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SMALL TIME EQUIVALENTS FOR THE DENSITY OF A PLANAR QUADRATIC LANGEVIN DIFFUSION

by Jacques Franchi

Abstract. — Exact small time equivalents for the density of the (heat kernel) semi-group, with a control of the error term, are obtained for a quadratic planar analogue of the Langevin diffusion, which is strictly hypoelliptic and non-Gaussian, and hence of a different nature from the known Riemannian, sub-Riemannian and linear-Gaussian cases. Two regimes are considered, an unscaled and a scaled one, where both can be seen as natural extensions beyond the degenerate Langevin-Gaussian framework. The result for the scaled regime seems to be the first such one in a non-Gaussian strictly hypoelliptic framework. The method is half-probabilistic, half-analytic.

Résumé (*Equivalents en temps petit pour la densité d'une diffusion de Langevin quadratique plane*). — Cet article fournit des équivalents exacts en temps petit, avec contrôle du terme d'erreur, relatifs à la densité (noyau de la chaleur) du semi-groupe associé à une diffusion quadratique plane, analogue non gaussien de la diffusion de Langevin. Dans ce cadre strictement hypoelliptique non gaussien, différent des cadres sous-riemannien et gaussien (linéaire), le régime de base et un régime rééchelonné sont considérés, qui sont tous deux des prolongements naturels du cas dégénéré Langevin-gaussien. L'étude du régime rééchelonné semble la première de ce type dans un tel cadre. La méthode suivie est mi-probabiliste mi-analytique.

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1. Introduction

The problem of estimating the heat kernel, or the density of a diffusion, particularly as time goes to zero, has been extensively studied for a long time, firstly in the elliptic case, and then largely solved and understood in the sub-Riemannian case too. We only reference articles [20, 2, 4, 5, 17], and the existence of other works on that subject by Azencott, Molchanov and Bismut, quoted in [4].

To summary roughly, a very classical question addresses the asymptotic behavior (as $s \searrow 0$) of the density $p_s(x,y)$ of the diffusion (x_s) solving a Stratonovich stochastic differential equation

$$x_s = x + \sum_{j=1}^{k} \int_0^s V_j(x_\tau) \circ dW_\tau^j + \int_0^s V_0(x_\tau)\, d\tau \,,$$

where the smooth vector fields V_j are supposed to satisfy a Hörmander condition.

The elliptic case (when $V_1, \ldots, V_k$ span the whole tangent space everywhere) being very well understood for a long time [20, 2], the studies focused then on the sub-elliptic case, that is to say, when the strong Hörmander condition (that the Lie algebra generated by the fields $V_1, \ldots, V_k$ has maximal rank everywhere) is fulfilled. In that case these fields generate a sub-Riemannian distance $d(x,y)$, defined as in control theory, by considering only C^1 paths whose tangent vectors are spanned by them. Then the wanted asymptotic expansion tends to have the following Gaussian-like form:

$$p_\varepsilon(x,y) = \varepsilon^{-d/2} \exp\big(-d(x,y)^2/(2\varepsilon)\big) \left(\sum_{\ell=0}^{n} \gamma_\ell(x,y)\, \varepsilon^\ell + \mathcal{O}(\varepsilon^{n+1}) \right) \tag{1}$$

for any $n \in \mathbb{N}^*$, with smooth γ_ℓ's and $\gamma_0 > 0$, provided x, y are not conjugate points (and uniformly within any compact set which does not intersect the cut-locus). See in particular ([4, theorem 3.1]). Note that the condition of remaining outside the cut-locus is necessary here, as shown in particular by [5].

The methods used to get this or a similar result have been of a different nature. In [4], G. Ben Arous proceeds by expanding the flow associated to the diffusion (in this direction, see also [7]) and using a Laplace method applied to the Fourier transform of x_s, then inverted by means of Malliavin's calculus (with a deterministic Malliavin matrix).

The strictly hypoelliptic case, i.e., when only the weak Hörmander condition (requiring the use of the drift vector field V_0 to recover the full tangent space) is fulfilled, remains much more problematic, and thus is rarely addressed. There is a priori no reason that the asymptotic behavior of $p_s(x,y)$ remains of the Gaussian-like type (1). Indeed this already fails for the mere Gaussian Langevin process $\big(\omega_s, \int_0^s \omega_\tau \, d\tau\big)$: the missing sub-Riemannian distance must be replaced by a time-dependent (actually Carnot-Carathéodory) distance $d_\varepsilon\big((\dot x, x); (\dot y, y)\big)$ which presents some degeneracy in one direction, namely the missing $d\big((\dot x, x); (\dot y, y)\big)^2/(2\varepsilon)$ must be replaced by

$$\begin{aligned}&\frac{6}{\varepsilon^3}\left|(x-y)-\frac{\varepsilon}{2}(\dot x-\dot y)\right|^2+\frac{1}{2\varepsilon}\left|\dot x-\dot y\right|^2\\&=\frac{1}{2\varepsilon}\Big(\left|\dot x-\dot y\right|^2+\frac{12}{\varepsilon^2}\left|(x-y)-\frac{\varepsilon}{2}(\dot x-\dot y)\right|^2\Big),\end{aligned}$$

and actually, for any $\varepsilon > 0$ and $\dot x, x, \dot y, y \in \mathbb{R}^d$ we have

$$\tag{2}\begin{aligned}&p_\varepsilon\big((\dot x, x); (\dot y, y)\big)\\&=\frac{3^{d/2}}{\pi^d\,\varepsilon^{2d}}\exp\left[-\frac{|\dot x-\dot y|^2+12\left|(x-y)-\varepsilon(\dot x-\dot y)/2\right|^2/\varepsilon^2}{2\,\varepsilon}\right].\end{aligned}$$

As in this Gaussian-Euclidean Langevin case this expression is actually an exact one, and holds not only asymptotically, we then have

$$\tag{3}\begin{aligned}&p_\varepsilon\big((\dot x, \varepsilon x); (\dot y, \varepsilon y)\big)\\&=\frac{3^{d/2}}{\pi^d\,\varepsilon^{2d}}\exp\left[-\frac{|\dot x-\dot y|^2+12\left|(x-y)-\frac{1}{2}(\dot x-\dot y)\right|^2}{2\,\varepsilon}\right],\end{aligned}$$

for any $\varepsilon > 0$ and $\dot x, x, \dot y, y \in \mathbb{R}^d$. Thus, in this scaled formulation, in the energy we recover a true, time-independent squared distance. So that, referring to the Riemannian and sub-Riemannian cases, there is no clear reason a priori to favour the unscaled version (2) to the scaled version (3). We shall emphasize this point of view below.

Barilari and Paoli [3] considers a general Gaussian hypoelliptic n-dimensional diffusion (X_t), solving a linear equation $dX_t = A\,X_t\,dt + B\,dW_t$, for a d-dimensional Brownian motion W. Taking advantage of the explicit exact expression for the heat kernel p_t which is computable in such a linear Gaussian case, the authors provide the full small time asymptotics for p_t.

Paoli [18] analyzes the small time asymptotics on the diagonal, relative to the heat kernel of a manifold-valued strictly hypoelliptic diffusion, in the spirit of previous works by Ben Arous and Léandre.

See also [9] and [19] for non-curved, strictly hypoelliptic, perturbed cases where Langevin-like estimates hold (without precise asymptotics), roughly having the following Li-Yau-like form:

$$\tag{4}\begin{aligned}C^{-1}\,\varepsilon^{-N}\,e^{-C\,d_\varepsilon(x_\varepsilon,y)^2}&\le p_\varepsilon(x,y)\\&\le C\,\varepsilon^{-N}\,e^{-C^{-1}\,d_\varepsilon(x_\varepsilon,y)^2},\quad\text{for } 0<\varepsilon<\varepsilon_0\,.\end{aligned}$$

In [11] a small time asymptotics of a simple model was only partly computed, similar to the one analyzed below but in five dimensions, in the specific off-diagonal regime of a dominant normalized Gaussian contribution. Thus the energy term appeared as given by the same squared time-dependent distance as in the Langevin case, the strictly second chaos coordinate appearing only in the off-exponent term, as a perturbative contribution.

A stronger interest lies in a significant strictly hypoelliptic diffusion, namely the relativistic diffusion, first constructed over Minkowski's space (see [10, 13]). It makes sense over a generic smooth Lorentzian manifold as well, see [12]. In the simplest case of Minkowski's space $\mathbb{R}^{1,d}$, it consists of the pair $(\dot{\xi}_s, \xi_s) \in \mathbb{H}^d \times \mathbb{R}^{1,d} \equiv T^1_+\mathbb{R}^{1,d}$ (parametrized by its proper time s, and analogous to a Langevin process), where the velocity $(\dot{\xi}_s)$ is a hyperbolic Brownian motion. Note that even there, a curvature constraint must be taken into account, namely that of the mass shell $\mathbb{H}^d$, at the heart of this framework.

This (Dudley) relativistic diffusion, even restricted to 3 dimensions which already contain the essence of the difficulty, constitutes a significant example, altogether explicit, physical and not too complicated, allowing a priori to progress towards the understanding of a more generic, but less accessible to begin with, strictly hypoelliptic (degenerate) case. However even this apparently simple example proves to be very delicate to analyze, regarding the small time asymptotics.

The present work handles a simpler example, also of the Langevin type, hence strictly hypoelliptic, degenerate but non-Gaussian. It also pertains to the Euclidean case of what [1] recently considered and called "kinetic Brownian motion", on a Riemannian manifold. In this setting we obtain the exact small time ($\varepsilon \searrow 0$) equivalent for the heat kernel density $p_\varepsilon(0;x)$, together with a control of the error term, for constant x and also in an interesting scaled case where x exhibits some dependence on ε. To the author's knowledge, the latter is the first example of a result of this type in a non-Gaussian strictly hypoelliptic framework. It reveals a different nature from the known Gaussian framework, see Remark 2.2 below.

The present work was influenced by a beautiful article [4], which decisively handled the off-cut-locus (hence in particular off-diagonal) generic sub-Riemannian framework, as far as the small-time asymptotics of the heat kernel is considered. Thus the strategy adopted below starts as the strategy followed by G. Ben Arous. However the present purpose is to deal with a strictly hypoelliptic situation, to which [4] does not apply. A main obstacle to handle a strictly hypoelliptic framework is the lack of sub-Riemannian distance. For that reason, a strategy adapted to such a degenerate framework can only partially follow the method of [4].

As the author is not yet ready to handle a generic strictly hypoelliptic framework, and actually doubts that a generic unified treatment (or even, unified generic result) be possible (different strictly hypoelliptic frameworks could produce different types of results; the present one already differs notably from the classical Gaussian Langevin case regarding the scaled energy, see Remark 2.2

below), the focus here is on a simple first example, which allows explicit computing of the Fourier transform of the heat kernel, and then concentrating on some finite-dimensional oscillatory integral. On the contrary, the choice of a 2-dimensional framework is unessential, but avoids even heavier notation and computations which higher dimensions would call for. According to the above remark about the Gaussian-Euclidean Langevin case, we consider both the unscaled and the scaled asymptotics, and the latter appears here as the most interesting, maybe indicating that the known settings are more interestingly extended to the strictly hypoelliptic framework in this way.

Besides, focusing on the present relatively simple example allows handling of the "pseudo-cut-locus" case, the analogue of conjugate points; which is delicate, even in the sub-Riemannian framework: the cut-locus constitutes a real difficulty in that case, see for example [5], and its case does not seem to be generically solved in a sub-elliptic framework. Furthermore the choice of a relatively simple particular framework allows expressing of all coefficients of the wanted equivalents, together with a control of the error, which will likely be out of reach in an even slightly more generic framework; however even in the present setting most functions that come into the scaled result remain implicit. In order to keep the already heavy enough computations within reasonable bounds, we focus on the asymptotics for the process started from 0.

Whereas the unscaled regime was already considered by V. Kolokoltsov in his book [16], by different, purely analytical means, see Remark 2.8 below, the scaled study seems to be the first one of this type in such a framework.

Organization of the content. In Section 2 the strictly hypoelliptic diffusion under consideration is described, and the central results, relating to both the unscaled case and the scaled case, are gathered in Theorem 2.1. Corollary 2.7 states that the squared Carnot-Carathéodory pseudo-distance yields the right exponent in the unscaled asymptotics, as in the (sub-)Riemannian [4] and Gaussian [3] frameworks.

Section 3 develops the leading strategy of the proof, which begins as that of [4]: the first main tool is a Fourier-Parseval expression for the density of the heat kernel under consideration, see Proposition 3.4. Here the lack of metric and geodesics forbids the use of a geodesic tube as in [4]. This is replaced by another key tool, which is the explicit computation of the Fourier transform, which is possible due to the choice of a Langevin-like diffusion of a quadratic type. The latter explains the reason for this choice, and why the present strategy could hardly be extended to non-quadratic examples, see Remark 2.5 below. The expression for the heat kernel density obtained in this way contains an oscillatory integral which is not computable, but which is no longer infinite-dimensional.

In Section 4 the oscillatory integral is analyzed which results from the preceding section, in the non-degenerate “off-cut-locus” case $w \neq 0$. As the dimension has now become finite, this does not need to resort to Malliavin calculus as in [4]. The saddle-point method is implemented, as described at the beginning of Section 4, after eqn (11). Since the saddle-point equation can not be fully solved in the complex plane, some implicit solution is exhibited. Several technical estimates are then needed, in particular to ensure that the implicit saddle point found in this way is either the only one or the dominant one, and then eventually yields the right equivalent.

Section 5 is devoted to the singular case $w = 0$, which is delicate too, and is analogous to the study at a sort of cut-locus, relating to some absent metric. The saddle-point method is implemented again, with technical difficulties more or less of the same nature as in the case $w \neq 0$, but not the same. A quasi saddle-point must be used. The control of the error term is somewhat looser than in the non-degenerate case.

Section 6 is devoted to the unscaled asymptotics, in both the pseudo-cut-locus and the non-pseudo-cut-locus cases. It specifies the previous analysis of ([16, Section 3.6]). The saddle-point method is implemented twice again, in a way somewhat resembling that of the preceding degenerate setting, though with different normalizations and other technical difficulties.

2. A planar Langevin diffusion, and the results

The planar Langevin diffusion we consider here reads as $x_s = (\omega_s, y_s)$, with a standard real Brownian motion (ω_s), a C^2 non-constant function $f(\mathbb{R} \to \mathbb{R})$ and

$$y_s := \int_0^s f(\omega_\tau)\, d\tau\,.$$

See [1] for the generalization to a generic Riemannian manifold, called “kinetic Brownian motion”.

Consider the scaled diffusion $s \longmapsto x_s^\varepsilon = \Big(\sqrt{\varepsilon}\,\omega_s, \varepsilon\int_0^s f\big[\sqrt{\varepsilon}\,\omega_\tau\big]d\tau\Big)$, which has the same law as $[s \mapsto x^1_{\varepsilon s} = x_{\varepsilon s}]$ and satisfies the stochastic differential equation

$$dx_s^\varepsilon = \sqrt{\varepsilon}\, V_1(x_s^\varepsilon)\, d\omega_s + \varepsilon\, V_0(x_s^\varepsilon)\, ds \quad \text{(analogous to (2.1) in [4])},$$

with $V_1 = \partial_\omega, V_0 = f(\omega)\,\partial_y, [V_1, V_0] = f'(\omega)\,\partial_y, \big[V_1, [V_1, V_0]\big] = f''(\omega)\,\partial_y$, so that $V_1, [V_1, V_0], \big[V_1, [V_1, V_0]\big]$ span $\mathbb{R}^2$ at any point, provided $f'(\omega) = f''(\omega) = 0$ cannot occur for some ω. Obvious examples are $f(\omega) = a\cos\omega + b\sin\omega$, $f(\omega) = a\,\mathrm{ch}\,\omega + b\,\mathrm{sh}\,\omega$ $\big($with $(a,b) \neq (0,0)\big)$.

Then the Hörmander hypoellipticity criterion ensures the existence of a smooth density $p_\varepsilon(\cdot,\cdot)$ with respect to the Lebesgue measure for the random variable x_ε. The density of x_1^ε is p_ε as well. We are interested in small values of ε.

We thus fix $(w,y)\in\mathbb{R}^2$, and look for the exact equivalents as time $\varepsilon\searrow 0$ of the generic value $p_\varepsilon\big(0;(w,y)\big)$ of the density of x_1^ε, and of the scaled $p_\varepsilon\big(0;(w,\varepsilon y)\big)$. We shall first consider the non-degenerate "off-pseudo-cut-locus" case $w\neq 0$, and then the degenerate case $w=0$.

Since the method used in this article relies among others on a specific quadratic calculation, namely Proposition 3.3 then yielding Proposition 3.4 below, from now on we focus on a quadratic function f.

2.1. The example of a quadratic function *f*. — Consider

$$f(x)=a\,x^2+c \quad \text{with } a\neq 0, \text{ so that} \quad \int_0^1 f\big(\sqrt{\varepsilon}\,\omega_\tau\big)\,d\tau = a\int_0^1 \varepsilon\,\omega_\tau^2\,d\tau + c\,,$$

and up to the mere affine transform $(y\mapsto ay+c)$, we can restrict to $a=1, c=0$. Thus

$$p_\varepsilon \equiv p_\varepsilon(0;(\cdot,\cdot)) \quad \text{is the density of} \quad \Big(\sqrt{\varepsilon}\,\omega_1, \varepsilon^2\int_0^1 \omega_s^2\,ds\Big),$$

$$\text{and of} \quad \Big(\omega_\varepsilon, \int_0^\varepsilon \omega_s^2\,ds\Big) \quad \text{as well.}$$

2.2. The results. — The central aim of this article is to establish the following exact equivalents, which address both scaled and unscaled heat kernels, with a control of the error term.

THEOREM 2.1. — *Denote by $p_\varepsilon(0;(\cdot,\cdot))$ the density of $\Big(\omega_\varepsilon, \int_0^\varepsilon \omega_s^2\,ds\Big)$ under the law of the standard real Brownian motion (ω_s) started from 0.*

A) *For all $(w,y)\in\mathbb{R}^*\times\mathbb{R}_+^*$ and any L_ε going to infinity, as $\varepsilon\searrow 0$ we have*

$$p_\varepsilon\big(0;(w,\varepsilon\,y)\big) = \sqrt{\frac{\sqrt{q(\tau)}}{w^2R(\tau)\,\mathrm{sh}\sqrt{q(\tau)}}}\times\frac{1+\mathcal{O}\Big(\sqrt{\varepsilon\log^3(\frac{1}{\varepsilon})\,L_\varepsilon}\Big)}{2\pi\,\varepsilon^2}$$
$$\times\exp\left[\frac{w^2(\varrho(\tau)-1)}{2\,\varepsilon}\right],$$

where $\tau := y\,w^{-2}$, $q\equiv q(\tau)\in]-\pi^2,\infty[$ is the unique solution to the saddle-point equation

$$2\tau = \frac{\mathrm{ch}\sqrt{q}\,\mathrm{sh}\sqrt{q}-\sqrt{q}}{\sqrt{q}\,\mathrm{sh}^2\sqrt{q}}\,;$$

$$\varrho(\tau) := 1+\tau\,q-\sqrt{q}\,\mathrm{coth}\sqrt{q} \quad \textit{and}$$

$$R(\tau) := \frac{\mathrm{ch}\sqrt{q}\,\mathrm{sh}\sqrt{q}-2q\,\mathrm{coth}\sqrt{q}+\sqrt{q}}{2\,q^{3/2}\,\mathrm{sh}^2\sqrt{q}}\,.$$

Moreover, the implicit functions q, R and ϱ are analytic on $\{0<\tau<\infty\}$,

(i) *q decreases from infinity to $-\pi^2$ and $q(1/3)=0$, $q'(1/3)=-\frac{45}{2}$;*

(ii) *R increases from 0 to infinity and $R(1/3)=\frac{4}{45}$, $R'(1/3)=\frac{32}{315}$;*

(iii) *ϱ is negative but $\varrho(1/3)=0$, satisfies $\lim\limits_0 \varrho=\lim\limits_\infty \varrho=-\infty$, and is strictly concave.*

Furthermore, $\varrho'(1/3)=0$, $\varrho''(1/3)=-15$, and

(iv) *as $\tau\searrow 0$: $q(\tau)\sim\tau^{-2}$, $\varrho(\tau)\sim\frac{-1}{4\tau}$, $R(\tau)\sim 4\tau^3$;*

(v) *as $\tau\nearrow\infty$: $q(\tau)=-\pi^2+\pi\sqrt{\frac{2}{\tau}+\frac{\mathcal{O}(1)}{\tau}}$, $\varrho(\tau)=-\pi^2\tau+2\pi\sqrt{2\tau}+\mathcal{O}(1)$, $R(\tau)=\frac{(2\tau)^{3/2}+\mathcal{O}(\tau)}{\pi}$.*

The graphs of the functions q,ϱ,R are depicted on Figure 2.1.

B) *Let $K=\displaystyle\int_0^{\pi/2}\cos\big(\tfrac{\operatorname{tg}\theta-3\theta}{2}\big)\sqrt{\tfrac{2}{\cos\theta}}\,d\theta\approx 2.15$. For any positive y, as $\varepsilon\searrow 0$ we have*

$$p_\varepsilon\big(0;(0,\varepsilon\,y)\big)=\big[1+\mathcal{O}\big(\varepsilon^{1/3}\big)\big]\,\frac{\sqrt{e}\,K}{\varepsilon^2\sqrt{8\pi\,y}}\,\exp\Big[-\frac{\pi^2 y}{2\,\varepsilon}\Big].$$

C) *For all $(w,y)\in\mathbb{R}^*\times\mathbb{R}_+^*$, as $\varepsilon\searrow 0$ we have*

$$p_\varepsilon\big(0;(w,y)\big)=\frac{1+\mathcal{O}(\sqrt{\varepsilon})}{\sqrt{8y\,\varepsilon^3}}\,\exp\Bigg[-\frac{\pi^2 y}{2\,\varepsilon^2}+\frac{\pi\sqrt{2w^2y}}{\varepsilon^{3/2}}-\frac{3w^2}{4\varepsilon}-\frac{(4\pi^2+3)|w|^3}{24\pi\sqrt{2y\,\varepsilon}}+\frac{(2\pi^2-3)w^4}{48\,\pi^2 y}\Bigg].$$

D) *For any positive y, as $\varepsilon\searrow 0$ we have*

$$p_\varepsilon\big(0;(0,y)\big)=\big[1+\mathcal{O}\big(\varepsilon^{2/3}\big)\big]\,\frac{\sqrt{e}\,K}{\sqrt{8\pi y\,\varepsilon^3}}\,\exp\Big[-\frac{\pi^2 y}{2\,\varepsilon^2}\Big].$$

Remark 2.2. — Part A of Theorem 2.1 handles the scaled non-pseudo-diagonal case $w\neq 0$, whereas Part B handles the scaled "pseudo-cut-locus" case $w=0$. Despite the lack of a metric, this second case is analogous to the cut-locus case of the sub-Riemannian setting, which is already rather specific in that context, see [5] for example. Of course the increase of the sub-process (y_s) excludes a truly diagonal case.

The term $S:=w^2(1-\varrho(y/w^2))$, which is the limit of $-2\varepsilon\log p_\varepsilon\big(0;(w,\varepsilon\,y)\big)$ and appears in the leading term $e^{-S/2\varepsilon}$ of Part A of the statement, is time-independent and is a strictly convex function of y with a minimum at $\frac{w^2}{3}$. Recall that in the Langevin-Gaussian setting (which corresponds to $f=Id$ and is strictly hypoelliptic too) the analogous term is the squared Carnot-Carathéodory distance $d_\varepsilon\big(0;(w,\varepsilon v)\big)^2=4w^2+12v^2-12vw$. Despite some

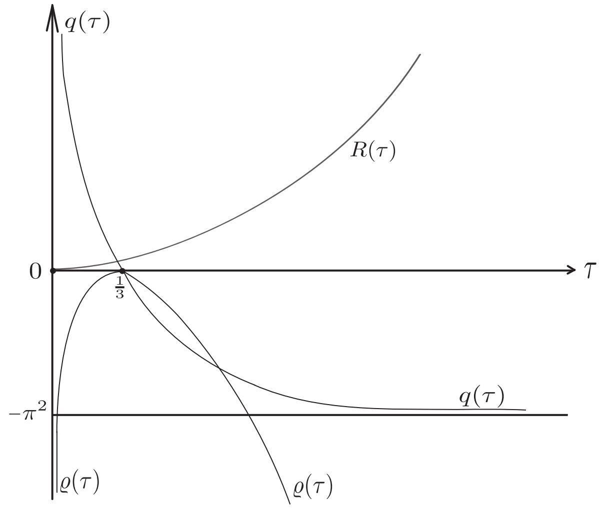

FIGURE 2.1. Graphs of the functions q, ϱ, R of Theorem 2.1.A

analogy, the present scaled setting (parts A and B) behaves in a different way: S exhibits a non-quadratic behaviour, by $[A(iv),(v)]$ and B: clearly for fixed w as $y \to 0$ or $y \to \infty$, and furthermore, for large $\frac{w^2}{y}$ we have $S \equiv S(w,y) \sim \frac{w^4}{4y}$.

REMARK 2.3. — Part C of Theorem 2.1 handles the unscaled non-pseudo-cut-locus case $w \neq 0$, whereas Part D handles the unscaled pseudo-cut-locus case $w = 0$, again analogous to the cut-locus case of the sub-Riemannian setting.

The time-dependent energy term which appears in Part C of the statement appears to be more complicated than its sub-Riemannian and Langevin-Gaussian analogues. But as in these two cases, and as in a generic Gaussian case as well, see [3], it happens to have the form $d_\varepsilon\big(0;(w,y)\big)^2/2\varepsilon$, where d_ε denotes the Carnot-Carathéodory (pseudo-)distance. See Remark 2.6 and Corollary 2.7 below.

REMARK 2.4. — The expression of the scaled pseudo-diagonal case B can be a posteriori derived from the expression of the unscaled pseudo-diagonal case D, by the mere transform $y \mapsto \varepsilon y$, except for the control on the error term. Owing to this and to the preceding remark 2.3, it could be tempting to compare the scaled squared Carnot-Carathéodory pseudo-distance $d_\varepsilon\big(0;(w,\varepsilon y)\big)^2$ derived from the exponent of the case C, to its counterpart $w^2\big(1-\varrho(y/w^2)\big)$ of the case A. Now, whereas on the one hand they are both time-independent functions, equal to the product of w^2 by a function of $\tau \equiv yw^{-2}$, since formally

$$d_\varepsilon\big(0;(w,\varepsilon y)\big)^2 w^{-2} = \pi^2\tau - 2\pi\sqrt{2\tau} + \tfrac{3}{2} + \tfrac{4\pi^2+3}{12\pi\sqrt{2\tau}} - \tfrac{2\pi^2-3}{24\pi^2\tau}\,,$$

but on the other hand, though they are both $\pi^2\tau - 2\pi\sqrt{2\tau} + \mathcal{O}(1)$ (hence positive) for large τ, for small τ the formal $d_\varepsilon\big(0;(w,\varepsilon y)\big)^2 w^{-2}$ does not even remain non-negative.

Remark 2.5. — A refinement of the calculations of this article, adding a component $v_s := \int_0^s \omega.$ to the Langevin diffusion (x_s), would yield the extension of Theorem 2.1 to the asymptotics of $p_\varepsilon\big(0;(w,v,y)\big)$, and then of $p_\varepsilon\big((w_0,y_0);(w,y)\big)$ and even $p_\varepsilon\big((w_0,v_0,y_0);(w,v,y)\big)$, but at the price of heavier computations, though the present ones are already heavy enough. On the contrary, any generalization to other strictly hypoelliptic (non-Gaussian) settings seems to remain problematic, in so far as the quadratic character of the present diffusion is crucial here (and in [16]).

Remark 2.6. — Since Stroock & Varadhan and then Bismut [6], the diffusion under study is classically associated with a control problem, where the control replaces the driving Brownian motion, and then a minimal action functional or energy $E_\varepsilon(x,x')$ is necessary for the controlled process to go from x to x' within a duration of ε. Moreover, $2\varepsilon\, E_\varepsilon(x,x')$ equals the squared Carnot-Carathéodory pseudo-distance. It is well known that in the (sub-)Riemannian setting the Carnot-Carathéodory pseudo-distance is the (sub-)Riemannian distance, and that for small ε this energy $E_\varepsilon(x,x')$ appears as the dominant term of $-\log p_\varepsilon(x,x')$ (see [4]). The latter remains true for a generic hypoelliptic linear Gaussian diffusion, as recently underlined by D. Barilari and E. Paoli ([3, Corollary 1.2]).

In the present case of the quadratic Langevin diffusion (x_s), the controlled (skeleton) equation reads

$$\begin{aligned} X_s^v &= \int_0^s V_0(X_\tau^v)\,d\tau + \int_0^s V_1(X_\tau^v)\,dv_\tau \\ &= \left(v_s, \int_0^s v_\tau^2\,d\tau\right), \quad \text{for any control } v \in H_0^1, \end{aligned}$$

and the associated energy functional is

$$E_\varepsilon\big(0;(w,y)\big) := \min\Big\{\tfrac{1}{2}\int_0^\varepsilon \dot v^2 \,\Big|\, v \in H_0^1\big([0,\varepsilon],\mathbb{R}\big), v_\varepsilon = w, \int_0^\varepsilon v^2 = y\Big\}.$$

Setting $\mu(t) := v(\varepsilon t)$, we equivalently have the squared Carnot-Carathéodory pseudo-distance:

$$(5)\quad 2\varepsilon\, E_\varepsilon\big(0;(w,y)\big) = \min\Big\{\int_0^1 \dot\mu^2 \,\Big|\, \mu \in H_0^1\big([0,1],\mathbb{R}\big), \mu_1 = w, \int_0^1 \mu^2 = y/\varepsilon\Big\}.$$

The following corollary states that the energy $E_\varepsilon(x,x')$ ($=\frac{1}{2\varepsilon}$ the squared Carnot-Carathéodory pseudo-distance) yields the correct exponent in the unscaled small time asymptotics of p_ε, in the case of the quadratic Langevin diffusion (x_s) as well, hence not only in the (sub-)Riemannian and in the generic hypoelliptic linear Gaussian settings. Extending that observation even under a small perturbation would demand caution, owing to the high sensitivity of oscillatory integrals, see [15].

