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PSEUDOVALUATIONS ON THE DE RHAM–WITT COMPLEX

by Rubén Muñoz--Bertrand

Abstract. — For a polynomial ring over a commutative ring of positive characteristic, we define on the associated de Rham–Witt complex a set of functions, and show that they are pseudovaluations in the sense of Davis, Langer and Zink. To achieve this, we explicitly compute products of basic elements on the complex. We also prove that the overconvergent de Rham–Witt complex can be recovered using these pseudovaluations.

Résumé (*Pseudovaluations sur le complexe de de Rham–Witt*). — Pour tout anneau polynomial sur un anneau commutatif de caractéristique strictement positive, on définit sur le complexe de de Rham–Witt associé un ensemble de fonctions, et l'on démontre que ce sont des pseudovaluations au sens de Davis, Langer et Zink. Pour y parvenir, on calcule explicitement des produits d'éléments basiques du complexe. On prouve également que le complexe de de Rham–Witt surconvergent peut être retrouvé en employant ces pseudovaluations.

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Rubén Muñoz--Bertrand, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405 Orsay, France • *E-mail :* ruben.munoz-bertrand@universite-paris-saclay.fr • *Url :* https://www.imo.universite-paris-saclay.fr/~munoz/

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Introduction

Davis, Langer and Zink introduced the overconvergent de Rham–Witt complex in [2]. It is a complex of sheaves defined on any smooth variety X over a perfect field k of positive characteristic. It can be used to compute both the Monsky–Washnitzer and the rigid cohomology of the variety. This comparison was first established by [2] for quasi-projective smooth varieties; the assumption of quasi-projectiveness was then removed by Lawless [7].

This complex is defined as a differential graded algebra (dga) contained in the de Rham–Witt complex $W\Omega^\bullet_{X/k}$ of Deligne and Illusie. In order to achieve this they defined for any $\varepsilon > 0$, in the case where X is the spectrum of a polynomial ring $k[\underline{X}]$ over k, an order function $\gamma_\varepsilon \colon W\Omega^\bullet_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\}$. The overconvergent de Rham–Witt complex of X is the set of all $x \in W\Omega^\bullet_{k[\underline{X}]/k}$, such that $\gamma_\varepsilon(x) \neq -\infty$, for some $\varepsilon > 0$. In the general case, it is defined as the functional image of this set for a surjective morphism of smooth commutative algebras over k.

In degree zero (that is, for Witt vectors), these maps have nice properties. One of these is that they are pseudovaluations. We recall the definition. A **pseudovaluation** on a ring R is a function $v \colon R \to \mathbb{R} \cup \{+\infty, -\infty\}$, such that:

$$v(0) = +\infty,$$
$$v(1) = 0,$$
$$\forall r \in R,\ v(r) = v(-r)\,,$$
$$\forall r, s \in R,\ v(r+s) \geqslant \min\{v(r)\,, v(s)\}\,,$$
$$\forall r, s \in R,\ (v(r) \neq -\infty) \wedge (v(s) \neq -\infty) \implies (v(rs) \geqslant v(r) + v(s))\,.$$

The last formula will be referred to as the product formula in the remainder of this article.

Pseudovaluations and their behaviour have been studied in [3]. It appears that they form a convenient framework to study the overconvergence of recursive sequences. However, there are counterexamples showing that in positive degree, the maps γ_ε are not pseudovaluations. This becomes an obstacle when one wants to study the local structure of the overconvergent de Rham–Witt complex, or when one tries to find an interpretation of F-isocrystals for the overconvergent de Rham–Witt complex following the work of [4].

In this paper, we define new mappings

$$\zeta_\varepsilon \colon W\Omega^\bullet_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\}\ ,$$

for all $\varepsilon > 0$ and prove that these are pseudovaluations. Moreover, we show that the set of all $x \in W\Omega^\bullet_{k[\underline{X}]/k}$, such that $\zeta_\varepsilon(x) \neq -\infty$, for some $\varepsilon > 0$, also define the overconvergent de Rham–Witt complex.

In order to do so, in the first section we recall the main results concerning the de Rham–Witt complex, especially in the case of a polynomial algebra.

The second section, which is the most technical one, consists of computations of products of specific elements of the de Rham–Witt complex. The results are explicit and proven in the case where k is any commutative $\mathbb{Z}_{\langle p\rangle}$-algebra. This enables us in the last section to define the pseudovaluations and prove that in the case of a perfect field of positive characteristic, we retrieve with these functions the overconvergent de Rham–Witt complex.

The product formula comes in handy to control the overconvergence of sequences defined by recursion. This is the main motivation for this work, which will allow us in subsequent papers to study the structure of the overconvergent de Rham–Witt complex and, eventually, to give an interpretation of F-isocrystals as overconvergent de Rham–Witt connections.

1. The de Rham–Witt complex for a polynomial ring

Let p be a prime number. Let k be a commutative $\mathbb{Z}_{\langle p\rangle}$-algebra. Throughout this article, for any $i, j \in \mathbb{N}$, we shall write:

$$[\![i, j]\!] := \mathbb{N} \cap [i, j] .$$

Let $n \in \mathbb{N}$ and write $k[\underline{X}] := k[X_1, \dots, X_n]$. We will first recall basic properties of the de Rham–Witt complex of $k[\underline{X}]$, denoted $\left(W\Omega^\bullet_{k[\underline{X}]/k}, d\right)$ (for an introduction, see [5] or [6]). In degree zero, $W\Omega^0_{k[\underline{X}]/k}$ is isomorphic as a $W(k)$-algebra to $W(k[\underline{X}])$, the ring of Witt vectors over $k[\underline{X}]$.

There is a morphism of graded rings $F\colon W\Omega^\bullet_{k[\underline{X}]/k} \to W\Omega^\bullet_{k[\underline{X}]/k}$ called the **Frobenius endormorphism**, a morphism $V\colon W\Omega^\bullet_{k[\underline{X}]/k} \to W\Omega^\bullet_{k[\underline{X}]/k}$ of graded groups called the ***Verschiebung*** **endormorphism**, as well as a morphism of monoids $[\bullet] : (k[\underline{X}], \times) \to (W(k[\underline{X}]), \times)$ called the **Teichmüller lift** such that:

$$\forall r \in k[\underline{X}] ,\ F([r]) = [r^p] , \tag{1}$$

$$\forall m \in \mathbb{N},\ \forall x, y \in W\Omega^\bullet_{k[\underline{X}]/k},\ V^m(xF^m(y)) = V^m(x)\, y, \tag{2}$$

$$\forall m \in \mathbb{N},\ \forall x \in W\Omega^\bullet_{k[\underline{X}]/k},\ d(F^m(x)) = p^m F^m(d(x)) , \tag{3}$$

$$\forall m \in \mathbb{N},\ \forall P \in k[\underline{X}] ,\ F^m(d([P])) = \left[P^{p^m-1}\right] d([P]) , \tag{4}$$

$$\forall i, j \in \mathbb{N},\ \forall x \in W\Omega^i_{k[\underline{X}]/k},\ \forall y \in W\Omega^j_{k[\underline{X}]/k},\quad d(xy) = (-1)^i\, xd(y) + (-1)^{(i+1)j}\, yd(x) , \tag{5}$$

$$\forall m \in \mathbb{N},\ \forall (x_i)_{i\in[\![1,m]\!]} \in (W(k[\underline{X}]))^m ,\quad d\left(\prod_{i=1}^m x_i\right) = \sum_{i=1}^m \left(\prod_{j\in[\![1,m]\!]\smallsetminus\{i\}} x_j\right) d(x_i) . \tag{6}$$

We shall introduce basic elements on the de Rham–Witt complex, called basic Witt differentials, and recall how any de Rham–Witt differential on $k[\underline{X}]$ can be expressed as a series using these elements. We mostly follow [6].

Definition 1.1. — A **weight function** is a mapping $a\colon [\![1,n]\!] \to \mathbb{N}\left[\frac{1}{p}\right]$; for all $i \in [\![1,n]\!]$, its values shall be written as a_i. We define:

$$|a| \coloneqq \sum_{i=1}^{n} a_i,$$

and:

$$\underline{X}^a \coloneqq \prod_{i=1}^{n} {X_i}^{a_i}.$$

For any weight function a and any $J \subset [\![1,n]\!]$, we denote by $a|_J$ the weight function that for all $i \in [\![1,n]\!]$ satisfies:

$$a|_J(i) = \begin{cases} a_i & \text{if } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

The **support** of a weight function a is the following set:

$$\operatorname{Supp}(a) \coloneqq \{i \in [\![1,n]\!] \mid a_i \neq 0\}.$$

A **partition** I of a weight function a is a subset $I \subset \operatorname{Supp}(a)$. Its **size** is its cardinal. We will denote by $\mathcal{P}$ the set of all pairs (a, I), where a is a weight function, and I is a partition of a.

