Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

THE BOGOMOLOV–BEAUVILLE–YAU DECOMPOSITION FOR KLT PROJECTIVE VARIETIES WITH TRIVIAL FIRST CHERN CLASS – WITHOUT TEARS

Frédéric Campana

Tome 149
Fascicule 1

2021

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 1-13

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 1, tome 149, mars 2021

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
149 (1), 2021, p. 1-13

THE BOGOMOLOV–BEAUVILLE–YAU DECOMPOSITION FOR KLT PROJECTIVE VARIETIES WITH TRIVIAL FIRST CHERN CLASS – WITHOUT TEARS

BY FRÉDÉRIC CAMPANA

ABSTRACT. — We give a simplified proof (in characteristic zero) of the decomposition theorem for connected complex projective varieties with klt singularities and a numerically trivial canonical bundle. The proof mainly consists in reorganizing some of the partial results obtained by many authors and used in the previous proof but avoids those in positive characteristic by S. Druel. The single, to some extent new, contribution is an algebraicity and bimeromorphic splitting result for generically locally trivial fibrations with fibers without holomorphic vector fields. We first give the proof in the easier smooth case, following the same steps as in the general case, treated next. The last two words of the title are plagiarized from [4].

RÉSUMÉ (*La décomposition de Bogomolov-Beauville-Yau des variétés projectives klt à première classe de Chern triviale – sans larmes*). — Nous donnons une preuve simplifiée (en caractéristique zéro) du théorème de décomposition des variétés connexes et projectives complexes à singularités klt et fibré canonique numériquement trivial.Cette preuve consiste essentiellement en une réorganisation de la preuve originale basée sur des résultats partiels obtenus par divers auteurs, mais évite d'utiliser ceux de caractéristique positive obtenus par S. Druel. Le seul résultat nouveau, dans une certaine mesure, établit l'algébricité et le scindage méromorphe pour les fibrations génériquement localement triviales dont les fibres n'ont pas de champ de vecteur holomorphe non nul. Nous donnons tout d'abord la preuve dans le cas lisse, plus simple, suivant les mêmes étapes que dans le cas général, traité ensuite. Les deux derniers mots du titre plagient [4].

Texte reçu le 17 avril 2020, modifié le 8 juillet 2020, accepté le 21 octobre 2020.

FRÉDÉRIC CAMPANA, Université de Lorraine, Institut Elie Cartan, Nancy • *E-mail :* `frederic.campana@univ-lorraine.fr`

Mathematical subject classification (2010). — 14J32, 14E99, 32J25, 32Q20, 32Q25.

Key words and phrases. — Kähler–Einstein metrics, First Chern class, Hyperkähler varieties, Calabi–Yau varieties, Holonomy, Algebraic foliations, Fundamental group.

doi:10.24033/bsmf.2823

1. Introduction

When X is smooth, connected, compact Kähler, with $c_1(X) = 0$, the classical, metric, proof of the Bogomolov–Beauville–(Yau) decomposition theorem, given in [2] (the arguments of [6] being Hodge-theoretic), starts with a Ricci-flat Kähler metric ([26]) and then decomposes the universal cover X' of X according to De Rham theorem, in its holonomy factors. The Cheeger–Gromoll theorem then distinguishes the flat Euclidian factor $\mathbb{C}^s$ of X' from the (simply-connected) product P of the others (which are compact and with holonomy either $SU(m)$ or $Sp(k)$). The compactness of P combined with Bieberbach's theorem now imply that a finite étale cover of X is the product of a complex torus $\mathbb{C}^s/\Gamma$ with P.

We shall first give a different proof, but only for X smooth projective, of this product decomposition, weaker in the sense that P is not shown to be simply connected (see Theorem 2.1 below). Indeed, the proof does not go through the universal cover and uses neither the De Rham nor the Cheeger–Gromoll theorems.

This allows for its extension (given next) to the singular case obtained in [21], which uses many other partial results, among which are those of [18] and [14] (which plays a rôle analogous to that played by the Cheeger–Gromoll theorem). Our proof makes the step involving the delicate positive characteristic arguments of [14] superfluous. We, indeed, deduce the algebraicity of the foliation given by the flat factor of the holonomy from the splitting result (see Theorem 3.4) below, instead of using the Albanese map. This splitting result can be applied once the algebraicity of the leaves of the foliations given by the nonflat factors of the holonomy have been shown to be algebraic and without nonzero vector fields.

The author thanks Benoît Claudon and Mihai Păun for their help in reading the text and several discussions. After this text was posted on arXiv, the author received useful comments by S. Druel and H. Guenancia and thanks both of them too. He also thanks the referee for his careful reading and suggestions for making some statements more precise.

2. The smooth case

We treat this case first in order to show the steps in the general case in a simpler context.

