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Charles Fougeron

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Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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DIFFUSION RATE OF WINDTREE MODELS AND LYAPUNOV EXPONENTS

BY CHARLES FOUGERON

ABSTRACT. — Consider a windtree model with several parallel arbitrary right-angled obstacles placed periodically on the plane. We show that its diffusion rate is the largest Lyapunov exponent of some stratum of quadratic differentials and exhibit a new general strategy to compute the generic diffusion rate in a family of such models. This result enables us to numerically compute the diffusion rates of a wide class of windtree models and to observe its asymptotic behavior according to the shape of the obstacles.

RÉSUMÉ (*Diffusion du vent dans les arbres et exposants de Lyapunov*). — Nous considérons un modèle de vent dans les arbres avec des obstacles reproduits périodiquement dans le plans. Les obstacles seront ici des polygones à angles droits dont un des côtés est parallèle au coté d'un autre obstacle. En introduisant une stratégie générale, nous montrons que le taux de diffusion pour un élément générique de cette famille de modèles est le plus grand exposant de Lyapunov associé à une strate de différentielles quadratiques. Celui permet un calcul numérique des taux de diffusion sur une grande variétés de modèles et nous observons dans un deuxième temps le comportement assymptotique de celui-ci en faisant varier la forme des obstacles.

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CHARLES FOUGERON, Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany • *E-mail :* `charles.fougeron@math.cnrs.fr` • *Url :* `http://fougeron.perso.math.cnrs.fr/index.html`

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1. Introduction

The windtree model was first introduced by Paul and Tatiana Ehrenfest in 1912 [7] as part of statistical physics investigations. In this book they set a simplified model for non-interacting light particles moving around massive particles that do not move, but on which the light particles collide with elastic collisions. We classically refer to the light particles as the *wind* and the static ones as *trees*. The motivation of the two physicists was to understand the kinetic behavior of such a system. They asked, among others, the following question: *for a generic disposition of square trees orientated in the same direction, does the speed of K light particles equidistribute asymptotically in the four possible directions?*

Plenty of questions have been studied on this model, in particular for the $\mathbb{Z}^2$-periodic case with square obstacles. The results feature alternatively elements of chaotic and periodic behavior. In [16] the recurrence of billiard flow was proven along with abnormal diffusion for special dimensions of the obstacles. In [12] the genericity of non-ergodic behavior was shown, and its diffusion rate was computed to be 2/3 in [5]. A positive answer to the original question has only been provided very recently by [17].

In parallel a similar model with smooth convex obstacles has been studied by a large amount of mathematicians throughout the twentieth century (see e.g. [2] or [19]). In this case, the billiards satisfy some hyperbolicity property and the behavior of its flow is closely related to a Brownian motion.

A good tool to check if a polygonal windtree model has such an hyperbolic behavior is provided by the diffusion rates which should be 1/2 in the case of Brownian-like motions. In particular the result of [5] destroys any hope of directly applying the methods of the smooth convex case to the rectangular model. The question is still open in the case of the asymptotic behavior of polygonal shapes approaching smooth convex ones, for example with the circle: is the diffusion rate of periodic windtree models with regular n-gons going to 1/2 when n goes to ∞? We hope that developing methods to compute these diffusion rates in more general settings will provide a first step to understanding this asymptotic behavior and the non-convex obstacles cases.

The arguments of [5] rely on a remarkable correspondence between the diffusion rate of an infinite periodic billiard table and the Lyapunov exponent of an associated translation surface. This computation was generalized in [6] to any $\mathbb{Z}^2$-periodic windtree whose trees have only right angles and are horizontally and vertically symmetric. In every of these cases, the corresponding Lyapunov exponent belongs to some 2 dimensional subbundle of the Hodge bundle. Moreover in all of these cases the Lyapunov exponent is rational and can be computed using some geometric arguments.

We introduce a general strategy to exhibit the Lyapunov exponent of some locus in a stratum that correspondsto the diffusion rate of a given periodic windtree model. First we identify a common orbit closure of almost all translation surfaces associated to a family of windtree tables; then we find an irreducible subbundle of the Hodge bundle on this locus whose top Lyapunov exponent is exactly the diffusion rate. The tools for the first craft are given by recent results of [10], [21] and [22] and are introduced in subsection 4.2. For the second one, we show an additional lemma to the work of [3] which yields the diffusion rate for any translation surface in a generic direction.

In particular, we show that computing the orbit closure of a generic element of a family of windtree models boils down to constructing good examples in the family, for which there are favorable cylinders. This enables us to show inductively that the initial orbit closure contains bigger and bigger families of translation surfaces, and eventually to prove the density of almost all surfaces in the family.

We apply this method to the case of a periodic windtree with several obstacles in its fundamental domain. We pick a family of $n \geq 2$ rectangular obstacles inside a rectangle, each rectangle having its sides parallel to one side of the fundamental rectangle, and repeat this table $\mathbb{Z}^2$-periodically in the plane. We can then show the following theorem,

Theorem 1.1. — *For all $n \geq 2$, and almost every length parameter in this family of windtree models, in almost every direction, the diffusion rate is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n})$.*

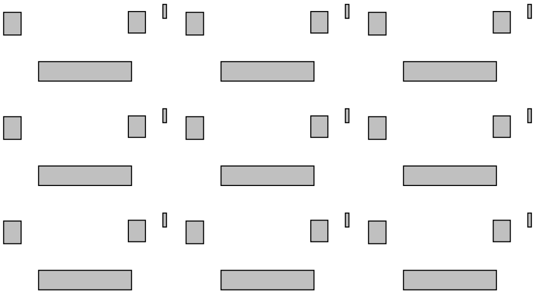

Figure 1.1. An example of a configuration of obstacles for which Theorem 1.1 applies for generic lengths

This theorem generalizes to obstacles with an arbitrary number of right angles, whose sides are parallel to the side of the rectangular fundamental domain. If n is the number of obstacles and p the total number of inward (concave) right angles in all the obstacles, we have a similar result,

Theorem 1.2. — *For all $n \geq 2$, $p \geq 0$, and almost every length parameter in this family of windtree models, in almost every direction, the diffusion rate is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n+p}, -1^p)$.*

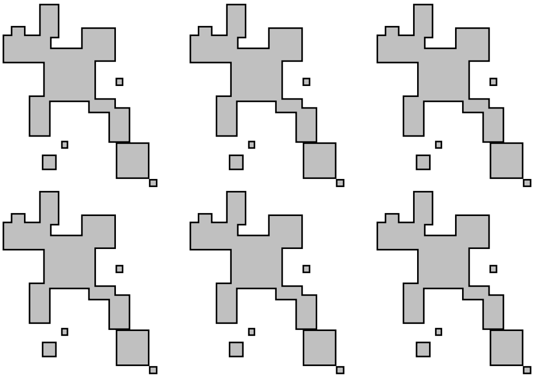

FIGURE 1.2. An example of a configuration of obstacles for which Theorem 1.2 applies for generic lengths

In the last section we discuss the value of these exponents by running numerical experiments with a Sage code developed by the author in a collaborative project [4]. These experiments give strong evidence that the family we have introduced above can approach arbitrarily close to any diffusion rate between 1/2 and 0. In particular it goes to 1/2 (*i.e.* the diffusion rate of the Brownian motion) when the number of obstacles goes to infinity.

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2. Translation surfaces

2.1. Definitions. — A *translation surface* is a surface whose change of charts are translations. Such a surface is endowed with a flat metric (the pull-back of the canonical metric on $\mathbb{R}^2$) and a canonical direction.

One way to think of these translation surfaces is by gluing the sides of a polygon via translations. Let P be a polygon with $2k$ edges and let $z_1, \dots, z_{2k}$ be complex numbers associated with the vectors of its sides. We assume that $z_i = z_{k+i}$, and glue the sides z_i and z_{k+i} to obtain a flat surface with conical singularities of angle multiples of 2π. These numbers are called *periods* of the translation surface.

