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A PLURALITY OF (NON)VISUALIZATIONS: BRANCH POINTS AND BRANCH CURVES AT THE TURN OF THE 19TH CENTURY

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Abstract. — This article deals with the different ways branch points and branch curves were visualized at the turn of the 19th century. On the one hand, for branch points of complex curves one finds an abundance of visualization techniques employed. German mathematicians such as Felix Klein or Walther von Dyck were the main promoters of these numerous forms of visualization, which appeared either as two-dimensional illustrations or three-dimensional material models. This plurality of visualization techniques, however, also resulted in inadequate images that aimed to show the varied ways branch points could possibly be represented. For branch (and ramification) curves of complex surfaces, on the other hand, there were hardly any representations. When the Italian school of algebraic geometry studied branch curves systematically only partial illustrations can be seen, and branch curves were generally made "invisible". The plurality of visualizations shifted into various forms of non-visualization. This can be seen in the different ways visualization techniques disappeared.

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RÉSUMÉ (Une diversité de visualisations et non-visualisations : points de branchement et courbes de ramification autour de 1900.)

L'article traite des différentes façons de visualiser les points et les courbes de branchement autour de 1900. De nombreuses techniques de visualisation ont été employées pour les points de branchement de courbes complexes. Des mathématiciens allemands comme Felix Klein ou Walther von Dyck ont été les principaux promoteurs de cette multitude de visualisations, que ce soit sous la forme d'illustrations ou de modèles matériels tridimensionnels. Cependant, cette pluralité de techniques a également été à l'origine d'images inadéquates visant à montrer les diverses manières possibles de représenter des points de branchement. Pour les courbes de branchement (et de ramification) de surfaces complexes, il est difficile de trouver une visualisation. Lorsque les courbes de branchement ont été systématiquement étudiées par l'école italienne de géométrie algébrique, seules des illustrations partielles ont pu être trouvées, et les courbes de branchement ont été généralement rendues « invisibles ». La pluralité des visualisations s'est transformée en une pluralité de non-visualisations, dont témoignent différents modes de disparition des techniques de visualisation.

INTRODUCTION

Ever since Bernhard Riemann (1826–1866) introduced the now well known *Riemann surfaces* in his 1851 doctoral dissertation on complex function theory, as the covering of the complex line (or of the projective complex line) for multi-valued analytic functions in a complex region, attempts have been made to visualize these coverings—and especially their branch points. The question concerning how to visualize these functions was also dealt with before Riemann's introduction of curves as covering: a complex valued curve $y = f(x)$ is embedded in a four dimensional space $\mathbb{C}^2$; every point (x_0, y_0), when $x_0, y_0 \in \mathbb{C}$ such that $y_0 = f(x_0)$ can be represented then in a four-dimensional real space $\mathbb{R}^4$ via a quadruple $(\mathrm{Re}(x_0), \mathrm{Im}(x_0), \mathrm{Re}(y_0), \mathrm{Im}(y_0))$. Hence visualizing these complex points (x_0, y_0) as a drawing on paper (by drawing for example only the *real* points in $\mathbb{R}^2$, i.e., the points for which $\mathrm{Im}(x_0) = \mathrm{Im}(y_0) = 0$) or as model in a three-dimensional space (by constructing models of surfaces whose points are either $(\mathrm{Re}(x_0), \mathrm{Im}(x_0), \mathrm{Im}(y_0))$ or $(\mathrm{Re}(x_0), \mathrm{Im}(x_0), \mathrm{Re}(y_0))$ would always risk being insufficient from a mathematical as well as from a visual point of view. [1] Notwithstanding this insufficiency, Riemann's con-

[1] To recall: for a given complex number $c = a + bi$, (where $i = \sqrt{-1}$), $\mathrm{Re}(c) = a$, $\mathrm{Im}(c) = b$.

cept of the complex curve as a covering, as I will show, prompted a variety of visualizations.

The present article deals with the various visualizations of a special phenomenon arising when considering these curves as covering of the complex line. To give an example, consider the function $y^2 = x - 1$ and its projection to the x-axis:

$$p : \{(x, y) \in \mathbb{C}^2 : y^2 = x - 1\} \to \mathbb{C}, (x, y) \mapsto x.$$

Generically, every point $x' \in \mathbb{C}$ has two different preimages (x', y_1), $(x', y_2) \in \mathbb{C}^2$ such that $(y_1)^2 = x' - 1$ and $(y_2)^2 = x' - 1$. However, for $x' = 1$, the number of the preimages is less than two (explicitly, there is only one preimage: $(1, 0)$). One might say that when considering the points $x'' \in \mathbb{C}$ which are close to $x' = 1$, the two preimages of x'' "come together," or "coincide" into one point when x'' approaches x'. Considering only *smooth* functions, these points, whose number of preimages is lower than the expected one, are called *branch points*;[2] while the points on the curve, for which few of the preimages "come together," are called—in current terminology—*ramification points*. However, as the terminology regarding these points was not standardized in the 19th century, they were also usually referred to as branch points ("Verzweigungspunkte" or "Windungspunkte" in German), a usage I will follow. It should also be noted that when n preimages "come together," one says that the branch point is of order $n - 1$.

The same phenomenon may also happen when considering complex surfaces as a cover of the complex plane $\mathbb{C}^2$, when in this situation, the collection of branch points is in fact a complex curve in $\mathbb{C}^2$, called the *branch curve* of the complex surface (when considered as a covering).[3] The question that this paper would like to answer concerns the nature of the various visualizations of branch points and branch curves during the 19th and the 20th century. More precisely, the paper, concentrat-

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2 In fact, the map p can be any surjective holomorphic map between a Riemann surface and the projective complex line (using current terminology).

3 And the corresponding curve on the surface is called in current terminology *ramification curve* (see Section II). The explicit computation of branch curves (and also of branch points) can be easily done—from a computational point of view—, at least when one deals with projections. For example, given a cubic surface: $f(z) = z^3 - 3az + b$, where a and b are homogeneous forms in (x, y, w) of degrees 2 and 3 respectively, and the projection is given by $(x, y, w, z) \to (x, y, w)$. In these coordinates, the ramification curve is given by the intersection of the surface and its derivative with respect to z, i.e., of $f = 0$ and $df/dz = 0 = z^2 - a = 0$ and the branch curve B is therefore given by $b^2 - 4a^3 = 0$, being a curve of degree 6.

ing on the years between 1874 and 1929, aims to show in Section I that while for *branch points* (of finite order)—either on the complex line or on the curve—there was an abundance of visualizations or a plurality of visual interpretations, for *branch curves*, the situation, as I will examine in Section II, was actually quite the opposite: while for branch points the different three-dimensional models and two-dimensional illustrations were at times epistemological and stimulated further research, for branch curves, similar illustrations—in the cases when they even existed—were mostly considered technically;[4] visualization techniques were ignored or considered unnecessary. It is here where one notices a shift in the mathematical practice of visualization: from a plurality of such techniques to either a rejection of them or partial visualization of an "auxiliary machinery," not, however, of the object itself. In some cases, this "auxiliary machinery" eventually became the object of research itself. In most cases, however, as will be elaborated in the concluding Section III, one can see that the plurality of visualizations was replaced by a plurality of non-visualizations, prompted by different modes of disappearance.

1. BRANCH POINTS: EPISTEMOLOGICAL VISUALIZATIONS

In this first section, I will deal with what may be thought of as a counter position to the situation concerning the visualization of branch curves, a topic that will be dealt with in the second section. This section will aim to show how branch points of complex curves were usually thought of during the second half of the 19th century as what could (and should) be visualized. This does not mean that all of the attempts at visualizing them

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[4] With these distinctions I follow throughout this article Hans-Jörg Rheinberger's differentiation between epistemic and technical objects. According to Rheinberger "epistemic objects [...] present themselves in a characteristic, irreducible vagueness. This vagueness is inevitable because, paradoxically, epistemic things embody what one does not yet know." [Rheinberger 1997, p. 28] These objects, their purpose, or the field of research that they open and the questions that they may propose are not yet defined or not yet canonically categorized. This is exactly what makes them into an epistemological object, as they are in the process of becoming "well-defined" or "stable." But "in contrast to epistemic objects, [...] experimental conditions"—and technical objects, as Rheinberger later adds—"tend to be characteristically determined within the given standards of purity and precision. [...] they restrict and constrain" the scientific objects [Rheinberger 1997, p. 29]. But while it seems that there is a clear distinction between the not yet defined epistemological object and the clearly defined technical one, Rheinberger immediately adds "The difference between experimental conditions and epistemic things, therefore, is functional rather than structural." [Rheinberger 1997, p. 30]

were considered successful, satisfactory or even accepted by the entire mathematical community. What I aim to show, by contrast, is how these attempts were directed at illustrating and showing what branch points *looked like*. Given that the research on the history of Riemann surfaces is vast, a full-blown examination of how branch points were visualized during this period and afterwards is beyond the scope of this paper.[5] Thus, for example, I will not deal with Hermann Weyl's influential book *Die Idee der Riemannschen Fläche* [Weyl 1913]. Rather I will examine a few different examples, especially from the last quarter of the 19th century and the first quarter of the 20th century, which indicate that the research of branch points of Riemann surfaces was coupled not only with analytical investigation within the domain of function theory, or with algebraic calculations, but also with visual practices.

1.1. *1850–1865: Puiseux, Riemann and Neumann*

A year before Riemann's presentation of his dissertation, Victor Puiseux (1820–1883) in 1850 published his manuscript *Recherches sur les fonctions algébriques*, dealing with complex functions defined by an equation $f(u,z) = 0$. Puiseux, one might say, viewed complex curves as a covering of the complex line $\mathbb{C}$, which would be defined, as noted above, using contemporary notation, as follows:

$$\mathrm{pr} : \{(u,z) \in \mathbb{C}^2 : f(u,z) = 0\} \to \mathbb{C}, (u,z) \mapsto z.$$

Assuming that for the function $f(u,z)$ the degree of z is p, given a complex point z_0 on the z-axis, Puiseux asks what would happen to the p solutions of the equation $f(u,z_0) = 0$, that is, to the points in the set $\mathrm{pr}^{-1}(z_0) = u_1(z_0), \ldots, u_p(z_0)$, when the point z_0 moves along a closed path, which avoids passing through points z' for which two or more values $u_i(z')$ coincide (recall that the z axis is the complex line $\mathbb{C}$, which is topo-

5 On the development of the concept of the n-dimensional manifold beginning from the 1850s with Riemann and his concept of covering, see e.g., [Scholz 1980; 1999] The topic concerning the various visualizations of Riemann surfaces (and not necessarily their branch points), starting from the second half of the 19th century, also deserves a more elaborate discussion than that presented here, one which would also take into account their digital visualization.

For an extensive survey of Riemann's work and the responses to it, see [Gray 2015, p. 153–194]; see also [Bottazzini & Gray 2013, p. 259–341] for a similar discussion, also containing other figures of branch points, similar to what is shown in this paper; Bottazzini and Gray show that Gustav Holzmüller in his 1882 book *Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildungen* [Holzmüller 1882, p. 271] and Felice Casorati with his sketches of Riemann surfaces in 1864 also drew figures of different visualizations of branch points [Neuenschwander 1998, p. 23].

logically a two-dimensional real space). Puiseux is explicitly interested in the following situation: "Let A be a point for which the p roots of the equation $f(u, z) = 0$ become equal."[6] [Puiseux 1850, p. 385] He initially looks at what happens on the z-axis: first denotes by a the z-value of the point A, that is $pr(A) = a$. Taking C, a point on the z-axis being close to a, he "draws" an infinitesimal loop called "$CLMC$" (see fig. 1.(I)), encircling the point a, beginning and ending at a point C.[7] Finally he examines, moving the z coordinate along this loop (beginning and ending at C), what happens to the solutions $u_1, \ldots, u_p$ of the equation $f(u, z) = 0$. As Puiseux describes, after one time encircling along this loop, the values $u_1, u_2, \ldots, u_p$ are permuted *cyclically* to $u_2, \ldots, u_p, u_1$ [Puiseux 1850, p. 387–388].

For Puiseux the way to understand this phenomenon was mainly by way of analytical reasoning, though he did refer to drawings. While Puiseux calculates algebraically what happens to the values $u_1, u_2, \ldots, u_p$ when the point z moves along the "contour fermé," the figures of this loop and of other loops that Puiseux refers to support his reasoning, although they are not essential to the argument. While one may suggest that for Puiseux, the way to understand the transformation and permutations of the points $u_1, u_2, \ldots, u_p$ was also enabled by means of a visualization of the system of loops, the drawing of the loops is extremely technical. This is to be seen with two other articles: the first, in another paper of Puiseux from 1851 "Nouvelles recherches sur les fonctions algébriques," which contains only a single drawing [Puiseux 1851, p. 230]. Although the paper is a continuation of the 1850 paper and often refers to it, the almost total lack of figures shows that for the simple case of cyclic permutation, a depiction of what a loop is actually unnecessary. The second indication of the marginality of these illustrations is to be noticed with the 1861 German translation of Puiseux's 1850 paper: "Untersuchungen über die algebraischen Functionen," which Hermann Fischer made [Puiseux 1861]. In contrast to Puiseux where the figures in the 1850 paper are to be found on a separate sheet, Fischer combines the figures adjacent to the text. Not underestimating this relocation of the figures, which certainly helped the reader to *see* what is meant directly, Figure 9 of the 1850 Puiseux's paper

6 "Soit maintenant A un point pour lequel p racines de l'équation $f(u, z) = 0$ deviennent égales." Later Puiseux [1850, p. 385–386] notes that this condition is equivalent to the vanishing of derivatives $d^i f/du^i$, for $i = 1, \ldots, p-1$ at the point A, when $d^p f/du^p(A)$ is not equal to 0.

7 "Décrivons autour de ce point [a] un contour fermé de dimensions infiniment petites CLMC, fig 9." [Puiseux 1850, p. 385]

was omitted. This was the figure that showed the basic loop encircling the branch point on the (complex) line.[8]

That said, during the investigation of the behavior of the preimages Puiseux neither distanced himself from any form of illustration nor did he attempt to dissuade the reader from visualizing certain aspects of this behavior. While investigating this behavior for curves with double point or with triple point, Puiseux analyzed loops, which—when deformed—either go around the image of the singular point or encircle it. Here the reference to the figures is essential, in order to understand the deformation of the loop and correspondingly, how the preimages of the curve change (see Fig. 1.(II), (III)) and how the preimages behave while encircling a singular point. Indeed, when investigating the case of a double point, where locally the degree of the curve is 2, as Puiseux explicitly notes, "Figures 18, 19 and 20 *show us* how [...] the points U_1 and U_2 [the preimages] are changed, while the loop CLMC, while being deformed, has to cross the point A [the image of the singular point]."[9] [Puiseux 1850, p. 423] In this case, Puiseux points the reader to the immediacy of the transmitted knowledge enabled by the figures.

Although he may seem to follow a similar reasoning, when turning to Riemann we can see how the role of visualization (and also of the senses) was given greater emphasis. This is apparent in his writings and in his lectures notes. In his 1851 dissertation, Riemann starts with the following definition of a surface as a covering: "For the following treatment we permit x, y to vary only over a finite region. The position of the point 0 is no longer considered as being in the plane A [i.e., on the complex line], but in a surface T spread out over the plane. We choose this wording since it is inoffensive to speak of one surface lying on another, to leave open the possibility that the position of 0 can extend more than once [mehrfach erstrecke] over a given part of the plane [...]."[10] [Riemann 1851, p. 7]

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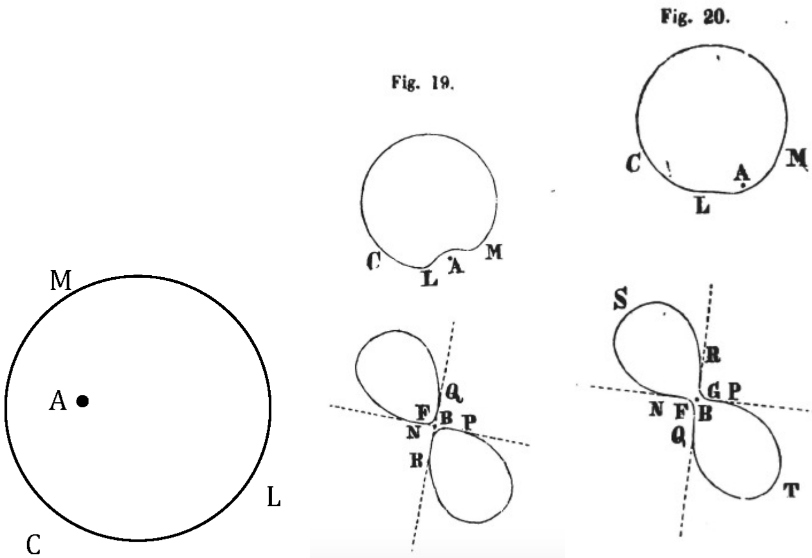
8 Although it is clear from the numeration of the figures in the 1861 translation that Figure 9 should have been added.

9 "Les Fig. 18, 19 et 20 montrent comment se modifient [...] les points U_1 et U_2, lorsque le contour *CLMC*, en se déformant, vient à franchir le point A." The German translation is as follows: "Aus den Figuren 18, 19 und 20 ist *sofort ersichtlich*, welche Modificationen die von den Punkten U_1, U_2 beschriebenen Curven erleiden, wenn die Curve *CLMC* bei der Verschiebung den Punkt A überschreitet."[Puiseux 1861, p. 65–66] The key expression is "*sofort ersichtlich*"—"immediately apparent".

10 "Für die folgenden Betrachtungen beschranken wir die Veränderlichkeit der Grössen x, y auf ein endliches Gebiet, indem wir als Ort des Punktes 0 nicht mehr die Ebene A selbst, sondern eine über dieselbe ausgebreitete Fläche T betrachten. Wir wählen diese Einkleidung, bei der es unanstössig sein wird, von auf einander liegenden Flächen zu reden, um die Möglichkeit offen zu lassen, dass der Ort des Punktes 0

Riemann implicitly suggests the imagining of the surface through the use of visual metaphor: "a surface T spread out over the plane [...] more than once." The question that then logically arises is how this surface behaves



(I) (II) (III)

FIGURE 1. (I): Depiction of Puiseux's *CLMC* loop, encircling the branch point A [Puiseux 1850, Planche I, fig. 9].[11] (II), (III): Puiseux's depiction of a deformation of a loop which originally went through A, being the image of the double (singular) point B [Puiseux 1861, p. 65, 66].[12]

at a neighborhood of a branch point. In his 1851 dissertation Riemann calls these points "turning points [Windungspunkte]" [Riemann 1851, p. 8], and his description is similar to Puiseux's: when a point on a plane A (i.e., on the complex line $\mathbb{C}$)[13] moves around ("umkreisen") the branch point, a permutation ("Anordnung") of the values of the surface occurs. Riemann asks the reader to fix a distinctive image in his mind by drawing

über denselben Theil der Ebene sich mehrfach erstrecke [...]." Translation taken from: [Baker et al. 2004, p. 4].

11 Drawing by M.F.

12 The figures are taken from [Puiseux 1861], as one can hardly recognize the deformation of the curve in the figures found in [Puiseux 1850, Planche I].

13 Or on the projective complex line $\mathbb{CP}^1$.

certain figures: when he describes the permutation of the values, he first indicates that in order to "fix *ideas* [Zur Fixierung der *Vorstellung*], *draw a circle* of radius R around the point 0 in the plane A and draw a diameter parallel to the x-axis [...]." [Riemann 1851, p. 26] [14] In 1857 he calls this point a branch point: "Verzweigungspunkte," using explicitly visual metaphors, indicating that the different parts of the function in the neighborhood of a branch point are called "branches [*Zweige*]". [15] He then defines a simple branch point and a branch point of multiplicity $\mu + 1$:

> A point of the surface T at which only two branches are connected, so that one branch continues into the other and vice versa around this point, is called a *simple branch point* [*einfacher Verzweigungspunkt*].
>
> A point of the surface around which it winds $\mu+1$ times can then be regarded as the equivalent of μ coincident (or infinitely near) simple branch points. [16] [Riemann 1857, p. 110]

Although Riemann drew neither a single sketch nor a single drawing of branch points *in his published papers* (either on the complex line or on the surface itself), one can discover a few drawings of the local behavior of the sheets of a Riemann surface in a neighborhood of a branch point in the lecture series "Theorie der Functionen complexer Variabeln". At the end of the summer semester of 1861 when these lectures were coming to a close, Riemann dealt with "multivalued functions" ["Mehrwerthige Functionen"] [Neuenschwander 1996]. Riemann considers these as a surface T covering the z-plane, and first defines the branch point on the complex line z as "a point around which one sheet continues into

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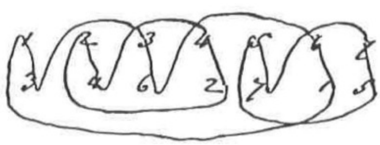
14 "Zur Fixirung der Vorstellungen denke man sich um den Punkt 0 in der Ebene A mit dem Halbmesser R einen Kreis beschrieben und parallel mit der x-Axe einen Durchmesser gezogen [...]." Translation taken from: [Baker et al. 2004, p. 23] (Cursive by M.F.).

15 [Baker et al. 2004, p. 80–81]: "[...] the different prolongations of a given function in a given region of the z-plane will be called *branches* of the original function and a point around which one branch continues into another a *branch-point* of the function." ([Riemann 1857, p. 90]: " [...] sollen die verschiedenen Fortsetzungen *einer* Function für denselben Theil der z-Ebene *Zweige* dieser Function genannt werden und ein Punkt, um welchen sich ein Zweig einer Function in einen andern fortsetzt, eine *Verzweigungsstelle* dieser Function.")

