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THE GENERALIZED INJECTIVITY CONJECTURE

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ABSTRACT. — We prove a conjecture of Casselman and Shahidi stating that the unique irreducible generic subquotient of a standard module is necessarily a subrepresentation for a large class of connected quasi-split reductive groups, in particular for those that have a root system of classical type (or product of such groups). To do so, we prove and use the existence of strategic embeddings for irreducible generic discrete series representations, extending some results of Moeglin.

RÉSUMÉ (*La conjecture d'injectivité généralisée*). — Nous prouvons la conjecture de Casselman-Shahidi, qui affirme que l'unique sous-quotient générique d'un module standard est nécéssairement une sous-représentation, pour une large classe de groupes réductifs, quasi-déployés et connexes, en particulier ceux qui ont un système de racines de type classique (ou produit de tels groupes). Pour se faire, nous prouvons l'existence de certains plongements particuliers de représentations séries discrètes, généralisant ainsi des résultats de Moeglin.

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1. Introduction

1.1. Let G be a quasi-split connected reductive group over a non-Archimedean local field F of characteristic zero. We assume that we are given a standard parabolic subgroup P with Levi decomposition $P = MU$, as well as an irreducible, tempered, generic representation τ of M. Now let ν be an element in the dual of the real Lie algebra of the split component of M; we take it in the positive Weyl chamber. The induced representation $I_P^G(\tau, \nu) := I_P^G(\tau_\nu)$, called the standard module, has a unique irreducible quotient, $J(\tau_\nu)$, often named the Langlands quotient. Since the representation τ is generic (for a non-degenerate character of U, see Section 2), i.e. has a Whittaker model, the standard module $I_P^G(\tau_\nu)$ is also generic. Further, by a result of Rodier [29] any generic induced module has a unique irreducible generic subquotient.

In their paper, Casselman and Shahidi [7] conjectured that:

(A) $J(\tau_\nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.
(B) The unique irreducible generic subquotient of $I_P^G(\tau_\nu)$ is a subrepresentation.

These questions were originally formulated for real groups by Vogan [38]. Conjecture (B), was resolved in [7] provided the inducing data is cuspidal. Conjecture (A), known as the standard module conjecture, was first proven for classical groups by Muić in [26] and was settled for quasi-split p-adic groups in [18] assuming the tempered L function conjecture proven a few years later in [19].

The second conjecture, known as the generalized injectivity conjecture was proved for classical groups $\mathrm{SO}(2n+1), Sp(2n)$, and $\mathrm{SO}(2n)$ for P a maximal parabolic subgroup, by Hanzer in [13].

In the present work, we prove the generalized injectivity conjecture (Conjecture (B)) for a large class of quasi-split connected reductive groups provided that the irreducible components of a certain root system (denoted Σ_σ) are of type A, B, C or D (see Theorem 1.1 below for a precise statement). Following the terminology of Borel–Wallach [4.10 in [3]], for a standard parabolic subgroup P, τ a tempered representation and $\eta \in (a_M^*)^+$, a positive Weyl chamber, (P, τ, η) is referred as Langlands data, and η is the Langlands parameter, see the Definition 2.8 herein.

We will study the unique irreducible generic subquotient of a standard module $I_P^G(\tau_\eta)$ and *first* make the following reductions:

- τ is a discrete series representation of the standard Levi subgroup M
- P is a maximal parabolic subgroup.
  Then, η is written $s\tilde{\alpha}$, see Section 1.4 for a definition of the latter.

Then, our approach has two layers. First, we realize the generic discrete series τ as a subrepresentation of an induced module $I_{P_1\cap M}^M(\sigma_\nu)$ for a unitary generic cuspidal representation of M_1 (using Proposition 2.5 of [19]), and the parameter

ν is dominant (i.e. in some positive closed Weyl chamber) in a sense later made precise; Using induction in stages, we can therefore embed the standard module $I_P^G(\tau_{s\tilde{\alpha}})$ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Let us denote $\nu + s\tilde{\alpha} := \lambda$. The unique generic subquotient of the standard module is also the unique generic subquotient in $I_{P_1}^G(\sigma_\lambda)$. By a result of Heiermann–Opdam [Proposition 2.5 of [19]], this generic subquotient appears as a subrepresentation of yet another induced representation $I_{P'}^G(\sigma'_{\lambda'})$ characterized by a parameter λ' in the closure of some positive Weyl chamber.

In an ideal scenario, λ and λ' are dominant with respect to P_1 (resp. P'), i.e. λ and λ' are in the closed positive Weyl chamber, and we may then build a bijective operator between those two induced representations using the dominance property of the Langlands parameters.

In case the parameter λ is not in the closure of the positive Weyl chamber, two alternatives procedures are considered: first, another strategic embedding of the irreducible generic subquotient in the representation induced from $\sigma''_{\lambda''}$ (relying on extended Moeglin's Lemmas) when the parameter λ'' (which depends on the form of λ) has a very specific aspect (this is Proposition 6.14); or (resp. and) showing the intertwining operator between $I_{P'}^G(\sigma'_{\lambda'})$ (or $I_{P_1}^G(\sigma''_{\lambda''})$) and $I_{P_1}^G(\sigma_\lambda)$ has a non-generic kernel.

1.2. In order to study a larger framework than the one of classical groups studied in [13], we will use the notion of *residual points* of the μ function (the μ function is the main ingredient of the Plancherel density for p-adic groups (see the Definition 2.4 and Section 2.2).

Indeed, as briefly suggested in the previous point, the triple (P_1, σ, λ), introduced above, plays a pivotal role in all the arguments developed thereafter, and of particular importance, the parameter λ is related to the μ function in the following ways:

- When σ_λ is a residual point for the μ function (abusively one says that λ is a residual point once the context is clear), the unique irreducible generic subquotient in the module induced from σ_λ is discrete series (a result of Heiermann in [15], see Proposition 2.6).
- Once the cuspidal representation σ is fixed, we attach to it the set Σ_σ, a root system in a subspace of $a^*_{M_1}$ defined using the μ function. More precisely, let α be a root in the set of reduced roots of A_{M_1} in Lie(G) and $(M_1)_\alpha$ be the centraliser of $(A_{M_1})_\alpha$ (the identity component of the kernel of α in A_{M_1}). We will consider the set

$$\Sigma_\sigma = \{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

  It is a subset of $a^*_{M_1}$ which is a root system in a subspace of $a^*_{M_1}$(cf. [35] 3.5) and we suppose the irreducible components of Σ_σ are of type A, B, C or D. Let us denote W_σ the Weyl group of Σ_σ.

This is where stands the particularity of our method, to deal with all possible standard modules, we needed an explicit description of this parameter λ lying in $a^*_{M_1}$. Thanks to Opdam's work in the context of affine Hecke algebras and Heiermann's one in the context of p-adic reductive groups such descriptive approach is made possible. Indeed, we have a bijective correspondence between the following sets explained in Section 4: {dominant residual point} $\leftrightarrow$ Weighted Dynkin diagram(s)}

The notion of Weighted Dynkin diagram is established and recalled in the Appendix A.1. We use this correspondence to express the coordinates of the dominant residual point and name this expression of the residual point a *residual segment* generalizing the classical notion of segments (of Bernstein–Zelevinsky). We associate to such a residual segment *set(s) of jumps* (a notion connected to that of Jordan block elements in the classical groups setting of Moeglin–Tadić in [23]).

Further, the μ function is intrinsically related to the *intertwining operators* mentioned in the previous subsection. A key aspect of this work is an appropriate use of (standard) intertwining operators, more precisely the use of intertwining operators with a non-generic kernel. Using the functoriality of induction, it is always possible to reduce the study of intertwining operators to *rank 1* intertwining operators (i.e. consider the well-understood intertwining operator $J_{s_{\alpha_i}P_1|P_1}$ between $I^{M_1}_{P_1\cap (M_1)_{\alpha_i}}(\sigma_\lambda)$ and $I^{M_1}_{\overline{P_1}\cap (M_1)_{\alpha_i}}(\sigma_\lambda)$); and in particular if σ is irreducible cuspidal (see Theorem 2.1). At the level of rank 1 intertwining operator (where $I^{M_1}_{P_1\cap (M_1)_{\alpha_i}}(\sigma_\lambda)$ is the direct sum of two non-isomorphic representations, see Theorem 2.1), determining the non-genericity of the kernel of the map $J_{s_{\alpha_i}P_1|P_1}$ reduces to a simple condition on the relevant coordinates (i.e. the coordinates determined by α_i) of $\lambda \in a^*_{M_1}$.

1.3. Having defined the root system Σ_σ let us present the main result of this paper:

Theorem 1.1 (Generalized injectivity conjecture for a quasi-split group). — *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero) such that its root system is of type A, B, C or D (or the product of these). Let π_0 be the unique irreducible generic subquotient of the standard module $I^G_P(\tau_\nu)$; then π_0 embeds as* a subrepresentation *in the standard module $I^G_P(\tau_\nu)$.*

Theorem 1.2 (Generalized injectivity conjecture for quasi-split group). — *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero). Let π_0 be the unique, irreducible generic subquotient of the standard module $I^G_P(\tau_\nu)$ and let σ be an irreducible, generic, cuspidal representation of*

M_1 such that a twist by an unramified real character of σ is in the cuspidal support of π_0.

Suppose that all the irreducible components of Σ_σ are of type A, B, C or D; then, under certain conditions on the Weyl group of Σ_σ (which is explained in Section 6.1, in particular Corollary 6.6), π_0 embeds as a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

Theorem 1.1 results from 1.2. Theorem 1.2 is true when the root system of the group G contains components of type E, F provided that Σ_σ is irreducible and of type A. We do not know if an analogue of Corollary 6.6 holds for groups whose root systems are of type E or F. Further, in the exceptional groups of type E or F, many cases where the cuspidal support of π_0 is (P_0, σ) (generalized principal series) cannot be dealt with using the methods proposed in this work; see Section 9 for details.

1.4. Let us briefly comment on the organisation of this manuscript, therefore giving a general overview of our results and the scheme of proof.

In Section 3, we formulate the problem in an as broad as possible context (any quasi-split reductive p-adic group G) and prove a few results on intertwining operators.

As M. Hanzer in [13], we distinguish two cases: the case of a generic discrete series subquotient and the case of a non-discrete series generic subquotient. As stated in 1.2, the case of a discrete series subquotient corresponds to σ_λ (in the cuspidal support of the generic discrete series) being a residual point.

As just stated in 1.2, our approach uses the bijection between Weyl group orbits of residual points and weighted Dynkin diagrams as studied in [27] and explained in the Appendix A.

Through this approach, we can make explicit the Langlands parameters of subquotients of the representations $I_{P_1}^G(\sigma_\lambda)$ induced from the generic cuspidal support σ_λ and classify them using the order on parameters in $a_{M_1}^*$ as given in Chapter XI, Lemma 2.13 in [3]. In particular, the minimal element for this order (in a sense that is later made precise) characterizes the unique irreducible generic non-discrete series subquotient; see Theorem 5.5.

Although requiring us to get acquainted with the notions of residual points, and then residual segments, our methods have two advantages.

The first is proving the generalized injectivity conjecture for a large class of quasi-split reductive groups (provided a certain construction of the standard Levi subgroup M_1 and the irreducible components of Σ_σ to be of type A, B, C or D; we have verified those conditions when the root system of the quasi-split (hence reductive) group is of type A, B, C or D), and recovering the results of Hanzer through alternative proofs. In particular, a key ingredient (which was not used by Hanzer in [13]) in our method is an embedding result of Heiermann–Opdam (Proposition 2.7). The second is a self-contained and uniform (in the sense that cases of root systems of type B, C and D are all treated in the same proofs) treatment.

Although based on the ideas of Hanzer in [13], our approach includes a much larger class of quasi-split groups and some cases of exceptional groups.

We separate this work into two different problems. The first problem is to determine the conditions on $\lambda \in a^*_{M_1}$ so that the unique generic subquotient of $I^G_{P_1}(\sigma_\lambda)$ with σ irreducible unitary generic cuspidal representation of a standard Levi M_1 is a subrepresentation. The results on this problem are presented in Theorem 6.3.

The second problem is to show that any standard module can be embedded in a module induced from cuspidal generic data, with $\lambda \in a^*_{M_1}$ satisfying one of the conditions mentioned in Theorem 6.3. This is done in the Section 7 and the following.

Regarding the first problem: in the Section 6.3, we present an embedding result for the unique irreducible generic discrete series subquotient of the generic standard module (see Proposition 6.14) relying on two extended Moeglin's lemmas (see Lemmas 6.12 and 6.13) and the result of Heiermann–Opdam (see Proposition 2.7). This embedding and the use of standard intertwining operators with a non-generic kernel allow us to prove the Theorem 6.3.

Once achieved the Theorem 6.3; it is rather straightforward to prove the generalized injectivity conjecture for the discrete series generic subquotient, first when P is a maximal parabolic subgroup and secondly for *any parabolic subgroup* in Section 7.2.

In Section 7.3, we continue with the case of a generic non-discrete series subquotient and further conclude with the case of the standard module induced from a tempered representation τ in Corollary 7.11 and Corollary 8.3.

The proof of Theorem 1.2 is done in several steps. First, we prove it for the case of an irreducible generic discrete series subquotient assuming τ discrete series and Σ_σ irreducible in Proposition 7.3.

We use this latter result for the case of a non-square integrable irreducible generic subquotient in Proposition 7.9; and also for the case of standard modules induced from non-maximal standard parabolic (Theorems 7.8 and 7.10). Then, the case of τ tempered follows (Corollary 7.11). The case of Σ_σ reducible is done in Section 8 and relies on the Appendix B.

The reader familiar with the work of Bernstein–Zelevinsky on GL_n (see [30] or [40]) may want to have a look at the author's PhD thesis where we treat independently the case of Σ_σ of type A to get a quicker overview on some tools used in this work.

From here, we use the following notations:

Notation. —
- *Standard module induced from a maximal parabolic subgroup:* Let $\Theta = \Delta - \{\alpha\}$ for α in Δ and let $P = P_\Theta$ be a maximal parabolic subgroup of G. We denote ρ_P the half sum of positive roots

in U, and for α the unique simple root for G, which is not a root for M,

$$\tilde{\alpha} = \frac{\rho_P}{(\rho_P, \alpha)} \quad \text{where } (\rho_P, \alpha) = \frac{2\langle \rho_P, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

(Rather than $\tilde{\alpha}$, in the split case, we could also take the fundamental weight corresponding to α). Since ν is in a_M^* (of dimension rank(G) − rank$(M) = 1$ since M is maximal) and should satisfy $\langle \nu, \check{\beta} \rangle > 0$, for all $\beta \in \Delta - \Theta = \{\alpha\}$, the standard module in this case is $I_P^G(\tau_{s\tilde{\alpha}})$, where $s \in \mathbb{R}$ such that $s > 0$, and τ is an irreducible tempered representation of M.

- For the sake of readability, we sometimes denote $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is expressed in terms of residual segments.
- Let σ be an irreducible cuspidal representation of a Levi subgroup $M_1 \subset M$ in a standard parabolic subgroup P_1, and let λ be in $(a_{M_1}^*)$; we denote $Z^M(P_1, \sigma, \lambda)$ the unique irreducible generic discrete series (or essentially square-integrable) in the standard module $I_{P_1 \cap M}^M(\sigma_\lambda)$.

  We will omit the index when the representation is a representation of G: $Z(P_1, \sigma, \lambda)$; often λ will be written explicitly with residual segments to emphasize the dependency on specific sequences of exponents.

2. Preliminaries

2.1. Basic objects. — Throughout this paper we will let F be a non-Archimedean local field of characteristic 0. We will denote by G the group of F-rational points of a quasi-split connected reductive group defined over F. We fix a minimal parabolic subgroup P_0 (which is a Borel B since G is quasi-split) with Levi decomposition $P_0 = M_0 U_0$ and A_0 a maximal split torus (over F) of M_0; P is said to be standard if it contains P_0. More generally, if P rather contains A_0, it is said to be semi-standard. Then P contains a unique Levi subgroup M containing A_0, and M is said to be semi-standard. For a semi-standard Levi subgroup M, we denote $\mathscr{P}(M)$ the set of parabolic subgroups P with Levi factor M.

We denote by A_M the maximal split torus in the center of M, $W = W^G$ the Weyl group of G defined with respect to A_0 (i.e. $N_G(A_0)/Z_G(A_0)$). The choice of P_0 determines an order in W, and we denote by w_0^G the longest element in W.

If Σ denote the set of roots of G with respect to A_0, the choice of P_0 also determines the set of positive roots (or negative roots, simple roots) which we denote by Σ^+ (or Σ^-, Δ).

To a subset $\Theta \subset \Delta$ we associate a standard parabolic subgroup $P_\Theta = P$ with Levi decomposition MU and denote A_M the split component of M. We will write a_M^* for the dual of the real Lie-algebra a_M of A_M, $(a_M)_{\mathbb{C}}^*$ for its complexification and a_M^{*+} for the positive Weyl chamber in a_M^* defined with respect to P. Further, $\Sigma(A_M)$ denotes the set of roots of A_M in $\mathrm{Lie}(G)$. It is a subset of a_M^*. For any root $\alpha \in \Sigma(A_M)$, we can associate a coroot $\check{\alpha} \in a_M$. For $P \in \mathcal{P}(M)$, we denote $\Sigma(P)$ the subset of positive roots of A_M relative to P.

Let $\mathrm{Rat}(M)$ be the group of F-rational characters of M; we have:

$$a_M^* = \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R} \text{ and } (a_M)_{\mathbb{C}}^* = a_M^* \otimes_{\mathbb{R}} \mathbb{C}.$$

For $\chi \otimes r \in a_M^*$, $r \in \mathbb{R}$, and λ in a_M, the pairing $a_M \times a_M^* \to \mathbb{R}$ is given by: $\langle \lambda, \chi \otimes r \rangle = \lambda(\chi) \cdot r$

Following [39] we define a map

$$H_M : M \to a_M = \mathrm{Hom}(\mathrm{Rat}(M), \mathbb{R})$$

such that

$$|\chi(m)|_F = q^{-\langle \chi, H_M(m) \rangle},$$

for every F-rational character χ in a_M^* of M, q being the cardinality of the residue field of F. Then H_P is the extension of this homomorphism to P, extended trivially along U.

We denote by $X(M)$ the group of unramified characters of M.

Let us assume that (σ, V) is an admissible complex representation of M. We adopt the convention that the isomorphism class of (σ, V) is denoted by σ. If χ_ν is in $X(G)$, with $\nu \in a_{G,\mathbf{C}}^*$, then we write $(\sigma_\nu, V_{\chi_\nu})$ for the representation $\sigma \otimes \chi_\nu$ on the space V.

Let (σ, V) be an admissible representation of finite length of M, a Levi subgroup containing M_0 a minimal Levi subgroup, centraliser of the maximal split torus A_0. Let P and P' be in $\mathcal{P}(M)$. Consider the intertwining integral:

$$(J_{P'|P}(\sigma_\nu) f)(g) = \int_{U \cap U' \backslash U'} f(u'g) du' \quad f \in I_P^G(\sigma_\nu),$$

where U and U' denote the unipotent radical of P and P', respectively.

For ν in $X(M)$ with $\mathrm{Re}(\langle \nu, \check{\alpha} \rangle) > 0$, for all α in $\Sigma(P) \cap \Sigma(P')$, the defining integral of $J_{P'|P}(\sigma_\nu)$ converges absolutely. Moreover, $J_{P'|P}$ defined in this way on some open subset of $\mathcal{O} = \{\sigma_\nu | \nu \in X(M)\}$ becomes a rational function on $\mathcal{O}$ ([39] Theorem IV 1.1). Outside its poles, this defines an element of

$$\mathrm{Hom}_G(I_P^G(V_\chi), I_{P'}^G(V_\chi)).$$

Moreover, for any χ in $X(M)$, there exists an element v in $I_P^G(V_\chi)$ such that $J_{P'|P}(\sigma_\chi) v$ is not zero ([39], IV.1 (10))

In particular, for all ν in an open subset of a_M^*, and $\overline{P}$ the opposite parabolic subgroup to P, we have an intertwining operator

$$J_{\overline{P}|P}(\sigma_\nu) : I_P^G(\sigma_\nu) \to I_{\overline{P}}^G(\sigma_\nu)$$

and for ν in $(a_M^*)^+$ far away from the walls it is defined by the convergent integral:

$$(J_{\overline{P}|P}(\sigma_\nu)f)(g) = \int_{\overline{U}} f(ug)du.$$

The intertwining operator is meromorphic in ν, and the map $J_{\overline{P}|P}J_{P|\overline{P}}$ is a scalar. Its inverse equals the Harish-Chandra μ function up to a constant and will be denoted $\mu^G(\sigma_\nu)$.

CONVENTION. — By [32] Sections 3.3 and 1.4, we can fix a non-degenerate character ψ of U which, for every Levi subgroup M, is compatible with $w_0^G w_0^M$. We will still denote ψ the restriction of ψ to $M \cap U$. Every generic representation π of M becomes generic with respect to ψ after changing the splitting in U. Throughout this paper, generic means ψ-generic. When the groups are quasi-split and connected, by a theorem of Rodier, the standard ψ-generic modules have exactly one ψ-generic irreducible subquotient. This unicity will be used in numerous proofs; we will use the name [U] to refer to this result.

2.2. The μ function. — Harish-Chandra's μ-function is the main ingredient of the Plancherel density for a p-adic reductive group G [39]. It assigns to every discrete series representation of a Levi subgroup a complex number and can be analytically extended to a meromorphic function on the space of essentially square-integrable representations of Levi subgroups.

Let $Q = NV$ be a parabolic subgroup of a connected reductive group G over F and σ an irreducible unitary cuspidal representation of N, then Harish-Chandra's μ-function μ^G corresponding to G defines a meromorphic function $a_{N,\mathbb{C}}^* \to \mathbb{C}$, $\lambda \to \mu^G(\sigma_\lambda)$ (cf. [15], Proposition 4.1, [34], 1.6), which (in a certain context, see Proposition 4.1 in [15]) can be written as:

$$\mu^G(\sigma_\lambda) = f(\lambda) \prod_{\alpha \in \Sigma(Q)} \frac{(1 - q^{\langle \check{\alpha}, \lambda \rangle})(1 - q^{-\langle \check{\alpha}, \lambda \rangle})}{(1 - q^{\epsilon_\alpha + \langle \check{\alpha}, \lambda \rangle})(1 - q^{\epsilon_\alpha - \langle \check{\alpha}, \lambda \rangle})},$$

where f is a meromorphic function without poles and zeroes on a_N^*, and the ϵ_α are non-negative rational numbers such that $\epsilon_\alpha = \epsilon_{\alpha'}$ if α and α' are conjugate. We refer the reader to Sections IV.3 and V.2 of [39] for some further properties of the Harish-Chandra μ function.

Clearly, the μ function denoted above μ^G can be defined with respect to any reductive group G; in particular, below we will use the functions μ^M for a Levi subgroup M.

Let $P_1 = M_1U_1$ be a standard parabolic subgroup. In [16] and [17], with the notations introduced in Section 3.2.1, the following results are mentioned:

THEOREM 2.1 (Harish-Chandra, see [17], 1.2). — *Fix a root $\alpha \in \Sigma(P_1)$ and an irreducible cuspidal representation σ of M_1.*

(a) *If* $\mu^{(M_1)_\alpha}(\sigma)=0$*, then there exists a unique (see Casselman's notes, 7.1 in* [6]*) non-trivial element* s_α *in* $W^{(M_1)_\alpha}(M_1)$ *so that* $s_\alpha(P_1\cap(M_1)_\alpha)=\overline{P_1}\cap(M_1)_\alpha$ *and* $s_\alpha\sigma\cong\sigma$*.*
(b) *If there exists a unique non-trivial element* s_α *in* $W^{(M_1)_\alpha}(M_1)$*, so* $s_\alpha(P_1\cap(M_1)_\alpha)=\overline{P_1}\cap(M_1)_\alpha$ *and* $s_\alpha\sigma\cong\sigma$*, then* $\mu^{(M_1)_\alpha}(\sigma)\neq 0\Leftrightarrow I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\sigma)$ *is reducible.*

If it is reducible, it is the direct sum of two non-isomorphic representations.

The μ function's factor in this setting is:

$$\mu^{(M_1)_\beta}(\sigma_\lambda)=c_\beta(\lambda)\cdot\frac{(1-q^{\langle\check\beta,\lambda\rangle})(1-q^{-\langle\check\beta,\lambda\rangle})}{(1-q^{\epsilon_{\check\beta}+\langle\check\beta,\lambda\rangle})(1-q^{\epsilon_{\check\beta}-\langle\check\beta,\lambda\rangle})}.$$

LEMMA 2.2 (Lemma 1.8 in [17]). — *Let* $\alpha\in\Delta_\sigma$*,* $s=s_\alpha$ *and assume* $(M_1)_\alpha$ *is a standard Levi subgroup of* G*. The operators* $J_{sP_1|P_1}$ *are meromorphic functions in* σ_λ *for* σ *unitary cuspidal representation and* λ *a parameter in* $(a_{M_1}^{(M_1)_\alpha}*)$*.*

The poles of $J_{sP_1|P_1}$ *are precisely the zeroes of* $\mu^{(M_1)_\alpha}$*. Any pole has order 1, and its residue is bijective. Furthermore,* $J_{P_1|sP_1}J_{sP_1|P_1}$ *equals* $(\mu^{(M_1)_\alpha})^{-1}$ *up to a multiplicative constant.*

Let us summarize the different cases:

- If $\mu^{(M_1)_\alpha}$ has a pole at σ_λ; then, the operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ (which are necessarily both non-zero) cannot be bijective. Indeed, at σ_λ, their product is zero; if any were bijective, it would imply that the other is zero.
- If $\mu^{(M_1)_\alpha}$ has a zero in σ_λ; it is Lemma 2.2 above.

Further, by a general result concerning the μ function, it has one and only one pole on the positive real axis if and only if, for σ a unitary irreducible cuspidal representation, $\mu(\sigma)=0$. Therefore, for each $\alpha\in\Sigma_\sigma$, by definition, there is be one λ on the positive real axis such that $\mu^{(M_1)_\alpha}$ has a pole.

EXAMPLE 2.3. — Consider the group $G=\mathrm{GL}_{2n}$ and one of its maximal Levi subgroups $M:=\mathrm{GL}_n\times\mathrm{GL}_n$. Set $\sigma_s:=\rho|\det|^s\otimes\rho|\det|^{-s}$ with ρ irreducible unitary cuspidal representation of GL_n. Then, $\mu(\rho\otimes\rho)=0$, and it is well known that at $s=\pm1/2$, $\mu(\sigma_s)$ has a pole and the operators $J_{P|\overline{P}}$ and $J_{\overline{P}|P}$ are not bijective.

2.3. Some results on residual points. — Let Q be any parabolic subgroup of G, with Levi decomposition $Q=LU$. We recall that the parabolic rank of G (with respect to L) is $\mathrm{rk}_{ss}(G)-\mathrm{rk}_{ss}(L)$, where rk_{ss} stands for the semi-simple rank. The following definition will be useful:

Definition 2.4 (residual point). — A point σ_ν for σ an irreducible unitary cuspidal representation of L is called a residual point for μ^G if

$$|\{\alpha \in \Sigma(Q) | \langle \check{\alpha}, \nu \rangle = \pm \epsilon_\alpha\}| - 2|\{\alpha \in \Sigma(Q) | \langle \check{\alpha}, \nu \rangle = 0\}| = \dim(a_L^*/a_G^*) = \mathrm{rk}_{ss}(G) - \mathrm{rk}_{ss}(L),$$

where ϵ_α appears in Section 2.2.

Remark 2.5. — Since the μ function depends only on a complex variable identified with $\sigma \otimes \chi_\lambda$, for $\lambda \in (a_L^G)^*$; once the unitary cuspidal representation σ is fixed, we will freely talk about λ (rather than σ_λ) as a residual point.

The main result of Heiermann in [15] is the following:

Theorem 2.6 (Corollary 8.7 in [15]). — *Let $Q = LU$ be a parabolic subgroup of G, σ a unitary cuspidal representation of L, and ν in a_L^*. For the induced representation $I_Q^G(\sigma_\nu)$ to have a discrete series subquotient, it is necessary and sufficient for σ_ν to be a residual point for μ^G and the restriction of σ_ν to A_G (the maximal split component in the center of G) to be a unitary character.*

We will also make a crucial use of the following result from [19]:

Proposition 2.7 (Proposition 2.5 in [19]). — *Let π be an irreducible generic representation that is a discrete series of G. There exists a standard parabolic subgroup $Q = LU$ of G and a unitary generic cuspidal representation (σ, E) of L, with $\nu \in \overline{(a_L^*)^+}$ such that π is a subrepresentation of $I_Q^G(\sigma_\nu)$.*

We need the following definition to recall the Langlands' classification (see, for instance, [3] Theorem 2.11 or [20]):

Definition 2.8. — A set of Langlands data for G is a triple (P, τ, ν) with the following properties:

1. $P = MU$ is a standard parabolic subgroup of G
2. ν is in $(a_M^*)^+$
3. τ is (the equivalence class of) an irreducible tempered representation of M.

Theorem 2.9 (Langlands' classification). —

1. *Let (P, τ, ν) be a set of Langlands data. Then the induced representation $I_P^G(\tau_\nu)$ has a unique irreducible quotient, the Langlands quotient denoted $J(P, \nu, \tau)$.*
2. *Let π be an irreducible admissible representation of G. Then there exists a unique triple (P, ν, τ) as in (1) such that $\pi = J(P, \nu, \tau)$. We call this triple the Langlands data, and ν is called the Langlands parameter of π.*

Theorem 2.10 (*Standard module conjecture* proved in [18] and [19]). — *Let $\nu \in a_M^{*+}$, and τ be an irreducible tempered generic representation of M. Denote $J(\tau, \nu)$ the Langlands quotient of the induced representation $I_P^G(\tau_\nu)$. Then, the representation $J(\tau, \nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.*

3. Setting and first results on intertwining operators

3.1. The setting. — Following [19], let us denote $a_{M_1}^{M*} = \mathbb{R}\Sigma^M \subset a_{M_1}^{G*}$, where Σ^M are the roots in Σ that are in M (with basis Δ^M) (see also [28] V.3.13).

With the setting and notations as given at the end of the Introduction (see 1.4), we consider τ a generic discrete series of M. By the above proposition (Proposition 2.7) there exists a standard parabolic subgroup $P_1 = M_1U_1$ of G, and we could further assume $M_1 \subset M$, σ_ν a cuspidal representation of M_1, a Levi subgroup of $M \cap P_1$, such that τ is a generic discrete series that appears as subrepresentation of $I_{M\cap P_1}^M(\sigma_\nu)$, with ν in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$. Moreover, σ_ν is a residual point for μ^M.

By transitivity of induction, we have:

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_P^G(I_{M\cap P_1}^M(\sigma_\nu))_{s\tilde{\alpha}} = I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}}),$$

where $s \in \mathbb{R}$ satisfies $s > 0$, and $\tilde{\alpha} = (\rho_P, \alpha)^{-1}\rho_P$ (rather than $\tilde{\alpha}$, we could also take the fundamental weight corresponding to α but we will rather follow a convention of Shahidi [see [7]]).

Convention. — The reader should note that our standard module $I_P^G(\tau_{s\tilde{\alpha}})$ is induced from an essentially square integrable representation $\tau_{s\tilde{\alpha}}$. The general case of a tempered representation τ will follow in the Corollary 7.11. Throughout this paper, we will adopt the following convention: τ will denote a discrete series representation and σ an (irreducible) cuspidal representation. Also, following notations (as for instance in [13] or [23]), $\pi \leq \Pi$ means π is realised as a subquotient of Π, whereas $\pi \hookrightarrow \Pi$ is stronger, which means it embeds as a subrepresentation.

In the following sections we will study the generic subquotient of $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and consider the cases where either there exists a discrete series subquotient, or there does not and, therefore, tempered or non-tempered generic (not square integrable) subquotients may occur.

Given a generic discrete series subquotient γ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$, using Proposition 2.7 above, it appears as a generic subrepresentation in some induced representation $I_{P'}^G(\sigma'_{\lambda'})$ for λ' in the closure of the positive Weyl chamber with respect to P', and σ' irreducible cuspidal generic.

The set-up is summarized in the following diagram:

$$\begin{array}{ccc} \gamma \leq I_P^G(\tau_{s\tilde{\alpha}}) & \hookrightarrow & I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}}) \\ & & \uparrow \\ \gamma & \hookrightarrow & I_{P'}^G(\sigma'_{\lambda'}) \end{array}$$

We will investigate the existence of a bijective up-arrow on the right-hand side of this diagram.

3.2. Intertwining operators. —

LEMMA 3.1. — *Let P_1 and Q be two parabolic subgroups of G having the same Levi subgroup M_1.*

Then, there exists an isomorphism $r_{P_1|Q}$ between the two induced modules $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$, for any irreducible unitary cuspidal representation σ, whenever λ is dominant for both P_1 and Q.

Proof. — We first assume that Q and P_1 are adjacent (two parabolic subgroups Q and P_1 are adjacent along α if $\Sigma(P_1) \cap -\Sigma(Q) = \{\alpha\}$). We denote β the common root of $\Sigma(\overline{Q})$ and $\Sigma(P_1)$; $\overline{Q}$ is the parabolic subgroup opposite Q with Levi subgroup M_1.

We have

$$I_Q^G(\sigma_\lambda) = I_{Q_\beta}^G(I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)),$$

where $(M_1)_\beta$ is the centraliser of A_β (the identity component in the kernel of β) in G, a semi-standard Levi subgroup (see Section 1 in [39]), and the same inductive formula holds replacing Q by P_1. Since λ is dominant for both Q and P_1, $\langle\lambda,\beta\rangle \geq 0$ (since β is a root in $\Sigma(P_1)$), but also $\langle\lambda,-\beta\rangle \geq 0$ since $-\beta$ is a root in $\Sigma(Q)$. Therefore, $\langle\check{\beta},\lambda\rangle = 0$.

We have λ in $a_{M_1}^*$, which decomposes as $(a_{M_1}^{(M_1)_\beta})^* \oplus (a_{(M_1)_\beta})^*$ and we write $\lambda = \mu \oplus \eta$. The dual of the Lie algebra, $(a_{M_1}^{(M_1)_\beta})^*$, is of dimension 1 (since M_1 is a maximal Levi subgroup in $(M_1)_\beta$) generated by $\check{\beta}$. If $\langle\check{\beta},\lambda\rangle = 0$, the projection of λ on $(a_{M_1}^{(M_1)_\beta})^*$ is also zero. That is, $\langle\check{\beta},\mu\rangle = 0$ or χ_μ is unitary.

Therefore, with σ unitary, and χ_μ a unitary character, the representations

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu) \qquad \text{and} \qquad I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu)$$

are unitary. Since they trivially satisfy the conditions (i) of Theorem 2.9 in [2] (see also [28] VI.5.4) they have an equivalent Jordan–Hölder composition series, and are, therefore, isomorphic (as unitary representations, having equivalent Jordan–Hölder composition series). Tensoring with χ_η preserves the isomorphism between

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu) \qquad \text{and} \qquad I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu).$$

That is, there exists an isomorphism between $I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and $I_{P\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$. The induction of this isomorphism, therefore, gives an isomorphism between $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$ that we call $r_{P_1|Q}$.

If we further assume that Q and P_1 are not adjacent but can be connected by a sequence of adjacent parabolic subgroups of G, $\{Q = Q_1, Q_2, Q_3, \ldots, Q_n = P_1\}$ with $\Sigma(Q_i) \cap \Sigma(\overline{Q_{i+1}}) = \{\beta_i\}$, we have the following set-up:

$$I_Q^G(\sigma_\lambda) \xrightarrow{r_{Q_2|Q}} I_{Q_2}^G(\sigma_\lambda) \xrightarrow{r_{Q_3|Q_2}} I_{Q_3}^G(\sigma_\lambda) \ldots \xrightarrow{r_{Q_n|Q_{n-1}}} I_{P_1}^G(\sigma_\lambda).$$

Again, under the assumption that λ is dominant for P_1 and Q, we have $\langle \beta_i, \lambda\rangle \geq 0$ and $\langle -\beta_i, \lambda\rangle \geq 0$, for each β_i in $\Sigma(P_1)\cap\Sigma(\overline{Q})$, and hence $\langle \check{\beta}_i, \lambda\rangle = 0$. Therefore, there exists an isomorphism between $I_{Q_i}^G(\sigma_\lambda)$ and $I_{Q_{i+1}}^G(\sigma_\lambda)$ denoted $r_{Q_{i+1}|Q_i}$. The composition of the isomorphisms $r_{Q_{i+1}|Q_i}$ will eventually give us the desired isomorphism between $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$. □

Proposition 3.2. — *Let $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$ be two induced modules with σ (or σ') irreducible cuspidal representation of M_1 (or M'), $\lambda \in a_{M_1}^*$, $\lambda' \in a_{M'}^*$, sharing a common subquotient; then:*

1. *There exists an element g in G such that ${}^gP' := gP'g^{-1}$ and P_1 have the same Levi subgroup.*
2. *If λ and λ' are dominant for P_1 (or P'), there exists an isomorphism R_g between $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$.*

Proof. — First, since the representations $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$ share a common subquotient by Theorem 2.9 in [2], there exists an element g in G such that $M_1 = gM'g^{-1}$, ${}^g\sigma'_{\lambda'} = \sigma_\lambda$ and $g\lambda' = \lambda$, where ${}^g\sigma(x) = \sigma(g^{-1}xg)$ for $x \in M_1$. The last point follows from the equality ${}^g\chi_{\lambda'} = \chi_{g\lambda'}$.

For the second point, we first apply the map $t(g)$ between $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{{}^gP'}^G({}^g\sigma'_{\lambda'})$, which is an isomorphism that sends f on $f(g^{-1})$.

As λ' is dominant for $P', g\lambda' = \lambda$ is dominant for ${}^gP'$, and we can further apply the isomorphism defined in the previous lemma (Lemma 3.1): $r_{P_1|{}^gP'}(\sigma_\lambda)$ (since P_1 and ${}^gP'$ have the same Levi subgroup: M_1); we will, therefore, have:

$$I_{P'}^G(\sigma'_{\lambda'}) \xrightarrow{t(g)} I_{{}^gP'}^G({}^g\sigma', g\cdot\lambda') \xrightarrow{r_{P_1|{}^gP'}} I_{P_1}^G(\sigma_\lambda),$$

and R_g is the isomorphism given by the composition of $t(g)$ and $r_{P_1|{}^gP'}$. □

3.2.1. *Intertwining operators with non-generic kernels.* — Our objective is to embed an irreducible generic subquotient as a subrepresentation in a module induced from the data (P_1, σ, λ) knowing it embeds in one with Langlands' data (P', σ', λ'). Notice that (P_1, σ, λ) is not necessarily a Langlands data since, as explained in the beginning of Section 4, the parameter λ is not necessarily in the positive Weyl chamber $(a_{M_1}^*)^+$. If the intertwining operator between those two induced modules has a non-generic kernel, the generic subrepresentation will necessarily appear in the image of the intertwining operator and will, therefore, appear as a *subrepresentation* in the induced module with Langlands' data (P_1, σ, λ). We detail the conditions to obtain the non-genericity of the kernel of the intertwining operator.