THEOREM 2.1. — *Let X be a smooth connected complex projective manifold with $c_1(X) = 0$. There exists a finite étale cover of X, which is a product of an abelian variety with projective manifolds that are either irreducible symplectic or Calabi–Yau.*

REMARK 2.2. — The notions of irreducible symplectic and Calabi–Yau manifolds are defined as in [2]: either by the values of $h^{p,0}$, or by the holonomy of any Ricci-flat Kähler metric. We need the projectivity of X, because the Kähler version of [13] is not known. Our proof also does not show the finiteness of the fundamental groups of symplectic or Calabi–Yau manifolds. A partial solution to this finiteness property is given in Proposition 2.7 below, based on more general L^2-methods. A complete solution is also given in Proposition 2.9, but it does not (in an obvious way) extend to the singular case.

Proof of Theorem 2.1. — We equip X with any Ricci-flat Kähler metric ([26]). Let Hol^0 (or Hol) be its restricted holonomy (or holonomy) representation and $T_X = F \oplus (\bigoplus_i T_i)$ be a (local near any given point of X) splitting of the tangent bundle of X into factors that are irreducible for the action of Hol^0. These local factors also correspond to a local splitting of X into a direct product of Kähler submanifolds. In particular, these local products are regular holomorphic foliations. Here, F is the "flat" factor consisting of restricted holonomy-invariant tangent vectors. Now, Hol^0 is a normal subgroup of Hol, and Hol/Hol^0 acts by permutation on the factors of the restricted holonomy decomposition. Because the action of Hol/Hol^0 is induced by a representation $\pi_1(X) \to Hol/Hol^0$, the local holonomy decomposition of T_X above holds globally on a suitable finite étale cover of X.

We now replace X by such a finite étale cover and obtain a global product decomposition $T_X = F \oplus (\bigoplus_i T_i)$ by regular holomorphic foliations, the restricted holonomy of F being trivial, while the ones of T_i are irreducible and of the form $SU(m_i)$ or $Sp(k_i)$.

LEMMA 2.3. — *Let $T_X = \bigoplus_j E_j$ be a direct sum decomposition by foliations E_j, with $c_1(X) = 0$. Then, $c_1(E_j) = 0, \forall j$.*

Proof. — Assume not and let H be a polarization on X, with $n := dim(X)$. Then, $c_1(E_j).H^{n-1} \neq 0$, for some j. Since $\sum_j c_1(E_j).H^{n-1} = 0$, we get $c_1(E_h).H^{n-1} > 0$ for some h. It then follows from [13], Lemma 4.10, that E_h contains a subfoliation G with $\mu_{H,min}(G) > 0$ and by [13], Theorem 4.1, that K_X is not pseudo-effective, contrary to the hypothesis $c_1(T_X) = 0$.

A second, shorter, proof (suggested by the referee) consists in invoking the semistability of T_X with respect to any polarization, so that $c_1(E_j).H^{n-1} \leq 0, \forall j$. □

From the preceding Lemma 2.3, if $T_X = F \oplus (\bigoplus_i T_i)$ is the holonomy decomposition of T_X considered above for X smooth projective with $c_1(X) = 0$, we get that $c_1(F) = c_1(T_i) = 0, \forall i$.

LEMMA 2.4. — *The dual $T_i^\star$ of each T_i is not pseudo-effective (which means that for any polarization H and any given $k > 0$, $h^0(X, Sym^m(T_i^\star) \otimes H^k) = \{0\}$ for $m \geq m(k)$).*

Proof. — We proceed in two steps. From [17], §15.3, and Proposition 24.22, it follows that $Sym^m(T_i), \forall i, \forall m > 0$ is an irreducible representation and, hence, stable. Next, [11], Theorem 1.3 (or alternatively [21], Theorem 1.1) implies that $T_i^\star$ is not pseudo-effective for each i. □

From [13], Theorem 4.2, Lemma 4.6, we now get the first claim of the next result[1]

Lemma 2.5. — *Each of the foliations T_i has algebraic leaves, which are compact*[2]*, since T_i is everywhere regular, and X is smooth. Thus, T_i defines a smooth (proper) fibration $f_i : X \to B_i$ on a smooth projective base B_i. Each of these fibrations is locally trivial with fiber F_i and becomes a product $X' = F_i \times B_i'$ after a suitable finite étale base-change $B_i' \to B_i$.*

Proof. — Second claim: let $C_i := F \oplus (\bigoplus_{\neq i} T_i)$ be the complement in T_X of T_i. This defines a regular holomorphic foliation locally over B_i, which is transversal to f_i, and thus shows that f_i is locally isotrivial over B_i. Third claim: it is sufficient to know that $Aut(F_i)$ is discrete or that $h^0(F_i, T_{F_i}) = 0$. However, this is easy, since F_i is a projective manifold with $c_1 = 0$ and irreducible nontrivial holonomy, which thus does not leave any tangent vector invariant, which implies the claimed vanishing by the Bochner principle. □

Consider any one of the projections $f_i : F_i \times B_i \to B_i$ (after a suitable finite étale cover). Then, $c_1(B_i) = 0$ and its holonomy decomposition is $F \oplus (\bigoplus_{j\neq i} T_i)$. Proceeding inductively on $dim(X)$, we obtain a decomposition in a product $X = (\times_i F_i) \times B$, where B is smooth projective with $c_1(B) = 0$ and trivial holonomy F.