16 "Ein Punkt der Fläche T, in welchem nur zwei Zweige einer Function zusammenhängen, so dass sich um diesen Punkt der erste in den zweiten und dieser in jenen fortsetzt, heisse ein *einfacher Verzweigungspunkt.* Ein Punkt der Fläche, um welchen sie sich $(\mu+1)$ mal windet, kann dann angesehen werden als μ zusammengefallene (oder unendlich nahe) einfache Verzweigungspunkte." Translation taken from: [Baker et al. 2004, p. 101].

another is called a branch value of the function" [17] [Neuenschwander 1996, p. 74]. He then goes on to describe the neighborhood of a branch point on the surface using a visual metaphor: "In the neighborhood of such a point, the surface T can be regarded as a screw surface of infinitely small height of the screw thread, the axis of which is perpendicular to the z-plane at that point." [18] [Neuenschwander 1996, p. 74] Riemann then considers a surface (covering) of degree n, and looks at the n preimages $w_1, \ldots, w_n$ of the variable z when this variable runs through a closed curve, which encircle *several* branch points in the z-plane (but which does not go through them). [Neuenschwander 1996, p. 75] Indicating that the permutation can be decomposed into a composition of cyclic permutation, Riemann's aim is to describe the permutation of these preimages—each of which corresponds to a specific branch point. He accompanies this explanation with *two* drawings: "Let for example the initial sequence of the roots $w_1, w_2, w_3, w_4, w_5, w_6, w_7$, which changes, as a result of returning to the starting point of a closed path on the z-plane, to the following sequence: $w_3, w_4, w_6, w_2, w_7, w_1, w_5$. Thereupon one has, following the diagram [Schema],



the permutation cycle:

$$\begin{array}{ccc|cc|cc} w_1 & w_3 & w_6 & w_2 & w_4 & w_5 & w_7 \\ w_3 & w_6 & w_1 & w_4 & w_2 & w_7 & w_5 \end{array}$$

Each cycle is distinct from the others and indicates a *special branch point.*" [Neuenschwander 1996, p. 75–76] To explicate what is meant, Riemann draws the following figure (see Figure 2) in his lecture: [19] As can be seen here Riemann associated *diagrammatic* visualization (of the interchanging of the sheets, see Figure 2) with a *symbolic* illustration of the permutation

[17] "Ein Punkt, um welchen sich ein Blatt in ein anderes fortsetzt heißt ein Verzweigungswert der Function."

[18] "In der Umgebung eines solchen Punktes kann die Fläche T als eine Schraubenfläche von unendlich kleiner Höhe des Schraubenganges betrachtet werden, deren Axe in jenem Punkte senkrecht zur z-Ebene steht."

[19] As can be seen from Neuenschwander's transcription [Neuenschwander 1996, p. 76], not all students in Riemann's course found it necessary to draw this image in their notebooks. However, Neuenschwander notes that "such representations [of Riemann surfaces] later became very widespread" [Neuenschwander 1998, p. 23].

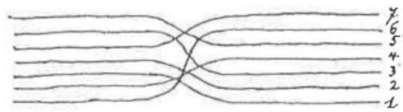
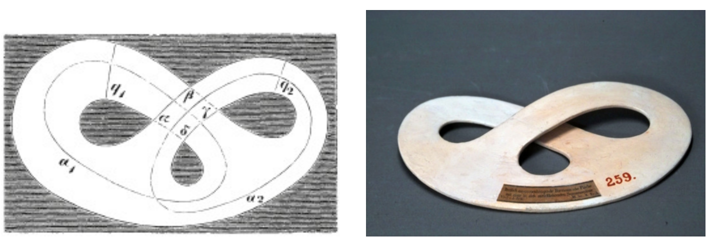


FIGURE 2. Riemann's drawing of seven strings, depicting, "geometrically speaking," "the order of the indices [...] how the layers [Blätter] of the surface T are intertwined with each other [ineinander übergehen]." [Neuenschwander 1996, p. 76]

cycle; contrary to Puiseux, diagrams have the same status as symbolic notation.

It is worth emphasizing that apart from Riemann's lecture notes and those of his students, he also drew figures to illustrate certain concepts in numerous of his papers. Riemann drew, for example, the double- and triply-connected surfaces—illustrations, according to which three dimensional models were prepared [see Figure 3.(I); Riemann 1857, p. 96].



(I) (II)

FIGURE 3. (I) Riemann's depiction of triply connected Riemann surface. The aim of these drawings is, according to Riemann, to make n-connected Riemann surfaces more "anschaulich" [Riemann 1857, p. 95]. (II) Material model of a triply connected Riemann surface. © 2019 Model collection of the Mathematical Institute, Göttingen University. The model was constructed in ca. 1888, and was mentioned in the 4th edition of Brill's catalogue *Catalog mathematischer Modelle* [Brill 1888, p. 36], where the author refers explicitly to Riemann's 1857 drawing. In addition, it is noted that "these models with thin walls are made of more durable material than plaster, in order to prevent accidental breakage."[20] [Brill 1888, p. 36]

20 "Diese Modelle mit dünnen Wandungen sind aus dauerhafterem Material als Gips hergestellt, um ein Zerbrechen bei zufälligem Auffallen zu verhüten."

What role did visualization play for Riemann? It is first useful to remember that in his 1854 talk *Über die Hypothesen, welche der Geometrie zu Grunde liegen* Riemann used the term *Mannigfaltigkeit* [manifold] in connection with 'magnitude'. He did this when he stated that he set himself "the task of constructing the notion of a multiply extended magnitude" [Riemann 1854, p. 273]—invoking various motivations when first using the term. When talking about continuous manifolds, the intuitions Riemann provides for choosing the term "Mannigfaltigkeit" are related to the positions of sensuous objects and colors:

> [...] occasions which give rise to notions whose measurement involves the consideration of continuous manifolds are so rarely encountered in everyday life that the location of material objects perceived through the senses, and colors [die Orte der Sinnengegenstände und die Farben], are perhaps the only simple examples of concepts whose modes of determination [Bestimmungsweisen] constitute a multi-dimensional manifold [Baker et al. 2004, p. 274, Riemann 1854, p. 258].

What Riemann meant by manifolds of color is clearly influenced by the philosopher Johann Friedrich Herbart.[21] Taking the concept of color, each particular color is a mode of determination [Bestimmungsweisen] of the general concept of color [Ferreirós 2007, p. 63], the totality of which forms a manifold. Therefore, as a colored point changes in the manifold of color, it does so continuously and "different colored points are thereby determined." [Banks 2013, p. 23] These references to colors as well as to objects of the senses already show a favorable attitude toward the role of visualization, as what is perceived through the sense of sight.

Secondly, it is in this context that Riemann sees his new conception of the concept "function" as what may also be regarded as "geometric investiture" ["geometrische Einkleidung"]—a means of "visualization" ["Zur Veranschaulichung"] [Riemann 1851, p. 40]. Needless to say, Puiseux in no way attempted to use such metaphors.[22] Riemann also mentions the "*anschauliche* meaning" of the principal curvatures of a surface [Riemann 1854, p. 281] or the "spatial *Anschauung*," respectively the "geometric in-

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21 "Herbart proposes a more or less psychological explanation of continuity, which emerges from the 'graded fusion' [abgestufte Verschmelzung] of some of our mental images [...]. His preferred examples were those of the [...] colors, which produce a triangle with blue, red and yellow at the vertices, and mixed colors in between." [Ferreirós 2007, p. 46]; See also: [Scholz 1982].

22 Note that Riemann did refer to Puiseux in his early lectures of functions of several variables [Neuenschwander 1996, p. 12]. Hence there was certainly a shift towards a more visual thinking, when comparing Riemann to Puiseux.

terpretation" of the Gaußian complex plane [Neuenschwander 1996, p. 5, Riemann 1851, p. 22]. Detlef Laugwitz notes that Riemann's treatment of branch points and branch cuts was done by means of a "visualization [of] the two leaves [Blätter], [which] penetrate along the branch cut". [Laugwitz 1996, p. 99] This is obviously noticeable when considering how Riemann visualized several branch points of the surface together (see Figure 2), pointing towards the way symbolic thinking (with the permutation group) and visual thinking are intertwined. The fact that Riemann had a favorable view regarding visualization and employing visual means and metaphors (such as "leaves" or "branches") to transfer his ideas was also noticed later, at the end of the 19th as well at the beginning of the 20th century. In 1886, Weierstraß criticizes Riemann's image of an n-layered Riemann surface as a "means of sensualisation" ["Versinnlichungsmittel"], which cannot be transferred to the case of functions with several variables [Weierstraß 1988, p. 144]. In 1925, Hermann Weyl indicates that Riemann used multi-layered Riemann surfaces for the "illustration of the multiple values of analytical functions" ["Veranschaulichung der Vieldeutigkeit analytischer Funktionen"] [Weyl 1988, p. 18].

However, it is also essential to emphasize that Riemann was also interested in the numerical relations between the different invariants of the curve. In his 1857 paper "Theorie der Abel'schen Functionen," Riemann notes that $2\,(p-1) = w - 2n$, where n is the degree of a covering $S \to \mathbb{CP}^1$, w the number of simple branch points, and p is the genus of the surface S [Riemann 1857, p. 114]. Zeuthen, Hurwitz, Chasles, Cayley, Brill and others gave proofs and generalizations of this numerical approach. The most well known generalization is the Riemann-Hurwitz formula for any covering of a Riemann surface $f: S \to T$,

$$2g - 2 = n \cdot (2g' - 2) + \sum_p (e_p - 1),$$

where g is the genus of S, g' is the genus of T, n the degree of f, and e_p the multiplicity of the branch point p. As we will see later at Section I.4, for Hurwitz these numerical relations were connected with algebraic reasoning, and not necessarily with visual reasoning.

* * *

It must be noted, however, that drawings of the *real part* of complex curves $f(x,y) = 0$ existed long before Riemann considered these curves as (branched) covering the complex line $\mathbb{C}$ or the projective line $\mathbb{CP}^1$—

such drawings were prompted by the introduction of analytic geometry.[23] These drawings also included drawings of *singular* curves in the real plane, and hence also of their singularities (nodes, cusps etc.). The problem was that once these curves were considered as Riemann surfaces, although the behavior of complex functions in the neighborhood of branch points (and hence also of singular points) became well understood from an analytical point of view, the two-dimensional drawings and three-dimensional material models that were made during the second half of the 19th century indicated that what was missing was a visualization of branch points as points of the Riemann surfaces, in addition to a clear depiction of a *neighborhood* of them.

It was Carl Gottfried Neumann (1832–1925), a German mathematician, known for his work on integral and differential equations and for his contributions to electromagnetism, who introduced, in his 1865 book *Vorlesungen über Riemann's Theorie der Abel'schen Integrale* a plurality of two-dimensional illustrations of branch points of Riemann surfaces, as well as of their neighborhood. As I will show in the later sections below, these figures were subsequently redrawn in different textbooks and manuscripts, and in addition, few three-dimensional models followed Neumann's two-dimensional figures while being constructed.

Neumann begins his book by indicating that the new theory of abelian and elliptical integrals is in somewhat of a poor state; however, this "grievance" can be resolved once "the new ways of *Anschauung* [neue Anschauungsweise]" of Riemann surfaces are taken into account. As we will see momentarily, these new "ways of *Anschauung*" are done with new methods of visualization.[24] Nevertheless, it also must be noted that this approach was not restricted to Riemann surfaces. Neumann explicitly notes that: "There are other parts of the mathematical science on which this way of *Anschauung* [jene Anschauungsweise] will probably have no less powerful effect." [Neumann 1865, p. iv] Thus, for example, Neumann mentions the Gauss plane of the complex numbers as what may enable not only the "obtaining [of] an *anschauliche* idea [Vorstellung]" of a pair of complex number, but also "to sensualize [versinnlichen] all pairs of values through it." [Neumann 1865, p. 45]. A similar expression appears later. When discussing an integral between two points, Neumann notes

23 Although these drawings were not necessarily thought of in terms of drawing a real part of a *complex curve*.

24 Indeed, Neumann calls for developing exactly these "new methods" [Neumann 1865, p. iv].

that one can “obtain an *anschauliches* image [Bild], when one is assisted with certain geometric ideas” [Neumann 1865, p. 63].[25]

If one considers how branch points were visualized, Neumann points not only towards new visualization techniques, but also towards the material construction of mathematical objects. Explicitly, chapter five of Neumann’s book deals exactly with the new ways of visualizing branch points, among other objects. Neumann presents in this chapter several illustrations, which depict branch points. One of them, being an illustration of a simple branch point, points towards the construction of a material model. Still using Riemann’s old coinage for the branch point, i.e., “Windungspunct” (and not “Verzweigungspunkt”), Neumann notes:[26]

> We shall [...] be able to obtain a restricted branched surface [Windungsfläche] by superimposing two flat circular surfaces (Fig. 34) [see Figure 4.(I)], slitting them along two superimposed radii, and then stitching together the opposite edges of the upper and lower slits namely the edge α with β' and β with α' [Neumann 1865, p. 165–166].

The point C, obtained via this cut and stitch process (see Figure 4.(I)), is the branch point. Here one cannot say that the “cut and stitch process” is simply a metaphor, since it clearly directs the reader to perform these material actions in order to “sensualize” the researched mathematical object. That said, the process of stitching does not have to be exact: Neumann then notes that the curve connecting the two edges should not be necessarily a straight line—it is in fact “inessential” that it would be straight—but rather the connecting curve can have an “arbitrary form [Gestalt]” [Neumann 1865, p. 166]. The illustration in Figure 4.(I) is therefore not exact and does not aim to be so; this is also clear from the fact that the Riemann surface does not intersect itself (although it might be so implied, when constructing the model materially; compare Section I.3, Figure 6).

Immediately after this section, Neumann suggests another way to visualize the behavior of the different sheets in the neighborhood of a branch point: this is done by sketching a circular loop on the Riemann surface, encircling the branch point on it [Neumann 1865, p. 167–168]. The loop obtained illustrates a path on the different sheets, describing their relation

25 “[Man kann] ein anschauliches Bild verschaffen, wenn man gewisse geometrische Vorstellungen zu Hülfe nimmt.“

26 “Wir würden [...] eine solche begrenzte Windungsfläche auch dadurch erhalten können, dass wir zwei ebene Kreisflächen (Fig. 34) übereinanderlegen, dieselben längs zweier über einanderliegenden Radien aufschlitzen, und sodann die entgegengesetzt liegenden Ränder des oberen und des unteren Schlitzes mit einander zusammenheften, nämlich den Rand α mit β' und β mit α'”.

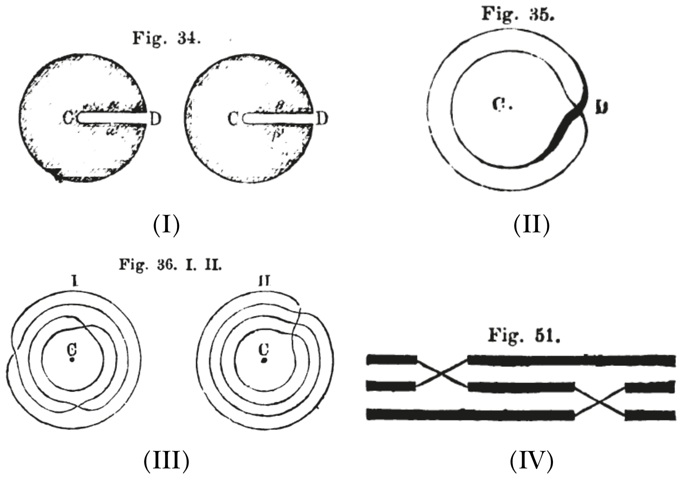

Figure 4. Three of Neumann's suggestions for visualizing branch points and their neighborhoods: (I): stitching together two circles to depict a simple branch point [Neumann 1865, p. 166]. (II), (III): Sketch a loop on the Riemann surface, encircling a simple branch point and two branch points of order 3 [Neumann 1865, p. 167, 168]. (IV): A section of a Riemann surface in the neighborhood of two branch points [Neumann 1865, p. 199].

to the branch point. While Figure 4.(II) describes this loop encircling a simple branch point, corresponding to Figure 4.(I), Figure 4.(III) depicts the behavior of the sheets in the neighborhood of two different branch points of order 3; producing a material model of these two types would be more problematic from a material point of view, and Neumann indeed does not even suggest it.

An even more simplified representation is later suggested by Neumann—indeed, he notes explicitly that the surfaces are "represented [dargestellt]" by these new figures [Neumann 1865, p. 198]. These figures represent the behavior of the sheets in a neighborhood of branch points, as can be seen in Figure 4.(IV) (compare Riemann's drawing in Figure 2). In this Figure Neumann draws a (plane) section of the Riemann surface of the function $\sqrt[3]{\frac{z-A}{z-B}}$ in the neighborhood of the two branch points A and B [Neumann 1865, p. 199]. One could say that Neumann draws a braid here. This method of presenting a section of a neighborhood of a branch point as a braid we already saw in Riemann's lectures and will be seen again in the works of Francesco Severi and Federigo Enriques, among other mathematicians, in the 20th century (see Section I.5). However, already here it is important to note that Neumann emphasizes that this depiction is not exact: the fact, that "the lines [in Figure 4.(IV)] are straight and lie one over the other is non essential." [27] [Neumann 1865]

1.2. *Klein and the new type of Riemann surfaces*

Despite the many new ways Neumann visualized branch points, these also indicated a need for more exact visual aids. What is clear, however, is that Neumann's approach towards visualization was more explicit than Riemann's and more sympathetic than Puiseux's. Neumann's attitude is clearly expressed in his attempts to "sensualize" the new *anschauliche* methods. This is done either by means of diagrams or materially, through cutting and gluing pieces of paper. Yet Neumann also emphasized that these new methods may be inaccurate—or to be more exact, that their accuracy of representation is not essential. This resulted in the above necessity, which is especially apparent in a series of papers Felix Klein wrote on Riemann surfaces, their singularities and their dual curves written between 1874 and 1876. [28] Klein notes at the beginning of his 1874 paper "Über eine neue Art der Riemannschen Flächen": [29]

> In the investigation of the algebraic functions y of a variable x, two different *illustrative aids* [*anschauungsmäßiger Hilfsmittel*] are employed. One represents either y and x uniformly as the coordinates of a point of the plane, where the real values of the plane alone are represented, and the image [das Bild] of the algebraic function becomes the algebraic curve; or that one spreads the complex values of the variable x over a plane [i.e., $\mathbb{C}$] and denotes the functional relations between y and x by the Riemann surface constructed over the plane. It must be desirable in many respects to have a transition between these two *illustrative images* [*Anschauungsbildern*] [Klein 1874, p. 558].

Presenting the two methods of visualizing algebraic functions: either considering only the real part of the complex curve in $\mathbb{R}^2$, or considering

27 "Dass diese Linien gerade über einander liegen, ist unwesentlich."

28 See: [Parshall & Rowe 1994, pp. 168–169].

29 "Bei der Untersuchung der algebraischen Funktionen y einer Veränderlichen x pflegt man sich zweier verschiedener *anschauungsmäßiger Hilfsmittel* zu bedienen. Man repräsentiert nämlich entweder y und x gleichmäßig als Koordinaten eines Punktes der Ebene, wo dann die reellen Werte derselben allein in Evidenz treten und das Bild der algebraischen Funktion die algebraische Kurve wird—oder man breitet die komplexen Werte der einen Variabeln x über eine Ebene aus und bezeichnet das Funktionsverhältnis zwischen y und x durch die über der Ebene konstruierte Riemannsche Fläche. Es muß in vielen Beziehungen wünschenswert sein, zwischen den beiden *Anschauungsbildern* einen Übergang zu besitzen.“ (Cursive by M.F.)

the surface embedded in a three-dimensional space obtained from considering x as a complex variable in the $\mathbb{C}$ (having two coordinates) and the real values of y—Klein finds both unsatisfying. The first is only an image ("Bild") of the real part of the function—hence incomplete; the second, though being a complete image ("ein vollständiges Bild" [Klein 1874, p. 559]), does not represent singularities (such as nodes) well, likewise neither inflection points nor branch points. The problem was therefore to give a satisfying "Anschauungsbild," that would account for the unique structure of the curve, and hence, for its branch points.

Klein's solution was to investigate, together with the given curve, another curve: the dual curve in the projective complex plane. The dual curve, whose points correspond to the set of lines tangent to the original curve C, is obtained by sending each point to the point dual to its tangent line.[30] Most of Klein's drawings did not consider the visualization of branch points and singular points as points on a Riemann surface, but rather followed Plücker's investigation of the connections between the invariants of algebraic curves to the corresponding invariants of their dual curves; specifically, Klein aimed also towards "visualizing [Veranschaulichung]" these relations [Klein 1876, p. 404]. However, at the end of his 1874 paper a sketch is given of how a branch point of a singular curve of the third degree looks like: "one awards to the [Riemann] surface an [...] outgoing branching [...], as illustrated [in Figure 5], for example, in a symmetrical manner, by the drawing."[31] [Klein 1874, p. 566]

However, from the sketch alone it is not clear how the different branches "interact" with each other or how one may pass from one branch to the other. Although the different directions of the drawn diagonal lines on each layer refer to different visual perceptions (or even to different haptic sensations, as if touching a similar engraved surface would transmit the difference between the layers), Klein in no way elaborates on this. This problematic, of how actually the different branches "behave," is only dealt with visually in the 1880s.

1.3. *Material models and the coloring of branch points*

Several years later, Klein's ideas were taken a step further. It is essential to note that the drawings in Klein's series of papers from 1874–1876 were not the only suggestion that he made for visualizing Riemann surfaces. At

30 Note that the dual curve $C^{\vee}$ is a curve in the dual projective space $(\mathbb{CP}^2)^{\vee}$.

31 "[...] man [erteilt] der Fläche eine [...] ausgehende Verzweigung [...], wie sie etwa, in symmetrischer Weise, durch die beigesetzte Zeichnung veranschaulicht ist."

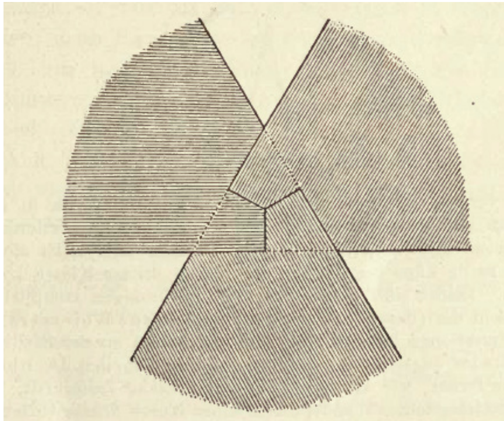

FIGURE 5. Klein's visualization of an order 2 branch point [Klein 1874, p. 566]. Compare Figure 6.

the end of this section and also in Section I.4, I will discuss Klein's other visualizations. One should also note that Klein's paper "Über die Transformation siebenter Ordnung der elliptischen Funktionen," appearing four years after his 1874 paper, points to a *three*-dimensional model made by the mathematician Walther von Dyck (1856–1934) in order to visualize a Riemann surface; this was done while investigating the simple group of 168 elements [Klein 1879, p. 132, footnote 29]. [32] In this article Klein also declares that his goal is to "design the most *anschauliches* image [Bild] of the branching of the Riemann surface" [Klein 1879, p. 91]. [33] To emphasize: in Neumann's figures (see Figure 4 in this present article) as well as in Klein's figure (see Figure 5), the neighborhood of the branch point (of the discussed Riemann surface) is visualized via *two*-dimensional images ["Bild"] of (the real part of) a Riemann surface, which is embedded in a three-dimensional space. One may ask, and it is something I will address later (see Section I.6), what Klein precisely means by "the most *anschauliches* image," when several models are given. Dyck's three-dimensional model indicates that it is with the popularization and spread of the material models of mathematical objects in Germany during the last quarter of the 19th

32 For a drawing of this model, see: [Gray 1982, p. 65]. The Riemann surface is now known as the Klein's quartic (having the equation $x^3y + y^3z + z^3x = 0$), and the group is the automorphism group of this curve.

