism of finite type $F_{\mathfrak{X}/\mathscr{S}} : \mathfrak{X} \to \mathfrak{X}'$ which lifts the relative Frobenius morphism $F_{X/k}$. Since X is smooth, the morphism $F_{X/k} : X \to X'$ is faithfully flat (cf. [**24**] 3.2). Hence $\{F_{\mathfrak{X}/\mathscr{S}} : \mathfrak{X} \to \mathfrak{X}'\}$ is an fppf covering in the sense of 7.11.

LEMMA 9.8. – *Let $(U', \mathfrak{T}, u)$ be an object of $\mathscr{E}'$, $U = F_{X/k}^{-1}(U')$. Suppose that U is affine and that there exists an étale k-morphism $U \to \mathrm{Spec}(k[T_1, \dots, T_d])$.*

(i) *There exists an object $(U, \mathfrak{Z})$ of $\underline{\mathscr{E}}$ and an element $\{f : \rho(U, \mathfrak{Z}) \to (U', \mathfrak{T})\}$ of $\mathrm{Cov}_{\mathrm{fppf}}(U', \mathfrak{T})$.*

(ii) *Let $g : (U_1', \mathfrak{T}_1) \to (U', \mathfrak{T})$ be a morphism of $\mathscr{E}'$. Then there exists a morphism $h : (U_1, \mathfrak{Z}_1) \to (U, \mathfrak{Z})$ of $\underline{\mathscr{E}}$ and an element $\{\varphi : \rho(U_1, \mathfrak{Z}_1) \to (U_1', \mathfrak{T}_1)\}$*

of $\mathrm{Cov}_{\mathrm{fppf}}(U_1', \mathfrak{X}_1)$ *such that the following diagram is Cartesian:*

(9.8.1)
$$\begin{tikzcd}
\rho(U_1, \mathfrak{Z}_1) \arrow[r, "\varphi"] \arrow[d, "\rho(h)"'] \arrow[dr, phantom, "\square"] & (U_1', \mathfrak{X}_1) \arrow[d, "g"] \\
\rho(U, \mathfrak{Z}) \arrow[r, "f"] & (U', \mathfrak{X}).
\end{tikzcd}$$

If g is a Cartesian morphism so is h.

Proof. – (i) We follow the proof of ([**32**] 1.4.5). Let $\mathfrak{U}$ be a smooth lifting of U over $\mathscr{S}$ and $F : \mathfrak{U} \to \mathfrak{U}'$ a lifting of $F_{U/k}$ as in 9.7. Note that U' is affine. Since the morphism u is affine, T and $\mathfrak{X}$ are affine. Since $\mathfrak{U}'$ is smooth over $\mathscr{S}$, there exists an $\mathscr{S}$-morphism $\tau : \mathfrak{X} \to \mathfrak{U}'$ which lifts u. We consider the following commutative diagram:

(9.8.2)
$$\begin{tikzcd}
& T \times_{U'} U \arrow[rr] \arrow[dl] \arrow[dd] & & \mathfrak{X} \times_{\mathfrak{U}'} \mathfrak{U} \arrow[dl] \arrow[dd] \\
T \arrow[rr, crossing over] \arrow[dd, "u"'] & & \mathfrak{X} & \\
& U \arrow[rr] \arrow[dl, "F_{U/k}"] & & \mathfrak{U} \arrow[dl, "F"] \\
U' \arrow[rr] & & \mathfrak{U}'. \arrow[from=uu, crossing over] &
\end{tikzcd}$$

We set $\mathfrak{Z} = \mathfrak{X} \times_{\mathfrak{U}'} \mathfrak{U}$, $Z = T \times_{U'} U$ and we denote the composition $\underline{Z} \to Z \to U$ by v. Then we obtain an object $(U, \mathfrak{Z}, v)$ of $\underline{\mathscr{E}}$. By (9.8.2), one verifies that the diagram

(9.8.3)
$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "f_{Z/k}"'] & T \arrow[d, "u"] \\
\underline{Z}' \arrow[r, "v'"] & U'
\end{tikzcd}$$

is commutative. Then, we obtain a morphism $\rho(U, \mathfrak{Z}, v) \to (U', \mathfrak{X}, u)$ of $\mathscr{E}'$. Since $\{F : \mathfrak{U} \to \mathfrak{U}'\}$ is an fppf covering, $\{\rho(U, \mathfrak{Z}, v) \to (U', \mathfrak{X}, u)\}$ is an element of $\mathrm{Cov}_{\mathrm{fppf}}(U', \mathfrak{X})$.

(ii) The morphism f is flat. We denote by $(U_1', \mathfrak{Z}_1)$ the fibered product $\rho(U, \mathfrak{Z}) \times_{(U', \mathfrak{X})} (U_1', \mathfrak{X}_1)$ in $\mathscr{E}$. By applying 9.5 to the projection $(U_1', \mathfrak{Z}_1) \to \rho(U, \mathfrak{Z})$, we obtain the Cartesian diagram (9.8.1). Since φ is the base change of f, φ is an element of $\mathrm{Cov}_{\mathrm{fppf}}(U_1', \mathfrak{X}_1)$. $\square$

The following lemma is a complement of 9.8 and will be used in the proof of 9.13 below.

Lemma 9.9. – *Let X be a k-scheme and $(U', \mathfrak{X})$ an object of $\mathscr{E}'$, $(U, \mathfrak{Z})$ an object of $\underline{\mathscr{E}}$ and $\{\rho(U, \mathfrak{Z}) \to (U', \mathfrak{X})\}$ an element of $\mathrm{Cov}_{\mathrm{fppf}}(U', \mathfrak{X})$. Then there exists an object $(U, \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Z})$ of $\underline{\mathscr{E}}$ and two morphisms $p_1, p_2 : (U, \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Z}) \to (U, \mathfrak{Z})$ such that*

$\rho(U,\mathfrak{Z}\times_{\mathfrak{X}}\mathfrak{Z}) = \rho(U,\mathfrak{Z})\times_{(U',\mathfrak{X})}\rho(U,\mathfrak{Z})$ and that $\rho(p_1)$ (resp. $\rho(p_2)$) is the projection $\rho(U,\mathfrak{Z})\times_{(U',\mathfrak{X})}\rho(U,\mathfrak{Z})\to\rho(U,\mathfrak{Z})$ on the first (resp. second) component.

Proof. – By applying 9.5 to the projection $\rho(U,\mathfrak{Z})\times_{(U',\mathfrak{X})}\rho(U,\mathfrak{Z}) \to \rho(U,\mathfrak{Z})$ on the first component, we obtain an object $(U,\mathfrak{Z}\times_{\mathfrak{X}}\mathfrak{Z})$ of $\underline{\mathscr{E}}$ and a morphism $p_1:(U,\mathfrak{Z}\times_{\mathfrak{X}}\mathfrak{Z})\to(U,\mathfrak{Z})$ as in the proposition. The existence of p_2 follows from the fullness of ρ (9.3(i)). □

We conclude Theorem 9.2 by a general result on topoi due to Oyama [**32**] which we do not repeat the proof.

Proposition 9.10 ([**32**] 4.2.1). – *Let $\mathscr{C}$ be a site, $\mathscr{D}$ a site whose topology is defined by a pretopology and $u:\mathscr{C}\to\mathscr{D}$ a functor. Assume that:*

(i) *u is fully faithful,*
(ii) *u is continuous and cocontinuous,*
(iii) *For every object V of $\mathscr{D}$, there exists a covering of V in $\mathscr{D}$ of the form $\{u(U_i)\to V\}_{i\in I}$ where U_i is an object of $\mathscr{C}$.*

Then the morphism of topoi $g:\widetilde{\mathscr{C}}\to\widetilde{\mathscr{D}}$ defined by $g^*=\widehat{u}^*$ and $g_*=\widehat{u}_*$ (7.15) is an equivalence of topoi.

9.11. – Let $\mathscr{F}$ be a sheaf of $\widetilde{\mathscr{E}}'$ (resp. $\widetilde{\mathscr{E}}'_{\mathrm{fppf}}$), $\{\mathscr{F}_{(U,\mathfrak{X})},\gamma_{f,\mathscr{F}}\}$ the descent data associated to $\mathscr{F}$ (7.7) and $\{\mathrm{C}^*(\mathscr{F})_{(U,\mathfrak{X})},\gamma_{f,\mathrm{C}^*(\mathscr{F})}\}$ the descent data associated to $\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{F})$ (resp. $\mathrm{C}^*_{X/\mathscr{S},\mathrm{fppf}}(\mathscr{F})$). Since ρ takes Cartesian morphisms to Cartesian morphisms (7.4), for any object $(U,\mathfrak{X})$ of $\underline{\mathscr{E}}$, we have

$$\mathrm{C}^*(\mathscr{F})_{(U,\mathfrak{X})} = \pi_{U*}(\mathscr{F}_{\rho(U,\mathfrak{X})}) \tag{9.11.1}$$

where π_U denotes the equivalence of topoi $U'_{\mathrm{zar}}\xrightarrow{\sim}U_{\mathrm{zar}}$. For any morphism $f:(U_1,\mathfrak{X}_1)\to(U_2,\mathfrak{X}_2)$ of $\underline{\mathscr{E}}$, we verify that

$$\gamma_{f,\mathrm{C}^*(\mathscr{F})} = \pi_{U_1*}(\gamma_{\rho(f),\mathscr{F}}). \tag{9.11.2}$$

We verify that, by definition

$$\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{O}_{\mathscr{E}',n}) = \mathscr{O}_{\underline{\mathscr{E}},n} \qquad (\text{resp. } \mathrm{C}^*_{X/\mathscr{S},\mathrm{fppf}}(\mathscr{O}_{\mathscr{E}',n}) = \mathscr{O}_{\underline{\mathscr{E}},n}). \tag{9.11.3}$$

The morphism $\mathrm{C}_{X/\mathscr{S}}$ (resp. $\mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}$) is therefore underlying a morphism of ringed topoi, which we denote also by

$$\mathrm{C}_{X/\mathscr{S}}:(\widetilde{\underline{\mathscr{E}}},\mathscr{O}_{\underline{\mathscr{E}},n})\to(\widetilde{\mathscr{E}}',\mathscr{O}_{\mathscr{E}',n}), \tag{9.11.4}$$

$$(\text{resp. } \mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}:(\widetilde{\underline{\mathscr{E}}}_{\mathrm{fppf}},\mathscr{O}_{\underline{\mathscr{E}},n})\to(\widetilde{\mathscr{E}}'_{\mathrm{fppf}},\mathscr{O}_{\mathscr{E}',n})). \tag{9.11.5}$$

Let $\mathscr{F}'$ be a quasi-coherent $\mathscr{O}_{\mathscr{E}',n}$-module of $\widetilde{\mathscr{E}}'$. By (9.11.1), $\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{F}')$ is also quasi-coherent. For any object $(U,\mathfrak{X})$ of $\underline{\mathscr{E}}$, we have an equality of $\mathscr{O}_{\mathfrak{X}_n}$-modules of $\mathfrak{X}_{n,\mathrm{zar}}$ (9.11.1)

$$(\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{F}'))_{(U,\mathfrak{X})} = \mathscr{F}'_{\rho(U,\mathfrak{X})}. \tag{9.11.6}$$

Theorem 9.12 (cf. **[32]** 1.4.3 for $n=1$). – *Let X be a smooth scheme over k. The inverse image and the direct image functors of* $\mathrm{C}_{X/\mathscr{S}}$ *induce equivalences of categories quasi-inverse to each other* (8.3)

$$\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E}',n}) \rightleftarrows \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n}). \tag{9.12.1}$$

The theorem follows from 8.9 and 9.13 below.

Proposition 9.13. – *Let X be a smooth k-scheme. The inverse image and the direct image functors of the morphism* $\mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}$ (9.11.5) *induce equivalences of categories quasi-inverse to each other:*

$$\mathscr{C}^{\mathrm{qcoh}}_{\mathrm{fppf}}(\mathscr{O}_{\mathscr{E}',n}) \rightleftarrows \mathscr{C}^{\mathrm{qcoh}}_{\mathrm{fppf}}(\mathscr{O}_{\underline{\mathscr{E}},n}). \tag{9.13.1}$$

Proof. – We write simply C for $\mathrm{C}_{X/\mathscr{S},\mathrm{fppf}}$. By 9.2, it suffices to show that the functors C^* and C_* preserve quasi-coherent crystals. The assertion for C^* follows from (9.11.1) and (9.11.2).

Let $\mathscr{F}$ be a quasi-coherent crystal of $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules of $\underline{\widetilde{\mathscr{E}}}_{\mathrm{fppf}}$ and $(U',\mathfrak{T},u)$ an object of $\mathscr{E}'$. We first show that $(\mathrm{C}_*(\mathscr{F}))_{(U',\mathfrak{T})}$ is quasi-coherent. The statement is local, therefore we may assume that $U=F^{-1}_{X/k}(U')$ satisifies the condition of 9.8, i.e., U is affine and there exists an étale k-morphism $U\to\mathbb{A}^d_k$. Then, by 9.8(i) and 9.9, there exist objects $(U,\mathfrak{Z})$ and $(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})$ of $\underline{\mathscr{E}}$, an element $\{f:\rho(U,\mathfrak{Z})\to(U',\mathfrak{T})\}$ of $\mathrm{Cov}_{\mathrm{fppf}}(U',\mathfrak{T})$ and two morphisms $p_1,p_2:(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})\to(U,\mathfrak{Z})$ such that $\rho(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})=\rho(U,\mathfrak{Z})\times_{(U',\mathfrak{T})}\rho(U,\mathfrak{Z})$ and that $\rho(p_1)$ and $\rho(p_2)$ are the canonical projections of $\rho(U,\mathfrak{Z})\times_{(U',\mathfrak{T})}\rho(U,\mathfrak{Z})$ (9.9). In particular, the morphism $\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}\to\mathfrak{Z}$ attached to p_1 (resp. p_2) is the projection on the first (resp. second) component.

Since the adjunction morphism $\mathrm{C}^*\mathrm{C}_*\to\mathrm{id}$ is an isomorphism (9.2), we have (9.11.1)

$$\pi_{U*}((\mathrm{C}_*(\mathscr{F}))_{\rho(U,\mathfrak{Z})})=\mathscr{F}_{(U,\mathfrak{Z})},\qquad \pi_{U*}((\mathrm{C}_*(\mathscr{F}))_{\rho(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})})=\mathscr{F}_{(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}. \tag{9.13.2}$$

By 8.4, we consider $\mathscr{F}_{(U,\mathfrak{Z})}$ (resp. $\mathscr{F}_{(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}$) as a quasi-coherent $\mathscr{O}_{\mathfrak{Z}_n}$-module of Z_{zar} (resp. $(\mathscr{O}_{(\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})_n})$-module of $(Z\times_T Z)_{\mathrm{zar}}$). Since $\mathscr{F}$ is a crystal, we have $(\mathscr{O}_{(\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})_n})$-linear isomorphisms

$$p^*_{2,n}(\mathscr{F}_{(U,\mathfrak{Z})})\xrightarrow[\sim]{c_{p_2}}\mathscr{F}_{(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}\xleftarrow[\sim]{c_{p_1}}p^*_{1,n}(\mathscr{F}_{(U,\mathfrak{Z})}). \tag{9.13.3}$$

We define $\varphi:p_2^*(\mathscr{F}_{(U,\mathfrak{Z})})\to p_1^*(\mathscr{F}_{(U,\mathfrak{Z})})$ to be the composition of c_{p_2} and the inverse of c_{p_1}. Thus, we obtain a effective descent datum $(\mathscr{F}_{(U,\mathfrak{Z})},\varphi)$ for the fppf covering $\{f_n:\mathfrak{Z}_n\to\mathfrak{T}_n\}$. Therefore there exists a quasi-coherent $\mathscr{O}_{\mathfrak{T}_n}$-module $\mathscr{M}$ of T_{zar}, a canonical $(\mathscr{O}_{\mathfrak{Z}_n})$-linear isomorphism

$$f_n^*(\mathscr{M})\xrightarrow{\sim}\mathscr{F}_{(U,\mathfrak{Z})} \tag{9.13.4}$$

and an exact sequence of U_{zar}

$$0\to\pi_{U*}(u_*(\mathscr{M}))\to\mathscr{F}_{(U,\mathfrak{Z})}\to\mathscr{F}_{(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}, \tag{9.13.5}$$

where $\mathscr{F}_{(U,\mathfrak{Z})}$ and $\mathscr{F}_{(U,\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}$ are now considered as sheaves of U_{zar}.

On the other hand, since $\mathrm{C}_*(\mathscr{F})$ is a sheaf, there exists an exact sequence of U'_{zar} (7.14.1)

$$0 \to (\mathrm{C}_*(\mathscr{F}))_{(U',\mathfrak{X})} \to (\mathrm{C}_*(\mathscr{F}))_{\rho(U,\mathfrak{Z})} \to (\mathrm{C}_*(\mathscr{F}))_{\rho(U,\mathfrak{Z}\times_{\mathfrak{X}}\mathfrak{Z})}. \tag{9.13.6}$$

By (9.13.2), we obtain an $u_*(\mathscr{O}_{\mathfrak{X}_n})$-linear isomorphism $u_*(\mathscr{M}) \xrightarrow{\sim} (\mathrm{C}_*(\mathscr{F}))_{(U',\mathfrak{X})}$ of U'_{zar}. In particular, $(\mathrm{C}_*(\mathscr{F}))_{(U',\mathfrak{X})}$ is quasi-coherent. Hence $\mathrm{C}_*(\mathscr{F})$ is quasi-coherent.

Let $g : (U'_1,\mathfrak{X}_1) \to (U'_2,\mathfrak{X}_2)$ be a morphism of $\mathscr{E}'$. We prove that the morphism (8.4.1)

$$c_g : g_n^*(\mathrm{C}_*(\mathscr{F})_{(U'_2,\mathfrak{X}_2)}) \to \mathrm{C}_*(\mathscr{F})_{(U'_1,\mathfrak{X}_1)} \tag{9.13.7}$$

associated to $\mathrm{C}_*(\mathscr{F})$ is an isomorphism. Since the problem is Zariski local, we may assume that U'_2 satisfies the condition of 9.8. By 9.8(ii), there exists a morphism $h : (U_1,\mathfrak{Z}_1) \to (U_2,\mathfrak{Z}_2)$ of $\underline{\mathscr{E}}$, an element $\{f_1 : \rho(U_1,\mathfrak{Z}_1) \to (U'_1,\mathfrak{X}_1)\}$ of $\mathrm{Cov}_{\mathrm{fppf}}(U'_1,\mathfrak{X}_1)$ and an element $\{f_2 : \rho(U_2,\mathfrak{Z}_2) \to (U'_2,\mathfrak{X}_2)\}$ of $\mathrm{Cov}_{\mathrm{fppf}}(U'_2,\mathfrak{X}_2)$ such that the following diagram is Cartesian

$$\begin{tikzcd} \rho(U_1,\mathfrak{Z}_1) \arrow[r, "f_1"] \arrow[d, "\rho(h)"'] & (U'_1,\mathfrak{X}_1) \arrow[d, "g"] \\ \rho(U_2,\mathfrak{Z}_2) \arrow[r, "f_2"] & (U'_2,\mathfrak{X}_2). \end{tikzcd} \tag{9.13.8}$$

By 9.9 and repeating the previous fppf descent argument, we have a canonical isomorphism (9.13.4)

$$f_{i,n}^*(\mathrm{C}_*(\mathscr{F})_{(U'_i,\mathfrak{X}_i)}) \xrightarrow{\sim} \mathscr{F}_{(U_i,\mathfrak{Z}_i)}, \tag{9.13.9}$$

for $i = 1,2$. Furthermore, since $\mathscr{F}$ is a crystal, we have an isomorphism

$$c_h : h_n^*(\mathscr{F}_{(U_2,\mathfrak{Z}_2)}) \xrightarrow{\sim} \mathscr{F}_{(U_1,\mathfrak{Z}_1)}. \tag{9.13.10}$$

In view of (9.11.2), (9.13.2), (9.13.8) and (9.13.9), the morphism

$$f_{1,n}^*(c_g) : f_{1,n}^*(g_n^*(\mathrm{C}_*(\mathscr{F})_{(U'_2,\mathfrak{X}_2)})) \to f_{1,n}^*(\mathrm{C}_*(\mathscr{F})_{(U'_1,\mathfrak{X}_1)}) \tag{9.13.11}$$

is identical to c_h (9.13.10) and hence is an isomorphism. Since $f_{1,n} : \mathfrak{Z}_{1,n} \to \mathfrak{X}_{1,n}$ is faithfully flat, we deduce that c_g is an isomorphism. The proposition follows. □

Definition 9.14. – Let X be a smooth k-scheme. We call *Cartier equivalence* (*modulo* p^n) the equivalence of categories (9.12.1)

$$\mathrm{C}^*_{X/\mathscr{S}} : \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E}',n}) \xrightarrow{\sim} \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n}). \tag{9.14.1}$$

The above equivalence depends only on X and is different from the Cartier transform of Ogus-Vologodsky [**31**] which depends on a lifting of X' to W_2. We will compare two constructions under the assumption that there exists a smooth lifting of X to W in § 12.

To simplify the notation, we write C^* for $\mathrm{C}^*_{X/\mathscr{S}}$ if there is no confusion.

9.15. – The Cartier equivalence is compatible with localization with respect to an open subscheme. More precisely, let U be an open subscheme of X. The category $\mathscr{E}(U/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(U/\mathscr{S})$) forms naturally a full subcategory of $\mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(X/\mathscr{S})$). Equipped with the Zariski topologies on both sides, the canonical functor $\mathscr{E}(U/\mathscr{S}) \to \mathscr{E}(X/\mathscr{S})$ (resp. $\underline{\mathscr{E}}(U/\mathscr{S}) \to \underline{\mathscr{E}}(X/\mathscr{S})$) is continuous and cocontinuous. It induces a morphism of topoi

$$j_U : \widetilde{\mathscr{E}}(U/\mathscr{S}) \to \widetilde{\mathscr{E}}(X/\mathscr{S}) \qquad (\text{resp. } \underline{j}_U : \widetilde{\underline{\mathscr{E}}}(U/\mathscr{S}) \to \widetilde{\underline{\mathscr{E}}}(X/\mathscr{S})) \tag{9.15.1}$$

such that the inverse image functor is given by resctricting a sheaf of $\widetilde{\mathscr{E}}(X/\mathscr{S})$ to $\mathscr{E}(U/\mathscr{S})$ (resp. $\widetilde{\underline{\mathscr{E}}}(X/\mathscr{S})$ to $\underline{\mathscr{E}}(U/\mathscr{S})$). The above morphisms fit into a commutative diagram

$$\begin{tikzcd} \widetilde{\underline{\mathscr{E}}}(U/\mathscr{S}) \arrow[r, "\mathrm{C}_{U/\mathscr{S}}"] \arrow[d, "\underline{j}_U"'] & \widetilde{\mathscr{E}}(U'/\mathscr{S}) \arrow[d, "j_{U'}"] \\ \widetilde{\underline{\mathscr{E}}}(X/\mathscr{S}) \arrow[r, "\mathrm{C}_{X/\mathscr{S}}"] & \widetilde{\mathscr{E}}(X'/\mathscr{S}). \end{tikzcd} \tag{9.15.2}$$

Then we have a commutative diagram

$$\begin{tikzcd} \mathscr{C}(\mathscr{O}_{\mathscr{E}'(X/\mathscr{S}),n}) \arrow[r, "\mathrm{C}^*_{X/\mathscr{S}}"] \arrow[d, "j_U^*"'] & \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}}(X/\mathscr{S}),n}) \arrow[d, "\underline{j}_U^*"] \\ \mathscr{C}(\mathscr{O}_{\mathscr{E}'(U/\mathscr{S}),n}) \arrow[r, "\mathrm{C}^*_{U/\mathscr{S}}"] & \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}}(U/\mathscr{S}),n}). \end{tikzcd} \tag{9.15.3}$$

9.16. – In the remainder of this section, $\mathfrak{X}$ denotes a smooth formal $\mathscr{S}$-scheme with special fiber X. We set $\mathfrak{X}' = \mathfrak{X} \times_{\mathscr{S},\sigma} \mathscr{S}$ (2.1) and we suppose that there exists an $\mathscr{S}$-morphism $F : \mathfrak{X} \to \mathfrak{X}'$ which lifts the relative Frobenius morphism $F_{X/k} : X \to X'$ of X. We show that the Cartier equivalence C^* (9.14.1) globalizes Shiho's local construction in § 6 defined by F.

The morphism F induces a morphism of $\mathscr{E}'$ that we denote also by

$$F : \rho(X, \mathfrak{X}) = (X', \mathfrak{X}, F_{X/k}) \to (X', \mathfrak{X}', \mathrm{id}) \tag{9.16.1}$$

Recall that (6.6) the morphism F induces a morphism of formal groupoids $\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$ above F.

The following result explains the relation between the Cartier equivalence C^* and the functor ψ_n^* induced by ψ (5.6).

Proposition 9.17. – *Keep the assumption of 9.16. The diagram* (9.13)*:*

(9.17.1)
$$\begin{tikzcd} \mathscr{C}(\mathscr{O}_{\mathscr{E}',n}) \arrow[r, "\mathrm{C}^*"] \arrow[d, "\wr"] & \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n}) \arrow[d, "\wr"] \\ \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{R}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \arrow[r, "\psi_n^*"] & \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{Q}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\}, \end{tikzcd}$$

where the vertical arrows are the equivalences of categories defined in 8.10, is 2-commutative. That is, there exists a functorial isomorphism of $\mathscr{O}_{\mathfrak{X}_n}$-modules, depending on F

(9.17.2)
$$\eta_F : \psi_n^*(\mathscr{M}_{(X',\mathfrak{X}')}) \xrightarrow{\sim} \mathrm{C}^*(\mathscr{M})_{(X,\mathfrak{X})}$$

compatible with the $\mathscr{Q}_{\mathfrak{X}}$-stratifications, for every crystal $\mathscr{M}$ of $\mathscr{O}_{\mathscr{E}',n}$-modules of $\widetilde{\mathscr{E}}'$.

Proof. – Let $\mathscr{M}$ be a crystal of $\mathscr{O}_{\mathscr{E}',n}$-modules of $\widetilde{\mathscr{E}}'$. By (9.11.1), we have

(9.17.3)
$$\mathrm{C}^*(\mathscr{M})_{(X,\mathfrak{X})} = \pi_{X*}(\mathscr{M}_{\rho(X,\mathfrak{X})}).$$

Since $\mathscr{M}$ is a crystal, F (9.16.1) induces a functorial isomorphism of $\mathscr{O}_{\mathfrak{X}_n}$-modules

(9.17.4)
$$\eta_F : F_n^*(\mathscr{M}_{(X',\mathfrak{X}')}) = \pi_{X*}(\widetilde{F}_n^*(\mathscr{M}_{(X',\mathfrak{X}')})) \xrightarrow{\sim} \mathrm{C}^*(\mathscr{M})_{(X,\mathfrak{X})}.$$

The composition $Q_{\mathfrak{X},1} \to \underline{Q_{\mathfrak{X},1}}' \to X'$ (9.1.1) identifies with the morphism $g : Q_{\mathfrak{X},1} \to X'$ induced by $\overline{F^2 : \mathfrak{X}^2} \to \mathfrak{X}'^2$ (6.6.2). Then, it follows from the proof of 6.6 that ψ induces a morphism of $\mathscr{E}'$ that we denote also by

(9.17.5)
$$\psi : \rho(X, Q_{\mathfrak{X}}) \to (X', R_{\mathfrak{X}'}),$$

which fits into the following commutative diagrams

(9.17.6)
$$\begin{tikzcd} \rho(X,Q_{\mathfrak{X}}) \arrow[r, "\psi"] \arrow[d, "\rho(q_1)"'] & (X',R_{\mathfrak{X}'}) \arrow[d, "q'_1"] \\ \rho(X,\mathfrak{X}) \arrow[r, "F"] & (X',\mathfrak{X}') \end{tikzcd} \qquad \begin{tikzcd} \rho(X,Q_{\mathfrak{X}}) \arrow[r, "\psi"] \arrow[d, "\rho(q_2)"'] & (X',R_{\mathfrak{X}'}) \arrow[d, "q'_2"] \\ \rho(X,\mathfrak{X}) \arrow[r, "F"] & (X',\mathfrak{X}'), \end{tikzcd}$$

where q_1, q_2 (resp. q'_1, q'_2) are the canonical projections of $(X, Q_{\mathfrak{X}})$ to $(X,\mathfrak{X})$ (resp. $(X', R_{\mathfrak{X}'})$ to $(X',\mathfrak{X}')$). Hence, ψ (9.17.5) induces an isomorphism

(9.17.7)
$$\pi_{X*}(\widetilde{\psi}_n^*(\mathscr{M}_{(X',R_{\mathfrak{X}'})})) \xrightarrow{\sim} \mathrm{C}^*(\mathscr{M})_{(X,Q_{\mathfrak{X}})}.$$

Recall that the left vertical functor of (9.17.1) is given by $\mathscr{M} \mapsto (\mathscr{M}_{(X',\mathfrak{X}')}, \varepsilon')$ where ε' is induced by isomorphisms $\widetilde{q}'^*_{2,n}(\mathscr{M}_{(X',\mathfrak{X}')}) \xrightarrow{\sim} \mathscr{M}_{(X',R_{\mathfrak{X}'})} \xleftarrow{\sim} \widetilde{q}'^*_{1,n}(\mathscr{M}_{(X',\mathfrak{X}')})$. By 5.6, we have

(9.17.8)
$$\psi_n^*(\mathscr{M}_{(X',\mathfrak{X}')}, \varepsilon') = (F_n^*(\mathscr{M}_{(X',\mathfrak{X}')}), \pi_{X*}(\widetilde{\psi}_n^*(\varepsilon'))).$$

On the other hand, the $\mathscr{O}_{\mathfrak{X}_n}$-module associated to $\mathrm{C}^*(\mathscr{M})$ is $\mathrm{C}^*(\mathscr{M})_{(X,\mathfrak{X})}$ and the associated $\mathscr{Q}_{\mathfrak{X}}$-stratification is induced by the isomorphisms $\widetilde{q}^*_{2,n}(\mathrm{C}^*(\mathscr{M})_{(X,\mathfrak{X})}) \xrightarrow{\sim}$

$\mathrm{C}^*(\mathscr{M})_{(X,Q_{\mathfrak{X}})} \xleftarrow{\sim} \tilde{q}_{1,n}^*(\mathrm{C}^*(\mathscr{M})_{(X,\mathfrak{X})})$. The proposition follows in view of (9.17.6), (9.17.2) and (9.17.7). □

9.18. – Keep the assumption of 9.16. Recall that the morphisms of formal $\mathfrak{X}$-groupoids $\varpi' : T_{\mathfrak{X}'} \to R_{\mathfrak{X}'}$ (5.13.1) and $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$ (5.12.1) induce functors

$$\varpi_n'^* : \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{R}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \to \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{T}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \tag{9.18.1}$$

$$\lambda_n^* : \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{Q}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\} \to \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{P}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\}. \tag{9.18.2}$$

By 5.11, 5.17 and 8.10, they further induce functors

$$\mu : \mathscr{C}(\mathscr{O}_{\mathscr{E}',n}) \to p\text{-}\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}'_n/\mathscr{S}_n), \tag{9.18.3}$$

$$\nu : \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n}) \to \mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n). \tag{9.18.4}$$

By (6.9.2), 6.10 and 9.17, the diagrams

$$\begin{tikzcd} \mathscr{C}(\mathscr{O}_{\mathscr{E}',n}) \arrow[r, "\mathrm{C}^*"] \arrow[d, "\wr"'] & \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n}) \arrow[d, "\wr"] \\ \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{R}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \arrow[r, "\psi_n^*"] \arrow[d, "\varpi_n'^*"'] & \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{Q}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\} \arrow[d, "\lambda_n^*"] \\ \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{T}_{\mathfrak{X}'}\text{-stratification}\end{matrix}\right\} \arrow[r, "\varphi_n^*"] \arrow[d, "\wr"'] & \left\{\begin{matrix} \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{P}_{\mathfrak{X}}\text{-stratification}\end{matrix}\right\} \arrow[d, "\wr"] \\ p\text{-}\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}'_n/\mathscr{S}_n) \arrow[r, "\Phi_n"] & \mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n), \end{tikzcd} \tag{9.18.5}$$

where the functors ψ_n^*, φ_n^* and Φ_n are induced by F, are commutative up to a functorial isomorphism of $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$. For every object $\mathscr{M}$ of $\mathscr{C}(\mathscr{O}_{\mathscr{E}',n})$, we have a functorial isomorphism

$$\eta_F : \Phi_n(\mu(\mathscr{M})) \xrightarrow{\sim} \nu(\mathrm{C}^*(\mathscr{M})). \tag{9.18.6}$$

We see that the Cartier equivalence C^* (9.14.1) is compatible with Shiho's functor Φ_n (6.1.5).

In the remainder of this section, we will explain how to relate Shiho's local constructions with respect to different liftings of Frobenius morphism using Cartier equivalence.

Let $F_1, F_2 : \mathfrak{X} \to \mathfrak{X}'$ denote two liftings of the relative Frobenius morphism $F_{X/k}$ of X.

LEMMA 9.19. – *The morphisms F_1, F_2 induce a morphism of $\mathscr{E}'$*

(9.19.1) $$\psi_{12} : \rho(X, Q_{\mathfrak{X}}) \to (X', R_{\mathfrak{X}'}).$$

Proof. – The proof is similar to that of 6.6. By the universal property of $R_{\mathfrak{X}'}$, it suffices to show that there exists a unique k-morphism $g : Q_{\mathfrak{X},1} \to X'$ which fits into a commutative diagram

(9.19.2)
$$\begin{tikzcd} Q_{\mathfrak{X},1} \arrow[r] \arrow[dd, "g"'] & Q_{\mathfrak{X}} \arrow[d] \\ & \mathfrak{X}^2 \arrow[d, "(F_1,F_2)"] \\ X' \arrow[r, "\Delta"] & \mathfrak{X}'^2. \end{tikzcd}$$

The problem being local on $\mathfrak{X}$, we can assume that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \ldots, T_d\})$. We put t_i the image of T_i in $\mathscr{O}_{\mathfrak{X}}$, $\xi_i = 1 \otimes t_i - t_i \otimes 1$, $t'_i = \pi^*(t_i) \in \mathscr{O}_{\mathfrak{X}'}$ and $\xi'_i = 1 \otimes t'_i - t'_i \otimes 1$ for all $1 \leqslant i \leqslant d$. Locally, there exists sections a_i, b_i of $\mathscr{O}_{\mathfrak{X}}$ such that $F_1^*(t'_i) = t_i^p + pa_i$, $F_2^*(t'_i) = t_i^p + pb_i$. By a similar calculation of (6.6.3), we have

(9.19.3) $$(F_1, F_2)^*(\xi'_i) = \xi_i^p + \sum_{j=1}^{p-1} \binom{p}{j} \xi_k^j (t_k \otimes 1)^{p-j} + p(1 \otimes b_k - a_k \otimes 1).$$

Since $\xi_i^p = p \cdot \left(\frac{\xi_i^p}{p}\right)$ in $\mathcal{Q}_{\mathfrak{X}}$, the assertion follows. ☐

9.20. – Keep the assumption of 9.19, we denote by α the composition

(9.20.1) $$\alpha : (X', \mathfrak{X}, F_{X/k}) = \rho(X, \mathfrak{X}) \xrightarrow{\rho(\iota_Q)} \rho(X, Q_{\mathfrak{X}}) \xrightarrow{\psi_{12}} (X', R_{\mathfrak{X}'}).$$

We set $q'_1, q'_2 : (X', R_{\mathfrak{X}'}) \to (X', \mathfrak{X}')$ the canonical morphisms of $\mathscr{E}'$ and we have $q'_i \circ \alpha = F_i$ (9.16.1).

Considering $\mathscr{R}_{\mathfrak{X}'}$ as a formal Hopf $\mathscr{O}_{\mathfrak{X}'}$-algebra of $\mathfrak{X}_{\mathrm{zar}}$, α induces a W-homomorphism of $\mathfrak{X}_{\mathrm{zar}}$:

(9.20.2) $$a : \mathscr{R}_{\mathfrak{X}'} \to \mathscr{O}_{\mathfrak{X}}.$$

Equipped with the left (resp. right) $\mathscr{O}_{\mathfrak{X}'}$-linear action on the source and the $\mathscr{O}_{\mathfrak{X}'}$-linear action induced by F_1 (resp. F_2) on the target, a is $\mathscr{O}_{\mathfrak{X}'}$-linear.

Suppose that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \ldots, T_d\})$ and we take again the notation of the proof of 9.19. In view of (9.19.3), the homomorphism a is determined by

(9.20.3) $$a\left(\frac{\xi'_i}{p}\right) = \frac{F_2^*(t'_i) - F_1^*(t'_i)}{p} \qquad \forall\ 1 \leqslant i \leqslant d.$$

9.21. – Let $\mathscr{M}$ be a crystal of $\mathscr{O}_{\mathscr{E}',n}$-modules, $\mathcal{M} = \mathscr{M}_{(X',\mathfrak{X}')}$ and $\varepsilon' : \widetilde{q}'^{*}_{2,n}(\mathcal{M}) \xrightarrow{\sim} \widetilde{q}'^{*}_{1,n}(\mathcal{M})$ the associated $\mathscr{R}_{\mathfrak{X}'}$-stratification (8.10). Recall (9.17) that the morphisms F_1 and F_2 induce morphisms $\psi_1, \psi_2 : \rho(X, Q_{\mathfrak{X}}) \to (X', R_{\mathfrak{X}'})$ of $\mathscr{E}'$ respectively. We associate to $(\mathcal{M}, \varepsilon')$ two different $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathcal{Q}_{\mathfrak{X}}$-stratification

$$(F^{*}_{1,n}(\mathcal{M}), \pi_{X*}(\widetilde{\psi}^{*}_{1,n}(\varepsilon'))) \quad \text{and} \quad (F^{*}_{2,n}(\mathcal{M}), \pi_{X*}(\widetilde{\psi}^{*}_{2,n}(\varepsilon'))). \tag{9.21.1}$$

Let $\Phi_{1,n}$ (resp. $\Phi_{2,n}$) be Shiho's functor induced by F_1 (resp. F_2) (6.1.5). We associate to $\mathscr{M}$ two different objects of $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ (9.18.5)

$$\Phi_{1,n}(\mu(\mathscr{M})) \quad \text{and} \quad \Phi_{2,n}(\mu(\mathscr{M})), \tag{9.21.2}$$

whose underlying $\mathscr{O}_{\mathfrak{X}_n}$-modules are $F^{*}_{1,n}(\mathcal{M})$ and $F^{*}_{2,n}(\mathcal{M})$ respectively.

The morphism α gives a natural way to glue $\Phi_{1,n}(\mu(\mathscr{M})), \Phi_{2,n}(\mu(\mathscr{M}))$.

Proposition 9.22. – *Keep the assumption of 9.21. The morphism α and the $\mathscr{R}_{\mathfrak{X}'}$-stratification ε' induce an isomorphism of $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathcal{Q}_{\mathfrak{X}}$-stratification:*

$$\alpha^{*}(\varepsilon') : (F^{*}_{2,n}(\mathcal{M}), \pi_{X*}(\widetilde{\psi}^{*}_{2,n}(\varepsilon'))) \xrightarrow{\sim} (F^{*}_{1,n}(\mathcal{M}), \pi_{X*}(\widetilde{\psi}^{*}_{1,n}(\varepsilon'))), \tag{9.22.1}$$

such that $\eta_{F_2} = \eta_{F_1} \circ \alpha^{*}(\varepsilon')$ (9.17.2).