The next lemma then concludes the proof of Theorem 2.1. □

Lemma 2.6. — *([5]) Let X be a connected compact Kähler manifold with $c_1(X) = 0$ and with trivial restricted holonomy representation (relative to some Ricci-flat Kähler metric). Then, X is covered by a torus.*

The symplectic and the even-dimensional Calabi–Yau manifolds can be shown to have a finite fundamental group by L^2-methods that extend to the singular case. Another approach is given right after this first proof, which works more generally, for compact Riemannian manifolds with nonnegative Ricci curvature and *maximal* b_1 vanishing, but does not extend in any obvious way to the singular case.

1. Although not explicitly stated in [13], this is a main step of the proof of 4.2 and is suggested by the proof of Lemma 4.6 there. The explicit formulation was first given in [14], §8. Since only the particular case of a polarization H^{n-1} is used here, one could even alternatively apply [7].

2. By contradiction: if not, the leaf through a regular point of the boundary of the closure of a leaf should be contained in this boundary, and of the same dimension. In the singular case, this compactness fails, and more delicate arguments are required.

Proposition 2.7. — *Let X be a connected compact Kähler manifold with $c_1(X)=0$ and $\chi(\mathcal{O}_X)\neq 0$. Then, $\pi_1(X)$ is finite.*

Proof. — We give two proofs, both relying on [1].

First proof. This is the proof given in [10], Corollary 5.3, and Remark 5.5. By [10], Theorem 4.1, it is sufficient to show that $\kappa^+(X)\leq 0$, that is, $\kappa(X,det(F))\leq 0$, for any subsheaf $F\subset\Omega^p_X,\forall p>0$. This follows from the semistability of $\wedge^r\Omega^p_X,\forall r,p>0$. Indeed, since K_X is trivial, $\Omega^p_X\cong(\Omega^{n-p}_X)^*$, and so any saturated subsheaf $D:=det(F)$ of rank 1 of $\wedge^r\Omega^p_X$ is numerically trivial, since both D and $D^*=det(\Omega^p/F)\subset(\wedge^{r'}\Omega^p)^*=\wedge^{r'}\Omega^{n-p}$, have nonpositive slope with respect to any polarization.

Second proof. If $X'\to X$ is the universal cover, and h an L^2-holomorphic p-form on X', then h is parallel (because the Laplacian of its squared norm equals the square norm of its covariant derivative, and so is nonnegative everywhere. Gaffney's integration trick implies that the Laplacian identically vanishes, since h is L^2, and X' is complete). Thus, h comes from X and vanishes, if X' is noncompact. By [1], one gets $0=\sum_{p\in\{0,n\}}(-1)^p h^0_{(2)}(X',\Omega^p_{X'})=\chi_{(2)}(X',\mathcal{O}_{X'})=\chi(X,\mathcal{O}_X)\neq 0$, which is a contradiction. □

Corollary 2.8. — *If X is a compact Kähler manifold of dimension n, and irreducible symplectic (or Calabi–Yau of even dimension), then $\pi_1(X)$ is finite, of cardinality dividing $(\frac{n}{2}+1)$ (resp. 2).*

Proof. — Let $X'\to X$ be the (compact) universal cover of X, of degree d. We then have $\chi(\mathcal{O}_{X'})=d.\chi(\mathcal{O}_X)$. On the other hand, X' is still irreducible symplectic (or Calabi–Yau), and so we have: $\chi(\mathcal{O}_{X'})=\sum_{p=0}^{p=\frac{n}{2}}(-1)^{2p}h^0(X',\Omega^{2p}_{X'})=\frac{n}{2}+1$ (or $\chi(\mathcal{O}_{X'})=\sum_{p\in\{0,n\}}(-1)^p h^0(X',\Omega^p_{X'})=2$). □

The following result applies to any Calabi–Yau manifold but does not immediately extend to the singular case.