We can define similar structures allowing the change of charts to be also translations composed with $-\mathrm{Id}$. The class of surfaces we obtain are called *half-translation* surfaces. The periods are still defined in the same way, but depend up to a sign on the choice of side.

Using triangulations Veech showed in [20] that this is a general construction with a notion of *pseudo-polygons* (in a much wider class of structures). The complex numbers $(z_i)_{1\leq i\leq k}$ (defined up to a sign in the case of half-translation surfaces) induce local coordinates in the moduli space of such structures, we call them *period coordinates*. We will introduce them as periods of abelian differentials below.

2.1.1. *Differentials and moduli spaces.* — There is a one-to-one correspondence between compact translation surfaces and Riemann surfaces equipped with a non-zero holomorphic 1-form. As well as between compact half-translation surfaces and Riemann surfaces equipped with quadratic differentials.

For $g \geq 1$ let $d_1, \dots, d_k \geq 0$ and $m_1, \dots, m_k \geq -1$ be integers such that $d_1 + \dots + d_k = 2g-2$ and $m_1 + \dots + m_k = 4g-4$. The strata $\mathcal{H}(d_1, \dots, d_k)$ and $\mathcal{Q}(m_1, \dots, m_k)$ are defined to be the sets of couples (S, ω) and (S, q) where S is a genus g closed Riemann surface, ω is a holomorphic 1-form on S, q is a quadratic differential form eventually with simple poles, and their zeros multiplicities are given respectively by $d_1, \dots, d_k$ and $m_1, \dots, m_k$ (a multiplicity of -1 corresponding to a simple pole). The conical points in a translation surface correspond to the zeros of the differential. If d is the multiplicity of the zero, the angle is equal to $2(d+1)\pi$ (and $(d+2)\pi$ for half-translation surfaces).

Given a translation surface (S, ω), let $\Sigma \subset S$ be the set of zeros of ω. Pick a basis $\{\xi_1, \dots, \xi_n\}$ for the relative homology group $H_1(S, \Sigma; \mathbb{Z})$. The map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^n$ defined by

$$\Phi(S, \omega) = \left(\int_{\xi_1} \omega, \dots, \int_{\xi_n} \omega \right)$$

redefines local period coordinates with translation as change of charts as above.

There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on connected components of strata coming from the linear action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$ in charts. For any translation surface in a stratum, its orbit closure via this action is some affine invariant manifold of the stratum: it is defined in local period coordinates by linear equations. They are endowed with a canonical measure supported on these manifolds called affine measures [9], [10].

In these coordinates, and for any affine subspace, we can define a notion of zero Lebesgue measure subsets. This is what we will refer to when saying Lebesgue-almost every surface. It can also be understood with respect to the Masur–Veech measure in the whole stratum (see [24] for a detailed introduction).

2.1.2. *Translation cover.* — To any half-translation surfaces S which is not a squared holomorphic form we associate its *translation cover* $\hat{S}$ corresponding to the subgroup of the fundamental group with holonomy equal to -1. It is a double cover. We endow $\hat{S}$ with the pulled-back metric of S which defines a translation surface structure for $\hat{S}$.

From a differential geometric point of view, we constructed a double cover of S on which the quadratic differential q can be written as ω^2 where ω is a holomorphic 1-form.

Let (S, q) be a half-translation surface in $\mathcal{Q}(m_1, \dots m_d)$, $\hat{S}$ its translation cover and $\hat{\Sigma}$ the preimage of its singular points. Following [1], assume there is a basis $\{a_1, b_1, \dots, a_g, b_g\}$ of $H_1(S; \mathbb{Z})$ which has trivial linear holonomy (this exists as long as there is a zero with odd multiplicity) and let $\gamma_1, \dots, \gamma_{d-2}$ be primitive non-crossing elements of $H_1(S, \Sigma; \mathbb{Z})$ representing a path from P_i to P_{i+1} where $\{P_1, \dots, P_d\} = \Sigma$.

Given a saddle connection or an absolute cycle with trivial linear holonomy γ, let γ', γ'' be its 2 lifts in $\hat{S}$ endowed with the orientation inherited from γ. Then we introduce

$$\hat{\gamma} := \gamma' - \gamma''.$$

By definition $\hat{\gamma}$ belongs to $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{C})$ the -1 eigenspace of the linear automorphism induced by the deck involution of the double cover.

Proposition 2.1. — *The family* $\{\hat{a}_1, \hat{b}_1, \dots, \hat{a}_g, \hat{b}_g, \hat{\gamma}_1, \dots, \hat{\gamma}_{d-2}\}$ *is a basis of* $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{C})$.

The dual family of $\{\hat{a}_1, \hat{b}_1, \dots, \hat{a}_g, \hat{b}_g, \hat{\gamma}_1, \dots, \hat{\gamma}_{d-2}\}$ forms a basis of the anti-invariant 1-forms,

$$H^1_-(\hat{S}, \hat{\Sigma}; \mathbb{C}) \subset H^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$$

where the relative cohomology is a local chart for some abelian stratum $\mathcal{H}(\alpha)$. This period is twice the polygonal periods we defined in Section 2.1 up to a sign.

This is the basis we will be using to express equations of billiard families.

2.2. Windtree tables.

— Let P be a filled polygon which does not self-intersect that will stand for the shape of the obstacles in our infinite billiard. Consider the plane $\mathbb{R}^2$ on which we place P periodically centered at each point of a lattice Λ as scatterers such that copies do not overlap. We denote the space consisting of the plane to which we removed the inside of every obstacle by $\mathcal{W}(P, \Lambda)$.

Definition 2.2. — We call $\mathcal{W}(P, \Lambda)$ a Λ-periodic windtree table with obstacle P.

Our purpose here is to understand the billiard flow on this infinite table and its asymptotic speed. We denote the billiard flow by

$$\phi_t^\theta : \mathcal{W}(P, \Lambda) \mapsto \mathcal{W}(P, \Lambda).$$

For $p \in \mathcal{W}(P,\Lambda)$ the point $\phi_t^\theta(p)$ is the position of the flow after time t starting from p in direction θ, which moves in straight lines until it encounters an obstacle on which it bounces according to Snell-Descartes law of reflection.

Definition 2.3. — In a windtree table $\mathcal{W}(P,\Lambda)$ for d the Euclidean distance on $\mathbb{R}$, $p \in \mathcal{W}(P,\Lambda)$ and $\theta \in [0,2\pi)$ the diffusion rate is the limit

$$\limsup_{t\to+\infty} \frac{\log d(p,\phi_t^\theta(p))}{\log t}.$$

2.2.1. *Associated flat surface.* — As the billiard table $\mathcal{W}(P,\Lambda)$ is Λ-periodic, we may consider its quotient

$$\mathcal{W}(P,\Lambda)/\Lambda \simeq \mathbb{R}^2/\Lambda - \mathring{P} =: \mathcal{T}_\Lambda(P)$$

which corresponds to playing billiards in a torus with one copy of the obstacle P placed in it. Then we associate to it a flat surface on which the linear flow corresponds to the billiard flow.

We take two copies of $\mathcal{T}_\Lambda(P)$ and glue the two copies of each side of P using an isometry fixing of the tangent vector and inverting the normal vector (the axial symmetry along this side). Now when the flow is bouncing in the billiard, the geodesic flow of the flat surface is simply a changing of the copy in the surface.

The gluing maps are changing orientation, hence in order for this surface to be a flat surface as defined above we choose as a convention two opposite orientations for the two copies. The change of charts now preserves orientation and is in $\mathrm{Iso}^+(\mathbb{C})$. We denote this flat surface by $S(P,\Lambda) = S$.