33 "[...] ein möglichst anschauliches Bild von der Verzweigung der Riemannschen Fläche zu entwerfen."

century that new methods of visualizing branch points were promoted, as already noted above.

The tradition of manufacturing models of mathematical objects (from plaster, wood, cardboard, strings etc.), which began in France from the second quarter of the 19th century, became abundant in Germany starting from the 1860s. [34] Several leading mathematicians—such as Klein, Alexander Brill and Dyck—supported and advocated the dissemination of such models, even if the entirety of the mathematical community in Germany did not support this *anschauliche* geometry. The aim of these models, as was the vision of Klein and Brill, was the transmission of mathematical knowledge visually as well as materially (and certainly not only symbolically). In that sense, the material models were epistemological, as they prompted the emergence of new knowledge. [35] This transmission—not via formulas, or via the proof of theorems—was considered not only as a legitimate activity, but also as what offers a complementary view concerning the mathematical object. [36] The influence on the research was clear, at least for Brill and Klein. Thus, for example, Klein remarked in 1872 that: "For geometry a model—be it realized and observed or only vividly imagined—is not a means to an end but the thing itself" [37] [Klein 1872,

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[34] A famous example is the various models from plaster of the cubic surface with the 27 real lines on it, made by, among others, Christian Wiener and Adolf Weiler. See: [Rowe 2013; Tobies 2017]. For recent studies on material mathematical models during the end of the 19th century, see e.g.,: [Mehrtens 2004; Rowe 2017; Sattelmacher 2013; Schubring 2017].

[35] See: [Mehrtens 2004, p. 289–291] regarding models as "epistemic things". See also: [Sattelmacher 2013].

[36] The clearest indication to that is the relatively high number of mathematical exhibitions, aimed either to scientists or also to the general public. These exhibitions took place during 1876 and 1925 in London, Munich, Chicago, Heidelberg and Edinburgh, among other places. In the German exhibitions especially, there was a greater emphasis on the situation of the then contemporary mathematics and making such mathematics visual and haptic. This occurred precisely during the period when mathematical objects were beginning to be considered in a more non-visual way. A key moment was the 1893 exhibition in Munich, as well as the later exhibition in Chicago in the same year. See: [Hashagen 2015; 2003, p. 425–436].

[37] "Ein Modell—mag es nun ausgeführt und angeschaut oder nur lebhaft vorgestellt sein—ist für diese Geometrie nicht ein Mittel zum Zwecke sondern die Sache selbst." Translation taken from [Mehrtens 2004, p. 289]. Mehrtens notes that with this remark, Klein distances himself from the strict abstract nature of his program. Moreover, Klein also emphasizes the pedagogical role of material models, as they serve the visualization of mathematical objects ("Die Anschauung hat für ihn [den mathematischen Inhalt] nur den Werth der Veranschaulichung, der allerdings in pädagogischer Beziehung sehr hoch anzuschlagen ist. Ein geometrisches Modell z. B. ist auf diesem Standpuncte sehr lehrreich und interessant. " [Klein 1872, p. 42])

p. 42]. At the end of the 1880s Brill notes the success of this tradition, essentially contributing to research on mathematical objects. Indeed, he claimed in 1887: "Often the model prompted conversely subsequent, retroactive investigations regarding the peculiarities of the presented form" [38] [Brill 1887, p. 77]. Klein was one of the supporters of the material model tradition in particular and of visualization in general. In a lecture given in Chicago in 1893, he notes that "mathematical models and courses in drawing are calculated to disarm [...] the hostility directed against the excessive abstractness of university instruction [in mathematics]" [Klein 1911, p. 109]. Two years later, in the lecture "Über Arithmetisierung der Mathematik" given on the 2 November 1895, Klein notes that for him, while—as noted above—one should look for the most "*anschauliches* Bild," namely models as well as two-dimensional drawings, this *Anschauung* is twofold: both cultivated (with the help and influence "of logical deduction") and "naive *Anschauung*, largely a natural gift, which is unconsciously increased by minute study of one branch or other of the science." Moreover, Klein maintains "that mathematical *Anschauung*—so understood—is always far in advance of logical reasoning and covers a wider field" [Klein 1896, p. 246]. Hinting towards the connection between the mathematical *Anschauung* and models and drawing, Klein adds: "Modern psychologists distinguish between visual, motor and auditory characteristics; mathematical *Anschauung* [...] appears to belong more closely to the first two classes [...]" [Klein 1896, p. 247]. It is no coincidence that Klein relates the visual and the haptic (i.e., the "motor") to the mathematical *Anschauung*. Indeed, in 1892, Walther von Dyck asked the physicist Ludwig Boltzmann, the editor of the catalogue of the exhibition *mathematisch-physikalischer Apparate, Modelle und Instrumente*, to write a contribution entitled "Über die Methoden der theoretischen Physik". Klein was heavily involved in the preparation of this exhibition, and wrote the opening article in the catalog, called "Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen" [Klein 1892]. In the same part of the catalogue, where Klein's paper appears, Boltzmann describes in his contribution that material mathematical models served "to make the results of our calculation perceptible [anschaulich zu machen] and that not merely by the imagination [Phantasie], but visible to the eye and at the same time palpable to the touch by means of gypsum and cardboard" [Boltzmann

[38] "Öfter veranlasste umgekehrt das Modell nachträgliche Untersuchungen über Besonderheiten des dargestellten Gebildes."

1892, p. 90; translation taken from: Boltzmann 1915, pp. 201–202].[39] What is evident in Boltzmann's 1892 conception—which, as I claim, Klein follows—is his differentiation between the senses: the seeing eye versus the touching hand. When a merger supposedly occurs, or at least an agreement, this happens between two senses with the help of the material model and the drawing.

Given this background, it is not at all surprising to find also material, three-dimensional models of branch points of Riemann surfaces; and indeed, two types of material models appeared starting from the middle of 1880s,[40] and both were presented during the 1893 exhibition in Munich and later in the catalogue of the exhibition. The exhibition took place during the third annual meeting of the German Mathematical Society (*Deutsche Mathematiker-Vereinigung*) organized by Dyck, which consisted mostly of mathematical models.

The first model (see Figure 6) might be considered as a three-dimensional realization of a Riemann surface in the neighborhood of a branch point of order 2, as Neumann depicted in Figure 4.(I) (for a simple branch point) and as Klein depicted in Figure 5. The model, described in the fourth edition of Brill's *Catalog of mathematical models* [Brill 1888, p. 48] and in Dyck's catalogue as one of three models of different Riemann surfaces, is described simply as a model of a "*three-leaves simply connected Riemann surface*, which has at its center a branch point of the second order."[41] [Dyck 1892, p. 176] Comparing it to the two-dimensional image that Klein drew, this three-dimensional model is a better visualization of what the Riemann surface looks like. The arrows drawn on this model also help to understand visually what happens on the surface, in terms of monodromy, when the Riemann surface is considered as a covering of the complex line; the arrows show the movement of the different points on the surface, while—when taking into account Puiseux's ideas—considering the preimages moving along a small loop (on the complex line) encircling the point 0 being the branch point. This model, though it is not known exactly when it was made, shows visually what Puiseux described algebraically.

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39 "[...] die Resultate des Calcüls anschaulich zu machen und zwar nicht blos für die Phantasie, sondern auch sichtbar für das Auge, greifbar für die Hand, mit Gips und Pappe."

40 It is essential to emphasize that the earliest catalog of mathematical models, which lists the models discussed in this section, is Brill's 1888 catalog. Brill's catalog of mathematical models from 1885 [Brill 1885] does not present these models.

41 "Modell einer *dreiblättrigen einfach zusammenhängenden Riemann'schen Fläche*, welche in ihren Innern einen Windungspunkt *zweiter* Ordnung enthält."

FIGURE 6. A model of a branch point of second order on a Riemann surface, preserved in the Göttingen Collection of Mathematical Models and Instruments (compare with Figure 2). A similar model of a branch point of first order on a Riemann surface was also produced. © 2019 Model collection of the Mathematical Institute, Göttingen University.

Nevertheless, this model might have been as problematic: an untrained student might have thought, seeing the model, that the Riemann surface intersects itself, which is not at all the case. The problem with this visualization stems, as noted above, from the attempt to visualize a complex curve in $\mathbb{C}^2$ as an object in a three-dimensional space $\mathbb{R}^3$. Whereas it might be thought that the curve intersects itself when only looking at the model, this confusion actually stems from ignoring the different colors drawn on the model. The "self intersection" is only due to the fact that for a complex curve, given by the equation $f(x, y) = 0$ (when $x, y \in \mathbb{C}$) only a three-dimensional section is (and can be) presented, i.e., of the points $(\mathrm{Re}(x), \mathrm{Im}(x), \mathrm{Re}(y))$. This mode of presentation causes points, which only differ in their $\mathrm{Im}(y)$-values, to coincide in the three-dimensional model. The way to show that these points are actually different was to color the model: in the neighborhood of the "intersection line" there are different colorings, i.e., black lines are drawn on one layer, in order to differentiate it from the second "intersecting" layer, which is white. This is done in order to make it visually clear that these layers do not really intersect, and the method shows that coloring the model had a unique function of its own: to designate the fourth coordinate.

The second model presented at Dyck's 1893 exhibition, which was already produced in 1886 under the guidance of Dyck in Munich at the *Technische Hochschule*, is somewhat more surprising. Under the heading "16 models for presenting of functions of complex variables," the description in Dyck's catalog begins with addressing the above-mentioned difficulty: [42]

> The present series of models was developed following an introductory lecture on function theory. The difficulty of a *visual description* [*anschaulichen Schilderung*] of the behavior of a function in the neighborhood of singular points led to the desire, to have also in this field, and at least for the most important singular points, the means of *spatial intuition* [*räumlicher Anschauung*], which has proven itself during teaching so appropriate and favorable in a number of other fields.
>
> In order to *visualize by a spatial representation* [um ... durch eine *räumliche Darstellung zu veranschaulichen*] the behavior of a function of a complex variable in the neighborhood of certain singular points, and also the whole behavior of certain types of functions of a complex variable, both the real and the imaginary part of the value of the function are considered as being a coordinate above the plane of the complex argument. Thus, every function of a complex argument is represented by two surfaces, which are denoted by R and I, the simultaneous consideration of which gives an *image* [*Bild*] of the behavior of the function [Dyck 1892, p. 176].

The goal of the models was to present visually and materially—in a more precise way than the depiction in Figures 1 or 2—the way complex functions behave in the neighborhood of branch points. Indeed, while Dyck notes explicitly that he is interested in the visualization of the neighborhood of "singular points," several of the functions, which are investigated, are not singular (e.g., the function $w^2 = z^2 - 1$) but rather

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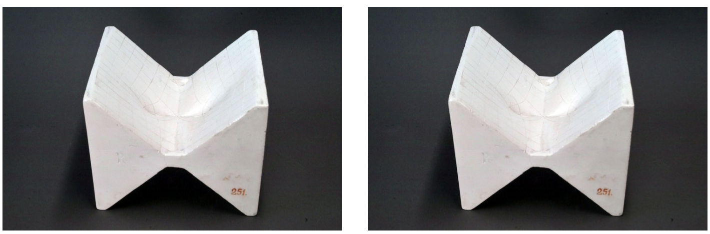
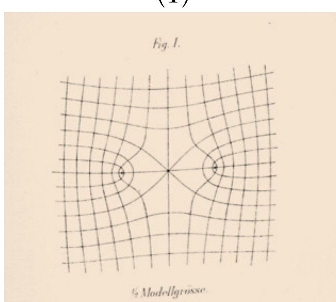
42 "Die vorliegende Serie von Modellen ist entstanden im Anschluss an eine einleitend Vorlesung über Functionentheorie. Die Schwierigkeit einer möglichst anschaulichen Schilderung des Verhaltens einer Function in der Umgebung singulärer Stellen liess den Wunsch aufkommen, auch auf diesem Gebiete und wenigstens für die wichtigsten singulären Vorkommnisse das Hilfsmittel räumlicher Anschauung zu besitzen, das schon auf einer Reihe anderer Gebiete so zweckmässig und fördernd im Unterricht sich erwiesen hat.

Um den Verlauf einer Function einer complexen Veränderlichen in der Umgebung gewisser singulärer Stellen und ebenso den Gesamtverlauf gewisser Typen von Functionen einer complexen Veränderlichen durch eine *räumliche Darstellung zu veranschaulichen*, sind in der bekannten Weise sowohl der reelle als auch der imaginäre Teil der Functionswerte über der Ebene des complexen Argumentes als Ordinaten aufgetragen. So wird jede Function eines complexen Argumentes durch zwei mit R und I bezeichnete Flächen versinnlicht, deren gleichzeitige Betrachtung ein *Bild* des Functionsverlaufes liefert." (cursive by M.F.) An almost identical text appears in Brill's 1888 *Catalog mathematischer Modelle*, when describing these models [Brill 1888, pp. 48–49]: models 173–182 are described as "16 Models for the representation of functions of a complex variable. Executed under the direction of Prof. Dr. Walther Dyck." ("16 Modelle zur Darstellung von Functionen einer complexen Veränderlichen. Ausgeführt unter Leitung von Prof. Dr. *Walther Dyck.*"). The passage that follows is identical to the second passage in the above citation.

have branch points when considered as a covering. Comparing this to Klein remarks in 1874, that the three-dimensional surface is a complete image ("ein vollständiges Bild"), it seems that Dyck rather urged that the consideration of *two complementary* images, or models, were needed. Given a complex function $f(w, z) = 0$, Dyck guided his students to construct two models, while considering w as a complex variable on the plane: the first, by considering the values $(\mathrm{Re}(w), \mathrm{Im}(w), \mathrm{Re}(z))$ as a three-dimensional surface (denoted by R in the above citation); the second, considering the values $(\mathrm{Re}(w), \mathrm{Im}(w), \mathrm{Im}(z))$ as a second three-dimensional surface (denoted by I in the above citation). Thus, *two* three-dimensional models are presented for several functions. For example, the two models for the function $w^2 = z^2 - 1$ are presented in Figure 7.(1), where the branch points are located above $z = 1$ and $z = -1$.



(1)



(2)

FIGURE 7. (1) The two models, produced by A. Wildbrett under the guidance of Dyck [Brill 1888, p. 49], of the surface $w^2 = z^2 - 1$, as preserved in Göttingen. Left: the real (R) part of the surface $w^2 = z^2 - 1$; right: the imaginary (I) part of the surface $w^2 = z^2 - 1$. © 2019 Model collection of the Mathematical Institute, Göttingen University. (2) The "orthogonal system" of the surface $w^2 = z^2 - 1$, apparently belongs to the real part, which was used to construct the two models [Dyck 1886, p. plate 1, Figure 1].

Indeed, these models are more surprising. Not only are they an alternative to Klein's proposals regarding the visualizations of Riemann surfaces (with the dual curve), but they also show the limits of the visualization of Riemann surfaces. Klein already noted the two common methods to visualize complex curves (drawing only their real part or producing the material model, i.e., the surface R in the above modeling) are insufficient. The above method attempts to fix this insufficiency, but it does so only by highlighting the fact that the complex curve defined by the equation $f(w, z) = 0$ in fact naturally "lives in" $\mathbb{C}^2$, that is, in a four-dimensional real space. Every one of the two models R and I are hence only partial models of the curve. The question thus rises—what in fact the visual relations between the two models and the curve itself are? What are the relations between these models and the (drawing of the) real part of this curve (i.e., when one considers the hyperbola $w^2 = z^2 - 1$ as a curve on the real plane)? And what is the role of the solid, plaster model, when it is clear that only the points lying *on* the surface are points that satisfy the equation of the (real or imaginary part of the) surface? While the analytical relations are clear, the models themselves need an elaborate explanation to enable a "transition" between them as Klein suggested, in order that they will actually serve the goals that the *anschauliche* geometry posited.

Indeed, while Dyck (or one of his students) writes in a manuscript that documents these models, that they should "sensualize [versinnlichen] the behavior in the multi-valued function in the neighborhood of a branching point" [43] [Dyck 1886, p. 4] the explanation that follows concentrates on the analytical description of the surfaces. As Dyck describes, for the surface $w^2 = z^2 - 1$, substituting $w = u + iv$, $z = x + iy$ in the equation and separating the real and the imaginary part of w, one obtains the surfaces R and I as two real surfaces (in a three-dimensional real space) of the 4th degree: [44]

$$(R.) \quad u^4 - (x^2 - y^2 - 1)u^2 - x^2y^2 = 0$$

$$(I.) \quad v^4 + (x^2 - y^2 - 1)v^2 - x^2y^2 = 0$$

As the manuals concisely describe, in order to construct these models an "orthogonal system" had to be drawn as a two-dimensional figure, taking

43 "[...] das Verhalten in der mehrwertigen Function in der Umgebung eines Verzweigungspunkts versinnlichen."

44 Moreover, as Gerd Fischer notes, the individual sheets of the real and the imaginary part of the models were indicated by coloring. However, these colorations have not survived into the 21st century, as can be seen in Figure 4. See [Fischer 1986b, p. 78]. See also: [Fischer 1986a, photos 123, 124, p. 120–121].

$u = constant$ and $v = constant$, and obtaining a series of curves which may be considered as level curves [Fischer 1986a]. An example for this system is presented in Figure 7.(2), where the branch points are the points, which in their neighborhood the squares of the orthogonal system become increasingly smaller.

To sum up this section, it is worth noting that in 1897 Klein himself presented another drawing of branch points, this time referring implicitly, via a two-dimensional drawing, to the three-dimensional model of the branch point (see Figure 6). In the first volume of the book *Vorlesungen über die Theorie der automorphen Functionen*, written by Klein and Robert Fricke, the authors sketch "inner branch points [innere Verzweigungspunkte]" [Fricke & Klein 1897, p. 372] (see Fig. 8). As can be seen from the drawing, from the inner point a line exits, one side of it is darkened, the other side is bright. This not only depicts another way of visualizing branch points, but also that the gradual darkening used differentiates between the different values around the branch point, and that the supposed self-intersection does not take place in the four-dimensional space.

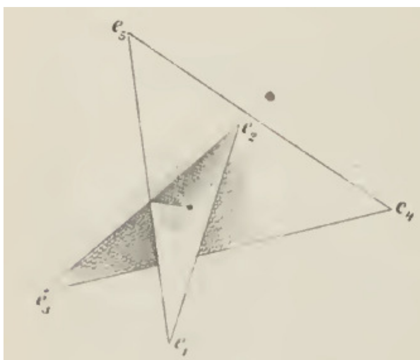


FIGURE 8. Different coloring in the neighborhood of a branch point (to be seen at the center of the image); Cf. the colored model in Figure 6.

These various illustrations and models [45] already indicate a plurality of two- and three-dimensional practices of visualization. I will discuss the epistemological implications of this plurality later (see Section I.6), but one may already ask what the implications of such plurality were. I claim that one possible implication might be a turn towards an approach being less

45 The same models and almost identical descriptions also appear in subsequent catalogs, see: [Schilling 1903, p. 119–120]; [Schilling 1911, p. 159].

visually polymorphic, i.e., an algebraic-symbolical approach, as can be seen with Adolf Hurwitz.

1.4. *Hurwitz and the shift to the algebraic approach*

Not every German mathematician during the 1890s who investigated branch points thought their visualization necessary, either materially or with a drawing. This is seen with Adolf Hurwitz's 1891 paper "Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten". The goal of the paper was "to investigate the totality of the n leafed Riemann surfaces, which are branched in a prescribed manner at w points,"[46] [Hurwitz 1891, p. 2] and one of the questions Hurwitz would like to answer is what is the number N of "n-leafed" Riemann surfaces (that is, of degree n), which are branched along the given w branch points.

It is already clear from the references Hurwitz mentions that he was well aware of the traditions mentioned above regarding the visualization of Riemann surfaces in general, and of branch points in particular. For example, Hurwitz refers at the beginning of his paper to Klein's article "Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale" [?]. In this article Klein used a new approach "to illustrate the intimate connection between the genus p of a surface and the number of crossing points that arise from the flows on it." [Parshall & Rowe 1994, p. 179] Inspired from potential theory, current flows on surfaces and Maxwell's *Treatise on Electricity and Magnetism*,[47] Klein presented new visualizations related to Riemann surfaces by means of numerous figures and sketches of

46 "Die Gesammtheit der n blättrigen Riemann'schen Flächen zu untersuchen, welche an w gegebenen Stellen in vorgeschriebener Weise verzweigt sind". I follow here also the analysis given in [Epple 1999, p. 186–192].

47 It is worth noting that Maxwell's concept of "analogy" between the physical phenomena he researched points to and echoes the plurality of visualization techniques and their epistemological role—not only in pure mathematics but also in mathematical physics—as treated thoroughly by Karin Krauthausen. Following Krauthausen [2014], while developing his theories on electricity and magnetism, extending Michael Faraday's work, Maxwell aimed in 1855 at finding a "geometrical *model* of the physical phenomena" [Maxwell 1856, p. 158] (cursive by M.F.), using analogies as a "method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science." [Maxwell 1856, p. 156] One might argue that Maxwell's conception of the model reflects the plurality of visualization techniques taking place in mathematical research (recall Maxwell's 1874 clay model of the thermodynamic surface [Maxwell 2002, p. 148]), as described in this paper for the case of branch points; however, the influences between mathematical physics and pure mathematics regarding these techniques of modeling and visualization is beyond the scope of the present paper. See however [Epple 2016, p. 15–18].

flows on various Riemann surfaces (see Figure 9) in his article.[48] Hurwitz also refers to a paper by Dyck from 1880, in which the latter offers a visual model of Riemann surfaces: "in so far as our subsequent investigations are essentially guided by Riemann's surface itself, it is first and foremost a question of their *visual [anschauliche] presentation*."[49] [Dyck 1880, p. 477]

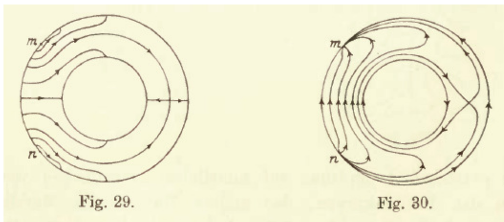


FIGURE 9. Two of Klein's drawings of currents on Riemann surfaces [?, p. 536].

