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TRANSFER FACTORS FOR JACQUET–MAO'S METAPLECTIC FUNDAMENTAL LEMMA

by Viet Cuong Do

Abstract. — In an earlier paper, we proved Jacquet–Mao's metaplectic fundamental lemma, which is the identity between two orbital integrals (one is defined on the space of symmetric matrices and the other one is defined on the twofold cover of the general linear group) corrected by a transfer factor. In this paper, we restricted our calculation to the case where the relevant representative is a diagonal matrix. The purpose of the present paper is to show that we can extend this result for the more general relevant representative. Our proof is based on the concept of Shalika germs for certain Kloosterman integrals.

Résumé (*Facteur de transfert pour le lemme fondamental métaplectique de Jacquet-Mao*). — Dans un article précédent, nous avons prouvé le lemme fondametal métaplectique de Jacquet–Mao qui est l'indentité entre deux intégrales orbitales (l'une est définie sur l'espace des matrices symétriques et l'autre est défine sur le revêtement à deux feuillets du groupe général linéaire) corrigée par un facteur de transfert. Dans cet article, nous avons limité notre calcul au cas où le représentant pertinent est une matrice diagonale. Le but du présent article est de montrer que nous pouvons étendre ce résultat au représentant pertinent plus général. Notre preuve est basée sur le concept des germes de Shalika pour certains intégraux Kloosterman.

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1. Introduction

Let K be a global field (that is, a number field or the function field of a curve over a finite field) and $\mathbb{A}$ be its ring of adeles. Jacquet conjectured that (cf. [4, 5]) the cuspidal representation of $\mathrm{GL}_r(\mathbb{A})$ distinguished by a general orthogonal subgroup should be the lifting of a cuspidal representation of its metaplectic cover $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (a certain twofold cover of GL_r). Jacquet and Mao have suggested that to solve this conjecture, we establish two relative trace formulas (one for the group GL_r and one for its metaplectic cover $\widetilde{\mathrm{GL}}_r$) and then compare them. Roughly speaking, the relative trace formula attached to a group is an identity between two expansions of a certain integral, known as "*geometric expansion*" and "*spectral expansion*". The terms of geometric expansion are quite explicit but complicated. The terms of spectral expansion contain information about automorphic representations. By comparing the geometric side of two relative trace formulas, we obtain a comparison between the two spectral sides.

One of the steps of this approach is precisely the fundamental lemma that we now state.

Let F be a non-Archimedean local field, $\mathcal{O}$ its valuation ring, and k its residue field. Assume that the cardinality q of k is odd. We choose once for all a uniformizer ϖ of $\mathcal{O}$ (i.e., a generator of the maximal ideal of $\mathcal{O}$). We write v for the valuation of F and $|.|$ for the norm, normalized such that $|x| = q^{-v(x)}$.

Let B_r be the standard Borel subgroup of GL_r (the subgroup of invertible upper triangular matrices) with unipotent radical N_r and let T_r be the maximal split torus contained in B_r. Let S_r be the variety $\{g \in \mathrm{GL}_r | {}^t g = g\}$ and W_r be the Weyl group of T_r.

Let $\psi : F \to \mathbb{C}^*$ be a nontrivial additive character of level 0 (i.e., ψ is trivial on $\mathcal{O}$, but nontrivial on $\varpi^{-1}\mathcal{O}$). We then define a character $\theta : N_r(F) \to \mathbb{C}^*$ of $N_r(F)$: $\theta(n) = \psi\left(\frac{1}{2}\sum_{i=2}^r n_{i-1,i}\right)$.

The local metaplectic cover $\widetilde{\mathrm{GL}}_r(F)$ of $\mathrm{GL}_r(F)$ is an extension of $\mathrm{GL}_r(F)$ by $\{\pm 1\}$ (cf. [8, page 40]). We can write the elements of $\widetilde{\mathrm{GL}}_r(F)$ in the form $\widetilde{g} = (g, z)$, with $g \in \mathrm{GL}_r(F)$ and $z \in \{\pm 1\}$, and the group multiplication is defined by

$$(g, z)(g, z') = (gg', \chi(g, g')zz'),$$

where χ is a certain cocycle (cf. loc. cit. for the definition of χ). This cover splits (canonically) over $N_r(F)$ (the splitting σ over $N_r(F)$ is simply defined by $\sigma(n) = (n, 1)$); it also splits over $\mathrm{GL}_r(\mathcal{O})$. Let κ^* be the canonical splitting over $\mathrm{GL}_r(\mathcal{O})$ (cf. [8, Propostition 0.1.3]). The function $\kappa : \mathrm{GL}_r(\mathcal{O}) \to \{\pm 1\}$ is then defined by $\kappa^*(g) = (g, \kappa(g))$. We denote by $\mathrm{GL}_r^*(\mathcal{O})$ the image of $\mathrm{GL}_r(\mathcal{O})$ via the splitting κ^*.

We say a function f on $\widetilde{\mathrm{GL}}_r(F)$ is *genuine*, if it satisfies $f(g, z) = f(g, 1).z$.

Let $\mathcal{H}_r$ be the set of the smooth functions with compact support on $\mathrm{GL}_r(F)$ which are bi-$\mathrm{GL}_r(\mathcal{O})$-invariant. This F-vector space is equipped an algebraic structure by the convolution

$$\phi * \phi'(x) = \int_{\mathrm{GL}_r(F)} \phi(g)\phi'(g^{-1}x)dg.$$

Its unit element is the function defined by

$$\phi_0(g) = \begin{cases} 1 & \text{if } g \in \mathrm{GL}_r(\mathcal{O}) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{\mathcal{H}}_r$ be the set of smooth functions with compact support on $\widetilde{\mathrm{GL}}_r(F)$, which are bi-$\mathrm{GL}_r^*(\mathcal{O})$-invariant and genuine. This F-vector space is equipped an algebraic structure by the convolution

$$f * f'(\widetilde{x}) = \int_{\mathrm{GL}_r(F)} f((g,1))f'((g,1)^{-1}\widetilde{x})dg.$$

Note that the function $\widetilde{g} \mapsto f(\widetilde{g})f'(\widetilde{g}^{-1}\widetilde{x})$ on $\widetilde{\mathrm{GL}}_r(F)$ is $\{\pm 1\}$-invariant and that the integration is over $\mathrm{GL}_r(F) \simeq \{\pm 1\} \setminus \widetilde{\mathrm{GL}}_r(F)$. Its unit element is the function defined by

$$f_0((g,1)) = \begin{cases} \kappa(g) & \text{if } g \in \mathrm{GL}_r(\mathcal{O}) \\ 0 & \text{otherwise.} \end{cases}$$

The group N_r acts on S_r by $n.s = {}^t nsn$ and $N_r \times N_r$ acts on GL_r by $(n,n').g = n^{-1}gn$. We say an orbit $N_r s$ (respectively, $(N_r \times N_r)g$) is *relevant* if the restriction of θ^2 (respectively, $(n,n') \mapsto \theta(n^{-1}n')$) on the stabilizer $(N_r)_s$ (respectively, $(N_r \times N_r)_g$) of s (respectively, of g) is trivial.

Consider the standard Levi subgroup M of GL_r of type $(r_1, \ldots, r_m)$. Thus, M is the group of matrices of the form $\mathrm{diag}(g_i)$ with $g_i \in \mathrm{GL}_{r_i}$. We denote by w_{GL_r} the longest Weyl element of GL_r (i.e., the $r \times r$ permutation matrix which entries are one on the second diagonal and which other entries are 0). Let $w_M = \mathrm{diag}(w_{\mathrm{GL}_{r_i}})$. Let T_M be the group of matrices of the form $\mathrm{diag}(a_i \mathrm{Id}_{r_i})$, where Id_{r_i} is an identity matrix of size r_i and $a_i \in F^*$ – the center of M. It follows from [9, Theorem 1] that the elements of the form $w_M \mathbf{t}$ (respectively, of the form $w_{\mathrm{GL}_r} w_M \mathbf{t}$) with $\mathbf{t} \in T_M$ (when M runs through the set of standard Levi subgroup of GL_r) form a system of representatives for the relevant orbits of N_r (respectively, of $N_r \times N_r$). So we then have a bijection between the sets of relevant orbits: $w_M \mathbf{t} \mapsto w_{\mathrm{GL}_r} w_M \mathbf{t}$. The elements of the form $w_M \mathbf{t}$ as above are called *relevant elements*. We denote by W_r^R the set of relevant elements in W_r (i.e., the intersection of the set of relevant elements and W_r). If $w \in W_r^R$, then the unique M such that $w = w_M$ is denoted by M_w. We also write T_w for T_M. For instance, if $w = \mathrm{Id}_r$, then $M_{\mathrm{Id}_r} = T_{\mathrm{Id}_r} = T_r$, and if $w = w_{\mathrm{GL}_r}$, then $M_{w_{\mathrm{GL}_r}} = \mathrm{GL}_r$, and $T_{w_{\mathrm{GL}_r}}(F) = T_{\mathrm{GL}_r} = \{\beta \mathrm{Id}_r | \beta \in F^*\} \simeq F^*$. The stabilizer of $\mathbf{t} \in T_r(F)$ (respectively, $w_{\mathrm{GL}_r}\mathbf{t}$) in N_r (respectively, $N_r \times N_r$) is trivial. In this sense, the diagonal matrices are representatives of the largest orbits. From now on, we shall drop the subscript M in the notation $w_M\mathbf{t}$ when we do not want to specify what the standard Levi subgroup is.

We denote by $\mathcal{C}_c^\infty(S_r(F))$ (respectively, $\mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$) the space of the smooth function of compact support on $S_r(F)$ (respectively, on $\widetilde{\mathrm{GL}}_r(F)$). Let ϕ be a function in $\mathcal{C}_c^\infty(S_r(F))$ and f be a genuine function in $\mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$. For each $w\mathbf{t}$ as above, we consider orbital integrals of the form

$$I(w\mathbf{t}, \phi) = \int_{N_r/(N_r)_{w\mathbf{t}}} \phi({}^t n w\mathbf{t} n)\theta^2(n) dn$$

and

$$J(w\mathbf{t}, f) = \int_{N_r \times N_r/(N_r \times N_r)_{w_{\mathrm{GL}_r} w\mathbf{t}}} f(\sigma(n)^{-1}(w_{\mathrm{GL}_r} w\mathbf{t}, 1)\sigma(n'))\theta(n^{-1}n') dn dn'.$$

If $\phi \in \mathcal{H}_r$, then the function $\phi_{|S_r(F)} \in \mathcal{C}_c^\infty(S_r(F))$. By abusing the notation $\mathcal{H}_r$ for the algebra $\{\phi_{|S_r(F)} | \phi \in \mathcal{H}_r\}$, our fundamental lemma is the following conjecture.

Conjecture 1.1 (Jacquet–Mao). — *There exists a homomorphism $h : \widetilde{\mathcal{H}}_r \to \mathcal{H}_r$, such that $J(w\mathbf{t}, f) = \Delta(w\mathbf{t}) I(w\mathbf{t}, h(f))$, where $f \in \widetilde{\mathcal{H}}_r$, and $\Delta(w\mathbf{t})$ is an explicit transfer factor.*

Since h is an homomorphism between two algebras, it should send the unit element of one to the unit element of the other. The (suggested) transfer factor for the largest orbit ($w = \mathrm{Id}_r$) is then calculated by the following propositions.

Proposition 1.2 (cf. [1],[2]). — *Let F be a local field of positive characteristic. Let $\boldsymbol{t} = \mathrm{diag}(t_1, \dots, t_r)$; we denote by $a_i = \prod_{j=1}^i t_j$. We then have*

$$J(\boldsymbol{t}, f_0) = \Delta(\boldsymbol{t}) I(\boldsymbol{t}, \phi_0),$$

where (agreeing that $a_0 = 1$)

$$\Delta(\boldsymbol{t}) = \zeta(-1)^{\sum_{j \not\equiv r (\mathrm{mod} 2)} v(a_j)} \left| \prod_{i=1}^{r-1} a_i \right|^{-1/2} \prod_{j \not\equiv r (\mathrm{mod} 2)} \gamma(a_j a_{j-1}^{-1}, \psi).$$

Here, $\zeta : k^* \to \{\pm 1\}$ is the nontrivial quadratic character, and $\gamma(\,\cdot\,, \psi)$ is the Weil constant defined by the following: given a compact open neigborhood Ω of 0 in F, for $|a|$ large enough, we have

$$\int_\Omega \psi\left(\frac{ax^2}{2}\right) dx = |a|^{-1/2}\gamma(a, \psi).$$

Proposition 1.3 (cf. [3]). — *Proposition 1.2 is still true when F is a local field of characteristic zero, with a sufficiently large residual characteristic.*

Similar identities are expected to be true for other relevant orbits.