Proposition 3.3. — *Let P_1 and Q be two parabolic subgroups of G having the same Levi subgroup M_1.*

Consider the two induced modules $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$ and assume σ is an irreducible generic cuspidal representation, and λ is dominant for P_1 and *anti-dominant for Q. Then there exists an intertwining map from $I_Q^G(\sigma_\lambda)$ to $I_{P_1}^G(\sigma_\lambda)$, which has a non-generic kernel.*

Proof. — We first assume that Q and P_1 are adjacent. We denote β the common root of $\Sigma(Q)$ and $\Sigma(\overline{P_1})$.

We have $I_Q^G(\sigma_\lambda) = I_{Q_\beta}^G(I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda))$, where $(M_1)_\beta$ is the centraliser of A_β (the identity component in the kernel of β) in G, a semi-standard Levi subgroup (see Section 1 in [39]), and the same inductive formula holds, replacing Q by P_1. Then, there are two cases: The case of $\langle\check{\beta},\lambda\rangle = 0$ is Lemma 3.1. If $\langle\check{\beta},\lambda\rangle > 0$, let us consider the intertwining operator defined in Section 2 between $I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and $I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and assume it is not an isomorphism. The representation σ being cuspidal, these modules are length 2 representations by the Corollary 7.1.2 of Casselman [6]. Let S be the kernel of this intertwining map and the Langlands quotient $J(\sigma, P_1\cap(M_1)_\beta,\lambda)$ its image. One has the exact sequences:

$$0 \to S \to I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda) \to J(\sigma, P_1\cap(M_1)_\beta,\lambda) \to 0,$$

$$0 \to J(\sigma, P_1\cap(M_1)_\beta,\lambda) \to I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda) \to S \to 0.$$

Further, the projection from

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$$

to

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)/J(\sigma, P_1\cap(M_1)_\beta,\lambda) \cong S \subset I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$$

defines a map whose kernel, $J(\sigma, P_1\cap(M_1)_\beta,\lambda)$, is not generic (by the main result of [18], which proves the standard module conjecture). In other words, we have the following exact sequence:

$$0 \to J(\sigma, P_1\cap(M_1)_\beta,\lambda) \to I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda) \xrightarrow{A} I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda).$$

Inducing from $(P_1)_\beta$ to G one observes that the kernel of the induced map $(I_{(P_1)_\beta}^G(A))$ is the induction of the kernel $J(\sigma, P_1\cap(M_1)_\beta,\lambda)$. Therefore, the kernel of the induced map is non-generic (here, we use the fact that there exists an isomorphism between the Whittaker models of the inducing and the induced representations, using result of [29] and [8]).

Assume now that Q and P_1 are not adjacent but can be connected by a sequence of adjacent parabolic subgroups of G,

$$\{Q = Q_1, Q_2, Q_3, \ldots, Q_n = P_1\} \quad \text{with } \Sigma(Q_i)\cap\Sigma(\overline{Q_{i+1}}) = \{\beta_i\}\,.$$

We have the following set-up:

$$I_Q^G(\sigma_\lambda) \xrightarrow{r_{Q_2|Q}} I_{Q_2}^G(\sigma_\lambda) \xrightarrow{r_{Q_3|Q_2}} I_{Q_3}^G(\sigma_\lambda)\ldots \xrightarrow{r_{Q_n|Q_{n-1}}} I_{P_1}^G(\sigma_\lambda).$$

Assume that certain maps $r_{Q_{i+1}|Q_i}$ have a kernel; by the same argument as above their kernels are non-generic, and, therefore, the kernel of the composite map is non-generic. Indeed, we have the next Lemma 3.4. □

LEMMA 3.4. — *The composition of intertwining operators with a non-generic kernel has a non-generic kernel.*

Proof. — Consider first the composition of two operators, A and B as follows:

$$I_Q^G(\sigma_\lambda) \xrightarrow{A} I_{Q_2}^G(\sigma_\lambda) \xrightarrow{B} I_{P_1}^G(\sigma_\lambda).$$

Clearly, the kernel of the composite $(B \circ A)$ contains the kernel of A and the elements in the space of the representation $I_Q^G(\sigma_\lambda)$, x, such that $A(x)$ is in the kernel of B.

This means that we have the following sequence of homomorphisms:

$$0 \to \ker(A) \to \ker(B \circ A) \xrightarrow{A} \ker(B) \cap \mathrm{Im}(A) \to 0,$$

the pull-back by A^{-1} of element in $\ker(B)$. The pull-back of a non-generic kernel yields a non-generic subspace in the pre-image. The fact that this sequence is exact is clear, except for the surjectivity of the map $\ker(B \circ A) \xrightarrow{A} \ker(B) \cap \mathrm{Im}(A)$. However, if $y \in \ker(B) \cap \mathrm{Im}(A)$, then there exists x such that $A(x) = y$, and we have $B \circ A(x) = B(y) = 0$ since $y \in \ker(B)$.

If both $\ker(B)$ and $\ker(A)$ are non-generic, the kernel of $(B \circ A)$ is itself non-generic. Extending the reasoning to a sequence of rank 1 operators with non-generic kernels yields the result. □

We have observed that the nature of intertwining operators relies on the dominance of the parameters λ and λ'. We now need a more explicit description of these parameters; to this end, we will call on a result that was first presented in [27] in the Hecke algebra context (Theorem Proposition 8.1 in [27], see also Appendix A) and further developed in [16].

4. Description of residual points via Bala–Carter

With the notations of Section 3, we will study generic subquotient in induced modules $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and $I_{P'}^G(\sigma'_{\lambda'})$.

One needs to observe, following the construction of our setting in Section 3, that ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$, whereas $s\tilde{\alpha}$ is in the positive Weyl chamber $(a_M^*)^+$, and, therefore, it is not expected that $\nu + s\tilde{\alpha}$ should be in the closure of the positive Weyl chamber $\overline{(a_{M_1}^*)^+}$.

In particular, let α be the only root in $\Sigma(A_0)$, which is not in $\mathrm{Lie}(M)$; we may have $\langle \nu, \check{\alpha} \rangle < 0$, and, therefore, for some roots $\beta \in \Sigma(A_{M_1})$, written as linear combination containing the simple root α, we may also have: $\langle \nu + s\tilde{\alpha}, \check{\beta} \rangle < 0$.

However, by the result presented in Appendix A, if $\nu + s\tilde{\alpha}$ is a residual point, it is in the Weyl group orbit of a dominant residual point (i.e. one whose expression can be directly deduced from a weighted Dynkin diagram). We therefore define:

DEFINITION 4.1 (dominant residual point). — A residual point σ_λ for σ an irreducible cuspidal representation is dominant if λ is in the closed positive Weyl chamber $\overline{(a_M^*)^+}$.

Bala–Carter theory allows us to describe explicitly the Weyl group orbit of a residual point. In the context of reductive p-adic groups studied in [16] (see, in particular, Proposition 6.2 in [16]), the fact that σ_λ lies in the cuspidal support of a discrete series can be translated somehow to the assertion that σ_λ corresponds to a distinguished nilpotent orbit in the dual of the Lie algebra ${}^L\mathfrak{g}$, and, therefore, by Proposition A.9 (see also [27], Appendices A and B, Proposition 8.1) to a weighted Dynkin diagram. Notice that Proposition A.9 requires G to be a semi-simple adjoint group, a certain parameter k_α to equal 1 for any root α in Φ, and further, it concerns only the case of unramified characters.

In the present work, we treat the case of weighted Dynkin diagrams of type A, B, C, D. The key proposition is Proposition 4.6 below.

Our setting. — Recall that in Section 3 we embedded the standard module as follows:

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_P^G(I_{M\cap P_1}^M(\sigma_\nu))_{s\tilde{\alpha}} = I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}}).$$

By hypothesis, σ_ν is a residual point for μ^M.

$\lambda = \nu + s\tilde{\alpha}$ is in $a_{M_1}^*$.

Describing the form of the parameter $\lambda \in a_{M_1}^*$ explicitly is essential for two reasons. First, to determine the nature (i.e. discrete series, tempered, or non-tempered representations) of the irreducible generic subquotients in the induced module $I_{P_1}^G(\sigma_\lambda)$; and secondly, to describe the intertwining operators and, in particular, the (non)-genericity of their kernels.

We will explain the following correspondences:

$$\text{(1)}\quad \{\text{dominant residual point}\} \leftrightarrow \{\text{weighted Dynkin diagram}\} \leftrightarrow \{\text{residual segments}\} \leftrightarrow \{\text{jumps of the residual segment}\}$$

The connection between residual points and root systems involved for weighted Dynkin diagrams requires a careful description of the participants involved.

The root system. — Let us now recall that $W(M_1)$ is the set of representatives in W of elements in the quotient group $\left\{w \in W | w^{-1}M_1w = M_1\right\}/W^{M_1}$ of minimal length in their right classes modulo W^{M_1}.

Assume σ is a unitary cuspidal representation of a Levi subgroup M_1 in G and let $W(\sigma, M_1)$ be the subgroup of $W(M_1)$ stabiliser of σ. The Weyl group of Σ_σ is W_σ, the subgroup of $W(M_1, \sigma)$ generated by the reflexions s_α.

Proposition 4.2 (3.5 in [35]). — *The set*

$$\Sigma_\sigma := \left\{\alpha \in \Sigma_{red}(A_{M_1}) \,|\, \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$$

is a root system.

For $\alpha \in \Sigma_\sigma$, let s_α the unique element in $W^{(M_1)_\alpha}(M_1, \sigma)$ that conjugates $P_1 \cap M_\alpha$ and $\overline{P_1} \cap (M_1)_\alpha$. The Weyl group W_σ of Σ_σ identifies to the subgroup of $W(M_1, \sigma)$ generated by reflexions s_α, $\alpha \in \Sigma_\sigma$. $\check{\alpha}$ the unique element in $a_{M_1}^{(M_1)_\alpha}$ that satisfies $\langle \check{\alpha}, \alpha \rangle = 2$.

Then $\Sigma_\sigma^\vee := \{\check{\alpha} | \alpha \in \Sigma_\sigma\}$ is the set of coroots of Σ_σ, the duality being that of a_{M_1} and $a_{M_1}^*$.

The set $\Sigma(P_1) \cap \Sigma_\sigma$ is the set of positive roots for a certain order on Σ_σ.

Remark 4.3. — An equivalent proposition is proved in [17] (Proposition 1.3). There, the author considers $\mathcal{O}$ the set of equivalence classes of representations of the form $\sigma \otimes \chi$, where χ is an unramified character of M_1. He proves that the set $\Sigma_{\mathcal{O},\mu} := \left\{\alpha \in \Sigma_{\mathrm{red}}(A_{M_1}) | \mu^{(M_1)_\alpha} \text{has a zero on } \mathcal{O}\right\}$ is a root system.

The Weyl group of G relative to a maximal split torus in M_1 acts on $\mathcal{O}$. The previous statement holds replacing W_σ by $W(M_1, \mathcal{O})$, the subgroup of $W(M_1)$ stabiliser of $\mathcal{O}$.

Lemma 4.4. — *If σ is the trivial representation of $M_1 = M_0$, the root system Σ_σ is the root system of the group G relative to A_0 (with length given by the choice of P_0).*

Proof. — Recall that $\Sigma_\sigma := \left\{\alpha \in \Sigma_{\mathrm{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$ is a root system. Apply this definition to the trivial representation. Clearly, for any $\alpha \in \Sigma(A_0)$, the trivial representation is fixed by any element in $W^{(M_0)_\alpha}(M_0)$, and therefore by s_α satisfying $s_\alpha(P_0 \cap (M_0)_\alpha) = \overline{P_0} \cap (M_0)_\alpha$. It is well known that the induced representation, $I_{P_0 \cap (M_0)_\alpha}^{(M_0)_\alpha}(\mathbf{1})$, is irreducible; therefore using Harish-Chandra's theorem (Theorem 2.1) above, $\mu^{(M_0)_\alpha}(\mathbf{1}) = 0$. Then

$$\begin{aligned}\left\{\alpha \in \Sigma_{\mathrm{red}}(A_0) | \mu^{(M_0)_\alpha}(\mathbf{1}) = 0\right\} &:= \left\{\alpha \in \Sigma(A_0) | \mu^{(M_0)_\alpha}(\mathbf{1}) = 0\right\} \\ &= \{\alpha \in \Sigma(A_0)\}. \end{aligned}$$

□

In general, the root system Σ_σ is the disjoint union of irreducible or empty components $\Sigma_{\sigma,i}$ for $i = 1, \ldots, r$. This will be detailed in Section 4.4.2.

Proposition 4.5. — *Let G be a quasi-split group whose root system Σ is of type A, B, C or D. Then the irreducible components of Σ_σ are of type A, B, C or D.*

Proof. — See the main result of the article [12] recalled in the Appendix B. □

How the root system Σ_σ determines the weighted Dynkin diagrams to be used in this work. —

Proposition 4.6. — *Assume G quasi-split over F. Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1. Put $\Sigma_\sigma = \{\alpha \in \Sigma_{\mathrm{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$. Let*

$$d = \mathrm{rk}_{ss}(G) - \mathrm{rk}_{ss}(M_1).$$

The set Σ_σ is a root system in a subspace of $a^*_{M_1}$ (cf. Proposition 4.2). Suppose that the irreducible components of Σ_σ are all of type A, B, C or D. Denote, for each irreducible component $\Sigma_{\sigma,i}$ of Σ_σ, by $a^{M^i*}_{M_1}$ the subspace of $a^{G*}_{M_1}$ generated by $\Sigma_{\sigma,i}$, by d_i its dimension and by $e_{i,1}, \dots, e_{i,d_i}$ a basis of $a^{M^i*}_{M_1}$ (or of a vector space of dimension $d_i + 1$ containing $a^{M^i*}_{M_1}$ if $\Sigma_{\sigma,i}$ is of type A), so that the elements of the root system $\Sigma_{\sigma,i}$ are written in this basis as in the work of Bourbaki [4].

For each i, there is a unique real number $t_i > 0$, such that, if $\alpha = \pm e_{i,j} \pm e_{i,j'}$ lies in $\Sigma_{\sigma,i}$; then $I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\frac{t_i}{2}(\pm e_{i,j} \pm e_{i,j'})})$ is reducible.

If $\Sigma_{\sigma,i}$ is of type B or C, then there is in addition a unique element $\epsilon_i \in \{1/2, 1\}$ such that $I^{(M_1)_{\alpha_{i,d_i}}}_{P_1 \cap (M_1)_{\alpha_{i,d_i}}}(\sigma_{\epsilon_i t_i e_{i,d_i}})$ is reducible.

Let $\lambda = \sum_i \sum_{j=1}^{d_i} \lambda_{i,j} e_{i,j}$ be in $\overline{a^{G*+}_{M_1}}$ with $\lambda_{i,j}$ real numbers.

Then σ_λ is in the cuspidal support of a discrete series representation of G, if and only if the following two properties are satisfied:

(i) *$d = \sum_i d_i$.*

(ii) *For all i, $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic of a simple complex adjoint group of*
   - *type D_{d_i} (or A_{d_i}) if $\Sigma_{\sigma,i}$ is of type D (or A);*

   otherwise:
   - *of type C_{d_i}, if $\epsilon_i = 1/2$;*
   - *of type B_{d_i}, if $\epsilon_i = 1$.*

Proof. — As λ lies in $a^{G*}_{M_1}$, σ_λ lies in the cuspidal support of a discrete series representation of G, if and only if it is a residual point of Harish-Chandra's μ-function.

Denote $e^{\pm}_{i,j;i',j'}$ the rational character of A_{M_1} whose dual pairing with an element x of $a^G_{M_1}$ with coordinates

$$(x_{1,1}, \dots, x_{1,d_1}, x_{2,1}, \dots, x_{2,d_2}, \dots, x_{r,1}, \dots, x_{r,d_r})$$

in the dual basis equals $x_{i,j} x^{\pm 1}_{i',j'}$ and by $e^{\pm}_{i,j}$ the one whose dual pair equals $x^{\pm 1}_{i,j}$.

The μ-function decomposes as $\prod_{\alpha\in\Sigma(P)}\mu^{M_\alpha}$. By assumption, the function $\lambda\mapsto\mu^{M_\alpha}(\sigma_\lambda)$ will not have a pole or zero on $a^*_{M_1}$ except if $\alpha\in\Sigma_\sigma$. This means that

(i) α is of the form $e^-_{i,j;i,j'}$, $j<j'$.
(ii) α is of the form $e^+_{i,j;i,j'}$, $j<j'$, and $\Sigma_{\sigma,i}$ of type B, C or D.
(iii) α is of the form $e^+_{i,j}$ or $2e^+_{i,j}$ and $\Sigma_{\sigma,i}$ of, respectively, type B or C.

Let $(\lambda_{i,j})_{i,j}$ be a family of real numbers as in the statement of the proposition and put $\lambda=\sum_i\sum_{j=1}^{d_i}\lambda_{i,j}e_{i,j}$. It follows from Langlands–Shahidi theory (cf. the proof of Theorem 5.1 in [19]) that there is, for each i, a real number $t_i>0$ and $\epsilon_i\in\{1/2,1\}$, so that:

- If $\alpha=e^{\pm}_{i,j;i,j'}\in\Sigma_\sigma$, $j<j'$, then
$$\mu^{M_\alpha}(\sigma_\lambda)=c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}})\frac{(1-q^{\lambda_{i,j}\pm\lambda_{i,j'}})(1-q^{-\lambda_{i,j}\mp\lambda_{i,j'}})}{(1-q^{t_i-\lambda_{i,j}\pm\lambda_{i,j'}})(1-q^{t_i+\lambda_{i,j}\mp\lambda_{i,j'}})},$$
where $c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}})$ denotes a rational function in $\sigma_{(\lambda_{i,j})_{i,j}}$, which is regular and non-zero for real $\lambda_{i,j}$.
- If $\alpha=e_{i,j}\in\Sigma_\sigma{}^*$ or $\alpha=2e_{i,j}\in\Sigma_\sigma$, then
$$\mu^{M_\alpha}(\sigma_{(\lambda_{i,j})_{i,j}})=c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}})\frac{(1-q^{\lambda_{i,j}})(1-q^{-\lambda_{i,j}})}{(1-q^{\epsilon_it_i-\lambda_{i,j}})(1-q^{\epsilon_it_i+\lambda_{i,j}})}$$
with $\epsilon_i=1,1/2$.

Put $\kappa_i^+=0$ if $\Sigma_{\sigma,i}$ is of type A and put $\kappa_i=0$ if $\Sigma_{\sigma,i}$ is of type A or D and otherwise $\kappa_i=\kappa_i^+=1$. As λ is in the closure of the positive Weyl chamber, it follows that, for σ_λ to be a residual point of Harish-Chandra's μ-function, it is necessary and sufficient, that for every i, one has

$$\begin{aligned}(2)\qquad d_i=&|\{(j,j')|j<j',\lambda_{i,j}-\lambda_{i,j'}=t_i\}|\\&+\kappa_i^+|\{(j,j')|j<j',\lambda_{i,j}+\lambda_{i,j'}=t_i\}|+\kappa_i|\{j|\lambda_{i,j}=\epsilon_it_i\}|\\&-2\big[|\{(j,j')|j<j',\lambda_{i,j}-\lambda_{i,j'}=0\}|\\&\qquad+\kappa_i^+|\{(j,j')|j<j',\lambda_{i,j}+\lambda_{i,j'}=0\}|+\kappa_i|\{j|\lambda_{i,j}=0\}|\big].\end{aligned}$$

If $\kappa_i=0$ or $\epsilon_i=1$, then this is the condition for $\frac{2}{t_i}(\lambda_{i,1},\dots,\lambda_{i,d_i})$ defining a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type A_{d_i}, D_{d_i} or B_{d_i} as in 5.7.5 in [5]. If $\epsilon_i=1/2$, one sees that $\frac{2}{t_i}(\lambda_{i,1},\dots,\lambda_{i,d_i})$ defines a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type C_{d_i}.

In other words, $\frac{2}{t_i}(\lambda_{i,1},\dots,\lambda_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of an adjoint simple complex group of type B_n, C_n or D_n, if $\kappa_i^+=1$ and $\kappa_i\epsilon_i$ is, respectively, 1, 1/2 or 0, and of type A_n if $\kappa_i=0$. □

Example 4.7 (See also Proposition 1.13 in [17] and the Appendix of the author's PhD thesis [11]). — In the context of classical groups, let us spell out the Levi subgroups and cuspidal representations of these Levi considered in the previous proposition.

Let M_1 be a standard Levi subgroup of a classical group G and σ a generic irreducible unitary cuspidal representation of M_1.

Then, up to conjugation by an element of G, we can assume:

$$M_1 = \underbrace{\mathrm{GL}_{k_1} \times \ldots \times \mathrm{GL}_{k_1}}_{d_1 \text{ times}} \times \underbrace{\mathrm{GL}_{k_2} \times \ldots \times \mathrm{GL}_{k_2}}_{d_2 \text{ times}} \\ \times \ldots \times \underbrace{\mathrm{GL}_{k_r} \times \ldots \times \mathrm{GL}_{k_r}}_{d_r \text{ times}} \times G(k),$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G, and

$$\sigma = \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots \otimes \sigma_2 \ldots\ldots \otimes \sigma_r \otimes \ldots \otimes \sigma_r \otimes \sigma_c.$$

Let us assume $k \neq 0$, and $\sigma_i \not\cong \sigma_j$ if $j \neq i$.

We identify A_{M_1} to $\mathbb{T} = \mathbb{G}_m^{d_1} \times \mathbb{G}_m^{d_2} \times \ldots \times \mathbb{G}_m^{d_r}$ and denote $\alpha_{i,j}$ the rational character of A_{M_1} (identified with $\mathbb{T}$), which sends an element

$$x = (x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}, \ldots, x_{r,1}, \ldots, x_{r,d_r})$$

to $x_{i,j}x_{i,j+1}^{-1}$ if $j < d_i$ and to x_{i,d_i} if $j = d_i$.

Let $(s_{i,j})_{i,j}$ be a family of non-negative real numbers, $1 \leq i \leq r$, $1 \leq j \leq d_i$ and $s_{i,j} \geq s_{i,j+1}$ for i fixed. Then,

$$\sigma_1|\det|^{s_{1,1}} \otimes \ldots \otimes \sigma_1|\det|^{s_{1,d_1}} \otimes \sigma_2|\det|^{s_{2,1}} \otimes \ldots \otimes \sigma_2|\det|^{s_{2,d_2}} \\ \otimes \ldots \otimes \sigma_r|\det|^{s_{r,1}} \otimes \ldots \otimes \sigma_r|\det|^{s_{r,d_r}} \otimes \sigma_c.$$

is in the cuspidal support of a discrete series representations of G, if and only if the following properties are satisfied:

(i) One has $\sigma_i \simeq \sigma_i^\vee$ for every i.

(ii) Denote by s_i the unique element in $\{0, 1/2, 1\}$ such that the representation of $G(k + k_i)$ parabolically induced from $\sigma_i|\cdot|^{s_i} \otimes \sigma_c$ is reducible (we use the result of Shahidi on reducibility points for generic cuspidal representations).

(iii) If, in addition, $G = \mathrm{SO}_{2n}(F)$, the situation can be a little subtler. For instance, in the maximal parabolic case, with $\sigma = \sigma_1 \otimes \sigma_c$ and k_1 odd, the long Weyl conjugate of $\sigma_1 \otimes \sigma_c$ is $\sigma_1^\vee \otimes c \cdot \sigma_c$ where c is a length zero representative of $O_{2n}(F)/\mathrm{SO}_{2n}(F)$. In particular, if $c \cdot \sigma_c \not\cong \sigma_c$, $\sigma_1^\vee \otimes c \cdot \sigma_c$ is not ramified, and no s_1 gives reducibility. However, this can still be the support of a discrete series.

Then, for all i, $2(s_{i,1},\dots,s_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of a simple complex adjoint group of

- type D_{d_i} if $s_i=0$; then $\Sigma_{\sigma,i}=\{\alpha_{i,1},\dots,\alpha_{i,d_i-1},\alpha_{i,d_i-1}+2\alpha_{i,d_i}\}$;
- type C_{d_i} if $s_i=1/2$; then $\Sigma_{\sigma,i}=\{\alpha_{i,1},\dots,\alpha_{i,d_i-1},2\alpha_{i,d_i}\}$;
- type B_{d_i} if $s_i=1$; then $\Sigma_{\sigma,i}=\{\alpha_{i,1},\dots,\alpha_{i,d_i-1},\alpha_{i,d_i}\}$.

For $i\neq j$, since $\sigma_i\ncong\sigma_j$, we have $\Sigma_{\sigma,i}\neq\Sigma_{\sigma,j}$.

Then M^i is isomorphic to

$$\underbrace{\mathrm{GL}_{k_1}\times\dots\times\mathrm{GL}_{k_1}}_{d_1\text{ times}}\times\dots\times\underbrace{\mathrm{GL}_{k_{i-1}}\times\dots\times\mathrm{GL}_{k_{i-1}}}_{d_{i-1}\text{ times}}\times\underbrace{\mathrm{GL}_{k_{i+1}}\times\dots\times\mathrm{GL}_{k_{i+1}}}_{d_{i+1}\text{ times}}$$
$$\times\dots\times\underbrace{\mathrm{GL}_{k_r}\times\dots\times\mathrm{GL}_{k_r}}_{d_r\text{ times}}\times G(k+d_ik_i).$$

4.1. From weighted Dynkin diagrams to residual segments. — The Dynkin diagram of a distinguished parabolic subgroup mentioned in the Proposition 4.6 is also called *weighted Dynkin diagrams*; a definition is given in Appendix A and their forms in A.1 .

Let a parameter $\nu\in a^*_{M_1}$ be written $(\nu_1,\nu_2,\dots,\nu_n)$ in a basis $\{e_1,e_2,\dots,e_n\}$ (or $\{e_1,e_2,\dots,e_n,e_{n+1}\}$ for type A) (such that this basis is the canonical basis associated to the classical Lie algebra a^*_0, as in [4] when $M_1=M_0$) and assume it is a dominant residual point. As it is dominant, observe that $\nu_1\geq\nu_2\geq\dots\geq\nu_n\geq 0$ (or $\nu_1\geq\nu_2\geq\dots\geq\nu_n$ for type A). Further, it corresponds by the previous Proposition (4.6) to a weighted Dynkin diagram of a certain type A,B,C or D (see also Bala–Carter theory presented in Appendix A).

Let us explain the following correspondence:

$$\{\text{weighted Dynkin diagram}\}\leftrightarrow\{\text{residual segment}\}\tag{3}$$

First, let us explain the following assignment:

$$\text{WDD}\to\nu,\quad\text{where }\nu\text{ is the vector with coordinates }\langle\nu,\alpha_i\rangle.$$

Let us start with a weighted Dynkin diagram of type A,B,C or D. The weights under roots α_i are 2 (or 0), which correspond to $\langle\nu,\alpha_i\rangle=1$ (or 0). See the weighted Dynkin diagrams given in Appendix A.1. Notice that we abusively use α_i rather than $\check{\alpha}_i$ in the product expression, to be consistent with the notations in the weighted Dynkin diagrams.

Using the expressions of α_i in the canonical basis (for instance $\alpha_i=e_i-e_{i+1}$, $2e_i$, or e_i), we compute the vector of coordinates $(\nu_1,\nu_2,\dots,\nu_n)$ with integers or half-integer entries. For instance, for $\alpha_i=e_i-e_{i+1}$, when $\langle\nu,\alpha_i\rangle=\langle\sum_{i=1}^n\nu_ie_i,\alpha_i\rangle=1$, we get $\nu_i-\nu_{i+1}=1$, whereas if $\langle\nu,\alpha_i\rangle=0$, then $\nu_i-\nu_{i+1}=0$. Conversely, let us be given a vector of coordinates $(\nu_1,\nu_2,\dots,\nu_n)$ with integers or half-integer entries and the type of root system (A,B,C or D). Using the relations ν_i and ν_{i+1} for any i, we deduce the weights under each root α_i and, therefore, obtain the weighted Dynkin diagram.

Definition 4.8 (residual segment). — The residual segment of type B, C, D associated to the dominant residual point $\nu := (\nu_1, \nu_2, \dots, \nu_n) \in \overline{a^{*+}_{M_1}}$ (depending on a fixed irreducible cuspidal representation σ of M_1) is the expression in coordinates of this dominant residual point in a particular basis of $a^*_{M_1}$ (the basis such that the roots in the weighted Dynkin diagram are canonically expressed as in [4]).

It is, therefore, a decreasing sequence of positive (half)-integers uniquely obtained from a weighted Dynkin diagram by the aforementioned procedure.

It is uniquely characterized by:

- An infinite tuple $(\dots, 0, n_{\ell+m}, \dots, n_\ell, n_{\ell-1}, \dots, n_0)$ or $(\dots, 0, n_{\ell+m}, \dots, n_\ell, n_{\ell-1}, \dots, n_{1/2})$, where n_i is the number of times the integer or half-integer value i appears in the sequence.
- The greatest (half)-integer in the sequence, ℓ, such that $n_\ell = 1, n_{\ell-1} = 2$, if it exists.
- The greatest integer, m, such that, for any $i \in \{1, \dots, m\}$, $n_{\ell+i} = 1$, and for any $i > m$, $n_{\ell+i} = 0$.

This residual segment uniquely determines the weighted Dynkin diagram of type B, C or D from which it originates.

Therefore, the values obtained for the n_i's depend on the weighted Dynkin diagram (see Appendix A.1) one observes the following relations:

- Type B: $n_\ell = 1, n_{\ell-1} = 2$, $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 = \frac{n_1-1}{2}$ if n_1 is odd or $n_0 = \frac{n_1}{2}$ if n_1 is even. (The regular orbit where $n_i = 1$, for all $i \geq 1$ is a special case.)
- Type C: $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$; $n_{1/2} = n_{3/2} + 1$, $n_\ell = 1, n_{\ell-1} = 2$. (The regular orbit, where $n_i = 1$, for all $i \geq 1/2$ is a special case.)
- Type D:
  1. $n_i = 1$ for all $i \geq \ell$ and $n_0 = 1$, $n_i = 2$ for all $i \in \{2, \dots, \ell - 1\}$.
  2. $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 \geq 2$, $n_0 = \left\{ \begin{array}{l} \frac{n_1}{2} \text{ if } n_1 \text{ is even} \\ \frac{n_1+1}{2} \text{ if } n_1 \text{ is odd} \end{array} \right\}$.

It will be denoted $(\underline{n})$.

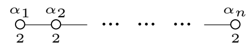
The residual segment of type A (we say *linear residual segment*, referring to the general *linear* group) is characterized with the same three objects and also corresponds bijectively to a weighted Dynkin diagram of type A. Then it is a decreasing sequence of (not necessarily positive) reals, and the infinite tuple given above is $(\dots, 0, 1, 1, 1, \dots, 1)$, i.e. $n_i \leq 1$, for all i. It is symmetrical around zero.

We will also abusively say *linear residual segment* for the translated version of a residual segment of type A; i.e. if it is not symmetrical around zero.

We usually do not write the commas to separate the (half)-integers in the sequence.

The use of the terminology "segments" is explained through the following example.

An example: Bernstein–Zelevinsky's segments. — Consider the weighted Dynkin diagram of type A:



As $\langle \nu, \alpha_i \rangle = 1$, for all $i \iff \nu_i - \nu_{i+1} = 1$, for all i; the vector of coordinates is, therefore, a strictly decreasing sequence of real numbers: $(\mathscr{a}, \mathscr{a} - 1, \mathscr{a} - 2, \ldots, \mathscr{b})$. Notice the specific font used to write linear residual segments.

The group GL_n is an example of reductive group whose root system is of type A. We may now recall the notions of segments for GL_n as defined in [2] and following the treatment in [30]. We fix an irreducible cuspidal representation ρ and denote $\rho(a) = \rho|\det|^a$. The representation $\rho_1 \times \rho_2$ denotes the parabolically induced representation from $\rho_1 \otimes \rho_2$.

Definition 4.9 (Segment, linked segments). — [Bernstein–Zelevinsky; following [30]] Let $r|n$. A segment is an isomorphism class of irreducible cuspidal representations of a group GL_n, of the form $\mathcal{S} = \{\rho, \rho(1), \rho(2), \ldots, \rho(r-1)\}$. We denote it $\mathcal{S} = [\rho, \rho(r-1)]$.

There is also a notion of intersection and union of two such segments explained in particular in [30]: the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is written $\mathcal{S}_1 \cap \mathcal{S}_2$, the union is written as $\mathcal{S}_1 \cup \mathcal{S}_2$.

Let $\mathcal{S}_1 = [\rho_1, \rho_1'], \mathcal{S}_2 = [\rho_2, \rho_2']$ be two segments. We say $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked if $\mathcal{S}_1 \not\subseteq \mathcal{S}_2, \mathcal{S}_2 \not\subseteq \mathcal{S}_1$ and $\mathcal{S}_1 \cup \mathcal{S}_2$ is a segment.

Once ρ is fixed, a segment is solely characterized by a string of (half)-integers; it therefore seems natural, in analogy with Bernstein–Zelevinsky's theory, to name any vector $(\nu_1, \ldots, \nu_k)$ corresponding to a dominant residual point and, therefore, by Proposition 4.6 (see also A.9 and [27], Proposition 8.1) to a weighted Dynkin diagram: *a residual segment*.

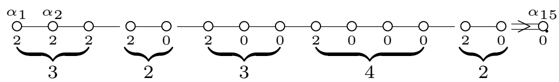
Let us define $\mathcal{S} = [\rho(r-1), \rho]$ a sequence of representations twisted by decreasing exponents and notice the difference with the definition of the segment as given in Bernstein–Zelevinsky where the exponents are increasing. The unique irreducible subrepresentation (ore quotient) of $\rho(r-1) \times \ldots \times \rho$ is denoted $Z(\mathcal{S})$ (or $L(\mathcal{S})$). If it is a subrepresentation, it is essentially square-integrable. Often, we denote it $Z(\rho, r-1, 0)$, and more generally $Z(\rho, \mathscr{a}, \mathscr{b})$ for $\mathscr{a}$ and $\mathscr{b}$ any two real numbers such that $\mathscr{a} - \mathscr{b} \in \mathbf{Z}$. In the literature, the generalized Steinberg is also denoted $\mathrm{St}_k(\varrho)$; it is the canonical discrete series associated to the segment $[\varrho(\frac{k-1}{2}), \ldots, \varrho(\frac{1-k}{2})]$, for an irreducible cuspidal representation ϱ. Often, $\mathrm{St}_k(\mathbf{1})$ will simply be denoted St_k.

This is a general phenomenon, since by Theorem 2.6, for any quasi-split reductive group, we associate to any residual segment an essentially square-integrable (or discrete series) representation. The well-known example of the

Steinberg representation of GL_k is also characteristic since the Steinberg is the unique irreducible *generic* subquotient in the parabolically induced representation $\varrho(\frac{k-1}{2}) \times \ldots \times \varrho(\frac{1-k}{2})$.

By Theorems 2.6 and 5.4, combined with Rodier's result, if the cuspidal support σ_λ, a residual point, is generic, then the induced representation is generic, and the unique irreducible generic subquotient is essentially square integrable. Therefore, the phenomenon presented here with the Steinberg subquotient, occurs more generally. When the generic representation σ_λ is a dominant residual point, the residual segment corresponding to λ characterizes the unique irreducible generic discrete series (or essentially square integrable) subquotient.

Example 4.10. — Consider B_{15} for instance (see A.1 to understand the relations between the p_i's), with $m = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 4$, $p_4 = 2$:



We have $\langle \nu, \alpha_{15} \rangle = \langle \nu, 2e_{15} \rangle = 0$, and, therefore, $\nu_{15} = 0$. $\langle \nu, \alpha_{14} \rangle = 0$, and, therefore, $\nu_{14} = \nu_{15} = 0$; $\langle \nu, \alpha_{13} \rangle = 1$, so $\nu_{13} - \nu_{14} = 1$. Eventually the vector of coordinates corresponding to a dominant residual point, ν is

$$(\nu_1, \nu_2, \nu_3, \ldots, \nu_{13}, \nu_{14}, \nu_{15}) = (765\ \ 43\ \ 322\ \ 2111\ \ 10\ \ 0)$$

4.2. Set of jumps associated to a residual segment. — In a following section (6.3), we will present certain embeddings of generic discrete series in parabolically induced modules. The proof of these embeddings necessitates the introduction of the definition of the *set of jumps* associated to a residual segment and, therefore, transitively, to an irreducible generic discrete series.

These *jumps* compose a finite set, *set of jumps*, of (half)-integers a_i's, such that the set of integers $2a_i + 1$ is of a given parity. In the context of classical groups, the latter set (composed of elements of a given parity) coincides with the *Jordan block* defined in [23]. We will also use the notion of Jordan block in this section.

Let us recall our steps so far.

If we are given π_0, an irreducible generic discrete series of G, by Proposition 2.7 and Theorem 2.6, it embeds as a subrepresentation in $I_P^G(\sigma'_{\lambda'})$ for $\sigma'_{\lambda'}$ a dominant residual point. Further, by the results of [16] (see, in particular, Proposition 6.2), $\sigma'_{\lambda'}$ corresponds to a distinguished unipotent orbit and, therefore, a weighted Dynkin diagram. Once $\Sigma_{\sigma'}$ is fixed (see Section 4 or the Introduction for the definition of $\Sigma_{\sigma'}$), and assuming it is irreducible, the type of weighted Dynkin diagram is given. All details will be given in Section 4.3. By the previous argumentation (Section 4.1), we associate a residual segment $(\underline{n_{\pi_0}})$ to the irreducible generic discrete series π_0.

We illustrate these steps in the following example:

Example 4.11 (classical groups). — Let σ_λ be in the cuspidal support of a generic discrete series π of a classical group (or its variants) $G(n)$, of rank n. First, assume $\sigma_\lambda := \rho|\cdot|^a \otimes \dots \rho|\cdot|^b \otimes \sigma_c$, where ρ is a unitary cuspidal representation of GL_k, and σ_c a generic cuspidal representation of $G(k'), k' < n$. Using Bala–Carter theory, since λ is a residual point, it is in the W_σ-orbit of a dominant residual point, which corresponds to a weighted Dynkin diagram of type B (or C, D), and, further, the above sequence of exponents $(a, \dots, b)$ is encoded $(\ell + m, \dots, \ell, \ell-1, \ell-1, \dots, 0) := (\underline{n})$ of type B (or C, D). The type of weighted diagram only depends on the reducibility point of the induced representation of $G(k+k') : I^{G(k+k')}(\rho|\cdot|^s \otimes \sigma_c)$, as explained in Proposition 4.6.

The bijective correspondence between residual segments *and* set of jumps. — Let us start with the bijective map:

$$(\underline{n}) \to \text{set of jumps of } (\underline{n}).$$

The length of a residual segment is the sum of the multiplicities: $n_{\ell+m} + n_{\ell+m-1} + \dots n_1 + n_0$.

We first write a length d residual segment $(\underline{n})$

$$\big((\ell+m), \dots, \underbrace{\ell}_{n_\ell \text{ times}}, \underbrace{\ell-1}_{n_{\ell-1} \text{ times}}, \dots, \underbrace{1}_{n_1 \text{ times}}, \underbrace{0}_{n_0 \text{ times}}\big)$$

as a length $2d+1$ (or $2d$) sequence of exponents (betokening an unramified character of the corresponding classical group, e.g. to B_d corresponds SO_{2d+1})

$$\big((\ell+m), \dots, \underbrace{\ell}_{n_\ell \text{ times}}, \underbrace{\ell-1}_{n_{\ell-1} \text{ times}}, \dots, \underbrace{1}_{n_1 \text{ times}}, \underbrace{0}_{n_0 \text{ times}}, \\ 0, \underbrace{0}_{n_0 \text{ times}}, \underbrace{-1}_{n_1 \text{ times}}, \dots, \underbrace{-\ell}_{n_\ell \text{ times}}, \dots, -(\ell+m)\big)$$

for type B_d only, we add the central zero. It is a decreasing sequence of $2d+1$ (for type B_d) or $2d$ (for type C_d, D_d) (half)-integers; from the previous Section 4.1, the reader has noticed that for C_d, $n_0 = 0$.