Proof. – We denote by $q_1, q_2 : (X, Q_{\mathfrak{X}}) \to (X, \mathfrak{X})$ the canonical morphisms of $\underline{\mathscr{E}}$, by $q'_1, q'_2 : (X', R_{\mathfrak{X}'}) \to (X', \mathfrak{X}')$ the canonical morphisms of $\mathscr{E}'$ and by $q'_{ij} : (X', R_{\mathfrak{X}'}(2)) \to (X', R_{\mathfrak{X}'})$ the morphism induced by $R_{\mathfrak{X}'}(2) \to \mathfrak{X}'^3 \xrightarrow{p'_{ij}} \mathfrak{X}'^2$ and the universal property of $R_{\mathfrak{X}}$ for all $1 \leqslant i < j \leqslant 3$ (cf. (4.11.1)).

Since $F_i = q'_i \circ \alpha$ for $i = 1, 2$, the isomorphism $\varepsilon' : \widetilde{q}'^{*}_{2,n}(\mathcal{M}) \xrightarrow{\sim} \widetilde{q}'^{*}_{1,n}(\mathcal{M})$ induces an isomorphism

$$\pi_{X*}(\widetilde{\alpha}^{*}_{n}(\varepsilon')) : \widetilde{F}^{*}_{2,n}(\mathcal{M}) \xrightarrow{\sim} \widetilde{F}^{*}_{1,n}(\mathcal{M}). \tag{9.22.2}$$

We write simply $\alpha^{*}(\varepsilon')$ for $\pi_{X*}(\widetilde{\alpha}^{*}_{n}(\varepsilon'))$. Then we have $\eta_{F_2} = \eta_{F_1} \circ \alpha^{*}(\varepsilon')$. It remains to show that the $\alpha^{*}(\varepsilon')$ is compatible with the $\mathcal{Q}_{\mathfrak{X}}$-stratifications on both sides.

The compositions of morphisms

$$\mathfrak{X}^2 \xrightarrow{p_1} \mathfrak{X} \xrightarrow{\Delta} \mathfrak{X}^2 \xrightarrow{(F_1,F_2)} \mathfrak{X}'^2 \xrightarrow{p'_2} \mathfrak{X}'$$

$$\mathfrak{X}^2 \xrightarrow{(F_2,F_2)} \mathfrak{X}'^2 \xrightarrow{p'_1} \mathfrak{X}'$$

are equal. We deduce that the compositions of morphisms of $\mathscr{E}'$

$$\rho(X, Q_{\mathfrak{X}}) \xrightarrow{\rho(q_1)} \rho(X, \mathfrak{X}) \xrightarrow{\alpha} (X', R_{\mathfrak{X}'}) \xrightarrow{q'_2} (X', \mathfrak{X}')$$

$$\rho(X, Q_{\mathfrak{X}}) \xrightarrow{\psi_2} (X', R_{\mathfrak{X}'}) \xrightarrow{q'_1} (X', \mathfrak{X}')$$

are equal. In view of the isomorphism $(X', R_{\mathfrak{X}'}(2)) \simeq (X', R_{\mathfrak{X}'}) \times_{(X', \mathfrak{X}')} (X', R_{\mathfrak{X}'})$ of $\mathscr{E}'$ (4.10.1), we obtain a morphism of $\mathscr{E}'$

$$u : \rho(X, Q_{\mathfrak{X}}) \to (X', R_{\mathfrak{X}'}(2)), \tag{9.22.3}$$

such that $\alpha\circ\rho(q_1) = q'_{12}\circ u$ and $\psi_2 = q'_{23}\circ u$. Symmetrically, we construct a morphism of $\mathscr{E}'$

$$v : \rho(X, Q_{\mathfrak{X}}) \to (X', R_{\mathfrak{X}'}(2)), \tag{9.22.4}$$

such that $\psi_1 = q'_{12}\circ v$ and $\alpha\circ\rho(q_2) = q'_{23}\circ v$. The compositions

$$\mathfrak{X}^2 \xrightarrow{(\iota_1,\iota_1,\iota_2)} \mathfrak{X}^3 \xrightarrow{(F_1,F_2,F_2)} \mathfrak{X}'^3 \xrightarrow{p'_{13}} \mathfrak{X}'^2$$
$$\mathfrak{X}^2 \xrightarrow{(\iota_1,\iota_2,\iota_2)} \mathfrak{X}^3 \xrightarrow{(F_1,F_1,F_2)} \mathfrak{X}'^3 \xrightarrow{p'_{13}} \mathfrak{X}'^2$$

are equal to $(F_1, F_2) : \mathfrak{X}^2 \to \mathfrak{X}'^2$. By the universal property of $R_{\mathfrak{X}'}$ (3.5), we deduce that $q'_{13}\circ u = q'_{13}\circ v = \psi_{12}$ (9.19), i.e., the compositions

$$\rho(X, Q_{\mathfrak{X}}) \underset{v}{\overset{u}{\rightrightarrows}} (X', R_{\mathfrak{X}'}(2)) \xrightarrow{q'_{13}} (X', R_{\mathfrak{X}'}) \tag{9.22.5}$$

are equal to ψ_{12}. By (9.17.6), we have $\widetilde{\psi}^*_{i,n}(\widetilde{q}'^*_{j,n}(\mathscr{M})) \simeq \widetilde{q}^*_{j,n}(\widetilde{F}^*_{i,n}(\mathscr{M}))$ for all $i, j = 1, 2$. By the cocycle condition $\widetilde{q}'^*_{12,n}(\varepsilon')\circ\widetilde{q}'^*_{23,n}(\varepsilon') = \widetilde{q}'^*_{13,n}(\varepsilon')$, we deduce a commutative diagram:

$$\begin{tikzcd}
\widetilde{\psi}^*_{2,n}(\widetilde{q}'^*_{2,n}(\mathscr{M})) \arrow[r, "\widetilde{\psi}^*_{2,n}(\varepsilon')"] \arrow[d, "\widetilde{q}^*_{2,n}(\widetilde{\alpha}^*_n(\varepsilon'))"'] & \widetilde{\psi}^*_{2,n}(\widetilde{q}'^*_{1,n}(\mathscr{M})) \arrow[d, "\widetilde{q}^*_{1,n}(\widetilde{\alpha}^*_n(\varepsilon'))"] \\
\widetilde{\psi}^*_{1,n}(\widetilde{q}'^*_{2,n}(\mathscr{M})) \arrow[r, "\widetilde{\psi}^*_{1,n}(\varepsilon')"] & \widetilde{\psi}^*_{1,n}(\widetilde{q}'^*_{1,n}(\mathscr{M})).
\end{tikzcd} \tag{9.22.6}$$

That is the isomorphism $\alpha^*(\varepsilon')$ is compatible with the $Q_{\mathfrak{X}}$-stratifications $\pi_{X*}(\widetilde{\psi}^*_{2,n}(\varepsilon'))$ and $\pi_{X*}(\widetilde{\psi}^*_{1,n}(\varepsilon'))$. □

Corollary 9.23. – *Keep the assumption of 9.21. The isomorphism* (9.22.1) *induces an isomorphism of* $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ (9.21.2)

$$\alpha^*(\varepsilon') : \Phi_{2,n}(\mu(\mathscr{M})) \xrightarrow{\sim} \Phi_{1,n}(\mu(\mathscr{M})) \tag{9.23.1}$$

such that $\eta_{F_2} = \eta_{F_1}\circ\alpha^*(\varepsilon')$ (9.18.6)*.*

It follows from (9.18.5) and 9.22.

Remark 9.24. – Given a lifting of the Frobenius morphism, Shiho's functor (6.1.4) applies to $\mathscr{O}_{\mathfrak{X}'_n}$-modules with integrable p-connection. However, the isomorphism (9.23.1), which glues local constructions of Shiho, depends on the $\mathscr{R}_{\mathfrak{X}'}$-stratification on the $\mathscr{O}_{\mathfrak{X}'_n}$-module $\mathscr{M}_{(X',\mathfrak{X}')}$.

CHAPTER 10

CARTIER TRANSFORM OF OGUS-VOLOGODSKY

For the convenience of the reader, we shall review the original construction of the Cartier transform of Ogus-Vologodsky [**31**]. In particular, we shall clarify some details, especially in regard to sheaf of affine functions on a torsor.

In this section, X denotes a scheme. Starting from 10.8, we will suppose that X is smooth over k.

10.1. – Let E a locally free $\mathscr{O}_X$-module of finite type. We denote by $\mathrm{S}(E)$ (resp. $\Gamma(E)$) the symmetric algebra (resp. PD-algebra) of E over $\mathscr{O}_X$ ([**23**] I 4.2.2.6) and for any integer $n \geqslant 0$, by $\mathrm{S}^n(E)$ (resp. $\Gamma_n(E)$) its homogeneous part of degree n. There exists a unique homomorphism of $\mathscr{O}_X$-algebras

$$\delta : \mathrm{S}(E) \to \mathrm{S}(E) \otimes_{\mathscr{O}_X} \mathrm{S}(E), \tag{10.1.1}$$

such that for every local section e of E, we have $\delta(e) = 1 \otimes e + e \otimes 1$. This homomorphisms makes $\mathrm{S}(E)$ into a Hopf commutative $\mathscr{O}_X$-algebra.

Let I (resp. J) be the ideal $\oplus_{n\geqslant 1} \mathrm{S}^n(E)$ of $\mathrm{S}(E)$ (resp. PD-ideal $\oplus_{n\geqslant 1}\Gamma_n(E)$ of $\Gamma(E)$). We denote by $\widehat{\mathrm{S}}(E)$ (resp. $\widehat{\Gamma}(E)$) the completion of $\mathrm{S}(E)$ (resp. $\Gamma(E)$) with respect to the filtration $\{I^n\}_{n\geqslant 1}$ (resp. PD-filtration $\{J^{[n]}\}_{n\geqslant 1}$).

Let M be a $\widehat{\Gamma}(E)$-module. We say that M is *quasi-nilpotent* if for any open subscheme U of X and any $e \in M(U)$, there exists a Zariski covering $\{U_i \to U\}_{i\in I}$ and a family of integers $\{N_i\}_{i\in I}$ such that for each $i \in I$, $e|_{U_i}$ is annihilated by the ideal $J^{[N_i]}$. For any integer $n \geqslant 0$, we say that M is *nilpotent of level* $\leqslant n$ if E is annihilated by $J^{[n+1]}$.

10.2. – We set $\Omega = \mathscr{H}\!om_{\mathscr{O}_X}(E, \mathscr{O}_X)$. The pairing $E \otimes_{\mathscr{O}_X} \Omega \to \mathscr{O}_X$ induces a canonical morphism

$$\Gamma_n(E) \otimes_{\mathscr{O}_X} \mathrm{S}^{m+n}(\Omega) \to \mathrm{S}^m(\Omega) \tag{10.2.1}$$

which is perfect if $m = 0$ ([**5**] A.10). If we equip $\Gamma(E)$ with the topology defined by the PD-filtration $\{J^{[n]}\}$ and $\mathrm{S}(\Omega)$ with the discrete topology, the $\mathscr{O}_X$-linear morphism $\Gamma(E) \otimes_{\mathscr{O}_X} \mathrm{S}(\Omega) \to \mathrm{S}(\Omega)$ is continuous. It extends by continuity to an action of $\widehat{\Gamma}(E)$ on $\mathrm{S}(\Omega)$:

$$\widehat{\Gamma}(E) \otimes_{\mathscr{O}_X} \mathrm{S}(\Omega) \to \mathrm{S}(\Omega). \tag{10.2.2}$$

We have an increasing exhaustive filtration of $\widehat{\Gamma}(E)$-submodules $\{\bigoplus_{m\leqslant n} \mathrm{S}^m(\Omega)\}_{n\geqslant 0}$ of $\mathrm{S}(\Omega)$ such that $\bigoplus_{m\leqslant n} \mathrm{S}^m(\Omega)$ is nilpotent of level $\leqslant n$. Then $\mathrm{S}(\Omega)$ is quasi-nilpotent. The above morphism induces an $\mathscr{O}_X$-linear isomorphism

$$\begin{aligned} \widehat{\Gamma}(E) &= \varprojlim_n \Big(\bigoplus_{m\leqslant n} \Gamma_m(E)\Big) \\ &\simeq \varprojlim_n \mathscr{H}\!om_{\mathscr{O}_X}\Big(\bigoplus_{m\leqslant n} \mathrm{S}^m(\Omega), \mathscr{O}_X\Big) \\ &= \mathscr{H}\!om_{\mathscr{O}_X}(\mathrm{S}(\Omega), \mathscr{O}_X). \end{aligned} \tag{10.2.3}$$

We equip $\mathscr{H}\!om_{\mathscr{O}_X}(\mathrm{S}(\Omega), \mathscr{O}_X)$ with the $\mathscr{O}_X$-algebra structure induced by the Hopf algebra $\mathrm{S}(\Omega)$ (4.4). The above isomorphism is an isomorphism of $\mathscr{O}_X$-algebras.

10.3. – Let $f : Y \to X$ be a morphism of schemes. We put $E_Y = f^*(E)$ and $\Omega_Y = f^*(\Omega)$. By the universal property of the symmetric algebra, we have a canonical isomorphism of $\mathscr{O}_Y$-algebras

$$f^*(\mathrm{S}_{\mathscr{O}_X}(\Omega)) \xrightarrow{\sim} \mathrm{S}_{\mathscr{O}_Y}(\Omega_Y). \tag{10.3.1}$$

Since $\mathrm{S}_{\mathscr{O}_X}(\Omega)$ is a direct sum of locally free $\mathscr{O}_X$-modules of finite type, by duality (10.2.3), we deduce a canonical isomorphism of $\mathscr{O}_Y$-algebras

$$\begin{aligned} \widehat{\Gamma}_{\mathscr{O}_Y}(E_Y) &\simeq \mathscr{H}\!om_{\mathscr{O}_Y}(\mathrm{S}_{\mathscr{O}_Y}(\Omega_Y), \mathscr{O}_Y) \\ &\simeq f^*(\mathscr{H}\!om_{\mathscr{O}_X}(\mathrm{S}_{\mathscr{O}_X}(\Omega), \mathscr{O}_X)) \\ &\simeq f^*(\widehat{\Gamma}_{\mathscr{O}_X}(E)). \end{aligned} \tag{10.3.2}$$

10.4. – Let $\mathscr{L}$ be an E-torsor of X_{zar}. *An affine function on* $\mathscr{L}$ is a morphism $f : \mathscr{L} \to \mathscr{O}_X$ of X_{zar} satisfying the following equivalent conditions ([**2**] II.4.7):

(i) For every open subscheme U of X and every $s \in \mathscr{L}(U)$, the morphism:

$$E(U) \to \mathscr{O}(U), \qquad t \mapsto f(s+t) - f(s) \tag{10.4.1}$$

is $\mathscr{O}_X(U)$-linear.

(ii) There exists a section $\omega_f \in \Omega(X)$, called the *linear term of* f, such that for every open subscheme U of X and all $s \in \mathscr{L}(U)$ and $t \in E(U)$, we have

$$f(s+t) = f(s) + \omega_f(t). \tag{10.4.2}$$

The condition (i) is clearly local for the Zariski topology on X. We denote by $\mathscr{F}$ the subsheaf of $\mathscr{H}\!om_{X_{\mathrm{zar}}}(\mathscr{L}, \mathscr{O}_X)$ consisting of affine functions on $\mathscr{L}$; in other words, for any open subscheme U of X, $\mathscr{F}(U)$ is the set of affine functions on $\mathscr{L}|_U$. It is naturally endowed with an $\mathscr{O}_X$-module structure. We call $\mathscr{F}$ *the sheaf of affine functions on* $\mathscr{L}$.

We have a canonical $\mathscr{O}_X$-linear morphism $c : \mathscr{O}_X \to \mathscr{F}$ whose image consists of constant functions. The "linear term" defines an $\mathscr{O}_X$-linear morphism $\omega : \mathscr{F} \to \Omega$. One verifies that the sequence

$$0 \to \mathscr{O}_X \xrightarrow{c} \mathscr{F} \xrightarrow{\omega} \Omega \to 0 \tag{10.4.3}$$

is exact. By ([**23**] I 4.3.1.7), the above sequence induces, for any integer $n \geqslant 1$, an exact sequence:

$$0 \to \mathrm{S}^{n-1}(\mathscr{F}) \to \mathrm{S}^n(\mathscr{F}) \to \mathrm{S}^n(\Omega) \to 0. \tag{10.4.4}$$

The $\mathscr{O}_X$-modules $(\mathrm{S}^n(\mathscr{F}))_{n\geqslant 0}$ form a inductive system. We denote its inductive limit by

$$\mathscr{A} = \varinjlim_{n\geqslant 0} \mathrm{S}^n(\mathscr{F}), \tag{10.4.5}$$

which is naturally endowed with a structure of an $\mathscr{O}_X$-algebra. For any integer $n \geqslant 0$, the canonical morphism $\mathrm{S}^n(\mathscr{F}) \to \mathscr{A}$ is injective. By letting $N_n(\mathscr{A}) = \mathrm{S}^n(\mathscr{F})$ for all $n \geqslant 0$, we obtain an increasing exhaustive filtration of $\mathscr{A}$.

There exists a unique homomorphism of $\mathscr{O}_X$-algebras

$$\mu : \mathscr{A} \to \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathscr{A}, \tag{10.4.6}$$

such that for every local section m of $\mathscr{F}$, we have $\mu(m) = 1 \otimes m + \omega(m) \otimes 1$. For $n \geqslant 0$, we have

$$\mu(N_n(\mathscr{A})) \subset \bigoplus_{i+j=n} \mathrm{S}^i(\Omega) \otimes N_j(\mathscr{A}). \tag{10.4.7}$$

By construction, the following diagram is commutative

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathscr{A} \arrow[d, "\delta\otimes\mathrm{id}"] \\ \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathscr{A} \arrow[r, "\mathrm{id}\otimes\mu"] & \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathscr{A}. \end{tikzcd} \tag{10.4.8}$$

10.5. – By (10.2.3) and (10.4.6), we have an $\mathscr{O}_X$-linear morphism:

$$\begin{array}{rcl} \widehat{\Gamma}(E) \otimes_{\mathscr{O}_X} \mathscr{A} & \to & \mathscr{A} \\ u \otimes a & \mapsto & (u \otimes \mathrm{id})(\mu(a)). \end{array} \tag{10.5.1}$$

By (10.4.8), the above morphism makes $\mathscr{A}$ into a $\widehat{\Gamma}(E)$-module. The action of E on $\mathscr{F}$ is given by ω (10.4.3) and duality. By (10.4.7), we see that $\mathscr{A}$ is quasi-nilpotent and that for any $n \geqslant 0$, $N_n(\mathscr{A})$ is a $\widehat{\Gamma}(E)$-submodule of $\mathscr{A}$ and is nilpotent of level $\leqslant n$.

The canonical $\mathrm{S}(\Omega)$-linear isomorphism $\Omega^1_{\mathrm{S}(\Omega)/\mathscr{O}_X} \xrightarrow{\sim} \Omega \otimes_{\mathscr{O}_X} \mathrm{S}(\Omega)$ induces an isomorphism

$$\Omega^1_{\mathscr{A}/\mathscr{O}_X} \xrightarrow{\sim} \Omega \otimes_{\mathscr{O}_X} \mathscr{A}. \tag{10.5.2}$$

We denote the universal $\mathscr{O}_X$-derivation by

$$d_{\mathscr{A}} : \mathscr{A} \to \Omega \otimes_{\mathscr{O}_X} \mathscr{A}. \tag{10.5.3}$$

For any local section m of $\mathscr{F}$, we have $d_{\mathscr{A}}(m) = \omega(m) \otimes 1$.

10.6. – Let $s \in \mathscr{L}(X)$ and let $\rho_s : \mathscr{F} \to \mathscr{O}_X$ be the associated splitting of the exact sequence (10.4.3). The morphism $\Omega \to \mathscr{F}$ deduced from $\mathrm{id} - c \circ \rho_s$ extends to an isomorphism of $\mathscr{O}_X$-algebras

$$\psi : \mathrm{S}(\Omega) \xrightarrow{\sim} \mathscr{A}, \tag{10.6.1}$$

which is compatible with the filtrations $(\bigoplus_{0 \leqslant i \leqslant n} \mathrm{S}^i(\Omega))_n$ and $(N_n(\mathscr{A}))_n$. The diagram (10.1.1)

$$\begin{tikzcd} \mathrm{S}(\Omega) \arrow[r, "\psi"] \arrow[d, "\delta"'] & \mathscr{A} \arrow[d, "\mu"] \\ \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathrm{S}(\Omega) \arrow[r, "\mathrm{id} \otimes \psi"] & \mathrm{S}(\Omega) \otimes_{\mathscr{O}_X} \mathscr{A} \end{tikzcd} \tag{10.6.2}$$

is commutative. Hence the isomorphism ψ is compatible with the $\widehat{\Gamma}(E)$-module structures (10.2).

10.7. – Let $f : Y \to X$ be a morphism of schemes and $\mathscr{L}$ an E-torsor of X_{zar}. For $\mathscr{O}_X$-modules, we will use the notation f^{-1} to denote the inverse image in the sense of abelian sheaves and will keep the notation f^* for the inverse image in the sense of modules. The *affine inverse image of* $\mathscr{L}$ *under* f, denoted by $f^+(\mathscr{L})$, is the $f^*(E)$-torsor of Y_{zar} deduced from the $f^{-1}(E)$-torsor $f^*(\mathscr{L})$ by extending its structural group by the canonical homomorphism $f^{-1}(E) \to f^*(E)$,

$$f^+(\mathscr{L}) = f^*(\mathscr{L}) \wedge^{f^{-1}(E)} f^*(E); \tag{10.7.1}$$

in other words, the quotient of $f^*(\mathscr{L}) \times f^*(E)$ by the diagonal action of $f^{-1}(E)$ (**[17]** III 1.4.6).

We denote by $\mathscr{F}$ the sheaf of affine functions on $\mathscr{L}$ (10.4) and by $\mathscr{F}^+$ the sheaf of affine functions on $f^+(\mathscr{L})$. Let $l : \mathscr{L} \to \mathscr{O}_X$ be an affine morphism, $\omega \in \Omega(X)$ its linear term and $\omega' = f^*(\omega) \in f^*(\Omega)(Y)$. Endowing $\mathscr{O}_Y$ with the structure of $f^*(E)$-object defined by ω' (**[2]** II.4.8), there exists a unique $f^*(E)$-equivariant morphism $l' : f^+(\mathscr{L}) \to \mathscr{O}_Y$ that fits into the commutative diagram

$$\begin{tikzcd} f^*(\mathscr{L}) \arrow[r, "l"] \arrow[d] & f^{-1}(\mathscr{O}_X) \arrow[d] \\ f^+(\mathscr{L}) \arrow[r, "l'"] & \mathscr{O}_Y, \end{tikzcd} \tag{10.7.2}$$

where the vertical arrows are the canonical morphisms (**[17]** III 1.3.6). The morphism l' is therefore affine, with linear term ω'. The resulting correspondence $l \mapsto l'$ induces an $\mathscr{O}_X$-linear morphism

$$\lambda_\sharp : \mathscr{F} \to f_*(\mathscr{F}^+). \tag{10.7.3}$$

Its adjoint morphism is an $\mathscr{O}_Y$-linear isomorphism ([2] II.4.13.4)

$$\lambda : f^*(\mathscr{F}) \xrightarrow{\sim} \mathscr{F}^+ \tag{10.7.4}$$

which fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_Y \arrow[r] \arrow[d] & f^*(\mathscr{F}) \arrow[r] \arrow[d, "\lambda"'] & f^*(\Omega) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_Y \arrow[r] & \mathscr{F}^+ \arrow[r] & f^*(\Omega) \arrow[r] & 0. \end{tikzcd} \tag{10.7.5}$$

In particular, the isomorphism λ is compatible with actions of $f^*(\widehat{\Gamma}_{\mathscr{O}_X}(E)) \simeq \widehat{\Gamma}_{\mathscr{O}_Y}(f^*(E))$ (10.3.2).

10.8. – In the remainder of this section, X denotes a *smooth scheme over* k. We denote by $\mathrm{Crys}(X/k)$ the crystalline site of X over k equipped with the PD-ideal 0, by $(X/k)_{\mathrm{crys}}$ the crystalline topos of X over k and by $\mathscr{O}_{X/k}$ the structure ring of $(X/k)_{\mathrm{crys}}$ defined for every object (U,T) of $\mathrm{Crys}(X/k)$, by $(U,T) \mapsto \Gamma(T, \mathscr{O}_T)$.

Let E be a crystal of locally free $\mathscr{O}_{X/k}$-modules of finite type on $\mathrm{Crys}(X/k)$ and $\mathscr{L}$ an E-torsor of $(X/k)_{\mathrm{crys}}$. For any object (U,T) of $\mathrm{Crys}(X/k)$, $\mathscr{L}_{(U,T)}$ is an $E_{(U,T)}$-torsor of T_{zar} ([4] III 3.5.1). We define $\mathscr{F}_{(U,T)}$ to be the sheaf of affine functions on $\mathscr{L}_{(U,T)}$ of T_{zar} (10.4).

Let $g : (U_1, T_1) \to (U_2, T_2)$ be a morphism of $\mathrm{Crys}(X/k)$ and $\jmath_g : |U_1|(= |T_1|) \to |U_2|(= |T_2|)$ the morphism of underlying topological spaces. The transition morphism of $\mathscr{L}$ associated to g ([5] 5.1)

$$c_g : \jmath_g^{-1}(\mathscr{L}_{(U_2,T_2)}) \to \mathscr{L}_{(U_1,T_1)} \tag{10.8.1}$$

is $\jmath_g^{-1}(E_{(U_2,T_2)})$-equivariant. By ([17] III 1.4.6(iii)), we obtain an $E_{(U_1,T_1)}$-equivariant isomorphism

$$\jmath_g^{+}(\mathscr{L}_{(U_2,T_2)}) \xrightarrow{\sim} \mathscr{L}_{(U_1,T_1)}. \tag{10.8.2}$$

By (10.7.4), we deduce an $\mathscr{O}_{T_1}$-linear isomorphism ([2] II 4.14.2)

$$\gamma_g : g^*(\mathscr{F}_{(U_2,T_2)}) \xrightarrow{\sim} \mathscr{F}_{(U_1,T_1)}. \tag{10.8.3}$$

In view of the compatibility conditions of c_g (10.8.1) and ([2] II 4.15), the data $\{\mathscr{F}_{(U,T)}, \gamma_g\}$ satisfy the compatibility conditions of ([5] 5.1). Hence, they define a crystal of $\mathscr{O}_{X/k}$-modules that we denote by $\mathscr{F}$ and call *the crystal of affine functions on* $\mathscr{L}$.

We denote by $\mathscr{A}$ the crystal of $\mathscr{O}_{X/k}$-algebras $\varinjlim_{n\geqslant 0} \mathrm{S}^n(\mathscr{F})$. It admits a increasing filtration of crystals of $\mathscr{O}_{X/k}$-modules $N_n(\mathscr{A}) = \mathrm{S}^n(\mathscr{F})$.

10.9. – Let (U, T, δ) be an object of $\mathrm{Crys}(X/k)$ and J_T the PD-ideal associated to the closed immersion $i : U \to T$. For any local section a of J_T, we have $a^p = 0$ and hence an isomorphism $U \xrightarrow{\sim} \underline{T}$. We denote by $\varphi_{T/k}$ the composition (2.2)

$$\varphi_{T/k} : T \xrightarrow{f_{T/k}} U' \longrightarrow X'. \tag{10.9.1}$$

The morphism $\varphi_{X/k}$ is equal to the relative Frobenius morphism $F_{X/k}$. If $g : (U_1, T_1) \to (U_2, T_2)$ is a morphism of $\mathrm{Crys}(X/k)$, then $\varphi_{T_2/k} \circ g = \varphi_{T_1/k}$.

Let M' be an $\mathscr{O}_{X'}$-module. There exists a canonical isomorphism

$$\widetilde{c}_g : g^*(\varphi^*_{T_2/k}(M')) \xrightarrow{\sim} \varphi^*_{T_1/k}(M'). \tag{10.9.2}$$

The data $\{\varphi^*_{T/k}(M'), \widetilde{c}_g\}$ defines a crystal of $\mathscr{O}_{X/k}$-modules that we denote by $\mathscr{M}$. Then the $\mathscr{O}_X$-module with integrable connection associated to $\mathscr{M}$ ([**5**] 6.8) is $F^*_{X/k}(M')$ and the Frobenius descent connection ∇_{can} (6.3.1) ([**31**] 1.1).

10.10. – We denote by $\mathrm{Crys}(X/\mathrm{W}_2)$ the crystalline site of X over W_2 equipped with the PD-structure γ_2 (5.7). Let $(U, \widetilde{T}, \delta)$ be an object of $\mathrm{Crys}(X/\mathrm{W}_2)$ and T the reduction modulo p of $\widetilde{T}$. Since δ and γ_2 are compatible ([**5**] 3.16), $(U, \widetilde{T}, \delta)$ induces an object $(U, T, \bar{\delta})$ of $\mathrm{Crys}(X/k)$.

Let $\mathrm{T}_{X/k}$ (resp. $\mathrm{T}_{X'/k}$) be the $\mathscr{O}_X$-dual of $\Omega^1_{X/k}$ (resp. $\Omega^1_{X'/k}$). Suppose we are given a smooth lifting $\widetilde{X}'$ of X'/k over W_2. Let $(U, \widetilde{T}, \delta)$ be an object of $\mathrm{Crys}(X/\mathrm{W}_2)$ such that $\widetilde{T}$ is flat over W_2, T the reduction modulo p of $\widetilde{T}$ and V an open subscheme of T. Then (U, T) is an object of $\mathrm{Crys}(X/k)$. We denote by $\widetilde{V}$ the open subscheme of $\widetilde{T}$ associated to V and by $\mathscr{L}_{\widetilde{X}', \varphi_{T/k}}(V)$ the set of W_2-morphisms $\widetilde{V} \to \widetilde{X}'$ which make the following diagram commute (10.9)

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\varphi_{T/k}|_V"'] & \widetilde{V} \arrow[d] \\ X' \arrow[r] & \widetilde{X}'. \end{tikzcd} \tag{10.10.1}$$

The functor $V \mapsto \mathscr{L}_{\widetilde{X}', \varphi_{T/k}}(V)$ defines a sheaf for the Zariski topology on T. The sheaf $\mathscr{L}_{\widetilde{X}', \varphi_{T/k}}$ is a torsor under the $\mathscr{O}_T$-module $\mathscr{H}om_{\mathscr{O}_T}(f^*_{T/k}(\Omega^1_{X'/k}), p\mathscr{O}_{\widetilde{T}}) \xrightarrow{\sim} \varphi^*_{T/k}(\mathrm{T}_{X'/k})$.

Proposition 10.11 ([**31**] Thm. 1.1). – *Suppose we are given a smooth lifting $\widetilde{X}'$ of X' over W_2. Let $\mathscr{T}_{X'/k}$ be the crystal of $\mathscr{O}_{X/k}$-modules associated to the $\mathscr{O}_{X'}$-module $\mathrm{T}_{X'/k}$ (10.9). Then, there exists a unique $\mathscr{T}_{X'/k}$-torsor $\mathscr{L}_{\widetilde{X}'}$ of $(X/k)_{\mathrm{crys}}$ satisfying following conditions:*

(i) *For every object (U, T) of $\mathrm{Crys}(X/k)$ admitting a flat lifting $(U, \widetilde{T})$ in $\mathrm{Crys}(X/\mathrm{W}_2)$, the abelian sheaf $\mathscr{L}_{\widetilde{X}', (U,T)}$ of T_{zar} is the sheaf $\mathscr{L}_{\widetilde{X}', \varphi_{T/k}}$ (10.10).*

(ii) *For every morphism $\widetilde{g} : (U_1, \widetilde{T}_1) \to (U_2, \widetilde{T}_2)$ of flat objects in $\mathrm{Crys}(X/\mathrm{W}_2)$ and any lifting $\widetilde{F} : \widetilde{T}_2 \to \widetilde{X}' \in \mathscr{L}_{\widetilde{X}', \varphi_{T_2/k}}(T_2)$, the transition morphism $c_g : \jmath_g^{-1}(\mathscr{L}_{\widetilde{X}', (U_2, T_2)}) \to \mathscr{L}_{\widetilde{X}', (U_1, T_1)}$ satisfies*

$$c_g(\jmath_g^{-1}(\widetilde{F})) = \widetilde{F} \circ \widetilde{g} : \widetilde{T}_1 \to \widetilde{X}'. \tag{10.11.1}$$

10.12. – We denote by $\mathscr{F}_{\widetilde{X}'}$ the crystal of affine functions on $\mathscr{L}_{\widetilde{X}'}$ and by $\mathscr{A}_{\widetilde{X}'}$ the quasi-coherent crystal of $\mathscr{O}_{X/k}$-algebras associated to $\mathscr{F}_{\widetilde{X}'}$ (10.8). We put $\mathcal{A}_{\widetilde{X}'} = \mathscr{A}_{\widetilde{X}',(X,X)}$. There exists an integrable connection ∇_A on $\mathcal{A}_{\widetilde{X}'}$. By (10.3.2) and 10.5, $\mathcal{A}_{\widetilde{X}'}$ is a quasi-nilpotent $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-module. The p-curvature

$$\psi : \mathcal{A}_{\widetilde{X}'} \to \mathcal{A}_{\widetilde{X}'} \otimes_{\mathscr{O}_X} F^*_{X/k}(\Omega^1_{X'/k})$$

of ∇_A is equal to the universal $\mathscr{O}_X$-derivation (10.5.3) (cf. [**31**] Prop. 1.5)

$$d : \mathcal{A}_{\widetilde{X}'} \to \mathcal{A}_{\widetilde{X}'} \otimes_{\mathscr{O}_X} F^*_{X/k}(\Omega^1_{X'/k}).$$

10.13. – A Higgs field (5.1) on an $\mathscr{O}_X$-module E relative to k is equivalent to an $\mathrm{S}(\mathrm{T}_{X/k})$-module structure on E, which extends its $\mathscr{O}_X$-module structure (cf. [**31**] 5.1). Let I_X be the ideal $\oplus_{m>0}\mathrm{S}^m(\mathrm{T}_{X/k})$ of $\mathrm{S}(\mathrm{T}_{X/k})$. For any integer $n \geqslant 0$, we say that a Higgs module E over X relative to k is *nilpotent of level* $\leqslant n$, if E is annihilated by I_X^{n+1} as an $\mathrm{S}(\mathrm{T}_{X/k})$-module.

We call *PD-Higgs module on X relative to k* a $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module E, and we say that the structure morphism $\psi : \widehat{\Gamma}(\mathrm{T}_{X/k}) \to \mathscr{H}om_{\mathscr{O}_X}(E,E)$ is the *PD-Higgs field* on E. For any local section ξ of $\widehat{\Gamma}(\mathrm{T}_{X/k})$, we set $\psi_\xi = \psi(\xi)$.

Let n be an integer $\geqslant 1$. We denote by $\mathrm{HIG}_\gamma(X/k)$ the category of $\widehat{\Gamma}(\mathrm{T}_{X/k})$-modules and by $\mathrm{HIG}^{\mathrm{qn}}_\gamma(X/k)$ (resp. $\mathrm{HIG}^n_\gamma(X/k)$) the full subcategory of $\mathrm{HIG}_\gamma(X/k)$ consisting of quasi-nilpotent objects (resp. nilpotent objects of level $\leqslant n$) (10.1).

Since $\mathrm{S}^n(\mathrm{T}_{X/k}) \simeq \Gamma^n(\mathrm{T}_{X/k})$ for all $0 \leqslant n \leqslant p-1$, a nilpotent Higgs module of level $\leqslant p-1$ induces naturally a nilpotent PD-Higgs module of level $\leqslant p-1$.

10.14. – Let (E_1,ψ_1) and (E_2,ψ_2) be two objects of $\mathrm{HIG}^{\mathrm{qn}}_\gamma(X/k)$ and ∂ a local section of $\mathrm{T}_{X/k}$. We define a PD-Higgs field ψ on $E_1 \otimes_{\mathscr{O}_X} E_2$ by ([**31**] 2.7.1)

$$\psi_{\partial^{[n]}} = \sum_{i+j=n} \psi_{1,\partial^{[i]}} \otimes \psi_{2,\partial^{[j]}}. \tag{10.14.1}$$

Let m,n be two integers, (E_1,ψ_1) an object of $\mathrm{HIG}^n_\gamma(X/k)$ and (E_2,ψ_2) an object of $\mathrm{HIG}^m_\gamma(X/k)$. There exists a unique PD-Higgs field ψ on $\mathscr{H}om_{\mathscr{O}_X}(E_1,E_2)$ defined, for every local sections h of $\mathscr{H}om_{\mathscr{O}_X}(E_1,E_2)$ and ∂ of $\mathrm{T}_{X/k}$ by (cf. [**31**] page 31)

$$\psi_{\partial^{[l]}}(h) = \sum_{i+j=l} (-1)^i \psi_{2,\partial^{[i]}} \circ h \circ \psi_{1,\partial^{[j]}} \qquad \forall\ l \geqslant 1. \tag{10.14.2}$$

10.15. – We denote by $\mathrm{D}_{X/k}$ the ring of PD-differential operators on X relative to k ([**5**], § 4). Let ∂ be a local section of $\mathrm{T}_{X/k}$ considered as a derivation of $\mathscr{O}_X$ over k and hence as a PD-differential operator of order $\leqslant 1$. The pth iterate $\partial^{(p)}$ of ∂ is again a derivation of $\mathscr{O}_X$ over k ([**25**] 5.0.2, [**5**] 4.5). We denote by ∂^p the pth power of ∂ in $\mathrm{D}_{X/k}$, which is an operator of order $\leqslant p$. The p-curvature morphism $c : F^*_X(\mathrm{T}_{X/k}) \to \mathrm{D}_{X/k}$ defined by $F^*_X(\partial) \mapsto \partial^p - \partial^{(p)}$, induces an isomorphism of $\mathscr{O}_{X'}$-algebras ([**6**] 2.2.3; [**31**] Thm. 2.1)

$$\mathrm{S}(\mathrm{T}_{X'/k}) \xrightarrow{\sim} F_{X/k*}(Z_{X/k}). \tag{10.15.1}$$

The above morphism makes $F_{X/k*}(\mathrm{D}_{X/k})$ into an Azumaya algebra over $\mathrm{S}(\mathrm{T}_{X'/k})$ of rank p^{2d}, where d is the dimension of X over k.

10.16. – We denote by $\mathrm{D}^{\gamma}_{X/k}$ the tensor product

$$\mathrm{D}_{X/k}\otimes_{\mathrm{S}(\mathrm{T}_{X'/k})}\widehat{\Gamma}(\mathrm{T}_{X'/k}) \tag{10.16.1}$$

via the morphism $\mathrm{S}(\mathrm{T}_{X'/k})\to F_{X/k*}(\mathrm{D}_{X/k})$ induced by the p-curvature morphism (10.15.1). To give a left $\mathrm{D}^{\gamma}_{X/k}$-module is equivalent to give an $\mathscr{O}_X$-module M with integrable connection ∇ and a homomorphism

$$\psi:\widehat{\Gamma}(\mathrm{T}_{X'/k})\to F_{X/k*}(\mathscr{E}nd_{\mathscr{O}_X}(M,\nabla)) \tag{10.16.2}$$

which extends the Higgs field $\mathrm{S}(\mathrm{T}_{X'/k})\to F_{X/k*}(\mathscr{E}nd_{\mathscr{O}_X}(M,\nabla))$ given by the p-curvature of ∇ (10.15.1) (cf. [**31**] page 32).