Proposition 2.9. — *Let M be a compact connected Riemannian manifold with nonnegative Ricci curvature, such that $b_1(M')=0$, for any finite étale cover M' of M. The fundamental group of M is finite.*

Proof. — By [24], the growth of $\pi_1(M)$ is polynomial (of degree bounded by the dimension of M). From [20], $\pi_1(M)$ is virtually nilpotent. Thus, $\pi_1(M')$ is nilpotent and torsion free for some finite étale cover M' of M. Thus, $\pi_1(M')$ is either trivial or has an abelianization of positive rank. Since $b_1(M')=0$, $\pi_1(M')=\{1\}$, hence the claim. □

3. The singular version

Let X be a complex projective variety with klt singularities whose first Chern class is zero, i.e. $c_1(X) = 0$. By [25], Chap. V, Corollary 4.9, the condition $K_X \equiv 0$ implies that K_X is $\mathbb{Q}$-trivial. We may, and shall, assume, by passing to an index-one cover, that the singularities of X are canonical, and that K_X is trivial. (Instead of [25], when the singularities are canonical, one could use either [22], Thm. 8.2, or [12], Thm. 3.1 applied to a resolution of X).

Remark 3.1. — Notice that passing to the index-one cover eliminates examples of rationally connected varieties with klt singularities and torsion canonical bundle, such as the Ueno surface, the quotient of $E \times E$ by $\mathbb{Z}_4$ acting diagonally by complex multiplication by $\sqrt{-1}$ on each factor, where E is the elliptic curve with this complex multiplication.

We denote by ω the unique Ricci-flat metric of X that belongs to a given Kähler class ([16]). We will see now that the steps of the previous proof extend to the singular context, using the results from [18], §8, 9 and [14], Prop. 4.10 and Prop. 3.13. The single new input here is the algebraicity criterion for foliations in Theorem 3.4 below, which makes superfluous the characteristic $p > 0$ methods and results by several authors used in [14]. The results of [22] used in [14] are also no longer needed.

Theorem 3.2 ([21]). — *Let X be a normal complex variety with klt singularities and with $c_1(T_X) = 0$. There exists a quasi étale cover $f : \widetilde{X} \to X$ with canonical singularities, which is a product $\widetilde{X} = \Pi_j Y_j \times A$, where A is an abelian variety, and Y_j's are varieties with canonical singularities, trivial canonical bundle, and irreducible restricted holonomy either $Sp(k_j)$, or $SU(m_j)$ (see §3.1 below). The Y_j' respectively are said to be irreducible symplectic (or Calabi–Yau).*

Since, by [18]*, there always exists a finite quasi étale cover with full holonomy either $Sp(k_j)$ or $SU(m_j)$, these notions coincide with the usual ones up to such a cover.*

3.1. Restricted holonomy cover. — We consider ω the "EGZ" Ricci-flat metric on X constructed in [16]. As shown[3] in [18], Prop. 7.3, after a quasi étale cover, obtained from the permutation representation of the holonomy on the factors of the restricted holonomy, the tangent sheaf T_X of X decomposes as follows:

$$T_X = \mathcal{F} \oplus \left(\bigoplus_i \mathcal{E}_i \right), \tag{1}$$

3. In the first version, Prop. 7.9 was quoted, instead of Prop. 7.3, which is sufficient for our purposes, as pointed out by S. Druel and H. Guenancia, whom the author thanks for this observation.

where the restricted holonomy of $\mathcal{F}$ is trivial, and the other ones are either $SU(n_i)$ or $Sp(k_i)$. The other properties of $\mathcal{E}_i$ used here are:

(i) *The sheaf $\mathcal{E}_i$ defines a nonsingular foliation of rank either n_i or $2k_i$. on X_{reg}.*
(ii) *The first Chern classes of $\mathcal{E}_i, \mathcal{F}$ are zero.*
(iii) *All the symmetric powers of $\mathcal{E}_i$ and their duals are irreducible representations of the holonomy factors and are stable, for any polarization on X.* The first property follows from standard representation theory, given the structure of the holonomy group. The stability is [18], Theorem 8.1, see also claim 9.17.
(iv) *In particular, we have $h^0(X, \mathcal{E}_i) = 0, \forall i$, by stability.*
(v) *The preceding properties still hold for any finite quasi étale cover of X.* Indeed, the Ricci-flat metric on X lifts to such covers, and the restricted holonomy decomposition lifts there too.
(vi) *The holonomy factors and their holonomy groups do not depend on the Ricci-flat Kähler metric chosen.*

3.2. Algebraic foliations. — Recall that a foliation on X is said to be algebraic if its leaves are so.

In the decomposition (1), the foliations $\mathcal{E}_i$ are algebraic. Indeed, by either [21] (or [11], Theorem 3.1) none of the $\mathcal{E}_i's$ is pseudo-effective[4]. We can, thus, apply [13], Theorem 4.2, Lemma 4.6, which implies that they are algebraic.