From now on, we focus only on the case where F is a local non-Archimedean field of characteristic zero and the residual characteristic of F is larger than $2r+1$ (the condition for the residual characteristic is needed in our calculation, cf. Propositions 2.2 and 3.3). For $\mathbf{t} = \operatorname{diag}(a_j a_{j-1}^{-1}, 1 \leq j \leq r) \in T_r(F)$, we introduce two new factors:

$$\Delta_r(\mathbf{t}) = \zeta(-1)^{\sum_{j \not\equiv r (\mathrm{mod} 2)} v(a_j)} \left| \prod_{i=1}^{r-1} a_i \right|^{-1/2} \times \prod_{j \not\equiv r (\mathrm{mod} 2)} \left[\gamma(a_j a_{j-1}^{-1}, \psi)(a_j a_{j-1}^{-1}, \varpi)^{v(a_j a_{j-1}^{-1})} \right]$$

and

$$\Delta'_r(\mathbf{t}) = \zeta(-1)^{\sum_{j \equiv r (\mathrm{mod} 2)} v(a_j)} \left| \prod_{i=1}^{r-1} a_i \right|^{-1/2} \times \prod_{j \equiv r (\mathrm{mod} 2)} \left[\gamma(a_j a_{j-1}^{-1}, \psi)(a_j a_{j-1}^{-1}, \varpi)^{v(a_j a_{j-1}^{-1})} \right].$$

Here, $(,) : F^* \times F^* \to \{\pm 1\}$ is the Hilbert symbol. It is a bilinear form on F^* that defines a nondegenerate bilinear form on $F^*/(F^*)^2$ and satisfies

$$(x, -x) = (x, y)(y, x) = 1.$$

Our main result is the following:

THEOREM 1.4 (The main theorem). — *Let ϕ be a smooth function of compact support over $S_r(F)$ and f be a smooth genuine function of compact support over $\widetilde{\mathrm{GL}}_r(F)$. If ϕ and f satisfy $J(\boldsymbol{t}, f) = \Delta_r(\boldsymbol{t}) I(\boldsymbol{t}, \phi)$ (respectively, $J(\boldsymbol{t}, f) = \Delta'_r(\boldsymbol{t}) I(\boldsymbol{t}, \phi)$) for all $\mathbf{t} \in T_r(F)$, then for all $w \in W_r^R$ and all $\mathbf{t} \in T_w(F)$ there exists $\Delta_w(\mathbf{t})$ (respectively, $\Delta'_w(\mathbf{t})$) (we refer to Theorem 4.7 for the formula of $\Delta_w(\mathbf{t})$ and of $\Delta'_w(\mathbf{t})$), such that $J(w\mathbf{t}, f) = \Delta_w(\boldsymbol{t}) I(w\boldsymbol{t}, \phi)$ (resp. $J(w\mathbf{t}, f) = \Delta'_w(\boldsymbol{t}) I(w\boldsymbol{t}, \phi)$). Moreover, the factors $\Delta_w(\mathbf{t})$ and $\Delta'_w(\mathbf{t})$ are independent of f and ϕ.*

It should be mentioned that to calculate Δ_w (and Δ'_w), we must use both Δ_i and Δ'_i. When $w = \mathrm{Id}_r$, we have $\Delta_w(\mathbf{t}) = \Delta_r(\mathbf{t})$ and $\Delta'_w(\mathbf{t}) = \Delta'_r(\mathbf{t})$.

Let $g \in \mathrm{GL}_r(F)$. We denote by $a_i(g)$ the determinant of the submatrix made of the first i lines and of the first i columns of the matrix g (with the convention that $a_0(g) = 1$). We consider a sequence $0 = r_0 < r_1 < \cdots < r_m = m$, such that the set $\{a_{r_i}(g)\}$ contains all the nonzero elements of $\{a_i(g) | 0 \leq i \leq r\}$ (m and r_i depend on g). Given $0 = r_0 < r_1 < \cdots < r_m = m$, we define a sequence

$(y_i)_{1\le i\le m} \in \{0,1\}^m$ as follows:

$$y_m = 0,\, y_i = \sum_{j=i+1}^{m} (r_j - r_{j-1}) \mod 2 \quad \forall 1 \le i \le m-1.$$

For instance when $g \in T_r(F)$, we have $m = r$, $r_i = i$ for all $i \in \{1,\dots,r\}$ and

$$y_i = \begin{cases} 1 & \text{if } i \not\equiv r \bmod 2 \\ 0 & \text{if } i \equiv r \bmod 2. \end{cases}$$

Now let ϕ_1 be the function $\phi_1 : \mathrm{GL}_r(F) \to \{\pm 1\}$ defined by

$$g \mapsto \prod_{j=1}^{m} \left((-1)^{\left[\frac{r_j - r_{j-1}}{2}\right]} a_{r_j}(g) a_{r_{j-1}}(g)^{-1}, \varpi \right)^{v(a_{r_j}(g) a_{r_{j-1}}(g)^{-1}) \left[\frac{r_j - r_{j-1} + y_j}{2}\right]}.$$

Here, m, r_i, y_i depend on g (defined as above; we use this kind of notation to simplify a formula), and $[x]$ is the integral part of a real number x. For instance, when $\mathbf{t} = \mathrm{diag}(a_j a_{j-1}^{-1}, 1 \le j \le r) \in T_r(F)$, we have

$$\phi_1(\mathbf{t}) = \prod_{j \not\equiv r (\mathrm{mod} 2)} (a_j a_{j-1}^{-1}, \varpi)^{v(a_j a_{j-1}^{-1})}.$$

Note that $a_i({}^t ngn) = a_i(g)$, for all $1 \le i \le r$. Using Propositions 1.2 and 1.3 we have

$$J(\mathbf{t}, f) = \Delta_r(\mathbf{t}) I(\mathbf{t}, \phi_0.\phi_1)$$

for all $\mathbf{t} = \mathrm{diag}(a_j a_{j-1}^{-1}, 1 \le j \le r)$.

Applying the main theorem for $\phi = \phi_0.\phi_1$ and $f = f_0$, then the (suggested) transfer factors of any kind of relevant orbits are calculated.

Corollary 1.5. — *Let F be a local field of characteristic zero with a sufficiently large residual characteristic. Let $w\mathbf{t}$ be a relevant orbit of GL_r. We then have*

$$J(w\mathbf{t}, f_0) = \Delta_w(\mathbf{t}) \phi_1(w\mathbf{t}) I(w\mathbf{t}, \phi_0).$$

The integral J above is, in fact, the Kloosterman integral, which is considered in [7, 6] (see Section 2). So we have the following density theorem for the Kloosterman integral:

Proposition 1.6 (cf. [6]). — *If the diagonal orbital integral $J(\mathbf{t}, f)$ of a function $f \in \mathcal{C}_c^\infty(\widetilde{\mathrm{GL}}_r(F))$ vanishes for all $\mathbf{t} \in T_r(F)$, then all the orbital integrals $J(w\mathbf{t}, f)$ with $w \in W_r^R$ and $\mathbf{t} \in T_w(F)$ of f vanish.*

Assume that a function $\phi \in \mathcal{C}_c^\infty(S_r(F))$ satisfies $I(\mathbf{t}, \phi) = 0$, for all $\mathbf{t} \in T_r(F)$. We then have

$$I(\mathbf{t}, \phi) = \Delta_r(\mathbf{t}) J(\mathbf{t}, 0)$$

for all $\mathbf{t} \in T_r(F)$. Applying the main theorem for the couple of functions $(\phi, 0)$, we obtain the following density theorem:

Proposition 1.7. — *If the diagonal orbital integrals $I(\mathbf{t}, \phi)$ of a function $\phi \in \mathcal{C}_c^\infty(S_r(F))$ vanishes for all $\mathbf{t} \in T_r(F)$, then all the orbital integrals $I(w\mathbf{t}, \phi)$ with $w \in W_r^R$ and $\mathbf{t} \in T_w(F)$ of ϕ vanish.*

We now state the organization of this manuscript. The main tool we use to prove the main theorem (Theorem 1.4) is Shalika germs, which describe the asymptotic behavior of the orbital integrals. In Section 2 (and Section 3), we recall this asymptotic behavior of the integral J (and the integral I, respectively) and do some calculation for the germ functions. The main theorem is proved by induction in Section 4. In this section, we introduce some intermediate integrals that are designed to use in inductive argument. The germ relations are used to handle cases when we cannot use these intermediate integrals directly.

The proof of the main theorem follows the guidelines of [6] closely; the new ingredient is the occurrence of the factors Δ_r and Δ'_r.

In the following sections, we need (to recall) some properties of the Hilbert symbol and of the Weil constant.

Proposition 1.8 (cf. [9, Proposition 1], [2]). — *For $a, b, c \in F^*$ we have:*

1. $(a, b) = (b, a)$.
2. $(a, bc) = (a, b)(a, c)$.
3. *If $v(a), v(b)$ are even, then $(a, b) = 1$.*
4. *If $v(a)$ is even, $v(b)$ is odd, then $(a, b) = (a, \varpi)$.*
5. *If $v(a), v(b)$ are odd, then $(a, b) = (-ab, \varpi)$.*
6. *If $a \in k^*$, then $(a, \varpi) = \zeta(a)$.*

Proposition 1.9 (cf. [9, Proposition 2]). — *Let ψ be a nontrivial additive character of level 0. We have:*

1. $\gamma(a, \psi) = 1$ *if $v(a)$ is even.*
2. $\gamma(ab, \psi) = \gamma(a, \psi)\gamma(b, \psi)(a, b)$.

2. Computation of the germ on the J side

For the convenience, we shall rewrite the orbital integral J. Suppose that $w \in W_r^R$ and $\mathbf{t} \in T_w$. Then let $P_w = M_w N_w$ be the standard parabolic subgroup, which has Levi factor M_w. Let $V_w = N_r \cap M_w$. We have (cf. [7, page 916])

$$(N_r \times N_r)_{w_{\mathrm{GL}_r} w\mathbf{t}} = N_r^w := \{(n_1, n_2) | w_{\mathrm{GL}_r} n_1^{-1} w_{\mathrm{GL}_r} w n_2 = w\}$$

for all $\mathbf{t} \in T_w$. Furthermore, N_r^w is the set of $(n_1, n_2) \in N_r \times N_r$ with $n_2 \in V_w$, ${}^t(n_1^{w_{\mathrm{GL}_r}}) := {}^t(w_{\mathrm{GL}_r} n_1 w_{\mathrm{GL}_r}) \in V_w$ and $n_2 = w n_1^{w_{\mathrm{GL}_r}} w$.

Lemma 2.1. — *Any point of the orbit of* $w_{\mathrm{GL}_r}w\mathbf{t}$ *under the action of* $N_r \times N_r$ *can be uniquely written in the following form*

$$\mu(u_1,u_2,v) = w_{\mathrm{GL}_r}{}^t u_1 w\mathbf{t} v u_2$$

with $u_i \in N_w(F)$ *and* $v \in V_w(F)$.

Proof. — Since $N_r \subset P_w$, by using the Levi decomposition for elements of N_r, we can rewrite any element of the orbit of $w_{\mathrm{GL}_r}w\mathbf{t}$ under the action of $N_r \times N_r$ as below:

$$\begin{aligned} n_1^{-1} w_{\mathrm{GL}_r} w\mathbf{t} n_2 &= w_{\mathrm{GL}_r}(w_{\mathrm{GL}_r} n_1^{-1} w_{\mathrm{GL}_r}) w\mathbf{t} n_2 \\ &= w_{\mathrm{GL}_r}{}^t u_1 {}^t v_1 w\mathbf{t} v_2 u_2 \\ &= w_{\mathrm{GL}_r}{}^t u_1 [{}^t v_1 w\mathbf{t}(w {}^t v_1^{-1} w)][(w {}^t v_1 w) v_2] u_2 \\ &= w_{\mathrm{GL}_r}{}^t u_1 w\mathbf{t} v u_2. \end{aligned}$$

Here, $u_i \in N_w(F)$, $v_i \in V_w(F)$, $v \in V_w(F)$, such that $v_1u_1 = {}^t(w_{\mathrm{GL}_r} n_1^{-1} w_{\mathrm{GL}_r})$, $v_2u_2 = n_2$ and $v = (w {}^t v_1 w)v_2$. The last identity follows: $(w_{\mathrm{GL}_r}{}^t v_1 w_{\mathrm{GL}_r}, w {}^t v_1^{-1} w) \in N_r^w$.