For each triple $(p,\theta,t) \in S \times [0,2\pi) \times \mathbb{R}_+$ we define an element $\gamma_t^\theta(p) \in H_1(S;\mathbb{Z})$ as follows. Consider the geodesic segment of lengths t starting from p in the direction θ and close it by a small piece of curve that does not cross h_κ and v_κ. The curve used to close the segment can be chosen to be uniformly bounded.

Let h, v be a horizontal and vertical simple loop in $\mathcal{T}_\Lambda(P)$ that generate the homology of the torus. Let $h^S = h_1 - h_2$ and $v^S = v_1 - v_2$ where h_1, h_2 (resp v_1, v_2) are the two lifts of h (resp. v) in S, and let a $f \in H^1(S;\mathbb{Z}^2)$ be a cocycle dual of (h^S, v^S) with respect to the intersection form.

The proposition below shows that the diffusion rate of a particle in a windtree table $\mathcal{W}(P,\Lambda)$ can be reduced to the study of the pairing of the approximate geodesic flow on S with f.

Proposition 2.4 ([5, Proposition 1]). — *The diffusion rate of $\phi_t^\theta(\tilde{p})$ is equal to*

$$\limsup_{t\to+\infty} \frac{\log\left|\langle f, \gamma_t^\theta(p)\rangle\right|}{\log t}$$

when p and $\tilde{p}$ project to the same point on $\mathcal{T}_\Lambda(P)$.

3. Lyapunov exponents

In [23], Zorich introduced Lyapunov exponents for translation surfaces. For any translation surfaces of genus g, we have g non-negative Lyapunov exponents

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0$$

that describe the dynamics of the linear flow on the surface. These Lyapunov exponents are associated, using Oseledets theorem, to a cocycle on the Hodge bundle over moduli space of the translation surfaces, whose fibers correspond to the cohomology group of the surface in the base. This cocycle is called the Kontsevich–Zorich cocyle.

In the case of half-translations surfaces, we consider the orientable 2-cover, which is endowed with a deck involution. This involution commutes with the cocycle, and the Lyapunov exponents spectrum splits into two parts: the invariant and anti-invariant part. Thus to any half-translation surfaces, we can associate g non-negative Lyapunov exponents from its invariant part,

$$\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq \lambda_g^+ \geq 0.$$

the top Lyapunov exponent appearing in Theorems 1.1 and 1.2 refers to λ_1^+ in the corresponding stratum of half-translation surfaces.

The reader can refer to the introduction of [5] or to the survey [24] for a detailed definition the Lyapunov exponents in this setting.

3.1. Relation to diffusion rates

In the previous section, we have seen that the diffusion rate on a windtree table is related to the asymptotic pairing of a cohomology class with a modified linear flow on the associated translation surface. We will consider throughout this section a translation S in some abelian stratum $\mathcal{H}(\alpha)$ which $\mathrm{SL}(2,\mathbb{R})$-orbit closure is the affine invariant subspace $\mathcal{M} \subset \mathcal{H}(\alpha)$ and $\nu_{\mathcal{M}}$ its invariant measure.

The following theorem relates the diffusion rate with a Lyapunov exponent,

Theorem ([5, Theorem 2]). — *Let $F_1 \supset F_2 \supset \cdots \supset F_k$ be the Oseledets flag decomposition of the Kontsevich-Zorich cocycle on $\mathcal{M}$, and λ be its top Lyapunov exponent. For every $\nu_{\mathcal{M}}$-Oseledets generic translation surface $S \in \mathcal{H}(\alpha)$, and for every point $p \in S$ with infinite forward orbit, for all $f \in F_1 \setminus F_2$,*

$$\limsup_{t \to +\infty} \frac{\log |\langle f, \gamma_t(p) \rangle|}{\log t} = \lambda.$$

In [3] it is proven that any translation surface $S \in \mathcal{H}(\alpha)$ such that its $\mathrm{SL}(2,\mathbb{R})$-orbit closure is $\mathcal{M}$ is Oseledets generic in Lebesgue-almost every direction. In particular they show the following theorem,

Theorem ([3, Theorem 1.5]). — *Fix $S \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R}) \cdot S}$ be the smallest affine invariant manifold containing S. Let V be a $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle which is defined and continuous on $\mathcal{M}$.*

Let $A_V : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{M} \to V$ *denote the restriction of the Kontsevich-Zorich cocycle to* V *and suppose that* A_V *is strongly irreducible with respect to the affine measure* $\nu_{\mathcal{M}}$ *whose support is* $\mathcal{M}$*. Then, for almost every* $\theta \in [0, 2\pi)$*,*

$$\lim_{t\to\infty} \frac{\log \|A_V(g_t r_\theta x)\|}{\log t} \to \lambda_1,$$

where λ_1 *is the top Lyapunov exponent of* A_V*.*

A little modification in their argument, that we provide in the appendix, enables us to show an additional lemma to this theorem.

Lemma 3.1. — *In the previous theorem, for any* $h \in V$ *and almost every* $\theta \in [0, 2\pi)$*,*

$$\lim_{t\to\infty} \frac{\log \|A_V(g_t r_\theta x)h\|}{\log t} \to \lambda_1,$$

where λ_1 *is the top Lyapunov exponent of* A_V*.*

This reduces the computation of the diffusion rate of a windtree model to determining irreducible components of the Kontsevich-Zorich cocycle along $\mathrm{SL}(2,\mathbb{R})$-orbits and to determine which of these contain the cohomology class f.

In our case this will be done by the following irreducibility lemma,

Lemma 3.2. — *In strata of quadratic differentials with at most simple poles, and more than 3 singularities that are not all of even order, the Kontsevich-Zorich cocycle is strongly irreducible on* H_1^+ *for the action of* $\mathrm{SL}(2,\mathbb{R})$*.*

Proof. — The tautological bundle generated by the real and imaginary part of the abelian form associated to a surface in the stratum is contained in H^- and not in H^+. Thus according to Theorem 1.1 of [8], the algebraic hull of the Kontsevich-Zorich cocycle is the Zariski closure of monodromy. But the monodromy on H^+ is Zariski dense in $Sp(2g, \mathbb{R})$ according to Section 6 in [15]. Hence H^+ cannot have invariant subspaces for the Kontsevich Zorich cocycle, and is strongly irreducible. □

This implies the following.

Corollary 3.3. — *Let* S *be a half-translation surface whose* $\mathrm{GL}(2,\mathbb{R})$ *orbit is dense in a quadratic stratum, then for almost all direction and every point* $p \in S$ *with infinite forward orbit, for all* $f \in H^1(S;\mathbb{R})$*,*

$$\limsup_{t\to+\infty} \frac{\log |\langle f, \gamma_t(p)\rangle|}{\log t} = \lambda_1$$

where λ_1 *is the top Lyapunov exponent of the quadratic stratum.*

4. Orbit closure

4.1. Definitions. — As mentioned previously, orbit closures of translation surfaces are affine invariant submanifolds of a stratum, *i.e.* they are defined by affine subspaces in period coordinates.

Following [22], we introduce

Definition 4.1. — Let $\mathcal{L}$ be an affine invariant submanifold. Two cylinders on a translation surface $S \in \mathcal{L}$ are $\mathcal{L}$-parallel if they are parallel at S and at every nearby surface $S' \in \mathcal{L}$.

Consider the matrices,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}.$$

Let $\mathcal{C}$ be a collection of horizontal cylinders on a translation surface S. We define $u_t^{\mathcal{C}}(S)$ (resp. $a_t^{\mathcal{C}}(S)$) as the translation surface obtained by applying the matrix u_t (resp. a_t) to the cylinders in $\mathcal{C}$ but not to the rest of the surfaces.