Returning to Hurwitz's paper, how does he approach the problem of counting the number N of Riemann surfaces of degree n branched over w points $a_1, \dots a_w$ lying at a plane E? As Moritz Epple notes [Epple 1999, p. 187], the beginning of Hurwitz's discussion is very similar to Puiseux's treatment. Hurwitz first chooses a point O on the two-dimensional real plane E and then draws non intersecting paths $l_1, \dots, l_w$ from O to $a_1, \dots, a_w$. It is essential to note here that in his paper Hurwitz mentions two other articles that dealt with the question of the construction and representation of Riemann surfaces. The first is Lüroth's "Note über Verzweigungsschnitte und Querschnitte in einer Riemann'schen Fläche" [Lüroth 1871], which constructs a set of loops in a *certain order* on the complex line (here the plane E), encircling the branch points. The permutations induced from the loops indicate how the Riemann surface is to be constructed. The second is Clebsch's "Zur Theorie der Riemann'schen Flächen" [Clebsch 1872], continuing Lüroth's work by proving that in fact any set of loops can be taken when considering effects on the induced permutations while changing the set of loops. In contrast

48 See: [Parshall & Rowe 1994, p. 177–182] regarding Klein's conception of Riemann surfaces in the above context.

49 "[...] insofern unsere folgenden Untersuchungen wesentlich an der Riemann'schen Flache selbst geführt werden, handelt es sich zuvörderst um deren *anschauliche Darstellung*." Dyck then offers to think a Riemann surface not as a covering of the plane, when the layers are one *over* the other, but rather that the different layers are *side by side*.

to Lüroth, Clebsch draws a figure, which illustrates the change of the loops in this set (see Figure 10).

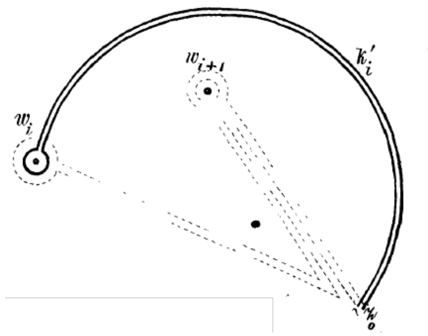


FIGURE 10. Given two branch points w_i and w_{i+1} on the complex line $\mathbb{C}$ and two loops (presented in dashed curves) encircling them, Clebsch [Clebsch 1872, p. 219] draws a new set of loops (k_i'), encircling w_i.

This visualization indicates that Clebsch maintained the position that Riemann surfaces and their construction according to the branch points should be visualized. Hurwitz transformed this point of view into an algebraic investigation.

Following Clebsch's and Lüroth's works, Hurwitz constructed the surface itself: "The Riemann surface is now formed by connecting the n leaves along the cuts $l_1, \dots, l_w$ in the following manner." [50] [Hurwitz 1891, p. 4] Hurwitz does draw several drawings of this system of paths [Hurwitz 1891, p. 34, 36], but as becomes clear, his intention *in this research* is *not* following the tradition of *anschauliche* geometry, [51] which might have led him to visualize the surface itself, but rather follows a more algebraic-combinatorial practice. Indeed, instead of continuing by describing the construction of the Riemann surface visually, Hurwitz expresses the problem using algebraic terms and conditions. For each of the paths $l_1, \dots, l_w$ Hurwitz assigns a permutation: S_i, where $1 \leqslant i \leqslant w$. If $S_i = \begin{pmatrix} 1, & 2, \dots, & n \\ \alpha_1 & \alpha_2, \dots, & \alpha_n \end{pmatrix}$ is the permutation sending 1 to α_1, 2 to α_2 etc., then along the path l_i the leaves numbered $1, \dots, n$ of the Riemann surface are connected to the

50 "Die Riemann'sche Fläche entsteht jetzt, indem man die n Blätter längs der Schnitte $l_1, \dots, l_w$ in folgender Weise mit einander verbindet."

51 In other mathematical investigations, Hurwitz did visualize his geometrical constructions, for example, by folding a paper. See [Oswald 2015].

leaves $\alpha_1, \ldots, \alpha_n$. Here Hurwitz expresses the necessary and sufficient conditions for the existence and construction of a Riemann surface in algebraic terms:

> The substitutions $S_1, S_2, \ldots, S_w$, which are chosen for the construction of the surface, should only satisfy the following two conditions: (I) The transition from any element to any other is to be possible by substitutions. (II) The composition of all substitutions is to give the identity, i.e., $S_1 S_2 \cdots S_w = 1$.[52] [Hurwitz 1891, p. 4]

The first condition forces the group to be transitive, which is equivalent to saying that the surface constructed would be connected. The second condition ensures that while circling the point O (which is not a branch point), the order of the leaves will not be permuted. What is essential to note is that via this algebraic formulation, it becomes irrelevant *where* the branch points $a_1, \ldots, a_w$ are located on the complex line $\mathbb{C}$, and the initial question was "reduced to a purely group theoretic question" [Epple 1999, p. 187]. Therefore Hurwitz has proved that while one can construct the Riemann surface topologically when giving a number of sheets, the position of its branch points, and the permutations describing the number of sheets, can thus only give the algebraic data *equivalently*.

This is clearly to be seen in the way Hurwitz considers a Riemann surface, as an *w-tuple* of permutations: $(S_1, S_2, \ldots, S_w)$, where "obviously, the substitution S_i immediately gives the 'type' of the branching at the point a_i."[53] [Hurwitz 1891, p. 6] How this branch point may be visualized is completely irrelevant, as this aspect is replaced with an algebraic element: a permutation. And to emphasize this point of view, the question whether two Riemann surfaces, symbolized by two w-tuples $(S_1, S_2, \ldots, S_w)$ and $(S'_1, S'_2, \ldots, S'_w)$, are topologically the same, is equivalent to answering the question whether there is a permutation T such that for every i, $1 \leqslant i \leqslant n$, $S_i = TS'_iT^{-1}$. Moreover, the question concerning the number N is answered via *combinatorial* arguments,[54] and the treatments of

52 "Die Substitutionen $S_1, S_2, \ldots, S_w$, welche man zur Herstellung der Flache wählt, sollen nur folgenden beiden Bedingungen genügen: I) Vermöge der Substitutionen soll ein Uebergang von jedem Element zu jedem andern möglich sein. II) Die Zusammensetzung aller Substitutionen soll die Identität ergeben, es soll also $S_1 S_2 \cdots S_w = 1$ sein."

53 "Offenbar ergiebt die Substitution S_i sofort die 'Art' der Verzweigung in dem Punkte a_i."

54 See for example: [Hurwitz 1891, p. 7–22]. According to [Epple 1999, p. 188–192], the other sections of the paper deal with what may be termed the first mathematical appearance of the braid group and the pure braid group, when considering the move-

Clebsch and Lüroth are re-presented in an algebraic way [Hurwitz 1891, p. 31–34]. Hence a shift clearly occurred: from a visual approach to the branch points towards an algebraic approach, which marginalized the different techniques of visualization. That said, as can be seen from Hurwitz methods, the existing techniques of visualization were unable to offer any solution to the questions he posed.

1.5. *Severi draws a "braid"*

Jumping ahead to the beginning of the 20th century, other visualization techniques appeared. Referencing Neumann's work, these attempted to be simpler than the then current three-dimensional models. In 1908 Francesco Severi's book *Lezioni de geometria algebrica* appeared, which was translated in 1921 to German as *Vorlesungen über Algebraische Geometrie* (see [Guerraggio & Nastasi 2006, p. 104–107]). Severi (1879–1961), one of the leading Italian mathematicians during the first half of the 20th century, specialized in algebraic geometry and the theory of complex functions. Chapter 7 of his book deals with the theme of algebraic functions as analytic functions and Riemann surfaces.

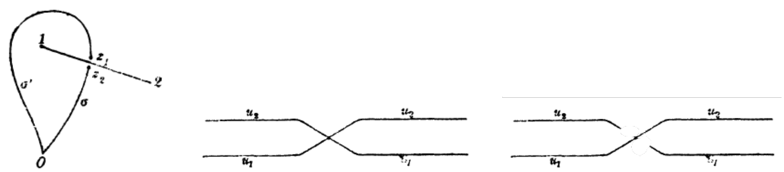
Concerning the behavior of functions in the vicinity of branch points, the first section of this chapter summarizes the research of Puiseux, Riemann, and Hurwitz, among others, and combines the more visual approaches of Puiseux and Riemann and the more algebraic approaches of Hurwitz. On the one hand, Severi draws loops on the complex line $\mathbb{C}$ [Severi 1921, p. 246, 256–257, Severi 1908, p. 198, 205]; on the other, he discusses the ways in which these loops can be presented as additions to other loops, implying their group-theoretical structure. Severi investigates how the permutations that encircle a loop around a branch point on $\mathbb{C}$ are induced, as well as the algebraic consequences (regarding the construction of Riemann surfaces) when choosing another set of loops encircling the branch points,[55] following Clebsch's treatment [Severi 1921, p. 256–258, Severi 1908, p. 205–208]. Severi combines visual and algebraic approaches, but he mainly summarizes the results described above.

Indications of a return to a simpler approach of representing curves and their branch points can be discovered, however, by means of a drawing Severi adds, when he describes—both in the 1908 Italian original and in

ment of the branch points $a_1, \ldots, a_w$ themselves. A discussion on this topic is outside the scope of this section; see however Section II.3.1.(I) below.

55 Severi also draws a sketch of the old and the new set of loops.

the 1921 translation—the construction of Riemann surfaces with the process of gluing two branches, called u_1 and u_2, along their "branch cut" ["taglio," "Verzweigungsschnitt"], i.e., a path on the surface that connects two (simple) branch points called 1 and 2 [Severi 1921, p. 261, Severi 1908, p. 211]. Very similar to Neumann's approach, starting from a loop on the complex line $\mathbb{C}$ encircling the branch point 1, Severi describes that when approaching the point z_1 (resp. z_2) on this loop (see Figure 11.(I)), the two leaves interchange. After depicting what happened on the complex plane, Severi draws a similar illustration to Neumann's. If we consider the loop encircling branch point number 1, the illustration displays how the leaves of the curve itself interchange while going along the loop. Severi notes the following: "[...] the figure [see Figure 11.(II))] [...] refers us to [...] the cut of two leaves with a plane perpendicular to the cut 1–2 [taglio 1–2]. The fact that the values taken by u on the first sheet can be transfered by continuous variation to the values taken by the function on the second sheet is thus *represented in a concrete way* [...]." [56]



(I) (II) (III)

FIGURE 11. (I) A loop around the branch point 1, dissected into two parts; (II) The interchanging of the leaves, presented through a section along the path. [Severi 1921, p. 261, Severi 1908, p. 211] (III) The correct depiction of the interchanging of the two leaves. [57]

56 [Severi 1908, p. 261–262]: "[...] come indica lo schema [...], che riferiscesi [...] dei due tagli con un piano perpendicolare al taglio 1–2. Il fatto che I valori assunti da u sul 1 foglio possono ricommettersi per variazione continua cui valori assunti dalla funzione stessa sul secondo foglio, viene cosi *rappresentato in modo concreto* [...]." (cursive by M.F.). The German translation is the following: "[...] sie [Die Skizze (in Figure 7.(II))] stellt einen Durchschnitt der beiden Blätter mit einer zum Verzweigungsschnitt 1–2 senkrechten Ebene dar. Die Tatsache, daß die Werte welche die Funktion u auf dem ersten Blatt annimmt, durch stetige Veränderung in die Werte übergeführt werden können, welche dieselbe Funktion auf dem zweiten Blatt annimmt, ist somit in konkreter Weise *veranschaulicht*." [Severi 1921, p. 211] That is, the German translation shifts the role of the diagram from the Italian "representation" to "visualization" ("*veranschaulicht*").

57 Figure changed by M.F.

The changing of values is illustrated with Figure 11.(II). Its presentation indicates the approach already presented in Riemann's lecture notes and in Neumann's 1865 book (see Figures 2 and 4.(IV)): this illustration—the representation "in a concrete way"—simplifies how the values change in the neighborhood of the point 1 on the surface, in the sense that the branches are presented as straight lines, creased at a certain point.[58] Also, it is obvious that the two branches *do not* intersect each other—otherwise the surface would be singular and the two branches would intersect at a node on the surface. However, as Severi describes, this is certainly not the case (see Figure 11.(III) for how Severi—and Neumann—should have drawn the two branches).

It is important to note that Severi hardly drew any diagrams or drawings in his book. Most of the drawings appear in the seventh chapter. Indeed, only in this chapter does he note that one can obtain a more concrete and more intuitive ["intuitivo," "anschaulich"] "model" of a Riemann surface of genus p, by thinking of it as a sphere with p handles attached to it [Severi 1921, p. 262, Severi 1908, p. 212]. That is, while Riemann surfaces could have been thought of as having a spatial, *anschaulich* model, other objects of algebraic geometry were considered more abstract, in the sense that a *concrete* model for them was unnecessary and hence a drawing was unnecessary as well. However, Severi's visualization of the behavior of curves at the neighborhood of branch points also appeared in other textbooks, extending his visualization that only applied to simple branch points. For example, Federigo Enriques, in the first volume of his book *Teoria Geometria delle Equazioni e delle funzioni algebriche*, published in 1915, presented Figure 12, indicating that it presents the case for a Riemann surface of degree 5 with a branch point of three leaves. Also here, the simplification may have caused a certain confusion just as in Severi's figure, implying (wrongly) that the various branches cut each other at the neighborhood of a branch point. Looking however at Figure 4.(IV), it is clear that Neumann was aware of this problematic representation since the width of the lines in this figure, connecting the different sheets of the Riemann surface, is narrower. This problematic visualization, seen already with the three-dimensional model of the branch point presented at Figure 6, was solved, among others, by Chisini's novel techniques of visualizing these points in the 1930s, as I will shortly survey in Section III.[59]

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58 Compare Figure 4.(II) and also Figure 6, where a circular section of the three-dimensional model would not be composed of straight lines.

59 See also Figure 21 in this paper.

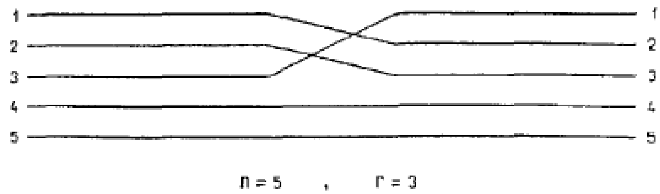

FIGURE 12. Enriques' illustration [1915, p. 361] of a cyclical interchanging of three leaves of a degree 5 Riemann surface.

What Severi's method and Enriques' generalization of it may have pointed to is a future investigation of curves using the techniques of braid theory. However, braid theory was only developed with Emil Artin in his paper "Theorie der Zöpfe" [Artin 1925].[60] Although Hurwitz already implicitly discussed braids (though from another point of view, see [Epple 1999, p. 189–192]), Severi and Enriques, with their explicit sketch—though *without* any algebraic theories resulting from it—did indeed indicate a possible way for future presentations of algebraic curves and their branch points.

1.6. *Visualization of branch points: between abundance and excess*

As was seen above, a plurality of visualization techniques existed for illustrating and rendering haptic and tangible branch points—either on the surface itself or on the complex line, when these surfaces are considered as covering and the complex line is considered as a two-dimensional (real) plane. This plurality of two-dimensional drawings as well as three-dimensional models existed side by side with other mathematical practices, which treated branched covering: the analytical and the algebraic.

That said, the plurality of visual and haptic approaches to the abstract mathematical object encountered resistance. Firstly, as I noted above, Weierstraß doubted the usefulness of Riemann's visualization. And while for Klein the models were considered epistemic things, Herbert Mehrtens remarks that Klein later "interpreted them [the models] as applied mathematics. But by the end of the century mathematicians took the models as imperfect representations of geometrical entities that could be used as an aid in communication about mathematics. The models were not 'evidence' in any sense of the term." [Mehrtens 2004, p. 301] Needless to say, other mathematicians in Germany rejected the appeal to the senses

60 See also [Epple 1999, p. 314–322].

or to the *Anschauung* as what might assist in mathematical understanding or research. Frege, for example, in 1884 claimed "[...] sensations are of absolutely no concern to arithmetic. No more are mental pictures, formed from the amalgamated traces of earlier sense-impressions." [Frege 1884, p. vi] Pasch also rejected the grounding of geometrical knowledge on the senses. As he noted, "after [the axioms of geometry] are established it is no longer necessary to resort to sense perceptions" [Pasch 1882, p. 17]. One may also claim that these objections come on the background of the crisis of the *Anschauung* [Volkert 1986]. Against this retroactive projection of the grand narrative of this crisis, however, it is essential to remember that most of the actors I have surveyed were affirmative and sympathetic towards the "*anschauliche* images" of branch points. It is unnecessary to add that branch points of Riemann surfaces were never thought of as one of the objects prompting this above-mentioned crisis. Hence when looking at the plurality of images as what can be classified under this crisis (either as an implicit initiator or as an outcome) would be an inadequate forcing of this narrative.

It is therefore instructive to examine the views of the individual actors themselves. As we have seen, various mathematicians presented various images of the branch point: various three-dimensional models (von Dyck), colored sketches (Klein), braids (Riemann, Neumann, Severi) or even knotted curves (Neumann). I would like to suggest that these different images of the same mathematical object caused its relativization. This plurality and the relativization it prompted stood in direct contrast to Klein's 1879 request to find "the most *anschauliches* image" of a branch point. Indeed, the problem with the different images of the branch points was that they presented different aspects of these points. For example, what did the different sheets look like around a section near the branch point (e.g., Figure 2) in contrast to how they appear at the neighborhood of this point, including this point itself (see Figure 6). These images did not only present different aspects of the branch point, they also had different functions with respect to how they visualized this point: they either aimed to be *exact*—i.e., represent exactly what the mathematical object was (as with the R and I models of Dyck) and, at the same time, attempt to aid the mathematical inference—, or they were *co-exact*, i.e., allowing manipulation, as with the diagrams of Neumann, Severi and Enriques. In this instance I follow Kenneth Manders' insight regarding exact and co-exact diagrams: while *exact* diagrams are unstable when subject to change (e.g., lengths of line segments, which cannot be varied without affecting the argument), *co-exact* ones are "insensitive to the effects of a range of variation in diagram entries" [Manders 2008,

p. 69] (e.g., inside-outside relations). While Manders treated the case of the Euclidean diagram, it is clear from the discussion above that this treatment can also be applied to other visualizations within other mathematical domains (see also [Larvor 2017]). The point is that several authors stated explicitly that their diagrams were not only capable of manipulation, but also "arbitrary" (e.g., Neumann), or simply did not represent the surfaces (as the straight lines in Severi's and Enriques' diagrams, which ignore the metrical properties). In that sense, this arbitrariness may have prompted a preference for a more algebraic or analytical approach.

This plurality of images—these visualizations—raises the question of epistemological relevance. Apart from the fact that Klein and Brill saw for a certain period of time the three-dimensional models as epistemic things (hence placing the images of branch points under a narrative in which mathematical *Anschauung* advances with the visual and motoric perceptions), the various visualizations of the branch points did not generally speaking result in new discoveries or novel research questions. By contrast they were regarded more as a means of illustrating and transmitting four-dimensional objects to the senses. That being said, there are two exceptions to this general claim: Firstly, Riemann, Neumann, Severi and Enriques presentation of the neighborhood of the branch point as a braid (although never using such terminology) led and prompted Chisini to look at algebraic plane curves (and at branch curves in particular) in terms of a factorization of braids; this resulted in several conjectures and a prospering field of research in Italy between 1930 and 1950 (see Section III). Secondly, one can also note that the inadequate plurality of visualization techniques possible caused a turn towards a more algebraic, non-visual approach (e.g., Hurwitz). To emphasize—Hurwitz was well aware of several of the different visualizations, but his line of investigation, which distances itself from this tradition, prompted new research questions and methods, which were not raised by these visualization techniques. Hence such distancing acted as an epistemic operation.

* * *

Considering a few of the problematic aspects that several of the mathematicians surveyed above encountered, while trying to visualize branch points of complex algebraic curves, it is not surprising that one hardly finds any visualization or drawing of branch *curves* of complex algebraic surfaces, when these surfaces were considered as a covering of the complex plane. The investigations of complex algebraic surfaces, done at the end of the 19th century,[61] showed that the algebraic-topological nature of these surface is much more complicated when compared to the corresponding situation with complex curves. Algebraic complex surfaces are four (real) dimensional objects, embedded in $\mathbb{C}^3$ or $\mathbb{CP}^3$, that is, in a six (real) dimensional space. The set of branch points of these surfaces, as mentioned in the introduction, is therefore not a set of isolated points, but rather an algebraic curve, called the branch curve, which is usually singular.

Without undermining this difficulty, as I will claim in Section II, visualization was not entirely abandoned or conceived of as useless. Although the branch curve itself (or the ramification curve on the surface) was never depicted, sketched or drawn at the turn of the century, visualization techniques were employed in order to make visible *other*, different mathematical machinery that was used to investigate the branch curve.

2. BRANCH CURVES: VISUALIZATION BETWEEN DISAPPEARANCE AND ABSENCE

Turning now to the mathematical research on the branch curve of a complex surface,[62] it is important to emphasize that three-dimensional models of such surfaces did exist, but mostly consisted of a model of the real part of the surface.[63] When considering only the branch curve, however, one notices an absence, resulting from obstacles in the visualization of these objects. One however should mention that the mathematical consideration of branch and ramification curves was initiated long before Riemann's investigation of branch points. Gaspard Monge, who looked at projections of three-dimensional bodies to the plane and considered under the context of tracing shadows of a body. Monge notes in 1785: "The projection of a body's shadow on any surface is therefore the figure

61 For a survey of the work of Castelnuovo and Enriques of algebraic surfaces, see [Gray 1999].

62 Note that I do not deal in this article with *real* branch curves that arise from the consideration of three-dimensional real manifolds as *real* covering of the 3-sphere (branched over a link or a knot), although these branched coverings and their branch curves were certainly visualized. This research also took place at the beginning of the 20th century, with Poul Heegaard, James W. Alexander, G. B. Briggs and Heinrich Tietze among others (see: [Epple 2004, p. 332–336; Stillwell 2012, Epple 1999]). To concentrate on this theme, however, would take us beyond the scope of this section that strictly concerns visualization techniques of *complex* branch curves.