Suppose that $w_{\mathrm{GL}_r}{}^t u_1 w\mathbf{t} v u_2 = w_{\mathrm{GL}_r}{}^t u_1' w\mathbf{t} v' u_2'$ with $u_i, u_i' \in N_w(F)$ and $v, v' \in V_w(F)$. We then have

$$[w_{\mathrm{GL}_r}{}^t(u_1^{-1}u_1')w_{\mathrm{GL}_r}] w_{\mathrm{GL}_r} w\mathbf{t}(v'u_2'u_2^{-1}v^{-1}) = w_{\mathrm{GL}_r} w\mathbf{t}.$$

Hence, $v'u_2'u_2^{-1}v^{-1} \in V_w(F)$ and $w_{\mathrm{GL}_r}{}^t(u_1^{-1}u_1')w_{\mathrm{GL}_r} \in V_w(F)$. This implies that $\{u_2'u_2^{-1}, u_1^{-1}u_1'\} \subset N_w(F) \cap V_w(F) = \{\mathrm{Id}_r\}$. Thus, $u_i = u_i'$ for all $i \in \{1,2\}$. As a consequence, we have $v = v'$. □

Note that the map μ is an isomorphism of analytic varieties (over F) $N_w(F) \times N_w(F) \times V_w(F)$ onto the orbit of $w\mathbf{t}$. Thus,

$$J(w\mathbf{t}, f) = \int_{N_w(F)\times N_w(F)\times V_w(F)} f((w_{\mathrm{GL}_r}{}^t u_1 w\mathbf{t} v u_2, 1))\theta(u_1u_2v)\, du_1 dv du_2.$$

Denote by f' the function $g \mapsto f((w_{\mathrm{GL}_r} g, 1))$, $\forall g \in \mathrm{GL}_r$; then $f' \in \mathcal{C}_c^\infty(\mathrm{GL}_r(F))$. The integral J is then the orbital Kloosterman integral $\mathrm{Kloos}(w\mathbf{t}; f')$, which is considered in [7] (in loc. cit., it is denoted by $I(w\mathbf{t}; f')$). This integral converges and defines a smooth function on $T_w(F)$.

We let M be the standard Levi subgroup of type $(r-1, 1)$. The corresponding element w_M is $\begin{pmatrix} w_{\mathrm{GL}_{r-1}} & 0 \\ 0 & 1 \end{pmatrix}$. We denote by $T_{w_M}^{w_{\mathrm{GL}_r}}$ the set of matrices $\mathbf{t} \in T_{w_M}(F)$, such that $\det(\mathbf{t}) = \det(w_M)\det(w_{\mathrm{GL}_r})$. There exists a smooth function $K_{w_M}^{w_{\mathrm{GL}_r}}$ on $T_{w_M}^{w_{\mathrm{GL}_r}}$ (cf. [7, 6]) with the following property: for any $h \in \mathcal{C}_c^\infty(\mathrm{GL}_r(F))$, there is a smooth function of compact support ω_h on T_M, such that

$$\mathrm{Kloos}(w_M\mathbf{t}, h) = \omega_h(\mathbf{t}) + \sum_{\alpha\beta = \mathbf{t}} K_{w_M}^{w_{\mathrm{GL}_r}}(\alpha)\mathrm{Kloos}(w_{\mathrm{GL}_r}\beta, h).$$

The sum is over all pairs in $\{(\alpha,\beta) \in (T^{w_{\mathrm{GL}_r}}_{w_M}, T_{\mathrm{GL}_r}(F)) \,|\, \alpha\beta = \mathbf{t}\}$. Taking $h = f'$, we have

$$J(w_M\mathbf{t}, f) = \omega_{f'}(\mathbf{t}) + \sum_{\alpha\beta=\mathbf{t}} K^{w_{\mathrm{GL}_r}}_{w_M}(\alpha) J(w_{\mathrm{GL}_r}\beta, f).$$

The function $K^{w_{\mathrm{GL}_r}}_{w_M}$ is the **germ** (for the side J) along the subset $T^{w_{\mathrm{GL}_r}}_{w_M}$.

Proposition 2.2 (cf. [6, Proposition 3.1]). — *Suppose that the residual characteristic of F is larger than r. Let*

$$\alpha = \mathrm{diag}(a, \ldots, a, a^{1-r}\det(w_M w_{\mathrm{GL}_r})).$$

Then, for $|a|$ sufficiently small,

$$K^{w_{\mathrm{GL}_r}}_{w_M}(\alpha) = |a|^{-\frac{(r-1)^2}{2}} \psi\left(\frac{r}{2a}\right)\left(\frac{r}{2^{r-1}}, a^{-1}\right)\gamma(a^{-1},\psi)^{r-1}.$$

Note that the characteristic of k is larger than r; we then have $\frac{i+1}{2i} \in \mathcal{O}^*$ for all $1 \leq i \leq r-1$. Since in our case $\theta(n) = \psi(\frac{1}{2}\sum_{i=2}^r n_{i-1,i})$ (not $\theta(n) = \psi(\sum_{i=2}^r n_{i-1,i})$ in loc. cit.), our formula (using the same processing) is a bit different from the one of Jacquet (in loc. cit.).

3. Computation of the germ on the I side

The discussion of [7, §2] applies to our situation, where $\mathrm{GL}_r(F)$ is replaced by $S_r(F)$, and the group $N_r \times N_r$ ($N_r \times N_r$ acts on GL_r by $g \mapsto {}^t n g n'$) by the group $N_r(F)$ acting on $S_r(F)$. Suppose that $w \in W_r^R$ and $\mathbf{t} \in T_w$. Then let $P_w = M_w N_w$ be the standard parabolic subgroup, which has Levi factor M_w. Let $V_w = N_r \cap M_w$. We have

$$(N_r)_{w\mathbf{t}} = (N_r)_w := \{n \in N_r | {}^t n w n = w\}$$

for all $\mathbf{t} \in T_w$. Furthermore, $(N_r)_w$ is the set of $n \in V_w$ such that ${}^t n w n = w$.

Any element of the orbit of $w\mathbf{t}$ can be written in the form (using the Levi decomposition for the elements of $N_r \subset P_w$)

$${}^t u {}^t v w \mathbf{t} v u,$$

with $u \in N_w(F)$ and $v \in V_w(F)$. Denote by $v_1 = w {}^t v w v$. We then have that v_1 is an element of $V_w^1 := \{v \in V_w | w {}^t v w = v\}$.

Lemma 3.1. — *Any point of the orbit of $w\mathbf{t}$ can be written uniquely in the form*

$$\nu(u, v_1) := {}^t u w \mathbf{t} v_1 u$$

with $u \in N_w(F)$ and $v_1 \in V_w^1(F)$.

Proof. — The proof is similar to the proof of Lemma 2.1. □

The map ν is a diffeomorphism of $N_r(F)\times V_w^1(F)$ onto the orbit $w\mathbf{t}$ in $S_r(F)$. Thus,

$$I(w\mathbf{t},\phi)=\int_{N_w(F)\times V_w^1(F)}\phi({}^tuw\mathbf{t}v_1u)\theta(u^2v_1)dudv_1.$$

Now, we let M be the standard Levi subgroup of type $(r-1,1)$. As before, there exists a smooth function $L_{w_M}^{w_{\mathrm{GL}_r}}$ on $T_{w_M}^{w_{\mathrm{GL}_r}}$ with the following property: for any $\phi\in\mathcal{C}_c^\infty(S_r(F))$, there is a smooth function of compact support ω_ϕ, such that

$$I(w_M\mathbf{t},\phi)=\omega_\phi(\mathbf{t})+\sum_{\alpha\beta=\mathbf{t}}L_{w_M}^{w_{\mathrm{GL}_r}}(\alpha)I(w_{\mathrm{GL}_r}\beta,\phi).$$

The sum is over all pairs in $\{(\alpha,\beta)\in(T_{w_M}^{w_{\mathrm{GL}_r}},T_{\mathrm{GL}_r}(F))\,|\,\alpha\beta=\mathbf{t}\}$. The function $L_{w_M}^{w_{\mathrm{GL}_r}}$ is the **germ** (for the side I) along the subset $T_{w_M}^{w_{\mathrm{GL}_r}}$.

Let $\mathbf{t}=\mathrm{diag}(a,\dots,a,a^{1-r}\det(w_Mw_{\mathrm{GL}_r}))$. Since ω_ϕ is a smooth function of compact support, we can choose $|a|$ small enough, such that $\omega_\phi(\mathbf{t})=0$. We consider the pair $(\alpha,\beta)\in(T_{w_M}^{w_{\mathrm{GL}_r}},T_{\mathrm{GL}_r}(F))$, such that $\alpha\beta=\mathbf{t}$. Since $\det(\alpha)=\det(w_Mw_{\mathrm{GL}_r})=\det(\mathbf{t})$ (by definition), we have $\det(\beta)=1$. Moreover, $\beta=\mathrm{diag}(z,z,\dots,z)$ with $z^r=1$.

We denote by $[x]$ the integral part of a real number x. Let $K_m:=\mathrm{Id}_r+\varpi^m\mathfrak{gl}_r(\mathcal{O})$ be the principal congruence subgroup of $\mathrm{GL}_r(F)$. We denote by $c_1(r)$ the scalar

$$c_1(r):=\mathrm{vol}(\varpi^m\mathcal{O})^{-\left[\frac{r^2}{4}\right]}.$$

Let ϕ be the product of the characteristic function of $w_{\mathrm{GL}_r}K_m\cap S_r(F)$ and the scalar $c_1(r)$. We have the following lemma:

Lemma 3.2. — *Let $\beta=\mathrm{diag}(z,z,\dots,z)$ with $z^r=1$ and ϕ as above. For m large enough, we then have*

$$I(w_{\mathrm{GL}_r}\beta,\phi)=\begin{cases}1, & \text{if } z=1,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. — Firstly, we calculate the integral $I(w_{\mathrm{GL}_r},\phi)$. After a unimodular change of variables, this integral has the form

$$I(w_{\mathrm{GL}_r},\phi)=\int\phi(x)\psi\left(\frac{\sum_{i+j=r+2}x_{i,j}}{2}\right)\otimes dx_{i,j},$$

where $x=(x_{i,j})$ is a symmetric matrix such that

$$x_{i,j}=\begin{cases}0, & \text{if } i+j<r+1,\\ 1, & \text{if } i+j=r+1.\end{cases}$$

The variables are the entries $x_{i,j}\in F$ with $i+j\geq r+2$, $i<j$ and the entries $x_{i,i}\in F$ with $2i\geq r+2$.

The number of entries $x_{i,j}$ with $i+j \geq r+2$ is $\frac{r(r-1)}{2}$. The number of entries $x_{i,i}$ with $2i \geq r+2$ is $r - \left[\frac{r+3}{2}\right] + 1 = \left[\frac{r}{2}\right]$. So the number of variables of the above integral is $\frac{\frac{r(r-1)}{2} - \left[\frac{r}{2}\right]}{2} + \left[\frac{r}{2}\right] = \left[\frac{r^2}{4}\right]$.

Now, with ϕ the product of the characteristic function of $w_{\mathrm{GL}_r} K_m$ and the scalar $c_1(r)$, this integral is equal to

$$c_1(r) \int \psi\left(\frac{\sum_{i+j=r+2} x_{i,j}}{2}\right) \otimes dx_{i,j}$$

integrated over the domain

$$x_{i,j} \equiv 0 \mod \varpi^m \mathcal{O} \text{ for } i+j \geq r+2.$$

Moreover, since ψ is of order 0, we have $\psi\left(\frac{\sum x_{i,j}}{2}\right) = 1$. This implies that

$$\int \psi\left(\frac{\sum_{i+j=r+2} x_{i,j}}{2}\right) \otimes dx_{i,j} = \mathrm{vol}(\varpi^m \mathcal{O})^{\left[\frac{r^2}{4}\right]} = c_1(r)^{-1}.$$

In consequence, the first assertion is proved.

We choose m large enough, such that $z \notin 1 + \varpi^m \mathcal{O}$, for all z that satisfy $z^r = 1$ and $z \neq 1$. We then have ${}^t n w_{\mathrm{GL}_r} \beta n \notin w_{\mathrm{GL}_r} K_m, \forall n \in N_r(F)$. The second assertion follows. □

As a consequence, we obtain

$$L_{w_M}^{w_{\mathrm{GL}_r}}(\alpha) = I(w_M \alpha, \phi), \tag{1}$$

where

$$\alpha = \mathrm{diag}(a, \ldots, a, a^{1-r} \det(w_M w_{\mathrm{GL}_r})),$$

and $|a|$ small enough.