Then we have

Theorem (The Cylinder Deformation Theorem [22]). — *Let $\mathcal{L}$ be an affine invariant manifold and S a translation surface in $\mathcal{L}$ and $\mathcal{C}$ an equivalence class of $\mathcal{L}$-parallel horizontal cylinders in S. Then for all $s, t \in \mathbb{R}$ the surfaces $a_s^{\mathcal{C}}(u_t^{\mathcal{C}}(S))$ remains in $\mathcal{L}$.*

4.2. Technical lemmas. — In this section we introduce some technical lemmas resulting from recent breakthroughs in the theory [10], [21] and [22]. They will be precious tools for computing orbit closures of translation surfaces associated with windtree models and show the main result of this article.

This first lemma shows the existence of one generic orbit closure which contains orbit closures of all surfaces in the family.

Lemma 4.2. — *Let $\mathcal{B}$ be a family of flat surfaces in a fixed stratum which is represented in some period coordinates by a real linear subspace B. Then for Lebesgue-almost every $S \in \mathcal{B}$, the orbit closure is an unique* $\mathrm{GL}_2(\mathbb{R})$*-invariant suborbifold $\mathcal{L}$ of the Teichmüller space. Moreover, in the above period coordinate, $\mathcal{L}$ is a linear subspace L such that $B \subset L$.*

Proof. — According to [10] Proposition 2.16 or alternatively [21] Corollary 1.9, there are countably many $\mathrm{GL}_2(\mathbb{R})$-invariant closed orbifolds in each stratum. Thus at least one orbit closure $\mathcal{L}$ of the family $\mathcal{B}$ intersects $\mathcal{B}$ with non-zero Lebesgue measure in $\mathcal{B}$. In period coordinates, if two linear subspace L and B intersect with non-zero Lebesgue measure in B, then $B \subset L$.

We now take the $\mathcal{L}_0$ intersection of all $\mathcal{L}$ as above. This intersection, as for any $\mathcal{L}$, is a closed $\mathrm{GL}_2(\mathbb{R})$-invariant subset which contains $\mathcal{B}$. Thus the orbit closure of any point of $\mathcal{L}_0$ is contained in $\mathcal{L}_0$. This implies that any $\mathcal{L}$

as above coincides with $\mathcal{L}_0$. Thus for any $S \in \mathcal{B}$ whose orbit closure has a non-zero-measure intersection with $\mathcal{B}$, $\overline{\mathrm{GL}_2(\mathbb{R}) \cdot S} = \mathcal{L}_0$.

Hence for any $S \in \mathcal{B}$ such that $\mathcal{N} := \overline{\mathrm{GL}_2(\mathbb{R}) \cdot S} \neq \mathcal{L}_0$, the Lebesgue measure of $\mathcal{N} \cap \mathcal{B}$ is zero. By taking out the countable number of such subsets from $\mathcal{B}$, the set of remaining points is of full Lebesgue measure, and the orbit closure of each of these points is $\mathcal{L}_0$. ☐

The following lemma will enable us to remove some *real* equations in the hyperplane defining the orbit closure of a given family.

Lemma 4.3. — *Suppose with the notation of the previous lemma, that B contains a $\mathbb{R}$-linear subspace D. Let D_{Re} and D_{Im} be the projections of D to $H^1(S;\mathbb{R})$ and $H^1(S;i\mathbb{R})$. Then $\mathrm{Vect}_{\mathbb{C}}(D_{Re}, D_{Im}) \subset L$ i.e. L contains the $\mathbb{C}$-linear span of D_{Re} and D_{Im}.*

Proof. — By Lemma 4.2, we know that $D \subset L$. By [21] the field of definition of such an affine manifold is real, in particular it is the complexification of L_{Re}. ☐

Once we have conjectured what the generic orbit closure of our billiard should be, our goal will consist of finding a surface in the family that has some good cylinder decomposition. The following lemma gives an other form of the cylinder deformation theorem, quoted in the previous subsection, that will be useful for us in this purpose.

Lemma 4.4. — *Let S be a half-translation surface, Σ the set of its singularities, and $\gamma_1, \gamma_2, \dots, \gamma_d$ a basis of $H_1(S, \Sigma; \mathbb{Z})$. We denote by $\hat{\gamma}_i$ their periods in S.*

If η is the homology class of the union of core curves of $\mathcal{L}$-parallel cylinders in the surface associated to periods $\hat{\gamma}_i$, then for all δ in a neighborhood of zero in $\mathbb{C}$, the surface with periods

$$\hat{\gamma}_i + \langle \eta, \gamma_i \rangle \, \delta$$

is in the orbit closure $\mathcal{L}$.

Proof. — According to the cylinder deformation theorem, it is sufficient to show that for one horizontal cylinder with core curve η, and a given simple curve γ the deformation $a_s^{\mathcal{C}}(u_t^{\mathcal{C}}(S))$ on the cylinder acts on the periods by adding $\langle \eta, \gamma \rangle \, \delta$ for δ in a neighborhood of 0 in $\mathbb{C}$.

The curve γ is only deformed by definition for its parts that are in the given cylinder. Choose a representation of γ such that each time it enters the cylinder, it leaves it on the opposite side in a straight vertical line. Then if h is the height of the cylinder, each upward vertical component of the path in the cylinder contributes ih to the period which by the cylinder deformation becomes $h(ie^s + t)$, so $h\,(t + i(e^s - 1))$ will be added. For downward components the opposite is added, thus in total

$$\langle \eta, \gamma \rangle \cdot h\,(t + i(e^s - 1))\,. \qquad ☐$$

Remark 4.5. — In the same article, A. Wright shows that the ratios of circumferences of an equivalence class of $\mathcal{L}$-parallel cylinders, generate the field of definition of $\mathcal{L}$ (Theorem 7.1 [22]). Moreover it was proved by the same author that this field of definition is a number field of degree at most the genus (Theorem 1.1 [21]). Thus if we have a cylinder decomposition such that the ratios of the circumferences of the cylinders are transcendent, they cannot be parallel in any affine invariant submanifold.

Consider a set of equations defining the family of translation surfaces $\mathcal{B}$ or more precisely its linear subspace B in period coordinates. Each equation is associated with a linear form $\phi_1, \dots, \phi_k$, such that $B = \bigcap_{i=1}^{k} \ker(\phi_i)$. Now if we manage to find a vector v in the subspace L defining the generic orbit closure of the family, such that v satisfies all but one of the above equations, *e.g.* $v \in \bigcap_{i=1}^{k-1} \ker(\phi_i)$, then

$$B \subset B' = \bigcap_{i=1}^{k-1} \ker(\phi_i) \subset L.$$

The generic orbit closure of B and B' will remain the same. Thus we can consider the new set of equations defining B' to compute the orbit closure. With such a process, we remove equations inductively, until B' is the whole ambient space.

Remark 4.6. — If we have a set of equations in a given basis, and we find a cylinder that has zero intersection number with all but one element of this basis, we can try to remove the equations in which this element appears. Of course we must be careful to break only one equation at each step. In particular it should concern at most two real equations.

4.3. Periodic windtree with several obstacles. — Choose a layout for n rectangular obstacles in the plane, all oriented in the same horizontal direction. Now repeat this pattern $\mathbb{Z}^2$-periodically in the plane, assuming as an initial step that they do not overlap. In other words, pick a square torus in which you place n horizontal rectangular obstacles. We call $\mathcal{B}_n$ this family of billiards. We investigate its generic $\mathrm{GL}(2,\mathbb{R})$ orbit closure to compute its diffusion rate. The case of $\mathcal{B}_1$ was done in [5] in which the authors proved that the diffusion rate is $2/3$.

As in 2.2 we associate to each windtree table in $\mathcal{B}_n$ a half-translation surface in $\mathcal{Q}(1^{4n})$. This yields an embedding

$$\mathcal{S} : \mathcal{B}_n \mapsto \mathcal{Q}(1^{4n}).$$

The rest of this section is devoted to proving the following theorem.

