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APPROXIMATION DIOPHANTIENNE ET DISTRIBUTION LOCALE SUR UNE SURFACE TORIQUE II

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RÉSUMÉ. — Nous proposons une formule empirique pour le problème de distribution locale des points rationnels de hauteur bornée. Il s'agit d'une version locale du principe de Batyrev-Manin-Peyre. Nous la vérifions pour une surface torique, sur laquelle des courbes rationnelles cuspidales et des courbes rationnelles nodales toutes les deux contribuent aux meilleures approximations en dehors d'un fermé de Zariski. Nous démontrons qu'en enlevant une partie mince, il existe une mesure limite et une formule asymptotique pour le grossissement critique.

ABSTRACT (*Diophantine approximation and local distribution on a toric surface II*). — We propose an empirical formula for the problem of local distribution of rational points of bounded height. This is a local version of the Batyrev-Manin-Peyre principle. We verify this for a toric surface, on which cuspidal rational curves and nodal rational curves all give the best approximations outside a Zariski closed subset. We prove the existence of a limit measure as well as an asymptotic formula for the critical zoom by removing a thin set.

1. Introduction

1.1. Contexte et heuristique. — Concernant les variétés ayant beaucoup de points rationnels, une question naturelle est combien il y en a de hauteur bornée et comment ils sont distribués. Dans des années 1990, Batyrev et Manin ont

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conjecturé une formule asymptotique (cf. [1, 3.11 & 3.12]) qui donne une prédiction pour l'ordre de croissance du cardinal de l'ensemble des points rationnels de hauteur bornée. Peyre (cf. [29, Conjecture 2.3.1]) a ensuite reformulé et raffiné leur conjecture sous une forme faisant intervenir des mesures, que nous appellerons *distribution globale* et nous énonçons comme suit. Soit X une «bonne» variété sur $\mathbf{Q}$ (cf. [30, Notations 2.1]). On note $X(\mathcal{A}_{\mathbf{Q}})^{\mathrm{Br}}$ l'ensemble des points adéliques de X pour lesquels l'obstruction de Brauer-Manin à l'approximation faible est triviale. On associe une hauteur de Weil exponentielle H au fibré anticanonique ω_X^{-1}.

Principe 1.1 (Batyrev-Manin-Peyre, [30] Répartition empirique 5.3). — Il existe un ouvert $U \subseteq X$ tel qu'en notant et $\kappa = \operatorname{rg}\operatorname{Pic}(X)$,

$$\delta_{U,B} = \sum_{\substack{P\in U(\mathbf{Q})\\ H(P)\leqslant B}} \delta_P, \tag{1}$$

on ait

$$\frac{1}{B(\log B)^{\kappa-1}}\delta_{U,B} \longrightarrow \mu_X^{\mathrm{Br}}, \quad B\to\infty$$

au sens de convergence vague pour certaine mesure μ_X^{Br} déduite d'un produit de mesures $\prod_{\nu\in\mathrm{Val}(\mathbf{Q})}\mu_\nu$ sur $X(\mathcal{A}_{\mathbf{Q}})$ par restriction à $X(\mathcal{A}_{\mathbf{Q}})^{\mathrm{Br}}$.

Remarquons qu'en fait, la partie réelle μ_∞ est une mesure à densité continue relativement à la mesure de Lebesgue sur $X(\mathbf{R})$ (cf. [29, 2.2.1]). Alors que la conjecture de Batyrev-Manin est vérifiée pour beaucoup de variétés presque de Fano [30, Définition 3.1], voire singulières, ce principe sous forme de mesure est rarement abordée dans la littérature. Pourtant, les travaux de Batyrev et Tschinkel [4], [2] semblent être en faveur de cela pour les variétés toriques.

Nous nous demandons dans quelle mesure une version plus forte de ce principe puisse être valide. À savoir, prenons un ouvert $D_Q(B)$ d'un point $Q\in X(\mathbf{R})$ pour la topologie réelle dont la taille dépend de B. Pourrions-nous espérer qu'il existe une mesure $\mu_{X,Q}$, définie localement sur le lieu réel de X que nous préciserons dans la suite, telle que pour certain $\kappa'\geqslant 1$,

$$\sum_{P\in D_Q(B)\cap X(\mathbf{Q}):H(P)\leqslant B} \delta_P \sim \mathrm{Vol}(D_Q(B))B(\log B)^{\kappa'-1}\mu_{X,Q}, \quad B\to\infty? \tag{2}$$

Une motivation de ce problème est l'étude de la *distribution locale* des points rationnels. Il fut considérée en premier par S. Pagelot [28], où il a pris pour $D_Q(B)$ des boules de rayon $\asymp B^{-\frac{1}{r}}$ (r sera appelé *facteur de zoom* dans la suite) et où il a constaté, sur certaines surfaces toriques, des phénomènes variés pour la distribution locale des points rationnels autour d'un point fixé (décrite par la mesure $\mu_{X,Q}$ dans (2)), même pour de différents r d'une variété fixée. Tout cela n'est pas *a priori* reflétée par la distribution globale (i.e. la mesure μ_X^{Br} dans (1)).

Pour établir l'existence de $\mu_{X,Q}$, il faut souvent retirer certaines sous-variétés *localement accumulatrices* et bien choisir le facteur de zoom r. Les travaux de D. McKinnon et M. Roth ([26] et [27]) concernant l'approximation diophantienne sur les variétés algébriques fournissent une constante α de nature arithmétique et géométrique, appelée *constante d'approximation* (Définition 2.1). S. Pagelot définit dans [28] *la constante essentielle* α_{ess} (Définition 2.2) qui caractérise l'approximation diophantienne générique. Par définition, on a $\alpha \leqslant \alpha_{\mathrm{ess}}$. En prenant le facteur de zoom r entre α et α_{ess}, la forme de $\mu_{X,Q}$ nous permet de récupérer plus d'informations qui sont «négligées» dans la considération (1). Voir des explications et des illustrations dans [19]. Nous espérons que les constantes $\alpha, \alpha_{\mathrm{ess}}$ jouent un rôle tout comme les invariants de Fujita (les invariants $\alpha(L), t(L)$ dans [1, 2.1 & 3.12]) dans le programme de Batyrev-Manin-Peyre (Principe 1.1) (cf. aussi le travail [23] et les références dedans).

Dans [18, (1.1)] et [19, (1.2)] nous avons défini une famille de mesures qui capturent les points dans $D_Q(B)$. Maintenant nous continuons à proposer des formules asymptotiques prédisant l'ordre de grandeur dans le cas $r = \alpha_{\mathrm{ess}}$. Nous désignons par $A_1(X)$ le groupe de Chow des 1-cycles modulo l'équivalence algébrique, et par $l(X)$ le rang du sous-groupe de $A_1(X)$ engendré par les classes des courbes rationnelles C vérifiant $\alpha(Q,C) = \alpha_{\mathrm{ess}}(Q)$. Tout au long de cet article, sauf si mentionné autrement, toutes les constantes d'approximation et tous les degrés seront calculés par rapport au fibré anti-canonique. Pour une partie Y de $X(\mathbf{Q})$, nous notons $\{\delta_{Y,Q,B,r}\}_B$ la famille de mesures de zoom de facteur r comptant les points rationnels sur Y de hauteur $\leqslant B$ (cf. §2.2) définie sur l'espace tangent $(T_QX)_{\mathbf{R}}$.

Formule empirique 1.2 (version faible). — *En dehors d'une partie mince M, nous avons que pour toute fonction f continue à support compact définie sur $(T_QX)_{\mathbf{R}}$,*

$$\delta_{X\setminus M,Q,B,\alpha_{\mathrm{ess}}}(f) = O_f\left(B^{1-\frac{\dim X}{\alpha_{\mathrm{ess}}}}(\log B)^{l(X)-1}\right). \tag{3}$$

Répartition empirique 1.3 (version forte). — *Si $\alpha_{\mathrm{ess}}(Q) > \dim X$, alors en dehors d'une partie mince M, nous avons*

$$\frac{1}{B^{1-\frac{\dim X}{\alpha_{\mathrm{ess}}}}(\log B)^{l(X)-1}}\delta_{X\setminus M,Q,B,\alpha_{\mathrm{ess}}} \to \delta_{\alpha_{\mathrm{ess}}}, \tag{4}$$

au sens de convergence vague pour une certaine mesure $\delta_{\alpha_{\mathrm{ess}}}$ sur $(T_QX)_{\mathbf{R}}$.

L'étude des formules 1.2 & 1.3 devrait ajouter une nouvelle évidence sur la connexion entre l'arithmétique des corps globaux et celle des corps de fonctions.

1.2. Résultats principaux. — Suite aux travaux [18] et [19] qui démontrent la formule 1.3 pour la surface X_3 et 1.2 pour Y_4 respectivement, dans cet article on considère la surface torique Y_3 dont l'éventail est représenté au milieu de la Figure 1.1 suivante. Admettant $\mathbf{P}^2$ et $\mathbf{P}^1 \times \mathbf{P}^1$ comme modèles minimaux, elle est une surface de del Pezzo généralisée (cf. [8]) (c'est-à-dire la désingularisation minimale d'une surface de del Pezzo singulière de degré 5). Le résultat sur ces trois surfaces donne des évidences fortes sur le principe que le couple $(\alpha, \alpha_{\text{ess}})$ et la façon dont elles sont calculées devraient caractériser l'accumulation locale des points rationnels.

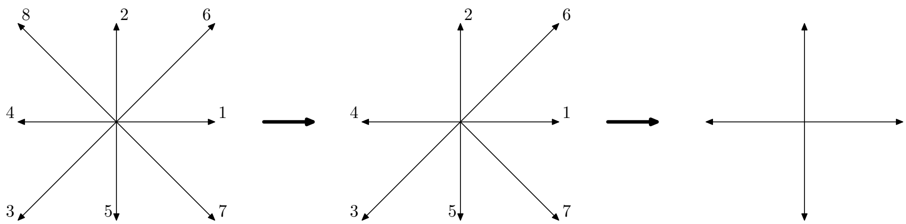

FIGURE 1.1. Les éventails de Y_4, de Y_3 et de $\mathbf{P}^1 \times \mathbf{P}^1$

Nous fixons tout au long ce travail le point central $Q = [1:1] \times [1:1]$. Notre premier résultat principal, concernant l'approximation de Q par d'autres points rationnels sur Y_3, dit que des courbes nodales, qui couvrent une partie dense de Y_3 et font déjà des objets centraux d'étude pour la surface Y_4 [19], et des courbes cuspidales donnent en même temps les meilleurs approximants en dehors d'un fermé de Zariski (cf. §3.4.1, §3.4.2). Autrement dit, ces ceux types de courbes achèvent la constante α.

THÉORÈME 1.4 (cf. Proposition 4.1, Corollaire 4.3). — *Nous avons*

- $\alpha(Q, Y_3) = 2$. *Elle s'obtient sur les trois courbes rationnelles lisses* $l_i, (1 \leqslant i \leqslant 3)$ *de degré minimal passant par* Q. *Ces courbes sont localement accumulatrices ;*
- $\alpha_{\text{ess}}(Q) = \frac{5}{2}$. *Elle peut être calculée sur des courbes nodales et des courbes cuspidales passant par* Q.

Les courbes $l_i, 1 \leqslant i \leqslant 3$ sont en fait les (transformations strictes des) sections de bidegré $(1,1)$ joignant Q et l'un des 3 points éclatés dans $\mathbf{P}^1 \times \mathbf{P}^1$ (cf. (13)). Il s'en suit que la surface Y_3 et la partie $Y_3 \setminus \bigcup_{i=1}^{3} l_i$ vérifie la conjecture de D. McKinnon [26, Conjecture 2.9]. Bien que des courbes nodales et des courbes cuspidales aient la même valeur de constante d'approximation, le point Q est approché de manière radicalement différente suivant elles (cf. §2.3), à cause de la façon dont α est calculée sur les courbes rationnelles (cf. Théorème 2.3). Le premier type correspond à l'approximation d'un nombre quadratique et le travail [19] montre que le nombre des points rationnels entrant dans le zoom critique est faible et il n'existe pas de mesure décrivant la distribution locale (cf. Théorème 2.4). Alors que pour le deuxième on approche un point rationnel sur le corps de base $\mathbf{Q}$. Dans ce cas l'ordre de grandeur est comparable avec la croissance de hauteur et les points sont répartis asymptotiquement suivant une mesure limite (cf. Théorème 2.5).

Notre deuxième théorème principal confirme que la Répartition Empirique 1.3 vaut pour Y_3.

Théorème 1.5 (cf. Théorèmes 7.6, 7.15, 7.16). — *Soient* $Z = \bigcup_{i=1}^{3} l_i, U = Y_3 \setminus Z$.

1. *Pour* $2 = \alpha(Q, Y_3) \leqslant r < \frac{5}{2}$, *nous avons*
$$\frac{1}{B^{1-\frac{1}{r}}} \delta_{Y_3,Q,B,r} \to \delta_r,$$
où δ_r *est une mesure à support dans* Z *;*
2. *Pour* $r = \alpha_{\mathrm{ess}}(Q) = \frac{5}{2}$, *il existe une partie mince, qui est la réunion de* Z *du type I et* M *du type II, telle qu'en notant* $V = Y_3(\mathbf{Q}) \setminus (Z(\mathbf{Q}) \cup M)$, *nous ayons*
$$\frac{1}{B^{\frac{1}{5}}} \delta_{V,Q,B,\frac{5}{2}} \longrightarrow \delta_{\frac{5}{2}},$$
où $\delta_{\frac{5}{2}}$ *est une mesure est absolument continue par rapport à la mesure de Lebesgue sur* $\mathbf{R}^2$.

C'est un fait empirique que la difficulté d'établir l'existence de la mesure limite augmente lorsque le degré de la surface baisse. Ceci est parallèle avec la conjecture de Batyrev-Manin sur la répartition globale (1). Même s'il n'est pas ardu d'établir une majoration uniforme d'ordre de grandeur $B^{\frac{1}{5}+\delta}$ (cf. Proposition 4.4) pour $r = \frac{5}{2}$, cependant, démontrer des formules asymptotiques ainsi que la convergence de mesures de zoom est un processus beaucoup plus délicat, et surtout on a dû surmonter la difficulté pour travailler avec le paramétrage donné par des courbes nodales dans [19, §5.2.2]. Nous renvoyons au Théorème 7.6 pour une formule asymptotique précise comprenant la mesure $\delta_{\frac{5}{2}}$ et un terme d'erreur. Ce serait intéressant de pouvoir interpréter le facteur arithmétique qui apparaît dans le terme principal de façon géométrique comme dans le Principe 1.1. Signalons que la densité de la mesure $\delta_{\frac{5}{2}}$ fait apparaître les trois courbes $l_i, 1 \leqslant i \leqslant 3$ qui sont localement accumulatrices. Ceci est analogue au résultat pour la surface X_3 (cf. [18, Théorème 1.3]).

Les *parties minces* (Définition 6.1), dont la contribution a été considérée comme négligeable dans plupart de cas (cf. le théorème de S.D. Cohen, [34, §13 Theorem 1]), se montrent parfois problématiques dans le programme de Batyrev-Manin (Principe 1.1). Elles ont été reprises depuis le premier contre-exemple de V. V. Batyrev et Yu. Tschinkel [3]. Mais c'est rare dans la littérature qu'on soit capable de contrôler le cardinal de la partie obtenue en retirant une partie mince. À la connaissance de l'auteur, les seules réussites jusqu'au présent sont le résultat du Rudulier [22] (cf. aussi [31, §8]) où elle a considéré des schémas de Hilbert des points sur des surfaces, et celui de Browning et

Heath-Brown [5] sur la variété bi-projective $\sum_{i=0}^{3} x_i y_i^2 = 0$ admettant une structure de fibration en quadratiques. Un fait intéressant pour la surface Y_3 est que la partie mince de type II M consiste en précisément les points sur des courbes cuspidales. Le Théorème 1.5 (2) dit que les parties minces ont aussi des influences non-négligeables pour le problème de distribution locale. Il nous fournit ainsi un autre exemple sur la gestion de ces ensembles.

Bien que la paire $(\alpha, \alpha_{\mathrm{ess}})$ soit un indicateur pour détecter les sous-variétés *localement accumulatrices* (Définition 2.2), on aurait besoin d'un critère vis-à-vis des points rationnels à retirer dans la démonstration des formules 1.2 et 1.3. Dans [31, §4], s'inspirant des études de familles des courbes rationnelles sur les variétés, Peyre a introduit la notion de *liberté* définie pour chaque point rationnel, analogue aux *pentes* à la J.-B. Bost. Ceci a pour but d'améliorer le Principe 1.1. Il semble raisonnable que cette notion puisse aussi s'insérer dans notre cadre (voir aussi [32, §7] pour une autre discussion).

1.3. Heuristique géométrique. — Pour soutenir les formules 1.2 & 1.3, nous tendons à donner, dans un travail à venir, une analogie géométrique s'inspirant celle du Principe 1.1. Il s'agit d'une sorte de «distribution» des courbes rationnelles sur une variété algébrique. À la lumière d'un théorème de McKinnon et Roth (Théorème 2.3) et à l'aide de la théorie de déformation, en prenant en compte seulement des courbes ayant des multiplicités suffisamment grandes en Q, cette heuristique nous fournit une interprétation des ordres de grandeur sur B et surtout sur $\log B$ compatible avec tous les résultats connus, comme l'avait déjà constaté par Pagelot [28] pour la surface $\mathbf{P}^1 \times \mathbf{P}^1$ que le bon exposant sur $\log B$ ne coïncide pas avec celui dans le cas global. En réalité, pour Y_3 étudiée dans ce texte, toutes telles courbes réalisant la constante essentielle appartiennent aux classes $m[\omega_{Y_3}^{-1}], m \in \mathbf{N}_{\geqslant 1}$, qui forment un sous-groupe de $A_1(Y_3)$ de rang 1 et donc $l(Y_3) = 1$ (ce qui suggère que la puissance sur $\log B$ devrait être $l(Y_3) - 1 = 0$).

1.4. Outils analytiques. — Expliquons les méthodes que l'on utilise, comment le zoom critique se traduit en des problèmes analytiques et d'où la partie mince intervient.

1.4.1. *Descente à la Legendre.* — En considérant la famille de courbes nodales $\{C_{a,b}\}$ (27) à paramètres $(a,b) \in \mathbf{Z}^2_{\mathrm{prem}}$ et en choisissant pour chaque (a,b) un paramétrage $\mathbf{P}^1 \to C_{a,b}$ (dont les paramètres sont notés $(x,y) \in \mathbf{Z}^2_{\mathrm{prem}}$), nous obtenons une application rationnelle $\mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow Y_3$ qui donne un paramétrage local des points rationnels sur Y_3 par les $C_{a,b}$. L'idée clef ici est que plutôt que de dénombrer les points sur chaque courbe $C_{a,b}$, c'est-à-dire fixer le coupe (a,b) en comptant (x,y), ce qui était la méthode adoptée dans [19], on compte directement les couples $(a,b) \times (x,y)$ et il se trouve que les (a,b) sont paramétrés par les (x,y) quand on restreigne à des équations de Pell-Fermat avec certaine condition de divisibilité que l'on précise tout de suite. Ce passage nous permet d'éliminer les paramètres (a,b). Il a des similarités avec la méthode de «descente» qui remonte à Legendre et Dirichlet (cf. par exemple [24] pour un survol historique et les références dedans) et est récemment reprises par Fouvry et Jouve dans les travaux [10], [11] et [12] pour traiter la solution fondamentale des équations de Pell-Fermat $x^2 - Dy^2 = 1$.

Plus précisément, il s'avère qu'il faut se concentrer sur les solutions des équations du type comme par exemple, (cf. (51) pour la forme la plus générale que l'on considère)

$$ax^2 - by^2 = b - a, \tag{5}$$

avec $b > a > 0, \mathrm{pgcd}(a,b) = 1$. En écrivant (5) en $a(x^2+1) = b(y^2+1)$, nous en déduisons que

$$a \mid y^2 + 1, \quad b \mid x^2 + 1,$$

puisque $\mathrm{pgcd}(a,b) = 1$. Nous arrivons donc au paramétrage

$$x^2 = bk - 1, \quad y^2 = ak - 1, \quad k \in \mathbf{N},$$

et nous voyons que $\mathrm{pgcd}(x^2+1, y^2+1) = k = k(x,y)$ et

$$a = \frac{y^2+1}{k(x,y)}, \quad b = \frac{x^2+1}{k(x,y)}.$$

En rajoutant la condition supplémentaire

$$b - a \mid x - y, \tag{6}$$

qui est équivalente à

$$(x^2 - y^2)/k(x,y) \mid x - y \Leftrightarrow x + y \mid k(x,y), \tag{7}$$

la résolution de l'équation (5) plus la condition (6) finalement se transforme en

$$x + y \mid k(x,y) \Leftrightarrow x^2 \equiv -1 \mod (x+y) \Leftrightarrow y^2 \equiv -1 \mod (x+y).$$

On est alors amené à un problème de distribution des racines de congruence polynomiale *quadratique*, pour lequel nous appuyons sur un résultat de Hooley (cf. [16, Theorem 3], [17, Theorem 2]).

La partie mince entre dans cette histoire en changeant le signe de l'équation (5) comme suit.

$$ax^2 - by^2 = a - b.$$

Dans ce cas nous sommes amené au problème de congruence

$$x^2 \equiv y^2 \equiv 1 \mod (x+y).$$

Ceci n'est plus quadratique mais *linéaire* puisque nous avons de manière équivalente,

$$(x+1)(x-1) \equiv 0 \mod (x+y),$$

qui admet en général plus de solutions.

En effet, définissons pour $F(X)\in\mathbf{Z}[X]$ un polynôme entier,

$$\varrho_F(n)=\#\{1\leqslant m\leqslant n : n\mid F(m)\}.$$

Lorsque $\deg F(X)=2$, il est classiquement connu d'après Ingham [20] et Erdős [9] que

(8)
$$\sum_{1\leqslant n\leqslant X}\varrho_F(n)\sim\begin{cases}c_F X; & \text{si } F(X) \text{ irréductible sur } \mathbf{Q};\\ c_F X\log X; & \text{si } F(X) \text{ a deux racines rationnelles distinctes.}\end{cases}$$

1.4.2. *Répartition modulo* 1 *des racines de congruence polynomiales.* — Pour obtenir une mesure limite de dimension 2, nous avons besoin en outre d'estimer des sommes plus raffinées que (8) suivantes qui ressemble à

$$\sum_{\substack{1\leqslant l\leqslant m\leqslant X\\ F(l)\equiv 0\mod m}}\mathbf{1}_{]\theta_1,\theta_2]}\left(\frac{l}{m}\right). \tag{9}$$

pour $0<\theta_1<\theta_2\leqslant 1$.

Quand $F(X)$ est irréductible, Hooley [16], [17] démontré en premier que la série $(\frac{l}{m})$, formée par des couples $(l,m)\in\mathbf{N}^2_{\geqslant 1}$ numérotés convenablement tels que $F(l)\equiv 0\mod m$, est *uniformément répartie* dans l'intervalle $]0,1]$. Par conséquent, la somme (9) ci-dessus se comporte comme $\sim c_F(\theta_2-\theta_1)X$. En pratique, le problème de comptage nous amène à une somme suivante plus générale que (9). Par un processus de «limite double», nous arrivons à démontrer que

$$\sum_{\substack{1\leqslant l\leqslant m\\ F(l)\equiv 0\mod m\\ G(\frac{l}{m})m\leqslant X}}\mathbf{1}_{]\theta_1,\theta_2]}\left(\frac{l}{m}\right)\sim XC_F\int_{\theta_1}^{\theta_2}\frac{\mathrm{d}\,x}{G(x)},$$

où $G:]0,1]\to\mathbf{R}_{>0}$ est une fonction continue bornée (cf. Proposition 7.13).

Le cas où $F(X)$ est réductible est récemment abordé par Dartyge et Martin [7]. Leur résultat implique que les points rationnels correspondant à ce cas ne se répartissent pas assez aléatoirement et donc sont à retirer dans notre dénombrement. Plus de détails se trouvent dans §A.2.2.

1.5. Organisation du texte. — Nous commençons par rappeler la définition des constantes approximation et de l'opération de zoom dans §2. Une discussion de la géométrie de Y_3 occupe de §3, où les équations de la famille de courbes nodales et celle de courbes cuspidales sont données en détails (§3.4). Nous démontrons le Théorème 1.4 et le Théorème 1.5 (1) dans §4 et nous donnons une majoration uniforme naïve (Proposition 4.4) dont la preuve est relativement courte. Pour démontrer la Répartition empirique 1.3, nous avons besoin du paramétrage des points rationnels par des courbes nodales, donné dans §5. La définition de partie mince est ensuite rappelée dans §6. En particulier pour la surface Y_3 nous démontrons que l'ensemble des points se situant sur des courbes cuspidales forme une partie mince (§6.2). Le dénombrement se déroule dans §7. Nous établions la convergence vague de la famille de mesures de zoom (§7.3 et §7.4) en dehors de la partie mince (§7.5). Dans l'Appendice A, nous étudions, en suivant Erdős et Hooley, des problèmes de congruence polynomiale dans les cas irréductibles (§A.1) et scindés (§A.2), avec une référence spéciale à l'équirépartition modulo 1 des racines de congruence (§A.1.2 et §A.2.2).

1.6. Notations. — $\mu(\cdot)$ désigne la fonction le Möbius, $\tau(\cdot)$ désigne la fonction donnant le nombre de diviseurs. La lettre p est réservé aux nombres premiers. La condition $p^\nu \| n$ signifie que $p^\nu \mid n$ et $p^{\nu+1} \nmid n$. On désigne par $a = \square$ pour un nombre réel a si $\sqrt{a} \in \mathbf{Q}$. Soit $E \subset \mathbf{Z}^n$, on note E_{prem} le sous-ensemble de E définie par

$$E_{\text{prem}} = \{(x_1, \cdots, x_n) \in E : \text{pgcd}(x_i, 1 \leqslant i \leqslant n) = 1\}.$$

Soient $b, c, m \in \mathbf{Z}$. On écrit désormais $b \equiv c[m]$ pour $b \equiv c \mod m$. Définissons une fonction arithmétique $g : \mathbf{N}_{\geqslant 1} \to \mathbf{N}_{\geqslant 1}$ donnée par

$$g(n) = \prod_p p^{\lceil \frac{v_p(n)}{2} \rceil}, \quad n \in \mathbf{N}. \tag{10}$$

Pour un domaine $I \subset \mathbf{R}^n$, $\mathbf{1}_I(\cdot)$ désigne sa fonction caractéristique.

2. Constante d'approximation et opération de grossissement

2.1. Constantes d'approximation et constantes essentielles. — La notion de constantes d'approximation fut introduite en premier par D. McKinnon dans [26], comme étant une généralisation de *mesure d'irrationalité* dans l'approximation diophantienne classique. Elle apparaît aussi dans le travail de pionnier de S. Pagelot [28] sur la distribution locale des points rationnels. Cette notion est ensuite reprise et réétudiée systématiquement par D. McKinnon et M. Roth dans [27]. Nous résumons brièvement quelques propriétés dont nous aurons besoin ici et nous renvoyons le lecteur à [27, §2], [18, §1.1.3] et [19, §2] pour plus de détails.

Soient X une variété projective irréductible définie sur $\mathbf{Q}$, $Q_0 \in X(\bar{\mathbf{Q}})$ un point algébrique réel fixé et L un fibré en droites gros tel qu'il induise une application birationnelle de X sur son image et bien définie sur un ouvert U_0 contenant Q_0. Nous y associons une hauteur de Weil exponentielle H et nous fixons une distance projective archimédienne $d(\cdot,\cdot)$ (cf. [27, §2]).

Définition 2.1 (McKinnon-Roth, cf. aussi [19], Définition 2.2). — Soit V une partie constructible de X. *La constante d'approximation* $\alpha(Q_0, V)$ est le supremum des $\delta > 0$ tels que l'inégalité du type de Liouville suivante

$$\exists C(\delta) > 0, \quad \forall y \in (V \cap U_0)(\mathbf{Q}) \setminus \{Q_0\}, \quad d(Q_0, y)^\delta H(y) > C(\delta),$$

soit valide.

D'après cette définition, la valeur de $\alpha(Q_0, V)$ est indépendante du choix de la distance et celui de la hauteur. Une plus petite valeur de la constante correspond à de meilleures approximations, contrairement à la notion classique de mesure d'irrationalité (puisque nous avons imposé l'exposant sur la distance).

Définition 2.2 (Pagelot). — Soit V une partie constructible de X. *La constante essentielle* (par rapport à V) est

$$\alpha_{\text{ess}}(Q_0, V) = \sup_{\substack{Y \subseteq V \text{partie constructible} \\ \text{dense pour la topologie de Zariski induite}}} \alpha(Q_0, Y).$$

Par convention, $\alpha_{\text{ess}}(Q_0) = \alpha_{\text{ess}}(Q_0, X)$. V est dite *localement accumulatrice* si elle vérifie

$$\alpha_{\text{ess}}(Q_0, V) < \alpha_{\text{ess}}(Q_0).$$

La constante essentielle mesure donc l'approximation générique du point Q_0, et les variétés localement accumulatrices contiennent des points plus proches de Q_0. Ces deux notions sont utiles pour interpréter le principe célèbre dans l'approximation diophantienne : toute approximation suffisamment «bonne» (i.e. dont la constante d'approximation est plus petite que la constante essentielle) devrait être faite sur certaines sous-parties strictes de la variété.

Les courbes rationnelles jouent un rôle très important dans le comportement local des points rationnels (comme conjecturé par McKinnon [26, Conjecture 2.7]). Leur constantes d'approximation sont calculées de la façon suivante.

Théorème 2.3 (McKinnon-Roth, [27], Theorem 2.16). — *Soient C une courbe rationnelle définie sur $\mathbf{Q}$ et L un faisceau inversible ample sur C et $Q_0 \in C(\bar{\mathbf{Q}})$. Soit $\phi : \mathbf{P}^1 \to C$ le morphisme de normalisation. Alors*

$$\alpha(Q_0, C) = \alpha_{\text{ess}}(Q_0, C) = \min_{P \in \phi^{-1}(Q_0)} \frac{d}{m_P r_P},$$

où $d = \deg_C(L)$, m_P est la multiplicité de la branche de C passant par Q_0 correspondant à P et

$$r(P) = \begin{cases} 0 & si\ k(P) \not\subset \mathbf{R}; \\ 1 & si\ k(P) = \mathbf{Q}; \\ 2 & sinon, \end{cases}$$

où par convention, $r_P = 0$ signifie que $\frac{d}{m_P r_P} = \infty$.

2.2. Opération de grossissement. — On résume la définition de l'opération de zoom. Pour plus de détails, voir [19, §2.2].

2.2.1. *Énoncé du problème.* — On note $\mathcal{C}^{\mathrm{b}}_{Q_0}X$ l'espace vectoriel des fonctions continues à support compact définie sur $(T_{Q_0}X)_{\mathbf{R}}$ (l'espace tangent de X en Q_0) à valeurs réelles. En ayant fixé un difféomorphisme local

$$\varrho : X(\mathbf{R}) \dashrightarrow (T_{Q_0}X)_{\mathbf{R}},$$

pour Y un sous-ensemble de $X(\mathbf{Q})$, la famille de mesures de zoom $\{\delta_{Y,Q_0,B,r}\}_B$ (de facteur r par rapport à Y) est définie par

$$\delta_{Y,Q_0,B,r}(f) = \sum_{x\in Y:H_L(x)\leqslant B} f(B^{\frac{1}{r}}\rho(x)), \quad \forall f \in \mathcal{C}^{\mathrm{b}}_{Q_0}X. \tag{11}$$

Si U est un sous-schéma de X, on écrit souvent $\delta_{U,Q_0,B,r}$ pour $\delta_{U(\mathbf{Q}),Q_0,B,r}$. Plus le facteur de zoom r est grand, plus le zoom est faible (c'est-à-dire on compte plus de points). Si $r < \alpha(U,Q_0)$, alors on a $\delta_{U,Q_0,B,r} \to \delta_{Q_0}$ la mesure de Dirac concentrée sur Q_0 ([19, Proposition 2.8]). Le cas le plus intéressant est quand U est une partie dense de X tel que $\alpha(Q_0,U) = \alpha_{\mathrm{ess}}(Q_0) < \infty$ et que l'on prend le facteur de zoom $r = \alpha_{\mathrm{ess}}(Q_0)$, appelé *critique*. Il semble exister souvent un «saut» de dimension du support de mesures de zoom quand le facteur r traverse $\alpha_{\mathrm{ess}}(Q_0)$. On espère aussi que pour $r > \alpha_{\mathrm{ess}}(Q_0)$, les points se distribuent de façon plus uniforme autour de Q_0. Le résultat [19, Théorème 1.2 (2)] est un premier pas dans cette direction.

Le problème (2) maintenant se pose de la manière suivante : Existe-t-il $\beta = \beta(r), \gamma = \gamma(r) \geqslant 0$ et δ_r une mesure sur $(T_{Q_0}X)_{\mathbf{R}}$ tels que

$$\frac{\delta_{U,Q_0,B,r}}{B^{\beta}(\log B)^{\gamma}} \longrightarrow \delta_r, \quad B \to \infty \tag{12}$$

au sens de *convergence vague* ?

Remarque. — La formulation de la série (11) dépende *a priori* de la hauteur et du difféomorphisme local choisis. La dernière est évidemment fonctorielle : un changement d'une carte locale devrait produire une autre mesure limite qui est la composée de l'ancienne avec cette application. Toutefois, dans tous les exemples connus, la fonction de densité de la mesure limite reste invariante et donc semble être une propriété intrinsèque et géométrique.

2.3. Comparaison entre les courbes nodales et les courbes cuspidales. — Comparons maintenant l'approximation sur les courbes cubiques ayant des singularités de types différents en le point à approcher. Lorsque les pentes des tangentes en ce point sont irrationnelles, il s'agit essentiellement de l'approximation des nombres *algébriques irrationnels* par les nombres rationnels. Lorsqu'elles sont rationnelles, on revient à l'approximation des nombres *rationnels*.

Par meilleur illustrer cette différence, nous prendrons des cubiques affines d'équation simple se plongeant dans $\mathbf{P}^2$. On munit le fibré $\mathcal{O}(1)$ d'une hauteur de Weil absolue exponentielle H. Choisissons sans perte de généralité le point $Q_0 = [0 : 0 : 1]$ à approcher et définissons la distance (pour $[u : v : w] \in \mathbf{P}^2(\mathbf{Q})$)

$$d([u : v : w]) = \max\left(\left|\frac{u}{w}\right|, \left|\frac{v}{w}\right|\right).$$

Cas I. — Considérons la courbe nodale $\mathcal{C} : y^2 = x^3 + ax^2$ pour $a \in \mathbf{Q}_{\geqslant 0}$. Les tangentes en points $(0,0)$ ont les pentes $\pm\sqrt{a}$. Un paramétrage pour $\mathcal{C}$ est donné par

$$k \longmapsto (k^2 - a, k(k^2 - a)).$$

On voit alors que la distance induite est $|k^2 - a|$, qui est équivalente à

$$\min(|k - \sqrt{a}|, |k + \sqrt{a}|).$$

Donc pour approcher le point $(0,0)$, il faut que k approche $\pm\sqrt{a}$ (correspondant aux deux branches dans la normalisation). La constante d'approximation est alors égale au degré (par rapport à $\mathcal{O}(1)$) de $\mathcal{C}$ divisé par 2 si $a \neq \square$. Car, si l'on note $\phi : \mathcal{C}_0 \to \mathcal{C}$ la normalisation, le sous-schéma $\phi^{-1}(Q_0) = \{\pm\sqrt{a}\}$ n'est pas défini sur $\mathbf{Q}$. On a alors $r_P = 2, \forall P$ dans le Théorème 2.3.

Théorème 2.4 ([19] Théorème 1.3). — *Nous avons*

$$\alpha(Q_0, \mathcal{C}) = \alpha_{\mathrm{ess}}(Q_0, \mathcal{C}) = \frac{3}{2},$$

et

$$\delta_{\mathcal{C}, Q_0, \frac{3}{2}, B}(\chi(\varepsilon)) = O_\varepsilon(1).$$

Cas II. — Considérons la courbe cuspidale $\mathcal{C}' : y^2 = x^3$ avec le point de rebroussement $Q_0 = (0,0)$ ayant la tangente $y = 0$. Un paramétrage usuel pour cette courbe est

$$t \longmapsto (t^2, t^3).$$

D'où la distance induite se calcule comme $|t^2|$. La constante d'approximation est alors divisée par 2 car $m_P = 2$ dans le Théorème 2.3.

Théorème 2.5 (Pagelot [28], cf. [19] Théorème A.1). — *Nous avons*

$$\alpha(Q_0, \mathcal{C}') = \alpha_{\mathrm{ess}}(Q_0, \mathcal{C}') = \frac{3}{2}$$

et que

$$\frac{1}{B^{\frac{3}{4}}}\delta_{\mathcal{C}', Q_0, \frac{3}{2}, B}(\chi(\varepsilon)) \to \delta_{\frac{3}{2}},$$

une mesure à support dans $\mathcal{C}'$.

On finit la discussion en observant que $\alpha(Q_0,\mathcal{C})=\alpha(Q_0,\mathcal{C}')$, mais

$$\delta_{\mathcal{C},Q_0,\frac{3}{2},B}(\chi(\varepsilon))=o(\delta_{\mathcal{C}',Q_0,\frac{3}{2},B}(\chi(\varepsilon))).$$

La manière dont on calcule α peut entraîner de différents ordres de grandeur dans le zoom.

3. Géométrie et courbes rationnelles sur Y_3

Nous considérons dans cet article la surface torique obtenue en éclatant 3 des 4 points invariants par l'action du tore de $\mathbf{P}^1\times\mathbf{P}^1$. Nous désignerons par Y_3 cette surface puisque Y_4 a été réservée à la surface étudiée dans [19]. Sans perte de généralité, on peut supposer que Y_3 est l'éclatement en

$$P_1=[1:0]\times[1:0],\quad P_2=[0:1]\times[1:0],\quad P_3=[1:0]\times[0:1].$$

On fixe désormais

$$Q=[1:1]\times[1:1]$$

le point à approcher.

3.1. Géométrie de Y_3. — Nous utilisons les coordonnées $[x:y]\times[s:t]$ de $\mathbf{P}^1\times\mathbf{P}^1$ pour les points différents de $P_i, 1\leqslant i\leqslant 3$. Rappelons l'éventail de Y_3 (Figure 1.1). Les courbes passant par Q de degré minimal sont les (transformations strictes des) courbes rationnelles $l_i, 1\leqslant i\leqslant 3$ définies par les équations

$$y=x,\quad t=s,\quad yt=xs. \tag{13}$$

On note $Z=\bigcup_{i=1}^3 l_i$ et $U=Y_3\setminus Z$. Il existe trois diviseurs exceptionnels E_i $(1\leqslant i\leqslant 3)$ (par rapport au modèle minimal $\mathbf{P}^1\times\mathbf{P}^1$). La classe du diviseur anticanonique est

$$[\omega_{Y_3}^{-1}]=\mathcal{O}(2,0)+\mathcal{O}(0,2)-[E_1]-[E_2]-[E_3].$$

Ses sections globales s'identifient à celles de $[\omega_{\mathbf{P}^1\times\mathbf{P}^1}^{-1}]$ qui s'annulent en les points éclatés. Une base S est donnée par

$$x^2st,\quad y^2st,\quad t^2xy,\quad s^2xy,\quad xyst,\quad y^2t^2. \tag{14}$$

On identifie localement T_QY_3 sur la carte $(y\neq 0)\cap(t\neq 0)$ à l'espace affine $\mathbf{R}^2$ via

$$\varrho:[x:y]\times[s:t]\longmapsto(w,z)=\left(\frac{x}{y}-1,\frac{s}{t}-1\right), \tag{15}$$

sur lequel on utilise la distance pour le dénombrement :

$$d((w,z))=\max(|w|,|z|). \tag{16}$$

La courbe $yt = xs$ s'écrivant $wz + w + z = 0$ (une hyperbole) sous ce difféomorphisme divise $\mathbf{R}^2$ en deux régions

$$R_1 = \{(w,z) : wz + w + z > 0\}, \quad R_2 = \{(w,z) : wz + w + z < 0\}. \tag{17}$$

3.2. Calcul de hauteur. — Nous utilisons la hauteur de Weil absolue associée à $\omega_{Y_3}^{-1}$ définie par

$$H(P) = \frac{\max_{f \in S}(|f(P)|)}{\operatorname{pgcd}(f(P), f \in S)}, \quad P \in \left(Y_3 \setminus \bigcup_{i=1}^{3} E_i\right)(\mathbf{Q}).$$

Pour un point $P = [x : y] \times [s : t]$ avec

$$\operatorname{pgcd}(x,y) = \operatorname{pgcd}(s,t) = 1,$$

on a

$$\begin{aligned} & \operatorname{pgcd}(x^2st, y^2st, t^2xy, s^2xy, xyst, y^2t^2) \\ = \; & \operatorname{pgcd}(\operatorname{pgcd}(x,s)\operatorname{pgcd}(x,t)\operatorname{pgcd}(y,s)\operatorname{pgcd}(y,t), y^2t^2) \\ = \; & \operatorname{pgcd}(x,t)\operatorname{pgcd}(y,s)\operatorname{pgcd}(y,t). \end{aligned} \tag{18}$$

3.3. Symétries. — De la structure de l'éventail de Y_3, nous constatons qu'il est symétrique par rapport à la droite engendrée par le 3-ième rayon. Cette surface admet donc l'automorphisme relevant celui de $\mathbf{P}^1 \times \mathbf{P}^1$ s'écrivant en coordonnées homogènes comme suit.

$$\Phi : [x : y] \times [s : t] \longmapsto [s : t] \times [x : y]. \tag{19}$$

Il fixe le diviseur exceptionnel E_1, échange E_2 avec E_3 et préserve le diviseur anticanonique $\omega_{Y_3}^{-1}$ ainsi que la hauteur associée définie dans §3.2. Dans les coordonnées (w,z) (15), l'application Φ n'est rien d'autre que la permutation de coordonnées

$$(\varrho \circ \Phi \circ \varrho^{-1})(w,z) = (z,w).$$

Cette symétrie nous permet de ramener l'étude aux régions

$$S_1 = \{(w,z) \in \mathbf{R}^2 : w > 0, z > w\}, \tag{20}$$

$$S_2 = \{(w,z) \in \mathbf{R}^2 : w < 0, wz + w + z > 0\}, \tag{21}$$

$$S_3 = \{(w,z) \in \mathbf{R}^2 : w > 0, wz + w + z < 0\}, \tag{22}$$

$$S_4 = \{(w,z) \in \mathbf{R}^2 : w < 0, z < w\}. \tag{23}$$

On a (rappelons (17))

$$S_1 \cup S_2 \subset R_1, \quad S_3 \cup S_4 \subset R_2.$$

3.4. Courbes nodales et courbes cuspidales sur Y_3. — Les équations à 3 paramètres $c, d, e \in \mathbf{Z}^3_{\text{prem}}$:

$$c(t-s)^2xy + d(y-x)^2st - eyt(y-x)(t-s) = 0,$$

définissent des courbes comme étant des sections de $\omega_{Y_3}^{-1}$. Pour une telle section irréductible ($\Leftrightarrow cd \neq 0$), un calcul de l'annulation de la matrice Hessienne donne que les pentes des tangentes en Q se coïncident si et seulement si $e^2 = 4cd$, ce qui revient à la condition qu'il existe $(r_1, r_2) \in (\mathbf{Z}^2_{\neq 0})_{\text{prem}}$ tel que

$$c = r_1^2, \quad d = r_2^2, \quad e = 2r_1r_2. \tag{24}$$

3.4.1. *Famille de courbes cuspidales.* — Nous obtenons donc la famille de courbes cuspidales

$$R_{r_1,r_2} : r_1^2(t-s)^2xy + r_2^2(y-x)^2st - 2r_1r_2yt(y-x)(t-s) = 0. \tag{25}$$

Dans les coordonnées (w, z), elles s'écrivent

$$(r_1z - r_2w)^2 + zw(r_1^2z + r_2^2w) = 0. \tag{26}$$

Nous avons (rappelons l'automorphisme Φ (19)) $\Phi(R_{r_1,r_2}) = R_{r_2,r_1}$ et la pente de la tangente en $Q = (0, 0)$ est $\frac{r_2}{r_1}$. Les points rationnels sur la famille $\{R_{r_1,r_2}\}_{r_1,r_2 \in \mathbf{Z}^2_{\text{prem}}}$ de courbes forment un ensemble *mince* de la région R_2. Nous en discutons plus loin dans §6.

3.4.2. *Famille de courbes nodales.* — Lorsque les coefficients c, d, e ne vérifient pas la condition (24), nous obtenons en général une famille de courbes nodales à 3 paramètres. La sous-famille qui sera utile est la suivante, également utilisée pour l'étude de la surface Y_4 dans [19] :

$$C_{a,b} : axy(s-t)^2 = bst(x-y)^2, \quad (a, b) \in \mathbf{Z}^2_{\text{prem}}. \tag{27}$$

Les tangentes en $Q = [1 : 1] \times [1 : 1]$ ont les pentes $\pm\sqrt{\frac{b}{a}}$. Elles sont donc irrationnelles si et seulement si $ab \neq \square$.

4. Déduction des constantes d'approximation et majoration naïve

Le but de cette section est consacrée à la démonstration du Théorème 1.4, puis sa conséquence, le Théorème 1.5 (1), et à celle d'une version faible du Théorème 1.5 (2). Elle s'agit d'une majoration naïve pour le zoom critique, qui est en faveur de l'heuristique de Batyrev-Manin (cf. [19, §2.2]).

4.1. Bornes inférieures uniformes. — La proposition suivante correspond à la première partie du Théorème 1.4 (1). Elle se découle par une minoration directe.

Proposition 4.1. — *Nous avons*

$$\alpha(Q, Y_3) = 2.$$

Démonstration. — Rappelons (15) et (16). Fixons $P = [x : y] \times [s : t] \neq Q$. Supposons sans perte de généralité que $s \neq t$. On a alors

$$\begin{aligned}
\mathrm{H}_{\omega_{Y_3}^{-1}}(P)d(P)^2 &\geqslant \frac{|t^2xy|}{\operatorname{pgcd}(x,t)\operatorname{pgcd}(y,s)\operatorname{pgcd}(y,t)}\left(\frac{s}{t}-1\right)^2 \\
&= \frac{|xy|}{\operatorname{pgcd}(x,t)\operatorname{pgcd}(y,s)\operatorname{pgcd}(y,t)}(s-t)^2 \\
&\geqslant 1.
\end{aligned}$$

Cela montre que $\alpha(Q, Y_3) \geqslant 2$ par la Définition 2.1. Mais les courbes spéciales $l_i, 1 \leqslant i \leqslant 3$ de (13) vérifient $\alpha(Q, l_i) = 2$. D'où $\alpha(Q, Y_3) \leqslant \alpha(Q, l_i) = 2$. Ceci clôt la démonstration. □

4.2. Constante essentielle. — Ensuite on démontre une borne inférieure pour les points généraux en augmentant la puissance 2 à $\frac{5}{2}$. On va l'utiliser pour obtenir la constante essentielle (Théorème 1.4 (2)).

PROPOSITION 4.2. — *Pour $P = [x : y] \times [s : t]$ n'appartenant pas aux trois courbes $l_i, 1 \leqslant i \leqslant 3$, i.e.*

$$x \neq y, \quad s \neq t, \quad xs \neq yt, \tag{28}$$

tel que $d(P) \leqslant C$ avec $0 < C < 1$, il existe alors $D = D(C) > 0$ tel que

$$\mathrm{H}_{\omega_{Y_3}^{-1}}(P)d(P)^{\frac{5}{2}} \geqslant D.$$

Démonstration. — Commençons par l'encadrement

$$\left|\frac{x}{y}-1\right| \leqslant C \Rightarrow 1 - C \leqslant \left|\frac{x}{y}\right| \leqslant 1 + C.$$

Par conséquent,

$$\left|\frac{y}{x}-1\right| = \left|\frac{x}{y}-1\right|\left|\frac{x}{y}\right|^{-1} \leqslant (1-C)^{-1}\left|\frac{x}{y}-1\right|.$$

Et de même pour $\left|\frac{t}{s}-1\right|$. On a aussi

$$\begin{aligned}
\left|\frac{xs}{yt}-1\right| &= \left|\frac{x}{y}\left(\frac{s}{t}-1\right)+\frac{x}{y}-1\right| \\
&\leqslant (1+C)\max\left(\left|\frac{x}{y}-1\right|,\left|\frac{s}{t}-1\right|\right) = (1+C)d(P).
\end{aligned}$$

En minorant $d(P)$ de la manière suivante,

$$d(P)^{\frac{5}{2}} \geqslant \left|\frac{x}{y}-1\right|\left|\frac{s}{t}-1\right|\left|\frac{xs}{yt}-1\right|^{\frac{1}{2}}(1+C)^{-\frac{1}{2}},$$

et en rappelant l'hypothèse (28), nous avons alors

$$
\begin{aligned}
&\mathrm{H}_{\omega_{Y_3}^{-1}}(P)d(P)^{\frac{5}{2}}\\
&\geqslant \frac{|xyst|}{\operatorname{pgcd}(x,t)\operatorname{pgcd}(y,t)\operatorname{pgcd}(y,s)}\left|\frac{y}{x}-1\right|\left|\frac{t}{s}-1\right|\left|\frac{xs}{yt}-1\right|^{\frac{1}{2}}\frac{(1-C)^2}{(1+C)^{\frac{1}{2}}}\\
&=\left(\frac{|y|}{\operatorname{pgcd}(y,t)\operatorname{pgcd}(y,s)}\frac{|t|}{\operatorname{pgcd}(y,t)\operatorname{pgcd}(x,t)}\frac{|yt-xs|}{\operatorname{pgcd}(x,t)\operatorname{pgcd}(y,s)}\right)^{\frac{1}{2}}\\
&\quad\times|x-y||t-s|(1-C)^2(1+C)^{-\frac{1}{2}}\\
&\geqslant(1-C)^2(1+C)^{-\frac{1}{2}}.
\end{aligned}
$$

□

Corollaire 4.3. — *Soit U l'ouvert $Y_3\setminus(Z=\bigcup_{i=1}^3 l_i)$. Nous avons*

$$\alpha(U,Q)=\alpha_{\mathrm{ess}}(Q)=\frac{5}{2}. \tag{29}$$

Les courbes (13) *sont les variétés localement accumulatrices.*

Démonstration. — Les courbes nodales $C_{a,b}, ab\neq\square$ vérifient $\alpha(C_{a,b},Q)=\frac{5}{2}$ d'après le Théorème 2.3 et leur réunion couvre l'ouvert dense U. D'où

$$\alpha_{\mathrm{ess}}(Q)\leqslant\alpha(Q,C_{a,b})=\frac{5}{2}.$$

D'après la Proposition 4.2, nous savons

$$\alpha_{\mathrm{ess}}(Q)\geqslant\alpha(Q,U)\geqslant\frac{5}{2}.$$

Cela démontre (29). Alors que les courbes $l_i, 1\leqslant i\leqslant 3$ de (13) sont lisse et de degré 2 et donc vérifient $\alpha(Q,l_i)=2$. □

Démonstration du Théorème 1.5 (1). — On applique [19, Proposition 2.7] et l'on obtient que pour $2\leqslant r<\frac{5}{2}$, pour tout $f\in\mathcal{C}^{\mathrm{b}}_Q(Y_3)$ et $B\gg 1$, on a

$$\delta_{Y_3,Q,B,r}(f)=\delta_{Z,Q,B,r}(f).$$

La convergence de la suite $\{B^{-(1-\frac{1}{r})}\delta_{Z,Q,B,r}\}_B$ résulte du théorème de Pagelot (Théorème 2.5). □

4.3. Majoration uniforme. — Sans utiliser les courbes nodales (27), nous démontrons le résultat faible suivant.

Proposition 4.4. — *Pour $\varepsilon>0$ fixé, on a que, pour tout B suffisamment grand et pour tout $\delta>0$,*

$$\#\{P\in U(\mathbf{Q}):\mathrm{H}_{\omega_{Y_3}^{-1}}(P)\leqslant B, d(P)\leqslant\varepsilon B^{-\frac{2}{5}}\}\ll_{\delta,\varepsilon}B^{\frac{1}{5}+\delta}.$$

Démonstration. — Introduisons les notations

$$e_1 = \mathrm{pgcd}(y,t), \quad e_2 = \mathrm{pgcd}(x,t), \quad e_3 = \mathrm{pgcd}(y,s); \tag{30}$$

$$f_1 = \frac{y}{e_1e_3}, \qquad f_2 = \frac{t}{e_1e_2}, \qquad f_3 = \frac{yt-xs}{e_2e_3}; \tag{31}$$

$$g_1 = \frac{x}{e_2}, \qquad g_2 = \frac{s}{e_3}. \tag{32}$$

Reprenons la démonstration de la proposition 4.2,

$$1 \gg_\varepsilon \mathrm{H}_{\omega_{Y_3}^{-1}}(P)d(P)^{\frac{5}{2}} \gg_\varepsilon |f_1f_2f_3|^{\frac{1}{2}}|x-y||t-s|.$$

On voit que

$$|x-y|, \quad |t-s|, \quad f_1, \quad f_2, \quad f_3 \quad \ll_\varepsilon 1. \tag{33}$$

On va ensuite déduire des encadrements pour les paramètres g_1, g_2, e_1, afin de démontrer qu'ayant les fixé, il n'y a qu'un nombre fini (dépendant de ε) de choix pour e_2, e_3. La condition de zoom

$$\left|\frac{x-y}{y}\right| \ll_\varepsilon B^{-\frac{2}{5}}, \qquad \left|\frac{s-t}{t}\right| \ll_\varepsilon B^{-\frac{2}{5}}$$

implique que

$$|t| \gg_\varepsilon B^{\frac{2}{5}}, \quad |y| \gg_\varepsilon B^{\frac{2}{5}}. \tag{34}$$

D'où l'on déduit

$$|s| \gg_\varepsilon B^{\frac{2}{5}}, \quad |x| \gg_\varepsilon B^{\frac{2}{5}}. \tag{35}$$

La condition que la hauteur soit bornée implique que

$$|xse_1^2e_2e_3| \leqslant |xsf_1f_2e_1^2e_2e_3| = |xyst| \leqslant Be_1e_2e_3.$$

D'où l'on déduit, grâce à (35),

$$e_1 \ll_\varepsilon B^{\frac{1}{5}}. \tag{36}$$

D'après (33), on a

$$f_3 = \frac{yt-xs}{e_2e_3} = f_1f_2e_1^2 - g_1g_2 \ll_\varepsilon 1.$$

Donc une fois que f_1, f_2 et e_1 sont choisis, il n'y a qu'un nombre fini de valeurs possibles pour le produit g_1g_2. Puisque

$$\#\{(n_1,n_2) \in \mathbf{N}^2_{\geqslant 1} : n_1n_2 = n\} = \tau(n) = O(n^\delta) \tag{37}$$

pour tout $\delta > 0$, on a donc démontré qu'ayant fixé f_1, f_2, e_1, le nombre des (g_1, g_2) est $O_{\delta,\varepsilon}(B^\delta)$.

Il nous reste à déterminer e_2, e_3. Fixons f_1, f_2, e_1 ainsi que g_1 et g_2. Comme $yt \neq xs$, on a

$$f_1f_2e_1^2 \neq g_1g_2 \Leftrightarrow f_3 \neq 0.$$

Soit $C_2(\varepsilon) > 0$ tel que

$$\max(|y-x|, |t-s|) \leqslant C_2(\varepsilon).$$

Le cardinal des (e_2, e_3) possibles est majoré par (compte tenu des signes possibles de x, y, s, t)

$$4\#\{(u,v) \in \mathbf{Z}^2_{\text{prem}} : |ug_1 - f_1e_1v| \leqslant C_2(\varepsilon) \text{ et } |vg_2 - f_2e_1u| \leqslant C_2(\varepsilon)\},$$

où on compte les points primitifs sur un réseau du déterminant $|f_3|$ dans un carré d'aire $4C_2(\varepsilon)^2$ dont l'intérieur contient l'origine. On en conclut qu'il n'y a qu'un nombre fini de choix pour e_2, e_3 une fois que f_1, f_2, e_1, g_1, g_2 sont choisis ([14, Lemma 2]). La majoration qui fallait démontrer résulte de (33), (36) et (37). □

5. Paramétrage par des courbes nodales

On rappelle brièvement le paramétrage donné par la famille de courbes nodales (27) utilisé de manière cruciale dans [19]. L'équation de $C_{a,b}$ s'écrit sous les coordonnées (w, z) comme

$$a(1+w)z^2 = b(1+z)w^2.$$

Le paramétrage rationnel que l'on utilise est (cf. [19, 5.3.1])

(38)

$$\begin{aligned}&\Psi : [a:b] \times [u:v] \\ &\longmapsto [x:y] \times [s:t] = \left[\frac{bv(u-v)}{D_1d_2d_3} : \frac{u(bv-au)}{D_1d_2d_3}\right] \times \left[\frac{au(u-v)}{d_1D_2d_3} : \frac{v(bv-au)}{d_1D_2d_3}\right],\end{aligned}$$

où

$$\begin{aligned}(39) \qquad d_1 &= \operatorname{pgcd}(u,b), \quad d_2 = \operatorname{pgcd}(v,a), \quad d_3 = \operatorname{pgcd}(u-v, b-a), \\ D_1 &= \operatorname{pgcd}(u^2, b), \quad D_2 = \operatorname{pgcd}(v^2, a).\end{aligned}$$

Nous avons (cf. [19, 5.12])

$$(40) \qquad \operatorname{pgcd}(x,t) = \frac{v}{d_2}, \quad \operatorname{pgcd}(y,s) = \frac{u}{d_1}, \quad \operatorname{pgcd}(y,t) = \frac{bv-au}{d_1d_2d_3}.$$

En le composant avec le difféomorphisme local ϱ, nous obtenons donc le paramétrage des points dans les régions $S_i, 1 \leqslant i \leqslant 4$ (20)-(23) comme suit :

$$(41) \qquad (\varrho \circ \Psi)((a,b) \times (u,v)) = (w,z) = \left(\frac{au^2 - bv^2}{u(bv-au)}, \frac{au^2-bv^2}{v(bv-au)}\right).$$

En particulier nous voyons que $\frac{z}{w} = \frac{u}{v}$, ce qui signifie que $\frac{u}{v}$ est la pente du point (w, z).

Soient

(42)
$$T_1 = \left\{(a,b)\times(u,v)\in \mathbf{N}^2_{\mathrm{prem}}\times\mathbf{Z}^2_{\mathrm{prem}} : b>a>0, u>v>0, \sqrt{\frac{b}{a}}<\frac{u}{v}<\frac{b}{a}\right\},$$

(43)
$$T_2 = \left\{(a,b)\times(u,v)\in \mathbf{N}^2_{\mathrm{prem}}\times\mathbf{Z}^2_{\mathrm{prem}} : b>a>0, u>-v>0, -\frac{u}{v}>\sqrt{\frac{b}{a}}\right\},$$

(44)
$$T_3 = \left\{(a,b)\times(u,v)\in \mathbf{N}^2_{\mathrm{prem}}\times\mathbf{Z}^2_{\mathrm{prem}} : b>a>0, u>-v>0, -\frac{u}{v}<\sqrt{\frac{b}{a}}\right\},$$

(45)
$$T_4 = \left\{(a,b)\times(u,v)\in \mathbf{N}^2_{\mathrm{prem}}\times\mathbf{Z}^2_{\mathrm{prem}} : b>a>0, u>v>0, \frac{u}{v}<\sqrt{\frac{b}{a}}\right\}.$$

Lemme 5.1. — *Pour $P=(a,b)\times(u,v)\in\bigcup_{i=1}^4 T_i$, nous avons $au^2-bv^2<0$ si et seulement si $P\in T_3\cup T_4$.*

Démonstration. — Un calcul élémentaire nous donne :

$$au^2-bv^2<0 \Leftrightarrow -\sqrt{\frac{b}{a}}<\frac{u}{v}<\sqrt{\frac{b}{a}} \Leftrightarrow (a,b)\times(u,v)\in T_3\cup T_4. \qquad \square$$

Proposition 5.2. — *L'application $\varrho\circ\Psi$ donne une bijection entre l'ensemble T_i et l'ensemble des $(w,z)\in\mathbf{Q}^2$ satisfaisant à $\max(|w|,|z|)<1$ dans S_i pour tout $1\leqslant i\leqslant 4$.*

Démonstration. — Tout d'abord pour $(a,b)\times(u,v)\in(\mathbf{N}_{\geqslant 1})^2_{\mathrm{prem}}\times(\mathbf{N}_{\geqslant 1}\times\mathbf{Z}_{\neq 0})_{\mathrm{prem}}$ l'application $\varrho\circ\Psi$ (41) nous donne (w,z) avec $\max(|w|,|z|)<1$. Réciproquement, étant donné $(w,z)\in\bigcup_{i=1}^4 S_i$ tel que $\max(|w|,|z|)<1$, les égalités

$$\frac{b}{a}=\frac{z^2(1+w)}{w^2(1+z)}\quad(>0),\qquad \frac{u}{v}=\frac{z}{w} \tag{46}$$

déterminent de façon unique les couples $(a,b)\in\mathbf{N}^2_{\mathrm{prem}}$ et $(u,v)\in(\mathbf{N}\times\mathbf{Z})_{\mathrm{prem}}$. De cette manière, nous avons que si $\max(|w|,|z|)<1$,

$$\begin{aligned}
&\frac{b}{a}=\frac{z^2(1+w)}{w^2(1+z)}>1\\
\Leftrightarrow&(z-w)(z+w+zw)>0\\
\Leftrightarrow&z>w \text{ et } z+w+zw>0 \quad\text{ou}\quad z<w \text{ et } z+w+zw<0,\\
\Leftrightarrow&(w,z)\in\bigcup_{i=1}^4 S_i.
\end{aligned}$$

De plus,

$$
\begin{array}{llll}
(w,z)\in S_1 & \Rightarrow & \sqrt{\dfrac{1+w}{1+z}}\dfrac{z}{w}<\dfrac{z}{w}<\dfrac{z^2(1+w)}{w^2(1+z)} & \Leftrightarrow \sqrt{\dfrac{b}{a}}<\dfrac{u}{v}<\dfrac{b}{a},\\
(w,z)\in S_2 & \Rightarrow & \sqrt{\dfrac{1+w}{1+z}}\Big(-\dfrac{z}{w}\Big)<-\dfrac{z}{w} & \Leftrightarrow -\dfrac{u}{v}>\sqrt{\dfrac{b}{a}},\\
(w,z)\in S_3 & \Rightarrow & \sqrt{\dfrac{1+w}{1+z}}\Big(-\dfrac{z}{w}\Big)>-\dfrac{z}{w}>1 & \Leftrightarrow 1<-\dfrac{u}{v}<\sqrt{\dfrac{b}{a}},\\
(w,z)\in S_4 & \Rightarrow & \sqrt{\dfrac{1+w}{1+z}}\dfrac{z}{w}>\dfrac{z}{w}>1 & \Leftrightarrow \sqrt{\dfrac{b}{a}}>\dfrac{u}{v}>1.
\end{array}
$$

Nous avons donc établi

$$(\varrho\circ\Psi)(T_i)=\{(w,z)\in\mathbf{Q}^2:(w,z)\in S_i,\max(|w|,|z|)<1\}.\qquad\square$$

La distance (16) se calcule sur $\bigcup_{i=1}^4 S_i$ par, d'après (41),

$$d((w,z))=\max\left(\left|\frac{au^2-bv^2}{u(bv-au)}\right|,\left|\frac{au^2-bv^2}{v(bv-au)}\right|\right)=\left|\frac{\frac{u^2}{v^2}-\frac{b}{a}}{\frac{b}{a}-\frac{u}{v}}\right|\tag{47}$$

puisque nous avons supposé $|\frac{b}{a}|>1$ et donc $|\frac{u}{v}|=|\frac{z}{w}|>1$.

6. Parties minces

6.1. Définition. — Le phénomène d'accumulation globale causé par des parties minces est déjà constaté dans plusieurs travaux, puisque le nombre des points de hauteur bornée qu'elles contiennent est parfois non-négligeable. Il s'avère qu'elles jouent aussi un rôle dans notre problème de comptage.

Définition 6.1 (Definition 3.1.1 [33]). — Soit X une variété intègre sur $\mathbf{Q}$. Soit M un sous-ensemble de $X(\mathbf{Q})$ vérifiant qu'il existe une variété V et un morphisme $f:V\to X$ tels que

1. $M\subseteq f(V(\mathbf{Q}))$;
2. Le morphisme f est génériquement fini et il n'admet pas de section rationnelle.

Alors M est dite du *type I* si la fibre générique de f est vide. Il est dit du *type II* si la variété V est intègre et le morphisme f est dominant. *Une partie mince* est une réunion finie d'ensembles du type I ou II.

Une partie mince du type I est donc contenue dans l'ensemble des points rationnels d'un fermé de Zariski. Alors que celle du type II est parfois dense dans X pour la topologie de Zariski.

6.2. Ensemble mince dans Y_3. — Considérons l'ensemble

$$M = \{(w,z) \in \mathbf{Q}^2 : -w-z-wz = \square \neq 0\} \subset R_2. \tag{48}$$

C'est un ensemble se trouvant dans la partie au-dessous de l'hyperbole $wz + w + z = 0$ et dense dans S_2 pour la topologie réelle.

D'après le paramétrage Ψ (38),

$$-w-z-wz = \square \Leftrightarrow -\frac{(au^2-bv^2)(b-a)}{(bv-au)^2} = \square \Leftrightarrow \left(\frac{u^2}{v^2} - \frac{b}{a}\right)\left(1 - \frac{b}{a}\right) = \square,$$

Soit V la sous-variété de $\mathbf{A}^3_{x_1,x_2,x_3}$ d'équation

$$x_3^2 = \left(x_2^2 - x_1\right)(1 - x_1).$$

Pour $y \in \mathbf{Q}$, soit $(y(z_1), y(z_2)) \in (\mathbf{N} \times \mathbf{Z}_{\neq 0})_{\text{prem}}$ défini par $y = \frac{y(z_1)}{y(z_2)}$. Considérons l'application rationnelle $\pi : V \dashrightarrow Y_3$ donnée par

$$\pi(x_1,x_2,x_3) = \Psi((x_1(z_1),x_1(z_2)) \times (x_2(z_1),x_2(z_2))).$$

Elle est donc génériquement de degré 2. Alors nous avons que $M \subset \mathrm{Im}(\pi)(\mathbf{Q})$. D'où M est une partie mince dans Y_3.

Proposition 6.2. — *Un point $P = (w,z) \in \mathbf{Q}^2$ est dans M si et seulement s'il existe une courbe cuspidale R_{r_1,r_2} (26) passant par P.*

Démonstration. — Une des courbes R_{r_1,r_2}, vue comme une équation en $\lambda = \frac{b}{a}$, s'écrit comme

$$w^2(1+z)\lambda^2 - 2zw\lambda + z^2(1+w) = 0, \tag{49}$$

avec le discriminant

$$\triangle = 4w^2z^2(1-(1+w)(1+z)) = -4w^2z^2(wz+w+z).$$

Donc un point $(w,z) \in \mathbf{Q}^2$ est sur R_{r_1,r_2} seulement si

$$\triangle = \square \Leftrightarrow -(wz+w+z) = \square. \tag{50}$$

Réciproquement, si un point $(w,z) \in M$, alors l'équation (49) admet deux solutions rationnelles distinctes, qui correspondent à deux courbes cuspidales passant par M. $\square$

7. Dénombrement d'approximants

Cette partie technique est consacrée à la démonstration du Théorème 1.5 (2). Après une étape préparatoire de finitude et de réduction (§7.1 & §7.2), on établit le résultat principal – le Théorème 7.6, dans §7.3 et §7.4 et le Théorème 1.5 (2) en résulte. À la fin dans §7.5 on finit par une discussion courte sur le nombre des points dans l'ensemble mince.

7.1. Finitude des paramètres. — Le lemme suivant détermine les paramètres possibles qui sont de nombre fini quand on regarde le zoom dans un voisinage borné fixé. Rappelons que $U = Y_3 \setminus \bigcup_{i=1}^{3} l_i$ (13) et les notations (39).

LEMME 7.1. — *Soient $\varepsilon > 0$. Nous avons que pour tout $P = [a:b] \times [u:v] \in U(\mathbf{Q})$ tel que $d(P)^{\frac{5}{2}} \mathrm{H}_{\omega_{Y_3}^{-1}}(P) \leqslant \varepsilon$,*

$$D_1, \quad D_2, \quad \frac{|b-a|}{d_3}, \quad \frac{|au^2-bv^2|}{D_1D_2d_3} \ll_{\varepsilon} 1.$$

Démonstration. — Reprenons les notations (30), (31) et rappelons encore la démonstration de la Proposition 4.4. Compte tenu du paramétrage dans §5, nous voyons que les quantités

$$\begin{aligned} |x-y| &= \frac{|au^2-bv^2|}{D_1d_2d_3}, \quad |t-s| = \frac{|au^2-bv^2|}{D_2d_1d_3}, \\ |f_1| &= \frac{|y|}{e_1e_3} = \frac{d_1^2}{D_1}, \quad |f_2| = \frac{|t|}{e_1e_2} = \frac{d_2^2}{D_2}, \\ |f_3| &= \frac{|yt-xs|}{e_2e_3} = \frac{|(b-a)(au^2-bv^2)|}{D_1D_2d_3^2}. \end{aligned}$$

sont finies dans tout voisinage borné grâce à (33). Il en est de même pour les quantités

$$\frac{|b-a|}{d_3}, \quad \frac{|au^2-bv^2|}{D_1D_2d_3}.$$

De plus, l'égalité

$$|x-y| \times |f_2| = \frac{|au^2-bv^2|}{D_1d_2d_3} \times \frac{d_2^2}{D_2} = \frac{|au^2-bv^2|}{D_1D_2d_3} \times d_2,$$

et la finitude ci-dessus impliquent que d_2 ainsi que D_2 sont finies grâce à la relation $d_2 \leqslant D_2 \leqslant d_2^2$. De même pour d_1 et D_1 en considérant $|t-s| \times |f_1|$. □

7.2. Réduction aux problèmes de congruences. — Pour $C_3 \in \mathbf{N}_{\geqslant 1}, D \in \mathbf{Z}_{\neq 0}$ avec $\mathrm{pgcd}(C_3, D) = 1$, on considère l'équation suivante en $(a,b) \times (u,v) \in \mathbf{Z}^2_{\mathrm{prem}} \times \mathbf{Z}^2_{\mathrm{prem}}$:

$$\mathcal{E}_{C_3,D} : C_3(au^2-bv^2) = D(b-a). \tag{51}$$

LEMME 7.2. — *Pour tout $(a,b) \times (u,v) \in (\mathbf{N}^2_{\geqslant 1})_{\mathrm{prem}} \times (\mathbf{Z}^2_{\neq 0})_{\mathrm{prem}}$ vérifiant $C_3 \mid b-a$ et (51), nous avons*

$$\mathrm{pgcd}(u^2,b)\,\mathrm{pgcd}(v^2,a) \mid D, \quad \mathrm{pgcd}(u-v, b-a) \mid \frac{b-a}{C_3}.$$

Démonstration. — Puisque

$$au^2 - bv^2 = a(u^2 - v^2) - (b-a)v^2 = b(u^2 - v^2) - u^2(b-a),$$

nous avons

$$\text{pgcd}(u^2, b)\,\text{pgcd}(v^2, a)\,\text{pgcd}(u-v, b-a) \mid au^2 - bv^2 = \frac{b-a}{C_3} \times D. \tag{52}$$

Grâce à (52), la première divisibilité est évidente puisque ces deux termes sont premiers avec $b-a$. Pour voir la deuxième, il suffit simplement d'observer que pour tout nombre premier $p \mid \text{pgcd}(u-v, b-a)$, écrivons $p^k \| \, \text{pgcd}(u-v, b-a)$ pour certain $k \geqslant 1$. Si $p \mid D$, comme $\text{pgcd}(C_3, D) = 1$ et $\text{pgcd}(u-v, b-a) \mid b-a$, on obtient

$$p^k \;\Big|\; \frac{b-a}{C_3}. \tag{53}$$

Pour les $p \nmid D$, (53) est automatique, grâce encore à (52). □

Le lemme suivant est une généralisation de la transformation faite dans l'introduction du texte pour l'équation (5).