63 That is, if the surface is given by $z = p(x, y)$, then three-dimensional models were representing the surfaces such that the points x, y, and $z = p(x, y)$ were real. A famous example is the model of the cubic surface with the 27 *real* lines on it; complex points of this surface could not have been visualized.

that the extensions of the rays of light tangent to the body's surface end on that surface."[64] He then notes: "In the following operations we will geometrically determine only the projections of the contours of the pure shadows, they are *the only ones* that it is necessary to have exactly in the drawings."[65] (see also Figure 13). The projection of the contour of the pure shadow is—in modern terminology—the branch curve, and Monge adds a figure—the first figure of a branch curve in a mathematical context, and almost surely the last one to explicitly appear during the 19th and the 20th centuries. Étienne Bobillier [1827–1828] found out during the late 1820s that ramification curves are on the intersection of the surface and its polar (see the following subsection),[66] though he did not investigate branch curves, and was not aware of Monge's research on them. It is only with George Salmon that one can find a systematic study of ramification curves, and as a result, of branch curves.

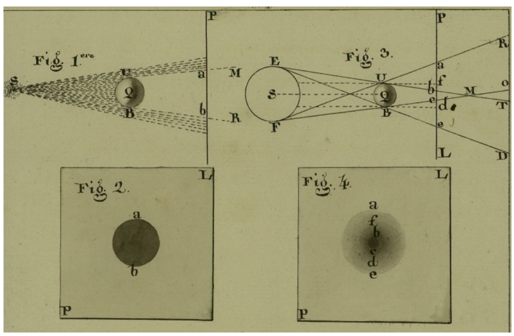

FIGURE 13. From Monge's *Des Ombres*: Figure 1 and 2 describe the illumination of a sphere from a point outside of it. © Collections École polytechnique-Palaiseau.

Since Monge, and later Bobillier, hardly or never considered explicitly branch curves as an object of mathematical investigation (and thus signaled the absence of more complex drawings or sketches of branch curves during the first quarter of the 19th century), I will begin with

64 "La projection de l'ombre d'un corps sur une surface quelconque est donc la figure que terminent sur cette surface les prolongements des rayons de lumière tangents à la surface du corps." [Monge 1847 [1785], p. 27].

65 "Dans les opérations suivantes nous ne déterminerons géométriquement que les projections des contours des ombres pures, *ce sont les seules* qu'il soit nécessaire d'avoir exactement dans les dessins [...]" [Monge 1847 [1785], p. 29].

66 On the work of Bobillier, see: [Haubrichs dos Santos 2015]. Bobillier however did coin the term "polar surface" [Bobillier 1827–1828, p. 181], which Salmon later used.

Salmon and his systematic investigation. This section will therefore survey the oscillation in visualization techniques between illustrating only the local behavior of the curve,[67] sketching other mathematical instruments or ignoring completely the possibility of visualizing any of the two options above. This ignorance may be termed "making invisible," an expression I will discuss more thoroughly later (see Section III). I begin with the third option, where the ignorance can be seen in one of the influential manuscripts written on surfaces in three-dimensional space: George Salmon's 1862 book *A treatise on the analytic geometry of three dimensions.*

2.1. *Salmon: The (almost) complete absence of illustrations*

George Salmon (1819–1904) was an Irish mathematician and theologian. He worked in the field of algebraic geometry for two decades, and then devoted the rest of his life to theology. Known especially today for his joint research, together with Cayley, on the 27 lines of the cubic surface,[68] his name and research, as Row Gow mentions, "would scarcely attract any attention among mathematicians so many years after his death if his reputation was based only on his research papers. [...] Salmon's lasting fame lies in the influence exerted by four textbooks he wrote. These were: *A Treatise on Conic Sections*; *A Treatise on the Higher Plane Curves*; *Lessons Introductory to the Modern Higher Algebra*; *A Treatise on the Analytic Geometry of Three Dimensions.*" [Gow 1997, p. 38][69]

How did these influential books treat surfaces and their branch curves? In order to understand Salmon's approach, it is instructive to take a step back and look at his approach to complex curves and their branch points. Salmon treats these subjects in his 1852 book *A Treatise on the Higher Plane Curves*, but from an entirely different point of view when compared to Riemann and Puiseux.

For Salmon, a plane curve, denoted by U, is represented by an algebraic equation as follows: "The general equation of the n-th degree between two variables may be written: $A + Bx + Cy + Dx^2 + Exy + Fy^2 + \cdots + Px^n + Qx^{n-1}y + \cdots + Rxy^{n-1} + Sy^n = 0$". [Salmon 1852, p. 18] Salmon does supply several drawings of what *singular* points look like; see Figure 14.(I) for illus-

67 At the neighborhood of its singular points, for example.

68 Salmon however was not involved in the preparation of the various models of the cubic surface and its lines.

69 See also [Gow 1997] for a summary of Salmon's mathematical work. See also: [Flood 2006, p. 208–209].

tration of a node and of a cusp.[70] However, illustrations of branch points are lacking. When reading how Salmon treats curves, it is also clear why. Salmon treated branch points on the curve only when considering a projection, similar to Puiseux and Riemann. But while the latter authors explicitly considered the curve (as covering) by taking into account the projection on the (complex) x-axis and drawing loops on it, investigating then the resulting permutations, Salmon's point of view was different. Salmon was interested in two types of projections: a projection from a point O on a curve, and a projection from a point O not on a curve (see Figure 14.(II)). However, Salmon uses neither the term "covering" nor the term "projection". Taking the context of Salmon's investigation into account, branch points *on the curve* would be, when considering the methods of Puiseux and Riemann, the points for which the lines exiting from O are tangent to the curve (see Figure 14.(II), when the point O is *not* located on the curve). While Puiseux and Riemann considered this point of view only implicitly, *starting* from an investigation of a neighborhood of the branch point on the complex line, Salmon concentrated on the branch points on the curve, seen as tangency points, *starting* with an investigation of a curve and the lines exiting from a point O and not on it. Since Salmon did not even consider the concept of the branches of a curve in the Riemannian sense or according to the theory of complex functions (Puiseux), the branch point for him was only a tangency point of a line exiting from a point O to the curve C. Hence, there was, one might say, nothing to visualize.

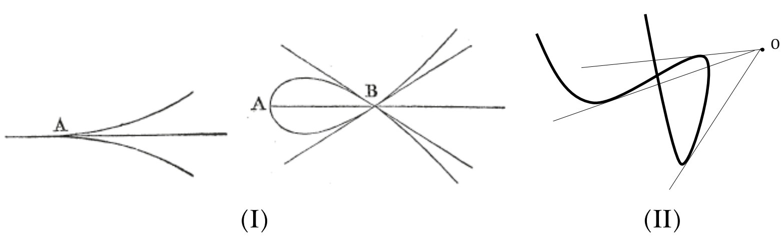

FIGURE 14. (I) Salmon's drawings of singular points: cusps (left) and a node (right). [Salmon 1852, p. 30] (II) a possible visualization of Salmon's conception of branch points on the curve.[71]

[70] A *node* is locally as the point *(0,0)* of the curve $(y-x)(y+x)=0$; a *cusp* is locally as the point *(0,0)* of the curve $y^2-x^3=0$.

[71] Drawing by M.F.

For Salmon, these branch points are given as the intersection points of the original curve with another curve, called the (first) polar curve. Salmon states that the equation of this polar curve, [72]

$$\Delta U = 0, \quad \text{or} \quad x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0$$

> would enable us to find the point of contact of tangents drawn through a given point. Were we given the point $x_2\, y_2\, z_2$, then the point of contact $x_1\, y_1\, z_1$ must satisfy the equation
>
> $$x_2\frac{dU}{dx} + y_2\frac{dU}{dy} + z_2\frac{dU}{dz} = 0.$$
>
> Hence the points of contact of tangents which can be drawn from a given point to a curve of the n-th degree lie on a curve of the $(n-1)$-th degree: viz., on the first polar of $x_2 y_2 z_2$, with regard to the given curve [...] [Salmon 1852, p. 62].

Hence, according to Bezout's theorem, [73] "from a given point [not on the curve] there can be drawn $n(n-1)$ tangents to a curve of the n-th degree." [74] [Salmon 1852]

Salmon does not mention the term "branch point" for obvious reasons—the term itself did not yet exist and was coined in German only by Riemann in 1857. However, even without the terminology, the point of view is completely different. And this point of view is carried out in the case of complex surfaces. Salmon already started in 1847 dealing with this subject. In a paper written in this year he gives a numerical analysis of the properties of the ramification curve (number of "cuspidal" and "ordinary double lines"), and notes, after discussing the formula for the number of tangent lines to a curve exiting from a given point: "[a]s I am not aware that the corresponding question as to reciprocal surfaces has been before investigated, I purpose in the present paper to enquire [this] [...]." [Salmon 1847, p. 65] A more detailed analysis appears in 1862—after Riemann's coinage already appeared—when Salmon publishes his book *A Treatise on the Analytic Geometry of Three Dimensions*, he now considers complex projective surfaces in the complex projective three-dimensional space $\mathbb{CP}^3$. His treatment follows a similar line of interpretation as his

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72 Recall that the projective (complex) plane has three coordinates x, y, z and hence a projective plane curve is expressed with three variables.

73 Bezout's theorem (for curves) was published in 1779. Étienne Bézout proves that given two complex projective curves of degree n and m, without a common component, these curves then intersect at mn points (counted with their multiplicities).

74 The theorem was already stated in 1818 by Poncelet [Poncelet 1817–1818, p. 215], whose writings Salmon knew well [Gow 1997, p. 53, 45–46].

treatment of complex curves: given a smooth surface U in $\mathbb{CP}^3$, choosing a point $O = (x' : y' : z' : w')$ not lying on the surface, and examining the tangent lines to U passing through O. Salmon calls this collection of all lines tangent to the surface the "tangent cone" [Salmon 1862, p. 190], and notes:

> [...] consider the case of tangents drawn through a point not on the surface. [...] we see that the points of contact of all tangent lines (or of all tangent planes) which can be drawn through $x'y'z'w'$, lie on the first polar [denoted by ΔU], which is of the degree $(n-1)$: viz.
>
> $$x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} + w'\frac{dU}{dw} = 0.$$
>
> And since the points of contact lie also on the given surface, their locus is the curve of the degree $n(n-1)$, which is the intersection of the surface with the polar. [Salmon 1862, p. 62]

Here Salmon implicitly considers projections of the surface from a point, though he does not say where the surface is projected. The "curve of the degree $n(n-1)$, which is the intersection of the surface with the polar" in contemporary terminology is called the *ramification curve*, although this term obviously does not stem from Salmon as such. Though being current terminology, I will use this term from now on, to distinguish between this curve (which is on the surface) and the branch curve (which is on the complex plane $\mathbb{C}^2$, being the image of the ramification curve, when one indeed considers a projection $U \to \mathbb{C}^2$ or $\mathbb{CP}^2$).

Salmon, as was indicated above, did not draw a single sketch to illustrate how this curve might look like on a surface.[75] Notwithstanding, and considering the fact that he did make drawings of curves in general, one might wonder why he did not draw any branch curves, being the projection of the ramification curve on a complex plane. This question is justified given that Salmon did take into account in his investigations the branch curve, as I will now show.

After defining the ramification curve, Salmon continues to investigate two types of special tangents to it (see an illustration in Figure 15). The first are tangent lines called "inflexional tangents," which are not only tangent to the surface, but in addition are also tangent to the ramification curve

[75] Salmon did not draw a single sketch of any complex surface in the 1862 book, but rather only partial images of concrete situations (e.g., tangent planes or tangent lines, for example; see [Salmon 1862, p. 274 or p. 296]). This might be also due to constraints on printing techniques during this period, but also in accordance to how Monge and his followers were using concrete images.

itself. Salmon proves that these points lie on the intersection of the ramification curve and the *second* polar of the surface, i.e., the surface $\Delta\,(\Delta U)$ or $\Delta^2 U$. Salmon indicates: "Through a point not on the surface can in general be drawn $n\,(n-1)\,(n-2)$ inflexional tangents. [...] consider the *xyzw* of any point on the tangent as known; its point of contact is determined as one of the intersections of the given surface U, which is of the n-th degree, with its first polar ΔU, which is of the $(n-1)$-th, and with the second polar $\Delta^2 U$, which is of the $(n-2)$-th. There are therefore $n(n-1)(n-2)$ such intersections." [Salmon 1862, p. 191] The second type of special tangents to the surface are lines which are tangent to the surface at two different points. Salmon indicates the following: "Through a point not on the surface can in general be drawn $\frac{1}{2}n(n-1)(n-2)(n-3)$ double tangents to it." [76] [Salmon 1862]

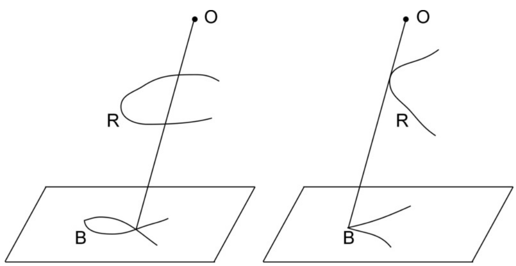

FIGURE 15. How special tangents to the surface S (not drawn) from a point O result in either a node (left: two points on the ramification curve R are projected to the same point) or a cusp of the branch curve B (right: the tangent to the surface is also tangent to the ramification curve R). (Figure drawn by M.F.)

It is after discussing these two special tangents that Salmon notes the special properties of the branch curve, i.e., the image of the ramification curve on a generic plane (which does not pass through O): "We have proved then that the tangent cone which is of the degree $n(n-1)$ has $n(n-1)(n-2)$ cuspidal edges, and $\frac{1}{2}n(n-1)(n-2)(n-3)$ double edges; that is to say, any plane meets the cone in a section having such a number of cusps and such a number of double points." [Salmon 1862, p. 192] The branch curve here, although, again, not termed as such, is the curve obtained by a "plane [that] meets the cone in a section". This curve, which hardly stands at the center of Salmon's investigation, therefore has

76 Identical calculations appear already in [Salmon 1874].

$n(n-1)(n-2)$ cusps and $\frac{1}{2}n(n-1)(n-2)(n-3)$ nodes.[77] However, one may wonder why Salmon did not take notice of the special properties of the branch curve of a cubic surface, a surface that was one of his specialties. As Salmon briefly indicates: "The general theory of surfaces [...] gives the following results, when applied to cubical surfaces. The tangent cone whose vertex is any point, and which envelops such a surface is, in general, of the sixth degree, having six cuspidal edges and no ordinary double edge." [Salmon 1862, p. 376] However, as we will see later (see Section II.3.2), when considering the branch curve of the (smooth) cubic surface, i.e., the intersection of "any plane" with the tangent cone (following Salmon's formulation), its six cusps lie on a conic; and Salmon indeed had the tools to discover it.[78]

Whereas Salmon was not interested in visualizing curves on surfaces, let alone ramification curves or their projection, attempts, however, were made to employ visual tools to study the branch curve. As we will see, Wilhelm Wirtinger was the first to *visually* consider a neighborhood of a singular point of the branch curve.

2.2. *Wirtinger draws a knot*

Wilhelm Wirtinger (1865–1945), an Austrian mathematician, was known for his work in complex analysis and knot theory, and especially for his presentation of the fundamental group of the complement of a knot in $\mathbb{R}^3$. At the end of the 19th century, as well as at the beginning of the 20th century, Wirtinger began research on branch curves of complex surfaces. While this research did not mature into a full-blown theory, it nevertheless prompted a re-contextualization of the research of singular complex plane curves by means of knots associated to their singular points.

When researching complex surfaces in the 1890s, Wirtinger considered projections of them. While Salmon only considered these projections implicitly, he was nevertheless aware that this projection could be done from any point—whether this point would lie on the surface or not. Viewing

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77 As we will see later these nodes and cusps will play a special role in Beniamino Segre's work.

78 The German translation of Salmon's book, called *Analytische Geometrie des Raumes. II. Theil. Analytische Geometrie der Curven im Raume und der algebraischen Flächen*, done by Wilhelm Fiedler does mention material models of several surfaces in the appendix, mostly made in Germany, [Salmon 1874, p. 622–623, 663, 667], but does not add in the book any figure or change the point of view regarding the treatment of branch curves.

these surfaces in the complex three-dimensional space $\mathbb{C}^3 = (x,y,z)$, Wirtinger solely thought about a very specific projection to complex plane $\mathbb{C}^2 = (x,y)$. That is, following Puiseux and Riemann and given a complex surface defined by an equation $f(x,y,z) = 0$, the projection was done from a point "in infinity," i.e.,:

$$p : \{(x,y,z) \in \mathbb{C}^3 : f(x,y,z) = 0\} \to \mathbb{C}^2, \quad (x,y,z) \mapsto (x,y).$$

How did Wirtinger think about branch curves? Wirtinger dealt with these curves in a letter written to Felix Klein on 22 December 1895 and in a 1905 lecture. I will first examine the letter,[79] and then analyze the shift Wirtinger made in his lecture ten years after.

As he explicates in the letter, Wirtinger's aim "is to show that an arbitrary system of $n-1$ dimensional algebraic varieties [Gebilde] with an associated branching scheme can always be understood as a system of branch manifolds of a function of n variables."[80] [Epple 1995, p. 398] As we will see with the investigations of Federigo Enriques and Beniamino Segre, the question that arose in the early 20th century was whether Wirtinger's statement here was correct. Wirtinger himself, as we will see, also indicates the possibility that not every $(n-1)$-dimensional manifold can be a branch manifold of an n-dimensional function. Explaining his motivation, he notes: "With one variable, the question is simple because in the neighborhood of a branch point the cyclic functions of the values are unique, that is, the n-th root of the variable itself is unique. *The germ of generalization lies in this setting*."[81] [Epple 1995, p. 398]

However, as Wirtinger immediately notes, the case of Riemann surfaces—i.e., when $n = 1$—does not entirely contain the germ of generalization, and one has to distinguish between two cases: when the branch manifold is smooth and when the branch manifold ["Verzweigungsmannigfaltigkeit"] is singular.[82] The distinction between the two cases does

79 The letter is to be found in Klein's Nachlass in Göttingen (Cod. Ms. Klein XII, 391). A transcription of it can be found in [Epple 1995, p. 397–399].

80 "Mein Ziel ist dabei zu erweisen, dass man ein beliebiges System algebraischer Gebilde von $n-1$ Dimensionen mit zugehörigem Verzweigungsschema *immer* als System von Verzweigungsmannigfaltigkeiten einer Function von n Variablen auffassen kann."

81 "Bei einer Variablen liegt die Frage deshalb so einfach, weil in der Nähe eines Verzweigungspunktes die cyclischen Functionen der Werte eindeutig werden, also die n-te Wurzel der Variablen selbst eindeutig ist. *In dieser Fassung liegt der Keim der Verallgemeinerung.*"

82 Note that Wirtinger usually uses the word "Verzweigungsmannigfaltigkeit" when talking about the branch variety, i.e., a singular one. The term "Gebilde" (variety) is used for more general descriptions, usually for smooth covering.

not arise in the case of Riemann surfaces—since there the branching manifold is a collection of points, and hence is always smooth.

As Wirtinger notes the analogy with Riemann and Puiseux's treatments is still valid for smooth branch manifolds: "The cyclic behavior remains true for arbitrary branching manifolds [...]."[83] [Epple 1995, p. 398] This means, as we saw with Puiseux, and in order to give an example, that the only permutations that can be induced by encircling a branch *point* are those that permute the branches cyclically, i.e., the values $u_1, u_2, \ldots, u_p$ are permuted to $u_2, \ldots, u_p, u_1$, or, in Wirtinger's words, the Galois group is cyclic (i.e., generated by one element, in this case, the above permutation). However, the situation is completely different when the branch manifold is singular. Wirtinger first determines, without giving any justification, that in this case, "then every Galois group is possible." [Epple 1995, p. 398] The example that Wirtinger gives is a cubic surface in $\mathbb{C}^3 : \{(x, y, z) \in \mathbb{C}^3 : z^3 + 3zx + 2y = 0\}$. The branch curve $\{x^3 + y^2 = 0\}$, which has a cusp at (0,0), is not specified by Wirtinger. However, he does note that the Galois group is not cyclic, but rather it is the whole symmetric group on three letters. Perhaps this is what led him to the above conclusion, that every Galois group is indeed possible.

Several passages later, however, Wirtinger formulates his statement as a question: "The kernel of the whole thing lies now for me in the investigation of the group of a branch point, that is to say [...] in the question: Can this group be arbitrarily pre-determined, or is it bound to conditions, so that associated functions exist?"[84] [Epple 1995, p. 398] Assuming that Wirtinger deals only with the projection of n-dimensional smooth manifolds, what he asks is what the possible singular points are that the $(n-1)$-dimensional branch manifold might have under a generic projection of an n-dimensional manifold. Or might the branch manifold have any type of singularity appear? Looking at branch curves of complex surfaces, Salmon already showed that the branch has (at least) two types of singular points: cusps and nodes.[85] Wirtinger ignores, however, the fact that a branch curve might also have nodes, and hardly deals with the investigation of branch curves in his letter to Felix Klein.

83 "Das cyclische Verhalten bleibt aufrecht für beliebige Verzweigungsmannigfaltigkeiten [...]."

84 "Der Kern der ganzen Sache liegt jetzt für mich in der Eruirung der Gruppe eines Verzweigungspunktes, also eigentlich [...] in der Frage: Kann man diese Gruppe willkürlich vorgeben, oder ist sie an Bedingungen gebunden, damit zugehörige Functionen existieren?"

85 Note that Salmon did not prove that these are the *only* types of singular points a branch curve of a smooth surface may have.

During the following years Wirtinger did not work on this problem. In 1901 in his article on algebraic functions and their integrals for the *Enzyklopädie der mathematischen Wissenschaften,* Wirtinger reformulated the question that he posed to Klein. For functions of several variables, as Wirtinger notes, "one has not yet succeeded in determining a given algebraic variety [algebraische Gebilde] by a finite number of data in a similar way as is possible with the different forms of a Riemann surface." [86] Referring directly to Hurwitz and the determination of a Riemann surface via a "finite number of data" (i.e., branch points, the number of sheets, and the permutations), Wirtinger asks whether this determination is possible for general branched covering (of dimension n). Taking Wirtinger's letter to Felix Klein into consideration, one might assume that what Wirtinger meant also involved considering the singularities of the $(n-1)$-dimensional branch manifold.