In all this paper, the p-adic valuation shall be denoted v_p. For a weight function a, we fix the following total order $\preceq$ on $\operatorname{Supp}(a)$:

$$\forall i, i' \in \operatorname{Supp}(a),\ i \preceq i' \\ \iff ((\mathrm{v}_p(a_i) \leqslant \mathrm{v}_p(a_{i'})) \wedge ((\mathrm{v}_p(a_i) = \mathrm{v}_p(a_{i'})) \implies (i \leqslant i'))).$$

We will denote by $\prec$ the associated strict total order and we also let $\min(a) \in \operatorname{Supp}(a)$ be the only element such that $\min(a) \preceq i$, for any $i \in \operatorname{Supp}(a)$.

Let $m \in [\![0,n]\!]$. Let $I \coloneqq \{i_j\}_{j \in [\![1,m]\!]}$ be a partition of a weight function a. We will always suppose that $i_j \prec i_{j'}$, for all $j, j' \in [\![1,m]\!]$, such that $j < j'$. By convention, we will say that $i_0 \preceq i$ and $i \prec i_{m+1}$ whenever $i \in \operatorname{Supp}(a)$. We define the following $m+1$ subsets of $\operatorname{Supp}(a)$ for any $l \in [\![0,m]\!]$:

$$I_l \coloneqq \{i \in \operatorname{Supp}(a) \mid i_l \preceq i \prec i_{l+1}\}.$$

Let a be a weight function. We set:

$$\mathrm{v}_p(a) \coloneqq \min\{\mathrm{v}_p(a_i) \mid i \in [\![1,n]\!]\},$$
$$u(a) \coloneqq \max\{0, -\mathrm{v}_p(a)\}.$$

If a is not the zero function, we put:

$$g(a) := F^{u(a)+\mathrm{v}_p(a)}\left(d\left(V^{u(a)}\left(\left[\underline{X}^{p^{-\mathrm{v}_p(a)}a}\right]\right)\right)\right).$$

Furthermore, if I is a partition of a, and η is any element in $W(k)$, we set:

(7)
$$e(\eta, a, I) := \begin{cases} d\left(V^{u(a)}\left(\eta\left[\underline{X}^{p^{u(a)}a|_{I_0}}\right]\right)\right)\prod_{l=2}^{\#I} g(a|_{I_l}) & \text{if } I_0 = \emptyset \text{ and } u(a) \neq 0, \\ V^{u(a)}\left(\eta\left[\underline{X}^{p^{u(a)}a|_{I_0}}\right]\right)\prod_{l=1}^{\#I} g(a|_{I_l}) & \text{otherwise.} \end{cases}$$

When $I_0 = \emptyset$ and $u(a) = 0$, then $V^{u(a)}\left(\eta\left[\underline{X}^{p^{u(a)}a|_{I_0}}\right]\right) = \eta$. So one can notice that, if one ignores η, the element defined above is a product of $\#I$ factors whenever $I_0 = \emptyset$, and of $\#I + 1$ factors otherwise, the factors being the images of an element through one of the functions d, g or V. We will use this fact later, when we define the pseudovaluations on the de Rham–Witt complex of a polynomial ring.

We recall the action of d, V and F on these elements.

Proposition 1.2. — *For any $(a, I) \in \mathcal{P}$ and any $\eta \in W(k)$, we have:*

$$d(e(\eta, a, I)) = \begin{cases} 0 & \text{if } I_0 = \emptyset, \\ e(\eta, a, I \cup \{\min(a)\}) & \text{if } I_0 \neq \emptyset \text{ and } \mathrm{v}_p(a) \leqslant 0, \\ p^{\mathrm{v}_p(a)} e(\eta, a, I \cup \{\min(a)\}) & \text{if } I_0 \neq \emptyset \text{ and } \mathrm{v}_p(a) > 0. \end{cases}$$

Proof. — See [6, proposition 2.6]. □

Proposition 1.3. — *For any $(a, I) \in \mathcal{P}$ and any $\eta \in W(k)$, we have:*

$$F(e(\eta, a, I)) = \begin{cases} e(\eta, pa, I) & \text{if } \mathrm{v}_p(a) < 0 \text{ and } I_0 = \emptyset, \\ e(p\eta, pa, I) & \text{if } \mathrm{v}_p(a) < 0 \text{ and } I_0 \neq \emptyset, \\ e(F(\eta), pa, I) & \text{if } \mathrm{v}_p(a) \geqslant 0. \end{cases}$$

Proof. — See [6, proposition 2.5]. □

Proposition 1.4. — *For any $(a, I) \in \mathcal{P}$ and any $\eta \in W(k)$, we have:*

$$V(e(\eta, a, I)) = \begin{cases} e\left(V(\eta), \frac{a}{p}, I\right) & \text{if } \mathrm{v}_p(a) > 0, \\ e\left(p\eta, \frac{a}{p}, I\right) & \text{if } \mathrm{v}_p(a) \leqslant 0 \text{ and } I_0 = \emptyset, \\ e\left(\eta, \frac{a}{p}, I\right) & \text{if } \mathrm{v}_p(a) \leqslant 0 \text{ and } I_0 \neq \emptyset. \end{cases}$$

Proof. — See [6, proposition 2.5]. □

The de Rham–Witt complex is endowed with a topology called the standard topology [5, I. 3.1.]. In this article, it will not be necessary to recall its definition, as we will only need the fact that a series of the form $\sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I)$, with $\eta_{a,I} \in W(k)$, for all $(a, I) \in \mathcal{P}$, converges in $W\Omega^{\bullet}_{k[\underline{X}]/k}$ if and only if for any $m \in \mathbb{N}$, we have $V^{u(a)}(\eta_{a,I}) \in V^m(W(k))$ except for a finite number of $(a, I) \in \mathcal{P}$.

The following theorem is essential to the definition of the overconvergent de Rham–Witt complex, and to its decomposition as a $W(k)$-module in the case of a polynomial algebra.

Theorem 1.5. — *For any differential $x \in W\Omega^{\bullet}_{k[\underline{X}]/k}$, there exists a unique function*

$$\eta\colon \begin{array}{c} \mathcal{P} \to W(k) \\ (a, I) \mapsto \eta_{a,I} \end{array}$$

such that:

$$x = \sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I)\,.$$

Proof. — See [6, theorem 2.8]. □

2. Computations

The goal of this section is to make explicit computations of the product of two basic Witt differentials, that is, elements of the form (7).

Let k be a commutative $\mathbb{Z}_{\langle p\rangle}$-algebra. Let $n \in \mathbb{N}$. In what follows, we shall denote $k[\underline{X}] \coloneqq k[X_1, \ldots, X_n]$. For any $(a, I) \in \mathcal{P}$ with a taking values in $\mathbb{N}$, we will write:

$$h(a, I) \coloneqq \prod_{i\in\operatorname{Supp}(a)\smallsetminus I} [X_i]^{k_i} \prod_{j\in I} g\big(a|_{\{j\}}\big)\,.$$

We will use the elements h defined above in order to achieve this, as they appear to be more convenient for calculations.

Lemma 2.1. — *Let R be a commutative k-algebra. Let $x \in R$ and let $m, m' \in \mathbb{N}$, such that $m + m' \neq 0$. Put $a \coloneqq \mathrm{v}_p(m + m')$ and $b \coloneqq p^{-a}(m + m')$. Then we have in the de Rham–Witt complex $W\Omega^{\bullet}_{R/k}$ of R:*

$$[x]^m\, d\Big([x]^{m'}\Big) = \frac{m'}{b} F^a\Big(d\Big([x]^b\Big)\Big)\,.$$

Proof. — Using (1) and (3), we get:

$$d\Big([x]^{m+m'}\Big) = d\Big(F^a\Big([x]^b\Big)\Big) = p^a F^a\Big(d\Big([x]^b\Big)\Big)\,.$$

Moreover, as:

$$(m + m')\,[x]^m\, d\Big([x]^{m'}\Big) = m' d\Big([x]^{m+m'}\Big)\,,$$

we obtain in the case where $k = \mathbb{Z}_{\langle p \rangle}$, $R = \mathbb{Z}_{\langle p \rangle}[X]$ and $x = X$ the formula:

$$p^a b [x]^m \, d\Big([x]^{m'}\Big) = p^a m' F^a \Big(d\Big([x]^b\Big)\Big) .$$

The ring $W\big(\mathbb{Z}_{\langle p \rangle}[X]\big)$ has no p-torsion, as $\mathbb{Z}_{\langle p \rangle}[X]$ itself has no p-torsion. So we deduce from theorem 1.5 that $W\Omega^\bullet_{k[\underline{X}]/k}$ also has no p-torsion, which allows us to conclude in this situation. For the general case, using the canonical commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_{\langle p \rangle}[X] & \longrightarrow & R \\ \uparrow & & \uparrow \\ \mathbb{Z}_{\langle p \rangle} & \longrightarrow & k, \end{array}$$

where the upper arrow sends X to x, we conclude using the morphism of $W\big(\mathbb{Z}_{\langle p \rangle}\big)$-dgas $W\Omega^\bullet_{\mathbb{Z}_{\langle p \rangle}[X]/\mathbb{Z}_{\langle p \rangle}} \to W\Omega^\bullet_{R/k}$ obtained by functoriality of $W\Omega^\bullet_{\bullet/\bullet}$. □

The next proposition gives a simple formula for products of values of h. The goal of the subsequent lemmas will be to use it in order to get a formula for products of elements of the form (7).