Then, we decompose this decreasing sequence as a multi-set of $2n_0 + 1$ (or $2n_1$ for type D_d or $2n_{1/2}$ for type C_d) (it is the number of elements in the Jordan block) linear residual segments symmetrical around zero:

$$\big\{(a_1, a_1 - 1, \dots, 0, \dots, -a_1); (a_2, a_2 - 1, \dots, 0, \dots, -a_2); \\ \dots; (a_{2n_0+1}, a_{2n_0+1} - 1, \dots, 0, \dots, -a_{2n_0+1})\big\}$$

resp.

$$\big\{(a_1, a_1 - 1, \dots, 1/2, -1/2, \dots, -a_1); (a_2, a_2 - 1, \dots, 1/2, -1/2, \dots, -a_2); \\ \dots; (a_{2n_{1/2}}, a_{2n_{1/2}} - 1, \dots, 1/2, -1/2, \dots, -a_{2n_{1/2}})\big\},$$

where a_1 is the largest (half)-integer in the above decreasing sequence, a_2 is the largest (half)-integer with multiplicity 2, and in general a_i is the largest (half)-integer with multiplicity i.

Definition 4.12 (set of jumps). — The *set of jumps* is the set $\{a_1, \dots, a_{2n_0+1}\}$ (or $\{a_1, \dots, a_{2n_{1/2}}\}$). As one notices, the terminology comes from the observation that multiplicities at each jump increases by 1: $n_{a_{i+1}} = n_{a_i} + 1$.

Let us make a parallel for the reader familiar with Moeglin–Tadić terminology for classical groups [23] (see also Tadić's notes [36] and [37] for an introductory summary of these notions). In such a context, the Jordan block of the irreducible discrete series π associated to the residual segment $(\underline{n})$ (denoted Jord_π) is constituted by the integers:

$$\{2a_1 + 1, 2a_2 + 1; \dots, 2a_{2n_0+1} + 1\}$$

(or $\left\{2a_1 + 1, 2a_2 + 1; \dots, 2a_{2n_{1/2}} + 1\right\}$). This is not a complete characterization of a Jordan block; for the correct use of the definition of Jordan block, we should also fix a self-dual irreducible cuspidal representation ρ of a general linear group and an irreducible cuspidal representation σ_c of a smaller classical group. We *abusively* use the terminology *Jordan block* to define one partition, but such a partition is only one of the constituents of the Jordan block as defined in [23]. Clearly, the Jordan block is a set of distinct odd (or even) integers. According to [23], the following condition should also be satisfied: $2d + 1 = \sum_i (2a_i + 1)$ for type B (or $2d = \sum_i (2a_i + 1)$ for type C).

Moreover, we are now going to explain there is a canonical way to obtain for a given type $(A, B, C$, or $D)$ and a fixed length d all distinguished nilpotent orbits, thus all weighted Dynkin diagrams and, therefore, all residual segments of these given type and length.

This is given by Bala–Carter theory (see Appendix A and, in particular, the Theorem A.7). First, one should partition the integer $2d + 1$ (or $2d$) into distinct odd (or even) integers (given $2d + 1$ or $2d$ there is a finite number of such partitions). Each partition corresponds to a distinguished orbit and further to a dominant residual point, hence a residual segment.

In fact, each partition corresponds to a Jordan block of an irreducible discrete series π (whose associated residual segment is $(\underline{n_\pi})$). Let us detail the three cases $(B, C$ and $D)$.

Let us, finally, illustrate the following correspondence:

$$\mathrm{Jord}_\pi \to \text{set of jumps } (\underline{n_\pi}) \to (\underline{n_\pi}).$$

- In the case of B_d, the set jumps of (n_π) derives easily from the choice of *one* partition of $2d + 1$ in distinct odd integers: $\mathrm{Jord}_\pi = \{2a_1 + 1, 2a_2 + 1, \dots, 2a_t + 1\}$. Then jumps of $(n_\pi) = \{a_1, a_2, \dots, a_t\}$.

Once this set of jumps has been identified, one writes the corresponding symmetrical around zero linear segments $(a_i, \dots, -a_i)$'s and by combining and reordering them, form a decreasing sequence of integers of length $2d+1$.

This length $2d+1$ sequence is symmetrical around zero, with a length d sequence of non-negative elements, a central zero, and the symmetrical sequence of negative elements. The length d sequence of positive elements is the residual segment $(\underline{n})$.

- Again, the case of C_d (by Theorem A.7 in Appendix A) $2d$ is partitioned into distinct even integers; each partition corresponds to a distinguished orbit and further to a dominant residual point, hence a residual segment.

  The correspondence is the following: to the Jordan block of a generic discrete series, π and its associated residual segment $\underline{n}_\pi$: $\mathrm{Jord}_\pi = \{2a_1+1, 2a_2+1, \dots, 2a_t+1\}$, for each a_i, one writes $(a_i, a_i-1, \dots, 1/2, -1/2, \dots -a_i)$. One takes all elements in all these sequences and reorders them to get a $2d$ decreasing sequence of half-integers. The length d sequence of positive half-integers corresponds to residual segment $(\underline{n})$ of type C_d.
- In the case of D_d, let $\mathrm{Jord}_\pi = \{2a_1+1, 2a_2+1, \dots, 2a_t+1\}$ be the Jordan block of a generic discrete series, π; then write the corresponding linear segments $(a_i, \dots, -a_i)$'s, with all these residual segments, and form a decreasing sequence of integers of length $2d$. This length $2d$ sequence is symmetrical around zero. The length d sequence of positive elements in chosen to form the residual segment $(\underline{n})$.

Example 4.13 (B_{14}). — Let us consider one partition of $2\cdot 14+1$ into distinct odd integers: $\{11, 9, 5, 3, 1\}$.

For each odd integer in this partition, write it as $2a_i+1$ and write the corresponding linear residual segments $(a_i, \dots, -a_i)$:

$$5\,4\,3\,2\,1\,0\,-1\,-2\,-3\,-4\,-5$$
$$4\,3\,2\,1\,0\,-1\,-2\,-3\,-4$$
$$2\,1\,0\,-1\,-2$$
$$1\,0\,-1$$
$$0$$

Re-assembling, we get

$$5\,4\,4\,3\,3\,2\,2\,2\,1\,1\,1\,1\,0\,0; 0; 0\,0\,-1\,-1\,-1\,-1\,-2\,-2\,-2\,-3\,-3\,-4\,-4\,-5$$

Then, the corresponding residual segment of length 14 ($29 = 2\cdot 14+1$) is:

$$5\,4\,4\,3\,3\,2\,2\,2\,1\,1\,1\,1\,0\,0.$$

Example 4.14 (C_9). — Then $2d'_i$ is 18, and we decompose 18 into distinct even integers: 18; $14+4$; $12+4+2$; $16+2$; $8+6+4$, $12+6$, $10+8$. To each of these partitions corresponds the Weyl group orbit of a residual point and, therefore, a residual segment. The regular orbit (since the exponents of the associated residual segment form a regular character of the torus) corresponds to 18. It is simply

$$(17/2, 15/2, 13/2, \ldots, 1/2).$$

The half-integer 17/2 is such that $2(17/2)+1=18$.

Let us consider the third partition, 12+4+2: $12=2(11/2)+1$; $4=2(3/2)+1$; $2=2(1/2)+1$. Each even integer gives a strictly decreasing sequence of half-integers $(11/2, 9/2, 7/2, 5/2, 3/2, 1/2)$; $(3/2, 1/2)$; $(1/2)$. Finally, we reorder the nine half-integers obtained as a decreasing sequence:

$$(11/2, 9/2, 7/2, 5/2, 3/2, 3/2, 1/2, 1/2, 1/2).$$

Remark 4.15. — Once given a residual segment, $(\underline{n})$, and its corresponding set of jumps $a_1 > a_2 > \ldots > a_n$, one observes that for any i, $(a_i, \ldots, -a_{i+1})(\underline{n_i})$ is in the W_σ-orbit of this residual segment, where $(a_i, \ldots, -a_{i+1})$ is a linear residual segment and $(\underline{n_i})$ a residual segment of the same type as $(\underline{n})$.

Therefore, a set of asymmetrical linear segments $(a_i, \ldots, -a_{i+1})$ along with the smallest residual segment of a given type (e.g. (100) for type B, or $(3/2, 1/2, 1/2)$ for type C) *or* a linear segments $(a_1, a_1-1, \ldots 0)$ (or $(a_1, a_1-1, \ldots 1/2)$ for type C) is in the W_σ-orbit of the residual segment $(\underline{n})$.

Clearly, a set of linear *symmetrical* segments cannot be in the W_σ-orbit of the residual segment $(\underline{n})$.

4.3. Application of the theory of residual segments. —

4.3.1. *Reformulation of our setting.* — Let us come back to our setting (recalled at the beginning of the Section 4).

Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1. Put $\Sigma_\sigma = \{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{M_{1,\alpha}}(\sigma) = 0\}$ (or $\Sigma_\sigma^M = \{\alpha \in \Sigma^M_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$). The set Σ_σ is a root system in a subspace of $(a^G_{M_1})^*$ (resp. $(a^M_{M_1})^*$)(cf. [35] 3.5).

Suppose that the irreducible components of Σ_σ are all of type A, B, C or D. First assume Σ_σ is irreducible, and let us denote $\mathcal{T}$ its type, and $\Delta_\sigma := \{\alpha_1, \ldots, \alpha_d\}$ the basis of Σ_σ (following our choice of basis for the root system of G).

We will consider maximal standard Levi subgroups of G, $M \supset M_1$, corresponding to sets $\Delta - \{\underline{\alpha_k}\}$, for a simple root $\underline{\alpha_k} \in \Delta$ (here we use the notation $\underline{\alpha_k}$ to avoid confusion with the roots in Δ_σ). Since $M \supseteq M_1 = M_\Theta$, $\Theta \subset \Delta - \{\underline{\alpha_k}\}$, or in other words, if we denote α_k the projection of $\underline{\alpha_k}$ on the orthogonal of Θ in $a^*_{M_1}$, then $\alpha_k \in \Sigma_\Theta$ (see Appendix B for the precise definition and analysis of this set), and even more, then $\alpha_k \in \Sigma_\sigma$. If $\underline{\alpha_k}$ is not a extremal root of the Dynkin diagram of G, Σ^M decomposes into two disjoint components.

REMARK 4.16. — The careful reader will have already noticed that it is possible that Σ^M breaks into *three* components rather than two; in the context Σ is of type D_n and $\underline{\alpha_k}$ in the above notation is the simple root $\alpha_{n-2} \in \Delta$. In this remark and in Appendix B, we rather use the notation α_i to denote the simple roots in Σ and $\overline{\alpha_i}$ their projections on the orthogonal to Θ. By the calculations done in [12], to obtain any root system in Σ_Θ for Σ of type D_n, we need either α_{n-1} *and* α_n in Δ to be in Θ, or only one of them in Θ. In the case that both of them are in Θ, but α_{n-2} is not, we are reduced to the case of B_{n-2}. Then $\overline{\alpha_{n-2}} = e_{n-2}$ would be the last root in Σ_σ. Therefore, if $M = M_{\Delta-\alpha_{n-2}}$, and therefore Σ_σ^M is irreducible, we treat the conjecture for this case in Section 7.1. In the case that only one of them (without loss of generality α_{n-1}) is in Θ, the projection $\overline{\alpha_{n-2}} = e_{n-2} - \frac{e_n+e_{n-1}}{2}$ has a squared norm equal to 3/2. This forbids this root to belong to Σ_Θ and, therefore, to be the root $\underline{\alpha_k}$ such that M is $M_{\Delta-\underline{\alpha_k}}$. Indeed, as explained at the very beginning of Section 4.3, since $M_1 = M_\Theta \subseteq M$, the root $\underline{\alpha_k}$ that *is not a root in* M is not a root in Θ either.

Then, Σ_σ^M is a disjoint union of two irreducible components $\Sigma_{\sigma,1}^M \bigcup \Sigma_{\sigma,2}^M$ of type A and $\mathscr{T}$, one of which may be empty (if we remove extremal roots from the Dynkin diagram). If we remove $\overline{\alpha_n}$, $\Sigma_{\sigma,2}^M$ is empty, and $\Sigma_{\sigma,1}^M$ is of type A, whereas if we remove $\overline{\alpha_1}$, $\Sigma_{\sigma,2}^M$ is of type $\mathscr{T}$ and $\Sigma_{\sigma,1}^M$ is empty.

Else we assume Σ_σ is not irreducible but a disjoint union of irreducible components or empty components $\Sigma_{\sigma,i}$ for $i = 1, \dots, r$ of type A, B, C or D: $\Sigma_\sigma = \bigcup_i \Sigma_{\sigma,i}$. Then, the basis of Σ_σ is

$$\Delta_\sigma := \{\alpha_{1,1}, \dots, \alpha_{1,d_1}; \alpha_{2,1}, \dots, \alpha_{2,d_2}, \dots, \alpha_{i,1}, \dots, \alpha_{i,d_i}, \dots, \alpha_{r,1}, \dots, \alpha_{r,d_r}\}.$$

Again, we will consider maximal standard Levi subgroup of G, $M \supset M_1$, corresponding to sets $\Delta - \{\overline{\alpha_k}\}$.

Then, for an index $j \in \{1, \dots, r\}$, $\Sigma_{\sigma,j}^M$ is a disjoint union of two irreducible components $\Sigma_{\sigma,j_1}^M \bigcup \Sigma_{\sigma,j_2}^M$ of type A and $\mathscr{T}$, one of which may be empty (if $\overline{\alpha_k}$ is an "extremal" root of the Dynkin diagram of G). If we remove the last simple root, $\overline{\alpha_n}$, of the Dynkin diagram, Σ_{σ,j_2}^M is empty, and Σ_{σ,j_1}^M is of type A, whereas if we remove α_1, Σ_{σ,j_2}^M is of type $\mathscr{T}$, and Σ_{σ,j_1}^M is empty. Therefore, it will be enough to prove our results and statements in the case of Σ_σ irreducible; since in case of reducibility, without loss of generality, we choose a component $\Sigma_{\sigma,j}$, and the same reasonings apply.

Now, in our setting (see the beginning of Section 4), σ_ν is a residual point for μ^M. Recall that Σ_σ is of rank $d = d_1 + d_2$. Therefore, the residual point is in the cuspidal support of the generic discrete series τ if and only if (applying Proposition 4.6 above): $\mathrm{rk}(\Sigma_\sigma^M) = d_1 - 1 + d_2$.

We write $\Sigma_\sigma^M := A_{d_1-1} \bigcup \mathscr{T}_{d_2}$, and ν corresponds to residual segments $(\nu_{1,1}, \dots, \nu_{1,d_1})$ and $(\nu_{2,1}, \dots, \nu_{2,d_2})$.

Let us assume that the representation σ_λ is in the cuspidal support of the essentially square integrable representation of M, $\tau_{s\tilde{\alpha}}$, where $\lambda = \nu + s\tilde{\alpha}$. We add the twist $s\tilde{\alpha}$ on the linear part (i.e. corresponding to A_{d_1-1}), and, therefore, $(\nu_{2,1}, \dots, \nu_{2,d_2})$ is left unchanged and is thus $(\lambda_{2,1}, \dots, \lambda_{2,d_2})$, whereas $(\nu_{1,1}, \dots, \nu_{1,d_1})$ becomes $(\lambda_{1,1}, \dots, \lambda_{1,d_1})$.

Then, we need to obtain from $(\lambda_{1,1}, \dots, \lambda_{1,d_1})(\lambda_{2,1}, \dots, \lambda_{2,d_2})$ a residual segment of length d and type $\mathscr{T}$. Indeed, it is the only option to ensure σ_λ is a residual point (applying Proposition 4.6) for μ^G, in particular, since $d = d_1 + d_2$ (and, therefore, writing $\Sigma_\sigma = A_{d_1-1} \bigcup \mathscr{T}_{d_2}$ does not satisfy the requirement of Proposition 4.6).

4.3.2. *Cuspidal strings.* — Assume that we remove a non-extremal simple root of the Dynkin diagram. The parameter λ in the cuspidal support is, therefore, constituted by a couple of residual segments, one of which is a linear residual segment: $(\mathscr{a}, \dots, \mathscr{b})$, and the other is denoted $(\underline{n})$. It will be convenient to define the cuspidal support to be given by the tuple $(\mathscr{a}, \mathscr{b}, \underline{n})$, where $\underline{n}$ is a tuple $(\dots, 0, n_{\ell+m}, \dots, n_\ell, n_{\ell-1}, \dots, n_1, n_0)$ uniquely characterizing the residual segment. We define:

Definition 4.17 (cuspidal string). — Given two residual segments, strings of integers (or half-integers): $(\mathscr{a}, \dots, \mathscr{b})(\underline{n})$. The tuple $(\mathscr{a}, \mathscr{b}, \underline{n})$, where $\underline{n}$ is the $(\ell + m + 1)$-tuple

$$(n_{\ell+m}, \dots, n_\ell, n_{\ell-1}, \dots, n_1, n_0),$$

is named a cuspidal string.

Recall that W_σ is the Weyl group of the root system Σ_σ.

Definition 4.18 (W_σ-cuspidal string). — Given a tuple $(\mathscr{a}, \mathscr{b}, \underline{n})$, where $\underline{n}$ is the $(\ell+m+1)$-tuple $(n_{\ell+m}, \dots, n_\ell, n_{\ell-1}, \dots, n_1, n_0)$, the set of all three-tuples $(\mathscr{a}', \mathscr{b}', \underline{n'})$, where $\underline{n'}$ is an $(\ell' + m' + 1)$-tuple $(n'_{\ell'+m'}, \dots, n'_{\ell'}, n'_{\ell'-1}, \dots, n'_1, n'_0)$ in the W_σ orbit of $(\mathscr{a}, \mathscr{b}, \underline{n})$ and is called W_σ-cuspidal string.

Remark 4.19. — These definitions can be extended to include the case of t linear residual segments (i.e. of type A): $(\mathscr{a}_1, \dots, \mathscr{b}_1)(\mathscr{a}_2, \dots, \mathscr{b}_2) \dots (\mathscr{a}_t, \dots, \mathscr{b}_t)$ and a residual segment $(\underline{n})$ of type B, C or D, then the parameter in the cuspidal support will be denoted $(\mathscr{a}_1, \mathscr{b}_1; \mathscr{a}_2, \mathscr{b}_2; \dots; \mathscr{a}_t, \mathscr{b}_t, \underline{n})$.

4.4. Application to the case of classical groups. — In the following section, we illustrate how these definitions naturally appear in the context of classical groups.

4.4.1. *Unramified principal series.* — Let τ be a generic discrete series of $M = M_L \times M_c$, the maximal Levi subgroup in a classical group G, then $M_L \subset P_L$ is a linear group, and $M_c \subset P_c$ is a smaller classical group. It is a tensor product of an essentially square integrable representation of a linear group and an irreducible generic discrete series π of a smaller classical group of the same type as G.

$$\tau := \mathrm{St}_{d_1}|\cdot|^s \otimes \pi, \text{ with } s = \frac{a + b}{2}.$$[1]

Further, let us assume that $(P_1, \sigma, \lambda) := (P_0, \mathbf{1}, \lambda)$. The twisted Steinberg is the unique subrepresentation in $I^{M_L}_{P_{0,L}}(a, \dots, b)$, whereas $\pi \hookrightarrow I^{M_c}_{P_{0,c}}(\underline{n})$.

Therefore,

$$I^G_P(\tau) \hookrightarrow I^G_{P_c \times P_L}(I^{M_L}_{P_{0,L}}(a, \dots, b) I^{M_c}_{P_{0,c}}(\underline{n})) \cong I^G_{P_0}((a, \dots, b)(\underline{n})).$$

4.4.2. *The general case.* — Assume τ is an irreducible generic essentially square integrable representation of a maximal Levi subgroup M of a classical group of rank $\sum_{i=1}^r d_i \cdot \dim(\sigma_i) + k$. Then $\tau := \mathrm{St}_{d_1}(\sigma_1)|\cdot|^s \otimes \pi$, with $s = \frac{a+b}{2}$.

We study the cuspidal support of the generic (essentially) square integrable representations $\mathrm{St}_{d_1}(\sigma_1)|\cdot|^s$ and π.

By Proposition 2.7, $\pi \hookrightarrow I^{M_c}_{P_{1,c}}(\sigma^c_{\nu_c})$ such that:

$$M_{1,c} = \underbrace{\mathrm{GL}_{k_2} \times \dots \times \mathrm{GL}_{k_2}}_{d_2 \text{ times}} \times \dots \times \underbrace{\mathrm{GL}_{k_r} \times \dots \times \mathrm{GL}_{k_r}}_{d_r \text{ times}} \times G(k),$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G.

We write the cuspidal representation $\sigma^c := \sigma_2 \otimes \dots \otimes \sigma_2 \otimes \dots \otimes \sigma_r \otimes \dots \otimes \sigma_r \otimes \sigma_c$ of $M_{1,c}$ and assume the inertial classes of the representations of GL_{k_i}, σ_i, are mutually distinct, and $\sigma_i \cong \sigma_i^\vee$ if $\sigma_i, \sigma_i^\vee$ are in the same inertial orbit.

The residual point ν_c is dominant: $\nu_c \in ((a^M_{M_1})^*+$. Applying Proposition 4.6 below with ν_c and the root system Σ^M_σ we have: $\nu_c := (\nu_2, \dots, \nu_r)$, where each ν_i for $i \in \{2, \dots, r\}$ is a residual point, corresponding to a residual segment of type $B_{d_i}, C_{d_i}, D_{d_i}$.

Further,

$$\mathrm{St}_{d_1}(\sigma_1)|\cdot|^s \hookrightarrow I^{M_L}_{P_{1,L}}(\sigma_1, \lambda_L) \cong I^{M_L}_{P_{1,L}}(\sigma_1|\cdot|^{a} \otimes \sigma_1|\cdot|^{a-1} \dots \sigma_1|\cdot|^{b}),$$

where λ_L is the residual segment of type A: $(a, a-1, \dots, b)$, and M_L is the linear part of Levi subgroup M. Such that eventually: $\sigma = \sigma_1 \otimes \sigma_1 \dots \sigma_1 \otimes \sigma_2 \otimes \dots \sigma_2 \otimes \dots \otimes \sigma_r \otimes \dots \sigma_r \otimes \sigma_c$, and σ_λ can be rewritten:

$$(4) \quad \sigma_1|\cdot|^{a} \otimes \sigma_1|\cdot|^{a-1} \dots \sigma_1|\cdot|^{b} \otimes \underbrace{\sigma_2|\cdot|^{\ell_2} \dots \sigma_2|\cdot|^{\ell_2}}_{n_{\ell_2,2} \text{ times}} \dots \underbrace{\sigma_2|\cdot|^{0} \dots \otimes \sigma_2|\cdot|^{0}}_{n_{0,2} \text{ times}} \dots$$
$$\underbrace{\sigma_r|\cdot|^{\ell_r} \dots \otimes \sigma_r|\cdot|^{\ell_r}}_{n_{\ell_r,r} \text{ times}} \dots \underbrace{\otimes \sigma_r|\cdot|^{0} \dots \otimes \sigma_r|\cdot|^{0}}_{n_{0,r} \text{ times}} \otimes \sigma_c.$$

1. It is worth noting that in the case of the Siegel parabolic for classical groups, $I_P(\tau_{s\tilde{\alpha}})$ is $\mathrm{Ind}^G_P(|\det|^{s/2}\tau)$ see p7, [31].

The character ν, representation of M_1, can be split into two parts ν_1 and $\underline{\nu} = (\nu_2, \dots, \nu_r)$, residual points, giving the discrete series denoted $\mathrm{St}_{d_1}(\sigma_1)$ in $I_{P_{1,L}}^{M_L}(\sigma_1)$ and π in $I_{P_{1,c}}^{M_c}(\sigma_c, \underline{\nu})$. By a simple computation, it can be shown that the twist $s\tilde{\alpha}$ will be added on the 'linear part' of the representation and leaves the semi-simple part (classical part) invariant.

Namely, ν is given by a vector $(\nu_1 = 0, \nu_2, \dots, \nu_r)$, and we add the twist $s\tilde{\alpha}$ on the first element to get the vector: $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, where each λ_i is a residual segment $(\underline{n_i})$ associated to the subsystem $\Sigma_{\sigma,i}$.

To use the bijection between W_σ orbits of residual points and weighted Dynkin diagrams, one needs to use a certain root system and its associated Weyl group. Then λ is a tuple of r residual segments of different types: $\{(\underline{n}_i)\}, i \in \{1, \dots, r\}$. If the parameter λ is written as a r-tuple: $(\lambda_1, \dots, \lambda_r)$, it is dominant if and only if each λ_i is dominant with respect to the subsystem $\Sigma_{\sigma,i}$.

We have not yet used the *genericity* property of the cuspidal support. This is where we use Proposition 4.6. The generic representation σ_c and the reducibility point of the representation induced from $\sigma_i|\cdot|^s \otimes \sigma_c$ determine the type of the residual segment $(\underline{n}_i)$ obtained.

5. Characterization of the unique irreducible generic subquotient in the standard module

5.1. Let us first outline the results presented in this section. Let us assume that the irreducible generic subquotient in the standard module is not a discrete series. We characterize the Langlands parameter of this unique irreducible non-square integrable subquotient using an order on Langlands parameters given in Lemma 5.2 below: more precisely, in Theorem 5.5, we prove that this unique irreducible generic subquotient is identified by its Langlands parameter being minimal for this order.

We then compare Langlands parameters in Section 5.3, and along those results and Theorem 5.5, we will prove a lemma (Lemma 5.16) in the vein of Zelevinsky's theorem at the end of this section.

Finally, before entering the next section we need to come back to the depiction of the intertwining operators used in our context. This Section 5.4 on intertwining operators also contains a lemma (Lemma 5.12) that is crucial in the proof of the main Theorem 6.3 in the following section.

5.2. An order on Langlands' parameters. — Using Langlands' classification (see Theorem 2.9) and the *standard module conjecture* (see Theorem 2.10), we can characterize the unique irreducible generic non-square integrable subquotient, denoted $I_{P'}^G(\tau'_{\nu'})$. In particular, on a given cuspidal support, we can characterize the form of the Langlands' parameter ν'. We introduce the necessary tools and results regarding this theory in this section.

To study subquotients in the standard module induced from a maximal parabolic subgroup P, $I_P^G(\tau_{s\tilde{\alpha}})$, we will use the following well-known lemma from [3]:

Let us recall their definition of the order:

Definition 5.1 (order). — $\lambda_\mu \leq \lambda_\pi$ if $\lambda_\pi - \lambda_\mu = \sum_i x_i \alpha_i$ for simple roots α_i in a_0^* and $x_i \geq 0$.

Lemma 5.2 (Borel-Wallach, 2.13 in Chapter XI of [3]). — *Let (P, σ, λ_π) be Langlands data. If μ is a constituent of $I_P^G(\sigma_{\lambda_\pi})$ the standard module, and if $\pi = J(P, \sigma, \lambda_\pi)$ is the Langlands quotient, then $\lambda_\mu \leq \lambda_\pi$, and equality occurs if and only if μ is $J(P, \sigma, \lambda_\pi)$.*

We will write this order on Langlands parameters:

$$\lambda_\mu {}_P{\leq} \lambda_\pi.$$

Lemma 5.3. — *Let $\nu = \sum_{i=1}^n a_i e_i$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$. $0_P{\leq} \nu$ if and only if $\sum_{i=1}^k a_i \geq 0$, for any k in non-D_n cases. In the case of D_n, one needs to specify $\sum_{i=1}^k a_i \geq 0$, for any $k \leq n-1$, $a_{n-1} \geq -a_n$ and $a_{n-1} \geq a_n$.*

Proof. — From the expression $\nu = \sum_{i=1}^n a_i e_i$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$, we can recover an expression of ν in the canonical basis of the Lie algebra a_0^*: $\nu = \sum_{i=1}^n x_i \alpha_i$.

Let us make explicit $\nu = \sum_i x_i \alpha_i$:

$$\begin{aligned}
\nu &= \sum_{i=1}^{n-1} x_i(e_i - e_{i+1}) + x_n \alpha_n \\
&= x_1(e_1 - e_2) + x_2(e_2 - e_3) + \ldots + x_{n-1}(e_{n-1} - e_n) = \\
\nu &= \sum_{i=1}^{n} a_i e_i \\
&= x_1 e_1 + (x_2 - x_1) e_2 + (x_3 - x_2) e_3 + \ldots \\
&\quad + \begin{cases} (x_{n-1} - x_{n-2}) e_{n-1} - x_{n-1} e_n & \text{for } A_{n-1} \\ (x_{n-1} + x_n) e_{n-1} + (x_n - x_{n-1}) e_n & \text{for } D_n \\ (x_{n-1} - x_{n-2}) e_{n-1} + (x_n - x_{n-1}) e_n & \text{for } B_n \\ (x_{n-1} - x_{n-2}) e_{n-1} + (2x_n - x_{n-1}) e_n & \text{for } C_n \end{cases}
\end{aligned}$$

$\nu = \sum_{i=1}^n x_i \alpha_i \geq 0 \Leftrightarrow x_i \geq 0 \ \forall i$.

From the above, $x_1 = a_1$, $x_2 - x_1 = a_2 \Leftrightarrow x_2 = a_1 + a_2$, … We have: $x_k = \sum_{i=1}^k a_i \ \forall k$, except for root systems of type D_n, where for index $n-1$ and n, $2x_n = \sum_{i=1}^{n-1} a_i + a_n$ and $2x_{n-1} = \sum_{i=1}^{n-1} a_i - a_n$, and for C_n, where $2x_n = \sum_{i=1}^n a_i$.

Notice that for A_{n-1}, $x_{n-1} = \sum_{i=1}^{n-1} a_i$ and $a_n = -x_{n-1}$ such that $\sum_{i=1}^n a_i = 0$.

Therefore, $0_P\leq \nu$ if and only if $\sum_{i=1}^{k} a_i \geq 0$ for any k in non-D_n cases. In the case of D_n, one needs to specify $\sum_{i=1}^{k} a_i \geq 0$, for any $k \leq n-1$, $\sum_{i=1}^{n-1} a_i \geq -a_n$ and $\sum_{i=1}^{n-1} a_i \geq a_n$. □

Our next result, Theorem 5.5, will be used in the course of the proof of the generalized injectivity conjecture for non-discrete series subquotients presented in Sections 7 and 7.3. We use the notations of Section 3. We will need the following theorem:

THEOREM 5.4 (Theorem 2.2 of [18]). — *Let $P = MU$ be a F-standard parabolic subgroup of G and σ an irreducible generic cuspidal representation of M. If the induced representation $I_P^G(\sigma)$ has a subquotient that lies in the discrete series of G (or is tempered), then the unique irreducible generic subquotient of $I_P^G(\sigma)$ lies in the discrete series of G (or is tempered).*

THEOREM 5.5. — *Let $I_P^G(\tau_\nu)$ be a generic standard module and (P', τ', ν') the Langlands data of its unique irreducible generic subquotient.*

If (P'', τ'', ν'') is the Langlands data of any other irreducible subquotient, then $\nu'\,_P\!\leq \nu''$. The inequality is strict if the standard module $I_{P''}^G(\tau''_{\nu''})$ is generic.

In other words, ν' is the smallest Langlands parameter for the order (defined in Lemma 5.2) among the Langlands parameters of standard modules having (σ, λ) as cuspidal support.

Proof. — First, using the result of Heiermann–Opdam (in [19]), we let $I_P^G(\tau_\nu)$ be embedded in $I_{P_1}^G(\sigma_{\nu_0+\nu})$ with cuspidal support $(\sigma, \lambda = \nu_0 + \nu)$.

Using Langlands' classification, we write $J(P', \tau', \nu')$ an irreducible generic subquotient of $I_P^G(\tau_\nu)$. Then the standard module conjecture claims that $J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'})$.

The first case to consider is a *generic* standard module $I_{P''}^G(\tau''_{\nu'})$. From the unicity of the generic irreducible module with cuspidal support (σ, λ) (Rodier's theorem, [U]), one sees that $J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'}) \leq I_{P''}^G(\tau''_{\nu''})$. Hence, $\nu'_{P''} < \nu''$.

Secondly, if the standard module $I_{P''}^G(\tau''_{\nu''})$ is any (non-generic) subquotient having (σ, λ) as cuspidal support, since this cuspidal support is generic one will see that one can replace τ'' by the generic tempered representation τ''_{gen} with the same cuspidal support and conserve the Langlands parameter ν'', and we are back to the first case. This is explained in the next paragraph. The lemma follows.

To replace the tempered representation τ'' of M'' the argument goes as follows: Since the representation σ in the cuspidal support of this representation is generic, by Theorem 5.4 the unique irreducible generic representation subquotient τ''_{gen} in the representation induced from this cuspidal support is tempered. As any representation in the cuspidal support of τ'' must lie in the cuspidal support of τ''_{gen}, any such representation must be conjugated to σ. That is, there exists a Weyl group element $w \in W$ such that if $\tau'' \hookrightarrow I_{P_1\cap M''}^{M''}(\sigma_{\nu_0})$, and then

$$\tau''_{\mathrm{gen}} \hookrightarrow I_{P_1\cap M''}^{M''}((w\sigma)_{w\nu_0}).$$

Twisting by $\nu'' \in a^*_{M'}$ comes second. Therefore, conjugation by this Weyl group element leaves invariant the Langlands parameter $\nu'' \in a^*_{M'}$, and $(\tau''_{\mathrm{gen}})_{\nu''}$ and $\tau''_{\nu''}$, therefore, share the same cuspidal support. □

5.3. Linear residual segments. — Let $I_P^G(\tau_{s\tilde{\alpha}})$ be a standard module; we call the parameter *$s\tilde{\alpha}$ the Langlands parameter of the standard module.* We have seen that this Langlands parameter (the *twist*) depends only on the linear (not semi-simple) part of the cuspidal support, i.e. the linear residual segment.

In this section and the following, we use the notation $\mathcal{S}$ (see Definition 4.9) to denote a *linear residual segment*; the underlying irreducible cuspidal representation ρ is implicit. A simple computation gives that if a standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where P is a maximal parabolic, embeds in $I_{P_1}^G(\sigma(\mathscr{a}, \mathscr{b}, \underline{n}))$ for a cuspidal string $(\mathscr{a}, \mathscr{b}, \underline{n})$, then $s = \frac{\mathscr{a}+\mathscr{b}}{2}$. The parameter $s\tilde{\alpha}$ is in $(a^*_M)^+$, but to use Lemma 5.2 we will need to consider it as an element of $a^*_{M_1}$.

Then, we say this Langlands parameter is *associated* to the linear residual segment $(\mathscr{a}, \ldots, \mathscr{b})$. In this section, we compare Langlands parameters associated to linear residual segments.

Lemma 5.6. — *Let γ be a real number such that $\mathscr{a} \geq \gamma \geq \mathscr{b}$.*

Splitting a linear residual segment $(\mathscr{a}, \ldots, \mathscr{b})$ whose associated Langlands parameter is $\lambda = \frac{\mathscr{a}+\mathscr{b}}{2} \in a^*_M$ into two segments: $(\mathscr{a}, \ldots, \gamma+1)(\gamma, \mathscr{b})$ yields necessarily a larger Langlands parameter, λ' for the order given in Lemma 5.2.

Proof. — We write $\lambda \in a^*_M$ as an element in $a^*_{M_1}$ to be able to use Lemma 5.2 (i.e. the Lemma 5.2 also applies with $a^*_{M_1}$):

$$\lambda = \Bigl(\underbrace{\frac{\mathscr{a}+\mathscr{b}}{2}, \ldots, \frac{\mathscr{a}+\mathscr{b}}{2}}_{\mathscr{a}-\mathscr{b}+1 \text{ times}}\Bigr) \quad \text{also,}$$

$$\lambda' = \Bigl(\underbrace{\frac{\mathscr{a}+(\gamma+1)}{2}, \ldots, \frac{\mathscr{a}+(\gamma+1)}{2}}_{\mathscr{a}-\gamma \text{ times}}, \underbrace{\frac{\gamma+\mathscr{b}}{2}, \ldots, \frac{\gamma+\mathscr{b}}{2}}_{\gamma-\mathscr{b}+1 \text{ times}}\Bigr),$$

$$\lambda' - \lambda = \Bigl(\underbrace{\frac{(\gamma+1)-\mathscr{b}}{2}, \ldots, \frac{(\gamma+1)-\mathscr{b}}{2}}_{\mathscr{a}-\gamma \text{ times}}, \underbrace{\frac{\gamma-\mathscr{a}}{2}, \ldots, \frac{\gamma-\mathscr{a}}{2}}_{\gamma-\mathscr{b}+1 \text{ times}}\Bigr).$$

Therefore, $x_1 = \frac{(\gamma+1)-\mathscr{b}}{2} > 0$. Since $x_k = \sum_{i=1}^k a_i$ as written in the proof of Lemma 5.3, one observes that $x_k > x_n$ for any $k < n = \mathscr{a} - \mathscr{b} + 1$, and $x_n = \frac{(\gamma+1)-\mathscr{b}}{2}(\mathscr{a}-\gamma) + \frac{\gamma-\mathscr{a}}{2}(\gamma - \mathscr{b} + 1) = (\mathscr{a}-\gamma)(\frac{(\gamma+1)-\mathscr{b}}{2} - \frac{-\gamma+\mathscr{b}-1}{2}) = 0$. Hence, $\lambda' \geq_P \lambda$ by Lemma 5.3. □

Proposition 5.7. — *Consider two linear (i.e. of type A) residual segments, i.e. strictly decreasing sequences of real numbers such that the difference between two consecutive reals is 1: $\mathcal{S}_1 := (a_1, \dots, b_1); \mathcal{S}_2 := (a_2, \dots, b_2)$. Typically, one could think of decreasing sequences of consecutive integers or consecutive half-integers.*

Assume $a_1 > a_2 > b_1 > b_2$ so that they are linked in the terminology of Bernstein–Zelevinsky. Taking intersection and union yields two unlinked residual segments $\mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathcal{S}_1 \cup \mathcal{S}_2$.

Denote $\lambda \in a_M^*$ the Langlands parameter $\lambda = (s_1, s_2)$ associated to $\mathcal{S}_1$ and $\mathcal{S}_2$, and expressed in the canonical basis associated to the Lie algebra a_0^*.

Denote $\lambda' \in a_M^* : \lambda' = (s_1', s_2')$ the one associated to the two unlinked segments $\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \cup \mathcal{S}_2$ ordered so that $s_1' > s_2'$.

Then, $\lambda' {}_P{\leq} \lambda$.