LEMME 7.3. — *Fixons* $(u, v) \in \mathbf{Z}^2_{\text{prem}}$ *tel que*

$$\min(u^2, v^2) > -\frac{D}{C_3}, \tag{54}$$

alors tout $(a, b) \in (\mathbf{N}^2_{\geqslant 1})_{\text{prem}}$ *vérifiant* (51) *s'écrit*

$$(a, b) = \left(\frac{C_3 v^2 + D}{k(u,v)}, \frac{C_3 u^2 + D}{k(u,v)} \right), \tag{55}$$

où on pose pour $(x, y) \in \mathbf{Z}^2$,

$$k(x, y) = \text{pgcd}(C_3 x^2 + D, C_3 y^2 + D).$$

Démonstration. — Étant donné un $(a, b) \in (\mathbf{N}^2_{\geqslant 1})_{\text{prem}}$ vérifiant $\mathcal{E}_{C_3,D}$ (51), nous avons $(C_3 u^2 + D)a = (C_3 v^2 + D)b$, et un raisonnement identique comme pour (5) donne bien (55). □

L'observation suivante est l'étape cruciale pour réduire le comptage aux problèmes de congruences. Elle joue un rôle tout comme le passage de (5) à (7) en supposant la condition (6). Introduisons la notation remplaçant désormais d_3 :

$$c_3 = \frac{b-a}{d_3} = \frac{b-a}{\text{pgcd}(u-v, b-a)}. \tag{56}$$

PROPOSITION 7.4. — *Fixons* $q \in \mathbf{N}_{\geqslant 1}$. *Pour tout* $(a, b) \times (u, v) \in (\mathbf{N}^2_{\geqslant 1})_{\text{prem}} \times (\mathbf{Z}^2_{\neq 0})_{\text{prem}}$ *vérifiant* (54) *et* (51), *la condition*

$$u + v \mid qk(u, v). \tag{57}$$

équivaut à l'assertion suivante :

$$c_3 \mid C_3 q. \tag{58}$$

Démonstration. — Grâce au Lemme 7.2, nous avons

$$\mathrm{pgcd}(u-v, b-a) = \frac{b-a}{c_3} \;\Big|\; \frac{b-a}{C_3},$$

et donc $C_3 \mid c_3$. Adaptons les notations introduites dans le Lemme 7.3. Soit $q' \mid q$ tel que $c_3 = q'C_3$. Nous avons

$$\begin{aligned}\frac{b-a}{C_3 q'} = \frac{u^2-v^2}{k(u,v)q'} &= \mathrm{pgcd}(u-v, b-a)\\ &= \mathrm{pgcd}\left(u-v, \frac{C_3(u^2-v^2)}{k(u,v)}\right)\\ &= \mathrm{pgcd}\left(\frac{k(u,v)}{u+v} \times \frac{u^2-v^2}{k(u,v)}, C_3 \times \frac{u^2-v^2}{k(u,v)}\right).\end{aligned}$$

Alors nous en déduisons que $u+v \mid q'k(u,v)$. En particulier si $q' \mid q$, on a *a fortiori* $u+v \mid qk(u,v)$.

Supposons maintenant (57) et écrivons

$$qk(u,v) = \lambda(u+v)$$

pour $\lambda \in \mathbf{N}_{\geqslant 1}$. Définissons

$$q' = \frac{q}{\mathrm{pgcd}(q,\lambda)}, \quad \lambda' = \frac{\lambda}{\mathrm{pgcd}(q,\lambda)}.$$

Nous avons $q'k(u,v) = \lambda'(u+v), \mathrm{pgcd}(\lambda', q') = 1$. D'où nous déduisons que $q' \mid u+v$ et $\lambda' \mid k(u,v)$. Cela nous permet d'écrire

$$\frac{b-a}{C_3} = \frac{u^2-v^2}{k(u,v)} = \frac{q'(u-v)}{\lambda'},$$

et nous avons donc $\lambda' \mid u-v$. Alors

$$\begin{aligned}\mathrm{pgcd}(u-v, b-a) &= \mathrm{pgcd}\left(u-v, \frac{C_3(u^2-v^2)}{k(u,v)}\right)\\ &= \mathrm{pgcd}\left(u-v, \frac{C_3 q'(u-v)}{\lambda'}\right)\\ &= \frac{u-v}{\lambda'}\,\mathrm{pgcd}(\lambda', C_3 q')\\ &= \frac{b-a}{C_3 q'},\end{aligned}$$

puisque $\mathrm{pgcd}(k(u,v), C_3) = 1, \lambda' \mid k(u,v)$. D'où $c_3 = C_3 q'$ avec $q' \mid q$. □

La prochaine proposition révèle que les courbes cuspidales interviennent dans le dénombrement en changeant le signe de (51).

Proposition 7.5. — *Rappelons l'ensemble mince M (48) et notons $A_{C_3,D}$ l'ensemble des $(a,b)\times(u,v)\in(\mathbf{N}^2_{\geqslant 1})_{\text{prem}}\times(\mathbf{Z}^2_{\neq 0})_{\text{prem}}$ vérifiant (54) et (51). Nous avons $(\varrho\circ\Psi)(A_{C_3,D})\cap M\neq\varnothing$ si et seulement si $C_3,-D=\square$. En particulier dans ce cas le polynomial C_3X^2+D est réductible dans $\mathbf{Z}[X]$. Par conséquent, $\varrho\circ\Psi$ restreinte à*

$$\bigsqcup_{\substack{(C_3,D)\in\mathbf{N}_{\geqslant 1}\times\mathbf{Z}_{<0}\\ \operatorname{pgcd}(C_3,D)=1\\ C_3,-D=\square}} A_{C_3,D}\cap(T_3\cup T_4)\to(T_QY_3)_{\mathbf{R}}$$

donne un paramétrage de l'ensemble mince

$$\{(w,z)\in M\cap(S_3\cup S_4):\max(|w|,|z|)<1\}.$$

D'après la Proposition 6.2, l'ensemble M est compris des points sur les courbes cuspidales R_{r_1,r_2}. Pour la topologie réelle, une telle R_{r_1,r_2} a plusieurs composantes connexes. Le résultat ci-dessus implique que celle qui passe par Q vivent seulement dans la région $(S_3\cup S_4)\cup\Phi(S_3\cup S_4)$ (Φ est l'automorphisme (19)).

Démonstration. — Prenons $(w,z)\in\mathbf{Q}^2\cap(\bigcup_{i=1}^4 S_i)$. Rappelons le paramétrage pour les coordonnées (w,z) (41). Nous calculons

$$\begin{aligned} wz+w+z&=\frac{(au^2-bv^2)^2}{uv(bv-au)^2}+\frac{au^2-bv^2}{u(bv-au)}+\frac{au^2-bv^2}{v(bv-au)}\\ &=\frac{(au^2-bv^2)(b-a)}{(bv-au)^2}. \end{aligned}$$

Par la définition de l'ensemble M, $\varrho\circ\Psi(\bigcup_{i=1}^4 T_i)\cap M\neq\varnothing$ si et seulement si

$$-(au^2-bv^2)(b-a)=\square.$$

En particulier nous voyons que $au^2-bv^2<0$ puisque $b>a$. Nous déduisons du Lemme 5.1 que l'image d'un tel point est dans $S_3\cup S_4$. De l'équation $\mathcal{E}_{C_3,D}$ (51), la condition ci-dessus est équivalente à

$$-(b-a)^2\frac{D}{C_3}=\square\Leftrightarrow-\frac{D}{C_3}=\square.$$

Le résultat en découle puisque nous avons imposé que D,C_3 soient premiers entre eux et que $C_3>0$. Le dernier énoncé découle de la Proposition 6.2. □

7.3. Région S_1. — Grâce à la similitude du calcul, nous nous bornons alors pour la suite de cette sous-section à la région S_1 (20). Comme expliqué précédemment, cette région n'intersecte pas l'ensemble mince M. Pour $\varepsilon_1>\varepsilon_2>0,\theta_1>\theta_2>1$ fixés, on désigne par $\underline{\varepsilon},\underline{\theta}$ pour ces paramètres et l'on considère

$$R(\underline{\varepsilon},\underline{\theta})=\{(w,z)\in S_1:\varepsilon_2<z\leqslant\varepsilon_1,\theta_2w<z\leqslant\theta_1w\},\tag{59}$$

une région trapézoïdale et $\chi_{\underline{\varepsilon},\underline{\theta}}(\cdot) = \mathbf{1}_{R(\underline{\varepsilon},\underline{\theta})}(\cdot)$ la fonction caractéristique. Elle sert d'une fonction «test». Pour déduire le Théorème 1.5 (2), il suffit d'établir la convergence de la suite $\{\delta_{\varrho^{-1}(S_1)\cap U,Q,B,\frac{5}{2}}(\chi_{\underline{\varepsilon},\underline{\theta}})\}_B$, car toute fonction continue à support compact est la limite uniforme d'une suite de fonctions caractéristique de la forme $\chi_{\underline{\varepsilon},\underline{\theta}}$.

Théorème 7.6. — *On a*

$$\begin{aligned}&\delta_{\varrho^{-1}(S_1)\cap U,Q,B,\frac{5}{2}}(\chi_{\underline{\varepsilon},\underline{\theta}})\\ &= B^{\frac{1}{5}}\left(\int \chi_{\underline{\varepsilon},\underline{\theta}}(w,z)\frac{\mathbb{E}(wz\sqrt{w+z})}{wz\sqrt{w+z}}\,\mathrm{d}\,w\,\mathrm{d}\,z + O_{\underline{\varepsilon},\underline{\theta}}\left(\frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}}\right)\right),\end{aligned}$$

où $\mathbb{E} :]0,\infty[\to [0,\infty[$ *est une fonction croissante en escalier.*

Remarque. — La mesure obtenue fait apparaître les courbes $l_i, 1 \leqslant i \leqslant 3$ (13) de degré 2 et la somme de leur puissances est égale exactement à la constante essentielle $\frac{5}{2}$.

7.3.1. *Déroulement du comptage.* — Pour $P = [x:y]\times[s:t]$ tel que $\varrho\circ\Psi(P) \in S_1$, on a

$$\max(|x^2st|, |y^2st|, |t^2xy|, |s^2xy|, |xyst|, |y^2t^2|) = |s^2xy|, \tag{60}$$

Pour $(a,b)\times(u,v) \in T_1$ (42), la formule (cf. §3.2) pour calculer la hauteur par rapport aux paramètres a,b,u,v est donnée, grâce à (18), (38) et (40), par

$$\begin{aligned}\mathrm{H}((\varrho\circ\Psi)(a,b)\times(u,v)) &= \frac{|s^2xy|}{\operatorname{pgcd}(x,t)\operatorname{pgcd}(y,s)\operatorname{pgcd}(y,t)}\\ &= \frac{a^2bu^2(u-v)^3}{D_1^2D_2^2d_3^3}.\end{aligned}$$

La distance (16) restreinte à S_1 est, d'après (47),

$$d((\varrho\circ\Psi)(a,b)\times(u,v)) = \frac{\frac{u^2}{v^2}-\frac{b}{a}}{\frac{b}{a}-\frac{u}{v}}. \tag{61}$$

L'équivalence établie dans la Proposition 7.4 nous permet de faire lien avec le problème de congruences polynomiales. Nous allons faire une partition des paramètres $(a,b)\times(u,v) \in T_1$ selon la famille des équations $(\mathcal{E}_{C_3,D})_{C_3\in\mathbf{N}_{\geqslant 1},D\in\mathbf{Z}_{\neq 0}}$ (51). D'après le Lemme 5.1, dans T_1 on a $au^2 - bv^2 > 0$, et donc $D > 0$ puisque $b > a$. Pour $D_1, D_2 \in \mathbf{N}_{\geqslant 1}, D_1D_2 \mid D, \operatorname{pgcd}(D_1,D_2) = 1$ et $W \in \mathbf{N}_{\geqslant 1}, \operatorname{pgcd}(W,D) = 1$, nous définissons $E^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D})$ ($\mathbf{D}$ désigne les paramètres D_1, D_2) l'ensemble des $(a,b)\times(u,v) \in T_1$ satisfaisant à l'équation $\mathcal{E}_{C_3,D}$ (51)

et vérifiant (62), (63),(64) et (65) suivantes.

$$D_1 = \mathrm{pgcd}(u^2,b), \quad D_2 = \mathrm{pgcd}(v^2,a), \tag{62}$$

$$\mathrm{pgcd}(u-v,b-a) = \frac{b-a}{C_3W}, \tag{63}$$

$$\theta_2 < \frac{u}{v} \leqslant \theta_1, \quad \varepsilon_2 B^{-\frac{2}{5}} < \frac{\frac{u^2}{v^2}-\frac{b}{a}}{\frac{b}{a}-\frac{u}{v}} \leqslant \varepsilon_1 B^{-\frac{2}{5}}, \tag{64}$$

$$\frac{a^2bu^2(u-v)^3}{D_1^2D_2^2d_3^3} = \frac{a^2bu^2(u-v)^3C_3^3W^3}{(b-a)^3D_1^2D_2^2} \leqslant B. \tag{65}$$

D'après le Lemme 7.2, (63) est bien définie. Un calcul donne que (54) est garantie par (64) quand $B \gg_{\overline{\varepsilon},\overline{\theta}} 1$. À l'aide du Lemme 7.3, nous pouvons éliminer les paramètres a,b dans (64) et nous obtenons

$$\theta_2 < \frac{u}{v} \leqslant \theta_1, \quad \varepsilon_2 v(C_3uv-D) \leqslant B^{\frac{2}{5}}D(u+v) \leqslant \varepsilon_1 v(C_3uv-D), \tag{66}$$

équivalente à (64).

Lemme 7.7. — *On a*

$$\delta_{\varrho^{-1}(S_1)\cap U,Q,B,\frac{5}{2}}(\chi_{\underline{\varepsilon},\underline{\theta}}) = \#\left(\bigsqcup_{\substack{C_3,D,W\in\mathbf{N}_{\geqslant 1},\mathrm{pgcd}(C_3W,D)=1\\ D_1D_2\mid D,\mathrm{pgcd}(D_1,D_2)=1}} E^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D})\right). \tag{67}$$

La réunion disjointe dans (67) *est finie.*

Démonstration. — Le Lemme 7.1 et les identités (51) et (63) nous révèlent que

$$\mathrm{pgcd}(u^2,b)\,\mathrm{pgcd}(v^2,a)c_3 \ll_{\varepsilon_1} 1,$$

et

$$\frac{|au^2-bv^2|}{\mathrm{pgcd}(u^2,b)\,\mathrm{pgcd}(v^2,a)\,\mathrm{pgcd}(u-v,b-a)} = \frac{c_3}{C_3}\times\frac{D}{\mathrm{pgcd}(u^2,b)\,\mathrm{pgcd}(v^2,a)} \ll_{\varepsilon_1} 1,$$

et donc $C_3,D,W = O_{\varepsilon_1}(1)$. □

7.3.2. *Conditions de seuils.* — Avant de poursuivre le dénombrement, nous allons premièrement trouver dans cette section pour chaque point $P=(a,b)\times(u,v)\in E^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D})$ une condition pour qu'il soit dénombré. Soit $(w,z) = \varrho\circ\Psi(P)$. Il s'avère que la condition (65) donne une *équation de seuil* (cf. (69)).

Lemme 7.8. — *Pour tout* $P=(a,b)\times(u,v)\in T_1$ *qui vérifie* (51), (62), (63) *et* (65), *notons*

$$(w_0,z_0) = B^{\frac{2}{5}}\varrho\circ\Psi(P) = (B^{\frac{2}{5}}w, B^{\frac{2}{5}}z). \tag{68}$$

Alors nous avons

$$z_0 w_0 \sqrt{z_0 + w_0} > \frac{D^{\frac{5}{2}} C_3^{\frac{1}{2}} W^3}{D_1^2 D_2^2}. \tag{69}$$

Démonstration. — D'après le Lemme 7.3,

$$a = \frac{C_3 v^2 + D}{k(u,v)}, \quad b = \frac{C_3 u^2 + D}{k(u,v)},$$

on a

$$au^2 - bv^2 = \frac{D(b-a)}{C_3} = \frac{D(u+v)(u-v)}{k(u,v)},$$

$$bv - au = \frac{(C_3 uv - D)(u-v)}{k(u,v)}.$$

Nous avons donc, d'après la définition du zoom et (41),

$$z_0 = B^{\frac{2}{5}} z = B^{\frac{2}{5}} \frac{au^2 - bv^2}{v(bv - au)} = B^{\frac{2}{5}} \frac{D(u+v)}{v(C_3 uv - D)}. \tag{70}$$

Nous obtenons que

$$\frac{z_0 C_3 uv^2}{D(u+v)} > B^{\frac{2}{5}}. \tag{71}$$

Avec la condition (65) et l'identité

$$\frac{b-a}{c_3} = \frac{u^2 - v^2}{k(u,v)},$$

on obtient

$$W^3 (C_3 v^2 + D)^2 (C_3 u^2 + D) u^2 \leqslant B(u+v)^3 D_1^2 D_2^2. \tag{72}$$

Cette borne supérieure nous donne la majoration

$$\frac{C_3^3 W^3 u^4 v^4}{D_1^2 D_2^2 (u+v)^3} < B. \tag{73}$$

Ces deux inégalités (71) & (73) donnent

$$\left(\frac{C_3^3 W^3}{D_1^2 D_2^2} \right)^{\frac{2}{5}} \frac{u^{\frac{8}{5}} v^{\frac{8}{5}}}{(u+v)^{\frac{6}{5}}} < \frac{z_0 C_3 uv^2}{D(u+v)},$$

qui elle-même implique

$$\frac{C_3^{\frac{1}{5}} W^{\frac{6}{5}} D}{D_1^{\frac{4}{5}} D_2^{\frac{4}{5}}} < z_0 \left(1 + \frac{v}{u}\right)^{\frac{1}{5}} \left(\frac{v}{u}\right)^{\frac{2}{5}}.$$

D'après la définition du paramétrage Ψ et du difféomorphisme local ϱ (15), nous avons la relation entre les paramètres (u,v) et les coordonnées (w,z),

$$\frac{u}{v}=\frac{z}{w}=\frac{z_0}{w_0}, \tag{74}$$

d'ou nous obtenons

$$\frac{DC_3^{\frac{1}{5}}W^{\frac{6}{5}}}{D_1^{\frac{4}{5}}D_2^{\frac{4}{5}}}<z_0\left(1+\frac{w_0}{z_0}\right)^{\frac{1}{5}}\left(\frac{w_0}{z_0}\right)^{\frac{2}{5}}=z_0^{\frac{2}{5}}w_0^{\frac{2}{5}}(z_0+w_0)^{\frac{1}{5}}.$$

En prenons la puissance $\frac{5}{2}$, nous obtenons la borne inférieure cherchée puisque $w_0,z_0>0$ dans la région S_1. $\square$

Lemme 7.9. — *Conservons la notation* (68). *Pour tout $\varepsilon>0$, il existe $\mu_0>0$ ne dépendant que de C_3,D,W,ε tel que pour tout $P=(a,b)\times(u,v)\in T_1$ vérifiant $z_0\leqslant\varepsilon$,* (51), (62), (63) *et*

$$z_0w_0\sqrt{z_0+w_0}\geqslant\frac{D^{\frac{5}{2}}C_3^{\frac{1}{2}}W^3}{D_1^2D_2^2}+\mu_0B^{-\frac{2}{5}}, \tag{75}$$

la condition (65) *soit vérifiée pour tout $B\gg_{C_3,D,W,\varepsilon}1$.*

Démonstration. — Notons

$$\theta_0=\frac{u}{v}=\frac{z_0}{w_0}>1.$$

La condition (75) implique (69), qui nous donne

$$\frac{z_0^{\frac{5}{2}}}{\sqrt{z_0+w_0}}=\frac{\theta_0^{\frac{5}{2}}}{\sqrt{1+\theta_0}}\leqslant\frac{z_0^{\frac{5}{2}}D_1^2D_2^2}{D^{\frac{5}{2}}C_3^{\frac{1}{2}}W^3}\ll_{C_3,D,W,\varepsilon}1,$$

et donc

$$\theta_0\ll_{C_3,D,W,\varepsilon}1. \tag{76}$$

D'après l'identité (70), nous avons

$$B^{\frac{2}{5}}=\frac{z_0v(C_3uv-D)}{D(u+v)}\geqslant\frac{z_0C_3uv^2}{u+v}-\varepsilon.$$

En combinant (76), ceci implique aussi

$$u,v\ll_{C_3,D,W,\varepsilon}B^{\frac{1}{5}}. \tag{77}$$

En utilisant l'inégalité (71) on obtient aussi

$$z_0\gg_{C_3,D,W,\varepsilon}1,\quad u,v\gg_{C_3,D,W,\varepsilon}B^{\frac{1}{5}}. \tag{78}$$

On peut alors déduire la condition de hauteur (65), ou de la manière équivalente à (72), d'une inégalité du type

$$W^3C_3^3u^4v^4+\mu_1u^4v^2\leqslant\left(\frac{z_0C_3uv^2}{u+v}-\varepsilon\right)^{\frac{5}{2}}(u+v)^3D_1^2D_2^2,$$

où $\mu_1=O_{C_3,D,W}(1)$. Pour achever cette inégalité avec la condition (75), il suffit d'utiliser les encadrements

$$1<\theta_0\ll_{C_3,D,W,\varepsilon}1,\quad 1\ll_{C_3,D,W,\varepsilon}z_0\leqslant\varepsilon,\quad u,v\gg\ll_{C_3,D,W,\varepsilon}B^{\frac{1}{5}}$$

qui rassemblent (74), (76), (77), (78) et de suivre la preuve du lemme précédent. □

Nous concluons des Lemmes 7.8 et 7.9 qu'en prenant une fonction test (59), nous pouvons remplacer la condition (65) par (69).

Corollaire 7.10. — *Nous avons*

$$\#E^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})=\#\widetilde{E}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})+O(\#\,\mathrm{Er}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})),\tag{79}$$

où $\widetilde{E}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})$ *consiste en les* $(a,b)\times(u,v)\in T_1$ *vérifiant* (51), (62), (63), (66), (69) *et* $\mathrm{Er}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})$ *est l'ensemble des* $(a,b)\times(u,v)\in T_1$ *satisfaisant aux mêmes conditions précédentes sauf* (69) *est remplacée par (cf.* (75)*)*

$$\frac{D^{\frac{5}{2}}C_3^{\frac{1}{2}}W^3}{D_1^2D_2^2}\leqslant z_0w_0\sqrt{z_0+w_0}\leqslant\frac{D^{\frac{5}{2}}C_3^{\frac{1}{2}}W^3}{D_1^2D_2^2}+\mu_0B^{-\frac{2}{5}}.\tag{80}$$

7.3.3. *Réduction du comptage.* — Le but dans cette section est d'utiliser la Proposition 7.4 pour réécrire le cardinal $\#\widetilde{E}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})$ dans le Corollaire 7.10 en une somme de racines de congruences quadratiques. Premièrement nous éliminons la condition de pgcd (63). Au vu de la deuxième divisibilité du Lemme 7.2 et de (58), on a $c_3=C_3W$ et une inversion de Möbius nous donne

$$\#\widetilde{E}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})=\sum_{q|W}\mu\left(\frac{W}{q}\right)\#B^{\varepsilon,\theta}_{C_3,D}(\mathbf{D},q)\tag{81}$$

où pour $q,D_1,D_2\in\mathbf{N}_{\geqslant1}$ fixés, $B^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D},q)$ est constitué des $(a,b)\times(u,v)\in T_1$ vérifiant (51), (57), (62), (66), (69).

Nous allons désormais nous concentrer sur un seul ensemble $B^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D},q)$ pour C_3,D,W,D_1,D_2,q fixés.

Nous avons, grâce à la co-primalité pré-supposée,

$$D_1=\mathrm{pgcd}(u^2,b)=\mathrm{pgcd}(u^2,au^2-bv^2)=\mathrm{pgcd}\left(u^2,\frac{b-a}{C_3}D\right)=\mathrm{pgcd}(u^2,D).$$

De la même manière,

$$D_2=\mathrm{pgcd}(v^2,a)=\mathrm{pgcd}(v^2,D).$$

Donc nous pouvons appliquer une inversion de Möbius pour éliminer ces conditions de co-primalité.

Lemme 7.11. — *Nous avons*

$$\#B^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D},q)=\sum_{\substack{h_1,h_2\in\mathbf{N}_{\geqslant 1}\\ D_1h_1,D_2h_2\mid D\\ \operatorname{pgcd}(h_1,h_2)=1}}\mu(h_1)\mu(h_2)\#\mathcal{B}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D},\mathbf{h},q)\tag{82}$$

où $\mathcal{B}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D},\mathbf{h},q)$ *est l'ensemble des* $(u,v)\in\mathbf{N}^2_{\text{prem}}$ *vérifiant* (57)*,* (69)*,* (66) *et*

$$h_1D_1\mid u^2,\quad h_2D_2\mid v^2\quad\Longleftrightarrow\quad g(h_1D_1)\mid u,\quad g(h_2D_2)\mid v,\tag{83}$$

La condition de divisibilité (57) maintenant s'écrit,

$$q(C_3u^2+D)\equiv 0[u+v],\quad q(C_3v^2+D)\equiv 0[u+v].\tag{84}$$

L'une de ces deux conditions implique l'autre. La restriction à la région S_1 implique $\frac{u}{v}>1$ et impose donc l'unicité des couples de solutions (u,v). Nous allons désormais nous concentrer sur u et $u+v$. Introduisons les notations m,n de sorte que

$$u=g(h_1D_1)n,\quad u+v=m.\tag{85}$$

Alors la condition de co-primalité de (u,v) implique celle de (m,n) :

$$\operatorname{pgcd}(u,v)=1\Longleftrightarrow\operatorname{pgcd}(m,n)=1\text{ et }\operatorname{pgcd}(m,g(h_1D_1))=1.\tag{86}$$

Maintenant (83) et (84) s'écrivent

$$m-g(h_1D_1)n\equiv 0[g(h_2D_2)],\quad q(C_3g(h_1D_1)^2n^2+D)\equiv 0[m].\tag{87}$$

Puisque $\operatorname{pgcd}(h_1D_1,h_2D_2)=1$, soient $1\leqslant\gamma_1<g(h_2D_2)$ tel que

$$\gamma_1g(h_1D_1)\equiv 1[g(h_2D_2)].$$

Nous obtenons

$$\gamma_1m-\gamma_1g(h_1D_1)n\equiv\gamma_1m-n\equiv 0[g(h_2D_2)].$$

Puisque $v=m-g(h_1D_1)n\geqslant 1$,

$$\gamma_1\leqslant\gamma_1m-\gamma_1g(h_1D_1)n\leqslant\gamma_1m-n.$$

Il existe donc un entier $l\in\mathbf{N}_{\geqslant 1}$ tel que

$$\gamma_1m-g(h_2D_2)l=n,\tag{88}$$

et la condition de congruence dans (87) maintenant devient

$$q(C_3g(h_1D_1h_2D_2)^2l^2+D)\equiv 0[m]\tag{89}$$

avec la condition de co-primalité pour (m,l) déduite de (86) :

$$\operatorname{pgcd}(m,g(h_1D_1h_2D_2))=\operatorname{pgcd}(m,l)=1.\tag{90}$$

Nous faisons une dernière inversion de Möbius qui élimine ces dernières conditions de pgcd. Soient

$$e_1 \mid \mathrm{pgcd}(m, g(h_1D_1)), \quad e_2 \mid \mathrm{pgcd}(m, g(h_2D_2)), \quad e_3 \mid \mathrm{pgcd}(m, l),$$

tels que $\mathrm{pgcd}(e_3, g(h_1D_1h_2D_2)) = 1$. Alors $e_1e_2e_3 \mid qD$ sinon la congruence (89) n'a pas de solution. Écrivons

$$m = e_1e_2e_3m', \quad l = e_3l'. \tag{91}$$

Nous pouvons enfin réécrire (89) comme

$$\frac{qC_3g(h_1D_1h_2D_2)^2e_3}{e_1e_2}(l')^2 + \frac{qD}{e_1e_2e_3} \equiv 0[m']. \tag{92}$$

Les relations entre (l', m') et (u, v) sont, d'après (85), (88) et (91)

$$u + v = e_1e_2e_3m', \quad u = g(h_1D_1)e_3(\gamma_1e_1e_2m' - g(h_2D_2)l'). \tag{93}$$

En résumé, nous avons établi la formule suivante.

Lemme 7.12. — *Nous avons (*$\mathbf{e}$ *désigne les paramètres* e_1, e_2, e_3*)*

$$\begin{aligned} &\#\mathcal{B}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D}, \mathbf{h}, q) \\ &\qquad = \sum_{\substack{e_1e_2 \mid g(h_1D_1h_2D_2), \mathrm{pgcd}(e_1,e_2)=1 \\ e_1e_2e_3 \mid qD, \mathrm{pgcd}(e_3, g(h_1D_1h_2D_2))=1}} \left(\prod_{j=1}^{3} \mu(\mathbf{e})\right)\#\mathcal{C}^{\varepsilon,\theta}_{C_3,D}(\mathbf{D}, \mathbf{h}, \mathbf{e}, q), \end{aligned} \tag{94}$$

où $\mathcal{C}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D}, \mathbf{h}, \mathbf{e}, q)$ *est l'ensemble des couples* $(l', m') \in \mathbf{N}_{\geqslant 1} \times \mathbf{N}_{\geqslant 1}$ *vérifiant les conditions* (93)*,* (66)*,* (69) *et*

$$\mathcal{F}(l') \equiv 0[m'], \tag{95}$$

où

$$\mathcal{F}(X) = \mathcal{F}_{C_3,D,\mathbf{hD},\mathbf{e},q}(X) = \frac{qC_3g(h_1D_1h_2D_2)^2e_3}{e_1e_2}X^2 + \frac{qD}{e_1e_2e_3} \in \mathbf{Z}[X]. \tag{96}$$

7.3.4. *Une étape clef.* — Nous allons démontrer la formule asymptotique suivante en appliquant la Proposition A.3, puisque la condition (66) ne donne pas directement une forme sommatoire souhaitée. Les notations dans cette proposition et sa preuve sont indépendantes de celles utilisées avant.

Proposition 7.13. — *Soient* $A, X > 0$ *et* $0 < \vartheta_2 < \vartheta_1 \leqslant 1$ *vérifiant* $A > \vartheta_1$*. Soit* $G :]0, \vartheta_1] \to \mathbf{R}_{>0}$ *une fonction continue. Soit* $F(Y) \in \mathbf{Z}[Y]$ *un polynôme irréductible de degré* $d \geqslant 2$*. Rappelons* α_d, β_d *dans la Proposition A.3. Alors nous avons*

$$\sum_{\substack{l,m \in \mathbf{N}, F(l) \equiv 0[m] \\ m^2G(A - \frac{l}{m}) \leqslant X}} \mathbf{1}_{]\vartheta_2,\vartheta_1]}\left(A - \frac{l}{m}\right) = X^{\frac{1}{2}}\left(Z_F\int_{\vartheta_2}^{\vartheta_1}\frac{\mathrm{d}\,x}{\sqrt{G(x)}} + O\left(\frac{(\log\log X)^{\frac{\alpha_d}{2}}}{(\log X)^{\frac{\beta_d}{2}}}\right)\right).$$

Démonstration. — Fixons $\alpha=\frac{\alpha_d}{2}$, $\beta=\frac{\beta_d}{2}$. Divisons l'intervalle $]\vartheta_2,\vartheta_1]$ en $O\big(\frac{(\log X)^\alpha}{(\log\log X)^\beta}\big)$ sous-intervalles $\{J_k\}_k$ où

$$J_k=]r_k,r_{k+1}]\,,\quad r_{k+1}=r_k+\frac{(\log\log X)^\beta}{(\log X)^\alpha}$$

et définissons la fonction en escalier H par

$$H(x)=\min_{y\in J_k}G(y),\quad \text{pour } x\in J_k.$$

Puisque G est uniformément continue sur $[\vartheta_2,\vartheta_1]$, il existe $c_0>0$ une constante absolue telle que

$$0\leqslant\sup_{x\in[\vartheta_2,\vartheta_1]}\left(\sqrt{G(x)^{-1}}-\sqrt{H(x)^{-1}}\right)<c_0\frac{(\log\log X)^\beta}{(\log X)^\alpha}.$$

On obtient donc

$$\begin{aligned}(97)\qquad &\sum_k\frac{|J_k|}{\sqrt{H(r_k)}}-\int_{\vartheta_2}^{\vartheta_1}\frac{\mathrm{d}\,x}{\sqrt{G(x)}}\\ &=\sum_k\left(\int_{r_k}^{r_{k+1}}(\sqrt{H(r_k)^{-1}}-\sqrt{G(x)^{-1}})\,\mathrm{d}\,x\right)\\ &=O\left(\Big(\sum_k|J_k|\Big)\frac{(\log\log X)^\beta}{(\log X)^\alpha}\right)=O\left(\frac{(\log\log X)^\beta}{(\log X)^\alpha}\right).\end{aligned}$$

Nous avons, grâce à la Proposition A.3 et (97),

$$\begin{aligned}&\sum_{\substack{l,m\in\mathbf{N}_{\geqslant1},F(l)\equiv0[m]\\ m^2H(A-\frac{l}{m})\leqslant X}}\mathbf{1}_{]\vartheta_2,\vartheta_1]}\left(A-\frac{l}{m}\right)\\ &=\sum_{\substack{l,m\in\mathbf{N}_{\geqslant1},F(l)\equiv0[m]\\ m^2H(A-\frac{l}{m})\leqslant X}}\sum_k\mathbf{1}_{J_k}\left(A-\frac{l}{m}\right)\\ &=\sum_{\substack{l,m\in\mathbf{N}_{\geqslant1},F(l)\equiv0[m]\\ m^2H(A-\frac{l}{m})\leqslant X}}\sum_k\mathbf{1}_{A-J_k}\left(\frac{l}{m}\right)\\ &=\sum_k\sum_{\substack{l,m\in\mathbf{N},F(l)\equiv0[m]\\ m\leqslant X^{\frac12}/\sqrt{H(r_k)}}}\mathbf{1}_{A-J_k}\left(\frac{l}{m}\right)\\ &=Z_FX^{\frac12}\left(\sum_k\frac{|J_k|}{\sqrt{H(r_k)}}\right)+O\bigg(X^{\frac12}\sum_k\frac{(\log\log X)^{\alpha_d}}{(\log X)^{\beta_d}}\bigg)\\ &=Z_FX^{\frac12}\int_{\vartheta_2}^{\vartheta_1}\frac{\mathrm{d}\,x}{\sqrt{G(x)}}+O\bigg(X^{\frac12}\frac{(\log\log X)^\alpha}{(\log X)^\beta}\bigg).\end{aligned}$$

Or nous avons aussi, en appliquant encore la Proposition A.3 et le raisonnement ci-dessus pour la somme entre la plus grosse parenthèse ci-dessous,

$$\begin{aligned}
&\sum_{\substack{l,m\in\mathbf{N}_{\geqslant 1},F(l)\equiv 0[m]\\ m^2G(A-\frac{l}{m})\leqslant X}} \mathbf{1}_{]\vartheta_2,\vartheta_1]}\left(A-\frac{l}{m}\right)\\
&=\sum_{\substack{l,m\in\mathbf{N}_{\geqslant 1},F(l)\equiv 0[m]\\ m^2H(A-\frac{l}{m})\leqslant X}} \mathbf{1}_{]\vartheta_2,\vartheta_1]}\left(A-\frac{l}{m}\right)\\
&\quad+O\left(\sum_{\substack{X^{\frac{1}{2}}/(\sqrt{H\left(A-\frac{l}{m}\right)}+c_0\frac{(\log\log X)^\beta}{(\log X)^\alpha})<m\leqslant X^{\frac{1}{2}}/\sqrt{H\left(A-\frac{l}{m}\right)}\\ F(l)\equiv 0[m]}} \mathbf{1}_{A-]\vartheta_2,\vartheta_1]}\left(\frac{l}{m}\right)\right)\\
&=Z_F X^{\frac{1}{2}}\int_{\vartheta_2}^{\vartheta_1}\frac{\mathrm{d}\,x}{\sqrt{G(x)}}+O\left(X^{\frac{1}{2}}\frac{(\log\log X)^\alpha}{(\log X)^\beta}\right).
\end{aligned}$$

Ceci achève la preuve de la formule énoncée. □

7.3.5. *Dénouement.* — Nous appliquons la Proposition 7.13 pour estimer l'ensemble $\mathcal{C}^{\underline{\varepsilon},\underline{\theta}}_{C_3,D}(\mathbf{D},\mathbf{h},\mathbf{e},q)$ dans le Lemme 7.12.

COROLLAIRE 7.14. — *Il existe une constante* $\Gamma=\Gamma_{C_3,D}(\mathbf{D},\mathbf{h},\mathbf{e},q)>0$ *telle que*

(98)

$$\begin{aligned}
&\#\mathcal{C}^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D},\mathbf{h},\mathbf{e},q)\\
&=B^{\frac{1}{5}}\Gamma Z_{\mathcal{F}}\iint\chi_{\underline{\varepsilon},\underline{\theta}}(w,z)\frac{\mathfrak{E}_{C_3,D,\mathbf{D},W}(zw\sqrt{z+w})\,\mathrm{d}\,z\,\mathrm{d}\,w}{zw\sqrt{z+w}}+O\left(B^{\frac{1}{5}}\frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}}\right),
\end{aligned}$$

où

$$\mathfrak{E}_{C_3,D,\mathbf{D},W}(x)=\mathbf{1}\left\{x:x>\frac{D^{\frac{5}{2}}C_3^{\frac{1}{2}}W^3}{D_1^2D_2^2}\right\},$$

$\chi_{\underline{\varepsilon},\underline{\theta}}$ *est définie par* (59), $\mathcal{F}(X)=\mathcal{F}_{C_3,D,\mathbf{hD},\mathbf{e},q}(X)$ *est le polynôme* (96) *et la constant* $Z_{\mathcal{F}}$ *est définie dans la Proposition A.1.*

Démonstration. — Quitte à décomposer la région $R(\underline{\varepsilon},\underline{\theta})$ (59) en des petites pièces, on peut supposer pour la suite que tous les points dans $\mathcal{C}^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D},\mathbf{h},\mathbf{e},q)$ (94) vérifient la condition de seuil donnée par les équations (69) dans les Lemmes 7.8 et 7.9 pour $B\gg_{C_3,D,\underline{\varepsilon},\underline{\theta}}1$.

Écrivons par simplicité

$$g_1=g(D_1h_1),\quad g_2=g(D_2h_2).$$

Avec le changement de paramètres (93), nous avons

$$v = e_1e_2e_3m' - u = e_3g_1g_2l' - e_1e_2e_3(\gamma_1g_1 - 1)m',$$

et d'où

(99)

$$\frac{C_3}{D}\frac{uv^2}{u+v} = \frac{C_3e_3^2g_1^3g_2^3}{De_1e_2}(m')^2\left(\frac{e_1e_2\gamma_1}{g_2} - \frac{l'}{m'}\right)\left(\frac{e_1e_2}{g_1g_2} - \left(\frac{e_1e_2\gamma_1}{g_2} - \frac{l'}{m'}\right)\right)^2.$$

Quitte à rajouter un terme d'erreur d'ordre grandeur $O(1)$, nous pouvons réécrire (66) comme

$$\begin{aligned} &\frac{e_1e_2\gamma_1}{g_2} - \frac{e_1e_2}{g_1g_2}\frac{\theta_1}{1+\theta_1} \leqslant \frac{l'}{m'} < \frac{e_1e_2\gamma_1}{g_2} - \frac{e_1e_2}{g_1g_2}\frac{\theta_2}{1+\theta_2}; \\ &\varepsilon_2(m')^2G\left(A - \frac{l'}{m'}\right) < B^{\frac{2}{5}} \leqslant \varepsilon_1(m')^2G\left(A - \frac{l'}{m'}\right). \end{aligned} \tag{100}$$

Remarquons que $e_1e_2 \mid g_1g_2$ et $\gamma_1g_1 \geqslant 1$. Nous avons donc

$$\begin{aligned} &\frac{e_1e_2\gamma_1}{g(h_2D_2)} - \frac{l'}{m'} \subset \left]\frac{e_1e_2}{g_1g_2}\frac{\theta_2}{1+\theta_2}, \frac{e_1e_2}{g_1g_2}\frac{\theta_1}{1+\theta_1}\right[\subset \left]0,1\right[, \\ &\frac{e_1e_2\gamma_1}{g_2} \geqslant \frac{e_1e_2}{g_1g_2} > \frac{e_1e_2}{g_1g_2}\frac{\theta}{1+\theta}, \quad \forall \theta \geqslant 0. \end{aligned}$$

Nous appliquons la Proposition 7.13 avec

$$\begin{aligned} G(x) &= \frac{C_3e_3^2g(h_1D_1h_2D_2)^3}{De_1e_2}x\left(\frac{e_1e_2}{g_1g_2} - x\right)^2, \quad F(Y) = \mathcal{F}_{C_3,D,\mathbf{hD},\mathbf{e},q}(Y) \\ A &= \frac{e_1e_2\gamma_1}{g_2}, \quad \vartheta_i = \frac{e_1e_2}{g_1g_2}\frac{\theta_i}{1+\theta_i}, \quad X = \frac{B^{\frac{2}{5}}}{\varepsilon_i}, \quad (i = 1,2), \end{aligned}$$

Il en découle que

$$\begin{aligned} &\#\mathcal{C}^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D},\mathbf{h},\mathbf{e},q) \\ &= \Bigg(\sum_{\substack{l',m'\in\mathbf{N},F(l')\equiv 0[m'] \\ (m')^2\varepsilon_2G(A-\frac{l'}{m'})\leqslant B^{\frac{2}{5}}}} - \sum_{\substack{l',m'\in\mathbf{N},F(l')\equiv 0[m'] \\ (m')^2\varepsilon_1G(A-\frac{l'}{m'})\leqslant B^{\frac{2}{5}}}}\Bigg)\mathbf{1}_{[\vartheta_2,\vartheta_1]}\left(A - \frac{l'}{m'}\right) \\ &= B^{\frac{1}{5}}\frac{(De_1e_2)^{\frac{1}{2}}Z_{\mathcal{F}}}{(C_3e_3^2g(h_1D_1h_2D_2)^3)^{\frac{1}{2}}}\left(\frac{1}{\sqrt{\varepsilon_2}} - \frac{1}{\sqrt{\varepsilon_1}}\right)\int_{\vartheta_1}^{\vartheta_2}\frac{\mathrm{d}\,x}{\sqrt{x\left(\frac{e_1e_2}{g_1g_2} - x\right)^2}} \\ &\quad + O\left(B^{\frac{1}{5}}\frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}}\right) \end{aligned}$$

$$= B^{\frac{1}{5}} \frac{D^{\frac{1}{2}} Z_{\mathcal{F}}}{2C_3^{\frac{1}{2}} e_3 g(D_1h_1D_2h_2)} \int_{\varepsilon_2}^{\varepsilon_1} \int_{\theta_2}^{\theta_1} \frac{\mathrm{d}\, z \,\mathrm{d}\, \theta}{z^{\frac{3}{2}}\sqrt{\theta(1+\theta)}} + O\left(B^{\frac{1}{5}} \frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}} \right)$$
$$= B^{\frac{1}{5}} \frac{D^{\frac{1}{2}} Z_{\mathcal{F}}}{2C_3^{\frac{1}{2}} e_3 g(D_1h_1D_2h_2)} \iint_{\substack{\varepsilon_2 \leqslant z \leqslant \varepsilon_1 \\ \theta_2 < \frac{z}{w} < \theta_1}} \frac{\mathrm{d}\, z \,\mathrm{d}\, w}{zw\sqrt{z+w}} + O\left(B^{\frac{1}{5}} \frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}} \right).$$

Nous obtenons donc la formule énoncée avec

$$\Gamma_{C_3,D}(\mathbf{D}, \mathbf{h}, \mathbf{e}, q) = \frac{D^{\frac{1}{2}}}{2C_3^{\frac{1}{2}} e_3 g(D_1h_1D_2h_2)}. \qquad \square$$

Il nous reste à traiter le terme d'erreur introduit dans (79) du Corollaire 7.10. Le même raisonnement comme ci-dessus nous donne l'estimation

$$(101) \qquad \# \operatorname{Er}^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})$$
$$\ll_{C_3,D,W,\varepsilon_i} B^{\frac{1}{5}} \iint \chi\left(R(\underline{\varepsilon}, \underline{\theta}) \cap \Omega\right) \frac{\mathrm{d}\, z \,\mathrm{d}\, w}{zw\sqrt{z+w}} + O\left(B^{\frac{1}{5}} \frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}} \right)$$
$$= O\left(B^{\frac{1}{5}} \frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}} \right),$$

où

$$\Omega = \left\{ (w,z) : \frac{D^{\frac{5}{2}} C_3^{\frac{1}{2}} W^3}{D_1^2 D_2^2} \leqslant zw\sqrt{z+w} \leqslant \frac{D^{\frac{5}{2}} C_3^{\frac{1}{2}} W^3}{D_1^2 D_2^2} + \mu_0(B^{-\frac{2}{5}}) \right\}.$$

Rassemblons les égalités (67), (79), (81), (82), (94), (98) et (101), nous sommes prêt à démontrer le Théorème 7.6.

Démonstration du Théorème 7.6. — Dans ce qui suit, pour éviter des formules superflues, on écrira le terme d'erreur sous la forme non-explicite $o(B^{\frac{1}{5}})$ car il sera clair d'où viennent ces contributions. Soient $C_3, D, W, D_1, D_2, q \in \mathbf{N}$ tels que

$$\operatorname{pgcd}(C_3W, D) = \operatorname{pgcd}(D_1, D_2) = 1, \quad D_1D_2 \mid D, \quad q \mid W.$$

En les fixant et en sommant sur les formules (82), (94), (98), nous obtenons la formule pour le cardinal de l'ensemble $B^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D}, q)$ comme suit. Il existe une constante

$$Z_{C_3,D,W}(\mathbf{D}, q) = \sum_{\substack{h_1,h_2 \in \mathbf{N}_{\geqslant 1} \\ D_1h_1, D_2h_2 \mid D \\ \operatorname{pgcd}(h_1,h_2)=1}} \mu(h_1)\mu(h_2)$$
$$\times \sum_{\substack{e_1e_2 \mid g(h_1D_1h_2D_2), \operatorname{pgcd}(e_1,e_2)=1 \\ e_1e_2e_3 \mid qD, \operatorname{pgcd}(e_3, g(h_1D_1h_2D_2))=1}} \left(\prod_{j=1}^{3} \mu(\mathbf{e}) \right) \frac{Z_{\mathcal{F}} D^{\frac{1}{2}}}{2C_3^{\frac{1}{2}} e_3 g(D_1h_1D_2h_2)}$$

telle que

$$\begin{aligned}&\#B^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D},q)\\&=B^{\frac{1}{5}}Z_{C_3,D,W}(\mathbf{D},q)\iint\chi_{\underline{\varepsilon},\underline{\theta}}(w,z)\frac{\mathfrak{E}_{C_3,D,\mathbf{D},W}\left(zw\sqrt{z+w}\right)}{zw\sqrt{z+w}}\,\mathrm{d}\,z\,\mathrm{d}\,w+o(B^{\frac{1}{5}}).\end{aligned}$$

Nous concluons de (81) qu'en fixant C_3,D,W,D_1,D_2,

$$\begin{aligned}\#\widetilde{E}^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D})&=\sum_{q\mid W}\mu\left(\frac{W}{q}\right)B^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}\left(\mathbf{D},q\right)\\&=B^{\frac{1}{5}}\iint\chi_{\underline{\varepsilon},\underline{\theta}}(w,z)\frac{\mathbf{E}_{C_3,D,\mathbf{D},W}\left(zw\sqrt{z+w}\right)}{zw\sqrt{z+w}}\,\mathrm{d}\,z\,\mathrm{d}\,w+o(B^{\frac{1}{5}}),\end{aligned}$$

où

$$\mathbf{E}_{C_3,D,\mathbf{D},W}(x)=\sum_{q\mid W}\mu\left(\frac{W}{q}\right)Z_{C_3,D,W}\left(\mathbf{D},q\right)\mathfrak{E}_{C_3,D,\mathbf{D},q}(x).$$

En reportant dans (67) et (79), rappelons l'estimation (101), nous obtenons finalement

$$\begin{aligned}\delta_{\varrho^{-1}(S_1)\cap U,Q,B,\frac{5}{2}}(\chi_{\underline{\varepsilon},\underline{\theta}})&=\sum_{\substack{C_3,D,W\in\mathbf{N}_{\geqslant1}\\\mathrm{pgcd}(C_3W,D)=1}}\ \sum_{\substack{D_1D_2\mid D\\\mathrm{pgcd}(D_1,D_2)=1}}\#E^{\underline{\varepsilon},\underline{\theta}}_{C_3,D,W}(\mathbf{D})\\&=B^{\frac{1}{5}}\iint\chi_{\underline{\varepsilon},\underline{\theta}}(w,z)\frac{\mathbb{E}\left(zw\sqrt{z+w}\right)}{zw\sqrt{z+w}}\,\mathrm{d}\,z\,\mathrm{d}\,w+o(B^{\frac{1}{5}}).\end{aligned}$$

où

$$\mathbb{E}(x)=\sum_{\substack{C_3,D,W\in\mathbf{N}_{\geqslant1}\\\mathrm{pgcd}(C_3W,D)=1}}\ \sum_{\substack{D_1D_2\mid D\\\mathrm{pgcd}(D_1,D_2)=1}}\mathbf{E}_{C_3,D,\mathbf{D},W}(x).\tag{102}$$

$\square$

7.4. Autre régions. — Dans les régions S_2,S_3,S_4 ((21)–(23)), la situation est similaire, malgré un changement mineur de hauteur et de distance. Nous rappelons le polynôme $\mathcal{F}(X)$ (96). En écrivant

$$\begin{aligned}&\frac{qC_3g(h_1D_1h_2D_2)^2e_3}{e_1e_2}X^2+\frac{qD}{e_1e_2e_3}\\&=\frac{q}{e_1e_2e_3}\left(C_3(g(h_1D_1h_2D_2)e_3X)^2+D\right)=\frac{q}{e_1e_2e_3}\mathcal{G}(X),\end{aligned}$$

le (non-)scindage du polynôme C_3X^2+D équivaut à celui de $\mathcal{G}(X)$. Rappelons la Proposition 7.5 et l'observation dans §7.4, les points rationnels dans la partie mince M (48) correspond à la famille d'équations $\mathcal{E}_{C_3,D}$ (51) avec $-C_3D=\square$. Alors en dehors de M, au moins un des $C_3,-D\neq\square$. Donc ces polynômes restent irréductibles et nous conduisent toujours au problème du congruence quadratique. Nous esquissons ici le résultat et nous ne rentrerons pas dans le détail puisque la méthode et les calculs sont presque pareils.

7.4.1. *Région* S_2. — Le Lemme 5.1 nous dit que pour $au^2 - bv^2 > 0$ pour $(a,b) \times (u,v) \in T_2$. En reportant dans (51), nous avons toujours $D > 0$. Donc le résultat sous-entendu ressemble au Théorème 7.6.

Théorème 7.15. — *Pour toute* $f \in \mathcal{C}_Q^{\mathrm{b}}(Y_3)$ *(cf. §2.2), on a*

$$\begin{aligned}&\delta_{\varrho^{-1}(S_2)\cap U,Q,B,\frac{5}{2}}(f)\\ &= B^{\frac{1}{5}}\left(\int f(w,z)\frac{\mathbb{E}'_{S_2}(-wz\sqrt{z+w})}{-wz\sqrt{z+w}}\,\mathrm{d}\,w\,\mathrm{d}\,z + O_f\left(\frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}}\right)\right),\end{aligned}$$

où $\mathbb{E}'_{S_i}(\cdot)$ *est une fonction en escalier définie de façon analogue à* $\mathbb{E}(\cdot)$ (102).

7.4.2. *Régions* S_3 *et* S_4. — Comme expliqué précédemment, on doit retirer dans la somme (67) les ensembles $E^{\varepsilon,\theta}_{C_3,D,W}(\mathbf{D})$ dont les paramètres vérifient $-C_3D = \square$. La mesure limite obtenue s'écrit de la façon suivante.

Théorème 7.16. — *Pour toute* $f \in \mathcal{C}_Q^{\mathrm{b}}(Y_3)$, *on a pour* $i = 3,4$,

$$\begin{aligned}&\delta_{\varrho^{-1}(S_3)\cap U,Q,B,\frac{5}{2}}(f)\\ &= B^{\frac{1}{5}}\left(\int f(w,z)\frac{\mathbb{E}''_{S_3}(w(-z)\sqrt{z+w})}{w(-z)\sqrt{z+w}}\,\mathrm{d}\,w\,\mathrm{d}\,z + O_f\left(\frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}}\right)\right);\\ &\delta_{\varrho^{-1}(S_4)\cap U,Q,B,\frac{5}{2}}(f)\\ &= B^{\frac{1}{5}}\left(\int f(w,z)\frac{\mathbb{E}''_{S_4}(wz\sqrt{-(w+z)})}{wz\sqrt{-(w+z)}}\,\mathrm{d}\,w\,\mathrm{d}\,z + O_f\left(\frac{(\log\log B)^{\frac{5}{6}}}{(\log B)^{\frac{2-\sqrt{2}}{6}}}\right)\right),\end{aligned}$$

où pour $i = 3,4$,

$$\mathbb{E}''_{S_i}(\cdot) = \sum_{\substack{C_3,W\in\mathbf{N}_{\geqslant 1},D\in\mathbf{Z}_{<0}\\ \mathrm{pgcd}(C_3W,D)=1,-DC_3\neq\square}}\ \sum_{\substack{D_1D_2|D\\ \mathrm{pgcd}(D_1,D_2)=1}} \mathbf{E}''_{S_i,C_3,D,\mathbf{D},W}(\cdot)$$

est définie de façon analogue à des quantités dans la preuve du Théorème 7.6.

7.5. Décompte de la partie mince. — Rappelons que M est entièrement contenues dans la région R_2. Dans l'esprit de l'équidistribution globale, il n'est pas raisonnable à croire que la distribution locale autour de Q soit décrite par une partie semi-algébrique qui n'est pas dense pour la topologie réelle. La majoration suivante améliore celle dans la Proposition 4.4.

Lemme 7.17. — *Pour toute région* R *à support compact dans* $S_3 \cup S_4$, *nous avons*

$$\delta_{M,Q,B,\frac{5}{2}}(\mathbf{1}_R) \ll_R B^{\frac{1}{5}}\log B.$$

Démonstration. — Par un argument comme le Lemme 7.1, nous n'avons qu'un nombre fini de paramètres C_3, D possibles paramétrant localement les points de M dans R. Soit $\varepsilon > 0$ tel que $R \subset \mathbb{B}(0,\varepsilon)$. Une borne de type (66) pour les points dans $T_3 \cup T_4$ nous donne que, compte-tenu de la Proposition 7.5, pour un certain $\gamma(C_3, D, \varepsilon) > 0$,

$$\delta_{M,Q,B,\frac{5}{2}}(\mathbf{1}_R) \leqslant \sum_{\substack{-C_3,D=\square\\ C_3,D\ll_\varepsilon 1}} \sum_{n\leqslant \gamma(C_3,D,\varepsilon)B^{\frac{1}{5}}} \varrho_{C_3X^2+D}(n),$$

chaque somme pour C_3, D fixés contribuant à l'ordre de grandeur $O_{C_3,D}(B^{\frac{1}{5}}\log B)$, en utilisant la Proposition A.7. D'où la majoration énoncée. $\square$

La Proposition A.7 et les raisonnements dans §7.3 nous suggèrent qu'on devrait avoir une minoration du type, pour tout $\varepsilon > 0$ suffisamment grand,

$$\delta_{M,Q,B,\frac{5}{2}}(\chi(\varepsilon)) \gg_\varepsilon B^{\frac{1}{5}} \log B.$$

(La minoration évidente $\gg_\varepsilon B^{\frac{1}{5}}$ peut être déduite du Théorème 2.5.) Cependant, le résultat de Dartyge-Martin [7, Theorem 1] (cf. §A.2.2) suggère aussi que, si l'on prend une fonction trigonométrique dite h, alors il devrait exister $C(h) > 0$ telle que

$$\delta_{M,Q,B,\frac{5}{2}}(h) \sim C(h)B^{\frac{1}{5}}.$$

Ces deux formules donnent une autre évidence en faveur de non-équirépartition des points dans M.

Annexe A. Congruences polynomiales et équidistribution modulo 1, d'après Erdős et Hooley

Nous considérons ici une version de congruences polynomiales à résidu fixé à la Hooley. Nous serons concernés par des dénombrements analogues au cardinal de l'ensemble des $(l,m) \in \mathbf{N}^2$ tels que

$$\lambda_2 B^{\frac{1}{5}} \leqslant m \leqslant \lambda_1 B^{\frac{1}{5}}, \quad \tau_2 \leqslant \frac{l}{m} \leqslant \tau_1, \quad F(l) \equiv 0[m], \tag{103}$$

où $0 < \lambda_2 < \lambda_1, 0 < \tau_2 < \tau_1 \leqslant 1$ et $F(X) \in \mathbf{Z}[X]$ est un polynôme de degré $\geqslant 2$. Nous allons distinguer deux cas dans la discussion selon que $F(X)$ est irréductible ou non dans $\mathbf{Z}[X]$.

Étant donné $F(X) \in \mathbf{Z}[X]$, on définit la fonction

$$\varrho_F(n) = \#\{1 \leqslant k \leqslant n : F(k) \equiv 0[n]\}. \tag{104}$$

C'est une fonction arithmétique multiplicative.

On y associe la série de Dirichlet (définie pour $\Re(s)$ suffisamment grand)

$$D_F(s) = \sum_{n=1}^{\infty} \frac{\varrho_F(n)}{n^s}. \tag{105}$$

A.1. Cas irréductible. — On discute premièrement le cas où $F(X)$ est irréductible sur $\mathbf{Z}[X]$.

A.1.1. *Ordre Moyen.* — Une observation qui remonte à Dedekind et Erdős dit que la série $D_F(s)$ se comporte de façon similaire à la fonction zêta de Dedekind associée au corps de nombres engendré par une racine de $F(X)$. En conséquence on en déduit l'ordre moyen de ϱ_F (Proposition A.1), à l'aide d'une méthode d'analyse complexe standard.

Proposition A.1 (Erdős, [9]). — *Supposons que $F(X) \in \mathbf{Z}[X]$ est irréductible. Soit θ une racine algébrique de $F(X)$ et notons $K_F = \mathbf{Q}(\theta)$ le corps de nombres qu'il génère. Alors nous avons*

$$D_F(s) = \zeta_{K_F}(s)\Psi(s),$$

où $\zeta_{K_F}(s)$ est la fonction zêta de Dedekind du corps K_F et $\Psi(s)$ est méromorphe et bornée dans le demi-plan $\Re(s) > \frac{1}{2} + \varepsilon, \forall \varepsilon > 0$. Par conséquent, la série $D_F(s)$ admet un pôle simple en $s = 1$. De plus, il existe $\lambda \in]0,1[$ dépendant du polynôme F tel que

$$\sum_{n \leqslant X} \varrho_F(n) = Z_F X + O(X^{\lambda}),$$

où,

$$Z_F = \Psi(1) \lim_{s \to 1} (s-1)\zeta_{K_F}(s). \tag{106}$$

Démonstration. — cf. e.g. [13, §5] ou [6, §7] □

Remarque. — Une formule asymptotique pour l'ordre moyen de ϱ_F est également obtenue par Hooley [15], [16] dans le cas quadratique.

A.1.2. *Équirépartion modulo* 1. — Nous continuons à supposer dans cette section que le polynôme $F(X)$ est irréductible. Hooley [17] a démontré le résultat suivant sur des sommes d'exponentielles. Ce théorème a été énoncé pour les polynômes primitifs, mais la même preuve s'applique aussi à ceux qui ne sont pas forcément primitifs.

Théorème A.2 (Hooley [17], Theorem 1). — *Soient $X > 1, h \in \mathbf{N}_{\geqslant 1}, d = \deg F(X) \geqslant 2$ et*

$$R(h, X) = \sum_{1 \leqslant k \leqslant X} \sum_{\substack{F(v) \equiv 0[k] \\ 1 \leqslant v \leqslant k}} \exp\left(\frac{2\pi i h v}{k}\right).$$

Alors

$$R(h,X)=O_F\left(\frac{h^{\frac{1}{2}}X(\log\log X)^{\frac{1}{2}(d^2+1)}}{(\log X)^{\delta_d}}\right)\quad où\quad \delta_d=\frac{d-\sqrt{d}}{d!}.$$

(N.B. Le résultat original de Hooley omet l'ordre de grandeur de h. Mais on le récupère facilement de sa preuve.)

Le but de cette section est de démontrer :

Proposition A.3. — *Pour tout intervalle compact $I\subseteq\mathbf{R}$, nous avons*

$$\sum_{1\leqslant k\leqslant X}\sum_{\substack{v\in\mathbf{Z}\\ F(v)\equiv 0[k]}}\mathbf{1}_I\left(\frac{v}{k}\right)=Z_F|I|X+O\left(X\frac{(\log\log X)^{\alpha_d}}{(\log X)^{\beta_d}}\right),$$

où Z_F est (106) *et*

$$\alpha_d=\frac{1}{3}(d^2+1),\quad \beta_d=\frac{1}{3}\delta_d.$$

Nous notons (s_n) la suite construite en numérotant les nombres rationnels (pas nécessairement réduit) $\frac{v}{k}\in[0,1[$ tels que $F(v)\equiv 0[k]$ par rapport à l'ordre croissant des dénominateurs. C'est-à-dire $\frac{v_1}{k_1}\prec\frac{v_2}{k_2}$ comme éléments de (s_n) si et seulement si

$$k_1\leqslant k_2,\quad \text{ou}\quad k_1=k_2 \text{ et } v_1\leqslant v_2.$$

Le théorème de Hooley implique que la suite (s_n) est équirépartie modulo 1, au sens de Weyl (cf. [21, Chapter 1]). Nous avons besoin d'une estimation de la discrépance $D_N(s_n)$ de cette suite. Pour la définition de *la discrépance*, voir par exemple [21, Chapter 2]. Les outils sont l'inégalité de Koksma-Denjoy (cf. [21, p. 143]) et celle de Erdős-Turán (cf. [21, Theorem 2.5 p. 112]).

Théorème A.4 (Koksma-Denjoy). — *Soient (x_n) une suite de nombres réels dans $[0,1[$ et $N\geqslant 1$. Soit ϕ une fonction mesurable à variation bornée définie sur $]0,1]$ (on note $V(\phi)$ la variation totale de ϕ). Alors*

$$\left|\frac{1}{N}\sum_{n=1}^{N}\phi(x_n)-\int_0^1\phi\right|\leqslant V(\phi)D_N(x_n).$$

Théorème A.5 (Erdős-Turán). — *Pour tout $m\in\mathbf{N}_{\geqslant 1}$, on a*

$$D_N(x_n)=O\left(\frac{1}{m}+\sum_{h=1}^{m}\frac{1}{h}\left|\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi i h x_n)\right|\right),$$

où la constante implicite est absolue.

Corollaire A.6. — *Avec les notations ci-dessus, nous avons*

$$D_N(s_n) = O_F\left(\frac{(\log\log N)^{\frac{1}{3}(d^2+1)}}{(\log N)^{\frac{2}{3}\delta_d}}\right).$$

Démonstration du corollaire. — Fixons $N \in \mathbf{N}_{\geqslant 1}$ et notons

$$S(h,N) = \sum_{n=1}^{N} \exp\left(2\pi i h x_n\right). \tag{107}$$

Soit M le dénominateur de x_N. D'après la Proposition A.1 et la relation suivante

$$\sum_{k<M} \varrho_F(k) < N \leqslant \sum_{k\leqslant M} \varrho_F(k),$$

il existe C_1, C_2 deux constantes absolues positives telles que

$$C_1 M \leqslant N \leqslant C_2 M. \tag{108}$$

Nous avons aussi la comparaison suivante

$$S(h,N) = R(h,M) + O(\varrho_F(M)) = R(h,M) + O(M^{\varepsilon}), \quad \forall\varepsilon > 0.$$

D'où, en utilisant le Théorème A.2 et l'estimation d'Erdős-Turán (Théorème A.5), nous calculons que pour tout $m \in \mathbf{N}_{\geqslant 1}$,

$$\begin{aligned}
D_N(s_n) &= O\left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h}\left|\frac{1}{N} S(h,N)\right|\right) \\
&= O\left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h}\left(\left|\frac{R(h,M)}{M}\right| + \frac{1}{M^{1-\varepsilon}}\right)\right) \\
&= O_F\left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h}\left(h^{\frac{1}{2}} \frac{(\log\log M)^{\frac{1}{2}(d^2+1)}}{(\log M)^{\delta_d}} + \frac{1}{M^{1-\varepsilon}}\right)\right) \\
&= O_F\left(\frac{1}{m} + \sqrt{m}\frac{(\log\log M)^{\frac{1}{2}(d^2+1)}}{(\log M)^{\delta_d}} + \frac{\log m}{M^{1-\varepsilon}}\right).
\end{aligned}$$

En prenant

$$m = \left\lfloor \left(\frac{(\log M)^{\delta_d}}{(\log\log M)^{\frac{1}{2}(d^2+1)}}\right)^{\frac{2}{3}} \right\rfloor,$$

nous obtenons, compte-tenu de (108),

$$D_N(s_n) = O_F\left(\frac{(\log\log M)^{\frac{1}{3}(d^2+1)}}{(\log M)^{\frac{2}{3}\delta_d}}\right) = O_F\left(\frac{(\log\log N)^{\frac{1}{3}(d^2+1)}}{(\log N)^{\frac{2}{3}\delta_d}}\right),$$

la majoration souhaitée. □

Démonstration de la Proposition A.3. — Quitte à décomposer l'intervalle I en une union de sous-intervalles disjoints, on peut supposer que $|I| \leqslant 1$ et $I \subseteq]n_0, n_0+1]$ avec $n_0 \in \mathbf{Z}$. Pour $k \in \mathbf{N}_{\geqslant 1}$ fixé, tout $v \in \mathbf{Z}$ tel que $\frac{v}{k} \in I$ correspond à un unique $0 \leqslant v' \leqslant k-1$ tel que $\frac{v'}{k} \in I - n_0$. Posons

$$N = N(X) = \sum_{k \leqslant X} \varrho_F(k).$$

D'après la Proposition A.1, $N \sim Z_F X$. Alors compte tenu de l'inégalité de Koksma-Denjoy (Théorème A.4) et du Corollaire A.6,

$$\begin{aligned}\sum_{k\leqslant X} \sum_{\substack{v\in\mathbf{Z}\\ F(v)\equiv 0[k]}} \mathbf{1}_I\left(\frac{v}{k}\right) &= \sum_{k\leqslant X} \sum_{\substack{F(v')\equiv 0[k]\\ 0\leqslant v'\leqslant k-1}} \mathbf{1}_{I-n_0}\left(\frac{v'}{k}\right)\\ &= \sum_{n=1}^{N} \mathbf{1}_{I-n_0}(s_n) = N\int_0^1 \mathbf{1}_{I-n_0}(x)\,\mathrm{d}\,x + O\left(ND_N(s_n)\right)\\ &= N|I| + O_F\left(\frac{N(\log\log N)^{\frac{1}{3}(d^2+1)}}{(\log N)^{\frac{2}{3}\delta_d}}\right)\\ &= Z_F|I|X + O_F\left(X\frac{(\log\log X)^{\frac{1}{3}(d^2+1)}}{(\log X)^{\frac{2}{3}\delta_d}}\right). \qquad \square\end{aligned}$$

A.2. Cas scindés. — Nous nous intéresserons exclusivement dans cette section aux polynômes quadratiques *scindés*, i.e. réductible sur $\mathbf{Z}[X]$. Fixons $a \in \mathbf{N}_{\geqslant 1}, c \in \mathbf{Z}_{\neq 0}$. Définissons

$$F(X) = aX^2 + c$$

La condition que $F(X)$ soit réductible revient à (on note $\Delta(F)$ le discriminant)

$$\Delta(F) = \square \Leftrightarrow -ac = \square.$$

A.2.1. *Ordre moyen.* — La fonction donnant le nombre de congruents ϱ_F ainsi que la série de Dirichlet $D_F(s)$ sont de nature différente de celles dans les cas irréductibles. En effet il s'agit ici d'un problème additif de diviseurs, qui fut considéré en premier par Ingham [20].