Proposition 2.2. — *Let $(a, I), (b, J) \in \mathcal{P}$, such that a and b take values in $\mathbb{N}$. There exists $m \in \mathbb{Z}_{\langle p \rangle}$, such that:*

$$h(a, I)\, h(b, J) = \begin{cases} m \, h(a + b, I \cup J) & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — First, we have by definition:

$$h(a, I)\, h(b, J) = \prod_{i \in \mathrm{Supp}(a) \smallsetminus I} [X_i]^{a_i} \prod_{i' \in I} g\big(a|_{\{i'\}}\big) \prod_{j \in \mathrm{Supp}(b) \smallsetminus J} [X_j]^{b_j} \prod_{j' \in J} g\big(b|_{\{j'\}}\big) .$$

Since $W\Omega^\bullet_{k[\underline{X}]/k}$ is alternating, this product is zero whenever $I \cap J \neq \emptyset$. Otherwise, we get:

$$h(a, I)\, h(b, J) = \prod_{i \in \mathrm{Supp}(a+b) \smallsetminus (I \cup J)} [X_i]^{a_i + b_i} \prod_{i' \in I} [X_{i'}]^{b_{i'}} \, g\big(a|_{\{i'\}}\big) \prod_{j' \in J} [X_{j'}]^{a_{j'}} \, g\big(b|_{\{j'\}}\big) .$$

Moreover, for any $i' \in I$ we obtain:

$$[X_{i'}]^{b_{i'}}\, g\big(a|_{\{i'\}}\big) \overset{(4)}{=} [X_{i'}]^{\left(1-p^{-\mathrm{v}_p\left(a|_{\{i'\}}\right)}\right)a_{i'}+b_{i'}}\, d\left([X_i]^{p^{-\mathrm{v}_p\left(a|_{\{i'\}}\right)}a_{i'}}\right)$$

$$\overset{2.1}{=} \frac{p^{-\mathrm{v}_p\left(a|_{\{i'\}}\right)}a_{i'}}{p^{-\mathrm{v}_p(a_{i'}+b_{i'})}(a_{i'}+b_{i'})} g\big((a+b)\,|_{\{i'\}}\big)\,.$$

Using the same argument, for any $j' \in J$, one successfully gets:

$$[X_{j'}]^{a_{j'}}\, g\big(b|_{\{j'\}}\big) = \frac{p^{-\mathrm{v}_p\left(b|_{\{j'\}}\right)}b_{j'}}{p^{-\mathrm{v}_p\left(a_{j'}+b_{j'}\right)}(a_{j'}+b_{j'})} g\big((a+b)\,|_{\{j'\}}\big)\,.$$

This concludes the proof, because:

$$\prod_{i\in \mathrm{Supp}(a+b)\smallsetminus(I\cup J)} [X_i]^{a_i+b_i} \prod_{i'\in I} g\big((a+b)\,|_{\{i'\}}\big) \prod_{j'\in J} g\big((a+b)\,|_{\{j'\}}\big)$$
$$= h(a+b, I\cup J)\,. \quad \square$$

Lemma 2.3. — *Let $(a,I)\in\mathcal{P}$, such that a takes values in $\mathbb{N}$. We have:*

$$g(a) = \sum_{j\in \mathrm{Supp}(a)} p^{\mathrm{v}_p(a_j)-\mathrm{v}_p(a)} h(a,\{j\})\,.$$

Proof. — Write $S \coloneqq \mathrm{Supp}(a)$ for simplicity. We compute:

$$\begin{aligned}
&F^{\mathrm{v}_p(a)}\Big(d\Big(\Big[\underline{X}^{p^{-\mathrm{v}_p(a)}a}\Big]\Big)\Big)\\
&\overset{(4)}{=} \Big[\underline{X}^{\left(1-p^{-\mathrm{v}_p(a)}\right)a}\Big]\, d\Big(\Big[\underline{X}^{p^{-\mathrm{v}_p(a)}a}\Big]\Big)\\
&\overset{(6)}{=} \Big[\underline{X}^{\left(1-p^{-\mathrm{v}_p(a)}\right)a}\Big] \sum_{j\in S}\left(\prod_{j'\in S\smallsetminus\{j\}}\Big[{X_{j'}}^{p^{-\mathrm{v}_p(a)}a_{j'}}\Big]\right) d\Big(\Big[{X_j}^{p^{-\mathrm{v}_p(a)}a_j}\Big]\Big)\\
&\overset{(4)}{=} \sum_{j\in S}\left(\prod_{j'\in S\smallsetminus\{j\}}\big[{X_{j'}}^{a_{j'}}\big]\right) F^{\mathrm{v}_p(a)}\Big(d\Big(\Big[{X_j}^{p^{-\mathrm{v}_p(a)}a_j}\Big]\Big)\Big)\\
&\overset{(1)}{\overset{(3)}{=}} \sum_{j\in S}\left(\prod_{j'\in S\smallsetminus\{j\}}\big[{X_{j'}}^{a_{j'}}\big]\right) p^{\mathrm{v}_p(a_j)-\mathrm{v}_p(a)} F^{\mathrm{v}_p(a_j)}\Big(d\Big(\Big[{X_j}^{p^{-\mathrm{v}_p(a_j)}a_j}\Big]\Big)\Big)\\
&= \sum_{j\in S} p^{\mathrm{v}_p(a_j)-\mathrm{v}_p(a)} h(a,\{j\})\,.
\end{aligned}$$

This ends the proof because by definition $g(a) = F^{\mathrm{v}_p(a)}\Big(d\Big(\Big[\underline{X}^{p^{-\mathrm{v}_p(a)}a}\Big]\Big)\Big)$. $\square$

The next lemma will be used to write any value of the function h defined above as a linear combination of elements of the form (7). The previous lemma can be seen as a kind of reciprocal.

LEMMA 2.4. — *Let $(a, I) \in \mathcal{P}$, such that a takes values in $\mathbb{N}$. Denote by P the set of partitions of $\operatorname{Supp}(a)$ of size $\#I$. Then there exists a function $s\colon P \to \mathbb{N} \subset W(k)$, such that:*

$$h(a, I) = \sum_{J \in P} e(s(J)\,, a, J)\,.$$

Proof. — Put $m \coloneqq \#I$. If $m = 0$, then obviously $h(a, I) = e(1, a, I)$. Thus, suppose that $m \neq 0$. Write $\{i_l\}_{l \in [\![1,m]\!]} \coloneqq I$, with $i_j \prec i_{j'}$, for any pair $j < j'$ in $[\![1, m]\!]$, and for all $j \in \operatorname{Supp}(a)$ put $v_j \coloneqq \mathrm{v}_p(a_j)$ and $b_j = p^{-v_j} a_j$. By definition:

$$h(a, I) = \prod_{i \in \operatorname{Supp}(a) \smallsetminus I} [X_i]^{a_i} \prod_{j \in I} F^{v_j}\Big(d\Big([X_j]^{b_j}\Big)\Big)\,.$$

So we can write:

$$h(a, I) = h\big(a|_{\operatorname{Supp}(a) \smallsetminus I_m}, I \smallsetminus \{i_m\}\big)\, F^{v_{i_m}}\Big(d\Big([X_{i_m}]^{b_{i_m}}\Big)\Big) \prod_{i \in I_m \smallsetminus \{i_m\}} [X_i]^{a_i}\,.$$

Moreover, we can compute:

$$\begin{aligned}
&F^{v_{i_m}}\Big(d\Big([X_{i_m}]^{b_{i_m}}\Big)\Big) \prod_{i \in I_m \smallsetminus \{i_m\}} [X_i]^{a_i} \\
&\overset{(1)}{=} F^{v_{i_m}}\left(d\Big([X_{i_m}]^{b_{i_m}}\Big) \prod_{i \in I_m \smallsetminus \{i_m\}} [X_i]^{p^{-v_{i_m}} a_i}\right) \\
&\overset{(5)}{=} F^{v_{i_m}}\left(d\left(\prod_{i \in I_m} [X_i]^{p^{-v_{i_m}} a_i}\right) - [X_{i_m}]^{b_{i_m}}\, d\left(\prod_{i \in I_m \smallsetminus \{i_m\}} [X_i]^{p^{-v_{i_m}} a_i}\right)\right) \\
&\overset{(1)\,(3)}{=} g(a|_{I_m}) - F^{v_{i_m}}\Big([X_{i_m}]^{b_{i_m}}\Big)\, p^{\mathrm{v}_p(a|_{I_m \smallsetminus \{i_m\}}) - v_{i_m}} g\big(a|_{I_m \smallsetminus \{i_m\}}\big) \\
&\overset{(1)}{=} g(a|_{I_m}) - p^{\mathrm{v}_p(a|_{I_m \smallsetminus \{i_m\}}) - v_{i_m}} [X_{i_m}]^{a_{i_m}}\, g\big(a|_{I_m \smallsetminus \{i_m\}}\big)\,.
\end{aligned}$$