Our goal of this section is to compute the germ $L_{w_M}^{w_{\mathrm{GL}_r}}$ or, which amounts to the same thing, the integral $I(w_M \alpha, \phi)$ where ϕ is the function defined above.

Let P be the parabolic subgroup of type $(r-1, 1)$ and N_M its unipotent radical. Then, $P = MN_M$. We denote by $V_M = N_r \cap M$ and $V_M^1 = \{v \in V_M | w_M {}^t v w_M = v\}$. If $u \in V_M$, then $u = (u_{i,j})$ is of the following form:

$$u_{i,j} = 0 \text{ for } (i,j) \notin \{(i,i), (i,r) | 1 \leq i \leq r\},$$
$$u_{i,i} = 1 \text{ for } 1 \leq i \leq r.$$

If $v \in V_M^1$, then $v = (v_{i,j})$ is of the following form:

$$v_{i,r} = 0 \text{ for } 1 \leq i \leq r-1,$$
$$v_{i,i} = 1 \text{ for } 1 \leq i \leq r,$$
$$v_{i,j} = 0 \text{ for } i > j,$$
$$v_{i,j} = v_{(r-j),(r-i)} \text{ for } 1 \leq i < j \leq r-1.$$

The orbital integral $I(w_M\alpha,\phi)$ can be written as follows:

$$I(w_M\alpha,\phi)=\int\phi({}^t uw\mathbf{t}vu)\theta(u^2v)(\otimes_{i=1}^{r-1}du_{i,r})\otimes dv_{i,j}. \tag{2}$$

The variables are the entries $u_{i,r}\in F$ with $1\leq i\leq r-1$, the entries $v_{i,r-i}\in F$ with $1\leq i<r-i$, and the entries $v_{i,j}\in F$ with $1\leq i<j\leq r-1$, $i\leq r-j$. Denote by $c_2(r)$ the number of variables of above the integral. We then have

$$c_2(r)=(r-1)+\left[\frac{r-1}{2}\right]+\frac{1}{2}\left(\frac{(r-2)(r-1)}{2}-\left[\frac{r-1}{2}\right]\right)=\left[\frac{r^2+2r-3}{4}\right].$$

After a unimodular change of variables, (2) can be written as

$$I(w_M\alpha,\phi)=|a|^{-c_2(r)}\int\phi(x)\psi\left(\frac{\sum_{i=1}^r x_{i,r+1-i}}{2a}\right)\otimes dx_{i,j},$$

where $x=(x_{i,j})$ denotes a matrix of the following form:

$$\begin{aligned} x_{i,j}&=0 \text{ for } i+j<r,\\ x_{i,j}&=a \text{ for } i+j=r,\\ x_{i,j}&=x_{j,i}.\end{aligned}$$

The variables are the entries $x_{i,j}\in F$ with $i+j\geq r+1$, $i<j$, the entries $x_{i,i}\in F$ with $2i\geq r+1$, the entry $x_{r,r}$ is a dependent variable. The entry $x_{r,r}$ can be computed (from the condition that the determinant of the matrix x be $\det(w_{\mathrm{GL}_r})$) by

$$a^{r-1}\det(w_M)x_{r,r}+\det\begin{pmatrix} 0 & \cdots & 0 & a & x_{1,r}\\ \vdots & \iddots & \iddots & \iddots & \vdots\\ 0 & \iddots & \iddots & x_{r-2,r-1} & x_{r-2,r}\\ a & \iddots & x_{r-1,r-2} & x_{r-1,r-1} & x_{r-1,r}\\ x_{r,1} & \cdots & x_{r,r-2} & x_{r,r-1} & 0\end{pmatrix}=\det(w_{\mathrm{GL}_r}).$$

Let $\mathcal{I}$ be a subset of $\{(i,j)|1\leq i\leq j\leq r\}$. We denote by $x^{\mathcal{I}}$ the matrix obtained from x by replacing the entries $x_{i,j}$ and $x_{j,i}$ by 0 with $(i,j)\in\mathcal{I}$. For instance, the above condition of $x_{r,r}$ can be written as follows:

$$a^{r-1}\det(w_M)x_{r,r}+\det(x^{\{(r,r)\}})=\det(w_{\mathrm{GL}_r}).$$

Since ϕ is the product of the characteristic function of $w_{\mathrm{GL}_r}K_m$ and the scalar $c_1(r)$, the above integral is equal to

$$|a|^{-c_2(r)}c_1(r)\int\psi\left(\frac{\sum_{i=1}^r x_{i,r+1-i}}{2a}\right)\otimes dx_{i,j}$$

integrated over the set:

$$\begin{aligned} x_{i,j} &\equiv 1 \mod \varpi^m\mathcal{O} \text{ for } i+j=r+1, \\ x_{i,j} &\equiv 0 \mod \varpi^m\mathcal{O} \text{ for } i+j>r+1,\ j>i,\ (i,j)\neq(r,r), \\ x_{r,r} &\equiv 0 \mod \varpi^m\mathcal{O}. \end{aligned}$$

The last condition can be written as follows:

$$\det(x^{\{(r,r)\}}) \equiv \det(w_{\mathrm{GL}_r}) \mod a^{r-1}\varpi^m\mathcal{O}.$$

By singling out the variable $x_{2,r}$ $(= x_{r,2})$ from the left-hand side, we have

(3) $$2x_{2,r}ay + x_{2,r}^2a^2z + \det(x^{\{(r,r),(2,r)\}}) \equiv \det(w_{\mathrm{GL}_r}) \mod a^{r-1}\varpi^m\mathcal{O},$$

where $y,z\in 1+\varpi^m\mathcal{O}$. Both y and z depend only on the variables $x_{i,j}$ with $(i,j)\neq(2,r)$.

We denote $T := \det(x^{\{(r,r),(2,r)\}}) - \det(w_{\mathrm{GL}_r})$. Since $x_{2,r}\equiv 0 \bmod \varpi^m\mathcal{O}$, we have $T\equiv 0 \bmod a\varpi^m\mathcal{O}$. We then have $y^2-zT\in 1+\varpi^m\mathcal{O}$. This implies that y^2-zT has a square root in F for m large enough. Assume that μ is a square root of y^2-zT. Since $y^2-zT\in 1+\varpi^m\mathcal{O}$, we have $(\mu-1)(\mu+1)\in\varpi^m\mathcal{O}$. This implies that $\max\{v(\mu-1),v(\mu+1)\}\geq\frac{m}{2}$. We can suppose that $v(\mu-1)=\max\{v(\mu-1),v(\mu+1)\}$ (otherwise, we change μ to $-\mu$). We get then $\mu=1+\varpi^{m/2}\mathcal{O}$ and $\mu+1=2+1+\varpi^{m/2}\mathcal{O}\in\mathcal{O}^*$. Thus, $v(\mu-1)\geq m$, i.e., $\mu\in 1+\varpi^m\mathcal{O}$. We have proved that there exists a square root μ of y^2-zT, such that $\mu\in 1+\varpi^m\mathcal{O}$.

The condition (3) can read

$$a^2z\left(x_{2,r}-\frac{-y-\mu}{az}\right)\left(x_{2,r}-\frac{-y+\mu}{az}\right)\equiv 0 \mod a^{r-1}\varpi^m\mathcal{O}.$$

For $|a|$ small enough (i.e., the valuation of a large enough), from the definition of y,z and T, we have $\frac{-y+\mu}{az}\in a^{-1}\varpi^m\mathcal{O}$ and $\frac{-y-\mu}{az}\in -2a^{-1}+a^{-1}\varpi^m\mathcal{O}$. Thus, the above condition is equivalent to

$$x_{2,r}\equiv\frac{-y+\mu}{az} \mod a^{r-2}\varpi^m\mathcal{O}.$$

Using this condition, we can integrate the variable $x_{2,r}$ away from the orbital integral I to obtain the scalar factor $|a|^{r-2}\mathrm{vol}(\varpi^m\mathcal{O})$ multiple by a new integral. The new integral has the same form as the old one, but the domain of integration is defined by

$$\begin{aligned} x_{i,j} &\equiv 1 \mod \varpi^m\mathcal{O} \text{ for } i+j=r+1, \\ x_{i,j} &\equiv 0 \mod \varpi^m\mathcal{O} \text{ for } i+j>r+1,\ (i,j)\neq(r,r),(2,r) \\ T &\equiv 0 \mod a\varpi^m\mathcal{O}. \end{aligned}$$

The determinant of $x^{\{(r,r),(2,r)\}}$ has the form

$$\prod_{i+j=r+1} x_{i,j}\det(w_{\mathrm{GL}_r}) + aX$$

with $X \in \varpi^m \mathcal{O}$. Thus, the condition on T can read

$$\prod_{i+j=r+1} x_{i,j} \equiv 1 \mod a\varpi^m \mathcal{O}.$$

Now, we integrate over the variables $x_{i,j}$ with $i+j > r+1$, $(i,j) \neq (r,r), (2,r)$ and get

$$I(w_M \alpha, \phi) = |a|^{r-2-c_2(r)} c_1(r) \mathrm{vol}(\varpi^m \mathcal{O})^{c_2(r)-\left[\frac{r+1}{2}\right]+1} I(a,r), \tag{4}$$

where the function $I(a,r)$ is defined as follows.

If $r = 2\ell$, then

$$I(a,r) \coloneqq \mathrm{vol}(\varpi^m \mathcal{O})^{-1} \int \psi \left(\frac{x_1 + x_2 + \cdots + x_\ell}{a} \right) \otimes dx_i.$$

The domain of integration is defined by

$$\begin{aligned} x_i &\equiv 1 \mod \varpi^m \mathcal{O} \\ x_1^2 x_2^2 \dots x_\ell^2 &\equiv 1 \mod a\varpi^m \mathcal{O}. \end{aligned}$$

If $r = 2\ell + 1$, then

$$I(a,r) \coloneqq \mathrm{vol}(\varpi^m \mathcal{O})^{-1} \int \psi \left(\frac{2x_1 + 2x_2 + \cdots + 2x_\ell + x_{\ell+1}}{2a} \right) \otimes dx_i.$$

The domain of integration is defined by

$$\begin{aligned} x_i &\equiv 1 \mod \varpi^m \mathcal{O} \\ x_1^2 x_2^2 \dots x_\ell^2 x_{\ell+1} &\equiv 1 \mod a\varpi^m \mathcal{O}. \end{aligned}$$

We have:

Proposition 3.3. — *Suppose that the residual characteristic of F is larger than $2r+1$. Then, if m is large enough,*

$$I(a,r) = |a|^{\frac{1}{2}\left[\frac{r-1}{2}\right]+1} \psi\left(\frac{r}{2a}\right) \left(\frac{r}{2^{r-1}}, a^{-1}\right) \gamma\left(a^{-1}, \psi\right)^{\left[\frac{r-1}{2}\right]},$$

for $|a|$ sufficiently small.