10.17. – There exists an isomorphism $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))\xrightarrow{\sim}\widehat{\Gamma}(F^*_{X/k}(\mathrm{T}_{X'/k}))$ (10.3.2). Let M be a left $\mathrm{D}^{\gamma}_{X/k}$-module and n an integer $\geqslant 0$. We say that M is *quasi-nilpotent* (resp. *nilpotent of level* $\leqslant n$) if M is quasi-nilpotent (resp. nilpotent of level $\leqslant n$) as a $\widehat{\Gamma}(F^*_{X/k}(\mathrm{T}_{X'/k}))$-module (10.1).

We denote by $\mathrm{MIC}_{\gamma}(X/k)$ the category of left $\mathrm{D}^{\gamma}_{X/k}$-modules and by $\mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)$ (resp. $\mathrm{MIC}^{n}_{\gamma}(X/k)$) the full subcategory of $\mathrm{MIC}_{\gamma}(X/k)$ consisting of quasi-nilpotent objects (resp. nilpotent objects of level $\leqslant n$).

Let (M,∇) be an $\mathscr{O}_X$-module with integrable connection whose p-curvature is nilpotent of level $\leqslant p-1$ ([**25**] 5.6). Since $\mathrm{S}^n(\mathrm{T}_{X'/k})\simeq\Gamma_n(\mathrm{T}_{X'/k})$ for all $0\leqslant n\leqslant p-1$, (M,∇) induces naturally an object of $\mathrm{MIC}^{p-1}_{\gamma}(X/k)$.

10.18. – Let (M_1,∇_1,ψ_1) and (M_2,∇_2,ψ_2) be two objects of $\mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)$. There exists a canonical integrable connection ∇ on $M_1\otimes_{\mathscr{O}_X}M_2$ ([**25**] 1.1.1). The morphisms ψ_1 and ψ_2 induce an action ψ of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$ on $M_1\otimes_{\mathscr{O}_X}M_2$ as in (10.14.1). Then we obtain an object $(M_1\otimes_{\mathscr{O}_X}M_2,\nabla,\psi)$ of $\mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)$.

Let m,n be two integers and (M_1,∇_1,ψ_1) an object of $\mathrm{MIC}^{m}_{\gamma}(X/k)$ and (M_2,∇_2,ψ_2) an object of $\mathrm{MIC}^{n}_{\gamma}(X/k)$. There exists an integrable connection ∇ on the $\mathscr{O}_X$-module $\mathscr{H}om_{\mathscr{O}_X}(M_1,M_2)$ defined, for every local sections ∂ of $\mathrm{T}_{X/k}$ and h of $\mathscr{H}om_{\mathscr{O}_X}(M_1,M_2)$ by ([**25**] 1.1.2)

$$\nabla_{\partial}(h)=\nabla_{2,\partial}\circ h-h\circ\nabla_{1,\partial}. \tag{10.18.1}$$

The morphisms ψ_1 and ψ_2 induce an action of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$ on $F_{X/k*}(\mathscr{H}om_{\mathscr{O}_X}(M_1,M_2))$ defined by the same formula as (10.14.2). These data make $\mathscr{H}om_{\mathscr{O}_X}(M_1,M_2)$ into an object of $\mathrm{MIC}^{m+n}_{\gamma}(X/k)$ ([**31**] 2.1).

10.19. – By 10.12, the $\mathscr{O}_X$-algebra $\mathscr{A}_{\widetilde{X}'}$ (10.12) is equipped with a quasi-nilpotent left $\mathrm{D}^{\gamma}_{X/k}$-module structure. Moreover, we have an exhaustive filtration

$\{N_n(\mathscr{A}_{\widetilde{X}'})\}_{n\geqslant 0}$ of left $\mathrm{D}^{\gamma}_{X/k}$-submodules of $\mathscr{A}_{\widetilde{X}'}$ such that $N_n(\mathscr{A}_{\widetilde{X}'})$ is nilpotent of level $\leqslant n$ (10.8). We define $(\mathscr{A}_{\widetilde{X}'})^{\vee}$ to be

$$(\mathscr{A}_{\widetilde{X}'})^{\vee} = \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{A}_{\widetilde{X}'}, \mathscr{O}_X) \simeq \varinjlim_{n\geqslant 0} \mathscr{H}\!om_{\mathscr{O}_X}(N_n(\mathscr{A}_{\widetilde{X}'}), \mathscr{O}_X). \tag{10.19.1}$$

By 10.17 and 10.18, we see that $(\mathscr{A}_{\widetilde{X}'})^{\vee}$ is an object of $\mathrm{MIC}_{\gamma}(X/k)$.

The involution morphism $\mathrm{T}_{X/k} \to \mathrm{T}_{X/k}$ defined by $x \mapsto -x$, induces a homomorphism

$$\iota : \widehat{\Gamma}(\mathrm{T}_{X'/k}) \to \widehat{\Gamma}(\mathrm{T}_{X'/k}). \tag{10.19.2}$$

Theorem 10.20 ([**31**] 2.8). – *Suppose we are given a smooth lifting $\widetilde{X}'$ of X' over W_2.*

(i) *The left $\mathrm{D}^{\gamma}_{X/k}$-module $(\mathscr{A}_{\widetilde{X}'})^{\vee}$ is a splitting module for the Azumaya algebra $F_{X/k*}(\mathrm{D}^{\gamma}_{X/k})$ over $\widehat{\Gamma}(\mathrm{T}_{X'/k})$.*

(ii) *The functors* (10.19.2)

$$\mathrm{C}_{\widetilde{X}'} : \mathrm{MIC}_{\gamma}(X/k) \xrightarrow{\sim} \mathrm{HIG}_{\gamma}(X'/k) \qquad E \mapsto \iota^*(\mathscr{H}\!om_{\mathrm{D}^{\gamma}_{X/k}}((\mathscr{A}_{\widetilde{X}'})^{\vee}, E)) \tag{10.20.1}$$

$$\mathrm{C}^{-1}_{\widetilde{X}'} : \mathrm{HIG}_{\gamma}(X'/k) \xrightarrow{\sim} \mathrm{MIC}_{\gamma}(X/k) \qquad E' \mapsto (\mathscr{A}_{\widetilde{X}'})^{\vee} \otimes_{\widehat{\Gamma}(\mathrm{T}_{X'/k})} \iota^*(E') \tag{10.20.2}$$

are equivalences of categories quasi-inverse to each other. Furthermore, they induce equivalences of tensor categories between $\mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)$ and $\mathrm{HIG}^{\mathrm{qn}}_{\gamma}(X'/k)$ (10.14), 10.18.

(iii) *Let (E,∇,ψ) be an object of $\mathrm{MIC}_{\gamma}(X/k)$ and $(E',\theta') = \mathrm{C}_{\widetilde{X}'}(E,\nabla,\psi)$. A lifting $\widetilde{F}$ of the relative Frobenius morphism $F_{X/k}$ induces a natural isomorphism of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules*

$$\eta_{\widetilde{F}} : (E,\psi) \xrightarrow{\sim} F^*_{X/k}(E',-\theta'). \tag{10.20.3}$$

We call $\mathrm{C}_{\widetilde{X}'}$ (resp. $\mathrm{C}^{-1}_{\widetilde{X}'}$) *Cartier transform (resp. inverse Cartier transform).*

Theorem 10.21 ([**31**] 2.17). – *Suppose we are given a smooth lifting $\widetilde{X}'$ of X' over W_2. Let (M',θ') be a nilpotent Higgs module on X'/k of level $\ell < p$* (10.13) *and $(M,\nabla) = \mathrm{C}^{-1}_{\widetilde{X}'}(M',\theta')$. Then, the lifting $\widetilde{X}'$ induces an isomorphism in the derived category $\mathrm{D}(\mathscr{O}_{X'})$*

$$\tau_{<p-\ell}(M' \otimes_{\mathscr{O}_{X'}} \Omega^{\bullet}_{X'/k}) \xrightarrow{\sim} F_{X/k*}(\tau_{<p-\ell}(M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k})), \tag{10.21.1}$$

where $M' \otimes_{\mathscr{O}_{X'}} \Omega^{\bullet}_{X'/k}$ is the Dolbeault complex of (M',θ'), $M \otimes_{\mathscr{O}_X} \Omega^{\bullet}_{X/k}$ is the de Rham complex of (M,∇) and $\tau_{<\bullet}$ denotes the truncation of a complex.

We will give a partial generalization of this result for certain p^n-torsion crystals (cf. 14.1, 14.19).

CHAPTER 11

PRELUDE ON RINGS OF DIFFERENTIAL OPERATORS

The purpose of this section is to review the description of crystals of Oyama site in term of modules of rings of differential rings introduced in § 10 following Oyama. It serves as a preparation for Section 12.

In this section, X denotes a smooth scheme over k. From 11.11 on, suppose we are given a smooth formal $\mathscr{S}$-scheme $\mathfrak{X}$ with special fiber X.

11.1. – Let $n \geqslant 1$ be an integer. We denote by P_X (resp. P_X^n) the PD-envelope (resp. the n^{th} PD-neighborhood) of the diagonal immersion $\Delta : X \to X^2$ with respect to the zero PD-ideal of k ([**5**] 3.31). We put $\mathscr{P}_X = \mathscr{O}_{P_X}$ and $\mathscr{P}_X^n = \mathscr{O}_{P_X^n}$ and we consider them as sheaves of X_{zar}. By 5.10, $\mathscr{P}_X$ is equipped with a Hopf $\mathscr{O}_X$-algebra structure (δ, π, σ) (4.2).

In the first part of this section, we study the $\mathscr{O}_X$-algebra $(\mathscr{P}_X)^\vee$ (4.4) of hyper PD-differential operators of X relative to k.

Assume that there exists an étale k-morphism $X \to \mathbb{A}_k^d = \operatorname{Spec}(k[T_1, \dots, T_d])$. We set t_i the image of T_i in $\mathscr{O}_X$ and $\xi_i = 1 \otimes t_i - t_i \otimes 1$. We consider the ξ_i's as sections of $\mathscr{P}_X$. Regarding $\mathscr{P}_X$ as a left (resp. right) $\mathscr{O}_X$-algebra, we have an isomorphism of PD-$\mathscr{O}_X$-algebras ([**4**] I 4.5.3)

$$\mathscr{O}_X\langle x_1, \dots, x_d\rangle \xrightarrow{\sim} \mathscr{P}_X, \tag{11.1.1}$$

where x_i is sent to ξ_i. The homomorphism of PD-algebras $\delta : \mathscr{P}_X \to \mathscr{P}_X \otimes_{\mathscr{O}_X} \mathscr{P}_X$ ([**4**] I 1.7.1) sends ξ_i to $\xi_i \otimes 1 + 1 \otimes \xi_i$. For any $\alpha \in \mathbb{N}^d$, we set $\xi^{[\alpha]} = \prod \xi_i^{[\alpha_i]}$. Then we deduce that

$$\delta(\xi^{[\alpha]}) = \sum_{\beta \in \mathbb{N}^d, \beta \leqslant \alpha} \xi^{[\beta]} \otimes \xi^{[\alpha - \beta]}. \tag{11.1.2}$$

The left (resp. right) $\mathscr{O}_X$-module $\mathscr{P}_X^n$ is free with a basis $\{\xi^{[\alpha]}, |\alpha| \leqslant n\}$ ([**4**] I 4.5.3).

11.2. – Let U be a open subscheme of X^2 such that $\Delta(X) \subset U$ and that $X \to U$ is a closed immersion. The canonical morphism $P_X \to X^2$ factors through an affine morphism $P_X \to U$. We denote by Z the scheme theoretic image of $P_X \to U$ ([**22**] 6.10.1 and 6.10.5). Note that the morphisms $X \to P_X$ and $P_X \to Z$ induce isomorphisms between the underlying topological spaces. Hence we regard $\mathscr{O}_Z$ as an $\mathscr{O}_X$-bialgebra of X_{zar}. We obtain an injective homomorphism of $\mathscr{O}$-bialgbras $\mathscr{O}_Z \to \mathscr{P}_X$ and we consider $\mathscr{O}_Z$ as a subalgebra of $\mathscr{P}_X$.

LEMMA 11.3. – *The scheme Z defined in 11.2 is independent of the choice of U up to canonical isomorphisms.*

Proof. – Let U_1 and U_2 be two open subscheme of X^2 such that $\Delta(X) \subset U_i$ and that $X \to U_i$ is a closed immersion $i = 1, 2$ and Z_1 (resp. Z_2) the scheme theoretic image of P_X in U_1 (resp. U_2). We can suppose that $U_1 \subset U_2$. The image of $|Z_1|$ and $|P_{\mathfrak{X}}|$ in $|U_1|$ (resp. $|U_2|$) are equal. Then the composition $Z_2 \to U_1 \to U_2$ is a closed immersion. By ([**22**] 6.10.3), we deduce a canonical isomorphism $Z_1 \xrightarrow{\sim} Z_2$. □

LEMMA 11.4. – *Assume that X is separated and that there exists an étale k-morphism $X \to \mathbb{A}^d_k$.*

(i) *The left (resp. right) $\mathscr{O}_X$-module $\mathscr{O}_Z$ is free with a basis $\{\xi^{[\alpha]}, \alpha \in \{0, 1, \ldots, p-1\}^d\}$* (11.1.1).

(ii) *The $\mathscr{O}_Z$-module $\mathscr{P}_X$ is free with a basis $\{\xi^{[pI]}, I \in \mathbb{N}^d\}$.*

Proof. – (i) It is clear that $\mathscr{O}_Z$ contains $\xi^{[\alpha]}$ for all $\alpha \in \{0, \ldots, p-1\}^d$. It suffices to show that $\mathscr{O}_Z$ is contained in the $\mathscr{O}_X$-submodule of $\mathscr{P}_X$ generated by $\{\xi^{[\alpha]}, \alpha \in \{0, 1, \ldots, p-1\}^d\}$.

The question being local, we suppose that X is quasi-compact. Let J be the ideal sheaf associated to the diagonal closed immersion $X \to X^2$ and $\varpi : P_X \to X^2$ the canonical morphism. By 11.3, we consider Z as the scheme theoretic image of ϖ. The ideal J being of finite type, we suppose that J is generated by m elements $x_1, \ldots, x_m$ of $J(X^2)$ for some integer $m \geqslant d$. Since $\mathscr{O}_{P_X}$ is a PD-algebra, the image of $x_1^p, \ldots, x_m^p$ in $\varpi_*(\mathscr{O}_{P_X})$ are zero. Put $N = (p-1)m$. Then the image of the ideal J^{N+1} in $\varpi_*(\mathscr{O}_{P_X})$ is zero, i.e., the morphism ϖ factors through the N^{th} order infinitesimal neighborhood $Y_N = \operatorname{Spec}(\mathscr{O}_{X^2}/J^{N+1})$ of the diagonal immersion $X \to X^2$. Then we obtain a homomorphism

$$\mathscr{O}_{Y_N} \to \mathscr{P}_X, \tag{11.4.1}$$

whose image is $\mathscr{O}_Z$. Recall ([**5**] 2.2) that, the left (resp. right) $\mathscr{O}_X$-module $\mathscr{O}_{Y_N}$ is free with a basis $\{\xi^I, |I| \leqslant N\}$. For any element $I = (i_1, \ldots, i_d) \in \mathbb{N}^d$, if one of the components i_j is $\geqslant p$, then the image of ξ^I in $\mathscr{P}_X$ is zero. Then the assertion follows.

(ii) The assertion follows from (i) and the local description of $\mathscr{P}_X$ (11.1.1). □

11.5. – The p-curvature morphism $c' : \mathrm{S}(\mathrm{T}_{X'/k}) \to F_{X/k*}(\mathrm{D}_{X/k})$ induces an isomorphism between $\mathrm{S}(\mathrm{T}_{X'/k})$ and the center $Z_{X/k}$ of $\mathrm{D}_{X/k}$ (10.15.1). It makes $\mathrm{D}_{X/k}$ into an $\mathrm{S}(\mathrm{T}_{X'/k})$-module of finite type. Let $I_{X'}$ be the ideal $\oplus_{m \geqslant 1} \mathrm{S}^m(\mathrm{T}_{X'/k})$ of $\mathrm{S}(\mathrm{T}_{X'/k})$. We denote by $\mathscr{K}$ the two-side ideal of $\mathrm{D}_{X/k}$ generated by $c'(I_{X'})$ and by $\widehat{\mathrm{D}}_{X/k}$ the completion of $\mathrm{D}_{X/k}$ with respect to the filtration $\{\mathscr{K}^m\}_{m \geqslant 1}$:

$$\widehat{\mathrm{D}}_{X/k} = \varprojlim_{m \geqslant 1} \mathrm{D}_{X/k} / \mathscr{K}^m. \tag{11.5.1}$$

The completion $\widehat{\mathrm{D}}_{X/k}$ is equal to the $(I_{X'})$-adic completion for the $\mathrm{S}(\mathrm{T}_{X'/k})$-module $\mathrm{D}_{X/k}$. Then we obtain an isomorphism of $\widehat{\mathrm{S}}(\mathrm{T}_{X'/k})$-algebras

$$\widehat{\mathrm{S}}(\mathrm{T}_{X'/k}) \otimes_{\mathrm{S}(\mathrm{T}_{X'/k})} \mathrm{D}_{X/k} \xrightarrow{\sim} \widehat{\mathrm{D}}_{X/k}. \tag{11.5.2}$$

11.6. – Recall that $\mathrm{D}_{X/k}$ is defined as $\varinjlim_{n\geqslant 1}(\mathscr{P}_X^n)^\vee$, where $(\mathscr{P}_X^n)^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}_X^n, \mathscr{O}_X)$. Then we have a canonical homomorphism of $\mathscr{O}_X$-algebras:

$$\mathrm{D}_{X/k} \to (\mathscr{P}_X)^\vee. \tag{11.6.1}$$

For any integer $n \geqslant 0$ and any open subscheme U of X, we define

$$\mathrm{F}^n(\mathscr{P}_X)(U) = \{a \in \mathscr{P}_X(U) | u(a) = 0 \quad \forall u \in \mathscr{K}^{n+1}(U)\}. \tag{11.6.2}$$

Then $\mathrm{F}^n(\mathscr{P}_X)$ is a left $\mathscr{O}_X$-submodule of $\mathscr{P}_X$. We set $(\mathrm{F}^n(\mathscr{P}_X))^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathrm{F}^n(\mathscr{P}_X), \mathscr{O}_X)$.

11.7. – Suppose that there exists an étale k-morphism $X \to \mathbb{A}_k^d = \mathrm{Spec}(k[t_1, \dots, t_d])$. Let t_i be the image of T_i in $\mathscr{O}_X$ and $\partial_i \in \mathrm{T}_{X/k}(X)$ the dual of dt_i that we consider as a PD-differential operator. For i, j, ∂_i and ∂_j commute. We set $\partial^I = \prod_{j=1}^d \partial_j^{i_j}$ for all $I = (i_1, \dots, i_d) \in \mathbb{N}^d$. Any local section of $\mathrm{D}_{X/k}$ can be written as a finite sum $\sum_I a_I \partial^I$. The p-curvature morphism c' sends ∂_i' to ∂_i^p and the ideal $\mathscr{K}$ is generated by $\{\partial_1^p, \dots, \partial_d^p\}$ over $\mathrm{D}_{X/k}$. Then, any local section of $\widehat{\mathrm{D}}_{X/k}$ can be written as an infinite sum:

$$\sum_{I \in \mathbb{N}^d} a_I \partial^I \qquad \text{with } a_I \in \mathscr{O}_X. \tag{11.7.1}$$

Since $c'(\partial'^\beta) = \partial^{p\beta}$ for $\beta \in \mathbb{N}^d$, the above section can be rewritten as

$$\sum_{\alpha \in \{0, \dots, p-1\}^d, \beta \in \mathbb{N}^d} b_{\alpha,\beta} \partial^\alpha \cdot c'(\partial'^\beta) \qquad \text{with } b_{\alpha,\beta} \in \mathscr{O}_X. \tag{11.7.2}$$

For any $I, J \in \mathbb{N}^d$, the image of ∂^I in $(\mathscr{P}_X)^\vee$ (11.6.1) satisfies $\partial^I(\xi^{[J]}) = \delta_{I,J}$.

Lemma 11.8. – (i) *For any $n \geqslant 0$, the sheaf* $\mathrm{F}^n(\mathscr{P}_X)$ *is an $\mathscr{O}_Z$-submodule of* $\mathscr{P}_X$ (11.2).

(ii) *With the assumption and notation of* 11.4, $\mathrm{F}^n(\mathscr{P}_X)$ *is a free $\mathscr{O}_Z$-module with basis* $\{\xi^{[pI]}, |I| \leqslant n\}$.

Proof. – (i) The question being local, we take again the assumption of 11.4. Since the ideal $\mathscr{K}$ is generated by $\{\partial_1^p, \dots, \partial_d^p\}$, a local section $a = \sum_I b_I \xi^{[I]}$ of $\mathscr{P}_X$ is annihilated by $\mathscr{K}$ if and only if $b_I = 0$ for all $I \in \mathbb{N}^d - \{0, \dots, p-1\}^d$. By 11.4(i), $\mathrm{F}^0(\mathscr{P}_X)$ is equal to the subsheaf $\mathscr{O}_Z$ of $\mathscr{P}_X$. Then the assertion follows.

(ii) The ideal $\mathscr{K}^{n+1}$ is generated by the set of PD-differential operators $\{\partial^{pI}, |I| = n+1\}$ over $\mathrm{D}_{X/k}$. Then the assertion follows from 11.4(ii) and the duality between ∂^I and $\xi^{[I]}$. □

11.9. – For any $n \geqslant 0$, since $\mathrm{F}^n(\mathscr{P}_X)$ is locally a direct summand of $\mathscr{P}_X$, the canonical morphism $(\mathscr{P}_X)^\vee \to (\mathrm{F}^n(\mathscr{P}_X))^\vee$ is surjective. By (11.1.2), 11.4(i) and 11.8(ii), the homomorphism $\delta : \mathscr{P}_X \to \mathscr{P}_X \otimes_{\mathscr{O}_X} \mathscr{P}_X$ (11.1) sends $\mathrm{F}^n(\mathscr{P}_X)$ to $\mathrm{F}^n(\mathscr{P}_X) \otimes_{\mathscr{O}_X} \mathrm{F}^n(\mathscr{P}_X)$. In view of (4.4), the $\mathscr{O}_X$-algebra $(\mathscr{P}_X)^\vee$ induces an $\mathscr{O}_X$-algebra structure on $(\mathrm{F}^n(\mathscr{P}_X))^\vee$.

PROPOSITION 11.10 (Berthelot[(1)]). – (i) *For any integer* $n \geqslant 1$, *the homomorphism* (11.6.1) *induces a canonical isomorphism of* $\mathscr{O}_X$*-algebras* $\mathrm{D}_{X/k} / \mathscr{K}^{n+1} \xrightarrow{\sim} (\mathrm{F}^n(\mathscr{P}_X))^\vee$.

(ii) *The homomorphism* (11.6.1) *induces a canonical isomorphism of* $\mathscr{O}_X$*-algebras* $\widehat{\mathrm{D}}_{X/k} \xrightarrow{\sim} (\mathscr{P}_X)^\vee$.

Proof. – (i) Since $\mathscr{K}^{n+1}$ acts trivially on $\mathrm{F}^n(\mathscr{P}_X)$, we obtain a homomorphism $\mathrm{D}_{X/k} / \mathscr{K}^{n+1} \to (\mathrm{F}^n(\mathscr{P}_X))^\vee$. In view of the local description of $\mathscr{K}^{n+1}$ and of $\mathrm{F}^n(\mathscr{P}_X)$ (11.8(ii)), this homomorphism is an isomorphism.

(ii) We have a canonical isomorphism $(\mathscr{P}_X)^\vee = \mathscr{H}om_{\mathscr{O}_X}(\varinjlim_{n\geqslant 0} \mathrm{F}^n(\mathscr{P}_X), \mathscr{O}_X) \simeq \varprojlim (\mathrm{F}^n(\mathscr{P}_X))^\vee$. Then the assertion follows from (i). □

11.11. – In the remainder of this section, suppose that there exists a smooth formal $\mathscr{S}$-scheme $\mathfrak{X}$ with special fiber X. We put $\mathscr{R}_{\mathfrak{X},1} = \mathscr{R}_{\mathfrak{X}}/p\mathscr{R}_{\mathfrak{X}}$ and $\mathscr{Q}_{\mathfrak{X},1} = \mathscr{Q}_{\mathfrak{X}}/p\mathscr{Q}_{\mathfrak{X}}$ (4.9). We review the Oyama's description of $\widehat{\Gamma}(\mathrm{T}_{X/k}), \mathrm{D}^\gamma_{X/k}$ in terms of rings of differential operators associated to Hopf algebras $\mathscr{R}_{\mathfrak{X},1}, \mathscr{Q}_{\mathfrak{X},1}$.

Recall that the left and the right $\mathscr{O}_X$-algebra structures of $\mathscr{R}_{\mathfrak{X},1}$ are equal (4.9). We denote by $q_1, q_2 : Q_{\mathfrak{X},1} \to X$ the canonical morphisms. Suppose that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We set t_i the image of T_i in $\mathscr{O}_{\mathfrak{X}}$ and $\xi_i = 1 \otimes t_i - t_i \otimes 1$. We take again the notation of 4.14 and of 4.15 and we denote by ζ_i the element $\frac{\xi_i}{p}$ of $\mathscr{R}_{\mathfrak{X},1}$ and by η_i the element $\frac{\xi_i^p}{p}$ of $\mathscr{Q}_{\mathfrak{X},1}$. We have isomorphisms of $\mathscr{O}_X$-algebras (4.14), 4.15

(11.11.1)
$$\mathscr{O}_X[x_1, \dots, x_d] \xrightarrow{\sim} \mathscr{R}_{\mathfrak{X},1}, \qquad x_i \mapsto \zeta_i;$$

(11.11.2)
$$\mathscr{O}_X[x_1, \dots, x_d, y_1, \dots, y_d]/(x_1^p, \dots, x_d^p) \xrightarrow{\sim} q_{j*}(\mathscr{Q}_{\mathfrak{X},1}), \quad x_i \mapsto \xi_i, \ y_i \mapsto \eta_i \quad j = 1, 2.$$

1. We learn the proof of a local version of (ii) from a talk note given by Berthelot on Cartier transform.

By 4.16, we have the following description of the Hopf algebra structures on $\mathscr{R}_{\mathfrak{X},1}$ and $\mathscr{Q}_{\mathfrak{X},1}$ ([**32**] 1.2.9)

(11.11.3)

$$\begin{cases} \delta : \mathscr{R}_{\mathfrak{X},1} \to \mathscr{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \mathscr{R}_{\mathfrak{X},1} & \zeta_i \mapsto 1 \otimes \zeta_i + \zeta_i \otimes 1 \\ \sigma : \mathscr{R}_{\mathfrak{X},1} \to \mathscr{R}_{\mathfrak{X},1} & \zeta_i \mapsto -\zeta_i \\ \pi : \mathscr{R}_{\mathfrak{X},1} \to \mathscr{O}_X & \zeta_i \mapsto 0 \end{cases}$$

$$\begin{cases} \delta : \mathscr{Q}_{\mathfrak{X},1} \to \mathscr{Q}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \mathscr{Q}_{\mathfrak{X},1} & \xi_i \mapsto 1 \otimes \xi_i + \xi_i \otimes 1 \\ & \eta_i \mapsto 1 \otimes \eta_i + \sum_{j=1}^{p-1} \frac{(p-1)!}{j!(p-j)!} \xi_i^j \otimes \xi_i^{p-j} + \eta_i \otimes 1 \\ \sigma : \mathscr{Q}_{\mathfrak{X},1} \to \mathscr{Q}_{\mathfrak{X},1} & \xi_i \mapsto -\xi_i, \eta_i \mapsto -\eta_i \\ \pi : \mathscr{Q}_{\mathfrak{X},1} \to \mathscr{O}_X & \xi_i \mapsto 0, \eta_i \mapsto 0 \end{cases}$$

PROPOSITION 11.12 ([**32**] 1.2.12). – *The Hopf $\mathscr{O}_X$-algebra $\mathscr{R}_{\mathfrak{X},1}$ is canonically isomorphism to* $\mathrm{S}(\Omega^1_{X/k})$ (10.1) *and the $\mathscr{O}_X$-algebra* $(\mathscr{R}_{\mathfrak{X},1})^\vee$ (4.4) *is canonically isomorphic to* $\widehat{\Gamma}(\mathrm{T}_{X/k})$ (10.1).

11.13. – In the following, we study the $\mathscr{O}_X$-algebra $(\mathscr{Q}_{\mathfrak{X},1})^\vee$. Suppose first that there exists an $\mathscr{S}$-morphism $F : \mathfrak{X} \to \mathfrak{X}'$ which lifts the relative Frobenius morphism $F_{X/k}$ of X. We denote by $\mathfrak{Y}$ (resp. Z) the fiber product of $F^2 : \mathfrak{X}^2 \to \mathfrak{X}'^2$ and $R_{\mathfrak{X}'} \to \mathfrak{X}'^2$ (resp. the diagonal immersion $X' \to \mathfrak{X}'^2$) and by S the fiber product of $F_{X/k} : X \to X'$ and $R_{\mathfrak{X}',1} \to X'$ (4.9). The morphism $F_{X/k} : X \to X'$ and the diagonal immersion $\Delta : X \to \mathfrak{X}^2$ induce a closed immersion $X \hookrightarrow Z$. We have a commutative diagram:

(11.13.1)
$$\begin{tikzcd}
S \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\square"] & Y \arrow[rr] \arrow[dr] \arrow[dd] & & R_{\mathfrak{X}',1} \arrow[dr] \arrow[dd] & \\
& & \mathfrak{Y} \arrow[rr] \arrow[dd] & & R_{\mathfrak{X}'} \arrow[dd] \\
X \arrow[r] & Z \arrow[rr] \arrow[dr] & & X' \arrow[dr] & \\
& & \mathfrak{X}^2 \arrow[rr, "F^2"] & & \mathfrak{X}'^2,
\end{tikzcd}$$

where $Y = \mathfrak{Y}_1$ is equal to $R_{\mathfrak{X}',1} \times_{X'} Z$ and the left square is Cartesian.

Let $\mathfrak{U}$ be an open formal subscheme of $\mathfrak{X}^2$ such that diagonal immersion $X \to \mathfrak{X}^2$ factors through a closed immersion $X \to \mathfrak{U}$ and put $\mathfrak{U}' = \mathfrak{U} \times_{\mathscr{S},\sigma} \mathscr{S}$. Let $\mathscr{I}$ (resp. $\mathscr{I}'$) be the ideal associated to the diagonal immersion $X \to \mathfrak{U}$ (resp. $X' \to \mathfrak{U}'$). For any local section x of $\mathscr{I}$, we have $x^p \in \mathscr{I}'\mathscr{O}_{\mathfrak{U}}$. Since Z is defined by $\mathscr{I}'\mathscr{O}_{\mathfrak{U}}$ and X is reduced, the closed immersion $X \hookrightarrow Z$ induces an isomorphism $X \xrightarrow{\sim} \underline{Z}$. Then, the closed immersion $\underline{Y} \to Y$ factors through $S \to Y$. By the universal property of $Q_{\mathfrak{X}}$ (3.5), we deduce an $\mathfrak{X}^2$-morphism

$$\nu : \mathfrak{Y} \to Q_{\mathfrak{X}}. \tag{11.13.2}$$

The composition $Q_{\mathfrak{X}} \to \mathfrak{X}^2 \xrightarrow{F^2} \mathfrak{X}'^2$ induces a morphism of formal groupoids above F (6.6.1):

$$\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}. \tag{11.13.3}$$

By (11.13.1) and the universal property of $R_{\mathfrak{X}'}$ (3.5), we deduce that the composition $\psi \circ \nu : \mathfrak{Y} \to Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$ is the canonical morphism

$$\mathfrak{Y} = R_{\mathfrak{X}'} \times_{\mathfrak{X}'^2} \mathfrak{X}^2 \to R_{\mathfrak{X}'}. \tag{11.13.4}$$

11.14. – Keep the assumption and notation of 11.13. The morphism ν (11.13.2) induces a morphism $S \to Q_{\mathfrak{X},1}$ which makes the following diagram commutes

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & Q_{\mathfrak{X},1} \arrow[r, "\psi_1"] \arrow[dd] & R_{\mathfrak{X}',1} \arrow[d] \\
X \arrow[rr] \arrow[dr] & & X' \arrow[d] \\
& X^2 \arrow[r] & X'^2.
\end{tikzcd} \tag{11.14.1}$$

Let g be the canonical morphism $S \to X$. Since $R_{\mathfrak{X}',1} \to X'$ is affine, we have an isomorphism $F^*_{X/k}(\mathscr{R}_{\mathfrak{X}',1}) \xrightarrow{\sim} g_*(\mathscr{O}_S)$ ([**22**] 9.3.2). Then the morphism $S \to Q_{\mathfrak{X},1}$ induces an $\mathscr{O}_X$-bilinear homomorphism

$$v : \mathscr{Q}_{\mathfrak{X},1} \to F^*_{X/k}(\mathscr{R}_{\mathfrak{X}',1}). \tag{11.14.2}$$

With the assumption and notation of 11.11, v sends ξ_i to 0 and η_i to $F^*_{X/k}(\zeta'_i)$ ([**32**] 1.2.14). Hence, v is independent of the choice of the lifting $F : \mathfrak{X} \to \mathfrak{X}'$ of $F_{X/k}$, and can be defined for a general smooth formal $\mathscr{S}$-scheme $\mathfrak{X}$ even if the relative Frobenius morphism cannot be lifted over $\mathscr{S}$.

By (11.11.3), v is compatible with Hopf $\mathscr{O}_X$-algebras structures. By taking $\mathscr{O}_X$-duals, we obtain a homomorphism of $\mathscr{O}_X$-algebras (10.3.2), 11.12

$$v^\vee : F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k})) \to (\mathscr{Q}_{\mathfrak{X},1})^\vee. \tag{11.14.3}$$

The morphism ψ_1 induces a homomorphism of Hopf algebras $s_F : \mathscr{R}_{\mathfrak{X}',1} \to F_{X/k*}(\mathscr{Q}_{\mathfrak{X},1})$ (4.3). By adjunction, we obtain a homomorphism of $\mathscr{O}_X$-algebras

$$s_F^\sharp : F^*_{X/k}(\mathscr{R}_{\mathfrak{X}',1}) \to \mathscr{Q}_{\mathfrak{X},1} \tag{11.14.4}$$

for the left $\mathscr{O}_X$-algebra structure on $\mathscr{Q}_{\mathfrak{X},1}$. The composition $v \circ s_F$ is the identical homomorphism.

Recall that we have a canonical morphism of formal $\mathfrak{X}$-groupoids $P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$ (5.12.1). Then we obtain a homomorphism of Hopf $\mathscr{O}_X$-algebras (4.3):

$$u : \mathscr{Q}_{\mathfrak{X},1} \to \mathscr{P}_X. \tag{11.14.5}$$

With the assumption and notation of 11.11, u sends ξ_i to ξ_i and η_i to $-\xi_i^{[p]}$ (11.1.1) ([**32**] 1.2.14).

By taking duals (4.4 and 11.10(ii)), u induces a homomorphism of $\mathscr{O}_X$-algebras

$$u^\vee : \widehat{\mathrm{D}}_{X/k} \to (\mathscr{Q}_{\mathfrak{X},1})^\vee. \tag{11.14.6}$$

Lemma 11.15 ([**32**] 1.2.15). – *Let $\sigma : F^*_{X/k}(\mathscr{R}_{\mathfrak{X},1}) \to F^*_{X/k}(\mathscr{R}_{\mathfrak{X},1})$ be the involution homomorphism defined in* (11.11.3).

(i) *The restriction of $u^\vee$ and $(\sigma \circ v)^\vee$ to $F^{-1}_{X/k}(\widehat{\mathrm{S}}(\mathrm{T}_{X'/k}))$ coincide.*

(ii) *The images of any local sections of $u^\vee(\widehat{\mathrm{D}}_{X/k})$ and of $(\sigma \circ v)^\vee(F^{-1}_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k})))$ commute in $(\mathscr{Q}_{\mathfrak{X},1})^\vee$.*

Proposition 11.16 ([**32**] 1.2.13). – *The homomorphisms $(\sigma \circ v)^\vee$ and $u^\vee$ induce an isomorphism of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-algebras* (10.16)

$$\mathrm{D}^{\gamma}_{X/k} \xrightarrow{\sim} (\mathscr{Q}_{\mathfrak{X},1})^\vee. \tag{11.16.1}$$

Proposition 11.17 ([**32**] 1.2.10). – *Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme, X its special fiber. There exists a canonical equivalence of tensor categories between* $\mathrm{HIG}^{\mathrm{qn}}_{\gamma}(X/k)$ (10.13) *(resp.* $\mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)$ (10.17)*) and the category of $\mathscr{O}_X$-modules with $\mathscr{R}_{\mathfrak{X}}$-stratification (resp. $\mathscr{Q}_{\mathfrak{X}}$-stratification)* (4.11), 5.4.

Proof. – We briefly recall here the main construction of the equivalence which will be used in the following and refer to [**32**] for details.

Let (M, ε) be an $\mathscr{O}_X$-module with $\mathscr{R}_{\mathfrak{X}}$-stratification and $\theta : M \to M \otimes_{\mathscr{O}_X} \mathscr{R}_{\mathfrak{X},1}$ the $\mathscr{O}_X$-linear morphism defined by $\theta(m) = \varepsilon(1 \otimes m)$ (5.5). By 5.5 and 11.12, we deduce a $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module ψ structure on M. To show that ψ is quasi-nilpotent, we suppose that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}}$. We take again notation of 11.11. For any $I \in \mathbb{N}^d$, we set $\partial^{[I]} = \prod_{j=1}^d \partial_j^{[i_j]} \in \widehat{\Gamma}(\mathrm{T}_{X/k})$. The action of $\partial^{[I]}$ on M is given by the composition

$$\psi_{\partial^{[I]}} : M \xrightarrow{\theta} M \otimes_{\mathscr{O}_X} \mathscr{R}_{\mathfrak{X},1} \xrightarrow{\mathrm{id} \otimes \partial^{[I]}} M. \tag{11.17.1}$$

Since $\mathscr{R}_{\mathfrak{X},1}$ is isomorphic to a polynomial algebra over $\mathscr{O}_X$ (11.11), for any local section m of M and any point x of X, there exists a neighborhood U of x such that $\theta(m)|_U$ is a section of $M(U) \otimes_{\mathscr{O}_X(U)} \mathscr{R}_{\mathfrak{X},1}(U)$ and can be written as a finite sum:

$$\varepsilon(1 \otimes m) = \theta(m) = \sum_{I \in \mathbb{N}^d} \psi_{\partial^{[I]}}(m) \otimes \zeta^I. \tag{11.17.2}$$

Hence, (M, ψ) is quasi-nilpotent (10.1).