Corollary 2.7. — *In the unscaled cases C, D of Theorem 2.1, for all $(w,y)\in \mathbb{R}^*\times\mathbb{R}_+^*$, as $\varepsilon\searrow 0$ we have*

$$p_\varepsilon\big(0;(w,y)\big) = \frac{1+\mathcal{O}(\sqrt{\varepsilon})}{\sqrt{8y\,\varepsilon^3}}\, e^{-E_\varepsilon(0;(w,y))};$$

$$p_\varepsilon\big(0;(0,y)\big) = \big[1+\mathcal{O}\big(\varepsilon^{2/3}\big)\big]\,\frac{\sqrt{e}\,K}{\sqrt{8\pi y\,\varepsilon^3}}\, e^{-E_\varepsilon(0;(0,y))}.$$

The technical proof is given in the appendix (Section 7). A noteworthy feature of this proof is to let the above constraint $\int_0^1\mu^2=\frac{y}{\varepsilon}$ appear exactly as the key saddle-point equation (28).

Remark 2.8. — During the revision process of this article, I. Bailleul drew attention to the book [16] by V. Kolokoltsov, that is devoted to asymptotic expansions, and particularly to small-time expansions of some types of heat kernels. V. Kolokoltsov systematically starts from a heat equation, and doesn't use any probability theory to express solutions (specifically, heat kernel densities $p_t(\cdot,\cdot)$, he calls "Green functions", though this often designates $\int_0^\infty(p_t-p_\infty)dt$) or approximate solutions. Thus the methods of [16] are purely analytical, mainly based on Hamiltonian calculus of variations, and then on approximation procedures in the spirit of the parametrix method.

As does the present article, V. Kolokoltsov observes in [16] that (roughly) a quadratic hypothesis on coefficients intervenes rather naturally, in order to allow efficient computations and approximations. Thus, his concrete and representative example (regarding small-time asymptotics), presented in ([16], Section 3.6), addresses the heat kernel equation

$$\frac{\partial p_t(w,y)}{\partial t} = \frac{h}{2}\frac{\partial^2 p_t(w,y)}{\partial w^2} - \frac{w^2}{2}\frac{\partial p_t(w,y)}{\partial y},$$

where h is some additional positive parameter (where it is enough to set equal to 1, regarding small-time asymptotics). In other words, as is done here, he actually considers the 2-dimensional diffusion-SDE:

$$x_s = \Big(\sqrt{h}\,\omega_s, \frac{h}{2}\int_0^s \omega_t^2\,dt\Big) = \sqrt{h}\int_0^s \partial_w(x_t)\,d\omega_t + \frac{h}{2}\int_0^s \omega_t^2\,\partial_y(x_t)\,dt\,.$$

Note that the correspondence between the present notation and that of [16] is $\big((y,x)$ in [16] $\equiv (w\sqrt{h}, -hy/2)$ here$\big)$; the minus sign in front of y being natural since the adjoint infinitesimal generator must be used. Note also that the additional parameter h doesn't help in handling the scaled cases A, B of Theorem 2.1, which are not considered in [16].

Thus in [16], Section 3.6, V. Kolokoltsov exactly addresses the unscaled cases C, D of Theorem 2.1; however by different means $\big($for example, although he also stresses the same saddle-point equation (12), his contour change is not at all like 4.2 or 5.1 below$\big)$ and by only sketching the proof and computations. In the degenerate unscaled case, his result is that of Case D, up to the multiplicative constant which appears not to be the right one and which Theorem 2.1.D corrects. And in the undegenerate unscaled case, his result is expressed in a less concrete and explicit way than C, closer to Corollary 2.7, again up to the multiplicative constant (which does not appear clearly in the expression of [16]).

3. Fourier expression of p_ε

3.1. Laplace transform of $\big(\omega_1, \int_0^1 \omega_s^2\, ds\big)$ under $\mathbb{P}_0$. — We perform the computation of a slightly more general Brownian Laplace transform. The principle of this type of computation goes back to Marc Yor [21].

Proposition 3.1. — *We have*

$$\mathbb{E}_0\left[\exp\left(\int_0^1 \left[\alpha_s\,\omega_s + \gamma_s\,\omega_s^2\right] ds\right)\right]$$

$$= \left[1 - g_1 \int_0^1 e^{2\int_1^s g} ds\right]^{-1/2} \exp\left[\frac{\int_0^1 \big(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau\big)^2 ds - \int_0^1 g}{2} + \frac{g_1\left(\int_0^1 \big(\int_s^1 e^{\int_\tau^s g}\alpha_\tau d\tau\big) e^{\int_1^s g} ds\right)^2}{2\left(1 - g_1\int_0^1 e^{2\int_1^s g} ds\right)}\right],$$

where α_s, γ_s are real deterministic, $\gamma_s \le 0$, and g solves the Riccati equation $g' = g^2 + 2\gamma$ $\big($equivalent to the linear equation $\frac{d^2}{ds^2}\exp\big(-\int_0^s g\big) = -2\gamma_s \exp\big(-\int_0^s g\big)\big)$ a.e. on $[0,1]$.

Proof. — Set

$$Y := \mathbb{E}_0\left[\exp\left(\int_0^1 \left[\alpha_s\,\omega_s + \gamma_s\,\omega_s^2\right] ds\right)\right]. \tag{6}$$

Consider the exponential $\mathbb{P}_0$-martingale defined (for some deterministic C^1 function g) by

$$\begin{aligned} M_s^g &:= \exp\Big(-\int_0^s g_\tau\, \omega_\tau\, d\omega_\tau - \tfrac12 \int_0^s g_\tau^2\, \omega_\tau^2\, d\tau \Big) \\ &= \exp\Big(\tfrac12 \int_0^s g - \tfrac12\, g_s\, \omega_s^2 + \tfrac12 \int_0^s (g'_\tau - g_\tau^2)\omega_\tau^2\, d\tau \Big). \end{aligned}$$

Denoting by $\mathbb{P}^g$ the new probability law having M_s^g as the density on $\mathcal{F}_s$ with respect to $\mathbb{P}_0$, we have:

$$\begin{aligned} Y\, e^{\int_0^1 g/2} &= \mathbb{E}^g \Big[\exp\Big(\tfrac12\, g_1\, \omega_1^2 + \int_0^1 \Big[\alpha_s\, \omega_s + \big(\gamma_s - \tfrac12 (g'_s - g_s^2) \big) \omega_s^2 \Big] ds \Big) \Big] \\ &= \mathbb{E}^g \Big[\exp\Big(\tfrac12\, g_1\, \omega_1^2 + \int_0^1 \alpha_s\, \omega_s\, ds \Big) \Big], \end{aligned} \tag{7}$$

by taking g almost everywhere solving the Ricatti equation $g' = g^2 + 2\gamma$.

On the other hand, the Girsanov formula provides a $(\mathbb{P}^g, \mathcal{F}_s)$ Brownian motion B such that $\omega_s = B_s - \int_0^s g_\tau\, \omega_\tau\, d\tau$, and then $\omega_s = \int_0^s \exp\Big(\int_s^\tau g \Big) dB_\tau$. Hence, for any real r:

$$\begin{aligned} &\mathbb{E}^g \Big[\Big(r\, \omega_1 + \int_0^1 \alpha_s\, \omega_s\, ds \Big)^2 \Big] \\ &= \mathbb{E}^g \Big[\Big(\int_0^1 \Big[r + \int_s^1 \alpha \Big] d\omega_s \Big)^2 \Big] \\ &= \mathbb{E}^g \Big[\Big(\int_0^1 \Big[r + \int_s^1 \alpha \Big] \Big[dB_s - g_s \Big(\int_0^s e^{\int_s^\tau g}\, dB_\tau \Big) ds \Big] \Big)^2 \Big] \\ &= \mathbb{E}^g \Big[\Big(\int_0^1 \Big[r + \int_s^1 \alpha - \int_s^1 \Big(r + \int_\tau^1 \alpha \Big) g_\tau\, e^{\int_\tau^s g} d\tau \Big] dB_s \Big)^2 \Big] \\ &= \int_0^1 \Big[r\, e^{\int_1^s g} + \int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau \Big]^2 ds\,. \end{aligned}$$

This yields the covariance matrix of the $\mathbb{P}^g$-Gaussian variable $(\omega_1, \int_0^1 \alpha_s\, \omega_s\, ds)$, namely

$$K = \begin{pmatrix} \int_0^1 e^{2\int_1^s g} ds & \int_0^1 \big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau \big) e^{\int_1^s g} ds \\ \int_0^1 \big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau \big) e^{\int_1^s g} ds & \int_0^1 \big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau \big)^2 ds \end{pmatrix},$$

whence the density of $\Big(\omega_1, \int_0^1 \alpha_s\, \omega_s\, ds\Big)$ with respect to $\mathbb{P}^g$. Thus

$$
\begin{aligned}
(8)\quad & \mathbb{E}^g\Big[\exp\Big(\int_0^1 \alpha_s\, \omega_s\, ds - r\, \omega_1^2\Big)\Big] \\
&= \int_{\mathbb{R}^2} e^{v - r\, u^2} \exp\Big[\frac{-1}{2\,\det K}\Big(u^2 \int_0^1 \Big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau\Big)^2 ds \\
&\quad - 2\, uv \int_0^1 \Big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau\Big) e^{\int_1^s g} ds + v^2 \int_0^1 e^{2\int_1^s g} ds\Big)\Big] \frac{du\, dv}{2\pi\sqrt{\det K}} \\
&= \int_{\mathbb{R}^2} \exp\Big[\tfrac{-1}{2}\Big(\Big[\frac{\int_0^1 \Big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau\Big)^2 ds}{\det K} + 2r\Big] u^2 + \Big[\frac{\int_0^1 e^{2\int_1^s g} ds}{\det K}\Big] v^2 - 2v \\
&\quad - 2\Big[\frac{\int_0^1 \Big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau\Big) e^{\int_1^s g} ds}{\det K}\Big] uv\Big)\Big] \frac{du\, dv}{2\pi\sqrt{\det K}} \\
&= \Big(1 + 2r \int_0^1 e^{2\int_1^s g} ds\Big)^{-1/2} \times \exp\left(\frac{\int_0^1 \big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau\big)^2 ds + 2r\, \det K}{2\Big(1 + 2r \int_0^1 e^{2\int_1^s g} ds\Big)}\right)
\end{aligned}
$$

by a classical Gaussian computation. Finally by (6), (7) and (8), taking $r = -g_1/2$ we have

$$
\begin{aligned}
Y_0 &= \mathbb{E}^g\Big[\exp\Big(\int_0^1 \alpha_s\, \omega_s\, ds + g_1\, \omega_1^2/2\Big)\Big] \times e^{-\int_0^1 g/2} \\
&= \Big(1 - g_1 \int_0^1 e^{2\int_1^s g} ds\Big)^{-1/2} \\
&\quad \times \exp\left(\frac{\int_0^1 \big(\int_s^1 e^{\int_\tau^s g}\, \alpha_\tau\, d\tau\big)^2 ds - g_1 \det K}{2\Big(1 - g_1 \int_0^1 e^{2\int_1^s g} ds\Big)} - \int_0^1 g/2\right) \\
&= \Big[1 - g_1 \int_0^1 e^{2\int_1^s g} ds\Big]^{-1/2} \\
&\quad \times \exp\left[\frac{\int_0^1 \big(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau\big)^2 ds - \int_0^1 g}{2} + \frac{g_1\Big(\int_0^1 \big(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau\big) e^{\int_1^s g} ds\Big)^2}{2\Big(1 - g_1 \int_0^1 e^{2\int_1^s g} ds\Big)}\right].
\end{aligned}
$$

□

Corollary 3.2. — *For any real deterministic continuous function α and constant γ, we have*

$$\mathbb{E}_0\Big[\exp\Big(\int_0^1 \big[\alpha_s\,\omega_s - \tfrac{1}{2}\gamma^2\,\omega_s^2\big]ds\Big)\Big]$$
$$= \frac{1}{\sqrt{\operatorname{ch}\gamma}}\,\exp\left(\frac{\int_0^1\Big(e^{\gamma s}\int_s^1 e^{-\gamma\tau}\,\alpha_\tau\,d\tau\Big)^2 ds}{2} + \frac{\Big(\int_0^1 \operatorname{sh}(\gamma\,s)\,\alpha_s\,ds\Big)^2}{2\gamma\,e^{\gamma}\operatorname{ch}\gamma}\right).$$

Moreover, for constant α,γ we have

$$\mathbb{E}_0\Big[\exp\Big(\alpha\,\omega_1 - \frac{\gamma^2}{2}\int_0^1 \omega_\cdot^2\Big)\Big] = \frac{1}{\sqrt{\operatorname{ch}\gamma}}\,\exp\Big(\frac{\alpha^2\operatorname{th}\gamma}{2\gamma}\Big).$$

Proof. — We apply Proposition 3.1 with $\gamma_s \equiv -\gamma^2/2$, so that we can take $g\equiv\gamma$, yielding:

$$\mathbb{E}_0\Big[\exp\Big(\int_0^1 \big[\alpha_s\,\omega_s - \tfrac{1}{2}\gamma^2\,\omega_s^2\big]ds\Big)\Big]$$
$$= \frac{\exp\left[\frac{\int_0^1\left(\int_s^1 e^{\gamma(s-\tau)}\,\alpha_\tau\,d\tau\right)^2 ds - \gamma}{2} + \frac{\gamma\left(\int_0^1\left(\int_s^1 e^{\gamma(s-\tau)}\,\alpha_\tau\,d\tau\right)e^{\gamma(s-1)}ds\right)^2}{2\left(1-\gamma\int_0^1 e^{2\gamma(s-1)}ds\right)}\right]}{\sqrt{1-\gamma\int_0^1 e^{2\gamma(s-1)}\,ds}}$$
$$= \frac{\exp\left(\frac{\int_0^1\left(e^{\gamma s}\int_s^1 e^{-\gamma\tau}\,\alpha_\tau\,d\tau\right)^2 ds - \gamma}{2} + \frac{\gamma\left(\int_0^1\left(\int_s^1 e^{-\gamma\tau}\,\alpha_\tau\,d\tau\right)e^{2\gamma s}ds\right)^2}{2\,e^{\gamma}\operatorname{ch}\gamma}\right)}{\sqrt{e^{-\gamma}\operatorname{ch}\gamma}}$$
$$= \frac{1}{\sqrt{\operatorname{ch}\gamma}}\,\exp\left(\frac{\int_0^1\Big(e^{\gamma s}\int_s^1 e^{-\gamma\tau}\,\alpha_\tau\,d\tau\Big)^2 ds}{2} + \frac{\Big(\int_0^1 \operatorname{sh}(\gamma\,s)\,\alpha_s\,ds\Big)^2}{2\gamma\,e^{\gamma}\operatorname{ch}\gamma}\right).$$

The second formula of the statement is deduced from the first one by taking $\alpha_s = \frac{\alpha}{\eta}\,1_{[1-\eta,1]}(s)$ and letting $\eta\searrow 0$. This yields

$$\int_0^1 \operatorname{sh}(\gamma\,s)\,\alpha_s\,ds \longrightarrow \alpha\operatorname{sh}\gamma \quad\text{and}$$
$$\int_0^1\Big(e^{\gamma\,s}\int_s^1 e^{-\gamma\,\tau}\,\alpha_\tau\,d\tau\Big)^2 ds = \frac{\alpha^2}{\eta^2}\int_0^1\Big(e^{\gamma\,s}\int_{s\vee(1-\eta)}^1 e^{-\gamma\,\tau}\,d\tau\Big)^2 ds$$
$$= \frac{\alpha^2}{\gamma^2\,\eta^2}\int_0^{1-\eta} e^{2\gamma s}\big(e^{-\gamma(1-\eta)} - e^{-\gamma}\big)^2 ds + \frac{\alpha^2}{\gamma^2\,\eta^2}\int_{1-\eta}^1 e^{2\gamma s}\big(e^{-\gamma s} - e^{-\gamma}\big)^2 ds$$
$$= \frac{\alpha^2}{2\gamma^3\,\eta^2}\big(e^{\gamma\eta}-1\big)^2\big(e^{-2\gamma\eta} - e^{-2\gamma}\big) + \mathcal{O}(\eta) \longrightarrow \frac{\alpha^2}{2\gamma}\big(1-e^{-2\gamma}\big) = \frac{\alpha^2\operatorname{sh}\gamma}{\gamma\,e^{\gamma}},$$

and then

$$\mathbb{E}_0\Big[\exp\Big(\alpha\,\omega_1 - \frac{\gamma^2}{2}\int_0^1 \omega_\cdot^2\Big)\Big]$$
$$= \frac{1}{\sqrt{\operatorname{ch}\gamma}}\exp\Big(\frac{\alpha^2\operatorname{sh}\gamma}{2\gamma\, e^\gamma} + \frac{\alpha^2\operatorname{sh}^2\gamma}{2\gamma\, e^\gamma\operatorname{ch}\gamma}\Big) = \frac{1}{\sqrt{\operatorname{ch}\gamma}}\exp\Big(\frac{\alpha^2\operatorname{th}\gamma}{2\gamma}\Big). \qquad \square$$

3.2. Fourier transform of $\big(\omega_1, \int_0^1 \omega_s^2\, ds\big)$ under $\mathbb{P}_0$. — It is given by the following expression, by means of the above and of analytic continuation. The detailed proof is given in the appendix (Section 7).

Proposition 3.3. — *For all $(\alpha,\xi)\in\mathbb{R}^2$, we have*

$$\mathbb{E}_0\Big[\exp\Big(i\,\alpha\,\omega_1 + i\,\xi\int_0^1 \omega_\cdot^2\Big)\Big]$$
$$= \Big(\operatorname{ch}^2\sqrt{\xi} - \sin^2\sqrt{\xi}\Big)^{-1/4}$$
$$\times \exp\Bigg[\frac{i}{4}\operatorname{sgn}(\xi)\int_0^{2\sqrt{|\xi|}} \frac{\operatorname{sh}\theta + \sin\theta}{\operatorname{ch}\theta + \cos\theta}\, d\theta$$
$$- \alpha^2\,\frac{\big(\operatorname{sh}(2\sqrt{\xi}) + \sin(2\sqrt{\xi})\big) + i\big(\operatorname{sh}(2\sqrt{\xi}) - \sin(2\sqrt{\xi})\big)}{4\sqrt{\xi}\,\big(\operatorname{ch}(2\sqrt{\xi}) + \cos(2\sqrt{\xi})\big)}\Bigg].$$

3.3. Fourier expression for the scaled $p_\varepsilon\big(0;(w,\varepsilon\, y)\big)$. — We deduce the Fourier expression for $p_\varepsilon\big(0;(w,\varepsilon\, y)\big)$ from the above Proposition 3.3.

Proposition 3.4. — *For all $(w,y,\varepsilon)\in\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{R}_+^*$, we have*

$$(9)\qquad p_\varepsilon\big(0;(w,\varepsilon\, y)\big)$$
$$= \frac{e^{-w^2/2\varepsilon}}{4\pi\sqrt{2\pi\,\varepsilon^5}}\int_0^\infty \Re\Bigg\{\exp\Bigg[\frac{i(h(x) - y\,x) - H(x)}{4\,\varepsilon}$$
$$+ \tfrac{i}{4}\int_0^{\sqrt{x}} \tfrac{\operatorname{sh} - \sin}{\operatorname{ch} - \cos}\Bigg]\Bigg\}\Big[\tfrac{x}{\operatorname{ch}\sqrt{x} - \cos\sqrt{x}}\Big]^{1/4} dx\,,$$

where we set:

$$(10)\qquad h(x) := \frac{\operatorname{sh}\sqrt{x} - \sin\sqrt{x}}{\operatorname{ch}\sqrt{x} - \cos\sqrt{x}}\times w^2\sqrt{x}\quad \text{and}$$
$$H(x) := \frac{\operatorname{sh}\sqrt{x} + \sin\sqrt{x}}{\operatorname{ch}\sqrt{x} - \cos\sqrt{x}}\times w^2\sqrt{x} - 2\,w^2\,.$$

Proof. — Since p_ε is the density of $\left(\sqrt{\varepsilon}\,\omega_1, \varepsilon^2 \int_0^1 \omega_s^2\,ds\right)$ under $\mathbb{P}_0$, we have

$$
\begin{aligned}
&2\pi\,\varepsilon^{5/2}\,p_\varepsilon\big(0;(w,y)\big)\\
&= \frac{\varepsilon^{5/2}}{2\pi}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} e^{i(\alpha\,v+\xi\,u)}\,p_\varepsilon\big(0;(v,u)\big)\,du\,dv\right] e^{-i(\alpha\,w+\xi\,y)}\,d\alpha\,d\xi\\
&= \frac{\varepsilon^{5/2}}{2\pi}\int_{\mathbb{R}^2}\mathbb{E}_0\!\left[\exp\!\left(i\,\alpha\sqrt{\varepsilon}\,\omega_1 + i\,\xi\,\varepsilon^2\int_0^1\omega_\cdot^2\right)\right] e^{-i(\alpha\,w+\xi\,y)}\,d\alpha\,d\xi\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathbb{E}_0\!\left[\exp\!\left(i\,\alpha\,\omega_1 + i\,\xi\int_0^1\omega_\cdot^2\right)\right] e^{-i\left(\frac{\alpha\,w}{\sqrt{\varepsilon}}+\frac{\xi\,y}{\varepsilon^2}\right)}\,d\alpha\,d\xi\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d\xi}{\big(\mathrm{ch}^2\sqrt{\xi}-\sin^2\sqrt{\xi}\big)^{\frac14}}\\
&\quad\times\left(\frac{\big(\mathrm{sh}(2\sqrt{\xi})+\sin(2\sqrt{\xi})\big)+i\big(\mathrm{sh}(2\sqrt{\xi})-\sin(2\sqrt{\xi})\big)}{2\sqrt{\xi}\,\big(\mathrm{ch}(2\sqrt{\xi})+\cos(2\sqrt{\xi})\big)}\right)^{-1/2}\\
&\quad\times\exp\left[\tfrac{i}{4}\,\mathrm{sgn}(\xi)\int_0^{2\sqrt{|\xi|}}\tfrac{\mathrm{sh}\,\theta+\sin\theta}{\mathrm{ch}\,\theta+\cos\theta}\,d\theta - \frac{i\,\xi y}{\varepsilon^2}\right.\\
&\qquad\left. - \frac{w^2\sqrt{\xi}\,\big(\mathrm{ch}(2\sqrt{\xi})+\cos(2\sqrt{\xi})\big)/\varepsilon}{\mathrm{sh}(2\sqrt{\xi})+\sin(2\sqrt{\xi})+i\big[\mathrm{sh}(2\sqrt{\xi})-\sin(2\sqrt{\xi})\big]}\right]\\
&= \frac{2^{3/4}}{\sqrt{2\pi}}\int_{\mathbb{R}}d\xi\,\Big[\mathrm{ch}(2\sqrt{\xi})+\cos(2\sqrt{\xi})\Big]^{1/4}\\
&\quad\times\sqrt{\frac{\sqrt{\xi}}{\mathrm{sh}(2\sqrt{\xi})+\sin(2\sqrt{\xi})+i\big[\mathrm{sh}(2\sqrt{\xi})-\sin(2\sqrt{\xi})\big]}}\\
&\quad\times\exp\left[\tfrac{i}{4}\,\mathrm{sgn}(\xi)\int_0^{2\sqrt{|\xi|}}\tfrac{\mathrm{sh}\,\theta+\sin\theta}{\mathrm{ch}\,\theta+\cos\theta}\,d\theta - \frac{i\,\xi y}{\varepsilon^2}\right.\\
&\qquad\left. - \frac{w^2\sqrt{\xi}\,\big(\mathrm{ch}(2\sqrt{\xi})+\cos(2\sqrt{\xi})\big)/\varepsilon}{\mathrm{sh}(2\sqrt{\xi})+\sin(2\sqrt{\xi})+i\big[\mathrm{sh}(2\sqrt{\xi})-\sin(2\sqrt{\xi})\big]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{3/4}}{\sqrt{2\pi}} \int_0^\infty \Big[\mathrm{ch}(2\sqrt{\xi}) + \cos(2\sqrt{\xi})\Big]^{1/4} \\
&\qquad \times \sqrt{\frac{\sqrt{\xi}}{\mathrm{sh}(2\sqrt{\xi}) + \sin(2\sqrt{\xi}) + i\big[\mathrm{sh}(2\sqrt{\xi}) - \sin(2\sqrt{\xi})\big]}} \\
&\qquad \times \exp\Bigg[\tfrac{i}{4}\int_0^{2\sqrt{\xi}} \tfrac{\mathrm{sh}\,\theta + \sin\theta}{\mathrm{ch}\,\theta + \cos\theta}\,d\theta - \frac{i\,\xi\,y}{\varepsilon^2} \\
&\qquad\qquad - \frac{w^2\sqrt{\xi}\,\big(\mathrm{ch}(2\sqrt{\xi}) + \cos(2\sqrt{\xi})\big)/\varepsilon}{\mathrm{sh}(2\sqrt{\xi}) + \sin(2\sqrt{\xi}) + i\big[\mathrm{sh}(2\sqrt{\xi}) - \sin(2\sqrt{\xi})\big]}\Bigg] d\xi \\
&\quad + \frac{2^{3/4}}{\sqrt{2\pi}} \int_0^\infty \Big[\mathrm{ch}(2\sqrt{\xi}) + \cos(2\sqrt{\xi})\Big]^{1/4} \\
&\qquad \times \sqrt{\frac{\sqrt{\xi}}{\mathrm{sh}(2\sqrt{\xi}) + \sin(2\sqrt{\xi}) - i\big[\mathrm{sh}(2\sqrt{\xi}) - \sin(2\sqrt{\xi})\big]}} \\
&\qquad \times \exp\Bigg[-\tfrac{i}{4}\int_0^{2\sqrt{\xi}} \tfrac{\mathrm{sh}\,\theta + \sin\theta}{\mathrm{ch}\,\theta + \cos\theta}\,d\theta + \frac{i\,\xi\,y}{\varepsilon^2} \\
&\qquad\qquad - \frac{w^2\sqrt{\xi}\,\big(\mathrm{ch}(2\sqrt{\xi}) + \cos(2\sqrt{\xi})\big)/\varepsilon}{\mathrm{sh}(2\sqrt{\xi}) + \sin(2\sqrt{\xi}) - i\big[\mathrm{sh}(2\sqrt{\xi}) - \sin(2\sqrt{\xi})\big]}\Bigg] d\xi \\
&= \frac{2^{3/4}}{\sqrt{2\pi}} \int_0^\infty (\mathrm{ch}\,t + \cos t)^{1/4} \\
&\qquad \times \Re\Bigg\{\sqrt{\frac{t/2}{\mathrm{sh}\,t + \sin t + i\big[\mathrm{sh}\,t - \sin t\big]}} \\
&\qquad\qquad \times \exp\Bigg[\tfrac{i}{4}\int_0^t \tfrac{\mathrm{sh}\,\theta + \sin\theta}{\mathrm{ch}\,\theta + \cos\theta}\,d\theta - \frac{i\,y\,t^2}{4\,\varepsilon^2} \\
&\qquad\qquad\qquad - \frac{w^2\,\big(\mathrm{ch}\,t + \cos t\big)\,t/2\varepsilon}{\mathrm{sh}\,t + \sin t + i\big[\mathrm{sh}\,t - \sin t\big]}\Bigg]\Bigg\}\,t\,dt \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Re\Bigg\{\exp\Bigg[\tfrac{i}{4}\int_0^t \tfrac{\mathrm{sh}\,\theta + \sin\theta}{\mathrm{ch}\,\theta + \cos\theta}\,d\theta - \tfrac{i}{2}\mathrm{Arctg}\Big(\tfrac{\mathrm{sh}\,t - \sin t}{\mathrm{sh}\,t + \sin t}\Big)\Bigg] \\
&\qquad\qquad \times \exp\Bigg[\frac{i(\mathrm{sh}\,t - \sin t) - (\mathrm{sh}\,t + \sin t)}{\mathrm{ch}\,t - \cos t} \times \frac{w^2\,t}{4\,\varepsilon} - \frac{i\,y\,t^2}{4\,\varepsilon^2}\Bigg]\Bigg\} \\
&\qquad \times \frac{t^{3/2}}{(\mathrm{ch}\,t - \cos t)^{1/4}}\,dt
\end{aligned}
$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Re\left\{\exp\left[\frac{i(\operatorname{sh} t - \sin t) - (\operatorname{sh} t + \sin t)}{\operatorname{ch} t - \cos t} \times \frac{w^2\, t}{4\,\varepsilon} - \frac{i\, y\, t^2}{4\,\varepsilon^2}\right]\right.$$
$$\left. \times \exp\left[\tfrac{i}{4}\int_0^t \tfrac{\operatorname{sh}\theta - \sin\theta}{\operatorname{ch}\theta - \cos\theta}\, d\theta\right]\right\} \times \frac{t^{3/2}}{(\operatorname{ch} t - \cos t)^{1/4}}\, dt\,.$$

Finally, changing t in $\sqrt{t}$ above, we obtain:

$$2\,(2\pi)^{3/2}\,\varepsilon^{5/2}\,p_\varepsilon\big(0;(w,y)\big)$$
$$= \int_0^\infty \Re\left\{ \exp\left[\frac{i(\operatorname{sh}\sqrt{t} - \sin\sqrt{t}) - (\operatorname{sh}\sqrt{t} + \sin\sqrt{t})}{\operatorname{ch}\sqrt{t} - \cos\sqrt{t}} \times \frac{w^2\,\sqrt{t}}{4\,\varepsilon} - \frac{i\,y\,t}{4\,\varepsilon^2}\right]\right.$$
$$\left. \times \exp\left[\tfrac{i}{4}\int_0^{\sqrt{t}} \tfrac{\operatorname{sh}\theta - \sin\theta}{\operatorname{ch}\theta - \cos\theta}\, d\theta\right]\right\} \times \frac{t^{1/4}}{\big(\operatorname{ch}\sqrt{t} - \cos\sqrt{t}\big)^{1/4}}\, dt\,.$$

Owing to the notation (10) and changing y into $\varepsilon\, y$, this is precisely the claim of (9). □

Remark 3.5. — Since $h(x^2)$ and $H(x^2)$ are even analytic functions of $x \in \mathbb{R}$, the functions h and H are analytical on $\mathbb{R}_+$.