In a 1905 lecture entitled "Über die Verzweigungen bei Funktionen von zwei Veränderlichen" a shift of context occurs in the way Wirtinger examines branch curves. Although the lecture itself is not available, Moritz Epple [1995] has reconstructed it and shown that Wirtinger, inspired by Poul Heegaard's 1898 dissertation, decided to consider only the local neighborhood of singular points of branch curves. Not only does Wirtinger ignore his former questions, such as those regarding the possible singular points of the branch curve or what is the necessary "finite number of data," but he also re-contextualizes the problem in his lecture. As Epple shows, Wirtinger now no longer considers the branch curve as a whole, but rather only the *intersection* of a neighborhood of a singular point of the branch curve with the 3-sphere. [87] For the case of the branch curve of the cubic surface presented by Wirtinger, one obtains the following: $\{x^3+y^2=0\}\cap\{|x|^2+|y|^2 = c\}$, for c small and positive number, and when $x, y \in \mathbb{C}$. Wirtinger recognized this intersection as the *trefoil knot,* and he most likely drew a figure (see Figure 16) during his lecture. With the help of this figure he then calculated the fundamental group of the complement of this knot (thought as embedded now in $\mathbb{R}^3$), as the Galois

86 Translation taken from: [Epple 1995, p. 383].

87 "Heegaard's idea was to study singular points of algebraic surfaces by looking at the restriction of the branched covering of defined by the equation of the surface to a 3-sphere bounding a small neighborhood of the singular point in question" [Epple 1995, p. 384]: For an analysis of Heegaard's thesis, see: [Epple 1999, p. 246–251]. It is essential to note that Heegaard called in his thesis explicitly for visualization of complex surfaces [Epple 1999, p. 247].

group of the cusp, proving that it is isomorphic to the symmetric group on three letters.[88]

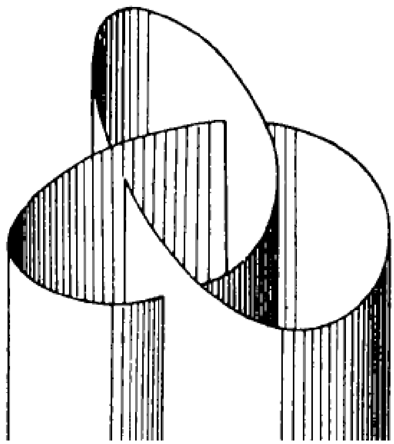

FIGURE 16. The drawing of Emil Artin of the trefoil knot, inspired from Wirtinger's lecture. Artin remarks: "One finds easily the generators of the fundamental group and the defining relations with the help of a method developed by Herr Wirtinger in his lectures [...]."[89] [Artin 1925, p. 58]

Wirtinger had "shown how to form a very intuitive picture of the topological situation around the branch point" [Epple 1995, p. 386]—hence following Klein's *anschauliche* geometry, and certainly supportive of the way visualization was used in mathematics. Indeed, Wirtinger, in a letter to Klein in 1896, "imagined the mathematician of the 20th century [...] like a painter who looks at the world with a painter's eyes, thinking about the way in which he would like to paint it. Correspondingly, the mathematician should try to 'see the mathematical problem' in whatever form she

88 The figure appeared several times in different contexts during the early 20th century. See: [Epple 1995, p. 384]: "There is a picture in Artin's article, which illustrates these techniques. The same picture had been described in words by Tietze, and it reappeared in Brauner's article. Finally, it was reprinted in Reidemeister's *Knotentheorie*. In all of these cases, the use of this picture in order to derive a presentation of the knot group is ascribed to Wirtinger."

89 "Die Erzeugenden der Fundamentalgruppe und die definierenden Relationen findet man nun wohl am einfachsten mit Hilfe einer Methode, die Herr Wirtinger in seinen Vorlesungen entwickelt hat [...]."

or he encounters it." [Epple 1995, p. 387] [90] Following Epple, Wiritinger calls not only for a concrete investigation, but also for a "productive imagination" [Epple 1999, p. 256] done with the help of a drawing, being not merely a "passive" visualization, but one, like the painter, which produces its own reality. This conception of visualization as epistemic and productive certainly aligns with Klein's ideas. That said, Wirtinger's novel method and the illustration that followed it have certainly led to a problematic approach in the research on branch curves. While Wirtinger tried to visualize how the singular points of the branch curve look like *locally*, he ignored the *global* question that he himself had indicated several years previously: can every curve with this local behavior be a branch curve? Although being one of the first attempts to actually illustrate a branch curve, the problematic issue lay in what Wirtinger did not consider and could not have considered with his illustration: that the singular points of the branch curve are to be found in a special position with respect to each other—i.e., if one would have liked to visualize the branch curve, then a global image would have been necessary. One may claim that the image of the local behavior in fact hindered this understanding and that Wirtinger's visualization may have led also to a dead lock in his research on branch curves. The importance of this point of view is hinted at unreservedly by Enriques and clearly expressed by Zariski and Segre at the end of the 1920s, as I will now show.

2.3. *Branch curves in Italy: Enriques, Zariski, Segre*

While discussing the attempts so far to visualize branch points and branch curves, most of the mathematicians I have dealt with were German as we saw in Section I.6. Whereas not all of the mathematicians in Germany had a favorable view towards what might be called *anschauliche* geometry, Riemann, Klein, Brill, Dyck and many more mathematicians in-

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90 Epple cites the following sentences from a letter from Wirtinger to Klein sent on 22 May 1896: "I imagine the mathematician of the 20th century in such a way that, like the painter, as often as he wants, he sees the world painterly and thinks how he would paint it (and not just of classical gallery paintings), also as often as he wants to see the mathematical problem, wherever and in whatever form it appears. As a result of general mathematical education, I now think of the ability of this seeing, at least in principle." ["Ich stelle mir den Mathematiker des 20 Jahrhunderts so vor, dass er, wie der Maler, so oft er will, die Welt malerisch sieht u. denkt wie er sie malen würde (u. nicht blos an classische Galeriebilder), auch so oft er will das mathematische Problem sieht, wo u. in welcher Gestalt immer es entgegentritt. Als Resultat der allgemeinen mathematischen Bildung, denke ich mir nun die Fähigkeit dieses Sehens, wenigstens im Princip."] [Epple 1995, p. 387]

fluenced by them, certainly had supported a concrete visualization of the geometrical objects in general and of branch points of Riemann surfaces in particular. When the discipline of algebraic geometry was developed in Italy, however, the approach towards visualization techniques began to change: these techniques, I claim, were increasingly marginalized, and when they were employed, they were mainly considered merely technical.

The reasons for this change in approach are diverse. While material models flourished in Germany for usage in research and even were mass-produced, the production of mathematical-physical models for the purpose of research had not taken root in Italy (see: [Giacardi 2015a;b; Palladino & Palladino 2009]). Models were indeed bought by Italian mathematicians for mainly pedagogical reasons, but there was hardly an equivalent tradition in Italy, which could be compared with the German one. The attempt of Giuseppe Veronese (1854–1917) to establish a national laboratory for the production of models failed. Nevertheless, a workshop for constructing models was founded (for teaching projective geometry) at the university of Naples. During the first decades of the 20th century, Guido Castelnuovo and his students also constructed models of surfaces. That said, these were by and large exceptions with respect to the situation in Italy as a whole. When models were needed they were acquired mainly from Germany. Livia Giacardi explains that this rejection is due to the mathematical tradition of geometry in Italy of the 19th and the 20th centuries, which consisted of several rather abstract mathematical approaches: the theoretical, analytical approach, the logical approach of the foundations of geometry, and the Italian school of algebraic geometry. One would expect that the usage of models would have been preferred by the algebraic geometry school, however, as Giacardi [Giacardi 2015a, p. 12] notes: "In spite of this, they [the members of this school] did not use physical models in their research work, but preferred to employ the *Gedankenexperiment*." [91]

However, as already seen above with Severi and Enriques, illustrations were used in published articles and books. As noted, Castelnuovo and Enriques did appreciate this mathematical tradition and also considered it

91 Giacardi notes that the models that Beltrami himself manufactured for the hyperbolic plane were an exception (see: [Capelo & Ferrari 1982]). Note moreover that although the political and scientific relations between the Berlin-Rome mathematical axes were more intensive starting from the 1920s onwards (see: [Remmert 2017]), at this time the tradition of model production in Germany was in decline.

even in an explorative way. The following 1928 citation from Castelnuovo serves as evidence of this Italian tradition: [92]

> We had constructed [...] a large number of surface models [...] [placed] in two showcases. One contained the regular surfaces for which everything proceeded as in the best of all possible worlds [...]. But when we tried to verify these properties on the surface of the other window, the irregular ones, trouble began and there were exceptions of every kind. In the end, the assiduous study of our models had led us to divine some properties that had to exist, with appropriate modifications, for the surfaces of both showcases; we then put these properties into practice with the construction of new models. If they resisted the test, we were looking for the logical justification for the last phase. With this procedure, which resembles the one carried in the *experimental sciences*, we have succeeded in establishing some distinctive traits for families of surfaces [Castelnuovo 1928, p. 194].

As the objects of algebraic geometry become more and more complex, however, fewer and fewer attempts at visualization were to be found; Norbert Schappacher indicates:

> "substantial basic knowledge required of any researcher preparing to work in Algebraic Geometry was invested with an essential illustrative component. More generally, there can be no doubt that basic objects of algebraic geometry [...] were naturally pictured (with or without actually drawing them) by all those working with them. [...] [However,] Italian geometers were led to analyzing constellations of objects which are increasingly difficult to visualize adequately [...]." [Schappacher 2015, p. 2806]

What Schappacher emphasizes is that several algebraic objects and methods could not be drawn at all and that algebraic arguments, though leading to results regarding curves and surfaces, were not followed by corresponding illustrations. In partially following Schappacher's view, when looking on the research on branch curves done in the Italian school of algebraic geometry, the role visualization played oscillated between three

92 "Avevamo costruito [...] un gran numero di modelli di superficie [...] e questi modelli avevamo distribuito [...] in due vetrine. Una conteneva le superficie regolari per le quali tutto procedeva come nel migliore dei mondi possibili [...]. Ma quando cercavamo di verificare queste proprietà sulle superficie dell'altra vetrina, le irregolari, cominciavano i guai e si presentavano eccezioni di ogni specie. Alla fine lo studio assiduo dei nostri modelli ci aveva condotto a divinare alcune proprietà che dovevano sussistere, con modificazioni opportune, per le superficie di ambedue le vetrine; mettevamo poi a cimento queste proprietà con la costruzione di nuovi modelli. Se resistevano alla prova, ne cercavamo, ultima fase, la giustificazione logica. Col detto procedimento, che assomiglia a quello tenuto nelle *scienze sperimentali*, siamo riusciti a stabilire alcuni caratteri distintivi tra le famiglie di superficie."

positions: an epistemological procedure, a technical tool and a complete absence.

2.4. *Enriques and the two visualizations*

Federigo Enriques (1871–1946) was one of the driving forces behind advancements in the school of algebraic geometry in Italy, especially in the field of birational geometry (See: [Brigaglia & Ciliberto 1995, esp. p. 97–124]). He was one of the first mathematicians to deal seriously with questions concerning branch curves, and especially on how constructions, which were done for branch points and Riemann surfaces (i.e., complex curves), can be generalized for the case of branch curves and complex surfaces.

Concerning branch curves, in 1923 Enriques is considered as the initiator of research on these questions. As Anatoly Libgober indicates, "[d]escribing the origins of the studies of the complements one should probably start with the work of Enriques [...] on multi-valued algebraic functions of several variables [...] since, it seems, they contain the earliest results on their fundamental groups of the complements." [Libgober 2011, p. 3] Libgober refers to Enriques' 1923 article entitled "Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curve diramazione". Yet it should be noted that Enriques in fact had similar questions in mind already at the close of the 19th century, and attempted to answer them in various ways.

2.4.1. *Enriques draws a rotating line*

Already in 1897, Enriques writes to Guido Castelnuovo (1865–1952), concerning an 1881 article by Leopold Kronecker, that "you yourself asked the question whether 'two multiple planes with the same branch curve can be represented [mapped] one to the other.' "[93] The question comes on the background of Enriques' investigation of complex surfaces that began in 1893 and also involved arguments regarding their branch curves, as can be seen in the correspondence with Castelnuovo [Bottazzini et al. 1996, e.g., p. 42, 56, 62, 70, 100]. These letters do not contain any visualization of the branch curve. Moreover, questions in some of them concern specific types of surfaces. The general question posed in 1897, which remains merely suggested, is whether two complex surfaces (called here "multiple

93 Letter of 5th June, 1897, in: [Bottazzini et al. 1996, p. 340]: "Tu stesso anzi ponevi la questione se 'due piani multipli colla stessa curva di diramazione sieno rappresentabili uno sull' altro'."

planes"),[94] having the same branch curves, are in fact equivalent (up to a certain transformation). Within a space of two years Enriques explicitly formulates this question, as we will see.

Kronecker's paper that Enriques refers to deals with the discriminant of an algebraic function with *one* variable.[95] Although the question Enriques and Castelnuovo are interested in concerns the branch curve of an algebraic function with two variables, the former claims that the question may be resolved by methods similar to Kronecker's. Enriques, however, indicates that it is easy to find the numerical invariants of the surface (among them, as he indicates, are the linear genus, the arithmetical genus and the geometrical genus) only by knowing the numerical invariants of a branch curve. Nevertheless, he remarks that he does not have time to deal with this question. In 1905 a similar approach is indicated. In a letter written to Castelnuovo on 1 February 1905, Enriques notes that once a branch curve, together with its degree and its number of nodes and cusps, are given one can then determine the degree of the branched surface from the relations between the various numerical invariants [Bottazzini et al. 1996, p. 603]. However, once again, Enriques does not develop his general remarks into a more comprehensive theory.[96]

As should be noted, the numerical approach to branch curves did not take into account any form of visual reasoning, and Enriques did not even treat the question of what the branch curve *looked like* or how it could be visualized. In 1899, however, he approached the research on the branch curve from another direction, employing a more visual form of reasoning. In another letter Enriques wrote to Castelnuovo on 26 February 1899 two questions are presented. The first: Can a branch curve be arbitrary? I.e., can any curve be a branch curve? The second: Given a branch curve, is there a *unique* complex surface of degree n branched along it? The arguments presented in the letter are concise but unclear, and Enriques indeed improved them in subsequent papers. I will cite the whole relevant section from this letter, however, as in it Enriques draws a sketch, which supports his argument.

94 The term "multiple plane" [piano multiplo] was used to denote surfaces covering the complex plane .

95 Indeed, the paper is called "Ueber die Discriminante algebraischer Functionen einer Variabeln" [1881].

96 This is only done in [Enriques 1912b], which deals only with numerical invariants of the branch curve and the constraints they impose on the surface. As the paper does not even hint towards visualization of branch curve, it is beyond the scope of the paper.

Answering an old question proposed by you, it seems to me that the branch curve of a multiple plane cannot be arbitrary, and that if a plane curve C is the branch curve of a multiple plane of degree n, it will define in general a unique n-multiple plane. This is why.

Take on the plane a pencil of lines passing through O and consider a (n degree) line a through O. It defines a certain number μ of objective curves $K_1, \ldots, K_\mu$; chose one curve K_1. Rotating the line a around O, we continually follow what happens to the curve K_1. After a whole turn, the curve K_1 generally permutes with $K_2 \ldots$ or K_μ. However, this must be ruled out if C is a branch curve of a degree n multiple plane. But if we impose the condition that K_1, for example, separates rationally between the μK's represented above the n-degree line, generally it will not happen the same for $K_2 \ldots$ or K_μ [Bottazzini et al. 1996, p. 400–401].[97]

Enriques' settings are as follows: a plane curve C is given (when C is not necessarily a branch curve), together with a point O, not on C, and a pencil of lines passing through O, which can be thought of as a rotating line: i.e., as the family of lines $\{y = tx : t \in \mathbb{R}\} \cup \{x = 0\}$, when $O = (0,0)$. Choosing one line a from this family, it intersects the curve C in several points. Enriques considers the μ Riemann surfaces of degree n which are ramified over these points, denoting these surfaces by $K_1, \ldots, K_\mu$. Enriques' claim is that once we rotate the line a (inside the pencil of lines) to do a full round, the movement of the intersection points with C will induce a permutation of the Riemann surfaces, i.e., a permutation of the set $K_1, \ldots, K_\mu$. However, Enriques claims that if C is a branch curve, then the induced permutation might be in fact the identity permutation—for example, choosing a Riemann surface K_1 it will not be permuted with any other of the Riemann surfaces $K_2, \ldots, K_\mu$.

97 "Ripensando ad una vecchia questione che tu proponevi, mi par di vedere che la curva di diramazione di un piano multiplo non possa darsi ad arbitrio, e che se una curva p[ia]na C è curva di diramazione d'un p[ia]no n-plo, essa definirà *in generate* un unico p[ia]no n-plo. Ed ecco perché. Prendi nel p[ia]no un fascia O e considera una retta (n-pla) per a [Lapsus per: una retta [...] a per O]. Essa definisce un certo numero μ di curve obiettive $K_1, \ldots, K_\mu$; scegliamone una K_1. Facciamo ruotare a attorno ad O, e seguiamo per continuità ciò che diventa K_1. Dopo un giro completo, in generale K_1 si permuterà con $K_2 \ldots$ o K_μ. Ciò invece deve escludersi se la C e curva di diramazione di un p[ia]no n-plo. Ma se imponiamo Ia condizione che K_1, ad es[empio], si separi raz[ionalmen]te fra le μK rappresentate sulla retta n-pla a, in generale non accadrà lo stesso per $K_2 \ldots$ o K_μ."

Enriques' arguments are at once both vague and condensed. It is clear that he relies on several arguments from Hurwitz's 1891 paper,[98] who also considered a similar situation. Given a set of points $a_1, \ldots, a_w$ in $\mathbb{C}$, Hurwitz sought to investigate what would happen to the set $F_1, F_2, \ldots$ of the Riemann surfaces of degree n, which are branched over these points, when the points $a_1, \ldots, a_w$ simultaneously start to move. Hurwitz found the following case—when the points eventually return to the initial position, though not necessarily in the same order—most interesting. For Hurwitz, "[e]ach closed path [geschlossenen Bahn] of the system of points $(a_1, a_2, \ldots, a_w)$ corresponds to a certain permutation

$$\begin{pmatrix} F_1 & F_2 & \ldots \\ F_1' & F_2' & \ldots \end{pmatrix}$$

of the Riemannian surfaces" [Hurwitz 1891, p. 23].

Enriques took a specific case of this "closed path," and visualized it, when the points $(a_1, a_2, \ldots, a_w)$ are actually intersection points of a line a with a given curve C. The "closed path" is formed when the line a performs a full rotation. It is clear that if C is a branch curve, then (at least) one of the Riemann surfaces $K_1, \ldots, K_\mu$ (branched along $a \cap C$) is a plane section of the complex surface, whose branch curve under a covering map is the given branch curve. However, it seems that what Enriques assumes is that there is only one complex surface of degree n branched along C—since in this case Enriques hints that the corresponding Riemann surface will not permute with any of the other of the Riemann surfaces, which is not a plane section a section of a complex surface.

Indeed, the visualization that Enriques proposed does not help to clarify his two claims that he sets up to explain: (1) why any plane complex curve cannot be a branch curve, and (2) why does a branch curve uniquely determine the surface.[99] There it is reasonable to pose the question, whether the sketch was indeed needed. Both Enriques and Castelnuovo knew that this sketch cannot represent in any way a branch curve, since a branch curve is always singular when the degree of the surface is higher

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98 Enriques mentions Hurwitz's research on Riemann surfaces in a letter to Castelnuovo on 7 June 1897 in: [Bottazzini et al. 1996, p. 341].

99 The claim that the branch curve determines the surface uniquely—i.e., that there are no two different surfaces branched over the same curve—was first conjectured by Oscar Chisini in 1944 for surfaces of degree greater than 4 [Chisini 1944], and only proved during the 21st century by Viktor S. Kulikov, with a single exception [Kulikov 1999a, 2008]. How Chisini investigated branch curves is beyond the scope of this paper; however, see the conclusion to Section III. Hence, it is not clear how Enriques thought about proving the uniqueness of a surface branched along a given curve.

than two. It seems, however, that it was essential for Enriques to present the new component—the rotating line—in his construction, as a new ingredient of an "experimental" visualization, anticipating Castelnuovo's reference to material models as "experimental sciences". Indeed, both Enriques and Castelnuovo were exposed to material models in Turin, since "in Turin the first acquisitions [of material mathematical models] thanks to Enrico D'Ovidio date from 1880–1881" [Giacardi 2015a, p. 20]. For Enriques, as someone who appreciated this tradition, (and for Castelnuovo, who obtained the sketch) the drawing was a way to prompt visual reasoning and connect the investigation of the branch curves with Hurwitz' more algebraic reasoning. It is important to note that Corrado Segre, who in 1907 was in charge of the Library in Turin instead of D'Ovidio, already commented in 1891—eight years before the letter from Enriques to Castelnuovo was sent—that "sometimes we [in Turin] even resorted to drawings or models of geometric figures to *see* certain properties [...] that could not be obtained with deductive reasoning."[100] [Segre 1963 (1891), p. 400] This approach, that there are types of reasoning beyond the merely deductive, certainly influenced Enriques, as we will also see in the following section when dealing with other attempts at drawing objects related to the branch curve.

Indeed, abandoning the second question (why does a branch curve uniquely determine the surface) and concentrating on the first, in two papers, written in 1912 and 1923, Enriques explains in more detail why branch curves cannot be just any curve. He does this by combining two types of reasoning: an algebraic one and a visual one, making his approach from the letter of 26 February 1899 more precise.

2.4.2. *Enriques and the loops encircling a branch curve*

In 1912 Enriques published his first paper on the subject of the branch curve, entitled "Sur le théorème d'existence pour les fonctions algébriques de deux variables indépendantes".[101] As Enriques later (in 1923) notes, the 1912 paper was not precise enough. I will review the 1923 paper in more detail later, but for the moment it is worth briefly examining the 1912 paper as a turn towards a more algebraic approach is taken.

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100 [...] "talvolta si è persino avuto ricorso a disegni o modelli di figure geometriche per *vedere* certe proprietà [...] che col solo ragionamento deduttivo non si sapevano ottenere."

101 Recall that a second paper on the numerical invariants of the branch curve and their connection to the numerical invariants of surfaces was also published in 1912 by Enriques [1912b].