Proof. — Let $(a_1, \dots, b_1)(a_2, \dots, b_2)$ be two segments with $a_1 > a_2 > b_1 > b_2$ so that the two segments are linked. The associated Langlands parameter is:

$$\lambda = \Bigg(\underbrace{\frac{a_1+b_1}{2}, \dots, \frac{a_1+b_1}{2}}_{a_1-b_1+1 \text{ times}}, \underbrace{\frac{a_2+b_2}{2}, \dots, \frac{a_2+b_2}{2}}_{a_2-b_2+1 \text{ times}}\Bigg).$$

Then taking the union and intersection of those two segments gives:

$$(a_1, \dots, b_2)(a_2, \dots, b_1) \quad \text{or} \quad (a_2, \dots, b_1)(a_1, \dots, b_2)$$

ordered so that $s_1' > s_2'$. The Langlands parameter will, therefore, be given by:

1. If $\frac{a_1+b_2}{2} \geq \frac{a_2+b_1}{2}$:
$$\lambda' = \Bigg(\underbrace{\frac{a_1+b_2}{2}, \dots, \frac{a_1+b_2}{2}}_{a_1-b_2+1 \text{ times}}, \underbrace{\frac{a_2+b_1}{2}, \dots, \frac{a_2+b_1}{2}}_{a_2-b_1+1 \text{ times}}\Bigg).$$
2. If $\frac{a_2+b_1}{2} > \frac{a_1+b_2}{2}$:
$$\lambda' = \Bigg(\underbrace{\frac{a_2+b_1}{2}, \dots, \frac{a_2+b_1}{2}}_{a_2-b_1+1 \text{ times}}, \underbrace{\frac{a_1+b_2}{2}, \dots, \frac{a_1+b_2}{2}}_{a_1-b_2+1 \text{ times}}\Bigg).$$

Then the difference $\lambda - \lambda'$ equals:

- In case (1),
$$\Bigg(\underbrace{\frac{b_1-b_2}{2}, \dots, \frac{b_1-b_2}{2}}_{a_1-b_1+1 \text{ times}}, \underbrace{\frac{a_2-a_1}{2}, \dots, \frac{a_2-a_1}{2}}_{b_1-b_2 \text{ times}},$$
$$\underbrace{\frac{b_2-b_1}{2}, \dots, \frac{b_2-b_1}{2}}_{a_2-b_1+1 \text{ times}}, 0, \dots, 0\Bigg).$$

First,

$$x_1 = \frac{\mathscr{b}_1 - \mathscr{b}_2}{2}.$$

Secondly, since $x_k = \sum_{i=1}^{k} a_i$ as written in the proof of Lemma 5.3, one observes that all subsequent x_k are greater than or equal to x_n, for $n = \mathscr{a}_1 - \mathscr{b}_1 + 1 + \mathscr{a}_2 - \mathscr{b}_2 + 1$.

Moreover,

$$\begin{aligned} x_n &= \frac{\mathscr{b}_1 - \mathscr{b}_2}{2}(\mathscr{a}_1 - \mathscr{b}_1 + 1) + \frac{\mathscr{a}_2 - \mathscr{a}_1}{2}(\mathscr{b}_1 - \mathscr{b}_2) + \frac{\mathscr{b}_2 - \mathscr{b}_1}{2}(\mathscr{a}_2 - \mathscr{b}_1 + 1) \\ &= \frac{\mathscr{b}_1 - \mathscr{b}_2}{2}(\mathscr{a}_1 - \mathscr{b}_1 + 1 + \mathscr{a}_2 - \mathscr{a}_1 - (\mathscr{a}_2 - \mathscr{b}_1 + 1)) = 0. \end{aligned}$$

- In case (2),

$$\begin{aligned} \lambda - \lambda' = \Bigg(&\underbrace{\frac{\mathscr{a}_1 - \mathscr{a}_2}{2}, \dots, \frac{\mathscr{a}_1 - \mathscr{a}_2}{2}}_{\mathscr{a}_2 - \mathscr{b}_1 + 1 \text{ times}}, \underbrace{\frac{\mathscr{b}_1 - \mathscr{b}_2}{2}, \dots, \frac{\mathscr{b}_1 - \mathscr{b}_2}{2}}_{\mathscr{a}_1 - \mathscr{a}_2 \text{ times}}, \\ &\underbrace{\frac{\mathscr{a}_2 - \mathscr{a}_1}{2}, \dots, \frac{\mathscr{a}_2 - \mathscr{a}_1}{2}}_{\mathscr{a}_2 - \mathscr{b}_2 + 1 \text{ times}}\Bigg). \end{aligned}$$

Here,

$$\begin{aligned} x_1 &= \frac{\mathscr{a}_1 - \mathscr{a}_2}{2} \\ x_n &= \frac{\mathscr{a}_1 - \mathscr{a}_2}{2}(\mathscr{a}_2 - \mathscr{b}_1 + 1) + \frac{\mathscr{b}_1 - \mathscr{b}_2}{2}(\mathscr{a}_1 - \mathscr{a}_2) + \frac{\mathscr{a}_2 - \mathscr{a}_1}{2}(\mathscr{a}_2 - \mathscr{b}_2 + 1) \\ &= \frac{\mathscr{a}_2 - \mathscr{a}_1}{2}(\mathscr{a}_2 - \mathscr{b}_1 + 1 + \mathscr{b}_1 - \mathscr{b}_2 - (\mathscr{a}_2 - \mathscr{b}_2 + 1)) = 0. \end{aligned}$$

□

PROPOSITION 5.8. — *The Langlands parameter λ', as defined in the previous Proposition 5.7, is the minimal Langlands parameter for the order given in Lemma 5.2 on this cuspidal support.*

Proof. — Let us consider a decreasing sequence of real numbers such that the difference between two consecutive elements is 1: $(\mathscr{a}_1, \mathscr{a}_1 - 1, \dots, \mathscr{a}_2, \dots, \mathscr{b}_1, \dots, \mathscr{b}_2)$, with the following conditions: $\mathscr{a}_1 > \mathscr{a}_2 > \mathscr{b}_1 > \mathscr{b}_2$ and all real numbers between $\mathscr{a}_2$ and $\mathscr{b}_1$ are repeated twice. Let us call this sequence c.

We consider the set $\mathscr{S}$ of tuple of linear segments $\mathcal{S}_i = (a_i, \dots, b_i)$ (strictly decreasing sequence of reals) such that if $s_i = \frac{a_i + b_i}{2} \geq s_j = \frac{a_j + b_j}{2}$ then the linear segment $\mathcal{S}_i$ is placed on the left of $\mathcal{S}_j$, i.e.:

$$(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k) \in \mathscr{S} \Leftrightarrow s_1 \geq s_2 \dots \geq s_k.$$

In this set $\mathscr{S}$, let us first consider the special case of a decreasing sequence $\delta \in \mathscr{S}$, where each segment is length 1 and $s_i = \mathcal{S}_i$. Then the Langlands parameter is just $\delta = (a_1, a_1 - 1, \ldots, a_2, a_2, \ldots, b_1, b_1, \ldots b_2)$.

Secondly, let us consider the case where all segments are mutually unlinked, then they have to be included in one another. The reader will readily notice that the only option is the following element in $\mathscr{S}$:

$$m := (a_1, \ldots, b_2)(a_2, \ldots, b_1).$$

Its Langlands parameter is

$$\lambda' = \Bigg(\underbrace{\frac{a_1 + b_2}{2}, \ldots, \frac{a_1 + b_2}{2}}_{a_1 - b_2 + 1\ \text{times}}, \underbrace{\frac{a_2 + b_1}{2}, \ldots, \frac{a_2 + b_1}{2}}_{a_2 - b_1 + 1\ \text{times}}\Bigg).$$

Let us show that $\delta \geq_P \lambda'$.

Clearly, on the vector $\delta - \lambda'$: $x_1 = a_1 - \frac{a_1+b_2}{2} > 0$, $x_k = \sum_{i=1}^{k} a_i$, and one observes that all subsequent x_k are greater than or equal to x_n, and x_n is the sum of the elements (counted with multiplicities) in the vector δ minus $\frac{a_1+b_2}{2}(a_1 - b_2 + 1) + \frac{a_2+b_1}{2}(a_2 - b_1 + 1)$; therefore, $x_n = 0$, as this sum ends up the same as in the proof of the previous proposition.

Let us show that m is the unique, irreducible element obtained in $\mathscr{S}$ when taking repeatedly intersection and union of any two segments in any element $s \in \mathcal{S}$. Let us write an arbitrary $s \in \mathscr{S}$ as $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_p)$; since we had a certain number of reals repeated twice in c, it is clear that some of the $\mathcal{S}_i$ are mutually linked.

For our purpose, we write the vector of lengths of the segments in s: $(k_1, k_2, \ldots, k_p)$. Let us assume, without loss of generality, that $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked. Taking intersection and union, we obtain two unlinked segments $\mathcal{S}_1' = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_2' = \mathcal{S}_1 \cap \mathcal{S}_2$. If $k_1 \geq k_2$, then $k_1' = k_1 + a$, and $k_2' = k_2 - a$, i.e. the greatest length necessarily increases. Therefore, the potential $\sum_i k_i^2$ is increasing, while the number of segments is non-increasing. The process ends when we can no longer take the intersection and union of linked segments; then the longest segment contains entirely the second longest. This is the element $m \in \mathscr{S}$ introduced above.

Since at each step (of taking intersection and union of two linked segments) the Langlands parameter $\lambda_{s'}$ of the element $s' \in \mathscr{S}$ is smaller than at the previous step (by Proposition 5.7), it is clear that λ' is the minimal element for the order on Langlands parameter. □

Remark 5.9. — Let us assume that we fix the cuspidal representation σ and two segments $(\mathcal{S}_1, \mathcal{S}_2)$. As a result of this proposition, the standard module $I_{P'}^G(\tau'_{\lambda'})$ induced from the unique irreducible generic, essentially square integrable representation $\tau'_{\lambda'}$ obtained when taking intersection and union $(\mathcal{S}_1 \cap \mathcal{S}_2)$

and $(\mathcal{S}_1 \cup \mathcal{S}_2)$ (i.e., which embeds in $I_{P_1}^G(\sigma((\mathcal{S}_1 \cap \mathcal{S}_2);(\mathcal{S}_1 \cup \mathcal{S}_2)))$ is irreducible by Theorem 5.5.

5.4. Intertwining operators. — In the following result, we play for the first time with cuspidal strings and intertwining operators. We fix a unitary irreducible cuspidal representation σ of M_1 and let $(\mathscr{a}, \mathscr{b}, \underline{n})$ and $(\mathscr{a}', \mathscr{b}', \underline{n'})$ be two elements in some W_σ-cuspidal string; i.e. there exists a Weyl group element $w \in W_\sigma$ such that $w(\mathscr{a}, \mathscr{b}, \underline{n}) = (\mathscr{a}', \mathscr{b}', \underline{n'})$.

For the sake of readability, we sometimes denote $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is expressed in terms of residual segments. We would like to study intertwining operators between $I_{P_1}^G(\sigma(\mathscr{a}, \mathscr{b}, \underline{n}))$ and $I_{P_1}^G(\sigma(\mathscr{a}', \mathscr{b}', \underline{n'}))$. As explained in Section 3 and Proposition 3.3, this operator can be decomposed into rank 1 operators. Let us recall how one can conclude on the non-genericity of their kernels in the two main cases.

Example 5.10 (Rank 1 intertwining operators with a non-generic kernel). — Let us assume Σ_σ is irreducible of type A, B, C or D. We fix a unitary irreducible cuspidal representation σ and let $\alpha = e_i - e_{i+1}$ be a simple root in Σ_σ. The element s_α operates on λ in $(a_{M_1}^G)^*$. In this first example, we illustrate the case where s_α acts as a coordinates' transposition on λ written in the standard basis $\{e_i\}_i$ of $(a_{M_1}^G)^*$.

Let us focus on two adjacent elements in the residual segment corresponding to λ (at the coordinates λ_i and λ_{i+1}): $\{a, b\}$, let us consider the rank 1 operator that goes from $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots\{a,b\}\ldots})$ to $I_{\overline{P_1} \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots\{a,b\}\ldots})$. By Proposition 3.3 it is an operator with a non-generic kernel if and only if $a < b$; Indeed, if we denote $\lambda := (\ldots, a, b, \ldots)$, then $\langle \check{\alpha}, \lambda \rangle = a - b < 0$ (The action of s_α on λ leaves fixed the other coordinates of λ that we simply denote by dots).

Since $\alpha \in \Sigma_\sigma$, by point (a) in Harish-Chandra's theorem [Theorem 2.1], there is a unique non-trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ such that $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$, and which operates as the transposition from (a, b) to (b, a). The rank 1 operator from $I_{\overline{P_1} \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,a,b,\ldots})$ to $I_{s_\alpha(\overline{P_1} \cap (M_1)_\alpha)}^{(M_1)_\alpha}(s_\alpha(\sigma_{\ldots,a,b,\ldots})) := I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,b,a,\ldots})$ is bijective. Eventually, we have shown that the composition of those two that goes from $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,a,b,\ldots})$ to $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots,b,a,\ldots})$ has a non-generic kernel.

If the Weyl group W_σ is isomorphic to $S_n \rtimes \{\pm 1\}$, the Weyl group element corresponding to $\{\pm 1\}$ is the sign change, and we operate this sign change on the latest coordinate of λ (the extreme right of the cuspidal string).

By the same argumentation as in the first example, for $a > 0$, the operator $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots -a})$ to $I_{P_1 \cap (M_1)_\alpha}^{(M_1)_\alpha}(\sigma_{\ldots a})$ has a non-generic kernel.

Example 5.11. — Let G be a classical group of rank n. Let us take σ an irreducible unitary generic cuspidal representation of M_1, a standard Levi subgroup of G. Let us assume Σ_σ is irreducible of type B, and take $\lambda := (s_1, s_2, \ldots, s_m)$ in $a^*_{M_1}$, ρ an irreducible unitary cuspidal representation of GL_k, and σ_c an irreducible unitary cuspidal representation of $G(k')$ $k' < n$. Then σ_λ is

$$\sigma_\lambda := \rho|\cdot|^{s_1} \otimes \rho|\cdot|^{s_2} \otimes \ldots \otimes \rho|\cdot|^{s_m} \otimes \sigma_c.$$

The element s_{α_i} operates as follows:

$$s_{\alpha_i}(\rho|\cdot|^{s_1} \otimes \ldots \rho|\cdot|^{s_i} \otimes \rho|\cdot|^{s_{i+1}} \ldots \otimes \rho|\cdot|^{s_m} \otimes \sigma_c) \\ = \rho|\cdot|^{s_1} \otimes \ldots \otimes \rho|\cdot|^{s_{i+1}} \otimes \rho|\cdot|^{s_i} \otimes \ldots \otimes \rho|\cdot|^{s_m} \otimes \sigma_c.$$

Indeed, for such α_i (which is in Σ_σ), one checks that property (a) in Theorem 2.1 holds: $s_{\alpha_i}(\sigma) \cong \sigma$. This is verified for any $i \in \{1, \ldots, n\}$. The intertwining operator usually considered in this manuscript is induced by functoriality from the application $\sigma_\lambda \to s_{\alpha_i}(\sigma_\lambda)$.

Lemma 5.12. — *Let $b' \leq \ell + m, b \leq a$. Fix a unitary irreducible cuspidal representation σ of a maximal Levi subgroup in a quasi-split reductive group G and two cuspidal strings $(a, b, \underline{n})$ and $(a, b', \underline{n'})$ in a W_σ-cuspidal string (notice that the right end of these are equals with value a). If $b' \geq b$, the intertwining operator between $I^G_{P_1}(\sigma(a, b, \underline{n}))$ and $I^G_{P_1}(\sigma(a, b', \underline{n'}))$ has a non-generic kernel.*

Proof. — In this proof, to detail the operations on cuspidal strings more explicitly we write the residual segments of type B, C, D defined in Definition 4.8 as

$$((\ell+m)(\ell+m-1)\ldots((\ell+1)\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

where n_i denote the number of times the (half)-integer i is repeated. We present the arguments for integers; the proof for half-integers follows the same argumentation.

First, assume $b \geq 0$, and consider changes on the cuspidal strings

$$(a, \ldots, b', b'-1, \ldots b)((\ell+m)\ldots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\ldots b^{n_b}\ldots 2^{n_2}1^{n_1}0^{n_0})$$

consisting in permuting successively all elements in $\{b, \ldots, b'-1\}$ with their right-hand neighbour, as soon as this right-hand neighbour is larger. We incorporate all elements starting with b until $b'-1$ from the left into the right-hand residual segment. The rank 1 intertwining operators associated to those permutations have a non-generic kernel (see Example 5.10); hence, the intertwining operator from $I^G_{P_1}(\sigma(a, b, \underline{n}))$ to $I^G_{P_1}(\sigma(a, b', \underline{n'}))$ as a composition of those rank 1 operators has a non-generic kernel.

Now assume that $b < 0$ and write $b = -\gamma$. Let us show that there exists an intertwining operator with a non-generic kernel from the module induced from $I^G_{P_1}(\sigma(a, -\gamma, \underline{n}))$ to the one induced from $I^G_{P_1}(\sigma(a, b', \underline{n'}))$. The decomposition in rank 1 operators has the following two steps (the details on the first step are given in the next paragraph):

1. (a) If $b' \geq 1 > b$ From the cuspidal string

$$(a, \dots, \gamma, \gamma-1, \dots, -\gamma) \\ ((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0})$$

to

$$(a, \dots, \gamma, \gamma-1, \dots, 1) \\ ((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \dots, -\gamma),$$

and then to

$$(a, \dots, \gamma, \gamma-1, \dots, 1) \\ ((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b+1}\dots 2^{n_2+1}1^{n_1+1}0^{n_0+1}).$$

(b) If $0 \geq b' \geq b$ From the cuspidal string

$$(a, \dots, \gamma, \gamma-1, \dots, -\gamma) \\ ((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0})$$

to

$$(a, \dots, \gamma, \gamma-1, \dots, b') \\ ((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, b'-1, \dots, -\gamma),$$

and then to

$$(a, \dots, \gamma, \gamma-1, \dots, b') \\ ((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b+1}\dots 2^{n_2+1}1^{n_1+1}0^{n_0+1}).$$

2. In case (a), from $(a, \dots, 1)(\underline{n''})$ to $(a, \dots, b')(\underline{n'})$ by the same arguments as in the case $b \geq 0$ treated in the first paragraph of this proof.

We detail the operations in step 1:

(i) Starting with $-\gamma$, all negative elements in $\{0, \dots, -\gamma\}$ are successively sent to the extreme right of the second residual segment $(\underline{n})$. At each step, the rank 1 intertwining operator between (a, p) and (p, a) where p is a negative integer (or half-integer) and $a > p$ has a non-generic kernel.

(ii) We use rank 1 operators of the second type (sign chance of the extreme right element of the cuspidal string). Since they intertwine cuspidal strings where the last element changes from negative to positive, they have non-generic kernels. Then, the positive element is moved up to the left. The right-hand residual segment goes from

$$((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \dots, -\gamma)$$

to

$$((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \dots, \gamma)$$

and then to

$$((\ell+m)\dots\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1},\gamma,-1,\dots,-(\gamma-1)).$$

Once changed to positive, permuting successively elements from right to left, one can reorganize the residual segment

$$(\ell+m)\dots\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1},\gamma,\dots 1)$$

such as it is a decreasing sequence of (half)-integers. Again, intertwining operators following these changes on the cuspidal string have non-generic kernels. □

Example 5.13. — Consider the cuspidal string

$$(5\,4\,3\,2\,1\,0\,{-1})(4\,3\;\;3\,2\,2\;\;2\,1\,1\;\;1\;\;0)$$

and the dominant residual point in its W_σ-cuspidal string:

$$(5\,4\;\;4\,3\,3\;\;3\,2\,2\,2\;\;2\,1\,1\,1\,1\;\;1\,0\;\;0).$$

To the Weyl group element $w \in W_\sigma$ associate an intertwining operator from the module induced with string

$$(5\,3\,4\,2\,1\,0\,{-1})(4\,3\;\;3\,2\,2\;\;2\,1\,1\;\;1\;\;0)$$

to the one induced with cuspidal string

$$(5\,4\;\;4\,3\,3\;\;3\,2\,2\,2\;\;2\,1\,1\,1\,1\;\;1\,0\;\;0),$$

which has a non-generic kernel.

Indeed, one will decompose it into transpositions s_α such as $(-1,4)$ to $(4,-1)$ and similarly for any $4>i\geq 0$: $(-1,i)$ to $(i,-1)$.

This process will result in

$$(5\,4\,3\,2\,1\,0)(4\,3\;\;3\,2\,2\;\;2\,1\,1\;\;1\;\;0\;\;{-1}).$$

Then one will change the -1 to 1, and by the above, the associated rank-one operator also has a non-generic kernel. Then use the rank 1 operators with a non-generic kernel such as: $(0,1)\to(1,0)$.

Then notice that the '4', '3' and '2' in the middle of the sequence can be moved to the left with a sequence of rank 1 operators with non-generic kernels such as: $(0,4)\to(4,0);\dots;(3,4)\to(4,3)$.

Lemma 5.14. — *Let $(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_t)$ be an ordered sequence of t linear segments and let us denote $\mathcal{S}_i=(a_i,\dots,b_i)$, for any i in $\{1,\dots,t\}$. This sequence is ordered so that, for any i in $\{1,\dots,t\}$, $s_i=\frac{a_i+b_i}{2}\geq s_{i+1}=\frac{a_{i+1}+b_{i+1}}{2}$. Let us assume that for some indices in $\{1,\dots,t\}$, the linear residual segments are linked.*

Let us denote $(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t)$ the ordered sequence corresponding to the end of the procedure of taking union and intersection of linked linear residual segments. This sequence is composed of at most t unlinked residual segments $\mathcal{S}'_i = (a'_i, \dots, b'_i)$, $i \in \{1, \dots, t\}$.

Repeatedly taking intersection and union yields smaller Langlands parameters for the order defined in Lemma 5.2, and we denote the smallest element for this order, $\underline{s'}$. It corresponds to the sequence $(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t)$, as explained in Proposition 5.8.

Then, there exists an intertwining operator with a non-generic kernel from the induced module $I_{P_1}^G(\sigma(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t; \underline{n}))$ to $I_{P_1}^G(\sigma(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t; \underline{n}))$.

Proof. — Let us first consider the case $t = 2$.

Consider two linear (i.e. of type A) residual segments, i.e. strictly decreasing sequences of either consecutive integers or consecutive half-integers $\mathcal{S}_1 := (a_1, \dots, b_1); \mathcal{S}_2 := (a_2, \dots, b_2)$.

Assume $a_1 > a_2 > b_1 > b_2$, so that they are linked in the terminology of Bernstein–Zelevinsky. Taking intersection and union yields two unlinked linear residual segments $\mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathcal{S}_1 \cup \mathcal{S}_2$: $(a_1, \dots, b_2)(a_2, \dots, b_1)$ or $(a_2, \dots, b_1)(a_1, \dots, b_2)$ ordered so that $s'_1 > s'_2$.

As in the proof of Lemma 5.12, because $a_2 > b_2$ and also $b_1 > b_2$, there exists an intertwining operator with a non-generic kernel from the module induced with cuspidal support $(a_1, \dots, b_2)(a_2, \dots, b_1)$ to the one induced with cuspidal support $(a_1, \dots, b_1)(a_2, \dots, b_2)$. This intertwining operator is a composition of rank 1 intertwining operators associated to permutations that have a non-generic kernel (see Example 5.10); as composition of those rank 1 operators, it has a non-generic kernel.

Similarly, because $a_1 > a_2$, there exists an intertwining operator with a non-generic kernel from the module induced with cuspidal support $(a_2, \dots, b_1)$ $(a_1, \dots, b_2)$ to the one induced with cuspidal support $(a_1, \dots, b_1)(a_2, \dots, b_2)$.

Let us now assume that the result of this lemma is true for t linear residual segments. Consequently, there exists an intertwining operator with a non-generic kernel from $I_{P_1}^G(\sigma(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t, \mathcal{S}_{t+1}; \underline{n}))$ to $I_{P_1}^G(\sigma(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t, \mathcal{S}_{t+1}; \underline{n}))$. In this case, $\mathcal{S}_{t+1}$ and $\mathcal{S}'_t$ may be linked and taking their union and intersection yields $\mathcal{S}'_{t+1}$ and $\mathcal{S}''_t$ and the existence of an intertwining operator with a non-generic kernel from $I_{P_1}^G(\sigma(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}''_t, \mathcal{S}'_{t+1}; \underline{n}))$ to $I_{P_1}^G(\sigma(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t, \mathcal{S}_{t+1}; \underline{n}))$. The latter argument is repeated if $\mathcal{S}''_t$ and $\mathcal{S}'_{t-1}$ are linked, and so on.

Another case to consider would be $\mathcal{S}_{t+1} \subset \mathcal{S}'_t$ with $s_{t+1} \leq s'_t$, and $\mathcal{S}_{t+1}$ linked to $\mathcal{S}'_{t-1}$. Then, using the irreducibility of the module induced from the two segments $\mathcal{S}'_t$ and $\mathcal{S}'_{t-1}$, one would interchange them, then deal with the intersection and union of $\mathcal{S}_{t+1}$ and $\mathcal{S}'_{t-1}$, obtain $\mathcal{S}'_{t+1}$ and $\mathcal{S}''_{t-1}$ and the existence of an intertwining operator from $I_{P_1}^G(\sigma(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t, \mathcal{S}''_{t-1}, \mathcal{S}'_{t+1}; \underline{n}))$ to $I_{P_1}^G(\sigma(\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_t, \mathcal{S}'_{t-1}, \mathcal{S}_{t+1}; \underline{n}))$.

Since the resulting segments $\mathcal{S}'_t$, $\mathcal{S}''_{t-1}$ and $\mathcal{S}'_{t+1}$ are unlinked, we can organize them so that their exponents are ordered. If $\mathcal{S}''_{t-1}$ is linked to any $\mathcal{S}'_i$, $i \neq t, t+1$, we repeat this argument.

Eventually, there exists an intertwining operator with a non-generic kernel from $I_{P_1}^G(\sigma(\mathcal{S}_1^*, \mathcal{S}_2^*, \dots, \mathcal{S}_t^*, \mathcal{S}_{t+1}^*; \underline{n}))$ to $I_{P_1}^G(\sigma(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t, \mathcal{S}_{t+1}; \underline{n}))$, where $(\mathcal{S}_1^*, \mathcal{S}_2^*, \dots, \mathcal{S}_t^*, \mathcal{S}_{t+1}^*)$ is the sequence of $t+1$ unlinked segments obtained at the end of the procedure of taking the intersection and union. □

5.5. A Lemma in the vein of Zelevinsky's theorem. — Recall this fundamental result of Zelevinsky, for the general linear group, which was also presented as Theorem 5 in [30]. We use the notation introduced in Definition 4.9.

Proposition 5.15 (Zelevinsky, [40], Theorem 9.7). — *If any two segments, $\mathcal{S}_i, \mathcal{S}_j$, j, i in $\{1, \dots, n\}$ of the linear group are not linked, we have the irreducibility of $Z(\mathcal{S}_1) \times Z(\mathcal{S}_2) \times \dots \times Z(\mathcal{S}_n)$ and, conversely, if $Z(\mathcal{S}_1) \times Z(\mathcal{S}_2) \times \dots \times Z(\mathcal{S}_n)$ is irreducible, then all segments are mutually unlinked.*

Here, we prove a similar statement in the context of any quasi-split reductive group of type A.

Lemma 5.16. — *Let τ be an irreducible generic discrete series of a standard Levi subgroup M in a quasi-split reductive group G. Let σ be an irreducible unitary generic cuspidal representation of a standard Levi subgroup M_1 in the cuspidal support of τ. Let us assume Σ_σ is irreducible of rank $d = \mathrm{rk}_{ss}(G) - \mathrm{rk}_{ss}(M_1)$ and type A.*

Let $\underline{s} = (s_1, s_2, \dots, s_t) \in a_{M_1}^*$ be ordered such that $s_1 \geq s_2 \geq \dots \geq s_t$ with $s_i = \frac{a_i + b_i}{2}$, for two real numbers $a_i \geq b_i$.

Then $I_P^G(\tau_{\underline{s}})$ is a generic standard module embedded in $I_{P_1}^G(\sigma_\lambda)$, and λ is composed of t residual segments $\{(a_i, \dots, b_i), i = 1, \dots, t\}$ of type A_{n_i}.

Let us assume that the t segments are mutually unlinked. Then λ is not a residual point, and, therefore, the unique irreducible generic subquotient of the generic module $I_{P_1}^G(\sigma_\lambda)$, is not a discrete series. This irreducible generic subquotient is $I_P^G(\tau_{\underline{s}})$. In other words, the generic standard module $I_P^G(\tau_{\underline{s}})$ is irreducible. Further, for any reordering $\underline{s'}$ of the tuple $\underline{s}$, which corresponds to an element $w \in W$ such that $w\underline{s} = \underline{s'}$ and discrete series τ' of M' such that $w\tau = \tau'$ and $wM = M'$, we have $I_{P'}^G(\tau'_{\underline{s'}}) \cong I_P^G(\tau_{\underline{s}})$.

Proof. — By the result of Heiermann–Opdam (Proposition 2.7), there exists a standard parabolic subgroup P_1, a unitary cuspidal representation σ, a parameter $\nu \in \overline{(a_{M_1}^M *)^+}$ such that the generic discrete series τ embeds in $I_{M_1 \cap M}^M(\sigma_\nu)$. By Heiermann's theorem (see Theorem 2.6), ν is a residual point, so it is composed of residual segments of type A_{n_i}. Then twisting by $\underline{s}$ and inducing to G, we obtain:

$$I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda) \text{ where } \lambda = (a_i, \dots, b_i)_{i=1}^t.$$

Let π be the unique irreducible generic subquotient of the generic standard module $I_P^G(\tau_{\underline{s}})$. Then using Langlands' classification and the standard module conjecture $\pi = J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'})$. Assume that τ' is a discrete series. We apply again the result of Heiermann–Opdam to this generic discrete series to embed $I_{P'}^G(\tau'_{\nu'})$ in $I_{P'_1}^G(\sigma'_{\lambda'})$.

As any representation in the cuspidal support of $\tau_{\underline{s}}$ must lie in the cuspidal support of π, any such representation much be conjugated to $\sigma'_{\lambda'}$, and, therefore, λ' is in the Weyl group orbit of λ. Let us consider this Weyl group orbit under the assumption that the t segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ are unlinked.

Whether the union of any two segments in $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ is not a segment, or the segments are mutually included in one another, it is clear there are no option to take intersections and unions to obtain new linear residual segments. Further, starting with λ, to generate new elements in its W_σ-orbit, one can split the segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$. By Lemma 5.6, this procedure necessarily yields larger Langlands parameters. Therefore, there is no option to reorganize them to obtain residual segments (a'_j, b'_j) of type $A_{n'_j}$ such that $n'_j \neq n_i$, for some $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, for some r such that $\sum_{j=1}^r n'_j = \sum_{i=1}^t n_i$.

The second option is to permute the order of the segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ to obtain any other parameter λ' in the Weyl group orbit of λ. From this λ', one clearly obtains the parameter $\nu' := \underline{s'}$ as a simple permutation of the tuple $\underline{s}$.

On the Langlands parameter $\underline{s}$, which is the unique one among the (ν')'s described in the previous paragraph in the Langlands situation (we consider all standard modules $I_{P'}^G(\tau'_{\nu'})$), we can use Theorem 5.5 to conclude that the generic standard module $I_P^G(\tau_{\underline{s}})$ for $\nu = \underline{s}$ is irreducible.

Now, we want to show $I_{P'}^G(\tau'_{\underline{s'}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$.

Looking at the cuspidal support, it is clear that there exists a Weyl group element in $W(M, M')$ sending σ_λ to $\sigma'_{\lambda'}$, and, therefore, $\tau_{\underline{s}}$ to the Langlands data $(w\tau)_{w\underline{s}} := \tau'_{\underline{s'}}$.

Consider first the case of a maximal parabolic subgroup P in G. Set $\underline{s} = (s_1, s_2)$, $\underline{s'} = (s_2, s_1)$, and τ' is a generic discrete series representation. We apply the map $t(w)$ between $I_P^G(\tau_{\underline{s}})$ and $I_{wP}^G((w\tau)_{w\underline{s}})$, which is an isomorphism. By definition, the parabolic wP has Levi M'. Then, by Lemma 5.4 [1] (see also Remark 2.10 in [2]) since the Levi subgroups and inducing representations are the same, the Jordan–Hölder composition series of $I_{wP}^G(\tau'_{\underline{s'}})$ and $I_{P'}^G(\tau'_{\underline{s'}})$ are the same, and since $I_P^G(\tau_{\underline{s}})$ is irreducible, they are isomorphic and irreducible.

Secondly, consider the case when the two parabolic subgroups P and P', with Levi subgroup M and M', are connected by a sequence of adjacent parabolic subgroups of G. Using Theorem 5.5 with any Levi subgroup in G, in particular a Levi subgroup M_α (containing M as a maximal Levi subgroup) shows that the representation $I_{P\cap M_\alpha}^{M_\alpha}(\tau_{\underline{s}})$ is irreducible.

Then, we are in the context of the above paragraph, and $I^{M_\alpha}_{s_\alpha(\overline{P}\cap M_\alpha)}((s_\alpha\tau)_{s_\alpha\underline{s}})$ (the image of the composite of the map $J_{\overline{P}\cap M_\alpha|P\cap M_\alpha}$ with the map $t(s_\alpha)$) is irreducible, and isomorphic to $I_{P\cap M_\alpha}^{M_\alpha}(\tau_{\underline{s}})$.

Let us denote Q the parabolic subgroup adjacent to P along α. Induction from M_α to G yields that $I_Q^G(s_\alpha\tau)_{s_\alpha\underline{s}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$. Writing the Weyl group element w in $W(M,M')$ such that $wM = M'$ as a product of elementary symmetries s_{α_i}, and applying a sequence of intertwining maps as above yields the isomorphism between $I_P^G(\tau_{\underline{s}})$ and $I_{P'}^G(\tau'_{\underline{s'}})$. □

Remark 5.17. — For an example, see [2], 2.6.

6. Conditions on the parameter λ so that the unique irreducible generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation

The goal of this section is to present specific forms of the parameter $\lambda \in a^*_{M_1}$ such that the unique irreducible generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ with σ irreducible unitary generic cuspidal representation of any standard Levi M_1 is a subrepresentation. There is an obvious choice of parameter satisfying this condition as it is proven in the following Lemma.

Lemma 6.1. — *Let σ be an irreducible generic cuspidal representation of M_1 and σ_λ be a dominant residual point and consider the generic induced module $I_{P_1}^G(\sigma_\lambda)$. Its unique irreducible generic square-integrable subquotient is a subrepresentation.*

Proof. — From Theorem 2.6, since λ is a residual point, $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient. From Rodier's theorem, it also has a unique irreducible generic subquotient; denote it γ.

From Theorem 5.4, this unique irreducible generic subquotient is a discrete series. Consider this unique generic discrete series subquotient; by Proposition 2.7, there exists a parabolic subgroup P' such that $\gamma \hookrightarrow I_{P'}^G(\sigma'_{\lambda'})$, and λ' dominant for P'. Then the lemma follows from Proposition 3.2 in Section 3. □

We need the following definition:

Definition 6.2. — Let (M_1,σ) be in the generic cuspidal support of an irreducible generic discrete series.

Let us denote $M_1 = M_\Theta$. Let us assume that $\Theta = \bigcup_{i=1}^n \Theta_i$, where, for any $i \in \{1,\dots,n-1\}$, Θ_i is an irreducible component of type A.

We say that this cuspidal support satisfies the conditions (CS) (given in Proposition 6.4 and Corollary 6.6) if:

- Σ_σ is irreducible of rank d.
- If $\Delta(P_1) = \{\alpha_1, \ldots, \alpha_{d-1}, \beta_d\}$ then $\Delta_\sigma = \{\alpha_1, \ldots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D.
- For any $i \in \{1, \ldots, n-1\}$, Θ_i has a fixed cardinal. Furthermore, the interval between any two disjoint consecutive components Θ_i, Θ_{i+1} is of length 1.

Our main result in this section is the following theorem.

Theorem 6.3. — *Let us consider $I_{P_1}^G(\sigma_\lambda)$ with σ an irreducible unitary generic cuspidal representation of a standard Levi M_1, and $\lambda \in a_{M_1}^*$ such that (M_1, σ) satisfies the conditions (CS) (see Definition 6.2). Let W_σ be the Weyl group of the root system Σ_σ. The unique irreducible generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ is necessarily a subrepresentation if the parameter λ is one of the following.*

1. *If λ is a residual point:*
   (a) *λ is a dominant residual point.*
   (b) *λ is a residual point of the form $(a, a_-)(\underline{n})$ with (a, a_-) two consecutive jumps in the jumps set associated to the dominant residual point in its W_σ-orbit.*
   (a) *λ is a residual point of the form $(a, b)(\underline{n})$ such that the dominant residual point in its W_σ-orbit has an associated jumps set containing (a, a_-) as two consecutive jumps and $b > a_-$.*
2. *If λ is not a residual point*
   (a) *λ is of the form $(a', b')(\underline{n'})$ such that the Langlands' parameter $\nu' = \frac{a'+b'}{2}$ is minimal for the order on Langlands parameter (see Section 5.2)*
   (b) *If λ is of the form $(a, b)(\underline{n})$ with $a = a', b' < b$ in the W_σ-orbit of a parameter as in* 2(a).

The proof of this theorem, given in Section 6.4, relies on Moeglin's extended lemmas and an embedding result (6.14).

6.1. On some conditions on the standard Levi M_1 and some relationships between $W(M_1)$ and W_σ. — Let G be a quasi-split reductive group over F (or a product of such groups) whose root system Σ is of type A, B, C or D, π_0 is an irreducible generic discrete series of G whose cuspidal support contains the representation σ_λ of a standard Levi subgroup M_1, where $\lambda \in a_{M_1}^*$ and σ is an irreducible unitary cuspidal generic representation.

Let

$$d = \mathrm{rk}_{ss}(G) - \mathrm{rk}_{ss}(M_1) = \dim a_{M_1} - \dim a_G.$$

Let us denote $M_1 = M_\Theta$. Then $\Delta - \Theta$ contains d simple roots.

Let us denote $\Delta(P_1)$ the set of non-trivial restrictions (or projections) to A_{M_1} (or to $a_{M_1}^G$) of simple roots in Δ such that elements in $\Sigma(P_1)$ (roots that are positive for P_1) are linear combinations of simple roots in $\Delta(P_1)$.

Let us denote $\Delta(P_1) = \{\alpha_1, \dots, \alpha_{d-1}, \beta_d\}$ and $\underline{\alpha}_i$ the simple root in Δ, which projects onto α_i in $\Delta(P_1)$.

As (M_1, σ_λ) is the cuspidal support of an irreducible discrete series, as explained in Proposition 4.6, the set Σ_σ is a root system of rank d in $\Sigma(A_{M_1})$ and its basis, when we set $\Sigma(P_1) \cap \Sigma_\sigma$ as the set of positive roots for Σ_σ, is Δ_σ.

Proposition 6.4. — *With the context of the previous paragraphs, let Σ_σ be irreducible. If $\Delta(P_1) = \{\alpha_1, \dots, \alpha_{d-1}, \beta_d\}$ then $\Delta_\sigma = \{\alpha_1, \dots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D.*

Proof. — This is a result of the case-by-case analysis conducted in the independent paper [12], where Δ_Θ denotes the $\Delta(P_1)$ considered in this proposition. From its definition, Σ_σ is a subsystem in Σ_Θ. If Σ_Θ contains a root system of type BC_d, it is clear that the last root, denoted α_d, of this system (which is either the short or long root depending on the reduced system chosen) can be different from β_d, if Σ_σ is of type D_d. □

We have not included the root β_d in Δ_σ because (as opposed to the context of classical groups) it is possible that there exists an irreducible cuspidal representation σ such that $s_{\beta_d}\sigma \ncong \sigma$.