Proposition A.7. — *Nous avons*

$$D_F(s) = \zeta(s)^2 \Phi(s),$$

où $\zeta(s)$ est la fonction zêta de Riemann et $\Phi(s)$ est holomorphe, bornée et sans zéros pour $\Re(s) > \frac{1}{2}+\varepsilon, \forall\varepsilon > 0$. Par conséquent, la série $D_F(s)$ admet en $s=1$ un pôle d'ordre 2 et converge absolument pour $\Re s > 1$. De plus, il existe une *constante* $C_F > 0$ *telle que*

$$\sum_{n \leqslant X} \varrho_F(n) \sim C_F X \log X.$$

A.2.2. *Répartition uniforme.* — Si l'on considère des sommes d'exponentielles $R(h, X)$ définies de façon similaire comme dans le Théorème A.2 pour un polynôme F quadratique scindé, Martin et Sitar ont démontré [25, Theorem 1.4] que $R(h, X) = O(X(\log X)^{\sqrt{2}-1+\varepsilon})$. Comme conjecturé dans [25, p. 14], on ne devrait pas espérer une majoration du type $R(h, X) = o_h(X)$. Récemment Dartyge et Martin [7, Theorem 1] réussissent à établir la formule asymptotique

$$R(h, X) = C(F, h)X + O(X^{\frac{4}{5}+\varepsilon}),$$

où $C(F, h) \neq 0$ dépend de F et de h. Une conséquence immédiate de leur résultat est que, en rappelant la somme d'exponentielle $S(h, X)$ (107),

$$\frac{1}{X} S(h, X) \to C(F, h) \neq 0, \quad X \to \infty.$$

Le critère de Weyl (cf. [21, Chapter 1, Theorem 2.1]) nous dit que la suite (s_n) formée par les racines de congruence n'est pas uniformément répartie modulo 1.

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CHAOS FOR CONVOLUTION OPERATORS ON THE SPACE OF ENTIRE FUNCTIONS OF INFINITELY MANY COMPLEX VARIABLES

by Blas M. Caraballo & Vinícius V. Fávaro

Abstract. — In sharp contrast to a classical result of Godefroy and Shapiro, Mujica and the second author showed that no translation operator on the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ of entire functions of infinitely many complex variables is hypercyclic. In an attempt to better understand the dynamics of such operators, in this work we show, firstly, that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is cyclic or n-supercyclic for any positive integer n. In the opposite direction, we show that every non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is mixing. Particularizing Arai's concept of Li-Yorke chaos to non-metrizable topological vector spaces, we show that non-trivial convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ are also Li-Yorke chaotic.

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Résumé (*Chaos pour les opérateurs de convolution sur l'espace des fonctions entières en une infinité de variables complexes*). — Contrastant fortement avec un résultat classique de Godefroy et Shapiro, Mujica et le deuxième auteur ont montré qu'aucun opérateur de translation sur l'espace $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ des fonctions entières en une infinité de variables complexes est hyper cyclique. Pour mieux comprendre la dynamique de tels opérateurs, dans ce travail, nous montrons premièrement qu'aucun opérateur de convolution sur $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ n'est cyclique ni n-supercyclique, quelque que soit l'entier positif n. Dans le sens opposé, nous montrons que tous les opérateurs de convolution non triviaux sur $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ sont mélangeant. En appliquant le concept, défini par Arai, de chaos de Li-Yorke sur des espaces vectoriels topologiques non métrisables, nous montrons que les opérateurs de convolution non triviaux sur $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ sont également Li-Yorke chaotiques.

Dedicated to the memory of Professor Jorge Mujica (1946–2017)

1. Introduction

In the last 30 years, the study of the dynamics of continuous linear operators (in short, operators) on topological vector spaces has been intensively explored. References [6, 25] provide deep and detailed surveys of the theory. In this paper, we are mainly interested in the linear dynamics of convolution operators on spaces of entire functions of infinitely many complex variables. We remark that several results about the linear dynamics of (not necessarily convolution) operators on spaces of entire functions of infinitely many complex variables have appeared in the last few decades, see, for instance, [3, 7, 9, 10, 11, 15, 18, 19, 20, 22, 24, 29, 31].

A classical result due to Godefroy and Shapiro [23] states that every nontrivial convolution operator on the space $\mathcal{H}(\mathbb{C}^n)$ of entire functions of several complex variables is hypercyclic (the definition will be given in Section 2). Moreover, Bonilla and Grosse-Erdmann [13] showed that these convolution operators are even frequently hypercyclic, which is a stronger notion than hypercyclicity. In sharp contrast with these results, Mujica and the second author [18] proved that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ can be hypercyclic. At first sight this result seems surprising, since it is well known that every $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ depends only on finitely many variables (see [17, p. 162]). Based on these facts, the following question arises:

> Which other dynamical concepts are satisfied by convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$?

The purpose of this paper is to answer this question, either positively or negatively, for the following concepts (the definitions will follow in Section 2): n-supercyclicity, cyclicity, Li-Yorke chaos (notions that are weaker than hypercyclicity), and mixing. In contrast with the aforementioned result of Godefroy and Shapiro, we will show that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ can be either cyclic or n-supercyclic for any positive integer n (Theorem 3.1), but every non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is mixing (Theorem 3.3).

Recall that the Birkhoff transitivity theorem gives the equivalence between hypercyclicity and topological transitivity for operators on Fréchet spaces. In particular, mixing implies hypercyclicity in this case. The Baire category theorem (the metrizability and completeness of the space) is the key to proving the Birkhoff transitivity theorem. For operators on a Fréchet space E, it is well known that hypercyclicity does not imply mixing. So, it is natural to ask if mixing implies hypercyclicity when E is not a Fréchet space. In [11, p. 254, Example 1], Bonet constructed the following example, which is mixing but not hypercyclic. Let $\mathcal{T} = \lambda B, \lambda > 1$, where B is the backward shift on ℓ_2. The operator $\mathcal{T}$ is mixing (hence, hypercyclic), and it has a dense set P of periodic points. Let e_i be the i-th canonical unit vector of ℓ_2. We denote by $\mathcal{E}$ the dense topological subspace of ℓ_2, which is the linear span of $\{e_i : i \in \mathbb{N}\} \cup P$. Clearly, $\mathcal{T}$ acts continuously from $\mathcal{E}$ into itself, it has a dense set of periodic points on $\mathcal{E}$, and it is mixing. Since every element of $\mathcal{E}$ has a finite orbit, $\mathcal{T}$ cannot be hypercyclic.

Note that the space in Bonet's example is metrizable (in fact, it is normed) but not complete. So, it is natural to ask if, in a non-metrizable complete, separable locally convex space, every mixing operator is hypercyclic. The answer is no, and the same example given by Grosse-Erdmann to answer [11, Open question 13.(2)] is a mixing non-hypercyclic operator. The details of this example will appear in Example 3.2. New examples of mixing non-hypercyclic operators on a non-metrizable complete, separable locally convex space will be provided by Theorem 3.3.

The notion of chaos in linear dynamics was introduced by Godefroy and Shapiro [23] in 1991. They adopted Devaney's definition of chaos. Recall that an operator on a Fréchet space is *chaotic* if it is hypercyclic and it has a dense set of periodic points. The notion of chaos for operators on an arbitrary topological vector space was given by Bonet [11]. He adopted Devaney's definition of chaos replacing the condition "hypercyclicity" with "topological transitivity", but both concepts coincide in Fréchet spaces (note that the Bonet and Grosse-Erdamnn examples above are chaotic). In addition to these notions explored in linear dynamics, we mention the first mathematical definition of chaos given in 1975 by Li and Yorke in [28], which is currently known as Li-Yorke chaos. The classical notion of Li-Yorke chaos was introduced for continuous maps defined on metric spaces. By [8, Theorem 9], hypercyclic operators on Fréchet spaces are Li-Yorke chaotic.

Since $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is a complete non-metrizable locally convex space, the classical notion of Li-Yorke chaos does not make sense for convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. Recently, Arai [2] introduced the notion of Li-Yorke chaos for an action of a group on a uniform space. Since every topological vector space is a uniform space, we will adopt Arai's definition of Li-Yorke chaos (the definition is given in Section 2). Using this definition we will prove that every hypercyclic operator on a topological vector space is Li-Yorke chaotic (Corollary 3.5), and every non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is Li-Yorke chaotic (Theorem 3.6). We will also observe in Remark 3.8 that Grosse-Erdmann's example is Li-Yorke chaotic, whereas Bonet's example is not.

It is worth mentioning that the criteria that appear in the literature to prove that an operator does or does not satisfy some notion of linear dynamics are, in general, for operators defined on F-spaces. Since $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is not a metric space, the known criteria are not useful to prove Theorems 3.1 and 3.3. However, to show Theorem 3.6, we will adapt a criterion obtained by Bernardes *et al* [8] for operators on Fréchet spaces to operators on Hausdorff topological vector spaces. This criterion is the key of the proof.

The following diagram summarizes the relation between the notions of linear dynamics raised in this paper for operators on an arbitrary topological vector space:

$$\begin{tikzcd}[arrows=Rightarrow]
& \text{Li-Yorke chaos} & \text{hypercyclicity} \arrow[l] \arrow[r] \arrow[d] & \text{supercyclicity} \arrow[r] \arrow[d] & \text{cyclicity} \\
\text{mixing} \arrow[r] & \text{topological transitivity} & & n\text{-supercyclicity} &
\end{tikzcd}$$

2. Preliminaries

Let V be a subset of a Hausdorff topological complex vector space E and let $T\colon E \to E$ be a continuous linear operator (from now on, we just write operator). The *orbit of V under T*, denoted by $\mathrm{orb}_T(V)$, is the subset of E given by

$$\mathrm{orb}_T(V) = \bigcup_{k=0}^{\infty} T^k(V).$$

If $V = \{x\}$ is a singleton, and $\mathrm{orb}_T(V) = \{T^k x : k \in \mathbb{N}_0\}$ is dense in E, where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, then T is said to be *hypercyclic* and x a *hypercyclic vector* for T. If the linear span of $\mathrm{orb}_T(V)$ is dense in E, then T is said to be *cyclic* and x a *cyclic vector* for T. If $V = \mathrm{span}\{x\}$ and $\mathrm{orb}_T(V) = \mathbb{C} \cdot \{T^k x : k \in \mathbb{N}_0\}$ is dense in E, then T is said to be *supercyclic* and x a *supercyclic vector* for T. If V is a vector subspace of dimension n and $\mathrm{orb}_T(V)$ is dense in E, then T is said to be *n-supercyclic* and V a *supercyclic subspace* for T. Note that the notions of 1-supercyclicity and supercyclicity are equivalent. Also, an n-supercyclic operator, for $n = 2, 3, \ldots$, need not be cyclic (for an infinite dimensional example, see [14]). Hilden and Wallen [27] proved that no operator on $\mathbb{C}^n$, $n = 2, 3 \ldots$, can be supercyclic. So, n-supercyclicity does not imply supercyclicity in general. For properties and results about supercyclicity and n-supercyclicity, we refer the reader to [14, 21, 26, 27]. We say that T is *mixing* (respectively *topologically transitive*) if for any two non-empty open sets $U_1, U_2 \subset E$, there is $n_0 \in \mathbb{N}$ such that $T^n(U_1) \cap U_2 \neq \emptyset$, for all $n \geq n_0$ (respectively, $T^{n_0}(U_1) \cap U_2 \neq \emptyset$).

As was stated in the Introduction, the classical notion of Li-Yorke chaos was introduced for maps defined on metric spaces. The definition is the following: given a metric space (M,d) and a continuous map $f\colon M \to M$, we recall that a pair $(x,y) \in M \times M$ is called a *Li-Yorke pair* for f if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A *scrambled set* for f is a subset S of M such that (x,y) is a Li–Yorke pair for f whenever x and y are distinct points in S. The map f is said to be *Li–Yorke chaotic* if there exists an uncountable scrambled set for f.

As we also mentioned in the Introduction, Arai [2] introduced the notion of Li-Yorke chaos for an action of a group on a uniform space. Adopting Arai's definition of Li-Yorke chaos in the particular case for operators on a Hausdorff topological vector space E, we have the following:

- A pair $(x,y) \in E \times E$ is said to be *asymptotic* for T if for any neighborhood of zero U, there exists $k \in \mathbb{N}$ such that $T^n(x-y) \in U$ for every $n \geq k$, that is, if $T^n(x-y) \to 0$. A pair $(x,y) \in E \times E$ is said to be *proximal* for T if for any neighborhood of zero U, there exists $n \in \mathbb{N}$ such that $T^n(x-y) \in U$, that is, if the sequence $\{T^n(x-y)\}_{n=1}^{\infty}$ has a subnet converging to zero.
- A pair $(x,y) \in E \times E$ is said to be a *Li-Yorke pair* for T if it is proximal but not asymptotic. In other words, (x,y) is a Li-Yorke pair for T if and only if the sequence $\{T^n(x-y)\}_{n=1}^{\infty}$ does not converge to zero but has a subnet converging to zero.

Using this definition of Li-Yorke pair, *scrambled set* and *Li-Yorke chaos* for operators on a Hausdorff topological vector space are defined as in the case of maps on a metric space.

It is easy to check that if E is metrizable, and we consider a translation-invariant metric (this metric exists by definition of metrizability), then both definitions of Li-Yorke chaos coincide.

2.1. Convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. — In this section, we prove some technical results about convolution operators that we need to show the main results of this work. First, we present some preliminary results about the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$.

Given the topological product $\mathbb{C}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{C}$, we consider the complex vector space of all entire functions $f\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}$, which is denoted by $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. It is well known that there are only two usual locally convex topologies on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$: the compact open topology τ_0 and its bornological associated topology τ_δ (see

[4, 17]). It is also known that, with both topologies, $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is separable. For details and properties of these topologies we refer the reader to [1, 4, 5].

For each $n \in \mathbb{N}$, we consider the canonical inclusion $J_n\colon \mathbb{C}^n \to \mathbb{C}^{\mathbb{N}}$, the canonical projection $\pi_n\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^n$ and the corresponding linear mappings

$$J_n^*\colon f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \to f \circ J_n \in \mathcal{H}(\mathbb{C}^n), \quad \pi_n^*\colon f_n \in \mathcal{H}(\mathbb{C}^n) \to f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}}).$$

Since $\pi_n \circ J_n = Id_{\mathbb{C}^n}$, it follows that

$$J_n^* \circ \pi_n^* = Id_{\mathcal{H}(\mathbb{C}^n)}, \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

So, $\mathcal{H}(\mathbb{C}^n)$ can be seen as the vector subspace of $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ of all entire functions on $\mathbb{C}^{\mathbb{N}}$ that depend only of the n first variables, through the injective map π_n^*, for each $n \in \mathbb{N}$. It is easy to check that

$$\pi_1^*(\mathcal{H}(\mathbb{C})) \subset \pi_2^*(\mathcal{H}(\mathbb{C}^2)) \subset \cdots \subset \pi_n^*(\mathcal{H}(\mathbb{C}^n)) \subset \cdots \subset \mathcal{H}(\mathbb{C}^{\mathbb{N}}). \tag{2}$$

By [17, p. 162] or [4, Corolário 38]

$$\mathcal{H}(\mathbb{C}^{\mathbb{N}}) = \bigcup_{n=1}^{\infty} \{f_n \circ \pi_n\colon f_n \in \mathcal{H}(\mathbb{C}^n)\} = \bigcup_{n=1}^{\infty} \pi_n^*(\mathcal{H}(\mathbb{C}^n)). \tag{3}$$

Also, by [1, Proposition 1.3] the topology τ_δ, which was independently introduced by Nachbin [30] and Couré [16] coincides with the inductive limit topology of the Fréchet spaces $\mathcal{H}(\mathbb{C}^n)$, $n \in \mathbb{N}$, that is,

$$(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau_\delta) = \mathrm{ind}_{n\in\mathbb{N}}\mathcal{H}(\mathbb{C}^n),$$

where $\mathcal{H}(\mathbb{C}^n)$ is endowed with its usual topology, the compact open topology. More precisely, τ_δ is the strongest locally convex topology on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, for which the maps π_n^* are continuous. If τ represents any of the topologies τ_0, τ_δ on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, then the linear operators

$$J_n^*\colon f \in (\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau) \to f \circ J_n \in \mathcal{H}(\mathbb{C}^n)$$
$$\pi_n^*\colon f_n \in \mathcal{H}(\mathbb{C}^n) \to f_n \circ \pi_n \in (\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$$

are continuous, and it follows from (1) that $\mathcal{H}(\mathbb{C}^n)$ is topologically isomorphic to a complemented subspace of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. In particular, $\pi_n^*(\mathcal{H}(\mathbb{C}^n))$ is a closed proper subspace of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. For background information on these topologies, we refer the reader to the book of Dineen [17].

Finally, we recall that the *translation operator by* $\xi \in \mathbb{C}^{\mathbb{N}}$,

$$\tau_\xi\colon \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \to \mathcal{H}(\mathbb{C}^{\mathbb{N}})$$

is given by $(\tau_\xi f)(x) = f(x - \xi)$, for every $x \in \mathbb{C}^{\mathbb{N}}$. Analogously, we define translation operators on $\mathcal{H}(\mathbb{C}^n)$ for each $n \in \mathbb{N}$.

REMARK 2.1. — It is interesting to note that, if $\xi \in \mathbb{C}^{\mathbb{N}}$ is such that $\pi_n(\xi) = 0$, then the translation operator τ_ξ on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ coincides with the identity operator on $\pi_n^*(\mathcal{H}(\mathbb{C}^n))$, that is, $\tau_\xi|_{\pi_n^*(\mathcal{H}(\mathbb{C}^n))} = Id$. Hence, $\pi_n^*(\mathcal{H}(\mathbb{C}^n))$ is a closed proper subspace of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ and τ_ξ-invariant.

Definition 2.2. — A *convolution operator* on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is a continuous linear mapping

$$L\colon \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \to \mathcal{H}(\mathbb{C}^{\mathbb{N}})$$

such that $L(\tau_\xi f) = \tau_\xi(Lf)$ for every $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ and $\xi \in \mathbb{C}^{\mathbb{N}}$. Analogously we define convolution operators on $\mathcal{H}(\mathbb{C}^n)$ for each $n \in \mathbb{N}$. We say that a convolution operator is *non-trivial* if it is not a scalar multiple of the identity.

Lemma 2.3. — *Let $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$, $k \in \mathbb{N}$ and T be a linear map on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. Then there is $N \in \mathbb{N}$ such that $T^i f = \pi_N^*((T^i f) \circ J_N)$ for every $i = 0, \ldots, k$. Moreover, if $f = f_n \circ \pi_n$ with $f_n \in \mathcal{H}(\mathbb{C}^n)$ for some $n \in \mathbb{N}$, and L is a linear map on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ that commutes with all the translation operators (in particular, if L is a convolution operator), then*

$$L^k f = \pi_n^*((L^k f) \circ J_n) = \pi_n^* \circ J_n^*(L^k f)$$

for every $k \in \mathbb{N}_0$.

Proof. — By (3) we may write

$$f = f_{n_0} \circ \pi_{n_0}, Tf = f_{n_1} \circ \pi_{n_1}, \ldots, T^k f = f_{n_k} \circ \pi_{n_k},$$

with $f_{n_i} \in \mathcal{H}(\mathbb{C}^{n_i})$ for every $i = 0, \cdots, k$. Let $N = \max\{n_i : i = 0, \ldots, k\}$ and $\xi \in \mathbb{C}^{\mathbb{N}}$ be such that $\pi_N(\xi) = 0$. Then $\pi_{n_i}(\xi) = 0$ for every $i = 0, \cdots, k$ and

$$\begin{aligned}\tau_\xi(T^i f)(x) &= (T^i f)(x - \xi) = f_{n_i} \circ \pi_{n_i}(x - \xi)\\ &= f_{n_i}(\pi_{n_i}(x) - \pi_{n_i}(\xi)) = f_{n_i}(\pi_{n_i}(x)) = (T^i f)(x)\end{aligned}$$

for every $x \in \mathbb{C}^{\mathbb{N}}$ and every $i = 0, \cdots, k$. On the other hand, given any $x = (x_j)_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}}$, if we take $\xi = (0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots) \in \mathbb{C}^{\mathbb{N}}$, then

$$\begin{aligned}(T^i f)(x) &= \tau_\xi(T^i f)(x) = (T^i f)(x - \xi) = (T^i f)(x_1, \ldots, x_N, 0, 0, \ldots)\\ &= (T^i f)(J_N \circ \pi_N(x)) = (T^i f) \circ J_N \circ \pi_N(x).\end{aligned}$$

Thus, $T^i f = \pi_N^*((T^i f) \circ J_N)$ for every $i = 0, \ldots, k$.

Now, suppose that $f = f_n \circ \pi_n$ and L is a linear map on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ that commutes with all the translation operators. Choosing $\xi \in \mathbb{C}^{\mathbb{N}}$ such that $\pi_n(\xi) = 0$, we get

$$\tau_\xi(f) = f, \quad \tau_\xi(Lf) = L(\tau_\xi f) = Lf, \ldots, \tau_\xi(L^k f) = L(\tau_\xi(L^{k-1} f)) = L^k f,$$

for every $k \in \mathbb{N}_0$. Using the same ideas as in the first part of the proof we obtain

$$L^k f = \pi_n^*((T^k f) \circ J_n),$$

for every $k \in \mathbb{N}_0$. □

This lemma tells us that the operator $\pi_n^* \circ J_n^*$ acts as the identity on $\mathrm{orb}_L(f)$, whenever $f = f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ and L is a convolution operator.

If $f = f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ with $f_n \in \mathcal{H}(\mathbb{C}^n)$ and $\xi \in \mathbb{C}^{\mathbb{N}}$ it is not difficult to verify that $\tau_\xi f = (\tau_{\pi_n(\xi)} f_n) \circ \pi_n$. In this sense, the following question is quite natural:

> Does every convolution operator L on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ satisfy $Lf = (L_n f_n) \circ \pi_n$, where L_n is a convolution operator on $\mathcal{H}(\mathbb{C}^n)$?

The following lemma gives a positive answer to this question.

LEMMA 2.4. — *Let L be a convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$.*

(a) *For any $n \in \mathbb{N}$, the mapping $L_n := J_n^* \circ L \circ \pi_n^* \colon \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^n)$, is a convolution operator. We say that L_n is* the convolution operator on $\mathcal{H}(\mathbb{C}^n)$ associated to L.

(b)
$$L(f_n \circ \pi_n) = (L_n f_n) \circ \pi_n, \quad \text{for every } f_n \in \mathcal{H}(\mathbb{C}^n) \text{ and } n \in \mathbb{N}.$$

(c) *L is a scalar multiple of the identity on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ if and only if L_n is a scalar multiple of the identity on $\mathcal{H}(\mathbb{C}^n)$, for every $n \in \mathbb{N}$.*

Proof. — Let $n \in \mathbb{N}$, $f_n \in \mathcal{H}(\mathbb{C}^n)$ and $f := f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$.

(a) Note that

$$L_n f_n = J_n^* \circ L \circ \pi_n^*(f_n) = J_n^* \circ L(f) = (Lf) \circ J_n. \tag{4}$$

Let $a \in \mathbb{C}^n$. We want to show that $\tau_a \circ L_n = L_n \circ \tau_a$. Applying (4) we have

$$\begin{aligned}[\tau_a \circ L_n](f_n)(z) = \tau_a(L_n f_n)(z) &= (L_n f_n)(z-a) = (Lf) \circ J_n(z-a) \\ &= (Lf)(J_n(z) - J_n(a)) = [\tau_{J_n(a)}(Lf)](J_n(z)),\end{aligned}$$

for every $z \in \mathbb{C}^n$ and so $[\tau_a \circ L_n](f_n) = [\tau_{J_n(a)}(Lf)] \circ J_n$. Using the fact that L is a convolution operator we get

$$\begin{aligned}[\tau_a \circ L_n](f_n) &= [L(\tau_{J_n(a)} f)] \circ J_n = [L((\tau_a f_n) \circ \pi_n)] \circ J_n \\ &= [L \circ \pi_n^*(\tau_a f_n)] \circ J_n = J_n^* \circ L \circ \pi_n^*(\tau_a f_n) = [L_n \circ \tau_a](f_n).\end{aligned}$$

(b) Applying Lemma 2.3 to the entire function $f_n \circ \pi_n$ and using (4) we get

$$L(f_n \circ \pi_n) = \pi_n^*((L(f_n \circ \pi_n)) \circ J_n) = \pi_n^*(L_n f_n) = (L_n f_n) \circ \pi_n. \tag{5}$$

(c) Let $\lambda \in \mathbb{C}$ be such that $Lg = \lambda g$ for every $g \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$. Then

$$(L_n f_n) \circ \pi_n = Lf = (\lambda f_n) \circ \pi_n.$$

Since π_n^* is injective, it follows that $L_n f_n = \lambda f_n$. Therefore, L_n is a scalar multiple of the identity.

Conversely, suppose that for each $n \in \mathbb{N}$, there exists $\lambda_n \in \mathbb{C}$ such that $L_n f_n = \lambda_n f_n$ for every $f_n \in \mathcal{H}(\mathbb{C}^n)$. It is not difficult to verify that, if $g \in \pi_n^*(\mathcal{H}(\mathbb{C}^n))$, then $Lg = \lambda_n g$. Note that to prove the assertion it suffices to show that $\lambda_n = \lambda_m$ for any $n, m \in \mathbb{N}$. So, let $n, m \in \mathbb{N}$ with $n \leq m$. By (2) we may choose $g \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ such that $g \neq 0$ and $g \in \pi_n^*(\mathcal{H}(\mathbb{C}^n)) \subset \pi_m^*(\mathcal{H}(\mathbb{C}^m))$. Thus, $\lambda_n g = Lg = \lambda_m g$ and, since $g \neq 0$, it follows that $\lambda_n = \lambda_m$. □

Below we list some remarks about the previous lemma.

Remark 2.5. — (1) For $\xi \in \mathbb{C}^{\mathbb{N}}$, the convolution operator $(\tau_\xi)_n = \tau_{\pi_n(\xi)}$ is a concrete example of a convolution operator associated to the translation τ_ξ.

(2) The hypothesis that L commutes with the translation operators is essential to show Lemma 2.4(b). In fact, consider the linear mapping $L\colon f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \to f \circ B \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$, where B is the backward shift on $\mathbb{C}^{\mathbb{N}}$. Since $\pi_1 \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$, we have $(L\pi_1)(\xi) = \xi_2$, for all $\xi = (\xi_j)_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}}$. Denoting the identity function on $\mathbb{C}$ by $Id_{\mathbb{C}}$, we have $\pi_1 = Id_{\mathbb{C}} \circ \pi_1$ and $L_1 Id_{\mathbb{C}} = 0$. Thus, $L(Id_{\mathbb{C}} \circ \pi_1) \neq (L_1 Id_{\mathbb{C}}) \circ \pi_1$.

(3) Lemma 2.4(b) allows us to write the orbit $\mathrm{orb}_L(f)$ in terms of convolution operators on $\mathcal{H}(\mathbb{C}^n)$, that is, if $f = f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$, and L is a convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$, then

$$Lf = (L_n f_n) \circ \pi_n, \quad L^2 f = L((L_n f_n) \circ \pi_n) = L_n(L_n f_n) \circ \pi_n = (L_n^2 f_n) \circ \pi_n,$$

and proceeding by induction it follows that

$$L^k f = (L_n^k f_n) \circ \pi_n,$$

for every $k \in \mathbb{N}_0$.

(4) By the proof of part (c), it is clear that we may rewrite it in the following way:

L is a scalar multiple of the identity on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ if and only if L_n is a scalar multiple of the identity on $\mathcal{H}(\mathbb{C}^n)$ for infinitely many $n \in \mathbb{N}$.

3. Linear dynamics of convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$

In this section, we will study the linear dynamics of convolution operators on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. We start by proving that convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ are neither cyclic nor n-supercyclic for any $n \in \mathbb{N}$. This result improves a result of Fávaro and Mujica [18], which states that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is hypercyclic.

Theorem 3.1. — (a) *No convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is cyclic.*

(b) *No convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is n-supercyclic, for any $n \in \mathbb{N}$.*

Proof. — Let L be a convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$.

(a) Let $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$. Then, we may write $f = f_n \circ \pi_n$ with $f_n \in \mathcal{H}(\mathbb{C}^n)$. By Lemma 2.3 the orbit of f under L is

$$\mathrm{orb}_L(f) = \{L^k f : k \in \mathbb{N}_0\} = \{\pi_n^*((L^k f) \circ J_n) : k \in \mathbb{N}_0\} \subset \pi_n^*(\mathcal{H}(\mathbb{C}^n)).$$

Since $\pi_n^*(\mathcal{H}(\mathbb{C}^n))$ is a closed proper subspace of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$, we have

$$\overline{\mathrm{span}\,\mathrm{orb}_L(f)}^{\tau} \subset \pi_n^*(\mathcal{H}(\mathbb{C}^n)).$$

Therefore, $\mathrm{span}\,\mathrm{orb}_L(f)$ cannot be a dense subset of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. So, there is no cyclic entire function for any convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. Hence, L is not cyclic.

(b) Let $n \in \mathbb{N}, n > 1$, and let V be an n-dimensional vector subspace of $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ with generators $f_1, \ldots, f_n$. Then $L^k(V)$ is a vector subspace of $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, generated by $L^k f_1, \cdots, L^k f_n$, and with dimension less than or equal to n, for every $k \in \mathbb{N}_0$. If we write $f_1 = f_{m_1} \circ \pi_{m_1}, \cdots, f_n = f_{m_n} \circ \pi_{m_n}$, with $f_{m_i} \in \mathcal{H}(\mathbb{C}^{m_i})$, for every $i = 1, \cdots, n$, then it follows from Lemma 2.3 and (2) that

$$L^k(V) \subset \pi_{m_1}^*(\mathcal{H}(\mathbb{C}^{m_1})) + \cdots + \pi_{m_n}^*(\mathcal{H}(\mathbb{C}^{m_n})) \subset \pi_m^*(\mathcal{H}(\mathbb{C}^m)),$$

for every $k \in \mathbb{N}_0$, where $m := \max\{m_i : i = 1, \ldots, n\}$. Therefore,

$$\mathrm{orb}_L(V) = \bigcup_{k=0}^{\infty} L^k(V) \subset \pi_m^*(\mathcal{H}(\mathbb{C}^m))$$

and so $\mathrm{orb}_L(V)$ cannot be dense in $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. Thus, no finite-dimensional subspace of $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is supercyclic for L. Hence, L is not n-supercyclic. □

Now we will prove that non-trivial convolution operators on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ are mixing and Li-Yorke chaotic. However, first, we will give the details about Grosse-Erdmann's example mentioned in the Introduction. Grosse-Erdmann kindly clarified some details about the example and communicated to us that it is a mixing operator. The details are as follows:

Example 3.2. — Consider the countable direct sum of ℓ_2, denoted by $\bigoplus_{n=1}^{\infty} \ell_2$, with its usual topology. This topology coincides with the inductive limit topology induced by the inclusions

$$\sigma_N : \bigoplus_{n=1}^{N} \ell_2 \to \bigoplus_{n=1}^{\infty} \ell_2, \quad (x_1, \ldots, x_N) \longmapsto (x_1, \ldots, x_N, 0, \ldots)$$

with $N \in \mathbb{N}$. Here, $\bigoplus_{n=1}^{N} \ell_2$ is considered with its usual topology. Consider a mixing operator on ℓ_2 (for example, the weighted backward shift $\mathcal{T}$ given in the Introduction). Thus,

$$\mathcal{S} : \bigoplus_{n=1}^{\infty} \ell_2 \to \bigoplus_{n=1}^{\infty} \ell_2, \quad (x_n)_{n=1}^{\infty} \longmapsto (\mathcal{T} x_n)_{n=1}^{\infty}$$

is mixing. In fact, let non-empty open subsets $U, V \subset \bigoplus_{n=1}^{\infty} \ell_2$, $x = (x_n)_{n=1}^{\infty} \in U$ and $y = (y_n)_{n=1}^{\infty} \in V$. Now, let $N \in \mathbb{N}$, such that $x_n = 0 = y_n$, for all $n > N$. Then, $\sigma_N^{-1}(U)$ and $\sigma_N^{-1}(V)$ are non-empty open subsets of $\bigoplus_{n=1}^{N} \ell_2$. Since the direct sum of N copies of $\mathcal{T}$ is mixing (see [25, Proposition 2.40]), there is $q \in \mathbb{N}$, such that

$$\left(\bigoplus_{n=1}^{N} \mathcal{T} \right)^k \left(\sigma_N^{-1}(U) \right) \cap \sigma_N^{-1}(V) \neq \emptyset,$$

for all $k \geq q$. Now, for each $k \geq q$, considering $\omega^{(k)} \in \sigma_N^{-1}(U)$, such that $\left(\bigoplus_{n=1}^{N} \mathcal{T} \right)^k \left(\omega^{(k)} \right) \in \sigma_N^{-1}(V)$, we get $\sigma_N(\omega^{(k)}) \in U$ and $\mathcal{S}^k \left(\sigma_N(\omega^{(k)}) \right) \in V$.

It is easy to see that $\mathcal{S}$ is not hypercyclic, since for any element $x = (x_n)_{n=1}^{\infty} \in \bigoplus_{n=1}^{\infty} \ell_2$, there is some N, such that $x_k = 0$, for all $k > N$, and the same is then true for all its iterates by $\mathcal{S}$.

THEOREM 3.3. — *Every non-trivial convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is mixing.*

Proof. — Let U and V be open non-empty subsets of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. Since

$$\mathcal{H}(\mathbb{C}^{\mathbb{N}}) = \bigcup_{n=1}^{\infty} \pi_n^*(\mathcal{H}(\mathbb{C}^n)) \quad \text{and} \quad \pi_n^*(\mathcal{H}(\mathbb{C}^n)) \subset \pi_m^*(\mathcal{H}(\mathbb{C}^m)), \quad n \leq m,$$

there exists $n_0 \in \mathbb{N}$, such that $U \cap \pi_n^*(\mathcal{H}(\mathbb{C}^n)) \neq \emptyset$ and $V \cap \pi_n^*(\mathcal{H}(\mathbb{C}^n)) \neq \emptyset$ for every $n \geq n_0$. Since L is a non-trivial convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$, it follows from Remark 2.5(4) that there is some $n \in \mathbb{N}$, $n \geq n_0$, such that L_n is a non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^n)$. It follows from the classical result of Godefroy and Shapiro that L_n is mixing. Now, since $(\pi_n^*)^{-1}(U)$ and $(\pi_n^*)^{-1}(V)$ are non-empty open subsets of $\mathcal{H}(\mathbb{C}^n)$, there exists $N \in \mathbb{N}$, such that

$$L_n^k \left((\pi_n^*)^{-1}(U) \right) \cap (\pi_n^*)^{-1}(V) \neq \emptyset$$

for every $k \geq N$. We claim that

$$L^k(U) \cap V \neq \emptyset$$

for every $k \geq N$. Indeed, let $k \in \mathbb{N}$, such that $k \geq N$. By choosing $f_{nk} \in (\pi_n^*)^{-1}(U)$, such that $L_n^k f_{nk} \in (\pi_n^*)^{-1}(V)$, we obtain that $f_{nk} \circ \pi_n \in U$ and

$$L^k(f_{nk} \circ \pi_n) = (L_n^k f_{nk}) \circ \pi_n = \pi_n^*(L_n^k f_{nk}) \in V.$$

Thus, $L^k(f_{nk} \circ \pi_n) \in L^k(U) \cap V$, and the assertion follows. □

Before proving that convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ are Li-Yorke chaotic, we will prove a characterization of Li-Yorke chaos that involves the existence of semi-irregular vectors. This characterization was obtained by Bernardes Jr. *et al.* [8] for operators on Fréchet spaces. We will generalize this fact for Hausdorff topological vector spaces. The definition below is known for Fréchet spaces.

Let E be a Hausdorff topological vector space and let T be an operator on E. A vector $x \in E$ is said to be a *semi-irregular vector* for T, if the sequence $(T^n x)_{n=1}^{\infty}$ does not converge to zero but has a subnet converging to zero. It is easy to see that $(x, y) \in E \times E$ is a Li-Yorke pair for T, if and only if $x - y$ is a semi-irregular vector for T.

As was mentioned in [8], the notion of semi-irregularity makes sense only for infinite-dimensional spaces. Indeed, an easy application of the Jordan form implies that there are no semi-irregular vectors for operators on finite-dimensional spaces.

Theorem 3.4. — *Let E be a Hausdorff topological vector space and let T be an operator on E. The following assertions are equivalent:*

(i) *T is Li-Yorke chaotic.*
(ii) *T admits a Li-Yorke pair.*
(iii) *T admits a semi-irregular vector.*

Proof. — Since the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are immediate, we just need to show that (iii) $\Rightarrow$ (i). Let x be a semi-irregular vector for T. Then for every $\alpha, \lambda \in \mathbb{C}$, with $\alpha \neq \lambda$, the sequence $\{T^n(\alpha x - \lambda x)\}_{n=1}^{\infty}$ does not converge to zero, but it has a subnet converging to zero. Hence, $\operatorname{span}\{x\}$ is an uncountable scrambled set for T and, thus, T is Li-Yorke chaotic. ☐

As an immediate consequence of this theorem we obtain the following:

Corollary 3.5. — *Every hypercyclic operator on a separable topological vector space is Li-Yorke chaotic.*

Theorem 3.6. — *Every non-trivial convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is Li-Yorke chaotic.*

Proof. — Let L be a non-trivial convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. We will show that L has a semi-irregular entire function. Since L is not a scalar multiple of the identity, it follows from Lemma 2.4(c) that there exists $n \in \mathbb{N}$ such that the convolution operator L_n associated to L is not a scalar multiple of the identity. So, L_n is a non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^n)$, and it follows from the classical result of Godefroy and Shapiro that L_n is hypercyclic. In particular, there exists a semi-irregular function $f_n \in \mathcal{H}(\mathbb{C}^n)$ for L_n. If we set

$f = f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$, then f is a semi-irregular function for L. In fact, since

$$L^k f = (L_n^k f_n) \circ \pi_n = \pi_n^* \left(L_n^k f_n\right), \quad \text{for every } k \in \mathbb{N}_0, \tag{6}$$

it follows immediately from (6) that the sequence $(L^k f)_{k=0}^{\infty}$ has a subsequence converging to zero. On the other hand, if $L^k f \to 0$ in the topology of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$, then

$$L_n^k f_n = J_n^* \circ \pi_n^* \left(L_n^k f_n\right) = J_n^*(L^k f) \to 0 \quad \text{in } \mathcal{H}(\mathbb{C}^n)$$

when $k \to \infty$, but this contradicts the fact that f_n is a semi-irregular function for L_n. Therefore, $(L^k f)_{k=0}^{\infty}$ does not converge to zero, and so f is a semi-irregular function for L. Applying Theorem 3.4 we obtain the desired result. $\square$

Corollary 3.7. — *Every non-trivial convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is mixing and Li-Yorke chaotic.*

Remark 3.8. — Bonet's example $\mathcal{T}$ acting on the non-complete normed space $\mathcal{E}$ is a mixing operator that is not Li-Yorke chaotic, because each vector of $\mathcal{E}$ has a finite orbit. Consequently, there is no semi-irregular vector for $\mathcal{T}$. On the other hand, Corollary 3.7 and Example 3.2 give operators on complete non-metrizable, locally convex spaces that are mixing and Li-Yorke chaotic (note that, in Grosse-Erdmann's example, if we choose a hypercyclic vector x for $\mathcal{T}$, then $(x, 0, 0, \ldots) \in \bigoplus_{n=1}^{\infty} \ell_2$ is a semi-irregular vector for $\mathcal{S}$). So, the following question arises:

> Is every mixing operator on a complete non-metrizable topological vector space Li-Yorke chaotic?

Alfred Peris Manguillot kindly communicated to us the following example, which shows that the answer is negative:

Consider

$$\varphi = \{(x_n) \in \mathbb{C}^{\mathbb{N}};\ \text{there is some } m \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n > m\}$$

equipped with its natural locally convex topology, which is the strongest one that can be defined on it. With this topology, φ is a complete non-metrizable locally convex space (for details, see [12, p. 591] or [32, p. 56] or [33, p. 200]). Consider the operator $T = I + B$ on φ, where I is the identity operator, and B is the backward shift on φ. By [25, Example 12.17] T is mixing. Note that T is not Li-Yorke chaotic, since there is no semi-irregular vector for T. Indeed, for $x \in \varphi$, $x = (x_1, \ldots, x_{k_0}, 0, 0, \ldots)$, with $x_{k_0} \neq 0$, it follows that the k_0-th coordinate of $T^n x$ is x_{k_0}, for all $n \in \mathbb{N}_0$. Hence, $(T^n x)_{n=0}^{\infty}$ does not have subnet converging to zero.

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SUR LA REPRÉSENTATION DES ENTIERS PAR LES FORMES CYCLOTOMIQUES DE GRAND DEGRÉ

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RÉSUMÉ. — Pour chaque entier $d \geq 4$, nous étudions la suite des entiers positifs représentés par une des formes binaires cyclotomiques $\Phi_n(X, Y)$ pour les n positifs tels que $\varphi(n) \geq d$. Le cas $d = 2$ a été étudié dans notre précédent texte avec C. Levesque (« Representation of integers by cyclotomic binary forms », *Acta Arith.* **184** (2018), no. 1, p. 67–86, `http://arxiv.org/abs/1701.01230`). Notre démonstration repose sur une variante d'un énoncé de C.L. Stewart and S.Y. Xiao (« On the representation of integers by binary forms », *Math. Ann.* **375** (2019), p. 133–163, `http://arxiv.org/abs/1605.03427`) concernant les valeurs communes prises par deux formes binaires de même degré et de discriminants non nuls. Toutes les constantes sont effectivement calculables.

ABSTRACT (*On the representation of integers by cyclotomic forms with large degree*). — For each integer $d \geq 4$, we study the sequence of positive integers which are represented by one at least of the cyclotomic binary forms $\Phi_n(X, Y)$, with n a positive integer satisfying $\varphi(n) \geq d$. The case $d = 2$ was studied in our previous work with C. Levesque ("Representation of integers by cyclotomic binary forms", *Acta Arith.* **184** (2018), no. 1, p. 67–86, `http://arxiv.org/abs/1701.01230`). Our proof is based on a variant of a statement of C.L. Stewart and S.Y. Xiao ("On the representation of integers by binary forms", *Math. Ann.* **375** (2019), p. 133–163, `http://arxiv.org/abs/1605.03427`) concerning the common values taken by two binary forms of the same degree and non-zero discriminants. All constants are effectively computable.

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1. Introduction

Rappelons ([6]) que la suite $(\Phi_n(X,Y))_{n\geq 1}$ des *formes cyclotomiques* est définie par la formule de récurrence

$$X^n - Y^n = \prod_{k\mid n} \Phi_k(X,Y).$$

Le polynôme $\Phi_n(X,Y)$ est homogène de degré $\varphi(n)$ (φ fonction indicatrice d'Euler) et il est relié au polynôme cyclotomique $\phi_n(t) \in \mathbb{Z}[t]$ par la formule

$$\Phi_n(X,Y) = Y^{\varphi(n)}\phi_n(X/Y).$$

Puisque, pour $n \geq 3$, le polynôme $\phi_n(t)$ n'a aucun zéro réel, on a donc l'inégalité

$$\Phi_n(x,y) \gg \max(|x|^{\varphi(n)}, |y|^{\varphi(n)}),$$

uniformément sur x et y réels.

Pour $N \geq 2$ et d entier pair, on désigne par $\mathcal{A}_d(N)$ le cardinal de l'ensemble des entiers $1 \leq m \leq N$ tels qu'il existe un entier n et des entiers (x,y) vérifiant les trois conditions

$$\begin{cases} \varphi(n) \geq d, \\ \Phi_n(x,y) = m, \\ \max(|x|,|y|) \geq 2. \end{cases} \tag{1}$$

On s'intéresse au comportement asymptotique de $\mathcal{A}_d(N)$ pour d fixé et N tendant vers l'infini. Il est alors sage d'introduire la dernière condition de (1) puisque pour tout p premier on a $\Phi_p(1,1) = p$ et le cardinal des $p \leq N$ masquerait le terme principal de l'estimation qui sera donnée en (6). Par convention, nous réservons la lettre p aux nombres premiers. Enfin, si n est un entier impair, on a l'égalité

$$\Phi_{2n}(X,Y) = \Phi_n(X,-Y).$$

On peut ainsi ajouter, aux conditions de (1), la condition de congruence

$$n \not\equiv 2 \bmod 4, \tag{2}$$

sans modifier l'étude de $\mathcal{A}_d(N)$.

Appelons *totient* toute valeur prise par la fonction φ. La suite croissante des totients est ainsi

$$\mathfrak{T} := \{1, 2, 4, 6, 8, 10, 12, 16, 18, 20, 22, 24, 28, 30, \ldots\}.$$

Cette suite contient la suite des $p-1$ mais reste mystérieuse à de nombreux points de vue (on se reportera avec profit aux articles de Ford [4] et [5] traitant, entre autres choses, de la fonction de comptage de la suite $\mathfrak{T}$ et du nombre de solutions à l'équation $\varphi(n) = d$). Il est naturel de restreindre l'étude de

$\mathcal{A}_d(N)$ au cas où d est un totient pair. L'étude de $\mathcal{A}_2(N)$ a été traitée dans [6, Théorème 1.3] où il est prouvé qu'il existe deux constantes $C_2 = 1,403132\dots$ et $C_2' = 0,302316\dots$ telles que, uniformément pour $N \geq 2$, on a l'égalité

$$\mathcal{A}_2(N) = C_2 \frac{N}{(\log N)^{\frac{1}{2}}} - C_2' \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right). \tag{3}$$

Les constantes C_2 et C_2' se définissent au moyen des valeurs, au point $s = 1$, de certaines fonctions de Dirichlet $L(s, \chi)$ où χ est le caractère de Kronecker attaché aux corps quadratiques de discriminants -4, -3 et 12.

Pour un totient $d \geq 4$, les outils de [6] conduisent à la majoration

$$\mathcal{A}_d(N) = O\big(N^{\frac{2}{d}} (\log N)^{1,161}\big), \tag{4}$$

(voir corollaire 4.11 et sa preuve ci–dessous).

Par rapport à [6], le présent travail innove en injectant résultats et méthodes de [9] que nous décrirons au §2. Contentons–nous pour l'instant de donner notre résultat principal qui améliore notablement (4). Pour son énoncé, nous introduisons les notations suivantes :

- si d est un totient, on note $d^\dagger$ le successeur immédiat de d dans la suite $\mathfrak{T}$.
- pour d entier ≥ 3, on pose

$$\eta_d = \begin{cases} 2/9 + 73/(108\sqrt{3}) & \text{si } d = 3, \\ (1/2 + 9/(4\sqrt{d}))/d & \text{si } 4 \leq d \leq 20, \\ 1/d & \text{pour } d \geq 21. \end{cases} \tag{5}$$

On prouvera donc le

Théorème 1.1. — *Soit $d \geq 4$ un totient. Alors, il existe une constante $C_d > 0$, telle que, pour tout $\varepsilon > 0$ et uniformément pour $N \geq 2$, on a l'égalité*

$$\mathcal{A}_d(N) = C_d N^{\frac{2}{d}} + O(N^{\frac{2}{d^\dagger}}) + O_\varepsilon(N^{\eta_d + \varepsilon}). \tag{6}$$

Remarque 1.2. — La formule (6) est d'autant plus précise que $d^\dagger - d$ est grand. Ainsi, dans le cas particulier où $d \geq 6$, la minoration triviale

$$d^\dagger \geq d + 2,$$

réduit la formule (6) en sa forme plus grossière

$$\mathcal{A}_d(N) = C_d N^{\frac{2}{d}} + O(N^{\frac{2}{d+2}}).$$

Remarque 1.3. — Le théorème 1.1 suppose $d \geq 4$. La formule (3) correspond donc au cas $d = 2$. Mais, par la présence au dénominateur du facteur $(\log N)^{\frac{1}{2}}$, elle diffère notablement de (6). Cette différence s'explique comme suit. Il y a trois formes cyclotomiques de degré 2. Ce sont les trois formes quadratiques binaires

$$\begin{aligned}(7)\qquad &\Phi_3(X,Y)=X^2+XY+Y^2,\quad \Phi_4(X,Y)=X^2+Y^2,\quad \text{et}\\ &\Phi_6(X,Y)=X^2-XY+Y^2.\end{aligned}$$

Puisque $\Phi_6(X,-Y)=\Phi_3(X,Y)$ les formes Φ_6 et Φ_3 représentent les mêmes entiers. Mais les formes Φ_3 et Φ_4 à la différence des formes cyclotomiques de degré au moins 4, ont un nombre infini d'automorphismes comme définis au §4.4. Par exemple on a $\mathrm{Aut}\Phi_4=\mathrm{O}(2,\mathbb{Q})$ (le groupe des matrices orthogonales 2×2 à coefficients dans $\mathbb{Q}$).

L'objet des théorèmes 1.4 et 1.6 est de compléter la formule (6). Nous précisons d'abord la constante C_d.

Théorème 1.4. — *Soit $d\geq 4$ un totient. La constante C_d de la formule* (6) *vérifie l'égalité*

$$C_d=\sum_{\substack{n\not\equiv 2 \bmod 4\\ \varphi(n)=d}} w_n\,A_{\Phi_n} \tag{8}$$

où

$$w_n:=\begin{cases}\frac14 & \textit{si } 4\nmid n,\\ \frac18 & \textit{si } 4\mid n,\end{cases} \tag{9}$$

et

$$A_{\Phi_n}=\iint_{\Phi_n(x,y)\leq 1}\mathrm{d}x\mathrm{d}y.$$

Voici deux exemples dans lesquels la formule (8) donnant la valeur de la constante C_d se simplifie.

1. Soit $p\geq 5$ un nombre premier de Sophie Germain, c'est-à-dire tel que le nombre $\ell=2p+1$ soit premier. Alors ℓ est l'unique entier $\not\equiv 2 \bmod 4$ tel que $\varphi(\ell)=2p$ et on a l'égalité
$$C_{2p}=\frac14 A_{\Phi_\ell}.$$
On conjecture qu'il y a une infinité de nombres premiers de Sophie Germain.
2. Supposons que $d\geq 4$ est une puissance de 2, disons $d=2^k$.
On désigne par $\mathcal{M}$ l'ensemble des nombres entiers $m\geq 1$ dont le développement binaire $m=2^{a_1}+2^{a_2}+\cdots+2^{a_r}$ est tel que chacun des nombres $F_{a_i}=2^{2^{a_i}}+1$ est premier (nombre premier de Fermat). L'ensemble $\mathcal{M}$ contient les entiers $1,2,3,\ldots,31$; on ne connaît pas d'autre

élément de $\mathcal{M}$. Pour chaque $m \in \mathcal{M}$ vérifiant $m \leq k$, on définit

$$\ell_k(m) = 2^{k-m+1} F_{a_1} F_{a_2} \cdots F_{a_r},$$

de sorte que $\varphi(\ell_k(m)) = 2^k$. Les entiers $n \not\equiv 2 \bmod 4$ tels que $\varphi(n) = d$ sont d'une part $n = 2d$, qui est multiple de 4, d'autre part les $\ell_k(m)$ avec $m < k$, qui sont aussi multiples de 4, et enfin, si $k \in \mathcal{M}$, $\ell_k(k)/2$ qui est impair, avec $A_{\Phi_{\ell_k(k)}} = A_{\Phi_{\ell_k(k)/2}}$. Alors

$$C_d = \begin{cases} \frac{1}{8} A_{\Phi_{2d}} + \frac{1}{8} \sum\limits_{\substack{m \in \mathcal{M} \\ m < k}} A_{\Phi_{\ell_k(m)}} & \text{si } k \notin \mathcal{M}, \\ \frac{1}{8} A_{\Phi_{2d}} + \frac{1}{8} \sum\limits_{\substack{m \in \mathcal{M} \\ m < k}} A_{\Phi_{\ell_k(m)}} + \frac{1}{4} A_{\Phi_{\ell_k(k)}} & \text{si } k \in \mathcal{M}, \end{cases}$$

avec

$$A_{\Phi_{2d}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(1+t^d)^{2/d}} = \frac{2}{d} \frac{\Gamma(1/d)^2}{\Gamma(2/d)}$$

(cf. §6.1 et [9, Corollaire 1.3 et § 5]).

Nous montrerons que l'on a

$$\lim_{n \to \infty} A_{\Phi_n} = 4.$$

C'est une conséquence de l'énoncé plus précis suivant, concernant le *domaine fondamental cyclotomique* $\mathcal{O}_n$ défini par

$$\mathcal{O}_n = \{(x, y) \in \mathbb{R}^2 \mid \Phi_n(x, y) \leq 1\}.$$

THÉORÈME 1.5. — *Soit $\varepsilon > 0$. Il existe $n_0 = n_0(\varepsilon)$ tel que, pour $n \geq n_0$, le domaine fondamental cyclotomique $\mathcal{O}_n$ d'indice n contient le carré centré en O de côté $2 - n^{-1+\varepsilon}$ et est contenu dans le carré centré en O de côté $2 + n^{-1+\varepsilon}$.*

Enfin nous discutons de l'optimalité de la formule (6)

THÉORÈME 1.6. — *On adopte les notations du théorème 1.1. Soit $d \geq 4$ un entier tel que d et $d+2$ soient des totients. Il existe une constante positive $v_d > 0$ telle que pour N suffisamment grand, on ait l'inégalité*

$$\mathcal{A}_d(N) \geq C_d N^{\frac{2}{d}} + v_d N^{\frac{2}{d+2}}.$$

REMARQUE 1.7. — Il est naturel de conjecturer qu'il y a une infinité de d tels que $d+2$ soit aussi un totient : c'est une conséquence de la conjecture des nombres premiers jumeaux. Enfin, on peut tout à fait envisager des énoncés analogues sous l'hypothèse $d^\dagger = d + \nu$ où ν est un entier pair fixé. Cette extension nécessiterait une adaptation des propriétés de confinement décrites aux §4.1, §4.2 et §4.3.

2. Valeurs prises par une forme binaire

L'objet de cette section est de décrire précisément le résultat de Stewart et Xiao [9]. Ce résultat déja mentionné plus haut est à la base de notre travail. Dans toute cette section $F = F(X,Y)$ est un polynôme homogène de $\mathbb{Z}[X,Y]$ de degré $d \geq 3$. On dit alors que F est une *forme binaire de degré d*. Un entier m est dit *représenté* par une forme binaire F s'il existe $(x,y) \in \mathbb{Z}^2$ tel que $F(x,y) = m$. On désigne par $R_F(N)$ le cardinal de l'ensemble des entiers m représentés par F et vérifiant $0 \leq |m| \leq N$. On appelle *automorphisme* de F toute matrice U de $\mathrm{Gl}(2,\mathbb{Q})$

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix},$$

telle que

$$F(X,Y) = F(u_1X + u_2Y, u_3X + u_4Y). \tag{10}$$

Muni de la multiplication des matrices, l'ensemble des automorphismes de F forme un sous–groupe fini de $\mathrm{Gl}(2,\mathbb{Q})$, noté $\mathrm{Aut}F$. Il existe alors une ensemble $\mathcal{G}$ de dix sous–groupes finis de $\mathrm{Gl}(2,\mathbb{Z})$ tel que que pour toute forme binaire F de degré $d \geq 3$, il existe $T \in \mathrm{Gl}(2,\mathbb{Q})$ et un unique $G \in \mathcal{G}$ vérifiant

$$\mathrm{Aut}F = TGT^{-1}.$$

L'ensemble $\mathcal{G}$ contient en particulier les deux groupes suivants

1. $\mathbb{D}_2$, groupe diédral à quatre éléments, engendré par
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ et } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
2. $\mathbb{D}_4$, groupe diédral à huit éléments, engendré par
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ et } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

À partir de la décomposition précédente, Stewart et Xiao construisent un nombre rationnel W_F (voir [9, Theorem 1.2]) dont la définition, de nature algébrique, est longue puisqu'elle envisage les dix possibilités pour G. Disons, pour mémoire, que W_F tient compte des déterminants des réseaux de $\mathbb{Z}^2$ dont l'image, par certains sous–groupes de $\mathrm{Aut}F$, est incluse dans $\mathbb{Z}^2$. En vue des applications, nous nous restreignons à deux cas particuliers

1. Cas où $G = \mathbb{D}_2$. Soit Λ le sous–réseau des éléments $(v,w) \in \mathbb{Z}^2$ tels que, pour tout $A \in \mathrm{Aut}F$ on ait $A\begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{Z}^2$. On pose alors

$$W_F = \frac{1}{2}\Big(1 - \frac{1}{2|\det(\Lambda)|}\Big). \tag{11}$$

2. Cas où $G = \mathbb{D}_4$. D'abord Λ est défini comme précédemment. Le groupe $\mathrm{Aut}F$ possède exactement trois sous–groupes de cardinal 4, notés G_1, G_2 et G_3. On désigne par Λ_i $(1 \leq i \leq 3)$ le sous–réseau des éléments $(v, w) \in \mathbb{Z}^2$ tels que, pour tout $A \in G_i$ on ait $A \begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{Z}^2$. On pose alors

$$W_F = \frac{1}{2}\Big(1 - \frac{1}{2|\det(\Lambda_1)|} - \frac{1}{2|\det(\Lambda_2)|} - \frac{1}{2|\det(\Lambda_3)|} + \frac{3}{4|\det(\Lambda)|}\Big). \tag{12}$$

Notons

$$A_F := \iint_{|F(x,y)|\leq 1} \mathrm{d}x\mathrm{d}y,$$

l'aire de la région fondamentale associée à F. Enfin, pour $d \geq 4$ entier pair, nous introduisons la constante β_d^* définie par

$$\beta_d^* = \begin{cases} 3/(d\sqrt{d}) & \text{pour } d = 4,\ 6,\ 8 \\ 1/d & \text{pour } d \geq 10. \end{cases} \tag{13}$$

Nous pouvons maintenant énoncer le résultat fondamental de [9, Theorem 1.1].

Théorème 2.1. — *Pour tout $d \geq 3$ il existe une constante $\beta_d < 2/d$ ayant la propriété suivante : Pour toute forme binaire F de degré d, de discriminant non nul, pour tout $\varepsilon > 0$, on a, uniformément pour $N \geq 2$ l'égalité*

$$R_F(N) = A_F\, W_F N^{\frac{2}{d}} + O_{F,\varepsilon}(N^{\beta_d+\varepsilon}). \tag{14}$$

Si, dans la formule (14)*, on se restreint aux formes binaires F de degré pair $d \geq 4$, de discriminant non nul, sans facteur linéaire réel, on peut donner à β_d la valeur β_d^*, définie en* (13).

3. Valeurs communes à deux formes binaires

Aux formes binaires $F_1 = F_1(X_1, X_2)$ et $F_2 = F_2(X_3, X_4)$ on associe les deux fonctions de comptage suivantes :

1. Pour $B \geq 2$, $\mathcal{N}_{F_1,F_2}(B)$ est le cardinal de l'ensemble
$$\Big\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid \max_{i=1,2,3,4} |x_i| \leq B, F_1(x_1, x_2) = F_2(x_3, x_4)\Big\}.$$
2. Pour $N \geq 2$, $R_{F_1,F_2}(N)$ est le cardinal de l'ensemble

$$\Big\{n \,\Big|\, |n| \leq N, n = F_1(x_1, x_2) = F_2(x_3, x_4), \text{pour certains } (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4\Big\}.$$

Généralisant la définition (10), on dit que F_1 et F_2 sont *isomorphes* s'il existe $U \in \mathrm{Gl}(2, \mathbb{Q})$ tel que

$$F_1(X_1, X_2) = F_2(u_1X_1 + u_2X_2, u_3X_1 + u_4X_2). \tag{15}$$

En particulier deux formes isomorphes ont même degré. Nous prouverons au §3.1 le

Théorème 3.1. — *Soient F_1 et F_2 deux formes non isomorphes de même degré $d \geq 3$. On suppose de plus que les discriminants de F_1 et de F_2 sont non nuls et qu'au moins une des formes F_i n'est pas divisible par une forme linéaire non nulle à coefficients rationnels. Alors, pour tout $\varepsilon > 0$ on a la majoration*

$$\mathcal{N}_{F_1,F_2}(B) = O(B^{d\eta_d+\varepsilon}), \tag{16}$$

où η_d est défini en (5).

Remarque 3.2. — La condition que l'une des formes F_1 et F_2 ne contient pas de facteur linéaire sur $\mathbb{Q}$ est importante. Considérons les deux formes

$$F_1(X_1, X_2) = X_1(X_1^2 + X_2^2)$$

et

$$F_2(X_3, X_4) = X_3(X_3^2 + 2X_4^2).$$

Elles ne sont pas isomorphes. L'égalité $F_1(0, x_2) = F_2(0, x_4) = 0$ implique l'inégalité $\mathcal{N}_{F_1,F_2}(B) \gg B^2$, ce qui est supérieur à la partie droite de (16).

Du théorème 3.1 nous déduirons le

Corollaire 3.3. — *Soient F_1 et F_2 deux formes vérifiant les hypothèses du théorème 3.1. On suppose de plus que les deux formes F_1 et F_2 sont définies positives. Alors pour tout $\varepsilon > 0$ on a l'inégalité*

$$R_{F_1,F_2}(N) \ll N^{\eta_d+\varepsilon}.$$

Démonstration du corollaire 3.3. — Les hypothèses impliquent les inégalités

$$|\, F_1(x_1, x_2)\,| \gg \max(|x_1^d|, |x_2^d|) \text{ et } |\, F_2(x_3, x_4)\,| \gg \max(|x_3^d|, |x_4^d|),$$

uniformément pour $x_i \in \mathbb{R}$. Par conséquent, si on a

$$-N \leq F_1(x_1, x_2) = F_2(x_3, x_4) \leq N,$$

on a alors

$$\max_{i=1,\,2,\,3,\,4} |x_i| \ll N^{\frac{1}{d}}.$$

Il suffit alors de remplacer B par $O(N^{\frac{1}{d}})$ dans la majoration (16). □

3.1. Preuve du théorème 3.1. — Puisque les formes F_1 et F_2 sont de discriminants non nuls, l'hypersurface $\mathbb{X}$ de $\mathbb{P}^3(\mathbb{C})$ définie par l'équation

$$\mathbb{X} : F_1(X_1, X_2) - F_2(X_3, X_4) = 0 \tag{17}$$

est lisse. Nous inspirant de [9], nous décomposons $\mathcal{N}_{F_1,F_2}(B)$ en

$$\mathcal{N}_{F_1,F_2}(B) = \mathcal{N}^{(1)}_{F_1,F_2}(B) + \mathcal{N}^{(2)}_{F_1,F_2}(B) + 1, \tag{18}$$

où

- $\mathcal{N}^{(1)}_{F_1,F_2}(B)$ est le nombre de quadruplets non nuls (x_1, x_2, x_3, x_4) de $\mathbb{Z}^4$, vérifiant $\max |x_i| \leq B$, et tels que le point projectif associé $(x_1 : x_2 : x_3 : x_4)$ appartienne à $\mathbb{X}$, mais ne se situe pas sur une droite (complexe) contenue dans $\mathbb{X}$,
- $\mathcal{N}^{(2)}_{F_1,F_2}(B)$ est le nombre de quadruplets non nuls (x_1, x_2, x_3, x_4) de $\mathbb{Z}^4$, vérifiant $\max |x_i| \leq B$, et tels que le point projectif associé $(x_1 : x_2 : x_3 : x_4)$ appartienne à une droite (complexe) contenue dans $\mathbb{X}$.

Nous prouverons d'abord la

Proposition 3.4. — *Pour tout $\varepsilon > 0$ et uniformément pour $B \geq 1$, on a l'inégalité*

$$\mathcal{N}^{(1)}_{F_1,F_2}(B) \ll B^{d\eta_d+\varepsilon}.$$

Puis la

Proposition 3.5. — *Uniformément pour $B \geq 1$, on a l'inégalité*

$$\mathcal{N}^{(2)}_{F_1,F_2}(B) \ll B.$$

En combinant ces deux propositions et la formule (18) on complète la preuve du théorème 3.1.

3.1.1. *Preuve de la Proposition 3.4.* — Si $\boldsymbol{x} = (x_1 : x_2 : x_3, : x_4)$ est un point de $\mathbb{P}^3(\mathbb{Q})$, on désigne par $h(\boldsymbol{x})$ la hauteur de $\boldsymbol{x}$, c'est–à–dire le maximum des $|x_i|$, si (x_1, x_2, x_3, x_4) est un quadruplet d'entiers premiers entre eux dans leur ensemble, représentant $\boldsymbol{x}$. Notons aussi $N^{(1)}(\mathbb{X}, B)$ le cardinal de l'ensemble des $\boldsymbol{x}$ de $\mathbb{P}^3(\mathbb{Q})$ appartenant à $\mathbb{X}$ mais non situés sur une droite contenue dans $\mathbb{X}$ et de hauteur $h(\boldsymbol{x}) \leq B$. En décomposant suivant la valeur δ du pgcd de (x_1, x_2, x_3, x_4) on déduit l'inégalité

$$\mathcal{N}^{(1)}_{F_1,F_2}(B) \leq \sum_{1\leq\delta\leq B} N^{(1)}(\mathbb{X}, B/\delta). \tag{19}$$

Un résultat de Salberger [7, Theorem 0.1], donne l'inégalité

$$N^{(1)}(\mathbb{Y}, B) \ll B^{d\eta_d+\varepsilon}$$

valable pour toute surface projective $\mathbb{Y} \subset \mathbb{P}^n(\mathbb{Q})$, géométriquement intègre de degré $d > 2$. La surface $\mathbb{X}$ vérifie cette propriété. En effet supposons qu'il existe des polynômes A et B de degré ≥ 1 tels que

$$F_1(X_1, X_2) - F_2(X_3, X_4) = A(X_1, X_2, X_3, X_4)B(X_1, X_2, X_3, X_4).$$

Calculant les dérivées partielles par rapport à chacun des X_i, on voit que tout point $(x_1 : x_2 : x_3 : x_4)$ de $\mathbb{P}^3(\mathbb{C})$ tel que

$$A(x_1, x_2, x_3, x_4) = B(x_1, x_2, x_3, x_4) = 0,$$

est un point singulier de $\mathbb{X}$. Puisqu'on est dans $\mathbb{P}^3(\mathbb{C})$, les deux surfaces d'équation $A = 0$ et $B = 0$ ont une intersection non vide. Ainsi $\mathbb{X}$ serait singulière, ce qui contredit la propriété de lissité énoncée au §3.1.

Il suffit de sommer sur $\delta < B$ l'inégalité (19) pour compléter la preuve de la Proposition 3.4.

3.1.2. *Droites contenues dans* $\mathbb{X}$. — Afin de démontrer la Proposition 3.5, nous donnons des conditions nécessaires pour qu'une droite projective de $\mathbb{P}^3(\mathbb{C})$ appartienne à $\mathbb{X}$ en exploitant le fait que dans l'équation (17) définissant $\mathbb{X}$, les paires de variables (X_1, X_2) et (X_3, X_4) sont séparées.

Lemme 3.6. — *Sous les hypothèses du théorème 3.1, si une droite projective de* $\mathbb{P}^3(\mathbb{C})$ *appartient à l'hypersurface* $\mathbb{X}$ *définie par* (17) *et contient un point rationnel, elle est définie par des équations*

$$\begin{cases} X_1 = u_1X_3 + u_2X_4 \\ X_2 = u_3X_3 + u_4X_4, \end{cases} \tag{20}$$

où l'un au moins des u_i *est irrationnel.*

Démonstration. — Pour $\mathbf{a} = (a_1, a_2, a_3, a_4)$ et $\mathbf{b} = (b_1, b_2, b_3, b_4)$ deux quadruplets non nuls et non proportionnels de nombres complexes, on suppose que la droite projective $\mathbb{D}_{\mathbf{a},\mathbf{b}}$ de $\mathbb{P}^3(\mathbb{C})$ définie par les équations

$$\mathbb{D}_{\mathbf{a},\mathbf{b}} \ : \ \begin{cases} a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = 0 \\ b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 = 0 \end{cases} \tag{21}$$

est contenue dans $\mathbb{X}$ et contient un point rationnel.

- Si $a_1 = a_2 = 0$ et $a_3 \neq 0$, on substitue $X_3 = -(a_4/a_3)X_4$ dans la deuxième équation de (21) qui devient ainsi

$$b_1X_1 + b_2X_2 = \frac{a_4b_3 - a_3b_4}{a_3}X_4. \tag{22}$$

  - Si $a_4b_3 - a_3b_4 = 0$, cela signifie que $\mathbb{X}$ contient une droite de la forme

$$\begin{cases} a_3X_3 + a_4X_4 = 0 \\ b_1X_1 + b_2X_2 = 0. \end{cases} \tag{23}$$

Exploitant la forme particulière de l'équation (17) définissant $\mathbb{X}$, on déduit

$$b_1X_1 + b_2X_2 \text{ divise } F_1(X_1, X_2) \text{ et } a_3X_3 + a_4X_4 \text{ divise } F_2(X_3, X_4).$$

Compte tenu de l'hypothèse sur les F_i, ces conditions de divisibilité contredisent l'hypothèse qu'il y a un point rationnel sur la droite définie par (23).

– Si $a_4b_3 - a_3b_4 \neq 0$, on remplace X_3 par la valeur donnée par la première équation de (21) et X_4 par la valeur donnée en (22) conduisant à

$$F_1(X_1, X_2) = F_2(X_3, X_4) = (a_4b_3 - a_3b_4)^{-d} F_2(-a_4, a_3)(b_1X_1 + b_2X_2)^d,$$

ce qui contredit l'hypothèse que le discriminant de F_1 est non nul.