Let (M,ε) be an $\mathscr{O}_X$-module with $\mathscr{Q}_{\mathfrak{X}}$-stratification and $\theta : M \to M\otimes_{\mathscr{O}_X}\mathscr{Q}_{\mathfrak{X},1}$ the morphism induced by ε. By 5.5 and 11.16, we associate to it a $\mathrm{D}^{\gamma}_{X/k}$-module (M,∇,ψ) (10.16). We take again the assumption and notation of the proof of 11.16. For $I\in\{0,\dots,p-1\}^d$, $J\in\mathbb{N}^d$, the action of the local section $\partial^I\otimes F^*_{X/k}(\partial'^{[J]})$ of $\mathrm{D}^{\gamma}_{X/k}$ on M is given by the composition

$$\nabla_{\partial^I}\circ\psi_{\partial'^{[J]}} :\ M \xrightarrow{\ \theta\ } M\otimes_{\mathscr{O}_X}\mathscr{Q}_{\mathfrak{X},1}\xrightarrow{\mathrm{id}\otimes w(\partial^I\otimes F^*_{X/k}(\partial'^{[J]}))} M. \tag{11.17.3}$$

Since $\mathscr{Q}_{\mathfrak{X},1}$ is of finite type over a polynomial algebra over $\mathscr{O}_X$ (11.11), for any local section m of M and any point x of X, there exists a neighborhood U of x such that $\theta(m)|_U$ is a section of $M(U)\otimes_{\mathscr{O}_X(U)}\mathscr{Q}_{\mathfrak{X},1}(U)$ and can be written as a finite sum (cf. **[32]** page 25)

$$\varepsilon(1\otimes m)=\theta(m)=\sum_{I\in\{0,\dots,p-1\}^d,J\in\mathbb{N}^d}(-1)^{|J|}\frac{1}{I!}(\nabla_{\partial^I}\circ\psi_{\partial'^{[J]}})(m)\otimes\xi^I\eta^J. \tag{11.17.4}$$

Hence (M,∇,ψ) is quasi-nilpotent (10.17). □

We deduce from 8.10 and 11.17 the following equivalences.

Corollary 11.18. – *Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme, X its special fiber, $\widetilde{\mathscr{E}}$ and $\underline{\widetilde{\mathscr{E}}}$ the Oyama topoi of X (7.9) and $\mathscr{O}_{\mathscr{E},1}$, $\mathscr{O}_{\underline{\mathscr{E}},1}$ the p-torsion structure rings of $\widetilde{\mathscr{E}}$ and $\underline{\widetilde{\mathscr{E}}}$ respectively* (8.1). *We denote by $\mathscr{C}(\mathscr{O}_{\mathscr{E},1})$ (resp. $\mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},1})$) the category of crystals of $\mathscr{O}_{\mathscr{E},1}$-modules of $\widetilde{\mathscr{E}}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},1}$-modules of $\underline{\widetilde{\mathscr{E}}}$)* (8.3). *Then, we have equivalences of tensor categories*

$$\mathrm{HIG}^{\mathrm{qn}}_{\gamma}(X/k)\simeq\mathscr{C}(\mathscr{O}_{\mathscr{E},1}),\qquad \mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)\simeq\mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},1}). \tag{11.18.1}$$

The following result is an analog of 10.20(iii) for Cariter equivalence.

Corollary 11.19. – *Let $\mathrm{C}_{X/\mathscr{S}} : \underline{\widetilde{\mathscr{E}}}\to\widetilde{\mathscr{E}}'$ be the morphism of topoi associated to X* (9.11.4), *$\mathscr{M}'$ a crystal of $\mathscr{O}_{\mathscr{E}',1}$-modules of $\widetilde{\mathscr{E}}'$* (8.3), *(M',θ) the associated $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module* (11.18) *and (∇,ψ) the $\mathrm{D}^{\gamma}_{X/k}$-module structure on $(\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}'))_{(X,\mathfrak{X})}$. Then a lifting $F:\mathfrak{X}\to\mathfrak{X}'$ of the relative Frobenius morphism $F_{X/k}$ of X induces a functorial isomorphism of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules:*

$$\eta_F:\iota^*(F^*_{X/k}(M',\theta))\xrightarrow{\sim}((\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}'))_{(X,\mathfrak{X})},\psi), \tag{11.19.1}$$

where ι denotes the involution homomorphism $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))\to F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$ (10.19.2).

Proof. – We take again the notation of 11.14. By 4.5, 11.12 and 11.16, the homomorphism $s^{\sharp}_F$ (11.14.4) induces a homomorphism of $\mathscr{O}_X$-algebras:

$$(s^{\sharp}_F)^{\vee}:\mathrm{D}^{\gamma}_{X/k}\to F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k})). \tag{11.19.2}$$

Since the composition $v \circ s_F^\sharp$ is the identity (11.14), the composition (11.14.3)

$$F_{X/k}^*(\widehat{\Gamma}(\mathrm{T}_{X'/k})) \xrightarrow{(\sigma\circ v)^\vee} \mathrm{D}_{X/k}^\gamma \xrightarrow{(s_F^\sharp)^\vee} F_{X/k}^*(\widehat{\Gamma}(\mathrm{T}_{X'/k})) \tag{11.19.3}$$

is the involution homomorphism $\iota : F_{X/k}^*(\widehat{\Gamma}(\mathrm{T}_{X'/k})) \to F_{X/k}^*(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$ (10.19.2).

Let ε be the associated $\mathscr{R}_{\mathfrak{X}'}$-stratification on M'. Recall 9.17 that the morphism F induces a functorial isomorphism of $\mathscr{O}_X$-modules compatible with $\mathscr{Q}_{\mathfrak{X}}$-stratifications:

$$\eta_F : (F_{X/k}^*(M'), s_F^*(\varepsilon)) \xrightarrow{\sim} (\mathrm{C}_{X/\mathscr{S}}^*(\mathscr{M}'))_{(X,\mathfrak{X})}. \tag{11.19.4}$$

In view of 11.17, the associated $\mathrm{D}_{X/k}^\gamma$-module structure on $F_{X/k}^*(M')$ in the left hand side is give by $F_{X/k}^*(\theta)$ via $(s_F^\sharp)^\vee$ that we denote by (∇_F, ψ_F). Then we obtain a functorial isomorphism of $\mathrm{D}_{X/k}^\gamma$-modules

$$\eta_F : (F_{X/k}^*(M'), \nabla_F, \psi_F) \xrightarrow{\sim} (\mathrm{C}_{X/\mathscr{S}}^*(\mathscr{M}'))_{(X,\mathfrak{X})}. \tag{11.19.5}$$

In view of (11.19.3), we have an equality of $F_{X/k}^*(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules

$$(F_{X/k}^*(M'), \psi_F) = \iota^*(F_{X/k}^*(M', \theta)). \tag{11.19.6}$$

This concludes the proof. $\square$

CHAPTER 12

COMPARISON WITH THE CARTIER TRANSFORM OF OGUS-VOLOGODSKY

In this section, we compare the Cartier equivalence modulo p and the Cartier transform of Ogus-Vologodsky. Our approach is different to that of Oyama. We interpret the Cartier equivalence as an admissibility condition à la Fontaine for a pair of crystals (12.17) and use it to compare with Cartier transform (12.22).

Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme, X its special fiber and $\mathrm{T}_{X/k}$ the $\mathscr{O}_X$-dual of the $\mathscr{O}_X$-module of differential forms $\Omega^1_{X/k}$. We set $\mathfrak{X}' = \mathfrak{X} \otimes_{\mathscr{S},\sigma} \mathscr{S}$.

12.1. – We first interpret the Cartier descent in the context of Cartier equivalence (12.3).

Let $\widetilde{\mathscr{E}}$ and $\underline{\widetilde{\mathscr{E}}}$ be the Oyama topoi of X (7.9) and $\mathscr{O}_{\mathscr{E},1}$, $\mathscr{O}_{\underline{\mathscr{E}},1}$ the p-torsion structure rings of $\widetilde{\mathscr{E}}$ and $\underline{\widetilde{\mathscr{E}}}$ respectively (8.1). We keep the conventions and notation of § 7-8. A morphism $g : (U_1, \mathfrak{T}_1, u_1) \to (U_2, \mathfrak{T}_2, u_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) induces a morphism of ringed topoi (8.2.1)

$$\widetilde{g}_s : (U_1, u_{1*}(\mathscr{O}_{T_1})) \to (U_2, u_{2*}(\mathscr{O}_{T_2})). \tag{12.1.1}$$

If we equip $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) with the Zariski topology, the functor (7.5.1)

$$\pi : \mathscr{E} \to \mathbf{Zar}_{/X} \quad (\text{resp. } \pi : \underline{\mathscr{E}} \to \mathbf{Zar}_{/X}) \quad (U, \mathfrak{T}) \to U \tag{12.1.2}$$

is cocontinuous. Since π commutes with the fibered product of a flat morphism and a morphism (7.3), one verifies that π is also continuous ([**3**] III 1.6). By 7.15, it induces a morphism of topoi that we denote by

$$v : \widetilde{\mathscr{E}} \to X_{\mathrm{zar}} \qquad (\text{resp. } v : \underline{\widetilde{\mathscr{E}}} \to X_{\mathrm{zar}}) \tag{12.1.3}$$

such that the inverse image functor is induced by the composition with π. Moreover, one verifies that the above morphisms fit into a commutative diagram

$$\begin{tikzcd} \underline{\widetilde{\mathscr{E}}} \arrow[r, "\mathrm{C}_{X/\mathscr{S}}"] \arrow[d, "v"'] & \widetilde{\mathscr{E}}' \arrow[d, "v'"] \\ X_{\mathrm{zar}} \arrow[r, "F_{X/k}"] & X'_{\mathrm{zar}}. \end{tikzcd} \tag{12.1.4}$$

Let $\mathscr{F}$ be a sheaf of X_{zar}. For any object $(U,\mathfrak{T})$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), we have $(v^*(\mathscr{F}))_{(U,\mathfrak{T})} = \mathscr{F}|_U$. For any morphism $f:(U_1,\mathfrak{T}_1)\to(U_2,\mathfrak{T}_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), the transition morphism γ_f of $v^*(\mathscr{F})$ (7.6.2) is the canonical isomorphism

$$\gamma_f : (\mathscr{F}|_{U_2})|_{U_1} \xrightarrow{\sim} \mathscr{F}|_{U_1}. \tag{12.1.5}$$

12.2. – For any object $(U,\mathfrak{T},u)$ of $\mathscr{E}$, the morphism $u:T\to U$ induces a canonical, functorial homomorphism

$$v^*(\mathscr{O}_X)(U,\mathfrak{T},u) = \mathscr{O}_X(U) \to \mathscr{O}_{\mathscr{E},1}(U,\mathfrak{T},u) = u_*(\mathscr{O}_T)(U). \tag{12.2.1}$$

Then the morphism of topoi $v:\widetilde{\mathscr{E}}\to X_{\mathrm{zar}}$ underlies a morphism of ringed topoi

$$\nu : (\widetilde{\mathscr{E}}, \mathscr{O}_{\mathscr{E},1}) \to (X_{\mathrm{zar}}, \mathscr{O}_X). \tag{12.2.2}$$

For any object $(U,\mathfrak{T},u)$ of $\underline{\mathscr{E}}$, we have a morphism $u'\circ f_{T/k}: T\to U'$ (9.1.1) and hence a canonical, functorial homomorphism

$$v^*(F^*_{X/k}(\mathscr{O}_{X'}))(U,\mathfrak{T},u) = \mathscr{O}_{X'}(U') \to \mathscr{O}_{\underline{\mathscr{E}},1}(U,\mathfrak{T},u) = (u'\circ f_{T/k})_*(\mathscr{O}_T)(U'). \tag{12.2.3}$$

The composition of morphisms $F_{X/k}\circ v:\widetilde{\underline{\mathscr{E}}}\to X_{\mathrm{zar}}\to X'_{\mathrm{zar}}$ underlies a morphism of ringed topoi

$$\mu : (\widetilde{\underline{\mathscr{E}}}, \mathscr{O}_{\underline{\mathscr{E}},1}) \to (X'_{\mathrm{zar}}, \mathscr{O}_{X'}). \tag{12.2.4}$$

By (12.1.4), the morphisms μ and ν' fit into a commutative diagram

$$\begin{tikzcd} (\widetilde{\underline{\mathscr{E}}}, \mathscr{O}_{\underline{\mathscr{E}},1}) \arrow[rr, "\mathrm{C}_{X/\mathscr{S}}"] \arrow[dr, "\mu"'] & & (\widetilde{\mathscr{E}}', \mathscr{O}_{\mathscr{E}',1}) \arrow[dl, "\nu'"] \\ & (X'_{\mathrm{zar}}, \mathscr{O}_{X'}). & \end{tikzcd} \tag{12.2.5}$$

Proposition 12.3. – *Let* $\mathbf{Mod}(\mathscr{O}_{X'})$ *be the category of* $\mathscr{O}_{X'}$*-modules and let* λ *be the functor*

$$\lambda : \mathbf{Mod}(\mathscr{O}_{X'}) \to \mathrm{MIC}^0_\gamma(X/k) \qquad M' \mapsto (F^*_{X/k}(M'), \nabla_{\mathrm{can}}, 0), \tag{12.3.1}$$

where $\mathrm{MIC}^0_\gamma(X/k)$ *denotes the category of nilpotent* $\mathrm{D}^\gamma_{X/k}$*-modules of level* $\leqslant 0$ (10.17). *Then, the following diagram is commutative up to a canonical isomorphism*

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{O}_{X'}) \arrow[r, "\lambda"] \arrow[d, "\nu'^*"'] & \mathrm{MIC}^0_\gamma(X/k) \arrow[d] \\ \mathscr{C}(\mathscr{O}_{\mathscr{E}',1}) \arrow[r, "\mathrm{C}^*_{X/\mathscr{S}}"] & \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},1}), \end{tikzcd} \tag{12.3.2}$$

where the right vertical arrow is given by (11.18).

The proposition follows from Lemmas 12.5, 12.6 below.

12.4. – Let M be an $\mathscr{O}_X$-module and $\mathscr{M} = \nu^*(M)$ (12.2.2). For any object $(U, \mathfrak{T}, u)$ of $\mathscr{E}$, we set $\widetilde{u}^*(M) = M|_U \otimes_{\mathscr{O}_U} u_*(\mathscr{O}_T)$. Recall (12.1) that we have $(v^*(M))_{(U,\mathfrak{T})} = M|_U$. In view of 7.10 and (12.2.1), we deduce that $\mathscr{M}_{(U,\mathfrak{T},u)} = \widetilde{u}^*(M)$. For any morphism $g : (U_1, \mathfrak{T}_1, u_1) \to (U_2, \mathfrak{T}_2, u_2)$ of $\mathscr{E}$, in view of (12.1.5), the transition morphism $c_g : \widetilde{g}_s^*(\mathscr{M}_{(U_2,\mathfrak{T}_2)}) \to \mathscr{M}_{(U_1,\mathfrak{T}_1)}$ is the canonical isomorphism (12.1.1)

$$\widetilde{g}_s^*(\widetilde{u}_2^*(M)) \xrightarrow{\sim} \widetilde{u}_1^*(M). \tag{12.4.1}$$

Hence $\mathscr{M}$ is a crystal of $\mathscr{O}_{\mathscr{E},1}$-modules of $\widetilde{\mathscr{E}}$. If M is moreover quasi-coherent, then so is $\mathscr{M}$ (8.3). In this case, for any object $(U, \mathfrak{T}, u)$ of $\mathscr{E}$, the $\mathscr{O}_T$-module $\mathscr{M}_{(U,\mathfrak{T})}$ of T_{zar} (8.4) is $u^*(M|_U)$.

LEMMA 12.5. – *Under the assumption of 12.4, the $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module associated to $\mathscr{M}$* (11.18) *is the $\mathscr{O}_X$-module M equipped with the zero PD-Higgs field* (10.13).

Proof. – The underlying $\mathscr{O}_X$-module is $\mathscr{M}_{(X,\mathfrak{X})} = M$. The reduction modulo p of the two canonical morphisms $q_1, q_2 : R_{\mathfrak{X}} \to \mathfrak{X}$ are equal. By (12.4.1), the $\mathscr{R}_{\mathfrak{X}}$-stratification on M associated to $\mathscr{M}$ (8.10)

$$\varepsilon : \widetilde{q}_{2,s}^*(M) \xrightarrow{\sim} \widetilde{q}_{1,s}^*(M)$$

is the identity morphism. In view of (11.17.1) and (11.17.2), the PD-Higgs field associated to ε is zero. □

LEMMA 12.6. – *Let M' be an $\mathscr{O}_{X'}$-module and $\mathscr{M} = \mu^*(M')$. Then $\mathscr{M}$ is a crystal of $\mathscr{O}_{\underline{\mathscr{E}},1}$-modules of $\underline{\widetilde{\mathscr{E}}}$ and the $\mathrm{D}^{\gamma}_{X/k}$-module associated to $\mathscr{M}$ is $(F^*_{X/k}(M'), \nabla_{\mathrm{can}}, 0)$* (11.18)*, where ∇_{can} denotes the Frobenius descent connection on $F^*_{X/k}(M')$* (6.3.1).

Proof. – We set $\mathscr{M}' = \nu'^*(M')$ (12.2.5). Then $\mathscr{M} = \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}')$ is a crystal by 12.4. For any object $(U, \mathfrak{T}, u)$ of $\underline{\mathscr{E}}$, we put $\phi_{T/k} = u' \circ f_{T/k} : T \to \underline{T}' \to U'$. Then we have $\rho(U, \mathfrak{T}, u) = (U', \mathfrak{T}, \phi_{T/k})$ (9.1.2) and (9.11.1)

$$\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}')_{(U,\mathfrak{T},u)} = \pi_{U*}(\mathscr{M}'_{(U',\mathfrak{T},\phi_{T/k})}) = \pi_{U*}(\widetilde{\phi}^*_{T/k}(M')). \tag{12.6.1}$$

The morphism $\phi_{X/k}$ associated to the object $(X, \mathfrak{X})$ of $\underline{\mathscr{E}}$ is $F_{X/k}$. Then we have $\mathscr{M}_{(X,\mathfrak{X})} = F^*_{X/k}(M')$. There exists a commutative diagram

$$\begin{tikzcd} Q_{\mathfrak{X},1} \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, "\phi_{Q_{\mathfrak{X},1}/k}"] & X \arrow[d, "F_{X/k}"] \\ X \arrow[r, "F_{X/k}"] & X'. \end{tikzcd} \tag{12.6.2}$$

The morphisms $q_1, q_2 : (X, Q_{\mathfrak{X}}) \to (X, \mathfrak{X})$ of $\underline{\mathscr{E}}$ induce isomorphisms (12.4.1), 12.6.1

$$\begin{aligned} c_{q_1} &: \widetilde{q}_{1,s}^*(F^*_{X/k}(M')) \xrightarrow{\sim} \pi_{X*}(\widetilde{\phi}^*_{Q_{\mathfrak{X},1}/k}(M')), \\ c_{q_2} &: \widetilde{q}_{2,s}^*(F^*_{X/k}(M')) \xrightarrow{\sim} \pi_{X*}(\widetilde{\phi}^*_{Q_{\mathfrak{X},1}/k}(M')). \end{aligned} \tag{12.6.3}$$

The $\mathscr{Q}_{\mathfrak{X}}$-stratification ε on $F^*_{X/k}(M')$ associated to the crystal $\mathscr{M}$ (8.10) is given by the composition of c_{q_2} and the inverse of c_{q_1}. In view of (12.6.2), for any local section m' of M', we have

$$\varepsilon(1\otimes F^*_{X/k}(m')) = F^*_{X/k}(m')\otimes 1. \tag{12.6.4}$$

Let $(F^*_{X/k}(M'),\nabla,\psi)$ be the $\mathrm{D}^{\gamma}_{X/k}$-module associated to $(F^*_{X/k}(M'),\varepsilon)$ (11.17). In view of (12.6.4), (11.17.3) and (11.17.4), we deduce that ∇ and ψ annihilate the subsheaf $F^{-1}_{X/k}(M')$ of $F^*_{X/k}(M')$. Hence ∇ is the Frobenius descent connection and $\psi=0$. □

12.7. – To compare the Cartier equivalence and the Cartier transform, we reconstruct the crystal $\mathscr{A}_{\mathfrak{X}'_2}$ (10.12) as a crystal in Oyama topos $\underline{\widetilde{\mathscr{E}}}$ (cf. 12.15 and 12.27) which allows us to compare Cartier transform and Cartier equivalence. Compared to Oyama's approach for this construction ([**32**] 1.5.2), our approach use certain torsor of liftings in Oyama topoi and is close the original construction of Ogus-Vologodsky.

Let $(U,\mathfrak{T},u)$ be an object of $\mathscr{E}$, W an open subscheme of T and $\mathfrak{W}_2$ the open subscheme of $\mathfrak{T}_2$ associated to W. We define $\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}(W)$ to be the set of $\mathscr{S}_2$-morphisms $\mathfrak{W}_2\to\mathfrak{X}_2$ which make the following diagram commutes

$$\begin{tikzcd} W \arrow[rr] \arrow[d, "u|_W"'] & & \mathfrak{W}_2 \arrow[d] \\ U \arrow[r] & X \arrow[r] & \mathfrak{X}_2. \end{tikzcd} \tag{12.7.1}$$

The functor $W\mapsto \mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}(W)$ defines a sheaf of T_{zar}. Since $\mathfrak{X}$ is smooth over $\mathscr{S}$, such morphisms exist locally. The sheaf $\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}$ is a torsor under the $\mathscr{O}_T$-module $\mathscr{H}om_{\mathscr{O}_{\mathfrak{T}_2}}(u^*(\Omega^1_{U/k}),p\mathscr{O}_{\mathfrak{T}_2})\xrightarrow{\sim} u^*(\mathrm{T}_{U/k})$ of T_{zar}.

12.8. – Let $g:(V,\mathfrak{Z},v)\to(U,\mathfrak{T},u)$ be a morphism of $\mathscr{E}$ and $g_s:Z\to T$ (resp. $g_2:\mathfrak{Z}_2\to\mathfrak{T}_2$) the reduction of the morphism $\mathfrak{Z}\to\mathfrak{T}$. For $\mathscr{O}_T$-modules, we will use the notation g_s^{-1} to denote the inverse image in the sense of abelian sheaves and will keep the notation g_s^* for the inverse image in the sense of modules. By adjunction, the isomorphism $g_s^*(u^*(\mathrm{T}_{U/k}))\xrightarrow{\sim} v^*(\mathrm{T}_{V/k})$ induces an $\mathscr{O}_T$-linear morphism:

$$\tau: u^*(\mathrm{T}_{U/k})\to g_{s*}(v^*(\mathrm{T}_{V/k})).$$

We have a canonical τ-equivariant morphism $\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}\to g_{s*}(\mathscr{R}_{\mathfrak{X},(V,\mathfrak{Z},v)})$ of T_{zar} defined for every local section $h:\mathfrak{W}_2\to\mathfrak{X}_2$ of $\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}$ by

$$\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}\to g_{s*}(\mathscr{R}_{\mathfrak{X},(V,\mathfrak{Z},v)}) \qquad h\mapsto h\circ g_2|_{g_2^{-1}(\mathfrak{W}_2)}. \tag{12.8.1}$$

By adjunction, we obtain a $g_s^{-1}(u^*(\mathrm{T}_{U/k}))$-equivariant morphism of Z_{zar}:

$$\gamma_g: g_s^*(\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)})\to\mathscr{R}_{\mathfrak{X},(V,\mathfrak{Z},v)}. \tag{12.8.2}$$

By ([**17**] III 1.4.6(iii)), we deduce a $v^*(\mathrm{T}_{V/k})$-equivariant isomorphism of Z_{zar} (10.7):

$$g_s^+(\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)})\xrightarrow{\sim}\mathscr{R}_{\mathfrak{X},(V,\mathfrak{Z},v)}. \tag{12.8.3}$$

One verifies that the data $\{u_*(\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}),\gamma_g\}$ satisfy the compatibility conditions of 7.7. Then it defines a sheaf of $\widetilde{\mathscr{E}}$ that we denote by $\mathscr{R}_{\mathfrak{X}}$ (7.10). We set $\mathscr{T}_{X/k}=\nu^*(\mathrm{T}_{X/k})$ (12.2.2), 12.5. In view of 12.7 and (12.8.2), $\mathscr{R}_{\mathfrak{X}}$ is a $\mathscr{T}_{X/k}$-torsor of $\widetilde{\mathscr{E}}$.

Proposition 12.9. – (i) *There exists a quasi-coherent crystal $\mathscr{F}_{\mathfrak{X}}$ of $\mathscr{O}_{\mathscr{E},1}$-modules of $\widetilde{\mathscr{E}}$ such that:*

(a) *For every object $(U,\mathfrak{T},u)$ of $\mathscr{E}$, $\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})}$ (8.4) is the sheaf of affine functions on the $u^*(\mathrm{T}_{U/k})$-torsor $\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}$ of T_{zar} (10.4).*

(b) *For every morphism $g:(V,\mathfrak{Z})\to(U,\mathfrak{T})$ of $\mathscr{E}$, any affine function $l:\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}\to\mathscr{O}_T$ and any section $h\in\mathscr{R}_{\mathfrak{X},(U,\mathfrak{T},u)}(T)$, the transition morphism $c_g:g_s^*(\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})})\xrightarrow{\sim}\mathscr{F}_{\mathfrak{X},(V,\mathfrak{Z})}$ (8.4) sends $g_s^*(l)$ to an affine function $l':\mathscr{R}_{\mathfrak{X},(V,\mathfrak{Z},v)}\to\mathscr{O}_Z$ such that*

(12.9.1) $$l'(h\circ g_2)=g_s^*(l(h))\in\mathscr{O}_Z.$$

(ii) *We have an exact sequence of crystals* (12.2.2)*:*

(12.9.2) $$0\to\mathscr{O}_{\mathscr{E},1}\to\mathscr{F}_{\mathfrak{X}}\to\nu^*(\Omega^1_{X/k})\to 0.$$

Proof. – (i) For any object $(U,\mathfrak{T},u)$ of $\mathscr{E}$, we define $\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})}$ as in (i). Recall (10.4.3) that we have an exact sequence of $\mathscr{O}_T$-modules of T_{zar}

(12.9.3) $$0\to\mathscr{O}_T\xrightarrow{c}\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})}\xrightarrow{\omega}u^*(\Omega^1_{U/k})\to 0.$$

For any morphism $g:(V,\mathfrak{Z})\to(U,\mathfrak{T})$ of $\mathscr{E}$, by (10.7.4) and (12.8.3), we obtain an $\mathscr{O}_Z$-linear isomorphism

(12.9.4) $$c_g:g_s^*(\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})})\xrightarrow{\sim}\mathscr{F}_{\mathfrak{X},(V,\mathfrak{Z})}$$

which fits into a commutative diagram (10.7.5)

(12.9.5) $$\begin{tikzcd} 0 \arrow[r] & g_s^*(\mathscr{O}_T) \arrow[r] \arrow[d, "\wr"] & g_s^*(\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})}) \arrow[r, "\omega"] \arrow[d, "c_g"'] & g_s^*(u^*(\Omega^1_{U/k})) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathscr{O}_Z \arrow[r] & \mathscr{F}_{\mathfrak{X},(V,\mathfrak{Z})} \arrow[r, "\omega"] & v^*(\Omega^1_{V/k}) \arrow[r] & 0. \end{tikzcd}$$

In view of the compatibility conditions of γ_g (12.8.2) and ([**2**] II 4.15), the data $\{\mathscr{F}_{\mathfrak{X},(U,\mathfrak{T})},c_g\}$ satisfy the compatibility conditions of 8.2. Hence, they define a quasi-coherent crystal of $\mathscr{O}_{\mathscr{E},1}$-modules of $\widetilde{\mathscr{E}}$ that we denote by $\mathscr{F}_{\mathfrak{X}}$. The equality (12.9.1) follows from (10.7.2) and (12.8.1).

The assertion (ii) follows from 12.4 and the diagram (12.9.5). □

12.10. – We denote by $\mathscr{B}_{\mathfrak{X}}$ the quasi-coherent crystal of $\mathscr{O}_{\mathscr{E},1}$-algebras $\varinjlim_{n\geqslant 1}\mathrm{S}^n_{\mathscr{O}_{\mathscr{E},1}}(\mathscr{F})$ of $\widetilde{\mathscr{E}}$. We set $\mathcal{F}_{\mathfrak{X}}=\mathscr{F}_{\mathfrak{X},(X,\mathfrak{X})}$ and $\mathcal{B}_{\mathfrak{X}}=\mathscr{B}_{\mathfrak{X},(X,\mathfrak{X})}$. Then we obtain an $\mathcal{R}_{\mathfrak{X}}$-stratification $\varepsilon_{\mathcal{F}}$ on $\mathcal{F}_{\mathfrak{X}}$ (resp. $\varepsilon_{\mathcal{B}}$ on $\mathcal{B}_{\mathfrak{X}}$) and a $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module structure $\psi_{\mathcal{F}}$ on $\mathcal{F}_{\mathfrak{X}}$ (resp. $\psi_{\mathcal{B}}$ on $\mathcal{B}_{\mathfrak{X}}$). On the other hand, $\mathcal{F}_{\mathfrak{X}}$ being the sheaf of affine functions on the

$\mathrm{T}_{X/k}$-torsor $\mathscr{R}_{\mathfrak{X},(X,\mathfrak{X})}$, $\mathscr{J}_{\mathfrak{X}}$ (resp. $\mathscr{B}_{\mathfrak{X}}$) is equipped by 10.5 with a $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module structure that we denote by $\kappa_{\mathscr{J}}$ (resp. $\kappa_{\mathscr{B}}$).

In the following, we show that $\psi_{\mathscr{J}}$ and $\kappa_{\mathscr{J}}$ are different by a sign (12.12).

Recall that we have an exact sequence (12.9.3)

$$0 \to \mathscr{O}_X \to \mathscr{J}_{\mathfrak{X}} \to \Omega^1_{X/k} \to 0. \tag{12.10.1}$$

The element $\mathrm{id}_{\mathfrak{X}_2} : \mathfrak{X}_2 \to \mathfrak{X}_2$ of $\mathscr{R}_{\mathfrak{X},(X,\mathfrak{X})}(X)$ induces a canonical splitting

$$s_{\mathrm{id}} : \mathscr{J}_{\mathfrak{X}} \xrightarrow{\sim} \mathscr{O}_X \oplus \Omega^1_{X/k}, \qquad l \mapsto (l(\mathrm{id}), \omega(l)). \tag{12.10.2}$$

Then it induces an isomorphism of $\mathscr{O}_X$-algebras

$$\mathscr{B}_{\mathfrak{X}} \xrightarrow{\sim} \mathrm{S}(\Omega^1_{X/k}). \tag{12.10.3}$$

By 10.6, the isomorphism s_{id} (12.10.2) (resp. (12.10.3)) is compatible with the action $\kappa_{\mathscr{J}}$ (resp. $\kappa_{\mathscr{B}}$) and the canonical action of $\widehat{\Gamma}(\mathrm{T}_{X/k})$ on $\mathscr{O}_X \oplus \Omega^1_{X/k}$ (resp. $\mathrm{S}(\Omega^1_{X/k})$) (10.2).

Proposition 12.11. – *Assume that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We take again the notation of 11.11.*

(i) *If l is a section of $\mathscr{J}_{\mathfrak{X}}$ such that $\omega(l) = 0$ (12.10.2), then $\varepsilon_{\mathscr{J}}(1 \otimes l) = l \otimes 1$.*

(ii) *For $1 \leqslant i \leqslant d$, let l_i be the section of $\mathscr{J}_{\mathfrak{X}}$ such that $l_i(\mathrm{id}) = 0$ and $\omega(l_i) = dt_i$. Then $\varepsilon_{\mathscr{J}}(1 \otimes l_i) = l_i \otimes 1 - 1 \otimes \zeta_i$.*

Proof. – Assertion (i) follows from 12.5 and 12.9(ii).

(ii) We denote by $q_s : R_{\mathfrak{X},1} \to X$ the canonical morphism. The canonical morphisms $q_{1,2}, q_{2,2} : R_{\mathfrak{X},2} \to \mathfrak{X}_2 \in \mathscr{R}_{\mathfrak{X},(X,R_{\mathfrak{X}},q_s)}(R_{\mathfrak{X},1})$ induce two splittings of $\mathscr{J}_{\mathfrak{X},(X,R_{\mathfrak{X}})}$ (12.9.3)

$$s_{q_1} : q_{s*}(\mathscr{F}_{\mathfrak{X},(X,R_{\mathfrak{X}})}) \xrightarrow{\sim} \mathcal{R}_{\mathfrak{X},1} \oplus (\mathcal{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \Omega^1_{X/k}) \quad f \mapsto (f(q_1), \omega(f)), \tag{12.11.1}$$

$$s_{q_2} : q_{s*}(\mathscr{F}_{\mathfrak{X},(X,R_{\mathfrak{X}})}) \xrightarrow{\sim} \mathcal{R}_{\mathfrak{X},1} \oplus (\mathcal{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \Omega^1_{X/k}) \quad f \mapsto (f(q_2), \omega(f)). \tag{12.11.2}$$

We identify $\mathcal{R}_{\mathfrak{X},1} \oplus (\mathcal{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \Omega^1_{X/k})$ with $\mathcal{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \mathscr{J}_{\mathfrak{X}}$ by $\mathrm{id}_{\mathcal{R}_{\mathfrak{X},1}} \otimes s_{\mathrm{id}}$ (12.10.2). In view of (12.9.1) and (12.9.5), the morphism s_{q_i} is inverse to $q_{s*}(c_{q_i})$ for $i = 1, 2$. Then, the $\mathcal{R}_{\mathfrak{X}}$-stratification ε on $\mathscr{J}_{\mathfrak{X}}$ is given by the composition of the inverse of s_{q_2} and s_{q_1}.

We denote by f the local section $s_{q_2}^{-1}(1 \otimes l_i)$ of $q_{s*}(\mathscr{F}_{\mathfrak{X},(X,R_{\mathfrak{X}})})$. Then we have $f(q_2) = 0$ and $\omega(f) = 1 \otimes dt_i$. To show the assertion, it suffices to prove that $f(q_1) = -\zeta_i \otimes 1$. The morphisms q_1, q_2 are induced by two homomorphisms

$$\iota_1 : \mathscr{O}_{\mathfrak{X}_2} \to \mathcal{R}_{\mathfrak{X},2}, \quad \iota_2 : \mathscr{O}_{\mathfrak{X}_2} \to \mathcal{R}_{\mathfrak{X},2}, \tag{12.11.3}$$

such that ι_1 is equal to ι_2 modulo p. Then they define a W_2-derivation

$$D = \iota_2 - \iota_1 : \mathscr{O}_{\mathfrak{X}_2} \to p\mathcal{R}_{\mathfrak{X},2}. \tag{12.11.4}$$

We denote by

$$\phi : \mathcal{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} \Omega^1_{X/k} \to p\mathcal{R}_{\mathfrak{X},2} \tag{12.11.5}$$

the $\mathscr{R}_{\mathfrak{X},1}$-linear morphism associated to D. Then we have

$$\phi(1\otimes dt_i)=1\otimes t_i-t_i\otimes 1=p\left(\frac{\xi_i}{p}\right)\in p\mathscr{R}_{\mathfrak{X},2}.$$

Identifying $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{R}_{\mathfrak{X},1}\otimes_{\mathscr{O}_X}\Omega^1_{X/k},p\mathscr{R}_{\mathfrak{X},2})\xrightarrow{\sim}\mathscr{R}_{\mathfrak{X},1}\otimes_{\mathscr{O}_X}\mathrm{T}_{X/k}$, we consider ϕ as a section of $q_s^*(\mathrm{T}_{X/k})$. Then we have $q_2=q_1+\phi\in\mathscr{R}_{\mathfrak{X},q_s}(R_{\mathfrak{X},1})$ and we deduce that (10.4(ii))

$$f(q_1)=f(q_2-\phi)=-\omega(f)(\phi)=-\zeta_i\otimes 1.$$

Then the proposition follows. □

Corollary 12.12. – *The $\widehat{\Gamma}(\mathrm{T}_{X/k})$-actions $\psi_{\mathscr{F}}$ and $\kappa_{\mathscr{F}}$ on $\mathscr{F}_{\mathfrak{X}}$ (resp. $\psi_{\mathscr{B}}$ and $\kappa_{\mathscr{B}}$ on $\mathscr{B}_{\mathfrak{X}}$) satisfy $\psi_{\mathscr{F}}=\iota^*(\kappa_{\mathscr{F}})$ (resp. $\psi_{\mathscr{B}}=\iota^*(\kappa_{\mathscr{B}})$), where $\iota:\widehat{\Gamma}(\mathrm{T}_{X/k})\to\widehat{\Gamma}(\mathrm{T}_{X/k})$ denotes the involution homomorphism* (10.19.2).

Proof. – The question being local, we take again the assumptions and notation of 12.11. Let $\partial_i\in\mathrm{T}_{X/k}(X)$ be the dual of dt_i. In view of (11.17.2) and 12.11(ii), the action of $\psi_{\mathscr{F},\partial_i}$ on $\mathscr{F}_{\mathfrak{X}}$ sends l_i to -1. We equip $\mathscr{O}_X\oplus\Omega^1_{X/k}$ (resp. $\mathrm{S}(\Omega^1_{X/k})$) with the canonical action of $\widehat{\Gamma}(\mathrm{T}_{X/k})$ (10.2). The isomorphism s_{id} (12.10.2) (resp. (12.10.3)) sends l_i to dt_i and induces an isomorphism of $\widehat{\Gamma}(\mathrm{T}_{X/k})$-modules

$$(\mathscr{F}_{\mathfrak{X}},\psi_{\mathscr{F}})\xrightarrow{\sim}\iota^*(\mathscr{O}_X\oplus\Omega^1_{X/k})\qquad(\text{resp. }((\mathscr{B}_{\mathfrak{X}},\psi_{\mathscr{B}})\xrightarrow{\sim}\iota^*(\mathrm{S}(\Omega^1_{X/k}))).\tag{12.12.1}$$

Then the assertion follows from 12.10. □

Corollary 12.13. – *We denote by ε^+_R (resp. ε^-_R) the $\mathscr{R}_{\mathfrak{X}}$-stratification on $\mathscr{R}_{\mathfrak{X},1}$ associated to the $\widehat{\Gamma}(\mathrm{T}_{X/k})$-action $\kappa_{\mathscr{B}}$ (resp. $\psi_{\mathscr{B}}$) on $\mathscr{B}_{\mathfrak{X}}$ via the isomorphisms of $\mathscr{O}_X$-algebras $\mathscr{B}_{\mathfrak{X}}\xrightarrow{\sim}\mathrm{S}(\Omega^1_{X/k})\xrightarrow{\sim}\mathscr{R}_{\mathfrak{X},1}$* (11.12), 12.10.3*:*

(i) *For any local section r of $\mathscr{R}_{\mathfrak{X},1}$, we have $\varepsilon^+_R(1\otimes r)=\delta(r)$* (11.11)*.*

(ii) *For any local section r of $\mathscr{R}_{\mathfrak{X},1}$, we have $\varepsilon^-_R(\delta(r))=r\otimes 1$.*

Proof. – The question being local, we take again the assumption and the notation of 12.11. Then for $1\leqslant i\leqslant d$, l_i is sent to ζ_i by the isomorphism $\mathscr{B}_{\mathfrak{X}}\xrightarrow{\sim}\mathscr{R}_{\mathfrak{X},1}$. Since ε^+_R, ε^-_R and δ are homomorphisms, it suffices to verify the assertion for the local sections ζ_i. By 12.11(ii) and 12.12(i), we have

$$\varepsilon^-_R(1\otimes\zeta_i)=\zeta_i\otimes 1-1\otimes\zeta_i,\qquad \varepsilon^+_R(1\otimes\zeta_i)=\zeta_i\otimes 1+1\otimes\zeta_i\tag{12.13.1}$$

Then, the assertion (i) follows from 11.11. The assertion (ii) follows from the relations

$$\varepsilon^-_R(\delta(\zeta_i))=\varepsilon^-_R(1\otimes\zeta_i+\zeta_i\otimes 1)=(\zeta_i\otimes 1-1\otimes\zeta_i)+1\otimes\zeta_i=\zeta_i\otimes 1.$$ □

With the above preparation, we can interpret a key calculation in Oyama's paper in the following forms.