A typical example of the above Proposition (6.4) is when Σ if of type B, C, and Σ_σ is of type D; then it occurs that $\Delta(P_1)$ contains $\beta_d = e_d$ or $\beta_d = 2e_d$, whereas Δ_σ contains $\alpha_d = e_{d-1} + e_d$.

This proposition allows us to use our results on intertwining operators with a non-generic kernel (see Proposition 3.3 and Example 5.10).

In the context of Harish-Chandra's theorem 2.1, the element denoted s_α corresponds to the element $\widetilde{w_0}^{(M_1)_\alpha}\widetilde{w_0^{M_1}}$ as defined in Chapter 1 in [32].

Let us describe it.

Let $P = MN$ be a standard parabolic. Let $\Theta \subset \Delta$, $M = M_\Theta$. In [32], Shahidi defines $\widetilde{w_0}$ as the element in $W(A_0, G)$, which sends Θ to a subset of Δ but every other root $\beta \in \Delta - \Theta$ to a negative root.

If $\widetilde{w_0^G}$, $\widetilde{w_0^M}$ are the longest elements in the Weyl groups of A_0 in G and M, respectively, then $\widetilde{w_0} = \widetilde{w_0^G}\widetilde{w_0^M}$. The length of this element in W is the difference of the lengths of each element in this composition. Therefore, if a representative of this element in G normalizes M, since it is of minimal length in its class in the quotient $\left\{w \in W | w^{-1}Mw = M\right\}/W^M$, this representative belongs to $W(M)$.

When P is maximal and self-associate (meaning $\widetilde{w_0}(\Theta) = \Theta$), then if α is the simple root of A_M in Lie(N), $\widetilde{w_0}(\alpha) = -\alpha$. In this case, $w_0 N w_0^{-1} = N^-$, the opposite of N for w_0 a representative of $\widetilde{w_0}$ in G.

Remark 6.5. — Applying the previous paragraph to the context of $P_1 \cap (M_1)_\beta$ and $(M_1)_\beta$, we first observe that $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}}(\Theta) = \Theta$. Then, one notices that $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}}$ sends β to $-\beta$.

In analogy with the notations of Theorem 2.1, let us denote $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}} = s_\beta$. We have: $s_\beta(P_1 \cap (M_1)_\beta) = \overline{P_1} \cap (M_1)_\beta$, then $s_\beta\lambda = \lambda$ if λ is in $\overline{(a_{M_1}^G *)^+}$ and is a residual point of type D.

By definition, if $\alpha \in \Sigma_\sigma$, by Harish-Chandra's theorem 2.1, $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$ and $s_\alpha \cdot M_1 = M_1$, and this means that s_α is a representative in G of a Weyl group element sending Θ on Θ.

Corollary 6.6. — *Let σ be an irreducible cuspidal representation of a standard Levi subgroup M_1 and let us assume that Σ_σ is an irreducible of rank $d = \mathrm{rk}_{ss}(G) - \mathrm{rk}_{ss}(M_1)$ and type A, B, C or D, then, we have the following:*

1. *For any α in $\Delta(P_1)$, $s_\alpha \in W(M_1)$.*
2. $W(M_1) = W_\sigma \cup \{s_{\beta_d} W_\sigma\}$.
3. *Let σ' (or σ) be an irreducible cuspidal representation of a standard Levi subgroup M_1' (or standard Levi subgroup M_1). Let us assume they are the cuspidal support of the same irreducible discrete series. Then $M_1' = M_1$.*

Proof. — 1. Let us assume Θ has the form given in Appendix B, Theorem B.1, which is a disjoint union of irreducible components: $\bigcup_{i=1}^n \Theta_i$. Then, let us show that for any α in $\Delta(P_1)$, $s_\alpha \in W(M_1)$.

By definition, s_α is a representative in G of the element $\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}$.

Let us first assume that α_i is the restriction of the simple root connecting Θ_i and Θ_{i+1}, both of type A, in the Dynkin diagram of G. Then

$$\Delta^{(M_1)_{\alpha_i}} = \Theta_i \cup \{\alpha_i\} \cup \Theta_{i+1} \bigcup_{j \neq i, i+1} \Theta_j.$$

The element $\widetilde{w_0^{M_1}}$ operates on each component as the longest Weyl group element for that component; it sends $\alpha_k \in \Theta_i$ to $-\alpha_{\ell_i+1-k}$, if ℓ_i is the length of the connected component Θ_i.

In a second time, $\widetilde{w_0^{(M_1)_{\alpha_i}}}$ operates on $\Theta_i \cup \{\alpha_i\} \cup \Theta_{i+1}$ in a similar fashion, and trivially on each component in $\bigcup_{j \neq i, i+1} \Theta_j$.

Secondly, let us assume that β is the restriction of the simple root connecting Θ_{n-1} of type A and Θ_n of type B, C or D in the Dynkin diagram of G.

$\widetilde{w_0^{(M_1)}}(\Theta_{n-1}) = \Theta_{n-1}$ (since this element simply permutes and multiplying by (-1) the simple roots in Θ_{n-1}), while $\widetilde{w_0^{(M_1)}}(\Theta_n) = -\Theta_n$. Further, $\widetilde{w_0^{(M_1)_\beta}}$ acts as (-1) on all the simple roots in $\Theta_{n-1} \cup \Theta_n$.

Eventually, $\widetilde{w_0^{(M_1)_\beta}}\widetilde{w_0^{(M_1)}}$ fixes Θ_n pointwise and sends each root in Θ_{n-1} to another root in Θ_{n-1}. It also fixes pointwise $\bigcup_{j\neq n-1,n}^n \Theta_j$. Therefore, for any α in $\Delta(P_1)$, $\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}(\Theta)=\Theta$, and hence $s_\alpha \in \left\{w\in W|w^{-1}M_1w=M_1\right\}$. Furthermore, since the length of this element is the difference of the lengths of each element in this composition, it is clear that s_α is of minimal length in its class in the quotient $\left\{w\in W|w^{-1}M_1w=M_1\right\}/W^{M_1}$, and hence this element is in $W(M_1)$.

2. Any element in $W(M_1)$ is a representative of minimal length in its class in the quotient $\left\{w\in W|w^{-1}M_1w=M_1\right\}/W^{M_1}$. The $s_\alpha=\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}$ described above where the elements $\alpha\in\Delta(P_1)$ are a set of generators of $W(M_1)$. Recall from Proposition 6.4 that $\Delta(P_1)=\{\alpha_1,\dots,\alpha_{d-1},\beta_d\}$ and $\Delta_\sigma=\{\alpha_1,\dots,\alpha_{d-1},\alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B,C,D. Therefore, $W(M_1)=W_\sigma\cup\{s_{\beta_d}W_\sigma\}$.

We also recall that in the context of Σ_σ of type D_d and $\Sigma(P_1)$ of type B_d or C_d: $s_{\alpha_d}=s_{\alpha_{d-1}}s_{\beta_d}s_{\alpha_{d-1}}s_{\beta_d}$.

3. Let us denote $M_1'=M_{\Theta'}$ and $M_1=M_\Theta$ and assume that Θ and Θ' are written as $\bigcup_{i=1}^n\Theta_i$, where, for any $i\in\{1,\dots,n-1\}$, Θ_i is an irreducible component of type A.

Since the cuspidal data are the support of the same irreducible discrete series, by Theorem 2.9 in [2], there exists $w\in W^G$ such that $M_1'=w\cdot M_1$, $\sigma'=w\cdot\sigma$. Since M_1' is isomorphic to M_1, Θ' is isomorphic to Θ.

Hence, applying the observations made in the first part of the proof of this proposition to M_1 and M_1', we observe Θ and Θ' share the same constraints: their components of type A are all of the same cardinal, and the interval between any two of these consecutive components is of length 1. Also, since Θ' is isomorphic to Θ, its last component Θ_m' is of the same type as Θ_m. Therefore, $\Theta'=\Theta$. Hence $M_1=M_1'$. □

REMARK 6.7. — This implies that if $P_1=M_1U_1$ and $P_1'=M_1'U_1'$ are both standard parabolic subgroups such that their Levi subgroups satisfy the conditions of the previous proposition, they are actually equal.

6.2. A few preliminary results for the proof of Moeglin's extended lemmas. — Let us recall Casselman's square-integrability criterion as stated in [39], whose proof can be found in ([6], (4.4.6)). Let $\Delta(P)$ be a set of simple roots, then ${}^+a_P^G*$, or ${}^+\overline{a}_P^G*$, denote the set of χ in a_M^* of the following form: $\chi=\sum_{\alpha\in\Delta(P)}x_\alpha\alpha$ with $x_\alpha>0$, or $x_\alpha\geq 0$. Further, denote π_P the Jacquet module of π with respect to P, and $\mathscr{E}xp$ the set of exponents of π as defined in Section I.3 in [39].

Proposition 6.8 (Propositions III.1.1 and III.2.2 in [39]). — *Let π be an irreducible representation with unitary central character. The following conditions are equivalent:*

1. *π is square-integrable (or tempered).*
2. *For any semi-standard parabolic subgroup $P = MU$ of G, and for any χ in $\mathscr{E}xp(\pi_P)$, $Re(\chi) \in {}^{+}a_P^G*$ (or $Re(\chi) \in {}^{+}\overline{a}_P^G*$).*
3. *For any standard parabolic subgroup $P = MU$ of G, proper and maximal, and for any χ in $\mathscr{E}xp(\pi_P)$, $Re(\chi) \in {}^{+}a_P^G*$ (or $Re(\chi) \in {}^{+}\overline{a}_P^G*$).*

In the following two lemmas, we will apply the previous proposition as follows:

Proposition 6.9. — *Let π_0 embed in $I_{P_1}^G(\sigma_\lambda)$. Let us write the parameter λ as a vector in the basis $\{e_i\}_{i\geq 0}$ (the basis of $a_{M_1}^*$ as chosen in the Definition 4.8, for instance) as $((x,y)+\underline{\lambda})$ for a linear residual segment (x,y) and assume $\sum_{k\in[x,y]} k \leq 0$. Then π_0 is not square-integrable.*

Proof. — Indeed, if

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma((x,y)+\underline{\lambda}))$$

by Frobenius reciprocity, the character χ_λ appears as exponent of the Jacquet module of π_0 with respect to P_1. Let us write λ as

$$\sum_i x_i(e_i - e_{i+1}) + \underline{\lambda} = \sum_i y_i e_i + \underline{\lambda},$$

it is clear that, for any integer j, $x_j = \sum_{i=1}^{j} y_j$, and notice there is an index j' such that $x_{j'} = \sum_{k\in[x,y]} k$. Therefore, using the hypothesis of the proposition, $x_{j'} = \sum_{k\in[x,y]} k \leq 0$. However, then χ_λ does not satisfy the requirement of Proposition 6.8 since $x_{j'}$ is negative. □

We will also use the following well-known result:

Theorem 6.10 ([28], Theorem VII.2.6). — *Let (π, V) be a admissible irreducible representation of G. Then (π, V) is tempered if and only if there exists a standard parabolic subgroup of G, $P = MN$, and a square integrable irreducible representation (σ, E) of M such that (π, V) is a subrepresentation of $I_P^G(\sigma)$.*

Lemma 6.11. — *Let $\beta \in \Delta(P_1)$, and assume $\beta \notin \Delta_\sigma$, then the elementary intertwining operator associated to $s_\beta \in W$ is bijective at σ_λ $\forall \lambda \in a_{M_1}^*$.*

Proof. — Set $s = s_\beta$ for $\beta \in \Delta(P_1)$, and $\beta \notin \Delta_\sigma$. Recall that we have $J_{P_1|sP_1} J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\beta})^{-1}$ up to a multiplicative constant.

Recall $\mathscr{O}$ denotes the set of equivalence classes of representations of the form $\sigma \otimes \chi$, where χ is an unramified character of M_1. The operator $\mu^{(M_1)_\beta} J_{P_1|s_\beta P_1}$ is regular at each unitary representation in $\mathscr{O}$ (see [39], V.2.3), $J_{s_\beta P_1|P_1}$ is itself regular on $\mathcal{O}$, since this operator is polynomial on $X^{nr}(G)$. By the general result mentioned after Lemma 2.2, the function $\mu^{(M_1)_\beta}$ has a pole at σ_λ for λ on the positive real axis, if $\mu^{(M_1)_\beta}(\sigma) = 0$. Therefore, by definition, since $\beta \notin \Delta_\sigma$, there is no pole at σ_λ. Further, since the regular operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ are non-zero at any point, if $\mu^{(M_1)_\beta}$ does not have a pole at σ_λ, these operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ are bijective. □

A consequence of this lemma is that for any root $\beta \in \Sigma(P_1)$ that admits a reduced decomposition without elements in Δ_σ, the intertwining operators associated to s_β are everywhere bijective.

6.3. Extended Moeglin's lemmas. — In this section and the following, the core of our argumentation relies on the form of the parameters λ; changes to the form of these parameters are induced by actions of Weyl group elements (see, for instance, Example 5.11). In fact, the Weyl group operates on σ_λ, and any Weyl group element decomposes in elementary symmetries s_{α_i} for $\alpha_i \in \Delta$. This kind of decomposition is explained in details in I.1.8 of the book [24]. If α_i is in Δ_σ, by Harish-Chandra's theorem (Theorem 2.1), $s_{\alpha_i}\sigma \cong \sigma$; however, recall that for $\beta_d \in \Delta(P_1)$ (see Proposition 6.4), we may not have $s_{\beta_d}\sigma \cong \sigma$.

The three next lemmas, inspired by Remark 3.2, page 154, and Lemma 5.1 in Moeglin [22] are used in our main embedding Proposition 6.14 (of the irreducible generic discrete series) result.

Recall that in general $P_{\Theta'}$ is the parabolic subgroup associated to the subset $\Theta' \subset \Delta$, and $M_{\Theta'}$ contains all the roots in Θ'. Recall that we denote $\underline{\alpha}_i$ the simple root in Δ that restricts to α_i in $\Delta(P_1)$.

Lemma 6.12. — *Let π_0 be a generic discrete series of a quasi-split reductive group G (of type A, B, C or D) whose cuspidal support (M_1, σ_λ) satisfies the condition (CS) (see Definition 6.2).*

Let

$$x, y \in \mathbb{R}, k - 1 = x - y \in \mathbb{N}.$$

This defines the integer k.

Let us denote

$$M' = M_{\Delta - \{\underline{\alpha}_1, \ldots, \underline{\alpha}_{k-1}, \underline{\alpha}_k\}}.$$

Let us assume that there exists $w_{M'} \in W^{M'}(M_1)$ and an irreducible generic representation τ, which is the unique generic subquotient of $I_{P_1 \cap M'}^{M'}(\sigma_{\lambda_1^{M'}})$, such that

$$(5) \qquad \pi_0 \hookrightarrow I_{P'}^G(\tau_{(x,y)}) \hookrightarrow I_{P_1}^G((w_{M'}\sigma)_{(x,y)+\lambda_1^{M'}});\ \lambda_1^{M'} \in a_{M_1}^{M'}.$$

Let us assume that y is minimal for this property.

Then τ is square integrable.

Proof. — Let us first remark that in Equation (5), the parameter in $a^*_{M_1}$ is decomposed as

$$\underbrace{(x,y)}_{\text{combination of } \alpha_1,\dots,\alpha_{k-1}} \quad + \quad \underbrace{\lambda_1^{M'}}_{\text{combination of } \alpha_{k+1},\dots,\beta_d} .$$

Let us denote τ the generic irreducible subquotient in $I^{M'}_{P_1\cap M'}(\sigma_{\lambda_1^{M'}})$ and let us show that τ is square integrable.

Assume, on the contrary, that τ is not square-integrable.

Then τ is tempered (but not square integrable) or non-tempered. Langlands' classification [Theorem 2.9] assures us that τ is a Langlands quotient $J(P'_L,\tau',\nu')$ for a parabolic subgroup $P'_L \supseteq P_1$ of M' or, equivalently, a subrepresentation in $I^{M'}_{P'_L}(\tau'_{\nu'})$, $\nu' \in \overline{((a^{M'}_{M'_L})^*)^-}$ (equivalently $\nu'_{P'_L}\leq 0$; the inequality is strict in the non-tempered case).

This is equivalent to claiming that there exists an irreducible generic cuspidal representation σ', (half)-integers ℓ, m with $\ell - m + 1 \in \mathbb{N}$ and $m \leq 0$ such that:

$$\tau \hookrightarrow I^{M'}_{P'_L}(\tau'_{\nu'}) \hookrightarrow I^{M'}_{P_1\cap M'}(\sigma'((\ell,m)+{\lambda_2}^{M'})), \tag{6}$$

$$\sum_{k\in[\ell,m]} k \leq 0. \tag{7}$$

We have extracted the linear segment (ℓ,m) out of the segment $\lambda_1^{M'}$ and named $\lambda_2^{M'}$ what is left.

Let us justify Equation (7). The parameter ν' reads

$$\Bigg(\underbrace{\dots,\frac{\ell+m}{2},\dots}_{\ell-m+1 \text{ times}},0,\dots,0\Bigg)$$

$$\nu'_{P'_L}\leq 0 \Leftrightarrow \frac{\ell+m}{2} \leq 0 \Leftrightarrow m \leq -\ell \Leftrightarrow \sum_{k\in[\ell,m]} k \leq 0.$$

From Equation (6)

$$\pi_0 \hookrightarrow I^G_{P'}(\tau_{(x,y)}) \hookrightarrow I^G_{P_1}(\sigma'((x,y)+(\ell,m)+\lambda_2^{M'})) \tag{8}$$

Since π_0 also embeds as a subrepresentation in $I^G_{P_1}(\sigma_\lambda)$, by Theorem 2.9 in [2] (see also [28] VI.5.4) there exists a Weyl group element w in W^G such that $w\cdot M_1 = M_1$, $w\cdot\sigma' = \sigma$ and $w((x,y)+(\ell,m)+\lambda_2^{M'}) = \lambda$. This means that we can take w in $W(M_1)$. However, we can be more precise on this Weyl group element: from Equation (6) and the hypothesis in the statement of the Lemma, we see we can take it in $W^{M'}(M_1)$, and this leaves the leftmost part of the cuspidal support, $\sigma_{(x,y)}$, invariant. This element, therefore, depends on x and y. We denote this element $w_{M'}$.

Let

$$M'' = M_{\Delta - \{\underline{\alpha}_q, \dots, \underline{\beta}_d\}} \quad \text{where } q = x - y + 1 + \ell - m + 1.$$

Now, let us consider two cases. First, let us assume $m \geq y$. If the two linear segments are unlinked, and the generic subquotient in $I_{P_1 \cap M''}^{M''}(\sigma'((x,y)+(\ell,m)))$ is irreducible, applying Lemma 5.16, we can interchange them in the above Equation (8) and we reach a contradiction to the Casselman square integrability criterion applied to the discrete series π_0 (considering its Jacquet module with respect to P_1, see Proposition 6.9 using $\sum_{k \in [\ell,m]} k \leq 0$).

By Proposition 5.8 and Remark 5.9, if the two linear segments are linked, the irreducible generic subquotient $\tau_{L,\mathrm{gen}}$ of $I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$ embeds in $I_{P_1 \cap M''}^{M''}((w \cdot w_{M'}\sigma)((\ell,y)+(x,m)))$ (for some Weyl group element $w \in W^{M''}(M_1)$, such that $w \cdot w_{M'}\sigma \cong w_{M'}\sigma$).

By Lemma 5.14 there exists an intertwining operator with a non-generic kernel sending $\tau_{L,\mathrm{gen}}$ to $I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$. Then by [U] in $I_{P_1}^{G}((w_{M'}\sigma)((x,y)+(\ell,m)+\lambda_2^{M'}))$, π_0 embeds in $I_{P''}^{G}((\tau_{L,\mathrm{gen}})_{\lambda_2^{M'}})$.

Therefore, inducing to G, we have

$$\pi_0 \hookrightarrow I_{P''}^{G}((\tau_{L,\mathrm{gen}})_{\lambda_2^{M'}}) \hookrightarrow I_{P'''}^{G}(I_{P_1 \cap M''}^{M'''}((w_{M'}\sigma)((\ell,y)+(x,m)+\lambda_2^{M'})),$$

but then since $\sum_{k \in [\ell,y]} k \leq 0$ (since $m \geq y$), we reach a contradiction to the Casselman square integrability criterion applied to the discrete series π_0 (considering its Jacquet module with respect to P_1).

Secondly, let us assume $m < y$. The induced representation

$$I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$

is reducible only if $\ell \in]x, y-1]$. Then using Proposition 5.8 and Remark 5.9 we know that the irreducible generic subquotient $\tau_{L,\mathrm{gen}}$ of

$$I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$

should embed in

$$I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,m)+(\ell,y)))$$

(or only in $I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,m)))$ if $\ell = y-1$).

Applying Lemma 5.16 we also know that it embeds in $I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)$ $((\ell,y)+(x,m)))$ (we can interchange the order of the two unlinked segments (ℓ,y) and (x,m)). Then, using Lemma 5.14 and [U] as above, we embed π_0 in $I_{P''}^{G}((\tau_{L,\mathrm{gen}})_{\lambda_2^{M'}}) \hookrightarrow I_{P_1}^{G}((w_{M'}\sigma)((x,y)+(\ell,m)+\lambda_2^{M'}))$.

However, π_0 does not embed in $I_{P_1}^{G}((w_{M'}\sigma)((x,m)+(\ell,y)+\lambda_2^{M'})))$ since y is minimal for such (embedding) properties.

Therefore, $\tau_{L,\mathrm{gen}}$ rather embeds in the quotient $I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((\ell,m)+(x,y)))$ of $I_{P_1 \cap M''}^{M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$.

Then π_0 embeds in

$$I_{P''}^G((\tau_{L,\mathrm{gen}})_{\lambda_2^{M'}}) \hookrightarrow I_{P''}^G(I_{P_1\cap M''}^{M''}((w_{M'}\sigma)((\ell,m)+(x,y))))_{\lambda_2^{M'}}$$
$$= I_{P_1}^G((w_{M'}\sigma)((\ell,m)+(x,y)+\lambda_2^{M'})).$$

Since $\sum_{k\in[\ell,m]} k \leq 0$, using Proposition 6.9, we reach a contradiction. □

LEMMA 6.13. — *Let π_0 be a generic discrete series of G whose cuspidal support satisfies the conditions (CS) (see Definition 6.2). Let a, a_- be two consecutive jumps in the set of jumps of π_0.*

Let us assume that there exists an irreducible representation π' of a standard Levi $M' = M_{\Delta-\{\underline{\alpha}_1,\ldots,\underline{\alpha}_{a-a_-}\}}$ such that

$$\pi_0 \hookrightarrow I_{P'}^G(\pi'_{(a,a_-+1)}) \hookrightarrow I_{P_1}^G(\sigma_{(a,a_-+1)+\lambda}). \tag{9}$$

Then there exists a generic discrete series π of $M'' = M_{\Delta-\{\overline{\alpha_{a+a_-+1}}\}}$ such that π_0 embeds in $I_{P''}^G((\pi)_{s\widetilde{\alpha}_{a+a_-}}) \hookrightarrow I_{P_1}^G(\sigma((a,-a_-)+(\underline{n})))$ with $s = \frac{a-a_-}{2}$ and $(\underline{n})$ a residual segment.

We split the proof in two steps:

Step A. — We first need to show that π' is necessarily tempered following the argumentation given in [22]. Assume, on the contrary, that π' is not tempered. Langlands' classification [Theorem 2.9] assures us that π' is a subrepresentation in $I_{P_L}^{M'}(\tau_\nu)$, for a parabolic standard subgroup $P_L \supseteq P_1$ and

$$\nu \in ((a_L^{M'})^*)^-.$$

This is equivalent to claiming that there exists x, y with $x-y+1\in\mathbb{N}$, and $y\leq 0$, a Levi subgroup

$$L = M_{\Delta-\{\underline{\alpha}_1,\ldots,\underline{\alpha}_{a-a_-}\}\cup\{\underline{\alpha}_{x-y}\}}$$

a unitary cuspidal representation $w_{M'}\sigma$ in the $W(M_1)^{M'}$ group orbit of σ, and the element $\lambda \in (a_{M_1}^{M'})^*$ decomposes as $(x,y)+\lambda_1^{M'}$ such that:

$$\pi' = I_{P_L}^{M'}(\tau_\nu) \hookrightarrow I_{P_1\cap M'}^{M'}((w_{M'}\sigma)((x,y)+\lambda_1^{M'})),$$
$$\sum_{k\in[x,y]} k < 0. \tag{10}$$

The first equality in the first equation is due to the standard module conjecture since π' is generic. The second equation results from the following sequences of equivalences: $\nu <_{P_L} 0 \Leftrightarrow \frac{x+y}{2} < 0 \Leftrightarrow y < -x \Leftrightarrow \sum_{k\in[x,y]} k < 0$.

The element $w_{M'}$ in $W(M_1)^{M'}$ leaves the leftmost part, $\sigma_{(a,a_-+1)}$, invariant. Then from Equation (9) and inducing to G:

$$\pi_0 \hookrightarrow I_{P_1}^G((w_{M'}\sigma)((a,a_-+1)+(x,y)+\lambda_1^{M'})).$$

We can change $(a, a_-+1)(x, y)$ to $(x, y)(a, a_-+1)$ if and only if the two segments $(a, \ldots, a_- + 1)$ and $(x, \ldots, y)$ are unlinked (see Lemma 5.16). As $y \leq 0$, this condition is equivalent to $x \notin\ ]a, a_-]$.

If we can change, since $\sum_{k\in[x,y]} k < 0$, by Proposition 6.9 we get a contradiction to the square integrability of π_0.

Assume, therefore, that we cannot change. Then the two segments are linked by Proposition 5.7.

Let

$$M''' = M_{\Delta-\{\underline{\alpha}_q,\ldots,\underline{\beta}_d\}} \quad \text{where } q = a - a_- + x - y + 1.$$

The induced representation

$$I^{M'''}_{P_1\cap M'''}((w_{M'}\sigma)((a, \ldots, a_- + 1) + (x, \ldots, y)))$$

has a generic submodule, which is:

$$Z^{M'''}(P_1, w_L \cdot w_{M'}\sigma, (a, \ldots, y)(x, \ldots, a_- + 1))$$

(for some Weyl group element w_L such that $w_L \cdot w_{M'}\sigma \cong w_{M'}\sigma$).

We twist these by the character $\lambda_1^{M'}$ central for M''', and therefore, by [U]:

$$\begin{aligned}\pi_0 &\hookrightarrow I^G_{P'''}(Z^{M'''}(P_1, w_{M'}\sigma, (a, \ldots, y)(x, \ldots, a_- + 1))_{\lambda_1^{M'}})\\ &\hookrightarrow I^G_{P'''}(I^{M'''}_{P_1\cap M'''}((w_{M'}\sigma)((a, \ldots, a_- + 1) + (x, \ldots, y)))_{\lambda_1^{M'}})\\ &= I^G_{P_1}((w_{M'}\sigma)((a, \ldots, y) + (x, \ldots, a_- + 1) + \lambda_1^{M'}).\end{aligned}$$

Let $Q' = L'U'$; we rewrite this as:

$$\begin{aligned}\pi_0 &\hookrightarrow I^G_{Q'}(Z^{L'}(P_1, w'_L \cdot w_{M'}\sigma, (a, \ldots, y)(\lambda_2^{M'})))\\ &\hookrightarrow I^G_{P_1}((w'_L \cdot w_{M'}\sigma)((a, \ldots, y) + \lambda_2^{M'}))\\ &:= I^G_{P_1}((w_{M'}\sigma)((a, \ldots, y) + \lambda_2^{M'}))\end{aligned}$$

for some Weyl group element w'_L such that $w'_L \cdot w_{M'}\sigma \cong w_{M'}\sigma$.

Further, we have $y < -a_-$ since y is negative, $x \geq a_-$ and $\sum_{k\in[x,y]} k < 0$. In this context, the above Lemma 6.12 claims there exists $y' \leq y$:

$$\pi_0 \hookrightarrow I^G_{P_1}((w_{M'}\sigma)((a, \ldots, y') + \lambda_3^{M'})).$$

Then the unique irreducible generic subquotient π'_0 of $I^{N'}_{P_1\cap N'}(\sigma_{\lambda_3^{M'}})$ is square-integrable, or equivalently $\sigma_{\lambda_3^{M'}}$ is a residual point for $\mu^{N'}$ (the type is given by $\Sigma_\sigma^{N'}$). Further, $\sigma_{(a,\ldots,y')+\lambda_3^{M'}}$ is a residual point for μ^G (the type ia given by Σ_σ), corresponding to the generic discrete series π_0.

Then the *set of jumps* of the residual segment associated to π_0 contains the *set of jumps* of the residual segment associated to π'_0 and two more elements a and $-y'$. However, then $a > -y' > a_-$, and this contradicts the fact that a and a_- are two consecutive jumps.

We have shown that π' is necessarily tempered.

Step B. — Let $(\underline{n}_{\pi_0})$ be the residual segment canonically associated to a generic discrete series π_0. Let us now denote a_{i+1} the greatest integer smaller than a_i in the *set of jumps* of $(\underline{n}_{\pi_0})$. Therefore, the half-integers, a_i and a_{i+1} satisfy the conditions of this lemma.

As the representation π' is tempered, by Theorem 6.10, there exists a standard parabolic subgroup $P_\#$ of M' and a discrete series τ' such that $\pi' \hookrightarrow I_{P_\#}^{M'}(\tau')$.

Again, as an irreducible generic discrete series representation of a not necessarily maximal Levi subgroup, using the result of Heiermann–Opdam (Proposition 2.7), there exists an irreducible cuspidal representation σ' and a standard parabolic $P_{1,\#}$ of $M_\#$ such that τ' embeds in $I_{P_{1,\#}}^{M_\#}(\sigma'((\frac{a-a_--1}{2}, -\frac{a-a_--1}{2}) + \bigoplus_j(\not{a}_j, -\not{a}_j) + (\underline{n}_{\pi_0''})))$, where $(\underline{n}_{\pi_0''})$ is a residual segment corresponding to an irreducible generic discrete series π_0'', and $(\frac{a-a_--1}{2}, -\frac{a-a_--1}{2})$ along with $(\not{a}_j, -\not{a}_j)$'s are linear residual segments for (half)-integers $\not{a}_j$.

Clearly, the point $(\frac{a-a_--1}{2}, \dots, -\frac{a-a_--1}{2}) + \bigoplus_j(\not{a}_j, -\not{a}_j) + (\underline{n}_{\pi_0''})$ is in $\overline{a_{M_1}^{M_\# *}}^{+}$.

Then,

$$\pi' \hookrightarrow I_{P_{1,\#}U_\#}^{M'}\left(\sigma'\left(\left(\frac{a-a_--1}{2}, \dots, -\frac{a-a_--1}{2}\right) + \bigoplus_j(\not{a}_j, \dots, -\not{a}_j) + (\underline{n}_{\pi_0''})\right)\right). \tag{11}$$

Since $P_{1,\#}U_\#$ is standard in P', which is standard in G, there exists a standard parabolic subgroup P_1' in G, such that, when inducing Equation (11), we obtain:

$$\pi_0 \hookrightarrow I_{P'}^G(\pi'_{(a,\dots,a_-+1)}) \hookrightarrow I_{P_1'}^G(\sigma'_{(a,\dots,a_-+1)+\bigoplus_j(\not{a}_j,\dots,-\not{a}_j)+(\underline{n}_{\pi_0''})}). \tag{12}$$

Let us denote $(a, \dots, a_- + 1) + \bigoplus_j(\not{a}_j, \dots, -\not{a}_j) + (\underline{n}_{\pi_0''}) := \lambda'$.

Since π_0 also embeds as a subrepresentation in $I_{P_1}^G(\sigma_{(a,\dots,a_-+1)+\lambda})$, by Theorem 2.9 in [2] (see also [28] VI.5.4) there exists a Weyl group element w in W^G such that $w \cdot M_1 = M_1'$, $w \cdot \sigma = \sigma'$ and $w((a, a_- + 1) + \lambda) = \lambda'$.

Since Σ_σ is irreducible and M_1' is standard, we have by Point (3) in Corollary 6.6 that $M_1' = M_1$, and we can take w in $W(M_1)$. Further, since P_1 and P_1' are standard parabolic subgroups of G, and Σ_σ is irreducible, they are actually equal (see Remark 6.7). Now, by Point (2) in Corollary 6.6 any element in $W(M_1)$ is either in W_σ or decomposes in elementary symmetries in W_σ and $s_{\beta_d}W_\sigma$ and

$$\sigma' = w\sigma = \begin{cases} \sigma & \text{if } w \in W_\sigma \\ s_{\beta_d}\sigma & \text{otherwise.} \end{cases}$$

Let us assume that we are in the context where $\sigma' = s_{\beta_d}\sigma \not\cong \sigma$. As explained in the first part of Section 6 (see Proposition 6.4), this happens if Σ_σ is of type D. Let us apply the bijective operator (see Lemma 6.11) from $I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ to $I^{(M_1)_{\beta_d}}_{\overline{P_1}\cap(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$, and then the bijective map $t(s_{\beta_d})$ (the definition of the map $t(g)$ has been given in the proof of Proposition 3.2) to $I^{(M_1)_{\beta_d}}_{s_{\beta_d}(\overline{P_1}\cap(M_1)_{\beta_d})}(\sigma_{s_{\beta_d}\lambda'}) = I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'})$.

As explained in Remark 6.5, $s_{\beta_d}\lambda' = \lambda'$ since λ' is a residual point of type D. Therefore, we have a bijective map from $I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ to $I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}(\sigma_{\lambda'})$. The induction of this bijective map gives a bijective map from $I^G_{P_1}(\sigma'_{(a,\dots,a_-+1)+\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi''_0})})$ to $I^G_{P_1}(\sigma_{(a,\dots,a_-+1)+\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi''_0})})$. Hence, we may write Equation (12) as:

$$\pi_0 \hookrightarrow I^G_{P'}(\pi'_{(a,\dots,a_-+1)}) \hookrightarrow I^G_{P_1}(\sigma_{(a,\dots,a_-+1)+\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi''_0})}). \tag{13}$$

Let us set $a = a_i$, $a_- = a_{i+1}$ for a_i, a_{i+1} two consecutive elements in the set of jumps of $(\underline{n}_{\pi_0})$. Therefore, $(a_i,\dots,a_{i+1}+1)\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi''_0})$ is in the Weyl group orbit of the residual segment associated to π_0: $(\underline{n}_{\pi_0})$.

Let us show that $(a_i,\dots,a_{i+1}+1)(a_{i+1},\dots,-a_{i+1})(\underline{n}^i)$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

One notices that in the tuple $\underline{n}_{\pi_0}$ of the residual segment $(\underline{n}_{\pi_0})$, the following relations are satisfied:

$$n_{a_i} = n_{a_{i+1}} - 1, \tag{14}$$

$$n_i = n_{i-1} - 1 \quad \text{or} \quad n_i = n_{i-1}, \quad \forall i > 0. \tag{15}$$

Consequently, when we withdraw $(a_i,\dots,a_{i+1}+1)$ from this residual segment, we obtain a segment $(\underline{n'})$ that cannot be a residual segment since $n'_{a_{i+1}} = n'_{a_{i+1}+1} + 2$, for $i \neq 1$; or if $i = 1$, $n'_{a_2} = 2$, but a_2 is now the greatest element in the set of jumps associated to the segment $(\underline{n'})$, so we should have $n'_{a_2} = 1$.

Therefore, to obtain a residual point (residual segment $(\underline{n}_{\pi''_0})$), we need to remove twice a_{i+1}.

Then, for any $0 < j < a_{i+1}$, if we remove twice j, $n'_j = n_j - 2$ and, for all i, the relations $n'_j = n'_{j-1} - 1$ or $n'_j = n'_{j-1}$ are still satisfied. As we also remove one zero, we have for $j = 0$, $n'_0 = n_0 - 1$, which is compatible with removing twice $j = 1$.

The residual segment left, thus obtained, will be denoted $(\underline{n}^i)$. We have shown that $(a_i,\dots,a_{i+1}+1)(a_{i+1},\dots,-a_{i+1})(\underline{n}^i)$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

Since $(\underline{n}^i)$ is a residual segment, from the conditions detailed in Equations (14) and (15) (see also Remark 4.15 in Section 4.2), no symmetrical linear residual segment $(a_k, -a_k)$ can be extracted from $(\underline{n}^i)$ to obtain another residual segment $(\underline{n}_{\pi_0''})$ such that $(a_i,\ldots,a_{i+1}+1)(a_{i+1},\ldots,-a_{i+1})(a_k,-a_k)(\underline{n}_{\pi_0''})$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

So, $(\underline{n}_{\pi_0''})=(\underline{n}^i)$ and

$$\pi'_{(a,a_-+1)}\hookrightarrow I_{P_1}^{M'}(\sigma((a_i,a_{i+1}+1)+(a_{i+1},-a_{i+1})+(\underline{n}^i))).$$

Eventually, using induction in stages, Equation (11) rewrites as:

$$\pi_0\hookrightarrow I_{P_1}^{G}(\sigma((a_i,a_{i+1}+1)+(a_{i+1},-a_{i+1})+(\underline{n}^i)))=\Theta,$$

and since the two segments $(a_i,\ldots,a_{i+1}+1)$ and $(a_{i+1},\ldots,-a_{i+1})$ are linked, we can take their union and deduce that there exists an irreducible generic essentially square integrable representation π_{a_i} of a Levi subgroup M^{a_i} in P^{a_i}, which once induced embeds as a subrepresentation in Θ and, therefore, by multiplicity one of the irreducible generic piece ([U], see 2.1) , π_0, we have:

$$\pi_0\hookrightarrow I_{P^{a_i}}^{G}(\pi_{a_i})\hookrightarrow I_{P_1}^{G}(\sigma((a_i,-a_{i+1})+(\underline{n}^i))).$$

Proposition 6.14. — *Let $(\underline{n}_{\pi_0})$ be a residual segment associated to an irreducible generic discrete series π_0 of G whose cuspidal support satisfies the conditions (CS) (see Definition 6.2).*

Let $a_1>a_2>\ldots>a_n$ be jumps *of this residual segment. Let $P_1=M_1U_1$ be a standard parabolic subgroup and σ be a unitary irreducible cuspidal representation of M_1 such that $\pi_0\hookrightarrow I_{P_1}^{G}(\sigma(\underline{n}_{\pi_0}))$.*

For any i, there exists a standard parabolic subgroup $P^{a_i}\supset P_1$ with Levi subgroup M^{a_i}, residual segment $(\underline{n}^i)$ and an irreducible generic essentially square-integrable representation $\pi_{a_i}=Z^{M^{a_i}}(P_1,\sigma,(a_i,-a_{i+1})(\underline{n}^i))$ such that π_0 embeds as a subrepresentation in

$$I_{P^{a_i}}^{G}(\pi_{a_i})\hookrightarrow I_{P_1}^{G}(\sigma((a_i,-a_{i+1})+(\underline{n}^i))).$$

Proof. — By the result of Heiermann–Opdam [Proposition 2.7] and Lemma 6.1, to any residual segment $(\underline{n}_{\pi_0})$ we associate the unique irreducible generic discrete series subquotient in $I_{P_1}^{G}(\sigma(\underline{n}_{\pi_0}))$.