The two following lemmas deal with the above functions h and H, respectively on $\mathbb{R}_+$ and on the imaginary interval $i\,]-\pi^2,\infty[$. Their technical proofs are given in the appendix (Section 7).

Lemma 3.6. — *The analytical functions h and H increase and are non-negative on $\mathbb{R}_+$, and satisfy:*

(i) $h(x) = \frac{w^2}{3}\, x \left[1 - \frac{x^2}{630} + \mathcal{O}(x^4)\right]$ *and* $H(x) = \frac{w^2}{90}\, x^2 - \frac{w^2}{36925}\, x^4 + \mathcal{O}(x^6)$ *near 0;*

(ii) $h(x) = w^2\sqrt{x}\big(1 + \mathcal{O}(e^{-\sqrt{x}})\big)$ *and* $H(x) = w^2\sqrt{x}\big(1 + \mathcal{O}(e^{-\sqrt{x}})\big) - 2w^2$ *near infinity.*

They admit increasing positive smooth reciprocal functions h^{-1} and H^{-1} on $\mathbb{R}_+^*$, such that

(iii) $h^{-1}(x) = \frac{3}{w^2}\, x\left[1 + \frac{x^2}{70\, w^4} + \mathcal{O}(x^4)\right]$ *and* $H^{-1}(x) = \sqrt{\frac{90\, x}{w^2}}\,(1 + \mathcal{O}(x))$ *near 0;*

(iv) $h^{-1}(x) = x^2\big(w^{-4} + \mathcal{O}(e^{-x})\big)$ *and* $H^{-1}(x) = \left[x + 2w^2\right]^2\big(w^{-4} + \mathcal{O}\left[e^{-x/w^2}\right]\big)$ *near infinity.*

Lemma 3.7. — *First consider the range $0 < x < \pi^2$. There,*

$$-i\,h(2ix) = \big(\coth\sqrt{x} - \operatorname{cotg}\sqrt{x}\big) w^2\sqrt{x} =: \lambda(x)$$

is an increasing positive smooth function of $x \in]0,\pi^2[$, *such that* $\lambda(x) = \frac{2w^2}{3}\,x + \mathcal{O}(x^3)$ *near 0 and* $\lambda(x) \sim \frac{\pi w^2}{\pi-\sqrt{x}}$ *near* π^2. *And*

$$H(2ix) = \big(\coth\sqrt{x} + \mathrm{cotg}\sqrt{x}\big)w^2\sqrt{x} - 2w^2 =: \Lambda(x)$$

is a decreasing negative smooth function of $x \in]0,\pi^2[$, *such that* $\Lambda(x) = -\frac{2w^2}{45}\,x^2 + \mathcal{O}(x^3)$ *near 0 and* $\Lambda(x) \sim \frac{-\pi w^2}{\pi-\sqrt{x}}$ *near* π^2. *Moreover, for any real* $t > -\pi^2$ *we have*

$$(i\,h - H)(2it) = 2w^2\big(1 - \sqrt{t}\coth\sqrt{t}\big);$$

for $-\pi^2 < t < 0$ *this means:*

$$(i\,h - H)(2it) = 2w^2\big(1 - \sqrt{|t|}\,\mathrm{cotg}\sqrt{|t|}\big).$$

Remark 3.8. — Lemma 3.7 entails that $\lambda'(x) = 2\,h'(2ix)$, and then that $h'(i\,\mathbb{R}_+) = \mathbb{R}_+^*$; and also $\Lambda'(x) = 2i\,H'(2ix)$.

4. Asymptotics for p_ε in the non-pseudo-cut-locus case

Recall the obvious necessary condition: $(w,y) \in \mathbb{R}\times\mathbb{R}_+^*$. We will now address the scaled $p_\varepsilon\big(0;(w,\varepsilon y)\big)$, in the non-pseudo-diagonal sub-case, which means the restriction of $w \in \mathbb{R}^*$. According to (9), we have to evaluate $\Re\{J_\varepsilon^y\}$, where

(11)

$$J_\varepsilon^y := \int_0^\infty \exp\left[\frac{i(h(x)-y\,x)-H(x)}{4\,\varepsilon} + \frac{i}{4}\int_0^{\sqrt{x}} \frac{\mathrm{sh}\,-\sin}{\mathrm{ch}\,-\cos}\right] \times \left[\frac{x}{\mathrm{ch}\sqrt{x}-\cos\sqrt{x}}\right]^{1/4} dx$$

$$= \int_0^\infty \exp\left[\frac{i(h(x)-y\,x)-H(x)}{4\,\varepsilon} + \int_0^{\sqrt{x}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)} + \tfrac{1}{2\theta}\Big)d\theta\right] dx\,.$$

Classically, see for example [8], such a one-dimensional oscillatory integral must be analyzed by the stationary phase method, or some variant of it, such as the most efficient saddle-point method (explained in [8]). The basic observation is that the dominant contribution necessarily arises from the neighborhoods of the stationary points, within the complex plane, i.e., the points at which the complex derivative of the phase $\big($here $z \mapsto i(h(z)-y\,z)-H(z)\big)$ vanishes. Thus these stationary or saddle-points solve the so-called saddle-point equation, namely: $h'(z) + i\,H'(z) = y$.

Then a convenient change of contour has to be found and performed, in order to integrate along a new contour which goes through the dominant saddle-point(s) (i.e., those which yield the dominant contribution to the integral).

A convenient neighborhood of the saddle-point(s) must then be specified, and the corresponding contribution evaluated.

Of course, a control of the contribution of the complement of the above neighborhood of the saddle-point(s) is necessary, and even more when it is not known, as will be the case below, whether all saddle-points have been detected. Thus a lot of estimates will be necessary, some of them being somewhat delicate, even more so as we shall have at our disposal only an implicit or an almost saddle-point.

Actually, in the present case the following supplementary feature deserves mention: the main contribution of J_ε^y happens to be possibly purely imaginary, while we want to analyze $\Re\{J_\varepsilon^y\}$; so that we have to overcome this difficulty by choosing a new contour which lets the possibly dominant, purely imaginary part, appear clearly.

This is the route we follow, from now on, to deal with the four cases of Theorem 2.1.

4.1. The saddle-point equation in the scaled sub-case $w \neq 0$. — This case, $w \neq 0$ corresponds to the off-cut-locus case in the sub-Riemannian setting, see in particular [4, 5]. Then the saddle-point equation corresponding to (11) is $h' + i\,H' = y$. The following lemma will provide an implicit solution. It's technical proof is given in the appendix (Section 7).

LEMMA 4.1. — *The smooth map $u \mapsto \frac{\operatorname{sh} u - u}{u(\operatorname{ch} u - 1)}$ has a negative derivative on $\mathbb{R}_+^*$, and maps $\mathbb{R}_+$ to $\left]0, \frac{1}{3}\right]$. The smooth map $v \mapsto \frac{v - \sin v}{v(1-\cos v)}$ has a positive derivative on $]0, 2\pi[$, and maps $[0, 2\pi[$ to $\left[\frac{1}{3}, \infty\right[$.*

By Lemma 3.7 we have $\lambda(q) = -i\,h(2iq)$ and $\Lambda(q) = H(2iq)$, whence $h'(2iq) + i\,H'(2iq) = \frac{\lambda' + \Lambda'}{2}(q)$, and then $2iq$ solves the above saddle-point equation if and only if $(\lambda' + \Lambda')(q) = 2y$, i.e., if and only if

$$\tau \equiv \frac{y}{w^2} = \frac{\operatorname{sh}(2\sqrt{q}) - 2\sqrt{q}}{2\sqrt{q}\left(\operatorname{ch}(2\sqrt{q}) - 1\right)}\,, \tag{12}$$

which by Lemma 4.1 has a unique solution $q \equiv q(\tau)$ in $\mathbb{R}_+$, for $0 < \tau \leq \frac{1}{3}$. And for $\tau \geq \frac{1}{3}$, $2iq$ solves the above saddle-point equation if and only if

$$\frac{y}{w^2} = \frac{2i\sqrt{q} - \sin(2i\sqrt{q})}{2i\sqrt{q}\left(1 - \cos(2i\sqrt{q})\right)} = \frac{2\sqrt{-q} - \sin(2\sqrt{-q})}{2\sqrt{-q}\,(1 - \cos(2\sqrt{-q}))}\,,$$

which by Lemma 4.1 has a unique solution $q \equiv q(\tau)$ in $[-\pi^2, 0]$, the limit case $q(\infty) = -\pi^2$ corresponding to the actual possibility $w = 0$.

Moreover, Lemma 4.1 shows that $\frac{\partial \tau}{\partial q} < 0$ on $\{-\pi^2 < q < \infty\}$, except possibly at $q = 0$. Now (12) expresses τ as an analytic function of $q \in]-\pi^2, \infty[$, which at 0 equals $\frac{1}{3} - \frac{2}{45}\,q + \mathcal{O}(q^2)$, so that $\frac{\partial_o \tau}{\partial q} = -\frac{2}{45} < 0$. This establishes the following.

LEMMA 4.2. — *The saddle-point equation (12) determines a decreasing analytic implicit function $q \equiv q(\tau)$ of $\tau > 0$. We have $\lim_0 q = \infty, q(\infty) = -\pi^2, q\left(\frac{1}{3}\right) = 0$ and $q'\left(\frac{1}{3}\right) = -\frac{45}{2}$.*

Note that the above does not exclude the eventuality of saddle-points outside $i\,\mathbb{R}$.

4.2. Change of contour (in the scaled sub-case $w \neq 0$). — The above leads to the necessity of changing the contour $\mathbb{R}_+$ for J_ε^y in (11), into

$$[0, 2iq] \bigcup \left(2iq + \mathbb{R}_+\right),$$

for any given $q \in]-\pi^2, \infty[$. Recall that we deal here with the case $w \neq 0$.

To this aim, we need the integrand in (11) to be holomorphic in the half-band $\mathcal{R} := \mathbb{R}_+ + i\,]-\pi^2, \infty[$, i.e., holomorphic in the interior of $\mathcal{R}$ and continuous on $\mathcal{R}$.

The only possible difficulty at this point could come from the ratio $\frac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{\mathrm{ch}\,\theta - \cos\theta}$.

Now its denominator $(\mathrm{ch}\,\theta - \cos\theta)$ vanishes only for $\theta \in (1 \pm i)\pi\mathbb{Z}$, which (within the image $\sqrt{\mathcal{R}}$ we have to consider, using the usual continuous determination of the square root) reduces the possible singularities to the set $\{\theta_k := (1+i)k\pi \,|\, k \in \mathbb{N}\}$.

But for $k = 0$, i.e., at 0, we have the integrand $\frac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta - \cos\theta)} + \frac{1}{2\theta} \sim \frac{i\theta}{12}$, and for $k \in \mathbb{N}^*$, in the vicinity of θ_k, setting $u := (\theta - \theta_k)$, an easy computation yields

$$\begin{aligned} \mathrm{ch}\,\theta - \cos\theta = (-1)^k \big[&\mathrm{ch}(k\pi)(\mathrm{ch}\,u - \cos u) \\ &+ \mathrm{sh}(k\pi)(\mathrm{sh}\,u + i\sin u)\big], \quad (i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta \\ = (-1)^k \big[&\mathrm{sh}(k\pi)(i-1)(\mathrm{ch}\,u - \cos u) + \mathrm{ch}(k\pi)((i-1)\,\mathrm{sh}\,u \\ &+ (i+1)\sin u)\big], \end{aligned}$$

and then

$$\tfrac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{\mathrm{ch}\,\theta - \cos\theta} = (i+1)\coth(k\pi) + \mathcal{O}(\theta - \theta_k).$$

This shows that there is no true singularity at all in $\sqrt{\mathcal{R}}$, and that we can apply the Cauchy theorem to perform the desired change of contour.

We then need to specify some values of the functions coming into the play. First, for any real (bounded) t and non-negative x we have:

$$\sqrt{x + it} = \sqrt{\frac{\sqrt{x^2 + t^2} + x}{2}} + i\,\mathrm{sgn}(t)\sqrt{\frac{\sqrt{x^2 + t^2} - x}{2}} =: a + ib\,, \tag{13}$$

from which elementary computations entail

$$\begin{aligned}
& i\,\frac{\operatorname{sh}\sqrt{x+it}-\sin\sqrt{x+it}}{\operatorname{ch}\sqrt{x+it}-\cos\sqrt{x+it}}-\frac{\operatorname{sh}\sqrt{x+it}+\sin\sqrt{x+it}}{\operatorname{ch}\sqrt{x+it}-\cos\sqrt{x+it}}\\
&=\frac{i\,(\operatorname{sh}(a+b)-\sin(a-b))-(\operatorname{sh}(a+b)+\sin(a-b))}{\operatorname{ch}(a+b)-\cos(a-b)}\,.
\end{aligned}$$

By the very definition (10) of (h,H), with $\tau\equiv y/w^2$, for any non-negative x we thus have

$$\begin{aligned}
& w^{-2}\big[i\,h(x+it)-H(x+it)-i\,y\,(x+it)\big]\\
&=2+\tau\,t-i\tau x\\
&\quad+\frac{i\big((a-b)\operatorname{sh}(a+b)-(a+b)\sin(a-b)\big)-\big((a+b)\operatorname{sh}(a+b)+(a-b)\sin(a-b)\big)}{\operatorname{ch}(a+b)-\cos(a-b)}\,.
\end{aligned}$$

Thus, setting $N(t,x):=w^{-2}\,\Re\big\{\big(i\,h-H-iy\,\mathrm{Id}\big)(x+it)\big\}$ and $\tilde N(t,x):=w^{-2}\,\Im\big\{\big(i\,h-H-iy\,\mathrm{Id}\big)(x+it)\big\}$, for any $(t,x)\in\mathbb{R}\times\mathbb{R}_+$ we have

$$w^{-2}\big[i\,h(x+it)-H(x+it)-iy\,(x+it)\big]=N(t,x)+i\,\tilde N(t,x)\,, \tag{14}$$

with

$$N(t,x)=2+\tau\,t-\frac{(a+b)\operatorname{sh}(a+b)+(a-b)\sin(a-b)}{\operatorname{ch}(a+b)-\cos(a-b)} \tag{15}$$

and

$$\tilde N(t,x)=\frac{(a-b)\operatorname{sh}(a+b)-(a+b)\sin(a-b)}{\operatorname{ch}(a+b)-\cos(a-b)}-\tau x\,.$$

Note in particular that we have the following.

LEMMA 4.3. — *As $x\to+\infty$ we have $h(x+it)\sim H(x+it)\sim w^2\sqrt{x}$ and $\int_0^{\sqrt{x+it}}\Big(\frac{(i-1)\operatorname{sh}\theta-(i+1)\sin\theta}{4\,(\operatorname{ch}\theta-\cos\theta)}+\frac{1}{2\theta}\Big)d\theta\sim\frac{\log x}{4}$, uniformly for bounded real t.*

Hence, for any positive R, changing the partial contour $[0,R]$ of J^y_ε into

$$[0,2iq]\bigcup\big(2iq+[0,R]\big)\bigcup\big(R+2i[0,q]\big)\,,$$

by (11) and Lemmas 3.7 and 4.3 we obtain

$$
\begin{aligned}
J_\varepsilon^y &= I_\varepsilon^y \\
&\quad + \lim_{R\to\infty}\int_0^R \exp\Bigg[\frac{i[h(2iq+x)-y(2iq+x)]-H(2iq+x)}{4\,\varepsilon} \\
&\qquad\qquad + \int_0^{\sqrt{2iq+x}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta\Bigg]dx \\
&\quad - 2i\lim_{R\to\infty}\int_0^q \exp\Bigg[\frac{i(h(2it+R)-y(2it+R))-H(2it+R)}{4\,\varepsilon} \\
&\qquad\qquad + \int_0^{\sqrt{2it+R}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta\Bigg]dt \\
&= I_\varepsilon^y + \tilde J_\varepsilon^y\,,
\end{aligned}
$$

with

$$
I_\varepsilon^y := 2i\int_0^q \exp\Bigg[\frac{y\,t+w^2\big(1-\sqrt{t}\coth\sqrt{t}\big)}{2\,\varepsilon} + \int_0^{\sqrt{2it}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta\Bigg]dt
$$

and

$$
\tilde J_\varepsilon^y := \int_0^\infty \exp\Bigg[\frac{i(h(2iq+x)-y(2iq+x))-H(2iq+x)}{4\,\varepsilon} + \int_0^{\sqrt{2iq+x}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta\Bigg]\,dx\,. \tag{16}
$$

Now to evaluate the first term I_ε^y, note that for $t>-\pi^2$, denoting $\sigma:=\mathrm{sign}(t)$ we have

$$
\begin{aligned}
&\int_0^{\sqrt{2it}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta \\
&= \int_0^{(1+\sigma i)\sqrt{|t|}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta \\
&= \int_0^{\sqrt{|t|}}\bigg(\frac{1+\sigma i}{4}\times\frac{(i-1)\,\mathrm{sh}[(1+\sigma i)\theta]-(i+1)\sin[(1+\sigma i)\theta]}{\mathrm{ch}[(1+\sigma i)\theta]-\cos[(1+\sigma i)\theta]}+\frac{1}{2\theta}\bigg)\,d\theta \\
&= \frac{\sigma}{4}\int_0^{\sqrt{|t|}}(\mathrm{cotg}\,\theta-\coth\theta)\,d\theta+\frac14\int_0^{\sqrt{|t|}}\bigg(\frac{2}{\theta}-\mathrm{cotg}\,\theta-\coth\theta\bigg)\,d\theta\ \in\mathbb{R}\,,
\end{aligned}
$$

so that the first term also reads

$$I_\varepsilon^y = 2i\int_0^q \exp\Bigg[\frac{y\,t + w^2\,\big(1-\sqrt{t}\coth\sqrt{t}\big)}{2\,\varepsilon} + \frac{\sigma}{4}\int_0^{\sqrt{|t|}} (\operatorname{cotg}\theta - \coth\theta)\,d\theta + \frac{1}{4}\int_0^{\sqrt{|t|}} \Big(\frac{2}{\theta} - \operatorname{cotg}\theta - \coth\theta\Big)\,d\theta\Bigg]\,dt$$

which belongs to $i\,\mathbb{R}$, and then will not contribute when taking the real part of J_ε^y.

Thus we are left with the determining contribution $\tilde{J}_\varepsilon^y$ of (16). We cut it into two parts:

$$\tilde{J}_\varepsilon^y = D_\varepsilon^y + R_\varepsilon^y\,, \tag{17}$$

the former corresponding to the integral from 0 to some small positive r_ε, to be specified later on, and the latter corresponding to the integral from r_ε to infinity.

4.3. Dealing with the dominant part D_ε^y (in the case $w \neq 0$). —

4.3.1. *Behaviour of the phase near the saddle-point.* — By Lemma 4.2 and the very choice of $q \equiv q(\tau) > -\pi^2$ solving (12), and by Lemma 3.7, for $0 \le x \le r_\varepsilon$ we have

$$\begin{aligned}
& i\big(h(2iq + x) - y(2iq + x)\big) - H(2iq + x)\\
&= i\big(h(2iq) - 2iq\,y\big) - H(2iq) + [i\,h''(2iq) - H''(2iq)]\,x^2/2 + \mathcal{O}(r_\varepsilon^3)\\
&= 2w^2\big(1 - \sqrt{q}\coth\sqrt{q}\big) + 2q\,y - \frac{d^2}{dq^2}\big(1-\sqrt{q}\coth\sqrt{q}\big)\,w^2x^2/4 + \mathcal{O}(r_\varepsilon^3)\\
&= 2w^2\big(1 + \tau q - \sqrt{q}\coth\sqrt{q}\big) + \frac{2q\coth\sqrt{q} - \sqrt{q} - \operatorname{ch}\sqrt{q}\operatorname{sh}\sqrt{q}}{16\,q^{3/2}\operatorname{sh}^2\sqrt{q}}\,w^2x^2 + \mathcal{O}(r_\varepsilon^3)\\
&= 2w^2\,\varrho(\tau) - w^2\,R(\tau)\,x^2/8 + \mathcal{O}(r_\varepsilon^3)\,,
\end{aligned}$$

where

$$\varrho(\tau) := 1 + \tau\,q - \sqrt{q}\coth\sqrt{q} \quad\text{and}\quad R(\tau) := \frac{\operatorname{ch}\sqrt{q}\operatorname{sh}\sqrt{q} - 2q\coth\sqrt{q} + \sqrt{q}}{2\,q^{3/2}\operatorname{sh}^2\sqrt{q}}\,. \tag{18}$$

4.3.2. *The case of $\tau \equiv \frac{y}{w^2} \le \frac{1}{3}$, i.e., $q \equiv q(\tau) \ge 0$.* — Setting $z := \sqrt{q(\tau)}$, we have $4\tau = \frac{\operatorname{sh}(2z)-2z}{z\operatorname{sh}^2 z}$ and then $\psi(z) := \frac{2z^2\operatorname{sh}^3 z}{z'(\tau)\operatorname{ch} z} = 2z^2 - z\operatorname{th} z - \operatorname{sh}^2 z$ is non-positive, since $\psi''(z)\frac{\operatorname{ch}^2 z}{2} = 3\operatorname{ch}^2 z - 2\operatorname{ch}^4 z - 1 + z\operatorname{th} z \le 3\operatorname{ch}^2 z - 2\operatorname{ch}^4 z - 1 + z \wedge z^2 \le 0$.

This shows that $z'(\tau)\le 0$ (even <0 for $q>0$) on the one hand. On the other hand,

$$\varrho(\tau)=1-z\coth z+\frac{z(\operatorname{ch}z\operatorname{sh}z-z)}{2\operatorname{sh}^2 z},$$
$$R(\tau)=\frac{\operatorname{ch}z\operatorname{sh}z-2z^2\coth z+z}{2\,z^3\operatorname{sh}^2 z}=\frac{-1}{z(\tau)\,z'(\tau)}>0\,,$$

which successively imply $R'(\tau)=R(\tau)^2(zz''(\tau)+z'(\tau)^2)$,

$$\varrho'(\tau)=(2z^2\coth z-\operatorname{ch}z\operatorname{sh}z-z)\frac{z'(\tau)}{2\operatorname{sh}^2 z}=z^2\ge 0,\quad \varrho''(\tau)=2zz'(\tau)<0\,,$$
$$\frac{2z^4\operatorname{sh}^4 z}{z'(\tau)}\,R'(\tau)=6z^3+4z^3\operatorname{sh}^2 z-3\operatorname{ch}z\operatorname{sh}^3 z-3z\operatorname{sh}^2 z=:\tilde\varphi(z)$$

such that

$$\begin{aligned}\varphi(z)&:=4\,\tilde\varphi(z/2)\\&=3z^3+(z^3-3z-3\operatorname{sh}z)\,(\operatorname{ch}z-1),\quad \varphi'(0)=\varphi''(0)=\varphi^{(3)}(0)=0\,,\\ \varphi^{(4)}(z)&=\operatorname{ch}z\operatorname{sh}z\left(3\,\frac{4z^2+5}{\operatorname{ch}z}+(z^2+33)\frac{z}{\operatorname{sh}z}-24\right)\\&<\operatorname{ch}z\operatorname{sh}z\left(\frac{12z^2+15}{1+\frac{z^2}{2}+\frac{z^4}{24}}+\frac{z^2+33}{1+\frac{z^2}{6}+\frac{z^4}{120}}-24\right)\\&=\;-z^4\operatorname{ch}z\operatorname{sh}z\left(\frac{12}{5}+\frac{47z^2}{120}+\frac{z^4}{60}\right)\left(1+\frac{z^2}{2}+\frac{z^4}{24}\right)^{-1}\left(1+\frac{z^2}{6}+\frac{z^4}{120}\right)^{-1}\\&<0\,,\end{aligned}$$

whence $\varphi(z)<0$ and then $\tilde\varphi(z)<0$.

Thus, for $0<\tau<\frac13$ we successively have $z'(\tau)<0<R'(\tau)$, with $z(\frac13)=0$, $z'(\frac13-)=-\infty$, $zz'(\frac13)=-\frac{45}{4}$, and as $\tau\nearrow\frac13$: $q(\tau)\sim\frac{15}{2}(1-3\tau)$, $\varrho(\tau)\sim-\frac56(1-3\tau)^2$ and $R(\tau)\nearrow\frac{4}{45}$; and as $\tau\searrow 0$: $z(\tau)\sim\frac{1}{2\tau}$, $q(\tau)\sim\frac{1}{4\tau^2}$, $\varrho(\tau)\sim\frac{-1}{4\tau}$ and $R(\tau)\sim 4\tau^3$.

4.3.3. *The case of* $\infty>\tau\equiv\frac{y}{w^2}>\frac13$, *i.e.*, $0>q\equiv q(\tau)>-\pi^2$. — In this case, according to Section 4.3.1 and setting $\zeta:=\sqrt{|q|}\in]0,\pi[$, on the one hand we have $4\tau=\frac{2\zeta-\sin(2\zeta)}{\zeta\sin^2\zeta}$ and then $\frac{2\zeta^2\sin^2\zeta}{\zeta'(\tau)}=\zeta+\cos\zeta\sin\zeta-2\zeta^2\operatorname{cotg}\zeta=:\theta(\zeta)>0$, since $\frac12\theta'(\zeta)=\cos^2\zeta+2\zeta^2+2\zeta^2\operatorname{cotg}^2\zeta-4\zeta\operatorname{cotg}\zeta$ is easily seen to be positive, by handling separately the sub-cases $\zeta\le\frac{\pi}{2}$ and $\zeta>\frac{\pi}{2}>1$. This shows that $\zeta'(\tau)<0$.

On the other hand we have

$$\varrho(\tau)=1-\tau\,\zeta^2-\zeta\operatorname{cotg}\zeta\quad\text{and}\quad R(\tau)=\frac{1+2\tau\,\zeta^2-\tau\,\sin^2\zeta-\zeta^2\sin^{-2}\zeta}{\zeta^2\,\sin^2\zeta}\,.$$

This entails $\varrho'(\tau) = -\zeta^2 < 0$, $\varrho''(\tau) = -\zeta(\tau)\,\zeta'(\tau) < 0$, and successively:

$$R(\tau) = \frac{\cos\zeta\sin^2\!\zeta + \zeta\sin\zeta - 2\zeta^2\cos\zeta}{2\,\zeta^3\sin^3\!\zeta} = \frac{\theta(\zeta)}{2\,\zeta^3\sin^2\!\zeta} > 0\,,$$

$$\frac{2\zeta^4\sin^4\zeta}{\zeta'(\tau)}\,R'(\tau) = 6\zeta^3 - 4\zeta^3\sin^2\!\zeta - 3\zeta\sin^2\!\zeta - 3\cos\zeta\sin^3\!\zeta =: \tilde\sigma(\zeta)$$

such that

$$\sigma(\zeta) := 4\,\tilde\sigma(\zeta/2) = 3\zeta^3 - (\zeta^3 + 3\zeta + 3\sin\zeta)(1-\cos\zeta),$$

$$\sigma'(0) = \sigma''(0) = \sigma^{(3)}(0) = 0\,,$$

$$\begin{aligned}
\sigma^{(4)}(\zeta) &= (\zeta^3 - 33\zeta + 33\sin\zeta)\cos\zeta + (12\zeta^2 - 15 + 15\cos\zeta)\sin\zeta \\
&> \frac{9\zeta^2}{2}(-\zeta\cos\zeta + \sin\zeta) \geq 0 \quad \text{for} \quad 0 < \zeta \leq \frac{\pi}{2} \\
&\qquad\left(\text{using } \cos\zeta > 1 - \frac{\zeta^2}{2} \text{ and } \sin\zeta > \zeta - \frac{\zeta^3}{6}\right)\,; \\
&> \left(33 + \frac{60}{\pi}\right)\zeta + 24\zeta^2 - \left(1 + \frac{24}{\pi}\right)\zeta^3 - 93 > 0 \quad \text{for} \quad \frac{\pi}{2} < \zeta < \pi \\
&\qquad\left(\text{using } -\cos\zeta < 1 > \sin\zeta > 2 - \frac{2\zeta}{\pi}\right).
\end{aligned}$$

whence $\sigma(\zeta) > 0$ and then $\tilde\sigma(\zeta) > 0$, for $0 < \zeta < \pi$, i.e., for $\tau > \frac{1}{3}$.