So we get:

$$\begin{aligned}
h(a, I) &= h\big(a|_{\operatorname{Supp}(a) \smallsetminus I_m}, I \smallsetminus \{i_m\}\big)\, g(a|_{I_m}) \\
&\quad - p^{\mathrm{v}_p(a|_{I_m \smallsetminus \{i_m\}}) - v_{i_m}} h\big(a|_{\{i_m\} \cup \operatorname{Supp}(a) \smallsetminus I_m}, I \smallsetminus \{i_m\}\big)\, g\big(a|_{I_m \smallsetminus \{i_m\}}\big)\,.
\end{aligned}$$

We can then deduce the lemma by induction on $m = \#I$. Indeed, if we suppose that $h\big(a|_{\operatorname{Supp}(a) \smallsetminus I_m}, I \smallsetminus \{i_m\}\big)$ can be written as a linear combination of elements of the form $e\big(1, a|_{\operatorname{Supp}(a) \smallsetminus I_m}, J'\big)$, where J' is a partition of

$\mathrm{Supp}(a) \smallsetminus I_m$ of size $m-1$, then since

$$e\big(1, a|_{\mathrm{Supp}(a)\smallsetminus I_m}, J'\big)\, g(a|_{I_m}) = e(1, a, J' \cup \{i_m\})\,,$$

the lemma is proven for the minuend of the above difference, and one can conclude for the subtrahend by using the same reasoning. □

Lemma 2.5. — *Let $(a, I), (b, J) \in \mathcal{P}$, such that a and b take values in $\mathbb{N}$. Let $\eta, \eta' \in W(k)$. Denoting by P the set of partitions of $\mathrm{Supp}(a+b)$ of size $\#I + \#J$, then there exists a function $s\colon P \to \mathbb{Z}_{\langle p\rangle}$ such that:*

$$e(\eta, a, I)\, e(\eta', b, J) = \sum_{L\in P} e(s(L)\,\eta\eta', a+b, L)\,.$$

Proof. — By definition we have $e(\eta, a, I) = \eta\Big[\underline{X}^{a|_{I_0}}\Big] \prod_{i=1}^{\#I} g(a|_{I_i})$. There is also a similar equation defining $e(\eta', b, J)$. Using lemma 2.3, for any $i \in [\![1, \#I]\!]$, we can write $g(a|_{I_i})$ as a linear combination of elements of the form $h(a|_{I_i}, \{j_i\})$ with $j_i \in I_i$. Also, by definition, $\Big[\underline{X}^{a|_{I_0}}\Big] = h(a|_{I_0}, \emptyset)$. Thus, we can write $e(\eta, a, I)$ as a linear combination of products of elements of the form $\eta h(a|_{I_0}, \emptyset) \prod_{i=1}^{\#I} h(a|_{I_i}, \{j_i\})$, where all $j_i \in I_i$, for any $i \in [\![1, \#I]\!]$. Again, we can do the same with $e(\eta', b, J)$. We can conclude by using proposition 2.2 and lemma 2.4. □

Lemma 2.6. — *Let $(a, I), (b, J) \in \mathcal{P}$, such that $u(a) \geqslant u(b)$ and $I_0 \neq \emptyset$. Denote by P the set of partitions of size $\#I + \#J$ of $\mathrm{Supp}(a+b)$, and put:*

$$v := \begin{cases} u(b) & \text{if } J_0 \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $\eta, \eta' \in W(k)$, there exists a function $s\colon P \to \mathbb{Z}_{\langle p\rangle}$ with:

$$\forall L \in P,\ \begin{cases} p^{v+u(a+b)} \mid s(L) & \text{if } L_0 = \emptyset, \\ p^{v} \mid s(L') & \text{otherwise,} \end{cases}$$

such that:

$$e(\eta, a, I)\, e(\eta', b, J) = \sum_{L\in P} e\Big(s(L)\, V^{u(a)-u(a+b)}\Big(\eta F^{u(a)-u(b)}(\eta')\Big), a+b, L\Big)\,.$$

Proof. — Put $\tilde{I} := \bigcup_{i\in[\![1,\#I]\!]} I_i$. We can compute:

$$\begin{aligned}
&e(\eta, a, I)\, e(\eta', b, J)\\
&\overset{(2)}{=} V^{u(a)}\Big(\eta\Big[\underline{X}^{p^{u(a)}(a|_{I_0})}\Big]\, F^{u(a)}(e(1, a|_{\tilde{I}}, I)\, e(\eta', b, J))\Big)\\
&\overset{1.3}{=} V^{u(a)}\Big(\eta\Big[\underline{X}^{p^{u(a)}(a|_{I_0})}\Big]\, e\Big(1, p^{u(a)}a|_{\tilde{I}}, I\Big)\, e\Big(p^v F^{u(a)-u(b)}(\eta')\,, p^{u(a)}b, J\Big)\Big)\\
&= V^{u(a)}\Big(e\Big(\eta, p^{u(a)}a, I\Big)\, e\Big(p^v F^{u(a)-u(b)}(\eta')\,, p^{u(a)}b, J\Big)\Big)\,.
\end{aligned}$$

These computations have been done so that the basic Witt differentials appearing in the last line are integral. In particular, we are now in position to apply lemma 2.5. That is, there is a function $s' \colon P \to \mathbb{Z}_{\langle p\rangle}$ such that:

$$e(\eta, a, I)\, e(\eta', b, J) = V^{u(a)}\left(\sum_{L\in P} e\Big(p^v s'(L)\, \eta F^{u(a)-u(b)}(\eta')\,, p^{u(a)}(a+b)\,, L\Big)\right).$$

We can conclude by using proposition 1.4 and the fact that the *Verschiebung* endomorphism is additive. □

In the last two statements of this section, we are interested in the case where k has characteristic p. The results become clearer in this situation because we have $p = V(F(1))$.

Lemma 2.7. — *Suppose k has characteristic p. Let $(a, I)\,, (b, J) \in \mathcal{P}$, such that $u(a) \geqslant u(b)$ and $I_0 \neq \emptyset$. Denote by P the set of partitions of size $\#I + \#J$ of* $\operatorname{Supp}(a+b)$*, and put:*

$$v := \begin{cases} u(b) & \text{if } J_0 \neq \emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\alpha, \beta \in \mathbb{N}$. Then for any $\eta \in V^\alpha(W(k))$ and any $\eta' \in V^\beta(W(k))$, there exists a function $s \colon P \to W(k)$ with:

$$\forall L \in P,\ \begin{cases} s(L) \in V^{\alpha+\beta+v+u(a)}(W(k)) & \text{if } L_0 = \emptyset,\\ s(L) \in V^{\alpha+\beta+v+u(a)-u(a+b)}(W(k)) & \text{otherwise,}\end{cases}$$

such that:

$$e(\eta, a, I)\, e(\eta', b, J) = \sum_{L\in P} e(s(L)\,, a+b, L)\,.$$

Proof. — This is a special case of lemma 2.6 when k has characteristic p, because in that case, we have $px = F(V(x)) = V(F(x))$ for any $x \in W(k)$, but also $\eta\eta' \in V^{\alpha+\beta}(W(k))$ [1, proposition 5. p. IX.15]. □

Proposition 2.8. — *Suppose k has characteristic p. Let $(a, I), (b, J) \in \mathcal{P}$ with $I_0 \neq \emptyset$. Denote by P the set of partitions of size $\#I + \#J$ of* $\mathrm{Supp}(a+b)$*. Let $\alpha, \beta \in \mathbb{N}$. Then, for any $\eta \in V^{\alpha}(W(k))$ and any $\eta' \in V^{\beta}(W(k))$, there exists a function $s \colon P \to W(k)$ with:*

$$\forall L \in P, \begin{cases} s(L) \in V^{\alpha+\beta+\min\{u(a),u(b)\}}(W(k)) & \textit{if } L_0 = \emptyset, \\ s(L) \in V^{\alpha+\beta+\max\{u(a),u(b)\}-u(a+b)}(W(k)) & \textit{otherwise,} \end{cases}$$

such that:

$$e(\eta, a, I)\, e(\eta', b, J) = \sum_{L \in P} e(s(L), a+b, L)\,.$$

Proof. — This statement is just a special case of lemma 2.7, except when $u(b) > u(a)$ and $J_0 = \emptyset$. In that situation, if $J' \coloneqq J \smallsetminus \{\min(b)\}$, we deduce from proposition 1.2 that:

$$\begin{aligned} e(\eta, a, I)\, e(\eta', b, J) & \\ &= e(\eta, a, I)\, d(e(\eta', b, J')) \\ &= (-1)^{\#I} \left(d(e(\eta, a, I)\, e(\eta', b, J')) - d(e(\eta, a, I))\, e(\eta', b, J')\right). \end{aligned}$$

This enables us to conclude using lemma 2.7 again. □

3. Pseudovaluations

We shall now consider the case where k is a commutative ring of characteristic p. Let $n \in \mathbb{N}$ and let $k[\underline{X}] \coloneqq k[X_1, \ldots, X_n]$. Recall that theorem 1.5 says that any $x \in W\Omega^{\bullet}_{k[\underline{X}]/k}$ can be uniquely written as $\sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I)$, where all $\eta_{a,I} \in W(k)$. This allows us to define specific $W(k)$-submodules of the de Rham–Witt complex.