- Si $a_1 = a_2 = 0$ et $a_4 \neq 0$, par le même type de raisonnement suivi précédemment, on est ramené au cas où $(a_1, a_2) \neq (0, 0)$.
- Par symétrie, on suit le même raisonnement dans les trois cas suivants : si $a_3 = a_4 = 0$, si $b_1 = b_2 = 0$ ou si $b_3 = b_4 = 0$.
- En conclusion de la discussion précédente, nous avons prouvé que si $\mathbb{D}_{\mathbf{a},\mathbf{b}}$, contenue dans $\mathbb{X}$, possède un point rationnel, on a nécessairement

$$(a_1, a_2),\ (a_3, a_4),\ (b_1, b_2),\ \text{et } (b_3,\ b_4) \text{ sont } \neq (0, 0). \tag{24}$$

- Supposons maintenant $a_1b_2 = a_2b_1$ et $a_1 \neq 0$. Multipliant la première équation de (21) par $-b_1$ et la seconde par a_1, on obtient que dans ce cas le système d'équations définissant $\mathbb{D}_{\mathbf{a},\mathbf{b}}$ est équivalent à

$$\begin{cases} a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = 0 \\ (a_1b_3 - a_3b_1)X_3 + (a_1b_4 - a_4b_1)X_4 = 0. \end{cases}$$

Or (24) a éliminé le cas $(b_1, b_2) = (0, 0)$. On en déduit que l'hypothèse $a_1b_2 = a_2b_1$ et $a_1 \neq 0$ est impossible.

- Supposons maintenant $a_1b_2 = a_2b_1$ et $a_2 \neq 0$. Mais ce cas est impossible par un raisonnement identique. Puisque par (24) le cas $(a_1, a_2) = (0, 0)$ est interdit, on est ramené à supposer que $a_1b_2 \neq a_2b_1$.
- Pour finir on suppose donc que $a_1b_2 \neq a_2b_1$. Par résolution d'un système (2,2) en les inconnues X_1 et X_2 et de déterminant non nul, on voit que le système (21) est équivalent à

$$\begin{cases} X_1 = u_1X_3 + u_2X_4 \\ X_2 = u_3X_3 + u_4X_4, \end{cases}$$

où les u_i sont des nombres complexes. On a donc l'égalité

$$F_2(X_3, X_4) = F_1(u_1X_3 + u_2X_4, u_3X_3 + u_4X_4). \tag{25}$$

– Si $\det\left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) = 0$, cela signifie que par exemple, on a, pour un certain λ complexe, l'égalité $u_1X_3 + u_2X_4 = \lambda(u_3X_3 + u_4X_4)$ donc en reportant, on déduit l'égalité
$$F_2(X_3, X_4) = (u_3X_3 + u_4X_4)^d F_1(\lambda, 1),$$
ce qui contredit l'hypothèse de non nullité du discriminant de F_2.

– Si $\det\left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) \neq 0$, par (25) on voit que les formes F_1 et F_2 sont isomorphes par un changement de variables linéaire à coefficients complexes. L'hypothèse de non isomorphisme, sur $\mathrm{Gl}(2, \mathbb{Q})$, de F_1 et F_2 implique que parmi les u_i l'un au moins est irrationnel.

Ceci termine la démonstration du lemme 3.6. □

3.1.3. *Preuve de la proposition 3.5.* — On sait que pour tout $d \geq 3$, il existe un entier $\ell(d)$, tel que, toute surface lisse de $\mathbb{P}^3(\mathbb{C})$ de degré d contient au plus $\ell(d)$ droites. Pour des études fines concernant cette constante $\ell(d)$ on se reportera à [8] et à [3] par exemple.

Grâce au lemme 3.6, on est ramené à dénombrer l'ensemble des quadruplets d'entiers $(x_1, \ldots, x_4)$ avec $\max |x_i| \leq B$, vérifiant
$$\begin{cases} x_1 = u_1x_3 + u_2x_4 \\ x_2 = u_3x_3 + u_4x_4, \end{cases}$$
sachant que l'un au moins des u_i est irrationnel. Disons que c'est u_1.

- si $\dim_{\mathbb{Q}}(1, u_1, u_2) = 3$, la seule solution en $(x_1, x_3, x_4) \in \mathbb{Z}^3$ de l'équation $x_1 = u_1x_3 + u_2x_4$ est $(0, 0, 0)$,
- si $\dim_{\mathbb{Q}}(1, u_1, u_2) = 2$, on exprime $u_2 = a + bu_1$, avec a et b rationnels et on est ramené à l'équation
$$x_1 - ax_4 = (x_3 + bx_4)u_1,$$
qui implique $x_1 - ax_4 = x_3 + bx_4 = 0$. Donc le système (20) admet $O(B)$ quadruplets $(x_1, \ldots, x_4)$ solutions de hauteur inférieure à B.

Ceci termine la preuve de la proposition 3.5.

4. Quelques propriétés des formes cyclotomiques

Dans ce paragraphe nous prouvons quelques résultats généraux concernant les formes cyclotomiques $\Phi_n(X, Y)$ dont le degré est $d = \varphi(n)$. Ces divers résultats seront nécessaires lors de la preuve des théorèmes 1.1, 1.4, 1.5 et 1.6. En particulier les résultats des sections §4.1, §4.2 et §4.3. ne seront utilisés que pour la preuve du Théorème 1.6. Nous rappelons d'abord plusieurs formules classiques sur les $\Phi_n(X, Y)$. Ces formules ne sont que la version homogène des formules correspondantes sur les $\phi_n(x)$.

Nous rappelons certaines notations et conventions : si $n \geq 1$ est un entier, on désigne par $\mu(n)$ la valeur de la fonction de Möbius, $\omega(n)$ est le nombre de facteurs premiers distincts de n, $\kappa(n)$ est le *radical* de n, c'est–à–dire le produit de tous les premiers divisant n, $d(n)$ le nombre de diviseurs. On dit que deux nombres rationnels u et v sont congrus modulo le nombre premier p si on a $u - v \in p\mathbb{Z}_p$, où $\mathbb{Z}_p$ est l'anneau des entiers p–adiques.

Si $n \geq 2$ est factorisé en $n := p^r m$, avec $p \nmid m$ et $r \geq 1$, la forme Φ_n vérifie les identités suivantes

$$\Phi_n(X,Y) = \prod_{d|n}(X^d - Y^d)^{\mu(n/d)},$$

$$\Phi_n(X,Y) = \frac{\Phi_m(X^{p^r}, Y^{p^r})}{\Phi_m(X^{p^{r-1}}, Y^{p^{r-1}})}, \tag{26}$$

et

$$\Phi_n(X,Y) = \Phi_{pm}(X^{p^{r-1}}, Y^{p^{r-1}}).$$

Par itération de cette dernière formule, on parvient à

$$\Phi_n(X,Y) = \Phi_{\kappa(n)}(X^{n/\kappa(n)}, Y^{n/\kappa(n)}). \tag{27}$$

Nous rappelons quelques valeurs de ϕ_n en certains points :

$$\phi_n(1) = \begin{cases} 0 & \text{si } n = 1, \\ p & \text{si } n = p^k \ (k \geq 1), \\ 1 & \text{si } \omega(n) \geq 2, \end{cases} \tag{28}$$

et

$$\phi_n(-1) = \begin{cases} -2 & \text{si } n = 1, \\ \phi_{n/2}(1) & \text{si } n \geq 2,\ 2\|n, \\ 1 & \text{si } n \geq 3,\ 2 \nmid n, \\ 1 & \text{si } n \geq 4,\ 4|n,\ n \neq 2^\ell, \\ 2 & \text{si } n \geq 4,\ n = 2^\ell. \end{cases} \tag{29}$$

Pour majorer les coefficients de Φ_n nous utiliserons le résultat suivant, dû à P. Bateman [1, p.1181]. Quand P est un polynôme, nous désignons par $\mathrm{L}(P)$ (*longueur de* P) la somme des valeurs absolues des coefficients de P.

Lemme 4.1. — *Pour tout* $n \geq 1$, *on a*

$$\mathrm{L}(\phi_n) \leq n^{d(n)/2}.$$

Les majorations classiques des fonctions arithmétiques $d(n)$ et $\varphi(n)$ impliquent alors que pour tout $\varepsilon > 0$ et pour n suffisamment grand, on a

$$\varphi(n)\mathrm{L}(\phi_n) \leq e^{n^\varepsilon}. \tag{30}$$

4.1. Propriétés de confinement modulo p. — Nous prouvons que pour tout a et b entiers $\Phi_n(a,b)$ est, pour tout m divisant n, restreint à quelques classes de congruence modulo m.

Proposition 4.2. — *Soient $n \geq 2$ et p un premier divisant n. Alors pour tout a et b de $\mathbb{Z}$ on a*

$$\Phi_n(a,b) \equiv 0,\ 1 \bmod p. \tag{31}$$

Démonstration. — Remarquons d'abord que lorsque $p = 2$ ou lorsque $b \equiv 0 \bmod p$, l'énoncé précédent est trivial. Enfin on peut se restreindre au cas

$$n \text{ sans facteur carré.}$$

C'est une conséquence de l'inclusion des images $\Phi_n(\mathbb{Z},\mathbb{Z}) \subset \Phi_{\kappa(n)}(\mathbb{Z},\mathbb{Z})$, qui se déduit directement de (27). Commençons par le cas où n est un nombre premier. On a

Lemme 4.3. — *Soient $p \geq 3$ un nombre premier, a et b deux entiers. Alors on a les congruences*

1. *Si $a \not\equiv b \bmod p$, on a*
$$\Phi_p(a,b) \equiv 1 \bmod p,$$
2. *Si $a \equiv b \bmod p$, on a*
$$\Phi_p(a,b) \equiv pa^{p-1} \bmod p^2.$$

Démonstration du lemme 4.3. — C'est une conséquence de la formule $a^p \equiv a \bmod p$. Dans le premier cas, on écrit

$$\Phi_p(a,b) = (a^p - b^p)/(a-b) \equiv (a-b)/(a-b) \equiv 1 \bmod p.$$

Dans le second cas, on écrit

$$\Phi_p(a,b) = a^{p-1} + a^{p-2}b + \cdots + ab^{p-2} + b^{p-1},$$

on pose $b = a + pt$ avec t entier et on développe suivant la formule du binôme pour obtenir le résultat. □

Poursuivons la démonstration de la proposition 4.2. On suppose donc que n est sans facteur carré et on pose

$$n = pm = pp_2 \cdots p_t.$$

Par itération de (26), on a l'égalité

$$\Phi_n(a,b) = \prod_{m_1 \mid m} \Phi_p(a^{m_1}, b^{m_1})^{\mu(m/m_1)}. \tag{32}$$

Ceci nous amène à décomposer le produit à droite de l'égalité (32) en

$$\Phi_n(a,b) = \Phi_n^{\dagger}(a,b)\tilde{\Phi}_n(a,b),$$

où $\Phi_n^\dagger$ correspond à la condition $a^{m_1} \not\equiv b^{m_1} \bmod p$ et $\tilde{\Phi}$ le produit complémentaire. Par le lemme 4.3, on a $\Phi_p(a^{m_1}, b^{m_1}) \equiv 1 \bmod p$ si et seulement si $a^{m_1} \not\equiv b^{m_1} \bmod\ p$. Ceci implique que $\Phi_n^\dagger(a,b)$ est un nombre rationnel qui est produit et quotient d'entiers congrus à 1 mod p. C'est donc un nombre rationnel congru à 1 mod p.

Par définition, on a l'égalité

$$\tilde{\Phi}_n(a,b) := \prod_{\substack{m_1 \mid m \\ (a/b)^{m_1} \equiv 1 \bmod p}} \Phi_p(a^{m_1}, b^{m_1})^{\mu(m/m_1)}. \tag{33}$$

D'après ce qui précède, pour compléter la preuve de la congruence (31), il reste à prouver que $\tilde{\Phi}_n(a,b)$ est un rationnel congru à 0 ou 1 mod p.

Soit ℓ l'ordre de (a/b) modulo p. Ainsi ℓ divise $p-1$ et le produit apparaissant dans (33) est sur les m_1 tels que

$$\ell \text{ divise } m_1 \text{ et } m_1 \text{ divise } m.$$

Ce produit est vide lorsque $\ell \nmid m$; c'est par exemple le cas si $a \not\equiv b \bmod p$ et si $(p-1, m) = 1$. On suppose donc que $\ell \mid m$. Quitte à réindicer les p_j, on peut supposer que ℓ est de la forme

$$\ell = p_2 \cdots p_r,$$

avec $1 \le r \le t$, avec la convention que $\ell = 1$ si $r = 1$. Posant alors $h = p_{r+1} \cdots p_t = m/\ell$ et $m_1 = \ell m_2$, on récrit la définition (33) comme

$$\tilde{\Phi}_n(a,b) := \prod_{m_2 \mid h} \Phi_p(a^{\ell m_2}, b^{\ell m_2})^{\mu(h/m_2)}. \tag{34}$$

On articule la discussion suivant plusieurs cas.

- si $h = 1$, le produit apparaissant dans (34) ne contient qu'un seul terme à savoir $\Phi_p(a^\ell, b^\ell)$. Il est congru à 0 mod p, d'après le lemme 4.3.2.
- si $h \neq 1$, en utilisant la formule de Möbius, on écrit (34) sous la forme

  $$\tilde{\Phi}_n(a,b) = \prod_{m_2 \mid h} \Big(\frac{\Phi_p(a^{\ell m_2}, b^{\ell m_2})}{p} \Big)^{\mu(h/m_2)},$$

  où maintenant chaque fraction du produit est congrue à $a^{\ell m_2 (p-1)}$ modulo p (lemme 4.3.2). Ainsi chacune de ces fractions est un entier premier à p. Le nombre rationnel $\tilde{\Phi}_n(a,b)$ vérifie donc

  $$\tilde{\Phi}_n(a,b) \equiv a^{\ell m_2 (p-1) \sum_{m_2 \mid h} \mu(h/m_2)} \equiv 1 \bmod p,$$

  en utilisant de nouveau la formule de Möbius. Ceci termine la preuve de la proposition 4.2. □

4.2. Propriétés de confinement modulo 9. — Par la proposition 4.2, on sait que si $3 \mid n$, on a $\Phi_n(a,b) \equiv 0$, $1 \bmod 3$. Ce n'est pas assez satisfaisant pour la future application. Nous utiliserons une version plus précise avec la

PROPOSITION 4.4. — *Soit $k \geq 2$. Alors pour tout a et tout b entiers, on a la congruence*

$$\Phi_{3^k}(a,b) \equiv 0,\ 1,\ 3 \bmod 9.$$

Démonstration. — Les cubes modulo 9 forment l'ensemble

$$\mathfrak{Q}(9) := \{0,\ 1,\ -1\}.$$

Or

$$\Phi_3(u,v) = u^2 + uv + v^2.$$

Si u et v parcourent l'ensemble $\mathfrak{Q}(9)$ on voit que $\Phi_3(u,v)$ parcourt l'ensemble

$$\mathcal{E} = \{0,1,3 \bmod 9\}.$$

Pour terminer la preuve de la proposition, il suffit d'utiliser le fait que

$$\Phi_{3^k}(a,b) = \Phi_3(a^{3^{k-1}}, b^{3^{k-1}}),$$

conséquence de (27). □

4.3. Propriétés de confinement modulo 4. Nous envisageons maintenant le cas où n est pair. Par la remarque (2), on peut même supposer que $4 \mid n$ et on écrit que $2^k \| n$ avec $k \geq 2$.

1. Soit a pair et b impair. Par application itérée de (26), on voit $\Phi_n(X,Y)$ est produit et quotient de polynômes de la forme $\Phi_{2^k}(X^\alpha, Y^\alpha)$ où α est un nombre impair. Puisque $\Phi_{2^k}(X,Y) = X^{2^{k-1}} + Y^{2^{k-1}}$ et $k \geq 2$, on déduit que $\Phi_{2^k}(a^\alpha, b^\alpha) \equiv 0 + 1 \equiv 1 \bmod 4$ et, par conséquent, que
$$\Phi_n(a,b) \equiv 1 \bmod 4.$$
2. Soit a et b pairs. Puisque $\Phi_n(X,Y)$ est somme de monômes de la forme $c_{\mu,\nu}X^\mu Y^\nu$ avec $\mu+\nu = \varphi(n)$ et $c_{\mu,\nu}$ entier, on voit que $4 \mid c_{\mu,\nu}a^\mu b^\nu$, d'où
$$\Phi_n(a,b) \equiv 0 \bmod 4.$$
3. Supposons $a \equiv b \equiv 1 \bmod 4$. On écrit $\Phi_n(a,b) = b^{\varphi(n)}\phi_n(a/b) \equiv \phi_n(1) \bmod 4$. Et, d'après (28), ceci vaut $2 \bmod 4$ si $n = 2^k$ avec $k \geq 2$ ou $1 \bmod 4$ si $n \neq 2^k$. D'où
$$\Phi_n(a,b) \equiv 1,\ 2 \bmod 4.$$
4. Supposons $a \equiv -b \equiv 1 \bmod 4$. On écrit $\Phi_n(a,b) = b^{\varphi(n)}\phi_n(a/b) \equiv \phi_n(-1) \bmod 4$. Il suffit d'appliquer les deux dernières lignes de (29) pour conclure que l'on a, dans ce cas
$$\Phi_n(a,b) \equiv 1,\ 2 \bmod 4.$$
5. Supposons $a \equiv -1 \bmod 4$. On utilise la relation $\Phi_n(a,b) = \Phi_n(-a,-b)$ valable pour $n \geq 3$.

Nous rassemblons ces divers résultats sous la forme de la

Proposition 4.5. — *Soit n un entier divisible par 4. Alors pour tout a et tout b entiers, on a la congruence*

$$\Phi_n(a,b) \equiv 0,\ 1,\ 2 \bmod 4.$$

4.4. Automorphismes des formes cyclotomiques. — Soit $\Phi_n(X,Y)$ une forme cyclotomique de degré $d=\varphi(n)$. Par la définition (10), rechercher les automorphismes de Φ_n consiste à rechercher les matrices U de $\mathrm{Gl}(2,\mathbb{Q})$

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix},$$

telles que

$$\Phi_n(X,Y) = \Phi_n(u_1X+u_2Y, u_3X+u_4Y). \tag{35}$$

Cette égalité formelle entraîne que l'ensemble $\mathbb{U}_n$ des racines primitives n–ièmes de l'unité est stable par l'application $\mathcal{H}$ de $\widehat{\mathbb{C}}$ dans $\widehat{\mathbb{C}}$ définie par

$$z \mapsto \mathcal{H}(z) := \frac{u_1z+u_2}{u_3z+u_4}. \tag{36}$$

Si $\mathbb{U}_n$ a au moins trois éléments (c'est–à–dire $n \geq 5$ et $n \neq 6$), il y a un cercle et un seul contenant $\mathbb{U}_n$. Il s'agit du cercle $\mathbb{S}^1$ et celui–ci est stable par $\mathcal{H}$. Les transformations de $\widehat{\mathbb{C}}$ de la forme $(az+b)/(cz+d)$ (avec $(a,b,c,d) \in \mathbb{C}^4$ et $ad-bc \neq 0$) laissant $\mathbb{S}^1$ globalement invariant sont connues : il s'agit des transformations $z \mapsto \rho z$ et $z \mapsto \rho/z$ avec ρ nombre complexe de module 1. Ainsi la fonction $\mathcal{H}$ définie en (36) a nécessairement une des quatre formes

$$\mathcal{H}(z) = z,\ -z,\ 1/z,\ -1/z,$$

puisque les u_i sont des rationnels. Enfin pour tout $n \geq 1$, on a l'équivalence

$$\xi \in \mathbb{U}_n \iff 1/\xi \in \mathbb{U}_n$$

et seulement pour $n \equiv 0 \bmod 4$, l'équivalence

$$\xi \in \mathbb{U}_n \iff -\xi \in \mathbb{U}_n.$$

Nous voyons donc que si

1. si $n \equiv 0 \bmod 4$, et $n \geq 8$, on a $\frac{u_1z+u_2}{u_3z+u_4} = z,\ -z,\ \frac{1}{z},\ -\frac{1}{z}$,
2. si $n \not\equiv 0 \bmod 4$, $n \geq 5$ et $n \neq 6$, on a $\frac{u_1z+u_2}{u_3z+u_4} = z,\ -z$.

Revenant à la définition (35) et rappelant que Φ_n est une forme homogène de degré $\varphi(n)$ on obtient les matrices U :

1. Pour $n \equiv 0 \bmod 4$, et $n \geq 8$, on a

$$U = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$$

2. Pour $n \not\equiv 0 \bmod 4$, $n \geq 5$ et $n \neq 6$, on a
$$U = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Les petites valeurs de n (celles vérifiant $\varphi(n) = 2$) se font directement grâce aux formes explicites données en (7). En notant $\mathbb{D}_k$ le groupe diédral à $2k$ éléments, on obtient la

Proposition 4.6. — *Soit $n \geq 3$ un entier. Alors le groupe des automorphismes* $\mathrm{Aut}\Phi_n$ *de* $\Phi_n(X,Y)$ *est*
$$\mathrm{Aut}\,\Phi_n = \begin{cases} \mathbb{D}_4 & \textit{si } 4 \textit{ divise } n, \\ \mathbb{D}_2 & \textit{dans le cas contraire.} \end{cases}$$

Nous en déduisons :

Corollaire 4.7. — *Pour $n \geq 3$, on a $W_{\Phi_n} = w_n$ où w_n est défini par* (9).

Démonstration. — Les groupes d'automorphismes des formes cyclotomiques sont constitués de matrices à coefficients entiers. Ainsi, dans les définitions (11) et (12), quand F est une forme cyclotomique, les réseaux Λ et Λ_i sont égaux à $\mathbb{Z}^2$, leur déterminant vaut 1. ☐

4.5. Isomorphismes entre formes cyclotomiques. — Rappelons que la définition d'isomorphisme entre deux formes binaires a été donnée en (15). La proposition suivante caractérise les formes binaires cyclotomiques isomorphes.

Proposition 4.8. — *Soient n_1 et n_2 deux entiers positifs avec $n_1 < n_2$. Les conditions suivantes sont équivalentes.*

(1) *On a $\varphi(n_1) = \varphi(n_2)$ et les deux formes binaires cyclotomiques Φ_{n_1} et Φ_{n_2} sont isomorphes.*
(2) *Les deux formes binaires cyclotomiques Φ_{n_1} et Φ_{n_2} représentent les mêmes entiers.*
(3) *n_1 est impair et $n_2 = 2n_1$.*

Les formes binaires cyclotomiques $\Phi_n(X,Y)$ avec $\varphi(n) = d$ et n non congru à 2 modulo 4 forment donc un système complet de représentants des classes d'isomorphisme des formes binaires cyclotomiques de degré d.

La démonstration de la proposition 4.8 utilisera le lemme suivant :

Lemme 4.9. — *Soit n un entier positif. Le groupe de torsion du corps cyclotomique $\mathbb{Q}(\zeta_n)$ est cyclique, d'ordre n si n est pair, d'ordre $2n$ si n est impair.*

Démonstration du lemme 4.9. — Le groupe de torsion du corps cyclotomique $\mathbb{Q}(\zeta_n)$ est cyclique, d'ordre multiple de n. S'il est d'ordre supérieur à n, alors il contient une racine primitive de l'unité d'ordre pn, avec p premier, dont le degré est $\varphi(pn)$. On en déduit $\varphi(pn) = \varphi(n)$, d'où il résulte que $p = 2$ et que n est impair. ☐

Démonstration de la proposition 4.8. — (1) $\Longrightarrow$ (3) Supposons Φ_{n_1} et Φ_{n_2} isomorphes avec $n_1 < n_2$. Il existe une matrice

$$\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$$

telle que

$$\zeta_{n_1} = \frac{u_1\zeta_{n_2} + u_2}{u_3\zeta_{n_2} + u_4}.$$

On en déduit que les corps cyclotomiques $\mathbb{Q}(\zeta_{n_1})$ et $\mathbb{Q}(\zeta_{n_2})$ coïncident, et le lemme 4.9 donne le résultat.

(3) $\Longrightarrow$ (2). Si n_1 est impair et $n_2 = 2n_1$, alors $\phi_{n_2}(t) = \phi_{n_1}(-t)$, donc les deux formes binaires cyclotomiques Φ_{n_1} et Φ_{n_2} représentent les mêmes entiers.

(2) $\Longrightarrow$ (1). En utilisant les notations du théorème 2.1 et du corollaire 3.3, nous avons, par hypothèse les égalités

$$R_{\Phi_1}(N) = R_{\Phi_2}(N) = R_{\Phi_1,\Phi_2}(N), \tag{37}$$

pour tout $N \geq 1$. Par le théorème 2.1 la première égalité de (37) implique $\varphi(n_1) = \varphi(n_2)$. Enfin si Φ_{n_1} et Φ_{n_2} n'étaient pas isomorphes, le corollaire 3.3 entraînerait que la deuxième égalité de (37) serait impossible pour N suffisamment grand. □

4.6. Résultats auxiliaires de comptage. — Notre démonstration du théorème 1.1 au §5 utilisera l'énoncé suivant [6, Theorem 1.1].

Théorème 4.10. — *Soit m un entier positif et soient n, x, y des entiers rationnels vérifiant $n \geq 3$, $\max\{|x|, |y|\} \geq 2$ et $\Phi_n(x, y) = m$. Alors*

$$\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} m^{\frac{1}{\varphi(n)}} \quad \textit{et par conséquent,} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

Nous en déduisons le

Corollaire 4.11. — *Pour $d \geq 2$ et $N \geq 1$, on a la majoration*

$$\mathcal{A}_d(N) \leq 29 N^{\frac{2}{d}} (\log N)^{1.161}.$$

Démonstration du corollaire 4.11. — D'après le théorème 4.10, les conditions $\varphi(n) \geq d$ et $\Phi_n(x, y) \leq N$ impliquent $\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} N^{\frac{1}{d}}$. Notons que $\mathcal{A}_d(N) = 0$ pour $N = 1$ et $N = 2$. La condition $\max\{|x|, |y|\} \geq 2$ permet d'obtenir $\varphi(n) \leq \frac{2}{\log 3} \log N$, donc $n \leq 5.383(\log N)^{1.161}$ (formule (1.1) de [6]). Il en résulte que le nombre de triplets (n, x, y) tels que $\varphi(n) \geq d$ et $\Phi_n(x, y) \leq N$ est majoré par

$$\frac{16}{3} 5.383 N^{\frac{2}{d}} (\log N)^{1.161}. \qquad \square$$

5. Démonstration des théorèmes 1.1 et 1.4

Pour n entier avec $\varphi(n) \geq 4$ et $N \geq 1$, on désigne par $\mathcal{B}_n(N)$ l'ensemble

$$\mathcal{B}_n(N) := \big\{m \leq N \mid m = \Phi_n(a,b) \text{ avec } \max(|a|,|b|) \geq 2\big\}. \tag{38}$$

Par le théorème 4.10, on a l'implication

$$\mathcal{B}_n(N) \neq \emptyset \Rightarrow \varphi(n) \ll \log N,$$

soit encore

$$\mathcal{B}_n(N) \neq \emptyset \Rightarrow n \ll \log N \log\log\log N,$$

uniformément pour $N > 10$. Ainsi, par la définition de $\mathcal{A}_d(N)$ et par la restriction (2), nous avons l'égalité

$$\mathcal{A}_d(N) = \Big|\bigcup_{\substack{n \not\equiv 2 \bmod 4 \\ \varphi(n) \geq d}} \mathcal{B}_n(N)\Big|, \tag{39}$$

où cette réunion porte sur un nombre fini de n.

Le terme principal dans l'estimation du cardinal de $\mathcal{A}_d(N)$ sera

$$\sum_{\substack{n \not\equiv 2 \bmod 4 \\ \varphi(n) = d}} \big|\mathcal{B}_n(N)\big|.$$

L'égalité $\big|\mathcal{B}_n(N)\big| = R_{\Phi_n}(N)$ permet d'appliquer le théorème 2.1 à chacun des termes de cette somme :

$$\big|\mathcal{B}_n(N)\big| = A_{\Phi_n} W_{\Phi_n} N^{\frac{2}{d}} + O_{\Phi_n,\varepsilon}(N^{\beta_d^*+\varepsilon}).$$

Grâce au corollaire 4.7, on a $W_{\Phi_n} = w_n$. On obtient

$$\sum_{\substack{n \not\equiv 2 \bmod 4 \\ \varphi(n) = d}} \big|\mathcal{B}_n(N)\big| = C_d N^{\frac{2}{d}} + O(N^{\beta_d^*+\varepsilon}) \tag{40}$$

avec la valeur de C_d annoncée dans la formule (8).

5.1. Minoration de $\mathcal{A}_d(N)$. — En restreignant le nombre de termes dans l'égalité (39), on a la minoration

$$\begin{aligned}\mathcal{A}_d(N) &\geq \Big|\bigcup_{\substack{n \not\equiv 2 \bmod 4 \\ \varphi(n) = d}} \mathcal{B}_n(N)\Big| \\ &\geq \sum_{\substack{n \not\equiv 2 \bmod 4 \\ \varphi(n) = d}} \big|\mathcal{B}_n(N)\big| - \sum\sum_{\substack{n_1 < n_2 \\ \varphi(n_1) = \varphi(n_2) = d \\ n_1,\ n_2 \not\equiv 2 \bmod 4}} \Big|\mathcal{B}_{n_1}(N) \cap \mathcal{B}_{n_2}(N)\Big|. \end{aligned} \tag{41}$$

La première partie du membre de droite de (41) est traitée dans (40).

Pour la seconde partie, on écrit l'égalité $\left|\mathcal{B}_{n_1}(N)\cap\mathcal{B}_{n_2}(N)\right| = R_{\Phi_{n_1},\Phi_{n_2}}(N)$. Par la proposition 4.8 les formes Φ_{n_1} et Φ_{n_2} ne sont pas isomorphes. Le corollaire 3.3 donne ainsi la majoration

$$\left|\mathcal{B}_{n_1}(N)\cap\mathcal{B}_{n_2}(N)\right| = O\big(N^{\eta_d+\varepsilon}\big).$$

En conclusion, nous avons prouvé la minoration suivante de $\mathcal{A}_d(N)$:

$$\mathcal{A}_d(N) \geq C_d N^{\frac{2}{d}} - O(N^{\beta_d^*+\varepsilon}) - O(N^{\eta_d+\varepsilon}),$$

qui se simplifie en

$$\mathcal{A}_d(N) \geq C_d N^{\frac{2}{d}} - O(N^{\eta_d+\varepsilon}), \tag{42}$$

puisque, d'après (5) et (13), on a pour tout $d \geq 4$ pair, l'inégalité

$$\eta_d \geq \beta_d^*. \tag{43}$$

5.2. Majoration de $\mathcal{A}_d(N)$. — On écrit maintenant

$$\mathcal{A}_d(N) \leq \Bigg|\bigcup_{\substack{n\not\equiv 2 \bmod 4\\ \varphi(n)=d}} \mathcal{B}_n(N)\Bigg| + \mathcal{A}_{d^\dagger}(N),$$

dont on déduit la majoration

$$\mathcal{A}_d(N) \leq \sum_{\substack{n\not\equiv 2 \bmod 4\\ \varphi(n)=d}} \left|\mathcal{B}_n(N)\right| + \mathcal{A}_{d^\dagger}(N). \tag{44}$$

Le premier terme de cette majoration a déja été traité en (40). Pour majorer $\mathcal{A}_{d^\dagger}(N)$ on utilise le corollaire 4.11 avec d remplacé par $d^\dagger$. Revenant en (44), on a donc prouvé la majoration

$$\mathcal{A}_d(N) \leq C_d N^{\frac{2}{d}} + O(N^{\beta_d^*+\varepsilon}) + O\big(N^{\frac{2}{d^\dagger}}(\log N)^{1.161}\big). \tag{45}$$

Cette formule appliquée en remplaçant d par $d^\dagger$ donne la majoration

$$\mathcal{A}_{d^\dagger}(N) \ll N^{\frac{2}{d^\dagger}},$$

où il n'y a plus de puissance de $\log N$ parasite. Reportant cette dernière majoration dans (44), l'inégalité (45) est améliorée en

$$\mathcal{A}_d(N) \leq C_d N^{\frac{2}{d}} + O(N^{\beta_d^*+\varepsilon}) + O(N^{\frac{2}{d^\dagger}}). \tag{46}$$

En combinant (42), (46) et l'inégalité (43), on termine la preuve de (6). Les preuves des théorèmes 1.1 et 1.4 sont complètes.

6. Preuve du théorème 1.5

6.1. Région fondamentale d'une forme binaire définie positive. — Quand F est une forme binaire définie positive de degré d, on désigne par $\mathcal{O}(F)$ l'ensemble des $(x,y) \in \mathbb{R}^2$ tels que $F(x,y) \leq 1$. C'est un compact du plan euclidien rapporté au repère orthonormé $(O, \vec{i}, \vec{j})$. Il est délimité par une courbe algébrique de degré d et de classe C^∞, d'équation $F(x,y) = 1$. Rappelons (§ 1) que A_F désigne l'aire de $\mathcal{O}(F)$. Le changement de variable $(x,y) \to (t,y)$ avec $x = ty$ donne

$$A_F = \iint_{F(x,y)\leq 1} \mathrm{d}x\mathrm{d}y = \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{F(t,1)^{2/d}}.$$

Quand F a ses coefficients algébriques, ce nombre est une période au sens de Kontsevich – Zagier. L'article [2] est consacré au calcul de A_F.

On désigne par $\mathrm{L}(F)$ la longueur du polynôme $F(X,1)$ et par $\mathrm{m}(F)$ le minimum de la fonction $F(t,1)$ sur $\mathbb{R}$. Comme F est homogène de degré d, on a, pour tout $(x,y) \in \mathbb{R}^2$,

$$\mathrm{m}(F) \max\{|x|,|y|\}^d \leq F(x,y) \leq \mathrm{L}(F) \max\{|x|,|y|\}^d.$$

Il en résulte que $\mathcal{O}(F)$ contient le carré centré en O de côté $\mathrm{L}(F)^{-1/d}$, à savoir

$$\{(x,y) \in \mathbb{R}^2 \ \mid \ \max\{|x|,|y|\} \leq \mathrm{L}(F)^{-1/d}\},$$

et qu'il est contenu dans le carré centré en O de côté $\mathrm{m}(F)^{-1/d}$:

$$\{(x,y) \in \mathbb{R}^2 \ \mid \ \max\{|x|,|y|\} \leq \mathrm{m}(F)^{-1/d}\}.$$

Par conséquent,

$$4\mathrm{L}(F)^{-2/d} \leq A_F \leq 4\mathrm{m}(F)^{-2/d}.$$

6.2. Le domaine fondamental cyclotomique $\mathcal{O}_n$ pour $n \geq 3$. — Pour $n \geq 3$, $\mathcal{O}_n = \mathcal{O}(\Phi_n)$ est la région fondamentale de la forme cyclotomique Φ_n et son aire est A_{Φ_n}.

Pour $n \geq 3$, $\mathcal{O}_n$ est symétrique par rapport à la première bissectrice et symétrique par rapport au point O. De plus, si n est divisible par 4, $\mathcal{O}_n$ est symétrique par rapport aux axes de coordonnées. Si n est impair, $\mathcal{O}_{2n}$ s'obtient à partir de $\mathcal{O}_n$ par symétrie par rapport à un des axes de coordonnées.

Pour $n = 4$, $\mathcal{O}_4$ est le disque $x^2 + y^2 \leq 1$ et

$$A_{\Phi_4} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{1+t^2} = \pi.$$

Quand p est un nombre premier impair on a

$$A_{\Phi_p} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{(1+t+t^2+\cdots+t^{p-1})^{(p-1)/2}}.$$

Par exemple $\mathcal{O}_3$ est l'intérieur de l'ellipse $x^2 + xy + y^2 = 1$ et

$$A_{\Phi_3} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{1 + t + t^2} = \frac{2\pi}{\sqrt{3}}.$$

6.3. Démontration du théorème 1.5. — La démonstration du théorème 1.5 repose sur des estimations de $\Phi_n(t)$: majorations et minorations. L'estimation de $\mathrm{m}(\phi_n)$ donnée dans [6] ne suffit pas pour démontrer le théorème 1.5.

Montrer que $\mathcal{O}_n$ contient le petit carré revient à démontrer, pour n suffisamment grand,

$$\Phi_n(x,y) < 1 \quad \text{quand} \quad \max\{|x|,|y|\} < 1 - n^{-1+\varepsilon}, \tag{47}$$

alors que montrer que $\mathcal{O}_n$ est contenu dans le grand carré revient à démontrer, pour n suffisamment grand,

$$\Phi_n(x,y) > 1 \quad \text{quand} \quad \max\{|x|,|y|\} > 1 + n^{-1+\varepsilon}. \tag{48}$$

Les relations $\Phi_n(x,y) = \Phi_n(y,x) = \Phi_n(-x,-y)$ (pour $n \geq 3$) permettent de se limiter au domaine $|y| \leq x$.

Si $P \in \mathbb{R}[X]$ est un polynôme de degré d, on a

$$|P(t)| \leq \mathrm{L}(P) \max\{1, |t|\}^d$$

pour tout $t \in \mathbb{R}$. En particulier

$$\phi_n(t) \leq \mathrm{L}(\phi_n) \max\{1, |t|\}^{\varphi(n)}$$

pour tout $t \in \mathbb{R}$.

De (30) on déduit, pour tout $\varepsilon > 0$, pour n suffisamment grand et pour tout $t \in \mathbb{R}$, l'inégalité

$$\phi_n(t) \leq e^{n^{\varepsilon}} \max\{1, |t|\}^{\varphi(n)}. \tag{49}$$

Montrons que cela implique (47). Soit n suffisamment grand et soit (x,y) satisfaisant

$$0 < |y| \leq x < 1 - n^{-1+\varepsilon}.$$

Posons $t = x/y$. On a $|t| \geq 1$ et, en utilisant (49) avec $\varepsilon/3$,

$$\Phi_n(x,y) = y^{\varphi(n)} \phi_n(t) \leq y^{\varphi(n)} e^{n^{\varepsilon/3}} t^{\varphi(n)} = x^{\varphi(n)} e^{n^{\varepsilon/3}} < \bigl(1 - n^{-1+\varepsilon}\bigr)^{\varphi(n)} e^{n^{\varepsilon/3}}.$$

Pour n suffisamment grand on a

$$\varphi(n) > n^{1-\varepsilon/3}, \qquad \log x \leq \log\bigl(1 - n^{-1+\varepsilon}\bigr) < -n^{-1+\varepsilon},$$

d'où

$$\varphi(n) \log x < -n^{2\varepsilon/3},$$

c'est-à-dire

$$x^{\varphi(n)} < e^{-n^{2\varepsilon/3}}.$$

Ceci complète la démonstration de (47).

Pour démontrer (48), on doit minorer $\phi_n(t)$. Nous utiliserons les estimations données par les deux lemmes suivants ; la première nous sera utile quand $|t|-1$ n'est pas trop petit, la suivante quand $|t|-1$ est positif et petit.

Lemme 6.1. — *Pour tout $t \in \mathbb{R}$ et pour tout $n \geq 3$, on a*

$$\phi_n(t) \geq |t|^{\varphi(n)-1}(|t|-1) \prod_{\substack{d|n \\ d\neq n}} \mathrm{L}(\phi_d)^{-1}.$$

Démonstration. — Le résultat est trivial si $|t| \leq 1$. Pour commencer prenons $t > 1$. De

$$t^n - 1 = \phi_n(t) \prod_{\substack{d|n \\ d\neq n}} \phi_d(t) \tag{50}$$

on déduit

$$t^n - 1 \leq \phi_n(t) \prod_{\substack{d|n \\ d\neq n}} (\mathrm{L}(\phi_d)t^{\varphi(d)}) = \phi_n(t)t^{n-\varphi(n)} \prod_{\substack{d|n \\ d\neq n}} \mathrm{L}(\phi_d). \tag{51}$$

On minore $t^n - 1$ par $(t-1)t^{n-1}$. Par conséquent,

$$t^{\varphi(n)-1}(t-1) \leq \phi_n(t) \prod_{\substack{d|n \\ d\neq n}} \mathrm{L}(\phi_d),$$

ce qui est la conclusion du lemme 6.1 pour $t > 1$.

Supposons $t < -1$ et n pair. Dans ce cas, $t^n - 1 = |t|^n - 1$; dans le produit (50) il y a deux facteurs négatifs, à savoir $\phi_1(t) = t - 1$ et $\phi_2(t) = t + 1$, et on remplace (51) par

$$|t|^n - 1 \leq \phi_n(t)|t|^{n-\varphi(n)} \prod_{\substack{d|n \\ d\neq n}} \mathrm{L}(\phi_d).$$

On conclut avec

$$|t|^n - 1 \geq (|t|-1)|t|^{n-1}.$$

Enfin pour $t < -1$ et n impair, le membre de gauche de (50) est $-|t|^n - 1$, et dans le membre de droite le seul facteur négatif est celui correspondant à $d = 1$, à savoir $\phi_1(t) = -|t| - 1$. Ainsi

$$|t|^n + 1 = \phi_n(t) \prod_{\substack{d|n \\ d\neq n}} |\phi_d(t)| \leq \phi_n(t)|t|^{n-\varphi(n)} \prod_{\substack{d|n \\ d\neq n}} \mathrm{L}(\phi_d).$$

Comme $|t|^n+1$ est minoré par $|t|^n$ on a une estimation plus précise que celle du lemme 6.1, à savoir

$$|t|^{\varphi(n)} \leq \phi_n(t) \prod_{\substack{d|n \\ d\neq n}} \mathrm{L}(\phi_d). \qquad \square$$

Lemme 6.2. — *Pour tout $t \in \mathbb{R}$ et pour tout $n \geq 1$, on a*

$$|\phi_n(t)-\phi_n(1)| \leq |t-1| \max\{1,|t|\}^{\varphi(n)-1} \varphi(n) \mathrm{L}(\phi_n)$$

et

$$|\phi_n(t)-\phi_n(-1)| \leq |t+1| \max\{1,|t|\}^{\varphi(n)-1} \varphi(n) \mathrm{L}(\phi_n).$$

Démonstration. — On pourrait faire intervenir la dérivée de ϕ_n dont la longueur est majorée par $\varphi(n)\mathrm{L}(\phi_n)$, mais on peut aussi faire un calcul direct comme ceci. Écrivons

$$\phi_n(t) = \sum_{j=0}^{\varphi(n)} a_j t^j.$$

Alors $a_0 + a_1 + \cdots + a_{\varphi(n)} = \phi_n(1)$, $|a_0| + |a_1| + \cdots + |a_{\varphi(n)}| = \mathrm{L}(\phi_n)$ et

$$\phi_n(t) - \phi_n(1) = \sum_{j=1}^{\varphi(n)} a_j (t^j - 1).$$

On écrit

$$\frac{t^j-1}{t-1} = \sum_{i=0}^{j-1} t^i$$

et

$$\frac{|t^j-1|}{|t-1|} \leq j \max\{1,|t|\}^{j-1} \leq \varphi(n) \max\{1,|t|\}^{\varphi(n)-1},$$

ce qui donne

$$|\phi_n(t)-\phi_n(1)| \leq |t-1| \max\{1,|t|\}^{\varphi(n)-1} \varphi(n) \sum_{j=1}^{\varphi(n)} |a_j|.$$

La même démonstration donne

$$|\phi_n(t)-\phi_n(-1)| \leq |t+1| \max\{1,|t|\}^{\varphi(n)-1} \varphi(n) \sum_{j=1}^{\varphi(n)} |a_j|. \qquad \square$$

Démonstration de (48). — Soit n un entier suffisamment grand et soit $(x,y) \in \mathbb{R}^2$ vérifiant $0 < |y| < x$ et

$$x > 1 + n^{-1+\varepsilon}.$$

On a
$$\log x > \frac{1}{2} n^{-1+\varepsilon} \quad \text{et} \quad \varphi(n) \log x > n^{2\varepsilon/3}.$$
On pose $t = x/y$, de sorte que $|t| > 1$. On écrit
$$\Phi_n(x,y) = y^{\varphi(n)} \phi_n(t)$$
et on minore $\phi_n(t)$ en considérant deux cas.

- *Premier cas.* Supposons
$$|t| > 1 + e^{-n^{\varepsilon/2}}.$$
Cette minoration implique
$$\frac{|t|-1}{|t|} > \frac{1}{2} e^{-n^{\varepsilon/2}}.$$
On utilise le lemme 6.1 et (30), où ε est remplacé par $\varepsilon/2$, pour obtenir
$$\phi_n(t) \geq |t|^{\varphi(n)-1}(|t|-1) e^{-n^{\varepsilon/2}} \geq \frac{1}{2} t^{\varphi(n)} e^{-2n^{\varepsilon/2}},$$
d'où
$$\Phi_n(x,y) \geq \frac{1}{2} x^{\varphi(n)} e^{-2n^{\varepsilon/2}}.$$
On a
$$\varphi(n) \log x > n^{2\varepsilon/3} > 2n^{\varepsilon/2} + \log 2,$$
ce qui donne $\Phi_n(x,y) > 1$ pour n suffisamment grand.
- *Deuxième cas.* Supposons maintenant
$$1 < |t| \leq 1 + e^{-n^{\varepsilon/2}}.$$
On a
$$\frac{|t|-1}{|t|} \leq |t| - 1 \leq e^{-n^{\varepsilon/2}} \text{ et } \log(|t|-1) - \log|t| \leq -n^{\varepsilon/2}.$$
On utilise les majorations
$$\varphi(n) \log|t| < n(|t|-1) \leq n e^{-n^{\varepsilon/2}}$$
et
$$\begin{aligned} &\log(|t|-1) + (\varphi(n)-1)\log|t| + n^{\varepsilon/3} + \log 2 \\ &\qquad \leq -n^{\varepsilon/2} + n e^{-n^{\varepsilon/2}} + n^{\varepsilon/3} + \log 2 < 0 \end{aligned}$$

pour n suffisamment grand. Le lemme 6.2 et (30), où ε est remplacé par $\varepsilon/3$, donnent

$$|\phi_n(t) - \phi_n(1)| < \frac{1}{2} \quad \text{si } t > 1,$$
$$|\phi_n(t) - \phi_n(-1)| < \frac{1}{2} \quad \text{si } t < -1,$$

ce qui implique $\phi_n(t) > 1/2$. Alors

$$\Phi_n(x,y) > \frac{1}{2} y^{\varphi(n)}$$

avec $|y| = x/|t|$. On a

$$\varphi(n)\log|y| = \varphi(n)\log x - \varphi(n)\log|t|,$$

$$\log|t| \le e^{-n^{\varepsilon/2}}, \quad \varphi(n)\log|t| \le n e^{-n^{\varepsilon/2}},$$

d'où

$$\varphi(n)\log x - \varphi(n)\log|t| \ge n^{2\varepsilon/3} - n e^{-n^{\varepsilon/2}} > \log 2$$

pour n suffisamment grand, ce qui implique $\Phi_n(x,y) > 1$. □

La preuve du théorème 1.5 est complète.

7. Preuve du théorème 1.6

On se place sous les hypothèses de ce théorème. Soit $d \ge 4$ un totient tel que $d+2$ soit aussi un totient. Soit $n_1 < n_2 < \cdots < n_t$ la liste des entiers tels que

$$n_i \not\equiv 2 \bmod 4 \text{ et } \varphi(n_i) = d,$$

et un entier m tel que $\varphi(m) = d+2$. On part de la minoration

$$\mathcal{A}_d(N) \ge \Big|\mathcal{B}_{n_1}(N) \cup \cdots \cup \mathcal{B}_{n_t}(N) \cup \mathcal{B}_m(N)\Big|$$

où on utilise la notation (38). Par le principe d'inclusion–exclusion on a la minoration

$$\mathcal{A}_d(N) \ge \Big|\mathcal{B}_{n_1}(N) \cup \cdots \cup \mathcal{B}_{n_t}(N)\Big| + \Big|\mathcal{C}_{\boldsymbol{n},m}(N)\Big|, \tag{52}$$

où $\mathcal{C}_{\boldsymbol{n},m}(N)$ est l'ensemble complémentaire

$$\mathcal{C}_{\boldsymbol{n},m}(N) := \big\{u \in \mathbb{Z} \ \mid \ u \in \mathcal{B}_m(N) \text{ et } u \notin B_{n_1}(N) \cup \cdots \cup \mathcal{B}_{n_t}(N)\big\}.$$

Le premier terme à droite de la minoration (52) est minoré en combinant (41) et (42) :

$$\Big|\mathcal{B}_{n_1}(N) \cup \cdots \cup \mathcal{B}_{n_t}(N)\Big| \ge C_d N^{\frac{2}{d}} - O(N^{\eta_d + \varepsilon}).$$

Pour minorer le cardinal de $\mathcal{C}_{\boldsymbol{n},m}(N)$, nous commençons par exhiber un ensemble de couples d'entiers (a,b) de densité positive dont l'image par Φ_m appartient à $\mathcal{C}_{\boldsymbol{n},m}(\infty)$. On a

LEMME 7.1. — *Soit d, t, n_1, …, n_t et m des entiers comme ci–dessus. Il existe alors un entier D et des classes de congruence a_0 et b_0* mod D *tels que*

$$a \equiv a_0 \text{ et } b \equiv b_0 \bmod D \Rightarrow \Phi_m(a,b) \notin \Big(\mathcal{B}_{n_1}(\infty) \cup \cdots \cup \mathcal{B}_{n_t}(\infty)\Big).$$

Démonstration du lemme 7.1. — À chaque entier n_i on associe l'entier $\varpi(n_i)$ défini comme suit :

- si n_i n'est pas de la forme $2^h 3^k$, alors $\varpi(n_i)$ est le plus petit diviseur premier ≥ 5 de n_i,
- si n_i est de la forme $2^h 3^k$ avec $h \geq 2$, alors $\varpi(n_i) = 4$,
- si n_i est de la forme 3^k avec $k \geq 2$, alors $\varpi(n_i) = 9$.

On pose alors

$$D := \mathrm{ppcm}\{\varpi(n_i)\}.$$

Pour définir a_0 et b_0 mod D, nous allons fixer leurs classes de congruence modulo chacun des $\varpi(n_i)$, avant d'appliquer le théorème chinois pour remonter en des classes modulo D :

- Si $\varpi(n_i)$ est un nombre premier ≥ 5, alors $\varpi(n_i)$ divise n_i et $\varphi(\varpi(n_i)) = \varpi(n_i) - 1$ divise d. On a alors pour $a \equiv 0 \bmod \varpi(n_i)$ et $b \equiv 2 \bmod \varpi(n_i)$ les congruences

  $$\Phi_m(a,b) \equiv b^{d+2} \equiv 4 \not\equiv 0,\ 1 \bmod \varpi(n_i);$$

  donc $\Phi_m(a,b)$ n'appartient pas à l'image de Φ_{n_i} d'après la proposition 4.2. On fixe $a_0 \equiv 0$ et $b_0 \equiv 2 \bmod \varpi(n_i)$.
- Si $\varpi(n_i) = 4$, c'est que n_i est de la forme $2^h 3^k$ avec $h \geq 2$. On remarque que d est divisible par 4 (rappelons que $\varphi(n_i) = d \geq 4$). Dans ce cas $d+2$ est congru à 2 modulo 4. Les seuls m tels que $\varphi(m) = d+2$ et $m \not\equiv 2 \bmod 4$ sont de la forme $m = p^s$ avec $p \equiv 3 \bmod 4$ et $s \geq 1$. Par la formule (28), on a pour a et b congru à 1 modulo 4, $\Phi_m(a,b) \equiv \phi_m(1) \equiv p \equiv 3 \bmod 4$, et $\Phi_m(a,b)$ n'est pas dans l'image de Φ_{n_i}, d'après la proposition 4.5. On fixe donc $a_0 \equiv b_0 \equiv 1 \bmod 4$.
- Si $\varpi(n_i) = 9$, alors n_i est de la forme 3^k avec $k \geq 2$. Par conséquent, $6 \mid \varphi(n_i) = d$. Soit m tel que $\varphi(m) = d+2$. Alors $\Phi_m(0,b) = b^{d+2}$. Donc si $3 \nmid b$, on a $\Phi_m(0,b) \equiv b^2 \bmod 9$. Si on impose $a \equiv 0 \bmod 9$ et $b \equiv 2 \bmod 9$, on voit que $\Phi_m(a,b) \equiv 4 \bmod 9$. Ce n'est pas une valeur prise par Φ_{3^k}, par la proposition 4.4. On fixe donc $a_0 \equiv 0 \bmod 9$ et $b_0 \equiv 2 \bmod 9$.

Le lemme 7.1 en résulte. □

Soient $M \geq 2$ et $\mathcal{E}(M)$ l'ensemble des couples d'entiers (a,b) tels que $|a|$, $|b| \leq M$ et $a \equiv a_0$ et $b \equiv b_0 \bmod D$, avec les notations du lemme 7.1. Il existe $c_0 > 0$ tel que

$$\Phi_m(\mathcal{E}(c_0 N^{\frac{1}{d+2}})) \subset \mathcal{C}_{\boldsymbol{n},m}(N). \tag{53}$$

Notons $\tilde\rho(n)$ le nombre de solutions de l'équation $n = \Phi_m(a,b)$ avec $(a,b) \in \mathcal{E}(c_0 N^{\frac{1}{d+2}})$. On a donc l'égalité

$$\sum_n \tilde\rho(n) = |\mathcal{E}(c_0 N^{\frac{1}{d+2}})| \sim (4c_0^2/D^2) N^{\frac{2}{d+2}}. \tag{54}$$

Pour appliquer l'inégalité de Cauchy–Schwarz, on écrit la partie gauche de l'équation (54) comme

$$\sum_n \tilde\rho(n) = \sum_{\tilde\rho(n)\geq 1} 1 \cdot \tilde\rho(n) \leq |\Phi_m(\mathcal{E}(c_0 N^{\frac{1}{d+2}}))|^{\frac12} \cdot \Big(\sum_n \rho^2(n)\Big)^{\frac12}, \tag{55}$$

où $\rho(n)$ est le nombre de solutions à l'équation $n = \Phi_m(a,b)$ avec $|a|$, $|b| \leq c_0 N^{\frac{1}{d+2}}$. Développant le carré, on voit que $\sum \rho^2(n)$ est le nombre de points entiers de hauteur $\ll N^{\frac{1}{d+2}}$ sur la surface de $\mathbb{P}^3(\mathbb{C})$ définie par

$$\Phi_m(X_1,X_2) - \Phi_m(X_3,X_4) = 0.$$

Cette surface est lisse de degré ≥ 3. Elle contient donc $O(1)$ droites. Sur chacune de ces droites il y a $O(N^{\frac{2}{d+2}})$ points de hauteur $\ll N^{\frac{1}{d+2}}$. Pour compter les points entiers non situés sur ces droites, on suit la même démonstration que pour la proposition 3.4 (voir aussi [9, Lemma 2.4]). Le nombre de ces points entiers est en $O(N^{\vartheta})$ pour un certain $\vartheta < 2/(d+2)$. Regroupant les deux contributions, on a donc la majoration

$$\sum_n \rho^2(n) \ll N^{\frac{2}{d+2}}.$$

Combinant (54) et (55) on obtient la minoration

$$|\Phi_m(\mathcal{E}(c_0 N^{\frac{1}{d+2}}))| \gg N^{\frac{2}{d+2}}.$$

Retournant à (53) puis à (52) on complète la preuve du théorème 1.6.

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A HENSTOCK-KURZWEIL TYPE INTEGRAL ON ONE-DIMENSIONAL INTEGRAL CURRENTS

BY ANTOINE JULIA

ABSTRACT. — We define a non-absolutely convergent integration method on integral currents of dimension 1 in Euclidean space. This integral is closely related to the Henstock-Kurzweil and Pfeffer integrals. Using it, we prove a generalized fundamental theorem of calculus on these currents. A detailed presentation of Henstock-Kurzweil integration is given in order to make the paper accessible to non-specialists.

RÉSUMÉ (*Une intégrale à la Henstock-Kurzweil sur les courants entiers de dimension* 1). — On définit une intégrale non absolument convergente sur les courants entiers euclidiens de dimension 1. Cette intégrale est inspirée des intégrales de Henstock-Kurzweil et de Pfeffer. Dans ce contexte, on démontre un théorème fondamental généralisé sur ces courants. On donne aussi une présentation détaillée de l'intégrale de Henstock et Kurzweil, pour les non-spécialistes.

1. Introduction

The goal of this paper is to present an integration method for functions defined on the support of an integral current of dimension 1 in Euclidean space. This method is inspired from the Henstock-Kurzweil (HK) and Pfeffer integrals

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[23, 20, 32], and, like them, tailored to the study of the fundamental theorem of calculus. The HK integral is a variant of the Riemann integral; it is more general than the Lebesgue integral, in the sense that all Lebesgue integrable functions are HK integrable, but *non-absolutely convergent*; there exist functions that are HK integrable, while their absolute value is not; in the same way that the series $\sum_k (-1)^k k^{-1}$ converges, while $\sum_k k^{-1}$ does not. A classical example of such functions is the derivative of $x \mapsto x^2 \sin(x^{-2})$ for $x \in (0, 1\,]$.

An integral that solves this problem was defined in two different ways by A. Denjoy and O. Perron in the early twentieth century, see [36]. R. Henstock and J. Kurzweil independently found a simpler, equivalent definition of this integral; we will use the HK formalism, but refer the reader to [18] for a comparison of these three approaches. For functions defined on a bounded interval $[\,a, b\,]$, the fundamental theorem of calculus of the integral of HK is the following:

Theorem A ([33, Theorem 6.1.2]). — *Let $f : [\,a, b\,] \to \mathbb{R}$ be a continuous function that is differentiable everywhere, then its derivative f' is HK integrable on $[\,a, b\,]$ and there holds:*

$$(HK) \int_a^b f' = f(b) - f(a). \tag{1}$$

Note that several other integration methods have been defined for which Theorem A holds, see, in particular, [2, 3, 7] and a "minimal" theory in [6]. The (Denjoy-Perron-)Henstock-Kurzweil integral is now well understood and its integrable functions and their primitives have been completely characterized [8]. It is also interesting to note that a small variation in the definition of the HK integral yields the McShane integral [28], which is equivalent to the Lebesgue integral. Lastly, although the present paper focuses on scalar valued functions, we mention that HK-like integration of Banach space valued functions has raised considerable interest [15, 37, 5], in particular with an application to the Cauchy problem in [10].

The Riemann-like formulation of the HK integral makes it straightforward to allow for singularities in the above theorem; if f is only differentiable at all but countably many points of $[\,a, b\,]$, the result still holds. This statement is, in some sense, optimal. Indeed, as shown by Z. Zahorsky in [41], the set of non-differentiability points of a continuous function is a countable union of G_δ sets. In particular, if it is uncountable, it must contain a Cantor subset by [30, Lemma 5.1], and to any Cantor subset of an interval having zero Lebesgue measure, one can associate a "Devil's Staircase," which has derivative equal to 0 outside of the set and is non-constant.

However, the differentiability condition can be relaxed and replaced by a pointwise Lipschitz condition. Thus, a more general statement is

THEOREM B ([33, Theorem 6.6.9]). — *Let $f : [a,b] \to \mathbb{R}$ be a continuous function that is pointwise Lipschitz at all but countably many points, then, it is differentiable almost everywhere in $[a,b]$, its derivative f' is HK integrable on $[a,b]$, and identity* (1) *holds.*

Natural extensions of the fundamental theorem of calculus include the Gauss-Green (or divergence) theorem and Stokes' theorem. For the former in bounded sets of finite perimeter an integral was developed by W.F. Pfeffer in [32], following, in particular, J. Mawhin [26] and J. Mařìk [25]. The results naturally extend to Stokes' theorem on smooth oriented manifolds. For singular varieties, an integral adapted to Stokes' theorem was defined by the author on certain types of integral currents in Euclidean spaces [21, 22]. Let us also mention works on integration on more fractal objects with different methods [40, 19, 42].

The present paper corresponds to the second chapter of the author's thesis [21], which focuses on one-dimensional integral currents. These are treated separately from the higher-dimensional ones, as they can be decomposed into a countable family of curves. We, thus, define an integral closer to the Henstock-Kurzweil one, which we call the $\mathcal{R}_1$ integral.

Given an integral current T of dimension 1 in $\mathbb{R}^n$, define $\operatorname{Indec}(T)$ to be the subset of $\operatorname{spt} T$ containing the points in the support of an indecomposable piece of T (see Section 3 for the notations on currents). Denote by $\|T\|$ the carrying measure of T, by $\overrightarrow{T}$ its tangent vector field and by $\operatorname{spt} T$ its support. The main result of this paper is the following:

THEOREM 1.1 (Fundamental theorem of calculus for 1-currents). — *Let T be a fixed integral current of dimension 1 in $\mathbb{R}^n$ and u be a continuous function on $\operatorname{spt} T$. Suppose that u is pointwise Lipschitz at all but countably many points in $\operatorname{Indec}(T)$ and that u is differentiable $\|T\|$ almost everywhere, then $x \mapsto \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$ is $\mathcal{R}_1$ integrable on T and*

$$(\partial T)(u) = (\mathcal{R}_1) \int_T \langle \mathrm{D}u, \overrightarrow{T}\rangle.$$

This theorem is equivalent to Theorem B when T represents a bounded interval.

Summary of the paper. — In Section 2, we define the integral of Henstock and Kurzweil and its main properties along with schemes of proofs of the main theorems. We also give an equivalent definition of integrability — inspired from the Pfeffer integral — which will be useful in the sequel. It is important to note that the Pfeffer integral is not equivalent to the HK integral.

In Section 3, we recall the definition of integral currents of dimension 1 in Euclidean spaces and define the main ingredients of $\mathcal{R}_1$ integration: pieces of a current and functions on the space of pieces of a current; we also study the derivation of these functions, following H. Federer [16, Section 2.9] and W.F. Pfeffer [33, Section 9.3] and [34]. Section 4 contains the definition of $\mathcal{R}_1$ integration and the proof of its main properties, as well as the proof of Theorem 1.1.

Possible generalizations. — First, one can ask if u could be allowed to be discontinuous outside of the set of positive lower-density points of $\|T\|$, yet remain bounded. Proposition 3.5 and Example 2.10 show that this is not straightforward.

A natural question would be whether Theorem 1.1 could be generalized to normal currents in Euclidean spaces. Indeed, by a Theorem of S.K. Smirnov [38], normal currents of dimension 1 also admit a decomposition into Lipschitz curves. More precisely, given a current T of dimension 1, with finite mass and finite boundary mass in $\mathbb{R}^n$, there exists a finite measure μ on the space of finite length Lipschitz curves in $\mathbb{R}^n$ such that

$$T = \int [\![\gamma]\!] \,\mathrm{d}\mu(\gamma), \tag{2}$$

where $[\![\gamma]\!]$ is the integral current of dimension 1 associated to the Lipschitz curve γ with multiplicity 1 and orientation given by the parameterization. However, there is no a-priori constraint on the measure μ; it can be somewhat diffuse, as the carrying measure of a normal current can be absolutely continuous with respect to the Lebesgue measure. It is, therefore, impossible to work with countable sums of pieces, and one would probably need another notion of piece of a normal current to define suitable Riemann sums. Recall that Fubini-type arguments do not work well with non-absolutely convergent integrals, as shown in [33, Section 11.1]. Note also that the space of curves, on which we would have to integrate is far from Euclidean.

Another natural idea would be to consider integral currents of dimension 1 in Banach spaces or complete metric spaces, following [1] or [13].

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Finally, I wish to mention that the notions of pieces and subcurrents studied in this paper and in [21, 22] are very close to the subcurrents defined by E. Paolini and E. Stepanov in [31] for normal currents in metric spaces.

2. The integral of Kurzweil and Henstock

Here, we give a short presentation of the Henstock-Kurzweil integral. The proofs of these results can be found in any treaty on the subject, such as [27, 33, 18, 17]. Let us also mention the very detailed recent book [29] — in French, and Appendix H to [11].

2.1. Definition and classical properties. — A non-negative function defined on a set $E \subseteq \mathbb{R}$ is called a **gauge** if its zero set is countable. In the classical definition of the Henstock-Kurzweil integral, gauges are always positive, but for our purposes it makes sense to allow the gauge to take on the value zero in a countable set. A **tagged family** in an interval $[\,a, b\,]$ is a finite collection of pairs $([\,a_j, b_j\,], x_j)_{j=1,2,\dots,p}$, where one has $a \leq a_1 < b_1 \leq a_2 < \cdots \leq a_p < b_p \leq b$ and for all j, $x_j \in [\,a_{j-1}, a_j\,]$. The **body** of a family $\mathcal{P}$ is the union denoted by $[\mathcal{P}]$ of all the intervals in $\mathcal{P}$. A **tagged partition** in $[\,a, b\,]$ is a tagged family whose body is $[\,a, b\,]$. If δ is a gauge on $[\,a, b\,]$, we say that a tagged family (or a tagged partition) is δ-fine, when for all j, $b_j - a_j < \delta(x_j)$. In particular, it holds that $\delta(x_j) > 0$, for all j.

Definition 2.1. — A function f defined on a compact interval $[\,a, b\,]$ is **Henstock-Kurzweil integrable on** $[\,a, b\,]$ if there exists a real number α, such that for all $\epsilon > 0$, there exists a positive gauge δ on $[\,a, b\,]$, such that for each δ-fine tagged partition $\mathcal{P} = \{([\,a_{j-1}, a_j\,], x_j)\}_{j=1,\dots,p}$, it holds that:

$$\left|\sum_{j=1}^{p} f(x_j)(a_j - a_{j-1}) - \alpha\right| < \epsilon.$$

In the following, we will write $\sigma(f, \mathcal{P})$ for the sum on the left-hand side, whenever $\mathcal{P}$ is a tagged family. If α exists as above, we denote it by $(HK)\int_a^b f$. This definition is well posed as a consequence of the following key result.

Lemma 2.2 (Cousin's lemma [18, Lemma 9.2]). — *If I is a closed bounded interval, and δ is a positive gauge on I, then a δ-fine tagged partition of I exists.*

To characterize integrability, the following proposition is useful:

Proposition 2.3 (Cauchy criterion for integrability [18, Theorem 9.7]). — *A function f is HK integrable on the interval $[\,a, b\,]$, if and only if for each $\epsilon > 0$ there exists a positive gauge δ on $[\,a, b\,]$, such that whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are δ-fine tagged partitions of $[\,a, b\,]$, it holds that*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \epsilon.$$

Let us also list some fundamental properties of HK integrable functions.

THEOREM 2.4 ([18, Theorems 9.8–9.13]). — *Let f be a Henstock-Kurzweil integrable function on the interval $[a,b]$:*

(i) If g is HK integrable on $[a,b]$, and λ is a real number, then $f+\lambda g$ is HK integrable, and

$$(HK)\int_a^b (f+\lambda g) = \left((HK)\int_a^b f\right) + \lambda\left((HK)\int_a^b g\right).$$

(ii) If a function g is equal to f almost everywhere on $[a,b]$, then g is also HK integrable and has the same integral.

(iii) If g is Lebesgue integrable, it is also HK integrable, and the two integrals coincide.

(iv) The restriction of f to a subinterval $[c,d] \subseteq [a,b]$ is HK integrable on $[c,d]$.

(v) (Saks-Henstock lemma) For $\epsilon > 0$ and δ, a positive gauge corresponding to ϵ in the definition of integrability of f, given any tagged family $(([a_j,b_j],x_j))_{j=1}^p$ in $[a,b]$ there holds

$$\sum_{j=1}^p \left| f(x_j)(b_j - a_j) - (HK)\int_{a_j}^{b_j} f \right| < 2\epsilon.$$

*(vi) The function $F : [a,b] \to \mathbb{R}; x \mapsto (HK)\int_a^x f$ is continuous; it is called the **indefinite HK integral of** F. Also, if f is non-negative, F is non-decreasing.*

(vii) The function F above is differentiable almost everywhere with the derivative equal to f.

(viii) f is Lebesgue measurable.

(ix) f is Lebesgue integrable if and only if $|f|$ is HK integrable.

Finally, we state three important convergence properties in the space of Henstock-Kurzweil integrable functions:

THEOREM 2.5. — *Let $(f_n)_n$ be a sequence of HK integrable functions on the interval $[a,b]$. Suppose that $f_n \to f$ pointwise almost everywhere. If any one of the following three conditions holds, then f is HK integrable, and $(HK)\int f = \lim_n (HK)\int f_n$:*

(i) (Monotone convergence theorem) For almost all x, for all n, $f_n(x) \leq f_{n+1}(x)$, and there holds $\sup_n (HK)\int f_n < +\infty$.

(ii) (Dominated convergence theorem) There exist HK integrable functions g and h, such that for all n, $g \leq f_n \leq h$ almost everywhere.

(iii) (Controlled convergence theorem [18, Theorem 13.16]*) $(f_n(x))_n$ is bounded for almost all $x \in [a,b]$, and for all $\epsilon > 0$, there exists a posi-*

tive gauge on $[a,b]$ *such that for all* n*, for all* δ*-fine tagged partition* $\mathcal{P}$ *of* $[a,b]$*:*

$$\left|\sigma(\mathcal{P}, f_n) - (HK)\int_a^b f_n\right| < \epsilon.$$

In the latter case, the sequence $(f_n)_n$ is called HK **equiintegrable**. The third statement has no equivalent in Lebesgue integration and relies strongly on the use of gauges.

The two first results are corollaries of the third, but they can also be proved using only the Saks-Henstock lemma and *purely HK* techniques; we will give such a proof for the monotone convergence theorem of $\mathcal{R}_1$ integration (see Theorem 4.16). However, when possible, it is quicker to rely on Lebesgue integration results and statement (ix) of Theorem 2.4. We conclude this section with a first version of the fundamental theorem of calculus for the Henstock-Kurzweil integral, which allows F to have points of non-differentiability.

Theorem 2.6 ([18, Theorem 9.6]). — *If* $F : [a,b] \to \mathbb{R}$ *is continuous and differentiable at all but countably many points, then* F' *is HK integrable on* $[a,b]$*, and* F *is the indefinite integral of* F'*.*

One can generalize this result to less regular functions F. This requires the introduction of another notion.

2.2. AC_* functions and the Fundamental Theorem of Calculus. — This section closely follows the presentation of Sections 1.9 to 1.11 in T. De Pauw's survey [12]. The proofs can be found there, and also in the book of W.F. Pfeffer [33].