Thus, for $\tau > \frac{1}{3}$ we successively have $\zeta'(\tau) > 0, R'(\tau) > 0$, with $\zeta(\frac{1}{3}+) = 0$, $\zeta'(\frac{1}{3}+) = \infty, \zeta\zeta'(\frac{1}{3}+) = \frac{45}{4}$, and as $\tau \searrow \frac{1}{3}$: $q(\tau) \sim \frac{15}{2}(1-3\tau)$, $\varrho(\tau) \sim -\frac{5}{6}(3\tau-1)^2$ and $R(\frac{1}{3}+) = \frac{4}{45}$; and as $\tau \nearrow \infty$: $\frac{1}{\sin^2\zeta} = 2\tau - \frac{\sqrt{2\tau}}{\pi} + \mathcal{O}(1)$, $\zeta(\tau) = \pi - \frac{1}{\sqrt{2\tau}} + \frac{\mathcal{O}(1)}{\tau}$, $q(\tau) = -\pi^2 + \pi\sqrt{\frac{2}{\tau}} + \frac{\mathcal{O}(1)}{\tau}$, $\varrho(\tau) = -\pi^2\tau + 2\pi\sqrt{2\tau} + \mathcal{O}(1)$ and $R(\tau) = \frac{1}{\pi}(2\tau)^{3/2} + \mathcal{O}(\tau)$.

Note the following consequence of Lemma 4.2 and of the above Sections 4.3.2 and 4.3.3.

Lemma 4.4. — *$\varrho(\tau)$ is a non-positive analytic function of $\tau > 0$, which increases from $-\infty$ to 0 on $]0, 1/3]$, decreases from 0 to $-\infty$ on $[1/3, \infty[$, and is strictly concave. $R(\tau)$ is an increasing positive analytic function of $\tau > 0$, with $R(0+) = 0, R(1/3) = 4/45, R(\infty) = \infty$.*

4.3.4. *Behaviour of the dominant part D^y_ε (case $w \neq 0$).* — For any positive τ and $0 \leq x \leq r_\varepsilon$, denoting $\sigma := \text{sign}(q) \equiv \text{sign}(q(\tau))$ we have

$$\sqrt{2iq + x} = (\sigma i + 1)\sqrt{|q|} + \frac{1 - \sigma i}{4\sqrt{|q|}}\,x + \mathcal{O}(r_\varepsilon^2)$$

and then

$$
\begin{aligned}
&\int_0^{\sqrt{2iq+x}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)\,d\theta \\
&= \int_0^{(\sigma i+1)\sqrt{|q|}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta+\mathcal{O}(r_\varepsilon^2) \\
&\quad+\frac{1-\sigma i}{4\sqrt{|q|}}\,x\times\left(\tfrac{(i-1)\,\mathrm{sh}\big[(\sigma i+1)\sqrt{|q|}\big]-(i+1)\sin\big[(\sigma i+1)\sqrt{|q|}\big]}{4\big(\mathrm{ch}\big[(\sigma i+1)\sqrt{|q|}\big]-\cos\big[(\sigma i+1)\sqrt{|q|}\big]\big)}+\tfrac{1}{2(\sigma i+1)\sqrt{|q|}}\right) \\
&= \frac{1}{2}\int_0^{\sqrt{|q|}}\left(\frac{1}{\theta}+\frac{\sigma-1}{2}\,\mathrm{cotg}\,\theta-\frac{\sigma+1}{2}\coth\theta\right)d\theta \\
&\quad+\left((\sigma+1)\coth\sqrt{|q|}+(\sigma-1)\,\mathrm{cotg}\sqrt{|q|}-\tfrac{2\sigma}{\sqrt{|q|}}\right)\frac{i\,x}{16\sqrt{|q|}}+\mathcal{O}(r_\varepsilon^2) \\
&= 1_{\{\tau<1/3\}}\left(\log\sqrt{\frac{\sqrt{q}}{\mathrm{sh}\sqrt{q}}}+i\,\frac{\sqrt{q}\coth\sqrt{q}-1}{8\,q}\,x\right)+1_{\{\tau=1/3\}}\,\frac{i\,x}{24} \\
&\quad+1_{\{\tau>1/3\}}\left(\log\sqrt{\frac{\sqrt{-q}}{\sin\sqrt{-q}}}+i\,\frac{\sqrt{-q}\,\mathrm{cotg}\sqrt{-q}-1}{8\,q}\,x\right)+\mathcal{O}(r_\varepsilon^2) \\
&= \log\sqrt{\frac{\sqrt{q}}{\mathrm{sh}\sqrt{q}}}+i\,\frac{\sqrt{q}\coth\sqrt{q}-1}{8\,q}\,x+\mathcal{O}(r_\varepsilon^2)\,.
\end{aligned}
$$

Therefore, according to Section 4.3.1, (11), (16) and (17) we obtain

$$
\begin{aligned}
D_\varepsilon^y &= \int_0^{r_\varepsilon}\exp\Bigg[\tfrac{2w^2\,\varrho(\tau)-w^2\,R(\tau)\,x^2/8+\mathcal{O}(r_\varepsilon^3)}{4\,\varepsilon} \\
&\qquad\qquad+\int_0^{\sqrt{2iq+x}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\Big)d\theta\Bigg]dx \\
&= \sqrt{\frac{\sqrt{q}}{\mathrm{sh}\sqrt{q}}}\,\exp\left[\frac{w^2\varrho(\tau)}{2\,\varepsilon}+\mathcal{O}\left(\frac{r_\varepsilon^3}{\varepsilon}+r_\varepsilon^2\right)\right] \\
&\quad\times\int_0^{r_\varepsilon}\exp\left[-\frac{w^2R(\tau)}{32\,\varepsilon}\,x^2+i\,\frac{\sqrt{q}\coth\sqrt{q}-1}{8\,q}\,x\right]dx\,.
\end{aligned}
$$

Then

$$\begin{aligned}
&\int_0^{r_\varepsilon} \exp\Big[-\frac{w^2 R(\tau)}{32\,\varepsilon}\,x^2 + i\,\frac{\sqrt{q}\coth\sqrt{q}-1}{8\,q}\,x\Big]\,dx \\
&= \int_0^{r_\varepsilon} \exp\Big[-\frac{w^2 R(\tau)}{32\,\varepsilon}\,x^2\Big] \times \Big[1 + i\,\frac{\sqrt{q}\coth\sqrt{q}-1}{8\,q}\,x + \mathcal{O}\big(r_\varepsilon^2\big)\Big]\,dx \\
&= \sqrt{\frac{16\,\varepsilon}{w^2 R(\tau)}}\int_0^{r_\varepsilon\sqrt{\frac{w^2 R(\tau)}{16\,\varepsilon}}} e^{-x^2/2}\Big[1 + i\sqrt{\varepsilon}\,\frac{\sqrt{q}\coth\sqrt{q}-1}{2\,q\sqrt{w^2 R(\tau)}}\,x + \mathcal{O}\big(r_\varepsilon^2\big)\Big]\,dx
\end{aligned}$$

has the real part

$$\begin{aligned}
&\sqrt{\frac{16\,\varepsilon}{w^2 R(\tau)}}\int_0^{r_\varepsilon\sqrt{\frac{w^2 R(\tau)}{16\,\varepsilon}}} e^{-x^2/2}\,dx\,\big(1+\mathcal{O}(r_\varepsilon^2)\big) \\
&= \sqrt{\frac{8\pi\,\varepsilon}{w^2 R(\tau)}}\Big(1+\mathcal{O}(r_\varepsilon^2) - \mathcal{O}\Big(\tfrac{\sqrt{\varepsilon}}{r_\varepsilon}\exp\Big[-\tfrac{w^2 R(\tau)\,r_\varepsilon^2}{32\,\varepsilon}\Big]\Big)\Big)\,.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\Re\{D_\varepsilon^y\} = &\sqrt{\frac{8\pi\,\varepsilon\,\sqrt{q}}{w^2 R(\tau)\,\mathrm{sh}\sqrt{q}}}\,\exp\Big[\frac{w^2\varrho(\tau)}{2\,\varepsilon}\Big] \\
&\times\Big(1+\mathcal{O}\Big(\tfrac{r_\varepsilon^3}{\varepsilon} + r_\varepsilon^2 - \tfrac{\sqrt{\varepsilon}}{r_\varepsilon}\exp\Big[-\tfrac{w^2 R(\tau)\,r_\varepsilon^2}{32\,\varepsilon}\Big]\Big)\Big)\,.
\end{aligned}$$

Choosing $r_\varepsilon = \sqrt{\varepsilon\log\frac{1}{\varepsilon}\times L_\varepsilon}$, for any L_ε going to infinity as $\varepsilon\searrow 0$, this yields (19)

$$\Re\{D_\varepsilon^y\} = \sqrt{\frac{8\pi\,\varepsilon\,\sqrt{q(\tau)}}{w^2 R(\tau)\,\mathrm{sh}\sqrt{q(\tau)}}}\,\exp\Big[\frac{w^2\varrho(\tau)}{2\,\varepsilon}\Big] \times \Big(1+\mathcal{O}\Big(\sqrt{\varepsilon\log^3(\tfrac{1}{\varepsilon})\,L_\varepsilon^3}\Big)\Big)\,.$$

4.4. Dealing with the residual part R_ε^y (in the case of $w\neq 0$). — Recall that $2\tau = \frac{\mathrm{sh}(2\sqrt{q})-2\sqrt{q}}{2\sqrt{q}\,\mathrm{sh}^2\sqrt{q}}$ ($q\equiv q(\tau)$), and by (11), (14), (16) and (17), that

$$R_\varepsilon^y = \int_{r_\varepsilon}^\infty \exp\Big[\frac{w^2(N+i\tilde{N})(2q,x)}{4\,\varepsilon} + \int_0^{\sqrt{2iq+x}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)} + \tfrac{1}{2\theta}\Big)d\theta\Big]\,dx\,.$$

Recall from the beginning of Section 4.2 that the integrand of the second term of the exponent is continuous on $\mathbb{R}_+$, and even bounded. Indeed its denominator vanishes only on $(1\pm i)\pi\mathbb{Z}$ on the one hand, and on the other hand, for large x we have $a\sim\sqrt{x}, b=\frac{q}{a}\sim\frac{q}{\sqrt{x}}$, so that $\cos\sqrt{2iq+x}$ is bounded, while $\mathrm{ch}\sqrt{2iq+x}\sim \mathrm{ch}\sqrt{x}+i\,\frac{\mathrm{sh}\sqrt{x}}{\sqrt{x}}\to\infty$.

In order to control the size of the phase we need the following. Recall that $r_\varepsilon = \sqrt{\varepsilon\log\frac{1}{\varepsilon}\times L_\varepsilon}$.

PROPOSITION 4.5. — *For small enough ε we have* $2\,\varrho(\tau) - \sup\limits_{x\ge r_\varepsilon} N\big(2\,q(\tau),x\big) > \frac{R(\tau)}{9}\, r_\varepsilon^2$.

This control is crucial, since it will in particular ensure that other saddle-points, if they exist, yield an asymptotically negligible contribution. The delicate technical proof of Proposition 4.5 is given in the appendix (Section 7).

4.5. Conclusion in the scaled sub-case $w \neq 0$. — The beginning of Step 1 of the proof of Proposition 4.5 shows that $N(2q,x) \sim -\sqrt{x}$ as $x \to \infty$.

Fix $T > 25\,\varrho(\tau)^2$ such that $N(2q,x) < -\sqrt{x}/2$ for $x \ge T$, and cut the expression of R^y_ε appearing at the beginning of Section 4.4 into $\int_{r_\varepsilon}^T + \int_T^\infty$. Then, using Lemma 4.3 and Proposition 4.5, we deduce that

$$\begin{aligned} R^y_\varepsilon &= \int_{r_\varepsilon}^T \mathcal{O}\bigg(\exp\bigg[\frac{w^2\,N(2q,r_\varepsilon)}{4\,\varepsilon}\bigg]\bigg) + \int_T^\infty e^{-\frac{w^2\sqrt{x}}{8\,\varepsilon}+\frac{\log x}{2}}\,dx \\ &= \mathcal{O}\bigg(\exp\bigg[\frac{w^2\varrho(\tau)}{2\,\varepsilon} - \frac{w^2 R(\tau)\,r_\varepsilon^2}{36\,\varepsilon}\bigg]\bigg) + \mathcal{O}(\varepsilon^3)\int_{w^2\sqrt{T}/8\varepsilon}^\infty e^{-x}\,x^2\,dx \\ &= e^{\frac{w^2\varrho(\tau)}{2\,\varepsilon}}\,\mathcal{O}\Big(\exp\Big[-\tfrac{w^2 R(\tau)}{36}\big(\log\tfrac{1}{\varepsilon}\big)L_\varepsilon\Big]\Big) + \mathcal{O}(\varepsilon)\,e^{-\frac{w^2\sqrt{T}}{8\,\varepsilon}} \\ &= \Re\big\{D^y_\varepsilon\big\}\times\mathcal{O}\Big(\varepsilon^{\sqrt{L_\varepsilon}} + \sqrt{\varepsilon}\,e^{\frac{w^2\varrho(\tau)}{8\,\varepsilon}}\Big) \end{aligned}$$

by (19) and Lemma 4.4. By (17) and Lemma 4.4, this entails

$$\Re\,\{J^y_\varepsilon\} = \Re\big\{D^y_\varepsilon\big\}\times\big(1+\mathcal{O}(\sqrt{\varepsilon})\big)\,. \tag{20}$$

Now, according to (9),(11), we have

$$p_\varepsilon\big(0;(w,\varepsilon\,y)\big) = \frac{e^{-w^2/2\varepsilon}}{4\pi\sqrt{2\pi\,\varepsilon^5}}\,\Re\,\{J^y_\varepsilon\}\,.$$

Finally, with (18), (19), Lemmas 4.2 and 4.4, this establishes Part A of Theorem 2.1.

5. The scaled pseudo-cut-locus sub-case $w = 0$

This case is analogous to the cut-locus case of the sub-Riemannian setting, which is already specific in that context, see [5]. For $w = 0$, (11) reads:

$$J^y_\varepsilon = \int_0^\infty \exp\bigg[-\frac{iy}{4\,\varepsilon}\,x + \int_0^{\sqrt{x}}\Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)} + \tfrac{1}{2\theta}\Big)d\theta\bigg]\,dx\,.$$

5.1. Change of contour. — As observed in Section 4.1, this case corresponds to $\tau = \infty$ and $q = -\pi^2$, roughly. Precisely, looking for a saddle-point under the form $\chi = -2i(\pi-\nu)^2$ and $\sqrt{\chi} = (1-i)(\pi-\nu)$, for a small positive $\nu \equiv \nu(\varepsilon, y)$, instead of Equation (12) we find:
(21)

$$\frac{i\,y}{2\,\varepsilon}\sqrt{\chi} = \frac{(i-1)\,\mathrm{sh}\,\sqrt{\chi} - (i+1)\sin\sqrt{\chi}}{4\,\big(\mathrm{ch}\,\sqrt{\chi} - \cos\sqrt{\chi}\big)} + \frac{1}{2\sqrt{\chi}}, \quad \text{hence}$$

$$\frac{y}{2\varepsilon}(1+i)(\pi-\nu) = \frac{(i+1)\,\mathrm{cotg}\,\nu}{4} + \frac{1+i}{4(\pi-\nu)}, \quad \text{i.e.,}$$

$$2y\,(\pi-\nu)^2 = \varepsilon\,(1 + (\pi-\nu)\,\mathrm{cotg}\,\nu)\,, \quad \text{i.e.,} \quad \mathrm{tg}\,\nu = \frac{\varepsilon\,(1-\nu/\pi)}{2\pi y\Big(1 - \frac{\varepsilon}{2\pi^2 y} - \frac{2\nu}{\pi} + \frac{\nu^2}{\pi^2}\Big)}\,,$$

which has a unique solution $\nu = \frac{\varepsilon}{2\pi y} + \mathcal{O}(\varepsilon^2)$, i.e., $\chi = -2i\big(\pi^2 - \frac{\varepsilon}{y} + \mathcal{O}(\varepsilon^2)\big)$.

Note that the justification for the change of contour given at the beginning of Section 4.2 still holds here. We change the contour $\mathbb{R}_+$ into $\big[0, -2i(\pi-\nu)^2\big] \bigcup \big(-2i(\pi-\nu)^2 + \mathbb{R}_+\big)$, as follows.

$$\begin{aligned} J^y_\varepsilon = &-2i\int_0^{(\pi-\nu)^2} \exp\Bigg[-\frac{y\,t}{2\,\varepsilon} + \int_0^{(1-i)\sqrt{t}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta - \cos\theta)} + \tfrac{1}{2\theta}\Big)d\theta\Bigg]\,dt \\ &+ \int_0^\infty \exp\Bigg[-\tfrac{iy}{4\,\varepsilon}\big[x - 2i(\pi-\nu)^2\big] \\ &\qquad + \int_0^{\sqrt{x-2i(\pi-\nu)^2}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta - \cos\theta)} + \tfrac{1}{2\theta}\Big)d\theta\Bigg]\,dx \\ &+ 2i\lim_{R\to\infty}\int_0^{(\pi-\nu)^2} \exp\Bigg[\frac{-y}{2\varepsilon}\big(t + \tfrac{iR}{2}\big) \\ &\qquad + \int_0^{\sqrt{R-2it}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta - \cos\theta)} + \tfrac{1}{2\theta}\Big)d\theta\Bigg]dt\,. \end{aligned}$$

Since

$$\begin{aligned} &\int_0^{(1-i)\sqrt{t}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{\mathrm{ch}\,\theta - \cos\theta} + \tfrac{2}{\theta}\Big)d\theta \\ &= 2\int_0^{\sqrt{t}} \big(\tfrac{1}{\theta} - \mathrm{cotg}\,\theta\big)d\theta = 2\log\Big[\tfrac{\sqrt{t}}{\sin\sqrt{t}}\Big] \in \mathbb{R}\,, \end{aligned}$$

the first term belongs to $i\,\mathbb{R}$ and does not contribute when taking the real part of J^y_ε.

Let us verify that the last term ($\lim\limits_{R\to\infty}\cdots$) vanishes: its modulus is controlled by

$$\begin{aligned}&\limsup_{R\to\infty}\ \sup_{0\le t\le(\pi-\nu)^2}\ \exp\left[\Re\left\{\int_0^{\sqrt{R-2it}}\left(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\right)d\theta\right\}\right]\\&=\limsup_{R\to\infty}\ e^{Z_R},\end{aligned}$$

with

$$\begin{aligned}Z_R&=\Re\left\{\int_0^{R+i\,\mathcal{O}(1/R)}\left(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\right)d\theta\right\}\\&=\int_0^R\left(\tfrac{1}{2\theta}-\tfrac{\mathrm{sh}\,\theta+\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}\right)d\theta+\Re\left\{\int_R^{R+i\,\mathcal{O}(R^{-1})}\left(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\right)d\theta\right\}\\&=\tfrac14\log\left[\tfrac{R^2}{\mathrm{ch}\,R-\cos R}\right]+\Re\left\{\tfrac{i}{2}\int_0^{\mathcal{O}(R^{-1})}\left(\tfrac{(i-1)\,\mathrm{sh}\,[R+it]-(i+1)\sin[R+it]}{2\,(\mathrm{ch}\,[R+it]-\cos[R+i])}+\tfrac{1}{R+it}\right)dt\right\}\\&=\tfrac14\log\left[\tfrac{R^2}{\mathrm{ch}\,R-\cos R}\right]+\mathcal{O}(R^{-1})\ \longrightarrow-\infty\quad\text{as } R\to\infty\,.\end{aligned}$$

Thus we have

$$\Re\{J_\varepsilon^y\}=\exp\left[-\frac{(\pi-\nu)^2y}{2\,\varepsilon}\right]\times\Big(\Re\{\tilde J_0^\varepsilon\}+\Re\{\tilde J_\varepsilon^\infty\}\Big),\tag{22}$$

where (for $\varepsilon\ll\tilde\varepsilon=o(1)$ to be specified later)

$$\begin{aligned}\tilde J_0^\varepsilon&:=\int_0^{\tilde\varepsilon}\exp\left[\frac{-y}{4\varepsilon}ix+\int_0^{\sqrt{x-2i(\pi-\nu)^2}}\left(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)}+\tfrac{1}{2\theta}\right)d\theta\right]dx\\&=\int_0^{\tilde\varepsilon}\exp\left[\frac{-y}{4\varepsilon}ix+\tfrac12\int_0^{(\pi-\nu)\sqrt{1+\frac{ix}{2(\pi-\nu)^2}}}\left(\tfrac1\theta-\mathrm{cotg}\,\theta\right)d\theta\right]dx\,,\end{aligned}\tag{23}$$

and $\tilde J_\varepsilon^\infty:=\int_{\tilde\varepsilon}^\infty$ *same integrand* is the residual integral. Note that as in the beginning of Section 4.4, the second term in the above exponent is bounded for $0\le x\le 1$.

5.2. Handling the dominant term $\tilde J_0^\varepsilon$. —

According to (23), we have

$$\tilde J_0^\varepsilon=\int_0^{\tilde\varepsilon}e^{\frac{-y}{4\varepsilon}\,ix}\left[\frac{\sqrt{(\pi-\nu)^2+\frac{ix}{2}}}{\sin\sqrt{(\pi-\nu)^2+\frac{ix}{2}}}\right]^{1/2}dx\,.$$

Set $\nu':=2\pi\nu-\nu^2=\varepsilon/y+\mathcal{O}(\varepsilon^2)$ and write

$$\sin\sqrt{(\pi-\nu)^2+\tfrac{ix}{2}}=\sin\left[\pi-\sqrt{(\pi-\nu)^2+\tfrac{ix}{2}}\right]=\sin\left[\frac{\nu'-\frac{ix}{2}}{\pi+\sqrt{(\pi-\nu)^2+\frac{ix}{2}}}\right].$$

Then, for any $\tilde{\varepsilon} = o(1)$, such that $\varepsilon = o(\tilde{\varepsilon})$, changing x into $2\nu' x$ we have

$$\begin{aligned}
\tilde{J}_0^\varepsilon &= 2\nu' \int_0^{\tilde{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon}} \left[\frac{\sqrt{(\pi-\nu)^2 + i\nu' x}}{\sin\left[\frac{\nu'(1-ix)}{\pi+\sqrt{(\pi-\nu)^2+i\nu' x}}\right]} \right]^{1/2} dx \\
&= 2\nu' \int_0^{\tilde{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon}} \sqrt{Y}\, dx\,,
\end{aligned}$$

where

$$X := \frac{\nu' x}{(\pi-\nu)^2},$$

$$U := \sqrt{(\pi-\nu)^2 + i\nu' x} = (\pi-\nu)\left[\sqrt{\frac{\sqrt{1+X^2}+1}{2}} + i\sqrt{\frac{\sqrt{1+X^2}-1}{2}}\right],$$

$$V := \frac{\nu'(1-ix)}{\pi+U}, \quad Y := \frac{U}{\sin V}, \quad Z := \frac{U}{V}.$$

We have $0 \le \nu' x \le \tilde{\varepsilon}/2$, $|U-\pi| = \mathcal{O}(\tilde{\varepsilon})$, $\frac{\varepsilon}{2y} < 2\pi|V| < \tilde{\varepsilon}$, and $\left|\sqrt{Y}-\sqrt{Z}\right| = \left|\frac{Y-Z}{\sqrt{Y}+\sqrt{Z}}\right| \le \frac{|V|^2}{|\sin V|\left|\sqrt{Y}+\sqrt{Z}\right|} = \mathcal{O}\big(|V|^{3/2}\big) = \mathcal{O}\big(\tilde{\varepsilon}^{3/2}\big)$. Hence, integrating by parts we obtain

$$\begin{aligned}
\tilde{J}_0^\varepsilon &= 2\nu' \int_0^{\tilde{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon}} \left[\sqrt{Z} + \mathcal{O}\big(\tilde{\varepsilon}^{3/2}\big)\right] dx \\
&= \mathcal{O}\big(\tilde{\varepsilon}^{5/2}\big) + 2\sqrt{\nu'} \int_0^{\tilde{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon}} \sqrt{\tfrac{U^2+\pi U}{1-ix}}\, dx \\
&= \mathcal{O}\big(\tilde{\varepsilon}^{5/2}\big) + \tfrac{4\varepsilon\, i}{y\sqrt{\nu'}} \left[e^{-\frac{iy\tilde{\varepsilon}}{4\varepsilon}} \sqrt{\tfrac{(U^2+\pi U)|_{\{\nu' x = \tilde{\varepsilon}/2\}}}{1-i\tilde{\varepsilon}/2\nu'}} - \sqrt{(\pi-\nu)(2\pi-\nu)} \right] \\
&\quad - \tfrac{4\varepsilon\, i}{y\sqrt{\nu'}} \int_0^{\tilde{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon}} \frac{d}{dx} \sqrt{\tfrac{U^2+\pi U}{1-ix}}\, dx \\
&= \mathcal{O}\big(\tilde{\varepsilon}^{5/2}\big) + \mathcal{O}\big(\varepsilon/\sqrt{\tilde{\varepsilon}}\big) - \tfrac{4\varepsilon\sqrt{2\pi^2-3\pi\nu+\nu^2}}{y\sqrt{\nu'}}\, i \\
&\quad - \frac{4\varepsilon\, i}{y\sqrt{\nu'}} \int_0^{\tilde{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon}} \frac{d}{dx} \sqrt{\tfrac{U^2+\pi U}{1-ix}}\, dx\,.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{dx}\sqrt{\tfrac{U^2+\pi U}{1-ix}} &= \frac{i\sqrt{U^2+\pi U}}{2(1-ix)^{3/2}} + \frac{\frac{d}{dx}(U^2+\pi U)}{2\sqrt{U^2+\pi U}\sqrt{1-ix}} \\
&= \frac{i(1+ix)^{3/2}\big(\pi\sqrt{2}+\mathcal{O}(\tilde{\varepsilon})\big)}{2\,(1+x^2)^{3/2}} + \mathcal{O}(\varepsilon) \\
&= \frac{(i-x)\big(\pi+\mathcal{O}(\tilde{\varepsilon})\big)}{\sqrt{2}\,(1+x^2)}\left[\sqrt{\frac{\sqrt{1+x^2}+1}{2\,(1+x^2)}} + i\sqrt{\frac{\sqrt{1+x^2}-1}{2\,(1+x^2)}}\right] + \mathcal{O}(\varepsilon) \\
&= \frac{\pi+\mathcal{O}(\tilde{\varepsilon})}{2\,(1+x^2)}\left[i\left(\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} - x\sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\right. \\
&\qquad\left. - \left(x\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} + \sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\right] + \mathcal{O}(\varepsilon)\,.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\Re\{\tilde{J}_0^\varepsilon\} &= \mathcal{O}\big(\tilde{\varepsilon}^{5/2}+\varepsilon/\sqrt{\tilde{\varepsilon}}\big) + \frac{2\pi\,\varepsilon}{y\sqrt{\nu'}}\int_0^{\tilde{\varepsilon}/2\nu'}\left[\frac{G(x)}{1+x^2}+\frac{\mathcal{O}(\tilde{\varepsilon})}{1+x^{3/2}}+\mathcal{O}(\varepsilon)\right]dx \\
&= \mathcal{O}\big(\tilde{\varepsilon}^{5/2}+\varepsilon/\sqrt{\tilde{\varepsilon}}+\sqrt{\varepsilon}\,\tilde{\varepsilon}\big) + \frac{2\pi\,\varepsilon}{y\sqrt{\nu'}}\int_0^{\tilde{\varepsilon}/2\nu'}\frac{G(x)}{1+x^2}\,dx\,,
\end{aligned}$$

with

$$\begin{aligned}
G(x) = &\left(\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} - x\sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\cos\left[\frac{y\nu' x}{2\,\varepsilon}\right] \\
&+\left(x\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} + \sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\sin\left[\frac{y\nu' x}{2\,\varepsilon}\right].
\end{aligned}$$

Since $\dfrac{y\nu' x}{2\,\varepsilon} = \dfrac{x}{2}+\mathcal{O}(\varepsilon)$, we also have $G(x) = F(x)+\mathcal{O}(\varepsilon)\sqrt{x}$, with

$$\begin{aligned}
F(x) &= \left(\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} - x\sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\cos\big(\tfrac{x}{2}\big) \\
&\quad+\left(x\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} + \sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\sin\big(\tfrac{x}{2}\big) \\
&= \sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}}\left(2-\sqrt{1+x^2}\right)\cos\big(\tfrac{x}{2}\big) \\
&\quad+\sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\left(2+\sqrt{1+x^2}\right)\sin\big(\tfrac{x}{2}\big)\,.
\end{aligned} \tag{24}$$

Moreover,

$$\int_{\tilde{\varepsilon}/2\nu'}^{\infty}\frac{F(x)}{1+x^2}\,dx = \mathcal{O}(\tilde{\varepsilon}/\nu')^{-3/2} = \mathcal{O}(\varepsilon/\tilde{\varepsilon})^{3/2}\,.$$

Therefore we obtain

$$\Re\{\tilde{J}_0^\varepsilon\} = 2\pi\sqrt{\frac{\varepsilon}{y}}\,K + \mathcal{O}\big(\tilde{\varepsilon}^{5/2} + \varepsilon/\sqrt{\tilde{\varepsilon}} + \varepsilon^2/\tilde{\varepsilon}^{3/2} + \sqrt{\varepsilon}\,\tilde{\varepsilon}\big),$$
$$\text{with} \quad K = \int_0^\infty \frac{F(x)}{1+x^2}\,dx\,.$$

Performing the change of variable $x = \operatorname{tg}\theta$, we find

$$K = \int_0^{\pi/2} \Big(\big[\cos\big(\tfrac{\theta}{2}\big) - \operatorname{tg}\theta\sin\big(\tfrac{\theta}{2}\big)\big]\cos\big(\tfrac{\operatorname{tg}\theta}{2}\big) + \big[\operatorname{tg}\theta\cos\big(\tfrac{\theta}{2}\big) + \sin\big(\tfrac{\theta}{2}\big)\big]\sin\big(\tfrac{\operatorname{tg}\theta}{2}\big)\Big)$$
$$\times \sqrt{2\cos\theta}\,d\theta$$
$$= \int_0^{\pi/2} \cos\big(\tfrac{\operatorname{tg}\theta - 3\theta}{2}\big)\sqrt{\tfrac{2}{\cos\theta}}\,d\theta\,.$$