Proposition 12.14 ([**32**] 1.5.3). – *Let M be a quasi-nilpotent $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module, ε the associated $\mathscr{R}_{\mathfrak{X}}$-stratification on M (11.17). We denote by M_0 the underlying $\mathscr{O}_X$-module of M equipped with the $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module structure defined by the zero PD-Higgs field. Then the stratification $\varepsilon : \mathscr{R}_{\mathfrak{X},1} \otimes_{\mathscr{O}_X} M \to M \otimes_{\mathscr{O}_X} \mathscr{R}_{\mathfrak{X},1}$ induces two isomorphisms of $\widehat{\Gamma}(\mathrm{T}_{X/k})$-modules* (10.14.2)

$$\text{(i)} \qquad (\mathscr{B}_{\mathfrak{X}}, \psi_{\mathscr{B}}) \otimes_{\mathscr{O}_X} M_0 \xrightarrow{\sim} M \otimes_{\mathscr{O}_X} (\mathscr{B}_{\mathfrak{X}}, \psi_{\mathscr{B}}),$$

$$\text{(ii)} \qquad (\mathscr{B}_{\mathfrak{X}}, \psi_{\mathscr{B}}) \otimes_{\mathscr{O}_X} \iota^*(M) \xrightarrow{\sim} M_0 \otimes_{\mathscr{O}_X} (\mathscr{B}_{\mathfrak{X}}, \psi_{\mathscr{B}}).$$

Proof. – We take again the notation of 12.13. To simplify the notation, we write $\mathscr{R}$ for $\mathscr{R}_{\mathfrak{X},1}$. The $\mathscr{R}_{\mathfrak{X}}$-stratification on M_0 is the identity morphism $\mathrm{id}_{\mathscr{R}\otimes M} : \mathscr{R} \otimes_{\mathscr{O}_X} M \to \mathscr{R} \otimes_{\mathscr{O}_X} M$ (cf. the proof of 12.5). We denote by θ_0 (resp. θ) the morphism $M \to M \otimes_{\mathscr{O}_X} \mathscr{R}$ defined by $m \mapsto m \otimes 1$ (resp. $m \mapsto \varepsilon(1 \otimes m)$) for every local section m of M.

(i) Since the action $\psi_{\mathscr{B}}$ on $\mathscr{B}_{\mathfrak{X}}$ is compatible with the ring structure of $\mathscr{B}_{\mathfrak{X}}$, it suffices to show that $\theta : M_0 \to M \otimes (\mathscr{B}_{\mathfrak{X}}, \psi_{\mathscr{B}})$ is $\widehat{\Gamma}(\mathrm{T}_{X/k})$-equivariant. In view of (5.4.2), it suffices to prove that the following diagram is commutative

(12.14.1)
$$\begin{tikzcd}
M \arrow[rr, "\theta_0"] \arrow[d, "\theta"'] & & M \otimes_{\mathscr{O}_X} \mathscr{R} \arrow[d, "\theta\otimes\mathrm{id}_{\mathscr{R}}"] \\
M \otimes_{\mathscr{O}_X} \mathscr{R} \arrow[r, "\theta\otimes\mathrm{id}_{\mathscr{R}}"] & M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R} \arrow[r, "\mathrm{id}_M \otimes \varepsilon_R^-"] & M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R}.
\end{tikzcd}$$

By condition (ii) of 5.5, the composition
(12.14.2)
$$M \xrightarrow{\theta} M \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\theta\otimes\mathrm{id}_{\mathscr{R}}} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\mathrm{id}_M \otimes \varepsilon_R^-} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R}$$

in the above diagram is equal to the composition
(12.14.3)
$$M \xrightarrow{\theta} M \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\mathrm{id}_M \otimes \delta} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\mathrm{id}_M \otimes \varepsilon_R^-} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R}.$$

By 12.13(ii), the above composition is equal to the composition

$$\text{(12.14.4)} \qquad M \xrightarrow{\theta_0} M \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\theta\otimes\mathrm{id}_{\mathscr{R}}} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R}.$$

The assertion follows.

(ii) By 12.12(i), it suffices to show that the morphism $(\mathscr{B}_{\mathfrak{X}}, \kappa) \otimes_{\mathscr{O}_X} M \to M_0 \otimes_{\mathscr{O}_X} (\mathscr{B}_{\mathfrak{X}}, \kappa_{\mathscr{B}})$ is $\widehat{\Gamma}(\mathrm{T}_{X/k})$-equivariant. Similarly to (i), by (5.4.2), it suffices to prove that the following diagram is commutative

(12.14.5)
$$\begin{tikzcd}
M \arrow[rr, "\theta"] \arrow[d, "\theta"'] & & M \otimes_{\mathscr{O}_X} \mathscr{R} \arrow[d, "\theta\otimes\mathrm{id}_{\mathscr{R}}"] \\
M \otimes_{\mathscr{O}_X} \mathscr{R} \arrow[r, "\theta_0\otimes\mathrm{id}_{\mathscr{R}}"] & M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R} \arrow[r, "\mathrm{id}_M \otimes \varepsilon_R^+"] & M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R}.
\end{tikzcd}$$

By condition (ii) of 5.5, the composition

$$M \xrightarrow{\theta} M \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\theta \otimes \mathrm{id}_{\mathscr{R}}} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R} \tag{12.14.6}$$

in the above diagram is equal to the composition

$$M \xrightarrow{\theta} M \otimes_{\mathscr{O}_X} \mathscr{R} \xrightarrow{\mathrm{id}_M \otimes \delta} M \otimes_{\mathscr{O}_X} \mathscr{R} \otimes_{\mathscr{O}_X} \mathscr{R}. \tag{12.14.7}$$

The assertion follows from 12.13(i). □

12.15. – The counterpart of the crystal $\mathscr{A}_{\mathfrak{X}'_2}$ (10.12) in Oyama topos $\underline{\widetilde{\mathscr{E}}}$ is defined by applying Cartier equivalence $\mathrm{C}^*_{X/\mathscr{S}}$ to $\mathscr{B}_{\mathfrak{X}'}$.

Let $\mathrm{C}_{X/\mathscr{S}} : \underline{\widetilde{\mathscr{E}}} \to \widetilde{\mathscr{E}}'$ be the morphism of topoi (9.1.3). We put $\underline{\mathscr{R}}_{\mathfrak{X}} = \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{R}_{\mathfrak{X}'})$ (12.8). For any object $(U, \mathfrak{T}, u)$ of $\underline{\mathscr{E}}$, we set $\phi_{T/k} = u' \circ f_{T/k} : T \to \underline{T}' \to U'$ so we have $\rho(U, \mathfrak{T}, u) = (U', \mathfrak{T}, \phi_{T/k})$ (9.1.2). By 9.1, the descent data of the sheaf $\underline{\mathscr{R}}_{\mathfrak{X}}$ of $\underline{\widetilde{\mathscr{E}}}$ is $\{u_*(\mathscr{R}_{\mathfrak{X}',(U',\mathfrak{T},\phi_{T/k})}), \gamma_{\rho(g)}\}$ (7.7, 12.7, 12.8).

We put $\underline{\mathscr{F}}_{\mathfrak{X}} = \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{F}_{\mathfrak{X}'})$ and $\underline{\mathscr{B}}_{\mathfrak{X}} = \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{B}_{\mathfrak{X}'})$. For any object $(U, \mathfrak{T}, u)$ of $\underline{\mathscr{E}}$, we have (9.11.6)

$$\underline{\mathscr{F}}_{\mathfrak{X},(U,\mathfrak{T},u)} = \mathscr{F}_{\mathfrak{X}',(U',\mathfrak{T},\phi_{T/k})}, \qquad \underline{\mathscr{B}}_{\mathfrak{X},(U,\mathfrak{T},u)} = \mathscr{B}_{\mathfrak{X}',(U',\mathfrak{T},\phi_{T/k})}. \tag{12.15.1}$$

The linearised descent data of the quasi-coherent crystal of $\mathscr{O}_{\underline{\mathscr{E}},1}$-modules $\underline{\mathscr{F}}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},1}$-algebras $\underline{\mathscr{B}}_{\mathfrak{X}}$) of $\underline{\widetilde{\mathscr{E}}}$ is $\{\mathscr{F}_{\mathfrak{X}',(U',\mathfrak{T},\phi_{T/k})}, c_{\rho(g)}\}$ (resp. $\{\mathscr{B}_{\mathfrak{X}',(U',\mathfrak{T},\phi_{T/k})}, c_{\rho(g)}\}$) (9.1), 8.2, 12.9.

We set $\underline{\mathcal{F}}_{\mathfrak{X}} = \underline{\mathscr{F}}_{\mathfrak{X},(X,\mathfrak{X})}$ and $\underline{\mathcal{B}}_{\mathfrak{X}} = \underline{\mathscr{B}}_{\mathfrak{X},(X,\mathfrak{X})}$. By 11.18, these $\mathscr{O}_X$-modules are equipped with $\mathrm{D}^{\gamma}_{X/k}$-module structures that we denote by $(\nabla_{\underline{\mathcal{F}}}, \psi_{\underline{\mathcal{F}}})$ (resp. $(\nabla_{\underline{\mathcal{B}}}, \psi_{\underline{\mathcal{B}}})$). We will show that $\underline{\mathcal{B}}_{\mathfrak{X}}$ is isomorphic to the algebra $\mathcal{A}_{\mathfrak{X}'_2}$ (10.12) introduced by Ogus-Vologodsky (12.26).

12.16. – We interpret the Cartier equivalence as an admissible isomorphism à la Fontaine for a pair of crystals with respect to the period ring $\underline{\mathcal{B}}_{\mathfrak{X}}$.

Let $\mathscr{M}'$ be a crystal of $\mathscr{O}_{\mathscr{E}',1}$-modules of $\widetilde{\mathscr{E}}'$, (M', θ'_M) the associated $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module, $\mathscr{M}'_0 = \nu^*(M')$ (12.4), $\mathscr{N} = \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}')$ and (N, ∇_N, ψ_N) the associated $\mathrm{D}^{\gamma}_{X/k}$-module. By 11.18, 12.5 and 12.14(i), we have an isomorphism of crystals of $\mathscr{O}_{\mathscr{E}',1}$-modules of $\widetilde{\mathscr{E}}'$

$$\mathscr{B}_{\mathfrak{X}'} \otimes_{\mathscr{O}_{\mathscr{E}',1}} \mathscr{M}'_0 \xrightarrow{\sim} \mathscr{M}' \otimes_{\mathscr{O}_{\mathscr{E}',1}} \mathscr{B}_{\mathfrak{X}'}. \tag{12.16.1}$$

Applying $\mathrm{C}^*_{X/\mathscr{S}}$, we obtain an isomorphism of crystals of $\mathscr{O}_{\underline{\mathscr{E}},1}$-modules of $\underline{\widetilde{\mathscr{E}}}$:

$$\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\underline{\mathscr{E}},1}} \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}'_0) \xrightarrow{\sim} \mathscr{N} \otimes_{\mathscr{O}_{\underline{\mathscr{E}},1}} \underline{\mathscr{B}}_{\mathfrak{X}}. \tag{12.16.2}$$

By 12.3, we deduce an isomorphism of $\mathrm{D}^{\gamma}_{X/k}$-modules

$$\lambda : \underline{\mathcal{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} F^*_{X/k}(M') \xrightarrow{\sim} N \otimes_{\mathscr{O}_X} \underline{\mathcal{B}}_{\mathfrak{X}}, \tag{12.16.3}$$

where the $\mathrm{D}^{\gamma}_{X/k}$-action on the left hand side is induced by $(\nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})$ on $\underline{\mathscr{B}}_{\mathfrak{X}}$ and $(\nabla_{\mathrm{can}}, 0)$ on $F^*_{X/k}(M')$ (10.18), that we still denote by $(\nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})$; and the $\mathrm{D}^{\gamma}_{X/k}$-action on the right hand side is induced by (∇_N, ψ_N) on N and $(\nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})$, that we denote by $(\nabla_{\mathrm{tot}}, \psi_{\mathrm{tot}})$.

The $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module structures $\psi_{\underline{\mathscr{B}}}$ on $\underline{\mathscr{B}}_{\mathfrak{X}}$ and $\theta_{M'}$ on M' define a $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module structure on $\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M'$ (10.14), that we denote by θ_{tot}. On the other hand, the zero PD-Higgs field on N and the action of $\psi_{\underline{\mathscr{B}}}$ on $\underline{\mathscr{B}}_{\mathfrak{X}}$ define a $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module structure on $N \otimes_{\mathscr{O}_X} \underline{\mathscr{B}}_{\mathfrak{X}}$, that we still denote by $\psi_{\underline{\mathscr{B}}}$.

Theorem 12.17. – *Let $\mathscr{M}'$ be a crystal of $\mathscr{O}_{\mathscr{E}',1}$-modules of $\widetilde{\mathscr{E}}'$, (M', θ'_M) the associated $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module, $\mathscr{N} = \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}')$ and (N, ∇_N, ψ_N) the associated $\mathrm{D}^{\gamma}_{X/k}$-module. Then, the isomorphism* (12.16.3)

$$\lambda : (\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M', (\nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}}), \theta_{\mathrm{tot}}) \xrightarrow{\sim} (N \otimes_{\mathscr{O}_X} \underline{\mathscr{B}}_{\mathfrak{X}}, (\nabla_{\mathrm{tot}}, \psi_{\mathrm{tot}}), \psi_{\underline{\mathscr{B}}}) \tag{12.17.1}$$

is compatible with the $\mathrm{D}^{\gamma}_{X/k}$-actions and the $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-actions defined on both sides in 12.16.

Proof. – We only need to prove the compatibility of the $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-actions. Let $\varepsilon' : \mathscr{B}_{\mathfrak{X}'} \otimes_{\mathscr{O}_{X'}} M' \xrightarrow{\sim} M' \otimes_{\mathscr{O}_{X'}} \mathscr{B}_{\mathfrak{X}'}$ be the $\mathscr{R}_{\mathfrak{X}'}$-stratification on M' (12.14), and $\psi_{\mathscr{B}'}$ the $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-action on $\mathscr{B}_{\mathfrak{X}'}$ defined in 12.10. The question is local. Since the Cartier equivalence is compatible with localisation (9.15.3), we can suppose that there exists a lifting $F : \mathfrak{X} \to \mathfrak{X}'$ of the relative Frobenius morphism $F_{X/k}$ of X. Then it induces a morphism $F : (X', \mathfrak{X}, F_{X/k}) \to (X', \mathfrak{X}')$ of $\mathscr{E}'$. The isomorphisms (12.16.1), (12.16.2) and the transition morphisms η_F associated to F (9.17.4) induce a commutative diagram

$$\begin{tikzcd}
F^*_{X/k}((\mathscr{B}_{\mathfrak{X}'} \otimes_{\mathscr{O}_{\mathscr{E}',1}} \mathscr{M}'_0)_{(X',\mathfrak{X}')}) \arrow[r, "\sim"] \arrow[d, "\eta_F"'] & F^*_{X/k}((\mathscr{M}' \otimes_{\mathscr{O}_{\mathscr{E}',1}} \mathscr{B}_{\mathfrak{X}'})_{(X',\mathfrak{X}')}) \arrow[d, "\eta_F"] \\
(\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\underline{\mathscr{E}},1}} \mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}'_0))_{(X,\mathfrak{X})} \arrow[r, "\sim"] & (\mathscr{N} \otimes_{\mathscr{O}_{\underline{\mathscr{E}},1}} \underline{\mathscr{B}}_{\mathfrak{X}})_{(X,\mathfrak{X})}.
\end{tikzcd}$$

Then we deduce a commutative diagram

$$\begin{tikzcd}
F^*_{X/k}(\mathscr{B}_{\mathfrak{X}'} \otimes_{\mathscr{O}_{X'}} M') \arrow[r, "F^*_{X/k}(\varepsilon')"] \arrow[d, "\eta_F"'] & F^*_{X/k}(M' \otimes_{\mathscr{O}_{X'}} \mathscr{B}_{\mathfrak{X}'}) \arrow[d, "\eta_F"] \\
\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M' \arrow[r, "\lambda"] & N \otimes_{\mathscr{O}_X} \underline{\mathscr{B}}_{\mathfrak{X}}.
\end{tikzcd} \tag{12.17.2}$$

By 11.19, the isomorphism $\eta_F : F^*_{X/k}(\mathscr{B}_{\mathfrak{X}'}) \xrightarrow{\sim} \underline{\mathscr{B}}_{\mathfrak{X}}$ underlies an isomorphism of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules

$$F^*_{X/k}(\mathscr{B}_{\mathfrak{X}'}, \iota^*(\psi_{\mathscr{B}'})) \xrightarrow{\sim} (\underline{\mathscr{B}}_{\mathfrak{X}}, \psi_{\underline{\mathscr{B}}}). \tag{12.17.3}$$

By 12.14(ii), the isomorphism $F^*_{X/k}(\varepsilon')$ is compatible with actions of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$:
(12.17.4)
$F^*_{X/k}(\varepsilon'): F^*_{X/k}(\mathscr{B}_{\mathfrak{X}'}\otimes_{\mathscr{O}_{X'}} M', \psi_{\mathscr{B}'}\otimes \mathrm{id}+\mathrm{id}\otimes\iota^*(\theta_{M'})) \xrightarrow{\sim} F^*_{X/k}(M'\otimes_{\mathscr{O}_{X'}} \mathscr{B}_{\mathfrak{X}'}, \mathrm{id}\otimes\psi_{\mathscr{B}'})$.
Then the assertion follows from (12.17.2), (12.17.3) and (12.17.4). □

REMARK 12.18. – In ([**31**], 2.23), Ogus and Vologodsky showed a similar result for certain filtered PD-Higgs modules.

Let (H,θ) (resp. (H,∇,ψ)) be a $\widehat{\Gamma}(\mathrm{T}_{X/k})$-module (resp. a $\mathrm{D}^\gamma_{X/k}$-module). We define its $\widehat{\Gamma}(\mathrm{T}_{X/k})$ invariants (resp. $\mathrm{D}^\gamma_{X/k}$ invariants) by
(12.18.1)
$$H^\theta = \mathscr{H}\!om_{\widehat{\Gamma}(\mathrm{T}_{X/k})}((\mathscr{O}_X,0),H) \qquad (\text{resp. } H^{(\nabla,\psi)} = \mathscr{H}\!om_{\mathrm{D}^\gamma_{X/k}}((\mathscr{O}_X,d,0),H)).$$

We can recover the Cartier equivalence by taking $\widehat{\Gamma}(\mathrm{T}_{X'/k})$ invariants (resp. $\mathrm{D}^\gamma_{X/k}$ invariants) for the isomorphism (12.17.1).

PROPOSITION 12.19. – *Keep the notation of 12.16 and suppose that $\mathscr{M}'$ is quasi-coherent. The isomorphism* (12.17.1) *induces*

(i) *a canonical isomorphism of* $\mathrm{D}^\gamma_{X/k}$*-modules*
$$(\underline{\mathscr{B}}_{\mathfrak{X}}\otimes_{\mathscr{O}_{X'}} M', \nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})^{\theta_{\mathrm{tot}}} \xrightarrow{\sim} (N,\nabla_N,\psi_N);$$

(ii) *a canonical isomorphism of* $\widehat{\Gamma}(\mathrm{T}_{X'/k})$*-modules*
$$(M',\theta_{M'}) \xrightarrow{\sim} (N\otimes_{\mathscr{O}_X} \underline{\mathscr{B}}_{\mathfrak{X}}, \psi_{\underline{\mathscr{B}}})^{(\nabla_{\mathrm{tot}},\psi_{\mathrm{tot}})}.$$

The assertion follows from 12.20 and 12.21.

LEMMA 12.20. – *Keep the notation of 12.16.*

(i) *The actions $(\nabla_{\underline{\mathscr{B}}},\psi_{\underline{\mathscr{B}}})$ of $\mathrm{D}^\gamma_{X/k}$ and θ_{tot} of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$ on $\underline{\mathscr{B}}_{\mathfrak{X}}\otimes_{\mathscr{O}_{X'}} M'$ commute with each other.*

(ii) *The actions $(\nabla_{\mathrm{tot}},\psi_{\mathrm{tot}})$ of $\mathrm{D}^\gamma_{X/k}$ and $\psi_{\underline{\mathscr{B}}}$ of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$ on $N\otimes_{\mathscr{O}_X}\underline{\mathscr{B}}_{\mathfrak{X}}$ commute with each other.*

Proof. – (i) By the Formula (10.14.1), one verifies that the action $\psi_{\underline{\mathscr{B}}}$ of $\widehat{\Gamma}(\mathrm{T}_{X'/k})\subset \mathrm{D}^\gamma_{X/k}$ and the action θ_{tot} of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$ on $\underline{\mathscr{B}}_{\mathfrak{X}}\otimes_{\mathscr{O}_{X'}} M'$ commute with each other. For any local sections D of $\mathrm{T}_{X/k}$, ξ' of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$, b of $\underline{\mathscr{B}}_{\mathfrak{X}}$ and m of M', by 10.14.1, we have
$$\begin{aligned}\nabla_{\underline{\mathscr{B}},D}(\theta_{\mathrm{tot},\xi'}(b\otimes m)) &= \nabla_{\underline{\mathscr{B}},D}(\psi_{\underline{\mathscr{B}},\xi'}(b))\otimes m + \nabla_{\underline{\mathscr{B}},D}(b)\otimes\theta_{M',\xi'}(m)\\ &= \psi_{\underline{\mathscr{B}},\xi'}(\nabla_{\underline{\mathscr{B}},D}(b))\otimes m + \nabla_{\underline{\mathscr{B}},D}(b)\otimes\theta_{M',\xi'}(m)\\ &= \theta_{\mathrm{tot},\xi'}(\nabla_{\underline{\mathscr{B}},D}(b\otimes m)).\end{aligned}$$

Since $\mathrm{D}^\gamma_{X/k}$ is generated by $\mathrm{T}_{X/k}$ and $\widehat{\Gamma}(\mathrm{T}_{X'/k})$, the assertion follows.

Assertion (ii) follows from (i) and 12.17. □

LEMMA 12.21. – *Keep the notation of 12.16 and suppose that $\mathscr{M}'$ is quasi-coherent. The canonical homomorphism of crystals $\mathscr{O}_{\underline{\mathscr{E}},1} \to \underline{\mathscr{B}}_{\mathfrak{X}}$ induces an isomorphism of $\mathscr{O}_X$-modules (resp. $\mathscr{O}_{X'}$-modules)*

$$N \xrightarrow{\sim} (N \otimes_{\mathscr{O}_X} \underline{\mathscr{B}}_{\mathfrak{X}})^{\psi_{\underline{\mathscr{B}}}}, \qquad (\text{resp. } M' \xrightarrow{\sim} (\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M')^{(\nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})}).$$

Proof. – The assertion in lemma being local, we may assume that there exists a lifting $F : \mathfrak{X} \to \mathfrak{X}'$ of the relative Frobenius morphism $F_{X/k}$ of X. By 11.19 and (12.12.1), we have isomorphisms of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules

$$\begin{aligned} \eta_F : F^*_{X/k}(\mathscr{B}_{\mathfrak{X}'} \otimes_{\mathscr{O}_{X'}} M', \iota^*(\psi_{\mathscr{B}'}) \otimes \mathrm{id}) &\xrightarrow{\sim} (N \otimes_{\mathscr{O}_X} \underline{\mathscr{B}}_{\mathfrak{X}}, \psi_{\underline{\mathscr{B}}}) \\ \mathrm{S}(\Omega^1_{X/k}) \otimes_{\mathscr{O}_X} F^*_{X/k}(M') &\xrightarrow{\sim} F^*_{X/k}(\mathscr{B}_{\mathfrak{X}'} \otimes_{\mathscr{O}_{X'}} M', \iota^*(\psi_{\mathscr{B}'}) \otimes \mathrm{id}), \end{aligned}$$

where $\mathrm{S}(\Omega^1_{X/k})$ is equipped with the canonical action of $\widehat{\Gamma}(\mathrm{T}_{X/k})$. A local section u of $\mathrm{S}(\Omega^1_{X/k}) \otimes_{\mathscr{O}_X} F^*_{X/k}(M')$ can be written as a finite sum

$$u = \sum_{i=0}^{m} \omega_i \otimes u_i, \tag{12.21.1}$$

with $u_i \in F^*_{X/k}(M')$ and $\omega_i \in \mathrm{S}^i(\Omega)$. In view of the perfect pairing $\Gamma_i(\mathrm{T}_{X/k}) \otimes_{\mathscr{O}_X} \mathrm{S}^i(\Omega^1_{X/k}) \to \mathscr{O}_X$ (10.2.1) for $i \geqslant 1$, the action of $\widehat{\Gamma}(\mathrm{T}_{X/k})$ on u is trivial if and only if $u_i = 0$ for $i \geqslant 1$, i.e., u belongs to the submodule $F^*_{X/k}(M')$ of $F^*_{X/k}(M') \otimes_{\mathscr{O}_X} \mathrm{S}(\Omega)$. The first isomorphism follows.

Equipped with the Frobenius descent connection ∇_{can} on $F^*_{X/k}(M')$, we have an injection of $\mathrm{D}^{\gamma}_{X/k}$-modules $(F^*_{X/k}(M'), \nabla_{\mathrm{can}}, 0) \to (\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M', \nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})$ and a canonical $\mathscr{O}_{X'}$-linear isomorphism ([**25**] 5.1)

$$M' \xrightarrow{\sim} F^*_{X/k}(M')^{(\nabla_{\mathrm{can}}, 0)}. \tag{12.21.2}$$

In view of the first isomorphism, we deduce that $(\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M')^{(\nabla_{\underline{\mathscr{B}}}, \psi_{\underline{\mathscr{B}}})}$ is contained in the image of $F^*_{X/k}(M')$ in $\underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_{X'}} M'$. Then the second isomorphism follows from (12.21.2). □

Now we use the above results to compare the Cartier equivalence $\mathrm{C}^*_{X/\mathscr{S}}$ and the Cartier transform of Ogus-Vologodsky.

THEOREM 12.22. – *Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme and X its special fiber. The following diagram is commutative up to a canonical isomorphism*

$$\begin{array}{ccc} \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},1}) & \xrightarrow{\mathrm{C}_{X/\mathscr{S}*}} & \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E}',1}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k) & \xrightarrow{\mathrm{C}_{\mathfrak{X}'_2}} & \mathrm{HIG}^{\mathrm{qn}}_{\gamma}(X/k), \end{array} \tag{12.22.1}$$

where $\mathrm{C}_{X/\mathscr{S}*}$ *is the direct image functor of the morphism of topoi* $\mathrm{C}_{X/\mathscr{S}}$ (9.12), *which depends only on* X, *and the vertical functors are equivalences of categories* (11.18) *and depend on* $\mathfrak{X}$.

Remark 12.23. – The top (resp. lower) horizontal functor in the above diagram depends only on X (resp. $\mathfrak{X}'_2$). However, the vertical functors (11.18.1) are constructed by a formal model $\mathfrak{X}$ of X. A natural question is: do vertical equivalences depend only on a lifting of X over the quotient of W by a power of p?

12.24. – Recall that Ogus and Vologodsky constructed a torsor $\mathscr{L}_{\mathfrak{X}'_2}$ of $(X/k)_{\mathrm{crys}}$ (10.11), the crystal of affine functions $\mathscr{F}_{\mathfrak{X}'_2}$ on $\mathscr{L}_{\mathfrak{X}'_2}$ of $(X/k)_{\mathrm{crys}}$ and the quasi-coherent crystal of $\mathscr{O}_{X/k}$-algebras $\mathscr{A}_{\mathfrak{X}'_2}$ associated to $\mathscr{F}_{\mathfrak{X}'_2}$ (10.12). We put $\mathscr{F}_{\mathfrak{X}'_2}=\mathscr{F}_{\mathfrak{X}'_2,(X,X)}$ and $\mathscr{A}_{\mathfrak{X}'_2}=\mathscr{A}_{\mathfrak{X}'_2,(X,X)}$, which are equipped with $\mathrm{D}^{\gamma}_{X/k}$-module structures (10.19). The Cartier transform is defined by (10.20(ii))

(12.24.1)
$$\mathrm{C}_{\mathfrak{X}'_2}:\mathrm{MIC}^{\mathrm{qn}}_{\gamma}(X/k)\xrightarrow{\sim}\mathrm{HIG}^{\mathrm{qn}}_{\gamma}(X'/k),\qquad N\mapsto \iota^*(\mathscr{H}\!om_{\mathrm{D}^{\gamma}_{X/k}}((\mathscr{A}_{\mathfrak{X}'_2})^{\vee},N)).$$

12.25. – Let (U,T,δ) be an object of $\mathrm{Crys}(X/k)$ such that there exists a flat formal $\mathscr{S}$-scheme $\mathfrak{T}$ with special fiber T. Recall (10.9) that the morphism $U\to T$ induces an isomorphism $U\xrightarrow{\sim}\underline{T}$. Then we obtain an object $(U,\mathfrak{T})$ of $\underline{\mathscr{E}}$. The morphism $\varphi_{T/k}$: $T\to X'$ (10.9.1) is the same as the composition $T\xrightarrow{\phi_{T/k}}U'\to X'$ (12.15). Moreover, the $\varphi^*_{T/k}(\mathrm{T}_{X'/k})$-torsor $\mathscr{L}_{\mathfrak{X}'_2,\varphi_{T/k}}$ of T_{zar} (10.10) is the same as the $\phi^*_{T/k}(\mathrm{T}_{U'/k})$-torsor $\mathscr{R}_{\mathfrak{X}',(U',\mathfrak{T},\phi_{T/k})}$ of T_{zar} (12.8). Recall that $\mathscr{F}_{\mathfrak{X}'_2,(U,T)}$ is the sheaf of affine functions on $\mathscr{L}_{\mathfrak{X}'_2,\varphi_{T/k}}$. By (12.15.1), we deduce a canonical isomorphism of $\mathscr{O}_T$-modules

$$\mathscr{F}_{\mathfrak{X}'_2,(U,T)}\xrightarrow{\sim}\underline{\mathscr{F}}_{\mathfrak{X},(U,\mathfrak{T})}. \tag{12.25.1}$$

Let $g:(U_1,T_1,\delta_1)\to(U_2,T_2,\delta_2)$ be a morphism of $\mathrm{Crys}(X/k)$. Suppose that there exists an $\mathscr{S}$-morphism $\mathfrak{g}:\mathfrak{T}_1\to\mathfrak{T}_2$ of flat formal $\mathscr{S}$-schemes with special fiber $g:T_1\to T_2$. Then we obtain a morphism $\mathfrak{g}:(U_1,\mathfrak{T}_1)\to(U_2,\mathfrak{T}_2)$ of $\underline{\mathscr{E}}$. In view of 10.11(ii), the transition morphism of the crystal of $\mathscr{O}_{\underline{\mathscr{E}},1}$-modules $\underline{\mathscr{F}}_{\mathfrak{X}}$ of $\widetilde{\underline{\mathscr{E}}}$ associated to $\mathfrak{g}$ (12.9.4), 12.15

$$c_{\mathfrak{g}}:g^*(\underline{\mathscr{F}}_{\mathfrak{X},(U_2,\mathfrak{T}_2)})\xrightarrow{\sim}\underline{\mathscr{F}}_{\mathfrak{X},(U_1,\mathfrak{T}_1)} \tag{12.25.2}$$

is compatible with the transition morphism of the crystal $\mathscr{F}_{\mathfrak{X}'_2}$ of $\mathscr{O}_{X/k}$-modules associated to g

$$c_g:g^*(\mathscr{F}_{\mathfrak{X}'_2,(U_2,T_2)})\xrightarrow{\sim}\mathscr{F}_{\mathfrak{X}'_2,(U_1,T_1)}. \tag{12.25.3}$$

Applying above facts to projections from the PD-envelop of diagonal immersion (X,P_X) to (X,X), we deduce the following lemma.

Lemma 12.26. – *The isomorphism* $\mathscr{F}_{\mathfrak{X}'_2}\xrightarrow{\sim}\underline{\mathscr{F}}_{\mathfrak{X}}$ (12.25.1) *is compatible with the actions of* $\mathrm{D}_{X/k}$.

COROLLARY 12.27. – *The isomorphism* $\mathscr{I}_{\mathfrak{X}'_2} \xrightarrow{\sim} \underline{\mathscr{I}}_{\mathfrak{X}}$ *induces a canonical isomorphism of* $\mathscr{O}_X$*-algebras* $\mathscr{A}_{\mathfrak{X}'_2} \xrightarrow{\sim} \underline{\mathscr{B}}_{\mathfrak{X}}$ *compatible with the actions of* $\mathrm{D}^{\gamma}_{X/k}$.

Proof. – By 12.26, we obtain an isomorphism $\mathscr{A}_{\mathfrak{X}'_2} \xrightarrow{\sim} \underline{\mathscr{B}}_{\mathfrak{X}}$ compatible with actions of $\mathrm{D}_{X/k}$. By 10.5 and (10.3.2), the $\mathscr{O}_X$-module $\underline{\mathscr{I}}_{\mathfrak{X}}$ (resp. $\mathscr{O}_X$-algebra $\underline{\mathscr{B}}_{\mathfrak{X}}$) is equipped with a $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-module structure that we denote by $\vartheta_{\underline{\mathscr{I}}}$ (resp. $\vartheta_{\underline{\mathscr{B}}}$). By 10.12 and the following lemma, the isomorphism $\mathscr{A}_{\mathfrak{X}'_2} \xrightarrow{\sim} \underline{\mathscr{B}}_{\mathfrak{X}}$ is compatible with actions of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$. Then the assertion follows. □

LEMMA 12.28. – *Two* $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$*-module structures* $\psi_{\underline{\mathscr{I}}}$ *and* $\vartheta_{\underline{\mathscr{I}}}$ *on* $\underline{\mathscr{I}}_{\mathfrak{X}}$ (12.15) *(resp.* $\psi_{\underline{\mathscr{B}}}$ *and* $\vartheta_{\underline{\mathscr{I}}}$ *on* $\underline{\mathscr{B}}_{\mathfrak{X}}$*) coincide.*

Proof. – It suffices to show the assertion for $\underline{\mathscr{I}}_{\mathfrak{X}}$. We have $\underline{\mathscr{I}}_{\mathfrak{X}} = \mathscr{F}_{\mathfrak{X}',(X',\mathfrak{X},F_{X/k})}$.

The question being local, we can reduce to case where there exists an $\mathscr{S}$-morphism $F : \mathfrak{X} \to \mathfrak{X}'$ which lifts the relative Frobenius morphism $F_{X/k}$ of X. Then we obtain a morphism $F : (X', \mathfrak{X}, F_{X/k}) \to (X', \mathfrak{X}')$ of $\mathscr{E}'$ and an isomorphism (12.9.4)

$$\eta_F : F^*_{X/k}(\mathscr{I}_{\mathfrak{X}'}) \xrightarrow{\sim} \underline{\mathscr{I}}_{\mathfrak{X}}. \tag{12.28.1}$$

Let $\psi_{\mathscr{I}'}$ be the $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module structure on $\mathscr{I}_{\mathfrak{X}'}$ induced by the crystal $\mathscr{F}_{\mathfrak{X}'}$. By 11.19, the isomorphism $\eta_F : F^*_{X/k}(\mathscr{I}_{\mathfrak{X}'}) \xrightarrow{\sim} \underline{\mathscr{I}}_{\mathfrak{X}}$ underlies an isomorphism of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules

$$\eta_F : F^*_{X/k}(\mathscr{I}_{\mathfrak{X}'}, \iota^*(\psi_{\mathscr{I}'})) \xrightarrow{\sim} (\underline{\mathscr{I}}_{\mathfrak{X}}, \psi_{\underline{\mathscr{I}}}). \tag{12.28.2}$$

On the other hand, regarding $\mathscr{I}_{\mathfrak{X}'}$ as a sheaf of affine functions, the $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-action on $\mathscr{I}_{\mathfrak{X}'}$ defined in 10.5 is equal to $\iota^*(\psi_{\mathscr{I}'})$ (12.12). By 10.7, η_F induces an isomorphism of $F^*_{X/k}(\widehat{\Gamma}(\mathrm{T}_{X'/k}))$-modules

$$\eta_F : F^*_{X/k}(\mathscr{I}_{\mathfrak{X}'}, \iota^*(\psi_{\mathscr{I}'})) \xrightarrow{\sim} (\underline{\mathscr{I}}_{\mathfrak{X}}, \vartheta_{\underline{\mathscr{I}}}). \tag{12.28.3}$$

The assertion follows. □

12.29. – *Proof of 12.22.* Let $\mathscr{N}$ be a quasi-coherent crystal of $\mathscr{O}_{\underline{\mathscr{E}},1}$-modules of $\widetilde{\underline{\mathscr{E}}}$, N the associated $\mathrm{D}^{\gamma}_{X/k}$-module, $\mathscr{M}' = \mathrm{C}_{X/\mathscr{S}*}(\mathscr{N})$ and M' the associated $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-module. By 9.12, we have a canonical isomorphism $\mathrm{C}^*_{X/\mathscr{S}}(\mathscr{M}') \xrightarrow{\sim} \mathscr{N}$. The $\mathrm{D}^{\gamma}_{X/k}$-module structure $(\nabla_{\underline{\mathscr{B}}_{\mathfrak{X}}}, \psi_{\underline{\mathscr{B}}_{\mathfrak{X}}})$ on $\underline{\mathscr{B}}_{\mathfrak{X}}$ induces a $\mathrm{D}^{\gamma}_{X/k}$-module structure on its dual $\underline{\mathscr{B}}^{\vee}_{\mathfrak{X}}$ (10.18). By (12.24.1) and 12.27, we have a canonical isomorphism of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-modules

$$\mathrm{C}_{\mathfrak{X}'_2}(N) \xrightarrow{\sim} \iota^*(\mathscr{H}om_{\mathrm{D}^{\gamma}_{X/k}}(\underline{\mathscr{B}}^{\vee}_{\mathfrak{X}}, N)), \tag{12.29.1}$$

where the $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-action on the right hand side is given by that of $\underline{\mathscr{B}}^{\vee}_{\mathfrak{X}}$. The canonical morphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} N) \to \mathscr{H}om_{\mathscr{O}_X}(\underline{\mathscr{B}}^{\vee}_{\mathfrak{X}}, N) \tag{12.29.2}$$

sends a local section φ of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} N)$ to $\chi : \underline{\mathscr{B}}_{\mathfrak{X}}^{\vee} \to N$ defined for every local section f of $\underline{\mathscr{B}}_{\mathfrak{X}}^{\vee}$ by

$$\chi(f) = (f \otimes \mathrm{id}_N)(\varphi(1)). \tag{12.29.3}$$

Since $\underline{\mathscr{B}}_{\mathfrak{X}}$ is locally a direct sum of free $\mathscr{O}_X$-modules of finite type (12.17.3), the morphism (12.29.2) is an isomorphism. We equip $\mathscr{O}_X$ the $\mathrm{D}^{\gamma}_{X/k}$-module structure $(d, 0)$. By (10.18.1) and (10.14.2), one verifies that a local section $\varphi : \mathscr{O}_X \to \underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} N$ is $\mathrm{D}^{\gamma}_{X/k}$-equivariant if and only if χ (12.29.3) is $\mathrm{D}^{\gamma}_{X/k}$-equivariant. Then we deduce an isomorphism

$$\mathscr{H}om_{\mathrm{D}^{\gamma}_{X/k}}((\mathscr{O}_X, d, 0), \underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} N) \xrightarrow{\sim} \mathscr{H}om_{\mathrm{D}^{\gamma}_{X/k}}(\underline{\mathscr{B}}_{\mathfrak{X}}^{\vee}, N). \tag{12.29.4}$$

In view of (10.14.2), the above isomorphism induces an isomorphism of $\widehat{\Gamma}(\mathrm{T}_{X'/k})$-modules (12.16)

$$(\mathscr{H}om_{\mathrm{D}^{\gamma}_{X/k}}(\mathscr{O}_X, \underline{\mathscr{B}}_{\mathfrak{X}} \otimes_{\mathscr{O}_X} N), \psi_{\underline{\mathscr{B}}_{\mathfrak{X}}}) \xrightarrow{\sim} \iota^*(\mathscr{H}om_{\mathrm{D}^{\gamma}_{X/k}}(\underline{\mathscr{B}}_{\mathfrak{X}}^{\vee}, N)). \tag{12.29.5}$$

Then the assertion follows from 12.19(ii) and (12.29.1). □

CHAPTER 13

FONTAINE MODULES

Inspired by the work of Fontaine-Laffaille [**15**], Faltings introduced a notion of *Fontaine module* on a smooth scheme over W [**13**]. In this section, we propose a new definition of Fontaine module using Cartier equivalence (13.7). Compared to Faltings' original definition, our definition avoids the choice of (local) liftings of Frobenius to talk about "divided Frobenius structures" (13.15) and encodes them in the Cartier equivalence. We also show the compatibility between various definitions in (13.20, 13.22, 13.30).