A function F defined on $[a,b]$ is AC_*, if for every set $D \subseteq [a,b]$ of zero Lebesgue measure and every $\epsilon > 0$, there exists a positive gauge δ on D such that whenever $\mathcal{P}$ is a δ-fine family in $[a,b]$ tagged in D, it holds that

$$\sum_{([c,d],x)\in\mathcal{P}} |F(d) - F(c)| < \epsilon. \tag{3}$$

In particular, an AC_* function is continuous. If f is HK integrable, then its indefinite integral F is AC_*, indeed, if D is a Lebesgue null set, we can consider the function $f_{D^c} := f\mathbb{1}_{D^c}$. As HK integration is insensitive to modifications on Lebesgue null sets, F is also the primitive of f_{D^c}, so for $\epsilon > 0$, we can apply the Saks-Henstock lemma 2.4(v) and find a gauge δ corresponding to $\epsilon/2$ on $[a,b]$. Considering the gauge $\delta_D = \delta|_D$ by the Saks-Henstock lemma for any δ_D-fine tagged family $\mathcal{P}$ in $[a,b]$, since f_D is equal to zero on D, condition (3) is satisfied. The following converse statement holds:

Proposition 2.7 ([33, Theorem 6.4.4]). — *If F is AC_* and almost everywhere differentiable in $[a,b]$, then F' is HK integrable, and*

$$F(b) - F(a) = (HK)\int_a^b F'.$$

We first find a general condition that ensures that a function is AC_*. Recall that a function F defined on an interval I is **pointwise Lipschitz** at a point $x \in I$, if

$$\operatorname{Lip}_x F := \limsup_{y \to x y \in I} \frac{|F(y) - F(x)|}{|y - x|} < +\infty.$$

Proposition 2.8 ([33, Proposition 6.6.3]). — *A continuous function F that is pointwise Lipschitz at all but countably many points is AC_*.*

Recall also Stepanoff's theorem:

Theorem 2.9 (Stepanoff [33, Theorem 6.6.8]). — *If $F : I \to \mathbb{R}$ is pointwise Lipschitz at all points of some set E in the interior of I, then F is differentiable almost everywhere in E.*

Combining Proposition 2.8, Theorem 2.9 and Proposition 2.7 yields

Theorem B ([33, Theorem 6.6.9]). — *Let $f : [a,b] \to \mathbb{R}$ be a continuous function that is pointwise Lipschitz at all but countably many points, then, it is differentiable almost everywhere in $[a,b]$, its derivative f' is HK integrable on $[a,b]$, and identity* (1) *holds.*

2.3. An equivalent definition of the HK integral. — All the above properties of the Henstock-Kurzweil integral can be extended to the case where the interval $[a,b]$ is replaced by a simple Lipschitz curve $\Gamma \subseteq \mathbb{R}^n$ (closed or not). Indeed, one can consider an arc-length parameterization γ of Γ and work on $f \circ \gamma$. If f is pointwise Lipschitz at $\gamma(x)$ along Γ, $f \circ \gamma$ is pointwise Lipschitz at x. The only thing that is not straightforward is relating differentiation in the ambient space $\mathbb{R}^n$ with differentiation along the curve. However, a Lipschitz curve has a tangent line at almost all points. In the next section, we consider countable sums of simple Lipschitz curves to develop Henstock-Kurzweil integration on integral currents of dimension 1. The sum of curves can often be decomposed in several ways, and Example 2.10 shows that the choice of the decomposition can have an effect on the integral, hence the need for a definition of integrability that does not depend on the decomposition.

Example 2.10. — In $\mathbb{R}^2$, consider the curve Γ^+ corresponding to the graph in $(0,1]$ of the function

$$x \mapsto f(x) := \operatorname{dist}(x, \{t \in (0,1], 2t\sin(t^{-2}) - 2t^{-1}\cos(t^{-2})\} = 0).$$

The curve Γ^+ is a Lipschitz curve and has length $\sqrt{2}$; orient Γ^+ towards the positive first coordinate. Let Γ^- be the reflection of Γ^+ across the horizontal axis. The union of curves Γ^+ and Γ^- can also be seen as the (closure of) the union of the graphs on $(0,1\,]$ of $x \mapsto \pm\,\mathrm{sgn}(x\sin(x^{-2}) - 2x^{-1}\cos(x^{-2}))f(x)$. Let Γ and $\tilde{\Gamma}$ be the corresponding curves. Let u be the function defined in $\mathbb{R}^2$ by

$$(x_1,x_2) \mapsto \begin{cases} 2\,\mathrm{sgn}(x_2)\left(x_1\sin(x_1^{-2}) - 2x_1^{-1}\cos(x_1^{-2})\right) & \text{if } x_1 > 0, x_2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If γ^+, γ^-, γ and $\tilde{\gamma}$ are the respective arc length parameterizations of the curves above, the functions $u \circ \gamma^+$ and $u \circ \gamma^-$ are HK integrable on $[\,0,\sqrt{2}\,]$ with the respective indefinite integrals $x \mapsto \pm\sqrt{2}x^2\sin(x^{-2})$. However, the functions $u \circ \gamma$ and $u \circ \tilde{\gamma}$ are, respectively, equal to $\pm|(u \circ \gamma^+)'|$, which are not HK integrable. These curves are plotted in Figure 2.1.

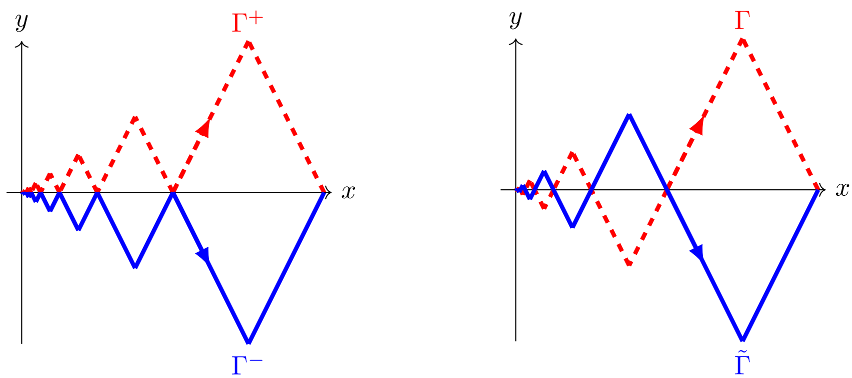

FIGURE 2.1. u is HK integrable on Γ^+ and Γ^- but not on Γ or $\tilde{\Gamma}$.

In order to generalize the Henstock-Kurzweil integral to other settings, it is necessary to use more flexible tools. In particular, we need to remove the dependency on the parameters and allow for families instead of partitions, so that some "small part" of the domain can be left out. The precise meaning of a "small part" is a key point here.

This will be formalized in the next section, but let us first state an equivalent definition of HK integrability on an interval. In order to define what "small" is we will consider functions F defined on the space of finite unions of disjoint compact intervals in $[\,a,b\,]$ (or equivalently on equivalence classes of bounded subsets of finite perimeter of $[\,a,b\,]$). Such a function is **subadditive** if given two families, $\mathcal{U}$ and $\mathcal{U}'$, of closed intervals of $[\,a,b\,]$, and it holds that

$$|F([\mathcal{U}] \cup [\mathcal{U}'])| \leq |F([\mathcal{U}])| + |F([\mathcal{U}'])|\,.$$

F is **additive** if for $\mathcal{U}$ and $\mathcal{U}'$ as above with $\mathcal{L}^1([\mathcal{U}] \cap [\mathcal{U}']) = 0$, it holds that

$$F\left([\mathcal{U}] \cup [\mathcal{U}']\right) = F([\mathcal{U}]) + F([\mathcal{U}']).$$

F is **continuous** on the space of finite unions of intervals if given a sequence $\mathcal{U}_j$ of families of intervals with $\#\mathcal{U}_j < C$ and $\mathcal{L}^1([\mathcal{U}_j]) \to 0$, and it holds that $F([\mathcal{U}_j]) \to 0$. In particular, there is a one-to-one equivalence between continuous functions $f : [\,a, b\,] \to \mathbb{R}$ with value 0 at a and continuous functions on the space of finite unions of intervals of $[\,a, b\,]$. Indeed, consider such an $f : [\,a, b\,] \to \mathbb{R}$. One can associate to it a function F on the space of finite unions of intervals by

$$F([\,x_1, y_1\,] \cup \cdots \cup [\,x_p, y_p\,]) = f(y_1) - f(x_1) + \cdots + f(y_p) - f(x_p).$$

As f is uniformly continuous on $[\,a, b\,]$, F is continuous; it is also additive by definition. Conversely, given a continuous additive F, one can define f on $[\,a, b\,]$ by $f : x \mapsto F([\,a, x\,])$. As F is additive, for $x < y$, there holds $f(y) - f(x) = F([\,x, y\,])$, and the continuity of F implies that of f.

Note that it is critical for the following to work on *finite* unions of intervals, and, hence we mention only *finite* additivity. One can associate such a function to a measure defined on an interval. Subadditivity is then always verified, but additivity is equivalent to the measure being non-atomic. Continuity holds when a measure is absolutely continuous and has bounded density with respect to the Lebesgue measure.

These notions seem impractical but we will see in the following section that they can be easily generalized. Indeed, while intervals are not well suited to algebraic operations, they can be seen as currents of dimension 1 in $\mathbb{R}$, using their canonical orientation and giving them multiplicity 1. The following property is a reformulation of HK integrability in the language of Pfeffer integration (see Theorem 6.7.5 in [33]).

THEOREM 2.11 (Equivalent integrability condition [33, Theorem 6.7.5]). — *A function f defined almost everywhere on $[\,a, b\,]$ is Henstock-Kurzweil integrable, if and only if there exists a non-negative subadditive continuous function G on the space of finite unions of intervals in $[\,a, b\,]$ and a real number I with the property that for all $\epsilon > 0$, there exists a gauge δ – not necessarily positive everywhere – and a positive number τ such that whenever $\mathcal{P}$ is a δ-fine tagged family in $[\,a, b\,]$ with*

$$G([\,a, b\,] \setminus [\mathcal{P}]) < \tau,$$

it holds that $|I - \sigma(f, \mathcal{P})| < \epsilon$.

A key result for this definition is the following. It implies that a tagged family satisfying the above constraints exists, it is a sort of generalization of Cousin's lemma (Lemma 2.2), where one considers families instead of partitions:

Lemma 2.12 (Howard-Cousin lemma for an interval [33, Proposition 6.7.6]). — *Let δ be a gauge on $[\,a,b\,]$ (with possibly countably many zeros). Let G be a non-negative, subadditive, continuous function on the space of a finite union of closed intervals in $[\,a,b\,]$. For every $\tau > 0$, there exists a δ-fine tagged family $\mathcal{P}$ in $[\,a,b\,]$ with*

$$G([\,a,b\,] \setminus [\mathcal{P}]) < \tau. \tag{4}$$

Remark 2.13. — The integral of Henstock and Kurzweil is **not** equivalent to that of Pfeffer. The integrability condition in the above statement differs from that of Pfeffer in that the latter considers families consisting of regular sets of finite perimeter – in one dimension; these are finite unions of intervals. See Example 12.3.5 in [33].

3. Integral currents of dimension 1 and their pieces

3.1. Notations. — In the following, $f|_A$ denotes the restriction of the function f to the set A, while $\mu \llcorner f$ and $\mu \llcorner A$ denote the multiplication of the (possibly vector valued) measure μ by the (scalar) function f or the indicator function of A; $\operatorname{spt} \mu$ is the support of μ. In $\mathbb{R}^n$, with the usual Euclidean metric, we denote the norm of a vector x by $|x|$ and the distance by $\operatorname{dist}(\cdot,\cdot)$. The usual scalar product of $x, y \in \mathbb{R}^n$ is $x \cdot y$, while the product of a vector v with a covector η is denoted $\langle \eta, v\rangle$; $\mathrm{U}(x,r)$ and $\mathrm{B}(x,r)$ are, respectively, the open and closed balls of center $x \in \mathbb{R}^n$ and radius $r > 0$.

The Hausdorff measure of dimension 1 is denoted by $\mathcal{H}^1$. If μ is a scalar measure, $\operatorname{set}_1 \mu$ denotes the points x where μ has positive density of dimension 1, i.e. where

$$\Theta^{1*}(\mu, x) := \limsup_{r\to 0} (2r)^{-1}\mu(\mathrm{B}(x,r)).$$

In the following, integrals in the sense of Lebesgue with respect to a measure μ will be denoted $(\mathcal{L})\int \mathrm{d}\mu$, omitting $(\mathcal{L})$ when there is no ambiguity.

A set $E \subseteq \mathbb{R}^n$ is **1-rectifiable** if there exists a countable collection of Lipschitz curves $\gamma_j : \mathbb{R} \to \mathbb{R}^n$, such that $\mathcal{H}^1(E \setminus \bigcup_j \gamma_j(\mathbb{R})) = 0$. A set is **0-rectifiable** if it is countable.

An **current** of dimension 1 in $\mathbb{R}^n$ is a continuous functional on the space of smooth differential forms of degree 1 with compact support: $\mathcal{D}^1(\mathbb{R}^n)$. The space of such currents is denoted by $\mathcal{D}_1(\mathbb{R}^n)$. A current of dimension 0 is a distribution, and a current of dimension 1 can be seen as a *vector valued distribution.* The mass of a current $T \in \mathcal{D}_1(\mathbb{R}^n)$ is the number

$$\mathbf{M}(T) := \sup\{T(\omega), \omega \in \mathcal{D}^1(\mathbb{R}^n), |\omega| \leq 1\} \in [0,+\infty].$$

Currents of finite mass are representable by integration: currents of dimension 0 with finite mass are signed measures, and currents of dimension 1 with finite mass are vector valued measures. We denote by $\|T\|$ the **carrying measure** of a current, i.e. the total variation of the associated signed or vector measure. The **support** of a current T is the smallest closed set $\operatorname{spt} T$, such that $T(\omega) = 0$ for all ω supported in the complement of $\operatorname{spt} T$. The **boundary** of a current $T \in \mathcal{D}_1(\mathbb{R}^n)$ is the current $\partial T \in \mathcal{D}_0$ defined by $\partial T(f) = T(\mathrm{d}f)$ for all $\omega \in \mathcal{D}^0(\mathbb{R}^n) = \mathcal{C}_c^\infty(\mathbb{R}^n)$. The **flat norm** of a current $T \in \mathcal{D}_1(\mathbb{R}^n)$ is the number

$$\mathbf{F}(T) := \sup\{T(\omega),\, \omega \in \mathcal{D}^1(\mathbb{R}^n), |\omega| \leq 1, |\,\mathrm{d}\omega| \leq 1\} \in [0, +\infty].$$

If $\gamma : [\,0, t_1\,] \to \mathbb{R}^n$ is a simple Lipschitz curve, we denote by $[\![\gamma]\!]$ the current of dimension 1 defined by

$$[\![\gamma]\!](\omega) = \int_0^{t_1} \langle \omega(\gamma(t)), \gamma'(t) \rangle \,\mathrm{d}t.$$

It holds that $\mathbf{M}([\![\gamma]\!]) = \int_0^{t_1} |\gamma'(t)|\,\mathrm{d}t$, and $\mathbf{M}(\partial[\![\gamma]\!])$ is either 0 or 2, depending on whether γ is an open curve or a closed curve. The carrying measure of $[\![\gamma]\!]$ is $\|[\![\gamma]\!]\| = \mathcal{H}^1 \llcorner \gamma([\,0, t_1\,])$. We work mostly with **integral currents of dimension** 1, which include currents representing curves of finite length. A current $T \in \mathcal{D}_1(\mathbb{R}^n)$ is integral ($T \in \mathbf{I}_1(\mathbb{R}^n)$) if it has compact support and can be written as a countable sum of simple Lipschitz curves $[\![\gamma_j]\!]$, such that

$$\sum_j \mathbf{M}([\![\gamma_j]\!]) = \mathbf{M}(T) \text{ and } \sum_j \mathbf{M}(\partial[\![\gamma_j]\!]) = \mathbf{M}(\partial T).$$

In particular, the density set of T, $\operatorname{set}_1 \|T\|$, is 1-rectifiable. This characterization of integral currents is very specific to the one-dimensional case.

An integral current T is **decomposable** if there exists two non-trivial integral currents Q and R with $Q + R = T$ and $\mathbf{M}(T) = \mathbf{M}(Q) + \mathbf{M}(R)$, $\mathbf{M}(\partial T) = \mathbf{M}(\partial Q) + \mathbf{M}(\partial R)$. If such a pair does not exist, T is called **indecomposable**. A current $T \in \mathbf{I}_1(\mathbb{R}^n)$ is indecomposable if and only if it is associated with an oriented simple Lipschitz curve with unit multiplicity.

3.2. Pieces of a current. — Let T be an integral current; an integral current S is a **piece of** T if

$$\|S\| \leq \|T\| \text{ and } \|T - S\| \leq \|T\|.$$

The notion of the piece of a current differs from that of the subcurrent defined in [21, 22] for integral currents in any dimension where the condition is $\|S\| \perp \|T - S\|$. Subcurrents of T are pieces of T, but the converse holds only if T has multiplicity 1 almost everywhere. However, our definition of piece coincides with the subcurrents defined by Stepanov and Paolini in [31].

Example 3.1. — Consider the current $T = 2[\![0, 2]\!] \in \mathbf{I}_1(\mathbb{R}^1)$, then

- The currents $[\![0, 2]\!]$ and $2[\![0, 1]\!]$ are pieces of T,
- $3[\![0, 2]\!]$, $3^{-1}[\![0, 2]\!]$ and $-[\![0, 2]\!]$ are not pieces of T.

PROPOSITION 3.2. — *An integral current S is a piece of $T \in \mathbf{I}_1(\mathbb{R}^1)$ if and only if there exists a $\|T\|$ measurable function $g : \mathbb{R}^n \to [0,1]$, such that $S = T \llcorner g$.*

Proof. — Suppose $S = T \llcorner g$, then $\|S\| = \|T\| \llcorner g \leq \|T\|$ and $\|T - S\| = \|T\| \llcorner (1-g) \leq \|T\|$.

Conversely, suppose S is a piece of T. Then S is of the form $\mathcal{H}^1 \llcorner (\theta_S \mathbb{1}_{M_S}) \overrightarrow{S}$, and $T = \mathcal{H}^1 \llcorner (\theta_T \mathbb{1}_{M_T}) \overrightarrow{T}$, where θ_S and θ_T are supposed to be non-negative, respectively, and $\mathcal{H}^1 \llcorner M_S$ and $\mathcal{H}^1 \llcorner M_T$ almost everywhere. By the hypotheses on S it holds that $\mathcal{H}^1(M_S \setminus M_T) = 0$, as well as

$$\begin{aligned} \theta_S \leq \theta_T, &\quad \mathcal{H}^1 \llcorner M_T \text{ almost everywhere,} \quad \text{and} \\ |\theta_T \overrightarrow{T} - \theta_S \overrightarrow{S}| \leq \theta_T, &\quad \mathcal{H}^1 \llcorner M_T \text{ almost everywhere.} \end{aligned}$$

This, in turn, implies that $\overrightarrow{T} = \overrightarrow{S}$ at $\mathcal{H}^1$ at almost all points where θ_S is positive. Define the function g by

$$g(x) = \begin{cases} 0 & \text{if } x \notin M_T, \text{ or } \theta_T(x) = 0, \\ \theta_S(x)/\theta_T(x) & \text{otherwise.} \end{cases}$$

Clearly, $g(x) \in [0,1]$ for all $x \in \mathbb{R}^n$ and $S = T \llcorner g$. □

In particular, elements of a decomposition of T are pieces of T; however, an indecomposable piece of T may not be a piece of any element of the decomposition of T (see Figure 3.1).

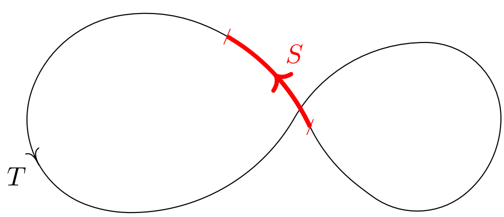

FIGURE 3.1. S is not a piece of an indecomposable element of T.

3.3. Continuous function on the space of pieces of T. — Denote by $\mathcal{S}_{\leq}(T)$ the collection of all pieces of T.

DEFINITION 3.3. — A function F on $\mathcal{S}_{\leq}(T)$ is **continuous**, if given a sequence $(S_j)_j$ in $\mathcal{S}_{\leq}(T)$ that converges to 0 in the flat norm with $\sup_j \mathbf{M}(\partial S_j) < +\infty$, we have $F(S_j) \to 0$; F is **additive** if whenever S_1 and S_2 are in $\mathcal{S}_{\leq}(T)$ with $S_1 + S_2 \in \mathcal{S}_{\leq}(T)$ (which is equivalent to $\|S_1\| + \|S_2\| \leq \|T\|$), it holds that $F(S_1 + S_2) = F(S_1) + F(S_2)$; F is **subadditive**, if instead for each S_1, S_2 as above, we have $F(S_1 + S_2) \leq F(S_1) + F(S_2)$.

Continuous additive functions on $\mathcal{S}_{\leq}(T)$ include the restriction of 1-charges as defined in [14]: 1-charges are continuous linear forms on the space of normal currents of dimension 1: $\mathbf{N}_1(\mathbb{R}^n)$ equipped with a certain topology. In particular, they include representatives of continuous functions f on $\mathbb{R}^n$ and continuous differential forms ω of degree 1 on $\operatorname{spt} T$, defined, respectively, as

$$\Theta_f : S \mapsto \partial S(f)$$

and

$$\Lambda_\omega : S \mapsto \int \langle \omega, \overrightarrow{S} \rangle \, \mathrm{d}\|S\|.$$

Another important example is the mass of pieces:

Proposition 3.4. — *For every $T \in \mathbf{I}_1(\mathbb{R}^n)$, the function $S \mapsto \mathbf{M}(S)$ is continuous and additive on $\mathcal{S}_{\leq}(T)$.*

Proof. — Additivity is clear. For the continuity, let $(S_j)_j$ be a sequence in $\mathcal{S}_{\leq}(T)$ converging in the flat norm to $S \in \mathcal{S}_{\leq}(T)$ with $\sup_j \mathbf{M}(\partial S_j) < +\infty$. First notice that $\mathbf{M}(S) \leq \liminf_j \mathbf{M}(S_j)$ by lower semi-continuity of mass in the flat norm topology. So, all we have to show is that $\liminf_j \mathbf{M}(S_j) \geq \mathbf{M}(S)$. In order to do this, for $\epsilon > 0$ define a smooth 1-form ω in $\mathbb{R}^n$, such that $|\omega(x)| \leq 1$ for all x and $R(\omega) \geq \mathbf{M}(R) - \epsilon$, for each $R \in \mathcal{S}_{\leq}(T)$. Such a form exists. Indeed, by the definition of mass, there exists a smooth form ω, such that $|\omega(x)| \leq 1$ for all $x \in \mathbb{R}^n$ and $T(\omega) \geq \mathbf{M}(T) - \epsilon$. Now, given $R \in \mathcal{S}_{\leq}(T)$, it holds that

$$R(\omega) = T(\omega) - (T-R)(\omega) \geq \mathbf{M}(T) - \epsilon - \mathbf{M}(T-R) \geq \mathbf{M}(R) - \epsilon.$$

By definition of flat convergence, $S_j(\tilde{\omega}) \to S(\tilde{\omega})$, which implies that $\mathbf{M}(S_j) \leq \mathbf{M}(S) - \epsilon - \epsilon$, for all j large enough. Since ϵ is arbitrary, $\mathbf{M}(S_j) \to \mathbf{M}(S)$. $\square$

As a consequence, to a $\|T\|$-Lebesgue integrable function f in $\mathbb{R}^n$, one can associate the continuous additive function on $\mathcal{S}_{\leq}(T)$:

$$\tilde{\Lambda}_f : S \mapsto \int f \, \mathrm{d}\|S\|.$$

In the definition of Θ_f, one can ask whether the continuity assumption of f on $\operatorname{spt} T$ can be relaxed. For instance, if f is continuous on $\operatorname{set}_1 \|T\|$, is that sufficient for Θ_f to be continuous? Clearly, if T is indecomposable, $\operatorname{set}_1 \|T\| = \operatorname{spt} T$, but if one considers a current that has a countable decomposition, things are different:

Proposition 3.5. — *There exists an integral current T of dimension 1 in $\mathbb{R}^2$ along with a bounded function f continuous on $\operatorname{set}_1 \|T\|$, but not on $\operatorname{spt} T$ such that the function on $\mathcal{S}_{\leq}(T)$ associated to the variation of f:*

$$\Theta_f : S \mapsto \partial S(f)$$

is not continuous.

Proof. — Consider a union of disjoint circles $\bigcup_{j=1}^{\infty} C_j$, where for $j = 1, 2, \ldots$, C_j is centered at $(a_j, 0) = (2^{-j}, 0)$ and has radius $r_j := 3^{-(j+1)}$. Define the function f piecewise on each C_j, so that $f = 1$ at the top (the point $(2^{-j}, 3^{-j-1})$) of each circle, $f = -1$ at the bottom $((2^{-j}, -3^{-j-1}))$ of each circle and f is smooth. A good choice is $f(x_1, x_2) = r_j^{-1} y$ if $(x_1, x_2) \in C_j$. Let $\overrightarrow{T}$ be a field of tangent unit vectors to the circles, oriented positively and

$$T := \left(\mathcal{H}^2 \llcorner \bigcup_j C_j \right) \overrightarrow{T}.$$

Clearly, $\operatorname{spt} T = \bigcup_j C_j \cup \{(0,0)\}$. Let us check that $\operatorname{set}_1 \|T\| = \bigcup_j C_j$: for $r > 0$ if $2^{1-j_0} \leq r \leq 2^{-j_0}$, it holds that

$$\|T\|(\mathrm{U}(0, r)) \leq \sum_{j \geq j_0} 2\pi r_j \leq 3^{j_0} \pi.$$

Thus $\Theta^{1*}(\|T\|, 0) = 0$ and $0 \notin \operatorname{set}_1 \|T\|$.

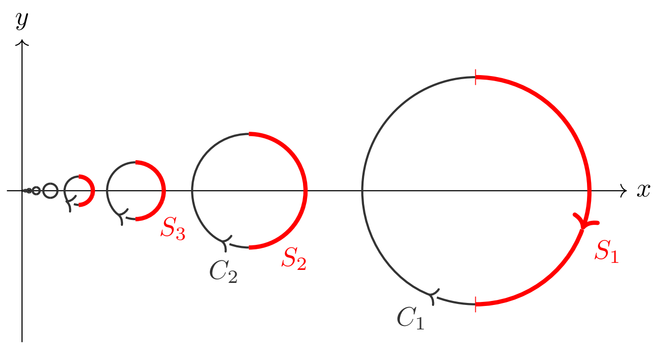

FIGURE 3.2. The current T and the sequence $(S_j)_j$ of pieces

Consider the sequence of pieces $S_j \in \mathcal{S}_{\leq}(T)$ corresponding to the half circles: $S_j = T \llcorner \{(x_1, x_2), 2^{-j} \leq x_1 \leq 2^{-j} + 3^{-j-1}\}$ (see Figure 3.2); S_j tends to 0 in mass and for all j, $\mathbf{M}(\partial S_j) = 2$. However, $\partial S_j(f) = 2 \not\to 0$. Therefore, the function $S \mapsto \partial S(f)$ is not continuous on $\mathcal{S}_{\leq}(T)$. $\square$

3.4. Derivation. — We use the terms derivation and derivate, following H. Federer [16, Section 2.9]. For a function on $\mathcal{S}_{\leq}(T)$, there is a notion of derivation along T, similar to the differentiation of measures in Radon-Nikodym theory:

Definition 3.6. — For x in $\operatorname{spt} T$ and $\delta > 0$, consider the subset $\mathcal{S}_{\leq}(T, x, \delta)$ of $\mathcal{S}_{\leq}(T)$ consisting of all pieces S of T such that

1. $x \in \operatorname{spt} S$,
2. S is indecomposable,
3. $\operatorname{diam} \operatorname{spt} S < \delta$.

If $\mathcal{S}_{\leq}(T, x, \delta)$ is not empty for some positive δ, the point x is called **good in** T. In this case, we can define the **upper and lower derivates** of F along T at x, respectively, as

$$\overline{\mathfrak{D}}_T F(x) := \inf_{\delta > 0} \sup_{\mathcal{S}_{\leq}(T,x,\delta)} \frac{F(S)}{\mathbf{M}(S)} \quad \text{and} \quad \underline{\mathfrak{D}}_T F(x) := \sup_{\delta > 0} \inf_{\mathcal{S}_{\leq}(T,x,\delta)} \frac{F(S)}{\mathbf{M}(S)}.$$

F is **derivable along** T **at** $x \in \operatorname{set}_1 \|T\|$ if the upper and lower derivates of F at x along T coincide, and the corresponding **derivate** is denoted $\mathfrak{D}_T F(x)$.

A related notion we will use is that of almost derivability: a function F on $\mathcal{S}_{\leq}(T)$ is **almost derivable** at $x \in \operatorname{set}_1 \|T\|$ if the upper and lower derivates of F along T at x are finite.

We denote by $\operatorname{Indec}(T)$ the set of points $x \in \mathbb{R}^n$, such that $\mathcal{S}_{\leq}(T, x, \delta)$ is not empty for some $\delta > 0$; T has upper density at least $1/2$ at a point of $\operatorname{Indec}(T)$, and thus it holds that $\operatorname{Indec}(T) \subseteq \operatorname{set}_1 \|T\|$ and

$$\mathcal{H}^1(\operatorname{set}_1 \|T\| \setminus \operatorname{Indec}(T)) = 0.$$

However, the latter set can be large, as we now show:

Proposition 3.7. — *There exists an integral current T of dimension 1 in $\mathbb{R}^2$ such that $\operatorname{set}_1 \|T\| \setminus \operatorname{Indec}(T)$ is uncountable.*

Proof. — A way to define such a set is to consider a fat Cantor subset of $[0, 1]$. For instance, one could let C be the set obtained by removing iteratively the middle intervals of length 4^{-k} for $k = 1, 2, \ldots$ from $[0, 1]$; C is a compact totally disconnected set with $\mathcal{L}^1(C) = 1/2 > 0$.

For each $k = 1, 2, \ldots$, there are 2^{k-1} segments of length 4^{-k} in the complement of C; denote them by S_k^j for $j = 1, 2, \ldots, 2^{k-1}$. In $\mathbb{R}^2$, let R_k^j be the square $S_k^j \times [0, 4^{-k}]$. We can consider the current defined by

$$T := \sum_{k,j} [\![\operatorname{bdry} R_k^j]\!],$$

where the boundary curves of the squares are given a canonical orientation (see Figure 3.3); T is a cycle that has a finite mass equal to four times the length of the complement of C in $[0, 1]$, and, therefore, $T \in \mathbf{I}_1(\mathbb{R}^2)$. Clearly, $\operatorname{spt} T = (C \times \{0\}) \cup \bigcup_{k,j} \operatorname{bdry} R_k^j \supseteq [0, 1]^2$. The question is how to characterize $\operatorname{set}_1 \|T\|$, and whether there exist points of $C \cap \operatorname{set}_1 \|T\|$, such that there is no indecomposable piece S of T with $x \in \operatorname{spt} S$.

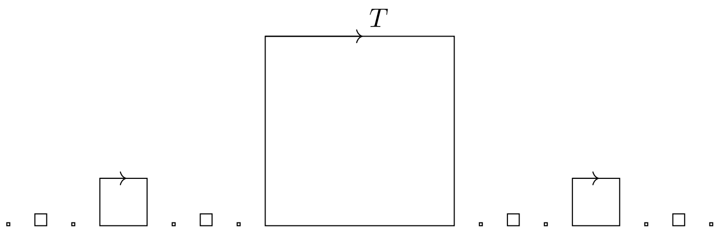

Figure 3.3. A current T for which $\mathrm{set}_1 \|T\| \setminus \mathrm{Indec}(T)$ is uncountable.

Claim 1. — Suppose S is an indecomposable piece of T, then S is a piece of $[\![\,\mathrm{bdry}\, R_k^j]\!]$ for some $k \in \{1, 2, \dots\}$ and $j \in \{1, \dots, 2^{k-1}\}$.

Proof. — By contradiction, let $S \in \mathcal{S}_{\leq}(T)$ be indecomposable and fix $x \in \mathrm{spt}\, S \cap \mathrm{bdry}\, R_k^j$ and $x' \in \mathrm{spt}\, S \cap \mathrm{bdry}\, R_{k'}^{j'}$ with $(k, j) \neq (k', j')$. Without loss of generality, we can suppose that $\partial S = \delta_{x'} - \delta_x$. We can also suppose that $x = (x_1, 0)$ and $x' = (x_1', 0)$ with $x_1 < x_1'$ and $x_1 = \max\{t \in S_k^j\}$, $x' = \min\{t, t \in S_{k'}^{j'}\}$. As S is indecomposable, and the differential form $(z_1, z_2) \mapsto \mathbf{e}_1^*$ is the differential of $(z_1, z_2) \mapsto z_1$, it holds that

$$\int \langle \mathbf{e}_1^*, \overrightarrow{S} \rangle \, \mathrm{d}\|S\| = x_1' - x_1.$$

However, since S is supported inside $[\, x_1, x_1' \,] \times \mathbb{R}$ and $\overrightarrow{S} = \overrightarrow{T}$, it holds that $\|S\|$ almost everywhere

$$\begin{aligned} \int \langle \mathbf{e}_1^*, \overrightarrow{S} \rangle \, \mathrm{d}\|S\| &\leq \|T\|(([\, x_1, x_1' \,] \times \mathbb{R}) \cap \{(z_1, z_2), \overrightarrow{T}(z_1, z_2) = \mathbf{e}_1\}) \\ &\leq \mathcal{L}^1(C^c \cap [\, x_1, x_1' \,]) < x_1' - x_1, \end{aligned}$$

where we used the fact that $C \cap [\, x_1, x_1' \,]$ contains a fat Cantor subset of C, which has positive Lebesgue measure. This is a contradiction. ■

The above claim implies that for all $x \in C \setminus \bigcup_{k,j} \mathrm{cl}\, S_k^j$, x is not in the support of any indecomposable piece of T. It remains to prove that $C \cap \mathrm{set}_1 \|T\| \setminus \bigcup_{k,j} \mathrm{cl}\, S_k^j$ is uncountable. For $x \in (0,1)$, $\Theta^{1*}(\|T\|, x) \geq \Theta^{*1}(\mathcal{H}^1 \llcorner E^c, x) = 1 - \Theta_*^1(\mathcal{H}^1 \llcorner C, x)$, so we only need to prove that $C \cap \{x, \Theta_*^1(\mathcal{H}^1 \llcorner C, x) < 1\}$ is uncountable.

In [9, Theorem 1], Buczolich proved that the set of points of a nowhere dense perfect set $P \subseteq \mathbb{R}$, where P has lower density larger than γ for any $\gamma > 0.5$ is always of first category in P. This implies that the set of points of density less than 1 is of second category in P, which, in turn, implies that it is uncountable (P is a Baire space with the topology inherited from $\mathbb{R}$, see, for instance, [30, Chapter 9]). Note that there are more precise ways to characterize the points of a Cantor set with given densities, see, for instance, the paper by Besicovitch [4]. □

Proposition 3.8. — *Let Λ_f be the function on $\mathcal{S}_\leq(T)$ associated to a Lebesgue $\|T\|$ integrable function f defined almost everywhere on* $\operatorname{set}_1\|T\|$ *by*

$$\Lambda_f : S \mapsto \int f \,\mathrm{d}\|S\|.$$

If f is continuous at $x \in \operatorname{set}_1\|T\|$, and x is good in T, then Λ_f is derivable at x along T with derivate $\mathfrak{D}_T\, F(x) = f(x)$.

Proof. — For a good point $x \in \operatorname{spt} T$, $\epsilon > 0$, choose $\delta > 0$, such that $|f(y) - f(x)| < \epsilon$ for all $y \in \mathrm{U}(x, \delta)$. For $S \in \mathcal{S}_\leq(T, x, \delta)$

$$|\Lambda_f(S) - f(x)\,\mathbf{M}(S)| \leq \int |f(y) - f(x)|\,\mathrm{d}\|S\|(y) \leq \epsilon\,\mathbf{M}(S).$$

Letting ϵ go to zero, we can conclude. □

Question 3.9. — *If F is a continuous function defined on $\mathcal{S}_\leq(T)$, are the extended real valued functions $\underline{\mathfrak{D}}_T F$, $\overline{\mathfrak{D}}_T F$ and $\mathfrak{D}_T\, F$ $\|T\|$ measurable? Are they Borel measurable?*

For Henstock-Kurzweil integration in one dimension and for Pfeffer integration on sets of finite perimeter, such results rely on the Vitali covering theorem and a derivation operation. A "covering" theorem using pieces of T would be useful. An alternative would be to study a suitable decomposition of T, but this approach is made difficult by the fact that there can be pieces of T that do not belong to any decomposition of T (recall Figure 3.1).

Definition 3.10. — Let T be an integral current of dimension 1 in $\mathbb{R}^n$ and let u be a function defined on $\operatorname{set}_1\|T\|$. Fix a good point $x \in \operatorname{set}_1\|T\|$. The function u is **differentiable along** T at x if there exists a linear form $\mathrm{D}u(x)$ on $\mathbb{R}^n$, such that for all $\epsilon > 0$, there exists $\delta > 0$, such that whenever $y \in \operatorname{set}_1\|T\| \cap \mathrm{U}(x, \delta)$, and there is an $S \in \mathcal{S}_\leq(T, x, 3\delta)$ with $y \in \operatorname{set}_1\|S\|$, it holds that

$$|u(y) - u(x) - \mathrm{D}u(x)\cdot(y - x)| \leq \epsilon|y - x|.$$

Note that if u is differentiable in $\mathbb{R}^n$ or differentiable on $\operatorname{spt} T$ in the sense of Whitney [39], then u is differentiable along T with the same differential.

Theorem 3.11. — *Suppose that u is a continuous function on* $\operatorname{spt} T$ *for some $T \in \mathbf{I}_1(\mathbb{R}^n)$. Fix $x \in \operatorname{set}_1\|T\|$, such that $\mathcal{S}_\leq(T, x, \delta) \neq \emptyset$ for some $\delta > 0$, then the following three statements hold:*

(i) If u has pointwise Lipschitz constant $\operatorname{Lip}_x u = 0$ at x, then Θ_u is derivable at x along T, and $\mathfrak{D}_T\,\Theta_u(x) = 0$.

(ii) If u is pointwise Lipschitz at x, then Θ_u is almost derivable at x with $-\operatorname{Lip}_x u \leq \underline{\mathfrak{D}}_T\Theta_u \leq \overline{\mathfrak{D}}_T\Theta_u \leq Lip_x u$.

(iii) If $\overrightarrow{T}$ has a $\|T\|$ approximately continuous representative at x (which we still denote by $\overrightarrow{T}$), $\|T\|$ has finite upper density at x and u is differentiable at x along T, then Θ_u is derivable at x along T, with $\mathfrak{D}_T\,\Theta_u(x) = \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$.

Remark 3.12. — The assumption that $\overrightarrow{T}$ has a $\|T\|$ approximately continuous representative at x is satisfied for $\|T\|$ almost all x. (We will prove that as Claim 1 in the proof of Proposition 4.3.)

Proof. — Let us start with (i) and (ii). For $\epsilon > 0$, there exists δ such that whenever $y \in \operatorname{spt} T$ with $|y-x| < \delta$,

$$|u(y) - u(x)| < (M+\epsilon)|y-x|,$$

with $M := \operatorname{Lip}_x u$. Given an indecomposable $S \in \mathcal{S}_{\leq}(T)$, with $x \in \operatorname{spt} S$ and $\operatorname{diam}\operatorname{spt} S < \delta$, S is of the form $\gamma_{\#}[\![0, \mathbf{M}(S)]\!]$ with $\gamma(0) = y_-$ and $\gamma(\mathbf{M}(S)) = y_+$. Since $|y_+ - x| + |x - y_-| \leq \mathbf{M}(S)$, we get

$$\begin{aligned}|\Theta_u(S)| &= |u(y_+) - u(y_-)|\\ &\leq |u(y_+) - u(x)| + |u(x) - u(y_-)|\\ &\leq (M+\epsilon)\,\mathbf{M}(S).\end{aligned}$$

As ϵ is arbitrary, this is enough to prove (ii) and (i), where we have $M = 0$. We turn to (iii).

If $\mathrm{D}u(x) = 0$, we can apply (i), and, thus, we can suppose $\mathrm{D}u(x) \neq 0$. Fix $\epsilon > 0$. There exists $\delta_1 > 0$ such that for any $r \in (0, \delta_1)$,

$$\frac{\|T\|(\mathrm{B}(x,r))}{2r} \leq 2\theta, \tag{5}$$

with $\theta := \Theta^{1*}(\|T\|, x) \in (0, +\infty)$. Replace $\overrightarrow{T}$ with its $\|T\|$ approximately continuous representative at x. Denote by $E_{x,\epsilon}$ the set

$$E_{x,\epsilon} := \operatorname{set}_1 \|T\| \cap \left\{ y, |\overrightarrow{T}(y) - \overrightarrow{T}(x)| > \frac{\epsilon}{2|\,\mathrm{D}u(x)|} \right\}.$$

There exists $\delta_2 > 0$, which we can suppose to be less than or equal to δ_1, such that whenever $r \in (0, \delta_2)$,

$$\frac{\|T\|(\mathrm{B}(x,r) \cap E_{x,\epsilon})}{\|T\|(\mathrm{B}(x,r))} < \frac{\epsilon}{4\theta|\,\mathrm{D}u(x)|}. \tag{6}$$

For $S \in \mathcal{S}_{\leq}(T, x, \delta_2)$, the vector field $\overrightarrow{S}$ is equal to $\overrightarrow{T}$ $\|S\|$ almost everywhere and if furthermore S represents a curve joining x to y, then we have $\partial S = \delta_y - \delta_x$. As for $j = 1, \ldots, n$ the 1 form $z \mapsto \mathbf{e}_j^*$ is the differential of the 0-form $z \mapsto z_j$, we can write:

$$\begin{aligned} y - x = (y_1 - x_1)\mathbf{e}_1 + \cdots + (y_n - x_n)\mathbf{e}_n &= \sum_{j=1}^{n} \partial S(z \mapsto z_j)\mathbf{e}_j \\ = \sum_{j=1}^{n} S(z \mapsto \mathbf{e}_j^*)\mathbf{e}_j &= \sum_{j=1}^{n} \int \langle \mathbf{e}_j^*, \overrightarrow{T} \rangle \, \mathrm{d}\|S\| \mathbf{e}_j = \int \overrightarrow{T} \, \mathrm{d}\|S\|. \end{aligned}$$

The same identity with opposite sign is true if $\partial S = \delta_x - \delta_y$ instead. Denote by d_S the diameter of $\operatorname{spt} S$. By (5) and (6),

$$\begin{aligned} |y - x - \mathbf{M}(S)\overrightarrow{T}(x)| &\leq \int |\overrightarrow{T}(x') - \overrightarrow{T}(x)| \, \mathrm{d}\|S\|(x') \\ &\leq 2\|S\| \left(E_{x,\epsilon} \cap \mathrm{B}(x, d_S)\right) + \frac{\epsilon}{2|\,\mathrm{D}u(x)|} \mathbf{M}(S), \end{aligned}$$

as $|\overrightarrow{T}(x') - \overrightarrow{T}(x)| \leq 2$ for $\|T\|$ almost all x', in particular in the exceptional set $E_{x,\epsilon}$. Furthermore, as $\|S\| \leq \|T\|$, and by (6) we have

$$\begin{aligned} |y - x - \mathbf{M}(S)\overrightarrow{T}(x)| &\leq \frac{\epsilon\|T\|(\mathrm{B}(x, d_s))}{2\theta|\,\mathrm{D}u(x)|} + \frac{\epsilon}{2|\,\mathrm{D}u(x)|} \mathbf{M}(S) \\ &\leq \frac{2\epsilon\theta d_S}{\theta|\,\mathrm{D}u(x)|} + \frac{\epsilon}{2|\,\mathrm{D}u(x)|} \mathbf{M}(S). \end{aligned}$$

Finally, as S is indecomposable, it holds that $d_S \leq \mathbf{M}(S)$ and

$$|y - x - \mathbf{M}(S)\overrightarrow{T}(x)| \leq \frac{5\epsilon}{2|\,\mathrm{D}u(x)|} \mathbf{M}(S). \tag{7}$$

By differentiability of u along T at x, there exists $\delta_3 > 0$, such that for $y \in \mathrm{U}(0, \delta_3) \cap \operatorname{set}_1 \|T\|$, such that there exists $S \in \mathcal{S}_{\leq}(T, x, \delta_3)$ with $y \in \operatorname{spt} S$,

$$|u(y) - u(x) - \langle \mathrm{D}u(x), y - x\rangle| < \epsilon|y - x|.$$

Let $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ and choose $S \in \mathcal{S}_{\leq}(T, x, \delta)$. We can write S as $S^+ + S^-$, where S^+ and S^- are indecomposable, $\partial S^+ = \delta_{y^+} - \delta_x$ and $\partial S^- = \delta_x - \delta_{y^-}$, with $\mathbf{M}(S) = \mathbf{M}(S^+) + \mathbf{M}(S^-)$, and we have

$$\Theta_u(S) = \Theta_u(S^+) + \Theta_u(S^-) = u(y^+) - u(x) + u(x) - u(y^-).$$

Thus, we can write

$$\begin{aligned}|\Theta_u(S) - \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle\, \mathbf{M}(S)| \\ &\leq |u(y^+) - u(x) - \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle\, \mathbf{M}(S^+)| \\ &\quad + |u(x) - u(y^-) - \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle\, \mathbf{M}(S^-)|\end{aligned}$$

and study only the first term of the right-hand side. We have

$$\begin{aligned}|u(y^+) - u(x) - \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle\, \mathbf{M}(S^+)| \\ &\leq |u(y^+) - u(x) - \langle \mathrm{D}u(x), y - x\rangle| \\ &\quad + |\langle \mathrm{D}u(x), y^+ - x\rangle - \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle\, \mathbf{M}(S^+)| \\ &\leq \epsilon|y^+ - x| + |\,\mathrm{D}u(x)||y^+ - x - \mathbf{M}(S^+)\overrightarrow{T}(x)| \\ &\leq 4\epsilon\, \mathbf{M}(S^+),\end{aligned}$$

by (7) applied to S^+. Doing the same with S^- and summing concludes the proof: there exists $\delta > 0$ such that for all $S \in \mathcal{S}_{\leq}(T, x, \delta)$,

$$|\Theta_u(S) - \mathbf{M}(S)\langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle| \leq \epsilon\, \mathbf{M}(S)$$

and Θ_u is, thus, differentiable along T at x. □

If one assumes only approximate continuity of the tangent – as we just did – the assumption that the currents S used in the derivation are indecomposable is necessary:

Example 3.13. — Consider the function $h : (x, y) \mapsto y$ and the current T associated to an infinite staircase with steps indexed by j, with height (y length) 3^{-j} and length (x-length) 2^{-j} symmetric in the x direction, converging at $(0, 0)$ (see Figure 3.4). If one considers a sequence of subcurrents S_j composed of a very small "intervals" (length 4^{-j}) around 0 and a vertical part of the step, it

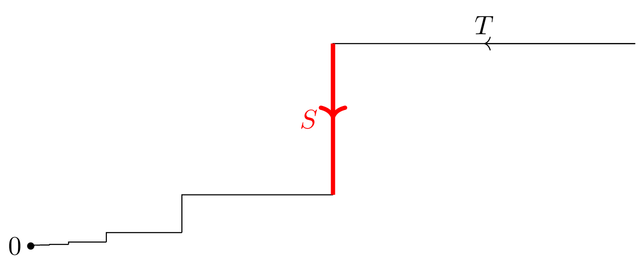

Figure 3.4. The piece S is not suitable for a differentiation basis of T at 0.

holds that

$$\Theta_h(S_j) = 3^{-j}C + o(3^{-j}).$$

Thus $\lim_j \Theta_h(S_j)/\mathbf{M}(S_j) = C > 0$.

However, if we consider a sequence of indecomposable currents R_j, with $0 \in \operatorname{spt} R_j$ and $\mathbf{M}(R_j) \to 0$, by the above Theorem we will get $\Theta_h(R_j)/\mathbf{M}(R_j) \to 0$.

An alternative restriction would be to bound the *regularity* of the pieces. This is actually how one proceeds in higher dimensions in [32, 22], as indecomposability is not a useful notion in dimensions strictly larger than 1.

4. Integration

We first need an analogue to Cousin's lemma in order to decompose a current of dimension 1 into small pieces.

4.1. Howard Cousin's lemma for currents of dimension 1. — Given a current $T \in \mathbf{I}_1(\mathbb{R}^n)$ and a gauge on $\operatorname{set}_1 \|T\|$, a **tagged family in** T is a finite collection $\mathcal{P}$ of pairs (S_j, x_j) for $j = 1, \ldots, p$, where

$$\begin{aligned}
&S_j \in \mathcal{S}_{\leq}(T) \text{ is indecomposable,}\\
&x_j \in \operatorname{set}_1 \|T\| \cap \operatorname{spt} S_j \quad \text{and}\\
&\sum_{j=1}^{p} \|S_j\| \leq \|T\|.
\end{aligned}$$

(If T has multiplicity 1 almost everywhere, the last condition prevents the pieces from overlapping.) Such a tagged family is **subordinate to a decomposition** $T_1, T_2, \ldots$ of T, if there exists a partition of $\mathcal{P}$ indexed by k into families $\mathcal{P}_k$ each in the respective T_k.

A **gauge** on a set E is a non-negative function δ such that the set $\{x \in E, \delta(x) = 0\}$ is countable. If δ is a gauge on a set $E \subseteq \operatorname{set}_1 \|T\|$, a δ**-fine** tagged family in T is a tagged family as above, such that for all $(S, x) \in \mathcal{P}$, it holds that $x \in E$ and $\operatorname{diam} \operatorname{spt} S < \delta(x)$. Furthermore, given a non-negative subadditive function G on $\mathcal{S}_{\leq}(T)$, and a positive real number τ, a tagged family $\mathcal{P}$ is (G, τ)**-full** if $G(T - [\mathcal{P}]) < \tau$.

Lemma 4.1 (Howard-Cousin lemma). — *Let T be an integral current of dimension 1 in $\mathbb{R}^n$. Let F be a subadditive continuous function on $\mathcal{S}_{\leq}(T)$. Given $\epsilon > 0$ and δ a gauge on $\operatorname{set}_1 \|T\|$, for any decomposition $T_1, T_2, \ldots$, there exists a $(|F|, \epsilon)$-full, δ-fine tagged family subordinate to this decomposition.*

Proof. — Fix a decomposition of T. For each k choose $\gamma_k : [\,0, \mathbf{M}(T_k)\,] \to \mathbb{R}^n$ to parameterize T_k by arc-length, so that $T_k = \gamma_{k\#}[\![0, \mathbf{M}(T_k)]\!]$. Let $\delta_k := \delta \circ \gamma_k$; it is a gauge on $I_k := [\,0, \mathbf{M}(T_k)\,]$.

Since T is integral, there exists k_0 such that, for all $k > k_0$, T_k is a cycle. Also, $\mathbf{M}(T_k) \to 0$ as $k \to \infty$. Since F is continuous and subadditive, there exists k_ϵ such that

$$\left| F\left(T - \sum_{k=1}^{k_\epsilon} T_k \right) \right| < \frac{\epsilon}{2}. \tag{8}$$

For $k = 1, 2, \ldots, k_0$, consider the interval $I_k = [\,0, \mathbf{M}(T_k)\,]$, along with the gauge δ_k and the continuous additive function $\gamma_k^{\#} F$ on $\mathcal{S}_{\leq}([\![I_k]\!])$ defined by $\gamma_k^{\#} F([\![a,b]\!]) = F(\gamma_{k\#}([\![a,b]\!])$, for $0 \leq a < b \leq \mathbf{M}(T)$. Note that it is enough to define $\gamma_k^{\#} F$ on indecomposable pieces of $[\![I_k]\!]$, as in this case, all pieces are a finite sum of disjoint indecomposable pieces. Apply Lemma 2.12 to I_k, $\delta \circ \gamma_k$, $|\gamma_k^{\#} F|$ and $\epsilon/(2k_0)$ to get a $\delta \circ \gamma_k$ fine $(\gamma_k^{\#}, \epsilon/(2k_0))$-full tagged family $\mathcal{P}_k$ in I_k.

The collection $\gamma_{\#}\mathcal{P}_k$ defined by $\{(\gamma_{k\#} S, \gamma_k(x)), (S,x) \in \mathcal{P}_k\}$ is a δ-fine tagged family in T_k (as γ_k has Lipschitz constant 1), which satisfies

$$|F(T_k - [\gamma_{\#}\mathcal{P}_k])| = (\gamma_k^{\#} F)([\![I_k]\!] - [\mathcal{P}_k])| < \frac{\epsilon}{2k_0}.$$

Summing over $k = 1, 2, \ldots, k_0$ and using (8) yields

$$\left| F\left(\sum_{k=1}^{k_0} T_k - \left[\bigcup_{k=1}^{k_0} \gamma_{k\#}\mathcal{P}_k \right] \right) \right| < \frac{\epsilon}{2}.$$

The collection $\mathcal{P} := \bigcup_{k=1}^{k_0} \gamma_{k\#}\mathcal{P}_k$ is, therefore, a tagged family in T that is δ-fine and (F, ϵ)-full. $\square$

4.2. *AC*$_*$ functions on $\mathcal{S}_{\leq}(T)$. — A function F on $\mathcal{S}_{\leq}(T)$ is AC_* if given a $\|T\|$ null set $E \subset \operatorname{set}_1 \|T\|$, for every $\epsilon > 0$, there exists a gauge δ on E with

$$|F([\mathcal{P}])| < \epsilon,$$

whenever $\mathcal{P}$ is a δ-fine tagged family in T. We say that a tagged family is **anchored** in a set E, if for all (S,x) in this tagged family, $x \in E$. Here, as the gauge δ is defined only on E, $\mathcal{P}$ is automatically anchored in E. The next two propositions are adapted from [34, Theorems 3.6.6. and 3.6.7].

Proposition 4.2. — *If F is a continuous additive function on $\mathcal{S}_{\leq}(T)$, which is AC_*, and such that $\underline{\mathfrak{D}}_T F(x) \geq 0$ almost everywhere, then F is non-negative, i.e. for all $S \in \mathcal{S}_{\leq}(T)$, $F(S) \geq 0$.*

Proof. — It is enough to prove that $F(T) \geq 0$, indeed, if T' is in $\mathcal{S}_{\leq}(T)$, the restriction of F to $\mathcal{S}_{\leq}(T')$ satisfies the hypothesis of the proposition. Let N be the set of points x, such that $\underline{\mathfrak{D}}_T F(x) < 0$. For $\epsilon > 0$, there exists a gauge δ_N on N, such that $|F([\mathcal{P}])| < \epsilon$, whenever $\mathcal{P}$ is a δ_N fine tagged family anchored in N. For each x at which $\underline{\mathfrak{D}}_T F(x) \geq 0$, there exists Δ_x, such that for all

$S \in \mathcal{S}_{\leq}(T, x, \Delta_x)$, $F(S) \geq -\epsilon\, \mathbf{M}(S)/\, \mathbf{M}(T)$. Define a gauge δ on $\mathrm{set}_1 \|T\|$ by letting

$$\delta(x) := \begin{cases} \delta_N(x) & \text{if } x \in N, \\ \Delta_x & \text{otherwise.} \end{cases}$$

Using Lemma 4.1, find a δ-fine tagged family $\mathcal{P}$ in T with $|F(T - [\mathcal{P}])| < \epsilon$. Let $\mathcal{P}_N$ be the subfamily of $\mathcal{P}$ consisting of all the elements anchored in N. Denoting by $\mathcal{P}^*$ the complement of $\mathcal{P}$ yields:

$$F(T) \geq F([\mathcal{P}]) - F(T - [\mathcal{P}]) \geq F([\mathcal{P}^*]) + F([\mathcal{P}_N]) - \epsilon \geq -3\epsilon.$$

Since ϵ is arbitrary, $F(T) \geq 0$. □

Proposition 4.3. — *If a continuous additive function F is almost derivable everywhere in* $\mathrm{set}_1 \|T\|$*, except in a countable set E_T, then F is AC_*.*

Proof. — Let N be a $\|T\|$ null set. For $\epsilon > 0$, and $k = 1, 2, \ldots$, let U_k be a neighbourhood of N with $\|T\|(U_k) < 2^{-k}\epsilon/k$. For $x \in N \setminus E_T$, choose a positive integer k_x and a positive Δ_x, such that $\mathrm{U}(x, \Delta_x) \subseteq U_{k_x}$, and for all $S \in \mathcal{S}_{\leq}(T, x, \Delta_x)$, $|F(S)| \leq k_x\, \mathbf{M}(S)$; k_x and Δ_x exist by almost derivability of F at x. Define a gauge δ on N by

$$\delta(x) := \begin{cases} 0 & \text{if } x \in E_T, \\ \Delta_x & \text{if } x \in N \setminus E_T. \end{cases}$$

Given a δ-fine tagged family $\mathcal{P}$ anchored in N, partition $\mathcal{P}$ into families $\mathcal{P}_k$ for $k = 1, 2, \ldots$, such that $(S, x) \in \mathcal{P}_k$ if and only if $k_x = k$. All but finitely many of these families are empty, and it holds that

$$|F([\mathcal{P}])| \leq \sum_{k=1}^{\infty} \sum_{(S,x)\in\mathcal{P}_k} |F(S)| \leq \sum_{k=1}^{\infty} k \sum_{(S,x)\in\mathcal{P}_k} \mathbf{M}(S) \leq \sum_{k=1}^{\infty} k\|T\|(U_k) < \epsilon. \qquad \square$$

4.3. The $\mathcal{R}_1$ integral on integral currents of dimension 1. —

Definition 4.4. — A function f defined $\|T\|$ almost everywhere on $\mathrm{set}_1 \|T\|$, is $\mathcal{R}_1$ **integrable on** T if there exists a continuous additive function F on $\mathcal{S}_{\leq}(T)$, and for every $\epsilon > 0$, there exists a gauge δ and a positive real number τ, such that whenever $\mathcal{P}$ is a δ-fine tagged family in T with $|F(T - [\mathcal{P}])| < \tau$, it holds that:

$$|F(T) - \sigma(f, \mathcal{P})| < \epsilon. \tag{9}$$

(Where $\sigma(f, \mathcal{P})$ denotes the Riemann sum $\sum_{(x,S)\in\mathcal{P}} f(x)\, \mathbf{M}(S)$.)

$F(T)$ is also the $\mathcal{R}_1$ integral of f on T, and we sometimes denote it $(\mathcal{R}_1) \int_T f$.

Question 4.5. — *Is it equivalent to asking that each family be subordinate to some decomposition? This is not clear because a piece of T can very well not be a piece of any decomposition (see Figure 3.1).*

According to Example 2.10, it is not sufficient to be integrable on all elements of one given decomposition to be integrable on the whole current. However, suppose f is integrable on each piece for two decompositions, is the integral the same?

We list the main basic properties of the integral. The proofs of the two first ones use elementary comparisons and the fact that given two gauges δ_1 and δ_2, the minimum of the two is a gauge and that if $\mathcal{P}$ is a $\min(\delta_1, \delta_2)$-fine family, it is also δ_1 and δ_2-fine. Similarly, if $\tau_1 \leq \tau_2$ and $\mathcal{P}$ is (G, τ_1)-full in T, then it is (G, τ_2)-full.

Proposition 4.6. — *The space of $\mathcal{R}_1$ integrable functions on T is a linear space and the integral $f \mapsto I(f, T)$ is linear on this space. Furthermore, if $f \leq g$ and f and g are $\mathcal{R}_1$ integrable on T, then $(\mathcal{R}_1) \int_T f \leq (\mathcal{R}_1) \int_T g$.*

Proposition 4.7 (Cauchy criterion). — *A function f is $\mathcal{R}_1$ integrable on T if and only if there is a continuous non-negative subadditive function G on $\mathcal{S}_{\leq}(T)$ and for every $\epsilon > 0$, there exists a gauge δ and a positive real number τ such that for any two δ-fine (G, τ)-full families $\mathcal{P}_1$ and $\mathcal{P}_2$,*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \epsilon. \tag{10}$$

Proposition 4.8. — *Let f be $\mathcal{R}_1$ integrable on the current $T \in \mathbf{I}_1(\mathbb{R}^n)$. For all $S \in \mathcal{S}_{\leq}(T)$, f is $\mathcal{R}_1$ integrable on S and $T - S$, and $I(f, S) + I(f, T - S) = I(f, T)$.*

Proof. — Let G be a continuous non-negative subadditive function on $\mathcal{S}_{\leq}(T)$ associated to the integrability of f on T. Fix $S \in \mathcal{S}_{\leq}(T)$ and notice first that $G \llcorner \mathcal{S}_{\leq}(S)$ and $G \llcorner \mathcal{S}_{\leq}(T - S)$ are also non-negative, continuous and subadditive. Given $\epsilon > 0$, choose a gauge δ on $\operatorname{set}_1 \|T\|$ and a positive τ associated to $\epsilon/2$ in the definition of integrability of f; $\delta \llcorner \operatorname{set}_1 \|T - S\|$ is a gauge on $\operatorname{set}_1 \|T - S\|$, so by Lemma 4.1, there exists a δ-fine $(G \llcorner \mathcal{S}_{\leq}(T - S), \tau/2)$-full tagged family $\mathcal{P}$ in $T - S$. Now, given two δ-fine $(G \llcorner \mathcal{S}_{\leq}(S), \tau/2)$-full families in S: $\mathcal{P}_1$ and $\mathcal{P}_2$, we define the concatenations $\mathcal{P} \cup \mathcal{P}_1$ and $\mathcal{P} \cup \mathcal{P}_2$. Since $[\mathcal{P}] \in \mathcal{S}_{\leq}(T - S)$ and $[\mathcal{P}_1], [\mathcal{P}_2] \in \mathcal{S}_{\leq}(S)$, we have $[\mathcal{P} \cup \mathcal{P}_1], [\mathcal{P} \cup \mathcal{P}_2] \in \mathcal{S}_{\leq}(T)$, so the concatenations are families in $\mathcal{S}_{\leq}(T)$. They are also δ-fine and for $j = 1, 2$,

$$\begin{aligned} G(T - [\mathcal{P} \cup \mathcal{P}_j]) &= G(T - S - [\mathcal{P}] + S - \mathcal{P}_j]) \\ &< G(T - S - [\mathcal{P}]) + G(S - \mathcal{P}_j) < \tau \end{aligned}$$

by subadditivity of G and definition of $\mathcal{P}$ an $\mathcal{P}_j$. Therefore, by Proposition 4.7

$$|\sigma(f, \mathcal{P} \cup \mathcal{P}_1) - \sigma(f, \mathcal{P} \cup \mathcal{P}_2)| = |\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \epsilon.$$

Thus, since ϵ, $\mathcal{P}_1$ and $\mathcal{P}_2$ are arbitrary, one can apply the Cauchy criterion lemma 4.7 to S, which proves that f is $\mathcal{R}_1$ integrable on S. By a similar argument f is $\mathcal{R}_1$ integrable on $T-S$. Therefore, for $\epsilon>0$, choosing a gauge δ and a positive τ adapted to the integrability of f on T, S and $T-S$ at the same time, yields for δ-fine $(G,T-S,\tau/2)$ and $(G,S,\tau/2)$-full families $\mathcal{P}$ and $\mathcal{P}'$ in $T-S$ and S, respectively,

$$\begin{aligned}
&|I(f,T)-(I(f,T-S)+I(f,S))|\\
&\quad\leq |I(f,T)-\sigma(f,\mathcal{P}\cup\mathcal{P}')|+|I(f,T-S)-\sigma(f,\mathcal{P})|+|I(f,S)-\sigma(f,\mathcal{P}')|\\
&\quad< 3\epsilon,
\end{aligned}$$

because $\mathcal{P}\cup\mathcal{P}'$ is a δ-fine (G,T,τ)-full tagged family in T. As ϵ is as small as we want, this concludes the proof. □

This allows us to define a function F on $\mathcal{S}_{\leq}(T)$ by $S\mapsto I(f,S)$, called the **indefinite integral of f (on T)**.

Proposition 4.9. — *The indefinite integral F of f defined above is additive and continuous on $\mathcal{S}_{\leq}(T)$.*

Proof. — For the additivity: let S_1 and S_2 be two pieces of T, such that $S_1+S_2\in\mathcal{S}_{\leq}(T)$. Clearly, S_1 and S_2 are pieces of S_1+S_2, so it suffices to apply Proposition 4.8 to see that $F(S_1)+F(S_2)=F(S_1+S_2)$.

For the continuity: if $(S_j)_j$ is a sequence of pieces of T converging to $0\in\mathcal{S}_w(T)$ with $\sup_j \mathbf{M}(\partial S_j)<\infty$, we want to show that $F(S_j)$ goes to zero as j tends to infinity. By additivity, it is equivalent to show that $I(f,T-S_j)\to I(f,T)$. For $\epsilon>0$ choose a gauge δ and a positive τ associated to the integrability of f on T. As can be seen above, for all j, δ and $\tau/2$ are associated to 2ϵ for the integrability of f on $T-S_j$. Let $\mathcal{P}$ be a δ-fine $(G,T-S_j,\tau/2)$-full tagged family in $T-S_j$. This satisfies

$$|\sigma(f,\mathcal{P})-F(T-S_j)|<2\epsilon.$$

By continuity of G, if j is large enough, we can suppose that

$$G(T-[\mathcal{P}])\leq G(T-S_j)+G(S_j-[\mathcal{P}])<\tau/2+\tau/2,$$

so $\mathcal{P}$ is (G,T,τ)-full, and

$$|\sigma(f,\mathcal{P})-F(T)|<\epsilon.$$

Therefore, for large enough j, $|F(T)-F(T-S_j)|<3\epsilon$, and we conclude that $F(S_j)\to 0$ as j tends to infinity. This proves that F is continuous on $\mathcal{S}_{\leq}(T)$. □

Theorem 4.10 (Saks-Henstock lemma). — *f is $\mathcal{R}_1$ integrable on T if and only if there exists a continuous additive function F on $\mathcal{S}_{\leq}(T)$ satisfying: for* *all $\epsilon > 0$, there exists a gauge δ on* $\mathrm{set}_1 \|T\|$*, such that whenever $\mathcal{P}$ is a δ-fine tagged family in T:*

$$\sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbf{M}(S)| < \epsilon. \tag{11}$$

Proof. — If the second condition in the statement is satisfied, it is straightforward to prove that f is $\mathcal{R}_1$ integrable on T, with integral $I(f,T)$ equal to $F(T)$, and the "control function" $G = |F|$, indeed, for $\epsilon > 0$, if δ is a gauge on T associated to $\epsilon/2$ in the statement of the theorem, and $\mathcal{P}$ is a δ-fine, $(G, \epsilon/2)$-full tagged family in T

$$|F(T) - \sigma(f,\mathcal{P})| \le \left| F(T) - \sum_{(S,x)\in\mathcal{P}} F(S) \right| + \sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbf{M}(S)| < \epsilon.$$

Similarly, one proves that F is the indefinite integral of f.

Conversely, suppose f is $\mathcal{R}_1$ integrable on T. The proof is very similar to that of the corresponding statement for the Henstock-Kurzweil integration. Suppose that f is $\mathcal{R}_1$ integrable on T and for $\epsilon > 0$, fix a positive number $\tau < \epsilon/4$ and a gauge δ on $\mathrm{set}_1 \|T\|$, such that whenever $\mathcal{P}$ is a δ-fine $(|F|, \tau)$-full tagged family in T,

$$|\sigma(f,\mathcal{P}) - F(T)| < \frac{\epsilon}{4}.$$

Let $\mathcal{P}$ be a δ-fine tagged family in T, without any hypothesis on the remainder $|F(T - [\mathcal{P}])|$. Notice first that since $T - [\mathcal{P}]$ is an integral current, there exists a δ-fine, $(|F|, \tau)$-full tagged family $\mathcal{Q}$ in $T - [\mathcal{P}]$, which implies that $\mathcal{P} \cup \mathcal{Q}$ is a δ-fine $(|F|, \tau/2)$-full tagged family in T, and

$$\sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbf{M}(S)| \le \sum_{(S,x)\in\mathcal{P}\cup\mathcal{Q}} |F(S) - f(x)\,\mathbf{M}(S)|. \tag{12}$$

Therefore, it is enough to prove that (11) holds for $(|F|, \tau)$-full families in T, and we suppose that $\mathcal{P}$ is $(|F|, \tau)$-full. We can write $\mathcal{P}$ as $\{(S_1, x_1), \ldots, (S_p, x_p)\}$ and, reordering, assume that for some $k_0 \le p$, if $1 \le j \le k_0$, $|F(S_j) - f(x_j)\,\mathbf{M}(S_j)| \ge 0$, whereas for $k_0 + 1 \le j \le p$, $||F(S_j) - f(x_j)\,\mathbf{M}(S_j)| < 0$. For $j = 1, \ldots, p$ use the $\mathcal{R}_1$ integrability of f on S_j to define a δ-fine, $(F \llcorner S_j, \tau/p)$-full tagged family $\mathcal{P}_j$, such that $|\sigma(f, \mathcal{P}_j) - F(S_j)| < \epsilon/(2p)$. Consider the families

$$\begin{aligned} \mathcal{P}^+ &:= \{(S_1, x_1), \ldots, (S_{k_0}, x_{k_0})\} \cup \mathcal{P}_{k_0+1} \cup \cdots \cup \mathcal{P}_p, \text{ and} \\ \mathcal{P}^- &:= \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{k_0} \cup \{(S_1, x_1), \ldots, (S_{k_0}, x_{k_0})\}. \end{aligned}$$

$\mathcal{P}^+$ and $\mathcal{P}^-$ are both δ-fine, $(|F|,\tau)$-full families in T, and, therefore, (12) holds for both. Furthermore, it holds that

$$\sum_{j=1}^{k_0} |F(S_j) - f(x_j)\,\mathbf{M}(S_j)| = \left|\sum_{j=1}^{k_0} F(S_j) - f(x_j)\,\mathbf{M}(S_j)\right|$$
$$\leq \left|\sigma(f,\mathcal{P}^+) - F(T)\right| + \sum_{j=k_0+1}^{p} |\sigma(f,\mathcal{P}_j) - F(S_j)| \leq \frac{\epsilon}{4} + \frac{(p-k_0)\epsilon}{2p}$$

and symmetrically

$$\sum_{j=k_0+1}^{p} |F(S_j) - f(x_j)\,\mathbf{M}(S_j)| = \left|\sum_{j=k_0+1}^{p} F(S_j) - f(x_j)\,\mathbf{M}(S_j))\right|$$
$$\leq \left|\sigma(f,\mathcal{P}^-) - F(T)\right| + \sum_{j=1}^{k_0} |\sigma(f,\mathcal{P}_j) - F(S_j)| \leq \frac{\epsilon}{4} + \frac{k_0\epsilon}{2p}.$$

Combining the two inequalities above yields

$$\sum_{j=1}^{p} |F(S_j) - f(x_j)\,\mathbf{M}(S_j)| < \epsilon. \qquad \square$$

Proposition 4.11. — *If f is $\mathcal{R}_1$ integrable on T, then given any decomposition $T = T_1 + T_2 + \dots$, f is $\mathcal{R}_1$ integrable on T_j, for all j with $I(f,T) = \sum_j I(f,T_j)$. In fact, $f \circ \gamma_j$ is HK integrable on $[\,0, \mathbf{M}(T_j)\,]$.*

Proof. — The first part of the statement is clear. For the second part, it suffices to notice that $\sum_{j=1}^k T_j \to T$ as k goes to infinity with $\sup_k \mathbf{M}(\partial(\sum_{j=1}^k T_j)) \leq \mathbf{M}(\partial T)$, for all k. By continuity of the indefinite integral F of f on T, $\sum_{j=1}^k F(T_j) \to \sum_{j=1}^\infty F(T_j) = F(T)$. $\square$

Proposition 4.12. — *If f is defined almost everywhere in $\operatorname{set}_1 \|T\|$ and is Lebesgue integrable with respect to $\|T\|$, then f is $\mathcal{R}_1$ integrable on T. As a consequence, the integral of a $\mathcal{R}_1$ integrable function does not depend on its values on a $\|T\|$ null set.*

Proof. — Let f be Lebesgue integrable with respect to $\|T\|$. Extend f by 0, so that it is defined everywhere in $\operatorname{spt} T$. Fix $\epsilon > 0$. By the Vitali-Carathéodory theorem (see [35, 2.24], there exists two functions g and h with $g \leq f \leq h$ almost everywhere, $(\mathcal{L}) \int (h-g)\,\mathrm{d}\|T\| < \epsilon$, and g and h are, respectively, upper and lower semi-continuous. By upper (or lower) semi-continuity of the function g (or h), for each $x \in \operatorname{spt} T$, there exists $\delta(x) > 0$, such that whenever $y \in \operatorname{spt} T \cap \mathrm{U}(x,\delta(x))$,

$$g(y) \leq f(x) - \epsilon \quad \text{(respectively } h(y) \geq f(x) - \epsilon).$$

Note that $\delta(x)$ can be chosen for g and h at the same time for each x. Suppose that $\mathcal{P}$ is a δ-fine tagged family in T, with $\mathbf{M}(T - [\mathcal{P}]) < \epsilon$,

$$(\mathcal{L}) \int g \, \mathrm{d}\|[\mathcal{P}]\| - \epsilon \, \mathbf{M}([\mathcal{P}]) \leq \sigma(f, \mathcal{P}) \leq (\mathcal{L}) \int g \, \mathrm{d}\|[\mathcal{P}]\| + \epsilon \, \mathbf{M}([\mathcal{P}]).$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two such families, it holds that

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| \leq (\mathcal{L}) \int (h - g) \, \mathrm{d}\|T\| + 2\epsilon \, \mathbf{M}(T).$$

As ϵ is arbitrary, we can use Proposition 4.7 to prove that f is $\mathcal{R}_1$ integrable. The $\mathcal{R}_1$ integral of f coincides with its Lebesgue integral. Indeed, choosing a sequence $(\mathcal{P}_j)_j$ of δ-fine families in T with the masses $\mathbf{M}(T - [\mathcal{P}_j])$ going to 0, it holds that

$$(\mathcal{L}) \int g \, \mathrm{d}\|[\mathcal{P}_j]\| \to (\mathcal{L}) \int g \, \mathrm{d}\|T\|$$

and the same holds for h.