Then, as explained in Section 4.2, this residual segment defines uniquely *jumps*: $a_1>a_2>\ldots>a_n$.

Start with the two elements $a_1=\ell+m$ and $a_2=\ell-1$ and consider the following induced representation:

$$\begin{aligned}(16)\quad & I_{P_1}^{G}(\sigma((\ell+m,a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0}))\\ &\quad = I_{P}^{G}(I_{P_1\cap M}^{M}(\sigma((\ell+m,a_2=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0}))).\end{aligned}$$

Let us denote $\nu:=(\ell+m,a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0}$.

The induced representation $I_{P_1\cap M}^{M}(\sigma((\ell+m,a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})):=I_{P_1\cap M}^{M}(\sigma_\nu)$ is a generic induced module.

The form of ν implies that σ_ν is not necessarily a residual point for μ^M. Indeed, the first linear residual segment $(\ell+m, a_2+1=\ell)$ is certainly a residual segment (of type A) but the second not necessarily.

Let π be the unique irreducible generic subquotient of $I^M_{P_1\cap M}(\sigma_\nu)$ (which exists by Rodier's theorem). We have:

$$\pi \leq I^M_{P_1\cap M}(\sigma_\nu) \quad \text{and} \quad I^G_P(\pi) \leq I^G_P(I^M_{P_1\cap M}(\sigma_\nu)) := I^G_{P_1}(\sigma_\lambda).$$

Assume $I^G_P(\pi)$ has an irreducible generic subquotient π'_0 different from π_0, then π'_0 and π_0 would be two generic irreducible subquotients in $I^G_{P_1}(\sigma_\lambda)$ contradicting Rodier's theorem. Hence, $\pi_0 \leq I^G_P(\pi)$.

Further, since π_0 embeds as a subrepresentation in

$$I^G_P(I^M_{P_1\cap M}(\sigma((\ell+m, a_2+1=\ell)+(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\dots 0^{n_0}))) := I^G_{P_1}(\sigma_\lambda),$$

it also has to embed as a subrepresentation in $I^G_P(\pi)$.

Therefore, applying Lemma 6.13, we conclude that there exists a residual segment $(\underline{n^1})$, an essentially square integrable representation π_{a_1} such that π_0 embeds as a subrepresentation in

$$I^G_{P^{a_1}}(\pi_{a_1}) \hookrightarrow I^G_{P_1}(\sigma((a_1,-a_2)+(\underline{n}^1))).$$

Let us consider now the elements $a_2=\ell-1$ and a_3. As in the proof of Lemma 5.16, since the linear residual segments $(a_1,\ell-1)$ and $(\ell-1)$ are unlinked, we apply a composite map from the induced representation $I^{M'}_{P_1\cap M'}(\sigma((a_1,\ell-1)+(\ell-1)+\dots 1^{n_1}0^{n_0}))$ to $I^{M'}_{P_1\cap M'}(\sigma((\ell-1)+(a_1,\ell-1)+\dots 1^{n_1}0^{n_0}))$. We can interchange the two segments and, as in the proof of Lemma 5.16, applying this intertwining map and inducing to G preserves the unique irreducible generic subrepresentation of $I^G_{P_1}(\sigma_\lambda)$.

We repeat this argument with

$$\begin{aligned}&I^{M''}_{P_1\cap M''}(\sigma((\ell-1)+(a_1,\ell-2)+(\ell-2)+\dots 1^{n_1}0^{n_0})) \quad \text{and}\\ &I^{M''}_{P_1\cap M''}(\sigma((\ell-1)+(\ell-2)+(a_1,\ell-2)+\dots 1^{n_1}0^{n_0})),\end{aligned}$$

and further repeat it with all exponents until a_3+1.

Eventually, the unique irreducible subrepresentation π_0 appears as a subrepresentation in $I^G_{P_1}(\sigma((a_2,a_3+1)+(a_1,a_3+1)+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0}))$

$$\begin{aligned}\pi_0 \hookrightarrow I^G_{P'^{a_2}}(I^{M'^{a_2}}_{P_1\cap M'^{a_2}}(&\sigma((a_2,a_3+1)+(a_1,a_3+1)\\ &+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0})))\\ := I^G_{P'^{a_2}}(I^{M'^{a_2}}_{P_1\cap M'^{a_2}}(&(w\sigma)_{w\nu})) \quad \text{where } w\in W_\sigma.\end{aligned}$$

Let π be the unique irreducible generic subquotient of $I_{P_1\cap M'^{a_2}}^{M'^{a_2}}(\sigma_{w\nu})$ (which exists by Rodier's theorem). We have: $\pi \leq I_{P_1\cap M'^{a_2}}^{M'^{a_2}}(\sigma_{w\nu})$ and

$$I_{P'^{a_2}}^G(\pi) \leq I_{P'^{a_2}}^G(I_{P_1\cap M'^{a_2}}^{M'^{a_2}}(\sigma_{w\nu})) := I_{P_1}^G(\sigma_{w\lambda}).$$

Assume $I_{P'^{a_2}}^G(\pi)$ has an irreducible generic subquotient π_0' different from π_0; then π_0' and π_0 would be two generic irreducible subquotients in $I_{P_1}^G((w\sigma)_{w\lambda})$, contradicting Rodier's theorem. Hence, $\pi_0 \leq I_{P'^{a_2}}^G(\pi)$. Further, since π_0 embeds as a subrepresentation in

$$I_{P'^{a_2}}^G(I_{P_1\cap M'^{a_2}}^{M'^{a_2}}(\sigma((a_2,a_3+1)+(a_1,a_3+1) \\ +(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0}))) := I_{P_1}^G(\sigma_{w\lambda}),$$

it also embeds as a subrepresentation in $I_{P'^{a_2}}^G(\pi)$.

Hence, applying Lemma 6.13, we conclude that there exists a residual segment $(\underline{n}^2)$ and an essentially square-integrable representation $\pi_{a_2} = Z^{M^2}(P_1\cap M^{a_2},\sigma,(a_2,-a_3)(\underline{n}^2))$ such that π_0 embeds as a subrepresentation in $I_{P^{a_2}}^G(\pi_{a_2}) \hookrightarrow I_{P_1}^G(\sigma((a_2,-a_3)+(\underline{n}^2)))$.

Similarly, for any two consecutive elements in the *set of jumps*, a_i and a_{i+1}, the same argumentation (i.e. first embedding π_0 as a subrepresentation in $I_{P'^{a_i}}^G(\pi)$ using intertwining operators, and concluding with Lemma 6.13) yields the embedding:

$$\pi_0 \hookrightarrow I_{P^{a_i}}^G(\pi_{a_i}) \hookrightarrow I_{P^{a_i}}^G(I_{P_1\cap M^i}^{P^{a_i}}(\sigma((a_i,-a_{i+1})+(\underline{n}^i)))$$

for an irreducible generic essentially square-integrable representation

$$\pi_{a_i} = Z^{M^{a_i}}(P_1\cap M^{a_i},\sigma,(a_i,-a_{i+1})(\underline{n}^i))$$

of the Levi subgroup M^{a_i}. □

6.4. Proof of the theorem 6.3. —

- 1(a) is the result of Lemma 6.1.
- 1(b) is the result of Proposition 6.14.
- 1(c) Let us denote π_0 the unique irreducible generic subquotient in $I_{P_1}^G(\sigma_{(a,b)\underline{n}})$. By Proposition 2.7, there exists a parabolic subgroup P' such that π_0 embeds as a subrepresentation in the induced module $I_{P'}^G(\sigma'_{\lambda'})$, for $\sigma'_{\lambda'}$ a dominant residual point for P'. Let $(w\sigma)_{w\lambda}$ be the dominant (for P_1) residual point in the W_σ-orbit of σ_λ, then (using Theorem 2.9 in [2] or Theorem VI.5.4 in [28]) π_0 is the unique irreducible generic subquotient in $I_{P_1}^G((w\sigma)_{w\lambda})$, and Proposition 3.2 gives us that these two ($I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G((w\sigma)_{w\lambda})$) are isomorphic.

  The point $(w\sigma)_{w\lambda}$ is a dominant residual point with respect to P_1: $w\lambda \in \overline{a_{M_1}^{*}}^{+}$, and there is a unique element in the orbit of the Weyl group W_σ of a residual point, which is dominant and is explicitly given by a residual segment using the correspondence of Section 2.5.1. We denote

$w\lambda := (\underline{n}_{\pi_0})$ this residual segment. Since $w \in W_\sigma$, $(w\sigma)_{w\lambda} \cong \sigma_{w\lambda}$. Hence, $\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\underline{n}_{\pi_0}))$.

Since $\mathscr{a} > \mathscr{b}$, and $(\underline{n_{\pi_0}})$ is a residual segment, it is clear that $\mathscr{a}$ is a jump. [Indeed, if you extract a linear residual segment $(\mathscr{a}, \dots, \mathscr{b})$ such that $\mathscr{a} > \mathscr{b}$ from $(\underline{n}_{\pi_0})$ such that what remains is a residual segment, then $\mathscr{a} = a$ has to be in the *set of jumps* of the residual segment $(\underline{n}_{\pi_0})$ as defined in Section 4.2]. Let us denote a_- the greatest integer smaller than a in the *set of jumps*. Therefore, the (half)-integers, a and a_- satisfy the conditions of Proposition 6.14. We will show below that $\mathscr{b} \geq -a_-$. Let $P_\flat = P_{\Delta - \{\alpha_{a+a_-+1}\}}$ be a maximal parabolic subgroup, with Levi subgroup $M_\flat$, which contains P_1.

Let $\pi_a = Z^{M_\flat}(P_1, \sigma, w_{a_-}\lambda)$, for $w_{a_-} \in W_\sigma$ be the generic essentially square integrable representation with cuspidal support $\sigma((a, -a_-)(\underline{n_{-a_-}}))$ associated to the residual segment $((a, -a_-) + (\underline{n_{-a_-}}))$ (in the W_σ-orbit of $(\underline{n}_{\pi_0})$). It is some discrete series twisted by the Langlands parameter $s_{-a_-}\widetilde{\alpha_{a+a_-+1}}$ with $s_{-a_-} = \frac{a-a_-}{2}$. By the Proposition 6.14 we can write

$$\pi_0 \hookrightarrow I_{P_\flat}^G(\pi_a) \hookrightarrow I_{P_1}^G(\sigma((a, -a_-)(\underline{n_{-a_-}}))). \tag{17}$$

Here, we need to justify that given a, for any $\mathscr{b}$ we have: $\mathscr{b} \geq -a_-$.

Consider again the residual segment $(\underline{n}_{\pi_0})$ and observe that by definition the sequence $(a, \dots, -a_-)$ is the longest linear segment with greatest (half)-integer a that one can withdraw from $(\underline{n}_{\pi_0})$ such that the remaining segment $(\underline{n_{-a_-}})$ is a residual segment of the same type, and $(a, \dots, -a_-)(\underline{n_{-a_-}})$ is in the Weyl group orbit of $(\underline{n}_{\pi_0})$.

Further, this is true for any couple (a, a_-) of elements in the *set of jumps* associated to the residual segment $(\underline{n}_{\pi_0})$. It is, therefore, clear that given a and a_- such that $s_{-a_-} = \frac{a-a_-}{2} > 0$ is the smallest positive (half)-integers possible, we have that $s_{\mathscr{b}} = \frac{a+b}{2} \geq s_{-a_-} = \frac{a-a_-}{2}$ and $\mathscr{b}$ is necessarily greater or equal to $-a_-$.

Once this embedding is given, using Lemma 5.12, there exists an intertwining operator with a non-generic kernel from the induced module $I_{P_1}^G(\sigma((a, -a_-)(\underline{n_{-a_-}})))$ given in Equation (17) to any other induced module from the cuspidal support $\sigma(a, \mathscr{b}, \underline{n_{\mathscr{b}}})$ with $\mathscr{b} \geq -a_-$.

Therefore, $\pi_0 \hookrightarrow I_{P_1}^G(\sigma(a, \mathscr{b}, \underline{n_{\mathscr{b}}})) = I_{P_1}^G(\sigma_\lambda)$.

- 2(a) Since λ is not a residual point, the generic subquotient is a non-discrete series. By Langlands' classification, Theorem 2.9, and the standard module conjecture, it has the form $J_{P'}^G(\tau'_{\nu'}) \cong I_{P'}^G(\tau'_{\nu'})$. By Theorem 5.5, ν' corresponds to the minimal Langlands parameter (this notion was introduced in Theorem 2.9) for a given cuspidal support.

For an explicit description of the parameter ν, given the cuspidal string $(a, b, \underline{n})$, the reader is encouraged to read the analysis conducted in the Appendix of the author's thesis [11].

The representation τ' (e.g. $\mathrm{St}_q|\cdot|^{\nu'} \otimes \pi'$ in the context of classical groups, for a given integer q) corresponds to a cuspidal string $(a', b', \underline{n'})$, and cuspidal representation σ', that is,

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P_1'}^G(\sigma'(a', b', \underline{n'})).$$

By Theorem 2.9 in [2], we know the cuspidal data $(P_1, \sigma, (a', b', \underline{n'}))$ and $(P_1', \sigma', \lambda' := (a', b', \underline{n'}))$ are conjugated by an element $w \in W^G$.

By Corollary 6.6 and since P_1 and P_1' are standard parabolic subgroups (see Remark 6.7), we have $P_1 = P_1'$, $w \in W(M_1)$. Any element in $W(M_1)$ decomposes in elementary symmetries with elements in W_σ and $s_{\beta_d} W_\sigma$:

$$\sigma' = w\sigma = \begin{cases} \sigma & \text{if } w \in W_\sigma \\ s_{\beta_d}\sigma & \text{otherwise.} \end{cases}$$

Let us assume that we are in the context where $\sigma' = s_{\beta_d}\sigma \ncong \sigma$. As explained in the first part of Section 6.3, this happens if Σ_σ is of type D.

Let us apply the bijective operator (see Lemma 6.11) from $I_{P_1 \cap (M_1)_{\beta_d}}^{(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ to $I_{\overline{P_1} \cap (M_1)_{\beta_d}}^{(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ and then the bijective map (the definition of the map $t(g)$ has been given in the proof of Proposition 3.2) $t(s_{\beta_d})$ to $I_{s_{\beta_d}(\overline{P_1} \cap (M_1)_{\beta_d})}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'}) = I_{P_1 \cap (M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'})$. As explained in Remark 6.5, $s_{\beta_d}\lambda' = \lambda'$ since λ' is a residual point of type D. Therefore, we have a bijective map from $I_{P_1 \cap (M_1)_{\beta_d}}^{(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ to $I_{P_1 \cap (M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{\lambda'})$ The induction of this bijective map gives a bijective map from $I_{P_1'}^G(\sigma'(a', b', \underline{n'}))$ to $I_{P_1'}^G(\sigma(a', b', \underline{n'}))$.

- 2(b) Assume now that we consider a tempered or non-tempered subquotient in $I_{P_1}^G(\sigma(a, b, \underline{n}))$. We first apply the argumentation developed in the previous point 2(a) to embed it in $I_{P_1'}^G(\sigma(a', b', \underline{n'}))$. Then it is enough to understand how one passes from the cuspidal string $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ to understand the strategy for embedding the unique irreducible generic subquotient as a subrepresentation $I_{P_1}^G(\sigma(a, b, \underline{n}))$.

  Starting from $(a, b, \underline{n})$, to minimize the Langlands parameter ν', we usually remove elements at the end of the first segment (i.e. the segment $(a, \ldots, b)$) to insert them on the second residual segment, or we enlarge the first segment on the right. This means either $a' < a$, or $b' < b$, or both.

If $\mathscr{a}' = \mathscr{a}$, and $\mathscr{b}' < \mathscr{b}$, in particular if $\mathscr{b}' < 0$, we have a non-generic kernel operator between $I_{P_1}^G(\sigma(\mathscr{a}', \mathscr{b}', \underline{n'}))$ and $I_{P_1}^G(\sigma(\mathscr{a}, \mathscr{b}, \underline{n}))$, as proved in Lemma 5.12. □

6.5. An order on the cuspidal strings in a W_σ-orbit. — It is possible to describe the set of points in the W_σ-orbit of a dominant residual point λ_D as follows.

Let us define a set of points $\mathscr{L}$ in the W_σ-orbit of a dominant residual point λ_D such that they are written as: $(\mathscr{a}, \mathscr{b})(\underline{n})$ with at most one linear residual segment $(\mathscr{a}, \mathscr{b})$ satisfying the condition $\mathscr{a} > \mathscr{b}$. Then $\mathscr{a}$ is a jump as explained in the proof of Theorem 6.3, point 1(c).

Let us attach a positive integer $C(1, \lambda) = \#\{\beta \in \Sigma_\sigma^+ | \langle \lambda, \check{\beta} \rangle < 0\}$ to any of these points.

By definition, $C(1, \lambda_D) = 0$. What are the points λ in $\mathscr{L}$ such that the function $C(1, \lambda)$ is maximal?

Lemma 6.15. — *The function $C(1, \cdot)$ on $\mathscr{L}$ is maximal for the points that are the form $(a, -a_-)(\underline{n})$, for (a, a_-) any two consecutive elements in the jumps sets associated to λ_D.*

Proof. — Let us choose a point in $\mathscr{L}$; since it is a point in $\mathscr{L}$, it uniquely determines a jump a (as its left end). For any fixed a, we show that the function $C(1, \lambda_a)$ is maximal for $\lambda_{a,-a_-} = (a, -a_-)(\underline{n})$. Let $\mathscr{L}_a$ denote the set of points in $\mathscr{L}$ such that the linear residual segment (if it exists) *has left end* a. The union of the $\mathscr{L}_a$ where a runs over the set of jumps is $\mathscr{L}$.

Let us choose a point $\lambda_{a,\mathscr{b}} = (a, \mathscr{b})(\underline{n})_\mathscr{b}$ in $\mathscr{L}_a$ and denote $L_\mathscr{b}$ the length of the residual segment $(\underline{n})_\mathscr{b}$. Recall also that $(\underline{n})_\mathscr{b} = (\ell, \dots \mathscr{b}^{n_\mathscr{b}} \dots 0^{n_0})$.

- Case $a > 0 > \mathscr{b}$

  Consider $\lambda_\mathscr{b}$ and $\lambda_{\mathscr{b}+1}$.

  Let us consider first those roots that are of the forms $e_i - e_j$, $i > j$. On $\lambda_\mathscr{b}$, the number of these roots that have non-positive scalar product is

$$(-b) \times L_\mathscr{b} + (L_\mathscr{b} - n_0) + (L_\mathscr{b} - (n_0 + n_1)) \\ + (L_\mathscr{b} - (n_0 + n_1 + n_2 + \dots + n_\mathscr{b})) + C_{\mathscr{b}+1},$$

  where $C_{\mathscr{b}+1}$ is some constant depending on the multiplicities n_i for $i \geq (\mathscr{b} + 1)$.

  Secondly, let us consider the roots of the forms $e_i + e_j$, $i > j$; on $\lambda_\mathscr{b}$, the number of these roots that have non-positive scalar product is

$$L_\mathscr{b} - (n_\mathscr{b} + n_{b+1} + n_\ell) + L_\mathscr{b} - (n_{b-1} + n_\mathscr{b} + n_{b+1} + \dots + n_\ell) \\ + L_\mathscr{b} - (n_{b-2} + n_{b-1} + n_\mathscr{b} + n_{b+1} + \dots + n_\ell) + \dots \\ + \mathscr{b} + \mathscr{b} - 1 + \mathscr{b} - 2 + \dots + 1.$$

Finally, one should also take into account the roots of type $e_i, 2e_i$ or $e_i + e_d$ if d is the dimension of Σ_σ and of type B, C or D. There are b such roots in our context.

$$\begin{aligned} C(1, \lambda_{\not b+1}) = {} & (-\not b - 1) \times (L_{\not b} + 1) + (L_{\not b} + 1 - n_0) + (L_{\not b} + 1 - (n_0 + n_1)) \\ & + \ldots + (L_{\not b} + 1 - (n_0 + n_1 + \ldots + n_{\not b})) + C_{b+1} + L_{\not b} + 1 \\ & - (n_{\not b-1} + n_{\not b} + n_{\not b+1} + \ldots + n_\ell) + L_{\not b} \\ & - (n_{b-2} + n_{b-1} + n_{\not b} + n_{\not b+1} + \ldots + n_\ell) + \ldots \\ & + \not b - 1 + \not b - 2 + \ldots + 1 + \not b - 1, \\ C(1, \lambda_{\not b}) - C(1, \lambda_{\not b+1}) = {} & L_{\not b} - (n_{\not b} + n_{b+1} + n_\ell) + \not b + \not b \\ & - (-L_{\not b} - \not b - 1 + \not b - 1 + \not b + \not b - 1), \\ C(1, \lambda_{\not b}) - C(1, \lambda_{\not b+1}) = {} & 2L_{\not b} - (n_{\not b} + n_{\not b+1} + n_\ell) + 3 \end{aligned}$$

Therefore

$$C(1, \lambda_b) > C(1, \lambda_{\not b+1}).$$

- Case $a > \not b > 0$

  Consider $\lambda_{\not b}$ and $\lambda_{\not b+1}$. The number $C(1, \lambda_{\not b})$ and $C(1, \lambda_{\not b+1})$ differ by $L_{\not b} - (n_0 + n_1 + \ldots + n_{\not b})$. As this number is clearly positive, we have $C(1, \lambda_{\not b}) > C(1, \lambda_{\not b+1})$.

This shows that $C(1, \cdot)$ decreases as the length of the linear residual segment (a, b) decreases. Furthermore, from the definition of the residual segment (Definition 4.8) and the observations made on cuspidal lines, the sequence $(a, \ldots, -a_-)$ is the *longest* linear segment with greatest (half)-integer a that one can withdraw from λ_D such that the remaining segment $\overline{(\underline{n}_{-a_-})}$ is a residual segment of the same type, and $(a, \ldots, -a_-)\overline{(\underline{n}_{-a_-})}$ is in the W_σ-orbit of λ_D. Therefore, $C(1, \lambda_{a,-a_-})$ is maximal on the set $\mathscr{L}_a$. ☐

As a consequence of this lemma, we will denote the points of maximal $C(1, \cdot)$, λ_{a_i}, for any a_i, in the jump set of λ_D.

The elementary symmetries associated to roots in Σ_σ permute the (half)-integers appearing in the cuspidal line $(\not a, \not b)(\underline{n})$.

We illustrate the set $\mathscr{L}$ with a picture. Let us assume that any two points in the W_σ-orbit are connected by an edge if they share the same parameter a, and/or the intertwining operator associated to the sequence of elementary symmetries connecting the two points has a non-generic kernel. Any point in $\mathscr{L}$ is on an edge joining the points of maximal $C(1, \cdot)$ to λ_D. We obtain the following picture.

Then the proof of the Theorem 6.3 could be thought about in the following way. Relying on the extended Moeglin's lemmas we obtain the embedding of the unique irreducible generic subquotient for a set of parameters $\{\lambda_{a_i}\}_i$. Those parameters are indexed by the jumps a_i in a (finite) *set of jumps* associated to

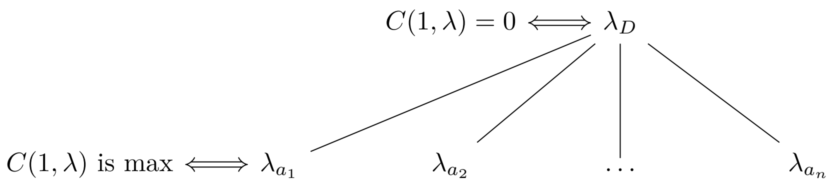


FIGURE 6.1. The set $\mathscr{L}$

the dominant residual point λ_D (they are in the W_σ-orbit of λ_D). Once this key embedding is given, for each jump a, we use intertwining operators with a non-generic kernel to send the unique irreducible generic subrepresentation that lies in $I_{P_1}^G(\sigma_{\lambda_a}) = I_{P_1}^G(\sigma((a, -a_-)(\underline{n}))$ to $I_{P_1}^G(\sigma((a, \mathscr{b})(\underline{n'}))$, for any $\mathscr{b} > -a_-$, where $(\underline{n'})$ is a residual segment of the same type as $(\underline{n})$.

7. Proof of the generalized injectivity conjecture for discrete series subquotients

Before entering the proof of the conjecture for discrete series subquotients, let us mention two other results. First, in order to use Theorem 2.6, let us first prove the following lemma:

LEMMA 7.1. — *Under the assumption that μ^G has a pole at $s\tilde{\alpha}$ (assumption 1) for τ, and μ^M has a pole at ν (for σ) of maximal order, for $\nu \in a_{M_1}^*$, $\sigma_{\nu+s\tilde{\alpha}}$ is a residual point.*

Proof. — We will use the multiplicativity formula for the μ function (see Section IV 3 in [39] or the earlier result (Theorem 1) in [33]):

$$\mu^G(\tau_{s\tilde{\alpha}}) = \frac{\mu^G}{\mu^M}(\sigma_{s\tilde{\alpha}+\nu}).$$

We first notice that if μ^M has a pole in ν (for σ) of maximal order, for $\nu \in a_{M_1}^*$, μ^M also has a pole of maximal order at $\nu + s\tilde{\alpha}$ (since $s\tilde{\alpha}$ is in a_M^*, we twist by a character of A_M, which leaves the function μ^M unchanged). Under assumption 1, the order of the pole at $\nu + s\tilde{\alpha}$ of the right-hand side of the equation is:

$$\text{ord}(\text{pole for} \mu^G \text{ at } \nu + s\tilde{\alpha}) - (\text{rk}_{ss}(M) - \text{rk}_{ss}(M_1)) \geq 1.$$

Since M is maximal we have $(\text{rk}_{ss}(G) - \text{rk}_{ss}(M)) = \dim(A_M) - \dim(A_G) = 1$, and then

$$\begin{aligned}(\text{rk}_{ss}(M) - \text{rk}_{ss}(M_1)) + 1 &= (\text{rk}_{ss}(M) - \text{rk}_{ss}(M_1)) + (\text{rk}_{ss}(G) - \text{rk}_{ss}(M)) \\ &= (\text{rk}_{ss}(G) - \text{rk}_{ss}(M_1)).\end{aligned}$$

Hence, ord(pole of μ^G in $\nu + s\tilde{\alpha}$) $\geq$ $(\text{rk}_{ss}(G) - \text{rk}_{ss}(M_1))$, and the lemma follows. □

The element $\nu + s\tilde{\alpha}$ being a residual point (a pole of maximal order for μ^G) for σ, by Theorem 2.6 we have a discrete series subquotient in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$. Further, consider the following classical lemma (see, for instance, [41]):

Lemma 7.2. — *Take τ a tempered representation of M, and ν_0 in the positive Weyl chamber. If ν_0 is a pole for μ^G, then $I_P^G(\tau_{\nu_0})$ is reducible.*

This lemma results from the fact that when τ is tempered and ν_0 in the positive Weyl chamber, $J_{\overline{P}|P}(\tau, \cdot)$ is holomorphic at ν_0. If the μ function has a pole at ν_0, then $J_{\overline{P}|P} J_{P|\overline{P}}(\tau, \cdot)$ is the zero operator at ν_0. The image of $J_{P|\overline{P}}(\tau, \cdot)$ would then be in the kernel of $J_{\overline{P}|P}(\tau, \cdot)$, a subspace of $I_P^G(\tau_{\nu_0})$, which is null if $I_P^G(\tau_{\nu_0})$ is irreducible. This would imply $J_{P|\overline{P}}$ is a zero operator, which is not possible. So $I_P^G(\tau_{\nu_0})$ must be reducible.

Under the hypothesis of Lemma 7.1, the module $I_P^G(\tau_{s\tilde{\alpha}})$ has a generic discrete series subquotient. In this section, we aim to prove that this generic subquotient is a subrepresentation.

We present here the proof of the generalized injectivity conjecture in the case of a standard module induced from a maximal parabolic $P = MU$. Then, the roots in Lie(M) are all the roots in Δ but α. We first present the proof in case α is not an extremal root in the Dynkin diagram of G, and secondly when it is an extremal root.

Proposition 7.3. — *Let π_0 be an irreducible generic representation of a quasi-split reductive group G of type A, B, C or D, which embeds as a subquotient in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, with $P = MU$ a maximal parabolic subgroup and τ discrete series of M.*

Let σ_ν be in the cuspidal support of the generic discrete series representation τ of the maximal Levi subgroup M; we take $s\tilde{\alpha}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and denote $\lambda = \nu + s\tilde{\alpha}$ in $\overline{a_{M_1}^{M}}^{+*}$.

Let us assume that the cuspidal support of τ satisfies the conditions (CS) (see Definition 6.2).

Let us assume that α is not an extremal simple root on the Dynkin diagram of Σ.

Let us assume σ_λ is a residual point for μ^G. This is equivalent to saying that the induced representation $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient. Then, π_0, which is discrete series embeds as a submodule in $I_{P_1}^G(\sigma_\lambda)$, and, therefore, in the standard module $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof. — First, notice that if $s = 0$, the induced module $I_P^G(\tau_{s\tilde{\alpha}})$ is unitary, and, hence, any irreducible subquotient is a subrepresentation; in the rest of the proof, we can, therefore, assume $s\tilde{\alpha}$ in $(a_M^*)^+$.

We are in the context of Section 4.3 and, therefore, we can write $\lambda := (a, \ldots, b)(\underline{n})$, for some (half)-integers $a > b$, and residual segment $(\underline{n})$. In this context, as we denote $s\tilde{\alpha}$ the Langlands parameter twisting the discrete series τ, then $s = s_{\mathscr{b}} = \frac{\mathscr{a}+\mathscr{b}}{2}$.

Notice that since σ_λ is in the W_σ-orbit of a dominant residual point whose parameter corresponds to a residual segment of type B, C or D, $\mathscr{a}$ and $\mathscr{b}$ are not only reals but *(half)-integers*. The conditions of application of Theorem 6.3 1(b) or 1(c) are satisfied, and, therefore, the unique irreducible generic subquotient in $I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation. By multiplicity 1, it will also embed as a subrepresentation in the standard module $I_P^G(Z^M(P_1,\sigma,\lambda))$. $\square$

Remark 7.4. — From the Theorem 6.3 and the argumentation given in the proof of the previous proposition, it is easy to deduce that if π_0 appears as a submodule in the standard module

$$I_{P_\flat}^G(Z^{M_\flat}(P_1,\sigma,w_{a_-}\lambda))$$

with Langlands' parameter $s_{a_-}\widetilde{\alpha_{a+a_-+1}}$, it also appears as a submodule in any standard module $I_P^G(Z^M(P_1,\sigma,(a,\mathscr{b},\underline{n_{\mathscr{b}}}))$ with Langlands' parameter $s_{\mathscr{b}}\tilde{\alpha} \geq s_{-a_-}\widetilde{\alpha_{a+a_-+1}}$, for the order defined in Lemma 5.2 as soon as $Z^M(P_1,\sigma,(a,\mathscr{b},\underline{n_{\mathscr{b}}}))$ has equivalent cuspidal support.

7.1. The case of Σ_σ^M irreducible. —

Proposition 7.5. — *Let π_0 be an irreducible generic discrete series of G with cuspidal support (M_1,σ), and let us assume that Σ_σ is irreducible. Let M be a standard maximal Levi subgroup such that Σ_σ^M is irreducible.*

Then, π_0 embeds as a subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where τ is an irreducible generic discrete series of M.

Proof. — Assume that Σ_σ is irreducible of rank d and let $\Delta_\sigma := \{\alpha_1,\ldots,\alpha_d\}$ be the basis of Σ_σ (following our choice of basis for the root system of G) and let us denote $\mathscr{T}$ its type.

We consider maximal standard Levi subgroups of G, $M \supset M_1$, such that the root system Σ_σ^M is irreducible. Typically, $M = M_{\Delta-\{\underline{\beta_d}\}}$.

Now, in our setting, σ_ν is a residual point for μ^M. It is in the cuspidal support of the generic discrete series τ if and only if (applying Proposition 4.6): $\mathrm{rk}(\Sigma_\sigma^M) = d-1$. Let us denote $(\nu_2,\ldots,\nu_d)$ the residual segment corresponding to the irreducible generic discrete series τ of M.

If $(\nu_2,\ldots,\nu_d)$ is a residual segment of type A to obtain a residual segment $(\nu_1,\nu_2,\ldots,\nu_d)$ of rank d and type:

- D: We need $\nu_d = 0$ and $\nu_1 = \nu_2 + 1$.
- B: We need $\nu_d = 1$ and $\nu_1 = \nu_2 + 1$.
- C: We need $\nu_d = 1/2$ and $\nu_1 = \nu_2 + 1$.

If $(\nu_2,\ldots,\nu_d)$ is a residual segment of type $\mathscr{T}$ (B, C, D) we need $\nu_1 = \nu_2+1$ to obtain a residual segment of type $\mathscr{T}$ and rank d.

In all these cases, the twist $s\tilde{\alpha}$ corresponds on the cuspidal support to adding one element on the left to the residual segment $(\nu_2, \dots, \nu_d)$; then the segment $(\nu_1, \nu_2, \dots, \nu_d) := (\lambda_1, \lambda_2, \dots, \lambda_d)$ is a residual segment:

$$\pi_0 \leq I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda).$$

This is equivalent to saying that σ_λ is a *dominant* residual point, and, therefore, by Lemma 6.1 π_0 embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$, and, therefore, in $I_P^G(\tau_{s\tilde{\alpha}})$ by [U] in the standard module. □

7.2. Not necessarily maximal parabolic subgroups. — In the course of the main theorem in this section, we will need the following result.

LEMMA 7.6. — *Let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t$ be t unlinked linear segments with $\mathcal{S}_i = (\mathscr{a}_i, \dots, \mathscr{b}_i)$ for any i. If*

$$(\mathscr{a}_1, \dots, \mathscr{b}_1)(\mathscr{a}_2, \dots, \mathscr{b}_2) \dots (\mathscr{a}_t, \dots, \mathscr{b}_t)(\underline{n})$$

is a residual segment $(\underline{n'})$, then at least one segment $(\mathscr{a}_i, \dots, \mathscr{b}_i)$ merges with $(\underline{n})$ to form a residual segment $(\underline{n''})$.

Proof. — Consider the case of t unlinked segments, with at least one disjoint from the others. We aim to prove that this segment can be inserted into $(\underline{n})$ independently of the others to obtain a residual segment. For each such (disjoint from the others) segment $(\mathscr{a}_i, \dots, \mathscr{b}_i)$ inserted, the following conditions are satisfied:

$$\begin{cases} n'_{\mathscr{a}_i+1} = n_{\mathscr{a}_i+1} = n'_{\mathscr{a}_i} - 1 = n_{\mathscr{a}_i} + 1 - 1 \\ n'_{\mathscr{b}_i} = n_{\mathscr{b}_i} + 1 = n_{\mathscr{b}_i - 1} - 1 + 1 = n_{\mathscr{b}_i-1} = n'_{\mathscr{b}_i-1}. \end{cases} \tag{18}$$

The relations $n'_{\mathscr{a}_i+1} = n_{\mathscr{a}_i+1}$ and $n'_{\mathscr{b}_i-1} = n_{\mathscr{b}_i-1}$ come from the fact that the elements $(\mathscr{a}_i + 1)$ and $(\mathscr{b}_i - 1)$ cannot belong to any other segment unlinked to $(\mathscr{a}_i, \dots, \mathscr{b}_i)$. If, for any i, those conditions are satisfied, $(\underline{n'})$ is a residual segment, by hypothesis.

Now, let us choose a segment that does not contain zero: $(\mathscr{a}_j, \mathscr{b}_j)$. Since by the Equation (18) $n_{\mathscr{a}_j+1} = n_{\mathscr{a}_j}$ and $n_{\mathscr{b}_j} = n_{\mathscr{b}_j-1} - 1$, adding only $(\mathscr{a}_j, \dots, \mathscr{b}_j)$ yields equations as (18) and, therefore, a residual segment.

If this segment contains zero and is disjoint from the others, then adding all segments or just this one yields the same results on the numbers of zeroes and ones: $n'_0 = n''_0$, $n'_1 = n''_1$; therefore, there is no additional constraint under these circumstances.

Secondly, let us consider the case of a chain of inclusions, which, without loss of generality, we denote $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \dots \supset \mathcal{S}_t$. Starting from $(\underline{n'})$, observe that adding the t linear residual segments yields the following conditions:

$$n'_{\mathscr{a}_i+1} = n_{\mathscr{a}_i+1} + i - 1 = n'_{\mathscr{a}_i} - 1 = n_{\mathscr{a}_i} + i - 1,$$
$$n'_{\mathscr{b}_i} = n_{\mathscr{b}_i} + i = n_{\mathscr{b}_i-1} - 1 + i = n'_{\mathscr{b}_i-1}.$$

Then, for any i, we clearly observe $n_{a_i+1} = n_{a_i}$; and $n_{b_i} = n_{b_i-1} - 1$. Assume that we only add the segment $(a_1, \ldots, b_1)$; then we observe $n''_{a_1+1} = n''_{a_1} - 1$ and $n''_{b_1} = n''_{b_1-1}$, satisfying the conditions for $(\underline{n''})$ to be a residual segment.

Assume that $\mathcal{S}_t$ contains zero; then any $\mathcal{S}_i$ also does. Assume that there is an obstruction at zero to form a residual segment when adding $t-1$ segments. If adding only $t-1$ zeroes does not form a residual segment, but t zeroes do, we have $n'_0 = \frac{n_1}{2}$. Then $n_0 + t = \frac{n_1}{2} + t = \frac{n_1+2t}{2}$ (the option $n'_1 = n_1 + 2t + 1$ is immediately excluded since there is at most two '1s' per segment $\mathcal{S}_i$).

We need to add $2t$ times '1'. Then we need at least $2t-1$ times '2' and $2t-2$ times '3' , ..., etc. Since $n'_1 = n_1 + 2t$, all $\mathcal{S}_i$'s will contain $(1 0 -1)$. There is no obstruction at zero while adding solely $\mathcal{S}_1$ (i.e. $n_0 + 1 = \frac{n_1+2}{2}$) and since $\mathcal{S}_1 \supset \mathcal{S}_2 \ldots \supset \mathcal{S}_t$ and $\mathcal{S}_1$ needs to contain $a_1 \geq \ell + m$, $\mathcal{S}_1$ can merge with $(\underline{n})$ to form a residual segment.

Finally, it would be possible to observe the case of a residual segment $\mathcal{S}_1$ containing $\mathcal{S}_2$ and $\mathcal{S}_3$ with $\mathcal{S}_2$ and $\mathcal{S}_3$ disjoint (or two – or more – disjoint chains of inclusions). Again, we have:

$$n'_{a_1+1} = n_{a_1+1} = n'_{a_1} - 1 = n_{a_1} + 1 - 1.$$

Assume that we only add the segment $(a_1, \ldots, b_1)$; then we observe $n''_{a_1+1} = n''_{a_1} - 1$ and $n''_{b_1} = n''_{b_1-1}$, satisfying the conditions for $(\underline{n''})$ to be a residual segment. □

Remark 7.7. — We show in this remark that if $s_i = \frac{a_i+b_i}{2} = s_j = \frac{a_j+b_j}{2}$, the linear segments $(a_i, \ldots, b_i)$ with $a_i > b_i$ and (a_j, b_j) with $a_j > b_j$ are such that one of them is included in the other (therefore unlinked).