Let $\mathfrak{X}$ denote a smooth formal $\mathscr{S}$-scheme and X its special fiber.

Definition 13.1. – Let n be an integer $\geqslant 1$. We define the category of filtered modules with quasi-nilpotent integrable connection $\mathrm{MIC}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n)$ as follows. *A filtered module with quasi-nilpotent integrable connection* is a triple $(M, \nabla, M^\bullet)$ consisting of an object (M, ∇) of $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ and a decreasing filtration $\{M^i\}_{i\in\mathbb{Z}}$

$$\cdots \subseteq M^2 \subseteq M^1 \subseteq M^0 = M = M^{-1} = \cdots \tag{13.1.1}$$

satisfying Griffiths' transversality

$$\nabla(M^i) \subset M^{i-1} \otimes \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \quad \forall\ i \geqslant 0. \tag{13.1.2}$$

Given two objects $(M_1, \nabla_1, M_1^\bullet)$ and $(M_2, \nabla_2, M_2^\bullet)$ of $\mathrm{MIC}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n)$, a morphism from $(M_1, \nabla_1, M_1^\bullet)$ to $(M_2, \nabla_2, M_2^\bullet)$ is a horizontal $\mathscr{O}_{\mathfrak{X}_n}$-linear morphism $f : M_1 \to M_2$ compatible with the filtrations.

For any $\ell \geqslant 0$, we denote by $\mathrm{MIC}^{\ell}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n)$ the full subcategory of $\mathrm{MIC}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n)$ consisting of objects with length $\leqslant \ell$ (i.e., the filtration satisfies $M^{\ell+1} = 0$).

13.2. – Let ℓ be an integer $\geqslant 0$, $(M, \nabla, M^\bullet)$ an object of $\mathrm{MIC}^{\ell}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n)$. We consider the $\mathscr{O}_{\mathfrak{X}_n}$-linear morphism

$$g : \bigoplus_{i=1}^{\ell} M^i \to \bigoplus_{i=0}^{\ell} M^i \tag{13.2.1}$$

defined for every local section m_i of M^i by $g(m_i) = (m_i, -pm_i)$ in $M^{i-1} \oplus M^i$. We set

$$\widetilde{M} = \mathrm{Coker}(g). \tag{13.2.2}$$

For any $0 \leqslant j \leqslant \ell$, we denote by $(-)_j$ the composition $M^j \to \bigoplus_{i=0}^{\ell} M^i \to \widetilde{M}$ and by $\widetilde{M}^{-j}$ the canonical image of $\bigoplus_{i=0}^{j} M^i$ in $\widetilde{M}$. We obtain a decreasing filtration

$$\widetilde{M}^0 \subseteq \widetilde{M}^{-1} \subseteq \cdots \subseteq \widetilde{M}^{-\ell} = \widetilde{M}. \tag{13.2.3}$$

We consider the W_n-linear morphism

$$h : \bigoplus_{i=0}^{\ell} M^i \to \Big(\bigoplus_{i=0}^{\ell} M^i\Big) \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \tag{13.2.4}$$

defined by

$$\begin{aligned} h|_{M^i} &= \nabla : M^i \to M^{i-1} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}, \quad \text{for } 1 \leqslant i \leqslant \ell, \\ h|_{M^0} &= p\nabla : M^0 \to M^0 \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}. \end{aligned} \tag{13.2.5}$$

LEMMA 13.3. – *The* W_n*-linear morphism* h *induces a quasi-nilpotent integrable* p*-connection* $\tilde{\nabla}$ *on* $\widetilde{M}$ *such that for any* $-\ell \leqslant i \leqslant -1$*, we have*

$$\tilde{\nabla}(\widetilde{M}^i) \subset \widetilde{M}^{i+1} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}. \tag{13.3.1}$$

Proof. – It follows from the definition that the composition

$$\bigoplus_{i=1}^{\ell} M^i \xrightarrow{g} \bigoplus_{i=0}^{\ell} M^i \xrightarrow{h} \Big(\bigoplus_{i=0}^{\ell} M^i\Big) \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \tag{13.3.2}$$

is zero. Hence the morphism h induces a W_n-linear morphism

$$\tilde{\nabla} : \widetilde{M} \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}. \tag{13.3.3}$$

We show that $\tilde{\nabla}$ is a p-connection. The restriction of h to M^0 is the p-connection $p\nabla$. Hence the restriction of $\tilde{\nabla}$ to $\widetilde{M}^0$ is a p-connection. Let f be a local section of $\mathscr{O}_{\mathfrak{X}}$, i an integer $\in [1, \ell]$ and m a local section of M^i. The morphism h sends $fm \in M^i$ to

$$\nabla(fm) = f\nabla(m) + m \otimes df \in M^{i-1} \otimes \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}. \tag{13.3.4}$$

Note that we have $(m)_{i-1} = (pm)_i$. Then we deduce that

$$\tilde{\nabla}(f(m)_i) = f\tilde{\nabla}((m)_i) + p(m)_i \otimes df. \tag{13.3.5}$$

The assertion follows. The integrability of ∇ implies easily that of $\tilde{\nabla}$.

It is clear from the definition that $\tilde{\nabla}$ satisfies the condition (13.3.1). Since M is p^n-torsion, the restriction of $\tilde{\nabla}$ to $\widetilde{M}^0$ is quasi-nilpotent. In view of (13.3.1), we deduce that $\tilde{\nabla}$ is quasi-nilpotent. $\square$

13.4. – Let $\varepsilon_{\widetilde{M},T}$ be the $\mathscr{T}_{\mathfrak{X}}$-stratification on $\widetilde{M}$ associated to $\tilde{\nabla}$ (5.17). We present its local description. Suppose that there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \mathrm{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We take again the notation of 5.2 and of 5.16. For any $0 \leqslant i \leqslant \ell$, any local section m of M^i and any $I \in \mathbb{N}$, in view of (13.2.5), we have

$$\tilde{\nabla}_{\partial^I}((m)_i) = \begin{cases} (\nabla_{\partial^I}(m))_{i-|I|} & \text{if } 0 \leqslant i \leqslant |I|; \\ p^{|I|-i}(\nabla_{\partial^I}(m))_0 & \text{if } |I| > i. \end{cases} \tag{13.4.1}$$

By (5.17.1), we deduce that
(13.4.2)

$$\varepsilon_{\widetilde{M},T}(1 \otimes (m)_i) = \sum_{|I| \leqslant i} \big(\nabla_{\partial^I}(m)\big)_{i-|I|} \otimes \left(\frac{\xi}{p}\right)^{[I]} + \sum_{|I|>i} p^{|I|-i}\big(\nabla_{\partial^I}(m)\big)_0 \otimes \left(\frac{\xi}{p}\right)^{[I]}.$$

PROPOSITION 13.5. – *Suppose that $\ell \leqslant p-1$. There exists an $\mathscr{R}_{\mathfrak{X}}$-stratification $\varepsilon_{\widetilde{M}}$ on $\widetilde{M}$, which induces the above $\mathscr{T}_{\mathfrak{X}}$-stratification $\varepsilon_{\widetilde{M},T}$ via the functor* (9.18.1).

Proof. – Let ε_M be the $\mathscr{P}_{\mathfrak{X}}$-stratification on M and $\theta_M : M \to M \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{P}_{\mathfrak{X}}$ the morphism defined by $\theta_M(m) = \varepsilon_M(1 \otimes m)$. We denote by J_P the PD-ideal of $\mathscr{P}_{\mathfrak{X}}$. By flatness of $J_P^{[\bullet]}$ over $\mathscr{O}_{\mathfrak{X}}$ (5.9), $M^i \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[j]}$ is a submodule of $M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{P}_{\mathfrak{X}}$ for any $i, j \geqslant 0$. By ([**29**] 3.1.3) and Griffiths' transversality, we have

$$\theta_M(M^i) \subset \sum_{j=0}^{i} M^j \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[i-j]}, \qquad \forall\ 0 \leqslant i \leqslant \ell. \tag{13.5.1}$$

We denote the target by $(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{P}_{\mathfrak{X}})^i$ and the induced morphism by $\theta_M^i : M^i \to (M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{P}_{\mathfrak{X}})^i$. Let $s : \mathscr{P}_{\mathfrak{X}} \to \mathscr{R}_{\mathfrak{X}}$ be the homomorphism defined in 5.19, and for all $j \leqslant p-1$, $s^j : J_P^{[j]} \to \mathscr{R}_{\mathfrak{X}}$ the induced morphism (5.19.2). For any $j \leqslant i$, we have a canonical morphism

$$(-)_j \otimes s^{i-j} : M^j \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[i-j]} \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{R}_{\mathfrak{X}}. \tag{13.5.2}$$

For any $0 \leqslant j' < j \leqslant i$, by flatness of $J_P^{[\bullet]}$ over $\mathscr{O}_{\mathfrak{X}}$ (5.9), we have the intersection in $M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{P}_{\mathfrak{X}}$

$$M^j \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[i-j]} \cap M^{j'} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[i-j']} = M^j \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[i-j']}.$$

Since $s^{i-j}|_{J_P^{[i-j']}} = p^{j-j'} s^{i-j'}$ (5.19.2), we deduce that the morphisms (13.5.2) are compatible and they induce an $\mathscr{O}_{\mathfrak{X}}$-linear morphism

$$u^i : (M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{P}_{\mathfrak{X}})^i \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{R}_{\mathfrak{X}}. \tag{13.5.3}$$

By construction, we have $u^i|_{(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{P}_{\mathfrak{X}})^{i+1}} = pu^{i+1}$. Then the morphism

$$\bigoplus_{i=0}^{\ell} u^i \circ \theta_M^i : \bigoplus_{i=0}^{\ell} M^i \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{R}_{\mathfrak{X}}$$

induces an $\mathscr{O}_{\mathfrak{X}}$-linear morphism

$$\theta_{\widetilde{M}}: \widetilde{M} \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{R}_{\mathfrak{X}}. \tag{13.5.4}$$

We will show that the above morphism satisfies the conditions of 5.5 and hence induces an $\mathscr{R}_{\mathfrak{X}}$-stratification. To do this, we can reduce to the case where there exists an étale $\mathscr{S}$-morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}} = \operatorname{Spf}(\mathrm{W}\{T_1, \dots, T_d\})$. We take again the notation of 5.11. For any $0 \leqslant i \leqslant \ell$, any local section m of M^i, we have

$$\theta^i_M(m) = \sum_{I \in \mathbb{N}^d} \nabla_{\partial^I}(m) \otimes \xi^{[I]} \in \sum_{j=0}^{i} M^j \otimes_{\mathscr{O}_{\mathfrak{X}_n}} J_P^{[i-j]}. \tag{13.5.5}$$

The p-adic valuation of $I!$ is less than $\sum_{k \geqslant 1} \lfloor \frac{|I|}{p^k} \rfloor$. If $i \leqslant p-1$ and $|I| \geqslant i$, $\frac{p^{|I|-i}}{I!}$ is an element of $\mathbb{Z}_p$. By (5.19.1), we have $s^i(\xi^{[I]}) = \frac{p^{|I|-i}}{I!}(\frac{\xi}{p})^I$. Then we deduce that

$$\theta_{\widetilde{M}}((m)_i) = \sum_{|I| \leqslant i} \frac{1}{I!} \big(\nabla_{\partial^I}(m)\big)_{i-|I|} \otimes \left(\frac{\xi}{p}\right)^I + \sum_{|I| > i} \frac{p^{|I|-i}}{I!} \big(\nabla_{\partial^I}(m)\big)_0 \otimes \left(\frac{\xi}{p}\right)^I. \tag{13.5.6}$$

It is clear that $\theta_{\widetilde{M}}$ verifies condition (i) of 5.5. By (13.5.6) and the local description (4.16) of $\delta: \mathscr{R}_{\mathfrak{X},n} \to \mathscr{R}_{\mathfrak{X},n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{R}_{\mathfrak{X},n}$, we deduce that

$$\begin{aligned} \theta_{\widetilde{M}} \otimes \mathrm{id}_{\mathscr{R}}(\theta_{\widetilde{M}}((m)_i)) &= \sum_{|I|+|J| \leqslant i} \frac{1}{I!J!} \big(\nabla_{\partial^{I+J}}(m)\big)_{i-|I|-|J|} \otimes \left(\frac{\xi}{p}\right)^I \otimes \left(\frac{\xi}{p}\right)^J \\ &\quad + \sum_{|I|+|J|>i} \frac{p^{|I|+|J|-i}}{I!J!} \big(\nabla_{\partial^{I+J}}(m)\big)_0 \otimes \left(\frac{\xi}{p}\right)^I \otimes \left(\frac{\xi}{p}\right)^J \\ &= \mathrm{id}_{\widetilde{M}} \otimes \delta(\theta_{\widetilde{M}}((m)_i)). \end{aligned}$$

By 5.5, we obtain an $\mathscr{R}_{\mathfrak{X}}$-stratification $\varepsilon_{\widetilde{M}}$ on $\widetilde{M}$. By comparing (13.4.2) and (13.5.6), $\varepsilon_{\mathscr{M}}$ extends $\varepsilon_{\widetilde{\mathscr{M}},T}$. □

13.6. – By 8.10, we associate to $(\widetilde{M}, \varepsilon_{\widetilde{M}})$ a crystal of $\mathscr{O}_{\mathscr{E},n}$-modules of $\widetilde{\mathscr{E}}$ that we denote by $\widetilde{\mathscr{M}}$.

We put $\mathfrak{X}' = \mathfrak{X} \times_{\mathscr{S},\sigma} \mathscr{S}$ (2.1) and we denote by $\pi: \mathfrak{X}' \to \mathfrak{X}$ the canonical morphism. In view of 3.8 and 4.11, the morphism π induces a morphism of formal groupoids $\pi_R: R_{\mathfrak{X}'} \simeq R_{\mathfrak{X}} \times_{\mathscr{S},\sigma} \mathscr{S} \to R_{\mathfrak{X}}$ above π (4.8). We denote by $\widetilde{\mathscr{M}}'$ the crystal of $\mathscr{O}_{\mathscr{E}',n}$-modules of $\widetilde{\mathscr{E}}'$ associated to the $\mathscr{O}_{\mathfrak{X}'_n}$-module with $\mathscr{R}_{\mathfrak{X}'}$-stratification $(\pi^*(\widetilde{M}), \pi_R^*(\varepsilon_{\widetilde{M}}))$ (5.6). The $\mathscr{O}_{\mathfrak{X}'_n}$-module with integrable p-connection associated to $\widetilde{\mathscr{M}}'$ (9.18.3) is the inverse image of $(\widetilde{M}, \widetilde{\nabla})$ by π ([**33**] page 6).

The previous construction is clearly functorial and it defines a functor (8.3)

$$\begin{array}{rcl} \mathrm{MIC}^{p-1}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n) & \to & \mathscr{C}(\mathscr{O}_{\mathscr{E}',n}) \\ (M, \nabla, M^{\bullet}) & \mapsto & \widetilde{\mathscr{M}}'. \end{array} \tag{13.6.1}$$

DEFINITION 13.7. – (i) A (p^n-torsion) *Fontaine module over* $\mathfrak{X}$ is a quadruple $(M, \nabla, M^\bullet, \varphi)$ consisting of an object $(M, \nabla, M^\bullet)$ of $\mathrm{MIC}_{\mathrm{F}}^{p-1}(\mathfrak{X}_n/\mathscr{S}_n)$ with quasi-coherent $\mathscr{O}_{\mathfrak{X}_n}$-module M^i, and a morphism of $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$

$$\varphi : \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'})) \to (M, \nabla) \tag{13.7.1}$$

where C^* is the Cartier equivalence (9.14.1) and $\nu : \mathscr{C}(\mathscr{O}_{\underline{\mathscr{E}},n}) \to \mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ is defined in (9.18.4).

(ii) We say that a Fontaine module is *strongly divisible* if φ (13.7.1) is an isomorphism.

(iii) Given two Fontaine modules $(M_1, \nabla_1, M_1^\bullet, \varphi_1)$ and $(M_2, \nabla_2, M_2^\bullet, \varphi_2)$, a morphism from $(M_1, \nabla_1, M_1^\bullet, \varphi_1)$ to $(M_2, \nabla_2, M_2^\bullet, \varphi_2)$ is a morphism $f : (M_1, \nabla_1, M_1^\bullet) \to (M_2, \nabla_2, M_2^\bullet)$ of $\mathrm{MIC}_{\mathrm{F}}^{p-1}(\mathfrak{X}_n/\mathscr{S}_n)$ such that the following diagram commutes

$$\begin{tikzcd} \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'_1})) \arrow[r, "\varphi_1"] \arrow[d, "\nu(\mathrm{C}^*(\tilde{f}'))"'] & (M_1, \nabla_1) \arrow[d, "f"] \\ \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'_2})) \arrow[r, "\varphi_2"] & (M_2, \nabla_2), \end{tikzcd} \tag{13.7.2}$$

where $\tilde{f}' : \widetilde{\mathscr{M}'_1} \to \widetilde{\mathscr{M}'_2}$ is the $\mathscr{O}_{\mathscr{E}',n}$-linear morphism induced by f (13.6.1).

We denote by $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X})$ the category of Fontaine modules over $\mathfrak{X}$ and by $\mathbf{MF}(\mathfrak{X})$ the full subcategory of $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X})$ of strongly divisible Fontaine modules $(M, \nabla, M^\bullet, \varphi)$ such that M is coherent.

REMARK 13.8. – In the p-torsion case, given an object $(M, \nabla, M^\bullet)$ of $\mathrm{MIC}_{\mathrm{F}}^{\ell}(X/k)$, $\widetilde{M} = \mathrm{gr}(M)$ and $\tilde{\nabla}$ is the Higgs field on $\mathrm{gr}(M)$ induced by ∇ and Griffiths' transversality which is of length $\leqslant \ell$. In ([**31**] 4.16), using their Cartier transform $\mathrm{C}_{\mathfrak{X}'_2}^{-1}$ (10.20), Ogus and Vologodsky define a p-torsion Fontaine module as an object $(M, \nabla, M^\bullet)$ of $\mathrm{MIC}_{\mathrm{F}}^{p-1}(X/k)$ together with a horizontal isomorphism

$$\varphi : \mathrm{C}_{\mathfrak{X}'_2}^{-1}(\pi^*(\mathrm{Gr}(M), \theta)) \xrightarrow{\sim} (M, \nabla). \tag{13.8.1}$$

By 12.22, our Definition 13.7 is compatible with theirs.

13.9. – In the remainder of this section, we compare Definition 13.7 with Faltings' definition [**13**] and Tsuji's definition (in a broader context) [**35**]. We will formulate their definitions using the notion of *(proto-)T-crystals*, the crystalline counterpart of filtered modules with quasi-nilpotent integrable connection (13.1), introduced by Ogus [**29**].

Let n be an integer and $\mathscr{X}$ a smooth scheme over $\mathscr{S}_n$. We denote by $\mathrm{Crys}(\mathscr{X}/\mathscr{S}_n)$ (resp. $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}$) the crystalline site (resp. topos) of $\mathscr{X}$ over $\mathscr{S}_n$, by $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$ the structural sheaf and by $J_{\mathscr{X}/\mathscr{S}_n}$ its PD-ideal.

Definition 13.10 ([**29**] 2.1.2 and 3.1). – (i) Let (U, T) be an object of $\mathrm{Crys}(\mathscr{X}/\mathscr{S}_n)$, J_T the PD-ideal of U in T and M an $\mathscr{O}_T$-module. We say that a decreasing filtration $\{M^i\}_{i\in\mathbb{Z}}$ of M by $\mathscr{O}_T$-submodules is *G-transversal to* J_T if for any $i \in \mathbb{Z}$, we have

$$J_T M \cap M^i = J_T^{[1]} M^{i-1} + J_T^{[2]} M^{i-2} + \cdots \tag{13.10.1}$$

In particular, we see that such a filtration is J_T-saturated, i.e., $J_T^{[i]} M^j \subset M^{i+j}$ for all $i \geqslant 0, j$.

(ii) Let $\mathscr{M}$ be a crystal of $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$-modules. We say that a decreasing filtration $\{\mathscr{M}^i\}_{i\in\mathbb{Z}}$ of $\mathscr{M}$ by $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$-submodules of $\mathscr{M}$ is *G-transversal to* $J_{\mathscr{X}/\mathscr{S}_n}$ if for every object T of $\mathrm{Crys}(\mathscr{X}/\mathscr{S}_n)$, the filtration $\{\mathscr{M}_T^i\}_{i\in\mathbb{Z}}$ of $\mathscr{M}_T$ is G-transversal to $J_{\mathscr{X}/\mathscr{S}_n,T}$.

Lemma 13.11 ([**29**] 3.1.1). – *Let $\mathscr{M}$ be a crystal of $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$-modules endowed with a filtration $\{\mathscr{M}^i\}_{i\in\mathbb{Z}}$ G-transversal to $J_{\mathscr{X}/\mathscr{S}_n}$. For any morphism $f : T_2 \to T_1$ of* $\mathrm{Crys}(\mathscr{X}/\mathscr{S}_n)$*, via the transition isomorphism $f^*(\mathscr{M}_{T_1}) \xrightarrow{\sim} \mathscr{M}_{T_2}$, $\mathscr{M}_{T_2}^i$ coincides with*

$$\sum_{i_1+i_2=i} J_{T_2}^{[i_1]} \operatorname{Im}(f^*(\mathscr{M}_{T_1}^{i_2}) \to f^*(\mathscr{M}_{T_1})). \tag{13.11.1}$$

Theorem 13.12 ([**29**] 3.1.2, 3.2.3). – *Let $\mathscr{Y}$ be a smooth $\mathscr{S}_n$-scheme, $\iota : \mathscr{X} \to \mathscr{Y}$ a closed $\mathscr{S}_n$-immersion and $\mathscr{D}$ the PD-envelope of ι compatible with γ. Let $\mathscr{M}$ be a quasi-coherent crystal of $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$-modules, $M = \mathscr{M}_{\mathscr{D}}$ and $\nabla : M \to M \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^1_{\mathscr{Y}/\mathscr{S}_n}$ the associated quasi-nilpotent integrable connection on M (as an $\mathscr{O}_{\mathscr{Y}}$-module) (*[**5**] *6.6). Then the evaluation of sheaves of $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}$ on $\mathscr{D}$ induces an equivalence of following sets of data:*

(i) *A decreasing filtration $\{\mathscr{M}^i\}_{i\in\mathbb{Z}}$ by quasi-coherent $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$-modules on $\mathscr{M}$ which is G-transversal to $J_{\mathscr{X}/\mathscr{S}_n}$.*

(ii) *A decreasing filtration $\{M^i\}_{i\in\mathbb{Z}}$ by quasi-coherent $\mathscr{O}_{\mathscr{D}}$-modules on M which is G-transversal to $J_{\mathscr{X}/\mathscr{S}_n,\mathscr{D}}$ and which satisfies Griffiths' transversality i.e., $\nabla(M^i) \subset M^{i-1} \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^1_{\mathscr{Y}/\mathscr{S}_n}$ for all i.*

We briefly review the construction of the data (i) from the data (ii) and we refer to [**29**] for more details. Let $\{M_i\}_{i\in\mathbb{Z}}$ be a filtration as in (ii). Let $\mathscr{D}(1)$ be the PD-envelope of the immersion $X \xrightarrow{\iota} Y \xrightarrow{\Delta} Y \times_{\mathscr{S}_n} Y$ compatible with γ, $p_1, p_2 : \mathscr{D}(1) \to \mathscr{D}$ the canonical projections and $\varepsilon : p_2^*(M) \xrightarrow{\sim} p_1^*(M)$ the $\mathscr{O}_{\mathscr{D}(1)}$-stratification induced by ∇. We define a filtration $\{A_j^i\}_{i\in\mathbb{Z}}$ on $p_j^*(M)$ by the Formula (13.11.1) for $j = 1, 2$. By Griffiths' transversality, one verifies that the isomorphism $\varepsilon : p_2^*(M) \xrightarrow{\sim} p_1^*(M)$ induces for any i, an isomorphism ([**29**] 3.1.3)

$$A_2^i \xrightarrow{\sim} A_1^i. \tag{13.12.1}$$

Given an object T of $\mathrm{Crys}(\mathscr{X}/\mathscr{S}_n)$, there exists locally a morphism $r : T \to \mathscr{D}$ of $\mathrm{Crys}(\mathscr{X}/\mathscr{S}_n)$. Using the Formula (13.11.1), we obtain a filtration $\{\mathscr{M}_T^i\}_{i\in\mathbb{Z}}$ on $r^*(M) \xrightarrow{\sim} \mathscr{M}_T$. Using the fact that ε is a filtered isomorphism, one verifies that the filtration $\{\mathscr{M}_T^i\}_{i\in\mathbb{Z}}$ on $\mathscr{M}_T$ is independent of the choice of r up to isomorphisms which come from the stratification and is well-defined. Then we obtain a filtration $\{\mathscr{M}^i\}_{i\in\mathbb{Z}}$ of $\mathscr{M}$ by quasi-coherent $\mathscr{O}_{\mathscr{X}/\mathscr{S}_n}$-modules. By ([**29**] 2.2.1.2, 2.3.1), one verifies that $\{\mathscr{M}^i\}_{i\in\mathbb{Z}}$ is G-transversal to $J_{\mathscr{X}/\mathscr{S}_n}$.

Definition 13.13 ([**29**] 3.2.1, 3.2.3). – (i) A *proto-T-crystal* is a pair $(\mathscr{M},(\mathscr{M}^i)_{i\in\mathbb{Z}})$ consisting of a quasi-coherent crystal $\mathscr{M}$ with a filtration $\{\mathscr{M}^i\}_{i\in\mathbb{Z}}$ G-transversal to $J_{\mathscr{X}/\mathscr{S}_n}$.

(ii) A proto-T-crystal $(\mathscr{M},\mathscr{M}^\bullet)$ is called *T-crystal* if for $m>0$ and i, the canonical morphism

$$\mathscr{M}^i_{\mathscr{X}}\otimes_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{X}}/p^m\mathscr{O}_{\mathscr{X}})\to\mathscr{M}_{\mathscr{X}}\otimes_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{X}}/p^m\mathscr{O}_{\mathscr{X}})$$

is injective.

We say a proto-T-crystal has length $\leqslant\ell$ if $\mathscr{M}=\mathscr{M}^0$ and the evaluation of $(\mathscr{M}^i)_{i\in\mathbb{Z}}$ at $\mathscr{X}$ has length $\leqslant\ell$ as in (13.1).

13.14. – Let $\mathfrak{Y}$ be a smooth formal $\mathscr{S}$-scheme, Y its special fiber and $\iota:\mathfrak{X}\hookrightarrow\mathfrak{Y}$ a closed $\mathscr{S}$-immersion. For any $n\geqslant 1$, we denote by $\mathfrak{D}_n$ the PD-envelope of ι_n compatible with γ, by $J_{\mathfrak{D}_n}$ the PD-ideal of $\mathscr{O}_{\mathfrak{D}_n}$.

Recall that σ denotes the Frobenius endomorphism of W. We suppose that there exists a σ-morphism $F_{\mathfrak{Y}}:\mathfrak{Y}\to\mathfrak{Y}$ lifting the Frobenius morphism $F_Y:Y\to Y$. For any $n\geqslant 1$, the $\mathscr{O}_{\mathfrak{Y}_n}$-linear morphism $\frac{dF_{\mathfrak{Y}}}{p}$ (6.1.1) induces by adjunction a semi-linear morphism with respect to $F_{\mathfrak{Y}}$ that we abusively denote by

$$\frac{dF_{\mathfrak{Y}}}{p}:\Omega^1_{\mathfrak{Y}_n/\mathscr{S}_n}\to F_{\mathfrak{Y}*}(\Omega^1_{\mathfrak{Y}_n/\mathscr{S}_n})=\Omega^1_{\mathfrak{Y}_n/\mathscr{S}_n}.\tag{13.14.1}$$

Since $\mathfrak{D}_n$ is equal to the PD-envelope of the immersion $X\to\mathfrak{Y}_n$ compatible with γ, the morphism $F_{\mathfrak{Y}}$ induces a σ-morphism $F_{\mathfrak{D}}:\mathfrak{D}\to\mathfrak{D}$ lifting the Frobenius morphism of $\mathfrak{D}_1$. We denote by $\varphi_{\mathfrak{D}_n}$ the homomorphism $\mathscr{O}_{\mathfrak{D}_n}=F_{\mathfrak{D}}^{-1}(\mathscr{O}_{\mathfrak{D}_n})\to\mathscr{O}_{\mathfrak{D}_n}$ induced by $F_{\mathfrak{D}}$. Since $\varphi_{\mathfrak{D}_1}(J_{\mathfrak{D}_1})=0$, we deduce that for any $0\leqslant r<p$, we have $\varphi_{\mathfrak{D}_n}(J^{[r]})\subset p^r\mathscr{O}_{\mathfrak{D}_n}$. Since $\mathfrak{D}_n$ is flat over $\mathscr{S}_n$ ([**4**] I 4.5.1), dividing $\varphi_{\mathfrak{D}_{n+r}}$ by p^r, we obtain a semi-linear morphism with respect to $F_{\mathfrak{D}}$

$$\varphi^r_{\mathfrak{D}_n}:J^{[r]}_{\mathfrak{D}_n}\to\mathscr{O}_{\mathfrak{D}_n}\qquad\forall\ 0\leqslant r\leqslant p-1.\tag{13.14.2}$$

Definition 13.15 ([**13**] II c), [**35**], 2.1.7). – (i) Let $(\mathscr{M},\mathscr{M}^\bullet)$ be a proto-T-crystal over $\mathfrak{X}_n/\mathscr{S}_n$ of length $<p$ (13.13), $(M=\mathscr{M}_{\mathfrak{X}_n},\nabla)$ the associated quasi-coherent $\mathscr{O}_{\mathfrak{D}_n}$-module with integrable connection and $M^\bullet$ the associated filtration (13.12). A family of *divided Frobenius morphisms* on $(\mathscr{M},\mathscr{M}^\bullet)$ with respect to $(\iota,F_{\mathfrak{Y}})$ is a family of semi-linear morphisms $\{\varphi^r_{\mathfrak{M}}:M^r\to M\}_{r<p}$ with respect to $F_{\mathfrak{D}}$ satisfying the following conditions:

(a) $\varphi^r_{\mathfrak{M}}|M^{r+1}=p\varphi^{r+1}_{\mathfrak{M}},\forall\ r<p-1$ (in particular, $\varphi^{-i}_{\mathfrak{M}}=p^i\varphi^0_{\mathfrak{M}}$ for $i>0$).

(b) For any integers $r,s\geqslant 0$ such that $r+s<p$ and any local sections a of $J^{[r]}_{\mathfrak{D}}$, x of M^s, we have

$$\varphi^{r+s}_{\mathfrak{M}}(ax)=\varphi^r_{\mathfrak{D}_n}(a)\varphi^s_{\mathfrak{M}}(x).$$

(c) The following diagram commutes for all $r < p$ (13.14.1):

(13.15.1)
$$\begin{tikzcd} M^r \arrow[r, "\nabla|_{M^r}"] \arrow[d, "\varphi^r_{\mathfrak{M}}"'] & M^{r-1} \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^1_{\mathfrak{Y}_n/\mathscr{S}_n} \arrow[d, "\varphi^{r-1}_{\mathfrak{M}} \otimes \frac{dF_{\mathfrak{Y}}}{p}"] \\ M \arrow[r, "\nabla"] & M \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^1_{\mathfrak{Y}_n/\mathscr{S}_n}. \end{tikzcd}$$

(ii) A *Fontaine module over $\mathfrak{X}$ with respect to* $(\iota, F_{\mathfrak{Y}})$ is a pair of a proto-T-crystal $(\mathscr{M}, \mathscr{M}^\bullet)$ over $\mathfrak{X}_n/\mathscr{S}_n$ of length $< p$ for some integer $n \geqslant 1$, together with a family of divided Frobenius morphisms with respect to $(\iota, F_{\mathfrak{Y}})$. We can equivalently write such a data as a quadruple $\mathfrak{M} = (M, \nabla, M^\bullet, \varphi^\bullet_{\mathfrak{M}})$ given by the evaluation of $(\mathscr{M}, \mathscr{M}^\bullet)$ at $\mathfrak{D}$ (13.12).

(iii) A morphism between two Fontaine modules $\mathfrak{M}$ and $\mathfrak{N}$ is a morphism of crystals compatible with the filtrations and the divided Frobenius morphisms.

We denote by $\mathbf{MF}_{\rm big}(\mathfrak{X}; \iota, F_{\mathfrak{Y}})$ the category of Fontaine modules over $\mathfrak{X}$ with respect to $(\iota, F_{\mathfrak{Y}})$. The quadruple $(\mathscr{O}_{\mathfrak{D}_n}, \nabla_{\mathfrak{D}_n}, J^{[\bullet]}_{\mathfrak{D}_n}, \varphi^\bullet_{\mathfrak{D}_n})$ is an object of $\mathbf{MF}_{\rm big}(\mathfrak{X}; \iota, F_{\mathfrak{Y}})$.

Let $\mathfrak{M} = (M, M^\bullet, \nabla_M, \varphi^\bullet_{\mathfrak{M}})$ be an object of $\mathbf{MF}_{\rm big}(\mathfrak{X}; \iota, F_{\mathfrak{Y}})$. Since M is an $\mathscr{O}_{\mathfrak{D}}$-module, the de Rham complexe $M \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^\bullet_{\mathfrak{Y}_n/\mathscr{S}_n}$ is concentrated on X. By Griffith's transversality, for any $r \leqslant p-1$, we have a subcomplex $(M^{r-q} \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^q_{\mathfrak{Y}_n/\mathscr{S}_n})_{q \geqslant 0}$ of the de Rham complex $M \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^\bullet_{\mathfrak{Y}_n/\mathscr{S}_n}$. By (13.15.1), the divided Frobenius morphisms $\{\varphi^\bullet_{\mathfrak{M}}\}$ and $\frac{dF_{\mathfrak{Y}}}{p}$ induce a W-linear morphism of complexes

(13.15.2)
$$(M^{r-\bullet} \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^\bullet_{\mathfrak{Y}_n/\mathscr{S}_n}) \otimes_{\sigma, \mathrm{W}} \mathrm{W} \to M \otimes_{\mathscr{O}_{\mathfrak{Y}_n}} \Omega^\bullet_{\mathfrak{Y}_n/\mathscr{S}_n}.$$

13.16. – Let $\mathfrak{M} = (M, M^\bullet, \nabla_M, \varphi^\bullet_{\mathfrak{M}})$ be a p^n-torsion object of $\mathbf{MF}_{\rm big}(\mathfrak{X}; \iota, \mathfrak{Y})$. We define the $\mathscr{O}_{\mathfrak{D}}$-module $\widetilde{\mathfrak{M}}$ as the quotient of $\oplus_{r<p} F^*_{\mathfrak{D}}(M^r)$ by the $\mathscr{O}_{\mathfrak{D}_n}$-submodule generated by local sections of the following forms:

(i) $(1 \otimes x)_{r-1} - (1 \otimes px)_r$ for all $x \in M^r, r < p$,

(ii) $(\varphi^r_{\mathfrak{D}_n}(a) \otimes x)_s - (1 \otimes ax)_{r+s}$ for all $a \in J^{[r]}_{\mathfrak{D}_n}, x \in M^s, r \geqslant 0, r + s < p$,

where $(-)_r$ denotes the canonical inclusion $F^*_{\mathfrak{D}}(M^r) \to \oplus_{r<p} F^*_{\mathfrak{D}}(M^r)$. In view of condition (d) of 13.15, the morphisms $\{\varphi^r_{\mathfrak{M}}\}_{r<p}$ induce an $\mathscr{O}_{\mathfrak{D}}$-linear morphism

(13.16.1)
$$\varphi_{\mathfrak{M}} : \widetilde{\mathfrak{M}} \to M.$$

Definition 13.17. – We say that an object $\mathfrak{M}$ of $\mathbf{MF}_{\rm big}(\mathfrak{X}; \iota, F_{\mathfrak{Y}})$ is *strongly divisible* if $\varphi_{\mathfrak{M}}$ is an isomorphism.

13.18. – In the case $\mathfrak{X} = \mathfrak{Y}$ and $\iota = \mathrm{id}$, we have $\mathfrak{D}_n = \mathfrak{X}_n$. We write simply $\mathbf{MF}_{\rm big}(\mathfrak{X}; F_{\mathfrak{X}})$ for $\mathbf{MF}_{\rm big}(\mathfrak{X}; \mathrm{id}, F_{\mathfrak{X}})$ and we denote by $\mathbf{MF}(\mathfrak{X}; F_{\mathfrak{X}})$ the full subcategory of $\mathbf{MF}_{\rm big}(\mathfrak{X}; F_{\mathfrak{X}})$ of strongly divisible objects whose underlying $\mathscr{O}_{\mathfrak{X}}$-modules are coherent.

Let $\mathfrak{M} = (M, \nabla, M^\bullet, \varphi^\bullet_{\mathfrak{M}})$ be an object of $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{\mathfrak{X}})$. The condition (i-b) of 13.15 and the relation (ii) of 13.16 are empty and we have $\widetilde{\mathfrak{M}} = F^*_{\mathfrak{X}}(\widetilde{M})$ (13.2.2).

13.19. – Suppose that there exists a σ-lifting $F_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}$ of the Frobenius morphism F_X. The morphism $F_{\mathfrak{X}}$ induces an $\mathscr{S}$-morphism $F : \mathfrak{X} \to \mathfrak{X}'$. Let $(M, \nabla, M^\bullet)$ be an object of $\mathrm{MIC}^{\mathrm{qn}}_{\mathrm{F}}(\mathfrak{X}_n/\mathscr{S}_n)$. By (9.18.6) and 13.6, the morphism F induces a functorial isomorphism of $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$:

$$\eta_F : \Phi_n(\pi^*(\widetilde{M}, \tilde{\nabla})) \xrightarrow{\sim} \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'})), \tag{13.19.1}$$

where Φ_n is Shiho's functor (6.1.4) defined by F. The underlying $\mathscr{O}_{\mathfrak{X}_n}$-module of $\Phi_n(\pi^*(\widetilde{M}, \tilde{\nabla}))$ is $F^*_{\mathfrak{X}}(\widetilde{M})$ (6.1.4). We denote the connection on $F^*_{\mathfrak{X}}(\widetilde{M})$ by ∇_F.