Our goal now is to show that $\mathcal{F}$ too is algebraic. This is true, if $T_X = \mathcal{F}$. We, thus, assume that some nonzero factor $\mathcal{E}_i$ appears in (1), and so:

$$T_X = \mathcal{G} \oplus \mathcal{E}\ , \tag{2}$$

where $\mathcal{E}$ has positive rank, and the properties (i)–(v) are satisfied. So, here we assume implicitly that $\mathcal{E}$ is one of the factors $\mathcal{E}_i$ in (1), and $\mathcal{G}$ is the sum of the other factors. Observe that $\mathcal{G}$ is a foliation, since the decomposition (1) is induced by the local holonomy splitting of X_{reg} (in general, the sum of two foliations need not be integrable).

LEMMA 3.3. — *Let X be an algebraic variety with canonical singularities and trivial first Chern class. Let ω be the* Ricci-*flat metric in some Kähler class on X, and $T_X = \mathcal{G} \oplus \mathcal{E}$ a corresponding decomposition as in the preceding lines. Assume that the foliation $\mathcal{G}$ is algebraic. Then:*

There exists a quasi étale cover $f : \widetilde{X} \to X$, where $\widetilde{X}$ has canonical singularities, and a product decomposition $\widetilde{X} = F \times Y$, which coincides at the tangent level with the decomposition $T_{\widetilde{X}} = f^{[*]}\mathcal{E} \oplus f^{[*]}\mathcal{G}$.

4. The result of [11] can, indeed, be applied on a resolution of the singularities of X, by lifting both the foliation and an ample class, since its argument deals with the general point of X only.

Proof. — The claim follows directly from [14], Prop. 4.10 (notice that the assumption $\tilde{q}(X) = 0$ there can be weakened to $\tilde{q}(F) = 0$, if F is the closure of a generic leaf of $\mathcal{E}$. The property $\tilde{q}(X) = 0$ is, indeed, used only to apply Prop. 4.8 of loc. cit., but 4.8 requires only the vanishing of $\tilde{q}$ for the fibers of $\mathcal{E}$). Now, $\tilde{q}(F) = 0$ follows from the properties (iii) and (v) of the holonomy factors quoted above. □

The algebraicity of $\mathcal{G}$ follows from Theorem 3.4 below, which, in fact, implies more: the bimeromorphic decomposition of X as a product, birationally, after a finite cover[5]. We may, and shall, assume that X has $\mathbb{Q}$-factorial terminal singularities by step 1 of the proof of Prop. 4.10 of [14]. By Prop. 3.13 of loc. cit, there is a Zariski open subset[6] X^0 of X, and a projective morphism $\varphi^0 : X^0 \to Y^0$, which is a locally trivial fibration in the analytic topology, its fibers being isomorphic to some F with $\tilde{q}(F) = 0$,by the properties (iv) and (v) of the holonomy factors quoted in §3.1 above. The conclusion then follows from the next algebraicity criterion for foliations.

Theorem 3.4. — *Let X and Y be two Kähler*[7] *normal spaces and let $f : X \to Y$ be a surjective and proper holomorphic map with connected fibers. We denote by $\mathcal{E} := T_{X/Y}$ the foliation on X induced by f. We assume that:*

(1) *f is a trivial fibration, locally in the analytic topology, with fiber F over some nonempty Zariski open set Y_0 of Y.*
(2) *$h^0(F, T_F) = 0$, and so the automorphism group of F is discrete.*

Then, there is a finite map $\vartheta : V \to Y$, étale over Y_0, such that base-changing $f : X \to Y$ and normalizing the fiber-product $X_V := X \times_Y V$, we have a birational decomposition $\delta : X_V \dashrightarrow F \times V$, isomorphic over Y_0.

Moreover, if $\mathcal{G}$ is any distribution on X, such that $T_X = T_{X/Y} \oplus \mathcal{G}$ over $f^{-1}(U)$, for some nonempty analytically open $U \subset Y_0$, then: $\delta_*((id_X \times \vartheta)^*(\mathcal{G})) = \mathcal{H}$, where $\mathcal{H} := T_{X_V/F} \subset T_{X_V}$ is the horizontal foliation defined by the product decomposition of $T_{F\times V}$. In particular, $\mathcal{G}$ is an algebraic foliation, and is the unique distribution on X, which is everywhere transversal to $T_{X/Y}$ over some open subset $U \subset Y_0$ as above.

Remark 3.5. — 1. The birational splitting after a generically finite base-change $V \to Y$ (but not necessarily étale over Y_0) always exists if X is projective (or Moishezon) under the single hypothesis (1) of Theorem 3.4. However, the algebraicity of $\mathcal{G}$ requires the hypothesis (2) as seen, for example, when $f : X \to Y$ is a morphism of Abelian varieties

5. S. Druel informed the author that one could also apply his Theorem 1.5 in [15]. Since the hypothesis, scope, and proofs of both results are different, it seems worth stating and proving Theorem 3.4.