Definition 3.1. — An element $x = \sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I) \in W\Omega^{\bullet}_{k[\underline{X}]/k}$ is said to be **integral** if $\eta_{a,I} = 0$, for all a with $u(a) \neq 0$. We denote by $W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$ the subset of all integral elements of the de Rham–Witt complex.

The element x is said to be **fractional** if $\eta_{a,I} = 0$ for all a with $u(a) = 0$. We denote by $W\Omega^{\mathrm{frac},\bullet}_{k[\underline{X}]/k}$ the subset of all fractional elements of the de Rham–Witt complex.

The element x is said to be **pure fractional** if $\eta_{a,I} = 0$ for all (a, I), such that $u(a) = 0$ or $I_0 = \emptyset$. We denote by $W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$ the subset of all pure fractional elements of the de Rham–Witt complex.

Notice that we have the following decomposition as $W(k)$-modules:

$$W\Omega^{\bullet}_{k[\underline{X}]/k} \cong W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k} \oplus W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k} \oplus d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)\,. \tag{8}$$

This is a refinement of Langer and Zink's decomposition into integral and fractional parts [6, (3.9)]. Indeed, we have:

$$W\Omega^{\mathrm{frac},\bullet}_{k[\underline{X}]/k} \cong W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k} \oplus d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)\,.$$

In the whole chapter, for any $x \in W\Omega^{\bullet}_{k[\underline{X}]/k}$, we will denote by $x|_{\mathrm{int}}$, $x|_{\mathrm{frac}}$, $x|_{\mathrm{frp}}$ and $x|_{\mathrm{d(frp)}}$ the obvious projections for these decompositions.

We will also denote by v_V the V-adic pseudovaluation on $W(k)$. Davis, Langer and Zink defined the following functions for any $\varepsilon > 0$ [2, (0.3)]:

$$\gamma_\varepsilon : \begin{array}{c} W\Omega^{\bullet}_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\} \\ \sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I) \mapsto \inf_{(a,I)\in\mathcal{P}}\{\mathrm{v}_V(\eta_{a,I}) + u(a) - \varepsilon|a|\}\,. \end{array}$$

To see that this definition coincides with the one given by Davis, Langer and Zink, it is necessary to see that in their definition of a basic Witt differential, they ask that $\mathrm{v}_V(\eta_{a,I}) \geqslant u(a)$, for all $(a, I) \in \mathcal{P}$ [6, p. 261]. The definition given in this article has been modified, which is why we need to add $u(a)$ in the definition of γ_ε.

The overconvergent de Rham–Witt complex of $k[\underline{X}]$ is the set of all $x \in W\Omega^{\bullet}_{k[\underline{X}]/k}$, such that there exists $\varepsilon > 0$ with $\gamma_\varepsilon(x) \neq -\infty$.

One of the main obstacles to studying the overconvergence of recursive sequences containing products of de Rham–Witt differentials is that these functions are not pseudovaluations. We will first study two counterexamples to the product rule in the case where $k[\underline{X}] \cong k[X, Y]$ as k-algebras. That is, we will find $x, y \in W\Omega^{\bullet}_{k[\underline{X}]/k}$, such that for all $\varepsilon > 0$, we have $\gamma_\varepsilon(x) \neq -\infty$, $\gamma_\varepsilon(y) \neq -\infty$ and $\gamma_\varepsilon(xy) < \gamma_\varepsilon(x) + \gamma_\varepsilon(y)$.

Example 3.2. — For any $m \in \mathbb{N}$, notice that:

$$\begin{aligned} V^m\Big(\Big[X^{p^m-1}\Big]\Big)\, d(V^m([X])) &= p^m d([X])\,, \\ \gamma_\varepsilon\Big(V^m\Big(\Big[X^{p^m-1}\Big]\Big)\Big) &= m - \frac{\varepsilon(p^m-1)}{p^m}, \\ \gamma_\varepsilon(d(V^m([X]))) &= m - \frac{\varepsilon}{p^m}, \\ \gamma_\varepsilon(p^m d([X])) &= m - \varepsilon < 2m - \varepsilon. \end{aligned}$$

This first counterexample illustrates what happens when one takes the product of two fractional elements. The phenomenon occurring here with $x = V^m\big(\big[X^{p^m-1}\big]\big)$ and $y = d(V^m([X]))$ is that the power of the denominator of the weight functions (which we denoted $a \mapsto u(a)$) can get smaller when taking products of differentials. Indeed, lemmas 2.6 and 2.7 and proposition 2.8 show that multiplying basic elements translates as an addition for weight functions. However, we notice in this example that the V-adic pseudovaluation we have to calculate gets bigger; it is just not big enough, so it compensates the decrease of u. In this example, to get a function satisfying the product formula of pseudovaluations, it seems to be enough to multiply the V-adic pseudovaluation by 2 in the definition of γ_ε. It is still not sufficient in general, as can be seen in the following counterexample.

EXAMPLE 3.3. — Let $m \in \mathbb{N}$. Then:

$$\gamma_\varepsilon\Big(V^m\Big(\Big[X^{p^m-1}\Big]\Big)\Big) = m - \frac{\varepsilon(p^m-1)}{p^m},$$
$$\gamma_\varepsilon(d(V^m([Y]))) = m - \frac{\varepsilon}{p^m},$$
$$\gamma_\varepsilon\Big(V^m\Big(\Big[X^{p^m-1}\Big]\Big)\, d(V^m([Y]))\Big) = m - \varepsilon < 2m - \varepsilon.$$

Another type of counterexample thus appears taking $x = V^m\big(\big[X^{p^m-1}\big]\big)$ and $y = d(V^m([Y]))$. In this situation, x, y and xy are basic Witt differentials, and the image through u of their associated weight functions is always m. This happens to be the main reason why the product formula fails with γ_ε in this context, as we need to add $2m$ when computing $\gamma_\varepsilon(x) + \gamma_\varepsilon(x)$, but m only appears once in the computation of $\gamma_\varepsilon(xy)$. So, in order for the product formula to work in general, we need to multiply the value of u in the definition of γ_ε by the number of factors in the definition of (7). As this number is smaller than n, as remarked after the first counterexample, we also have to multiply the V-adic pseudovaluation by $2n$.

This leads us to the definition below, which is a modification of Davis, Langer and Zink's definition. From now, $n \in \mathbb{N}$ is an arbitrary integer.

DEFINITION 3.4. — For any $\varepsilon > 0$ put:

$$\zeta_\varepsilon : \begin{array}{l} W\Omega^\bullet_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\} \\ x \mapsto \begin{cases} \inf_{(a,I)\in\mathcal{P}}\{2n\, \mathrm{v}_V(\eta_{a,I}) + \#I u(a) - \varepsilon|a|\} & \text{if } I_0 = \emptyset, \\ \inf_{(a,I)\in\mathcal{P}}\{2n\, \mathrm{v}_V(\eta_{a,I}) + (\#I + 1)\, u(a) - \varepsilon|a|\} & \text{if } I_0 \neq \emptyset, \end{cases} \end{array}$$

for $x = \sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I)$.

We will prove that these functions are pseudovaluations. Before we demonstrate the product formula, we first give a few basic properties. It is, for instance, immediate that:

$$\forall x, y \in W\Omega^\bullet_{k[\underline{X}]/k},\ \zeta_\varepsilon(x+y) \geqslant \min\{\zeta_\varepsilon(x)\, , \zeta_\varepsilon(y)\}\, . \tag{9}$$

Also, a consequence of proposition 1.2 is that:

$$\forall x \in W\Omega^\bullet_{k[\underline{X}]/k},\ \zeta_\varepsilon(d(x)) \geqslant \zeta_\varepsilon(x)\, . \tag{10}$$

The following proposition tells us that we recover the definition of the overconvergent de Rham–Witt complex with these functions.