In particular, if f is $\mathcal{R}_1$ integrable on T, and g is equal to f, $\|T\|$ almost everywhere, then $g - f$ is equal to zero $\|T\|$ almost everywhere and is, therefore, Lebesgue integrable with respect to $\|T\|$, and, thus, $\mathcal{R}_1$ integrable on T, and $g = (g - f) + f$ is also $\mathcal{R}_1$ integrable with the same integral (and indefinite integral) as f. □

Proposition 4.13. — *If f is $\mathcal{R}_1$ integrable on T, then its indefinite integral F is AC_*.*

Proof. — Let N be a $\|T\|$ null set. By the Saks-Henstock lemma, for $\epsilon > 0$, there exists a gauge δ on $\operatorname{set}_1 \|T\|$, such that

$$\sum_{(S,x) \in \mathcal{P}} |F(S) - f(x) \, \mathbf{M}(S)| < \epsilon,$$

for every δ-fine tagged family $\mathcal{P}$ in T. As F does not depend on the value of f on N, we can suppose that $f(x) = 0$, for all $x \in N$. If $\mathcal{P}$ is anchored in N, we have

$$|F([\mathcal{P}])| \leq \sum_{(S,x) \in \mathcal{P}} |F(S)| < \epsilon,$$

which proves that F is AC_* on T. □

Proposition 4.14. — *If f is $\mathcal{R}_1$ integrable, then it is $\|T\|$ measurable.*

Proof. — Consider a decomposition of T: $T_1, T_2, \ldots$ and a representative of f, still denoted by f. The function f is $\mathcal{R}_1$ integrable on each $T_k =: [\![\gamma_k]\!]$ and, therefore, $f \circ \gamma_k$ is HK integrable on $[\,0, \mathbf{M}(T_k)\,]$, and, thus, Lebesgue measurable. As γ_k is a homeomorphism from $(\,0, \mathbf{M}(T_k)\,)$ to its image, f is $\|T_k\|$ measurable, and also, $f_k := f \llcorner \operatorname{spt} \|T_k\|$ is $\|T\|$ measurable. Consider the function $\tilde{f} : x \mapsto \sup_k f_k(x)$; $\tilde{f}$ is $\|T\|$ measurable as a pointwise supremum of measurable functions. The function $f-\tilde{f}$ is equal to zero at each point of $\operatorname{spt} T_1 \cup \operatorname{spt} T_2 \cup \cdots \subseteq \operatorname{set}_1 \|T\|$. By definition of decomposition of currents, $\|T\| = \sum_{k=1}^{\infty} \|T_k\|$ and for all k, as T_k is indecomposable, $\operatorname{spt} T_k = \operatorname{set}_1 \|T_k\|$, and, therefore,

$$\|T\| \left(\mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} \operatorname{spt} T_k \right) = 0,$$

thus, $f = \tilde{f}$, $\|T\|$ almost everywhere. This proves that f is $\|T\|$ measurable. □

Proposition 4.15. — *A function f is Lebesgue integrable with respect to $\|T\|$ if and only if f and $|f|$ are $\mathcal{R}_1$ integrable on T.*

Proof. — Without loss of generality, we can suppose that f is non-negative and $\mathcal{R}_1$ integrable, we also fix a representative of f with respect to $\|T\|$. It suffices to show that f is Lebesgue integrable with respect to $\|T\|$. For $k = 1, 2, \dots$, consider the function $f_k := f \mathbb{1}_{\{x, f(x) \leq k\}}$. Since by Proposition 4.14, f is $\|T\|$ measurable, f_k is $\|T\|$ measurable and bounded and thus Lebesgue integrable with respect to $\|T\|$ (which is a finite measure). The sequence f_k is non-decreasing and converges pointwise to f. Furthermore, the sequence $\big((\mathcal{L}) \int f_k \, \mathrm{d}\|T\| \big)_k = \big((\mathcal{R}_1) \int_T f_k \big)_k$ is bounded from above by $(\mathcal{R}_1) \int_T f$. The Lebesgue monotone convergence theorem implies that f is Lebesgue integrable with respect to $\|T\|$. □

Theorem 4.16 (Monotone convergence theorem for the $\mathcal{R}_1$ integral.) — *Suppose that $(f_k)_{k=1,2,\dots}$ is a $\|T\|$ almost everywhere non-decreasing sequence of $\mathcal{R}_1$ integrable functions on T. If there exists $f : \operatorname{set}_1 \|T\| \to \mathbb{R}$, such that $f_k(x)$ converges to $f(x)$ $\|T\|$ almost everywhere, and if, furthermore, the sequence of integral: $(\mathcal{R}_1) \int_T f_k$ for $k = 1, 2, \dots$ is bounded from above. Then, f is $\mathcal{R}_1$ integrable on T with*

$$(\mathcal{R}_1) \int_T f = \lim (\mathcal{R}_1) \int_T f_k.$$

This result can be easily proved using the Lebesgue convergence theorem. However, in order to further illustrate gauge integration techniques, we give a proof that does not rely on the measurability of f or on Lebesgue integration results.

Proof. — Since the $\mathcal{R}_1$ integral of a function does not depend on its values in a $\|T\|$ null set, we can suppose that f_k converges pointwise to f everywhere and that for all $x \in \operatorname{set}_1 \|T\|$, the sequence $(f_k(x))_k$ is non-decreasing. Up to subtracting f_1, we can also suppose that all the f_k are non-negative (by linearity of the integral). For $k = 1, 2, \dots$, let F_k be the indefinite $\mathcal{R}_1$ integral of f_k on T; it is non-negative. Notice also that for all $S \in \mathcal{S}_{\leq}(T)$, and for $k \leq k'$, $F_k(S) \leq F_{k'}(S)$ by the last part of Proposition 4.6. Since $F_k(T)$ is bounded from above, it converges to a limit $F(T)$; similarly, we can define $F(S)$ for any $S \in \mathcal{S}_{\leq}(T)$, as both $(F_k(T-S))_k$ and $(F_k(S))$ are non-decreasing sequences bounded from above by $F(T) \geq F_k(S) + F_k(T-S)$; F is non-negative. The function F on $\mathcal{S}_{\leq}(T)$ is also additive, indeed, supposing S, S' and $S+S'$ are in $\mathcal{S}_{\leq}(T)$, we have

$$F(S+S') = \lim_{k\to\infty} F_k(S+S') = \lim_{k\to\infty} (F_k(S) + F_k(S')) = F(S) + F(S').$$

Let us now prove that F is continuous. Fix sequence $(S_j)_j$ in $\mathcal{S}_{\leq}(T)$ with $\sup_j \mathbf{M}(\partial S_j) < \infty$ and $\mathbf{F}(S_j) \to 0$. For each k, the sequence $(F_k(S_j))_j$ goes to 0 as j goes to ∞ and, similarly, $F_k(T-S_j) \to F_k(T)$ as $j \to \infty$. Thus, since for all k and j, $F(T) \geq F(T-S_j) \geq F_k(T-S_j)$, given $\epsilon > 0$, there exists k_0 such that for all $k \geq k_0$, $F_k(T) \geq F(T) - \epsilon/2$.

There exists also j_0, such that for all $j \geq j_0$, $F_{k_0}(T-S_j) \geq F_{k_0}(T) - \epsilon/2$. This implies that for all $j \geq j_0$ and all $k \geq k_0$,

$$F(T) \geq F(T-S_j) \geq F_k(T-S_j) \geq F_{k_0}(T-S_j) \geq F_{k_0}(T) - \frac{\epsilon}{2} \geq F(T) - \epsilon.$$

Thus, F is non-negative, additive and continuous on $\mathcal{S}_{\leq}(T)$. Since $F(S) \geq F_k(S)$ for all k, if $\mathcal{P}$ is an (F,τ)-full tagged family in T for some $\tau > 0$, $\mathcal{P}$ is also (F_k,τ)-full for all k.

From now on the argument follows the method of [29, 4.42]. Fix $\epsilon > 0$; there exists l, such that for all $k \geq l$, $F(T) - F_k(T) < \epsilon/4$. For each $k \geq l$, fix a gauge δ'_k on $\operatorname{set}_1 \|T\|$, such that for all δ'_k-fine, $(|F_k|, \epsilon/4)$-full families $\mathcal{P}$ in T,

$$\sum_{(x,S)\in\mathcal{P}} |F_k(S) - f_k(x)\,\mathbf{M}(S)| < \frac{\epsilon}{4^{k+2}}.$$

Define a new series of gauges $(\delta_k)_k$ such that for $x \in \operatorname{set}_1 \|T\|$,

$$\delta_k(x) := \min_{1\leq j\leq k} \delta'_k(x).$$

Note that δ_k is, indeed, a gauge, as a finite union of countable sets is countable. For each $x \in \operatorname{set}_1 \|T\|$, fix $l(x) \geq l$, so that $0 \leq f(x) - f_k(x) < \epsilon/(4\,\mathbf{M}(T))$ whenever $k \geq l(x)$. Moreover, let $\delta(x) := \delta_{l(x)}(x)$ be a gauge on $\operatorname{set}_1 \|T\|$. To check that the zero set of δ is countable, notice that it is contained in the countable union of the zero sets of the gauges δ'_k.

Let $\mathcal{P}$ be a δ/fine, $(F,\epsilon/4)$-full tagged family in T. It is also $(F_k,\epsilon/4)$-full, as we said above. Let l' be the maximum of the indices $l(x)$ over $(x,S) \in \mathcal{P}$. For $l \leq k \leq l'$, let $\mathcal{P}_k$ be the subfamily of $\mathcal{P}$ consisting of all the $(x,S) \in \mathcal{P}$ with

$l(x) = k$. We can write

$$\sigma(f, \mathcal{P}) - F(T) = \sum_{k=l}^{l'} \sigma(f, \mathcal{P}_k) - F(T)$$

$$= \sum_{k=l}^{l'} (\sigma(f, \mathcal{P}_k) - \sigma(f_j, \mathcal{P}_k)) + \sum_{k=l}^{l'} (\sigma(f_k, \mathcal{P}_k) - F_k([\mathcal{P}_k])) + \sum_{k=l}^{l'} F_k([\mathcal{P}_k]) - F(T).$$

To control the first term, by the choice of $l(x)$, for all k, we have

$$0 \leq \sigma(f, \mathcal{P}_k) - \sigma(f_k \mathcal{P}_k) < \frac{\mathbf{M}([\mathcal{P}_k])}{\mathbf{M}(T)} \frac{\epsilon}{4}.$$

Sum over $k = l, \dots, l'$ to obtain

$$0 \leq \sum_{k=l}^{l'} \sigma(f, \mathcal{P}_k) - \sigma(f_k \mathcal{P}_k) < \frac{\mathbf{M}([\mathcal{P}])}{\mathbf{M}(T)} \frac{\epsilon}{4} \leq \frac{\epsilon}{4}.$$

For the second term, for any k by the Saks-Henstock lemma applied to f_k and $\mathcal{P}_k$, we have

$$|\sigma(f_k, \mathcal{P}_k) - F_k([\mathcal{P}_k])| \leq \frac{\epsilon}{4^{k+2}}.$$

Summing over k yields

$$\sum_{k=l}^{l'} |\sigma(f_k, \mathcal{P}_k) - F_k([\mathcal{P}_k])| \leq \frac{\epsilon}{4}.$$

Finally, for the third term, notice that for all $k \geq l$,

$$F_k([\mathcal{P}_k]) \geq F_l([\mathcal{P}_k])$$

and, summing, we obtain

$$F(T) \geq F([\mathcal{P}]) \geq \sum_{k=l}^{l'} F_k([\mathcal{P}_k]) \geq F_l([\mathcal{P}]) \geq F_l(T) - \frac{\epsilon}{4} \geq F(T) - \epsilon/2,$$

as $\mathcal{P}$ is $(F_l, \epsilon/4)$-full in T. Combining the three above estimates we get

$$|\sigma(f, \mathcal{P}) - F(T)| < \epsilon,$$

which proves that f has $\mathcal{R}_1$ integral $F(T)$ on T. By the same reasoning one can prove that f is $\mathcal{R}_1$ integrable on $S \in \mathcal{S}_{\leq}(T)$ with integral $F(S)$ and that, thus, F is the indefinite integral of f on T. $\square$

4.4. Fundamental theorem of calculus for the $\mathcal{R}_1$ integral. —

Proposition 4.17. — *If F is a continuous additive function on $\mathcal{S}_{\leq}(T)$, which is AC_* and derivable $\|T\|$ almost everywhere, then $x \mapsto \mathfrak{D}_T F(x)$ is $\mathcal{R}_1$ integrable on T with indefinite integral F.*

Proof. — Let N be the set of non-derivability points of F in $\operatorname{set}_1 \|T\|$. Let f be the function defined on $\operatorname{set}_1 \|T\|$ by $f(x) = 0$ if $x \in N$ and $f(x) = \mathfrak{D}_T F(x)$ otherwise. For $\epsilon > 0$, let δ be a gauge on $\operatorname{set}_1 \|T\|$, such that whenever $\mathcal{P}$ is a δ-fine tagged family in T anchored in N, $|F([\mathcal{P}])| < \epsilon$ and for all $x \in \operatorname{set}_1 \|T\| \setminus N$, $\delta(x)$ is a positive number, such that for all $S \in \mathcal{S}_{\leq}(T, x, \delta(x))$

$$|F(S) - f(x)\,\mathbf{M}(S)| < \epsilon\,\mathbf{M}(S).$$

If $\mathcal{P}$ is a δ-fine tagged family in T with $|F(T - [\mathcal{P}])| < \epsilon$, let $\mathcal{P}_N$ be the subfamily of $\mathcal{P}$ containing all the pairs $(S, x) \in \mathcal{P}$ with $x \in N$. It holds that

$$\begin{aligned}
&|F(T) - \sigma(f, \mathcal{P})| \\
&\quad \leq |F(T - [\mathcal{P}])| + |F([\mathcal{P}_N])| + \sum_{(S,x) \in \mathcal{P}, x \notin N} |F(S) - f(x)\,\mathbf{M}(S)| \\
&\quad < 3\epsilon
\end{aligned}$$

Thus, f is $\mathcal{R}_1$ integrable in T with $I(f, T) = F(T)$. Since $F|_{\mathcal{S}_{\leq}(S)}$ satisfies the hypothesis of the theorem for any $S \in \mathcal{S}_{\leq}(T)$, $I(f, S) = F(S)$, and F is the indefinite integral of $\mathfrak{D}_T F$ on T. □

Proposition 4.18. — *If u is a continuous function on $\operatorname{spt} T$, differentiable $\|T\|$ almost everywhere and such that Θ_u is AC_*, then the function*

$$x \mapsto \mathfrak{D}_T \Theta_u(x) = \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle$$

is $\mathcal{R}_1$ integrable on T with indefinite integral Θ_u.

Proof. — Using Proposition 4.17 it suffices to prove that the set

$$\{x, \Theta_u \text{ is not derivable at } x\} \cup \{x, \mathfrak{D}_T \Theta_u(x) \neq \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle\}$$

is $\|T\|$ negligible. By Theorem 3.11 (iii), u is differentiable $\|T\|$ almost everywhere, and, thus, we only need to prove that the set of points x at which $\overrightarrow{T}$ has a $\|T\|$ approximately continuous representative is $\|T\|$ negligible.

Claim 1. — The function $x \mapsto \overrightarrow{T}$ is $\|T\|$ approximately continuous $\|T\|$ almost everywhere, i.e. for $\|T\|$ almost every x, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\Theta^{m*}(\|T\| \llcorner \{y, |\overrightarrow{T}(x) - \overrightarrow{T}(y)| \geq \delta\}, x) < \epsilon.$$

Proof. — The measure $\|T\|$ in $\mathbb{R}^n$ is finite and Borel regular, and, therefore, the Besicovitch covering theorem holds for $\|T\|$ (see [24, Theorem 2.7]). Or, in the words of H. Federer [16, 2.8.9, 2.8.18], the ambient space $\mathbb{R}^n$ is directionally limited, and the collection of balls

$$\{(x, \mathrm{U}(x,r) \mid x \in \mathbb{R}^n, r > 0\},$$

forms a Vitali relation for the measure $\|T\|$. Furthermore, the function $\overrightarrow{T}$: $\mathrm{set}_1 \|T\| \to \Lambda_1(\mathbb{R}^n)$ is $\|T\|$ measurable. Thus, by [16, 2.9.13], the vector function $\overrightarrow{T}$ is $\|T\|$ approximately continuous $\|T\|$ almost everywhere. ■

□

We can finally restate and prove our main result:

Theorem 1.1 (Fundamental theorem of calculus for 1-currents). — *Let T be a fixed integral current of dimension 1 in $\mathbb{R}^n$ and u be a continuous function on* $\operatorname{spt} T$. *Suppose that u is pointwise Lipschitz at all but countably many points in* $\mathrm{Indec}(T)$ *and that u is differentiable $\|T\|$ almost everywhere, then $x \mapsto \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$ is $\mathcal{R}_1$ integrable on T and*

$$(\partial T)(u) = (\mathcal{R}_1) \int_T \langle \mathrm{D}u, \overrightarrow{T}\rangle.$$

Proof. — Let Θ_u be the function on $\mathcal{S}_{\leq}(T)$ associated to the variations of u. By Theorem 3.11(ii), Θ_u is almost derivable at all points of $\mathrm{set}_1 \|T\|$ except for a countable set. By Theorem 4.3, Θ_u is AC_*. By Theorem 3.11 (iii), Θ_u is derivable $\|T\|$ almost everywhere along T with a derivative equal to $\langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$. Use Propositions 4.17 and 4.18 to conclude. □

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LOWER BOUNDS ON THE DIMENSION OF THE RAUZY GASKET

BY RODOLFO GUTIÉRREZ-ROMO & CARLOS MATHEUS

ABSTRACT. — The Rauzy gasket R is the maximal invariant set of a certain renormalization procedure for special systems of isometries naturally appearing in the context of Novikov's problem in conductivity theory for monocrystals.

It was conjectured by Novikov and Maltsev in 2003 that the Hausdorff dimension $\dim_{\mathrm{H}}(R)$ of the Rauzy gasket lies strictly between 1 and 2.

In 2016, Avila, Hubert and Skripchenko confirmed that $\dim_{\mathrm{H}}(R) < 2$. In this note, we use some results by Cao–Pesin–Zhao in order to show that $\dim_{\mathrm{H}}(R) > 1.19$.

RÉSUMÉ (*Bornes inférieures pour la dimension de la baderne de Rauzy*). — La baderne de Rauzy R est l'ensemble maximal invariant pour une certaine procédure de renormalisation sur les systèmes d'isométries speciaux issus du problème de Novikov en théorie de conductivité des monocristaux.

Il fut conjecturé par Novikov et Maltsev en 2003 que la dimension de Hausdorff $\dim_{\mathrm{H}}(R)$ de la baderne de Rauzy est strictement comprise entre 1 et 2.

En 2016, Avila, Hubert et Skripchenko ont confirmé que $\dim_{\mathrm{H}}(R) < 2$. Dans cette note, on utilise des résultats par Cao–Pesin–Zhao afin de montrer que $\dim_{\mathrm{H}}(R) > 1.19$.

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1. Introduction

The Rauzy gasket is a fractal subset of the standard 2-simplex. It was given life by Arnoux and Rauzy [1] in the context of representing low-complexity subshifts as interval exchange maps, although it was only given a name later. It is also related to frequencies of letters in ternary episturmian words [2], dynamics of special systems of isometries [7], and a particular case of Novikov's problem around the trajectories of electrons on Fermi surfaces in the presence of constant magnetic fields [6, 3]. It is depicted in Figure 1.1.

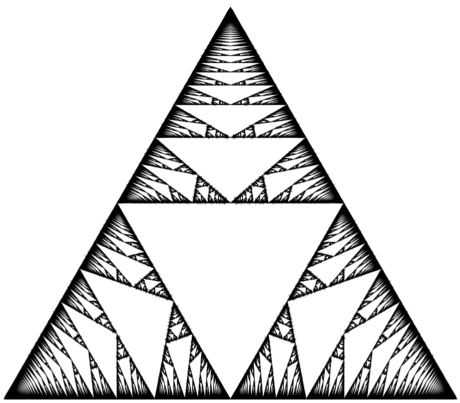

Figure 1.1. The Rauzy gasket.

Concretely, the Rauzy gasket is defined as follows. Consider the standard 2-simplex $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$. We decompose Δ into three simplices $\Delta_j = \{(x_1, x_2, x_3) \in \Delta : x_j \geq \sum_{k \neq j} x_k\}$ and a hole $\Delta \setminus \bigcup_{j=1}^3 \Delta_j$. The projectivizations of the matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

induce weakly contracting maps $f_j \colon \Delta \to \Delta_j$, $j = 1, 2, 3$. In this context, recall from [2] that the *Rauzy gasket* is the unique non-empty compact subset of Δ such that

$$R = f_1(R) \cup f_2(R) \cup f_3(R).$$

The fact that the Rauzy gasket has zero Lebesgue measure has been proved by several authors, including Levitt [9][1], Arnoux–Starosta [2] and De Leo–Dynnikov [6].

1. Using an argument attributed to Yoccoz.

A general conjecture by Novikov and Matlsev [10] from 2003 implies the following particular statement about the Rauzy gasket:

Conjecture 1.1 (Novikov–Maltsev). — $1 < \dim_{\mathrm{H}}(R) < 2$.

Some numerical experiments by De Leo and Dynnikov [6] suggest that $1.7 < \dim_{\mathrm{H}}(R) < 1.8$, and Avila–Hubert–Skripchenko [4] established that $\dim_{\mathrm{H}}(R) < 2$.

The main result of this note is the following theorem:

Theorem 1.2. — $\dim_{\mathrm{H}}(R) > 1.19$.

The proof of this result occupies the remainder of this text.

2. Lower bounds on the Hausdorff dimension of the Rauzy gasket

In this section, we give a lower bound on $\dim_{\mathrm{H}}(R)$ via the construction of appropriate uniformly expanding repellers inside R.

2.1. General framework. — We will use somewhat general methods to obtain bounds for the Hausdorff dimension of a uniformly expanding repeller in dimension 2. These methods rely on estimating the singular values of the derivatives of the maps defining the set. More precisely, given n uniformly contracting maps $T_1, \dots, T_n \colon X \to X$, where $X \subseteq \mathbb{R}^2$ is a compact set, and a repeller K defined as the unique non-empty compact set such that $K = \bigcup_{k=1}^n T_k(K)$, we need to estimate quantities of the form $\max_{x\in X} \|D_xT_k\|$ and $\min_{x\in X} \|(D_xT_k)^{-1}\|$, where $\|\cdot\|$ denotes the largest singular value. Since for any $a,b,c,d \in \mathbb{R}$ one has that

$$\left\|\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right\| = \sqrt{\frac{a^2+b^2+c^2+d^2+\sqrt{(a^2+b^2+c^2+d^2)^2-4(ad-bc)^2}}{2}},$$

we obtain the simple estimates

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{2}} \leq \left\|\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right\| \leq \sqrt{a^2+b^2+c^2+d^2},$$

which we will write as $\|\cdot\|^- \leq \|\cdot\| \leq \|\cdot\|^+$.

There are several methods in the literature to obtain lower bounds on the Hausdorff dimension of repellers. For our purposes, the thermodynamical method of Cao–Pesin–Zhao [5] is quite useful. In a nutshell, they consider a repeller Λ of a C^2-expanding map g on a surface, a parameter $1 \leq s \leq 2$, and the potential $\psi^s(x,g) = \log\alpha_1(x,g) + (s-1)\log\alpha_2(x,g)$, where $\alpha_1(x,g) \geq \alpha_2(x,g)$ are the singular values of D_xg. Observe that $\|D_xg\|^- \leq \alpha_1(x,g) \leq \|D_xg\|^+$ and $\|(D_xg)^{-1}\|^- \leq \alpha_2(x,g)^{-1} \leq \|(D_xg)^{-1}\|^+$.

By Corollary 3.1 of [5], one has that

$$\dim(\Lambda) \geq s_1,$$

where s_1 is the unique root of the equation $P(g, -\psi^s(\cdot, g)) = 0$ and $P(g, \theta)$ stands for the topological pressure of the potential θ, i.e.,

$$P(g, \theta) := \sup \left\{ h_\mu(g) + \int \theta \, d\mu(x) : \mu \text{ is } g\text{-invariant} \right\}$$

(see (3.2) and (2.4) in [5]). The theory of (subadditive) thermodynamical formalism (as explained[2] in Section 3 of [8], for instance) states that

$$P(g, \theta) < 0 \iff \sum_{m \geq 1} \sum_{x \in \mathrm{Fix}(g^m)} \exp(\theta_m(x)) < \infty,$$

where $\theta_m(x) := \sum_{j=0}^{m-1} \theta(g^j(x))$.

In general, $s \mapsto P(g, -\psi^s(\cdot, g))$ is a continuous and strictly decreasing function of s. Therefore, $s_1 \geq s_0$ for all s_0 with

$$\sum_{m \geq 1} \sum_{x \in \mathrm{Fix}(g^m)} \exp\left(-\sum_{j=0}^{m-1} \psi^{s_0}(g^j(x), g) \right) = \infty.$$

2.2. The Rauzy gasket. — Observe that each composition $f_k \circ f_j$, with $k \neq j$, is a contraction on Δ (cf. Lemma 2 in [2]). Thus, for each integer $n \geq 2$, the unique non-empty compact subset K_n such that

$$K_n = \bigcup_{i \in S_n} f_{i_n} \circ \cdots \circ f_{i_1}(K_n),$$

where $S_n = \{1, 2, 3\}^n \setminus \{(1, \ldots, 1), (2, \ldots, 2), (3, \ldots, 3)\}$, is a uniformly expanding repeller contained in R.

In the following, we consider the Riemannian metric on $T\Delta = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$ induced by the usual Euclidean scalar product of $\mathbb{R}^3$ normalized so that the vectors $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{-1, 0, 1\}$ have norm 1. In particular, $\mathcal{B} = \{(1, -1, 0), (-1, -1, 2)/\sqrt{3}\}$ is an orthonormal basis of $T\Delta$.

Remark 2.1. — A natural alternative is to consider the Fubini–Study metric $d(\mathbb{R}x, \mathbb{R}y) = \frac{\|x \wedge y\|}{\|x\| \|y\|}$ on the projective space $P\mathbb{R}^3$. However, we chose the *ad hoc* Riemannian metric above because the operation of taking exterior powers would lead to heavier calculations.

The repeller K_{13}, defined by g_{13} sending each $\Delta_i = f_{i_{13}} \circ \cdots \circ f_{i_1}(\Delta)$, with $i \in S_{13}$, onto Δ, is uniformly expanding with respect to this Riemannian metric.

2. Cf. Lemma 3.2 of [8] in particular.

Indeed, we can estimate the smallest expansion factor as

$$\frac{1}{\max_{i\in S_{13}}\max_{x\in\Delta_i}\|(D_xg_{13})^{-1}\|^+}$$

to obtain a value of at least $\sqrt{3}$.

Now, denote by

$$a=\log\left(\max_{i\in S_{13}}\max_{x\in\Delta_i}\|D_xg_{13}\|^+\right),\quad b=\log\left(1\Big/\min_{i\in S_{13}}\min_{x\in\Delta_i}\|(D_xg_{13})^{-1}\|^-\right)$$

and $\mathrm{Fix}(g_{13}^m)=\exp(cm)$ for all m (i.e., $c=\log(|S_{13}|)=\log(3^{13}-3)$). Observe that

$$\log\alpha_1(x,g)\le a\quad\text{ and }\quad\log\alpha_2(x,g)\le b$$

for every $x\in\bigcup_{i\in S_{13}}\Delta_i$. Hence,

$$\sum_{j=0}^{m-1}\psi^{s_0}(g_{13}^j(x),g_{13})\le(a+b(s_0-1))m,$$

and we deduce that

$$\begin{aligned}\sum_{m\ge1}\sum_{x\in\mathrm{Fix}(g_{13}^m)}\exp&\left(-\sum_{j=0}^{m-1}\psi^{s_0}(g_{13}^j(x),g_{13})\right)\\&\ge\sum_{m\ge1}\exp((c-a-b(s_0-1))m)=\infty,\end{aligned}$$

if $c-a-b(s_0-1)>0$, i.e., $s_0<1+(c-a)/b$.

This way, we obtain the bound

$$\dim_{\mathrm{H}}(K_{13})\ge s_1\ge1+\frac{c-a}{b}.$$

With the help of a computer, we can find the exact values of a and b. We obtain:

$$a=\log\left(3208\sqrt{\frac{86185}{3}}\right),\quad b=\log\left(4917248\sqrt{\frac{2}{1595}}\right)\quad\text{and}\quad c=\log(1594320),$$

which yields $\dim_{\mathrm{H}}(K_{13})\ge1+\frac{c-a}{b}>1.08$.

This lower bound can be improved by restricting it to a smaller fractal. Indeed, instead of using every sequence in S_{13}, we can take a subset of such sequences designed to optimize the previous bound by decreasing the values of a and b while trying to maintain a large value of c. The heuristic we use is as follows:

1. Sort the $i\in S_{13}$ according to $\max_{x\in\Delta_i}\|D_xg_{13}\|^+$ in ascending order, assigning a number $r^+(i)$ to each $i\in S_n$.

2. Find the $i \in S_{13}$ that maximizes $\log(r^+(i)) - \log(\max_{x\in\Delta_i} \|D_x g_{13}\|^+)$ and denote it by i^*. Let $S_{13}^+ = \{i \in S_n : r^+(i) \leq r^+(i^*)\}$. For the remaining steps, we ignore the elements of $S_{13} \setminus S_{13}^+$. Let $a' = \log \max_{x\in\Delta_{i^*}} \|D_x g_{13}\|^+$.
3. Sort the $i \in S_{13}^+$ according to $\min_{x\in\Delta_i} \|(D_x g_{13})^{-1}\|^-$ in descending order, assigning a number $r^-(i)$ to each $i \in S_{13}^+$.
4. Find the $i \in S_{13}^+$ that maximizes $\frac{\log(r^-(i))-a'}{\log(1/\min_{x\in\Delta_i} \|(D_x g_{13})^{-1}\|^-)}$ and denote it by i^{**}. Our new set of sequences is now $S_{13}^{+-} = \{i \in S_{13}^+ : r^-(i) \leq r^-(i^{**})\}$, and we have $b = \log(1/\min_{x\in\Delta_{i^{**}}} \|(D_x g_{13})^{-1}\|^-)$ and $c = \log(r^-(i^{**}))$. We also define $a = \log \max_{i\in S_{13}^{+-}} \max_{x\in\Delta_i} \|D_x g_{13}\|^+$ (it may happen that $a < a'$, as we have removed more sequences).

We repeat this heuristic until the list of sequences does not change. We get the following final values:

$$a = \log\left(6800\sqrt{\frac{829}{3}}\right), \quad b = \log\left(615627\sqrt{\frac{3}{515}}\right) \quad \text{and} \quad c = \log(898224).$$

Thus, we obtain the bound $\dim_{\mathrm{H}}(K_{13}) \geq 1 + \frac{c-a}{b} > 1.19$ and we establish the lower bound $\dim_{\mathrm{H}}(R) > 1.19$.

Remark 2.2. — The lower bound on $\dim_{\mathrm{H}}(R)$ produced by this method can be improved by taking larger values of n. Nevertheless, some numerical experiments suggest that such bounds will not exceed 1.3. Since we have used very coarse approximations, it is quite likely that these numbers are far from the actual value of $\dim_{\mathrm{H}}(R)$.

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ON THE EQUIVARIANT BLOW-NASH CLASSIFICATION OF SIMPLE INVARIANT NASH GERMS

BY FABIEN PRIZIAC

ABSTRACT. — We make progress towards the classification of simple Nash germs invariant under the involution changing the sign of the first coordinate, with respect to equivariant blow-Nash equivalence, which is an equivariant Nash version of blow-analytic equivalence, taking advantage of invariants for this relation, the equivariant zeta functions.

RÉSUMÉ (*Sur l'équivalence Nash après éclatements équivariante des germes Nash invariants simples*). — On effectue des avancées en direction de la classification des germes Nash simples invariants sous l'involution changeant le signe de la première coordonnée, par rapport à l'équivalence Nash après éclatements équivariante, qui est une version Nash équivariante de l'équivalence analytique après éclatements, en tirant parti d'invariants pour cette relation, les fonctions zêta équivariantes.

1. Introduction

The classification of real analytic germs requires us to carefully choose the equivalence relation used. One may think about the (right) C^1-equivalence. However, this is too strong, as illustrated by the example of the Whitney

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family $f_t(x,y) = xy(y-x)(y-tx)$, $t > 1$ (f_t and $f_{t'}$ are C^1-equivalent if and only if $t = t'$), while the topological equivalence is too rough. In [19], T.-C. Kuo suggested an equivalence relation for which the Whitney family has only one equivalence class: the blow-analytic equivalence. More generally, any analytically parameterized family of isolated singularities has a locally finite classification with respect to blow-analytic equivalence.

Two real analytic germs are said to be blow-analytically equivalent if, roughly speaking, they become analytically equivalent after compositions with real modifications, e.g. compositions of blowings-up along smooth centers. From the definition of this equivalence relation, further studies on real analytic germs were stimulated. In particular, invariants have been constructed for blow-analytic equivalence, like the Fukui invariants ([15]), as well as the zeta functions of S. Koike and A. Parusiński ([18]), inspired by the motivic zeta functions of J. Denef and F. Loeser ([8]), using the Euler characteristic with compact supports as a motivic measure.

A refinement of blow-analytic equivalence was defined for Nash germs, that is, germs of real analytic functions with a semialgebraic graph, by G. Fichou in [10]: the blow-Nash equivalence, that is, the Nash equivalence after compositions with Nash modifications. The algebraicity involved allowed Fichou to use the virtual Poincaré polynomial ([22] and [9]), which is an additive and multiplicative invariant on $\mathcal{AS}$ sets ([20] and [21]) encoding more information than the Euler characteristic with compact supports, in order to define new zeta functions, invariant for the blow-Nash equivalence of Nash germs. Recently, in [6] J.-B. Campesato gave an equivalent alternative definition of blow-Nash equivalence as arc-analytic equivalence, proving that the blow-Nash equivalence of [10] is, indeed, an equivalence relation; he defined a more general invariant for it, the motivic local zeta function.

In [13], G. Fichou used his zeta functions of [10] to classify simple Nash germs (a germ is called simple if sufficiently small perturbations provide only finitely many analytic classes) with respect to blow-Nash equivalence. He showed that this classification actually coincides with the real analytic one, that is, the ADE-classification of [2]. An analog result for blow-analytic equivalence is not known.

In this paper, we are interested in real analytic germs invariant under right composition with the action of the group $G = \mathbb{Z}/2\mathbb{Z}$ only changing the sign of the first coordinate (which we will simply call invariant germs). In [24], we defined the equivariant blow-Nash equivalence for invariant Nash germs, which is, roughly speaking, an equivariant Nash equivalence after compositions with equivariant Nash modifications. Using the equivariant virtual Poincaré series ([14]), which is an additive invariant on G-$\mathcal{AS}$ sets, as a motivic measure, we constructed "equivariant" zeta functions, which are invariants for the equivariant blow-Nash equivalence.

Similarly to the nonequivariant frame, we ask if the equivariant blow-Nash classification of invariant Nash germs could coincide with the equivariant Nash classification for sufficiently "tame" invariant singularities. The equivariant analytic classification of simple invariant real analytic germs was established by V.I. Arnold in [1] (see also [16]). The representatives for this classification are the invariant singularities A_k, B_k, C_k, D_k, E_6, E_7, E_8 and F_4 (see Theorem 2.1 below). We will first show that a simple invariant Nash germ is G-blow-Nash equivalent (and even G-Nash equivalent) to one of these germs. The largest part of our study will then consist in trying to distinguish, with respect to G-blow-Nash equivalence, the invariant $ABCDEF$-singularities, notably using the equivariant zeta functions.

For some cases, we will be faced with either the equality of the respective equivariant zeta functions of a couple of germs or the fact that they are equal if and only if the respective equivariant virtual Poincaré series of specific sets are equal. The former situation is, in particular, due to the fact that the equivariant virtual Poincaré series can not distinguish two different algebraic actions on the same sphere as soon as there is at least one fixed point. As for the latter situation, we do not know if the invariance of the virtual Poincaré polynomial under bijection with the $\mathcal{AS}$ graph (see [23]) "generalizes" to an invariance of the equivariant virtual Poincaré series under equivariant bijection with the $\mathcal{AS}$ graph. If this is proven to be true, it should allow us to compute all the coefficients of the equivariant zeta functions considered.

The next section is devoted to the equivariant Nash classification of simple invariant Nash germs; we prove that it coincides with the equivariant real analytic classification of [1] and [16]. Indeed, two invariant Nash germs are equivariantly Nash equivalent if and only if they are equivariantly analytically equivalent (Proposition 2.3). This can be deduced from an equivariant Nash approximation theorem of E. Bierstone and P. Milman in [4].

In Section 3, we justify the fact that a germ G-Nash equivalent to a germ of the list $ABCDEF$ is, in particular, G-blow-Nash equivalent to it. On the other hand, one can notice that, forgetting the G-action, the invariant singularities A_k and B_k, or C_k and D_k, E_6 and F_4, are both A-, or D-, E-singularities. Since equivariant blow-Nash equivalence is a particular case of blow-Nash equivalence, and because the ADE-singularities are not blow-Nash equivalent to one another ([13]), we are reduced to comparing, with respect to G-blow-Nash equivalence, the invariant germs of the families A_k and B_k, or C_k and D_k, E_6 and F_4.

Section 4 recalls the definition of the tools we are going to use to do so: the equivariant zeta functions. Sections 6, 7 and 8 are devoted to the comparison of the invariant germs of a specific couple of families (A_k and B_k, C_k and D_k, and finally E_6 and F_4). We proceed as follows. We begin by computing the first coefficients of the equivariant zeta functions (that is, the coefficients of degree strictly smaller than the degree of the germs) in order to extract first cases of non-G-blow-Nash equivalence. Reducing our study to the remaining cases, we then compute the coefficient of degree equal to the degree of the germs.

Finally, for the cases for which it is not sufficient, we compare the coefficients of higher degrees of the respective equivariant zeta functions.

These comparisons lead to interesting examples of computations of equivariant virtual Poincaré series. The first one, to which Section 5 is devoted, is the computation of the equivariant virtual Poincaré series of the fibers over 0, -1 and $+1$ of the quadratic forms $Q_{p,q}(y) := \sum_{i=1}^{p} y_i^2 - \sum_{j=1}^{q} y_{p+j}^2$, equipped with four different actions of G.

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2. Equivariant Nash classification of invariant simple Nash germs

Consider the affine space $\mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$. We denote by s the involution of $\mathbb{R}^n$ changing the sign of the first coordinate x_1:

$$s : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, x_2, \ldots, x_n) & \mapsto & (-x_1, x_2, \ldots, x_n) \end{array}$$

This equips $\mathbb{R}^n$ with a linear action of the group $G = \{id_{\mathbb{R}^n}, s\}$.

In this paper, a function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ will be said to be invariant if f is invariant under the right composition with s, that is, if f is the germ of an equivariant function (we equip $\mathbb{R}$ with the trivial action of G).

In [1] and [16], the classification of invariant, simple, real analytic germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with respect to equivariant analytic equivalence, that is, right equivalence via an equivariant analytic diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$:

Theorem 2.1 ([1], [16]). — *An invariant simple real analytic function germ $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is equivariantly analytically equivalent to one and only one invariant germ of the following list:*

$$\begin{array}{llll}
A_k,\ k \geq 0\colon & \pm x_1^2 \pm x_2^{k+1} + Q, & E_6\colon & \pm x_1^2 + x_2^3 \pm x_3^4 + Q, \\
B_k,\ k \geq 2\colon & \pm x_1^{2k} \pm x_2^2 + Q, & E_7\colon & \pm x_1^2 + x_2^3 + x_2 x_3^3 + Q, \\
C_k,\ k \geq 3\colon & x_1^2 x_2 \pm x_2^k + Q, & E_8\colon & \pm x_1^2 + x_2^3 + x_3^5 + Q, \\
D_k,\ k \geq 4\colon & \pm x_1^2 + x_2^2 x_3 \pm x_3^{k-1} + Q, & F_4\colon & \pm x_1^4 + x_2^3 + Q,
\end{array}$$

where $Q = \pm x_s^2 \pm \cdots \pm x_n^2$, with $s = 4$ for singularities D_k and E_k, and $s = 3$ in the other cases.

Remark 2.2. — If we forget the action of the involution s on $\mathbb{R}^n$, we notice that the families A_k and B_k, and C_k and D_k, E_6 and F_4, E_7, E_8, of Theorem 2.1 are singularities A, and D, E_6, E_7, E_8.

In this paper, we are interested in the classification of invariant Nash germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, that is, germs of equivariant analytic functions with semialgebraic graphs. Recall (see, for instance, [5] Corollary 8.1.6) that a Nash germ can be considered as an algebraic power series, via its Taylor series. The above classification is also valid for invariant simple Nash germs with respect to equivariant Nash equivalence, that is, right equivalence via an equivariant Nash diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, according to the following proposition:

PROPOSITION 2.3. — *Let $f, h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two invariant Nash germs. Then f and h are equivariantly Nash equivalent if and only if they are equivariantly analytically equivalent.*

This property is a particular case of the following result:

THEOREM 2.4. — *Let G be a reductive algebraic group acting linearly on $\mathbb{R}^n$ and $\mathbb{R}^p$. Consider two equivariant Nash germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. If f and h are equivariantly analytically equivalent, then they are equivariantly Nash equivalent.*

REMARK 2.5. —
- Since a Nash diffeomorphism is, in particular, analytic, the converse is obviously true.
- Any finite group is reductive.

Proof (of Theorem 2.4). — Suppose there exists an equivariant analytic diffeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f \circ \phi = h$. Denote $F(x, y) := f(y) - h(x)$ for $x, y \in \mathbb{R}^n$. Then $F : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^p, 0)$ is a Nash germ and can be considered as an algebraic power series in $\mathbb{R}_{alg}[[x, y]]^p$, and $\phi(x)$ as an equivariant convergent power series in $\mathbb{R}\{x, y\}$ such that $F(x, \phi(x)) = 0$.

Therefore, by Theorem A of [4] and Example 11.3 of [26], we can approximate $\phi(x)$ by an equivariant algebraic power series $\widetilde{\phi}(x)$ such that $F(x, \widetilde{\phi}(x)) = 0$, and we do the approximation closely enough so that $\widetilde{\phi}(x)$ remains a diffeomorphism. As a consequence, $\widetilde{\phi} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is an equivariant Nash diffeomorphism such that $f \circ \widetilde{\phi} = h$.

Notice that, actually, Theorem A of [4] is about approximation of equivariant formal solutions of polynomial equations by equivariant algebraic power series, but it is also true for algebraic power series equations. Indeed, following G. Rond's ideas, it is possible to reduce it to the case of polynomial equations, as in [3] Lemma 5.2 and [7] Reduction (2) of the proof of Theorem 1.1, using arguments of the proof of Lemma 8.1 in [25], along with the fact that the morphism $\mathbb{R}[x, y]_{(x,y)} \to \mathbb{R}_{alg}[[x, y]]$ is faithfully flat by [5] Corollary 8.7.16. □

3. Equivariant blow-Nash equivalence

Now, we want to study the classification of invariant simple Nash germs with respect to G-blow-Nash equivalence via an equivariant blow-Nash isomorphism

$(G = \{id_{\mathbb{R}^n}, s\})$. We first recall the definition of G-blow-Nash equivalence via an equivariant blow-Nash isomorphism of [24]:

Definition 3.1. — Let $f, h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two invariant Nash germs. We say that f and h are G-blow-Nash equivalent via an equivariant blow-Nash isomorphism, if there exist

- two equivariant Nash modifications $\sigma_f : (M_f, \sigma_f^{-1}(0)) \to (\mathbb{R}^n, 0)$ and $\sigma_h : (M_h, \sigma_h^{-1}(0)) \to (\mathbb{R}^n, 0)$ of f and h, respectively (we refer to [24] for the definition of an equivariant Nash modification),
- an equivariant Nash isomorphism Φ between G-globally stabilized semi-algebraic and analytic neighbourhoods $(M_f, \sigma_f^{-1}(0))$ and $(M_h, \sigma_h^{-1}(0))$ such that Φ preserves the multiplicities of the Jacobian determinant of σ_f and σ_g along their exceptional divisors,
- an equivariant homeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$,

such that the following diagram commutes:

$$\begin{tikzcd}
(M_f, \sigma_f^{-1}(0)) \arrow[rr, "\Phi"] \arrow[d, "\sigma_f"'] & & (M_h, \sigma_h^{-1}(0)) \arrow[d, "\sigma_h"] \\
(\mathbb{R}^n, 0) \arrow[rr, "\phi"] \arrow[dr, "f"] & & (\mathbb{R}^n, 0) \arrow[dl, "h"'] \\
& (\mathbb{R}, 0) &
\end{tikzcd}$$

We have the following:

Proposition 3.2. — *An invariant simple Nash germ* $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *is* G-*blow-Nash equivalent via an equivariant blow-Nash isomorphism to an invariant germ of the list of Theorem 2.1.*

Proof. — This comes from the fact that if f and h are equivariantly Nash equivalent invariant Nash germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, then they are G-blow-Nash equivalent via an equivariant blow-Nash isomorphism.

Indeed, if f and h are equivariantly Nash equivalent via an equivariant Nash isomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, then we have the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{R}^n, 0) \arrow[rr, "\Phi"] \arrow[d, "id_{\mathbb{R}^n}"'] & & (\mathbb{R}^n, 0) \arrow[d, "id_{\mathbb{R}^n}"] \\
(\mathbb{R}^n, 0) \arrow[rr, "\Phi"] \arrow[dr, "f"] & & (\mathbb{R}^n, 0) \arrow[dl, "h"'] \\
& (\mathbb{R}, 0) &
\end{tikzcd}$$

Now, if $f^{-1}(0)$, or $h^{-1}(0)$, has only one irreducible component at $0 \in \mathbb{R}^n$, the identity $id_{\mathbb{R}^n} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is an equivariant Nash modification of f, or g, so that f and g are G-blow-Nash equivalent via an equivariant blow-Nash isomorphism. If this is not the case, we have to perform a composition $\sigma_f : (M_f, \sigma_f^{-1}(0)) \to (\mathbb{R}^n, 0)$, or $\sigma_h : (M_h, \sigma_h^{-1}(0)) \to (\mathbb{R}^n, 0)$, of successive equivariant blowings-up along G-invariant smooth Nash centres, such that

- the irreducible components of the strict transform of $f^{-1}(0)$ by σ_f, or of $h^{-1}(0)$ by σ_h, do not intersect,
- $f \circ \sigma_f$ and $jac\ \sigma_f$, or $h \circ \sigma_h$ and $jac\ \sigma_h$, have only normal crossings simultaneously,
- there exists a finite collection of G-invariant affine charts for σ_f, or for σ_h, such that, on each of these charts, the action of G is of the form

$$(x_1, x_2, \ldots, x_n) \mapsto (\epsilon_1 x_1, \epsilon_2 x_2, \ldots, \epsilon_n x_n),$$

  where $\epsilon_i \in \{\pm 1\}$ (so that the action of G on M_f, or on M_h, can be locally linearized on the normal crossings, in the sense of [24]),
- after each blowing-up, f and h remain equivariantly Nash equivalent

(perform σ_h with such properties with respect to h, then construct σ_f as the pullback of σ_h via the equivariant Nash diffeomorphism Φ). $\square$

The second step of the study will consist in understanding the relations, with respect to G-blow-Nash equivalence via an equivariant blow-Nash isomorphism, between the invariant Nash germs of the list of Theorem 2.1.

Equivariant blow-Nash equivalence (or equivariant blow-Nash equivalence via an equivariant blow-Nash isomorphism) is a particular case of the blow-Nash equivalence (or blow-Nash equivalence via a blow-Nash isomorphism) defined in [10]. In [13], Fichou proved that the classification of simple Nash germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with respect to blow-Nash equivalence via a blow-Nash isomorphism is the same as Arnold's ADE-classification of real analytic germs with respect to right analytic equivalence.

As a consequence, the A, D, E-singularities, belonging to different blow-Nash classes, cannot be G-blow-Nash-equivalent via an equivariant blow-Nash isomorphism either. We are then reduced to trying to distinguish the invariant germs of the families A_k and B_k, or C_k and D_k, E_6 and F_4.

For this purpose, we will use the equivariant zeta functions defined in [24], which are invariants for equivariant blow-Nash equivalence via an equivariant blow-Nash isomorphism.

4. Equivariant zeta functions

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an invariant Nash germ. We recall the definition given in [24] of the equivariant zeta functions of f.

Denote $\mathcal{L} := \{\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0) \mid \gamma(t) = a_1 t + a_2 t^2 + \ldots, a_i \in \mathbb{R}^n\}$ the space of formal arcs at the origin of $\mathbb{R}^n$. The action of G on $\mathbb{R}^n$ naturally induces an action of G on $\mathcal{L}$, by left composition with s. For $m \in \mathbb{N} \setminus \{0\}$, the space

$$\mathcal{L}_m := \{\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0) \mid \gamma(t) = a_1 t + a_2 t^2 + \ldots + a_m t^m\}$$

of arcs truncated at order $m+1$ is globally stable under this action, as well as the spaces

$$\begin{aligned} A_m(f) &:= \{\gamma \in \mathcal{L}_m \mid f \circ \gamma(t) = c t^m + \ldots, c \neq 0\}, \\ A_m^+(f) &:= \{\gamma \in \mathcal{L}_m \mid f \circ \gamma(t) = +t^m + \ldots\} \quad \text{and} \\ A_m^-(f) &:= \{\gamma \in \mathcal{L}_m \mid f \circ \gamma(t) = -t^m + \ldots\}. \end{aligned}$$

These latter sets are Zariski constructible sets equipped with an algebraic action of G, and we define

$$Z_f^G(u, T) := \sum_{m \geq 1} \beta^G(A_m(f)) u^{-mn} T^m \in \mathbb{Z}[u][[u^{-1}]][[T]]$$

and

$$Z_f^{G,\pm}(u, T) := \sum_{m \geq 1} \beta^G(A_m^{\pm}(f)) u^{-mn} T^m \in \mathbb{Z}[u][[u^{-1}]][[T]],$$

the naive equivariant zeta function and the equivariant zeta functions with sign of f, respectively.

Here, $\beta^G(\cdot)$ denotes the equivariant virtual Poincaré series on G-$\mathcal{AS}$ sets of [14]; it is an additive invariant with respect to equivariant isomorphisms, with values in $\mathbb{Z}[[u]]$, such that, if X is a compact nonsingular G-$\mathcal{AS}$ set, $\beta^G(X) = \sum_{i \in \mathbb{Z}} dim_{\mathbb{Z}_2} H_i(X; G)\, u^i$, where $H_*(X; G)$ denotes the equivariant Borel-Moore homology of X with coefficients in $\mathbb{Z}_2$ defined in [17].

Remark 4.1. —
- By an isomorphism between arc-symmetric sets we mean a birational map containing the arc-symmetric sets in its support.
- The equivariant virtual Poincaré series of a point is $\frac{u}{u-1}$, the equivariant virtual Poincaré series of two fixed points is $2\frac{u}{u-1}$ and the equivariant virtual Poincaré series of two points exchanged by G is 1; see [14] Example 3.12.
- If S^d denotes the unit sphere in $\mathbb{R}^d$, then

$$\beta^G(S^d) = \begin{cases} 1 + u + \ldots + u^d & \text{if } G \text{ acts via the central symmetry of } \mathbb{R}^d, \\ 2\frac{u}{u-1} + u + \ldots + u^d & \text{if } G \text{ acts with a fixed point} \end{cases}$$

  (see [14] Example 3.12).
- If $\mathbb{R}^d$ is equipped with any algebraic action of G, then $\beta^G(\mathbb{R}^d) = \frac{u^{d+1}}{u-1}$: see [14] Example 3.12.

- If X is a G-$\mathcal{AS}$ set and if the affine space $\mathbb{R}^d$ is equipped with any algebraic action of G, then $\beta^G(X \times \mathbb{R}^d) = u^d \beta^G(X)$ (the product $X \times \mathbb{R}^d$ is equipped with the diagonal action of G): see [14] Proposition 3.13.
- If X is a G-$\mathcal{AS}$ set and if the affine line $\mathbb{R}$ is equipped with an algebraic action of G stabilizing 0, then $\beta^G(X \times (\mathbb{R}^*)^d) = (u-1)^d \beta^G(X)$; see [24] Lemma 3.9.
- If X is a G-$\mathcal{AS}$ set, then the coefficients of the negative powers of u in $\beta^G(X)$ are all equal to $\sum_{i \geq 0} \beta_i(X^G)$, where X^G is the fixed point set of X and $\beta_i(\cdot)$ denotes the i^{th} virtual Betti number ([22]); see [14] Proposition 4.5.

Theorem 4.2 (Theorem 4.1 of [24]). — *Let $f, h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two invariant Nash germs. If f and h are G-blow-Nash equivalent via an equivariant blow-Nash isomorphism, then $Z_f^G(u,T) = Z_h^G(u,T)$ and $Z_f^{G,\pm}(u,T) = Z_h^{G,\pm}(u,T)$.*

Remark 4.3. — In the rest of this paper, we will simply talk about equivariant blow-Nash equivalence to refer to equivariant blow-Nash equivalence via an equivariant blow-Nash isomorphism.

In the next parts of the paper, we are then going to use the equivariant zeta functions in order to try to distinguish the families A_k and B_k, and C_k and D_k, E_6 and F_4, with respect to G-blow-Nash equivalence. More precisely, we will show that, in some cases, some terms of the respective equivariant zeta functions of the germs considered are different.

On the other hand, we will prove that, in some other cases, the equivariant zeta functions are equal.

Before this, in the following section, we compute equivariant virtual Poincaré series associated to the quadratic form

$$Q_{p,q}(y) := \sum_{i=1}^{p} y_i^2 - \sum_{j=1}^{q} y_{p+j}^2,$$

where $p, q \in \mathbb{N}$, $(y_1, \dots, y_{p+q}) \in \mathbb{R}^{p+q}$. More precisely, we compute the equivariant virtual Poincaré series of the algebraic sets

$$Y_{p,q} := \{Q_{p,q} = 0\} \text{ and } Y_{p,q}^{\xi} := \{Q_{p,q} = \xi\},$$

for $\xi = \pm 1$, in the cases where the action of G on $\mathbb{R}^{p+q}$ is given by

1. $(y_1, \dots, y_p, y_{p+1}, \dots, y_{p+q}) \mapsto (-y_1, \dots, y_p, y_{p+1}, \dots, y_{p+q})$,
2. $(y_1, \dots, y_p, y_{p+1}, \dots, y_{p+q}) \mapsto (y_1, \dots, y_p, -y_{p+1}, \dots, y_{p+q})$,
3. $(y_1, \dots, y_p, y_{p+1}, \dots, y_{p+q}) \mapsto (-y_1, \dots, -y_p, -y_{p+1}, \dots, -y_{p+q})$,
4. or $(y_1, \dots, y_p, y_{p+1}, \dots, y_{p+q}) \mapsto (y_1, \dots, y_p, y_{p+1}, \dots, y_{p+q})$.

This will turn out to be useful in the comparisons of the equivariant zeta functions.

5. Computation of $\beta^G(Y_{p,q})$ and $\beta^G(Y^{\xi}_{p,q})$

We have the following result:

Proposition 5.1. — 1. *If $0 < p < q$, then*

$$\beta^G(Y_{p,q}) = \begin{cases} \frac{u^{p+q}-u^q+u^{p-1}}{u-1} & \text{in the case } n^o 1, \\ \frac{u^{p+q}-u^q+u^{p+1}}{u-1} & \text{in the three other cases.} \end{cases}$$

2. *If $p = q \neq 0$, then*

$$\beta^G(Y_{p,q}) = \begin{cases} \frac{u^{2p}-u^p+u^{p-1}}{u-1} & \text{in the cases } n^o 1 \text{ and } n^o 2, \\ \frac{u^{2p}-u^p+u^{p+1}}{u-1} & \text{in the two other cases.} \end{cases}$$

3. *If $p = 0$, then*

$$\beta^G(Y_{p,q}) = \frac{u}{u-1}.$$

Remark 5.2. — If $q < p$, just exchange the roles of p and q along with the actions of the cases n°1 and n°2.

Proof (of Proposition 5.1). — If $p = 0$, then $Y_{p,q} = \{0\}$ and $\beta^G(Y_{p,q}) = \frac{u}{u-1}$ by Remark 4.1.

If $0 < p < q$, as in [12] Proof of Proposition 2.1 and [13] Proof of Lemma 3.1, we apply the equivariant change of variables $u_i = y_i + y_{i+p}$, $v_i = y_i - y_{i+p}$ for $i = 2, \ldots, p$ and the equation $Q_{p,q} = 0$ becomes

$$y_1^2 - y_{p+1}^2 + \sum_{i=2}^{p} u_i v_i - \sum_{j=2p+1}^{p+q} y_j^2 = 0,$$

the action of G on the new coordinates u_i, v_i being trivial in the cases n°1, n°2 and n°4, and changing their signs in the case n° 3.

As in [12] and [13], we write, by additivity of the equivariant virtual Poincaré series,

$$\beta^G(Y_{p,q}) = \beta^G(Y_{p,q} \cap \{u_2 \neq 0\}) + \beta^G(Y_{p,q} \cap \{u_2 = 0\}).$$

Because, if $u_2 \neq 0$, the coordinate v_2 is determined via an equivariant isomorphism by u_2 and the other variables that are free, we have $\beta^G(Y_{p,q} \cap \{u_2 \neq 0\}) = \beta^G(\mathbb{R}^* \times \mathbb{R}^{p+q-2}) = (u-1)\frac{u^{p+q-1}}{u-1}$ (see Remark 4.1). Furthermore, the equation describing $Y_{p,q} \cap \{u_2 = 0\}$ is

$$y_1^2 - y_{p+1}^2 + \sum_{i=3}^{p} u_i v_i - \sum_{j=2p+1}^{p+q} y_j^2 = 0$$

(notice that the variable v_2 is then free) and, by an induction, we obtain

$$\begin{aligned}\beta^G(Y_{p,q}) &= \sum_{i=2}^{p} u^{p+q+1-i} + u^{p-1}\beta^G\left(\left\{y_1^2 - y_{p+1}^2 - \sum_{j=2p+1}^{p+q} y_j^2 = 0\right\}\right)\\ &= u^{q+1}\frac{u^{p-1}-1}{u-1} + u^{p-1}\beta^G\left(\left\{y_1^2 - y_{p+1}^2 - \sum_{j=2p+1}^{p+q} y_j^2 = 0\right\}\right).\end{aligned}$$

Now, in order to compute $\beta^G\left(\left\{y_1^2 - y_{p+1}^2 - \sum_{j=2p+1}^{p+q} y_j^2 = 0\right\}\right)$, we equivariantly blow up the latter algebraic set at the origin of $\mathbb{R}^{q-p+2}$; in the chart $y_1 = w$, $y_i = wz_i$, $i = p+1, 2p+1, \ldots, p+q$, the blown-up variety is defined by

$$w^2\left(1 - z_{p+1}^2 - \sum_{j=2p+1}^{p+q} z_j^2\right) = 0,$$

the action of G being given by

- $(w, z_{p+1}, z_{2p+1}, \ldots, z_{p+q}) \mapsto (-w, -z_{p+1}, -z_{2p+1}, \ldots, -z_{p+q})$ in the case n°1,
- $(w, z_{p+1}, z_{2p+1}, \ldots, z_{p+q}) \mapsto (w, -z_{p+1}, z_{2p+1}, \ldots, z_{p+q})$ in the case n°2,
- $(w, z_{p+1}, z_{2p+1}, \ldots, z_{p+q}) \mapsto (-w, z_{p+1}, z_{2p+1}, \ldots, z_{p+q})$ in the case n° 3,
- $(w, z_{p+1}, z_{2p+1}, \ldots, z_{p+q}) \mapsto (w, z_{p+1}, z_{2p+1}, \ldots, z_{p+q})$ in the case n°4.

We have

$$\begin{aligned}&\beta^G\left(\left\{y_1^2 - y_{p+1}^2 - \sum_{j=2p+1}^{p+q} y_j^2 = 0\right\} \setminus \{0\}\right)\\ &\quad= \beta^G\left(\left\{1 - z_{p+1}^2 - \sum_{j=2p+1}^{p+q} z_j^2 = 0\right\} \setminus \{w=0\}\right)\\ &\quad= \beta^G(\mathbb{R}^* \times S^{q-p})\\ &\quad= (u-1)\beta^G(S^{q-p})\end{aligned}$$

Finally, since the action of G on the sphere S^{q-p} is the central symmetry in the case n°1 and admits a fixed point in the three other cases, we have

$$\beta^G(S^{q-p}) = \begin{cases}\frac{u^{q-p+1}-1}{u-1} & \text{in the case n°1,}\\ \frac{u^{q-p+1}+u}{u-1} & \text{in the three other cases}\end{cases}$$

(see Remark 4.1). Using the additivity relation

$$\beta^G\left(\left\{y_1^2 - y_{p+1}^2 - \sum_{j=2p+1}^{p+q} y_j^2 = 0\right\} \setminus \{0\}\right) = \beta^G\left(\left\{y_1^2 - y_{p+1}^2 - \sum_{j=2p+1}^{p+q} y_j^2 = 0\right\}\right) - \beta^G(\{0\})$$

and the equality $\beta^G(\{0\}) = \frac{u}{u-1}$, we obtain the desired result.

If $p = q \in \mathbb{N} \setminus \{0\,;1\}$, we do as before in order to obtain the equality

$$\beta^G(Y_{p,q}) = u^{p+1}\frac{u^{p-1}-1}{u-1} + u^{p-1}\beta^G\left(\left\{y_1^2 - y_{p+1}^2 = 0\right\}\right)$$

(notice that the quantity $\beta^G(\{y_1^2 - y_{p+1}^2 = 0\})$ is the same in the cases n°1 and n°2). Now, as above, we equivariantly blow up at the origin of $\mathbb{R}^2$ and look in the chart $y_1 = u_1$, $y_{p+1} = u_1 v_{p+1}$; the blown-up variety is given by the equation

$$u_1^2(1 - v_{p+1}^2) = 0$$

and the action of G is given by

- $(u_1, v_{p+1}) \mapsto (-u_1, -v_{p+1})$ in the case n°1,
- $(u_1, v_{p+1}) \mapsto (u_1, -v_{p+1})$ in the case n°2,
- $(u_1, v_{p+1}) \mapsto (-u_1, v_{p+1})$ in the case n°3,
- $(u_1, v_{p+1}) \mapsto (u_1, v_{p+1})$ in the case n°4.

As a consequence,

$$\beta^G(\{1 - v_{p+1}^2 = 0\}) = \begin{cases} 1 & \text{in the cases n°1 and n°2,} \\ 2\frac{u}{u-1} & \text{in the two other cases} \end{cases}$$

(see Remark 4.1) and we obtain the desired result.

If $p = q = 1$, we have $\beta^G(Y_{p,q}) = \beta^G\left(\left\{y_1^2 - y_{p+1}^2 = 0\right\}\right)$ and we can use the previous computation. $\square$

This proposition can be used to compute the quantities $\beta^G(Y_{p,q}^{\xi})$. Indeed:

Proposition 5.3. — *We have*

$$\beta^G(Y_{p,q}^{+1}) = \frac{1}{u-1}\left(\beta^G(Y_{p,q+1}) - \beta^G(Y_{p,q})\right)$$

and

$$\beta^G(Y_{p,q}^{-1}) = \frac{1}{u-1}\left(\beta^G(Y_{p+1,q}) - \beta^G(Y_{p,q})\right)$$

Proof. — We show the first equality, the proof of the second one being similar.

Denote $Z_{p,q}$ the projective algebraic set

$$\left\{ [Y_1 : \ldots : Y_{p+q}] \in \mathbb{P}^{p+q-1}(\mathbb{R}) \mid \sum_{i=1}^{p} Y_i^2 - \sum_{j=1}^{q} Y_{p+j}^2 = 0 \right\}$$

As in [11] Proof of Corollary 2.5, we can equivariantly compactify $Y_{p,q}^{+1}$ into the projective algebraic set $Z_{p,q+1}$, the part at infinity being equivariantly isomorphic to $Z_{p,q}$ (we equip $\mathbb{P}^{p+q}(\mathbb{R})$ and $\mathbb{P}^{p+q-1}(\mathbb{R})$ with the actions of G naturally induced from the considered action on the variables of $\mathbb{R}^{p+q}$).

Now, we compute $\beta^G(Z_{p,q})$, using, as in [11] Proof of Proposition 2.1, the fact that the projection

$$p: \begin{array}{ccc} Y_{p,q} \setminus \{0\} & \to & Z_{p,q} \\ (y_1, \ldots, y_{p+q}) & \mapsto & [y_1 : \ldots : y_{p+q}] \end{array}$$

is a piecewise algebraically trivial fibration, compatible with the respective considered actions of G. More precisely, we can cover $Z_{p,q}$ by the globally G-invariant open subvarieties

$$U_i := Z_{p,q} \cap \{Y_i \neq 0\}, i \in \{1, \ldots, p+q\},$$

and, for each $i \in \{1, \ldots, p+q\}$, we can define the isomorphism

$$\varphi_i : \begin{array}{ccc} p^{-1}(U_i) = Y_{p,q} \cap \{y_i \neq 0\} & \to & U_i \times \mathbb{R}^* \\ (y_1, \ldots, y_i, \ldots, y_{p+q}) & \mapsto & ([y_1 : \ldots : y_i : \ldots : y_{p+q}], y_i) \end{array}$$

For $i \in \{1, \ldots, p+q\}$, if the sign of the coordinate y_i is changed under the action of G, we equip $\mathbb{R}^*$ with the action of G given by the involution $z \mapsto -z$. If y_i remains unchanged under the action of G, we equip $\mathbb{R}^*$ with the trivial action of G. Furthermore, equipping the product $U_i \times \mathbb{R}^*$ with the diagonal action, this makes the isomorphism φ_i equivariant.

By the additivity of the equivariant virtual Poincaré series, the quantity $\beta^G(Y_{p,q} \setminus \{0\})$ can be written as the alternating sum of the terms

$$\sum_{J \subset \{1, \ldots, p+q\},\, Card(J)=r} \beta^G \left(p^{-1} \left(\bigcap_{m \in J} U_m \right) \right), 1 \leq r \leq p+q,$$

and, via the equivariant isomorphisms φ_i, we have

$$\beta^G \left(p^{-1} \left(\bigcap_{m \in J} U_m \right) \right) = \beta^G \left(\left(\bigcap_{m \in J} U_m \right) \times \mathbb{R}^* \right) = (u-1)\beta^G \left(\bigcap_{m \in J} U_m \right).$$

As a consequence, once again due to the additivity of the equivariant virtual Poincaré series,

$$\beta^G(Y_{p,q} \setminus \{0\}) = (u-1)\beta^G(Z_{p,q}).$$

Therefore,

$$\begin{aligned}\beta^G(Y^{+1}_{p,q}) &= \beta^G(Z_{p,q+1}) - \beta^G(Z_{p,q}) \\ &= \frac{1}{u-1}\left(\beta^G(Y_{p,q+1}\setminus\{0\}) - \beta^G(Y_{p,q}\setminus\{0\})\right) \\ &= \frac{1}{u-1}\left(\beta^G(Y_{p,q+1}) - \beta^G(Y_{p,q})\right). \end{aligned}$$ $\square$

6. The germs A_k and B_k

In this section, we want to study the relations with respect to G-blow-Nash equivalence between the invariant germs of the families

$$f_k^{\epsilon_k}(x) := \pm x_1^2 + \epsilon_k x_2^{k+1} + Q \quad \text{and} \quad g_k^{\epsilon_k}(x) := \epsilon_k x_1^{2k} \pm x_2^2 + Q',$$

where $\epsilon_k \in \{-1\,;+1\}$.