6. Up to a finite étale cover of X^0, by shrinking the open set Y^0 of the proof.

7. Or in the class $\mathcal{C}$.

with positive-dimensional fibers, which has many horizontal nonalgebraic foliations.

2. There is certainly a bimeromorphic version of Theorem 3.4, where the generic fibers of f are assumed to be bimeromorphically equivalent, by similar arguments.
3. A global Kähler condition is required for the algebraicity of $\mathcal{G}$ to hold, as the following simple example shows. Let F be a projective $K3$ surface with an infinite and finitely generated group of automorphisms $G \subset Aut(F)$. Let $C = C'/\pi_1(C)$ be a curve of genus $g \geq 1$ with universal cover C', such that $\pi_1(C)$ admits a surjective group morphism $\rho : \pi_1(C) \to G$. Such a C exists when $g \geq m$, is the cardinality of some set of generators of G. One can choose $g = 1$ if and only if G is abelian, generated by at most 2 elements. Let $X := (C' \times F)/\pi_1(C)$, where $p \in \pi_1(C)$ acts on the right on $(C' \times F)$ by: $(c', f).p := (p^{-1}.c', \rho(p^{-1}).f)$. The foliation $\mathcal{G}'$ with leaves $C' \times \{f\}$ on X' induces a foliation $\mathcal{G}$ on X, which is not algebraic. In this case, $Iso(Z, X/C) = Iso^{\star}(Z, X/C)$ is irreducible, noncompact, isomorphic to $C" := C'/Ker\rho$, the Galois cover of C with group G. The leaves of $\mathcal{G}$ are isomorphic to $C"$.

Proof. — Let $\varphi : Z := F \times Y \to Y, \psi : F \times Y \to F$ be the projections onto the second (resp. first) factor. As in [8], §8, we define:

$$\mathrm{Iso}^{*}(Z, X/Y) \subset \mathcal{C}(Z \times_Y X/Y) \tag{3}$$

to be the subset of the relative Chow variety of $Z \times_Y X$ over Y parameterizing the graphs of isomorphisms of F-seen as a fiber of φ over a point $y \in Y_0$- to X_y, the fiber of f over y. According to [8], §8, $\mathrm{Iso}^{*}(Z, F)$ is a Zariski open subset (with countably many components if $Aut(F)$, which is here discrete, infinite) of the relative Chow scheme of $(Z \times_Y X/Y)$, which consists of cycles contained in one of the fibers of the fiber product over Y.

Let $\mathrm{Iso}(Z, X/Y)$ be the topological (i.e., Zariski here) closure of $\mathrm{Iso}^{\star}(Z, X/Y)$ in $\mathcal{C}(Z \times_Y X/Y)$. It consists of the union of the closures of the components of $\mathrm{Iso}^{\star}(Z, X/Y)$, all of these closures being proper over Y, and irreducible components of the Chow–Barlet scheme of $(Z \times_Y X/Y)$. It is equipped with a projection to Y, by restriction of the one on $\mathcal{C}(Z \times_Y X/Y)$. Since f is locally trivial over Y_0, the projection $\mathrm{Iso}^{*}(Z, X/Y) \to Y$ is open over Y_0. This projection is proper on each component of $\mathrm{Iso}^{\star}(Z, X/Y)$, since these irreducible components of $\mathrm{Iso}(Z, X/Y)$ are compact (essentially by a general result, [23]) of D. Lieberman, based on E. Bishop's theorem). Moreover, by the assumption (2) of Theorem 3.4, the fibers of $\mathrm{Iso}(Z, X/Y)$ to Y are discrete over Y_0. If V is an irreducible component of $\mathrm{Iso}(Z, X/Y)$, its projection $\vartheta : V \to Y$ is, thus, onto, and finite étale over, Y_0. Indeed, if $Y' \subset Y_0$ is any small analytic open subset over which $X_{Y'} := f^{-1}(Y') \cong Y' \times F$ is given, $Iso^{\star}(Z, X/Y)$ identifies naturally with $Y' \times Aut(F)$ over Y' and shows that $\vartheta : V \to Y$ is étale over Y_0.

We, thus, get a fiber product $X_V := X \times_Y V$, with the obvious projections $f_V : X_V \to V, g : X_V \to X$. Let $V_0 := \vartheta^{-1}(Y_0)$. Any $v \in V_0$ is, thus, equipped naturally with an isomorphism $ev_v : F \cong X_y, y := \vartheta(v)$. This evaluation map extends (see [8], §8, Prop. 1) meromorphically to $ev : F \times V \to X_V$, which is, thus, bimeromorphic and isomorphic over V_0.

In order to simplify notation, we replace X, Y, f by X_V, V, f_V, respectively, and identify via ev X_V with $F \times V = F \times Y$ (recall that ev is isomorphic over $V_0 = Y_0$), and all the assumptions of Theorem 3.4 are preserved. The projections of $X = F \times Y$ onto its second (or first) factor are denoted $f = \varphi$ and ψ.