If the length of the segments are the same, they are equal. Without loss of generality, let us consider the following case of different lengths:

$$a_i - b_i + 1 > a_j - b_j + 1. \tag{19}$$

Since $\frac{a_i+b_i}{2} = \frac{a_j+b_j}{2}$, $a_i + b_i = a_j + b_j$ and from Equation (19) $a_i - a_j > b_i - b_j$ replacing b_i by $a_j + b_j - a_i$, and further a_i by $a_j + b_j - b_i$, we obtain:

$$a_j + b_j - b_i - a_j > b_i - b_j \Leftrightarrow b_j > b_i$$

Therefore, $a_i > a_j > b_j > b_i$.

Consequently, the content of the proofs of the next theorem (Theorem 7.8), when considering the case of equal parameters $s_i = s_j$, remain the same.

Theorem 7.8. — *Let us assume σ_ν is in the cuspidal support of a generic discrete series representation τ of a standard Levi subgroup M of a quasi-split reductive group G. Let us assume that the cuspidal support of τ satisfies the conditions (CS) (see Definition 6.2). Let us take $\underline{s}$ in $\overline{(a^*_M)^+}$, such that $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$ and denote $\lambda = \nu + \underline{s}$ in $\overline{a_{M_1}^{M}}^{+*}$. Let us assume that*

σ_λ is a residual point for μ^G. Let π_0 be an irreducible generic discrete series representation of G, which is a subquotient in $I_P^G(\tau_{\underline{s}})$. Then, the unique irreducible generic square-integrable subquotient, π_0, in the standard module $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation.

Proof. — Let us assume that Σ_σ^M is a disjoint union of t subsystems of type A and a subsystem of type $\mathscr{T}$. Let $\underline{s} = (s_1, s_2, \ldots, s_t)$ be ordered such that $s_1 \geq s_2 \geq \ldots \geq s_t \geq 0$ with $s_i = \frac{a_i + b_i}{2}$, for two (half)-integers $a_i \geq b_i$.

Using the depiction of residual points in Section 4.3, we write the residual point

$$\sigma\left(\bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n})\right) \quad \text{where } \lambda \text{ reads } \bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n}).$$

Let us denote the linear residual segments $(a_i, \ldots, b_i) := \mathcal{S}_i$ and assume that for some indices $i, j \in \{1, \ldots, t\}$, the segments $\mathcal{S}_i, \mathcal{S}_j$ are linked. By Lemma 5.14 there exists an intertwining operator with a non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}_1', \mathcal{S}_2', \ldots, \mathcal{S}_t'; \underline{n})))$ to $I_{P_1}^G(\sigma(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t; \underline{n}))$. Therefore, if we prove that the unique irreducible discrete series subquotient appears as a subrepresentation in $I_{P_1}^G(\sigma(\mathcal{S}_1', \mathcal{S}_2', \ldots, \mathcal{S}_t'; \underline{n}))$, it will consequently appears as a subrepresentation in $I_{P_1}^G(\sigma(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t; \underline{n}))$. This means that we are reduced to the case of the cuspidal support σ_λ being constituted by t *unlinked* segments.

Further, notice that by the above Remark 7.7, when $s_i = s_j$, the segments $\mathcal{S}_i$, and $\mathcal{S}_j$ are *unlinked.* This allows us to treat the case $s_1 = s_2 = \ldots = s_t > 0$ and $s_1 > s_2 = \ldots = s_t = 0$.

So let us assume that all linear segments $(a_i, \ldots, b_i)$ are unlinked.

We prove the theorem by induction on the number t of linear residual segments.

First, $t = 0$, let $P_0 = G$ and π be the generic irreducible square integrable representation corresponding to the dominant residual point $\sigma_\lambda := \sigma(\underline{n}_{\pi_0})$,

$$I_{P_0}^G(\pi) \hookrightarrow I_{P_1}^G(\sigma(\underline{n}_{\pi_0})).$$

By Lemma 6.1, λ being in the closure of the positive Weyl chamber, the unique irreducible generic discrete series subquotient is necessarily a subrepresentation.

The proof of the step from $t = 0$ to $t = 1$ is Proposition 7.3.

Assume that the result is true for any standard module

$$I_{P'_{\Theta_{\leq t}}}^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n})\right)\right),$$

with t or less than t linear residual segments, where $P'_{\Theta_{\leq t}}$ is any standard parabolic subgroup whose Levi subgroup is obtained by removing t or less than t simple (non-extremal) roots from Δ.

We consider now π_0 the unique irreducible generic discrete series subquotient in

$$I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}}) \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_{i=1}^{t}(a_i,\dots,b_i)(a_{t+1},\dots,b_{t+1})(\underline{n'})\right)\right).$$

To distinguish this from the case of a discrete series τ of P_{Θ_t}, we denote τ' the irreducible generic discrete series and s' in $a_{M_{\Theta_{t+1}}}{}^{*+}$.

Using Lemma 7.6, we know there is at least one linear segment with index $j \in [1, t+1]$ such that $(a_j,\dots,b_j)$ can be inserted in $(\underline{n'})$ to form a residual segment. Without loss of generality, let us choose this index to be $t+1$ (or else we use bijective intertwining operators on the unlinked segments to set $(a_j,\dots,b_j)$ in the last position). Then, there exists a Weyl group element w such that $w((a_{t+1},\dots,b_{t+1})(\underline{n'})) = (\underline{n})$ for a residual segment $(\underline{n})$.

Let $M_1 = M_\Theta$ with $\Theta = \bigcup_{i=1}^{s}\Theta_i$, for some $s > t$ and $M' = M_{\Theta'}$, where $\Theta' = \bigcup_{i=1}^{s-2}\Theta_i \cup \Theta_t \cup \{\underline{\alpha}_t\} \cup \Theta_{t+1}$, if we assume (by convention) that the root $\underline{\alpha}_t$ connects the two connected components Θ_t and Θ_{t+1}.

Since $M' \cap P$ is a maximal parabolic subgroup in M', we can apply the result of Proposition 7.3 to π' the unique irreducible discrete series subquotient in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1}, b_{t+1})(\underline{n'}))$.

Notice that $\Sigma^{M'}$ is a reducible root system, and, therefore, so is $\Sigma_\sigma^{M'}$. This is because we choose an irreducible component of $\Sigma^{M'}$ so that we can apply the result of Proposition 7.3. It appears as a subrepresentation in $I_{P_1\cap M'}^{M'}(\sigma(\underline{n}))$.

Then, since the parameter $\bigoplus_{i=1}^{t}(a_i,\dots,b_i)$ corresponds to a central character χ for M', we have:

$$I_{P'}^G(\pi'_\chi) \hookrightarrow I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(\underline{n}))_{\bigoplus_{i=1}^{t}(a_i,\dots,b_i)}) \cong I_{P_1}^G\left(\sigma\left(\bigoplus_{i=1}^{t}(a_i,\dots,b_i)(\underline{n})\right)\right).$$

By Proposition 7.3, the subquotient π' appears as a subrepresentation in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},\dots,b_{t+1})(\underline{n'}))$, and, therefore, in the standard module embedded in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},\dots,b_{t+1})(\underline{n'}))$ by [U].

Since the parameter $\bigoplus_{i=1}^{t}(a_i,\dots,b_i)$ correspond to a central character for M', we have:

$$\begin{aligned}I_{P'}^G(\pi'_\chi) &\hookrightarrow I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(a_{t+1}, b_{t+1})(\underline{n'}))_{\bigoplus_{i=1}^{t}(a_i,\dots,b_i)})\\ &\cong I_{P_1}^G\left(\sigma\left(\bigoplus_{i=1}^{t}(a_i,\dots,b_i)(a_{t+1},\dots,b_{t+1})(\underline{n'})\right)\right).\end{aligned}$$

We, therefore, have two options: Either $I_{P'}^G(\pi'_\chi)$ is irreducible and then it is the unique irreducible generic subrepresentation in

$$I_{P'}^G(I_{P_1\cap M'}^{M'}\left(\sigma\left(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})\right)\right),$$

$$= I_{P_1}^G\left(\sigma\left(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})\right)\right),$$

and by multiplicity 1 in $I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}})$. Otherwise, it is reducible, but then its unique irreducible generic subquotient is also the unique irreducible generic subquotient in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\ldots,b_i)(\underline{n})))$.

Then, by induction hypothesis, it embeds as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\ldots,b_i)(\underline{n})))$; and by [U], also in $I_{P'}^G(\pi'_\chi)$. Hence, it embeds in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})))$, and, therefore, in $I_{P_{\Theta_{t+1}}}^G(\tau'_{s'})$, which concludes this induction argument and the proof. □

7.3. Proof of the generalized injectivity conjecture for non-discrete series subquotients. — We could have $I_P^G(\tau_{s\tilde{\alpha}})$ reducible without having hypothesis 1 in Lemma 7.2 satisfied, that is, without having $s\tilde{\alpha}$ a pole of the μ function for τ; i.e. the converse of the Lemma 7.2 does not necessarily hold.

It is only in this case that a non-tempered or tempered (but not square-integrable) generic subquotient may occur in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Proposition 7.9. — *Let σ_ν be in the cuspidal support of a generic discrete series representation τ of a maximal Levi subgroup M of a quasi-split reductive group G. Let us take $s\tilde{\alpha}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and denote $\lambda = \nu + s\tilde{\alpha}$ in $\overline{a_{M_1}^{M}}^{+*}$.*

Let us assume that the cuspidal support of τ satisfies the conditions (CS) (see Definition 6.2).

Let us assume that σ_λ is not a residual point for μ^G, and, therefore, the unique irreducible generic subquotient in $I_P^G(\tau_{s\tilde{\alpha}})$ is essentially tempered but not square integrable or not essentially tempered.

Then, this unique irreducible generic subquotient embeds as a submodule in $I_{P_1}^G(\sigma_\lambda)$ and, therefore, in the standard module $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof. — First, notice that if $s = 0$, the induced module $I_P^G(\tau_{s\tilde{\alpha}})$ is unitary, and, hence, any irreducible subquotient is a subrepresentation. In the rest of the proof we can, therefore, assume $s\tilde{\alpha}$ in $(a_M^*)^+$.

Let us denote π_0 the irreducible generic tempered or non-tempered representation that appears as a subquotient in a standard module $I_P^G(\tau_{s\tilde{\alpha}})$ induced from a maximal parabolic subgroup P of G.

We are in the context of Section 4.3 and, therefore, we can write $\lambda := (a, \dots, b) + (\underline{n})$, for some $a > b$, and residual segment $(\underline{n})$. Here, we assume σ_λ is not a residual point. Then $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma(a, b, \underline{n}))$ has a unique irreducible generic subquotient that is tempered or non-tempered.

Following the proof of Theorem 6.3 2(a) and 2(b), we can write this unique irreducible generic subquotient $I_{P'}^G(\tau'_{\nu'})$, which either embeds in an induced module that satisfies the conditions 2(a) or 2(b) of Theorem 6.3, and then we can conclude by [U]. This is the context of existence of an intertwining operator with a non-generic kernel between the induced module with cuspidal strings $(a', b', \underline{n'})$ and $(a, b, \underline{n})$.

Otherwise, one observes that passing from $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ requires certain elements γ, with $a \geq \gamma > a'$, to move up, i.e. from right to left. This means using rank 1 operators, which change $(\gamma + n, \gamma)$ to $(\gamma, \gamma + n)$ for integers $n \geq 1$, those rank 1 operators may clearly have generic kernels.

In this context, we will rather use the results of Proposition 7.3.

Consider again $I_{P'}^G(\tau'_{\nu'})$ embedded in $I_{P_1}^G(\sigma(a', b', \underline{n'}))$. Let us denote π' the unique irreducible generic discrete series subquotient corresponding to the dominant residual point $\sigma((\underline{n'}))$. Let $M'' = M_{\Delta - \{\underline{\alpha_1, \dots, \alpha_{a-b+1}}\}}$ be a standard Levi subgroup; we have:

$$\pi' \hookrightarrow I_{P_1 \cap M''}^{M''}((\sigma((\underline{n'}))).$$

Since the character corresponding to the linear residual segment $(a', \dots, b')$ is central for M'', we write:

$$\pi'_{(a', \dots, b')} \hookrightarrow I_{P_1 \cap M''}^{M''}(\sigma((a', \dots, b') + (\underline{n'}))) \cong I_{P_1 \cap M''}^{M''}(\sigma(\underline{n'}))_{(a', \dots, b')}.$$

Since $\tau'_{\nu'}$ is irreducible (and generic), we also have

$$\tau'_{\nu'} \hookrightarrow I_{P_1 \cap M'}^{M'}(\sigma((a', \dots, b') + (\underline{n'}))).$$

We know:

$$\tau'_{\nu'} \hookrightarrow I_{P''}^{M'}(\pi'_{(a', \dots, b')}) \hookrightarrow I_{P_1 \cap M'}^{M'}(\sigma((a', \dots, b') + (\underline{n'}))). \tag{20}$$

By the generalized injectivity conjecture for square-integrable subquotient (Proposition 7.3), any standard module embedded in $I_{P_1 \cap M''}^{M''}(\sigma((\underline{n'})))$ has π' as subrepresentation. We may, therefore, embed π' as a subrepresentation in

$$I_{P_1 \cap M''}^{M''}((w_\flat \sigma)((a^\flat, b^\flat, \underline{n}^\flat))),$$

with $w_\flat \sigma \cong \sigma$, and therefore inducing Equation (20) to G,

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P_1}^G((w_\flat \sigma)((a', \dots, b') + (a^\flat, b^\flat)(\underline{n}^\flat)).$$

The sequence $(a^\flat, b^\flat, \underline{n}^\flat)$ is chosen appropriately to have an intertwining operator with a non-generic kernel from $I_{P_1}^G(\sigma((a', \dots, b') + (a^\flat, b^\flat, \underline{n}^\flat))$ to $I_{P_1}^G(\sigma(a, b, \underline{n}))$.

The unique irreducible generic subrepresentation $I_{P'}^G(\tau'_{\nu'})$ in $I_{P_1}^G(\sigma(\mathscr{a}, \mathscr{b}, \underline{n}))$ cannot appear in the kernel and, therefore, appears in the image of this operator. It therefore appears as a subrepresentation in $I_{P_1}^G(\sigma(\mathscr{a}, \mathscr{b}, \underline{n}))$ and concluded by [U]. □

Theorem 7.10. — *Let σ_ν be in the cuspidal support of a generic discrete series representation τ of a standard Levi subgroup M of a quasi-split reductive group.*

Let us take $\underline{s}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$ and denote $\lambda = \nu + \underline{s}$ in $\overline{a_{M_1}^M}^{+*}$. Let us assume that σ_λ is not a residual point for μ^G and that the unique irreducible generic subquotient satisfies the conditions (CS) (see Definition 6.2).

Then, the unique irreducible generic in $I_P^G(\tau_{\underline{s}})$ (which is essentially tempered or non-tempered) embeds as a subrepresentation in $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof. — First, notice that by Remark 7.7, when $s_i = s_j$, the segments $\mathcal{S}_i$, and $\mathcal{S}_j$ are *unlinked.* Using the argument given in Section 4.3, we write σ_λ as $\sigma(\bigoplus_{i=1}^t(\mathscr{a}_i, \dots, \mathscr{b}_i)(\underline{n}))$, where λ reads $\bigoplus_{i=1}^t(\mathscr{a}_i, \dots, \mathscr{b}_i)(\underline{n})$.

The proof goes along the same inductive line as in the proof of Proposition 7.8.

The case of $t = 1$ is Proposition 7.9. That is, given a cuspidal support (P_1, σ_λ), for any standard module induced from a maximal parabolic subgroup P: $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$, the unique irreducible generic subquotient is a subrepresentation. We use an induction argument on the number t of linear residual segments obtained when removing t simple roots to define the Levi subgroup $M \subset P$. Considering that an essentially tempered or non-tempered irreducible generic subquotient in a standard module with t linear residual segments $I_{P_{\Theta_t}}^G(\tau_{\underline{s}})$ is necessarily a subrepresentation, one uses the same arguments as in the proof of Theorem 7.8 to conclude that a tempered or non-tempered irreducible generic subquotient in a standard module with $t+1$ linear residual segments $I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}})$ is a subrepresentation, therefore proving the theorem. □

Eventually, we now consider the generic subquotients of $I_P^G(\gamma_{s\tilde{\alpha}})$ when γ is a generic irreducible tempered representation.

Corollary 7.11 (Standard modules). — *Let G be a quasi-split reductive group of type A, B, C or D and let us assume that Σ_σ is irreducible.*

The unique irreducible generic subquotient of $I_P^G(\gamma_{\underline{s}})$ when γ is a generic irreducible tempered representation of a standard Levi M is a subrepresentation.

Proof. — Let $P = MU$. By Theorem 6.10, as a tempered representation of M, γ appears as a subrepresentation of $I_{P_3 \cap M}^M(\tau)$ for some discrete series τ and standard parabolic $P_3 = M_3U$ of G; τ is a generic irreducible representation of the Levi subgroup M_3, and, therefore,

$$I_P^G(\gamma_{\underline{s}}) \hookrightarrow I_P^G(I_{M\cap P_3}^M(\tau))_{\underline{s}} \cong I_{P_3}^G(\tau_{\underline{s}}),$$

where P_3 is not necessarily a maximal parabolic subgroup of G. Since $\underline{s}$ is in $(a_M^*)^+$, $\underline{s}$ is in $\overline{(a_{M_3}^*)^+}$. Let us write this parameter $\overline{\underline{s}}$ when it is in $\overline{(a_{M_3}^*)^+}$.

The unique irreducible generic subquotients of $I_P^G(\gamma_{\underline{s}})$ are the unique irreducible generic subquotients of $I_{P_3}^G(\tau_{\overline{\underline{s}}})$, where $\overline{\underline{s}}$ is in $\overline{(a_{M_3}^*)^+}$. Since P_3 is not a maximal parabolic subgroup of G, we may now use Theorems 7.8 and 7.10 with $\overline{\underline{s}}$ in $\overline{(a_{M_3}^*)^+}$ to conclude that these unique irreducible generic subquotients, whether square-integrable or not, are subrepresentations. □

8. The case Σ_σ reducible

Let us recall that the set Σ_σ is a root system in a subspace of $a_{M_1}^*$ (cf. [35], 3.5). We assume that the irreducible components of Σ_σ are all of type A, B, C or D. In Proposition 4.6, we denoted for each irreducible component $\Sigma_{\sigma,i}$ of Σ_σ, by $a_{M_1}^{M_i*}$ the subspace of $a_{M_1}^{G*}$ generated by $\Sigma_{\sigma,i}$, by d_i its dimension and by $e_{i,1},\dots,e_{i,d_i}$ a basis of $a_{M_1}^{M_i*}$ (or of a vector space of dimension d_i+1 containing $a_{M_1}^{M_i*}$ if $\Sigma_{\sigma,i}$ is of type A), so that the elements of the root system $\Sigma_{\sigma,i}$ are written in this basis as in the work of Bourbaki, [4].

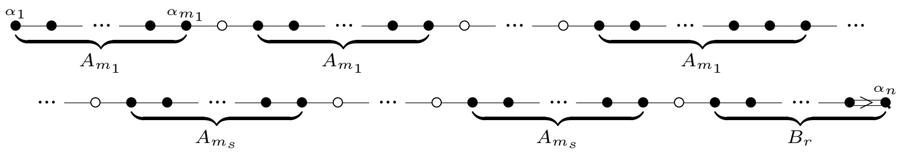
The following result is analogous to Proposition 1.10 in [17]. Recall that $\mathcal{O}$ denotes the set of equivalence classes of representations of the form $\sigma\otimes\chi$, where χ is an unramified character of M_1.

Proposition 8.1. — *Let $P_1' = M_1U_1'$, and $P_1 = M_1U_1$. If the intersection of $\Sigma(P_1)\cap\Sigma(\overline{P_1'})$ with Σ_σ is empty, the operator $J_{P_1'|P_1}$ is well defined and bijective on $\mathcal{O}$.*

Proof. — The operator $J_{P_1'|P_1}$ is decomposed in elementary operators that come from intertwining operators relative to $(M_1)_\alpha$ with $\alpha\notin\Sigma_\sigma$, so it is enough to consider the case where P_1 is a maximal parabolic subgroup of G and $P_1' = \overline{P_1}$. Then, if $\alpha\notin\Sigma_\sigma$ and by the same reasoning as in the Lemma 6.11, the operator $J_{P_1'|P_1}$ is well defined and bijective at any point on $\mathcal{O}$. □

Let G, π_0 σ_λ, $\lambda\in a_{M_1}^*$ be defined as in the main Theorem 1.2.

In this section, we consider the case of a *reducible* root system Σ_σ. As explained in Appendix B, this case occurs in particular when Σ_Θ (see the notations in Appendix B) is reducible, and then Θ has connected components of type A of different lengths. An example is the following Dynkin diagram for Θ:



Let us assume that Θ is a disjoint union of components of type A_{m_i}, $i = 1 \dots s$ and $m_i \neq m_{i+1}$ for any i, where each component of type A_{m_i} appears d_i times. Set $m_i = k_i - 1$.

Let us denote $\Delta^i_{M_1} = \{\alpha_{i,1}, \dots, \alpha_{i,d_i}\}$ the non-trivial restrictions of roots in Σ, generating the set $a^{M^i}_{M_1}*$. Similarly to the case of Σ_σ irreducible, we may have $\Delta_{\sigma_i} = \{\alpha_{i,1}, \dots, \beta_{i,d_i}\}$, where β_{i,d_i} can be different from α_{i,d_i} in the case of type B, C or D. For any $i \neq s$, the pre-image of the root α_{i,d_i} is *not* simple.

Indeed, for instance, in the above Dynkin diagram, the first root 'removed' is $e_{k_1} - e_{k_1+1}$, the second is $e_{2k_1} - e_{2k_1+1}$, etc. They are simple roots, and their restrictions to A_{M_1} are roots of $\Delta^1_{M_1}$ (the generating set of $a^{M^1}_{M_1}*$); the last root to consider is $e_{k_1d_1} - e_{n-r+1}$, which restricts to $e_{k_1d_1}$; then the pre-image of $e_{k_1d_1}$ is not simple.

However, since $e_{n-r} - e_{n-r+1}$ restricts to e_{n-r}, the pre-image of α_{s,d_s} is simple.

The Levi subgroup M^i is defined such that $\Delta^{M^i} = \Delta^{M_1} \cup \{\underline{\alpha_{i,1}}, \dots, \underline{\alpha_{i,d_i}}\}$ where $\Delta^i_{M_1} = \{\alpha_{i,1}, \dots, \alpha_{i,d_i}\}$.

It is a *standard* Levi subgroup for $i = s$. This is quite an important remark since most of our results in the previous sections were conditional on having *standard* parabolic subgroups.

Furthermore, since $\Sigma_{\sigma,i}$ generates $a^{M^i}_{M_1}*$ and is of rank d_i, the semi-simple rank of M^i is $d_i + \mathrm{rk}_{ss}(M_1)$. Since $\Sigma_{\sigma,i}$ is irreducible, an equivalent of Proposition 6.4 is satisfied for M^i.

Proposition 8.2. — *Let π_0 be an irreducible generic representation of a quasi-split reductive group G and assume it is the unique irreducible generic subquotient in the standard module $I^G_P(\tau_{s\tilde{\alpha}})$, where M is a maximal Levi subgroup (and α is not an extremal simple root on the Dynkin diagram of Σ) of G, and τ is an irreducible generic discrete series of M. Let us assume that Σ_σ is reducible.*

Then π_0 is a subrepresentation in the standard module $I^G_P(\tau_{s\tilde{\alpha}})$.

Proof. — The proof starts with the setting of Section 3: τ is an irreducible generic discrete series of a maximal Levi subgroup, and by Heiermann–Opdam's result, $\tau \hookrightarrow I^M_{P_1\cap M}(\sigma_\nu)$, for $\nu \in (a^{M}_{M_1}{}^*)^+$. Then, ν is a residual point for μ^M.

Let us write $\Sigma_\sigma^M = \bigcup_{i=1}^{r+1} \Sigma_{\sigma,i}^M$, then the residual point condition is

$$\dim((a_{M_1}^M)^*) = \mathrm{rk}(\Sigma_\sigma^M) = \sum_{i=1}^{r+1} d_i^M,$$

where d_i^M is the dimension of $(a_{M_1}^{M^i})^*$ generated by $\Sigma_{\sigma,i}^M$. The residual point ν decomposes into $r+1$ disjoint residual segments: $\nu = (\nu_1, \ldots, \nu_{r+1}) := (\underline{n_1}, \underline{n_2}, \ldots, \underline{n_{r+1}})$.

Since Σ^M decomposes into two disjoint irreducible components, one of them being of type A, the restrictions of simple roots of this irreducible component of type A in Δ^M generates an irreducible component of Σ_σ of type A. Let us denote this A component $\Sigma_{\sigma,I}^M$ for $I \in \{1, \ldots, r+1\}$, $d_I = \mathscr{b} - \gamma$ and denote $\nu_I + s\tilde{\alpha} := (\mathscr{b}, \ldots, \gamma)$ the twisted residual segment of type A.

Let us further assume that there is one index j such that there exists a residual segment $(\underline{n_j'})$ of length $\mathscr{b} - \gamma + 1 + d_j$ and type $\mathscr{T}$ $(B, C$ or $D)$ in the W_σ-orbit of $(\mathscr{b}, \gamma)\underline{(n_j)}$, where the residual segment $(\underline{n_j})$ is of the same type as $\mathscr{T}$.

Since all intertwining operators corresponding to rank 1 operators associated to s_β for $\beta \notin \Delta_\sigma$ are bijective (see Lemma 6.11), all intertwining operators interchanging any two residual segments $(\underline{n_k})$ and $(\underline{n_{k'}})$ are bijective. Therefore, we can interchange the positions of all residual segments (or said differently, interchange the order of the irreducible components for $i = 1, \ldots, r+1$) and, therefore, set $(\mathscr{b}, \ldots, \gamma)(\underline{n_j})$ in the last position, i.e. we set $I = r, j = r+1$. This flexibility is quite powerful since it allows us to circumvent the difficulty arising with M^i not being standard for any $i \neq r$.

When adding the root α to Θ (when inducing from M to G), we form from the disjoint union $\Sigma_{\sigma,r}^M \bigcup \Sigma_{\sigma,r+1}^M$ the irreducible root system that we denote $\Sigma_{\sigma,r}$.

The Levi subgroup M^r is the smallest standard Levi subgroup of G containing M_1, the simple root α and the set of simple roots whose restrictions to A_{M_1} lie in $\Delta_{M_1}^r$. It is a group of semi-simple rank $d_r + \mathrm{rk}_{ss}(M_1)$. We may, therefore, apply the results of the previous sections with Σ_σ irreducible to this context. Let us assume first the unique irreducible generic subquotient π is discrete series. From the result of Heiermann–Opdam, we have:

$$\pi \hookrightarrow I_{P_1 \cap M^r}^{M^r}(\sigma(\underline{n_r'})),$$

where the residual segment $(\underline{n_r'})$ is the dominant residual segment in the W_σ-orbit of $(\mathscr{b}, \gamma, \underline{n_r})$. The unramified character χ corresponding to the remaining residual segments (n_k)'s, $k \neq r, r+1$ is a central character of M^r (since its expression in the $a_{M_1}^*$ is orthogonal to all the roots in Δ^{M^r}). Then:

$$\pi_\chi \hookrightarrow I_{P_1 \cap M^r}^{M^r}(\sigma(\underline{n_r'}))\bigoplus\nolimits_{j \neq r, r+1}(\underline{n_j})$$

As a result:

$$\pi_0 \hookrightarrow I_{P^r}^G(\pi_\chi) \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_{j\neq r}(\underline{n_j}) + (\underline{n'_r})\right)\right). \tag{21}$$

In Equation (21), we claim that π_0 embeds first in $I_{P_1}^G(\sigma(\bigoplus_{j\neq r}\underline{n_j}) + (\underline{n'_r})))$ by the Heiermann–Opdam embedding result (since the residual segment $\bigoplus_{j\neq r}(\underline{n_j})+(\underline{n'_r})$ corresponds to a parameter in $\overline{(a_{M_1}^*)^+}$), and therefore it should embed in $I_{P^r}^G(\pi_\chi)$ by [U].

Applying our conclusion in the case of an irreducible root system (in Proposition 7.3) to $\Sigma_{\sigma,r}$, we embed π in the induced module $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))$ as a subrepresentation (and, therefore, in a standard module $I_{P\cap M^r}^{M^r}(\tau_{\frac{\mathscr{b}+\gamma}{2}})$ embedded in $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r})))$,

$$\pi_\chi \hookrightarrow I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))_{\bigoplus_{j\neq r,r+1}(\underline{n_j})} \cong I_{P_1\cap M^r}^{M^r}\left(\sigma(\mathscr{b},\gamma,\underline{n_r}) + \bigoplus_{j\neq r,r+1}(\underline{n_j})\right).$$

Therefore:

$$\pi_0 \hookrightarrow I_{P^r}^G(\pi_\chi) \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_{j\neq r}(\underline{n_j}) + (\mathscr{b},\gamma,\underline{n_r})\right)\right).$$

When π is non-(essentially) square integrable, i.e. tempered or non-tempered, and embeds in $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b}',\gamma',\underline{n'_r}))$ (see the construction in Section 6.4, 2(a)), we showed in Proposition 7.9 that there exists an intertwining operator with a non-generic kernel sending π in $I_{P_1\cap M^r}^{M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))$. Since the other remaining residual segments (n'_k)'s, $k\neq r,r+1$ do not contribute when minimizing the Langlands parameter ν', the unique irreducible generic subquotient in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k}) + (\mathscr{b},\gamma,\underline{n_r})))$ embeds in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k}) + (\mathscr{b}',\gamma',\underline{n'_r})))$, and we can use the inducting of the previously defined (at the level of M^r) intertwining operator to send this generic subquotient as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k}) + (\mathscr{b},\gamma,\underline{n_r})))$. We conclude the argument as usual by [U]. □

Proposition 8.3. — *Let π_0 be an irreducible generic representation and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\tau_s)$, where the set of simple roots in M (Δ^M) is the set of simple roots Δ minus t simple roots, $s=(s_1,\ldots,s_t)$ such that $s_1\geq s_2\geq\ldots\geq s_t$, and τ is an irreducible generic discrete series.*

Then it is a subrepresentation.

Proof. — The representation τ is an irreducible generic discrete series of a non-maximal Levi subgroup M such that $I_P^G(\tau_s)$ is a standard module. By Heiermann-Opdam's result, $\tau\hookrightarrow I_{P_1\cap M}^M(\sigma_\nu)$, for $\nu\in\overline{(a_{M_1}^{M\,*})^+}$. Then, ν is a residual point for μ^M.

Let us denote $M = M_\Theta$. Then $\Theta = \bigcup_{i=1}^{t+1}\Theta_i$, where Θ_i, for $i\in\{1,\ldots,t\}$, is of type A.

Since M_1 is a standard Levi subgroup of G contained in M, we can write $\Sigma_\sigma^M = \bigcup_{i=1}^{t+r}\Sigma_{\sigma,i}^M$, and then the residual point condition is $\dim((a_{M_1}^M)^*) = \mathrm{rk}(\Sigma_\sigma^M) = \sum_{i=1}^{r+t} d_i^M$, where d_i^M is the dimension of $(a_{M_1}^{M^i})^*$ generated by $\Sigma_{\sigma,i}^M$. The residual point ν decomposes in t linear residual segments along with r residual segments: $\nu = (\nu_1,\ldots,\nu_{r+t}) := (\underline{n_1},\underline{n_2},\ldots,\underline{n_{r+t}})$.

Adding the twist $s=(s_1,\ldots,s_t)$ we obtain a parameter λ in $(a_{M_1}^G)^*$ composed of t twisted linear residual segments $\{(a_i,\ldots,b_i)\}_{i=1}^t$ and r residual segments $(\underline{n_{t+1}},\underline{n_{t+2}},\ldots,\underline{n_{t+r}})$.

Let us first assume that λ is a residual point.

This means all linear residual segments can be incorporated in the r residual segments of type $\mathscr{T}$ to form residual segments $\{(\underline{n'_j})\}_{j=1}^r$ of type $\mathscr{T}$ and length d_i such that $\sum_i d_i = d$, where d is $\mathrm{rk}_{ss}(G)-\mathrm{rk}_{ss}(M_1) = \dim a_{M_1} - \dim a_G$. It is also possible that, as twisted linear residual segments, they are already in a form as in Proposition 7.5. In that case, the linear residual segment need not be incorporated in any residual segment of type $\mathscr{T}$.

Furthermore, as in the proof of Theorem 7.8, we can reduce our study to the case of unlinked residual linear segments.

By Heiermann–Opdam's Proposition (2.7):

$$\pi_0 \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_j \underline{n'_j}\right)\right).$$

Let us consider the last irreducible component $\Sigma_{\sigma,r}$ of Σ_σ and the residual segment $(\underline{n'_r})$ associated to it.

Let us assume this irreducible subsystem is obtained from some subsystems $\Sigma_{\sigma,i}^M$ of type A denoted $A_q,\ldots,A_s$ and one of type $\mathscr{T}$ when inducing from M to G

$$\{A_q,\ldots,A_s\}\leftrightarrow\{\mathscr{T}\} \qquad \{(b_{r,q},\ldots,\gamma_{r,q}),\ldots,(b_{r,s},\ldots,\gamma_{r,s})\}\leftrightarrow\{(\underline{n_r})\}.$$

The Levi subgroup M^r is the smallest standard Levi subgroup of G containing M_1, s simple roots (among the t simple roots in $\Delta-\Theta$) and the set of roots whose restrictions to A_{M_1} lie in $\Delta_{M_1}^r$. It is a group of semi-simple rank $d_r + \mathrm{rk}_{ss}(M_1)$.

We may, therefore, apply the results of the previous sections with Σ_σ irreducible to this context: the unique irreducible generic discrete series, π, in the induced module $I_{P_1\cap M^r}^{M^r}(\sigma(\bigoplus_{j=q}^s(b_{r,j},\gamma_{r,j})+(\underline{n_r})))$ is a subrepresentation.

As in the proof of Proposition 8.2, since π also embeds in $I_{P_1\cap M^r}^{M^r}(\sigma(\underline{n'_r}))$, when we add the twist by the central character corresponding to $\bigoplus_{k\neq r}\underline{(n'_k)}$, we obtain:

$$\pi_0 \hookrightarrow I_P^G(\pi_\chi) \hookrightarrow I_{P^r}^G\left(I_{P_1\cap M^r}^{M^r}\left(\sigma\left(\bigoplus_{j=k}^{s}(\mathscr{b}_{r,j},\ldots,\gamma_{r,j}) + (\underline{n_r})\right)_{\bigoplus_{k\neq r,r+1}(\underline{n'_k})}\right)\right).$$

When π is non-tempered and embeds (as a subrepresentation) in $I_{P_1\cap M^r}^{M^r}((\sigma(\mathscr{b}',\gamma',\underline{n''_r}))$, we showed in Proposition 7.9 that there exists an intertwining operator with a non-generic kernel sending π in $I_{P_1\cap M^r}(\sigma(\mathscr{b},\gamma,\underline{n_r}))$.

Since the other remaining residual segments (n'_k)'s, $k\neq r$ do not contribute when minimizing the Langlands parameter ν', the unique irreducible generic subquotient in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k}) + (\mathscr{b},\gamma,\underline{n_r})))$ embeds in $I_{P^r}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k}) + (\mathscr{b}',\gamma',\underline{n'_r})))$, and we can use the inducting of the previously defined intertwining operator to send this generic subquotient as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n'_k}) + (\mathscr{b},\gamma,\underline{n_r})))$.

Then

$$\pi_0 \hookrightarrow I_{P^r}^G(\pi_\chi) \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_{k\neq r}(\underline{n'_k}) + \bigoplus_{j=q}^{s}(\mathscr{b}_{r,j},\gamma_{r,j}) + (\underline{n_r})\right)\right).$$

We conclude the argument with [U] as usual.

Using bijective intertwining operators, we now reorganize this cuspidal support so as to put the linear residual segments $\bigoplus_{j=q}^{s}(\mathscr{b}_{r,j},\gamma_{r,j})$ on the left-most part and $\Sigma_{\sigma,r-1}$ on the right-most part. The residual segment $(\underline{n'_{r-1}})$ is (possibly) again formed of some linear residual segments $(\mathscr{b}_i,\gamma_i)$ and the residual segment $(\underline{n_{r-1}})$. We argue just as above. Since the linear residual segments are unlinked, we can reorganize them so as to ensure $s_1\geq s_2\geq \ldots s_t$.

Eventually, repeating this procedure,

$$\pi_0 \hookrightarrow I_{P_1}^G\left(\sigma\left(\bigoplus_{i=1}^{t}(\mathscr{b}_i,\gamma_i) + \bigoplus_{j=1}^{r}(\underline{n_j})\right)\right).$$

Further, by [U] the generic piece also embeds as a subrepresentation in the standard module. □

Corollary 8.4. — *Let π_0 be an irreducible generic representation of G and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\gamma_{\underline{s}})$, where M is a standard Levi subgroup of G. Let us assume that Σ_σ is reducible.*

Then it is a subrepresentation.

Proof. — Let $P=MU$. We argue as in the Corollary 7.11: using Theorem 6.10, the tempered representation of M, γ, appears as a subrepresentation of $I_{P_3\cap M}^M(\tau)$ for some discrete series τ and standard parabolic subgroup $P_3=M_3U$ of G; τ is a generic irreducible representation of the standard Levi subgroup

M_3, and, therefore,

$$I_P^G(\gamma_{\underline{s}}) \hookrightarrow I_P^G(I_{M\cap P_3}^M(\tau))_{\underline{s}} \cong I_{P_3}^G(\tau_{\underline{s}}),$$

where P_3 is not necessarily a maximal parabolic subgroup of G.

Since $\underline{s}$ is in $(a_M^*)^+$, $\underline{s}$ is in $\overline{(a_{M_3}^*)^+}$. Let us write this parameter $\overline{\underline{s}}$ when it is in $\overline{(a_{M_3}^*)^+}$.

The unique irreducible generic subquotients of $I_P^G(\gamma_{\underline{s}})$ are the unique irreducible generic subquotients of $I_{P_3}^G(\tau_{\overline{\underline{s}}})$, where $\overline{\underline{s}}$ is in $\overline{(a_{M_3}^*)^+}$. Since P_3 is not a maximal parabolic subgroup of G, we use the result of Proposition 8.3. □

9. Exceptional groups

The arguments developed in the context of reductive groups whose roots systems are of classical type may apply in the context of exceptional groups provided the set $W(M_1)$ is equal to the Weyl group W_σ or differ by one element as in Corollary 6.6. However, this hypothesis shall not necessarily be satisfied, as the E_8 Example 5.3.3 in [35] illustrates. In this example, where W_σ, the Weyl group of Σ_σ, is of type D_8, it shall be rather different from $W(A_0)$.

In an auxiliary work [12], we observed that in most cases where a root system of rank $d = \dim(a_{M_1}^*/a_G^*)$ occurs in Σ_Θ, it is of type A or D, or of very small rank (such as in F_4). Further, the main result of [12] (Theorem 2) is that only classical root systems occur in Σ_Θ, except when G is of type E_8, and Θ contains one (any) root of E_8.

This latter case along with the case of $\Theta = \emptyset$ (in the context of exceptional groups), $\Sigma_\Theta = \Sigma$, $M_1 = P_0 = B$ and σ a generic irreducible representation of P_0 (in particular the case of trivial representation σ) shall be treated in an independent work since the combinatorial arguments given in this work do not apply as easily.