First, if any two invariant Nash germs are G-blow-Nash equivalent, they are, in particular, blow-Nash equivalent and then, according to [11] Theorem 2.5, they have the same co-rank and index.

Therefore, if two germs $f_k^{\epsilon_k}$ and $f_l^{\epsilon_l}$ are G-blow-Nash equivalent, they have the same quadratic part up to permutation of the variables $x_1, x_3, \ldots, x_n$. Furthermore, we know, by [13] Proposition 3.4, that $k=l$ and, if $k=l$ is odd, that $\epsilon_k = \epsilon_l$. If k is even, $f_k^{+1}(x_1,x_2,x_3,\ldots,x_n) = f_k^{-1}(x_1,-x_2,x_3,\ldots,x_n)$, and the (linear) change of variables is equivariant with respect to the involution s on $\mathbb{R}^n$; f_k^{+1} and f_k^{-1} are, then, G-Nash equivalent, in particular G-blow-Nash equivalent.

As a conclusion, inside the family A_k, we are reduced to trying to distinguish the germs

$$f_k^{\epsilon_k,+}(x) := +x_1^2 + \epsilon_k x_2^{k+1} + Q \quad \text{and} \quad f_k^{\epsilon_k,-}(x) := -x_1^2 + \epsilon_k x_2^{k+1} + Q',$$

where $\epsilon_k \in \{-1\,;+1\}$ and $+x_1^2 + Q$ and $-x_1^2 + Q'$ are the same quadratic part up to permutation of the variables $x_1, x_3, \ldots, x_n$.

Similarly, if two germs $g_k^{\epsilon_k}$ and $g_l^{\epsilon_l}$ are G-blow-Nash equivalent, they have the same quadratic part, up to permutation of the variables $x_2, \ldots, x_n$, and $k=l$ and $\epsilon_k = \epsilon_l$. As a consequence, the classification of the germs B_k up to G-blow-Nash equivalence is the same as their classification up to equivariant analytic equivalence.

Finally, if two germs $f_k^{\epsilon_k}$ and $g_{k'}^{\epsilon_{k'}}$ are blow-Nash equivalent, then $k = 2k'-1$ and $\epsilon_k = \epsilon_{k'}$, and, furthermore, $\pm x_1^2 + Q$ and $\pm x_2^2 + Q'$ are the same quadratic part up to permutation of all variables. Consequently, it remains to look at the relation between the germs

$$f_{2k-1} = \epsilon x_2^{2k} + \eta x_1^2 + Q \quad \text{and} \quad g_k = \epsilon x_1^{2k} + \eta' x_2^2 + Q',$$

where $\epsilon, \eta, \eta' \in \{1, -1\}$ and $\eta x_1^2 + Q = \eta' x_2^2 + Q'$ up to permutation of all variables.

In the following parts of this section, we will compute some terms of the equivariant zeta functions of f_k and g_k. In virtue of Theorem 4.2, this will allow us to make further distinctions inside each of the above couples of germs in some cases.

6.1. Computation of the first terms of the equivariant zeta functions. — If h is an invariant Nash germ $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, recall that, for $m \geq 1$,

$$\begin{aligned} A_m(h) &= \{\gamma(t) = a_1 t + \cdots + a_m t^m \in \mathcal{L}_m \mid h \circ \gamma(t) = ct^m + \cdots, c \neq 0\} \\ &= \{\gamma \in \mathcal{L}_m \,|\, h \circ \gamma(t) = ct^m + \cdots, c \in \mathbb{R}\} \setminus \{\gamma \in \mathcal{L}_m \mid h \circ \gamma(t) = 0 \times t^m + \cdots\} \end{aligned}$$

Since h is an invariant germ, the latter sets are both globally stable under the action of G on $\mathcal{L}_m$ and, by the additivity of the equivariant virtual Poincaré series, the quantity $\beta^G(A_m(h))$ is equal to the difference $\beta^G({}^0A_m(h)) - \beta^G(A_m^0(h))$, where

$$\begin{aligned} {}^0A_m(h) &:= \{\gamma \in \mathcal{L}_m \mid h \circ \gamma(t) = ct^m + \cdots, c \in \mathbb{R}\} \quad \text{and} \\ A_m^0(h) &:= \{\gamma \in \mathcal{L}_m \mid h \circ \gamma(t) = 0 \times t^m + \cdots\}. \end{aligned}$$

Fix $k \geq 0$ and consider the invariant germ $f_k^{\epsilon,\eta}(x_1, \ldots, x_n) = \eta x_1^2 + \epsilon x_2^{k+1} + Q$. We denote $x_2 = x$ and $\eta x_1^2 + Q = Q_{p,q} = \sum_{i=1}^p y_i^2 - \sum_{j=1}^q y_{p+j}^2$ in such a way that G acts on the renamed coordinates via the involution n°1 or n°2, depending on the sign of η.

We first compute $\beta^G(A_m^0(f_k^{\epsilon,\eta}))$ for $m < k+1$. Notice that the set $A_1(f_1^{\epsilon,\eta})$ is empty and, consequently, $\beta^G(A_1(f_1^{\epsilon,\eta})) = 0$.

Proposition 6.1. — *Suppose $k \geq 2$ and $m < k+1$.*

1. *If $pq = 0$, then*

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = \begin{cases} \frac{u^{m+(r+1)(p+q)+1}}{u-1} & \text{if } m = 2r+1, \\ \frac{u^{m+r(p+q)+1}}{u-1} & \text{if } m = 2r. \end{cases}$$

2. *If $(p,q) = (1,1)$, then*

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = \begin{cases} ru^{2m}\beta^G(Y_{1,1} \setminus \{0\}) + \frac{u^{4(r+1)}}{u-1} & \text{if } m = 2r+1, \\ (r-1)u^{2m}\beta^G(Y_{1,1} \setminus \{0\}) + u^{4r}\beta^G(Y_{1,1}) & \text{if } m = 2r, \end{cases}$$

3. *If $pq \neq 0$ and $(p,q) \neq (1,1)$, then*

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = \begin{cases} u^m u^{(r+1)(p+q)-1} \frac{u^{r(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q} \setminus \{0\}) \\ \quad + \frac{u^{(r+1)(2+p+q)}}{u-1} & \text{if } m = 2r+1, \\ u^m u^{(r+1)(p+q)-2} \frac{u^{(r-1)(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q} \setminus \{0\}) \\ \quad + u^{r(2+p+q)}\beta^G(Y_{p,q}) & \text{if } m = 2r. \end{cases}$$

Proof. — We follow the computation steps of [13], keeping the track of the action of G in our context.

An arc γ of $\mathcal{L}_m$ can be written as

$$\begin{aligned}\gamma(t) &= (a_1 t + \cdots + a_m t^m, c_1^1 t + \cdots + c_m^1 t^m, \ldots, c_1^{p+q} t + \cdots + c_m^{p+q} t^m)\\ &= \begin{pmatrix} a_1 \\ c_1^1 \\ \vdots \\ c_1^{p+q} \end{pmatrix} t + \cdots + \begin{pmatrix} a_m \\ c_m^1 \\ \vdots \\ c_m^{p+q} \end{pmatrix} t^m = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} t + \cdots + \begin{pmatrix} a_m \\ c_m \end{pmatrix} t^m\end{aligned}$$

if $c_i := (c_i^1, \ldots, c_i^{p+q})$. The group G acts on $\mathcal{L}_m$ changing the sign of the variables c_i^1, or c_i^{p+1}, in the case n°1, or n°2.

We begin with the case $pq \neq 0$, $(p,q) \neq (1,1)$ and $m = 2r+1$ odd. An arc γ of $\mathcal{L}_m$ belongs to $A_m^0(f_k^{\epsilon,\eta})$ if and only if

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ \Phi_{p,q}(c_1, c_2) = 0, \\ \cdots \\ Q_{p,q}(c_r) + \sum_{t=1}^{r-1} \Phi_{p,q}(c_t, c_{2r-t}) = 0, \\ \sum_{t=1}^{r} \Phi_{p,q}(c_t, c_{2r+1-t}) = 0, \end{cases}$$

where $\Phi_{p,q}$ is the function on $\mathbb{R}^{p+q} \times \mathbb{R}^{p+q}$ defined by $\Phi_{p,q}(u,v) = 2\sum_{i=1}^{p} u_i v_i - 2\sum_{j=1}^{q} u_{p+j} v_{p+j}$.

The first equality of the system means $c_1 \in Y_{p,q}$ by definition. Now, if $c_1^1 \neq 0$, the variables $c_2^1, \ldots, c_{2r}^1$ are determined by c_1^1, and the other (free) variables via an equivariant morphism. Therefore,

$$\begin{aligned}\beta^G(A_m^0(f_k^{\epsilon,\eta})) &= \beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 \neq 0\}\right) + \beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = 0\}\right)\\ &= \beta^G\left((Y_{p,q} \setminus (\{0\} \times Y_{p-1,q})) \times \mathbb{R}^{m+(m-1)(p+q-1)+1}\right)\\ &\quad + \beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = 0\}\right)\\ &= u^{m+(m-1)(p+q-1)+1} \beta^G\left(Y_{p,q} \setminus (\{0\} \times Y_{p-1,q})\right)\\ &\quad + \beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = 0\}\right)\end{aligned}$$

Next, we have

$$\begin{aligned}\beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = 0\}\right) &= u^{m+(m-1)(p+q-1)+1} \beta^G\left(Y_{p-1,q} \setminus (\{0\} \times Y_{p-2,q})\right)\\ &\quad + \beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = c_1^2 = 0\}\right),\end{aligned}$$

and we obtain by induction

$$\begin{aligned}\beta^G(A_m^0(f_k^{\epsilon,\eta})) &= u^{m+(m-1)(p+q-1)+1} \beta^G(Y_{p,q} \setminus \{0\})\\ &\quad + \beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = \ldots = c_1^p = 0\}\right).\end{aligned}$$

If $c_1^1 = \ldots = c_1^p = 0$, then $c_1^{p+1} = \ldots = c_1^{p+q} = 0$ (since $Q_{p,q}(c_1) = 0$), and the other variables verify the system

$$\begin{cases} Q_{p,q}(c_2) = 0, \\ \Phi_{p,q}(c_2, c_3) = 0, \\ \ldots \\ Q_{p,q}(c_r) + \sum_{t=2}^{r-1} \Phi_{p,q}(c_t, c_{2r-t}) = 0, \\ \sum_{t=2}^{r} \Phi_{p,q}(c_t, c_{2r+1-t}) = 0. \end{cases}$$

Noticing that the vector c_m as well as the variables a_{m-1}, a_m are free and renaming the remaining variables, we have $\beta^G\left(A_m^0(f_k^{\epsilon,\eta}) \cap \{c_1^1 = \ldots = c_1^p = 0\}\right) = u^{2+(p+q)}\beta^G(A_{m-2}^0(f_k^{\epsilon,\eta}))$. Consequently,

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = u^{m+(m-1)(p+q-1)+1}\beta^G(Y_{p,q} \setminus \{0\}) + u^{2+(p+q)}\beta^G(A_{m-2}^0(f_k^{\epsilon,\eta}))$$

and, by an induction on the index m,

$$\begin{aligned} \beta^G(A_m^0(f_k^{\epsilon,\eta})) = \beta^G(Y_{p,q} \setminus \{0\}) \sum_{t=0}^{r-1} u^{t(2+p+q)} u^{m-2t+(m-2t-1)(p+q-1)+1} \\ + u^{(r-1)(2+p+q)}\beta^G(A_3^0(f_k^{\epsilon,\eta}) \cap \{c_1 = 0\}) \end{aligned}$$

As a conclusion, since the system describing $A_3^0(f_k^{\epsilon,\eta}) \cap \{c_1 = 0\}$ is trivial, the variables a_i, as well as the vectors c_2 and c_3, are free and

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = u^m u^{(r+1)(p+q)-1} \frac{u^{r(p+q-2)} - 1}{u^{p+q-2} - 1}\beta^G(Y_{p,q} \setminus \{0\}) + \frac{u^{(r+1)(2+p+q)}}{u-1}.$$

If m is even, $m = 2r$, the system describing $A_m^0(f_k^{\epsilon,\eta})$ is

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ \Phi_{p,q}(c_1, c_2) = 0, \\ \ldots \\ \sum_{t=1}^{r-1} \Phi_{p,q}(c_t, c_{2r-1-t}) = 0, \\ Q_{p,q}(c_r) + \sum_{t=1}^{r-1} \Phi_{p,q}(c_t, c_{2r-t}) = 0. \end{cases}$$

Therefore, by similar computations, we obtain

$$\begin{aligned} \beta^G(A_m^0(f_k^{\epsilon,\eta})) = \beta^G(Y_{p,q} \setminus \{0\}) \sum_{t=0}^{r-2} u^{t(2+p+q)} u^{m-2t+(m-2t-1)(p+q-1)+1} \\ + u^{(r-1)(2+p+q)}\beta^G(A_2^0(f_k^{\epsilon,\eta})) \end{aligned}$$

Since $A_2^0(f_k^{\epsilon,\eta})$ is described by the equation $Q_{p,q}(c_1)=0$, the vector c_2 as well as the variables a_1 and a_2 being free,

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = u^m u^{(r+1)(p+q)-2}\frac{u^{(r-1)(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q}\setminus\{0\}) + u^{r(2+p+q)}\beta^G(Y_{p,q}).$$

Finally, if $(p,q)=(1,1)$, the same process gives

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = \begin{cases}\beta^G(Y_{1,1}\setminus\{0\})\sum_{t=0}^{r-1}u^{2m} + \frac{u^{4(r+1)}}{u-1} & \text{if } m=2r+1,\\ \beta^G(Y_{1,1}\setminus\{0\})\sum_{t=0}^{r-2}u^{2m} + u^{4r}\beta^G(Y_{1,1}) & \text{if } m=2r.\end{cases}$$

If $pq=0$, since $Y_{p,q}=\{0\}$, the equations $Q_{p,q}(c_1)=\ldots=Q_{p,q}(c_r)=0$ impose $c_1,\ldots,c_r$ to be zero vectors, and, the other variables being free, we have

$$\beta^G(A_m^0(f_k^{\epsilon,\eta})) = \begin{cases}\frac{u^{m+(r+1)(p+q)+1}}{u-1} & \text{if } m=2r+1,\\ \frac{u^{m+r(p+q)+1}}{u-1} & \text{if } m=2r.\end{cases} \qquad \square$$

Remark 6.2. — We obtain the same quantities for $\beta^G(A_m^0(g_l^\epsilon))$ with $m<2l$, providing that we equip the set $Y_{p,q}$ with the trivial action of G. Indeed, the computation steps above remain equivariant if the group G acts on $\mathcal{L}_m$ changing the sign of the variables a_i.

Proposition 6.3. — *Let h be an invariant Nash germ $(\mathbb{R}^n,0)\to(\mathbb{R},0)$ and $m\geq 2$. Then*

$$\beta^G({}^0A_m(h)) = u^n\beta^G(A_{m-1}^0(h)).$$

Proof. — Notice that ${}^0A_m(h)=\{\gamma\in\mathcal{L}_m \mid h\circ\gamma(t)=0\times t+\cdots+0\times t^{m-1}+ct^m+\cdots\}$. Therefore, the system describing ${}^0A_m(h)$ is the same as the system describing $A_{m-1}^0(h)$, the last n variables being free.

As a consequence, we have an equivariant isomorphism between ${}^0A_m(h)$ and the product $\mathbb{R}^n\times A_{m-1}^0(h)$ (this set is equipped with the diagonal action of G, the first term being equipped with the involution s), and, consequently,

$$\beta^G({}^0A_m(h)) = u^n\beta^G(A_{m-1}^0(h)). \qquad \square$$

We also compute $\beta^G(A_m^\xi(f_k^{\epsilon,\eta}))$ for $m<k+1$:

Proposition 6.4. — *Suppose $k\geq 2$ and $m<k+1$.*

1. *If $pq=0$, then*

$$\beta^G(A_m^\xi(f_k^{\epsilon,\eta})) = \begin{cases}0 & \textit{if } m=2r+1,\\ u^{m+r(p+q)}\beta^G(Y_{p,q}^\xi) & \textit{if } m=2r.\end{cases}$$

2. *If* $(p,q)=(1,1)$, *then*

$$\beta^G(A_m^\xi(f_k^{\epsilon,\eta}))=\begin{cases} ru^{2m}\beta^G(Y_{1,1}\setminus\{0\}) & \textit{if } m=2r+1,\\ (r-1)u^{2m}\beta^G(Y_{1,1}\setminus\{0\})+u^{4r}\beta^G(Y_{1,1}^\xi) & \textit{if } m=2r,\end{cases}$$

3. *If* $pq\neq 0$ *and* $(p,q)\neq(1,1)$, *then*

$$\beta^G(A_m^\xi(f_k^{\epsilon,\eta}))=\begin{cases} u^m u^{(r+1)(p+q)-1}\frac{u^{r(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q}\setminus\{0\}) & \textit{if } m=2r+1,\\ u^m u^{(r+1)(p+q)-2}\frac{u^{(r-1)(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q}\setminus\{0\}) & \\ \quad +u^{r(2+p+q)}\beta^G(Y_{p,q}^\xi) & \textit{if } m=2r.\end{cases}$$

Proof. — We first deal with the case $pq\neq 0$, $(p,q)\neq(1,1)$ and $m=2r$ even. Keeping the notations of the proof of Proposition 6.1, the system describing $A_m^\xi(f_k^{\epsilon,\eta})$ is

$$\begin{cases} Q_{p,q}(c_1)=0,\\ \Phi_{p,q}(c_1,c_2)=0,\\ \dots\\ \sum_{t=1}^{r-1}\Phi_{p,q}(c_t,c_{2r-1-t})=0,\\ Q_{p,q}(c_r)+\sum_{t=1}^{r-1}\Phi_{p,q}(c_t,c_{2r-t})=\xi.\end{cases}$$

The computation steps are the same as in the proof of Proposition 6.1, and we have

$$\begin{aligned}\beta^G(A_m^\xi(f_k^{\epsilon,\eta}))=u^m u^{(r+1)(p+q)-2}\frac{u^{(r-1)(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q}\setminus\{0\})\\ +u^{(r-1)(2+p+q)}\beta^G(A_2^\xi(f_k^{\epsilon,\eta})).\end{aligned}$$

Since the set $A_2^\xi(f_k^{\epsilon,\eta})$ is described by the equation $Q_{p,q}(c_1)=\xi$ and the other variables being free, we obtain the result.

If m is odd, $m=2r+1$, as in the proof of Proposition 6.1, we obtain

$$\begin{aligned}\beta^G(A_m^\xi(f_k^{\epsilon,\eta}))=u^m u^{(r+1)(p+q)-1}\frac{u^{r(p+q-2)}-1}{u^{p+q-2}-1}\beta^G(Y_{p,q}\setminus\{0\})\\ +u^{(r-1)(2+p+q)}\beta^G(A_3^\xi(f_k^{\epsilon,\eta})\cap\{c_1=0\})\end{aligned}$$

and the set $A_3^\xi(f_k^{\epsilon,\eta})\cap\{c_1=0\}$ is empty.

Similar considerations provide the results for the cases $(p,q)=(1,1)$ and $pq=0$. □

Remark 6.5. — Again, we have the same quantities for $\beta^G(A_m^\xi(g_l^\epsilon))$ with $m<2l$, providing that we equip the sets $Y_{p,q}$ and $Y_{p,q}^\xi$ with the trivial action of G.

Now, we are ready to deduce distinctions, with respect to G-blow-Nash equivalence, between $f_k^{\epsilon,+}$ and $f_k^{\epsilon,-}$, or between f_{2k-1} and g_k, in some cases:

Corollary 6.6. — *Let $k \geq 1$. Suppose that the invariant germs*

$$f_k^{\epsilon,+}(x) := +x_1^2 + \epsilon x_2^{k+1} + Q \text{ and } f_k^{\epsilon,-}(x) := -x_1^2 + \epsilon x_2^{k+1} + Q'$$

have the same quadratic part up to permutation of the variables $x_1, x_3, \dots, x_n$. Then they are not G-blow-Nash equivalent.

As a consequence, the classification of the germs A_k up to G-blow-Nash equivalence is the same as their classification up to equivariant analytic equivalence.

Proof (of Corollary 6.6). — We begin by assuming $k \geq 2$. We first compare $\beta^G(A_2(f_k^{\epsilon,+}))$ and $\beta^G(A_2(f_k^{\epsilon,-}))$. Since $\beta^G({}^0A_2(f_k^{\epsilon,\eta})) = u^{1+p+q}\beta^G(A_1^0(f_k^{\epsilon,\eta}))$ (by Proposition 6.3) and $A_1^0(f_k^{\epsilon,\eta}) = \mathcal{L}_1$, we are reduced to comparing $\beta^G(A_2^0(f_k^{\epsilon,+}))$ and $\beta^G(A_2^0(f_k^{\epsilon,-}))$.

Denote by p the number of signs $+$ and by q the number of signs $-$ in the quadratic part of $f_k^{\epsilon,+}$ and $f_k^{\epsilon,-}$ (notice that $pq \neq 0$). Then, according to Proposition 6.1,

$$\beta^G(A_2^0(f_k^{\epsilon,\eta})) = u^{2+p+q}\beta^G(Y_{p,q}).$$

Therefore, by Proposition 5.1,

- if $p < q$, then

$$\beta^G(A_2^0(f_k^{\epsilon,+})) = u^{2+p+q}\frac{u^{p+q} - u^q + u^{p-1}}{u-1} \quad \text{and}$$
$$\beta^G(A_2^0(f_k^{\epsilon,-})) = u^{2+p+q}\frac{u^{p+q} - u^q + u^{p+1}}{u-1},$$

- if $q < p$, then

$$\beta^G(A_2^0(f_k^{\epsilon,+})) = u^{2+p+q}\frac{u^{p+q} - u^p + u^{q+1}}{u-1} \quad \text{and}$$
$$\beta^G(A_2^0(f_k^{\epsilon,-})) = u^{2+p+q}\frac{u^{p+q} - u^p + u^{q-1}}{u-1}$$

In particular, $\beta^G(A_2^0(f_k^{\epsilon,+})) \neq \beta^G(A_2^0(f_k^{\epsilon,-}))$ if $p \neq q$. Consequently, if $p \neq q$, the naive equivariant zeta functions of $f_k^{\epsilon,+}$ and $f_k^{\epsilon,-}$ are different, and, by Theorem 4.2, these germs are not G-blow-Nash equivalent.

If $p = q$, $\beta^G(A_2^0(f_k^{\epsilon,+})) = \beta^G(A_2^0(f_k^{\epsilon,-}))$, and we look at the term $\beta^G(A_2^{+1}(f_k^{\epsilon,\eta}))$ of the equivariant zeta functions with sign $+$. According to Proposition 6.4,

$$\beta^G(A_2^{+1}(f_k^{\epsilon,\eta})) = u^{2+2p}\beta^G(Y_{p,p}^{+1})$$

and, by 5.3, $\beta^G(Y_{p,p}^{+1}) = \frac{1}{u-1}\left(\beta^G(Y_{p,p+1}) - \beta^G(Y_{p,p})\right)$. Since the quantity $\beta^G(Y_{p,p})$ is the same in either of the cases n°1 and n°2, we are reduced to comparing the quantities $\beta^G(Y_{p,p+1})$ in the cases n°1 and n°2.

We have

$$\beta^G(Y_{p,p+1}) = \begin{cases} \frac{u^{2p+1}-u^{p+1}+u^{p-1}}{u-1} & \text{in the case n}^\circ 1, \\ \frac{u^{2p+1}-u^{p+1}+u^{p+1}}{u-1} & \text{in the case n}^\circ 2, \end{cases}$$

and, as a consequence, $\beta^G(A_2^{+1}(f_k^{\epsilon,+})) \neq \beta^G(A_2^{+1}(f_k^{\epsilon,-}))$, so $f_k^{\epsilon,+}$ and $f_k^{\epsilon,-}$ are not G-blow-Nash equivalent in the case $p = q$ either.

If $k = 1$, notice that $f_1^{\epsilon,\eta}(x,y) = \epsilon x^2 + Q_{p,q}(y)$, and we are reduced to comparing $\beta^G(A_2^0(f_1^{\epsilon,+}))$ and $\beta^G(A_2^0(f_1^{\epsilon,-}))$ as well. We have $\beta^G(A_2^0(f_1^{\epsilon,\eta})) = u^{1+p+q}\beta^G(Y_{p+1,q})$ if $\epsilon = +1$, and $\beta^G(A_2^0(f_1^{\epsilon,\eta})) = u^{1+p+q}\beta^G(Y_{p,q+1})$ if $\epsilon = -1$. As above, we can show, for instance if $\epsilon = +1$, that $\beta^G(A_2^0(f_1^{\epsilon,+})) \neq \beta^G(A_2^0(f_1^{\epsilon,-}))$ when $p+1 \neq q$, and $\beta^G(A_2^{+1}(f_1^{\epsilon,+})) \neq \beta^G(A_2^{+1}(f_1^{\epsilon,-}))$ if $p+1 = q$. □

Remark 6.7. — If $k = 0$, $f_0^{\epsilon,\eta}(x,y) = \epsilon x + Q_{p,q}(y)$ and, using the notations of the proof of Proposition 6.1, the left members of all the equations describing $A_m^0(f_0^{\epsilon,\eta})$, or $A_m^{\xi}(f_0^{\epsilon,\eta})$, for $m \geq 1$, contain a term $\epsilon a_i + \ldots$, so that each of these sets is equivariantly isomorphic to an affine space. As a consequence (see Remark 4.1), the respective equivariant zeta functions of $f_0^{\epsilon,+}$ and $f_0^{\epsilon,-}$ are equal.

Corollary 6.8. — *Let $k \geq 2$. Suppose that the invariant germs*

$$f_{2k-1} = \epsilon x_2^{2k} + \eta x_1^2 + Q \text{ and } g_k = \epsilon x_1^{2k} + \eta' x_2^2 + Q'$$

have, up to permutation of all variables, the same quadratic part, with p signs $+$ and q signs $-$.

If $p \leq q$ and $\eta = +1$ or $q \leq p$ and $\eta = -1$, then f_{2k-1} and g_k are not G-blow-Nash equivalent.

If $p = q+1$ or $q = p+1$, then f_{2k-1} and g_k are not G-blow-Nash equivalent.

Proof. — We first deal with the case $p \leq q$ and $\eta = +1$ (notice that $p \neq 0$); the case $q \leq p$ and $\eta = -1$ is symmetric.

As in the proof of previous Corollary 6.6, we have

$$\beta^G(A_2^0(f_{2k-1})) = u^{2+p+q}\beta^G(Y_{p,q}) \text{ and } \beta^G(A_2^0(g_k)) = u^{2+p+q}\beta^G(Y_{p,q})$$

where, in the left equality, the set $Y_{p,q}$ is equipped with the action n°1 and, in the right one, with the trivial action of G. Since the corresponding equivariant virtual Poincaré series are different by Proposition 5.1, $\beta^G(A_2(f_{2k-1})) \neq \beta^G(A_2(g_k))$, and the naive equivariant zeta functions of f_{2k-1} and g_k are different. As a consequence, f_{2k-1} and g_k are not G-blow-Nash equivalent.

Now we suppose $p = q+1$ (the case $q = p+1$ is symmetric). In particular $q < p$, so we can assume $\eta = +1$ (the case $q < p$ and $\eta = -1$ was handled above, being symmetric to the case $p < q$ and $\eta = +1$).

We consider $\beta^G(A_2^{+1}(f_{2k-1})) = u^{2+p+q}\beta^G(Y_{p,q}^{+1})$ and $\beta^G(A_2^{+1}(g_k)) = u^{2+p+q}\beta^G(Y_{p,q}^{+1})$ (Proposition 6.4). Due to Proposition 5.3, we know that

$\beta^G(Y^{+1}_{p,q}) = \frac{1}{u-1}\left(\beta^G(Y_{p,q+1}) - \beta^G(Y_{p,q})\right)$. By Proposition 5.1, the respective quantities $\beta^G(Y_{p,q})$ for f_{2k-1} and g_k are equal, whereas the quantities $\beta^G(Y_{p,q+1}) = \beta^G(Y_{p,p})$ are different. Consequently, the equivariant zeta functions with sign $+$ of f_{2k-1} and g_k are different, and, therefore, the latter germs are not G-blow-Nash equivalent. □

REMARK 6.9. — In the other cases, the quantities $\beta^G(Y_{p,q})$ and $\beta^G(Y^{\xi}_{p,q})$ are the same for f_{2k-1} and g_k.

6.2. Computation of $\beta^G(A_{2k}(f_{2k-1}))$ and $\beta^G(A_{2k}(g_k))$. — For the continuation of the section, due to Corollaries 6.6 and 6.8, we only need to consider the germs

$$f_{2k-1} = \epsilon x_2^{2k} + \eta x_1^2 + Q \text{ and } g_k = \epsilon x_1^{2k} + \eta' x_2^2 + Q',$$

which are assumed to have the same quadratic part $Q_{p,q}$, such that $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$.

In order to prove that the germs f_{2k-1} and g_k are not G-blow-Nash equivalent in some of these cases either, we will compute the coefficients $\beta^G(A_{2k}(f_{2k-1}))$ and $\beta^G(A_{2k}(g_k))$ of their respective naive equivariant zeta functions:

PROPOSITION 6.10. — *Suppose $k \geq 2$.*

1. *If $pq = 0$, then*

$$\beta^G(A^0_{2k}(f_{2k-1})) = u^{2k-1+k(p+q)}\beta^G(\{f_{2k-1} = 0\}) \quad \text{and}$$
$$\beta^G(A^0_{2k}(g_k)) = u^{2k-1+k(p+q)}\beta^G(\{g_k = 0\}).$$

2. *If $pq \neq 0$, then*

$$\beta^G(A^0_{2k}(f_{2k-1})) = u^{2k-2}u^{(p+q)(k+1)}\beta^G(Y_{p,q} \setminus \{0\})\frac{u^{(p+q-2)(k-1)} - 1}{u^{p+q-2} - 1} + u^{k(p+q)+2k-1}\beta^G(\{f_{2k-1} = 0\})$$

(the group G acts on $Y_{p,q}$ via the involution n°1 or n°2 depending on the sign of η) and

$$\beta^G(A^0_{2k}(g_k)) = u^{2k-2}u^{(p+q)(k+1)}\beta^G(Y_{p,q} \setminus \{0\})\frac{u^{(p+q-2)(k-1)} - 1}{u^{p+q-2} - 1} + u^{k(p+q)+2k-1}\beta^G(\{g_k = 0\})$$

(the group G acts trivially on $Y_{p,q}$).

Proof. — We keep the notations of the proof of Proposition 6.1 and we proceed as in [13] Proof of Lemma 3.3. First, suppose that $pq \neq 0$. An arc γ of $\mathcal{L}_{2k}$

belongs to $A^0_{2k}(f_{2k-1})$ if and only if

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ \Phi_{p,q}(c_1, c_2) = 0, \\ \dots \\ \sum_{t=1}^{k-1} \Phi_{p,q}(c_t, c_{2k-1-t}) = 0, \\ \epsilon a_1^{2k} + Q_{p,q}(c_k) + \sum_{t=1}^{k-1} \Phi_{p,q}(c_t, c_{2k-t}) = 0. \end{cases}$$

We have

$$\begin{aligned} \beta^G(A^0_{2k}(f_{2k-1})) = {} & u^{2k+(2k-1)(p+q-1)+1} \beta^G(Y_{p,q} \setminus \{0\}) \\ & + \beta^G(A^0_{2k}(f_{2k-1}) \cap \{c_1^1 = \ldots = c_1^p = 0\}), \end{aligned}$$

and $\beta^G(A^0_{2k}(f_{2k-1}) \cap \{c_1^1 = \ldots = c_1^p = 0\}) = u^{2+p+q} \beta^G(C^0_{2k-2})$, if C^0_{2k-2} denotes the algebraic set described by the equations

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ \Phi_{p,q}(c_1, c_2) = 0, \\ \dots \\ \sum_{t=1}^{k-2} \Phi_{p,q}(c_t, c_{2k-3-t}) = 0, \\ \epsilon a_1^{2k} + Q_{p,q}(c_{k-1}) + \sum_{t=1}^{k-2} \Phi_{p,q}(c_t, c_{2k-2-t}) = 0. \end{cases}$$

By an induction, we obtain

$$\begin{aligned} \beta^G(A^0_{2k}(f_{2k-1})) = {} & \beta^G(Y_{p,q} \setminus \{0\}) \sum_{t=0}^{k-2} u^{t(2+p+q)} u^{2k-2t+(2k-2t-1)(p+q-1)+1} \\ & + u^{(k-1)(2+p+q)} \beta^G(C^0_2). \end{aligned}$$

Since C^0_2 is defined by the equation $\epsilon a_1^{2k} + Q_{p,q}(c_1) = 0$, and since the vector c_2 and the variable a_2 are free, we deduce the desired expression for $\beta^G(A^0_{2k}(f_{2k-1}))$. The steps of computation are the same for $\beta^G(A^0_{2k}(g_k))$.

If $pq = 0$, the vectors $c_1, \ldots, c_{k-1}$ are zero vectors, and the system is reduced to the equation $\epsilon a_1^{2k} + Q_{p,q}(c_k) = 0$, the other variables being free. $\square$

Since $p > q + 1$ and $\eta = +1$ or $q > p + 1$ and $\eta = -1$, the quantity $\beta^G(Y_{p,q})$ is the same for f_{2k-1} and g_k. As a consequence, in order to compare $\beta^G(A_{2k}(f_{2k-1}))$ and $\beta^G(A_{2k}(g_k))$, we are reduced to considering the quantities $\beta^G(\{f_{2k-1} = 0\})$ and $\beta^G(\{g_k = 0\})$ (notice that $\beta^G({}^0A_{2k}(f_{2k-1})) = \beta^G({}^0A_{2k}(g_k))$ by the results of Section 6.1). We compute these equivariant virtual Poincaré series for all $k \geq 2$, $p, q \in \mathbb{N}$ and $\eta \in \{1, -1\}$:

Lemma 6.11. — *We have*

$$\begin{aligned} \beta^G(\{f_{2k-1} = 0\}) = {} & \beta^G(\{\epsilon x_2^2 + \eta x_1^2 + Q = 0\}) \\ & - (k-1)\beta^G(\{\eta x_1^2 + Q = 0\}) + (k-1)\beta^G(\{0\}), \end{aligned}$$

where the second set in the right member is considered as an algebraic subset of $\mathbb{R}^{n-1}$*, and* G *acts on the considered sets via the involution n*^{o} *1 or n*^{o} *2 depending on the sign of* η*, and*

$$\beta^G(\{g_k = 0\}) = \beta^G(\{\epsilon x_1^2 + \eta' x_2^2 + Q' = 0\}) - \rho\beta^G(\{\eta' x_2^2 + Q' = 0\}) \\ - \tau\beta^G(\{\eta' x_2^2 + Q' = 0\}) + (k-1)\beta^G(\{0\}),$$

where the second and third sets in the right member are considered as algebraic subsets of $\mathbb{R}^{n-1}$*, the group* G *acts on the second set via the involution n*^{o} *4 (trivial action), on the third set via the involution n*^{o} *3 (change of signs of all coordinates) and*

1. *if* $k = 2l+1$ *is odd, then* $\rho = \tau = l$ *and* G *acts on the first set in the right member via the involution n*^{o} *1 or n*^{o} *2 depending on the sign of* ϵ*,*
2. *if* $k = 2l$ *is even, then* $\rho = l$*,* $\tau = l-1$ *and* G *acts on the first set in the right member via the involution n*^{o} *3.*

Proof. — We begin with $\beta^G(\{f_{2k-1} = 0\})$. Recall that

$$f_{2k-1}(x_1, x_2, x_3, \ldots, x_n) = \epsilon x_2^{2k} + \eta x_1^2 + Q(x_3, \ldots, x_n).$$

We proceed to an equivariant blowing-up of the algebraic set $\{f_{2k-1} = 0\}$ at the origin of $\mathbb{R}^n$. In the chart $x_2 = u$, $x_i = uv_i$, $i = 1, 3, \ldots, n$, the blown-up variety is defined by the equation

$$u^2 f_{2k-3}(v_1, u, v_3, \ldots, v_n) = 0,$$

the action of G being given by the involution

$$(v_1, u, v_3, \ldots, v_n) \mapsto (-v_1, u, v_3, \ldots, v_n).$$

We have $\beta^G(\{f_{2k-1} = 0\} \setminus \{0\}) = \beta^G(\{f_{2k-3} = 0\} \setminus \{u = 0\})$, therefore,

$$\beta^G(\{f_{2k-1} = 0\}) = \beta^G(\{f_{2k-3} = 0\}) - \beta^G(\{\eta v_1^2 + Q(v_3, \ldots, v_n) = 0, u = 0\}) \\ + \beta^G(\{0\}).$$

We then obtain the desired result by an induction.

For the computation of $\beta^G(\{g_k = 0\})$, recall that $g_k(x_1, x_2, x_3, \ldots, x_n) = \epsilon x_1^{2k} + \eta' x_2^2 + Q'(x_3, \ldots, x_n)$ and proceed to an equivariant blowing-up of the set $\{g_k = 0\}$ at the origin of $\mathbb{R}^n$, seen in the chart $x_1 = u$, $x_i = uv_i$, $i = 2, 3, \ldots, n$. In this chart, the blown-up variety is defined by

$$u^2 g_{k-1}(u, v_2, v_3, \ldots, v_n) = 0,$$

the action of G being given by the involution

$$(u, v_2, v_3, \ldots, v_n) \mapsto (-u, -v_2, -v_3, \ldots, -v_n),$$

and we have

$$\beta^G(\{g_k = 0\}) = \beta^G(\{g_{k-1} = 0\}) - \beta^G\{\eta' x_2^2 + Q'(x_3, \ldots, x_n) = 0\}) + \beta^G(\{0\}).$$

One further equivariant blowing-up of $\{g_{k-1} = 0\}$ provides the equation

$$u^2 g_{k-1}(u, v_2, v_3, \dots, v_n) = 0,$$

the group G acting via the involution $(u, v_2, v_3, \dots, v_n) \mapsto (-u, v_2, v_3, \dots, v_n)$. The desired expression is then obtained by an induction. $\square$

Remark 6.12. — According to Proposition 5.1, the quantity $\beta^G(\{\eta' x_2^2 + Q' = 0\})$ is the same if G acts via the involution n°4 or via the involution n°3. Therefore, in the previous Lemma 6.11, we can simply write $\rho\beta^G(\{\eta' x_2^2 + Q' = 0\}) + \tau\beta^G(\{\eta' x_2^2 + Q' = 0\})$ as $(k-1)\beta^G(\{\eta' x_2^2 + Q' = 0\})$ with G acting trivially on the latter set.

Because $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$, we have $\beta^G(\{\eta x_1^2 + Q = 0\}) = \beta^G(\{\eta' x_2^2 + Q' = 0\})$, and we are finally reduced to comparing $\beta^G(\{\epsilon x_2^2 + \eta x_1^2 + Q = 0\})$ and $\beta^G(\{\epsilon x_1^2 + \eta' x_2^2 + Q' = 0\})$. The cases where these quantities are different are those where the germs f_{2k-1} and g_k are not G-blow-Nash-equivalent:

Corollary 6.13. — *If k is odd and if $p > q+1$, $\eta = +1$ and $\epsilon = -1$ or $q > p+1$, $\eta = -1$ and $\epsilon = +1$, then the germs f_{2k-1} and g_k are not G-blow-Nash-equivalent.*

Proof. — Assume that k is odd and suppose that $p > q+1$, $\eta = +1$ and $\epsilon = -1$ (the case $q > p+1$, $\eta = -1$ and $\epsilon = +1$ is symmetric). We have $\{\epsilon x_2^2 + \eta x_1^2 + Q = 0\} = Y_{p,q+1}$, where $Y_{p,q+1}$ is equipped with the involution n°1, and $\{\epsilon x_1^2 + \eta' x_2^2 + Q' = 0\} = Y_{p,q+1}$, where $Y_{p,q+1}$ is equipped with the involution n°2. Then, by Proposition 5.1, $\beta^G(\{\epsilon x_2^2 + \eta x_1^2 + Q = 0\}) \neq \beta^G(\{\epsilon x_1^2 + \eta' x_2^2 + Q' = 0\})$ and $\beta^G(A_{2k}(f_{2k-1})) \neq \beta^G(A_{2k}(g_k))$. $\square$

In the remaining cases, the quantities $\beta^G(\{\epsilon x_2^2 + \eta x_1^2 + Q = 0\})$ and $\beta^G(\{\epsilon x_1^2 + \eta' x_2^2 + Q' = 0\})$ are equal, so $\beta^G(A_{2k}(f_{2k-1})) = \beta^G(A_{2k}(g_k))$. As a consequence, for these cases, we are led to look at the remaining coefficients of the equivariant zeta functions of f_{2k-1} and g_k. In the following section, we begin with the computation of the terms $\beta^G(A_{2k}^{\xi}(f_{2k-1}))$ and $\beta^G(A_{2k}^{\xi}(g_k))$.

6.3. Computation of $\beta^G(A_{2k}^{\xi}(f_{2k-1}))$ and $\beta^G(A_{2k}^{\xi}(g_k))$. — We assume that we are not in one of the previous cases for which we showed that f_{2k-1} and g_k are not G-blow-Nash-equivalent. In particular, we have

$$\beta^G(A_m(f_{2k-1})) = \beta^G(A_m(g_k)) \quad \text{and} \quad \beta^G(A_m^{\xi}(f_{2k-1})) = \beta^G(A_m^{\xi}(g_k))$$

for $m < 2k$, and

$$\beta^G(A_{2k}(f_{2k-1})) = \beta^G(A_{2k}(g_k)).$$

Now, the same steps of computation as in the proof of Proposition 6.10 provide the following formulae for $\beta^G(A_{2k}^{\xi}(f_{2k-1}))$ and $\beta^G(A_{2k}^{\xi}(g_k))$:

Proposition 6.14. — *Suppose $k \geq 2$.*

1. *If $pq = 0$, then*

$$\begin{aligned}\beta^G(A^{\xi}_{2k}(f_{2k-1})) &= u^{2k-1+k(p+q)}\beta^G(\{f_{2k-1}=\xi\}) \quad \textit{and}\\ \beta^G(A^{\xi}_{2k}(g_k)) &= u^{2k-1+k(p+q)}\beta^G(\{g_k=\xi\}).\end{aligned}$$

2. *If $pq \neq 0$, then*

$$\begin{aligned}\beta^G(A^{\xi}_{2k}(f_{2k-1})) = u^{2k-2}u^{(p+q)(k+1)}\beta^G(Y_{p,q}\setminus\{0\})\frac{u^{(p+q-2)(k-1)}-1}{u^{p+q-2}-1}\\ + u^{k(p+q)+2k-1}\beta^G(\{f_{2k-1}=\xi\})\end{aligned}$$

(the group G acts on $Y_{p,q}$ via the involution $n^o 1$ or $n^o 2$ depending on the sign of η) and

$$\begin{aligned}\beta^G(A^{\xi}_{2k}(g_k)) = u^{2k-2}u^{(p+q)(k+1)}\beta^G(Y_{p,q}\setminus\{0\})\frac{u^{(p+q-2)(k-1)}-1}{u^{p+q-2}-1}\\ + u^{k(p+q)+2k-1}\beta^G(\{g_k=\xi\})\end{aligned}$$

(the group G acts trivially on $Y_{p,q}$).

As in the previous Section 6.2, we are reduced to considering the quantities $\beta^G(\{f_{2k-1}=\xi\})$ and $\beta^G(\{g_k=\xi\})$. Below, we give the first steps of computation of these equivariant virtual Poincaré series for all $k\geq 2$, $(p,q)\in\mathbb{N}^2\setminus\{(0,0)\}$ and $\eta\in\{1,-1\}$. We write $f_{2k-1}=\epsilon x_2^{2k}+\eta x_1^2+Q=\epsilon x_2^{2k}+\sum_{i=1}^p y_i^2-\sum_{j=1}^q y_{p+j}^2$ and $g_k=\epsilon x_1^{2k}+\eta' x_2^2+Q'=\epsilon x_1^{2k}+\sum_{i=1}^p y_i^2-\sum_{j=1}^q y_{p+j}^2$. Then:

Lemma 6.15. — *We have*

$$\beta^G(\{f_{2k-1}=\xi\}) = \begin{cases} u^{q+2}\frac{u^{p-1}-1}{u-1}+u^{p-1}\beta^G\left(\left\{\epsilon x_2^{2k}+y_1^2-y_{p+1}^2-\sum_{j=2p+1}^{p+q}y_j^2=\xi\right\}\right) & \\ \hfill \textit{if } 0<p<q,\\ u^{p+2}\frac{u^{q-1}-1}{u-1}+u^{q-1}\beta^G\left(\left\{\epsilon x_2^{2k}+y_1^2-y_{p+1}^2+\sum_{j=q+1}^{p}y_j^2=\xi\right\}\right) & \\ \hfill \textit{if } 0<q<p,\\ u^{p+2}\frac{u^{p-1}-1}{u-1}+u^{p-1}\beta^G\left(\left\{\epsilon x_2^{2k}+y_1^2-y_{p+1}^2=\xi\right\}\right) & \textit{if } p=q,\\ \beta^G\left(\left\{\epsilon x_2^{2k}-y_1^2-\sum_{j=2}^{q}y_j^2=\xi\right\}\right) & \textit{if } p=0,\\ \beta^G\left(\left\{\epsilon x_2^{2k}+y_1^2+\sum_{i=2}^{p}y_i^2=\xi\right\}\right) & \textit{if } q=0,\end{cases}$$

the group G acting only changing the sign of y_1 or y_{p+1} depending on the sign of η, and

$$\beta^G(\{g_k=\xi\}) = \begin{cases} u^{q+1}\frac{u^p-1}{u-1} + u^p\beta^G\big(\big\{\epsilon x_1^{2k} - \sum_{j=2p+1}^{p+q} y_j^2 = \xi\big\}\big) & \text{if } 0<p<q,\\ u^{p+1}\frac{u^q-1}{u-1} + u^q\beta^G\big(\big\{\epsilon x_1^{2k} + \sum_{j=q+1}^{p} y_j^2 = \xi\big\}\big) & \text{if } 0<q<p,\\ u^{p+1}\frac{u^p-1}{u-1} + u^p\beta^G\big(\big\{\epsilon x_1^{2k} = \xi\big\}\big) & \text{if } p=q,\\ \beta^G\big(\big\{\epsilon x_1^{2k} - \sum_{j=1}^{q} y_j^2 = \xi\big\}\big) & \text{if } p=0,\\ \beta^G\big(\big\{\epsilon x_1^{2k} + \sum_{i=1}^{p} y_i^2 = \xi\big\}\big) & \text{if } q=0,\end{cases}$$

the group G acting only changing the sign of x_1.

Proof. — We focus on the case $2\leq p\leq q$ and proceed as in the proof of Proposition 5.1; in order to compute $\beta^G(\{f_{2k-1}=\xi\})$, we apply the (equivariant) change of variables $u_i=y_i+y_{i+p}$, $v_i=y_i-y_{i+p}$ for $i=2,\dots,p$, and the equation $f_{2k-1}=\xi$ becomes

$$\epsilon x_2^{2k}+y_1^2-y_{p+1}^2+\sum_{i=2}^{p}u_iv_i-\sum_{j=2p+1}^{p+q}y_j^2=\xi.$$

Then, as in the proof of Proposition 5.1, we use the stratification by the globally G-stable subsets $\{f_{2k-1}=\xi\}\cap\{u_2=\ldots=u_i=0, u_{i+1}\neq 0\}$, along with the additivity of the equivariant virtual Poincaré series, to obtain the desired formula for $\beta^G(\{f_{2k-1}=\xi\})$.

As for $\beta^G(\{g_k=\xi\})$, we can apply the equivariant change of variables $u_i=y_i+y_{i+p}$, $v_i=y_i-y_{i+p}$ for $i=1,\dots,p$ (the strata $\{g_k=\xi\}\cap\{u_1=\ldots=u_i=0,u_{i+1}\neq 0\}$ are G-globally stable). □

Remark 6.16. — Regarding the equation $f_{2k-1}=\xi$, we could also have applied the change of variables $u_1=y_1+y_{p+1}$, $v_1=y_1-y_{p+1}$, provided that G acts on these new coordinates via the involution $(u_1,v_1)\mapsto(-v_1,-u_1)$ or $(u_1,v_1)\mapsto(v_1,u_1)$ (depending on the sign of η). However, the stratum $\{f_{2k-1}=\xi\}\cap\{u_1\neq 0\}$ is not globally stable under this action of G.

From these formulae, among the remaining cases for which we did not establish that the germs f_{2k-1} and g_k are not G-blow-Nash equivalent, we first extract the cases for which $\beta^G(A_{2k}^{\xi}(f_{2k-1}))=\beta^G(A_{2k}^{\xi}(g_k))$:

Proposition 6.17. — *If $p>q+1$ and $\eta=\epsilon=+1$ or $q>p+1$ and $\eta=\epsilon=-1$, we have $\beta^G(A_{2k}^{\xi}(f_{2k-1}))=\beta^G(A_{2k}^{\xi}(g_k))$.*

Proof. — Similarly to the previous proofs, we focus on the case $p>q+1$, $q\neq 0$ and $\eta=\epsilon=+1$. Then, $\beta^G(\{f_{2k-1}=\xi\})=u^{p+2}\frac{u^{q-1}-1}{u-1}+u^{q-1}\beta^G\big(\big\{+x_2^{2k}+y_1^2-y_{p+1}^2+\sum_{j=q+1}^{p}y_j^2=\xi\big\}\big)$. On the latter set, the action of G only changes the sign of y_1, so that we can use the equivariant change of variables $u = y_{q+1} + y_{p+1}$, $v = y_{q+1} - y_{p+1}$ in order to obtain the equality

$$\beta^G(\{f_{2k-1} = \xi\}) = u^{p+1}\frac{u^q - 1}{u-1} + u^q \beta^G\left(\left\{+x_2^{2k} + y_1^2 + \sum_{j=q+2}^{p} y_j^2 = \xi\right\}\right).$$

Therefore, $\beta^G(A_{2k}^{\xi}(f_{2k-1})) = \beta^G(A_{2k}^{\xi}(g_k))$ if and only if $\beta^G\big(\big\{+x_2^{2k} + y_1^2 + \sum_{j=q+2}^{p} y_j^2 = \xi\big\}\big) = \beta^G\big(\big\{+x_1^{2k} + \sum_{j=q+1}^{p} y_j^2 = \xi\big\}\big)$ (recall that, on the latter set, the action of G only changes the sign of x_1).

Now, if $\xi = -1$, both sets are empty, and if $\xi = +1$, they are compact, nonsingular and equivariantly homeomorphic to spheres having a nonempty fixed point set. As a consequence, for $\xi = \pm 1$, $\beta^G\big(\big\{+x_2^{2k} + y_1^2 + \sum_{j=q+2}^{p} y_j^2 = \xi\big\}\big) = \beta^G\big(\big\{+x_1^{2k} + \sum_{j=q+1}^{p} y_j^2 = \xi\big\}\big)$ (see Remark 4.1) and $\beta^G(A_{2k}^{\xi}(f_{2k-1})) = \beta^G(A_{2k}^{\xi}(g_k))$. □

Finally, we give the cases for which the equality $\beta^G(A_{2k}^{\xi}(f_{2k-1})) = \beta^G(A_{2k}^{\xi}(g_k))$ depends on the equality of two equivariant virtual Poincaré series:

Proposition 6.18. —
- *If k is even, and if $p > q+1$, $\eta = +1$ and $\epsilon = -1$, the equality $\beta^G(A_{2k}^{\xi}(f_{2k-1})) = \beta^G(A_{2k}^{\xi}(g_k))$ is true if and only if the equivariant virtual Poincaré series of the algebraic subsets $\big\{-x_2^{2k} + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\big\} \subset \mathbb{R}^{K+1}$, $K := p - q$, equipped with the action of G only changing the sign of y, and $\big\{-x_1^{2k} + \sum_{i=1}^{K} z_i^2 = \xi\big\} \subset \mathbb{R}^{K+1}$, equipped with the action of G only changing the sign of x_1, are equal.*
- *If k is even, and if $q > p+1$, $\eta = -1$ and $\epsilon = +1$, we have $\beta^G(A_{2k}^{\xi}(f_{2k-1})) = \beta^G(A_{2k}^{\xi}(g_k))$ if and only if $\beta^G\big(\big\{x_2^{2k} - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\big\}\big) = \beta^G\big(\big\{x_1^{2k} - \sum_{i=1}^{K} z_i^2 = \xi\big\}\big)$.*

Proof. — If we focus on the case $p > q+1$, $q \neq 0$, $\eta = +1$ and $\epsilon = -1$, the same computation as in the proof of the previous Proposition 6.17 provides the equivalence $\beta^G(A_{2k}^{\xi}(f_{2k-1})) = \beta^G(A_{2k}^{\xi}(g_k))$ if and only if $\beta^G\big(\big\{-x_2^{2k} + y_1^2 + \sum_{j=q+2}^{p} y_j^2 = \xi\big\}\big) = \beta^G\big(\big\{-x_1^{2k} + \sum_{j=q+1}^{p} y_j^2 = \xi\big\}\big)$. □

Remark 6.19. —
1. Recall that we showed in Corollary 6.13 that the germs f_{2k-1} and g_k are not G-blow-Nash equivalent in the case k odd and $p > q+1$, $\eta = +1$, $\epsilon = -1$ or $q > p+1$, $\eta = -1$, $\epsilon = +1$ (notice that in the previous proof of Proposition 6.18, we did not use the fact that k was even).
2. Forgetting the action of G, the virtual Poincaré polynomials of the algebraic subsets $\big\{x^{2k} - \sum_{i=1}^{K} y_i^2 = \xi\big\}$, $\xi = \pm 1$, of $\mathbb{R}^{K+1}$ can be computed using the invariance of the virtual Poincaré polynomial under bijection

with the $\mathcal{AS}$ graph (see [23]). However, we do not know if the equivariant virtual Poincaré series is invariant under equivariant bijection with the $\mathcal{AS}$ graph.

As a consequence of the results of this Section 6.3, we will then consider the other coefficients $\beta^G(A_M(f_{2k-1}))$ and $\beta^G(A_M(g_k))$, and $\beta^G(A^{\xi}_M(f_{2k-1}))$ and $\beta^G(A^{\xi}_M(g_k))$, $M > 2k$, of the equivariant zeta functions of f_{2k-1} and g_k, in the cases of Propositions 6.17 and 6.18. In the next section, we will show that the comparison of these quantities reduces to the comparison of the equivariant virtual Poincaré series of $\{f_{2k-1} = \xi\}$ and $\{g_k = \xi\}$ as well.

6.4. The last terms of the equivariant zeta functions. — Suppose $p > q + 1$, $\eta = \epsilon = +1$ or k even, $p > q + 1$, $\eta = +1$, $\epsilon = -1$. The following results will also be true for the respective symmetric cases.

We first establish the equality between the last coefficients of the naive equivariant zeta functions of f_{2k-1} and g_k (and, therefore, the equality of $Z^G_{f_{2k-1}}(u,T)$ and $Z^G_{g_k}(u,T)$):

Proposition 6.20. — *For all $M > 2k$, we have*

$$\beta^G(A_M(f_{2k-1})) = \beta^G(A_M(g_k)).$$

Proof. — Let M be greater than $2k$. We prove that

$$\beta^G(A^0_M(f_{2k-1})) = \beta^G(A^0_M(g_k))$$

(this will give the desired result because of Proposition 6.3 and the additivity of the equivariant virtual Poincaré series).

As in the proofs of Propositions 6.1 and 6.10, consider the system of equations defining $A^0_M(f_{2k-1})$. The same computations will bring, in the expression of $\beta^G(A^0_M(f_{2k-1}))$, a contribution of (a multiple in $\mathbb{Z}[u][[u^{-1}]]$ of) $\beta^G(Y_{p,q}\setminus\{0\})$ and a contribution of the equivariant virtual Poincaré series of a set defined by a system whose first equation is $\epsilon a_1^{2k} + Q_{p,q}(c_1) = 0$. Stratifying this last algebraic set with the subsets $\{c_1^1 = \ldots = c_1^{i-1} = 0, c_1^i \neq 0\}$, $i = 1,\ldots,p+q$, and $\{c_1 = 0\}$ provides a contribution of $\beta^G(\{f_{2k-1} = 0\} \setminus \{0\})$ and a new system where $c_1 = 0$, $a_1 = 0$ and whose first (nontrivial) equations are the ones defining $A^0_m(f_{2k-1})$, for $m = min(M - 2k, 2k)$.

As a consequence, we can repeat the same steps of computations on this system, and this will give further contributions of $\beta^G(Y_{p,q} \setminus \{0\})$ (provided by the equations $Q_{p,q}(c_1) = 0$) and $\beta^G(\{f_{2k-1} = 0\} \setminus \{0\})$ (provided by the equations $\epsilon a_j^{2k} + Q_{p,q}(c_1) = 0$).

Since these systems and these operations are also valid for the computation of $\beta^G(A^0_M(g_k))$ and because, in the considered cases, the quantities $\beta^G(Y_{p,q})$ are equal for f_{2k-1} and g_k and $\beta^G(\{f_{2k-1} = 0\}) = \beta^G(\{g_k = 0\})$, the expressions of $\beta^G(A^0_M(f_{2k-1}))$ and $\beta^G(A^0_M(g_k))$ are identical. □

Similar considerations bring the following results for the last coefficients of the equivariant zeta functions with signs:

Proposition 6.21. — 1. *If* $p > q+1$, $\eta = \epsilon = +1$, *then for all* $M > 2k$, *we have* $\beta^G(A_M^{\xi}(f_{2k-1})) = \beta^G(A_M^{\xi}(g_k))$, *and, consequently,* $Z^{G,\pm}_{f_{2k-1}}(u,T) = Z^{G,\pm}_{g_k}(u,T)$.

2. *If* k *is even, and if* $p > q+1$, $\eta = +1$ *and* $\epsilon = -1$, *we have the equality* $Z^{G,\xi}_{f_{2k-1}}(u,T) = Z^{G,\xi}_{g_k}(u,T)$ *if and only if* $\beta^G\big(\{-x_2^{2k} + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{-x_1^{2k} + \sum_{i=1}^{K} z_i^2 = \xi\}\big)$ *(the former set is a subset of* $\mathbb{R}^{K+1}$ *equipped with the action of* G *only changing the sign of* y *and the latter set is a subset of* $\mathbb{R}^{K+1}$ *equipped with the action of* G *only changing the sign of* x_1*).*

Proof. — Let M be greater than $2k$. The system defining $A_M^{\xi}(f_{2k-1})$ is obtained by replacing 0 by ξ in the right member of the last equation of the system defining $A_M^0(f_{2k-1})$. Consequently, the same argument works as in the proof of previous Proposition 6.20, and $\beta^G(A_M^{\xi}(f_{2k-1})) = \beta^G(A_M^{\xi}(g_k))$ if and only if the contribution given by the very last equation provided by the computation is the same for f_{2k-1} and g_k.

As in the proof of Proposition 6.4, if M is odd, this contribution is the equivariant virtual Poincaré series of an empty set, and if M is even and not a multiple of $2k$, it is $\beta^G(Y^{\xi}_{p,q})$; in both cases, $\beta^G(A_M^{\xi}(f_{2k-1})) = \beta^G(A_M^{\xi}(g_k))$. Finally, if M is a multiple of $2k$, the respective contributions are $\beta^G(\{f_{2k-1} = \xi\})$ and $\beta^G(\{g_k = \xi\})$, hence the result by Lemma 6.15 (see also the proofs of Propositions 6.17 and 6.18). ☐

6.5. Conclusion. — As a conclusion, we summarize and gather the results of the previous sections in the following theorem:

Theorem 6.22. — *Let* $k \geq 2$. *Suppose that the invariant germs*

$$f_{2k-1} = \epsilon x_2^{2k} + \eta x_1^2 + Q \text{ and } g_k = \epsilon x_1^{2k} + \eta' x_2^2 + Q'$$

have, up to permutation of all variables, the same quadratic part, with p *signs* $+$ *and* q *signs* $-$.

1. *If*
   - $p \leq q$, $\eta = +1$ *or* $q \leq p$, $\eta = -1$,
   - $p = q+1$ *or* $q = p+1$,
   - k *is odd, and if* $p > q+1$, $\eta = +1$, $\epsilon = -1$ *or* $q > p+1$, $\eta = -1$, $\epsilon = +1$,

   then f_{2k-1} *and* g_k *are not* G*-blow-Nash equivalent.*
2. *If* $p > q+1$, $\eta = \epsilon = +1$ *or* $q > p+1$, $\eta = \epsilon = -1$, *then* $Z^G_{f_{2k-1}}(u,T) = Z^G_{g_k}(u,T)$ *and* $Z^{G,\xi}_{f_{2k-1}}(u,T) = Z^{G,\xi}_{g_k}(u,T)$.

3.
- *If k is even, and if $p > q+1$, $\eta = +1$, $\epsilon = -1$, then $Z^G_{f_{2k-1}}(u,T) = Z^G_{g_k}(u,T)$. Furthermore, $Z^{G,\xi}_{f_{2k-1}}(u,T) = Z^{G,\xi}_{g_k}(u,T)$ if and only if $\beta^G\big(\{-x_2^{2k} + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{-x_1^{2k} + \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.*
- *If k is even, and if $q > p+1$, $\eta = -1$, $\epsilon = +1$, then $Z^G_{f_{2k-1}}(u,T) = Z^G_{g_k}(u,T)$. Furthermore, $Z^{G,\xi}_{f_{2k-1}}(u,T) = Z^{G,\xi}_{g_k}(u,T)$ if and only if $\beta^G\big(\{x_2^{2k} - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{x_1^{2k} - \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.*

In particular, in the cases 2 and 3, we are not able to determine whether or not the germs f_{2k-1} and g_k are G-blow-Nash equivalent.

Remark 6.23. — 1. As one can see from the computations, the fact that the equivariant Poincaré series of a given sphere is the same for any action of G on it with a nonempty fixed point set (see Remark 4.1) induces equalities between coefficients of the respective equivariant zeta functions of f_{2k-1} and g_k.

2. If the equivariant virtual Poincaré series was proved to be an invariant under equivariant bijection with the $\mathcal{AS}$ graph, this could allow us to compute (and compare) the quantities $\beta^G\big(\{-x_2^{2k} + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}\big)$ and $\beta^G\big(\{-x_1^{2k} + \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.

7. The germs C_k and D_k

In a second time, we plan to make progress towards the classification with respect to G-blow-Nash equivalence of the invariant germs of the families

$$h_k^{\epsilon_k}(x) := \pm x_1^2 + x_2^2 x_3 + \epsilon_k x_3^{k-1} + Q \text{ and } r_k^{\epsilon_k}(x) := x_1^2 x_2 + \epsilon_k x_2^k + \pm x_3^2 + Q',$$

where $\epsilon_k \in \{-1; +1\}$.

By the same arguments as in the introduction of Section 6, we know that if two germs $h_k^{\epsilon_k}$ and $h_l^{\epsilon_l}$ are G-blow-Nash equivalent, they have the same quadratic part up to permutation of the variables $x_1, x_4, \dots, x_n$, and, by [13] Proposition 3.11, that $k = l$ and $\epsilon_k = \epsilon_l$. Therefore, inside the family D_k, it remains to show that the germs

$$h_k^{\epsilon_k,+}(x) := +x_1^2 + x_2^2 x_3 + \epsilon_k x_3^{k-1} + Q \quad \text{and}$$
$$h_k^{\epsilon_k,-}(x) := -x_1^2 + x_2^2 x_3 + \epsilon_k x_3^{k-1} + Q',$$

where $\epsilon_k \in \{-1; +1\}$ and $+x_1^2 + Q$ and $-x_1^2 + Q'$ are the same quadratic part up to permutation of the variables $x_1, x_4, \dots, x_n$, and are not G-blow-Nash equivalent. As a consequence, the classification of the germs C_k up to G-blow-Nash equivalence is the same as their classification up to equivariant analytic equivalence.

As for the family C_k, if two germs $r_k^{\epsilon_k}$ and $r_l^{\epsilon_l}$ are G-blow-Nash equivalent, they have the same quadratic part up to permutation of the variables $x_3,\dots,x_n$, $k=l$ and $\epsilon_k=\epsilon_l$.

On the other hand, if two germs $h_k^{\epsilon_k}$ and $r_{k'}^{\epsilon_{k'}}$ are G-blow-Nash equivalent then $k=k'+1$, $\epsilon_k=\epsilon_{k'}$ and $\pm x_1^2+Q$ and $\pm x_2^2+Q'$ are the same quadratic part up to permutation of all variables. As a consequence, we focus on the comparison of the germs

$$h_{k+1}=x_2^2x_3+\epsilon x_3^k+\eta x_1^2+Q \quad \text{and} \quad r_k=x_1^2x_2+\epsilon x_2^k+\eta' x_3^2+Q'$$

where $\epsilon,\eta,\eta'\in\{1,-1\}$ and $\eta x_1^2+Q=\eta' x_3^2+Q'$ up to permutation of all variables.

In the following, as we did for the families A_k and B_k, we study and compare the respective equivariant zeta functions of h_k and r_k; using Theorem 4.2, this allows us to extract further cases of non-G-blow-Nash equivalence.

7.1. Computation of the first terms of the equivariant zeta functions. — Fix $k\geq 4$ and consider the invariant germ $h_k^{\epsilon,\eta}(x_1,\dots,x_n)=\eta x_1^2+x_2^2x_3+\epsilon x_3^{k-1}+Q$. Denote $x_2=x$, $x_3=z$ and $\eta x_1^2+Q=Q_{p,q}=\sum_{i=1}^p y_i^2-\sum_{j=1}^q y_{p+j}^2$ (G acts on the renamed coordinates via the involution n°1 or n°2 depending on the sign of η), so that $h_k^{\epsilon,\eta}(x,z,y)=x^2z+\epsilon z^{k-1}+Q_{p,q}(y)$.

The following proposition gives the computed expressions for $\beta^G(A_m^0(h_k^{\epsilon,\eta}))$ (see the beginning of Section 6.1 for the definition of $A_m^0(h)$ for h an invariant Nash germ) if $m<k-1$. The same expressions can be obtained for $\beta^G(A_m^0(r_{k-1}^{\epsilon}))$, providing that $Y_{p,q}$ is equipped with the trivial action in this case.

Proposition 7.1. — *Suppose* $2\leq m<k-1$.

1. *If* $p+q=1$, *then*

$$\beta^G(A_m^0(h_k^{\epsilon,\eta}))=\begin{cases} ru^{2m+1}+\frac{u^{4r+4}}{u-1} & \text{if } m=2r+1,\\ (r-1)u^{2m+1}+\frac{u^{4r+2}}{u-1} & \text{if } m=2r.\end{cases}$$

2. *If* $p+q\neq 1$, *then*

$$\beta^G(A_m^0(h_k^{\epsilon,\eta}))=\begin{cases} u^{3r+2+(r+1)(p+q)}\frac{u^{r(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) & \\ \quad +\frac{u^{3r+3+(r+1)(p+q)}}{u-1} & \text{if } m=2r+1,\\ u^{3r+(r+1)(p+q)}\frac{u^{(r-1)(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) & \\ \quad +u^{3r+1+r(p+q)}\beta^G(Y_{p,q}) & \text{if } m=2r.\end{cases}$$

Proof. — As in Section 6.1, we follow the computations of [13], paying attention to their equivariance with respect to the considered actions of G.

Here, we write an arc γ of $\mathcal{L}_m$ as

$$\gamma(t)=\begin{pmatrix} a_1t+\cdots+a_mt^m\\ b_1t+\cdots+b_mt^m\\ c_1^1t+\cdots+c_m^1t^m\\ \vdots\\ c_1^{p+q}t+\cdots+c_m^{p+q}t^m\end{pmatrix}$$
$$=\begin{pmatrix} a_1\\ b_1\\ c_1^1\\ \vdots\\ c_1^{p+q}\end{pmatrix}t+\cdots+\begin{pmatrix} a_m\\ b_m\\ c_m^1\\ \vdots\\ c_m^{p+q}\end{pmatrix}t^m=\begin{pmatrix} a_1\\ b_1\\ c_1\end{pmatrix}t+\cdots+\begin{pmatrix} a_m\\ b_1\\ c_m\end{pmatrix}t^m$$

(the group G acts on $\mathcal{L}_m$ changing the sign of the variables c_i^1, resp. c_i^{p+1}, in the case n°1, or n°2).