To establish the last claim of Theorem 3.4, we only have to check that $\mathcal{G}$ coincides over Y_0 with the sheaf $T_{X/F} := \mathcal{H}$, which will also prove the algebraicity of $\mathcal{G}$.

We restrict everything over the open set $U \subset Y_0$ appearing in the last assumption of Theorem 3.4, so we assume that $X_U := f^{-1}(U) = F \times U$ and, thus, have a first decomposition $T_{X_U} = \psi^*(T_F) \oplus \mathcal{H}$, where $\mathcal{H}$ is the kernel of the map $d\psi : T_{X_U} \to \psi^*(T_F)$.

The second decomposition $T_{X_U} = \psi^*(T_F) \oplus \mathcal{G}$ gives equivalently an isomorphism $df_{|\mathcal{G}} : \mathcal{G} \to f^*(T_U)$ over X_U. Let $(df_{|\mathcal{G}})^{-1} : f^*(T_U) \to \mathcal{G}$ be its inverse. Let $\gamma := d\psi \circ (df_{|\mathcal{G}})^{-1} : f^*(T_U) \to \psi^*(T_F)$ be the composite map, seen as an element $\gamma \in H^0(X_U, f^*(\Omega^1_U) \otimes \psi^*(T_F))$. We have the following equalities:

$$\begin{aligned} H^0(X_U, f^*(\Omega^1_U) \otimes \psi^*(T_F)) &= H^0(U, \Omega^1_U \otimes f_*(\psi^*(T_F))) \\ &= H^0(U, \Omega^1_U \otimes \{0\}) = \{0\}. \end{aligned}$$

The last two equalities follow from assumption (2) of Theorem 3.4, which implies that $f_*(\psi^*(T_F)) = \{0\}$. This shows that $\mathcal{G} = \mathcal{H} = T_{Z/F}$ over X_U and so everywhere by analytic continuation. □

We can now conclude the proof of Theorem 3.2 by induction on $\dim(X)$, since we now know that (up to quasi étale covers) $X = Y \times Z$, in which Y is a product of varieties with canonical singularities, $c_1 = 0$, restricted holonomy either SU or Sp, and Z is in the same class of varieties but with trivial restricted holonomy (i.e., $T_Z = \mathcal{F}$). Theorem 3.2 then follows from:

3.3. A singular Bieberbach theorem. — Assume now that only the factor $\mathcal{F}$ appears in the decomposition (1). We are reduced to showing that if Z has canonical singularities, trivial first Chern class, and a trivial restricted holonomy group, it is covered by an abelian variety. However, this is just Corollary 1.16 in [19].

3.4. The fundamental group. — Let X be a complex projective variety with klt singularities and $K_X \equiv 0$. Recall that X is said to be irreducible symplectic (or Calabi–Yau) if its restricted holonomy representation for any, or some, EGZ

Ricci-flat Kähler metric is irreducible and of the form $Sp(m)$ (or $SU(n)$). In this situation, we have the following result, which is entirely similar to the smooth case:

THEOREM 3.6. — *(See* [18]*, 13.1) If* $\chi(\mathcal{O}_X) \neq 0$*, then* $\pi_1(X')$ *is finite, if* $\rho : X' \to X$ *is any resolution. This applies to irreducible symplectic varieties and to even-dimensional Calabi–Yau varieties: the cardinality of* $\pi_1(X')$ *lies in* $[1, \frac{n}{2}+1]$ *in the first case (or in* $[1,2]$ *in the second case).*

Since the map $\rho_*(\pi_1(X')) \to \pi_1(X)$ *is surjective (by* [9]*, Proposition 1.3), this implies the same statement for* $\pi_1(X')$*.*

Proof. — We apply [10], which says that $\pi_1(X')$ is finite if $\kappa(X', det(\mathcal{F})) \leq 0$ for any $\mathcal{F} \subset \Omega^p_{X'}), \forall p > 0$. Since the sections of $det(\mathcal{F})^{\otimes m}$ are sections of $Sym^m(\Omega^p_{X'})$, and the restrictions of these are reflexive sections, hence parallel over the regular locus of X by [18], Theorem 8.2, these sections are determined by their value in one single point of X_{reg}. Thus, $\kappa(X', det(\mathcal{F})) \leq 0$.

One could also argue as in the first proof of Proposition 2.7.

The invariant $\chi(\mathcal{O}_X)$ behaves as in the smooth case when X has klt and, thus, rational singularities. It is, in particular, multiplicative under finite étale covers.