Proof. — Firstly, we consider the case $r = 2\ell+1$. We change variables and set

$$x_{\ell+1} = \left(\prod_{i=1}^{\ell} x_i^{-2} \right) t$$

with $t \in 1 + a\varpi^m \mathcal{O}$ and then integrate over t. We obtain

$$I(a, 2\ell+1) = |a| \int \psi \left(\frac{\delta}{2a} \right) \otimes dx_i,$$

where the phase function δ is given by

$$\delta = \sum_{i=1}^{\ell} 2x_i + \frac{1}{\prod_{i=1}^{\ell} x_i^2}.$$

We set $x_i = 1 + u_i$ with $u_i \in \varpi^m \mathcal{O}$. This function can be written as

$$\delta = 2\ell + \sum_{i=1}^{\ell} 2u_i + \prod_{i=1}^{\ell} \frac{1}{1 + 2u_i + u_i^2}.$$

We now consider the Taylor expansion of this function at the origin. It has the form

$$(2\ell + 1) + 3\sum_{i=1}^{\ell} u_i^2 + 4 \sum_{1 \leq i < j \leq \ell} u_i u_j + \text{higher degree terms.}$$

Since the quadratic form

$$3\sum_{i=1}^{\ell} X_i^2 + 4 \sum_{1 \leq i < j \leq \ell} X_i X_j$$

is equivalent to the quadratic form

$$\sum_{i=1}^{\ell} \frac{2i+1}{2i-1} Y_i^2$$

by a unipotent transformation (see [6, Lemma 5.1]), this Taylor expansion may be written in the form (after a unimodular change of coordinates)

$$\delta = (2\ell + 1) + \sum_{i=1}^{\ell} \frac{2i+1}{2i-1} y_i^2 + \text{higher degree terms.}$$

We choose m large enough, such that the origin is the only critical point in the domain of integration. Using the principle of stationary phase, there exists a neighborhood $0 \in \Omega$ in F, such that for $|a|$ small enough $I(a, 2\ell+1)$ is the product of the factors

$$|a|\psi\left(\frac{2\ell+1}{2a}\right),$$

$$\int_{\Omega} \psi\left(\frac{2i+1}{2(2i-1)a} y_i^2\right) dy_i = \left|\frac{2i+1}{(2i-1)a}\right|^{-1/2} \gamma\left(\frac{2i+1}{(2i-1)a}, \psi\right), \quad 1 \leq i \leq \ell.$$

Using the property of the Weil constant that $\gamma(bc, \psi) = \gamma(b, \psi)\gamma(c, \psi)(b, c)$, we have

$$\gamma\left(\frac{2i+1}{(2i-1)a}, \psi\right) = \gamma\left(\frac{2i+1}{2i-1}, \psi\right) \gamma(a^{-1}, \psi) \left(\frac{2i+1}{2i-1}, a^{-1}\right).$$

Since the residual characteristic of F is larger than $2r+1$, we have $\frac{2i+1}{2i-1} \in \mathcal{O}^*$ for all $1 \leq i \leq \ell$. So the above equation simplifies to

$$\gamma\left(\frac{2i+1}{(2i-1)a}, \psi\right) = \gamma(a^{-1}, \psi)\left(\frac{2i+1}{2i-1}, a^{-1}\right).$$

Using the formula $(b,c)(b',c) = (bb',c)$ and $\frac{2i+1}{2i-1} \in \mathcal{O}^*$, for all $1 \leq i \leq \ell$, we have

$$\begin{aligned} I(a, 2\ell+1) &= |a|^{\frac{\ell}{2}+1}\psi\left(\frac{2\ell+1}{2a}\right)(2\ell+1, a^{-1})\gamma(a^{-1},\psi)^{\ell} \\ &= |a|^{\frac{\ell}{2}+1}\psi\left(\frac{2\ell+1}{2a}\right)(2\ell+1, a^{-1})\gamma(a^{-1},\psi)^{\ell}\left(\frac{1}{2}, a^{-1}\right)^{2\ell} \\ &= |a|^{\frac{\ell}{2}+1}\psi\left(\frac{2\ell+1}{2a}\right)\left(\frac{2\ell+1}{2^{2\ell}}, a^{-1}\right)\gamma(a^{-1},\psi)^{\ell}. \end{aligned}$$

For the case $r = 2\ell$. We set

$$x_\ell = t\left(\prod_{i=1}^{\ell-1} x_i\right)^{-1}.$$

Since $\prod_{i=1}^{\ell} x_i^2 \equiv 1 \bmod a\varpi^m\mathcal{O}$, we have $t^2 \equiv 1 \bmod a\varpi^m\mathcal{O}$. For m large enough and $|a|$ small enough, this condition is equivalent to $t \equiv \pm 1 \bmod a\varpi^m\mathcal{O}$. Moreover $t = \prod_{i=1}^{\ell} x_i \equiv 1 \bmod \varpi^m\mathcal{O}$, so $t \equiv 1 \bmod a\varpi^m\mathcal{O}$. We now integrate over t to get

$$I(a, 2\ell) = |a| \int \psi\left(\frac{\delta}{a}\right) \otimes dx_i,$$

where the phase function δ is given by

$$\delta = \sum_{i=1}^{\ell-1} x_i + \prod_{i=1}^{\ell-1} \frac{1}{x_i}.$$

We set $x_i = 1 + u_i$ with $u_i \in \varpi^m\mathcal{O}$. This function can be written as

$$\delta = \ell - 1 + \sum_{i=1}^{\ell-1} u_i + \prod_{i=1}^{\ell-1} \frac{1}{1+u_i}.$$

The Taylor expansion of this function at the origin has the form

$$\ell + \sum_{i=1}^{\ell-1} u_i^2 + \sum_{1\leq i<j\leq \ell-1} u_i u_j + \text{ higher degree terms.}$$

Since the quadratic form

$$\sum_{i=1}^{\ell-1} X_i^2 + \sum_{1\leq i<j\leq \ell-1} X_i X_j$$

is equivalent to the quadratic form

$$\sum_{i=1}^{\ell-1} \frac{i+1}{2i} Y_i^2$$

by a unipotent transformation (see [6, Lemma 2.1]), this Taylor expansion may be written in the form (after a unimodular change of coordinates)

$$\delta = \ell + \sum_{i=1}^{\ell-1} \frac{i+1}{2i} y_i^2 + \text{higher degree terms}.$$

We choose m large enough, such that the origin is the only critical point in the domain of integration. Using the principle of stationary phase, there exists a neighborhood $0 \in \Omega$ in F, such that for $|a|$ small enough, $I(a, 2\ell)$ is the product of the factors

$$|a|\psi\left(\frac{\ell}{a}\right),$$

$$\int_\Omega \psi\left(\frac{i+1}{2ai} y_i^2\right) dy_i = \left|\frac{i+1}{ai}\right|^{-1/2} \gamma\left(\frac{i+1}{ai}, \psi\right), \ 1 \leq i \leq \ell - 1.$$

Using the property of the Weil constant that $\gamma(bc, \psi) = \gamma(b, \psi)\gamma(c, \psi)(b, c)$ we have

$$\gamma\left(\frac{i+1}{ai}, \psi\right) = \gamma\left(\frac{i+1}{i}, \psi\right) \gamma(a^{-1}, \psi) \left(\frac{i+1}{i}, a^{-1}\right).$$

Since the residual characteristic of F is larger than $2r+1$, we have $\frac{i+1}{i} \in \mathcal{O}^*$ for all $1 \leq i \leq \ell - 1$. So the above equation simplifies to

$$\gamma\left(\frac{i+1}{ai}, \psi\right) = \gamma(a^{-1}, \psi) \left(\frac{i+1}{i}, a^{-1}\right).$$

Using the formula $(b, c)(b', c) = (bb', c)$ and $\frac{i+1}{i} \in \mathcal{O}^*$, for all $1 \leq i \leq \ell - 1$, we have

$$\begin{aligned} I(a, 2\ell) &= |a|^{\frac{\ell+1}{2}} \psi\left(\frac{\ell}{a}\right) (\ell, a^{-1})\gamma(a^{-1}, \psi)^{\ell-1} \\ &= |a|^{\frac{\ell+1}{2}} \psi\left(\frac{\ell}{a}\right) (\ell, a^{-1})\gamma(a^{-1}, \psi)^{\ell-1} \left(\frac{1}{2}, a^{-1}\right)^{2\ell-2} \\ &= |a|^{\frac{\ell-1}{2}+1} \psi\left(\frac{2\ell}{2a}\right) \left(\frac{2\ell}{2^{2\ell-1}}, a^{-1}\right) \gamma(a^{-1}, \psi)^{\ell-1}. \end{aligned}$$

In summary, we get

$$I(a, r) = |a|^{\frac{1}{2}\left[\frac{r-1}{2}\right]+1} \psi\left(\frac{r}{2a}\right) \left(\frac{r}{2^{r-1}}, a^{-1}\right) \gamma\left(a^{-1}, \psi\right)^{\left[\frac{r-1}{2}\right]}. \qquad \square$$

Combining (1) and (4) and Propositions 3.3 and 2.2 we obtain the following proposition:

PROPOSITION 3.4. — *Suppose that the residual characteristic of F is larger than $2r+1$. For*

$$\alpha = \mathrm{diag}(a, a, \ldots, a^{1-r}\det(w_M)\det(w_{\mathrm{GL}_r})$$

and $|a|$ is small enough,

$$L_{w_M}^{w_{\mathrm{GL}_r}}(\alpha) = |a|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\gamma(a^{-1},\psi)^{-\left[\frac{r}{2}\right]}K_{w_M}^{w_{\mathrm{GL}_r}}(\alpha).$$

Note that to simplify the formula, we used the following identities:

$$-\left[\frac{r^2}{4}\right] + \left[\frac{r^2+2r-3}{4}\right] - \left[\frac{r+1}{2}\right] + 1 = 0$$

and

$$(r-2) - \left[\frac{r^2+2r-3}{4}\right] + \frac{1}{2}\left[\frac{r-1}{2}\right] + 1 + \frac{(r-1)^2}{2} = \left[\frac{r^2}{4}\right] - \frac{1}{2}\left[\frac{r}{2}\right].$$

4. Proof of the main theorem

We shall prove Theorem 1.4 by induction on r. It is trivial when $r=1$. We suppose that it holds for $1 \leq r' < r$.

Firstly, we consider $W_r^R \ni w \neq w_{\mathrm{GL}_r}$. The relevant element $w\mathbf{t}$ then has the following form

$$w\mathbf{t} = \begin{pmatrix} w_1\mathbf{t}_1 & 0 \\ 0 & w_2\mathbf{t}_2 \end{pmatrix},$$

with $w_i\mathbf{t}_i$ the relevant element of GL_{r_i}. For convenience, we shall introduce some intermediate orbital integrals.

On the side J, for a function $f \in \mathcal{C}_c^\infty(\mathrm{GL}_{r_1+r_2}(F))$, we define the intermediate integral

$$\begin{aligned} &J_{r_2}^{r_1}\left[\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, f\right] \\ &:= \int f\left[w_{\mathrm{GL}_{r_1+r_2}}\begin{pmatrix} \mathrm{Id}_{r_1} & \\ {}^tX & \mathrm{Id}_{r_2} \end{pmatrix}\begin{pmatrix} w_{\mathrm{GL}_{r_1}}g_1 & \\ & w_{\mathrm{GL}_{r_2}}g_2 \end{pmatrix}\begin{pmatrix} \mathrm{Id}_{r_1} & Y \\ & \mathrm{Id}_{r_2} \end{pmatrix}\right] \\ &\qquad \times \theta\left[\begin{pmatrix} \mathrm{Id}_{r_1} & X+Y \\ & \mathrm{Id}_{r_2} \end{pmatrix}\right] dXdY \\ &= \int f\left[w_{\mathrm{GL}_{r_1+r_2}}\begin{pmatrix} A_{r_1} & A_{r_1}Y \\ {}^tXA_{r_1} & {}^tXA_{r_1}Y + B_{r_2} \end{pmatrix}\right]\theta\left[\begin{pmatrix} \mathrm{Id}_{r_1} & X+Y \\ & \mathrm{Id}_{r_2} \end{pmatrix}\right] dXdY, \end{aligned}$$

where $A_{r_1} := w_{\mathrm{GL}_{r_1}}g_1 \in \mathrm{GL}_{r_1}(F)$, $B_{r_2} := w_{\mathrm{GL}_{r_2}}g_2 \in \mathrm{GL}_{r_2}(F)$, and the domain of integration is $M_{r_1\times r_2}(F)$ – the set of matrices of size $r_1 \times r_2$.

Fixing $f \in \mathcal{C}_c^\infty(\mathrm{GL}_{r_1+r_2}(F))$, the function $\mathfrak{J}_{r_2}^{r_1}(g_1, g_2) := J_{r_2}^{r_1}\left[\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, f\right]$ is a smooth function of support compact on $\mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F)$.