Theorem 4.7. — *For all $n \geq 2$, the image of almost all half-translation surface in $\mathcal{Q}(1^{4n})$ associated to a windtree table in B_n has a dense* $\mathrm{GL}(2,\mathbb{R})$*-orbit.*

The first two paragraphs will introduce affine equations that define the subset of translation surfaces in a stratum associated to the family of windtree models.

The next three paragraph will show inductively that the $\mathrm{GL}(2,\mathbb{R})$-orbit closure of this subset in the stratum is constrained by no equation.

The first argument will rely on Lemma 4.3, to show a real-like equation does does not constrains the orbit closure. The rest of the proof relies on Lemma 4.4 and constructing elements of the orbit closure that have a cylinder with nice properties.

Theorem 4.7 together with Lemma 4.2 and Corollary 3.3 imply the following,

Theorem (Theorem 1.1). — *For all $n \geq 2$, and for Lebesgue-almost every length parameter of windtree models in B_n, in Lebesgue-almost every direction, the diffusion rate is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n})$.*

We will then show how our methods generalize to prove,

Theorem (Theorem 1.2). — *For all $n \geq 2$, $k_1, \dots k_n \geq 0$, $p = \sum k_i$, and for Lebesgue-almost every length parameter in $B_n(k_1, \dots, k_n)$, in, Lebesgue-almost every direction, the diffusion rate is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n+p}, -1^p)$.*

4.3.1. *Relative homology basis.* — Let S be a surface in $\mathcal{S}(\mathcal{B}_n)$, it has genus $n+1$ and the stratum $\mathcal{Q}(1^{4n})$ has dimension $6n$. We consider a_1, b_1, a_2, b_2 simple loops which generate the homology of the two copies of the torus, and take $c_1, d_1, \dots, c_{n-1}, d_{n-1}$ as the loops around the obstacles and between two consecutive obstacles. These generate the absolute homology of S. Now for each obstacle i, start at the lower left corner and browse the rectangle clockwise, we denote by $\alpha_i, \beta_i, \alpha'_i$ the three saddle connections we cover until reaching the lower right corner. Let γ_i be a path from the lower right corner of obstacle i to the lower left corner of obstacle $i+1$.

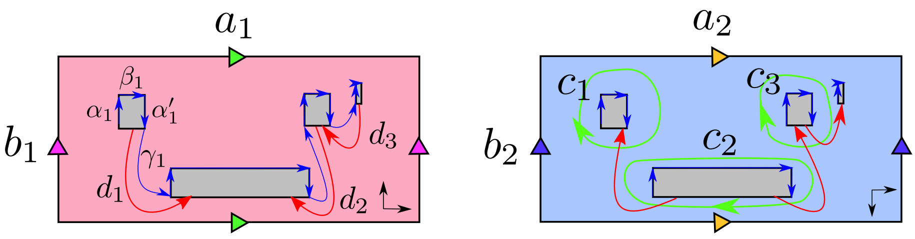

Figure 4.1. Basis for relative homology

These paths form a basis of the relative homology group of S. According to Lemma 2.1 if we take the hat image of these homology elements besides from $\hat{\alpha}'_{n-1}$ they form a basis of $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{C})$ which induce local coordinates in the stratum (see e.g. [1]) called period coordinates.

We also introduce β'_i for $1 \leq i \leq n-1$, the last side of the rectangle that closes obstacle i. In other term, the class that satisfies $\alpha_i + \beta_i + \alpha'_i + \beta'_i = c_i$. For ways of intersection numbers with cylinders we will construct later, we prefer to replace c_i by β'_i in the basis and equations.

4.3.2. *Equations defining the family.* — To write down equations in period coordinates we need to eliminate an ambiguity given by the non-trivial holonomy of the surface. We choose a fundamental domain for the action of this holonomy given by the two copies glued along the vertical sides to which we remove the horizontal sides. This corresponds to drawing the copies reflected along the horizontal axis. Now the family $\mathcal{S}$ is defined locally by the following equations, by an abuse of notation we write the homology class while meaning their period,

$$\begin{gathered} \operatorname{Re}(a_1) = -\operatorname{Re}(a_2) \\ \operatorname{Im}(b_1) = -\operatorname{Im}(b_2) \\ \operatorname{Im}(a_1) = 0,\ \operatorname{Im}(a_2) = 0 \\ \operatorname{Re}(b_1) = 0,\ \operatorname{Re}(b_2) = 0 \end{gathered} \tag{1}$$

$$\begin{gathered} \operatorname{Im}(\beta_i) = 0,\ \operatorname{Re}(\alpha_i) = 0 \text{ for all } 1 \leq i \leq n \\ \alpha_i = -\alpha'_i \text{ for all } 1 \leq i \leq n-1 \\ \beta_i = -\beta'_i \text{ for all } 1 \leq i \leq n-1 \end{gathered} \tag{2}$$

$$d_i = \gamma_i - \overline{\gamma}_i \text{ for all } 1 \leq i \leq n-1. \tag{3}$$

There are $2n+6$ real equations and $3n-3$ complex equations which are clearly independent in this basis. The quadratic stratum is of complex dimension $6n$, thus the induced subspace is of real dimension $12n - 2n - 6n = 4n$. On the other hand for the family of billiards, we have $2n$ variables for the size of each obstacle, $2n-2$ for the relative position of the obstacles, and 2 dimensions for the size of the rectangular torus. Thus we have indeed listed all the equations that define our billiard family.

4.3.3. *Removing equation (3).* — Notice that the periods appearing in equation (3) appear in no other equation. By applying Lemma 4.3, we see that $\operatorname{Vect}_{\mathbb{C}}(d_i)$ and $\operatorname{Vect}_{\mathbb{C}}(d_i + 2\gamma_i)$ are contained in L. Thus L is contains the hyperspace defined by equations (2) and (3).

4.3.4. *Removing equations (1).* — We now aim at removing the equation that bind b_1 and b_2. The first naive attempt to consider a horizontal cylinder in one torus that does not meet any obstacle, always has a parallel cylinder in the other copy of the torus that change the periods such that the equation on b_1 and b_2 remains satisfied.

We make use of the fact proven above, that in the orbit closure γ_i and d_i have no correlation thus we can move the "obstacles" in the two copies independently. Figure 4.2 shows how to have a cylinder in one torus and not in the other: by placing the obstacles in one copy such that there is no horizontal loop in one torus but there is in the other.

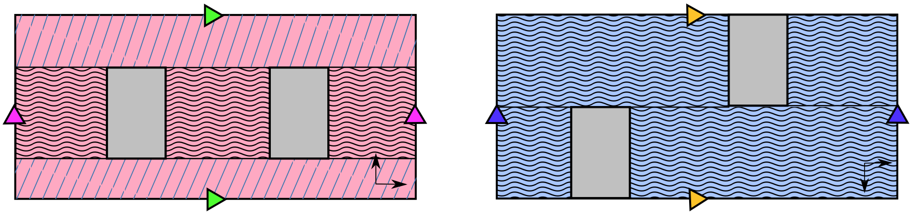

FIGURE 4.2. A good cylinder decomposition to change only b_1

For a generic choice of lengths, the hatched cylinder is not commensurable to any other cylinder and its core curve intersects only b_1. The cylinder deformation states that the orbit closure contains any surface with the same periods as the initial surfaces, except for b_1 to which any complex number can be added. Thus by independently adding a real and a pure imaginary number, we show that the equations $\mathrm{Re}(b_1) = 0$, and $\mathrm{Im}(b_1) = \mathrm{Im}(b_2)$ can be removed from the list of equations defining L.

By making a symmetric construction in the other copy of the torus, we can change the period of b_2 and show that the equation $\mathrm{Re}(b_2) = 0$ can be removed.

The same construction in the vertical direction breaks the equations on a_1 and a_2.

As a result, the affine space L contains the space defined by equations (1) and (2) minus the equations where a and b appear.