At the beginning of this paper, Enriques describes the state of the research regarding branch points of his period and the construction of Riemann surface branched above them: given $2m$ simple branch points, n the degree of the to be constructed Riemann surface, and the $2m$ transpositions[102] corresponding to the branch points, forming a transitive group inside the symmetric group of n elements, a Riemann surface exists branched over these $2m$ points [Enriques 1912a, p. 419]. Like Hurwitz, Enriques algebraically formulates the conditions for the construction of the Riemann surface, where the only essential condition is that the set of transpositions corresponding to the branch points would be transitive and that their product would be the identity permutation. The question that Enriques poses is whether the same situation can be generalized to branch curves and the construction of complex surfaces.

Enriques immediately notes that the branch curve can be considered in two different ways: either as a complex *curve* in the (x, y) plane, "in the sense of algebraic geometry," or as a surface in a (real) 4-dimensional space, which "supplies the representation of complex points on the plane (x, y)" [Enriques 1912a, p. 419]. Indeed, Enriques hints at the two different approaches of representing complex curves, but does not elaborate on them. Instead, the main question is whether from a given (complex) curve, as discussed above, one can construct an n-degree complex surface branched along this curve. Enriques repeats his claim that the curve cannot be arbitrary, now explaining more clearly why: "given the invariants $(p_a, p^{(1)})$ of the [surface[103] given by the] equation $F(xyz) = 0$ one finds that in general the [branch] curve $f(xy) = 0$ has a certain number of nodes which are > 0 for $n > 3$ and a certain number of cusps which are > 0 for $n > 2$, when these numbers can be calculated with the help of the known formulas" [Enriques 1912a, pp. 419–420]. Enriques then states two claims, which are given offhand, without any proof. The first, that branch curves do not have higher singularities;[104] the second, that given two plane curves of the same degree, with the same number of nodes and cusps, it may be that one would be a branch curve, while the other not [Enriques 1912a, p. 420]. This is an important statement, as it

102 A transposition is a permutation, which permutes only two elements. A simple branch point may be defined as a branch point whose corresponding permutation is transposition.

103 Here, p_a is the arithmetic genus of the surface, and $p^{(1)}$ is the linear genus of the surface. See: [Hazewinkel 1995, pp. 111–112]

104 Ibid.: "on exclura qu'il y ait des singularités plus élevées." This was only proved in 2011, in: [Enriques 1912a, pp. 419–420].

indicates that the variety $V_{n,\delta,\varkappa}$,[105] which parameterizes all plane curves with degree n with δ nodes and $\varkappa$ cusps, with no other singularities (as a subvariety in the projective complex space $\mathbb{CP}^{\frac{n(n+3)}{2}}$), may be, for given n, δ, and $\varkappa$, reducible: i.e., having *several* non-intersecting components. Such a claim is given—surprisingly—in a very casual way, without any proof or example. The first example of this reducibility involving branch curves is Zariski's from 1929, proving that $V_{6,0,6}$ has two disjoint components (see Section II.3.2).

It seems that what led Enriques to make the second claim are the necessary and sufficient conditions that he proposed for a curve as a branch curve. These conditions are stated in a completely algebraic way, and, as Enriques claims, due to their complexity, do not hold for every curve. Specifically, when Enriques denoted the degree of the branch curve as $2m$, the question is how to assign transpositions for each of the $2m$ branches (of the branch curve), transpositions that would describe the way the complex surface is branched. In order to find the sufficient and necessary conditions, Enriques considers two types of critical points of the branch curve: simple ones ("which correspond to parallel lines to the y axis, tangent to $f = 0$ at a simple point" [Ciliberto & Flamini 2011]) and cusps. Enriques then poses three purely algebraic conditions for the associated transpositions,[106] indicating that if they are satisfied then the curve is a branch curve and one can construct a complex surface branched along it. The claim, however, is made without proving that those conditions

[105] I am using modern notation here.

[106] Enriques calls the set of simple branch points set (1), and the set of cusps set (2). He then states the following: "If there exists an irreducible algebraic function $z(xy)$ of $n > 2$ branches $z_1, \ldots, z_n$ corresponding to the branch curve $f(xy) = 0$, we have $2m = 2n + 2p - 2$, $p \geq 0$ [where p is the genus of the branch curve, $2m$ the degree of it] and the following conditions are satisfied:

1. The transpositions between the $2m$ branches, corresponding to the points in set (1), form an intransitive group. More precisely, the $2m$ branches $y_1, \ldots, y_{2m}$ would be divided in relation to this group in a certain number ρ, $\rho \geq n - 1$, of intransitive systems, comprised respectively of $t_1, t_2, \ldots, t_\rho$ elements, when $t_1 + t_2 + \cdots + t_\rho = 2m$.

2. To these ρ intransitive systems, one can associate ρ pairs of [different] number $(i\ s)$, when i, s belong to the set $1, 2, \ldots, n$, in the following manner: every transposition between $y_1, \ldots, y_{2m}$, corresponding to a point in set (2), will take a branch y corresponding to a system $(i\ s)$ into a branch corresponding to a system $(r\ h)$, where the two couples $(i\ s)$ and $(r\ h)$ have a joint element (either $h = i$ or $h = s$).

3. The transposition $(i\ s)$ corresponding to the $2m$ points $y_1, \ldots, y_{2m}$ generate a transitive group regarding [the permutation group of] $1, 2, \ldots, n$ and the product of all the $2m$ transpositions, taken in a suitable order, is reduced to the identity.

Conversely, *if conditions* 1, 2, 3 *are satisfied, there will always be an algebraic function* $z(xy)$ *with* n *branches, of which* $f(xy) = 0$ *is the branch curve.*" [Enriques 1912a, pp. 420–421].

are necessary and sufficient. Surprisingly, the nodes of the branch curve and the conditions they might have implied regarding their associated transpositions go unmentioned.

However, this pure algebraic consideration is re-formulated in a 1923 paper where it shifts into a more topological context that increasingly relies on visualization. I claim that this 1923 paper is a combination of the 1912 paper and Enriques insights from the 1899 letter.

* * *

In the 1923 paper entitled "Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curva di diramazione," Enriques declares at the outset that while submitting the 1912 paper, he already encountered several difficulties. As he states, it was precisely due to such difficulties that he now wished to resubmit the paper [Enriques 1923, p. 185].

As Enriques notes in the 1923 paper, not just any plane curve can be a branch curve [Enriques 1923, p. 185]—but once more he fails to give even a single example. As in the 1912 paper, Enriques' aim is to provide a proof for the necessary and sufficient conditions for a curve being a branch curve, formulating more precisely and more extensively the algebraic conditions of eleven years ago. The paper begins with the same setting of the 1899 letter: a plane curve C is given, assuming it is a branch curve of a surface F; a point O not on C is also given, and a pencil of lines $y = tx$ on this plane, passing through O. The parameter t is a complex one, whose values vary in the (complex) plane denoted by τ. Taking $t = 0$, the line $y = 0$ cuts the curve C in m points: $A_1, \dots, A_m$. When considering the section of the surface F above this line,[107] The above-mentioned section is the intersection of P and F. one obtains a (smooth) Riemann surface, denoted by K_0, which is considered as a branched cover of the line $y = 0$, branched over $A_1, \dots, A_m$, which are simple branch points.

Enriques' key move that enables him to later visualize his arguments is to note that the line $y = 0$ is a complex line, homeomorphic to a two-dimensional real plane. Hence, following the common method of investigation of a Riemann surface, one can look at a "system of loops" [Enriques 1923, p. 187] which are denoted by l_i, in the two-dimensional real plane $y = 0$, going out from O and encircling the points $A_1, \dots, A_m$. Every loop then corresponds to a permutation $S_i = (r_i\ s_i)$,[108] which describes the

[107] Denoting by O' the point from which one projects F to the complex plane, one considers a plane P passing through O' and $y = 0$.

[108] The notation $(r\ s)$ stands for a permutation, which permutes between r and s and leaves all the other numbers as they are.

permutation of the sheets r_i and s_i of the Riemann surface: while moving along l_i (inside the real plane $y = 0$), the n preimages of the starting point of l_i: $p_1, \dots, p_{r_i}, \dots, p_{s_i}, \dots p_n$ permute between themselves such that after one circling (when returning to the initial point of l_i) the values permuted equal to $p_1, \dots, p_{s_i}, \dots, p_{r_i}, \dots p_n$. Following the arguments from 1899, Enriques then moves the line $y = 0$ (as a member in the family of lines $y = tx$), claiming that after moving t along a loop on the plane τ, the resulting permutations should be identical. This results in conditions of invariance that should hold concerning these permutations, which Enriques claims are true for every branch curve. Enriques' task is to explicitly formulate these conditions.

This is done in the main section (Section III) of Enriques' paper. Enriques starts by noting that while rotating the line $y = 0$ in the pencil of lines, the rotated line might intersect three types of critical points of the curve C: simple tangent point (denoted by $T_1, \dots, T_\mu$ on the lines in the family $y = tx$ with C), nodes of C (denoted by $D_1, \dots, D_\delta$) and cusps of C (denoted by $Q_1, \dots, Q_\chi$). Before analyzing what happens to the above-mentioned permutations S_i while approaching one of these critical points, Enriques makes an observation regarding the loops l_i. This observation, and the argument that follows it, is mainly visual in orientation.

The question Enriques poses is the following: when moving t along a loop on the plane τ, what happens to the loops l_i when approaching one of these critical points? Assuming that $y = t_c x$ is a line which intersects C at a critical point, then there are two intersection points A_{r_i} and A_{s_i} which are merged (see Figure 17).[109] Remunerating the points of intersection, one can assume that the points A_1 and A_2 are merged. But if one takes a value t_0 very close to t_c, what would be the relative position of the loops l_1 and l_2?

Denoting by L_0 the complex line $y = t_0 x$, the question that Enriques poses concerns, in current terminology, the finding of a well-ordered basis for the fundamental group of the complement of the intersection points $A_1, A_2, \dots$ of L_0 with $C : \pi_1 (L_0 - \{L_0 \cap C\}) \simeq \pi_1 (\mathbb{C} - \{A_1, A_2, \dots\})$. Enriques then asks if one can always rearrange the loops in this group such that l_1 and l_2 would be "fairly close" ["onestamente vicini"], meaning that they "tend to merge without including any other point A or crossing other

109 The fact that there are *two* points that coincide is due to the nature of the critical points (tangent point, node or cusp). Were there other types of singular points, this might not have been the situation. However, Enriques does not prove—also in this paper—that the only singular points of a branch curve are nodes and cusps.

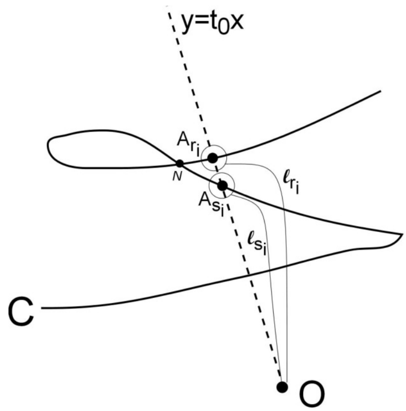

FIGURE 17. The complex (dashed) line $y = t_0x$ intersects the curve C near a singular point (here: the node N), and the points of intersection A_{r_i} and A_{s_i} will be merged at this point. The complex line $y = t_0x$ is homeomorphic to a two-dimensional real plane, and the loops ℓ_{r_i} and ℓ_{s_i}, which surround the intersection points, are drawn in this plane (figure drawn by M.F.).

loops."[110] [Enriques 1923, p. 189] Enriques suggests two cases: either the loops are already in this situation, i.e., already "fairly close" or, although being close to each other, other loops interrupt them to be merged while approaching the critical points. He then gives two drawings as examples for the second situation (see Figure 18):

What Enriques immediately notes is that "for simplicity, in this elementary case, we can demonstrate that one can transform the loops, permitting us to reduce it to the case when l_1 and l_2 would be fairly close"[111] [Enriques 1923, p. 190]. The transformation is shown in Figure 19.

Immediately afterwards Enriques proves that this transformation does not change the corresponding permutations S_1 and S_2. The above transformation, however, relies completely on a visual argument. Taking only these two cases into account, in Figure 19 Enriques solely *draws* the transformations of the loops. The argument of how to transform the basis of the above fundamental group is completely visual, relying eventually on the

110 "[...] può accadere che l_1 e l_2, per $t = t_c$, diventino o possan farsi diventare *onestamente vicini*, cioè tendenti a confondersi senza includere alcun altro punto A o attraversare altri cappi."

111 "Ma, riferendoci per semplicità a questo case elementare, possiamo dimostrare ehe, in ogni case, una conveniente trasformazione dei cappi, permette di ridursi al ease in cui l_1 e l_2, diventino onestamente vicini."

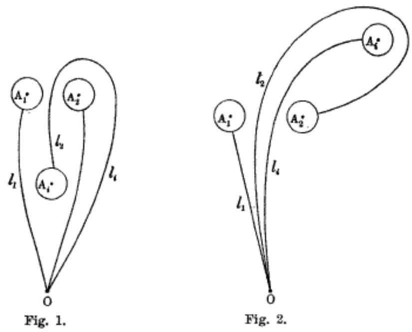

FIGURE 18. Enriques' depiction of two cases of a loop "interrupting" the loops encircling the points A_1 and A_2 to come together [Enriques 1923, p. 190].

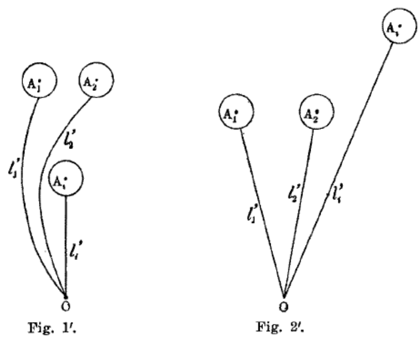

FIGURE 19. Enriques' depiction of the transformation of the loops from Figure 18 [Enriques 1923, p. 190].

ability of the reader to *imagine* more complex cases when another, more complicated loop l_i would be between l_1 and l_2.

With these assumptions, denoting as above the loops in the vicinity of a critical point as l_1 and l_2, Enriques deals with the three types of critical points, investigating what the relations between the induced permutation S_1 and S_2 are. He starts by investigating what happens in the neighborhood of a tangent point T. Enriques then asks what happens to the loops on the (complex) line L_0 (given by $y = tx$) and their corresponding permutations when the line encircles the point T (recall that the loops l_1 and l_2 encircle the points A_1 and A_2). When encircling the point T, Enriques subsequently indicates that it is clear from the analytical power series around this point[112] that what happens is a 180° turn of the points A_1 and A_2 [Enriques 1923, p. 191],[113] which results in a deformation of the loops. Enriques then adds a figure in order to depict this deformation (see Figure 20).

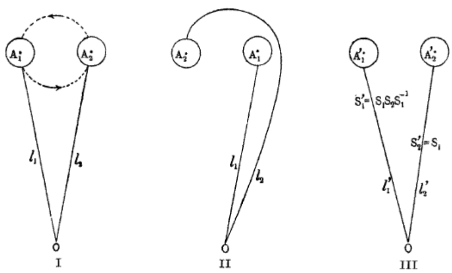

Fig. 3.

FIGURE 20. Enriques' three drawings of the change of the loops, which encircle the points A_1 and A_2, while the complex line $y = tx$ encircles the branch point T [Enriques 1923, p. 191].

While the passage from the first part of the figure (see Figure 20, part I) to the second (part II) describes the changing basis of the loops (from l_1 and l_2 to l_1' and l_2') and can be derived from the analytical power series, the third part of the figure indicates the algebraic relation between the permutations S_1 and S_2 corresponding to l_1 and $l_2$and the permutations S_1' and S_2' corresponding to l_1' and l_2'. "[…] [L]ooking at fig. 3 [Figure 20.III here]," Enriques subsequently notes, "where the three drawings I, II, III, which represent the sucessive states of the transformation, where the transition from II to III consists in transporting l_2 from the right to the left of l_1: we see that the permutations S_1 and S_2 change respectively in $S_1' = S_1S_2S_1^{-1}, S_2' = S_1$."[114] [Enriques 1923, p. 192].

While the argument here is also visual, there is no explicit algebraic proof. Enriques implies that any automorphism of the group $\pi_1(L_0 - \{L_0 \cap C\}) \simeq \pi_1(\mathbb{C} - \{A_1, A_2, \ldots\})$, being in this case the automorphism sending l_1 to $l_1' = l_1l_2l_1^{-1}$ and $l_2$to $l_2' = l_1$, also occurs at the

112 Being "$x - x_c = \lambda(t - t_c)^{1/2} + \cdots$" [Enriques 1923, p. 190].

113 "È chiaro che tale effetto si riduce a quello di un cerchio infinitesimo che circondi T, a cui risponde un cerchietto descritto dai punti A_1 e A_2, ognuno dei quali percorre uno dei due archi $A_1 A_2$, indicati nella fig. 3.I [here Figure 18.I]."

114 "Ebbene, osserviamo nella fig. 3 i tre disegni I, II, III, che rappresentano gli stati successivi della trasformazione, dove il passaggio dalla II alla III consiste nel trasportare l_2 dalla destra alia sinistra di l_1: vediamo cosi che le sostituzioni S_1 e S_2, si cambiano rispettivamente in $S_1' = S_1S_2S_1^{-1}$, $S_2' = S_1$."

level of the corresponding permutations to the l_i's. This claim, however, deserves either an algebraic proof or at least a reference to where it is proved. Yet Enriques provides neither of these, giving a visual justification instead. This might suggest that the visualization here plays a double role: both functioning as an indispensible inference method, and at the same time concealing the fact that an algebraic-symbolical proof may also be given. What this implicitly indicates is that an algebraic proof is unnecessary even though, as we will see later, Zariski followed this algebraic line of thought. This perspective, however, misses the bigger framework within which Enriques worked. Taking into account his familiarity with Hurwitz's algebraic formulation of the same procedure, Enriques rather decided to give more weight to the visualization of the loops themselves encircling the branch curve. Indeed, Hurwitz already provided the algebraic proof in 1891 [Hurwitz 1891, p. 28–31].

After using this visual argument, Enriques then employs algebraic inference methods, proving that in this case of the tangency point, $S_1' = S_1, S_2' = S_2$. Using the same argumentation, though without drawing any figures, he subsequently investigates what happens in the neighborhood of a node and a cusp. In the case of a node, Enriques proves that locally the corresponding permutations should be disjoint (e.g., $S_1 = (12), S_2 = (34)$) while in the case of a cusp, the corresponding permutations should have only one index in common (e.g., $S_1 = (12), S_2 = (23)$).

The recapitulation of the theorem, appearing at Section IV of Enriques' paper, summarizes the necessary conditions for a curve to be a branch curve, i.e., what results when it is known that C is a branch curve. The theorem might be thought of as algebraic, presenting the map sending the loops (in the complex line $y = tx$, encircling the points $A_1, \ldots, A_m$) to their corresponding permutations. While the language that Enriques employs is partially algebraic (describing the relations between the permutations), it is also to a certain degree visual-topological: the algebraic map determined while one follows "any path [cammino] of t that approaches a critical point $t = t_c$" [Enriques 1923, p. 196]. In other words, at *every* neighborhood of a critical point there is a (possible) visualization of the path (of t) approaching it as well as of the corresponding loops (though a loop of the t path is never drawn). Eventually, as Enriques described in Section III of his paper, one considers a loop of t at the complex line τ (thought as a two dimensional real plane). While the algebraic conditions are presented as local ones (what happens at the neighborhood of each critical point), the visual argumentation hints at a global consideration: one should consider the branch curve in its entirety, as the line $y = tx$

performs a complete rotation while moving t along a loop on the plane τ. Proving afterwards that the conditions presented above are not only necessary but also sufficient, Enriques formulates his theorem in terms of "elementary loops" ["giri elementari" or "sistema primitive di cappi"] [Enriques 1923, p. 198]. This already indicates a more algebraic formulation, using the tools of group theory. Nevertheless, Enriques fails to develop this any further.

2.5. *Zariski and the group-theoretic approach*

Such a development takes place only several years later, by Oscar Zariski (1899–1986), in two papers written in 1928 and 1929 respectively.[115] As we will see, his approach is almost purely algebraic: i.e., when Zariski does draw a figure, it is purely technical.

In the 1928 paper "Sopra il teorema d'esistenza per le funzioni algebriche di due variabili" Zariski already notes that while Enriques was the first to pose the problem of the necessary and sufficient conditions of a curve to be a branch curve, the answer he gave did not explicitly introduce "the concept of the fundamental group" [Zariski 1928, p. 134].[116] By the "concept of the fundamental group," denoted in Zariski's paper by G, he means the group of loops in the "residual space $S_4 - D$," [Zariski 1929, p. 306] when S_4 is a real 4-dimensional space (i.e., $\mathbb{R}^4$), and D is the branch curve. The same formulation appears in the 1929 paper "On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve," a translation of the 1928 paper with new results also added. Zariski in fact shifts the mathematical domain in which the problem was normally situated. "The complete solution of the existence problem depends upon the solution of the following purely topological problem: *Given an algebraic curve, to find its fundamental group.* In this paper we attempt to throw some light upon the structure of the fundamental group." [Zariski 1929, p. 305] Zariski therefore points out that the investigation of branch curves should be focused on finding "the structure of the fundamental group" and not on visualizing the curve or this group.

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115 For a detailed biography of Zariski and his work, see [Parikh 1991; Slembek 2002].

116 See also: [Libgober 2011, p. 4]: "Today we recognize Enriques relations among the permutations as the same as the relations satisfied by the generators of the fundamental group".

His investigation is therefore completely algebraic, and more specifically, group theoretic.[117]

Indeed, Zariski reformulates Enriques' results in an algebraic way. According to Zariski, "[a] finite set of generators of G [the fundamental group] can easily be constructed," using Solomon Lefschetz's method of finding a more convenient basis to work with,[118] when "the generators g_i [of the group G] satisfy several relations, called *generating relations*." [Lefschetz 1924, p. 307] These generators are what Enriques called a "primitive system of loops" ("sistema primitivo di cappi" [Enriques 1923, p. 198]), but Zariski takes this vague formulation and makes it precise using group theory. By looking at the relations between the generators, relations which are induced from the critical points of the branch curve, Zariski turns Enriques' visual investigation into an algebraic one: the relations of the corresponding permutations, induced from Enriques' analytical-visual argument, turn into algebraic relations between the different generators g_i. [Zariski 1929, p. 310–311] The only figure that appears in this context is a merger of Enriques' three figures (see Figure 20) into one figure (see Figure 21), while the argument is completely independent of any visual demonstration: Zariski notes that for a tangent critical point, "g_1 is transformed into g_2 and g_2 is transformed into $g_2^{-1}g_1g_2$ [...] (see Fig. 1). This leads to the generating relation: $g_1 = g_2$." [Zariski 1929, p. 310] Stating also the relations induced from a node and a cusp of the branch curve, Zariski reformulates Enriques' results in an algebraic way:

> The following theorem is an implicit consequence of the existence theorem, as it is stated by Enriques: THEOREM 4. The elementary generating relations together with the relation $g_1 g_2 \cdots g_n = 1$, form a complete set of generating relations, i.e., every relation between the generators is a consequence of them. [Zariski 1929, p. 312]

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117 This group theoretical approach highlights Zariski's growing interest in the algebraization of geometry, which culminates in the latter half of the 1930s. As Carol Parikh notes, "[Zariski] began [during the early 1930s] with the books of two algebraists who had been deeply influenced by Emmy Noether in Göttingen, B. L. van der Waerden's *Modern Algebra* and Wolfgang Krull's *The Theory of Ideals*." [Parikh 1991, p. 52]. See also [Parikh 1991, pp. 51–57]. Zariski's connection with Emmy Noether and the German algebraic school mark an important turning point in his conception of algebraic geometry.