Given a horizontal morphism $\varphi : \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'})) \to (M, \nabla)$, we obtain a horizontal morphism φ_F and a family of morphisms $\{\varphi^i_F : M^i \to M\}_{i=0}^{p-1}$:

$$\varphi_F = \varphi \circ \eta_F : (F^*_{\mathfrak{X}}(\widetilde{M}), \nabla_F) \to (M, \nabla), \tag{13.19.2}$$

$$\varphi^i_F : M^i \xrightarrow{(-)_i} \widetilde{M} \xrightarrow{\varphi_F} M. \tag{13.19.3}$$

For $i > 0$, we set $\varphi^{-i}_F = p^i \varphi^0_F$. Then we obtain a functor

$$\lambda_F : \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}) \to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{\mathfrak{X}}) \qquad (M, \nabla, M^\bullet, \varphi) \mapsto (M, \nabla, M^\bullet, \varphi^\bullet_F). \tag{13.19.4}$$

Conversely, a family of divided Frobenius morphisms $\{\varphi^i_F : M^i \to M\}_{i \leqslant p-1}$ satisfying (i-a,c) of 13.15 induces an $\mathscr{O}_{\mathfrak{X}_n}$-linear morphism $\varphi_F : F^*_{\mathfrak{X}}(\widetilde{M}) \to M$ (13.16.1). Then we obtain a morphism $\varphi : \mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,\mathfrak{X})} \to M$ by composing with η_F^{-1}. We define a functor

$$\chi_F : \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{\mathfrak{X}}) \to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}) \qquad (M, \nabla, M^\bullet, \varphi^\bullet_F) \mapsto (M, \nabla, M^\bullet, \varphi). \tag{13.19.5}$$

Proposition 13.20. – *The functors λ_F (13.19.4) and χ_F (13.19.5) are well-defined. They induce equivalences of categories quasi-inverse to each other and preserve the strong divisibility condition (13.7), 13.17.*

Proof. – Let $(M, \nabla, M^\bullet, \varphi)$ be an object of $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X})$. It follows from the definition of $\widetilde{M}$ that the morphisms $\{\varphi^\bullet_F\}$ satisfy condition (i-b) of 13.15. We show that they also satisfy condition (i-c). Recall 6.1 that for any local sections m of $\widetilde{M}$ and f of $\mathscr{O}_{\mathfrak{X}_n}$, we have (6.1.3)

$$\nabla_F(f F^*_{\mathfrak{X}}(m)) = f \zeta\big(F^*_{\mathfrak{X}}(\tilde{\nabla}(m))\big) + F^*_{\mathfrak{X}}(m) \otimes df, \tag{13.20.1}$$

where ζ denotes the composition

$$F^*_{\mathfrak{X}}(\widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}) \xrightarrow{\sim} F^*_{\mathfrak{X}}(\widetilde{M}) \otimes_{\mathscr{O}_{\mathfrak{X}_n}} F^*_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}) \xrightarrow{\mathrm{id} \otimes dF_{\mathfrak{X}}/p} F^*_{\mathfrak{X}}(\widetilde{M}) \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}.$$

For any $0 \leqslant i \leqslant p-1$ and any local section m of M^i, we denote by $(\nabla(m))_{i-1}$ the image of $\nabla(m)$ via $M^{i-1} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n} \to \widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}$ for $1 \leqslant i \leqslant p-1$

and by $(\nabla(m))_{-1}$ the image of $p\nabla(m)$ in $\widetilde{M}\otimes_{\mathscr{O}_{\mathfrak{X}_n}}\Omega^1_{\mathfrak{X}_n/\mathscr{S}_n}$ for $i=0$. In view of the definition of $\widetilde{\nabla}$ (13.3), we have $\widetilde{\nabla}((m)_i)=(\nabla(m))_{i-1}$ and

$$
\begin{aligned}
\nabla(\varphi_F^i(m)) &= \varphi_F\otimes\mathrm{id}(\nabla_F(F_{\mathfrak{X}}^*((m)_i))) && (13.19.3)\\
&= \varphi_F\otimes\mathrm{id}(\zeta(F_{\mathfrak{X}}^*(\widetilde{\nabla}((m)_i)))) && (13.20.1)\\
&= \varphi_F\otimes\frac{dF_{\mathfrak{X}}}{p}((\nabla(m))_{i-1})\\
&= \varphi_F^{i-1}\otimes\frac{dF_{\mathfrak{X}}}{p}(\nabla(m)).
\end{aligned}
\tag{13.20.2}
$$

The commutativity of (13.15.1) follows. The functor (13.19.4) is well-defined and preserves the strong divisibility conditions.

Conversely, let $(M,\nabla,M^\bullet,\varphi_F^\bullet)$ be an object of $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{\mathfrak{X}})$. In view of (13.20.2), the associated morphism $\varphi_F:F_{\mathfrak{X}}^*(\widetilde{M})\to M$ is compatible with the connections ∇_F and ∇. Via η_F^{-1} (13.19.1), we obtain a morphism $\varphi:\nu(\mathrm{C}^*(\widetilde{\mathscr{M}'}))\to(M,\nabla)$ as (13.7.1). Hence the functor χ_F (13.19.5) is well-defined and is clearly quasi-inverse to λ_F. □

Remark 13.21. – In particular, we see that the notion of Fontaine module over $\mathscr{S}$ (13.7) is compatible with the notion of Fontaine modules over W introduced by Fontaine and Laffaille (cf. [**15**] 1.2 or [**36**] 2.2.1).

13.22. – Let $F_1,F_2:\mathfrak{X}\to\mathfrak{X}'$ be two liftings of the relative Frobenius morphism $F_{X/k}$ of X and let $F_{i,\mathfrak{X}}=\pi\circ F_i:\mathfrak{X}\to\mathfrak{X}$. In ([**13**] proof of Thm. 2.3), Faltings proposed a Taylor formula to construct an equivalence of categories between $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{2,\mathfrak{X}})$ and $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{1,\mathfrak{X}})$. We will show that this Taylor Formula (13.22.6) is naturally encoded in the Cartier equivalence.

We present an explicit description of the equivalence of categories

$$\lambda_{F_2}\circ\chi_{F_1}:\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{1,\mathfrak{X}})\xrightarrow{\sim}\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{2,\mathfrak{X}}).\tag{13.22.1}$$

In particular, we will see that Faltings' construction coincides with the above equivalence.

The morphisms F_1 and F_2 induce a morphism of $\mathscr{E}'$ (9.20.1)

$$\alpha:\rho(X,\mathfrak{X})\to(X',R_{\mathfrak{X}'}).\tag{13.22.2}$$

Let $(M,\nabla,M^\bullet)$ be an object of $\mathrm{MIC}_{\mathrm{F}}^{p-1}(\mathfrak{X}_n/\mathscr{S}_n)$. Recall that the morphism α induces a functorial isomorphism of $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$ (9.23.1)

$$\alpha^*(\pi_R^*(\varepsilon_{\widetilde{M}})):(F_{2,\mathfrak{X}}^*(\widetilde{M}),\nabla_{F_2})\xrightarrow{\sim}(F_{1,\mathfrak{X}}^*(\widetilde{M}),\nabla_{F_1}),\tag{13.22.3}$$

such that $\eta_{F_2}=\eta_{F_1}\circ\alpha^*(\pi_R^*(\varepsilon_{\widetilde{M}}))$ (9.22). In view of the proof of 13.20, a family of divided Frobenius morphisms $\{\varphi_{F_j}^i\}_{i\leqslant p-1}$ is equivalent to a horizontal morphism φ_{F_j} (13.19.2) for $j=1,2$. Then the functor (13.22.1) is given by

$$
\begin{aligned}
&\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{1,\mathfrak{X}})\to\mathbf{MF}_{\mathrm{big}}(\mathfrak{X};F_{2,\mathfrak{X}})\\
&(M,\nabla,M^\bullet,\varphi_{F_1})\mapsto(M,\nabla,M^\bullet,\varphi_{F_1}\circ\alpha^*(\pi_R^*(\varepsilon_{\widetilde{M}}))).
\end{aligned}
\tag{13.22.4}
$$

Let us describe the isomorphism (13.22.3) in terms of a system of local coordinates. Assume that there exists an étale $\mathscr{S}$-morphism $f:\mathfrak{X}\to\widehat{\mathbb{A}}^d_{\mathscr{S}}=\mathrm{Spf}(\mathrm{W}\{T_1,\dots,T_d\})$ and put t_i the image of T_i in $\mathscr{O}_{\mathfrak{X}}$ and $\xi_i=1\otimes t_i-t_i\otimes 1\in\mathscr{O}_{\mathfrak{X}^2}$. Let i be an integer $\in[0,p-1]$, m an element of M^i and $(m)_i$ its image in $\widetilde{M}$. We have (13.5.6)

(13.22.5)

$$\varepsilon_{\widetilde{M}}(1\otimes(m)_i)=\sum_{|I|\leqslant i}\frac{1}{|I|!}\big(\nabla_{\partial^I}(m)\big)_{i-|I|}\otimes\left(\frac{\xi}{p}\right)^I+\sum_{|I|>i}\frac{p^{|I|-i}}{|I|!}\big(\nabla_{\partial^I}(m)\big)_0\otimes\left(\frac{\xi}{p}\right)^I.$$

Recall that the morphism $\alpha:\mathfrak{X}\to\mathscr{R}_{\mathfrak{X}'}$ (13.22.2) induces a homomorphism $a:\mathscr{R}_{\mathfrak{X}'}\to\mathscr{O}_{\mathfrak{X}}$ (9.20.2) which sends $\frac{\xi_i'}{p}$ to $\frac{F^*_{2,\mathfrak{X}}(t_i)-F^*_{1,\mathfrak{X}}(t_i)}{p}$ (9.20.3). We deduce that

(13.22.6)

$$\begin{aligned}\alpha^*(\pi_R^*(\varepsilon_{\widetilde{M}}))(1\otimes_{F_2}(m)_i)=&\sum_{|I|\leqslant i}\left(\frac{F^*_{2,\mathfrak{X}}(\underline{t})-F^*_{1,\mathfrak{X}}(\underline{t})}{p}\right)^I\bigotimes_{F_1}\frac{1}{|I|!}\big(\nabla_{\partial^I}(m)\big)_{i-|I|}\\&+\sum_{|I|>i}\left(\frac{F^*_{2,\mathfrak{X}}(\underline{t})-F^*_{1,\mathfrak{X}}(\underline{t})}{p}\right)^I\bigotimes_{F_1}\frac{p^{|I|-i}}{|I|!}\big(\nabla_{\partial^I}(m)\big)_0,\end{aligned}$$

where

$$\left(\frac{F^*_{2,\mathfrak{X}}(\underline{t})-F^*_{1,\mathfrak{X}}(\underline{t})}{p}\right)^I=\prod_{j=1}^d\left(\frac{F^*_{2,\mathfrak{X}}(t_j)-F^*_{1,\mathfrak{X}}(t_j)}{p}\right)^{i_j}\qquad\forall\ I=(i_1,\dots,i_d)\in\mathbb{N}^d.$$

We review some basic properties about Fontaine modules. The following result is fisrt showed in **[13]** and we refer to **[29]** for another approach.

Proposition 13.23 (**[13]** 2.1; **[29]** 5.3.3). – *Suppose that there exists a σ-lifting $F_{\mathfrak{X}}:\mathfrak{X}\to\mathfrak{X}$ of the Frobenius morphism. Let $(M,\nabla,M^\bullet,\varphi)$ be an object of* $\mathbf{MF}(\mathfrak{X};F_{\mathfrak{X}})$ (13.18).

(i) *Then each M^i is locally a direct sum of sheaves of the form $\mathscr{O}_{\mathfrak{X}_n}$; each morphism $M^i\to M^{i-1}$ locally split. In particular, $(M,\nabla,M^\bullet)$ forms a T-crystal* (13.13).

(ii) *Any morphism of $\mathbf{MF}(\mathfrak{X};F_{\mathfrak{X}})$ is strictly compatible with the filtrations (***[9]** *1.1.5).*

(iii) *The category $\mathbf{MF}(\mathfrak{X};F_{\mathfrak{X}})$ is abelian.*

Then we deduce the corresponding statement for (global) Fontaine modules.

Corollary 13.24. – (i) *For every object $(M,\nabla,M^\bullet,\varphi)$ of $\mathbf{MF}(\mathfrak{X})$, $(M,\nabla,M^\bullet)$ forms a T-crystal.*

(ii) *Any morphism of $\mathbf{MF}(\mathfrak{X})$ is strictly compatible with the filtrations.*

(iii) *The category $\mathbf{MF}(\mathfrak{X})$ is abelian.*

Proof. – Assertions (i) and (ii) being local, they follow from 13.20 and 13.23.

For any morphism $f : (M_1, \nabla_1, M_1^\bullet, \varphi_1) \to (M_2, \nabla_2, M_2^\bullet, \varphi_2)$ of p^n-torsion objects of $\mathbf{MF}(\mathfrak{X})$, we denote by (L, ∇_L) and (N, ∇_N) the kernel and the cokernel of f in $\mathrm{MIC}^{\mathrm{qn}}(\mathfrak{X}_n/\mathscr{S}_n)$. We denote by $L^\bullet$ (resp. $N^\bullet$) the filtration on L (resp. N) induced by $M_1^\bullet$ (resp. $M_2^\bullet$) ([**9**] 1.1.8). Since f is strictly compabitlbe, for any $i < p$, we have $f(M_1^i) = f(M_1) \cap M_2^i$ ([**9**] 1.1.11) and an exact sequence $0 \to L^i \to M_1^i \to M_2^i \to N^i \to 0$. By the snake lemma, we deduce an exact sequence

$$0 \to \widetilde{L} \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{N} \to 0.$$

Then we deduce a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \nu(\mathrm{C}^*(\widetilde{\mathscr{L}'})) \arrow[r] & \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'_1})) \arrow[r] \arrow[d, "\varphi_1"', "\wr"] & \nu(\mathrm{C}^*(\widetilde{\mathscr{M}'_2})) \arrow[r] \arrow[d, "\wr"', "\varphi_2"] & \nu(\mathrm{C}^*(\widetilde{\mathscr{N}'})) \arrow[r] & 0 \\
0 \arrow[r] & (L, \nabla_L) \arrow[r] & (M_1, \nabla_1) \arrow[r] & (M_2, \nabla_2) \arrow[r] & (N, \nabla_N) \arrow[r] & 0.
\end{tikzcd}$$

Hence we can define $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ in $\mathbf{MF}(\mathfrak{X})$. We deduce that the category $\mathbf{MF}(\mathfrak{X})$ is abelian. □

13.25. – In the following, we compare categories $\mathbf{MF}(\mathfrak{X}; \iota, F_{\mathfrak{Y}})$ with respect to different choice of data $(\iota : \mathfrak{X} \to \mathfrak{Y}, F_{\mathfrak{Y}})$ following Tsuji [**35**].

Let $(\iota_1 : \mathfrak{X} \to \mathfrak{Y}_1, F_{\mathfrak{Y}_1})$ and $(\iota_2 : \mathfrak{X} \to \mathfrak{Y}_2, F_{\mathfrak{Y}_2})$ be two data as in 13.14 and suppose that there exists a *smooth* $\mathscr{S}$-morphism $g : \mathfrak{Y}_2 \to \mathfrak{Y}_1$ compatible with ι_1, ι_2 and the Frobenius morphisms $F_{\mathfrak{Y}_1}$ and $F_{\mathfrak{Y}_2}$. Then g induces a PD-morphism $g_D : \mathfrak{D}_2 \to \mathfrak{D}_1$ compatible with $F_{\mathfrak{D}_2}$ and $F_{\mathfrak{D}_1}$. Note that g_D induces an isomorphism on the underlying topological spaces.

LEMMA 13.26 ([**35**] 2.2.2). – *Let x be a point of $\mathfrak{X}_n$ and let $t_1, \dots, t_d$ be a family of local sections of $\mathscr{O}_{\mathfrak{Y}_{2,n}}$ in a neighborhood of $\iota_2(x)$ such that $\{dt_1, \dots, dt_d\}$ form a basis of $\Omega^1_{\mathfrak{Y}_{2,n}/\mathfrak{Y}_{1,n},x}$ and that $\iota_2^*(t_i) = 0$ (the existence follows from the fact that ι_1 is a closed immersion). Then there exists an $\mathscr{O}_{\mathfrak{D}_{1,n},x}$-PD-isomorphism*

$$\mathscr{O}_{\mathfrak{D}_{1,n},x}\langle T_1, \dots, T_d\rangle \xrightarrow{\sim} \mathscr{O}_{\mathfrak{D}_{2,n},x}, \tag{13.26.1}$$

which sends T_i to t_i.

PROPOSITION 13.27 ([**35**] Proof of 2.2.1). – *Keep the assumption of 13.25. The morphism $g : \mathfrak{Y}_2 \to \mathfrak{Y}_1$ induces equivalences of categories quasi-inverse to each other:*

$$\begin{aligned} g^* &: \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \iota_1, F_{\mathfrak{Y}_1}) \to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \iota_2, F_{\mathfrak{Y}_2}), \\ g_* &: \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \iota_2, F_{\mathfrak{Y}_2}) \to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \iota_1, F_{\mathfrak{Y}_1}). \end{aligned} \tag{13.27.1}$$

We present the construction of pull-back functor and we refer to [**35**] for the construction of the push-forward functor. Let $\mathfrak{M} = (M_1, \nabla_1, M_1^\bullet, \varphi_1^\bullet)$ be a p^n-torsion object of $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \iota_1, F_{\mathfrak{Y}_1})$ and $(\mathscr{M}, \mathscr{M}^\bullet)$ the associated proto-T-crystal.

The evaluation $(M_2, \nabla_2, M_2^\bullet)$ of $(\mathscr{M}, \mathscr{M}^\bullet)$ at $\mathfrak{D}_2$ is given by $M_2 = \mathscr{O}_{\mathfrak{D}_{2,n}} \otimes_{\mathscr{O}_{\mathfrak{D}_{1,n}}} M_1$ and for $r \in \mathbb{Z}$, $M_2^r = \sum_{r_1 \geqslant 0, r_1 + r_2 = r} \mathrm{Im}(J^{[r_1]}_{\mathfrak{D}_{2,n}} \otimes_{\mathscr{O}_{\mathfrak{D}_{1,n}}} M_1^{r_2} \to M_2)$ (13.11).

The connection $\nabla_2 : M_2 \to M_2 \otimes_{\mathscr{O}_{\mathfrak{Y}_{2,n}}} \Omega^1_{\mathfrak{Y}_{2,n}/\mathscr{S}_n}$ is defined, for any local sections a of $\mathscr{O}_{\mathfrak{D}_{2,n}}$ and m of M, by

$$\nabla_2(a \otimes m) = a \cdot g^*(\nabla_1(m)) + m \otimes \nabla_{\mathfrak{D}_{2,n}}(a). \tag{13.27.2}$$

Let x be a point of $\mathfrak{X}$. With the notation and assumption of 13.26, for any $I = (i_1, \dots, i_d) \in \mathbb{N}^d$, we set $\underline{T}^{[I]} = \prod_{j=1}^d T_j^{[i_j]}$. In view of 13.15(i-c) and (13.26.1), $M^r_{2,x}$ can be written as a direct sum of $\mathscr{O}_{\mathfrak{D}_{1,n},x}$-modules $M^r_{2,x} = \bigoplus_{|I|=s} M^{r-s}_{1,x} \cdot \underline{T}^{[I]}$. For any $r < p$, we deduce that the semi-linear morphisms

$$\varphi^{r_1}_{J_{\mathfrak{D}_{2,n}}} \otimes \varphi^{r_2}_1 : J^{[r_1]}_{\mathfrak{D}_{2,n}} \otimes_{\mathscr{O}_{\mathfrak{D}_{1,n}}} M^{r_2}_1 \to M_2, \qquad r_1 + r_2 = r,\ 0 \leqslant r_1 \leqslant p-1$$

are compatible and induce a family of divided Frobenius morphisms with respect to $F_{\mathfrak{D}_2}$:

$$\varphi^r_2 : M^r_2 \to M_2. \tag{13.27.3}$$

Conditions 13.15(i a-c) follow from those of $\varphi^\bullet_1$ and of $\varphi^\bullet_{\mathfrak{D}_{2,n}}$.

Remark 13.28. – By (13.27.2), the morphism g induces a morphism of de Rham complexes

$$M_1 \otimes_{\mathscr{O}_{\mathfrak{Y}_{1,n}}} \Omega^\bullet_{\mathfrak{Y}_{1,n}/\mathscr{S}_n} \to M_2 \otimes_{\mathscr{O}_{\mathfrak{Y}_{2,n}}} \Omega^\bullet_{\mathfrak{Y}_{2,n}/\mathscr{S}_n}. \tag{13.28.1}$$

For any $r \leqslant p-1$, we have a commutative diagram (13.15.2)

$$\begin{tikzcd}
(M_1^{r-\bullet} \otimes_{\mathscr{O}_{\mathfrak{Y}_{1,n}}} \Omega^\bullet_{\mathfrak{Y}_{1,n}/\mathscr{S}_n}) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \arrow[r] \arrow[d] & M_1 \otimes_{\mathscr{O}_{\mathfrak{Y}_{1,n}}} \Omega^\bullet_{\mathfrak{Y}_{1,n}/\mathscr{S}_n} \arrow[d] \\
(M_2^{r-\bullet} \otimes_{\mathscr{O}_{\mathfrak{Y}_{2,n}}} \Omega^\bullet_{\mathfrak{Y}_{2,n}/\mathscr{S}_n}) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \arrow[r] & M_2 \otimes_{\mathscr{O}_{\mathfrak{Y}_{2,n}}} \Omega^\bullet_{\mathfrak{Y}_{2,n}/\mathscr{S}_n}.
\end{tikzcd} \tag{13.28.2}$$

Lemma 13.29 ([**35**] 2.3.2). – *The functor* g^* (13.27.1) *sends strongly divisible objects to strongly divisible objects* (13.17).

Proof. – The canonical morphisms $\mathscr{O}_{\mathfrak{D}_{2,n}} \otimes_{\mathscr{O}_{\mathfrak{D}_{1,n}}} M^r_1 \to M^r_2$ induce an $\mathscr{O}_{\mathfrak{D}_n}$-linear morphism (13.16)

$$u_g : g^*_D(\widetilde{\mathfrak{M}}) \to \widetilde{g^*(\mathfrak{M})}. \tag{13.29.1}$$

In view of condition 13.16(ii), the above morphism is surjective. We have a commutative diagram:

$$\begin{tikzcd}
g^*_D(\widetilde{\mathfrak{M}}) \arrow[r, two heads, "u_g"] \arrow[d, "g^*_D(\varphi_{\mathfrak{M}})"'] & \widetilde{g^*(\mathfrak{M})} \arrow[d, "\varphi_{g^*(\mathfrak{M})}"] \\
g^*_D(M_1) \arrow[r, equal] & M_2.
\end{tikzcd} \tag{13.29.2}$$

If $\varphi_{\mathfrak{M}}$ is an isomorphism, then u_g is an isomorphism and so is $\varphi_{g^*(\mathfrak{M})}$. The lemma follows. $\square$

In the end, we construct a natural functor from $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X})$ to the category of Fontaine modules with respect to the diagonal immersion and two liftings of Frobenius morphism on $\mathfrak{X}$.

Proposition 13.30. – *Suppose that $\mathfrak{X}$ is separated over $\mathscr{S}$. Take again the notation of 13.22 and let $\Delta : \mathfrak{X} \to \mathfrak{X}^2$ be the diagonal immersion, $q_1, q_2 : \mathfrak{X}^2 \to \mathfrak{X}$ the canonical projections and $F_{\mathfrak{X}^2} = (F_{1,\mathfrak{X}}, F_{2,\mathfrak{X}}) : \mathfrak{X}^2 \to \mathfrak{X}^2$. Then the diagram*

(13.30.1)
$$\begin{tikzcd} \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}) \arrow[r, "\lambda_{F_1}"] \arrow[d, "\lambda_{F_2}"'] & \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{1,\mathfrak{X}}) \arrow[d, "q_1^*"] \\ \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{2,\mathfrak{X}}) \arrow[r, "q_2^*"] & \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \Delta, F_{\mathfrak{X}^2}) \end{tikzcd}$$

is commutative up to a canonical isomorphism.

Proof. – Let $\mathfrak{M} = (M, \nabla, M^\bullet, \varphi)$ be a Fontaine module over $\mathfrak{X}$ and $\varphi_{F_i} = \varphi \circ \eta_{F_i} : F_{i,\mathfrak{X}}^*(\widetilde{M}) \to M$ (13.19.2) for $i = 1, 2$. Then we have $\lambda_{F_i}(\mathfrak{M}) = (M, \nabla, M^\bullet, \varphi_{F_i}) \in \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{i,\mathfrak{X}})$. We denote abusively by q_i the composition $P_{\mathfrak{X}} \to \mathfrak{X}^2 \xrightarrow{q_i} \mathfrak{X}$ and we set $\mathfrak{M}_i = q_i^*(\lambda_{F_i}(\mathfrak{M})) = (q_i^*(M), \nabla_i, A_i^\bullet, \varphi_{\mathfrak{M}_i}^\bullet) \in \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \Delta, F_{\mathfrak{X}^2})$.

Let $\varepsilon : q_2^*(M) \xrightarrow{\sim} q_1^*(M)$ be the $\mathscr{P}_{\mathfrak{X}}$-stratification on M. By 13.12, it is a filtered isomorphism with respect to the filtrations $A_2^\bullet$ and $A_1^\bullet$. It remains to show that ε is compatible with the divided Frobenius morphisms $\varphi_{\mathfrak{M}_i}^\bullet$ on both sides.

Since $\varphi : \mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,\mathfrak{X})} \to M$ is a horizontal morphism, the following diagram commutes

(13.30.2)
$$\begin{tikzcd} q_2^*(\mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,\mathfrak{X})}) \arrow[r, "\sim"] \arrow[d, "q_2^*(\varphi)"'] & \mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,P_{\mathfrak{X}})} & q_1^*(\mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,\mathfrak{X})}) \arrow[l, "\sim"'] \arrow[d, "q_1^*(\varphi)"] \\ q_2^*(M) \arrow[rr, "\varepsilon", "\sim"'] & & q_1^*(M). \end{tikzcd}$$

We have $\varphi_{F_i} = \eta_{F_i} \circ \varphi$ and the following commutative diagrams:
(13.30.3)
$$\begin{tikzcd} q_2^*(F_{2,\mathfrak{X}}^*(\widetilde{M})) \arrow[r, "q_2^*(\eta_{F_2})", "\sim"'] \arrow[dr, "q_2^*(\varphi_{F_2})"'] & q_2^*(\mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,\mathfrak{X})}) \arrow[d, "q_2^*(\varphi)"] \\ & q_2^*(M), \end{tikzcd} \qquad \begin{tikzcd} q_1^*(F_{1,\mathfrak{X}}^*(\widetilde{M})) \arrow[r, "q_1^*(\eta_{F_1})", "\sim"'] \arrow[dr, "q_1^*(\varphi_{F_1})"'] & q_1^*(\mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,\mathfrak{X})}) \arrow[d, "q_1^*(\varphi)"] \\ & q_1^*(M). \end{tikzcd}$$

The filtered isomorphism $\varepsilon : A_2^\bullet \xrightarrow{\sim} A_1^\bullet$ induces an isomorphism $\widetilde{\varepsilon} : \widetilde{\mathfrak{M}_2} \xrightarrow{\sim} \widetilde{\mathfrak{M}_1}$ (13.16). In the diagrams

(13.30.4)

$$\begin{tikzcd} q_2^*(F_{2,\mathfrak{X}}^*(\widetilde{M})) \arrow[r, "\sim"] \arrow[d, "u_{q_2}"] \arrow[dd, bend right=60, "q_2^*(\varphi_{F_2})"'] & \mathrm{C}^*(\widetilde{\mathscr{M}'})_{(X,P_{\mathfrak{X}})} & q_1^*(F_{1,\mathfrak{X}}^*(\widetilde{M})) \arrow[l, "\sim"'] \arrow[d, "u_{q_1}"'] \arrow[dd, bend left=60, "q_1^*(\varphi_{F_1})"] \\ \widetilde{\mathfrak{M}_2} \arrow[rr, "\widetilde{\varepsilon}"] \arrow[d, "\varphi_{\mathfrak{M}_2}"] & (1) \quad (2) & \widetilde{\mathfrak{M}_1} \arrow[d, "\varphi_{\mathfrak{M}_1}"'] \\ q_2^*(M) \arrow[rr, "\varepsilon"] & & q_1^*(M) \end{tikzcd}$$

the left-hand side diagram and the right-hand side diagram are commutative (13.29.2).

To prove the assertion, it suffices to show that diagram (2) is commutative. By (13.30.2) and (13.30.3), the outer diagram of (13.30.4) commutes. Since u_{q_i} is surjective (13.29.1), it suffices to prove the following lemma. □

Lemma 13.31. – *Diagram* (1) *of* (13.30.4) *is commutative.*

Proof. – Recall (9.19) that F_1, F_2 induce a $\mathscr{S}$-morphism $Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$. We denote the composition $\rho(X, P_{\mathfrak{X}}) \to \rho(X, Q_{\mathfrak{X}}) \to (X', R_{\mathfrak{X}'})$ of morphisms of $\mathscr{E}'$ (9.1.2) by f. It fits into the following commutative diagram:

$$\begin{tikzcd} \rho(X,\mathfrak{X}) \arrow[d, "F_2"'] & \rho(X, P_{\mathfrak{X}}) \arrow[l, "\rho(q_2)"'] \arrow[r, "\rho(q_1)"] \arrow[d, "f"'] & \rho(X,\mathfrak{X}) \arrow[d, "F_1"] \\ (X', \mathfrak{X}') & (X', R_{\mathfrak{X}'}) \arrow[l, "q_2'"'] \arrow[r, "q_1'"] & (X', \mathfrak{X}'). \end{tikzcd}$$

Hence the composition $q_2^*(F_{2,\mathfrak{X}}^*(\widetilde{M})) \xrightarrow{\sim} q_1^*(F_{1,\mathfrak{X}}^*(\widetilde{M}))$ of the upper arrows of diagram (1) coincides with the pull-back of the $\mathscr{R}_{\mathfrak{X}}$-stratification $\varepsilon_{\widetilde{M}}$ on $\widetilde{M}$ (13.5) via the composition $P_{\mathfrak{X}} \xrightarrow{f} R_{\mathfrak{X}'} \xrightarrow{\pi_R} R_{\mathfrak{X}}$ (13.6):

(13.31.1) $$f^*(\pi_R^*(\varepsilon_{\widetilde{M}})) : \mathscr{P}_{\mathfrak{X}} \otimes_{\mathscr{R}_{\mathfrak{X}}} (\mathscr{R}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widetilde{M}) \xrightarrow{\sim} \mathscr{P}_{\mathfrak{X}} \otimes_{\mathscr{R}_{\mathfrak{X}}} (\widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \mathscr{R}_{\mathfrak{X}}).$$

To show the lemma, we may suppose that there exists an étale morphism $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathscr{S}}$ and we take again the notation of 13.22. For $1 \leqslant k \leqslant d$, we set $F_1^*(t'_k) = t_k^p + pa_k$ and $F_2^*(t'_k) = t_k^p + pb_k$. By (9.19.3), the homomorphism $\mathscr{R}_{\mathfrak{X}'} \to \mathscr{P}_{\mathfrak{X}}$ induced by f sends

(13.31.2) $$\left(\frac{\xi'_k}{p}\right) \mapsto z_k = (p-1)!\xi_k^{[p]} + \sum_{j=1}^{p-1} \frac{(p-1)!}{j!(p-j)!} \xi_k^j (t_k \otimes 1)^{p-j} + (1 \otimes b_k - a_k \otimes 1).$$

For any multi-index $I = (i_1, \dots, i_d)$, we set $z^I = \prod_{k=1}^d z_k^{i_k}$. Let i be an integer $\in [0, p-1]$, m a local section of M^i and $(m)_i$ its image in $\widetilde{M}$. By (13.22.5) and

(13.31.2), the isomorphism (13.31.1) sends
(13.31.3)
$$1\otimes(1\otimes_{q_2}(m)_i)\mapsto \sum_{|I|\leqslant i}\frac{z^I}{I!}\otimes\big((\nabla_{\partial^I}(m))_{i-|I|}\otimes_{q_1}1\big)+\sum_{|I|>i}\frac{p^{|I|-i}z^I}{I!}\otimes\big((\nabla_{\partial^I}(m))_0\otimes_{q_1}1\big).$$

The $\mathscr{P}_{\mathfrak{X}}$-linear morphism u_{q_2} (resp. u_{q_1}) (13.29.1) sends
(13.31.4)
$$1\otimes(1\otimes_{q_2}(m)_i)\mapsto\big(1\otimes_{F_{P_{\mathfrak{X}}}}(1\otimes_{q_2}m)\big)_i\qquad(\text{resp. } 1\otimes((m)_i\otimes_{q_1}1)\mapsto\big(1\otimes_{F_{P_{\mathfrak{X}}}}(m\otimes_{q_1}1)\big)_i,$$
where $(-)_i : F^*_{P_{\mathfrak{X}}}(A^i_j)\to\widetilde{\mathfrak{M}_j}$ denotes the canonical morphism for $j=1,2$.

On the other hand, the isomorphism $\varepsilon : A^i_2\xrightarrow{\sim}A^i_1$ sends $1\otimes m$ to $\sum_I\nabla_{\partial^I}(m)\otimes\xi^{[I]}$. Hence the isomorphism $\widetilde{\varepsilon}:\widetilde{\mathfrak{M}_2}\xrightarrow{\sim}\widetilde{\mathfrak{M}_1}$ in diagram (1) sends

$$\big(1\otimes_{F_{P_{\mathfrak{X}}}}(1\otimes_{q_2}m)\big)_i\mapsto\sum_I\big(1\otimes_{F_{P_{\mathfrak{X}}}}(\nabla_{\partial^I}(m)\otimes_{q_1}\xi^{[I]})\big)_i. \tag{13.31.5}$$

With the notation of 13.14, the divided Frobenius morphisms on $(\mathscr{O}_{P_{\mathfrak{X}_n}},J^{[\bullet]}_{P_{\mathfrak{X}_n}})$ satisfies
(13.31.6)
$$\varphi_{P_{\mathfrak{X}_n}}(\xi_k)=pz_k,\ \forall\ 1\leqslant k\leqslant d,\qquad \varphi^i_{P_{\mathfrak{X}_n}}(\xi^{[I]})=\frac{p^{|I|-i}z^I}{I!},\quad\forall\ i<p,|I|\geqslant i.$$
By conditions (i) and (ii) of 13.16, we deduce that

$$(1\otimes_{F_{P_{\mathfrak{X}}}}(\nabla_{\partial^I}(m)\otimes\xi^{[I]}))_i=\begin{cases}(\frac{z^I}{I!}\otimes_{F_{P_{\mathfrak{X}}}}(\nabla_{\partial^I}(m)\otimes1))_{i-|I|} & \text{if } |I|\leqslant i,\\ (\frac{p^{|I|-i}z^I}{I!}\otimes_{F_{P_{\mathfrak{X}}}}(\nabla_{\partial^I}(m)\otimes1))_0 & \text{if } |I|>i.\end{cases} \tag{13.31.7}$$

By comparing (13.31.3), (13.31.4), (13.31.5) and (13.31.7), the assertion follows. □

CHAPTER 14

THE FONTAINE MODULE STRUCTURE ON THE CRYSTALLINE COHOMOLOGY OF A FONTAINE MODULE

To illustrates our definition of Fontaine modules, we reprove the following result of Faltings on the crystalline cohomology of a Fontaine module.

THEOREM 14.1 ([**13**] IV 4.1). – *Let $\mathfrak{X}$ be a smooth proper formal $\mathscr{S}$-scheme of relative dimension d and $(M, \nabla, M^\bullet, \varphi)$ a p^n-torsion object of* $\mathbf{MF}(\mathfrak{X})$ (13.7) *of length $\leqslant \ell \leqslant p-1$ (i.e., $M^{\ell+1}=0$). We denote by F^i the subcomplex $M^{i-\bullet} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n}$ of the de Rham complex $M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n}$.*

(i) *Let m be an integer such that $\min\{m, d\} + \ell \leqslant p-1$ and i an integer $\leqslant p-1$. Then the canonical morphism $\mathbb{H}^m(\mathrm{F}^i) \to \mathbb{H}^m(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n})$ is injective.*

(ii) *Let m be an integer such that $\min\{m, d-1\} + \ell \leqslant p-2$ and i an integer $\leqslant p-1$. The isomorphism φ induces a family of semi-linear morphisms $\phi^{m,i}_{\mathrm{H}} : \mathbb{H}^m(\mathrm{F}^i) \to \mathbb{H}^m(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n})$ (with respect to σ). Then the data*

$$(\mathbb{H}^m(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n}), (\mathbb{H}^m(\mathrm{F}^i))_{i=0}^{p-1}, (\phi^{m,i}_{\mathrm{H}})_{i=0}^{p-1})$$

form an object of $\mathbf{MF}(\mathscr{S})$ (13.21).

(iii) *In the spectral sequence of the filtered de Rham complex $(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n}, \mathrm{F}^i)$* ([**9**] 1.4.5)

$$\mathrm{E}_1^{r,s} = \mathbb{H}^{r+s}(\mathrm{gr}^r_F(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n})) \Rightarrow \mathbb{H}^{r+s}(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n}), \tag{14.1.1}$$

the differential morphism $d_1^{r,s}$ vanishes for $\min\{r+s, d-1\} + \ell \leqslant p-2$.

The data $(\mathscr{O}_{\mathfrak{X}_n}, d)$ defines a Fontaine module of length 0 over $\mathfrak{X}$. By 14.1(iii), we deduce that:

COROLLARY 14.2 ([**16**] 2.8; [**13**] IV 4.1). – *If $d \leqslant p-1$, the Hodge to de Rham spectral sequence of $\mathfrak{X}_n/\mathscr{S}_n$ degenerates at E_1.*

We defer the proof of the theorem to 14.18 and we begin with some preparations on crystalline cohomology of T-cyrstals (13.13) following Ogus [**29**].

14.3. – Let $\iota : \mathscr{X} \to \mathscr{Y}$ a closed immersion of smooth $\mathscr{S}_n$-schemes and $\mathscr{D}$ the PD-envelope of ι compatible with γ. Let $(\mathscr{M}, \mathscr{M}^\bullet)$ a T-crystal of $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}$ (13.13) and (M, ∇) (resp. $M^\bullet$) the associated $\mathscr{O}_{\mathscr{D}}$-module with integrable connection (resp. filtration) (13.12). Since $M^\bullet$ satisfy Griffiths' transversality, it induces a filtration on the de Rham complex $M \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^\bullet_{\mathscr{Y}/\mathscr{S}_n}$ defined for every $i \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geqslant 0}$ by

$$\mathrm{F}^i(M \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^q_{\mathscr{Y}/\mathscr{S}_n}) = M^{i-q} \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^q_{\mathscr{Y}/\mathscr{S}_n}. \tag{14.3.1}$$

Since M is an $\mathscr{O}_{\mathscr{D}}$-module, the de Rham complex $M \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^\bullet_{\mathscr{Y}/\mathscr{S}_n}$ is concentrated on $\mathscr{X}$.

Ogus showed that the above subcomplex computes the crystalline cohomology of $\mathscr{M}^i$.