4.3.5. *Removing equations (2).* — Consider now a billiard with the same square obstacles of transcendent side length such that all the obstacles are aligned horizontally in the same order used for defining the basis. The distances between the obstacles are chosen to be algebraically independent. On these surfaces there is a full decomposition in cylinders and all of the cylinders are not parallel in any affine invariant submanifold. The cylinder going from the right of the last obstacle to the left of the first intersects b_1, α_1 and b_2 in the chosen basis (with a wavy pattern in Figure 4.3). Where we use the fact that the right side of the last obstacle is not taken in our basis.

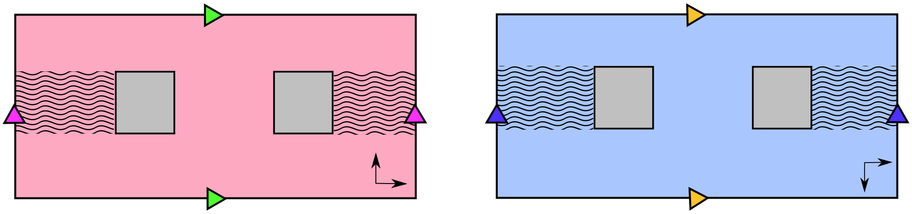

FIGURE 4.3. A good cylinder decomposition to change α_1

The number of intersection of the core curve with each one of these curves is one. The previous argument has eliminated the constrains on b_1 and b_2 thus this cylinder deformation breaks the equations $\mathrm{Re}(\alpha_1) = 0$ and $\mathrm{Im}(\alpha_1) = \mathrm{Im}(\alpha'_1)$. Only $\mathrm{Re}(\alpha'_1) = 0$ remains.

To get rid of the equation $\mathrm{Re}(\alpha'_1) = 0$ we consider another surface in which the obstacles are aligned vertically and whose width are algebraically independent. The horizontal cylinder joining the two sides of the first obstacle is intersecting α'_1, b_1, b_2 and α_1, and is not parallel to any other cylinder (in a wavy pattern in Figure 4.4). Only α'_1 appears in the remaining equations, and the cylinder deformation breaks this equation.

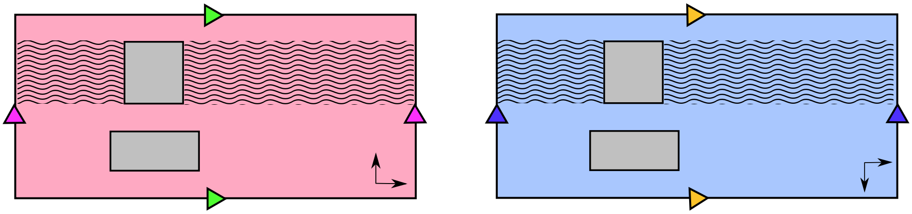

FIGURE 4.4. A good cylinder decomposition to change α'_1

Now we show by induction on i that no equation with α_i and α'_i constrains the orbit closure.

Take the cylinder intersecting α'_i and α_{i+1}. By the previous steps α'_i does not appear in any equation defining the orbit closure, so we can break the equations $\mathrm{Re}(\alpha_{i+1}) = 0$ and $\mathrm{Im}(\alpha_{i+1}) = \mathrm{Im}(\alpha'_{i+1})$.

We take a cylinder joining the sides of the $(i + 1)$-th obstacle in the second surface, as in Figure 4.4, to break the equation $\mathrm{Re}(\alpha_{i+1})'$.

The same argument can be applied in the vertical direction for β_i and β'_i. This ends the proof of generic density for billiards in $\mathcal{S}(\mathcal{B}_n)$.

4.3.6. *Obstacles with many right angles.* — Consider now a more general periodic windtree table with n obstacles which are horizontal polygons with right angles. For each obstacle i there are k_i inward (concave) and $4 + k_i$ outward (convex) right angles. This implies that the obstacle has $2 + k_i$ vertical and $2 + k_i$ horizontal sides. We denote this family by $\mathcal{B}_n(k_1, \ldots, k_n)$.

The associated quadratic differential has simple zeros at the outward right angles and poles at the inward. It has genus $n + 1$ and is in the stratum $\mathcal{Q}(1^{4n+p}, -1^p)$, where $p = \sum k_i$.

To construct a basis of homology of the associated translation surface, we start from the left point of the lowest horizontal side and browse the obstacle boundary clockwise until we come back to the starting point. This yields saddle connections $\alpha_i^1, \beta_i^1, \alpha_i^2, \ldots, \alpha_i^{4+k_i}, \beta_i^{4+k_i}$. The classes $\alpha_n^{4+k_n}$ and $\beta_n^{4+k_n}$ are not taken into consideration to yield a basis of $H_1^-(S, \Sigma; \mathbb{C})$. Let γ_i be the path joining the starting points of two consecutive obstacles i and $i + 1$, and define as in the previous section absolute homology classes a, b and d.

The equations in period coordinates are very similar as in the previous case, we only need to adapt equations on the obstacles.

$$\begin{aligned}
&\operatorname{Re}(a_1) = -\operatorname{Re}(a_2),\ \operatorname{Im}(b_1) = -\operatorname{Im}(b_2)\\
&\operatorname{Im}(a_1) = 0,\ \operatorname{Im}(a_2) = 0,\ \operatorname{Re}(b_1) = 0,\ \operatorname{Re}(b_2) = 0
\end{aligned}$$

$$\begin{aligned}
&\operatorname{Re}(\alpha_i^j) = 0 \text{ for all } 1 \le i \le n \text{ and } 1 \le j \le 2 + k_i - 1\\
&\operatorname{Im}(\beta_i^j) = 0 \text{ for all } 1 \le i \le n \text{ and } 1 \le j \le 2 + k_i - 1\\
&\sum_{j=1}^{4+k_i} \alpha_i^j = 0,\ \sum_{j=1}^{4+k_i} \beta_i^j = 0 \text{ for all } 1 \le i \le n - 1
\end{aligned}$$

$$d_i = \gamma_i - \overline{\gamma}_i \text{ for all } 1 \le i \le n - 1.$$

There are now $\sum(4 + 2k_i - 2) + 6 = 2n + 2p + 6$ real equations and $3n - 3$ complex equations that are clearly independent. The quadratic stratum is of complex dimension

$$2(n + 1) + 4n + 2p - 2 = 6n + 2p,$$

thus the induced subspace is of real dimension

$$12n + 4p - 2n - 2p - 6 - 6n + 6 = 4n + 2p.$$

On the other hand for the family of billiards, we have $\sum(4+2k_i-2) = 2n+2p$ variables for the size of each obstacle, $2n - 2$ for the relative position of the obstacles, and 2 dimensions for the size of the rectangular torus. Thus we have indeed listed all the equations that define our billiard family.

The first part of the previous argument applies *verbatim* to this case with the real and imaginary part equations. For the second part we need to exhibit a similar construction of cylinders. The construction of Figure 4.2 is straightforward enough to generalize to any shape of obstacle. We will detail the generalization of the construction in Figure 4.3.

We start with the vertical side that does not appear in the basis. Now we can find an element of the family such that the obstacle n is in the neighborhood of a rectangle as in Figure 4.5, making every other side very small, and similarly for the first obstacle. There is a horizontal cylinder joining the given side of obstacle n with a side of the first obstacle. This surface will be completely decomposed into horizontal cylinders and the lengths are chosen to be all algebraically independent.