Hence, taking $\tilde{\varepsilon} = \varepsilon^{1/3}$ we obtain

$$\Re\{\tilde{J}_0^\varepsilon\} = 2\pi\sqrt{\frac{\varepsilon}{y}}\,K + \mathcal{O}\big(\varepsilon^{5/6}\big), \tag{25}$$
$$\text{with} \quad K = \int_0^\infty \frac{F(x)}{1+x^2}\,dx = \int_0^{\pi/2}\cos\big(\tfrac{\operatorname{tg}\theta-3\theta}{2}\big)\sqrt{\tfrac{2}{\cos\theta}}\,d\theta\,.$$

LEMMA 5.1. — *We have $K>0$, and actually $K \approx 2.15$.*

Proof. — As θ runs $[0,\pi/2[$, $(\operatorname{tg}\theta - 3\theta)$ decreases from 0 to $\sqrt{2} - 3\operatorname{arctg}(\sqrt{2}) > -1.46$, and then increases to ∞, with $\operatorname{tg}(1.4) - 3(1.4) < 1.6$. Hence for $0 \le \theta \le 1.4$ we have $|\operatorname{tg}\theta - 3\theta| < 1.6$ and then $\cos\big(\frac{\operatorname{tg}\theta-3\theta}{2}\big) > \cos(0.8) > 0.69$. Therefore,

$$\begin{aligned}
K &> 0.69\int_0^{1.4}\sqrt{\tfrac{2}{\cos\theta}}\,d\theta - \int_{1.4}^{\pi/2}\sqrt{\tfrac{2}{\cos\theta}}\,d\theta \\
&= 0.69\int_{\frac{\pi}{2}-1.4}^{\frac{\pi}{2}}\sqrt{\tfrac{2}{\sin\theta}}\,d\theta - \int_0^{\frac{\pi}{2}-1.4}\sqrt{\tfrac{2}{\sin\theta}}\,d\theta \\
&> 0.69\int_{\frac{\pi}{2}-1.4}^{1}\sqrt{\tfrac{2}{\theta}}\,d\theta + 0.69\int_1^{\frac{\pi}{2}}\sqrt{2}\,d\theta - \int_0^{\frac{\pi}{2}-1.4}\sqrt{\tfrac{\pi-2.8}{\theta\sin\left(\frac{\pi}{2}-1.4\right)}}\,d\theta \\
&= 1.38\,(\sqrt{2}-\sqrt{\pi-2.8}) + 0.69\,\big(\tfrac{\pi}{2}-1\big)\sqrt{2} - (\pi-2.8)\sqrt{\tfrac{2}{\sin\left(\frac{\pi}{2}-1.4\right)}} > 0.53\,.
\end{aligned}$$

□

REMARK 5.2. — Integrating by parts using

$$-2\frac{d}{dx}\sqrt{\tfrac{\sqrt{1+x^2}\pm1}{1+x^2}} = \sqrt{\tfrac{\sqrt{1+x^2}\mp1}{(1+x^2)^3}}\big(\sqrt{1+x^2}\pm2\big),$$

we also have

$$K = \int_0^\infty \left[\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}}\, \cos\left(\tfrac{x}{2}\right) + \sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\, \sin\left(\tfrac{x}{2}\right) \right] dx$$
$$= \int_0^{\pi/2} \cos\left(\tfrac{\operatorname{tg}\theta-\theta}{2}\right) \sqrt{\tfrac{2}{\cos^3\theta}}\, d\theta\,.$$

5.3. Control of the residual term $\tilde{J}_\varepsilon^\infty$. — For any positive x, set

$$\alpha := \sqrt{\frac{\sqrt{1+\frac{x^2}{4(\pi-\nu)^4}}+1}{2}} \quad \text{and} \quad \beta := \sqrt{\frac{\sqrt{1+\frac{x^2}{4(\pi-\nu)^4}}-1}{2}}\,,$$

so that $\sqrt{1+\frac{i\,x}{2(\pi-\nu)^2}} = \alpha + i\beta$, and parametrize the integral in the exponent of (23) by

$$x = 2(\pi-\nu)^2\operatorname{sh}(2t), \quad \alpha = \operatorname{ch} t, \quad \beta = \operatorname{sh} t, \quad t\in\mathbb{R}_+\,;$$

so that $x \geq \varepsilon^{1/3} \Leftrightarrow t > \varepsilon' = \frac{\varepsilon^{1/3}}{4\,\pi^2} + \mathcal{O}(\varepsilon)$, and setting $U_t := (\pi-\nu)(\operatorname{ch} t + i\operatorname{sh} t)$, the residual integral in (22) reads

(26)

$$\tilde{J}_\varepsilon^\infty = 4(\pi-\nu)^2 \int_{\varepsilon'}^\infty \exp\left[\frac{(\pi-\nu)^2 y}{2ie}\operatorname{sh}(2t) + \frac{1}{2}\int_0^{U_t}\left(\tfrac{1}{\theta} - \operatorname{cotg}\theta\right)d\theta\right]\operatorname{ch}(2t)\,dt\,.$$

Now

$$\int_0^{U_t}\left(\tfrac{1}{\theta}-\operatorname{cotg}\theta\right)d\theta$$
$$\equiv \int_0^{U_t} d\left(\log\tfrac{\theta}{\sin\theta}\right)$$
$$= \int_0^t \left(\frac{1}{\operatorname{ch} s + i\operatorname{sh} s} - (\pi-\nu)\,\frac{\sin[2(\pi-\nu)\operatorname{ch} s] - i\operatorname{sh}[2(\pi-\nu)\operatorname{sh} s]}{\operatorname{ch}[2(\pi-\nu)\operatorname{sh} s] - \cos[2(\pi-\nu)\operatorname{ch} s]}\right)$$
$$\times (\operatorname{sh} s + i\operatorname{ch} s)\,ds$$
$$= \int_0^t \Bigg(\tfrac{\operatorname{sh}(2s)+i}{\operatorname{ch}(2s)}$$
$$- \tfrac{\sin[2(\pi-\nu)\operatorname{ch} s]\operatorname{sh} s + \operatorname{sh}[2(\pi-\nu)\operatorname{sh} s]\operatorname{ch} s + i[\sin(2[\pi-\nu)\operatorname{ch} s]\operatorname{ch} s - \operatorname{sh}[2(\pi-\nu)\operatorname{sh} s]\operatorname{sh} s]}{\left(\operatorname{ch}[2(\pi-\nu)\operatorname{sh} s] - \cos[2(\pi-\nu)\operatorname{ch} s]\right)/(\pi-\nu)}\Bigg)\,ds\,,$$

and then
(27)

$$
\begin{aligned}
&\Re\left\{\int_0^{U_t} d\left(\log\tfrac{\theta}{\sin\theta}\right)\right\}\\
&= \int_0^t \left(\mathrm{th}(2s) - (\pi-\nu)\,\frac{\sin[2(\pi-\nu)\mathrm{ch}\, s]\,\mathrm{sh}\, s + \mathrm{sh}[2(\pi-\nu]\mathrm{sh}\, s)\,\mathrm{ch}\, s}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\, s] - \cos[2(\pi-\nu)\,\mathrm{ch}\, s]}\right) ds\\
&= \frac{\log\mathrm{ch}(2t)}{2} - \frac{\pi-\nu}{2}\, I_t,\\
&\qquad \text{with}\quad I_t := 2\int_0^t \frac{\sin[2(\pi-\nu)\mathrm{ch}\, s]\,\mathrm{sh}\, s + \mathrm{sh}[2(\pi-\nu)\mathrm{sh}\, s]\,\mathrm{ch}\, s}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\, s] - \cos[2(\pi-\nu)\,\mathrm{ch}\, s]}\, ds\,,\\
&\Im\left\{\int_0^{U_t} d\left(\log\tfrac{\theta}{\sin\theta}\right)\right\}\\
&= \mathrm{Arctg}(\mathrm{th}\, t) + (\pi-\nu)\int_0^t \frac{\mathrm{sh}[2(\pi-\nu)\mathrm{sh}\, s]\,\mathrm{sh}\, s - \sin[2(\pi-\nu)\mathrm{ch}\, s]\,\mathrm{ch}\, s}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\, s] - \cos[2(\pi-\nu)\,\mathrm{ch}\, s]}\, ds\,.
\end{aligned}
$$

The proof of the following technical lemma is given in the appendix (Section 7).

Lemma 5.3. — *For any positive constant $T>0$, there exists a positive constant ε_0 such that for $0<\varepsilon<\varepsilon_0$ and $t\geq T$, we have $\Re\{U_t\,\mathrm{cotg}\, U_t\} \geq \frac{\pi-\nu}{2}\,\mathrm{sh}\, t$. We also have $I_t \geq \mathrm{sh}\, t$ for $0<\varepsilon<\varepsilon_0$ and for any $t\geq 0$.*

Integrating then by parts in (26), we obtain

$$
\begin{aligned}
\tilde{J}_\varepsilon^\infty &= \frac{4i\varepsilon}{y}\int_{\varepsilon'}^\infty \exp\left[\tfrac12\int_0^{U_t}\left(\tfrac{1}{\theta} - \mathrm{cotg}\,\theta\right)d\theta\right] d\left(\exp\left[\frac{(\pi-\nu)^2 y}{2i\varepsilon}\,\mathrm{sh}(2t)\right]\right)\\
&= \frac{4i\varepsilon}{y}\lim_{T\to\infty}\exp\left[\frac{(\pi-\nu)^2 y}{2i\varepsilon}\,\mathrm{sh}(2T) + \tfrac12\int_0^{U_T} d\left(\log\tfrac{\theta}{\sin\theta}\right)\right]\\
&\quad - \frac{4i\varepsilon}{y}\exp\left[\frac{(\pi-\nu)^2 y}{2i\varepsilon}\,\mathrm{sh}(2\varepsilon') + \tfrac12\int_0^{U_{\varepsilon'}} d\left(\log\tfrac{\theta}{\sin\theta}\right)\right]\\
&\quad + \frac{2(\pi-\nu)\,\varepsilon}{iy}\int_{\varepsilon'}^\infty \exp\left[\frac{(\pi-\nu)^2 y}{2i\varepsilon}\,\mathrm{sh}(2t) + \tfrac12\int_0^{U_t} d\left(\log\tfrac{\theta}{\sin\theta}\right)\right](\mathrm{sh}\, t + i\,\mathrm{ch}\, t)\\
&\qquad\qquad \times\left(\tfrac{1}{U_t} - \mathrm{cotg}\, U_t\right) dt\,,
\end{aligned}
$$

so that, using (27) and then Lemma 5.3 (and the expression of $\operatorname{cotg} U_t$ in its proof), we obtain

$$\begin{aligned}\big|\tilde{J}_\varepsilon^\infty\big| &= \mathcal{O}(\varepsilon)\lim_{T\to\infty}\Big\{\big[\mathrm{ch}(2T)\big]^{1/4}\,e^{-\frac{\pi-\nu}{4}\,I_T}\Big\} + \mathcal{O}(\varepsilon)\big[\mathrm{ch}(2\varepsilon')\big]^{1/4}\,e^{-\frac{\pi-\nu}{4}\,I_{\varepsilon'}}\\ &\quad + \mathcal{O}(\varepsilon)\int_{\varepsilon'}^{\infty} e^{-\frac{\pi-\nu}{4}\,I_t}\,\big[\mathrm{ch}(2t)\big]^{3/4}\Big(\big|U_t^{-1}\big| + \big|\operatorname{cotg} U_t\big|\Big)\,dt\\ &= \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)\int_{\varepsilon'}^{\infty} e^{\frac{3t}{2}-\frac{\pi-\nu}{4}\,\mathrm{sh}\,t}\left(\tfrac{1}{\sqrt{\mathrm{ch}(2t)}} + \sqrt{\tfrac{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\,t]+\cos[2(\pi-\nu)\,\mathrm{ch}\,t]}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\,t]-\cos[2(\pi-\nu)\,\mathrm{ch}\,t]}}\right)dt\\ &= \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)\int_{\eta}^{\infty} e^{\frac{3t}{2}-\frac{\pi-\nu}{4}\,\mathrm{sh}\,t}\,dt\\ &\quad + \mathcal{O}(\varepsilon)\int_{\frac{\varepsilon^{1/3}}{4\,\pi^2}+\mathcal{O}(\varepsilon)}^{\eta}\frac{dt}{\sqrt{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\,t]-\cos[2(\pi-\nu)\,\mathrm{ch}\,t]}} = \mathcal{O}\big(\varepsilon\log\tfrac{1}{\varepsilon}\big)\end{aligned}$$

since we have $\cos\big[2(\pi-\nu)\,\mathrm{ch}\,t\big] = 1-2\nu^2+2\pi\nu t^2+\mathcal{O}\big(t^4+\nu^2t^2+\nu^4\big)$ and then

$$\mathrm{ch}\;[2(\pi-\nu)\,\mathrm{sh}\,t] - \cos[2(\pi-\nu)\,\mathrm{ch}\,t] = 2\pi^2t^2+2\nu^2-2\pi\nu t^2+\mathcal{O}\big(t^2+\nu^2\big)^2$$

$\sim 2\pi^2t^2$ near $\varepsilon^{1/3}$ and $\geq \pi^2t^2$ on $[\varepsilon^{1/2},\eta]$, for any small enough positive constant η.

Thus, by (22), (25) and the above, and since $\nu = \dfrac{\varepsilon}{2\pi y} + \mathcal{O}(\varepsilon^2)$, we have

$$\Re\{J_\varepsilon^y\} = e^{-\frac{(\pi-\nu)^2 y}{2\,\varepsilon}}\left(2\pi K\sqrt{\frac{\varepsilon}{y}} + \mathcal{O}\big(\varepsilon^{5/6}\big)\right) = 2\pi K\sqrt{\frac{e\,\varepsilon}{y}}\,e^{-\frac{\pi^2 y}{2\,\varepsilon}}\,\big[1+\mathcal{O}\big(\varepsilon^{1/3}\big)\big].$$

According to (9) and (11), we thus have established Part B of Theorem 2.1.

6. The unscaled asymptotics of $p_\varepsilon\big(0;(w,y)\big)$

6.1. Unscaled saddle points. — We again use Proposition 3.4, but under the change $y\mapsto y/\varepsilon$, to switch from the previous scaled case to the present unscaled one. According to (9), we have to evaluate $\Re\Big\{J_\varepsilon^{y/\varepsilon}\Big\}$, given by (11). Then the corresponding saddle-point equation, in the sub-case $w\neq 0$, is $h'+i\,H' = y/\varepsilon$. Thus Equation (12) becomes: $2iq$ is a saddle-point if and only if

$$\frac{y}{w^2\,\varepsilon} = \frac{\mathrm{sh}(2\sqrt{q})-2\sqrt{q}}{2\sqrt{q}\,\big(\mathrm{ch}(2\sqrt{q})-1\big)}\,. \tag{28}$$

As observed in Section 4.1 and already implemented in Section 5.1, this corresponds to $q\approx -\pi^2$, roughly. So that the unscaled sub-case $w\neq 0$ shares features of both scaled sub-cases $w\neq 0$ and $w=0$. Precisely, looking for a saddle-point under the form $\chi = -2i(\pi-\nu)^2$ and $\sqrt{\chi} = (1-i)(\pi-\nu)$, for a small positive $\nu \equiv \nu(\varepsilon, y, w)$, we find that (28) reads:

$$\sin\nu = \sqrt{\frac{w^2\varepsilon}{2\,y}}\sqrt{1+\frac{\sin\nu\cos\nu}{\pi-\nu}}, \quad \text{whence}$$

$$\nu = \sqrt{\frac{w^2\varepsilon}{2\,y}} + \frac{w^2\varepsilon}{4\pi y} + \tfrac{4\pi^2+15}{24\,\pi^2}\left[\frac{w^2\varepsilon}{2\,y}\right]^{\frac{3}{2}} + \tfrac{w^4\varepsilon^2}{4\pi^3 y^2} + \mathcal{O}\big(\varepsilon^{\frac{5}{2}}\big).$$

As to the unscaled sub-case $w = 0$, Equation (21) becomes

$$(29)\quad 2y\,(\pi-\nu)^2 = \varepsilon^2\,(1+(\pi-\nu)\operatorname{cotg}\nu)\,, \quad \text{i.e.,} \quad \operatorname{tg}\nu = \frac{\varepsilon^2\,(1-\nu/\pi)}{2\pi y\left(1-\frac{\varepsilon^2}{2\pi^2 y}-\frac{2\nu}{\pi}+\frac{\nu^2}{\pi^2}\right)}\,,$$

which has a unique solution $\nu = \frac{\varepsilon^2}{2\pi y} + \frac{\varepsilon^4}{2\pi^3 y^2} + \mathcal{O}(\varepsilon^6)$, $\chi = -2i\big(\pi^2 - \frac{\varepsilon^2}{y} + \mathcal{O}(\varepsilon^4)\big)$.

6.2. The unscaled pseudo-cut-locus sub-case $w = 0$. — By performing the same change of contour as in Section 5.1, with the above value for ν replacing the previous one, we get the following analogue of (22):

$$(30)\qquad \Re\{J_\varepsilon^{y/\varepsilon}\} = \exp\left[\frac{-y}{2\,\varepsilon^2}(\pi-\nu)^2\right] \times \left(\Re\{\hat{J}_0^\varepsilon\} + \Re\{\hat{J}_\varepsilon^\infty\}\right),$$

with (for $\hat{\varepsilon} = o(1)$ to be specified below)

$$(31)\qquad \hat{J}_0^\varepsilon := \int_0^{\hat{\varepsilon}} \exp\left[-\frac{ixy}{4\varepsilon^2} + \tfrac{1}{2}\int_0^{(\pi-\nu)\sqrt{1+\frac{ix}{2(\pi-\nu)^2}}} \left(\tfrac{1}{\theta} - \operatorname{cotg}\theta\right)d\theta\right]dx\,,$$

and $\hat{J}_\varepsilon^\infty := \displaystyle\int_{\hat{\varepsilon}}^\infty$ *same integrand* is the residual integral. Then we follow the analogous Section 5.2. According to (31) we have

$$\hat{J}_0^\varepsilon = \int_0^{\hat{\varepsilon}} e^{-\frac{ixy}{4\varepsilon^2}}\left[\frac{\sqrt{(\pi-\nu)^2+\frac{ix}{2}}}{\sin\sqrt{(\pi-\nu)^2+\frac{ix}{2}}}\right]^{1/2} dx\,.$$

Again set $\nu' = 2\pi\nu - \nu^2 = \varepsilon^2/y + \mathcal{O}(\varepsilon^4)$. For any $\hat{\varepsilon} = o(1)$ such that $\nu = o(\hat{\varepsilon})$, we have

$$\hat{J}_0^\varepsilon = 2\nu'\int_0^{\hat{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon^2}}\left[\frac{\sqrt{(\pi-\nu)^2+i\nu' x}}{\sin\left[\frac{\nu'(1-ix)}{\pi+\sqrt{(\pi-\nu)^2+i\nu' x}}\right]}\right]^{1/2} dx = 2\nu'\int_0^{\hat{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon^2}}\sqrt{Y}\,dx$$

$$= 2\nu'\int_0^{\hat{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon^2}}\left[\sqrt{Z} + \mathcal{O}\big(\hat{\varepsilon}^{3/2}\big)\right]dx$$

$$= \mathcal{O}\big(\hat{\varepsilon}^{5/2}\big) + 2\sqrt{\nu'}\int_0^{\hat{\varepsilon}/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon^2}}\sqrt{\frac{U^2+\pi U}{1-ix}}\,dx$$

$$= \mathcal{O}\big(\hat\varepsilon^{5/2}\big) + \tfrac{4\varepsilon^2 i}{y\sqrt{\nu'}}\left[e^{-\frac{iy\hat\varepsilon}{4\varepsilon^2}}\sqrt{\tfrac{(U^2+\pi U)|_{\{\nu' x=\hat\varepsilon/2\}}}{1-i\hat\varepsilon/2\nu'}} - \sqrt{(\pi-\nu)(2\pi-\nu)}\right]$$

$$- \tfrac{4\varepsilon^2 i}{y\sqrt{\nu'}}\int_0^{\hat\varepsilon/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon^2}}\,\frac{d}{dx}\sqrt{\tfrac{U^2+\pi U}{1-ix}}\,dx$$

$$= \mathcal{O}\big(\hat\varepsilon^{5/2} + \varepsilon^2/\sqrt{\hat\varepsilon}\big) - \tfrac{4\varepsilon^2\sqrt{2\pi^2-3\pi\nu+\nu^2}}{y\sqrt{\nu'}}\,i$$

$$- \tfrac{4\varepsilon^2 i}{y\sqrt{\nu'}}\int_0^{\hat\varepsilon/2\nu'} e^{-\frac{ixy\nu'}{2\varepsilon^2}}\,\frac{d}{dx}\sqrt{\tfrac{U^2+\pi U}{1-ix}}\,dx\,.$$

Now the control of the last term above is also as in Section 5.2, up to replacing $\mathcal{O}(\varepsilon)$ by $\mathcal{O}(\nu) = \mathcal{O}(\varepsilon^2)$. Hence we obtain

$$\Re\{\hat J_0^\varepsilon\} = \mathcal{O}\big(\hat\varepsilon^{5/2} + \varepsilon^2/\sqrt{\hat\varepsilon}\big) + \frac{2\pi\,\varepsilon^2}{y\sqrt{\nu'}}\int_0^{\hat\varepsilon/2\nu'}\left[\frac{\hat G(x)}{1+x^2} + \frac{\mathcal{O}(\hat\varepsilon)}{1+x^{3/2}} + \mathcal{O}\big(\varepsilon^2\big)\right]dx\,,$$

with

$$\hat G(x) = \left(\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} - x\sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\cos\left[\tfrac{y\nu' x}{2\,\varepsilon^2}\right]$$
$$+ \left(x\sqrt{\tfrac{\sqrt{1+x^2}+1}{1+x^2}} + \sqrt{\tfrac{\sqrt{1+x^2}-1}{1+x^2}}\right)\sin\left[\tfrac{y\nu' x}{2\,\varepsilon^2}\right].$$

Since $\frac{y\nu' x}{2\,\varepsilon^2} = \frac{x}{2} + \mathcal{O}(\varepsilon^2)$, we have $\hat G(x) = F(x) + \mathcal{O}(\varepsilon^2)\sqrt{x}$, with the same function F as in Section 5.2, and $\int_{\hat\varepsilon/2\nu'}^\infty \frac{F(x)}{1+x^2}\,dx = \mathcal{O}(\hat\varepsilon/\nu')^{-3/2} = \mathcal{O}\big(\varepsilon^2/\hat\varepsilon\big)^{3/2}$. Therefore we obtain

$$\Re\{\hat J_0^\varepsilon\} = \frac{2\pi\,\varepsilon}{\sqrt{y}}\,K + \mathcal{O}\big(\hat\varepsilon^{5/2} + \varepsilon^2/\sqrt{\hat\varepsilon} + \varepsilon^3/\hat\varepsilon^{3/2} + \varepsilon\,\hat\varepsilon\big).$$

Hence, taking $\hat\varepsilon = \varepsilon^{2/3}$ we obtain

$$\Re\{\hat J_0^\varepsilon\} = \frac{2\pi\,K\,\varepsilon}{\sqrt{y}} + \mathcal{O}\big(\varepsilon^{5/3}\big), \quad \text{with} \quad K \text{ as before, in (25)}\,. \tag{32}$$

Then the control of the residual term $\hat J_\varepsilon^\infty$ is straight forwardly adapted from Section 5.3, to yield $|\hat J_\varepsilon^\infty| = \mathcal{O}\big(\varepsilon^2\log\frac{1}{\varepsilon}\big)$. By (30), (32), and since $\nu = \dfrac{\varepsilon^2}{2\pi y} + \mathcal{O}(\varepsilon^4)$, we have

$$\Re\{J_\varepsilon^{y/\varepsilon}\} = e^{\frac{-y}{2\varepsilon^2}(\pi-\nu)^2}\left(\frac{2\pi K\,\varepsilon}{\sqrt{y}} + \mathcal{O}\big(\varepsilon^{5/3}\big)\right) = \frac{2\pi K\,\varepsilon}{\sqrt{y/e}}\,e^{\frac{-\pi^2 y}{2\,\varepsilon^2}}\big[1 + \mathcal{O}\big(\varepsilon^{2/3}\big)\big].$$

Hence, according to (9) and (11), we have established Part D of Theorem 2.1.

6.3. The unscaled sub-case $w \neq 0$. — Perform the same change of contour as in Sections 4.2 and 5.1, starting from (11) written as

$$J_\varepsilon^{y/\varepsilon} = \int_0^\infty \exp\left[\frac{i(h(x) - x\,y/\varepsilon) - H(x)}{4\,\varepsilon} + \int_0^{\sqrt{x}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta - (i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta - \cos\theta)} + \tfrac{1}{2\theta}\Big) d\theta\right] dx\,,$$

in which $w \neq 0$, and with $\nu = \sqrt{\frac{w^2\varepsilon}{2\,y} + \frac{w^2\,\varepsilon}{4\pi\,y} + \frac{4\pi^2+15}{24\,\pi^2}\big[\frac{w^2\varepsilon}{2\,y}\big]^{3/2} + \frac{w^4\varepsilon^2}{4\pi^3 y^2}} + \mathcal{O}\big(\varepsilon^{5/2}\big)$ of Section 6.1, which was derived from the saddle-point equation (28) by substituting $-q = (\pi-\nu)^2$.

The validity of Section 4.2 is not modified by the mere change of y into y/ε (for any fixed positive ε), so that, as seen with (16) and (17), the real part for $J_\varepsilon^{y/\varepsilon}$ is again unchanged under the replacement of its integration path $\mathbb{R}_+$ by $\big[-2i(\pi-\nu)^2 + \mathbb{R}_+\big[$.

Then Section 4.3 (actually 4.3.1 and 4.3.3) is adapted in the following way. According to Remark 3.5 and Lemma 3.7, for $0 \leq x \leq \check{\varepsilon} = o(\sqrt{\varepsilon})$ (to be specified later) we have

$$(i\,h - H)\big[x - 2i(\pi-\nu)^2\big] = 2w^2\Big[1 - \sqrt{(\pi-\nu)^2 + \tfrac{ix}{2}}\,\mathrm{cotg}\,\sqrt{(\pi-\nu)^2 + \tfrac{ix}{2}}\Big]\,.$$

Thus proceeding as in Section 5.1, with

$$\begin{aligned}\nu' = 2\pi\nu - \nu^2 &= \pi\sqrt{\frac{2w^2\varepsilon}{y}} + \tfrac{4\pi^2+3}{12\,\pi}\Big(\frac{w^2\varepsilon}{2\,y}\Big)^{3/2} - \tfrac{2\pi^2-3}{6\,\pi^2}\Big(\frac{w^2\varepsilon}{2\,y}\Big)^2 + \mathcal{O}\big(\varepsilon^{5/2}\big)\\ &= \pi\sqrt{\frac{2w^2\varepsilon}{y}}\left(1 + \tfrac{4\pi^2+3}{48\pi^2 y}\,w^2\varepsilon - \tfrac{2\pi^2-3}{12\,\pi^3}\Big(\frac{w^2\varepsilon}{2\,y}\Big)^{3/2} + \mathcal{O}(\varepsilon^2)\right)\end{aligned}$$

and $\check{\varepsilon} = o(\sqrt{\varepsilon})$ to be specified below, we have the following analogue of (22) and (30):

$$\Re\{J_\varepsilon^{y/\varepsilon}\} = \exp\left[-\frac{(\pi-\nu)^2 y}{2\,\varepsilon^2}\right] \times \Re\{\check{J}_0^\varepsilon + \check{J}_1^\varepsilon + \check{J}_2^\varepsilon\}\,, \tag{33}$$

6.3.1. *Handling the dominant term $\check{J}_0^\varepsilon$.* — with

$$\check{J}_0^\varepsilon = \int_0^{\check{\varepsilon}} \exp\left[\tfrac{(i\,h-H)\left[x-2i(\pi-\nu)^2\right]}{4\,\varepsilon} - \tfrac{i\,y\,x}{4\,\varepsilon^2} + \int_0^{\sqrt{x-2i(\pi-\nu)^2}} \Big(\tfrac{(i-1)\,\mathrm{sh}\,\theta-(i+1)\sin\theta}{4\,(\mathrm{ch}\,\theta-\cos\theta)} + \tfrac{1}{2\theta}\Big) d\theta\right] dx$$

(and $\check{J}_1^\varepsilon, \check{J}_2^\varepsilon = \int_{\check{\varepsilon}}^T, \int_T^\infty$ *same integrand*, for some small enough constant $T > 0$)

$$= 2\nu' \int_0^{\check{\varepsilon}/2\nu'} \exp\left[\tfrac{(i\,h-H)\left[2\nu' x - 2i(\pi-\nu)^2\right]}{4\,\varepsilon} - \tfrac{iy\nu' x}{2\,\varepsilon^2} + \tfrac{1}{2}\int_0^{(\pi-\nu)\sqrt{1+\frac{i\nu' x}{(\pi-\nu)^2}}} \left(\tfrac{1}{\theta} - \operatorname{cotg}\theta\right) d\theta \right] dx$$

$$= 2\nu' \int_0^{\check{\varepsilon}/2\nu'} \exp\left[\frac{w^2}{2\varepsilon}(1 - U \operatorname{cotg} U) - \frac{iy\nu' x}{2\,\varepsilon^2} + \tfrac{1}{2}\log\left(\frac{U}{\sin U}\right) \right] dx\,,$$

with the notation of Section 5.2. As in the beginning of Section 5.2, we then obtain

(34)

$$\check{J}_0^\varepsilon = 2\sqrt{\nu'}\, e^{\frac{w^2}{2\varepsilon}} \int_0^{\check{\varepsilon}/2\nu'} \exp\left[\frac{w^2 U}{2\varepsilon} \operatorname{cotg} V - \frac{iy\nu' x}{2\,\varepsilon^2} \right] \times \left[\sqrt{\frac{U^2 + \pi U}{1 - ix}} + \mathcal{O}\big(\check{\varepsilon}^{3/2}\sqrt{\nu'}\,\big) \right] dx$$

$$= 2\sqrt{\nu'}\, e^{\frac{w^2}{2\varepsilon}} \int_0^{\check{\varepsilon}/2\nu'} \exp\left[\frac{w^2 U}{2\varepsilon V}\left[1 - \tfrac{V^2}{3} + \mathcal{O}(|V|^4)\right] - \frac{iy\nu' x}{2\,\varepsilon^2} \right] \times \left[\sqrt{\tfrac{U^2+\pi U}{1-ix}} + \mathcal{O}\big(\check{\varepsilon}^{3/2}\varepsilon^{1/4}\big) \right] dx$$

$$= 2\sqrt{\nu'}\, e^{\frac{w^2}{2\varepsilon}} \int_0^{\check{\varepsilon}/2\nu'} e^{\Theta_\varepsilon(x)} \left[1 + \mathcal{O}\big(\check{\varepsilon}^3/\varepsilon\big)\right] \left[\sqrt{\frac{U^2 + \pi U}{1 - ix}} + \mathcal{O}\big(\check{\varepsilon}^{3/2}\,\varepsilon^{1/4}\big) \right] dx\,,$$

with

$$\Theta_\varepsilon(x) = \frac{\pi^2\, w^2}{\varepsilon\nu'(1 - ix)} \left[\frac{U(U+\pi)}{2\pi^2} - \frac{y\,\nu'^2(i+x)\,x}{2\pi^2 w^2\,\varepsilon} - \frac{U\nu'^2(1 - 2ix - x^2)}{6\pi^2\,(U+\pi)} \right].$$

The computational proof of the following is given in the appendix (Section 7).