Associated with the action of $(N_{r_1}\times N_{r_1})\times(N_{r_2}\times N_{r_2})$ on $\mathrm{GL}_{r_1}\times\mathrm{GL}_{r_2}$ we consider the double orbital integral

$$J(w_1\mathbf{t}_1,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2}) \\ := \iint \mathfrak{I}^{r_1}_{r_2}(n_1^{-1}w_{\mathrm{GL}_{r_1}}w_1\mathbf{t}_1n_1',n_2^{-1}w_{\mathrm{GL}_{r_2}}w_2\mathbf{t}_2n_2')\theta(n_1n_1')dn_1dn_1'\theta(n_2n_2')dn_2dn_2',$$

where the (n_i,n_i') is integrated over $N_{r_i}\times N_{r_i}$ divided by the stabilizer of $w_{\mathrm{GL}_{r_i}}w_i\mathbf{t}_i$. We then have

$$J\left(\begin{pmatrix} w_1\mathbf{t}_1 & \\ & w_2\mathbf{t}_2\end{pmatrix},f\right)=J(w_1\mathbf{t}_1,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2}). \tag{5}$$

We can also define the partial orbital integrals $J_1(w_1\mathbf{t}_1,g_2;\mathfrak{I}^{r_1}_{r_2})$ and $J_2(g_1,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2})$. For example,

$$J_1(w_1\mathbf{t}_1,g_2;\mathfrak{I}^{r_1}_{r_2}):=\int \mathfrak{I}^{r_1}_{r_2}(n_1^{-1}w_{\mathrm{GL}_{r_1}}w_1\mathbf{t}_1n_1',g_2)\theta(n_1n_1')dn_1dn_1',$$

where the (n_1,n_1') is integrated over $N_{r_1}\times N_{r_1}$ divided by the stabilizer of $w_{\mathrm{GL}_{r_1}}w_1\mathbf{t}_1$. If we fix $w_1\mathbf{t}_1$, this integral defines a smooth function of compact support on $\mathrm{GL}_{r_2}(F)$. Moreover, we have

$$J(w_1\mathbf{t}_1,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2})=J(w_2\mathbf{t}_2,J_1(w_1\mathbf{t}_1,\cdot\,;\mathfrak{I}^{r_1}_{r_2}))$$

and

$$J(w_1\mathbf{t}_1,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2})=J(w_1\mathbf{t}_1,J_2(\,\cdot\,,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2})).$$

On the side I, for a function $\phi\in\mathcal{C}_c^\infty(\mathrm{GL}_{r_1+r_2}(F))$, we define the intermediate integral

$$\begin{aligned} &I^{r_1}_{r_2}\left[\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix},\phi\right] \\ &:=\int\phi\left[\begin{pmatrix}\mathrm{Id}_{r_1} & \\ {}^tX & \mathrm{Id}_{r_2}\end{pmatrix}\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix}\begin{pmatrix}\mathrm{Id}_{r_1} & X\\ & \mathrm{Id}_{r_2}\end{pmatrix}\right]\theta\left[\begin{pmatrix}\mathrm{Id}_{r_1} & 2X\\ & \mathrm{Id}_{r_2}\end{pmatrix}\right]dX \\ &=\int\phi\left[\begin{pmatrix} g_1 & g_1X\\ {}^tXg_1 & {}^tXg_1X+g_2\end{pmatrix}\right]\theta\left[\begin{pmatrix}\mathrm{Id}_{r_1} & 2X\\ & \mathrm{Id}_{r_2}\end{pmatrix}\right]dX,\end{aligned}$$

where $g_i\in\mathrm{GL}_{r_i}(F)$ and the domain of integration is $M_{r_1\times r_2}(F)$ – the set of matrices of size $r_1\times r_2$.

Fixing $\phi\in\mathcal{C}_c^\infty(\mathrm{GL}_{r_1+r_2}(F))$, the function $\mathfrak{I}^{r_1}_{r_2}(g_1,g_2):=I^{r_1}_{r_2}\left[\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix},\phi\right]$ is a smooth function of support compact on $\mathrm{GL}_{r_1}(F)\times\mathrm{GL}_{r_2}(F)$.

Associated with the action of $N_{r_1}\times N_{r_2}$ on $\mathrm{GL}_{r_1}\times\mathrm{GL}_{r_2}$ we consider the double orbital integral

$$I(w_1\mathbf{t}_1,w_2\mathbf{t}_2;\mathfrak{I}^{r_1}_{r_2}):=\int\mathfrak{I}^{r_1}_{r_2}({}^tn_1w_1\mathbf{t}_1n_1,{}^tn_2w_2\mathbf{t}_2n_2)\theta_{r_1}(n_1)dn_1\theta_{r_2}(n_2)dn_2,$$

where the n_i is integrated over N_{r_i} divided by the stabilizer of $w_i\mathbf{t}_i$. We then have

$$I\left(\begin{pmatrix} w_1\mathbf{t}_1 & \\ & w_2\mathbf{t}_2\end{pmatrix}, \phi\right) = I(w_1\mathbf{t}_1, w_2\mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2}). \tag{6}$$

We can also define the partial orbital integrals $I_1(w_1\mathbf{t}_1, g_2; \mathfrak{I}^{r_1}_{r_2})$ and $I_2(g_1, w_2\mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})$. For example,

$$I_1(w_1\mathbf{t}_1, g_2; \mathfrak{I}^{r_1}_{r_2}) := \int \mathfrak{I}^{r_1}_{r_2}({}^t n_1 w_1\mathbf{t}_1 n_1, g_2)\theta_{r_1}(n_1)dn_1,$$

where the n_1 is integrated over N_{r_1} divided by the stabilizer of $w_1\mathbf{t}_1$. If we fix $w_1\mathbf{t}_1$, this integral defines a smooth function of compact support on $\mathrm{GL}_{r_2}(F)$. Moreover, we have

$$I(w_1\mathbf{t}_1, w_2\mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2}) = I(w_2\mathbf{t}_2, I_1(w_1\mathbf{t}_1, \cdot\,; \mathfrak{I}^{r_1}_{r_2}))$$

and

$$I(w_1\mathbf{t}_1, w_2\mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2}) = I(w_1\mathbf{t}_1, I_2(\,\cdot\,, w_2\mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})).$$

Now we can continue with the induction argument. If $w_i = \mathrm{Id}_{r_i}$, then $w = \mathrm{Id}_r$. Suppose that $J(\mathrm{diag}(\mathbf{t_1}, \mathbf{t_2}), f) = \Delta_r(\mathrm{diag}(\mathbf{t}_1, \mathbf{t}_2))I(\mathrm{diag}(\mathbf{t_1}, \mathbf{t_2}), \phi)$. Using the identities (5) and (6) we have

$$J(\mathbf{t}_1, \mathbf{t}_2; \mathfrak{J}^{r_1}_{r_2}) = \Delta_r(\mathrm{diag}(\mathbf{t}_1, \mathbf{t}_2))I(\mathbf{t}_1, \mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})$$

Using the relation between the double integrals and the partial integral, this implies

$$J(\mathbf{t}_1, J_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{J}^{r_1}_{r_2})) = \Delta_r(\mathrm{diag}(\mathbf{t}_1, \mathbf{t}_2))I(\mathbf{t}_1, I_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})) \tag{7}$$

- If $r \equiv r_1 \bmod 2$ (it is equivalent to $r_2 \equiv 0 \bmod 2$), we have

  $$\frac{\Delta_r(\mathrm{diag}(\mathbf{t}_1, \mathbf{t}_2))}{\Delta_{r_1}(\mathbf{t}_1)} = \zeta(-1)^{\frac{r_2}{2}v(\det(\mathbf{t}_1))}|\det(\mathbf{t}_1)|^{-\frac{r_2}{2}}\Delta_{r_2}(\mathbf{t}_2).$$

  For fixed $\mathbf{t}_2 \in T_{r_2}(F)$, we define

  $$I'_2(g_1, \mathbf{t_2}; \mathfrak{I}^{r_1}_{r_2}) = \zeta(-1)^{\frac{r_2}{2}v(\det(g_1))}|\det(g_1)|^{-\frac{r_2}{2}}\Delta_{r_2}(\mathbf{t}_2)I_2(g_1, \mathbf{t_2}; \mathfrak{I}^{r_1}_{r_2}).$$

  This is a smooth function of compact support on $\mathrm{GL}_{r_1}(F)$.

  The identity (7) is true for all $\mathbf{t}_i \in T_{r_i}$, so when we fix $\mathbf{t}_2 \in T_{r_2}(F)$, we obtain the matching relation over $\mathrm{GL}_{r_1}(F)$:

  $$J(\mathbf{t}_1, J_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{J}^{r_1}_{r_2})) = \Delta_{r_1}(\mathbf{t}_1)I(\mathbf{t}_1, I'_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})) \quad \forall\, \mathbf{t_1} \in T_{r_1}(F).$$

  By induction, there exists Δ_{w_1}, such that

  $$\begin{aligned} &J(w_1\mathbf{t}_1, J_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{J}^{r_1}_{r_2})) = \Delta_{w_1}(\mathbf{t}_1)I(w_1\mathbf{t}_1, I'_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})) \quad \forall\, \mathbf{t_1} \in T_{w_1}(F) \\ &= \Delta_{w_1}(\mathbf{t}_1)\zeta(-1)^{\frac{r_2}{2}v(\det(w_1\mathbf{t}_1))}|\det(w_1\mathbf{t}_1)|^{-\frac{r_2}{2}}\Delta_{r_2}(\mathbf{t}_2)I(w_1\mathbf{t}_1, I_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})) \\ &= \Delta_{r_2}(\mathbf{t}_2)\zeta(-1)^{\frac{r_2}{2}v(\det(\mathbf{t}_1))}|\det(\mathbf{t}_1)|^{-\frac{r_2}{2}}\Delta_{w_1}(\mathbf{t}_1)I(w_1\mathbf{t}_1, I_2(\,\cdot\,, \mathbf{t}_2; \mathfrak{I}^{r_1}_{r_2})). \end{aligned}$$

Reusing the relation between the double integrals and the partial integrals, we obtain the matching relation over $\mathrm{GL}_{r_2}(F)$

$$J(\mathbf{t_2}, J_1(w_1\mathbf{t}_1, \cdot\,; \mathfrak{I}^{r_1}_{r_2})) = \Delta_{r_2}(\mathbf{t}_2) I(\mathbf{t}_2, I'_1(w_1\mathbf{t}_1, \cdot\,; \mathfrak{I}^{r_1}_{r_2})) \quad \forall\, \mathbf{t_2} \in T_{r_2}(F),$$

where

$$I'_1(w_1\mathbf{t}_1, g_2; \mathfrak{I}^{r_1}_{r_2}) = \zeta(-1)^{\frac{r_2}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta_{w_1}(\mathbf{t}_1) I_1(w_1\mathbf{t}_1, g_2; \mathfrak{I}^{r_1}_{r_2}).$$

We should note that for fixed $w_1\mathbf{t_1}$, the function $I'_1(w_1\mathbf{t}_1, g_2; \mathfrak{I}^{r_1}_{r_2})$ is a smooth function of compact support over $\mathrm{GL}_{r_2}(F)$.

By induction, there exists Δ_{w_2} such that

$$\begin{aligned} &J(w_2\mathbf{t_2}, J_1(w_1\mathbf{t}_1, \cdot\,; \mathfrak{I}^{r_1}_{r_2})) = \Delta_{w_2}(\mathbf{t}_2) I(w_2\mathbf{t}_2, I'_1(w_1\mathbf{t}_1, \cdot\,; \mathfrak{I}^{r_1}_{r_2})) \quad \forall\, \mathbf{t_2} \in T_{w_2}(F) \\ &= \zeta(-1)^{\frac{r_2}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2) I(w_2\mathbf{t_2}, I_1(w_1\mathbf{t}_1, \cdot\,; \mathfrak{I}^{r_1}_{r_2})). \end{aligned}$$

We then have $\Delta_w(\mathbf{t}) = \zeta(-1)^{\frac{r_2}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2)$.

- If $r \not\equiv r_1 \bmod 2$ (it is equivalent to $r_2 \equiv 1 \bmod 2$), we then have

$$\frac{\Delta_r(\mathrm{diag}(\mathbf{t}_1, \mathbf{t}_2))}{\Delta'_{r_1}(\mathbf{t}_1)} = \zeta(-1)^{\frac{r_2-1}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta_{r_2}(\mathbf{t}_2).$$

By doing the same (as in the above case), we obtain

$$\Delta_w(\mathbf{t}) = \zeta(-1)^{\frac{r_2-1}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta'_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2).$$

We have proved the following proposition:

Proposition 4.1. — *Let* $w\mathbf{t} = \begin{pmatrix} w_1\mathbf{t}_1 & 0 \\ 0 & w_2\mathbf{t}_2 \end{pmatrix}$ *be a relevant element of* GL_r. *We can define*

$$\Delta_w(\mathbf{t}) = \begin{cases} \zeta(-1)^{\frac{r_2}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2) & \text{if } r_2 \equiv 0 \bmod 2 \\ \zeta(-1)^{\frac{r_2-1}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta'_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2) & \text{if } r_2 \not\equiv 0 \bmod 2 \end{cases},$$

which validates the assertions in Theorem 1.4.