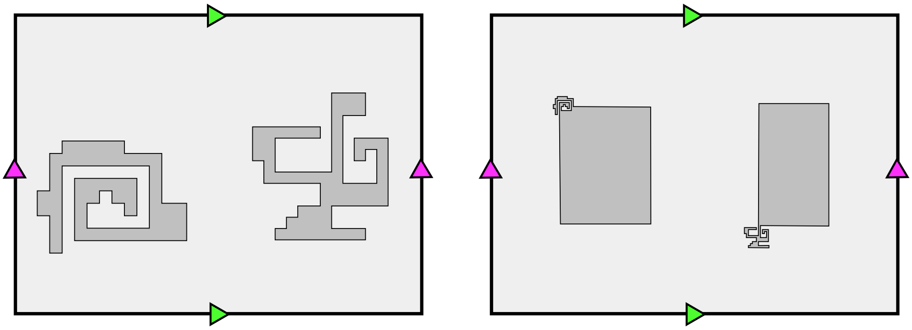

FIGURE 4.5. Example of deformation

This enables us to remove the equations constraining α_1^j. Then by induction we show that a generic billiard in $\mathcal{B}_n(k_1, \ldots, k_n)$ induces a quadratic differential with dense orbits in the stratum. This concludes the proof of Theorem 4.7.

5. Some numerical computations

Figure 5.1 shows numerical approximations of the principal Lyapunov exponent of strata $\mathcal{Q}(1^{4n})$. We observe that it goes to $1/2$ when $n \to \infty$.

In Figure 5.2, we represent a computation of the principal Lyapunov exponent for $\mathcal{Q}(1^{4n+10}, -1^{10})$. When we fix the number of simple poles and increase the number of simple zeros, the diffusion rate again goes to $1/2$ but now by smaller values.

The $1/2$ value is also the diffusion rate for the Brownian motion. Intuitively, these convex angles scatter the linear flow which follows completely different paths from one side to the other of the singularity. They mimic the hyperbolic behavior of smooth convex obstacles.

An opposite behavior is given by the concave right angles of the obstacles. In Figure 5.3, we present the largest Lyapunov exponent of strata corresponding to windtrees with two obstacles with an increasing number of concave angles.

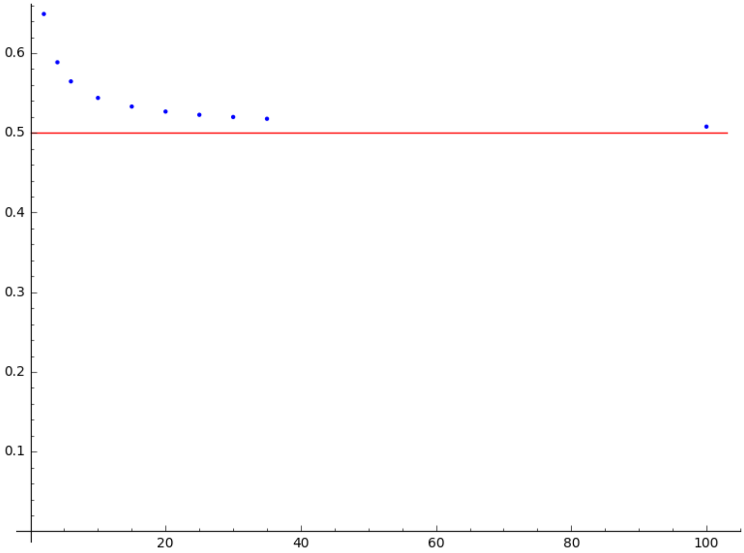

FIGURE 5.1. Principal Lyapunov exponent for $\mathcal{Q}(1^{4n})$

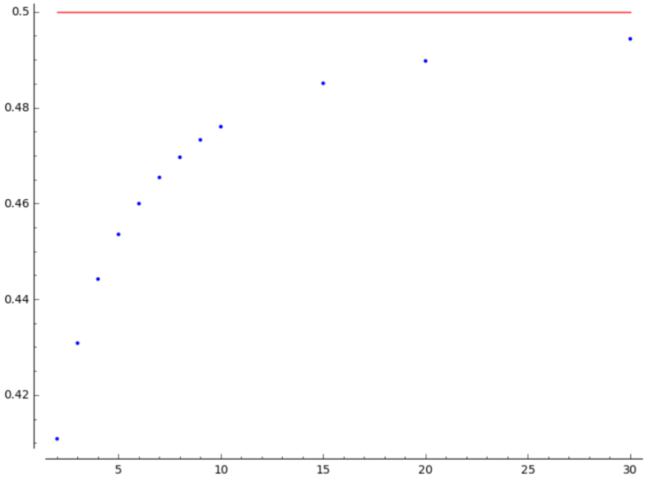

FIGURE 5.2. Principal Lyapunov exponent for $\mathcal{Q}(1^{4n+10}, -1^{10})$

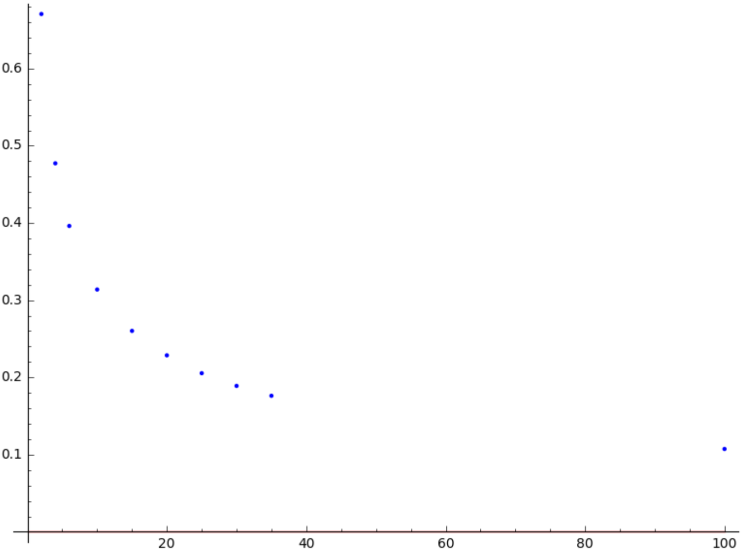

FIGURE 5.3. Principal Lyapunov exponent for $\mathcal{Q}(1^{8+p}, -1^p)$

Further experiments show that contrary to the previous case, for a fixed number of simple zeros and a number of simple poles going to infinity, the principal Lyapunov exponent is going to zero. A heuristic explanation for this phenomenon is that when the flow hits the obstacle close to a concave right angle in the billiard it comes back on its steps slightly shifted as drawn in Figure 5.4. This enters into resonance with the result of [6] which states that when we increase the number of concave right angles of a single obstacle for a periodic windtree, the diffusion rate goes to zero. This also enters in the frame of the more general Grivaux-Hubert conjecture that we explore and reformulate in [11].

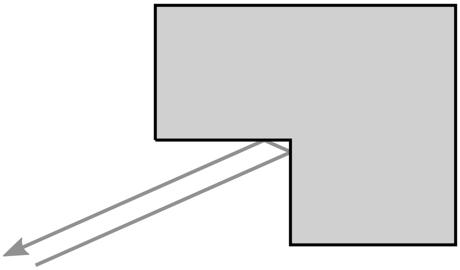

FIGURE 5.4. Flow bouncing close to a concave right angle

Appendix A. Generic Lyapunov exponent

In this section we follow the proof of [3] which shows that any translation surface is Lyapunov and Birkhoff generic in its orbit closure for almost every direction. We will focus on one of the key results in this article about Lyapunov genericity on a irreducible component for the Kontsevich-Zorich cocycle.

THEOREM (1.5 in [3]). — *Fix* $x \in \mathcal{H}_1(\alpha)$ *and let* $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R})x}$ *be the smallest affine invariant manifold containing* x*, let* V *be a* $\mathrm{SL}(2,\mathbb{R})$ *invariant subbundle of the Hodge bundle which is defined and continuous on* $\mathcal{M}$*. Let* $A_V : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{M} \to V$ *denote the restriction of the Kontsevich-Zorich cocycle to* V *and suppose that* A_V *is strongly irreducible with respect to the affine measure* $\nu_{\mathcal{M}}$ *whose support is* $\mathcal{M}$*. Then, for almost every* $\theta \in [0, 2\pi)$*,*

$$\lim_{t\to\infty} \frac{\log \|A_V(g_t r_\theta x)\|}{\log t} \to \lambda_1$$

where λ_1 *is the top Lyapunov exponent of* A_V*.*

Our purpose here is to show the following additional lemma to this theorem, introduced as Lemma 3.1 in section 3.