Lemma 6.1. — *We have the following expansion: for $0 \le \nu' x \le \check{\varepsilon}/2$,*

$$\Theta_\varepsilon(x) = \pi\sqrt{\frac{w^2 y}{2\,\varepsilon^3}} - \frac{3w^2}{4\varepsilon} - \frac{(4\pi^2+3)|w|^3}{16\pi\sqrt{2y\,\varepsilon}} + \frac{(2\pi^2-3)w^4}{24\,\pi^2 y} + \mathcal{O}\big(\tfrac{\check{\varepsilon}^3}{\varepsilon^{3/2}} + \tfrac{\check{\varepsilon}^2}{\sqrt{\varepsilon}} + \check{\varepsilon} + \sqrt{\varepsilon}\big) + \frac{i\,w^4\,x}{24\,y\,(1-ix)} - \left[1 - \tfrac{4\pi^2+3}{48\pi^2 y}\,w^2\varepsilon + \tfrac{2\pi^2-3}{6\,\pi^3}\left(\tfrac{w^2\varepsilon}{2\,y}\right)^{3/2}\right]\sqrt{\frac{w^2 y}{2\varepsilon^3}} \times \frac{\pi\,x^2}{1-ix}\,.$$

Therefore, setting

$$D_\varepsilon(w,y) := \pi\sqrt{\frac{w^2 y}{2\,\varepsilon^3}} - \frac{w^2}{4\varepsilon} - \frac{(4\pi^2+3)|w|^3}{16\pi\sqrt{2y\,\varepsilon}} + \frac{(2\pi^2-3)w^4}{24\,\pi^2 y} = \frac{\pi\,w^2}{2\,\varepsilon\,\nu}\left[1 - \frac{w^2\varepsilon}{6\,y} + \mathcal{O}(\varepsilon^2)\right] \tag{35}$$

and changing x into $\varepsilon^{3/4}\,x$, according to (34) we have

$$\begin{aligned}\check{J}_0^\varepsilon &= 2\sqrt{\nu'} \times e^{D_\varepsilon(w,y)} \times \varepsilon^{3/4}\int_0^{\frac{\check\varepsilon\varepsilon^{-3/4}}{2\nu'}} \exp\left[\tfrac{i\,w^4\varepsilon^{3/4}\,x}{24\,y\,(1-i\,\varepsilon^{3/4}x)} - [1-\mathcal{O}(\varepsilon)]\tfrac{\sqrt{w^2y/2}\,\pi\,x^2}{1-i\,\varepsilon^{3/4}x}\right]\\ &\quad\times\left[\sqrt{\tfrac{2\pi^2}{1-i\,\varepsilon^{3/4}x}} + \mathcal{O}\big(\check\varepsilon^{3/2}\,\varepsilon^{1/4}\big)\right]\times\left[1+\mathcal{O}\big(\tfrac{\check\varepsilon^3}{\varepsilon^{3/2}} + \tfrac{\check\varepsilon^2}{\sqrt{\varepsilon}} + \check\varepsilon + \sqrt{\varepsilon}\big)\right]dx\\ &= (2\pi)^{3/2}\left(\tfrac{2w^2}{y}\right)^{1/4}\varepsilon \times e^{D_\varepsilon(w,y)} \times \check{I}_0^\varepsilon\,,\end{aligned}$$

and by taking $\check\varepsilon = \varepsilon^{3/4}$,

$$\begin{aligned}\check{I}_0^\varepsilon = \int_0^{\frac{1}{2\nu'}} &\exp\left[\tfrac{i\,w^4\varepsilon^{3/4}\,x}{24\,y\,(1-i\,\varepsilon^{3/4}x)} - [1-\mathcal{O}(\varepsilon)]\tfrac{\sqrt{w^2y/2}\,\pi\,x^2}{1-i\,\varepsilon^{3/4}x}\right]\\ &\times\left[\sqrt{\tfrac{1}{1-i\,\varepsilon^{3/4}x}} + \mathcal{O}_x\big(\varepsilon^{\,11/8}\big)\right]\left[1+\mathcal{O}_x\big(\varepsilon^{1/2}\big)\right]dx\end{aligned}$$

(the notation $\mathcal{O}_x(\cdot)$ emphasizes a dependence upon x, opposite to constant $\mathcal{O}(\cdot)$)

$$\begin{aligned}&= \int_0^{\frac{1+\mathcal{O}(\varepsilon)}{2\nu'}} \exp\left[-\frac{i\,\frac{w^4\varepsilon^{3/4}[1+\mathcal{O}(\varepsilon)]}{24\,y}\,x + \sqrt{\frac{w^2y}{2}}\,\pi\,x^2}{1-i\,\varepsilon^{3/4}[1+\mathcal{O}(\varepsilon)]\,x}\right]\frac{\left[1+\mathcal{O}_x\big(\sqrt{\varepsilon}\big)\right]dx}{\sqrt{1-i\,\varepsilon^{3/4}[1+\mathcal{O}(\varepsilon)]\,x}}\\ &= \int_0^{\frac{1+\mathcal{O}(\varepsilon)}{2\pi\sqrt{\frac{2w^2\varepsilon}{y}}}} \exp\left[\frac{\big(\varepsilon^{3/2}x^2 - i\,\varepsilon^{3/4}x\big)\Big(\frac{w^4[1+\mathcal{O}(\varepsilon)]}{24\,y} + \sqrt{\frac{w^2y}{2}}\,\pi\,x^2\Big)}{1+\varepsilon^{3/2}[1+\mathcal{O}(\varepsilon)]\,x^2}\right]\\ &\qquad\times\frac{e^{-\sqrt{\frac{w^2y}{2}}\pi x^2}\left[1+\mathcal{O}_x\big(\sqrt{\varepsilon}\big)\right]dx}{\sqrt{1-i\,\mathcal{O}_x\big(\varepsilon^{1/4}\big)}}\\ &= \int_0^{\frac{1+\mathcal{O}(\varepsilon)}{2\pi\sqrt{\frac{2w^2\varepsilon}{y}}}} \exp\left[\frac{\big(\mathcal{O}_x(\sqrt{\varepsilon}) - i\,\mathcal{O}_x(\varepsilon^{1/4})\big)\big(\mathcal{O}(1) + x^2\big)}{1+\mathcal{O}_x\big(\sqrt{\varepsilon}\big)}\right]\\ &\qquad\times\frac{e^{-\sqrt{\frac{w^2y}{2}}\,\pi\,x^2}\left[1+\mathcal{O}_x\big(\sqrt{\varepsilon}\big)\right]dx}{1-i\,\mathcal{O}_x\big(\varepsilon^{1/4}\big)}\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{1+\mathcal{O}(\varepsilon)}{\pi\sqrt{\frac{8w^2\varepsilon}{y}}}} \exp\Big[\Big(-\sqrt{\tfrac{w^2y}{2}}\,\pi + \mathcal{O}_x(\sqrt{\varepsilon}) - i\,\mathcal{O}_x(\varepsilon^{1/4})\Big)x^2\Big] \\
&\qquad \times \big[1+\mathcal{O}_x\big(\sqrt{\varepsilon}\big) + i\,\mathcal{O}_x\big(\varepsilon^{1/4}\big)\big]\,dx \\
&= \int_0^{\frac{1+\mathcal{O}(\varepsilon)}{2\sqrt{\pi\varepsilon}}\left[\frac{y^3}{2w^2}\right]^{1/4}} \exp\Big[-\Big(1-\mathcal{O}_x(\sqrt{\varepsilon}) + i\,\mathcal{O}\big(\varepsilon^{1/4}\big)\Big)\frac{x^2}{2}\Big] \\
&\qquad \times \frac{1+\mathcal{O}_x\big(\sqrt{\varepsilon}\big) + i\,\mathcal{O}_x\big(\varepsilon^{1/4}\big)}{\sqrt{\pi}\,(2w^2y)^{1/4}}\,dx \\
&= \frac{1+\mathcal{O}\big(\sqrt{\varepsilon}\big) + i\,\mathcal{O}\big(\varepsilon^{1/4}\big)}{\sqrt{\pi}\,(2w^2y)^{1/4}} \int_0^\infty \exp\Big[-\Big(1-\mathcal{O}_x(\sqrt{\varepsilon}) + i\,\mathcal{O}\big(\varepsilon^{1/4}\big)\Big)\frac{x^2}{2}\Big]\,dx \\
&= \frac{1+\mathcal{O}\big(\sqrt{\varepsilon}\big) + i\,\mathcal{O}\big(\varepsilon^{1/4}\big)}{(8w^2y)^{1/4}}\,.
\end{aligned}
$$

The last equality is precisely justified as follows: with all the above $\mathcal{O}_x(\cdot)$ being real, firstly,

$$
\begin{aligned}
&\bigg|\int_0^\infty \exp\Big[-\big(1-\mathcal{O}_x(\sqrt{\varepsilon}) + i\,\mathcal{O}_x(\varepsilon^{1/4})\big)\,x^2/2\Big]\,dx \\
&\qquad - \int_0^\infty \exp\Big[-\big(1+i\mathcal{O}_x(\varepsilon^{1/4})\big)\,x^2/2\Big]\,dx\bigg| \\
&\le \int_0^\infty \Big|\exp\big[\mathcal{O}_x(\sqrt{\varepsilon})\,x^2\big]-1\Big| e^{-x^2/2}\,dx \\
&\le \int_0^\infty e^{-[1-\mathcal{O}(\sqrt{\varepsilon})]\,x^2/2}\,dx - \int_0^\infty e^{-x^2/2}\,dx = \mathcal{O}(\sqrt{\varepsilon})\,,
\end{aligned}
$$

and secondly,

$$
\begin{aligned}
&\int_0^\infty \exp\Big[-\big(1+i\,\mathcal{O}_x(\varepsilon^{1/4})\big)\,x^2/2\Big]\,dx \\
&= \int_0^\infty \cos\Big[\mathcal{O}_x\big(\varepsilon^{1/4}\big)\,x^2\Big]\,e^{-x^2/2}\,dx - i\int_0^\infty \sin\Big[\mathcal{O}_x\big(\varepsilon^{1/4}\big)\,x^2\Big]\,e^{-x^2/2}\,dx \\
&= \sqrt{\pi/2}\,\Big[1-\mathcal{O}\big(\sqrt{\varepsilon}\big) + i\,\mathcal{O}\big(\varepsilon^{1/4}\big)\Big]\,.
\end{aligned}
$$

Thus we obtain

$$
\Re\{\check{J}_0^\varepsilon\} = \big[1+\mathcal{O}\big(\sqrt{\varepsilon}\big)\big]\,\frac{2\pi^{3/2}\,\varepsilon}{\sqrt{y}}\,\exp\big[D_\varepsilon(w,y)\big]\,. \tag{36}
$$

6.3.2. *Control of the first residual term* $\check{J}_1^\varepsilon$. — According to (33) and the beginning of the above Section 6.3.1, and parameterizing by $x = 2(\pi-\nu)^2\,\mathrm{sh}(2t)$

as in Section 5.3, we have the following analogue of (26):

$$\check{J}_1^\varepsilon, \check{J}_2^\varepsilon = 4(\pi-\nu)^2 \left\{ \int_\eta^T , \int_T^\infty \right\} \tag{37}$$

$$\exp\left[\frac{w^2}{2\varepsilon}\,\big(1-U_t \operatorname{cotg} U_t\big) - \frac{i\,y}{2\,\varepsilon^2}(\pi-\nu)^2 \operatorname{sh}(2t) + \tfrac{1}{2}\int_0^{U_t} \big(\tfrac{1}{\theta} - \operatorname{cotg}\theta\big) d\theta\right] \operatorname{ch}(2t)\, dt\,.$$

The computational proof of the following is given in the appendix (Section 7).

Lemma 6.2. — *We have*

$$1-\Re\{U_t \operatorname{cotg} U_t\} = \frac{\pi}{\nu} \times \left(1 - \frac{\pi^2 t^2}{\nu^2+\pi^2 t^2}\left[1+\mathcal{O}\big(\nu+t^2\big)\right] + \mathcal{O}\big(\nu^2\big)\right).$$

Therefore, for some small enough constant $T>0$, $\varepsilon>0$ and for $\eta\le t\le T$, we have

$$\begin{aligned}
1-\Re\{U_t \operatorname{cotg} U_t\} &\le \frac{\pi}{\nu}\left[1-\frac{4\pi^2t^2/5}{\nu^2+\pi^2t^2}+\mathcal{O}\big(\nu^2\big)\right]\\
&\le \frac{\pi}{\nu}\left[1-\frac{4\pi^2\eta^2/5}{\nu^2+\pi^2\eta^2}+\mathcal{O}\big(\nu^2\big)\right]\\
&= \frac{\pi}{\nu}\left[1-\frac{\frac{\varepsilon^{3/2}}{20\pi^2}\left[1+\mathcal{O}(\nu)\right]}{\nu^2+\frac{\varepsilon^{3/2}}{16\pi^2}\left[1+\mathcal{O}(\nu)\right]}+\mathcal{O}(\nu^2)\right]\\
&= \frac{\pi}{\nu}\left[1-\frac{1+\mathcal{O}(\nu)}{20\pi^2\,\nu^2\,\varepsilon^{-3/2}\left[1+\mathcal{O}(\nu)\right]+5/4}+\mathcal{O}(\nu^2)\right]\\
&= \frac{\pi}{\nu}\left[1-\frac{1+\mathcal{O}(\nu)}{20\pi^2(\frac{w^2}{2y})^{3/2}\,\nu^{-1}\left[1+\mathcal{O}(\nu)\right]+5/4}+\mathcal{O}\big(\nu^2\big)\right]\\
&= \frac{\pi}{\nu}\left[1-\big(\tfrac{2y}{w^2}\big)^{3/2}\times\frac{\nu\left[1+\mathcal{O}(\nu)\right]}{20\pi^2+\mathcal{O}(\nu)}+\mathcal{O}\big(\nu^2\big)\right]\\
&\le \frac{\pi}{\nu}\left[1-\big(\tfrac{2y}{w^2}\big)^{3/2}\frac{\nu}{20\pi^2}+\mathcal{O}\big(\nu^2\big)\right],
\end{aligned}$$

whence by (35):

$$\begin{aligned}
\frac{w^2}{2\varepsilon}\,\Re\big\{1-U_t \operatorname{cotg} U_t\big\} &\le \frac{\pi w^2}{2\varepsilon\nu}\left[1-\sqrt{\varepsilon}\left(\frac{y}{10\pi^2w^2}-T^2\sqrt{\tfrac{2w^2}{y}}\right)+\mathcal{O}(\varepsilon)\right]\\
&= D_\varepsilon(w,y)\left[1-\sqrt{\varepsilon}\left(\tfrac{y}{10\pi^2w^2}-T^2\sqrt{\tfrac{2w^2}{y}}\right)+\mathcal{O}(\varepsilon)\right]\\
&= D_\varepsilon(w,y)-\frac{w^2}{10\pi\,\varepsilon}\left(\sqrt{\tfrac{y^3}{2w^6}}-10\pi^2T^2+\mathcal{O}\big(\sqrt{\varepsilon}\big)\right).
\end{aligned}$$

Taking $T \le \frac{1}{16}\big(\frac{y^3}{2w^6}\big)^{1/4}$, we have $10\pi^2\,T^2 < \frac{25}{64}\sqrt{\frac{y^3}{2w^6}}$ and then

$$\frac{w^2}{2\varepsilon}\,\Re\big\{1-U_t\,\mathrm{cotg}\,U_t\big\} \le D_\varepsilon(w,y) - \frac{\sqrt{y^3/w^2}}{100\,\varepsilon} + \mathcal{O}\big(\varepsilon^{-1/2}\big)\,.$$

Owing to (37) and (36), we thus obtain

$$\begin{aligned}\big|\check{J}_1^\varepsilon\big| &\le 4(\pi-\nu)^2\int_\eta^T \exp\Big[\frac{w^2}{2\varepsilon}\,\Re\big\{1-U_t\,\mathrm{cotg}\,U_t\big\}\\ &\qquad\qquad + \tfrac{1}{2}\,\Re\Big\{\int_0^{U_t}\big(\tfrac{1}{\theta}-\mathrm{cotg}\,\theta\big)d\theta\Big\}\Big]\mathrm{ch}(2t)\,dt\\ &= \mathcal{O}(1)\,\exp\Big[D_\varepsilon(w,y) - \frac{\sqrt{y^3/w^2}}{100\,\varepsilon} + \mathcal{O}\big(\varepsilon^{-1/2}\big)\Big] = \Re\{\check{J}_0^\varepsilon\}\times\mathcal{O}(\varepsilon)\,.\end{aligned}$$

6.3.3. *Control of the second residual term* $\check{J}_2^\varepsilon$. — According to (37), we are left with

$$\big|\check{J}_2^\varepsilon\big| \le \mathcal{O}(1)\int_T^\infty \exp\Big[\frac{w^2}{2\varepsilon}\,\Re\big\{1-U_t\,\mathrm{cotg}\,U_t\big\}+\Re\left\{\int_0^{U_t} d\left(\log\sqrt{\tfrac{\theta}{\sin\theta}}\right)\right\}\Big]\mathrm{ch}(2t)\,dt\,.$$

Recall $U_t = (\pi-\nu)\big(\mathrm{ch}\,t+i\,\mathrm{sh}\,t\big)$, (27) $\Re\left\{\int_0^{U_t} d\left(\log\sqrt{\frac{\theta}{\sin\theta}}\right)\right\} = \frac{1}{4}\log\mathrm{ch}(2t) - \frac{\pi-\nu}{4}\,I_t$.

By Lemma 5.3 we have

$$\begin{aligned}\big|\check{J}_2^\varepsilon\big| &= \mathcal{O}(1)\,e^{\frac{w^2}{2\varepsilon}}\int_T^\infty \exp\Big[-\frac{(\pi-\nu)w^2}{4\,\varepsilon}\,\mathrm{sh}\,t - \frac{\pi-\nu}{4}\,\mathrm{sh}\,t\Big]\mathrm{ch}(2t)^{5/4}\,dt\\ &= \mathcal{O}(1)\,e^{\frac{w^2}{2\varepsilon}}\int_{\mathrm{sh}\,T}^\infty e^{-\frac{(\pi-\nu)(w^2+\varepsilon)}{4\,\varepsilon}\,u}\,\frac{(1+2u^2)^{5/4}}{\sqrt{1+u^2}}\,du\\ &= \mathcal{O}(1)\,e^{\frac{w^2}{2\varepsilon}}\int_{\mathrm{sh}\,T}^\infty e^{-\pi w^2 u/(5\varepsilon)}\big(1+u^{3/2}\big)du\\ &= \mathcal{O}(\varepsilon)\,e^{\frac{w^2}{2\varepsilon}-\frac{\pi w^2\mathrm{sh}\,T}{5\,\varepsilon}} = \Re\big\{\check{J}_0^\varepsilon\big\}\,e^{-\pi\sqrt{\frac{w^2 y}{2\,\varepsilon^3}}\,[1+\mathcal{O}(\sqrt{\varepsilon})]} = \Re\big\{\check{J}_0^\varepsilon\big\}\,\mathcal{O}(\varepsilon)\end{aligned}$$

by (35) and (36).

6.3.4. *Conclusion of the unscaled sub-case $w \neq 0$.* — Using (9), (11), (33), (35), (36) and the above Sections 6.3.2 and 6.3.3, we finally derive:

$$
\begin{aligned}
p_\varepsilon\big(0;(w,y)\big) &= \frac{e^{-w^2/2\varepsilon}}{4\pi\sqrt{2\pi\,\varepsilon^5}} \times e^{-\frac{(\pi-\nu)^2 y}{2\,\varepsilon^2}}\, \Re\{\check{J}_0^\varepsilon + \check{J}_1^\varepsilon + \check{J}_2^\varepsilon\} \\
&= \frac{1+\mathcal{O}(\sqrt{\varepsilon})}{2\sqrt{2y\,\varepsilon^3}} \times \exp\left[-\frac{w^2}{2\varepsilon} - \frac{(\pi^2-\nu')y}{2\,\varepsilon^2} + D_\varepsilon(w,y)\right] \\
&= \frac{1+\mathcal{O}(\sqrt{\varepsilon})}{2\sqrt{2y\,\varepsilon^3}}\, \exp\left[-\frac{\pi^2 y}{2\,\varepsilon^2} + \frac{\nu' y}{2\,\varepsilon^2} + \pi\sqrt{\frac{w^2 y}{2\varepsilon^3}} - \frac{3w^2}{4\varepsilon}\right. \\
&\qquad\qquad \left. - \frac{(4\pi^2+3)|w|^3}{16\pi\sqrt{2y\,\varepsilon}} + \frac{(2\pi^2-3)w^4}{24\,\pi^2 y}\right] \\
&\Bigg(\text{recall } \nu' = 2\pi\nu - \nu^2 \\
&\qquad = \pi\sqrt{\frac{2w^2\varepsilon}{y}} + \tfrac{4\pi^2+3}{12\,\pi}\left(\frac{w^2\varepsilon}{2\,y}\right)^{3/2} - \tfrac{2\pi^2-3}{6\,\pi^2}\left(\frac{w^2\varepsilon}{2\,y}\right)^2 \\
&\qquad\quad + \mathcal{O}\big(\varepsilon^{5/2}\big)\Bigg) \\
&= \frac{1+\mathcal{O}(\sqrt{\varepsilon})}{\sqrt{8y\,\varepsilon^3}} \times \exp\left[-\frac{\pi^2 y}{2\,\varepsilon^2} + \frac{\pi\sqrt{2w^2 y}}{\varepsilon^{3/2}} - \frac{3w^2}{4\varepsilon} - \frac{(4\pi^2+3)|w|^3}{24\pi\sqrt{2y\,\varepsilon}}\right. \\
&\qquad\qquad \left. + \frac{(2\pi^2-3)w^4}{48\,\pi^2 y}\right] \\
&= \frac{1+\mathcal{O}(\sqrt{\varepsilon})}{\sqrt{8y\,\varepsilon^3}}\, \exp\left[-\frac{\pi^2 y}{2\,\varepsilon^2}\left(1 - 4\sqrt{\frac{w^2\varepsilon}{2\pi^2 y}} + \frac{3\,w^2\varepsilon}{2\pi^2 y}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{4\pi^2+3}{6}\left[\frac{w^2\varepsilon}{2\pi^2 y}\right]^{3/2} - \frac{2\pi^2-3}{6}\left[\frac{w^2\varepsilon}{2\pi^2 y}\right]^2\right)\right].
\end{aligned}
$$

This ends the proof of Part C of Theorem 2.1, and the whole proof, except for the following last section.

7. Appendix: Some technical proofs

Here are the technical proofs which were postponed so far, and even when non-trivial or delicate, only require elementary means.

Proof of Corollary 2.7. — We have to evaluate the energy $E_\varepsilon\big(0;(w,y)\big)$ defined by (5), and to compare it with the exponents in Parts C, D of Theorem 2.1.

According to the Euler-Lagrange equation we must have $\ddot{\mu} = \lambda\,\mu$, hence

$$\begin{aligned}
&\mu_t = \frac{w\,\mathrm{sh}(\alpha t)}{\mathrm{sh}\,\alpha} \text{ or } \frac{w\sin(\alpha t)}{\sin\alpha} \quad \text{for } w\neq 0, \quad \text{or}\\
&\mu_t = \beta\sin(k\pi t) \quad \text{for } w=0,\\
\frac{2\,y}{w^2\varepsilon} &= \frac{2}{w^2}\int_0^1 \mu^2 = \frac{\mathrm{ch}\,\alpha\,\mathrm{sh}\,\alpha-\alpha}{\alpha\,\mathrm{sh}^2\alpha} \quad\text{or}\quad \frac{\alpha-\cos\alpha\sin\alpha}{\alpha\,\sin^2\alpha}, \quad\text{or}\\
\frac{y}{\varepsilon} &= \int_0^1\mu^2 = \frac{\beta^2}{2} \quad \text{for } w=0,
\end{aligned}$$

whence

$$\begin{aligned}
&\int_0^1 \dot{\mu}^2 = \frac{\mathrm{ch}\,\alpha\,\mathrm{sh}\,\alpha+\alpha}{2\,\mathrm{sh}^2\alpha/(w^2\alpha)} = \alpha^2\left(\frac{w^2}{\mathrm{sh}^2\alpha}+\frac{y}{\varepsilon}\right) \quad\text{or}\quad \alpha^2\left(\frac{w^2}{\sin^2\alpha}-\frac{y}{\varepsilon}\right) \quad\text{or}\\
&\int_0^1 \dot{\mu}^2 = \frac{\beta^2k^2\pi^2}{2} = \frac{\pi^2 y}{\varepsilon} \text{ for } w=0.
\end{aligned}$$

Note that by substituting $\alpha=\sqrt{\pm q}$ into the above, the condition $\int_0^1\mu^2 = y/\varepsilon$ reads exactly as the saddle-point equation (28).

Thus, for $w=0$ we get $E_\varepsilon\big(0;(0,y)\big) = \frac{\pi^2 y}{2\,\varepsilon^2}$ for any $\varepsilon>0$, namely exactly the energy which appears in Part D of Theorem 2.1.