Proposition 3.5. — *Let $x \in W\Omega^\bullet_{k[\underline{X}]/k}$. There exists $\varepsilon > 0$, such that $\gamma_\varepsilon(x) \neq -\infty$ if and only if $\zeta_{\varepsilon'}(x) \neq -\infty$, for some $\varepsilon' > 0$.*

Proof. — Notice that whenever $n \neq 0$, we have:

$$\forall x \in W\Omega^\bullet_{k[\underline{X}]/k},\ 2n\gamma_{\frac{\varepsilon}{2n}}(x) \geqslant \zeta_\varepsilon(x) \geqslant \gamma_\varepsilon(x)\,.$$

This ends the proof except when $n = 0$. However, when $n = 0$, then $W\Omega^\bullet_{k[\underline{X}]/k} \cong W(k)$ as $W(k)$-dgas, so there is nothing to do. □

We will now prove the product formula. We are doing this by exhaustion using the decomposition (8). Even though most of the proofs below follow the same, simple strategy, it is still interesting to carry them out in detail as one gets stronger formulas in some cases.

Proposition 3.6. — *For any $\varepsilon > 0$ and any $x, y \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$, we have:*

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon(xy) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y)\,.$$

Proof. — By definition of $W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$, we know that for all $(a, I), (b, J) \in \mathcal{P}$, there exists $\eta_{a,I}, \eta'_{b,J} \in W(k)$, such that:

$$x = \sum_{\substack{(a,I)\in\mathcal{P}\\ u(a)=0}} e(\eta_{a,I}, a, I)\,,$$

$$y = \sum_{\substack{(b,J)\in\mathcal{P}\\ u(b)=0}} e\big(\eta'_{b,J}, b, J\big)\,.$$

For any $(a, I), (b, J) \in \mathcal{P}$, such that $u(a) = u(b) = 0$, using lemma 2.5 we get:

$$\zeta_\varepsilon\big(e(\eta_{a,I}, a, I)\, e\big(\eta'_{b,J}, b, J\big)\big) \geqslant 2n\,\mathrm{v}_V\big(\eta_{a,I}\eta'_{b,J}\big) + (\#I + \#J)\,u(a+b) - \varepsilon|a+b|.$$

Since $u(a+b) = 0$ and $\mathrm{v}_V\Big(\eta_{a,I}\eta'_{b,J}\Big) \geqslant \mathrm{v}_V(\eta_{a,I}) + \mathrm{v}_V\Big(\eta'_{b,J}\Big)$ because k has characteristic p [1, proposition 5. p. IX.15], we can conclude. □

Proposition 3.7. — *Let $\varepsilon > 0$. For any $x \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$ and any $y \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$, we have:*

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon\big((xy)\,|_{\mathrm{d(frp)}}\big) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1.$$

Proof. — By definition of integral and pure fractional elements, we know that, for all $(a, I), (b, J) \in \mathcal{P}$, there exists $\eta'_{a,I}, \eta_{b,J} \in W(k)$ such that:

$$x = \sum_{\substack{(b,J)\in\mathcal{P}\\ u(b)=0}} e(\eta_{b,J}, b, J),$$
$$y = \sum_{\substack{(a,I)\in\mathcal{P}\\ u(a)>0\\ I_0\neq\emptyset}} e(\eta'_{a,I}, a, I).$$

Then, for any $(a, I), (b, J) \in \mathcal{P}$, such that $u(a) > 0$, $I_0 \neq \emptyset$ and $u(b) = 0$, lemma 2.7 gives us:

$$\zeta_\varepsilon\big(\big(e(\eta_{b,J}, b, J)\, e\big(\eta'_{a,I}, a, I\big)\big)\,|_{\mathrm{d(frp)}}\big) \geqslant 2n\big(\mathrm{v}_V(\eta_{b,J}) + \mathrm{v}_V\big(\eta'_{a,I}\big) + u(a)\big) + (\#I + \#J)\, u(a+b) - \varepsilon|a+b|.$$

However, $u(a+b) = u(a)$, so

$$\zeta_\varepsilon\big(\big(e(\eta_{b,J}, b, J)\, e\big(\eta'_{a,I}, a, I\big)\big)\,|_{\mathrm{d(frp)}}\big) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1,$$

as needed. □

Proposition 3.8. — *For any $\varepsilon > 0$, any $j \in \mathbb{N}$, any $x \in W\Omega^{\mathrm{int},j}_{k[\underline{X}]/k}$ and any $y \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$, we get:*

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon((xy)\,|_{\mathrm{frp}}) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y) + j.$$

Proof. — By definition of integral and pure fractional elements, we know that for all $(a, I), (b, J) \in \mathcal{P}$, there exists $\eta'_{a,I}, \eta_{b,J} \in W(k)$, such that:

$$x = \sum_{\substack{(b,J)\in\mathcal{P}\\ u(b)=0\\ \#J=j}} e(\eta_{b,J}, b, J),$$
$$y = \sum_{\substack{(a,I)\in\mathcal{P}\\ u(a)>0\\ I_0\neq\emptyset}} e(\eta'_{a,I}, a, I).$$

Using lemma 2.7, we know that for any $(a, I), (b, J) \in \mathcal{P}$, such that $u(a) > 0$, $I_0 \neq \emptyset$ and $u(b) = 0$, we have:

$$\zeta_\varepsilon\big(\big(e(\eta_{b,J}, b, J)\, e\big(\eta'_{a,I}, a, I\big)\big)\,|_{\mathrm{frp}}\big) \geqslant 2n\big(\mathrm{v}_V(\eta_{b,J}) + \mathrm{v}_v\big(\eta'_{a,I}\big)\big) + (\#I + \#J + 1)\, u(a+b) - \varepsilon|a+b|.$$

Furthermore, notice that $u(a+b) = u(a) > 0$. Therefore, we obtain that $\zeta_\varepsilon\big(\big(e(\eta_{b,J}, b, J)\, e\big(\eta'_{a,I}, a, I\big)\big)\,|_{\mathrm{frp}}\big) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y) + \#J$, which ends this proof. □

Proposition 3.9. — *Let $\varepsilon > 0$. For any $x \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$ and any $y \in W\Omega^{\bullet}_{k[\underline{X}]/k}$, we have:*

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon(xy) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y)\,.$$

Proof. — Recall that $y = y|_{\mathrm{int}} + y|_{\mathrm{frp}} + y|_{\mathrm{d(frp)}}$ and notice that $\zeta_\varepsilon(y|_{\mathrm{int}}) \geqslant \zeta_\varepsilon(y)$, $\zeta_\varepsilon(y|_{\mathrm{frp}}) \geqslant \zeta_\varepsilon(y)$ and $\zeta_\varepsilon\big(y|_{\mathrm{d(frp)}}\big) \geqslant \zeta_\varepsilon(y)$. Therefore, using proposition 3.7 we get:

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon\big((x(y|_{\mathrm{frp}}))\,|_{\mathrm{d(frp)}}\big) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1.$$

Applying proposition 3.8 and (9) yields:

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon((x(y|_{\mathrm{frp}}))\,|_{\mathrm{frp}}) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y)\,.$$

Using lemma 2.7, we obtain $x(y|_{\mathrm{frp}}) \in W\Omega^{\mathrm{frac},\bullet}_{k[\underline{X}]/k}$. Thus, formula (9) implies that:

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon(x(y|_{\mathrm{frp}})) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y)\,.$$

Moreover, using proposition 3.6 we get:

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon(x(y|_{\mathrm{int}})) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y)\,.$$

By applying (9) once more, we see that it only remains to show that:

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon\big(x\big(y|_{\mathrm{d(frp)}}\big)\big) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y)\,.$$

Again by (9), it is sufficient to prove this in the case where $x \in W\Omega^{i}_{k[\underline{X}]/k}$, for some $i \in \mathbb{N}$, and $y \in W\Omega^{j}_{k[\underline{X}]/k}$, for some $j \in \mathbb{N}$. Let $y' \in W\Omega^{\mathrm{frp},j-1}_{k[\underline{X}]/k}$ be the element such that $d(y') = y|_{\mathrm{d(frp)}}$. Using proposition 1.2 we get $\zeta_\varepsilon(y') = \zeta_\varepsilon\big(y|_{\mathrm{d(frp)}}\big)$. However, by (5) we find:

$$x\big(y|_{\mathrm{d(frp)}}\big) = xd(y') = (-1)^i\left(d(xy') - (-1)^{(i+1)(j-1)}\,y'd(x)\right).$$