LEMMA. — *In the previous theorem, for any* $h \in V$ *and almost every* $\theta \in [0, 2\pi)$

$$\lim_{t\to\infty} \frac{\log \|A_V(g_t r_\theta x)h\|}{\log t} \to \lambda_1.$$

In [3] intuition of the result is provided by analogy with random walks. We start by showing the analog of Lemma 3.1 for random walks.

A.1. Random walks.

— Let μ be a $\mathrm{SO}(2,\mathbb{R})$-invariant compactly supported measure on $\mathrm{SL}(2,\mathbb{R})$ which is absolutely continuous with respect to Haar measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called μ-stationary if

$$\mu * \nu = \int_{\mathrm{SL}(2,\mathbb{R})} (g_*\nu) d\mu(g) = \nu.$$

By a theorem of Furstenberg [14], [13], restated in [18, Theorem 1.4], there exists a probability measure ρ on $\mathrm{SL}(2,\mathbb{R})$ such that the map $\nu \to \rho * \nu$ is a bijection between ergodic measures for the action of an upper triangular subgroup of $\mathrm{SL}(2,\mathbb{R})$ and ergodic μ stationary measures which are $\mathrm{SL}(2,\mathbb{R})$-invariant affine measures according to [9, Theorem 1.4].

This is a first step for an analogy between Teichmüller flow in some affine invariant locus and a random walk with the associated measure.

Let Gr_s denote the Grassmanmian of s-dimensional subspaces in the $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle V. Let $\tilde{\mathcal{H}} = \mathcal{H}_1(\alpha) \times \mathrm{Gr}_s$ and $\tilde{\nu}$ be the μ stationary measure on it; we may write $d\tilde{\nu}(x,U) = d\nu(x)d\eta_x(U)$.

The measure η_x on Gr_s heuristically corresponds to the mean position of any linear subspace carried along the Teichmüller flow using a Gauss-Manin connection. Let h be some vector in $V \setminus 0$ and $I(h) \subset \mathrm{Gr}_s$ be the set of s-dimensional subspaces containing h.

Lemma (C.10 in [9]). — *If the cocycle A_V is strongly irreducible on V then for almost every $x \in \mathcal{H}_1(\alpha)$ and any vector $h_x \in V$, $\delta_x(I(h_x)) = 0$*

In particular if we consider some Oseledets flag this Lemma yields that generically they do not contain a fixed vector h along random walks.

We showed a random walk version of the theorem in the previous paragraph.

Theorem A.1 (Theorem 2.6 and Lemma 2.9 of [3]). — *Fix $x \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R}) \cdot x}$ be the smallest affine invariant manifold containing x, let V be a $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle which is defined and continuous on $\mathcal{M}$. Let $A_V : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{M} \to V$ denote the restriction of the Kontsevich-Zorich cocycle to V and suppose that A_V is strongly irreducible with respect to the affine measure $\nu_{\mathcal{M}}$ whose support is $\mathcal{M}$. Then for a fixed $h \in V$ and for $\mu^{\mathbb{N}}$-almost every $\overline{g} = (g_1, \dots, g_n, \dots)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_V(g_n \dots g_1, x)h\| \to \lambda_1$$

where λ_1 is the top Lyapunov exponent of A_V.

This theorem already appears in [3] as a remark to a more general theorem. We reformulate the proof in this specific case for convenience to the reader.

A.2. Proof of Theorem A.1. — We fix $\mathcal{M}$ and V as in the theorem. Pick an arbitrary $v_0 \in V$ and let $v_i(\overline{g}) = A_V(g_i \dots g_1, x)v_0$. The key tool to prove this theorem is a decomposition lemma for the sequences of cocycle in the case of strong irreducibility.

Lemma (2.11 and 2.16 in [3]). — *For all $\epsilon > 0$, there exists an integer L such that for every $x \in \mathcal{M}$, for almost every $\overline{g}$, we have that all but a set of $\mathbb{N}$ of density 4ϵ is in disjoint blocks $[i+1, i+L]$ so that*

$$\exp(\lambda_1 - \epsilon)^L \leq \frac{\|A_v(g_{i+L} \dots g_{i+1}, y)v\|}{\|v\|} \leq \exp(\lambda_1 + \epsilon)^L.$$

Proof. — Refer to section 2.3 of [3]. □

Now let $\overline{g}$ be in the full measure set as above, K be the subset of density 4ϵ and I the set of indices i in the blocks $[i+1, i+L]$. Then for $n \gg L$,

$$\begin{aligned}\log\|v_n\| &= \sum_{i=1}^{n} \log\frac{\|v_i\|}{\|v_{i-1}\|} \\ &= \underbrace{\sum_{i\in I\cap[1,n-L]} \log\frac{\|v_{i+L}\|}{\|v_i\|}}_{S_1} + \underbrace{\sum_{i\in K\cap[1,n-L]} \log\frac{\|v_i\|}{\|v_{i-1}\|}}_{S_2} + \underbrace{\sum_{i=n-L'+1}^{n} \log\frac{\|v_i\|}{\|v_{i-1}\|}}_{S_3}\end{aligned}$$

where $n - L' = \max\{n-L, I+L\}$.

Let C be such that for all g in the support of μ and all $y \in \mathcal{M}$, $\|A_V(g,y)\| \le C$. Then $|S_3| \le L\log C$, and $|S_2| \le 4\epsilon n \log C$.

Moreover

$$\frac{|I\cap[1,\dots,n]|\cdot L}{n} \ge 1-4\epsilon$$

Hence

$$S_1 \ge |I\cap[1,\dots,n]|\cdot(\lambda_1-\epsilon)L \ge (1-4\epsilon)n(\lambda_1-\epsilon)$$

and

$$\frac{1}{n}\log\|v_n\| \ge (1-4\epsilon)(\lambda_1-\epsilon) - 4\epsilon\log C - \frac{L}{n}\log C$$

for almost every $\overline{g}$ and any $n \gg L$.

Since $\epsilon \ge 0$ is arbitrary, we get for all $h \in V$ and almost every $\overline{g}$,

$$\liminf_{n\to\infty} \log\|A_V(g_n\dots g_1, x)h\| \ge \lambda_1.$$

And with a similar argument we get an upper bound

$$\limsup_{n\to\infty} \log\|A_V(g_n\dots g_1, x)h\| \le \lambda_1.$$

Which implies Theorem A.1.

A.3. Proof of Lemma 3.1. — According to the sublinear tracking Lemma of [3], for almost every $\theta \in [0, 2\pi)$, there exists $\overline{g} = (g_1, \dots, g_n, \dots)$ satisfying Theorem A.1 such that we can write

$$g_{\lambda n} r_\theta = \epsilon_n g_n \dots g_1$$

with $\epsilon_n \in \mathrm{SL}(2,\mathbb{R})$ satisfying

$$\lim_{n\to\infty} \frac{1}{n}\log\|\epsilon_n\| = 0.$$

By the cocycle relation we have

$$A_V(g_{\lambda n}, r_\theta x) = A_V(\epsilon_n, g_n\dots g_1 x)A_V(g_n\dots g_1, x).$$

But there exists $C > 0$ and $N < \infty$ so that for all $g \in \mathrm{SL}(2, \mathbb{R})$ and all $x \in \mathcal{H}_1(\alpha)$,

$$\|A_V(g, x)\| \leq C\|g\|^N.$$

Hence

$$\log \|A_V(g_{\lambda n}, r_\theta x)h\| = \log \|A_V(g_n \dots g_1, x)\|h + o(n).$$

This proves the lemma.

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