And as to Part C of Theorem 2.1, for $w\neq 0$ and small ε, we get some small $\nu>0$ such that

$$\frac{2\,y}{w^2\varepsilon} = \frac{\pi-\nu+\cos\nu\sin\nu}{(\pi-\nu)\sin^2\nu}\,,$$

and then successively:

$$\begin{aligned}
\frac{w^2\varepsilon}{2\,y} &= \nu^2\Big(1-\frac{\nu}{\pi}-\frac{\nu^2}{3}+\frac{\nu^3}{\pi}+\big[\tfrac{2}{45}-\tfrac{2}{3\pi^2}\big]\nu^4+\mathcal{O}\big(\nu^5\big)\Big),\\
\tilde{\varepsilon} := \sqrt{\frac{w^2\varepsilon}{2\,y}} &= \nu\,\Big(1-\frac{\nu}{2\pi}-\big[\tfrac{1}{6}+\tfrac{1}{8\pi^2}\big]\nu^2+\big[\tfrac{5}{12\pi}-\tfrac{1}{16\pi^3}\big]\nu^3\\
&\qquad +\big[\tfrac{1}{120}-\tfrac{7}{48\pi^2}-\tfrac{5}{128\pi^4}\big]\nu^4+\mathcal{O}\big(\nu^5\big)\Big),\\
\nu &= \tilde{\varepsilon}\,\Big(1+\frac{\tilde{\varepsilon}}{2\pi}+\big[\tfrac{1}{6}+\tfrac{5}{8\pi^2}\big]\tilde{\varepsilon}^2+\frac{\tilde{\varepsilon}^3}{\pi^3}+\big[\tfrac{3}{40}-\tfrac{5}{48\pi^2}+\tfrac{231}{128\pi^4}\big]\tilde{\varepsilon}^4\\
&\qquad +\mathcal{O}\big(\tilde{\varepsilon}^5\big)\Big) \text{ as in Sections 6.1 \& 6.3},\\
\nu^2 &= \frac{w^2\varepsilon}{2\,y}\,\Big(1+\frac{\tilde{\varepsilon}}{\pi}+\big[\tfrac{1}{3}+\tfrac{3}{2\pi^2}\big]\tilde{\varepsilon}^2+\mathcal{O}\big(\tilde{\varepsilon}^3\big)\Big),\\
\nu^{-1} &= \tilde{\varepsilon}^{-1}\,\Big(1-\frac{\tilde{\varepsilon}}{2\pi}-\big[\tfrac{1}{6}+\tfrac{3}{8\pi^2}\big]\tilde{\varepsilon}^2+\big[\tfrac{1}{6\pi}-\tfrac{1}{2\pi^3}\big]\tilde{\varepsilon}^3+\mathcal{O}\big(\tilde{\varepsilon}^4\big)\Big),
\end{aligned}$$

$$\frac{w^2}{\nu^2} = \frac{2y}{\varepsilon}\Big(1 - \frac{\tilde{\varepsilon}}{\pi} - \big[\tfrac{1}{3} + \tfrac{1}{2\pi^2}\big]\tilde{\varepsilon}^2 + \big[\tfrac{1}{2\pi} - \tfrac{5}{8\pi^3}\big]\tilde{\varepsilon}^3 - \big[\tfrac{1}{15} - \tfrac{1}{3\pi^2} + \tfrac{1}{\pi^4}\big]\tilde{\varepsilon}^4 + \mathcal{O}\big(\tilde{\varepsilon}^5\big)\Big),$$

$$\frac{w^2}{\sin^2\nu} = \frac{w^2}{\nu^2} + \frac{w^2}{3} + \frac{w^2\nu^2}{15} + \mathcal{O}\big(\nu^4\big),$$

so that

$$\begin{aligned} E_\varepsilon\big(0;(w,y)\big) &= \frac{(\pi-\nu)^2}{2\,\varepsilon}\Big(\frac{w^2}{\sin^2\nu} - \frac{y}{\varepsilon}\Big) \\ &= \frac{\pi^2 w^2}{2\varepsilon\,\nu^2} - \frac{\pi^2 y}{2\,\varepsilon^2} + \big[\tfrac{\pi^2}{3} + 1\big]\frac{w^2}{2\varepsilon} + \frac{\pi y \nu}{\varepsilon^2} - \frac{\pi w^2}{\varepsilon\,\nu} - \frac{\pi w^2 \nu}{3\,\varepsilon} - \frac{y\,\nu^2}{2\,\varepsilon^2} \\ &\quad + \big[\tfrac{\pi^2}{5} + 1\big]\frac{w^2\nu^2}{6\,\varepsilon} + \mathcal{O}(\sqrt{\varepsilon}) \\ &= \frac{\pi^2 y}{2\,\varepsilon^2} - \frac{\pi\sqrt{2w^2 y}}{\varepsilon^{3/2}} + \frac{3w^2}{4\,\varepsilon} + \big[\tfrac{\pi}{6} + \tfrac{1}{8\pi}\big]\frac{|w|^3}{\sqrt{2y\varepsilon}} - \big[\tfrac{1}{3} - \tfrac{1}{2\pi^2}\big]\frac{w^4}{8y} \\ &\quad + \mathcal{O}(\sqrt{\varepsilon}). \end{aligned}$$

Thus $E_\varepsilon\big(0;(w,y)\big)$ precisely gives the energy appearing in Part C of Theorem 2.1. □

Proof of Proposition 3.3. — Taking $\alpha = 0$ and $\gamma = i\sqrt{2t}$ in Corollary 3.2, for small positive t we have:

$$\mathbb{E}_0\Big[\exp\Big(t\int_0^1 \omega_\cdot^2\Big)\Big] = \frac{1}{\sqrt{\cos\sqrt{2t}}},$$

which, by analytic continuation and monotone convergence, is valid for any positive $t < \pi^2/8$, in any region where the square roots of the right hand side admit a continuous determination. This yields the same domain of validity for the formula

$$\mathbb{E}_0\Big[\exp\Big(\alpha\,\omega_1 + t\int_0^1 \omega_\cdot^2\Big)\Big] = \frac{1}{\sqrt{\cos\sqrt{2t}}}\,\exp\Big(\frac{\alpha^2\,\mathrm{tg}\sqrt{2t}}{2\sqrt{2t}}\Big),$$

and as well, by analytic continuation with respect to α, for the formula

$$\mathbb{E}_0\Big[\exp\Big(i\,\alpha\,\omega_1 + t\int_0^1 \omega_\cdot^2\Big)\Big] = \frac{1}{\sqrt{\cos\sqrt{2t}}}\,\exp\Big(-\frac{\alpha^2\,\mathrm{tg}\sqrt{2t}}{2\sqrt{2t}}\Big).$$

The vertical segment $\frac{i\,\pi^2}{4}]-1,1[$ is contained in a domain of validity for t. Indeed, taking $t = i\,\xi$ and $\sqrt{2t} = (1 + i\,\mathrm{sgn}(\xi))\sqrt{|\xi|} = (1+i)\sqrt{\xi}$ we successively have:

$$\begin{aligned}
\frac{\mathrm{tg}\sqrt{2t}}{\sqrt{2t}} &= \frac{\mathrm{tg}\sqrt{|\xi|}+i\,\mathrm{sgn}(\xi)\mathrm{th}\sqrt{|\xi|}}{\big(1-i\,\mathrm{sgn}(\xi)\mathrm{tg}\sqrt{|\xi|}\,\mathrm{th}\sqrt{|\xi|}\big)(1+i\,\mathrm{sgn}(\xi))\sqrt{|\xi|}}\\
&= \frac{\big(\frac{\mathrm{th}}{\cos^2}\sqrt{|\xi|}+\frac{\mathrm{tg}}{\mathrm{ch}^2}\sqrt{|\xi|}\big)+i\,\mathrm{sgn}(\xi)\big(\frac{\mathrm{th}}{\cos^2}\sqrt{|\xi|}-\frac{\mathrm{tg}}{\mathrm{ch}^2}\sqrt{|\xi|}\big)}{\big(1+\mathrm{tg}^2\sqrt{|\xi|}\,\mathrm{th}^2\sqrt{|\xi|}\big)2\sqrt{|\xi|}}\\
&= \frac{\big(\mathrm{sh}(2\sqrt{|\xi|})+\sin(2\sqrt{|\xi|})\big)+i\,\mathrm{sgn}(\xi)\big(\mathrm{sh}(2\sqrt{|\xi|})-\sin(2\sqrt{|\xi|})\big)}{2\sqrt{|\xi|}\,\big(\mathrm{ch}(2\sqrt{|\xi|})+\cos(2\sqrt{|\xi|})\big)}\\
&= \frac{\big(\mathrm{sh}(2\sqrt{\xi})+\sin(2\sqrt{\xi})\big)+i\,\big(\mathrm{sh}(2\sqrt{\xi})-\sin(2\sqrt{\xi})\big)}{2\sqrt{\xi}\,\big(\mathrm{ch}(2\sqrt{\xi})+\cos(2\sqrt{\xi})\big)}\,,\\
\cos\sqrt{2t} &= \cos\sqrt{\xi}\,\mathrm{ch}\sqrt{\xi}-i\sin\sqrt{\xi}\,\mathrm{sh}\sqrt{\xi}\\
&= \cos\sqrt{|\xi|}\,\mathrm{ch}\sqrt{|\xi|}-i\,\mathrm{sgn}(\xi)\sin\sqrt{|\xi|}\,\mathrm{sh}\sqrt{|\xi|}\,,
\end{aligned}$$

and for $|\xi|<\pi^2/4$,

$$\begin{aligned}
&\mathrm{Arg}\big(\cos\sqrt{|\xi|}\,\mathrm{ch}\sqrt{|\xi|}-i\,\mathrm{sgn}(\xi)\sin\sqrt{|\xi|}\,\mathrm{sh}\sqrt{|\xi|}\big)\\
&= -\mathrm{sgn}(\xi)\,\mathrm{Arctg}\big(\mathrm{tg}\sqrt{|\xi|}\,\mathrm{th}\sqrt{|\xi|}\big)\\
&= -\frac{\mathrm{sgn}(\xi)}{2}\int_0^{2\sqrt{|\xi|}}\frac{\mathrm{sh}\,\theta+\sin\theta}{\mathrm{ch}\,\theta+\cos\theta}\,d\theta\,,
\end{aligned}$$

$$\begin{aligned}
\sqrt{\cos\sqrt{2t}} &= \sqrt{\cos\sqrt{\xi}\,\mathrm{ch}\sqrt{\xi}-i\sin\sqrt{\xi}\,\mathrm{sh}\sqrt{\xi}}\\
&= \sqrt{\cos\sqrt{|\xi|}\,\mathrm{ch}\sqrt{|\xi|}-i\,\mathrm{sgn}(\xi)\sin\sqrt{|\xi|}\,\mathrm{sh}\sqrt{|\xi|}}\\
&= \Big(\mathrm{ch}^2\sqrt{|\xi|}-\sin^2\sqrt{|\xi|}\Big)^{1/4}\exp\bigg[-\frac{i}{4}\,\mathrm{sgn}(\xi)\int_0^{2\sqrt{|\xi|}}\frac{\mathrm{sh}\,\theta+\sin\theta}{\mathrm{ch}\,\theta+\cos\theta}\,d\theta\bigg].
\end{aligned}$$

These functions are well defined and analytic, at least for $|\xi|<\pi^2/4$. This establishes Proposition 3.3, first for $\alpha\in\mathbb{R}$ and $|\xi|<\pi^2/4$, and then for all $(\alpha,\xi)\in\mathbb{R}^2$ by analytic continuation. □

Proof of Lemma 3.6. — First,

$$\frac{d}{dx}\bigg[\frac{\mathrm{sh}\,x-\sin x}{\mathrm{ch}\,x-\cos x}\,x\bigg]=\frac{2-2\,\mathrm{ch}\,x\cos x}{(\mathrm{ch}\,x-\cos x)^2}\,x+\frac{\mathrm{sh}\,x-\sin x}{\mathrm{ch}\,x-\cos x}=\frac{h_1(x)}{(\mathrm{ch}\,x-\cos x)^2}$$

with $h_1(x):=2x-2x\,\mathrm{ch}\,x\cos x+\mathrm{ch}\,x\,\mathrm{sh}\,x-\mathrm{ch}\,x\sin x-\mathrm{sh}\,x\cos x+\cos x\sin x$,

$$\begin{aligned}
h_1'(x) &= 2x\,\mathrm{ch}\,x\sin x-2x\,\mathrm{sh}\,x\cos x+2-4\,\mathrm{ch}\,x\cos x+\mathrm{ch}(2x)+\cos(2x)\\
&= 2x\,(\mathrm{ch}\,x\sin x-\mathrm{sh}\,x\cos x)+2(\mathrm{ch}\,x-\cos x)^2>0\,.
\end{aligned}$$

h_1' is positive for $0<x\le\pi$ since in that range $\frac{d}{dx}(\mathrm{ch}\,x\sin x-\mathrm{sh}\,x\cos x)=2\,\mathrm{sh}\,x\sin x>0$ and then $(\mathrm{ch}\,x\sin x-\mathrm{sh}\,x\cos x)>0$; and for $x\ge\pi$: since

$\frac{d}{dx}\Big[\frac{(\operatorname{ch} x-1)^2}{x\,e^x}\Big]$ has the sign of $2x\operatorname{sh}x-(x+1)(\operatorname{ch}x-1)>(x-1)\operatorname{sh}x>0$, we have

$$\begin{aligned}\frac{h_1'(x)}{2x\,e^x} > \frac{(\operatorname{ch}x-1)^2-x\,(\operatorname{ch}x+\operatorname{sh}x)}{x\,e^x} &= \frac{(\operatorname{ch}x-1)^2}{x\,e^x}-1\\ &> \frac{(\operatorname{ch}\pi-1)^2}{\pi\,e^\pi}-1>1/2\,.\end{aligned}$$

Therefore h_1' and h_1 are positive, which entails that the positive smooth function h increases on $\mathbb{R}_+$. The remaining about h is banal. Similarly,

$$\frac{d}{dx}\left[\frac{\operatorname{sh}x+\sin x}{\operatorname{ch}x-\cos x}\,x\right]=\frac{-2\operatorname{sh}x\sin x}{(\operatorname{ch}x-\cos x)^2}\,x+\frac{\operatorname{sh}x+\sin x}{\operatorname{ch}x-\cos x}=\frac{h_2(x)}{(\operatorname{ch}x-\cos x)^2}$$

with $h_2(x):=-2x\operatorname{sh}x\sin x+\operatorname{ch}x\operatorname{sh}x+\operatorname{ch}x\sin x-\operatorname{sh}x\cos x-\cos x\sin x$, and then $\frac12\,h_2'(x)=\operatorname{sh}^2x+\sin^2x-x\,(\operatorname{ch}x\sin x+\operatorname{sh}x\cos x)>\operatorname{sh}^2x-x\,e^x$. Now $\frac{d}{dx}\big[2\operatorname{sh}x\sin x-x\,(\operatorname{ch}x\sin x+\operatorname{sh}x\cos x)\big]=(\operatorname{tg}x+\operatorname{th}x-2x)\operatorname{ch}x\cos x$ is positive on $]0,\pi]$, since $\frac{d}{dx}(\operatorname{tg}x+\operatorname{th}x-2x)=\operatorname{tg}^2x-\operatorname{th}^2x>0$ on $]0,\frac{\pi}{2}]$ and $\operatorname{tg}x+\operatorname{th}x-2x<0$ on $]\frac{\pi}{2},\pi]$.

Hence, on $]0,\pi]$, $\big[2\operatorname{sh}x\sin x-x\,(\operatorname{ch}x\sin x+\operatorname{sh}x\cos x)\big]$ and then $\frac12\,h_2'(x)>(\operatorname{sh}x-\sin x)^2$ are positive. Then for $x>\pi$: $\frac{d}{dx}\Big[\frac{\operatorname{sh}^2x}{x\,e^x}-1\Big]$ has the sign of $\frac{2x}{x+1}-\operatorname{th}x$, which is positive since it has derivative $\frac{\operatorname{ch}(2x)-x^2-2x}{(x+1)^2\operatorname{ch}^2x}>\frac{(x-1)^2}{(x+1)^2\operatorname{ch}^2x}>0$; therefore $\frac{h_2'(x)}{2x\,e^x}>\frac{\operatorname{sh}^2\pi}{\pi\,e^\pi}-1>\frac45$.

This proves that h_2 is positive, hence that the function $x\mapsto\frac{\operatorname{sh}x+\sin x}{\operatorname{ch}x-\cos x}x$ increases on $\mathbb{R}_+$.

Then $\frac{d}{dx}\log\big[\frac{\operatorname{ch}x-\cos x}{x^2}\big]=\frac{\operatorname{sh}x+\sin x}{\operatorname{ch}x-\cos x}-\frac{2}{x}$ has the sign of $\frac{x}{2}-\frac{\operatorname{ch}x-\cos x}{\operatorname{sh}x+\sin x}$, whose derivative is $\frac{(\operatorname{sh}x-\sin x)^2}{2(\operatorname{sh}x+\sin x)^2}>0$. Hence the smooth function H increases and is non-negative on $\mathbb{R}_+$.

The remaining about H is banal. □

Proof of Lemma 3.7. — We successively have

$$\begin{aligned}\operatorname{sh}\big(\sqrt{2ix}\big)&=\operatorname{sh}\sqrt{x}\,\cos\sqrt{x}+i\operatorname{ch}\sqrt{x}\,\sin\sqrt{x},\\ \sin\big(\sqrt{2ix}\big)&=\operatorname{ch}\sqrt{x}\,\sin\sqrt{x}+i\operatorname{sh}\sqrt{x}\,\cos\sqrt{x}\,,\\ \cos\big(\sqrt{2ix}\big)&=\operatorname{ch}\sqrt{x}\,\cos\sqrt{x}-i\operatorname{sh}\sqrt{x}\,\sin\sqrt{x},\\ \operatorname{ch}\big(\sqrt{2ix}\big)&=\operatorname{ch}\sqrt{x}\,\cos\sqrt{x}+i\operatorname{sh}\sqrt{x}\,\sin\sqrt{x}\,,\end{aligned}$$

and then

$$\begin{aligned}\frac{\operatorname{sh}\sqrt{x}-\sin\sqrt{x}}{\operatorname{ch}\sqrt{x}-\cos\sqrt{x}}&=(1-i)\,\frac{\operatorname{ch}\sqrt{x}\,\sin\sqrt{x}-\operatorname{sh}\sqrt{x}\,\cos\sqrt{x}}{2\operatorname{sh}\sqrt{x}\,\sin\sqrt{x}}\\ &=(1-i)\,\big(\coth\sqrt{x}-\operatorname{cotg}\sqrt{x}\big)\,.\end{aligned}$$

The expression displayed in the statement for $\lambda(x)$ follows easily. Then,

$$w^{-2}\,\frac{d}{dx}\lambda(x^2) = \coth x - \operatorname{cotg} x + \frac{x}{\sin^2 x} - \frac{x}{\operatorname{sh}^2 x}$$

is a sum of positive terms $\big($since $\frac{d}{dx}\big[\operatorname{ch} x \sin x - \operatorname{sh} x \cos x\big] = 2 \operatorname{sh} x \sin x > 0\big)$. The case of H is very similar, but with

$$w^{-2}\,\frac{d}{dx}\Lambda(x^2) = \coth x + \operatorname{cotg} x - \frac{x}{\sin^2 x} - \frac{x}{\operatorname{sh}^2 x}\,.$$

The decreasing property of Λ now follows from the fact that, for $0 < x < \pi$, we have

$$\coth x + \operatorname{cotg} x < \frac{2}{x} < \frac{x}{\sin^2 x} + \frac{x}{\operatorname{sh}^2 x}\,.$$

Indeed, the latter is equivalent to the positivity of $\ell(x) := \frac{x^2}{\sin^2 x} + \frac{x^2}{\operatorname{sh}^2 x} - 2$, which holds (since $\ell(0+) = 0$ and) by the following:

$$\frac{\ell'(x)}{2x} = \frac{\sin x - x\cos x}{\sin^3 x} + \frac{\operatorname{sh} x - x \operatorname{ch} x}{\operatorname{sh}^3 x}, \qquad \frac{d}{dx}\,\frac{\ell'(x)}{2x} = \frac{\sigma(x)}{\sin^4 x} + \frac{\tau(x)}{\operatorname{sh}^4 x},$$

with $\sigma(x) := x(1+2\cos^2 x) - 3\cos x \sin x$ and $\tau(x) := x(1+2\operatorname{ch}^2 x) - 3 \operatorname{ch} x \operatorname{sh} x$, which are positive since

$$\sigma'(x) = 4\sin^2 x - 2x\sin(2x), \quad \sigma''(x) = 2\sin(2x) - 4x\cos(2x),$$
$$\sigma'''(x) = 8x\sin(2x) > 0\,;$$
$$\tau'(x) = 2x\operatorname{sh}(2x) - 4\operatorname{sh}^2 x, \quad \tau''(x) = 4x\operatorname{ch}(2x) - 2\operatorname{sh}(2x),$$
$$\tau'''(x) = 8x\operatorname{sh}(2x) > 0\,.$$

Finally,

$$\frac{d}{dx}\left[\coth x + \operatorname{cotg} x - \frac{2}{x}\right] = \frac{2}{x^2} - \frac{1}{\sin^2 x} - \frac{1}{\operatorname{sh}^2 x} < 0$$

ends the proof. □

Proof of Lemma 4.1. — The derivative of the first map, multiplied by $2(\operatorname{ch} u - 1)^2$, equals

$$2\operatorname{sh} u - \frac{\operatorname{ch} u - 1}{u^2/2}(u + \operatorname{sh} u) \le \varphi(u) := \big(1 - \tfrac{u^2}{12}\big)\operatorname{sh} u - u - \tfrac{u^3}{12}\,,$$

which satisfies $\varphi'(u) = \big(1 - \frac{u^2}{12}\big)\operatorname{ch} u - \frac{u}{6}\operatorname{sh} u - 1 - \frac{u^2}{4}$, $\varphi''(u) = \big(\frac{5}{6} - \frac{u^2}{12}\big)\operatorname{sh} u - \frac{u}{3}\operatorname{ch} u - \frac{u}{2}$, $\varphi'''(u) = \big(\frac{1}{2} - \frac{u^2}{12}\big)\operatorname{ch} u - \frac{u}{2}\operatorname{sh} u - \frac{1}{2}$, $\varphi''''(u) = -\frac{u^2}{12}\operatorname{sh} u - \frac{2u}{3}\operatorname{ch} u < 0$, so that $\varphi(u) < 0$.

The derivative of the second map, multiplied by $v^2(1-\cos v)^2$, equals

$$(1-\cos v)(v + \sin v) - v^2 \sin v =: \tilde{\varphi}(v)\,,$$

which satisfies $\tilde\varphi(0) = \tilde\varphi(2\pi) = 0$ and

$$\begin{aligned}\tilde\varphi(\pi + x) &= (1 + \cos x)(\pi + x - \sin x) + (\pi + x)^2 \sin x\\ &\geq \big(2 - \tfrac{x^2}{2}\big)(\pi + x - \sin x) + (\pi + x)^2 \sin x\\ &= \tfrac{\pi+x}{2}(4 - x^2) + \big(\pi^2 - 2 + 2\pi x + \tfrac{3x^2}{2}\big) \sin x\\ &\qquad \text{which is clearly positive for } 0 \leq x \leq 2, \text{ and is}\\ &> \tfrac{\pi+x}{2}(4 - x^2) + \big(\pi^2 - 2 + 2\pi x + \tfrac{3x^2}{2}\big)x\\ &= 2\pi + \pi^2 x + \tfrac{3\pi}{2}\, x^2 + x^3 > 0 \quad \text{for } -1 \leq x < 0,\end{aligned}$$

showing that $\tilde\varphi > 0$ on $[\pi - 1, \pi + 2]$. Then for $0 < x \leq \pi$ we have

$$\begin{aligned}\tilde\varphi(2\pi - x) &= (1 - \cos x)(2\pi - x - \sin x) + (2\pi - x)^2 \sin x\\ &\geq \big(\tfrac{x^2}{2} - \tfrac{x^4}{24}\big)(2\pi - x - \sin x) + (2\pi - x)^2 \sin x\\ &= \big(\pi - \tfrac{x}{2}\big)\big(1 - \tfrac{x^2}{12}\big)x^2 + \big(4\pi(\pi - x) + \tfrac{x^2}{2} + \tfrac{x^4}{24}\big) \sin x > 0\,.\end{aligned}$$

Then for $0 < v \leq \sqrt{8}$ we have

$$\begin{aligned}\tfrac{2}{v^2}\, \tilde\varphi(v) &\geq \tfrac{2}{v^2}\big(\tfrac{v^2}{2} - \tfrac{v^4}{24}\big)(v + \sin v) - 2 \sin v = \big(1 - \tfrac{v^2}{12}\big)v - \big(1 + \tfrac{v^2}{12}\big) \sin v\\ &> \big(1 - \tfrac{v^2}{12}\big)v - \big(1 + \tfrac{v^2}{12}\big)\big(v - \tfrac{v^3}{6} + \tfrac{v^5}{120}\big) = \tfrac{v^5(8-v^2)}{1440} > 0\,.\end{aligned}$$

Since $\sqrt{8} > \pi - 1$, this ends the proof that $\tilde\varphi(v) > 0$ for $0 < v < 2\pi$. □

Proof of Proposition 4.5. —

- *Step 1.* By (15) we have

$$N(2\,q, 0) - N(2\,q, x) = \frac{(a + b)\,\mathrm{sh}(a + b) + (a - b) \sin(a - b)}{\mathrm{ch}(a + b) - \cos(a - b)} - 2\sqrt{q}\,\coth\sqrt{q}\,.$$

By Section 4.3.1 we have $N(2q, 0) = 2\,\varrho(\tau)$ and $N(2q, r_\varepsilon) = 2\,\varrho(\tau) - R(\tau)\, r_\varepsilon^2/8 + \mathcal{O}(r_\varepsilon^3)$.

Then by (13) (with $t \equiv 2\,q(\tau)$) and setting $r := \sqrt{4q^2 + x^2}$, we successively have

$$a = \sqrt{\tfrac{r+x}{2}}, \quad b = \mathrm{sign}(q)\sqrt{\tfrac{r-x}{2}},$$

$$ab = q > -\pi^2, \quad (a + b)(a - b) = x \geq 0, \quad a^2 + b^2 = r\,,$$

$$\frac{\partial a}{\partial x} = \frac{1}{4a}\Big[\frac{x}{r} + 1\Big] = \frac{a}{2r}, \quad \frac{\partial b}{\partial x} = \frac{1}{4b}\Big[\frac{x}{r} - 1\Big] = \frac{-b}{2r}, \quad a\,\frac{\partial b}{\partial x} + b\,\frac{\partial a}{\partial x} = 0\,,$$

$$\frac{\partial}{\partial x} = \frac{\partial a}{\partial x}\,\frac{\partial}{\partial a} + \frac{\partial b}{\partial x}\,\frac{\partial}{\partial b} = \frac{1}{2r}\left[a\,\frac{\partial}{\partial a} - b\,\frac{\partial}{\partial b}\right], \quad \frac{\partial(a \pm b)}{\partial x} = \frac{a \mp b}{2\,r}, \quad \frac{\partial r}{\partial x} = \frac{x}{r}\,,$$

$$\frac{\partial}{\partial x}N(2q, x) = \frac{\mathrm{sh}(a + b) \sin(a - b)}{\big[\mathrm{ch}(a + b) - \cos(a - b)\big]^2} - \frac{(a + b) \sin(a - b) + (a - b)\,\mathrm{sh}(a + b)}{2\,r\,\big[\mathrm{ch}(a + b) - \cos(a - b)\big]}\,.$$

Now the wanted result will follow from the negativity of $\frac{\partial}{\partial x}N(2q,x)$ for positive x. Thus the proof reduces to proving that

$$\psi := \big[(a+b)\sin(a-b)+(a-b)\,\mathrm{sh}(a+b)\big]\big[\mathrm{ch}(a+b)-\cos(a-b)\big] \\ -2r\,\mathrm{sh}(a+b)\,\sin(a-b)>0\,.$$

To study of the sign of ψ, which is a function of the two variables $-\pi^2<q<\infty, 0<x<\infty$, we shall use the alternative (positive) coordinates

$$\theta := a-b \quad \text{and} \quad t := a+b\,.$$

We thus have $t^2-\theta^2=4q>-4\pi^2$, $t^2+\theta^2=2r$, $\theta t=x>0$, and we must show that

$$\psi \equiv \Phi(\theta,t) := \big[t\sin\theta+\theta\,\mathrm{sh}\,t\big]\big[\mathrm{ch}\,t-\cos\theta\big]-(\theta^2+t^2)\,\mathrm{sh}\,t\sin\theta>0\,, \tag{38}$$

within the range under consideration, which now reads $\big\{\theta>0, t>0, t^2>\theta^2-4\pi^2\big\}$.

- *Step 2.* Clearly, $\Phi(\theta,t)>0$ for $(2m+1)\pi\le\theta\le 2(m+1)\pi$.

Therefore we have to consider the cases $2m\,\pi<\theta<(2m+1)\pi$, for $m\in\mathbb{N}$.

Expanding $\Phi(\theta,t)$ with respect to t, we have $\Phi(\theta,t)=\sum\limits_{n\in\mathbb{N}}\frac{t^{2n+1}}{(2n+1)!}\,\Phi_n(\theta)$, with

$$\Phi_n(\theta) := 4^n\theta-(4n^2-1)\sin\theta-\theta\cos\theta-\theta^2\sin\theta-\cos\theta\sin\theta\,1_{\{n=0\}}\,.$$

Then for $n\ge 1$: $\Phi_{n+1}(\theta)-\Phi_n(\theta)=4^n\times 3\theta-4(2n+1)\sin\theta>4\sin\theta\times\big(3\times 4^{n-1}-2n-1\big)\ge 0$, and $\Phi_1(\theta)-\Phi_0(\theta)=3\theta-4\sin\theta+\cos\theta\sin\theta$ is positive too, since $(\Phi_1-\Phi_0)(2m\,\pi)=(\Phi_1-\Phi_0)'(2m\,\pi)=0$ and $(\Phi_1-\Phi_0)''(\theta)=4(1-\cos\theta)\sin\theta>0$.

Moreover, we have

$$\Phi_0(\theta)=\theta+\sin\theta-\theta\cos\theta-\cos\theta\sin\theta-\theta^2\sin\theta=2\sin\big(\tfrac{\theta}{2}\big)\,\tilde{\Phi}_0(\theta)\,,$$

with

$$\begin{aligned}\tilde{\Phi}_0(\theta) &:= (\theta+\sin\theta)\sin\big(\tfrac{\theta}{2}\big)-\theta^2\cos\big(\tfrac{\theta}{2}\big)\\ &= \theta\sin\big(\tfrac{\theta}{2}\big)+\big(\tfrac{1}{2}-\theta^2\big)\cos\big(\tfrac{\theta}{2}\big)-\tfrac{1}{2}\cos\big(\tfrac{3\theta}{2}\big)\\ &= \sum_{k\in\mathbb{N}}\frac{(-1)^k\,(\theta/2)^{2k}}{(2k)!}\,c_k=\sum_{k\ge 3}\frac{(-1)^k\,(\theta/2)^{2k}}{(2k)!}\,c_k,\\ &\qquad\text{with}\quad c_k=4k(4k-3)-\frac{9^k-1}{2}\,.\end{aligned}$$

Now for any $k \geq 4$ we have $-256 = c_3 \geq c_{k-1} > c_k$ since $\frac{c_k}{c_{k-1}} = 9 \times \frac{9^k-32k^2-24k-1}{9^k-288k^2+792k-513} > 9$, and for $0 < \theta < \pi$:

$$\frac{\frac{(\theta/2)^{2k}}{(2k)!}\,c_k}{\frac{(\theta/2)^{2k-2}}{(2k-2)!}\,c_{k-1}} = \frac{\theta^2\,c_k}{8k(2k-1)c_{k-1}} = \frac{9\,\theta^2}{8k(2k-1)} \times \frac{9^k-32k^2-24k-1}{9^k-288k^2+792k-513}$$
$$< \frac{45}{4k(2k-1)} \times \frac{9^k-32k^2-24k-1}{9^k-288k^2+792k-513}$$
$$\leq \frac{45}{112} \times \frac{9^k-32k^2-24k-1}{9^k-288k^2+792k-513} < 1\,,$$

since

$$\begin{aligned} &112(9^k-288k^2+792k-513)-45(9^k-32k^2-24k-1)\\ &= 67\times 9^k - 30816k^2 + 89784k - 57411\\ &> 67\times(9^k-460k^2+1340k-857) \geq 67\times(9^4-460\times 4^2+1340\times 4-857)\\ &= 67\times 3704 > 0\,. \end{aligned}$$

This shows the positivity of $\tilde\Phi_0(\theta)$ and then of all $\Phi_n(\theta)$, hence the wanted positivity of $\Phi(\theta,t)$, for $0<\theta\leq\pi$ and all positive t. This proves (38) in the case where $m=0$.