118 [Zariski 1929, p. 307]: "It can be shown,* that any circuit g is equivalent to a circuit g' belonging to a generic 'line' l [...] through O (a two-dimensional manifold, homeomorphic to a sphere)." In the footnote * Zariski refers to: [Lefschetz 1924, p. 33]. Lefschetz's analysis at Chapter III "the topology of algebraic surfaces" (to which Zariski refers) does not contain a single illustration.

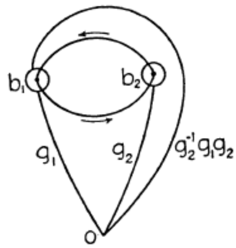

FIGURE 21. Zariski's depiction of the change of the loops while encircling a branch point. [Zariski 1929, p. 310]

Another additional development can be seen in Zariski's papers when compared to Enriques' work. While Enriques only hinted at the fact that there are curves that are not branch curves but have the same numeric invariants as branch curves, Zariski clearly formulates this claim. In 1928, Zariski asks "is the number of the cusps enough for determining the fundamental group of a given curve, or does this group depend also on the position of the cusps?" [Zariski 1928, p. 137] Zariski gives as an example two curves of order 6 with 6 cusps: the first, a branch curve of a complex surface of degree 3: [119] the branch curve is of degree 6, has 6 cusps lying on a conic, [120] and the fundamental group of the complement of this curve is isomorphic to a group generated by two elements g_1 and g_2 such that $(g_1)^3 = 1$ and $(g_2)^2 = 1$. [121] The second curve is also of degree 6, when the 6 cusps are in general position. Nevertheless, Zariski asks whether the two curves have the same fundamental group [Zariski 1928, p. 138]. Initially there is a negative answer to this question [Zariski 1929, p. 320], but only in few years later Zariski proves that "the fundamental group of a sextic with six cusps not on a conic is cyclic" [Zariski 1937, p. 357], that is, this group is generated by one element g, with only one relation $g^6 = 1$. The computations that Zariski performs in both cases are completely algebraic, and he does

119 Under a generic projection from a point not on a surface.

120 This result was in no way obvious since a unique conic passes through 5 points in a generic position, i.e., there is no conic that passes through 6 points in a *generic* position.

121 See: [Zariski 1929, p. 325]: "The fundamental group of a sextic curve f possessing 6 cusps on a conic (branch curve of the general cubic surface) is generated by two elements of orders 2 and 3 respectively." In: [Zariski 1928, p. 137] a different yet equivalent description for the relations is given: the relations presented there are $g_2g_1g_2 = g_1g_2g_1$ and $(g_1g_2)^3 = 1$.

not visualize the corresponding curves or the loops involved. Moreover, as we will see in the next subsection, in 1930 by using completely different methods Segre proved the fact that the second curve is *not* a branch curve.

If the global position of the cusps played such a crucial role for Zariski, why did he not offer a visualization of the global position of these cusps to the reader? [122] The answer is already hinted at above: for Zariski the drawings related to the branch curve were technical, and their investigation was found in another—algebraic—mathematical context. Heisuke Hironaka, a student of Zariski, notes that for his teacher, "you don't get algebraic intuition from the geometric intuition" [Parikh 1991, p. 81]. In so doing Hironaka indicates Zariski's preference for not relying on figures and drawings. While it might seem that at least in the case of the branch curve of the cubic surface, visualization was perhaps possible, it is essential to recall that in order to compute the fundamental group of a sextic with six cusps not on a conic, Zariski used a deformation argument by "remov[ing] a certain number of cusps" [Zariski 1937, p. 356] from a generic sextic with 9 cusps. These deformation processes were only visualized in exceptional cases.

With Zariski's emphasis on the group-theoretical investigation, as well as on transforming the visualization into a technical or even an obsolete method, one can note a shift in the way visualizations were considered. When Zariski posed his questions on the positional aspect of the singular points of the branch curve, he did so without even implying their possible visualization. Segre further advanced this approach, as we will see in the next section.

2.6. *Segre and special position of the singular points*

A year after the publication of Zariski's 1929 paper, which presented the example of the branch curve of a surface of degree 3, Beniamino Segre (1903–1977) published his paper "Sulla Caratterizzazione delle curve di diramazione dei piani multipli generali". Segre was an extremely productive algebraic geometer. Though Segre published only one paper in 1930 dealing with the branch curve, this paper pointed towards a different direction for research and generalized Zariski's results in two ways. Firstly, while Zariski showed that the cusps of the branch curve of a (smooth) complex

122 Interestingly, in his 1928 paper Zariski does mention Wirtinger's construction—which *was* accompanied by a drawing—concerning the intersection of a three-dimensional sphere with a neighborhood of a cusp, resulting in a "Dreiblattschlinge" [Zariski 1928, p. 137] (though the usual term was and is "Kleeblattschlinge"). However, Zariski was not interested in the local investigation of the cusps, but rather in their global behavior.

surface of degree 3 are in a special position (i.e., all of them lie on a conic), Segre shows the singular points of any branch curve of a (smooth) complex surface of degree n—*for any* n, $n \geq 3$—are in a special position. This means that the position of these singular points is not generic. Secondly, Segre pursues the questions posed by Enriques and Zariski: what are the necessary and sufficient conditions for a nodal cuspidal plane curve to be a branch curve of a smooth complex surface embedded in the (projective complex) three-dimensional space $\mathbb{CP}^3$. Segre, however, does not follow their methods. [123]

The way to this generalization for Segre involved a shift in the methods of inference used as well as of the mathematical domain, in which the problem was located. Segre notes at the beginning of his paper that with the approaches of Zariski and Enriques "group-theoretic and topological considerations are essentially involved—while leading to remarkable results, these do not exhaust the argument. The difficulties encountered in the above approaches depend on the fact that not every algebraic plane curve is a branch curve of a (non cyclic) multiple plane: it is a matter of *characterizing* branch curves of such [multiple] planes." [124] [Segre 1930, p. 97]

How does Segre characterize branch curves? The method Segre presents is completely novel, when compared to the methods of Enriques and Zariski. Concentrating only on complex smooth surfaces embedded in the $\mathbb{CP}^3$, Salmon already knew the numerical invariants of a branch curve of a surface of degree n (see Section II.1), and Segre references his work: [Segre 1930, p. 99] The degree of the branch curve is $n(n-1)$, the number of nodes is $\frac{1}{2}n(n-1)(n-2)(n-3)$ and the number of cusps is $n(n-1)(n-2)$. Segre then uses the machinery of *adjoint curves*: Given a plane curve C, a second curve A is said to be *adjoint* to C if it contains each singular point of C of multiplicity r with multiplicity at least $r-1$. In

[123] While Segre refers explicitly to Enriques' and Zariski's papers, his motivation (as well as Enriques' and Zariski's) also lied in the investigation of the variety $V_{n,\delta,\varkappa}$ (of curves of degree n with δ nodes and $\varkappa$ cusps). Zariski proved that $V_{6,0,6}$ has two disjoint components, and Segre was inspired by this discovery to see whether one can obtain—via an investigation of branch curves—decompositions of other varieties $V_{n,\delta,\varkappa}$ (i.e., for different n, δ and $\varkappa$). Regarding the investigation of this variety, Aldo Brigaglia and Ciro Ciliberto note, that due to remarks also made by Zariski regarding the difficulties investigating this variety, "the research on moduli [i.e., the variety $V_{n,\delta,\varkappa}$] suffered the same fate as that concerned with the fundamental theorem of irregular surfaces, that is, it was left incomplete and inconclusive." [Brigaglia & Ciliberto 1995, p. 102]

[124] " [...] nelle quali entrano in gioco in modo essenziale considerazioni gruppali e topologiche—pur conducendo a risultati notevolissimi, non esauriscono l'argomento. Le difficoltà che s'incontrano nella suddetta estensione, dipendon dal fatto che non ogni curva piana algebrica è curva di diramazione d'un piano multiplo (non ciclico): si tratta dunque di *caratterizzare* le curve di diramazione di tali piani."

particular, A is adjoint to a nodal-cuspidal curve C if it passes through all nodes and all cusps of C. For example, for the branch curve of a surface of degree 3, the six cusps lie on a conic; hence the conic is an adjoint curve to this branch curve. Already in the first section of the paper, one sees that Segre considers neither the visualization of curves nor their fundamental group. His main results in this section involve proving—using *non-visual*, algebraic-geometric methods, such as the existence and properties of linear series, equivalence of divisors and Noether's $AF + BG$ theorem—that, for example, the following adjoint curves to the branch curve exist:

(1) Two adjoint curves of degrees $(n-1)(n-2)$ and $(n-1)(n-2)+1$ passing smoothly through the nodes and the cusps of the branch curve. [Segre 1930, p. 100, 102]

(2) An adjoint curve of degree $n(n-1)-2$, having nodes at the cusps of the branch curve and passing smoothly through the nodes of the branch curve. [Segre 1930, p. 101]

Nevertheless, Segre's main result runs in the opposite direction: that is, he proves the following theorem:

> A nodal-cuspidal plane curve B of degree $n(n-1)$ with $\frac{1}{2}n(n-1)(n-2)(n-3)$ nodes and $n(n-1)(n-2)$ cusps is the branch curve of a generic projection of a smooth surface of degree n in $\mathbb{CP}^3$ *if and only if* there are two adjoint curves of degrees $(n-1)(n-2)$ and $(n-1)(n-2)+1$ passing through the nodes and the cusps of the curve [Segre 1930, p. 111].

Just as before the proof uses tools and methods from algebraic geometry, which Segre failed to visualize generally. He also ignored the possibility of drawing the special relations between the singular points of the branch curve. This is no surprise: taking a look at the three volumes of Segre's *Opere scelte* [Segre 1987a;b; 2000] there is not a single sketch or figure in his papers. Beniamino Segre's preference for the symbolical algebraic-geometrical method over the more visual-topological one is clear. This stands in sharp contrast to his uncle, Corrado Segre, "the leader of the Italian School of algebraic geometry," who, as Giacardi notes, "increased the collection of models [in Turin]. In fact he believed that the models could sometimes pave the way to discovery" [Giacardi 2015b, p. 2785]. Beyond this personal preference, however, one should note two fundamental differences when comparing to the visualizations of Enriques and Zariski. Firstly, all of Enriques' sketches and most of Zariski's sketches and figures related to branch curves, were of local nature: that is, what was drawn was a depiction of what occurs (to certain loops) in a local neighborhood of the singular points of the branch curve. An attempt to construct a three-dimensional model or to sketch a two-dimensional figure of the branch curve entirely, as was done in the case of Riemann surfaces and their branch points, was not even attempted.

If one compares the German to the Italian tradition of constructing material models, a second difference can be seen. The construction of a three-dimensional model of the branch curve (as, for example, the real part of a singular Riemann surface) required practical specialty and expertise in handcraft that the Italian school of algebraic geometric did not have. As mentioned above, while models of surfaces were mostly bought from Germany, they were hardly produced in Italy. However, this only adds up to an imperfect explanation as to why the lack of visualization methods of branch curves is linked to the almost non-existent expertise of the Italians in making models. Certainly, a more elaborate explanation is required.

Indeed, it should also be remembered that the thesis advisor of B. Segre in Turin was his uncle, Corrado Segre, who, as noted above, supported the construction of material mathematical models and even considered them epistemic things, carrying their own reasoning. As already mentioned, these models were also constructed in Turin. Even if he did not include a single sketch or reference to models in his writings, B. Segre must have been at least aware of this tradition and also of the ideas of his uncle regarding such models. However, to emphasize—the production of the models of branch curves had to take into account the special position of the nodes and the cusps. As remarked above, it is unclear whether in Turin the necessary expertise for constructing such models existed. This is to be contrasted with the situation in Germany. As David Rowe notes [Rowe 2018], models of surfaces were produced in Germany that visualized the special position of singular points of surfaces. To be more precise, such models visualized how six nodes of a quartic surface lie on a conic [Rowe 2018, p. 63]—a situation which was almost identical to the special position of the six cusps of the branch curve of a surface of the third degree. Hence, one may claim that either the Italian mathematicians, who in practice did construct or buy models, were unaware of *these* German models, or that they did not have sufficient expertise in model construction to visualize the special position of the singular points of the branch curve.

3. CONCLUSION: THE MANY FACES OF BRANCH POINTS AND BRANCH CURVES

After his 1930 paper, Segre did not investigate branch curves any further. As Edoardo Vesentini remarks [Vesentini 2005, p. 188], the "memoir by B. Segre on the characterization of the branch curve of a multiple plane, that appeared in 1930 and was inspired by a paper of Enriques, followed shortly by a paper by O. Zariski on the same topic. But a critical remark made by O. Zariski on an infinitesimal method used by Enriques in earlier papers on the moduli of an algebraic surface, set some doubts on the validity of Enriques' argument and, consequently on the papers that Segre devoted to this topics."[125] In addition, as Edoardo Sernesi notes, other mistakes in some of Segre's related research were discovered.[126] These mistakes may explain why Segre failed to develop his research on branch curves further.

Research on branch curves did not stop, however, and with it new horizons of visualization appeared after Segre's work. Nevertheless, such developments coincided with new forms of inexperience and the eventual disappearance of visualization. Before concluding, I would like to survey briefly the historical development that took place in the years following—the 1930s.

On the one hand, Egbert van Kampen (1908–1942) in 1933 presented a precise algebraic computation of the fundamental group of the complement of a complex plane curve;[127] his treatment, however, does not contain a single drawing. Zariski's 1935 treatment of branch curves in his book *Algebraic Surfaces* followed van Kampen's formulation.[128] Additionally, Zariski emphasized the topological nature of Enriques' discoveries and less their algebraic context.[129]

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125 Zariski's critical remark concerns an implicit assumption Enriques made regarding the completeness of the "characteristic system of a complete continuous system of surfaces," a proof of which, according to Zariski, "is not likely to be an easy undertaking". [Zariski 1935, p. 99] This assumption was eventually disproved by Wahl, by finding a counterexample, in 1974. See [Wahl 1974, p. 573].

126 As Edoardo Sernesi notes, another paper by Segre [Segre 1929] deals with the construction of "new components" [of $V_{n,\delta,\varkappa}$] starting from those given and aiming to establish for which values of $n, \delta, \varkappa$, the variety $V_{n,\delta,\varkappa}$ is not empty [Sernesi 2012, p. 446]. However, Sernesi adds that while "Segre's procedure seems to be correct," "his conclusions, as they stand, are incorrect." [Sernesi 2012, p. 447].

127 Van Kampen remarks, while discussing Enriques' results, that "as the resulting proof [of Enriques for finding the relations of the fundamental group of , when is a complex plane curve] seemed too algebraic for this simple and nearly purely topological question, Dr. Zariski asked me to publish a topological proof which is contained in this paper" [van Kampen 1933, p. 255].

128 See: [Parikh 1991, p. 49]: "Most valuable to Zariski was the hiring of E. R. van Kampen, a gifted topologist from Holland. Warm and charming, part Indonesian, he shared with Zariski a lively interest in fundamental groups."

129 [Zariski 1935, p. 162]: "The following comment on the theorem of Enriques may be of interest. From a purely topological point of view the theorem of Enriques says

The works of Oscar Chisini (1889–1967), one of Enriques' students, should also be taken into account, on the other hand. Already in 1920–1921, probably following Enriques, he initiated a research on branch curves, which dealt with the question of the birational equivalence of two complex surfaces having the same branch curve; nevertheless, this paper did not contain a single drawing or a sketch [Chisini 1921]. In the 1930s, 1940s and the 1950s, however, while returning to research complex surfaces as coverings of the complex projective plane and their branch curves, he arrived at the idea of "realizing a visible model of the fundamental group of the complement of an algebraic curve in the complex projective plane [...] being particularly relevant in the theory of multiple planes. The model in question is that by Chisini called the *characteristic braid* of the algebraic curve, and allowed him to place in evidence the topological-combinatorial aspects of the theory of curve singularities and multiple planes." [Brigaglia & Ciliberto 1995, p. 113–114] In 1933 Chisini first integrated braids (in Italian "treccia" [Chisini 1933, p. 1151]) as a visual aid into the research of complex curves and their branch points. Given a complex curve as a cover of the complex line, Chisini investigated the preimages of a loop (on the complex line) surrounding the image of singular and branch points of the curve (see for example Figure 22). [130] However, how Chisini continued to research braid theory and branch curves—leading him in 1944 to conjecture that the branch curve uniquely determines the associated branched covering once the degree of the cover is larger than 4—is beyond the scope of the current paper. [131]

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that the relations [Enriques found] give a complete set of conditions for the existence of k-fold covering manifolds with f as branch curve (4-dimensional Riemannian varieties consisting of k samples of the projective plane P connected in a proper manner along the curve f). However, from this does not follow immediately the completeness of the set of generating relations [...] for the fundamental group G, proved by van Kampen."

130 Chisini, it is essential to note, did not use in 1933 the machinery of algebraic braid theory, as developed by Artin in 1925.

131 Chisini conjecture [Chisini 1944] is as follows: Let B be the branch curve of a generic ramified covering of degree at least 5. Then the branched covering is uniquely determined by the branch curve. This stands in contrast to the situation of Riemann surfaces and Hurwitz's formulation. Recall that Hurwitz noted that by specifying the number of sheets, the position of its branch points, and the local monodromy behavior at these branch points, one can determine the Riemann surface. Chisini conjectured *almost the contrary*: that, in fact, for complex algebraic surfaces, once the degree of the branch curve is big enough, one does not need to specify the number of sheets (i.e., the degree of the surface) or the permutations of the sheets (i.e., the epimorphism to the symmetric group)—and only the "position" of the branch curve is enough to determine uniquely the surface.

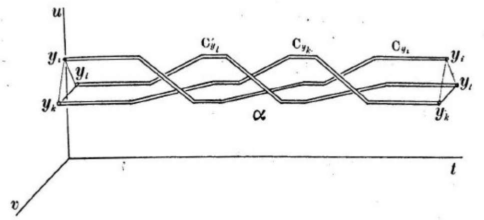

FIGURE 22. Chisini's "model" [Chisini 1933, p. 1141] for depicting the preimages of a loop encircling a branch point (figure taken from: [Chisini 1933, p. 1153]).

* * *

For branch points, as was seen in the first section, there was an abundance or rather plurality of visualizations both two- and three-dimensional. To emphasize the obvious: two-dimensional sketches were more easily produced, usually not requiring any special training and being a part of the writing process—as can be seen with Enriques' sketch; this stands in opposition to the production of three-dimensional models, which did require special craftmanship. Nevertheless, the various illustrations and three-dimensional models, which at times were inadequate to each other, prompted indirectly, as I suggested, along with the various inherent (dimensional) restrictions (i.e., visualizing the ramification curve as a curve on a four-dimensional object (the surface) in a six-dimensional space), an "invisibility" of the branch curve, when mathematicians came to deal with them. [132] Only a partial visualization of the entire branch curve was undertaken (or it was considered in its entirety to be non-visualizable). And this happened in several modes: Wirtinger, to give a first example, made only the local behavior of the singular points of the branch curve visible, prompting incomprehension of the global behavior of these points. Although the real part of the branch curve could have been drawn, all of the actors surveyed in Section II decided not to do so. Even if we consider Enriques' 1899 sketch as an exception, he was not even attempting to illustrate the unique characteristics of this curve. Besides the fact that they considered visualization unnecessary or technical, the reason why Segre and Zariski did not make a single drawing might be due to the special position of nodes and cusps. If one wanted to draw the (real part of the) branch curve, one had to find an equation for this curve where all of the singular points were real (in order to show their special position), a task that would result in tedious calculations. When Zariski finally

132 Regarding making invisible scientific objects and procedures, see: [Nasim 2018].

posed the question regarding the position of the cusps with respect to each other, to give a second example, he did not draw a single illustration of the global position of these points. Zariski also shifted the research from one on branch curves to an algebraic research on the fundamental group of a complement of a curve as such. Visualization as a mean of mathematical reasoning or as an inference step, as was seen with Enriques' loops, led to another kind of invisibility. Not only one can consider certain loops as those, which should have been drawn (recall Enriques' loop of the parameter t on the complex plane τ) but were not; but this invisibility occurred in two additional ways: firstly, it concealed the necessity of an algebraic argument; secondly, and precisely due to this lack of consideration when it came to algebraic arguments, it may have resulted in a more algebraic approach, which can be seen in Zariski's work. And eventually, with Segre and Zariski, a process involving the differentiation of research traditions and mathematical practices took place, leading to a diminishment in the epistemological advantages that visualization techniques may have provided, and favoring instead group-theoretic or algebraic-geometrical methods.

Becoming technical—as with Zariski's usage of diagrams—points towards another mode of becoming invisible, one in which the epistemological aspect of the visualized object disappears. This can be seen with Zariski's illustration in particular, and with the Italian usage of three-dimensional material models in general, located in an essentially different culture of visualization than that which occurred in Germany. As was noted, in Germany there were models that showed singular points in a special position, but those models were probably not a part of the German-Italian exchange

To conclude, I would like to return shortly to the differentiation I mentioned at the conclusion of Section I.6. There I discussed how visualization techniques oscillated between being exact and co-exact. That is, they fluctuated between being the *exact* material or illustrated representation of the mathematical object (for example, the three-dimensional models of a branch point of Riemann surfaces) and between being a *co-exact* visualization of partial, non-metrical (i.e., non-sensitive to quantitative parameters) (for example, the loops of Enriques or the braids drawn by Sevei, Enriques and Chisini). The oscillation of these techniques, between being exact and co-exact, between being epistemological and technical, created not only new mathematical approaches to the visualization of branch curves and branch points, but also engendered new approaches—algebraic or analytical, for example—which rendered the mathematical object—in this case, the branch curve—invisible.

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