If X is irreducible symplectic (or even-dimensional Calabi–Yau) and n-dimensional, we have: $h^0(X, \Omega^{[p]}_X) \leq h_{n,p}$, where $h_{n,p} = 0$ for p odd, and $h_{n,p} = 1$ for $p \leq n$ even (or $h_{n,p} = 0$ for $p \neq 0, n$, and $h_{n,p} = 1$ for $p = 0, n$), and so $\chi(\mathcal{O}_X)$ lies in $[1, \frac{n}{2}+1]$ (or in $[1,2]$). This shows the claim, since these inequalities still hold on the universal cover $X"$ of X, and $\chi(\mathcal{O}_{X"}) = d.\chi(\mathcal{O}_X)$, where d is the degree of $X"$ over X and also the cardinality of $\pi_1(X)$. □

REMARK 3.7. — Our proof of Theorem 3.6 differs slightly from the one in [18], 13.1, both relying on [10], and thus on [1]. See [10], §.5 for further remarks on this topic.

BIBLIOGRAPHY

[1] M. ATIYAH – "Elliptic operators, discrete groups, and Von Neumann algebras", *Astérisque* **32–33** (1976), p. 43–72.

[2] A. BEAUVILLE – "Variétés Kählériennes dont la première classe de Chern est nulle", *J. Diff. Geom.* **18** (1983), p. 755–782.

[3] A. BESSE – *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 10, Springer Verlag, 1987.

[4] F. BEUKERS, A. J. VAN DER POORTEN & R. VAN DER YAGER – "Dyson's lemma without tears", *Indagationes Mathematicae* **2** (1991), no. 1, p. 19–28.

[5] L. Bieberbach – “Über die Bewegungsgruppen der Euklidischen Raüme. I, II”, *Math. Ann.* **70** (1911), p. 297–336, **72** (1912), p. 400-412.
[6] F. Bogomolov – “Hamiltonian Kähler manifolds”, *Sov. Math. Dokl.* **19** (1978), p. 1462–1465.
[7] F. Bogomolov & M. McQuillan – “Rational curves on foliated varieties”, 2001, IHES/M/01/07.
[8] F. Campana – “Réduction d’Albanese d’un morphisme propre et faiblement kählérien, ii”, *Comp. Math.* **54** (1985), p. 373–398.
[9] ______, “On twistor spaces of class $\mathcal{C}$”, *J. Diff. Geom.* **33** (1991), p. 451–459.
[10] ______, “Fundamental group and positivity of the cotangent bundles of compact Kähler manifolds”, *J. Alg. Geom.* **4** (1995), p. 487–502.
[11] F. Campana, J. Cao & M. Păun – “Subharmonicity of direct images and applications”, ArXiv.
[12] F. Campana & T. Peternell – “Geometric stability of the cotangent bundle and the universal cover of a projective manifold”, *Bull. SMF* (2011).
[13] F. Campana & M. Păun – “Foliations with positive movable slope”, (2019).
[14] S. Druel – “A decomposition theorem for singular spaces with trivial canonical class of dimension at most five”, *Inv. math.* **211** (2018), p. 245–296.
[15] ______, “Codimension one foliations with numerically trivial canonical class on singular spaces”, mars 2020, arXiv:1809.06905 (2018).
[16] P. Eyssidieux, V. Guej & A. Zeriahi – “Singular Kähler-Einstein metrics”, *J. AMS.* **22** (2009), p. 607–639.
[17] W. Fulton & J. Harris – *Representation theory. A first course*, Springer-Verlag, 1991.
[18] D. Greb, H. Guenancia & S. Kebekus – “Klt varieties with trivial canonical class, holonomy, differential forms, and fundamental groups”, *Geometry and Topology* **23** (2019), no. 4, p. 2051–2124.
[19] D. Greb, S. Kebekus & T. Peternell – “Etale fundamental group of Kawamata log-terminal spaces, flat sheaves and quotients of abelian varieties”, *Duke Math. J.* **165** (2016), no. 10, p. 1965–2004.
[20] M. Gromov – “Groups of polynomial growth and expanding maps”, *Publ. IHES* **53** (1981), p. 53–78.
[21] A. Höring & T. Peternell – “Algebraic integrability of foliations with numerically trivial canonical bundle”, *Inv. Math.* **216** (2019), p. 395–419.
[22] Y. Kawamata – “Minimal models and the kodaira dimension of algebraic fiber spaces”, *Journal für die reine und angewandte Mathematik* **363** (1985), p. 1–46.

[23] D. Lieberman – "Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds", in *Séminaire François Norguet III*, Lecture Notes in Mathematics, no. 670, 1978, p. 140–186.

[24] J. Milnor – "A note on curvature and fundamental group", *J. Diff. Geom.* **2** (1968), p. 1–7.

[25] N. Nakayama – *Zariski decomposition and abundance*, MSJ Memoirs, no. 14, The Mathematical Society of Japan, Tokyo, 2004.

[26] S. Yau – "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I", *Comm. Pure and Appl. Math.* **31** (1978), p. 339–411.