Similarly, we can also obtain the following proposition:

Proposition 4.2. — *Let* $w\mathbf{t} = \begin{pmatrix} w_1\mathbf{t}_1 & 0 \\ 0 & w_2\mathbf{t}_2 \end{pmatrix}$ *be a relevant element of* GL_r. *We can define*

$$\Delta'_w(\mathbf{t}) = \begin{cases} \zeta(-1)^{\frac{r_2}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta'_{w_1}(\mathbf{t}_1)\Delta'_{w_2}(\mathbf{t}_2) & \text{if } r_2 \equiv 0 \bmod 2 \\ \zeta(-1)^{\frac{r_2+1}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_2}{2}} \Delta_{w_1}(\mathbf{t}_1)\Delta'_{w_2}(\mathbf{t}_2) & \text{if } r_2 \not\equiv 0 \bmod 2 \end{cases},$$

which validates the assertions in Theorem 1.4.

Corollary 4.3. — *Let M be the standard parabolic subgroup of type $(r-1,1)$ of GL_r. Let $\mathbf{t}=\mathrm{diag}(a,\dots,a,b)\in T_{w_M}$. We then have*

$$\Delta_{w_M}(\mathbf{t})=|a|^{-\frac{r-1}{2}}\Delta'_{w_{\mathrm{GL}_{r-1}}}(\mathrm{diag}(a,\dots,a))$$

and

$$\Delta'_{w_M}(\mathbf{t})=\zeta(-1)^{v(\det(\mathbf{t}))}|a|^{-\frac{r-1}{2}}\gamma(b,\psi)(b,\varpi)^{v(b)}\Delta_{w_{\mathrm{GL}_{r-1}}}(\mathrm{diag}(a,\dots,a)).$$

The remaining relevant elements are of the type $\Delta_{w_{\mathrm{GL}_r}}\mathbf{t}$. To work with this type of relevant element, we shall use the germ relations mentioned in Sections 2 and 3.

Let M be the standard parabolic subgroup of type $(r-1,1)$ of GL_r. Recall that we have

$$J(w_M\mathbf{t},f)=\omega_f(\mathbf{t})+\sum_{\alpha\beta=\mathbf{t}}K_{w_M}^{w_{\mathrm{GL}_r}}(\alpha)J(w_{\mathrm{GL}_r}\beta,f)$$

and

$$I(w_M\mathbf{t},\phi)=\omega_\phi(\mathbf{t})+\sum_{\alpha\beta=\mathbf{t}}L_{w_M}^{w_{\mathrm{GL}_r}}(\alpha)I(w_{\mathrm{GL}_r}\beta,\phi),$$

where w_f,w_ϕ are smooth functions of compact support, $\mathbf{t}\in T_{w_M}$, and the sums are over all pairs in $\mathfrak{S}:=\{(\alpha,\beta)\in(T_{w_M}^{w_{\mathrm{GL}_r}},T_{w_{\mathrm{GL}_r}})|\alpha\beta=\mathbf{t}\}$. Given $(\alpha,\beta)\in(T_{w_M}^{w_{\mathrm{GL}_r}},T_{w_{\mathrm{GL}_r}})$, then all pairs $(\alpha',\beta')\in(T_{w_M}^{w_{\mathrm{GL}_r}},T_{w_{\mathrm{GL}_r}})$ satisfying $\alpha'\beta'=\alpha\beta$ have the form $(z^{-1}\alpha,z\beta)$, with z an r-th root of unity.

Given $\beta\in T_{w_{\mathrm{GL}_r}}$, we choose $\alpha=\mathrm{diag}(a,\dots,a,a^{1-r}\det(w_Mw_{\mathrm{GL}_r}))$ with $|a|$ so small that $\omega_f(\alpha\beta)=\omega_\phi(\alpha\beta)=0$. We then get

$$J(w_M\alpha\beta,f)=\sum_{z|z^r=1}K_{w_M}^{w_{\mathrm{GL}_r}}(z^{-1}\alpha)J(w_{\mathrm{GL}_r}z\beta,f)$$

and

$$I(w_M\alpha\beta,\phi)=\sum_{z|z^r=1}L_{w_M}^{w_{\mathrm{GL}_r}}(z^{-1}\alpha)I(w_{\mathrm{GL}_r}z\beta,\phi).$$

Moreover, with $|a|$ small enough, using Proposition 3.4, we have

$$L_{w_M}^{w_{\mathrm{GL}_r}}(z^{-1}\alpha)=c(a,z).K_{w_M}^{w_{\mathrm{GL}_r}}(z^{-1}\alpha),$$

with $c(a,z):=|az^{-1}|^{[\frac{r^2}{4}]-\frac{1}{2}[\frac{r}{2}]}\gamma((az^{-1})^{-1},\psi)^{-[\frac{r}{2}]}$. Combining them with the matching relation of ϕ and f on the orbit of $w_M\alpha\beta$, we obtain

$$\sum_{z|z^r=1}K_{w_M}^{w_{\mathrm{GL}_r}}(z^{-1}\alpha)(J(w_{\mathrm{GL}_r}z\beta,f)-c(a,z).\Delta^{\cdot}_{w_M}(\alpha\beta)I(w_{\mathrm{GL}_r}z\beta,\phi))=0, \tag{8}$$

where $\Delta^{\cdot}_{w_M}(\alpha\beta)$ is $\Delta_{w_M}(\alpha\beta)$ or $\Delta'_{w_M}(\alpha\beta)$ depend on the matching relation.

We hope that this condition implies

$$J(w_{\mathrm{GL}_r}z\beta,f)-c(a,z).\Delta^{\cdot}_{w_M}(\alpha\beta)I(w_{\mathrm{GL}_r}z\beta,\phi)=0$$

for all z satisfied $z^r = 1$. In particularly, we have

$$J(w_{\mathrm{GL}_r}\beta, f) = c(a,1).\Delta^{\cdot}_{w_M}(\alpha\beta) I(w_{\mathrm{GL}_r}\beta, \phi).$$

For fixed β, f, and ϕ, the two integrals I and J do not depend on the choice of a, so we can require that $v(a)$ is even. With this addition condition, we define that

$$\Delta^{\cdot}_{w_{\mathrm{GL}_r}}(\beta) := c(a,1).\Delta^{\cdot}_{w_M}(\alpha\beta). \tag{9}$$

Now we shall prove that this is well defined (i.e., the definition does not depend on a). More precisely, we prove the following proposition:

Proposition 4.4. — *Let $\beta \in F^\times$. Viewing β as an element of $T_{w_{\mathrm{GL}_r}}(F)$; we then have*

$$\Delta_{w_{\mathrm{GL}_r}}(\beta) = \zeta(-1)^{[\frac{r}{2}][\frac{r+1}{2}]v(\beta)} |\beta|^{\frac{1}{2}[\frac{r}{2}]-[\frac{r^2}{4}]} \gamma(\beta,\psi)^{[\frac{r}{2}]} (\beta,\varpi)^{[\frac{r}{2}]v(\beta)}$$

and

$$\Delta'_{w_{\mathrm{GL}_r}}(\beta) = \zeta(-1)^{[\frac{r+1}{2}][\frac{r+2}{2}]v(\beta)} |\beta|^{\frac{1}{2}[\frac{r}{2}]-[\frac{r^2}{4}]} \gamma(\beta,\psi)^{[\frac{r+1}{2}]} (\beta,\varpi)^{[\frac{r+1}{2}]v(\beta)}.$$

Proof. — We shall prove this proposition by induction on r.

- For $r = 1$, it is trivial. In this case, we have

  $$\Delta_{w_{\mathrm{GL}_1}}(\beta) = \Delta_1(\beta) = 1$$

  and

  $$\Delta'_{w_{\mathrm{GL}_1}}(\beta) = \Delta'_1(\beta) = \zeta(-1)^{v(\beta)}\gamma(\beta,\psi)(\beta,\varpi)^{v(\beta)}.$$

- Assume that this proposition holds for $r-1$. Combining equation (9) and the formula in Corollary 4.3 we have

$$\begin{aligned}
\Delta_{w_{\mathrm{GL}_r}}(\beta) &= |a|^{[\frac{r^2}{4}]-\frac{1}{2}[\frac{r}{2}]} \gamma(a^{-1},\psi)^{-[\frac{r}{2}]} \Delta_M(\alpha\beta) \\
&= |a|^{[\frac{r^2}{4}]-\frac{1}{2}[\frac{r}{2}]} \Delta_M(\alpha\beta) \quad (\text{since } v(a) \text{ is even then } \gamma(a^{-1},\psi) = 1) \\
&= |a|^{[\frac{r^2}{4}]-\frac{1}{2}[\frac{r}{2}]} |a\beta|^{-\frac{r-1}{2}} \Delta'_{w_{\mathrm{GL}_{r-1}}}(\mathrm{diag}(a\beta,\dots,a\beta)) \\
&= |a|^{[\frac{r^2}{4}]-\frac{1}{2}[\frac{r}{2}]} |a\beta|^{-\frac{r-1}{2}} \\
&\quad \times \zeta(-1)^{[\frac{r}{2}][\frac{r+1}{2}]v(a\beta)} |a\beta|^{\frac{1}{2}[\frac{r-1}{2}]-[\frac{(r-1)^2}{4}]} \gamma(a\beta,\psi)^{[\frac{r}{2}]} (a\beta,\varpi)^{[\frac{r}{2}]v(a\beta)} \\
&= \zeta(-1)^{[\frac{r}{2}][\frac{r+1}{2}]v(\beta)} |\beta|^{\frac{1}{2}[\frac{r}{2}]-[\frac{r^2}{4}]} \gamma(a\beta,\psi)^{[\frac{r}{2}]} (a\beta,\varpi)^{[\frac{r}{2}]v(\beta)}. \qquad (10)
\end{aligned}$$

  The last equation is obtained by using $v(a) \equiv 0 \bmod 2$ and the identity

  $$\left[\frac{r^2}{4}\right] - \frac{1}{2}\left[\frac{r}{2}\right] = \left[\frac{(r-1)^2}{4}\right] - \frac{1}{2}\left[\frac{r-1}{2}\right] + \frac{r-1}{2}.$$

Moreover, we have

$$\begin{aligned}
&\gamma(a\beta,\psi)(a\beta,\varpi)^{v(\beta)}\\
&=\gamma(a,\psi)\gamma(\beta,\psi)(a,\beta)(a\beta,\varpi)^{v(\beta)}\\
&=\gamma(\beta,\psi)(a,\beta)(a\beta,\varpi)^{v(\beta)} \text{ (since } a \text{ is even then } \gamma(a,\psi)=1)\\
&=\gamma(\beta,\psi)(a,\varpi)^{v(\beta)}(a\beta,\varpi)^{v(\beta)} \quad (11)\\
&=\gamma(\beta,\psi)(\beta,\varpi)^{v(\beta)}. \quad (12)
\end{aligned}$$

Since $v(a)$ is even, we have $(a,\beta)=\begin{cases}1 & \text{if } v(\beta) \text{ is even}\\ (a,\varpi) & \text{if } v(\beta) \text{ is odd}\end{cases} = (a,\varpi)^{v(\beta)}$. This implies the relation (11).

Combining the relations (10) and (12) we get

$$\Delta_{w_{\mathrm{GL}_r}}(\beta)=\zeta(-1)^{\left[\frac{r}{2}\right]\left[\frac{r+1}{2}\right]v(\beta)}|\beta|^{\frac{1}{2}\left[\frac{r}{2}\right]-\left[\frac{r^2}{4}\right]}\gamma(\beta,\psi)^{\left[\frac{r}{2}\right]}(\beta,\varpi)^{\left[\frac{r}{2}\right]v(\beta)}.$$

Similarly, we have

$$\begin{aligned}
&\Delta'_{w_{\mathrm{GL}_r}}(\beta)\\
&=|a|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\gamma(a^{-1},\psi)^{-\left[\frac{r}{2}\right]}\Delta'_M(\alpha\beta)\\
&=|a|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\Delta'_M(\alpha\beta) \quad \big(\text{since } a \text{ is square then } \gamma(a^{-1},\psi)=1\big)\\
&=|a|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\zeta(-1)^{v((-1)^{r-1}\beta^r)}|a\beta|^{-\frac{r-1}{2}}\gamma((-a)^{1-r}\beta,\psi)\\
&\quad\times((-a)^{1-r}\beta,\varpi)^{v((-a)^{1-r}\beta)}\Delta_{w_{\mathrm{GL}_{r-1}}}(a\beta)\\
&=|a|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\zeta(-1)^{v(\beta^r)}|a\beta|^{-\frac{r-1}{2}}\gamma((-a)^{1-r}\beta,\psi)((-a)^{1-r}\beta,\varpi)^{v(\beta)}\\
&\quad\times\zeta(-1)^{\left[\frac{r-1}{2}\right]\left[\frac{r}{2}\right]v(a\beta)}|a\beta|^{\frac{1}{2}\left[\frac{r-1}{2}\right]-\left[\frac{(r-1)^2}{4}\right]}\gamma(a\beta,\psi)^{\left[\frac{r-1}{2}\right]}(a\beta,\psi)^{\left[\frac{r-1}{2}\right]v(a\beta)}\\
&=\zeta(-1)^{\left[\frac{r+1}{2}\right]\left[\frac{r+2}{2}\right]v(\beta)}|\beta|^{\frac{1}{2}\left[\frac{r}{2}\right]-\left[\frac{r^2}{4}\right]}\gamma(\beta,\psi)^{\left[\frac{r+1}{2}\right]}(\beta,\varpi)^{\left[\frac{r+1}{2}\right]v(\beta)}.
\end{aligned}$$

The last equation is obtained by using the relation (12) (note that it will follow $v(a)\equiv 0 \bmod 2$) and the identity

$$\left[\frac{r-1}{2}\right]\left[\frac{r}{2}\right]+r=\left[\frac{r+1}{2}\right]\left[\frac{r+2}{2}\right]. \qquad \square$$

Lemma 4.5. — *Suppose that $v(a)$ is even. We have*

$$\Delta_{w_{\mathrm{GL}_r}}(z\beta)=c(a,z)\Delta_{w_M}(\alpha\beta)$$

and

$$\Delta'_{w_{\mathrm{GL}_r}}(z^{-1}\beta)=c(a,z)\Delta'_{w_M}(\alpha\beta)$$

for all $z\in F^*$ satisfied $z^r=1$ and $\beta\in F^*$.