So one can conclude using (9), (10) as well as propositions 3.7 and 3.8. □

Proposition 3.10. — *For any $\varepsilon > 0$ and any $x, y \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$, we get:*

$$(\zeta_\varepsilon(x) \neq -\infty \wedge \zeta_\varepsilon(y) \neq -\infty) \implies \zeta_\varepsilon((xy)\,|_{\mathrm{frp}}) \geqslant \zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1.$$

Proof. — Reasoning as in the proof of proposition 3.8, it is enough to prove that for any $(a, I), (b, J) \in \mathcal{P}$ with $u(a) \neq 0$, $I_0 \neq \emptyset$, $u(b) \neq 0$ and $J_0 \neq \emptyset$, and any Witt vectors $\eta_{a,I}, \eta_{b,J} \in W(k)$, we have:

$$\zeta_\varepsilon((e(\eta_{a,I}, a, I)\, e(\eta_{b,J}, b, J))\,|_{\mathrm{frp}}) \geqslant \zeta_\varepsilon(e(\eta_{a,I}, a, I)) + \zeta_\varepsilon(e(\eta_{b,J}, b, J)) + 1.$$

A consequence of lemma 2.7 is that:

$$\begin{aligned}\zeta_\varepsilon((e(\eta_{a,I}, a, I)\, e(\eta_{b,J}, b, J))\,|_{\mathrm{frp}}) &\\ \geqslant 2n(\mathrm{v}_V(\eta_{a,I}) &+ \mathrm{v}_V(\eta_{b,J}) + u(a) + u(b) - u(a+b))\\ &+ (\#I + \#J + 1)\,u(a+b) - \varepsilon|a+b|.\end{aligned}$$

Therefore, one can conclude if one has:

$$2n(u(a)+u(b)-u(a+b))+(\#I+\#J+1)\,u(a+b) \geqslant (\#I+1)\,u(a)+(\#J+1)\,u(b)+1.$$

Notice that $\#I+1\leqslant n$ and $\#J+1\leqslant n$ because we assumed that $I_0\neq\emptyset$ and $J_0\neq\emptyset$. Since $u(a+b)\leqslant\max\{u(a)\,,u(b)\}$, we get:

$$2n(u(a)+u(b))+(\#I+\#J+1-2n)\,u(a+b) \geqslant 2n\min\{u(a)\,,u(b)\}+(\#I+\#J+1)\max\{u(a)\,,u(b)\}\,.$$

This ends the proof whenever $n\neq 0$. If $n=0$, there is nothing to show. □

Proposition 3.11. — *Let $\varepsilon>0$, $x\in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$ and $y\in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$. We have:*

$$(\zeta_\varepsilon(x)\neq-\infty\wedge\zeta_\varepsilon(y)\neq-\infty)\implies\zeta_\varepsilon((xy)\,|_{\mathrm{frp}})\geqslant\zeta_\varepsilon(x)+\zeta_\varepsilon(y)\,.$$

Proof. — We only have to demonstrate that for any $(a,I)\,,(b,J)\in\mathcal{P}$ with $u(a)\neq 0$, $I_0\neq\emptyset$, $u(b)\neq 0$ and $J_0=\emptyset$, and any Witt vectors $\eta_{a,I},\eta_{b,J}\in W(k)$, we have $\zeta_\varepsilon((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{frp}})\geqslant\zeta_\varepsilon(e(\eta_{a,I},a,I))+\zeta_\varepsilon(e(\eta_{b,J},b,J))$. Using proposition 2.8, one obtains:

$$\begin{aligned}\zeta_\varepsilon((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{frp}}) &\geqslant 2n(\mathrm{v}_V(\eta_{a,I})+\mathrm{v}_V(\eta_{b,J})+\max\{u(a)\,,u(b)\}-u(a+b))\\ &\quad+(\#I+\#J+1)\,u(a+b)-\varepsilon|a+b|.\end{aligned}$$

So we can conclude if we show:

$$2n(\max\{u(a)\,,u(b)\}-u(a+b))+(\#I+\#J+1)\,u(a+b) \geqslant(\#I+1)\,u(a)+\#Ju(b)\,.$$

Notice that $\#I+1\leqslant n$ because we assumed that $I_0\neq\emptyset$ and $\#J\leqslant n$. As $u(a+b)\leqslant\max\{u(a)\,,u(b)\}$, we can see that:

$$2n\max\{u(a)\,,u(b)\}+(\#I+\#J+1-2n)\,u(a+b) \geqslant(\#I+\#J+1)\max\{u(a)\,,u(b)\}\,.\quad\square$$

Proposition 3.12. — *For any $\varepsilon>0$ and any $x,y\in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$, we have:*

$$(\zeta_\varepsilon(x)\neq-\infty\wedge\zeta_\varepsilon(y)\neq-\infty)\implies\zeta_\varepsilon\big((xy)\,|_{\mathrm{d(frp)}}\big)\geqslant\zeta_\varepsilon(x)+\zeta_\varepsilon(y)+3.$$

Proof. — We only have to verify that for any $(a,I)\,,(b,J)\in\mathcal{P}$, such that $u(a)\neq 0$, $I_0\neq\emptyset$, $u(b)\neq 0$ and $J_0\neq\emptyset$, and any $\eta_{a,I},\eta_{b,J}\in W(k)$, we have

$\zeta_\varepsilon\big((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{d(frp)}}\big) \geqslant \zeta_\varepsilon(e(\eta_{a,I},a,I))+\zeta_\varepsilon(e(\eta_{b,J},b,J))+3$. Due to lemma 2.7 we can see that:

$$\begin{aligned}&\zeta_\varepsilon\big((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{d(frp)}}\big)\\ &\quad\geqslant 2n(\mathrm{v}_V(\eta_{a,I})+\mathrm{v}_V(\eta_{b,J})+u(a)+u(b))+(\#I+\#J)\,u(a+b)-\varepsilon|a+b|.\end{aligned}$$

Therefore, the proof is complete if:

$$2n(u(a)+u(b))+(\#I+\#J)\,u(a+b)\geqslant(\#I+1)\,u(a)+(\#J+1)\,u(b)+3.$$

In the fractional part that we are studying, we necessarily have $u(a)>0$, $u(b)>0$ and $u(a+b)>0$. Moreover, $\#I+1\leqslant n$ and $\#J+1\leqslant n$ as we assumed that $I_0\neq\emptyset$ and $J_0\neq\emptyset$. So we get:

$$\begin{aligned}&2n(u(a)+u(b))+(\#I+\#J)\,u(a+b)\\ &\qquad\qquad\geqslant(\#I+1)\,u(a)+(\#J+1)\,u(b)+2+\#I+\#J.\end{aligned}$$

If $\#I+\#J\neq 0$, the proof is complete. Otherwise, it means that we are multiplying two Witt vectors. In particular, the projection on $d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$ is 0, but $\zeta_\varepsilon(0)=+\infty$, so the proof becomes obvious. □

Proposition 3.13. — *For any $\varepsilon>0$ and any $x,y\in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$, we have:*

$$(\zeta_\varepsilon(x)\neq-\infty\wedge\zeta_\varepsilon(y)\neq-\infty)\implies\zeta_\varepsilon((xy)\,|_{\mathrm{int}})\geqslant\zeta_\varepsilon(x)+\zeta_\varepsilon(y)+2.$$

Proof. — It is enough to prove that for any $(a,I),(b,J)\in\mathcal{P}$, such that $u(a)\neq0$, $I_0\neq\emptyset$, $u(b)\neq0$ and $J_0\neq\emptyset$, and for any $\eta_{a,I},\eta_{b,J}\in W(k)$, we have $\zeta_\varepsilon((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{int}})\geqslant\zeta_\varepsilon(e(\eta_{a,I},a,I))+\zeta_\varepsilon(e(\eta_{b,J},b,J))+2$. However, using lemma 2.7 one gets:

$$\begin{aligned}&\zeta_\varepsilon((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{int}})\\ &\qquad\qquad\geqslant 2n(\mathrm{v}_V(\eta_{a,I})+\mathrm{v}_V(\eta_{b,J})+u(a)+u(b))-\varepsilon|a+b|\end{aligned}$$

because in the integral part, we always have $u(a+b)=0$, which ends the proof as $n\geqslant\#I+1$ and $n\geqslant\#J+1$ since we assumed that $I_0\neq\emptyset$ and $J_0\neq\emptyset$. □

Proposition 3.14. — *For any $\varepsilon>0$ and any $x,y\in W\Omega^{\mathrm{frac},\bullet}_{k[\underline{X}]/k}$, we have:*

$$(\zeta_\varepsilon(x)\neq-\infty\wedge\zeta_\varepsilon(y)\neq-\infty)\implies\zeta_\varepsilon((xy)\,|_{\mathrm{int}})\geqslant\zeta_\varepsilon(x)+\zeta_\varepsilon(y)\,.$$

Proof. — We will first show that for any $(a,I),(b,J)\in\mathcal{P}$, such that $u(a)\neq0$, $I_0\neq\emptyset$ and $u(b)\neq0$, and any $\eta_{a,I},\eta_{b,J}\in W(k)$, we always have:

$$\zeta_\varepsilon((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{int}})\geqslant\zeta_\varepsilon(e(\eta_{a,I},a,I))+\zeta_\varepsilon(e(\eta_{b,J},b,J))\,.$$

Due to proposition 2.8 we can see that:

$$\begin{aligned}&\zeta_\varepsilon((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{int}})\\ &\qquad\qquad\geqslant 2n(\mathrm{v}_V(\eta_{a,I})+\mathrm{v}_V(\eta_{b,J})+\min\{u(a)\,,u(b)\})-\varepsilon|a+b|\end{aligned}$$

because in the integral part we have $u(a+b)=0$, which is only possible if $u(a)=u(b)$. This proves this specific case because $2n \geqslant (\#I+1+\#J)$ if $J_0=\emptyset$, and $2n \geqslant (\#I+1+\#J+1)$ otherwise.