We focus on the generic case $pq\neq 0$, $p+q\neq 1$. First suppose that m is odd, $m=2r+1$. Then an arc γ of $\mathcal{L}_m$ belongs to $A_m^0(h_k^{\epsilon,\eta})$ if and only if

$$\begin{cases} Q_{p,q}(c_1)=0,\\ a_1^2b_1+\Phi_{p,q}(c_1,c_2)=0,\\ a_1^2b_2+2a_1a_2b_1+Q_{p,q}(c_2)+\Phi_{p,q}(c_1,c_3)=0,\\ \cdots\\ \sum_{t=1}^{r-1}a_t^2b_{2r-2t}+2\sum_{t=1}^{r-1}a_t\sum_{\delta=t+1}^{2r-(t+1)}a_\delta b_{2r-\delta-t}+Q_{p,q}(c_r)\\ \quad+\sum_{t=1}^{r-1}\Phi_{p,q}(c_t,c_{2r-t})=0,\\ \sum_{t=1}^{r}a_t^2b_{2r+1-2t}+2\sum_{t=1}^{r-1}a_t\sum_{\delta=t+1}^{2r+1-(t+1)}a_\delta b_{2r+1-\delta-t}\\ \quad+\sum_{t=1}^{r}\Phi_{p,q}(c_t,c_{2r+1-t})=0.\end{cases}$$

Stratifying $A_m^0(h_k^{\epsilon,\eta})$ with the G-globally invariant subsets $\{c_1^1=\ldots=c_1^{i-1}=0,c_1^i\neq 0\}$, $i=1,\ldots,p$, and $\{c_1^1=\ldots=c_1^p=0\}=\{c_1=0\}$, as we did in the proof of Proposition 6.1, we obtain, by additivity of the equivariant virtual Poincaré series,

$$\beta^G(A_m^0(h_k^{\epsilon,\eta}))=u^{2\times(2r+1)+2r(p-1)+2rq+1}\beta^G(Y_{p,q}\setminus\{0\})$$
$$+\beta^G(A_m^0(h_k^{\epsilon,\eta})\cap\{c_1=0\}),$$

the algebraic set $A^0_m(h_k^{\epsilon,\eta}) \cap \{c_1=0\}$ being described by the system

$$\begin{cases} a_1^2 b_1 = 0, \\ a_1^2 b_2 + 2a_1a_2b_1 + Q_{p,q}(c_2) = 0, \\ \cdots \\ \sum_{t=1}^{r-1} a_t^2 b_{2r-2t} + 2\sum_{t=1}^{r-1} a_t \sum_{\delta=t+1}^{2r-(t+1)} a_\delta b_{2r-\delta-t} + Q_{p,q}(c_r) \\ \quad + \sum_{t=2}^{r-1} \Phi_{p,q}(c_t, c_{2r-t}) = 0, \\ \sum_{t=1}^{r} a_t^2 b_{2r+1-2t} + 2\sum_{t=1}^{r-1} a_t \sum_{\delta=t+1}^{2r+1-(t+1)} a_\delta b_{2r+1-\delta-t} \\ \quad + \sum_{t=2}^{r} \Phi_{p,q}(c_t, c_{2r+1-t}) = 0. \end{cases}$$

Now, if $a_1 \neq 0$, then $b_1 = 0$, and the coordinates $b_2, \ldots, b_{2r-1}$ are determined by a_1 and the other variables (via an equivariant morphism), and thus

$$\beta^G(A^0_m(h_k^{\epsilon,\eta}) \cap \{c_1=0\}) = (u-1)\frac{u^{[2r+2+2r(p+q)]+1}}{u-1} \\ + \beta^G(A^0_m(h_k^{\epsilon,\eta}) \cap \{c_1=0, a_1=0\}).$$

If $c_1 = 0$ and $a_1 = 0$, the remaining coordinates verify the system

$$\begin{cases} Q_{p,q}(c_2) = 0, \\ a_2^2 b_1 + \Phi_{p,q}(c_2,c_3) = 0, \\ a_2^2 b_2 + 2a_2a_3b_1 + Q'_{p,q}(c_3) + \Phi_{p,q}(c_2,c_4) = 0, \\ \cdots \\ \sum_{t=2}^{r-1} a_t^2 b_{2r-2t} + 2\sum_{t=2}^{r-1} a_t \sum_{\delta=t+1}^{2r-(t+1)} a_\delta b_{2r-\delta-t} + Q_{p,q}(c_r) \\ \quad + \sum_{t=2}^{r-1} \Phi_{p,q}(c_t, c_{2r-t}) = 0, \\ \sum_{t=2}^{r} a_t^2 b_{2r+1-2t} + 2\sum_{t=2}^{r-1} a_s \sum_{\delta=t+1}^{2r+1-(t+1)} a_\delta b_{2r+1-\delta-t} \\ \quad + \sum_{t=2}^{r} \Phi_{p,q}(c_t, c_{2r+1-t}) = 0. \end{cases}$$

Notice that the vector c_m and the variables a_m, b_{m-1} and b_m are free and that, if we rename the variables, these equations define the set $A^0_{m-2}(h_k^{\epsilon,\eta})$, so that

$$\beta^G(A^0_m(h_k^{\epsilon,\eta}) \cap \{c_1=0, a_1=0\}) = u^{3+p+q}\beta^G(A^0_{m-2}(h_k^{\epsilon,\eta})).$$

By an induction process, we then obtain

$$\beta^G(A^0_m(h_k^{\epsilon,\eta})) = \beta^G(Y_{p,q}\setminus\{0\}) \left[\sum_{t=0}^{r-1} u^{t(3+p+q)} u^{2\times(m-2t)+(m-2t-1)(p+q-1)+1}\right] \\ + \left[\sum_{t=0}^{r-1} u^{t(3+p+q)} u^{(m-1-2t)(p+q+1)+3}\right] \\ + u^{(r-1)(3+p+q)}\beta^G(A^0_3(h_k^{\epsilon,\eta}) \cap \{c_1=0, a_1=0\}),$$

the equations for $A^0_3(h_k^{\epsilon,\eta}) \cap \{c_1=0, a_1=0\}$ becoming trivial. As a consequence (notice that for all $t = 0, \ldots, r-1$, $2\times(m-2t)+(m-2t-1)(p+q-1)+1 =$

$(m-1-2t)(p+q+1)+3)$,

$$\beta^G(A^0_m(h_k^{\epsilon,\eta})) = u^{3r+2+(r+1)(p+q)}\frac{u^{r(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) + \frac{u^{3r+3+(r+1)(p+q)}}{u-1}.$$

If m is even, $m=2r$, the system defining $A^0_m(h_k^{\epsilon,\eta})$ is

$$\begin{cases} Q_{p,q}(c_1)=0,\\ a_1^2b_1+\Phi_{p,q}(c_1,c_2)=0,\\ \dots\\ \sum_{t=1}^{r-1} a_t^2 b_{2r-1-2t} + 2\sum_{t=1}^{r-2} a_s \sum_{\delta=t+1}^{2r-1-(t+1)} a_\delta b_{2r-1-\delta-t}\\ \quad +\sum_{t=1}^{r-1}\Phi_{p,q}(c_t,c_{2r-1-t})=0,\\ \sum_{t=1}^{r-1} a_t^2 b_{2r-2t} + 2\sum_{t=1}^{r-1} a_t \sum_{\delta=t+1}^{2r-(t+1)} a_\delta b_{2r-\delta-t} + Q_{p,q}(c_r)\\ \quad +\sum_{t=1}^{r-1}\Phi_{p,q}(c_t,c_{2r-t})=0, \end{cases}$$

and we have

$$\begin{aligned}\beta^G(A^0_m(h_k^{\epsilon,\eta})) &= \beta^G(Y_{p,q}\setminus\{0\})\left[\sum_{t=0}^{r-2} u^{t(3+p+q)}u^{2\times(m-2t)+(m-2t-1)(p+q-1)+1}\right]\\ &\quad+\left[\sum_{t=0}^{r-2} u^{t(3+p+q)}u^{(m-1-2t)(p+q+1)+3}\right]\\ &\quad+u^{(r-1)(3+p+q)}\beta^G(A^0_2(h_k^{\epsilon,\eta}))\end{aligned}$$

Because $A^0_2(h_k^{\epsilon,\eta})$ is described by the equation $Q_{p,q}(c_1)=0$, and since the vector c_2 as well as the variables a_1,a_2,b_1 and b_2 are free, we obtain

$$\beta^G(A^0_m(h_k^{\epsilon,\eta})) = u^{3r+(r+1)(p+q)}\frac{u^{(r-1)(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) + u^{3r+1+r(p+q)}\beta^G(Y_{p,q}). \qquad \square$$

As for $\beta^G(A^\xi_m(h_k^{\epsilon,\eta}))$, we have the following expressions if $m<k+1$:

Proposition 7.2. — *Suppose $m<k-1$.*

1. *If $(p,q)=(0,1)$, then*

$$\beta^G(A^\xi_m(h_k^{\epsilon,\eta})) = \begin{cases} ru^{2m+1} & \text{if } m=2r+1,\\ (r-1)u^{2m+1}+u^{4r+1}\beta^G(Y^\xi_{0,1}) & \text{if } m=2r.\end{cases}$$

2. *If $(p,q)=(1,0)$, then*

$$\beta^G(A^\xi_m(h_k^{\epsilon,\eta})) = \begin{cases} ru^{2m+1} & \text{if } m=2r+1,\\ (r-1)u^{2m+1}+u^{4r+1}\beta^G(Y^\xi_{1,0}) & \text{if } m=2r.\end{cases}$$

3. *If $p+q \neq 1$, then*

$$\beta^G(A_m^{\xi}(h_k^{\epsilon,\eta})) = \begin{cases} u^{3r+2+(r+1)(p+q)} \frac{u^{r(p+q-1)}-1}{u^{p+q-1}-1} \big(\beta^G(Y_{p,q} \setminus \{0\})+1\big) \\ \hfill \text{if } m = 2r+1, \\ u^{3r+(r+1)(p+q)} \frac{u^{(r-1)(p+q-1)}-1}{u^{p+q-1}-1} \big(\beta^G(Y_{p,q} \setminus \{0\})+1\big) \\ \quad +u^{3r+1+r(p+q)} \beta^G(Y_{p,q}^{\xi}) \hfill \text{if } m = 2r. \end{cases}$$

Proof. — If we keep the notations of the proof of Proposition 7.1, the system defining $A_m^{\xi}(h_k^{\epsilon,\eta})$ is obtained by replacing 0 by ξ in the right member of the last of the equations describing $A_m^0(h_k^{\epsilon,\eta})$. Furthermore, the system for $A_3^{\xi}(h_k^{\epsilon,\eta}) \cap \{c_1 = 0, a_1 = 0\}$ has no solution, whereas $A_2^{\xi}(h_k^{\epsilon,\eta})$ is described by the equation $Q_{p,q}(c_1) = \xi$. □

We are now able to show that the germs $h_k^{\epsilon,+}$ and $h_k^{\epsilon,-}$ are not G-blow-Nash equivalent:

Corollary 7.3. — *Let $k \geq 4$. Suppose that the invariant Nash germs*

$$h_k^{\epsilon,+}(x) := +x_1^2 + x_2^2 x_3 + \epsilon x_3^{k-1} + Q \text{ and } h_k^{\epsilon,-}(x) := -x_1^2 + x_2^2 x_3 + \epsilon x_3^{k-1} + Q'$$

have the same quadratic part up to permutation of the variables $x_1, x_4, \dots, x_n$. Then they are not G-blow-Nash equivalent. As a consequence, the classification of the germs D_k up to G-blow-Nash equivalence is the same as their classification up to equivariant analytic equivalence.

Proof. — We compare $\beta^G(A_2(h_k^{\epsilon,+}))$ and $\beta^G(A_2(h_k^{\epsilon,-}))$. Because

$$\beta^G(A_2(h_k^{\epsilon,\eta})) = \beta^G({}^0A_2(h_k^{\epsilon,\eta})) - \beta^G(A_2^0(h_k^{\epsilon,\eta})),$$
$$\beta^G({}^0A_2(h_k^{\epsilon,\eta})) = u^n \beta^G(A_1^0(h_k^{\epsilon,\eta})) \quad \text{(by Proposition 6.3)}$$

and $A_1^0(h_k^{\epsilon,\eta}) = \mathcal{L}_1$, we are reduced to comparing the quantities $\beta^G(A_2^0(h_k^{\epsilon,+}))$ and $\beta^G(A_2^0(h_k^{\epsilon,-}))$.

Now, if p denotes the number of signs $+$ and q the number of signs $-$ in the quadratic part of $h_k^{\epsilon,+}$ and $h_k^{\epsilon,-}$ (notice that $p+q \neq 1$), we have, by Proposition 7.1,

$$\beta^G(A_2^0(h_k^{\epsilon,\eta}))) = u^{4+p+q} \beta^G(Y_{p,q})$$

Consequently, if $p \neq q$, we can use the same arguments as in the proof of Corollary 6.6 to conclude that the naive equivariant zeta functions of $h_k^{\epsilon,+}$ and $h_k^{\epsilon,-}$ are different and, therefore, that the germs $h_k^{\epsilon,+}$ and $h_k^{\epsilon,-}$ are not G-blow-Nash equivalent.

If $p = q$, we compare $\beta^G(A_2^{+1}(h_k^{\epsilon,+}))$ and $\beta^G(A_2^{+1}(h_k^{\epsilon,-}))$. Since, by Proposition 7.2,

$$\beta^G(A_2^{+1}(h_k^{\epsilon,+})) = u^{4+2p} \beta^G(Y_{p,q}^{+1}),$$

we can, in this case as well, use the arguments of the proof of Corollary 6.6, in order to conclude that the equivariant zeta functions with signs $+$ of $h_k^{\epsilon,+}$ and $h_k^{\epsilon,-}$ are different. □

Using again the formulae of Propositions 7.1 and 7.2, we then extract cases for which the germs h_{k+1} and r_k are not G-blow-Nash equivalent:

Corollary 7.4. — *Let $k \geq 3$. Suppose that the invariant germs*

$$h_{k+1} = x_2^2 x_3 + \epsilon x_3^k + \eta x_1^2 + Q \quad \textit{and} \quad r_k = x_1^2 x_2 + \epsilon x_2^k + \eta' x_3^2 + Q'$$

have, up to permutation of all variables, the same quadratic part, with p signs $+$ and q signs $-$.

If $p \leq q$ and $\eta = +1$ or $q \leq p$ and $\eta = -1$, then h_{k+1} and r_k are not G-blow-Nash equivalent. If $p = q+1$ or $q = p+1$, then h_{k+1} and r_k are not G-blow-Nash equivalent.

Proof. — For the first point, focus, for instance, on the case $p \leq q$ and $\eta = +1$. As in the proof of Corollary 7.3, we consider $\beta^G(A_2^0(h_{k+1})) = u^{4+p+q}\beta^G(Y_{p,q})$ and $\beta^G(A_2^0(r_k)) = u^{4+p+q}\beta^G(Y_{p,q})$. Since the action of G on the former set $Y_{p,q}$ is the action n°1 and the action on the latter set $Y_{p,q}$ is the trivial action, we obtain $\beta^G(A_2(h_{k+1})) \neq \beta^G(A_2(r_k))$.

For the second point, assume, for instance, $p = q+1$. Suppose furthermore that $\eta = +1$ and consider the quantities $\beta^G(A_2^{+1}(h_{k+1})) = u^{4+p+q}\beta^G(Y_{p,q}^{+1})$ and $\beta^G(A_2^{+1}(r_k)) = u^{4+p+q}\beta^G(Y_{p,q}^{+1})$. By Proposition 5.3, $\beta^G(Y_{p,q}^{+1}) = \frac{1}{u-1}(\beta^G(Y_{p,p}) - \beta^G(Y_{p,q}))$. Since $q < p$ and $\eta = +1$, the quantity $\beta^G(Y_{p,q})$ is the same for h_{k+1} and r_k, while the quantities $\beta^G(Y_{p,p})$ are different (see Proposition 5.1). As a consequence, $\beta^G(A_2^{+1}(h_{k+1})) \neq \beta^G(A_2^{+1}(r_k))$. □

Now, we are going to study the other coefficients of the equivariant zeta functions of h_{k+1} and r_k in the remaining cases, that is, if $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$. Notice that, in these cases, the quantities $\beta^G(Y_{p,q})$ and $\beta^G(Y_{p,q}^{\xi})$ are identical for h_{k+1} and r_k.

7.2. Computation of $\beta^G(A_k(h_{k+1}))$ and $\beta^G(A_k(r_k))$. — Assuming that the Nash germs h_{k+1} and r_k have the same quadratic part $Q_{p,q}$, with $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$, we first compute the coefficients $\beta^G(A_k(h_{k+1}))$ and $\beta^G(A_k(r_k))$ of their respective naive equivariant zeta functions. Bearing in mind Proposition 6.3, we actually give formulae for $\beta^G(A_k^0(h_{k+1}))$ and $\beta^G(A_k^0(r_k))$:

Proposition 7.5. — *Suppose $k \geq 3$. Then*

$$\beta^G(A_k^0(h_{k+1})) = \begin{cases} u^{3l+2+(l+1)(p+q)}\frac{u^{l(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})\right) \\ +u^{3l+1+(l+2)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1} \\ \quad +u^{3l+1+(l+1)(p+q)}\beta^G(\{h_{k+1}(x_2,x_3,0)=0\}) & \text{if } k=2l+1, \\ u^{3l+(l+1)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) \\ \quad +u^{3l+l(p+q)}\beta^G(\{h_{k+1}(0,x_3,y)=0\}) & \text{if } k=2l, \end{cases}$$

and

$$\beta^G(A_k^0(r_k)) = \begin{cases} u^{3l+2+(l+1)(p+q)}\frac{u^{l(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})\right) \\ \quad +u^{3l+1+(l+2)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1} \\ \quad +u^{3l+1+(l+1)(p+q)}\beta^G(\{r_k(x_1,x_2,0)=0\}) & \text{if } k=2l+1, \\ u^{3l+(l+1)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) \\ \quad +u^{3l+l(p+q)}\beta^G(\{r_k(0,x_2,y)=0\}) & \text{if } k=2l. \end{cases}$$

Proof. — We do the computations for $\beta^G(A_k^0(h_{k+1}))$.

First, suppose k to be odd, $k=2l+1$. Keeping the notations of the proof of Proposition 7.1, the set $A_k^0(h_{k+1})$ is defined by the system

$$\begin{cases} Q_{p,q}(c_1)=0, \\ a_1^2b_1+\Phi_{p,q}(c_1,c_2)=0, \\ a_1^2b_2+2a_1a_2b_1+Q_{p,q}(c_2)+\Phi_{p,q}(c_1,c_3)=0, \\ \cdots \\ \sum_{t=1}^{l-1}a_t^2b_{2l-2t}+2\sum_{t=1}^{l-1}a_t\sum_{\delta=t+1}^{2l-(t+1)}a_\delta b_{2l-\delta-t}+Q_{p,q}(c_l) \\ \quad +\sum_{t=1}^{l-1}\Phi_{p,q}(c_t,c_{2l-t})=0, \\ \epsilon b_1^{2l+1}+\sum_{t=1}^{l}a_t^2b_{2l+1-2t}+2\sum_{t=1}^{l-1}a_t\sum_{\delta=t+1}^{2l+1-(t+1)}a_\delta b_{2l+1-\delta-t} \\ \quad +\sum_{t=1}^{l}\Phi_{p,q}(c_t,c_{2l+1-t})=0. \end{cases}$$

Proceeding as in the proof of Proposition 7.1 (see also the proof of Proposition 6.10), we obtain

$$\begin{aligned} \beta^G(A_k^0(h_{k+1})) &= u^{3l+2+(l+1)(p+q)}\frac{u^{l(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})\right) \\ &\quad +\sum_{t=0}^{l-2}u^{t(3+p+q)}u^{(k-1-2t)(p+q+1)+3}+u^{(l-1)(3+p+q)}\beta^G(S_3^0), \end{aligned}$$

if S_3^0 denotes the algebraic set defined by the equation $\epsilon b_1^{2l+1}+a_1^2b_1=0$, the variables a_2,a_3, b_2,b_3 as well as the vectors c_2,c_3 being free. Hence the desired result.

If we suppose k even, $k = 2l$, the set $A_k^0(h_{k+1})$ is described by the system

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ a_1^2 b_1 + \Phi_{p,q}(c_1, c_2) = 0, \\ \dots \\ \sum_{t=1}^{l-1} a_t^2 b_{2l-1-2t} + 2\sum_{t=1}^{l-2} a_t \sum_{\delta=t+1}^{2l-1-(t+1)} a_\delta b_{2l-1-\delta-t} \\ \quad + \sum_{t=1}^{l-1} \Phi_{p,q}(c_t, c_{2l-1-t}) = 0, \\ \epsilon b_1^{2l} + \sum_{t=1}^{l-1} a_t^2 b_{2l-2t} + 2\sum_{t=1}^{l-1} a_t \sum_{\delta=t+1}^{2l-(t+1)} a_\delta b_{2l-\delta-t} + Q_{p,q}(c_l) \\ \quad + \sum_{t=1}^{l-1} \Phi_{p,q}(c_t, c_{2l-t}) = 0, \end{cases}$$

and we have

$$\begin{aligned} \beta^G(A_k^0(h_{k+1})) = u^{3l+(l+1)(p+q)} \frac{u^{(l-1)(p+q-1)} - 1}{u^{p+q-1} - 1} \left(\beta^G(Y_{p,q} \setminus \{0\}) + 1\right) \\ + u^{(l-1)(3+p+q)} \beta^G(S_2^0), \end{aligned}$$

where S_2^0 is the algebraic set given by the equation $\epsilon b_1^{2l} + Q_{p,q}(c_1) = 0$, with free variables a_1, a_2, b_2 and free vector c_2. $\square$

Since, in our present framework, the quantity $\beta^G(Y_{p,q})$ is the same for h_{k+1} and r_k, we are reduced to studying the equivariant virtual Poincaré series of the G-algebraic sets $\{h_{k+1}(x_2, x_3, 0) = 0\}$ and $\{r_k(x_1, x_2, 0) = 0\}$ if k is odd, or $\{h_{k+1}(0, x_3, y) = 0\}$ and $\{r_k(0, x_2, y) = 0\}$ if k is even.

Notice that, if k is even, $\beta^G(\{h_{k+1}(0, x_3, y) = 0\})$, and $\beta^G(\{r_k(0, x_2, y) = 0\})$ have been already computed in Lemma 6.11: if k is even, and if $p > q + 1$ and $\eta = +1$ or $q > p + 1$ and $\eta = -1$, the equivariant virtual Poincaré series $\beta^G(\{h_{k+1}(0, x_3, y) = 0\})$ and $\beta^G(\{r_k(0, x_2, y) = 0\})$ are equal, and, therefore, $\beta^G(A_k(h_{k+1})) = \beta^G(A_k(r_k))$.

In the next lemma, we compute $\{h_{k+1}(x_2, x_3, 0) = 0\}$ and $\{r_k(x_1, x_2, 0) = 0\}$ if k is odd, $k = 2l + 1$:

Lemma 7.6. — *We have*

$$\beta^G(\{h_{2l+2}(x_2, x_3, 0) = 0\}) = \beta^G(\{x_2^2 + \epsilon x_3^2 = 0\}) - \frac{u}{u-1} + \frac{u^2}{u-1},$$

where the latter set is considered as an algebraic subset of $\mathbb{R}^2$ on which the group G acts trivially, and

$$\beta^G(\{r_{2l+1}(x_1, x_2, 0) = 0\}) = \beta^G(\{x_1^2 + \epsilon x_2^2 = 0\}) - \frac{u}{u-1} + \frac{u^2}{u-1},$$

where the latter set is considered as an algebraic subset of $\mathbb{R}^2$ on which the group G acts only changing the sign of the coordinate x_1.

Proof. — We make the computation for $\beta^G(\{r_{2l+1}(x_1,x_2,0)=0\})$.

Consider the equation $x_1^2x_2+\epsilon x_2^{2l+1}=0$. If $x_2\neq 0$, it is equivalent to $x_1^2+\epsilon x_2^{2l}=0$, and if $x_2=0$, it becomes trivial. Consequently,

$$\beta^G(\{r_{2l+1}(x_1,x_2,0)=0\})=\beta^G(\{x_1^2+\epsilon x_2^{2l}=0\}\setminus\{(0,0)\})+\frac{u^2}{u-1},$$

and we use Lemma 6.11 to write $\beta^G(\{x_1^2+\epsilon x_2^{2l}=0\})=\beta^G(\{x_1^2+\epsilon x_2^2=0\})-(l-1)\beta^G(\{x_1^2=0\})+(l-1)\beta^G(\{(0,0)\})=\beta^G(\{x_1^2+\epsilon x_2^2=0\})$ (recall also that the equivariant virtual Poincaré series of a point is $\frac{u}{u-1}$). $\square$

If $\epsilon=+1$, the sets $\{x_2^2+x_3^2=0\})$ and $\{x_1^2+x_2^2=0\}$ are both reduced to a single point. On the other hand, if $\epsilon=-1$, we have $\beta^G(\{x_2^2-x_3^2=0\})=\frac{2u^2-u}{u-1}$, whereas $\beta^G(\{x_1^2-x_2^2=0\})=\frac{u^2-u+1}{u-1}$ (see Proposition 5.1). As a consequence:

Corollary 7.7. — *If k is odd and if $\epsilon=-1$, the germs h_{k+1} and r_k are not G-blow-Nash equivalent.*

If $p>q+1$ and $\eta=+1$ or $q>p+1$ and $\eta=-1$, and if k is even or k is odd and $\epsilon=+1$, the coefficients $\beta^G(A_k(h_{k+1}))$ and $\beta^G(A_k(r_k))$ of the respective naive equivariant zeta functions of h_{k+1} and r_k are equal. We are then led to look at the coefficients $\beta^G(A_k^{\xi}(h_{k+1}))$ and $\beta^G(A_k^{\xi}(r_k))$ of their respective equivariant zeta functions with signs.

7.3. Computation of $\beta^G(A_k^{\xi}(h_{k+1}))$ and $\beta^G(A_k^{\xi}(r_k))$. — For the cases listed above, we consider the quantities $\beta^G(A_k^{\xi}(h_{k+1}))$ and $\beta^G(A_k^{\xi}(r_k))$, expressed by the following formulae (just follow the steps of computation of the proof of Proposition 7.5):

Proposition 7.8. — *Suppose $k\geq 3$. Then,*

$$\beta^G(A_k^{\xi}(h_{k+1}))=\begin{cases} u^{3l+2+(l+1)(p+q)}\frac{u^{l(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})\right) & \\ \quad +u^{3l+1+(l+2)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1} & \\ \quad +u^{3l+1+(l+1)(p+q)}\beta^G(\{h_{k+1}(x_2,x_3,0)=\xi\}) & \text{if } k=2l+1,\\ u^{3l+(l+1)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) & \\ \quad +u^{3l+l(p+q)}\beta^G(\{h_{k+1}(0,x_3,y)=\xi\}) & \text{if } k=2l,\end{cases}$$

and

$$\beta^G(A_k^{\xi}(r_k))=\begin{cases} u^{3l+2+(l+1)(p+q)}\frac{u^{l(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})\right) & \\ \quad +u^{3l+1+(l+2)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1} & \\ \quad +u^{3l+1+(l+1)(p+q)}\beta^G(\{r_k(x_1,x_2,0)=\xi\}) & \text{if } k=2l+1,\\ u^{3l+(l+1)(p+q)}\frac{u^{(l-1)(p+q-1)}-1}{u^{p+q-1}-1}\left(\beta^G(Y_{p,q}\setminus\{0\})+1\right) & \\ \quad +u^{3l+l(p+q)}\beta^G(\{r_k(0,x_2,y)=\xi\}) & \text{if } k=2l.\end{cases}$$

If k is even, we can use the formulae of Lemma 6.15 and the same arguments as in the proofs of Propositions 6.17 and 6.18 in order to establish the following facts:

PROPOSITION 7.9. — *Suppose k is even, $k = 2l \geq 4$.*

1. *If $p > q+1$ and $\eta = \epsilon = +1$ or $q > p+1$ and $\eta = \epsilon = -1$, then $\beta^G(A_k^\xi(h_{k+1})) = \beta^G(A_k^\xi(r_k))$.*
2. - *If $p > q+1$, $\eta = +1$ and $\epsilon = -1$, the equality $\beta^G(A_k^\xi(h_{k+1})) = \beta^G(A_k^\xi(r_k))$ is true if and only if the equivariant virtual Poincaré series of the algebraic subsets $\left\{-x_3^{2l} + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\right\} \subset \mathbb{R}^{K+1}$, $K := p-q$, equipped with the action of G changing only the sign of y, and $\left\{-x_2^{2l} + \sum_{i=1}^{K} z_i^2 = \xi\right\} \subset \mathbb{R}^{K+1}$, equipped with the trivial action of G, are equal.*
   - *If $q > p+1$, $\eta = -1$ and $\epsilon = +1$, we have $\beta^G(A_k^\xi(h_{k+1})) = \beta^G(A_k^\xi(r_k))$ if and only if $\beta^G\left(\left\{x_3^{2l} - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\right\}\right) = \beta^G\left(\left\{x_2^{2l} - \sum_{i=1}^{K} z_i^2 = \xi\right\}\right)$.*

If k is odd, $k = 2l+1$ and $\epsilon = +1$, and if $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$, we are reduced to comparing $\beta^G(\{h_{k+1}(x_2, x_3, 0) = \xi\}) = \beta^G(\{x_2^2 x_3 + x_3^{2l+1} = \xi\})$ and $\beta^G(\{r_k(x_1, x_2, 0) = \xi\}) = \beta^G(\{x_1^2 x_2 + x_2^{2l+1} = \xi\})$. We are going to show that these two quantities are equal and, therefore:

PROPOSITION 7.10. — *If $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$, and if k is odd and $\epsilon = +1$, then $\beta^G(A_k^\xi(h_{k+1})) = \beta^G(A_k^\xi(r_k))$.*

Proof. — We compute the equivariant virtual Poincaré series of the nonsingular curve $C := \{x_0^2 y_0 + y_0^{2l+1} = \xi\}$ of $\mathbb{R}^2$, on which the group G acts only changing the sign of the first coordinate x_0, or trivially.

First suppose the action of G is the former one. Suppose also $l \geq 2$. We equivariantly compactify C in the projective space $\mathbb{P}^2(\mathbb{R})$ with homogeneous coordinates $[X : Y : Z]$, on which G acts via the involution $[X : Y : Z] \mapsto [-X : Y : Z] = [X : -Y : -Z]$. We denote by $\Gamma := \{X^2 Y Z^{2l-2} + Y^{2l+1} = \xi Z^{2l+1}\}$ this compactification, and by $p := [1 : 0 : 0]$ the point at infinity.

The equivariant compactification Γ is singular at the fixed point p, as one can see in the globally invariant chart $X \neq 0$. If (y_0, z_0) are the coordinates in this chart, the group G acting via the involution $(y_0, z_0) \mapsto (-y_0, -z_0)$, we denote by C' the curve $\Gamma \cap \{X \neq 0\} = \{y_0 z_0^{2l-2} + y_0^{2l+1} = \xi z_0^{2l+1}\}$ (the point at infinity is the fixed point $q = [0 : \xi : 1]$ of C).

Equivariantly blowing-up Γ at p resolves the singularity; in the chart $y_0 = u_0 v_0$, $z_0 = v_0$, where the action of G is given by $(u_0, v_0) \mapsto (u_0, -v_0)$, the equation of the strict transform is $u_0 + u_0^{2l+1} v_0^2 - \xi v_0^2 = 0$, and it intersects the exceptional divisor at the single point p_0 with coordinates $(u_0, v_0) = (0, 0)$.

The resolved compact G-variety, denoted by $\widetilde{\Gamma}$, is equivariantly homeomorphic to a circle equipped with an action of G fixing the two points p_0 and q.

As a conclusion, we have

$$\begin{aligned}\beta^G(C) &= \beta(\Gamma \setminus \{p\}) = \beta^G(\widetilde{\Gamma} \setminus \{p_0\}) = \beta^G(\widetilde{\Gamma}) - \beta^G(\{p_0\}) \\ &= u + 2\frac{u}{u-1} - \frac{u}{u-1} = \frac{u^2}{u-1}\end{aligned}$$

(see Remark 4.1).

If $l = 1$, the point p of Γ is not singular, and Γ is a compact nonsingular G-variety equivariantly homeomorphic to a circle with two fixed points, p and q.

If now we suppose that the affine space $\mathbb{R}^2$ with coordinates (x_0, y_0) is equipped with the trivial action of G, we will obtain the same expression for $\beta^G(C)$, since the equivariant homology of a circle is the same as soon as there is at least one fixed point.

Consequently, the equivariant virtual Poincaré series $\beta^G(\{h_{k+1}(x_2, x_3, 0) = \xi\}) = \beta^G(\{x_2^2 x_3 + x_3^{2l+1} = \xi\})$ and $\beta^G(\{r_k(x_1, x_2, 0) = \xi\}) = \beta^G(\{x_1^2 x_2 + x_2^{2l+1} = \xi\})$ are equal, and, then, $\beta^G(A_k^{\xi}(h_{k+1})) = \beta^G(A_k^{\xi}(r_k))$. $\square$

In the next section, we will look at the last part of the respective equivariant zeta functions of h_{k+1} and r_k. Still supposing that $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$, we will show that, if k is even or k is odd and $\epsilon = +1$, their comparison reduces as in Proposition 7.9.

7.4. The last terms of the equivariant zeta functions. — Suppose $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$. Suppose that k is even or that k is odd and $\epsilon = +1$. The naive equivariant zeta functions of h_{k+1} and r_k are equal:

Proposition 7.11. — *For all $M > k$, we have*

$$\beta^G(A_M(h_{k+1})) = \beta^G(A_M(r_k)).$$

Proof. — Let M be greater than k. We prove that

$$\beta^G(A_M^0(h_{k+1})) = \beta^G(A_M^0(r_k)).$$

Suppose k to be even, $k = 2l$. Consider the system of equations describing $A_M^0(h_{k+1})$ and $A_M^0(r_k)$. The same computations as in the proofs of Propositions 7.1 and 7.5 bring, in both expressions of $\beta^G(A_M^0(h_{k+1}))$ and $\beta^G(A_M^0(r_k))$, an equal contribution of $\beta^G(Y_{p,q} \setminus \{0\})$ and a contribution of the equivariant virtual Poincaré series of a set defined by a system whose first equation is $\epsilon b_1^k + Q_{p,q}(c_1) = 0$. Stratifying this last algebraic set with the subsets $\{c_1^1 = \ldots = c_1^{i-1} = 0, c_1^i \neq 0\}$, $i = 1, \ldots, p+q$, and $\{c_1 = 0\}$ provides a contribution of $\beta^G(\{h_{k+1}(0, x_3, y) = 0\} \setminus \{0\})$, or $\beta^G(\{r_k(0, x_2, y) = 0\} \setminus \{0\})$ (it is the same quantity in our hypothesis), and we are led to the further condition $c_1 = 0$, and then $b_1 = 0$, in the previous system.

Now, stratifying with the subsets $\{a_1 \neq 0\}$ (this will provide an equal contribution for h_{k+1} and r_k) and $\{a_1 = 0\}$. If $a_1 = 0$, shifting by -1 the indices of the remaining variables a_i and c_i, we obtain a new system, whose first equations are, if $M \geq 2k$:

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ \Phi_{p,q}(c_1, c_2) = 0, \\ a_1^2 b_2 + Q_{p,q}(c_2) + \Phi_{p,q}(c_1, c_3) = 0, \\ a_1^2 b_3 + 2a_1 a_2 b_2 + \Phi_{p,q}(c_1, c_4) + \Phi_{p,q}(c_2, c_3) = 0, \\ \dots \\ \sum_{t=1}^{l-2} a_t^2 b_{2l-1-2t} + 2\sum_{t=1}^{l-2} a_s \sum_{\delta=t+1}^{2l-2-(t+1)} a_\delta b_{2l-1-\delta-t} \\ \quad + \sum_{t=1}^{l-1} \Phi_{p,q}(c_t, c_{2l-1-t}) = 0, \\ \epsilon b_2^{2l} + \sum_{t=1}^{l-1} a_t^2 b_{2l-2t} + 2\sum_{t=1}^{l-2} a_t \sum_{\delta=t+1}^{2l-1-(t+1)} a_\delta b_{2l-\delta-t} + Q_{p,q}(c_l) \\ \quad + \sum_{t=1}^{l-1} \Phi_{p,q}(c_t, c_{2l-t}) = 0. \end{cases}$$

These equations can be obtained from the system defining $A_k^0(h_{k+1})$, by replacing the term ϵb_1^{2l} with ϵb_2^{2l} in the last equation and imposing b_1 to be 0 in the other ones.

Therefore, a similar process to the one outlined above can be applied, which provides further equal contributions for $\beta^G(A_M^0(h_{k+1}))$ and $\beta^G(A_M^0(r_k))$. In any case, the final equation will be either $Q_{p,q}(c_1) = 0$, $\epsilon b_j^{2l} + Q_{p,q}(c_1) = 0$ or trivial, so that the induced respective contributions are equal as well.

As a consequence, $\beta^G(A_M^0(h_{k+1})) = \beta^G(A_M^0(r_k))$ and $\beta^G(A_M(h_{k+1})) = \beta^G(A_M(r_k))$.

If k is odd, $k = 2l+1$, and $\epsilon = +1$, from the initial system of equations defining $A_M^0(h_{k+1})$ and $A_M^0(r_k)$, we are reduced to considering a system whose first equation is $b_1^{2l+1} + a_1^2 b_1 = 0$ (see the proof of Proposition 7.5). Therefore, $b_1 = 0$. Stratifying with the subsets $\{a_1 \neq 0\}$ and $\{a_1 = 0\}$, we then get, after a renaming of the variables, a system whose first nontrivial equations are, if $M \geq 2k$:

$$\begin{cases} Q_{p,q}(c_1) = 0, \\ \Phi_{p,q}(c_1, c_2) = 0, \\ a_1^2 b_2 + Q_{p,q}(c_2) + \Phi_{p,q}(c_1, c_3) = 0, \\ a_1^2 b_3 + 2a_1 a_2 b_2 + \Phi_{p,q}(c_1, c_4) + \Phi_{p,q}(c_2, c_3) = 0, \\ \dots \\ \sum_{t=1}^{l-1} a_t^2 b_{2l+1-2t} + 2\sum_{t=1}^{l-1} a_s \sum_{\delta=t+1}^{2l-(t+1)} a_\delta b_{2l+1-\delta-t} \\ \quad + \sum_{t=1}^{l} \Phi_{p,q}(c_t, c_{2l+1-t}) = 0, \\ b_2^{2l+1} + \sum_{t=1}^{l} a_t^2 b_{2l+2-2t} + 2\sum_{t=1}^{l-1} a_t \sum_{\delta=t+1}^{2l+1-(t+1)} a_\delta b_{2l+2-\delta-t} + Q_{p,q}(c_{l+1}) \\ \quad + \sum_{t=1}^{l} \Phi_{p,q}(c_t, c_{2l+2-t}) = 0. \end{cases}$$

Repeating the process provides further equal contributions for $\beta^G(A^0_M(h_{k+1}))$ and $\beta^G(A^0_M(r_k))$ and, if $M \geq 2k$, we are led to a new system whose first equation is $b_2^{2l+1} + Q_{p,q}(c_1) = 0$. We will show in Lemma 7.12 below that the respective induced contributions are equal.

In any case, these repeated steps of computations will eventually allow us to consider a single equation, which will be either $Q_{p,q}(c_1) = 0$, $b_j^{2l+1} + a_1^2 b_j = 0$, $b_j^{2l+1} + Q_{p,q}(c_1) = 0$ or trivial.

Consequently, if k is odd, $\beta^G(A^0_M(h_{k+1})) = \beta^G(A^0_M(r_k))$ and $\beta^G(A_M(h_{k+1})) = \beta^G(A_M(r_k))$ as well. □

Lemma 7.12. — *Suppose that k is odd, $k = 2l+1$, $p > q+1$ and $\eta = +1$ (the property will also be true if $q > p+1$ and $\eta = -1$). Then*

$$\beta^G(\{h_{k+1}(0, x_3, y) = 0\}) = \beta^G(\{r_k(0, x_2, y) = 0\}).$$

Proof. — Applying successive blowings-up as in the proof of Lemma 6.11, we obtain

$$\begin{aligned}\beta^G(\{h_{k+1}(0, x_3, y) = 0\}) &= \beta^G(\{\epsilon x_3 + Q_{p,q}(y) = 0\}) \\ &\qquad - k\beta^G(\{Q_{p,q}(y) = 0\}) + k\beta^G(\{0\}) \\ &= \beta^G(\mathbb{R}^{p+q}) - k\beta^G(\{Q_{p,q}(y) = 0\}) + k\beta^G(\{0\}).\end{aligned}$$

We have the same expression for $\beta^G(\{r_k(0, x_2, y) = 0\})$ and, therefore, since $p > q+1$ and $\eta = +1$, the two quantities are equal. □

As for the last part of the equivariant zeta functions with signs of h_{k+1} and r_k, adapting the computations of the proof of Proposition 7.11, we obtain the following (still under the hypothesis at the beginning of Section 7.4):

Proposition 7.13. — 1. *Suppose k is even.*

- *If $\eta = \epsilon$, then for all $M > k$, $\beta^G(A^\xi_M(h_{k+1})) = \beta^G(A^\xi_M(r_k))$, and, consequently, the respective equivariant zeta functions with signs of h_{k+1} and r_k are equal.*
- *If $\eta = +1$, $\epsilon = -1$, we have the equality $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ if and only if $\beta^G\big(\big\{-x_3^k + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\big\}\big) = \beta^G\big(\big\{-x_2^k + \sum_{i=1}^{K} z_i^2 = \xi\big\}\big)$ (the former set is a subset of $\mathbb{R}^{K+1}$ equipped with the action of G only changing the sign of y, and the latter set is a subset of $\mathbb{R}^{K+1}$ equipped with the trivial action of G).*
- *If $\eta = -1$, $\epsilon = +1$, we have the equality $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ if and only if $\beta^G\big(\big\{x_3^k - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\big\}\big) = \beta^G\big(\big\{x_2^k - \sum_{i=1}^{K} z_i^2 = \xi\big\}\big)$.*

2. *Suppose k is odd and $\epsilon = +1$.*
   - *If $\eta = +1$, the equality $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ is true if and only if the quantities $\beta^G\big(\{x_3^k + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}\big)$ and $\beta^G\big(\{x_2^k + \sum_{i=1}^{K} z_i^2 = \xi\}\big)$ are equal.*
   - *If $\eta = -1$, the equality $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ is true if and only if the quantities $\beta^G\big(\{x_3^k - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}\big)$ and $\beta^G\big(\{x_2^k - \sum_{i=1}^{K} z_i^2 = \xi\}\big)$ are equal.*

Proof. — Let M be greater than k. Since the system describing $A^{\xi}_M(h_{k+1})$ and $A^{\xi}_M(r_k)$ is obtained from the one defining $A^0_M(h_{k+1})$ and $A^0_M(r_k)$ by replacing 0 by ξ in the right member of the last equation, we are reduced, as in the proof of Proposition 7.11, to considering a single equation.

If k is even, $k = 2l$, this equation is either $Q_{p,q}(c_1) = \xi$, $\epsilon b_j^{2l} + Q_{p,q}(c_1) = \xi$ or an equation with no solution. Under our current hypothesis, the quantity $\beta^G(Y^{\xi}_{p,q})$ is the same for h_{k+1} and r_k. If $\epsilon = \eta$, we can show, as in the proof of Proposition 6.17, using the formulae of Lemma 6.15, that $\beta^G(\{h_{k+1}(0,x_3,y) = \xi\}) = \beta^G(\{r_k(0,x_2,y) = \xi\})$. If $\epsilon = -\eta$, we also use Lemma 6.15 to obtain the desired equivalences.

If k is odd and $\epsilon = +1$, the final equation is either $Q_{p,q}(c_1) = \xi$, $b_j^{2l+1} + a_1^2 b_j = \xi$, $b_j^{2l+1} + Q_{p,q}(c_1) = \xi$ or an equation with no solution. The quantity $\beta^G(Y^{\xi}_{p,q})$ is the same for h_{k+1} and r_k, and we showed in Proposition 7.10 that $\beta^G(\{h_{k+1}(x_2,x_3,0) = \xi\}) = \beta^G(\{r_k(x_1,x_2,0) = \xi\})$. Finally, we can obtain formulae similar to the ones in Lemma 6.15 for $\beta^G(\{h_{k+1}(0,x_3,y) = \xi\})$ and $\beta^G(\{r_k(0,x_2,y) = \xi\})$ if k is odd, and this provides the desired equivalences. $\square$

7.5. Conclusion. — We gather the results obtained in the following statement:

THEOREM 7.14. — *Let $k \geq 3$. Suppose that the invariant germs*

$$h_{k+1} = x_2^2 x_3 + \epsilon x_3^k + \eta x_1^2 + Q \text{ and } r_k = x_1^2 x_2 + \epsilon x_2^k + \eta' x_3^2 + Q'$$

have, up to permutation of all variables, the same quadratic part, with p signs $+$ and q signs $-$.

1. *If*
   - *$p \leq q$, $\eta = +1$ or $q \leq p$, $\eta = -1$,*
   - *$p = q+1$ or $q = p+1$,*
   - *k is odd, $\epsilon = -1$,*

   then h_{k+1} and r_k are not G-blow-Nash equivalent.
2. *If k is even, and if $p > q+1$, $\eta = +1$, $\epsilon = +1$ or $q > p+1$, $\eta = -1$, $\epsilon = -1$, then $Z^G_{h_{k+1}}(u,T) = Z^G_{r_k}(u,T)$ and $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$.*

3.
- *If k is even, and if $p > q+1$, $\eta = +1$, $\epsilon = -1$, then* $Z^G_{h_{k+1}}(u,T) = Z^G_{r_k}(u,T)$. *Furthermore,* $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ *if and only if* $\beta^G\big(\{-x_3^k + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{-x_2^k + \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.
- *If k is even, and if $q > p+1$, $\eta = -1$, $\epsilon = +1$, then* $Z^G_{h_{k+1}}(u,T) = Z^G_{r_k}(u,T)$. *Furthermore,* $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ *if and only if* $\beta^G\big(\{x_3^k - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{x_2^k - \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.
- *If k is odd, and if $p > q+1$, $\eta = +1$, $\epsilon = +1$, then* $Z^G_{h_{k+1}}(u,T) = Z^G_{r_k}(u,T)$. *Furthermore,* $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ *if and only if* $\beta^G\big(\{x_3^k + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{x_2^k + \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.
- *If k is odd, and if $q > p+1$, $\eta = -1$, $\epsilon = +1$, then* $Z^G_{h_{k+1}}(u,T) = Z^G_{r_k}(u,T)$. *Furthermore,* $Z^{G,\xi}_{h_{k+1}}(u,T) = Z^{G,\xi}_{r_k}(u,T)$ *if and only if* $\beta^G\big(\{x_3^k - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}\big) = \beta^G\big(\{x_2^k - \sum_{i=1}^{K} z_i^2 = \xi\}\big)$.

In particular, in the cases 2 and 3, we are not able to determine whether or not the germs h_{2k+1} and r_k are G-blow-Nash equivalent.

Remark 7.15. — If we forget the G-actions, the virtual Poincaré polynomials of the algebraic subsets $\{x^{2l+1} + \sum_{i=1}^{K} y_i^2 = \xi\}$ and $\{x^{2l+1} - \sum_{i=1}^{K} y_i^2 = \xi\}$ of $\mathbb{R}^{K+1}$, $\xi = \pm 1$, can also be computed using the invariance of the virtual Poincaré polynomial under bijection with the $\mathcal{AS}$ graph (see Remark 6.19).

8. The germs E_6 and F_4

Finally, we study the classification with respect to G-blow-Nash equivalence of the families

$$\varphi^\epsilon(x) := \pm x_1^2 + x_2^3 + \epsilon x_3^4 + Q$$

and

$$\omega^\epsilon(x) := \epsilon x_1^4 + x_2^3 + \pm x_3^2 + Q'$$

where $\epsilon \in \{-1; +1\}$.

If two germs φ^ϵ and $\varphi^{\epsilon'}$ are G-blow-Nash equivalent, they have the same quadratic part up to permutation of the variables $x_1, x_4, \ldots, x_n$ and, by [13] Proposition 3.14, $\epsilon = \epsilon'$. Furthermore, we will show in Corollary 8.3 below that the germs

$$\varphi^{\epsilon,+}(x) := +x_1^2 + x_2^3 + \epsilon x_3^4 + Q \text{ and } \varphi^{\epsilon,-}(x) := -x_1^2 + x_2^3 + \epsilon x_3^4 + Q',$$

where $\epsilon \in \{-1\,; +1\}$ and $+x_1^2 + Q$ and $-x_1^2 + Q'$ are the same quadratic part up to permutation of the variables $x_1, x_4, \ldots, x_n$, are not G-blow-Nash equivalent, so that the classification of the germs E_6 up to G-blow-Nash equivalence is the same as their classification up to equivariant analytic equivalence.

If two germs ω^ϵ and $\omega^{\epsilon'}$ are G-blow-Nash equivalent, they also have the same quadratic part, up to permutation of the variables $x_3, \dots, x_n$, and $\epsilon = \epsilon'$ as well; the classification of the germs F_4 up to G-blow-Nash equivalence is the same as their classification up to equivariant analytic equivalence.

If now two germs φ^ϵ and $\omega^{\epsilon'}$ are G-blow-Nash equivalent, then $\epsilon = \epsilon'$ and $\pm x_1^2 + Q$ and $\pm x_3^2 + Q'$ are the same quadratic part up to permutation of all variables, so that we intend to compare the germs

$$\varphi(x) = x_2^3 + \epsilon x_3^4 + \eta x_1^2 + Q \text{ and } \omega(x) = x_2^3 + \epsilon x_1^4 + \eta' x_3^2 + Q',$$

where $\epsilon, \eta, \eta' \in \{1, -1\}$ and $\eta x_1^2 + Q = \eta' x_3^2 + Q'$ up to permutation of all variables.

As in the previous two parts, we will consider the respective equivariant zeta functions of φ and ω, along with Theorem 4.2, to try to distinguish these invariant germs with respect to G-blow-Nash equivalence.

We begin with the computation of the first coefficients $\beta^G(A_2(\varphi))$, $\beta^G(A_3(\varphi))$, $\beta^G(A_4(\varphi))$ of the naive equivariant zeta function of φ (notice that the set $A_1(\varphi)$ is empty so that $\beta^G(A_1(\varphi)) = 0$). Due to Proposition 6.3, we can focus on the quantities $\beta^G(A_m^0(\varphi))$, $m \leq 4$. The corresponding expressions for ω are similar, in this case, equipping the set $Y_{p,q}$ with the trivial action of G.

Proposition 8.1. — *Write $\varphi = \varphi(x, z, y) = x^3 + \epsilon z^4 + Q_{p,q}(y)$. We have $\beta^G(A_2^0(\varphi)) = u^{4+p+q} \beta^G(Y_{p,q})$, $\beta^G(A_3^0(\varphi)) = u^{2(p+q)+5} \beta^G(Y_{p,q} \setminus \{0\}) + \frac{u^{2(p+q)+6}}{u-1}$ and $\beta^G(A_4^0(\varphi)) = u^{3(p+q)+6} \beta^G(Y_{p,q} \setminus \{0\}) + u^{2(p+q)+6} \beta^G(\{\varphi(0, z, y) = 0\})$.*

Proof. — If $m \geq 1$, we write an arc γ of $\mathcal{L}_m$ as

$$\gamma(t) = \begin{pmatrix} a_1 t + \cdots + a_m t^m \\ b_1 t + \cdots + b_m t^m \\ c_1^1 t + \cdots + c_m^1 t^m \\ \vdots \\ c_1^{p+q} t + \cdots + c_m^{p+q} t^m \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ b_1 \\ c_1^1 \\ \vdots \\ c_1^{p+q} \end{pmatrix} t + \cdots + \begin{pmatrix} a_m \\ b_m \\ c_m^1 \\ \vdots \\ c_m^{p+q} \end{pmatrix} t^m = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} t + \cdots + \begin{pmatrix} a_m \\ b_1 \\ c_m \end{pmatrix} t^m$$

(the group G only acts in changing the sign of the coordinates c_i^1, or c_i^{p+1}, in the case n°1, or n°2).

The set $A_2^0(\varphi)$ is described by the single equation $Q_{p,q}(c_1)=0$, the other variables remaining free. The set $A_3^0(\varphi)$ is defined by the system

$$\begin{cases} Q_{p,q}(c_1)=0, \\ a_1^3+\Phi_{p,q}(c_1,c_2)=0, \end{cases}$$

and, stratifying with the G-globally invariant subsets $\{c_1^1=\ldots=c_1^{i-1}=0, c_1^i\neq 0\}$, $i=1,\ldots,p$, and $\{c_1^1=\ldots=c_1^p=0\}=\{c_1=0\}$, we obtain

$$\beta^G(A_3^0(\varphi))=u^{6+2(p-1)+2q+1}\beta^G(Y_{p,q}\setminus\{0\})+\beta^G(A_3^0(\varphi)\cap\{c_1=0\}).$$

If $c_1=0$, then $a_1=0$ and the other variables are free, hence the desired expression.

Finally, $A_4^0(\varphi)$ is described by the system of equations

$$\begin{cases} Q_{p,q}(c_1)=0, \\ a_1^3+\Phi_{p,q}(c_1,c_2)=0, \\ \epsilon b_1^4+3a_1^2a_2+Q_{p,q}(c_2)+\Phi(c_1,c_3)=0. \end{cases}$$

Equivariantly stratifying $A_4^0(\varphi)$ as we did for $A_3^0(\varphi)$, we get the equality

$$\beta^G(A_4^0(\varphi))=u^{8+3(p-1)+3q+1}\beta^G(Y_{p,q}\setminus\{0\})+\beta^G(A_4^0(\varphi)\cap\{c_1=0,a_1=0\}),$$

the set $A_4^0(\varphi)\cap\{c_1=0,a_1=0\}$ being given by the equation $\epsilon b_1^4+Q_{p,q}(c_2)=0$. □

Using the same way of computation, we obtain the following expressions for the first terms of the equivariant zeta functions with signs of φ:

Proposition 8.2. — *We have* $\beta^G(A_2^\xi(\varphi))=u^{4+p+q}\beta^G(Y_{p,q}^\xi)$, $\beta^G(A_3^\xi(\varphi))=u^{2(p+q)+5}\beta^G(Y_{p,q}\setminus\{0\})+\frac{u^{2(p+q)+6}}{u-1}$ *and* $\beta^G(A_4^\xi(\varphi))=u^{3(p+q)+6}\beta^G(Y_{p,q}\setminus\{0\})+u^{2(p+q)+6}\beta^G(\{\varphi(0,x_3,y)=\xi\})$.

As we did in Sections 6.1, 6.2, 7.1 and 7.2, we deduce the following distinctions:

Corollary 8.3. —
1. *The germs $\varphi^{\epsilon,+}$ and $\varphi^{\epsilon,-}$ are not G-blow-Nash equivalent.*
2. *If $p\leq q$ and $\eta=+1$ or $q\leq p$ and $\eta=-1$, then the germs φ and ω are not G-blow-Nash equivalent.*
3. *If $p=q+1$ or $q=p+1$, then φ and ω are not G-blow-Nash equivalent.*

If $p>q+1$ and $\eta=+1$ or $q>p+1$ and $\eta=-1$, the respective quantities $\beta^G(Y_{p,q})$ and $\beta^G(Y_{p,q}^\xi)$ are identical for φ and ω. Furthermore, notice that, equivariantly, $\{\varphi(0,x_3,y)=0\}=\{f_3(x_2,y)=0\}$, or $\{\varphi(0,x_3,y)=\xi\}=\{f_3(x_2,y)=\xi\}$, and $\{\omega(0,x_1,y)=0\}=\{g_2(x_1,y)=0\}$, or $\{\omega(0,x_1,y)=\xi\}=\{g_2(x_1,y)=\xi\}$. Therefore, due to the computations of Section 6.2, we can state the following:

Proposition 8.4. — *Suppose that $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$.*

1. *For $m \leq 4$, $\beta^G(A_m(\varphi)) = \beta^G(A_m(\omega))$.*
2. *For $m \leq 3$, $\beta^G(A^\xi_m(\varphi)) = \beta^G(A^\xi_m(\omega))$.*
3.
   - *If $\eta = \epsilon$, then $\beta^G(A^\xi_4(\varphi)) = \beta^G(A^\xi_4(\omega))$.*
   - *If $\eta = +1$ and $\epsilon = -1$, then $\beta^G(A^\xi_4(\varphi)) = \beta^G(A^\xi_4(\omega))$ if and only if the equivariant virtual Poincaré series of the algebraic subsets $\big\{-x_3^4 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\big\} \subset \mathbb{R}^{K+1}$, $K := p-q$, equipped with the action of G only changing the sign of y, and $\big\{-x_1^4 + \sum_{i=1}^{K} z_i^2 = \xi\big\} \subset \mathbb{R}^{K+1}$, equipped with the action of G only changing the sign of x_1, are equal.*
   - *If $\eta = -1$ and $\epsilon = +1$, then we have $\beta^G(A^\xi_4(\varphi)) = \beta^G(A^\xi_4(\omega))$ if and only if $\beta^G\big(\big\{x_3^4 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\big\}\big) = \beta^G\big(\big\{x_1^4 - \sum_{i=1}^{K} z_i^2 = \xi\big\}\big)$.*

For these cases, we then have to look at the other coefficients of the equivariant zeta functions of φ and ω. We begin by showing that, under this hypothesis $p > q+1$, $\eta = +1$ or $q > p+1$, $\eta = -1$, the respective naive equivariant zeta functions of φ and ω are equal:

Proposition 8.5. — *If $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$, then for all $M > 4$, we have $\beta^G(A_M(\varphi)) = \beta^G(A_M(\omega))$.*

Proof. — Letting M be greater than 4, we prove that $\beta^G(A^0_M(\varphi)) = \beta^G(A^0_M(\omega))$. If we consider the system defining the two latter sets, the same computations as in Proposition 8.1 provide an equal (under our current hypothesis) contribution of $\beta^G(Y_{p,q} \setminus \{0\})$, and we are reduced to considering a system whose first condition is $a_1 = 0$, and the next equation is (after a shift of indices) $\epsilon b_1^4 + Q_{p,q}(c_1) = 0$. This equation induces equal contributions for $\beta^G(A^0_M(\varphi))$ and $\beta^G(A^0_M(\omega))$ as well (recall that $\{\varphi(0,x_3,y) = 0\} = \{f_3(x_2,y) = 0\}$, and $\{\omega(0,x_1,y) = 0\} = \{g_2(x_1,y) = 0\}$).

We then stratify with respect to the coordinates of c_1 as we did in the proofs of Propositions 6.20 and 7.11, and we obtain a further condition $b_1 = 0$. If $M \geq 8$, the first subsequent equations become

$$\begin{cases} a_2^3 + Q_{p,q}(c_1) = 0, \\ 3a_2^2 a_3 + \Phi_{p,q}(c_1,c_2) = 0, \\ \epsilon b_2^4 + 3a_2 a_3^2 + 3a_2^2 a_4 + Q_{p,q}(c_2) + \Phi(c_1,c_3) = 0. \end{cases}$$

Another stratification with respect to the vector c_1 provides an equal (by Lemma 7.12) contribution of $\beta^G(\{\varphi(x_2,0,y) = 0\}) = \beta^G(\{h_4(0,x_3,y) = 0\})$, or $\beta^G(\{\omega(x_2,0,y) = 0\}) = \beta^G(\{r_3(0,x_2,y) = 0\})$, and the condition $a_2 = 0$.

Carrying on with the computation, we obtain the equivalence $\beta^G(A^0_M(\varphi)) = \beta^G(A^0_M(\omega))$ if and only if $\beta^G(\{\varphi = 0\}) = \beta^G(\{\omega = 0\})$, from the equations of the form $\epsilon b_j^4 + a_{j'}^3 + Q_{p,q}(c_1) = 0$ with $4j = 3j'$. In Lemma 8.6, we prove that $\beta^G(\{\varphi = 0\}) = \beta^G(\{\omega = 0\})$. □

Lemma 8.6. — *Suppose that $p > q+1$ and $\eta = +1$ or $q > p+1$ and $\eta = -1$. Then*

$$\beta^G(\{\varphi = 0\}) = \beta^G(\{\omega = 0\}).$$

Proof. — Suppose that $p > q+1$ and $\eta = +1$. Considering an equivariant resolution of singularities of the G-algebraic set $\{\omega = 0\}$, we compare the quantities $\beta^G(\{\omega = 0\})$ and $\beta^G(\{\varphi = 0\})$.

Write $\omega(x) = x^3 + \epsilon z^4 + Q_{p,q}(y)$ (the group G acts via the involution $(x,z,y) \mapsto (x,-z,y)$). Using an equivariant change of coordinates as in the proof of Proposition 5.1, we can assume $q = 0$. We then equivariantly blow up the G-algebraic set $\{\omega = 0\}$ at the origin of $\mathbb{R}^n$:

- in the chart $x = u$, $z = uv$, $y_i = uw_i$, with G-action $(u,v,w_i) \mapsto (u,-v,w_i)$, the equation of the blown-up variety is
$$u^2[u + \epsilon u^2 v^4 + Q_{p,q}(w)] = 0,$$
- in the chart $x = vu$, $z = v$, $y_i = vw_i$, with G-action $(u,v,w_i) \mapsto (-u,-v,-w_i)$, it is
$$v^2[vu^3 + \epsilon v^2 + Q_{p,q}(w)] = 0,$$
- in the respective charts $x = w_j u$, $z = w_j v$, $y_j = w_j$, $y_i = w_j w_i$ for $i \neq j$, with G-action $(u,v,w_i) \mapsto (u,v,w_i)$, it is
$$w_j^2[w_j u^3 + \epsilon w_j^2 v^4 + 1 + Q(\widehat{w})] = 0.$$

The set of points of the strict transform of $\{\omega = 0\}$, which are in the first chart but not in the second one, is given by $v = 0, u + Q_{p,q}(w) = 0$, and, therefore, it is equivariantly isomorphic to an affine space; the respective induced contributions for $\beta^G(\{\omega = 0\})$ and $\beta^G(\{\varphi = 0\})$ are equal. Now, the set of points of the strict transform, which are in one of the last charts but not in the second and the first ones, is given by $v = 0, u = 0, 1 + Q(\widehat{w}) = 0$; it is the empty set $(q = 0)$.

Furthermore, notice that the intersection of the strict transform of $\{\omega = 0\}$ with the exceptional divisor is a circle with a nonempty fixed-point set.

Consequently, we are reduced to considering the equivariant virtual Poincaré series of the algebraic set of $\mathbb{R}^n$ defined by the equation $zx^3 + \epsilon z^2 + Q_{p,q}(y) = 0$, G acting via $(x,z,y) \mapsto (-x,-z,-y)$ (for φ, the involution would have been $(x,z,y_1,y_i) \mapsto (x,z,-y_1,y_i)$). We equivariantly blow up this G-algebraic set at the origin of $\mathbb{R}^n$ as well:

- in the chart $x = u$, $z = uv$,$y_i = uw_i$, with G-action $(u,v,w_i) \mapsto (-u,v,w_i)$, the equation of the blown-up variety is
$$u^2[vu^2 + \epsilon v^2 + Q_{p,q}(w)] = 0,$$

- in the chart $x = vu$, $z = v$, $y_i = vw_i$, with G-action $(u, v, w_i) \mapsto (u, -v, w_i)$, it is
$$v^2[v^2u^3 + \epsilon + Q_{p,q}(w)] = 0,$$
- in the respective charts $x = w_j u$, $z = w_j v$, $y_j = w_j$, $y_i = w_j w_i$ for $i \neq j$, with G-action $(u, v, w_j, w_i) \mapsto (u, v, -w_j, w_i)$, it is
$$w_j^2[vw_j^2u^3 + \epsilon v^2 + 1 + Q(\widehat{w})] = 0.$$

The set of points of the strict transform, which are in the second chart but not in the first one, is given by $u = 0, \epsilon + Q_{p,q}(w) = 0$; it is the Cartesian product of an affine line and the set $Y_{p,q}^{-\epsilon}$, and, therefore, it induces an equal contribution for $\beta^G(\{\omega = 0\})$ and $\beta^G(\{\varphi = 0\})$ under our current hypothesis. As for the set of points of the strict transform, which are in one of the last charts but not in the first and the second ones, it is given by $u = 0, v = 0, 1+Q(\widehat{w}) = 0$, and, thus, it is empty.

On the other hand, the intersection of the strict transform with the exceptional divisor provides equal contributions for $\beta^G(\{\omega = 0\})$ and $\beta^G(\{\varphi = 0\})$ as well.

As a consequence, we can focus on the equation $zx^2 + \epsilon z^2 + Q_{p,q}(y) = 0$ in $\mathbb{R}^n$, the group G acting via $(x, z, y) \mapsto (-x, z, y)$ for ω (respectively via $(x, z, y_1, y_i) \mapsto (x, z, -y_1, y_i)$ for φ). We equivariantly blow up once again:

- in the chart $x = u$, $z = uv$,$y_i = uw_i$, with G-action $(u, v, w_i) \mapsto (-u, -v, -w_i)$, the equation of the blown-up variety is
$$u^2[uv + \epsilon v^2 + Q_{p,q}(w)] = 0,$$
- in the chart $x = vu$, $z = v$, $y_i = vw_i$, with G-action $(u, v, w_i) \mapsto (-u, v, w_i)$, it is
$$v^2[vu^2 + \epsilon + Q_{p,q}(w)] = 0,$$
- in the respective charts $x = w_j u$, $z = w_j v$, $y_j = w_j$, $y_i = w_j w_i$ for $i \neq j$, with G-action $(u, v, w_i) \mapsto (-u, v, w_i)$, it is
$$w_j^2[vw_ju^2 + \epsilon v^2 + 1 + Q(\widehat{w})] = 0.$$

By similar arguments to those above, we are reduced to considering the equation $uv+\epsilon v^2+Q_{p,q}(w) = 0$. We can then stratify with respect to v and show that the induced respective contributions for $\beta^G(\{\varphi = 0\})$ and $\beta^G(\{\omega = 0\})$ are also the same. This finally proves the equality $\beta^G(\{\varphi = 0\}) = \beta^G(\{\omega = 0\})$. □

Similarly to what we did in the proofs of Propositions 6.21 and 7.13, we can adapt the proof of Proposition 8.5 in order to state a sufficient and necessary condition for the equality of the respective equivariant zeta functions with signs of φ and ω to be true:

Proposition 8.7. — 1. *Suppose* $p > q+1$ *and* $\eta = +1$. *Then,* $Z_\varphi^{G,\xi}(u,T) = Z_\omega^{G,\xi}(u,T)$ *if and only if we have the equalities*

- $\beta^G(\{x_2^3 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 + \sum_{i=1}^{K} z_i^2 = \xi\})$,
- $\beta^G(\{\epsilon x_3^4 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{\epsilon x_1^4 + \sum_{i=1}^{K} z_i^2 = \xi\})$,
- *and* $\beta^G(\{x_2^3 + \epsilon x_3^4 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 + \epsilon x_1^4 + \sum_{i=1}^{K} z_i^2 = \xi\})$,

where, in the left members of the equalities, the considered sets are algebraic subsets of $\mathbb{R}^{K+2}$ *equipped with the action of* G *only changing the sign of* y, *and, in the right members, the sets are subsets of* $\mathbb{R}^{K+2}$ *equipped with the action of* G *only changing the sign of* x_1.

2. *Suppose* $q > p+1$ *and* $\eta = -1$. *Then,* $Z_\varphi^{G,\xi}(u,T) = Z_\omega^{G,\xi}(u,T)$ *if and only if we have the equalities*
   - $\beta^G(\{x_2^3 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 - \sum_{i=1}^{K} z_i^2 = \xi\})$,
   - $\beta^G(\{\epsilon x_3^4 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{\epsilon x_1^4 - \sum_{i=1}^{K} z_i^2 = \xi\})$,
   - *and* $\beta^G(\{x_2^3 + \epsilon x_3^4 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 + \epsilon x_1^4 - \sum_{i=1}^{K} z_i^2 = \xi\})$.

Remark 8.8. — As we showed in the proof of Proposition 6.17, we have $\beta^G(\{+x_3^4 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{+x_1^4 + \sum_{i=1}^{K} z_i^2 = \xi\})$ and $\beta^G(\{-x_3^4 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{-x_1^4 - \sum_{i=1}^{K} z_i^2 = \xi\})$ for $\xi = \pm 1$.

We finally gather the results of this section in one theorem:

Theorem 8.9. — *Suppose that the invariant germs*

$$\varphi(x) = x_2^3 + \epsilon x_3^4 + \eta x_1^2 + Q \text{ and } \omega(x) = x_2^3 + \epsilon x_1^4 + \eta' x_3^2 + Q'$$

have, up to permutation of all variables, the same quadratic part, with p *signs* $+$ *and* q *signs* $-$.

1. *If*
   - $p \leq q$, $\eta = +1$ *or* $q \leq p$, $\eta = -1$,
   - $p = q+1$ *or* $q = p+1$,

   then φ *and* ω *are not* G*-blow-Nash equivalent.*
2. - *If* $p > q+1$, $\eta = +1$, *then* $Z_\varphi^G(u,T) = Z_\omega^G(u,T)$. *Furthermore,* $Z_\varphi^{G,\xi}(u,T) = Z_\omega^{G,\xi}(u,T)$ *if and only if* $\beta^G(\{x_2^3 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 + \sum_{i=1}^{K} z_i^2 = \xi\})$, $\beta^G(\{\epsilon x_3^4 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{\epsilon x_1^4 + \sum_{i=1}^{K} z_i^2 = \xi\})$ *and* $\beta^G(\{x_2^3 + \epsilon x_3^4 + y^2 + \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 + \epsilon x_1^4 + \sum_{i=1}^{K} z_i^2 = \xi\})$.
   - *If* $q > p+1$, $\eta = -1$, *then* $Z_\varphi^G(u,T) = Z_\omega^G(u,T)$. *Furthermore,* $Z_\varphi^{G,\xi}(u,T) = Z_\omega^{G,\xi}(u,T)$ *if and only if* $\beta^G(\{x_2^3 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 - \sum_{i=1}^{K} z_i^2 = \xi\})$, $\beta^G(\{\epsilon x_3^4 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) =$

$\beta^G(\{\epsilon x_1^4 - \sum_{i=1}^K z_i^2 = \xi\})$ *and* $\beta^G(\{x_2^3 + \epsilon x_3^4 - y^2 - \sum_{i=1}^{K-1} y_i^2 = \xi\}) = \beta^G(\{x_2^3 + \epsilon x_1^4 - \sum_{i=1}^K z_i^2 = \xi\})$.

In particular, in the case 2, we are not able to determine whether or not the germs φ and ω are G-blow-Nash equivalent.

Remark 8.10. — Forgetting the G-action, the respective virtual Poincaré polynomials of the algebraic subsets $\{x^3 + \epsilon z^4 + \sum_{i=1}^K y_i^2 = \xi\}$ and $\{x^3 + \epsilon z^4 - \sum_{i=1}^K y_i^2 = \xi\}$, $\epsilon = \pm 1$, $\xi = \pm 1$, of $\mathbb{R}^{K+1}$ can be computed using the invariance of the virtual Poincaré polynomial under bijection with the $\mathcal{AS}$ graph (see also Remarks 6.19 and 7.15). If the equivariant virtual Poincaré series was shown to be an invariant under equivariant $\mathcal{AS}$ bijection, it should be possible to compute the quantities considered above.

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