Proof. — We prove only the first relation (the second one can be done similarly). We should mention that we use $v(z)=0$ in some steps of the following calculation.

$$\begin{aligned}
&c(a,z)\Delta_{w_M}(\alpha\beta)\\
&=|az^{-1}|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\gamma((az^{-1})^{-1},\psi)^{-\left[\frac{r}{2}\right]}\Delta_M(\alpha\beta)\\
&=|az^{-1}|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}\Delta_M(\alpha\beta)\quad\big(\text{since } v(az^{-1}) \text{ is even}\big)\\
&=|az^{-1}|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}|a\beta|^{-\frac{r-1}{2}}\Delta'_{w_{\mathrm{GL}_{r-1}}}(\mathrm{diag}(a\beta,\dots,a\beta))\\
&=|az^{-1}|^{\left[\frac{r^2}{4}\right]-\frac{1}{2}\left[\frac{r}{2}\right]}|a\beta|^{-\frac{r-1}{2}}\times\\
&\quad\times\zeta(-1)^{\left[\frac{r}{2}\right]\left[\frac{r+1}{2}\right]v(a\beta)}|a\beta|^{\frac{1}{2}\left[\frac{r-1}{2}\right]-\left[\frac{(r-1)^2}{4}\right]}\gamma(a\beta,\psi)^{\left[\frac{r}{2}\right]}(a\beta,\psi)^{\left[\frac{r}{2}\right]v(a\beta)}\\
&=\zeta(-1)^{\left[\frac{r}{2}\right]\left[\frac{r+1}{2}\right]v(\beta)}|z\beta|^{\frac{1}{2}\left[\frac{r}{2}\right]-\left[\frac{r^2}{4}\right]}\gamma(\beta,\psi)^{\left[\frac{r}{2}\right]}(\beta,\varpi)^{v(\beta)}\\
&=\zeta(-1)^{\left[\frac{r}{2}\right]\left[\frac{r+1}{2}\right]v(z\beta)}|z\beta|^{\frac{1}{2}\left[\frac{r}{2}\right]-\left[\frac{r^2}{4}\right]}\gamma(z\beta,\psi)^{\left[\frac{r}{2}\right]}(z\beta,\varpi)^{v(\beta)}\\
&=\Delta_{w_{\mathrm{GL}_r}}(z\beta).
\end{aligned}$$

□

The relation (8) reads

$$\sum_{z|z^r=1}K_{w_M}^{w_{\mathrm{GL}_r}}(z^{-1}\alpha)(J(w_{\mathrm{GL}_r}z\beta,f)-\Delta^{\cdot}_{w_{\mathrm{GL}_r}}(z\beta)I(w_{\mathrm{GL}_r}z\beta,\phi))=0,$$

for $|a|$ small enough and $v(a)\equiv 0 \bmod 2$. If we set

$$m^{\cdot}(z)=J(w_{\mathrm{GL}_r}z\beta,f)-\Delta^{\cdot}_{w_{\mathrm{GL}_r}}(z\beta)I(w_{\mathrm{GL}_r}z\beta,\phi),$$

we can see that the above relation reads

$$\sum_{z|z^r=1}\psi\left(\frac{rz}{2a}\right)m^{\cdot}(z)=0,$$

for $|a|$ small enough and $v(a)\equiv 0 \bmod 2$. We have to see that $m^{\cdot}(z)=0$,for all z. Thus, it follows from the following lemma.

Lemma 4.6. — *Given x_i, which are distinct points in F. Suppose that for each index i, there is a constant m_i, such that*

$$\sum_i m_i\psi(x_ix)=0,$$

for all even x (i.e., $v(x)\equiv 0 \bmod 2$) with $|x|$ large enough. Then $m_i=0$, for all i.

Proof. — Suppose that $m_{i_0}\neq 0$. At the cost of multiplying by $\psi(-x_{i_0}x)$ we may assume that our relation takes the form

$$1=\sum m_i\psi(x_ix),$$

where $x_i \neq 0$. We choose a b with $|b|$ large and integrate this identity over the set $|x| = |b^2|$ against the multiplicative Haar measure. The left-hand side gives a positive value. On the other hand, for fixed $x_i \neq 0$ and $|b|$ large, the integral

$$\int_{|x|=|b^2|} \psi(x_i x) d^\times x$$

vanishes. Thus, the terms on the right-hand side are zero. We then have a contradiction. □

For $x \in \{0,1\}$, we denote by

$$\Delta_{x,r}(\beta) = \zeta(-1)^{\left[\frac{r+x}{2}\right]\left[\frac{r+x+1}{2}\right]v(\beta)} |\beta|^{\frac{1}{2}\left[\frac{r}{2}\right]-\left[\frac{r^2}{4}\right]} \gamma(\beta,\psi)^{\left[\frac{r+x}{2}\right]} (\beta,\varpi)^{\left[\frac{r+x}{2}\right]v(\beta)}.$$

We then have $\Delta_{0,r}(\beta) = \Delta_{w_{\mathrm{GL}_r}}(\beta)$ and $\Delta_{1,r}(\beta) = \Delta'_{w_{\mathrm{GL}_r}}(\beta)$.

Let M be the standard Levi subgroup of GL_r of type $(r_1,\dots,r_m)$. Let $\mathbf{t} = \mathrm{diag}(t_i \mathrm{Id}_{r_i}, 1 \leq i \leq m) \in T_M$. For $x \in \{0,1\}$, we define a sequence $(y^m_{x,i})_{1\leq i\leq m} \in \{0,1\}^m$ as follows:

- $y^m_{x,m} = x$,
- $y^m_{x,i} = (y^m_{x,i+1} + r_{i+1}) \bmod 2 \ \forall 1 \leq i \leq m-1$.

We denote by $\Delta_{x,M}$ the function over $T_M(F)$:

$$\Delta_{x,M}(\mathbf{t}) \coloneqq \left[\zeta(-1)^{\sum_{i=1}^{m-1} r_i \left(\sum_{j=i+1}^{m} \left[\frac{r_j + y^m_{x,j}}{2}\right]\right) v(t_i)}\right] \left[\prod_{i=1}^{m-1} |t_i|^{\frac{-r_i}{2}(\sum_{j=i+1}^m r_j)}\right] \times \prod_{i=1}^{m} \Delta_{y^m_{x,i}, r_i}(t_i).$$

Theorem 4.7. — *Let M be the standard Levi subgroup of GL_r of type $(r_1,\dots,r_m)$. Let $\mathbf{t} = \mathrm{diag}(t_i \mathrm{Id}_{r_i}, 1 \leq i \leq m) \in T_M$. We then have*

$$\Delta_{w_M}(\mathbf{t}) = \Delta_{0,M}(\mathbf{t}) \quad \text{and} \quad \Delta'_{w_M}(\mathbf{t}) = \Delta_{1,M}(\mathbf{t}).$$

Proof. — We prove this theorem by induction over m.

- If $m = 1$, then $M = \mathrm{GL}_r$, and $\mathbf{t} = \mathrm{diag}(\beta,\dots,\beta)$. We have

$$\Delta_{x,\mathrm{GL}_r}(\mathbf{t}) = \Delta_{x,r}(\beta).$$

- Suppose that the theorem holds for $m-1$. We denote by M_1 the standard Levi subgroup of $\mathrm{GL}_{r_1+\dots+r_{m-1}}$ of type $(r_1, r_2, \dots, r_{m-1})$. Using Proposition 4.1 for $w = w_M$, $w_1 = w_{M_1}$, $w_2 = w_{\mathrm{GL}_{r_m}}$; $\mathbf{t}_1 = \mathrm{diag}(t_i \mathrm{Id}_{r_i}, 1 \leq i \leq m-1)$ and $\mathbf{t_2} = t_m \mathrm{Id}_{r_m}$, we have

$$\Delta_{w_M}(\mathbf{t}) = \begin{cases} \zeta(-1)^{\frac{r_m}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_m}{2}} \Delta_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2) & \text{if } r_m \equiv 0 \bmod 2 \\ \zeta(-1)^{\frac{r_m-1}{2} v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_m}{2}} \Delta'_{w_1}(\mathbf{t}_1)\Delta_{w_2}(\mathbf{t}_2) & \text{if } r_m \not\equiv 0 \bmod 2 \end{cases}$$
$$= \zeta(-1)^{\left[\frac{r_m}{2}\right] v(\det(\mathbf{t}_1))} |\det(\mathbf{t}_1)|^{-\frac{r_m}{2}} \Delta_{r_m \bmod 2, M_1}(\mathbf{t_1}) \Delta_{0,r_m}(t_m).$$

Note that $y^{m-1}_{r_m \bmod 2,i} = y^m_{0,i}$, so the last equation can be rewritten as

$$\begin{aligned}
\Delta_{w_M}(\mathbf{t}) &= \zeta(-1)^{\left[\frac{r_m}{2}\right]v(\det(\mathbf{t}_1))}|\det(\mathbf{t}_1)|^{-\frac{r_m}{2}}\Delta_{r_m \bmod 2, M_1}(\mathbf{t_1})\Delta_{y^m_{0,m},r_m}(t_m)\\
&= \zeta(-1)^{\left[\frac{r_m}{2}\right]v(\det(\mathbf{t}_1))}|\det(\mathbf{t}_1)|^{-\frac{r_m}{2}}\Delta_{y^m_{0,m},r_m}(t_m)\\
&\quad\times\left[\zeta(-1)^{\sum_{i=1}^{m-2} r_i\left(\sum_{j=i+1}^{m-1}\left[\frac{r_j+y^{m-1}_{r_m \bmod 2,j}}{2}\right]\right)v(t_i)}\right]\\
&\quad\times\left[\prod_{i=1}^{m-2}|t_i|^{\frac{-r_i}{2}(\sum_{j=i+1}^{m-1} r_j)}\right]\prod_{i=1}^{m-1}\Delta_{y^{m-1}_{r_m \bmod 2,i},r_i}(t_i)\\
&= \zeta(-1)^{\left[\frac{r_m+y^m_{0,m}}{2}\right]\left(\sum_{i=1}^{m-1}(r_i v(t_i))\right)}|\prod_{i=1}^{m-1} t_i^{r_i}|^{-\frac{r_m}{2}}\Delta_{y^m_{0,m},r_m}(t_m)\\
&\quad\times\left[\zeta(-1)^{\sum_{i=1}^{m-2} r_i\left(\sum_{j=i+1}^{m-1}\left[\frac{r_j+y^m_{0,j}}{2}\right]\right)v(t_i)}\right]\\
&\quad\times\left[\prod_{i=1}^{m-2}|t_i|^{\frac{-r_i}{2}(\sum_{j=i+1}^{m-1} r_j)}\right]\prod_{i=1}^{m-1}\Delta_{y^m_{0,i},r_i}(t_i)\\
&= \Delta_{0,M}(\mathbf{t}).
\end{aligned}$$

We do similarly for $\Delta'_{w_M}(\mathbf{t})$. □

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