For the general case, notice that if $I_0=\emptyset$, then proposition 1.2 gives us the equality:

$$e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J)=d(e(\eta_{a,I},a,I\smallsetminus\{\min(a)\})\,e(\eta_{b,J},b,J)) \\ -(-1)^{\#I-1}\,e(\eta_{a,I},a,I\smallsetminus\{\min(a)\})\,d(e(\eta_{b,J},b,J))\,.$$

Therefore, we can conclude using the first paragraph as well as (10). □

Proposition 3.15. — *Let $\varepsilon>0$, $x\in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$ and $y\in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$. We have:*

$$(\zeta_\varepsilon(x)\neq-\infty\wedge\zeta_\varepsilon(y)\neq-\infty)\implies\zeta_\varepsilon\big((xy)\,|_{\mathrm{d(frp)}}\big)\geqslant\zeta_\varepsilon(x)+\zeta_\varepsilon(y)+1.$$

Proof. — We only have to show that for any $(a,I),(b,J)\in\mathcal{P}$, such that $u(a)\neq0$, $I_0\neq\emptyset$, $u(b)\neq0$ and $J_0=\emptyset$, and any $\eta_{a,I},\eta_{b,J}\in W(k)$, we have $\zeta_\varepsilon\big((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{d(frp)}}\big)\geqslant\zeta_\varepsilon(e(\eta_{a,I},a,I))+\zeta_\varepsilon(e(\eta_{b,J},b,J))+1$. Using proposition 2.8 one finds that:

$$\zeta_\varepsilon\big((e(\eta_{a,I},a,I)\,e(\eta_{b,J},b,J))\,|_{\mathrm{d(frp)}}\big) \\ \geqslant 2n(\mathrm{v}_V(\eta_{a,I})+\mathrm{v}_V(\eta_{b,J})+\min\{u(a)\,,u(b)\}) \\ +(\#I+\#J)\,u(a+b)-\varepsilon|a+b|.$$

So the proof is over if:

$$2n\min\{u(a)\,,u(b)\}+(\#I+\#J)\,u(a+b)\geqslant(\#I+1)\,u(a)+\#Ju(b)+1.$$

Since $\#I+1\leqslant n$ and $1\leqslant\#J\leqslant n$ because we assumed that $I_0\neq\emptyset$ and $J_0=\emptyset$, and since $u(a+b)\neq0$ because we study the fractional part, this inequality becomes obvious whenever $u(a)=u(b)$; if not, then $u(a+b)=\max\{u(a)\,,u(b)\}$, and we are done. □

Proposition 3.16. — *For any $\varepsilon>0$ and any $x,y\in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$, we get:*

$$(\zeta_\varepsilon(x)\neq-\infty\wedge\zeta_\varepsilon(y)\neq-\infty)\implies\zeta_\varepsilon\big((xy)\,|_{\mathrm{d(frp)}}\big)\geqslant\zeta_\varepsilon(x)+\zeta_\varepsilon(y)\,.$$

Proof. — One can suppose without any loss of generality that $x\in W\Omega^i_{k[\underline{X}]/k}$, for some $i\in\mathbb{N}$. Put $y'\in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$, such that $d(y')=y$. Using proposition 1.2 we get $\zeta_\varepsilon(y')=\zeta_\varepsilon(y)$. However, $xy=(-1)^i\,d(xy')$, so we can conclude due to (9), (10) as well as proposition 3.11. □

We will now study the cases not treated in the previous statements of this section. Notice that if one takes $x, y \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$, then $xy \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$. In particular, $(xy)\,|_{\mathrm{frp}} = (xy)\,|_{\mathrm{d(frp)}} = 0$, which implies:

$$\forall \varepsilon > 0,\ \zeta_\varepsilon((xy)\,|_{\mathrm{frp}}) = \zeta_\varepsilon\big((xy)\,|_{\mathrm{d(frp)}}\big) = +\infty.$$

In a similar fashion, if $x \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$ and $y \in W\Omega^{\mathrm{frac},\bullet}_{k[\underline{X}]/k}$, lemma 2.7 implies that:

$$\forall \varepsilon > 0,\ \zeta_\varepsilon((xy)\,|_{\mathrm{int}}) = +\infty.$$

Also, if $x, y \in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$, then as xy lies in the image of d, we get $(xy)\,|_{\mathrm{frp}} = 0$, which in turn implies that:

$$\forall \varepsilon > 0,\ \zeta_\varepsilon((xy)\,|_{\mathrm{frp}}) = +\infty.$$

The following table compiles all of the propositions that we have shown concerning the function ζ_ε, for any $\varepsilon > 0$ (we will always suppose that $\zeta_\varepsilon(x) \neq -\infty$ and $\zeta_\varepsilon(y) \neq -\infty$).

	$\zeta_\varepsilon((xy)\,\vert_{\mathrm{int}}) \geqslant$	$\zeta_\varepsilon((xy)\,\vert_{\mathrm{frp}}) \geqslant$	$\zeta_\varepsilon\big((xy)\,\vert_{\mathrm{d(frp)}}\big) \geqslant$
$x \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$ $y \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$	$+\infty$	$+\infty$
$x \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$ $y \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$	$+\infty$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1$
$x \in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$ $y \in W\Omega^{\mathrm{int},\bullet}_{k[\underline{X}]/k}$	$+\infty$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$
$x \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$ $y \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 2$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 3$
$x \in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$ $y \in W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y) + 1$
$x \in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$ $y \in d\Big(W\Omega^{\mathrm{frp},\bullet}_{k[\underline{X}]/k}\Big)$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$	$+\infty$	$\zeta_\varepsilon(x) + \zeta_\varepsilon(y)$

In particular, this proves the main theorem of this paper:

Theorem 3.17. — *For any $\varepsilon > 0$, the function ζ_ε is a pseudovaluation.*

Proof. — This is straightforward using (9) and the previous table. □

Corollary 3.18. — *Let $\varepsilon > 0$. Let $\varphi\colon k[\underline{X}] \to R$ be a surjective morphism of commutative k-algebras. Then:*

$$\zeta_{\varepsilon,\varphi}\colon \begin{array}{c} W\Omega^\bullet_{R/k} \to \mathbb{R} \cup \{+\infty, -\infty\} \\ x \mapsto \sup\{\zeta_\varepsilon(y) \mid y \in \varphi^{-1}(\{x\})\} \end{array}$$

is a pseudovaluation.

Proof. — According to [3, p. 4], this map is a pseudovaluation if and only if $\zeta_{\varepsilon,\varphi}(1) \neq +\infty$. We will show that $\zeta_{\varepsilon,\varphi}(1) \leqslant 0$. Let $y \in W\Omega^\bullet_{k[\underline{X}]/k}$ such that $\zeta_\varepsilon(y) > 0$. Write $y = \sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I)$ with $\eta_{a,I} \in W(k)$, for all $(a, I) \in \mathcal{P}$ using theorem 1.5. Then, by definition of ζ_ε, for all $(a, I) \in \mathcal{P}$, such that $\eta_{a,I} \neq 0$, we must have $2n\,\mathrm{v}_V(\eta_{a,I}) + u(a) > 0$. If $n = 0$, this cannot happen, otherwise then either $\mathrm{v}_V(\eta_{a,I}) > 0$ or $u(a) > 0$. In all cases, this implies that $e(\eta_{a,I}, a, I)$ is in the image of V. In turn, y is also in the image of V, and by functoriality of $W\Omega^\bullet_{\bullet/k}$, so is $\varphi(y)$. In particular, $\varphi(y) \neq 1$, so $\zeta_{\varepsilon,\varphi}(1) \leqslant 0$, and we are finished. □

In subsequent papers, we will use these results and this table in order to study the local structure of the overconvergent de Rham–Witt complex, and give an interpretation of F-isocrystals in this context.

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