- *Step 3.* We are left with the cases $2m\,\pi < \theta < (2m+1)\pi, m\in\mathbb{N}^*$, of the preceding step.

Setting $\theta =: 2m\,\pi+\alpha$ and $t =: \sqrt{\theta^2-4\pi^2}+s$, we have to consider the range $s>0<\alpha<\pi$.

We set $t_\alpha := \sqrt{4(m^2-1)\pi^2+4m\,\pi\alpha+\alpha^2} \geq \sqrt{4\pi\alpha+\alpha^2}$. We have

$$\begin{aligned} \Phi(\theta,t) &= t\,\mathrm{ch}\,t\sin\alpha + (m\pi+\alpha/2)\,\mathrm{sh}(2t) - (t/2)\sin(2\alpha) - (2m\pi+\alpha)\cos\alpha\,\mathrm{sh}\,t\\ &\quad - (\theta^2+t^2)\,\mathrm{sh}\,t\sin\alpha\\ &= (t_\alpha+s)(\mathrm{ch}\,t_\alpha\,\mathrm{ch}\,s+\mathrm{sh}\,t_\alpha\,\mathrm{sh}\,s)\sin\alpha + (m\pi+\tfrac{\alpha}{2})\big(\mathrm{ch}(2t_\alpha)\,\mathrm{sh}(2s)\\ &\quad + \mathrm{sh}(2t_\alpha)\,\mathrm{ch}(2s)\big) - \sin(2\alpha)\tfrac{t_\alpha+s}{2}\\ &\quad - (2m\pi+\alpha)\cos\alpha\,(\mathrm{ch}\,t_\alpha\,\mathrm{sh}\,s+\mathrm{sh}\,t_\alpha\,\mathrm{ch}\,s)\\ &\quad - (4\pi^2+2t_\alpha^2+2t_\alpha s+s^2)\sin\alpha\,(\mathrm{ch}\,t_\alpha\,\mathrm{sh}\,s+\mathrm{sh}\,t_\alpha\,\mathrm{ch}\,s)\\ &= \sum_{n\geq 0}\left(A_n\,\frac{s^{2n}}{(2n)!}+B_n\,\frac{s^{2n+1}}{(2n+1)!}\right), \end{aligned}$$

with

$$\begin{aligned} A_n &= 4^n(m\,\pi+\tfrac{\alpha}{2})\,\mathrm{sh}(2t_\alpha) - (2m\,\pi+\alpha)\cos\alpha\,\mathrm{sh}\,t_\alpha - (4n-1)\sin\alpha\,t_\alpha\,\mathrm{ch}\,t_\alpha\\ &\quad - 1_{\{n=0\}}\,t_\alpha\,\cos\alpha\sin\alpha - 2\big[n(2n-1)+2\pi^2+t_\alpha^2\big]\sin\alpha\,\mathrm{sh}\,t_\alpha \end{aligned}$$

and

$$\begin{aligned}B_n &= 4^n(2m\,\pi+\alpha)\,\mathrm{ch}(2t_\alpha) - (2m\,\pi+\alpha)\cos\alpha\,\mathrm{ch}\,t_\alpha - (4n+1)\sin\alpha\,t_\alpha\,\mathrm{sh}\,t_\alpha\\ &\qquad - 1_{\{n=0\}}\cos\alpha\sin\alpha - \big[4n^2-1+4\pi^2+2t_\alpha^2\big]\sin\alpha\,\mathrm{ch}\,t_\alpha\,.\end{aligned}$$

Thus for any $n\in\mathbb{N}^*$ we have:

$$\begin{aligned}A_{n+1}-A_n &= 4^n\times 3(2m\,\pi+\alpha)\,\mathrm{ch}\,t_\alpha\,\mathrm{sh}\,t_\alpha - 4\sin\alpha\,t_\alpha\,\mathrm{ch}\,t_\alpha - 2(4n+1)\sin\alpha\,\mathrm{sh}\,t_\alpha\\ &\geq \alpha\,\mathrm{ch}\,t_\alpha\big(4^n\times 3\,\mathrm{sh}\,t_\alpha - 4t_\alpha\big) + 2\,\mathrm{sh}\,t_\alpha\big(4^n\times 3m\,\pi\,\mathrm{ch}\,t_\alpha - (4n+1)\alpha\big)\\ &\geq \alpha\,t_\alpha\,\mathrm{ch}\,t_\alpha(3\times 4^n-4)\\ &\quad + 2\,\mathrm{sh}\,t_\alpha\big(2(n+1)\times 3\pi(1+2\pi\alpha+\alpha^2/2) - 4n\,\alpha-\alpha\big) > 0\end{aligned}$$

and

$$\begin{aligned}B_{n+1}-B_n &= 4^n\times 3(2m\,\pi+\alpha)\,\mathrm{ch}(2t_\alpha) - 4\sin\alpha\,t_\alpha\,\mathrm{sh}\,t_\alpha - 4(2n+1)\sin\alpha\,\mathrm{ch}\,t_\alpha\\ &\geq 4\alpha\big(3\,\mathrm{ch}(2t_\alpha) - t_\alpha\,\mathrm{sh}\,t_\alpha\big) + 2(2n+1)\big(3\pi\,\mathrm{ch}(2t_\alpha) - 2\,t_\alpha\,\mathrm{ch}\,t_\alpha\big) > 0\,.\end{aligned}$$

Then

$$\begin{aligned}A_1-A_0 &= 3(2m\,\pi+\alpha)\,\mathrm{ch}\,t_\alpha\,\mathrm{sh}\,t_\alpha - 2\sin\alpha\,\big(2\,t_\alpha\,\mathrm{ch}\,t_\alpha+\mathrm{sh}\,t_\alpha\big) + t_\alpha\,\cos\alpha\sin\alpha\\ &\geq 3(2\pi+\alpha)\,\mathrm{ch}\,t_\alpha\,\mathrm{sh}\,t_\alpha - 2\alpha\,\big(2\,t_\alpha\,\mathrm{ch}\,t_\alpha+\mathrm{sh}\,t_\alpha\big) - t_\alpha/2\\ &> \mathrm{sh}\,t_\alpha\big(6\pi\,\mathrm{ch}\,t_\alpha - 2\alpha - \alpha\,\mathrm{ch}\,t_\alpha - \tfrac{1}{2}\big) > \mathrm{sh}\,t_\alpha\big((\pi+4\alpha)\,\mathrm{ch}\,t_\alpha - 2\alpha - \tfrac{1}{2}\big)\\ &> 2(1+\alpha)\,\mathrm{sh}\,t_\alpha > 0\,,\\ B_1-B_0 &= 3(2m\,\pi+\alpha)\,\mathrm{ch}(2t_\alpha) - 4\sin\alpha\,t_\alpha\,\mathrm{sh}\,t_\alpha - 4\sin\alpha\,\mathrm{ch}\,t_\alpha + \cos\alpha\sin\alpha\\ &\geq 3(2\pi+\alpha)\,\mathrm{ch}(2t_\alpha) - 4\alpha\,t_\alpha\,\mathrm{sh}\,t_\alpha - 4\alpha\,\mathrm{ch}\,t_\alpha - \alpha\\ &> 9\alpha\,(\mathrm{ch}^2 t_\alpha + \mathrm{sh}^2 t_\alpha) - 4\alpha\,\mathrm{sh}^2 t_\alpha - 4\alpha\,\mathrm{ch}\,t_\alpha - \alpha\\ &> 2\alpha\,(2\,\mathrm{ch}^2 t_\alpha + 3\,\mathrm{sh}^2 t_\alpha) > 0\,.\end{aligned}$$

Moreover,

$$\begin{aligned}A_0 &= (m\,\pi+\tfrac{\alpha}{2})\,\mathrm{sh}(2t_\alpha) - (2m\,\pi+\alpha)\cos\alpha\,\mathrm{sh}\,t_\alpha + \sin\alpha\,t_\alpha\,\mathrm{ch}\,t_\alpha - t_\alpha\cos\alpha\sin\alpha\\ &\qquad - 2\big[2\pi^2+t_\alpha^2\big]\sin\alpha\,\mathrm{sh}\,t_\alpha\\ &\geq \big((2m\,\pi+\alpha)[\mathrm{ch}\,t_\alpha-1] - 2\alpha\big[2\pi^2+t_\alpha^2\big]\big)\,\mathrm{sh}\,t_\alpha + (\mathrm{ch}\,t_\alpha-\cos\alpha)\,t_\alpha\sin\alpha\\ &> \big((2\pi+\alpha)\big[\tfrac{t_\alpha^2}{2}+\tfrac{t_\alpha^4}{24}\big] - 2\big[2\pi^2+t_\alpha^2\big]\alpha\big)\,\mathrm{sh}\,t_\alpha\\ &> \big(2\pi(4\pi^2-15) + (8\pi^2-9)\alpha\big)\,\frac{\alpha^2}{6}\,\mathrm{sh}\,t_\alpha > 0\,,\end{aligned}$$

and

$$
\begin{aligned}
B_0 &= (2m\,\pi+\alpha)\,\mathrm{ch}(2t_\alpha) - (2m\,\pi+\alpha)\cos\alpha\,\mathrm{ch}\,t_\alpha - \sin\alpha\, t_\alpha\,\mathrm{sh}\,t_\alpha - \tfrac{\sin(2\alpha)}{2} \\
&\quad - \big[4\pi^2 - 1 + 2t_\alpha^2\big]\sin\alpha\,\mathrm{ch}\,t_\alpha \\
&> (2m\,\pi+\alpha)\,\mathrm{ch}(2t_\alpha) - (2m\,\pi+\alpha)\,\mathrm{ch}\,t_\alpha - \alpha\, t_\alpha\,\mathrm{sh}\,t_\alpha - \alpha \\
&\quad - \big[4\pi^2 - 1 + 2t_\alpha^2\big]\alpha\,\mathrm{ch}\,t_\alpha \\
&\geq (2\pi+\alpha)(\mathrm{ch}\,t_\alpha - 1)(2\,\mathrm{ch}\,t_\alpha + 1) - \alpha\, t_\alpha\,\mathrm{sh}\,t_\alpha - \alpha - \big[4\pi^2 - 1 + 2t_\alpha^2\big]\alpha\,\mathrm{ch}\,t_\alpha \\
&> (2\pi+\alpha)\big[\tfrac{t_\alpha^2}{2} + \tfrac{t_\alpha^4}{24}\big](\mathrm{ch}\,t_\alpha + \mathrm{sh}\,t_\alpha + 1) - \alpha\, t_\alpha\,\mathrm{sh}\,t_\alpha - \alpha \\
&\quad - \big[4\pi^2 - 1 + 2t_\alpha^2\big]\alpha\,\mathrm{ch}\,t_\alpha \\
&> \Big((2\pi+\alpha)\big[\tfrac{t_\alpha^2}{2} + \tfrac{t_\alpha^4}{24}\big] - \big[4\pi^2 - 1 + 2t_\alpha^2\big]\alpha\Big)\,\mathrm{ch}\,t_\alpha \\
&\quad + \Big((2\pi+\alpha)\big[\tfrac{t_\alpha^2}{2} + \tfrac{t_\alpha^4}{24}\big] - \alpha\, t_\alpha\Big)\,\mathrm{sh}\,t_\alpha + \pi t_\alpha^2 - \alpha \\
&> \Big(4\pi^2 + \pi\alpha + \big(\tfrac{2\pi^2+3}{6}\big)\,t_\alpha^2 - 4\pi^2 + 1 - 2t_\alpha^2\Big)\alpha\,\mathrm{ch}\,t_\alpha \\
&\quad + \Big(4\pi^2 + \big(\tfrac{2\pi^2+3}{6}\big)\,t_\alpha^2 - t_\alpha\Big)\alpha\,\mathrm{sh}\,t_\alpha + (4\pi^2-1)\alpha \\
&> \Big(1 + \pi\alpha + \big(\tfrac{2\pi^2-9}{6}\big)\,t_\alpha^2\Big)\alpha\,\mathrm{ch}\,t_\alpha + \Big(4\pi^2 - \tfrac12 + \tfrac{\pi^2}{3}\,t_\alpha^2\Big)\alpha\,\mathrm{sh}\,t_\alpha + (4\pi^2-1)\,\alpha \\
&> 4\pi^2\,\alpha > 0\,.
\end{aligned}
$$

This ends the proof of (38), hence of Lemma 4.5. □

Proof of Lemma 5.3. — Fix $T>0$, set $a := 2(\pi-\nu)$, and consider

$$
\begin{aligned}
\lambda_t &:= 2\,\mathrm{sh}\,t\,\sin[a\,\mathrm{ch}\,t] + 2\,\mathrm{ch}\,t\,\mathrm{sh}[a\,\mathrm{sh}\,t] - \big(\mathrm{ch}[a\,\mathrm{sh}\,t] - \cos[a\,\mathrm{ch}\,t]\big)\,\mathrm{ch}\,t\,, \\
\varrho_t &:= 2\,\mathrm{ch}\,t\,\sin[a\,\mathrm{ch}\,t] + 2\,\mathrm{sh}\,t\,\mathrm{sh}[a\,\mathrm{sh}\,t] - \big(\mathrm{ch}[a\,\mathrm{sh}\,t] - \cos[a\,\mathrm{ch}\,t]\big)\,\mathrm{sh}\,t\,.
\end{aligned}
$$

Since we have

$$
\mathrm{cotg}\,U_t = \frac{\sin[2(\pi-\nu)\,\mathrm{ch}\,t] - i\,\mathrm{sh}[2(\pi-\nu)\,\mathrm{sh}\,t]}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\,t] - \cos[2(\pi-\nu)\,\mathrm{ch}\,t]} \quad \text{and}
$$

$$
\frac{U_t\,\mathrm{cotg}\,U_t}{(\pi-\nu)} = \frac{\mathrm{ch}\,t\sin[2(\pi-\nu)\mathrm{ch}\,t] + \mathrm{sh}\,t\,\mathrm{sh}[2(\pi-\nu)\mathrm{sh}\,t] + i(\mathrm{sh}\,t\sin[2(\pi-\nu)\mathrm{ch}\,t] - \mathrm{ch}\,t\,\mathrm{sh}[2(\pi-\nu)\mathrm{sh}\,t])}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\,t] - \cos[2(\pi-\nu)\,\mathrm{ch}\,t]},
$$

hence

$$
\frac{\Re\big\{U_t\,\mathrm{cotg}\,U_t\big\}}{(\pi-\nu)/2} - \mathrm{sh}\,t = \frac{\varrho_t}{\mathrm{ch}[2(\pi-\nu)\,\mathrm{sh}\,t] - \cos[2(\pi-\nu)\,\mathrm{ch}\,t]}\,,
$$

then we only have to verify, for small enough $\varepsilon>0$, that $\varrho_t>0$ for $t\geq T$, and that $\lambda_t\geq 0$.

We have $\varrho_t' = A_t \operatorname{ch} t + B_t \operatorname{sh} t$, with $A_t := 2\operatorname{sh}[a\operatorname{sh} t] + \cos[a\operatorname{ch} t] - \operatorname{ch}[a\operatorname{sh} t]$, and

$$B_t := 2a\operatorname{ch} t\cos[a\operatorname{ch} t] + 2a\operatorname{ch} t\operatorname{ch}[a\operatorname{sh} t] - a\operatorname{ch} t\operatorname{sh}[a\operatorname{sh} t] - a\operatorname{sh} t\sin[a\operatorname{ch} t] \\ + 2\sin[a\operatorname{ch} t]\,.$$

Then $A_t'/a = 2\operatorname{ch} t\operatorname{ch}[a\operatorname{sh} t] - \operatorname{sh} t\sin[a\operatorname{ch} t] - \operatorname{ch} t\operatorname{sh}[a\operatorname{sh} t] \geq 0$ entails $A_t \geq 0$, and since $\operatorname{ch} t\operatorname{sh}[a\operatorname{sh} t] + \operatorname{sh} t \leq \operatorname{ch} t\operatorname{ch}[a\operatorname{sh} t]$, we have

$$B_t \geq a\big(\operatorname{ch}[a\operatorname{sh} t]+2\cos[a\operatorname{ch} t]\big)\operatorname{ch} t-2 \geq a(1+2\cos a)\operatorname{ch} t-2 \geq 2a\operatorname{ch} t-2 > 10\,.$$

Hence $\varrho_t' \geq 10\,t$, whence for $\varepsilon < \varepsilon_0$ and $t \geq T$: $\varrho_t \geq 5t^2 - 2\sin(2\nu) > 5T^2 - 4\nu > 0$.

Then we have $\lambda_t \geq \alpha_t \operatorname{ch} t - 2\operatorname{sh} t$, with

$$\alpha_t := 2\operatorname{sh}[a\operatorname{ch} t] - \operatorname{ch}[a\operatorname{sh} t] + \cos[a\operatorname{ch} t] \quad \text{and}$$
$$a^{-1}\,\alpha_t' = (2\operatorname{ch}[a\operatorname{ch} t] - \operatorname{sh}[a\operatorname{sh} t])\operatorname{ch} t + \sin[a\operatorname{ch} t]\operatorname{sh} t \geq \operatorname{ch}[a\operatorname{ch} t]\operatorname{ch} t - \operatorname{sh} t > 0\,.$$

Whence $\alpha_t \geq \alpha_0 = 2\operatorname{sh} a - 1 + \cos a > 2a - 1 > 11$ for any small positive ε, and then

$$\lambda_t \geq 9\operatorname{ch} t > 0 \text{ (which entails } I_t' > \operatorname{ch} t) \text{ for any } t \geq 0. \qquad \square$$

Proof of Lemma 6.1. — We successively have:

$$\nu'^2 = \tfrac{2\pi^2 w^2\varepsilon}{y}\left(1 + \tfrac{4\pi^2+3}{24\pi^2 y}\,w^2\varepsilon - \tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2} + \mathcal{O}(\varepsilon^2)\right),$$
$$U = \pi - \nu + \tfrac{i\nu' x}{2(\pi-\nu)} + \tfrac{(\nu' x)^2}{8(\pi-\nu)^3} + \mathcal{O}(\nu' x)^3,$$
$$U(U+\pi) = 2\pi^2 - 3\pi\nu + \nu^2 + \tfrac{i(3\pi-2\nu)\nu' x}{2(\pi-\nu)} + \tfrac{\pi(\nu' x)^2}{8(\pi-\nu)^3} + \mathcal{O}(\nu' x)^3\,,$$
$$\tfrac{2\,U}{U+\pi} = 1 - \tfrac{\nu}{2\pi} - \tfrac{\nu^2}{4\pi^2} + \tfrac{i\,\nu' x}{4\pi^2} + \tfrac{(2\pi-\nu)(\nu' x)^2}{16\pi^2(\pi-\nu)^3} \\ + \mathcal{O}\big[(\nu' x)^3 + \nu(\nu' x)^2 + \nu^2(\nu' x) + \nu^3\big],$$

$$
\begin{aligned}
&\frac{\varepsilon\,\nu'(1-ix)}{\pi^2\,w^2}\,\Theta_\varepsilon(x)\\
&=\frac{U(U+\pi)}{2\pi^2}-\frac{y\,\nu'^2(i+x)\,x}{2\pi^2 w^2\,\varepsilon}-\frac{U\nu'^2(1-2ix-x^2)}{6\pi^2\,(U+\pi)}\\
&=1-\tfrac{3}{2\pi}\Big(\sqrt{\tfrac{w^2\varepsilon}{2\,y}}+\tfrac{w^2\varepsilon}{4\pi y}+\tfrac{4\pi^2+15}{24\,\pi^2}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{\frac32}\Big)\\
&\quad+\tfrac{1}{2\pi^2}\Big(\tfrac{w^2\varepsilon}{2y}+\tfrac{1}{\pi}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{\frac32}\Big)+\mathcal{O}\big(\varepsilon^2+\check\varepsilon^3\big)\\
&\quad+i\Big[\tfrac{(3\pi-2\nu)\nu'}{4\pi^2(\pi-\nu)}-1-\tfrac{4\pi^2+3}{24\pi^2 y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}\big(\varepsilon^2\big)\Big]\,x\\
&\quad+\Big[\tfrac{\nu'^2}{16\pi(\pi-\nu)^3}-1-\tfrac{4\pi^2+3}{24\pi^2 y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}\big(\varepsilon^2\big)\Big]\,x^2\\
&\quad-\tfrac{w^2\varepsilon}{6\,y}\,\big(1-2ix-x^2\big)\Big[1-\tfrac{\nu}{2\pi}-\tfrac{\nu^2}{4\pi^2}+\tfrac{i\,\nu' x}{4\pi^2}+\tfrac{(2\pi-\nu)(\nu' x)^2}{16\pi^2(\pi-\nu)^3}\\
&\qquad\qquad+\mathcal{O}\big(\check\varepsilon^3+\check\varepsilon^2\sqrt{\varepsilon}+\varepsilon\big)\Big]\\
&=1-\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{w^2\varepsilon}{8\pi^2 y}-\tfrac{4\pi^2+7}{16\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{\frac32}\\
&\quad-\Big[1+\tfrac{w^2\varepsilon}{6y}-\tfrac{4\pi^2+3}{12\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}\big(\varepsilon^2\big)\Big]\,x^2\\
&\quad+i\Big[-1+\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{4\pi^2-3}{24\pi^2 y}\,w^2\varepsilon+\tfrac{28\pi^2+21}{48\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}\big(\varepsilon^2\big)\Big]\,x\\
&\quad+\mathcal{O}\big(\varepsilon^2+\check\varepsilon^3\big)\\
&\quad-\big(1-2ix-x^2\big)\,\tfrac{w^2\varepsilon}{6\,y}\Big[1-\tfrac{1}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}+\tfrac{ix}{2\pi}\Big(\sqrt{\tfrac{w^2\varepsilon}{2\,y}}+\mathcal{O}(\varepsilon)\Big)\\
&\qquad\qquad+\mathcal{O}(\varepsilon)\,x^2+\mathcal{O}\big(\check\varepsilon^3+\check\varepsilon^2\sqrt{\varepsilon}+\varepsilon\big)\Big],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\frac{\varepsilon\,\nu'(1-ix)}{\pi^2\,w^2}\,\Theta_\varepsilon(x)&=1-\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{4\pi^2+3}{24\pi^2 y}\,w^2\varepsilon-\tfrac{4\pi^2+21}{48\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{\frac32}\\
&\quad-i\Big[1-\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{4\pi^2+3}{24\pi^2 y}\,w^2\varepsilon-\tfrac{4\pi^2+7}{16\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\Big]\,x\\
&\quad-\Big[1+\tfrac{2\pi^2-3}{12\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\Big]\,x^2\\
&\quad+\mathcal{O}\big(\check\varepsilon^3+\check\varepsilon^2\varepsilon+\check\varepsilon\,\varepsilon^{3/2}+\varepsilon^2\big).
\end{aligned}
$$

Since

$$
\nu'^{-1}=\tfrac{1}{\pi}\sqrt{\tfrac{y}{2w^2\varepsilon}}\,\Big(1-\tfrac{4\pi^2+3}{48\pi^2 y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{12\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}(\varepsilon^2)\Big)\,,
$$

we equivalently have

$$\begin{aligned}
&\frac{\varepsilon^{3/2}(1-ix)}{\pi\sqrt{w^2y/2}}\,\Theta_\varepsilon(x)\\
&=\frac{\varepsilon\,\nu'(1-ix)}{\pi^2\,w^2}\,\Theta_\varepsilon(x)\Big(1-\tfrac{4\pi^2+3}{48\pi^2y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{12\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}(\varepsilon^2)\Big)\\
&=1-\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{4\pi^2+3}{16\pi^2y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}+\mathcal{O}\big(\check\varepsilon^3+\check\varepsilon^2\varepsilon+\check\varepsilon\,\varepsilon^{3/2}+\varepsilon^2\big)\\
&\quad-i\Big[1-\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{4\pi^2+3}{16\pi^2y}\,w^2\varepsilon+\tfrac{\pi^2-3}{6\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\Big]x\\
&\quad-\Big[1-\tfrac{4\pi^2+3}{48\pi^2y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\Big]x^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\varepsilon^{3/2}\,\Theta_\varepsilon(x)}{\pi\sqrt{w^2y/2}}&=1-\tfrac{3}{2\pi}\sqrt{\tfrac{w^2\varepsilon}{2\,y}}-\tfrac{4\pi^2+3}{16\pi^2y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\\
&\quad+\mathcal{O}\big(\check\varepsilon^3+\check\varepsilon^2\varepsilon+\check\varepsilon\,\varepsilon^{3/2}+\varepsilon^2\big)\\
&\quad+\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\frac{i\,x}{6\pi(1-ix)}\\
&\quad-\Big[1-\tfrac{4\pi^2+3}{48\pi^2y}\,w^2\varepsilon+\tfrac{2\pi^2-3}{6\,\pi^3}\big(\tfrac{w^2\varepsilon}{2\,y}\big)^{3/2}\Big]\frac{x^2}{1-ix}\,,
\end{aligned}$$

which is directly equivalent to the formulation of Lemma 6.1. □

Proof of Lemma 6.2. — From the above proof of Lemma 5.3, we see that

$$\frac{\Re\{U_t\operatorname{cotg}U_t\}}{(\pi-\nu)}=\frac{\operatorname{ch}t\,\sin[2(\pi-\nu)\operatorname{ch}t]+\operatorname{sh}t\operatorname{sh}[2(\pi-\nu)\operatorname{sh}t]}{\operatorname{ch}[2(\pi-\nu)\operatorname{sh}t]-\cos[2(\pi-\nu)\operatorname{ch}t]}\,.$$

Then for small t we successively have:

$$\begin{aligned}
\cos\big[2(\pi-\nu)\operatorname{ch}t\big]&=1-2\nu^2+2\pi\nu t^2+\mathcal{O}\big(t^4+\nu^2t^2+\nu^4\big)\,;\\
\operatorname{ch}\big[2(\pi-\nu)\operatorname{sh}t\big]&=1+2\pi^2t^2-6\pi\nu t^2+\mathcal{O}\big(t^4+\nu^2t^2\big)\,;\\
\operatorname{ch}[2(\pi-\nu)\operatorname{sh}t]&-\cos[2(\pi-\nu)\operatorname{ch}t]\\
&=2\big(\nu^2+\pi^2[1-4\nu/\pi]\,t^2\big)\big[1+\mathcal{O}\big(t^2+\nu^2\big)\big]\,;\\
\operatorname{sh}\big[2(\pi-\nu)\operatorname{sh}t\big]&=2(\pi-\nu)t+\mathcal{O}\big(t^3\big);\\
\operatorname{sh}t\operatorname{sh}[2(\pi-\nu)\operatorname{sh}t]&=2(\pi-\nu)t^2+\mathcal{O}\big(t^4\big)\,;\\
\sin[2(\pi-\nu)\operatorname{ch}t]&=-2\nu+\pi t^2+\tfrac{\pi}{12}\,t^4+\mathcal{O}\big(t^6+\nu t^2+\nu^3\big)\\
&=\big(\pi t^2-2\nu\big)\big[1+\mathcal{O}\big(t^2+\nu^2\big)\big]\,;\\
\operatorname{ch}t\sin[2(\pi-\nu)\operatorname{ch}t]&=-2\nu+\pi t^2+\tfrac{7\pi}{12}\,t^4+\mathcal{O}\big(t^6+\nu t^2+\nu^3\big)\,;\\
\operatorname{ch}t\sin[2(\pi-\nu)\operatorname{ch}t]+\operatorname{sh}t\operatorname{sh}[2(\pi-\nu)\operatorname{sh}t]&=\big(3\pi t^2-2\nu\big)\big[1+\mathcal{O}\big(t^2+\nu^2\big)\big].
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\nu}{(\pi-\nu)}\,\Re\{U_t \operatorname{cotg} U_t\} &= \frac{\nu \times \big(3\pi t^2 - 2\nu\big)\big[1+\mathcal{O}\big(t^2+\nu^2\big)\big]}{2\big(\nu^2+\pi^2[1-4\nu/\pi]\,t^2\big)\big[1+\mathcal{O}\big(t^2+\nu^2\big)\big]} \\
&= \frac{-1+\frac{3\pi}{2\nu}t^2+\mathcal{O}\big(\nu^2+t^2+t^4/\nu\big)}{1+\pi^2t^2/\nu^2-4\pi t^2/\nu+\mathcal{O}\big(\nu^2+t^2+t^4/\nu^2\big)} \\
&= -1+\frac{\pi^2[1-5\nu/2\pi]\,t^2/\nu^2+\mathcal{O}\big(\nu^2+t^2+t^4/\nu^2\big)}{\big(1+\pi^2[1-4\nu/\pi]\,t^2/\nu^2\big)\big[1+\mathcal{O}\big(\nu^2+t^2\big)\big]} \\
&= -1+\frac{\pi^2[1-5\nu/2\pi]\,t^2/\nu^2\big[1+\mathcal{O}\big(t^2+\nu^2\big)\big]+\mathcal{O}\big(\nu^2\big)}{\big(1+\pi^2[1-4\nu/\pi]\,t^2/\nu^2\big)\big[1+\mathcal{O}\big(\nu^2+t^2\big)\big]} \\
&= -1+\frac{\pi^2[1-5\nu/2\pi]\,t^2}{\nu^2+\pi(\pi-4\nu)t^2}\big[1+\mathcal{O}\big(\nu^2+t^2\big)\big]+\mathcal{O}\big(\nu^2\big)\,,
\end{aligned}$$

whence the expansion of Lemma 6.2. □

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