Theorem 14.4 ([**29**] 6.1.1). – *We keep the above notation and we denote by $u_{\mathscr{X}/\mathscr{S}_n}$ the canonical morphism of topoi $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}} \to \mathscr{X}_{\mathrm{zar}}$.*

(i) *There exists an isomorphism in the derived category $\mathrm{D}(\mathscr{X}_{\mathrm{zar}}, \mathrm{W}_n)$*

$$\mathrm{R}\, u_{\mathscr{X}/\mathscr{S}_n *}(\mathscr{M}) \xrightarrow{\sim} \mathscr{M}_{\mathscr{D}} \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^\bullet_{\mathscr{Y}/\mathscr{S}_n}. \tag{14.4.1}$$

(ii) *For every i, there exists an isomorphism in $\mathrm{D}(\mathscr{X}_{\mathrm{zar}}, \mathrm{W}_n)$ compatible with* (14.4.1)

$$\mathrm{R}\, u_{\mathscr{X}/\mathscr{S}_n *}(\mathscr{M}^i) \xrightarrow{\sim} \mathrm{F}^i(\mathscr{M}_{\mathscr{D}} \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^\bullet_{\mathscr{Y}/\mathscr{S}_n}). \tag{14.4.2}$$

Corollary 14.5 ([**29**] 6.1.7). – *Let $\iota_1 : \mathscr{X} \to \mathscr{Y}_1$ and $\iota_2 : \mathscr{X} \to \mathscr{Y}_2$ be two closed $\mathscr{S}_n$-immersions of $\mathscr{X}$ into smooth $\mathscr{S}_n$-schemes, $\mathscr{D}_1$, $\mathscr{D}_2$ the PD-envelopes of ι_1, ι_2 compatible with γ and $f : \mathscr{Y}_2 \to \mathscr{Y}_1$ an $\mathscr{S}_n$-morphism such that $\iota_2 = f \circ \iota_1$. Then the morphisms of complexes induced by f*

$$\mathscr{M}_{\mathscr{D}_1} \otimes_{\mathscr{O}_{\mathscr{Y}_1}} \Omega^\bullet_{\mathscr{Y}_1/\mathscr{S}_n} \to \mathscr{M}_{\mathscr{D}_2} \otimes_{\mathscr{O}_{\mathscr{Y}_2}} \Omega^\bullet_{\mathscr{Y}_2/\mathscr{S}_n},$$
$$\mathrm{F}^i(\mathscr{M}_{\mathscr{D}_1} \otimes_{\mathscr{O}_{\mathscr{Y}_1}} \Omega^\bullet_{\mathscr{Y}_1/\mathscr{S}_n}) \to \mathrm{F}^i(\mathscr{M}_{\mathscr{D}_2} \otimes_{\mathscr{O}_{\mathscr{Y}_2}} \Omega^\bullet_{\mathscr{Y}_2/\mathscr{S}_n}),\ \forall\, i \in \mathbb{Z},$$

are quasi-isomorphisms and compatible with (14.4.1) *and* (14.4.2).

14.6. – We suppose that $\mathscr{X}$ is separated over $\mathscr{S}_n$ and we take a Zariski covering $\mathscr{U} = \{U_i\}_{i \in I}$ of $\mathscr{X}$ consisting of affine schemes. For any integer $r \geqslant 0$ and any element $J = (j_0, j_1, \ldots, j_r)$ of I^{r+1}, we denote by U^J the intersection $\bigcap_{i=0}^r U_{j_i}$, by U_J the product $\{U_{j_i}\}_{i=0}^r$ over W_n and by P_J the PD-envelope of the diagonal immersion $U^J \to U_J$ compatible with γ. Note that U^J and P_J are also affine. Then we obtain two compatible simplicial objects:

$$\textstyle\coprod_{i \in I} U_i \underset{s}{\overset{d}{\leftleftarrows\!\!\rightarrow}} \coprod_{J \in I^2} U^J \underset{s}{\overset{d}{\Lleftarrow\!\!\rightrightarrows}} \coprod_{J \in I^3} U^J \cdots \tag{14.6.1}$$

$$\textstyle\coprod_{i \in I} U_i \underset{s}{\overset{d}{\leftleftarrows\!\!\rightarrow}} \coprod_{J \in I^2} U_J \underset{s}{\overset{d}{\Lleftarrow\!\!\rightrightarrows}} \coprod_{J \in I^3} U_J \cdots, \tag{14.6.2}$$

where d, s denote the faces and degeneracy morphisms. Then the faces and degeneracy morphisms of (14.6.2) induce a simplicial object compatible with (14.6.1):

$$\coprod_{i\in I} U_i \overset{d}{\underset{s}{\leftleftarrows\!\!\rightarrow}} \coprod_{J\in I^2} P_J \overset{d}{\underset{s}{\leftleftarrows\!\!\rightrightarrows}} \coprod_{J\in I^3} P_J \cdots . \tag{14.6.3}$$

Let $(\mathscr{M}, \mathscr{M}^{\bullet})$ be a T-crystal of $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}$. We associate to $(\mathscr{M}, \mathscr{M}^{\bullet})$ a bicomplex $C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})$ and for any $i \in \mathbb{Z}$, a bicomplex $\mathrm{F}^i\big(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\big)$ by setting

$$C^{r,s}_{\mathscr{U}}(\mathscr{M}) = \bigoplus_{J\in I^{r+1}} \Gamma(U^J, \mathscr{M}_{P_J} \otimes_{\mathscr{O}_{U_J}} \Omega^s_{U_J/\mathscr{S}_n}), \quad r, s \geqslant 0 \tag{14.6.4}$$

$$\mathrm{F}^i\big(C^{r,s}_{\mathscr{U}}(\mathscr{M})\big) = \bigoplus_{J\in I^{r+1}} \Gamma(U^J, \mathscr{M}^{i-s}_{P_J} \otimes_{\mathscr{O}_{U_J}} \Omega^s_{U_J/\mathscr{S}_n}), \quad r, s \geqslant 0, \tag{14.6.5}$$

the horizontal differential morphism $\partial_1^{r,s}$ is the alternating sum of the resctriction morphisms induced by the faces morphisms (14.6.3) and the vertical differential morphism $\partial_2^{r,s}$ is given by the connection on $\mathscr{M}_{P_J}$.

Proposition 14.7 ([**13**] IV a)). – *There exist canonical isomorphisms of cohomology groups:*

$$\begin{aligned} \mathrm{H}^{\bullet}((\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}, \mathscr{M}) &\xrightarrow{\sim} \mathrm{H}^{\bullet}(\mathrm{Tot}(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M}))) \\ \mathrm{H}^{\bullet}((\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}, \mathscr{M}^i) &\xrightarrow{\sim} \mathrm{H}^{\bullet}(\mathrm{Tot}(\mathrm{F}^i\big(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\big))), \ \forall i \in \mathbb{Z}. \end{aligned} \tag{14.7.1}$$

Proof. – Let e be the final object of $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}$. An open subscheme U of $\mathscr{X}$ defines a subobject $\widetilde{U}$ of e by

$$\widetilde{U}(V,T) = \begin{cases} e(V,T) & \text{if } V \subset U, \\ \varnothing & \text{otherwise.} \end{cases}$$

Moreover, there exists a canonical equivalence of topoi ([**4**] IV 3.1.2)

$$(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}/\widetilde{U}} \xrightarrow{\sim} (U/\mathscr{S}_n)_{\mathrm{crys}}, \tag{14.7.2}$$

which identifies the localisation morphism with respect to $\widetilde{U}$ and the functoriality morphism induced by $U \to X$. Then, the morphisms $\{\widetilde{U}_i \to e\}_{i\in I}$ form a covering in $(\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}$. With the notation of 14.6, we have $\prod_{j\in J} \widetilde{U}_j = \widetilde{U^J}$. By cohomological descent, we have a spectral sequence ([**10**] 5.3.3.2, [**3**] Vbis 2.5.5)

$$\mathrm{E}_1^{r,s} = \bigoplus_{J\in I^{r+1}} \mathrm{H}^s((U^J/\mathscr{S}_n)_{\mathrm{crys}}, \mathscr{M}|\widetilde{U^J}) \Rightarrow \mathrm{H}^{r+s}((\mathscr{X}/\mathscr{S}_n)_{\mathrm{crys}}, \mathscr{M}), \tag{14.7.3}$$

whose differential morphism $d_1^{r,s} : \mathrm{E}_1^{r,s} \to \mathrm{E}_1^{r+1,s}$ is the alternating sum of the morphisms induced by the faces morphisms of (14.6.1).

On the other hand, we calculate $\mathrm{H}^{\bullet}(\mathrm{Tot}(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})))$ by filtering this bicomplex by rows ([**19**] 0.11.3.2). By 14.4(i), the vertical cohomology of $C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})$ is isomorphic to a direct sum of crystalline cohomology groups

$$\begin{aligned}\mathrm{H}_{\mathrm{I}}^{r,s} &= \operatorname{Ker} \partial_2^{r,s}/\operatorname{Im}(\partial_2^{r,s-1}) \\ &\simeq \bigoplus_{J\in I^{r+1}} \mathrm{H}^s((U^J/\mathscr{S}_n)_{\mathrm{crys}}, \mathscr{M}|\widetilde{U^J}).\end{aligned} \tag{14.7.4}$$

Recall that we have a spectral sequence ([**37**] 5.6.1)

$$\mathrm{E}_1'^{r,s} = \mathrm{H}_{\mathrm{I}}^{r,s} \Rightarrow \mathrm{H}^{r+s}(\mathrm{Tot}(C_{\mathscr{U}}^{\bullet,\bullet}(\mathscr{M}))), \tag{14.7.5}$$

whose differential morphism $d_1'^{r,s} : \mathrm{E}_1'^{r,s} \to \mathrm{E}_1'^{r+1,s}$ is induced by the morphism of complexes

$$\partial_1^{r,\bullet} : C_{\mathscr{U}}^{r,\bullet}(\mathscr{M}) \to C_{\mathscr{U}}^{r+1,\bullet}(\mathscr{M}). \tag{14.7.6}$$

In view of 14.5 and 14.6, the morphism $d_1'^{r,s}$ coincides with $d_1^{r,s}$ (14.7.3). Then the assertion for $\mathscr{M}$ follows.

Using 14.4(ii) and 14.5, one verifies the assertion for $\mathscr{M}^i$ in the same way. □

14.8. – We consider the injective $\mathscr{O}_{\mathfrak{X}/\mathscr{S}_n}$-linear morphism

$$g : \bigoplus_{i=1}^{p-1} \mathscr{M}^i \to \bigoplus_{i=0}^{p-1} \mathscr{M}^i \tag{14.8.1}$$

defined for every local section m of $\mathscr{M}^i$ by $g(m) = (m, -pm)$ in $\mathscr{M}^{i-1} \oplus \mathscr{M}^i$. We set $\Lambda_{\mathscr{M}} = \operatorname{Coker}(g)$. For any $r, s \in \mathbb{Z}$, the morphism g induces an injective morphism

$$\bigoplus_{i=1}^{p-1} \bigoplus_{J\in I^{r+1}} \Gamma(U^J, \mathscr{M}_{P_J}^{i-s} \otimes_{\mathscr{O}_{U_J}} \Omega^s_{U_J/\mathscr{S}_n}) \to \bigoplus_{i=0}^{p-1} \bigoplus_{J\in I^{r+1}} \Gamma(U^J, \mathscr{M}_{P_J}^{i-s} \otimes_{\mathscr{O}_{U_J}} \Omega^s_{U_J/\mathscr{S}_n}) \tag{14.8.2}$$

compatible with the differential morphisms and hence an injective morphism of bicomplexes:

$$g_C : \bigoplus_{i=1}^{p-1} \mathrm{F}^i\big(C_{\mathscr{U}}^{\bullet,\bullet}(\mathscr{M})\big) \to \bigoplus_{i=0}^{p-1} \mathrm{F}^i\big(C_{\mathscr{U}}^{\bullet,\bullet}(\mathscr{M})\big). \tag{14.8.3}$$

We denote its quotient by $C_{\mathscr{U}}^{\bullet,\bullet}(\Lambda_{\mathscr{M}})$. By 14.7, the crystalline cohomology groups of $\Lambda_{\mathscr{M}}$ are canonically isomorphic to cohomology groups of $\mathrm{Tot}(C_{\mathscr{U}}^{\bullet,\bullet}(\Lambda_{\mathscr{M}}))$.

14.9. – Let $\mathfrak{X}$ be a smooth and separated formal $\mathscr{S}$-scheme. In the following, we will use 14.7 to construct a Fontaine module structure on the crystalline cohomology of an object of $\mathbf{MF}(\mathfrak{X})$ (13.7). For this purpose, we first show how to associate an object of $\mathbf{MF}_{\mathrm{big}}(\mathfrak{X})$ to a Fontaine module with respect to a family of Frobenius liftings (13.15).

Suppose that there exist $m+1$ liftings $F_1, \dots, F_{m+1}$ of the relative Frobenius morphism $F_{X/k}$ of X. We set $\pi : \mathfrak{X}' \to \mathfrak{X}$ the canonical morphism, $F_{\mathfrak{X}^{m+1}} = \pi^{m+1} \circ (F_1, \dots, F_{m+1}) : \mathfrak{X}^{m+1} \to \mathfrak{X}^{m+1}$ and $\Delta : \mathfrak{X} \to \mathfrak{X}^{m+1}$ the diagonal closed immersion.

For $1 \leqslant i \leqslant m+1$, the projection $q_i : \mathfrak{X}^{m+1} \to \mathfrak{X}$ on the i-th component induces a functor (13.27.1)

$$q_i^* : \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; F_{i,\mathfrak{X}}) \to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \Delta, F_{\mathfrak{X}^{m+1}}).$$

By composing with the functor λ_{F_i} (13.19.4), we obtain a functor

$$q_i^* \circ \lambda_{F_i} : \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}) \to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \Delta, F_{\mathfrak{X}^{m+1}}). \tag{14.9.1}$$

We denote by $P_{\mathfrak{X}}(m)$ the PD-envelope of the diagonal immersion Δ compatible with γ (5.7) and by $F_{P_{\mathfrak{X}}(m)} : P_{\mathfrak{X}}(m) \to P_{\mathfrak{X}}(m)$ the lifting of Frobenius morphism induced by $F_{\mathfrak{X}^{m+1}} : \mathfrak{X}^{m+1} \to \mathfrak{X}^{m+1}$. We denote abusively the composition $P_{\mathfrak{X}}(m) \to \mathfrak{X}^{m+1} \xrightarrow{q_i} \mathfrak{X}$ by q_i.

Proposition 14.10. – *Let n be an integer $\geqslant 1$, $\mathfrak{M} = (M, \nabla, M^\bullet, \varphi)$ a p^n-torsion Fontaine module over $\mathfrak{X}$, $\mathscr{M}$ the crystal of $\mathscr{O}_{\mathfrak{X}_n/\mathscr{S}_n}$-modules associated to (M, ∇) and $\{\mathscr{M}^i\}$ the filtration on $\mathscr{M}$ associated to $\{M^i\}$ (13.12). Then there exists a functor*

$$\begin{aligned} \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}) &\to \mathbf{MF}_{\mathrm{big}}(\mathfrak{X}; \Delta, F_{\mathfrak{X}^{m+1}}) \\ (M, \nabla, M^\bullet, \varphi) &\mapsto (\mathscr{M}_{P_{\mathfrak{X}}(m)}, \nabla_{P_{\mathfrak{X}}(m)}, \mathscr{M}^\bullet_{P_{\mathfrak{X}}(m)}, \varphi^\bullet_{P_{\mathfrak{X}}(m)}), \end{aligned} \tag{14.10.1}$$

which is isomorphic to the functor $q_i^* \circ \lambda_{F_i}$ (14.9.1) via the transition morphism

$$c_{q_i} : q_i^*(M) \xrightarrow{\sim} \mathscr{M}_{P_{\mathfrak{X}}(m)}. \tag{14.10.2}$$

Proof. – It suffices to show that the divided Frobenius morphisms $\varphi^\bullet_{\mathfrak{M}_i}$ constructed by $q_i^* \circ \lambda_{F_i}$ (14.9.1) are compatible via q_i (14.10.2). We can reduce to the case $m = 1$. In this case, the proposition follows from 13.30. □

14.11. – In the rest of this section, we suppose that $\mathfrak{X}$ is a smooth proper formal $\mathscr{S}$-scheme of relative dimension d. Let n an integer $\geqslant 1$, $(M, \nabla, M^\bullet, \varphi)$ a p^n-torsion object of $\mathbf{MF}(\mathfrak{X})$ of length $\ell < p$ (i.e., $M^\ell \neq 0$, $M^{\ell+1} = 0$) and $(\mathscr{M}, \mathscr{M}^\bullet)$ the associated T-crystal (13.24). We write simply $\mathrm{H}^\bullet_{\mathrm{crys}}(-)$ for the crystalline cohomology groups $\mathrm{H}^\bullet((\mathfrak{X}_n/\mathscr{S}_n)_{\mathrm{crys}}, -)$.

Lemma 14.12. – *Keep the notation of 14.8 and of 14.11. The morphism φ induces* W*-linear morphisms of bicomplexes*

$$\phi^i_C : \mathrm{F}^i\left(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\right) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \to C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M}), \quad \forall\ 0 \leqslant i \leqslant p-1, \tag{14.12.1}$$

$$\psi_C : C^{\bullet,\bullet}_{\mathscr{U}}(\Lambda_{\mathscr{M}}) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \to C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M}), \tag{14.12.2}$$

which are functorial in $(M, \nabla, M^\bullet, \varphi) \in \mathbf{MF}(\mathfrak{X})$. In particular, we obtain for $m \geqslant 0$, W*-linear morphisms*

$$\phi^{m,i}_{\mathrm{H}} : \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \to \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}), \quad \forall\ 0 \leqslant i \leqslant p-1, \tag{14.12.3}$$

$$\psi^m : \mathrm{H}^m_{\mathrm{crys}}(\Lambda_{\mathscr{M}}) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \to \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}). \tag{14.12.4}$$

Proof. – We take a Zariski covering $\mathscr{U} = \{\mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ consisting of affine formal schemes and for each $i \in I$ a lifting $F_i : \mathfrak{U}_i \to \mathfrak{U}_i'$ of the relative Frobenius morphism of $\mathfrak{U}_{i,1}$. For any integer $r \geqslant 0$ and $J = (j_0, \ldots, j_r) \in I^{r+1}$, we denote by $\mathfrak{U}^J$ the intersection $\bigcap_{i=0}^{r} \mathfrak{U}_{j_i}$, by $\mathfrak{U}_J$ the product of $(r+1)$-copies of $\mathfrak{U}^J$ and by P_J the PD-envelope of the diagonal closed immersion $\mathfrak{U}^J \to \mathfrak{U}_J$. Note that P_J is equal to the PD-envelope of the immersion of $\mathfrak{U}^J$ in the product of $\{\mathfrak{U}_{j_i}\}_{i=0}^r$ over $\mathscr{S}$. We denote by $F_{\mathfrak{U}_J} : \mathfrak{U}_J \to \mathfrak{U}_J$ the morphism induced by $\{F_{j_i}\}_{i=0}^n$ and by $F_{P_J} : P_J \to P_J$ the lifting of the Frobenius morphism induced by $F_{\mathfrak{U}_J}$. By 14.10, we associate to $(M, \nabla, M^\bullet, \varphi)$ a family of divided Frobenius morphisms with respect to $(\mathfrak{U}^J \to \mathfrak{U}_J, F_{\mathfrak{U}_J})$:

$$\varphi^i_{P_J} : \mathscr{M}^i_{P_J} \to \mathscr{M}_{P_J} \qquad \forall\, i \leqslant p-1. \tag{14.12.5}$$

For any $r, s \geqslant 0$, in view of (13.15.2) and (13.28.2), the W-linear morphism

$$\bigoplus_{J\in I^{r+1}} \varphi^{i-s}_{P_J} \otimes \wedge^s\left(\frac{dF_{U_J}}{p}\right) : \bigoplus_{J\in I^{r+1}} \Gamma(\mathfrak{U}^J, \mathscr{M}^{i-s}_{P_J} \otimes_{\mathscr{O}_{\mathfrak{U}_{J,n}}} \Omega^s_{\mathfrak{U}_{J,n}/\mathscr{S}_n}) \otimes_{\sigma,\mathrm{W}} \mathrm{W}$$
$$\to \bigoplus_{J\in I^{r+1}} \Gamma(\mathfrak{U}^J, \mathscr{M}_{P_J} \otimes_{\mathscr{O}_{\mathfrak{U}_{J,n}}} \Omega^s_{\mathfrak{U}_{J,n}/\mathscr{S}_n})$$

is compatible with differential morphisms $\partial_1^{\bullet,\bullet}, \partial_2^{\bullet,\bullet}$ of $\mathrm{F}^i\big(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\big)$, $C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})$ respectively (14.6). Then we obtain a W-linear morphism of bicomplexes

$$\phi^i_C : \mathrm{F}^i\big(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\big)\otimes_{\sigma,\mathrm{W}} \mathrm{W} \to C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M}). \tag{14.12.6}$$

By condition (i-a) of 13.15, we see that the composition (14.8.3)

(14.12.7)

$$\bigoplus_{i=1}^{p-1} \mathrm{F}^i\big(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\big)\otimes_{\sigma,\mathrm{W}} \mathrm{W} \xrightarrow{g_C} \bigoplus_{i=0}^{p-1} \mathrm{F}^i\big(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})\big)\otimes_{\sigma,\mathrm{W}} \mathrm{W} \xrightarrow{\oplus\phi^i_C} C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})$$

vanishes. Then we obtain a W-linear morphism of bicomplexes

$$\psi_C : C^{\bullet,\bullet}_{\mathscr{U}}(\Lambda_{\mathscr{M}}) \otimes_{\sigma,\mathrm{W}} \mathrm{W} \to C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M}). \tag{14.12.8}$$

It is clear that the above constructions are functorial.

By 14.7 and 14.8, we obtain morphisms of cohomology groups (14.12.3) and (14.12.4). □

Proposition 14.13. – *If $pM = 0$ and $\min\{m, d-1\} + \ell \leqslant p-2$, the morphism ψ^m (14.12.4) is an isomorphism.*

Proof. – We use $\mathrm{H}^{r,s}_{\mathrm{I}}(-)$ to denote the vertical cohomology of a bicomplex. Recall (14.7.5) that we have two spectral sequences:

$$\mathrm{E}'^{r,s}_1 = \mathrm{H}^{r,s}_{\mathrm{I}}(C^{\bullet,\bullet}_{\mathscr{U}}(\Lambda_{\mathscr{M}})) \Rightarrow \mathrm{H}^{r+s}(\mathrm{Tot}(C^{\bullet,\bullet}_{\mathscr{U}}(\Lambda_{\mathscr{M}}))) \tag{14.13.1}$$

$$\mathrm{E}^{r,s}_1 = \mathrm{H}^{r,s}_{\mathrm{I}}(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})) \Rightarrow \mathrm{H}^{r+s}(\mathrm{Tot}(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M}))). \tag{14.13.2}$$

The morphism of bicomplexes ψ_C induces a k-linear morphism of spectral sequences $\mathrm{E}'\otimes_{\sigma,k} k \to \mathrm{E}$. Then it is enough to prove that for every $r \geqslant 0$ and s satisfying $\min\{s, d-1\} + \ell \leqslant p-2$, the induced morphism

$$\mathrm{H}^{r,s}_{\mathrm{I}}(C^{\bullet,\bullet}_{\mathscr{U}}(\Lambda_{\mathscr{M}})) \otimes_{\sigma,k} k \to \mathrm{H}^{r,s}_{\mathrm{I}}(C^{\bullet,\bullet}_{\mathscr{U}}(\mathscr{M})) \tag{14.13.3}$$

is an isomorphism. Since $\mathscr{M}$ is p-torsion, we have $\Lambda_{\mathscr{M}}=\bigoplus_{i=0}^{p-2}\mathrm{gr}^i(\mathscr{M})\oplus\mathscr{M}^{p-1}$. For any $s\geqslant 0$, we set $\Lambda_{\mathscr{M}}^{-s}=\bigoplus_{i=0}^{p-2}\mathrm{gr}^{i-s}(\mathscr{M})\oplus\mathscr{M}^{p-1-s}$. Since $\mathfrak{U}^J$ is affine, $C_{\mathscr{U}}^{r,s}(\Lambda_{\mathscr{M}})$ can be written as a direct sum

$$\bigoplus_{J\in I^{r+1}}\Gamma(\mathfrak{U}^J,\Lambda_{\mathscr{M},P_J}^{-s}\otimes_{\mathscr{O}_{\mathfrak{U}_{J,1}}}\Omega^s_{\mathfrak{U}_{J,1}/k}).$$

Recall (14.12.6) that the divided Frobenius morphisms $\varphi_{P_J}^i:\mathscr{M}_{P_J}^i\to\mathscr{M}_{P_J}$ and the semi-linear morphism $\wedge^s(\frac{dF_{\mathfrak{U}}}{p}):\Omega^s_{\mathfrak{U}_{J,1}/k}\to\Omega^s_{\mathfrak{U}_{J,1}/k}$ induce a k-linear morphism

$$\psi_{J,C}^{r,s}:\Gamma(\mathfrak{U}^J,\Lambda_{\mathscr{M},P_J}^{-s}\otimes_{\mathscr{O}_{\mathfrak{U}_{J,1}}}\Omega^s_{\mathfrak{U}_{J,1}/k})\otimes_{\sigma,k}k\to\Gamma(\mathfrak{U}^J,\mathscr{M}_{P_J}\otimes_{\mathscr{O}_{\mathfrak{U}_{J,1}}}\Omega^s_{\mathfrak{U}_{J,1}/k})$$

and that the morphism $\psi_C^{r,s}$ (14.12.2) is defined by a direct sum of morphisms $\bigoplus_{J\in I^{r+1}}\psi_{J,C}^{r,s}$.

Then the assertion follows from the following lemma. $\square$

LEMMA 14.14. – *For any $r\geqslant 0$, $J\in I^{r+1}$, the morphism of complexes*

$$\psi_{J,C}^{r,\bullet}:\Gamma(\mathfrak{U}^J,\Lambda_{\mathscr{M},P_J}^{-\bullet}\otimes_{\mathscr{O}_{\mathfrak{U}_{J,1}}}\Omega^\bullet_{\mathfrak{U}_{J,1}/k})\otimes_{\sigma,k}k\to\Gamma(\mathfrak{U}^J,\mathscr{M}_{P_J}\otimes_{\mathscr{O}_{\mathfrak{U}_{J,1}}}\Omega^\bullet_{\mathfrak{U}_{J,1}/k})\tag{14.14.1}$$

induces an isomorphism on the m-th cohomology group for any integer m satisfying $\min\{m,d-1\}+\ell\leqslant p-2$.

Proof. – In view of (13.28.2) and (14.5), we can reduce to the case where $r=0$, $J\in I$. To simplify the notation, we write $\mathfrak{U}$ for $\mathfrak{U}_J$, F for the lifting of Frobenius $F_J:\mathfrak{U}\to\mathfrak{U}'$, U the special fiber of $\mathfrak{U}$ and M (resp. M^i) for $\mathscr{M}_{\mathfrak{U}}=M|U$ (resp. $\mathscr{M}^i_{\mathfrak{U}}=M^i|U$).

We set $\mathrm{gr}(M)=\bigoplus_{i=0}^{\ell}M^i/M^{i+1}$. By Griffiths' transversality, ∇ induces a Higgs field on $\mathrm{gr}(M)$:

$$\theta:\mathrm{gr}(M)\to\mathrm{gr}(M)\otimes_{\mathscr{O}_X}\Omega^1_{X/k}.$$

The source of (14.14.1) can be written as

$$\Gamma(U,(\bigoplus_{i=0}^{p-2-s}\mathrm{gr}^i(M)\oplus M^{p-1-s})\otimes_{\mathscr{O}_U}\Omega^s_{U/k})\otimes_{\sigma,k}k,\tag{14.14.2}$$

which is equal to

$$\Gamma(U,\mathrm{gr}(M)\otimes_{\mathscr{O}_U}\Omega^s_{U/k})\otimes_{\sigma,k}k\qquad\text{if } s\leqslant p-1-\ell.\tag{14.14.3}$$

The differential morphism of the source is induced by θ for $s\leqslant p-1-\ell$.

Since $pM=0$, we have $(\widetilde{M},\widetilde{\nabla})=(\mathrm{gr}(M),\theta)$ (13.8). The isomorphism φ and the lifting F induce an isomorphism of $\mathrm{MIC}(\mathfrak{X}_n/\mathscr{S}_n)$ (13.19.2)

$$\varphi_F:\Phi_1\big((\mathrm{gr}(M),\theta)\otimes_{\sigma,k}k\big)\xrightarrow{\sim}(M,\nabla).\tag{14.14.4}$$

Recall (13.20) that φ_F induces a family of divided Frobenius morphisms $\varphi_F^\bullet$. The morphism $\psi_{J,C}^{r,\bullet}$ (14.14.1), which is induced by $\varphi_F^\bullet$, coincides with the composition of morphism of complexes (6.5.1) induced by Φ_1 and the isomorphism of de Rham complexes induced by φ_F

$$(\mathrm{gr}(M)\otimes_{\mathscr{O}_U}\Omega^\bullet_{U/k})\otimes_{\sigma,k}k\to M\otimes_{\mathscr{O}_U}\Omega^\bullet_{U/k}\tag{14.14.5}$$

in degrees $\leqslant p-1-\ell$. Then the lemma follows from 6.5. $\square$

Remark 14.15. – Suppose that $pM = 0$. Let $\mathrm{gr}(M) = \bigoplus_{i=0}^{\ell} M^i/M^{i+1}$ and θ the Higgs field on $\mathrm{gr}(M)$ induced by ∇ and Griffiths' transversality. By 14.4 and a similar argument of 14.14, we deduce for $m \leqslant p-2-\ell$, an isomorphism

$$\mathrm{H}^m_{\mathrm{crys}}(\Lambda_{\mathscr{M}}) \xrightarrow{\sim} \mathbb{H}^m(\mathrm{gr}(M)\otimes\Omega^{\bullet}_{X/k}). \tag{14.15.1}$$

Proposition 14.16. – (i) *If $d \leqslant p-1-\ell$, the morphism ψ^m* (14.12.4) *is an isomorphism for all m.*

(ii) *If $d > p-1-\ell$, the morphism ψ^m* (14.12.4) *is an isomorphism for $m+\ell \leqslant p-3$, and is a monomorphism for $m+\ell = p-2$.*

Proof. – We prove it by induction on n. In the case $n=1$, i.e., M is p-torsion, it follows from 14.13. Suppose that the proposition is true for $n-1$. By 13.24, the quadruple $(pM, \nabla|_{pM}, pM^{\bullet}, \varphi|_{p\,\mathrm{C}^*(\widetilde{\mathscr{M}'})})$ is a subobject of $(M,\nabla,M^{\bullet},\varphi)$ in $\mathbf{MF}(\mathfrak{X})$ and we denote its quotient by $(\overline{M},\overline{\nabla},\overline{M}^{\bullet},\overline{\varphi})$. If $\overline{\mathscr{M}}$ and $\overline{\mathscr{M}}^{\bullet}$ denote the crystal of $\mathscr{O}_{\mathfrak{X}_n/\mathscr{S}_n}$-modules and the filtration associated to $(\overline{M},\overline{\nabla},\overline{M}^{\bullet})$ (13.12), we have an exact sequence:

$$0 \to p\mathscr{M}^i \to \mathscr{M}^i \to \overline{\mathscr{M}}^i \to 0 \qquad \forall\ i \leqslant p-1. \tag{14.16.1}$$

By the snake lemma, we have a commutative diagram:

(14.16.2)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=1}^{p-1} p\mathscr{M}^i \arrow[r] \arrow[d] & \bigoplus_{i=0}^{p-1} p\mathscr{M}^i \arrow[r] \arrow[d] & \Lambda_{p\mathscr{M}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{p-1} \mathscr{M}^i \arrow[r] \arrow[d] & \bigoplus_{i=0}^{p-1} \mathscr{M}^i \arrow[r] \arrow[d] & \Lambda_{\mathscr{M}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{p-1} \overline{\mathscr{M}}^i \arrow[r] \arrow[d] & \bigoplus_{i=0}^{p-1} \overline{\mathscr{M}}^i \arrow[r] \arrow[d] & \Lambda_{\overline{\mathscr{M}}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Since ψ_m is functorial in $\mathbf{MF}(\mathfrak{X})$, the assertion follows by dévissage from the induction hypothesis. □

Proposition 14.17. – (i) *Let m be an integer such that* $\min\{m, d-1\} \leqslant p-2-\ell$, *the exact sequence* (14.8)

$$0 \to \bigoplus_{i=1}^{p-1} \mathscr{M}^i \xrightarrow{g} \bigoplus_{i=0}^{p-1} \mathscr{M}^i \to \Lambda_{\mathscr{M}} \to 0$$

induces an exact sequence of cohomology groups

$$0 \to \bigoplus_{i=1}^{p-1} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i) \to \bigoplus_{i=0}^{p-1} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i) \to \mathrm{H}^m_{\mathrm{crys}}(\Lambda_{\mathscr{M}}) \to 0. \tag{14.17.1}$$

(ii) *If* $m+\ell = p-2$, *morphism* ψ^m (14.12.4) *is an isomorphism.*

Proof. – We prove the proposition by induction on m. The case $m=-1$ is trivial. Suppose that the assertion is true for $m-1$ and we will prove it for m. By hypothesis of induction, we have an exact sequence (which automatically exists for $m=0$)

$$0 \to \bigoplus_{i=1}^{p-1} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i) \to \bigoplus_{i=0}^{p-1} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i) \to \mathrm{H}^m_{\mathrm{crys}}(\Lambda_{\mathscr{M}}). \tag{14.17.2}$$

Since $\mathscr{M}, \mathscr{M}^i$ are coherent, by 14.4, we see that the cohomology groups in the above sequence are finite type W_n-modules. By 14.16, we have the following inequalities on the length of W_n-modules

$$\begin{aligned}\mathrm{lg}_{\mathrm{W}_n} \mathrm{H}^m_{\mathrm{crys}}(\Lambda_{\mathscr{M}}) &\leqslant \mathrm{lg}_{\mathrm{W}_n} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}) \\ &= \sum_{i=0}^{p-1} \mathrm{lg}_{\mathrm{W}_n} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i) - \sum_{i=1}^{p-1} \mathrm{lg}_{\mathrm{W}_n} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}^i).\end{aligned} \tag{14.17.3}$$

Then we deduce the surjectivity of the last arrow of (14.17.2), and that the above inequality is an equality. The assertion (i) follows. The assertion (ii) follows from 14.16 and the equality (14.17.3). □

14.18. – *Proof of 14.1.* Assertion (i) follows from 14.4 and 14.17.

We take for $\phi^{m,i}_{\mathrm{H}}$ the morphism (14.12.3). Then assertion (ii) follows from (i), 14.16 and 14.17.

Note that the complex $\mathrm{F}^i = 0$ if $i > d+\ell$. For any r, s satisfying $\min\{r+s, d-1\}+\ell \leqslant p-2$, we deduce from (i) that

$$\mathbb{H}^{r+s}(\mathrm{gr}^r_{\mathrm{F}}(M \otimes_{\mathscr{O}_{\mathfrak{X}_n}} \Omega^\bullet_{\mathfrak{X}_n/\mathscr{S}_n})) \xrightarrow{\sim} \mathbb{H}^{r+s}(\mathrm{F}^r)/\mathbb{H}^{r+s}(\mathrm{F}^{r+1}). \tag{14.18.1}$$

Then assertion (iii) follows by comparing the W_n-length of $\mathrm{E}_1^{r,s}$ and of E^{r+s}. □

Remark 14.19. – Using the comparison isomorphism between the de Rham and the Dolbeault complexes (10.21), Ogus and Vologodsky proved 14.1 for p-torsion Fontaine modules (13.8) (**[31]** 4.17). More precisely, let $(M, \nabla, M^\bullet, \varphi)$ be a p-torsion object of $\mathbf{MF}(\mathfrak{X})$ of length ℓ and θ the Higgs field on $\mathrm{gr}(M)$ induced by ∇ and Griffiths' transversality. By 10.21, the isomorphism (10.21.1) induces via (13.8.1)

$$\varphi : \mathrm{C}^{-1}_{\mathfrak{X}'_2}(\pi^*(\mathrm{Gr}(M), \theta)) \xrightarrow{\sim} (M, \nabla),$$

for $m \leqslant p-1-\ell$, an isomorphism:

$$\mathbb{H}^m(\mathrm{gr}(M) \otimes \Omega^\bullet_{X/k}) \otimes_{\sigma,k} k \xrightarrow{\sim} \mathbb{H}^m(M \otimes \Omega^\bullet_{X/k}). \tag{14.19.1}$$

By (14.15.1), we obtain for $m \leqslant p-2-\ell$, an isomorphism

$$\mathrm{H}^m_{\mathrm{crys}}(\Lambda_{\mathscr{M}}) \otimes_{\sigma,k} k \xrightarrow{\sim} \mathrm{H}^m_{\mathrm{crys}}(\mathscr{M}). \tag{14.19.2}$$

The above isomorphism is an analog of the isomorphism ψ^m (14.12.4), 14.13 and allows us to deduce 14.1 for p-torsion objects. We don't know whether these two isomorphisms coincide or not.

Theorem 14.1 provides a generalization of Ogus-Vologodsky's Result (10.21) for p^n-torsion Fontaine modules. However, we don't know how to generalize 10.21 for the Cartier equivalence modulo p^n.

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Let W be the ring of the Witt vectors of a perfect field of characteristic p, $\mathfrak{X}$ a smooth formal scheme over W, $\mathfrak{X}'$ the base change of $\mathfrak{X}$ by the Frobenius morphism of W, $\mathfrak{X}'_2$ the reduction modulo p^2 of $\mathfrak{X}'$ and X the special fiber of $\mathfrak{X}$. We lift the Cartier transform of Ogus-Vologodsky defined by $\mathfrak{X}'_2$ modulo p^n. More precisely, we construct a functor from the category of p^n-torsion $\mathscr{O}_{\mathfrak{X}'}$-modules with integrable p-connection to the category of p^n-torsion $\mathscr{O}_{\mathfrak{X}}$-modules with integrable connection, each subject to suitable nilpotence conditions. Our construction is based on Oyama's reformulation of the Cartier transform of Ogus-Vologodsky in characteristic p. If there exists a lifting $F : \mathfrak{X} \to \mathfrak{X}'$ of the relative Frobenius morphism of X, our functor is compatible with a functor constructed by Shiho from F. As an application, we give a new interpretation of Faltings' relative Fontaine modules and of the computation of their cohomology.

Soient W *l'anneau des vecteurs de Witt d'un corps parfait de caractéristique* $p > 0$, $\mathfrak{X}$ *un schéma formel lisse sur* W, $\mathfrak{X}'$ *le changement de base de* $\mathfrak{X}$ *par l'endomorphisme de Frobenius de* W, $\mathfrak{X}'_2$ *la réduction modulo* p^2 *de* $\mathfrak{X}'$ *et* X *la fibre spéciale de* $\mathfrak{X}$. *On relève la transformée de Cartier d'Ogus-Vologodsky définie par* $\mathfrak{X}'_2$. *Plus précisément, on construit un foncteur de la catégorie des* $\mathscr{O}_{\mathfrak{X}'}$*-modules de* p^n*-torsion à* p*-connexion intégrable dans la catégorie des* $\mathscr{O}_{\mathfrak{X}}$*-modules de* p^n*-torsion à connexion intégrable, chacune étant soumise à des conditions de nilpotence appropriées. S'il existe un relèvement* $F : \mathfrak{X} \to \mathfrak{X}'$ *du morphisme de Frobenius relatif de* X, *notre foncteur est compatible avec une construction « locale » de Shiho définie par* F. *Comme application de la transformée de Cartier modulo* p^n, *on donne une nouvelle interprétation des modules de Fontaine relatifs introduits par Faltings et du calcul de leur cohomologie.*