Furthermore, it might be necessary for the case E_8 and Θ containing only one root to obtain a result analogous to Proposition 4.6, *which includes the exceptional root systems.* This would allow us to use weighted Dynkin diagrams (of exceptional type) to express the coordinates of residual points.

1. Let us assume that Σ_Θ contains Σ_σ of type A and the basis of Δ_Θ contains at least two projections of simple roots in Δ: α and β. Let us assume that the standard module is $I_P^G(\tau_{s\tilde{\alpha}})$, such that τ is a discrete series of M, and $\Delta_M = \Delta - \{\alpha\}$. The proof of the generalized injectivity conjecture for Σ_σ of type A (see [11]) carries over this context if the Levi M' given there is such that $\Delta_{M'} = \Delta - \{\beta\}$, and one should pay attention to the choice of (the order of simple roots in the) basis Δ_Θ to ensure that the parameter ν' for the root system $\Sigma_\sigma^{M'}$ splits into two residual segments appropriately (hence, also an appropriate choice of M determining Σ_σ^M). Let us simply recall that from the Lemmas 5.16

and 5.7, we know that if there is an embedding of the irreducible generic subquotient $I_{P'}^G(\tau'_{\underline{s'}})$ into $I_{P'_1}^G(\sigma'_{\lambda'})$, the parameter λ' is in the W_σ-orbit of λ, and hence $M'_1 = w \cdot M_1 = M_1$ and $\sigma' = w \cdot \sigma = \sigma$ since $w \in W_\sigma$.

2. Under the assumption that $W(M_1)$ equals W_σ or $W(M_1) = W_\sigma \cup \{s_{\beta_d} W_\sigma\}$ (see Corollary 6.6) and $s\beta_d\lambda = \lambda$, the cases where Σ_σ is irreducible of type D_d in Σ_Θ can be dealt with using the methods proposed in this work.

Proposition 9.1. — *Let G be a quasi-split reductive group of exceptional type, Σ its root system and Δ a basis of Σ. Let P be a standard parabolic subgroup $P = MU$ of G.*

Let us consider $I_P^G(\tau_{\underline{s}})$ with τ an irreducible discrete series of M, $\underline{s} \in (a_M^*)^+$. Let σ be a unitary cuspidal representation of M_1 in the cuspidal support of τ and assume Σ_σ (defined with respect to G) is of type A and irreducible of rank $d = \mathrm{rk}_{ss}(G) - \mathrm{rk}_{ss}(M_1)$. Further assume that Δ_σ contains at least two restrictions of simple roots in Δ.

Then, the unique irreducible generic subquotient of $I_P^G(\tau_{\underline{s}})$ is a subrepresentation.

9.1. Generalized injectivity in G_2. —

Theorem 9.2. — *Let G be of type G_2. Let π_0 be the unique irreducible generic subquotient of a standard module $I_P^G(\tau_s)$, then it is a subrepresentation.*

We follow the parametrization of the root system of G_2 as in Muić [25]: α is the short root and β the long root. We have $M_\alpha \cong \mathrm{GL}_2$, $M_\beta \cong \mathrm{GL}_2$. Without loss of generality, let us assume τ is a discrete series representation of $M = M_\alpha$; the reasoning is the same for M_β. As τ is a discrete series for GL_2, $\tau = \mathrm{St}_2$.

$$\tau \hookrightarrow I_B^{M_\alpha}(|\cdot|^{1/2}|\cdot|^{-1/2}).$$

We twist τ with $s\tilde{\alpha}$; $\tau_{s\tilde{\alpha}} \hookrightarrow I_B^{M_\alpha}(|\cdot|^{1/2}|\cdot|^{-1/2}) \otimes |\cdot|^s$,

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_B^G(|\cdot|^{s+1/2}|\cdot|^{s-1/2}).$$

Conjecturally for two values of s (since there are only two weighted Dynkin diagrams conjecturally in bijection with dominant residual points) we obtain a dominant residual point of type G_2. Since they are dominant residual points, the unique generic subquotient in $I_B^G(|\cdot|^{s+1/2}|\cdot|^{s-1/2})$ is a subrepresentation and, therefore, appears as subrepresentation in $I_P^G(\tau_{s\tilde{\alpha}})$.

Suppose the value of s is such that $(s+1/2, s-1/2)$ is not a dominant residual point. The set-up considered is that of $\mathrm{St}_2 \hookrightarrow I_B^M(|\cdot|^{1/2}|\cdot|^{-1/2})$ twisted by $|\cdot|^s$ so that it embeds in $I_B^M(|\cdot|^{s+1/2}|\cdot|^{s-1/2})$. Then, $I_P^G(\mathrm{St}_2|\cdot|^s) \hookrightarrow I_B^G(|\cdot|^{s+1/2}|\cdot|^{s-1/2})$. Using the result of Casselman–Shahidi (generalized injectivity conjecture for cuspidal inducing data) it is clear that the generic irreducible subquotient in $I_B^G(|\cdot|^{s+1/2}|\cdot|^{s-1/2})$ embeds as a subrepresentation.

9.1.1. *The case of a non-discrete series-induced representation.* — We now consider the general case of a standard module, with τ a tempered representation of $M \cong \mathrm{GL}_2$. As an irreducible tempered representation of GL_2, $\tau \cong I_B^{\mathrm{GL}_2}(\mathbf{1} \otimes \mathbf{1})$. Then the standard module is $I_P^G(\tau_s) \cong I_P^G(I_B^{\mathrm{GL}_2}(\mathbf{1} \otimes \mathbf{1}) \otimes |\cdot|^s) \cong I_B^G((\mathbf{1} \otimes \mathbf{1}) \otimes |\cdot|^s) = I_B^G(|\cdot|^s |\cdot|^s)$. Since $I_B^G(\mathbf{1} \otimes \mathbf{1})$ is unitary, its unique generic subquotient is itself, and there is nothing to prove.

9.1.2. *Residual segments.* — As an aside, we compute the residual segments of type G_2 here. The weighted Dynkin diagrams for G_2 are:

Let $\lambda = (\lambda_1, \lambda_2)$ means that $\lambda = \lambda_1(2\alpha + \beta) + \lambda_2(\alpha + \beta)$. On the other hand, it is known that

$$\langle 2\alpha + \beta, \alpha^\vee \rangle = 1,\ \langle \alpha + \beta, \alpha^\vee \rangle = -1,\ \langle 2\alpha + \beta, \beta^\vee \rangle = 0,\ \langle \alpha + \beta, \beta^\vee \rangle = 1. \tag{22}$$

From the first weighted Dynkin diagram above, the parameter λ satisfies:

$$\langle \lambda, \alpha^\vee \rangle = 1, \quad \langle \lambda, \beta^\vee \rangle = 1.$$

From the above relations 22, one should be able to compute that the residual segment is $\lambda = (2, 1)$.

In the second weighted Dynkin diagram, the parameter λ satisfies:

$$\langle \lambda, \alpha^\vee \rangle = 0, \quad \langle \lambda, \beta^\vee \rangle = 1.$$

Using the above relations 22, we conclude that the residual segment is $(1, 1)$.

Appendix A. Bala–Carter theory

This short appendix is written with considerably more detail in the author's PhD thesis [11].

Let $\mathcal{N} = \mathcal{N}_\mathfrak{g}$ be the cone of nilpotent elements in $\mathfrak{g}$. This cone is the disjoint union of a finite number of G-orbits. Let $\mathcal{O}$ be a nilpotent orbit in $G \backslash \mathcal{N}$ and let $x \in \mathcal{O}$ be a representative element. A theorem of Jacobson–Morozov extends x to a standard ($\mathfrak{sl}_2$) triple $\{e, h, f\} \in \mathfrak{g}$, where h can be chosen to lie in the fundamental dominant Weyl chamber:

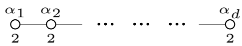
$$\{h' \in \mathfrak{g} \,|\, \mathrm{Re}(\alpha(h')) \geq 0, \forall \alpha \in \Delta \text{ and whenever } \mathrm{Re}(\alpha(h')) = 0, \mathrm{Im}(\alpha(h')) \geq 0\}.$$

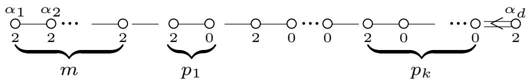
THEOREM A.1 (Kostant,[21]). — *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. A nilpotent orbit $\mathcal{O}$ is completely determined by the values $[\alpha_1(h), \alpha_2(h), \ldots, \alpha_n(h)]$.*

For every simple root α in Δ, we have $\langle \alpha, h \rangle \in \{0, 1, 2\}$ (see Section 3.5 in [9]).

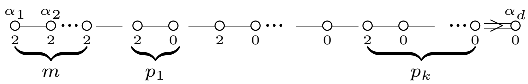
If we label every node of the Dynkin diagram of $\mathfrak{g}$ with the eigenvalues $\alpha(h) = \langle \alpha, h \rangle$ of h on the corresponding simple root space $\mathfrak{g}_\alpha$, then all labels are 0,1 or 2. We call such a labeled Dynkin diagram, **a weighted Dynkin diagram**.

A.1. Weighted Dynkin diagrams. — The diagrams presented here are also presented in Carter's book [5], page 175.

A_d 

C_d 

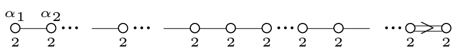
with $m + p_1 + \dots p_k + 1 = d$, $p_1 = 2, p_{i+1} = p_i$ or $p_i + 1$ for each i. ($k = 0$, $m = l - 1$ is a special case)

B_d 

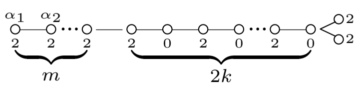
with $m + p_1 + \dots p_k = d$, $p_1 = 2, p_{i+1} = p_i$ or $p_i + 1$ for $i = 1, 2, \dots, k - 2$ and

$$p_k = \begin{cases} \frac{p_{k-1}}{2} & \text{if } p_{k-1} \text{ is even} \\ \frac{p_{k-1}-1}{2} & \text{if } p_{k-1} \text{ is odd} \end{cases}$$

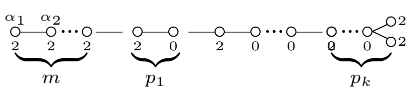
In addition the diagram:



is distinguished.

D_d 

with $m + 2k + 2 = d$, and those of the form



with $m + p_1 + \dots p_k = l$, $p_1 = 2, p_{i+1} = p_i$ or $p_i + 1$ for $i = 1, 2, \dots, k - 2$ and

$$p_k = \begin{cases} \frac{p_{k-1}}{2} & \text{if } p_{k-1} \text{ is even} \\ \frac{p_{k-1}+1}{2} & \text{if } p_{k-1} \text{ is odd} \end{cases}$$

The key notion used by Bala and Carter was the notion of distinguished nilpotent element. It is an element that is not contained in any proper Levi subalgebra. Alternatively, a nilpotent element $n \in \mathfrak{g}$ is called distinguished if it does not commute with any non-zero semi-simple element of $\mathfrak{g}$. Or also, a nilpotent element X (or orbit $\mathcal{O}_X$) is distinguished if the only Levi subalgebra containing X (or meeting $\mathcal{O}_X$) is $\mathfrak{g}$ itself.

Definition A.2 (distinguished parabolic subalgebra). — A parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ is called distinguished if $\dim \mathfrak{l} = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$, in which $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition of $\mathfrak{p}$, with Levi part $\mathfrak{l}$.

The Theorem 5.9.5 in [5] implies the following correspondence:

$$\{\text{Nilpotent } \mathrm{Ad}(G)\text{-orbits of } \mathfrak{g}\} \leftrightarrow \begin{Bmatrix} G \text{ conjugacy classes} \\ \text{of pairs } (\mathfrak{p}, \mathfrak{m}) \text{ of } \mathfrak{g} \end{Bmatrix} \tag{23}$$

in which $\mathfrak{m}$ is a Levi factor, $\mathfrak{p} \subseteq \mathfrak{m}'$ is a distinguished parabolic subalgebra of the semi-simple part of $\mathfrak{m}$.

Let us give a few more results on distinguished orbits, in particular the Theorem A.7 explains the partitions used in 4.2:

We need to introduce a grading: given a non-zero nilpotent element in $\mathfrak{g}$, using the standard triple above, the Jacobson–Morozov Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ satisfies $\phi(e) = n \in \mathfrak{n}$ and $\phi(h) = \gamma$ is in the dominant chamber of $\mathfrak{t}$.

The adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ yields a grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ in which

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid \mathrm{ad}(\gamma)(x) = ix\}\,; \qquad [\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$$

and $n \in \mathfrak{g}(2)$. Further, we set

$$\begin{cases} \mathfrak{p} = \mathfrak{p}(\gamma) = \displaystyle\bigoplus_{i \geq 0} \mathfrak{g}(i) \\ \mathfrak{u} = \displaystyle\bigoplus_{i > 0} \mathfrak{g}(i) \\ \mathfrak{l} = \mathfrak{g}(0) \end{cases} \tag{24}$$

The Lie subalgebra $\mathfrak{p}$ contains $\mathfrak{b}$ and is thus a parabolic subalgebra whose Levi decomposition is $\mathfrak{p} = \mathfrak{u} \oplus \mathfrak{l}$.

On the other hand, starting with a subset $J \subseteq \Delta$ and denoting $\mathfrak{p}_J$ the standard parabolic subalgebra, one defines a function $\eta_J : \Phi_0 \to \mathbb{Z}$ defined on roots of Δ as twice the indicator function of J and extended linearly to all roots.

We obtain a grading: $\mathfrak{g} = \bigoplus_{i \geq 0} \mathfrak{g}_J(i)$ by declaring $\mathfrak{g}_J(0) = \mathfrak{t} \oplus \sum_{\eta_J(\alpha)=0} \mathfrak{g}_\alpha$ and otherwise $\mathfrak{g}_J(i) = \sum_{\eta_J(\alpha)=i} \mathfrak{g}_\alpha$. Then, $\mathfrak{p}_J = \bigoplus_{i \geq 0} \mathfrak{g}_J(i)$ and its nilpotent radical is $\mathfrak{n}_J = \bigoplus_{i > 0} \mathfrak{g}_J(i)$.

To summarize, to the standard triple containing n one attaches a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$.

If $\dim \mathfrak{g}(1) = 0$, then we call n (or $\mathcal{O}_n$) an even nilpotent element (even nilpotent orbit, respectively).

Proposition A.3 (Corollary 3.8.8 in [9]). — *A weighted Dynkin diagram has labels only 0 or 2 if and only if it corresponds to an even nilpotent orbit (i.e, if* $\dim \mathfrak{g}(1) = 0$*).*

The two following propositions are taken from Chapter 2 of Di Martino's thesis [10]:

Proposition A.4. — *The standard parabolic subalgebra* $\mathfrak{p}_J$ *is distinguished if and only if* $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$. *In this case, if* n *is any element in the unique open orbit of the parabolic subgroup* P_J *on its nilpotent radical* $\mathfrak{n}_J$, *then the parabolic subalgebra associated to* n *as in (24) equals* $\mathfrak{p}_J$.

A distinguished nilpotent element also satisfies the following:

Proposition A.5. — *A nilpotent element* $n \in \mathfrak{g}$ *is distinguished if and only if* $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$. *Moreover, if* $n \in \mathfrak{g}$ *is distinguished, then* $\dim \mathfrak{g}(1) = 0$.

Theorem A.6 (Theorem 8.2.3 in [9]). — *Any distinguished orbit in* $\mathfrak{g}$ *is even.*

Theorem A.7 (Theorem 8.2.14 in [9]). —

1. *If* $\mathfrak{g}$ *is of type A, then the only distinguished orbit is principal.*
2. *If* $\mathfrak{g}$ *is of type B, C or D, then an orbit is distinguished if and only if its partition has no repeated parts. Thus, the partition of a distinguished orbit in types B, D has only odd parts, each occurring once, while the partition of a distinguished orbit in type C has only even parts, each occurring once.*

A.2. Distinguished nilpotent orbits and residual points. — The connection with the notion of residual point is now made accessible.

Let G be a Chevalley (semi-simple) group and $T \subseteq B$ a maximal split torus and a Borel subgroup. We have a root datum $\mathcal{R}(G, B, T)$. By reversing the role of $X^*(T)$ and $X_*(T)$, we obtain a new root datum $\mathcal{R}^\vee = (X_*(T), \Delta, X^*(T), \Delta^\vee)$. Let $({}^LG, {}^LB, {}^LT)$ be the triple with root datum $\mathcal{R}^\vee$. The L-group ${}^LG$ is the dual group, with maximal torus ${}^LT$, and Borel subgroup ${}^LB$. Denote the respective Lie algebra ${}^L\mathfrak{g}, {}^L\mathfrak{t}$ and ${}^L\mathfrak{b}$. Let $(V^*, \langle,\rangle)$ be a finite dimensional Euclidean space containing and spanned by the root system: $\Delta \subseteq V^*$, the canonical pairing between V and V^* is denoted by $\langle,\rangle$. We fix an inner product on V by transport of structure from $(V^*, \langle,\rangle)$ via the canonical isomorphism $V^* \to V$ associated with $\langle,\rangle$. Thus, this map becomes an isometry, and for each $\alpha \in \Delta$, the coroot $\check{\alpha} \in V$ is given as the image of $2\langle\alpha,\alpha\rangle^{-1}\alpha \in V^*$.

To this data we associate the Weyl group W_0 generated by the reflexions s_α ($s_\alpha(x) = x - \langle x, \check{\alpha}\rangle\alpha$ and $s_\alpha(y) = y - \langle\alpha, y\rangle\check{\alpha}$) over the hyperplanes $H_\alpha \subseteq V^*$ consisting of elements $x \in V^*$, which are orthogonal to $\check{\alpha}$ with respect to $\langle,\rangle$.

Let us make a remark before stating the correspondence result related to our use in this manuscript.

Remark A.8. — The bijective correspondence (below) is originally formulated for residual subspaces. Let k be the "coupling parameter" as defined in [14].

An affine subspace $L \subseteq V$ is called residual if, for a root system Φ (in a root datum)

$$\#\{\alpha \in \Phi \mid \langle \alpha, L\rangle = k\} = \#\{\alpha \in \Phi \mid \langle \alpha, L\rangle = 0\} + \operatorname{codim} L$$

(If $\mathscr{R}$ is semi-simple, there exist residual subspaces that are singletons $\{\lambda\} \subseteq V$, the residual points).

For example, when the parameter k (called "coupling parameter" in [14]) equals 1, the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha\in\Phi}\alpha$ is a residual point, since the above equation is verified. More generally, for any $k = (k_\alpha)_{\alpha\in\Phi}$, the vector $\rho(k) = \frac{1}{2}\sum_{\alpha\in\Phi} k_\alpha \alpha$ is a residual point.

Then the bijective correspondence is given between the set of nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0- orbits of residual subspaces.

We mention the following result partially related to Proposition 4.6. The bijective correspondence concerns only unramified characters, and we fix the parameter $k_\alpha = 1$, for all $\alpha \in \Phi_0$.

Proposition A.9. — *There is a bijective correspondence $\mathcal{O}_{W_0\lambda(\mathcal{O})} \leftrightarrow W_0\lambda(\mathcal{O})$ between the set of distinguished nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0-orbits of residual points.*

Proof. — This particular bijection is a specific case of the larger bijective correspondence given between the set of nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0-orbits of residual subspaces. It is discussed in detail in [27], Appendices A and B, but also in [16], Proposition 6.2. □

Let $({}^L\mathfrak{m}, {}^L\mathfrak{p})$ be a representative of a class, for which ${}^L\mathfrak{m} = {}^L\mathfrak{g}$ and ${}^L\mathfrak{p} \subseteq {}^L\mathfrak{g}$ is a standard distinguished parabolic subalgebra. We have a corresponding distinguished nilpotent orbit $\mathcal{O}$. With Proposition A.4, the data ${}^L\mathfrak{p}$ is equivalent to the assignment of an even weighted Dynkin diagram: $2\lambda(\mathcal{O})$.

Since we have $\dim\mathfrak{g}(0) = \dim\mathfrak{h} + \#\{\alpha \in \Phi \mid \langle\check{\alpha}, 2\lambda(\mathcal{O})\rangle = 0\}$ and $\dim\mathfrak{g}(2) = \#\{\alpha \in \Phi \mid \langle\check{\alpha}, 2\lambda(\mathcal{O})\rangle = 2\}$, the assignment of an even weighted Dynkin diagram implies $\dim\mathfrak{g}(0) = \dim\mathfrak{g}(2)$, and this equality sets $\lambda(\mathcal{O})$ as a residual point.

The definition of $\lambda(\mathcal{O})$ depends on the choice of positive roots and Borel subgroup ${}^LB$. A different choice yields a different element on the same W_0-orbit. See [27], Appendices A and B, and particularly Proposition 8.1 in [27].

Appendix B. Projections of root systems

Let us first follow the notations of the book of [28], Chapter V. We will also use the notations of Section 2. Let $X^*(G)$ denote the group of rational characters of G; its dual is $X_*(G)$. We denote $a_0 = X_*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$ and $a_0^* = X^*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$.

The duality between $X^*(A_0)$ and $X_*(A_0)$ extends to a duality (canonical pairing) between the vector spaces a_0 and a_0^* (see Chapter V of [28] or the author's PhD thesis).

Because of the existence of the scalar product (sustaining the duality), the restriction map from $(a_0^G)^*$ to $(a_\Theta^G)^*$ is a *projection* map from (a_0^G) to (a_Θ^G). With the notations of Section 6, the roots in $\Delta(P_1)$ generating $(a_{M_1})^*$ are non-trivial restrictions of roots in $\Delta \setminus \Delta^{M_1}$ (recall that in the notations of [24], I.1.6, Δ^{M_1} are the roots of Δ that are in M_1), and (a_{M_1}) is generated by the projection of roots in $\Delta^\vee \setminus \Delta^{M_1\ \vee}$. In the article [12], we rather considered projections of roots. We studied the set Σ_Θ, projections of roots onto the orthogonal to Θ. Let us denote d the dimension of a_Θ, i.e. the cardinal of $\Delta - \Theta$.

Theorem B.1 (see [12]). — *Let Σ be an irreducible root system of classical type (i.e. of type A, B, C or D). The subsystems in Σ_Θ are necessarily of classical type. In addition, if the irreducible (connected) components of Θ of type A are all of the same length, and the interval between each of them of length 1, then Σ_Θ contains an irreducible root system of rank d (not necessarily reduced).*

We have used the following observation, from [4, Equation (10) in VI.3, Proposition 12 in VI.4, Chapter VI]: Let α and β be two non-orthogonal elements of a root system. Set

$$C = \left(\frac{1}{\cos(\alpha,\beta)}\right)^2 \qquad \text{and} \qquad R = \frac{\|\alpha\|^2}{\|\beta\|^2}\ .$$

Thus, if $\|\alpha\| \geq \|\beta\|$,

$$\frac{C}{R} \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = 4\ .$$

The case of reducible $\boldsymbol{\Sigma_\Theta}$. — In [12] we saw that in order to obtain a projected root system irreducible and of rank d, we had to impose several constraints. Let us explain once more some of them. Let us first consider two components A_m and A_q of (the Dynkin diagram of) Θ, let e_r and e_s be the vectors in the basis vectors of smallest index such $\Xi_r = \{e_r, \dots, e_{r+m}\}$ corresponds to A_m and Ξ_s to A_q. Let us assume that two simple consecutive roots α_{k-1} and α_k are outside of Θ and $k = r+m+1 = s-1$. Then $\Xi_k = \{e_k\}$. Let us consider the projections of α_{k-1} and α_k. Since e_k is orthogonal to all roots in Θ, $\overline{e_k} = e_k$. Therefore,

$$\|\overline{\alpha_{k-1}}\|^2 = \|\overline{e_{k-1}} - \overline{e_k}\|^2 = \frac{1}{m+1} + 1, \quad \|\overline{\alpha_k}\|^2 = \|\overline{e_k} - \overline{e_{k+1}}\|^2 = 1 + \frac{1}{q+1}.$$

Then

$$C = \left(\frac{1}{\cos(\overline{\alpha_{k-1}}, \overline{\alpha_k})}\right)^2 = \left(\frac{1}{m+1} + 1\right)\left(1 + \frac{1}{q+1}\right),$$

and if we assume $\|\overline{\alpha_{k-1}}\| \geq \|\overline{\alpha_k}\|$ i.e. $m \geq q$, we have:

$$R = \frac{\|\overline{\alpha_{k-1}}\|^2}{\|\overline{\alpha_k}\|^2} = \frac{\frac{1}{m+1} + 1}{1 + \frac{1}{q+1}}.$$

If α_k and α_{k-1} were to be part of a root system, we would need

$$\frac{C}{R} = \left(1 + \frac{1}{q+1}\right)^2 \in \{2^2, 1, (2/3)^2\} \quad \text{and} \quad CR = \left(1 + \frac{1}{m+1}\right)^2 = 4.$$

This implies that $q = 0$ and $m = 0$ or $m = -\frac{4}{3}$ leading to a contradiction with the setting above. This illustrates the fact that in the main theorem (Theorem B.1), the intervals between the irreducible connected components of Θ need to be of length 1, and *at most 1*.

In general, the complement of Theorem B.1 above is the following:

Theorem B.2. — *Let Σ be an irreducible root system of type B, C or D. If the irreducible (connected) components of Θ of type A are all* not *of the same length, and the interval between each of them of length 1, then Σ_Θ contains a reducible root system of rank d (not necessarily reduced); $\Sigma_\Theta = \bigoplus_i \Sigma_{\Theta,i}$, and if d_i is the rank of the irreducible i-th component, then $\sum_i d_i = d$.*

The number of irreducible components $(\Sigma_{\Theta,i})$ is as many as there are changes of length *plus 1. That is, if there are d_1 components of type A_{m_1}, followed by d_2 components of type A_{m_2}, et cetera until d_s components of type A_{m_s}, such that $m_i \neq m_{i+1}$ for any i, and one last component of type B or C or D, there are $s-1$ changes in the length (m_i) and, therefore, s irreducible connected components in Σ_Θ. The set Σ_Θ is composed of irreducible components of type A and possibly one component of type B, C or D.*

Proof. — We have explained the condition on the interval being of at most length 1 in the paragraph preceding the statement of the theorem. We do not repeat here the methods of proof for the case of Σ_Θ irreducible that apply here; in particular, the treatment of the case $e_n \notin \Theta$, the reduction to this case's argumentation when $e_n \in \Theta$, and the argumentation showing that the components of type A of Θ should be of the same length to obtain a root system in the projection.

We consider the case of a root system of type B, C, D. Let us then assume that we have d_1 components of type A_{m_1} in Θ, by the argumentation given in [12], we obtain a root system of type BC_{d_1}. Let us assume that these d_1 components of type A_{m_1} are followed by d_2 components of type A_{m_2}, $m_2 \neq m_1$. Let us denote by e_{1,d_1} the vector associated to the last component of type A_{m_1} and by $e_{2,1}$ the vector associated to the first component of type A_{m_2}.

The projection

$$\overline{e_{1,d_1}} - \overline{e_{2,1}} = \frac{e_{(d_1-1)m_1+1} + e_{(d_1-1)m_1+2} + \ldots + e_{(d_1-1)m_1+m_1}}{m_1+1} - \frac{e_{d_1m_1+1} + e_{d_1m_1+2} + \ldots + e_{d_1m_1+m_2}}{m_2+1}$$

of $e_{1,d_1} - e_{2,1}$ cannot be a root in Σ_Θ (it would contradict the conditions of validity of the value C and R when calculated with respect to the last root of the previously considered BC_{d_1}).

However, the projections of the roots corresponding to the intervals between any two of the d_2 components of type A_{m_2} (say of $\overline{e_s} - \overline{e_t}$) along with all roots of the form $\pm e_s$ or $\pm e_t$ (or $\pm 2e_s$ or $\pm 2e_t$) form a root system of type BC_{d_2}. Some specificities, such as a root system of type C appearing in the projection for certain cases under Σ of type C or D carry over here (see [12]).

The key mechanism ensuring that the sum of the d_i equals d is the observation that one needs *three* consecutive components of type A_q of a given length q (followed by components of length $m \neq q$) to obtain in the projection a BC_3 *(hence of rank 3!)*, whereas one would obtain only an A_2 type of root system. This means that even if the root connecting the A_q to A_m is not a root in the projection, i.e. "we are missing a simple root", we get a simple root of type $\overline{e_i}$ or $\overline{2e_i}$.

One may notice that another possibility would be to obtain a reducible root system such as $A_1 \times A_1 \times \ldots \times A_1$. This case is not excluded, but it would not be possible to find such a system of *maximal rank*.

Indeed, let us briefly recall the formulas written for the case of Σ of type A in [12], where we consider three vectors e_r, e_s and e_t whose projections are associated to three components of Θ of type A_m, A_p and A_q. Let $\alpha = e_i - e_j$ be a root whose projection is $\overline{\alpha} = \pm(\overline{e_r} - \overline{e_s})$ and $\beta = e_k - e_l$ a root whose projection is $\overline{\beta} = \pm(\overline{e_s} - \overline{e_t})$, then the square of the scalar product of $\overline{\alpha}$ and $\overline{\beta}$ is

$$\left(\langle \overline{\alpha}, \overline{\beta} \rangle\right)^2 = \frac{1}{(p+1)^2} \; .$$

This excludes the possibility of α and β being orthogonal. Therefore, for two consecutive roots in the projection (projections of simple roots), it is not possible to obtain a system of type $A_1 \times A_1$.

If there is a sequence of connected consecutive components of Θ of type A that we index by an integer i (in increasing order) and length q_i with $q_i \neq q_{i+1}$ for any i, let us denote $\overline{\alpha_i} = \overline{e_r} - \overline{e_s}$, where $e_r \in A_{q_i}$ and $e_s \in A_{q_{i+1}}$.

Further, let us denote $\overline{\alpha_{i+2}} = \overline{e_t} - \overline{e_z}$, where $e_t \in A_{q_{i+2}}$ and $e_z \in A_{q_{i+3}}$. The orthogonal roots $\overline{\alpha_i}$ and $\overline{\alpha_{i+2}}$ form a root system of type $A_1 \times A_1$.

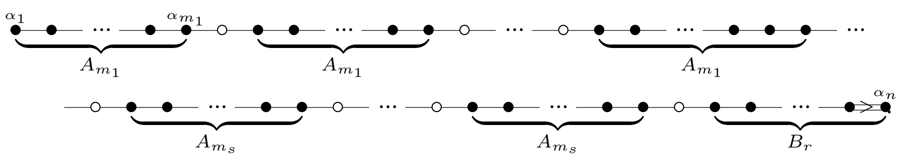
The root $\overline{\alpha_{i+1}} = \overline{e_s} - \overline{e_t}$ does not contribute to this subsystem.

Therefore, the maximal number of A_1 factor such that the reducible root system $A_1 \times A_1$ appear in Σ_Θ is $d/2$.

By a similar reasoning, it would be possible to obtain a reducible system of type $A_2 \times A_2 \times \ldots \times A_2$ if Θ is composed of a succession of connected components of type A such that the three first ones are of length m, the three next ones of length $q \neq m$, etc. Then the projection of the root connecting A_m and A_q would not contribute to this subsystem. Again, this would never give any reducible system of *maximal rank d*.

Because to any change of length of the A components, the corresponding root (connecting the two components of different length) cannot appear as a (simple) root in the projection, we are missing a root (of the set $\Delta - \Theta$ of size d) at any change of length. In the case Σ is of type A, this 'missing' root is not replaced by any short or long root (e_i or $2e_i$), and therefore it is impossible to obtain a basis of the root system in the projection. In other words, there does not exist any reducible root system of maximal rank in the projection Σ_Θ of Σ of type A. □

Let us illustrate one case of the previous theorem with a Dynkin diagram of Σ of type B:



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THE NAKAYAMA FUNCTOR AND ITS COMPLETION FOR GORENSTEIN ALGEBRAS

by Srikanth B. Iyengar & Henning Krause

To Bill Crawley-Boevey on his 60th birthday.

Abstract. — Duality properties are studied for a Gorenstein algebra that is finite and projective over its center. Using the homotopy category of injective modules, it is proved that there is a local duality theorem for the subcategory of acyclic complexes of such an algebra, akin to the local duality theorems of Grothendieck and Serre in the context of commutative algebra and algebraic geometry. A key ingredient is the Nakayama functor on the bounded derived category of a Gorenstein algebra and its extension to the full homotopy category of injective modules.

Résumé (*Le foncteur de Nakayama et sa complétion pour les algèbres de Gorenstein*). — Des propriétés de dualité sont étudiées pour une algèbre de Gorenstein finie et projective sur son centre. En utilisant la catégorie homotopique des modules injectifs, il est démontré qu'il existe un théorème de dualité locale pour la sous-catégorie des objets acycliques d'une telle algèbre, semblable aux théorèmes de dualité locale de Grothendieck et Serre dans le cadre de l'algèbre commutative et de la géométrie algébrique. Un ingrédient clé est le foncteur de Nakayama sur la catégorie dérivée bornée d'une algèbre de Gorenstein, et son extension à toute la catégorie homotopique des modules injectifs.

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1. Introduction

This work is a contribution to the representation theory of Gorenstein algebras, both commutative and noncommutative, with a focus on duality phenomena. The notion of a Gorenstein variety was introduced by Grothendieck [26, 25, 29, 30] and grew out of his reinterpretation and extension of Serre duality [43] for projective varieties. A local version of his duality is that over a Cohen–Macaulay local algebra R of dimension d, with maximal ideal $\mathfrak{m}$, and for complexes F, G with F perfect, there are natural isomorphisms

$$\operatorname{Hom}_R(\operatorname{Ext}^i_R(F,G), I(\mathfrak{m})) \cong \operatorname{Ext}^{d-i}_R(G, R\varGamma_{\mathfrak{m}}(\omega_R \otimes^{\mathrm{L}}_R F)),$$

where ω_R is a dualizing module, and $I(\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. The functor $R\varGamma_{\mathfrak{m}}$ represents local cohomology at $\mathfrak{m}$. Serre duality concerns the case where R is the local ring at the vertex of the affine cone of a projective variety. The ring R (equivalently, the variety it represents) is said to be Gorenstein if, in addition, the R-module ω_R is projective. Serre observed that this property is characterized by R having a finite self-injective dimension. This result appears in the work of Bass [4], who gave numerous other characterizations of Gorenstein rings.

Iwanaga [31] launched the study of Noetherian rings, not necessarily commutative, having finite self-injective dimension on both sides. Now known as Iwanaga–Gorenstein rings, these form an integral part of the representation theory of algebras. In that domain, the principal objects of interest are maximal Cohen–Macaulay modules and the associated stable category. Auslander [1] and Buchweitz [13] have proved duality theorems for the stable category of a Gorenstein algebra with *isolated* singularities. The driving force behind our work was to understand what duality phenomena can be observed for general Gorenstein algebras. Theorem 1.2 below is what we found, following Grothendieck's footsteps.

We set the stage to present that result and begin with a crucial definition.

Definition 1.1. — Let R be a commutative Noetherian ring. An R-algebra A is called *Gorenstein* if

(1) the R-module A is finitely generated and projective, and
(2) for each $\mathfrak{p}$ in $\operatorname{Spec} R$ with $A_{\mathfrak{p}} \neq 0$ the ring $A_{\mathfrak{p}}$ has finite injective dimension as a module over itself, on the left and on the right.

A Gorenstein R-algebra A itself need not be Iwanaga–Gorenstein. Indeed, for A commutative and Gorenstein, the injective dimension of A is finite precisely when its Krull dimension is finite, and there exist rings locally of finite injective dimension but of infinite Krull dimension. There are precedents to the study of Gorenstein algebras, starting with [4] and more recently in the work of Goto and Nishida [24]. Our work differs from theirs in its focus on duality. We refer to [22] for a discussion of examples and natural constructions preserving the Gorenstein property.

Let A be a Gorenstein R-algebra and $\omega_{A/R} := \operatorname{Hom}_R(A,R)$ the dualizing *bimodule*. Unlike in the commutative case, $\omega_{A/R}$ does not need to be projective (neither on the left nor on the right), and the bimodule structure can be complicated. Nevertheless, it is a tilting object in $\mathbf{D}(\operatorname{Mod} A)$, the derived category of A-modules, inducing a triangle equivalence

$$\operatorname{RHom}_A(\omega_{A/R}, -)\colon \mathbf{D}(\operatorname{Mod} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} A)\,;$$

see Section 4. The representation theory of a Gorenstein algebra A is governed by its maximal Cohen–Macaulay modules, namely, finitely generated A-modules M with $\operatorname{Ext}^i_A(M,A) = 0$ for $i \geq 1$. For our purposes, their infinitely generated counterparts are also important. Thus, we consider Gorenstein projective A-modules (abbreviated to G-projective), which are by definition A-modules occurring as syzygies in acyclic complexes of projective A-modules [13, 19]. The G-projective modules form a Frobenius exact category, and so the corresponding stable category, is triangulated. Its inclusion into the usual stable module category has a right adjoint, the Gorenstein approximation functor, $\operatorname{GP}(-)$. The functor

$$S := \operatorname{GP}(\omega_{A/R} \otimes_A -)\colon \underline{\operatorname{GProj}} A \longrightarrow \underline{\operatorname{GProj}} A$$

is an equivalence of triangulated categories and plays the role of a Serre functor on the subcategory of finitely generated G-projectives. This is spelled out in the result below. Here, the $\widehat{\operatorname{Ext}}^i_A(-,-)$ are the Tate cohomology modules, which compute morphisms in $\underline{\operatorname{GProj}} A$.

Theorem 1.2. — *Let A be a Gorenstein R-algebra and let M, N be G-projective A-modules with M finitely generated. For each $\mathfrak{p} \in \operatorname{Spec} R$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\widehat{\operatorname{Ext}}^i_A(M,N), I(\mathfrak{p})) \cong \widehat{\operatorname{Ext}}^{d(\mathfrak{p})-i}_A(N, \varGamma_{\mathfrak{p}} S(M))\,,$$

where $d(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) - 1$.

This is the duality theorem we seek; it is proved in Section 9. It is new even for commutative rings. The parallel to Grothendieck's duality theorem is clear.

In the following, we explain the strategy for proving this theorem and some essential ingredients. The functor $\varGamma_{\mathfrak{p}}$ is analogous to the local cohomology functor encountered above. It is constructed in Section 7 following the recipe in [7], using the natural R-action on $\underline{\operatorname{GProj}} A$. Even if N is finitely generated, $\varGamma_{\mathfrak{p}}(N)$ need not be, which is one reason we have to work with infinitely generated modules in the first place. If R is local with maximal ideal $\mathfrak{p}$, and A has isolated singularities, $\varGamma_{\mathfrak{p}}$ is the identity, and the duality statement above is precisely the one discovered by Auslander and Buchweitz.

For a Gorenstein algebra, the stable category of G-projective modules is equivalent to $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, the homotopy category of acyclic complexes of injective A-modules. This connection is explained in Section 6 and builds on the results from [33, 35]. In fact, much of the work that goes into proving Theorem 1.2 deals with $\mathbf{K}(\operatorname{Inj} A)$, the full homotopy category of injective A-modules; see Section 2. A key ingredient in all this is the Nakayama functor on the category of A-modules:

$$\mathbf{N}\colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} A \quad \text{where} \quad \mathbf{N}(M) = \operatorname{Hom}_A(\omega_{A/R}, M).$$

As noted above, its derived functor induces an equivalence on $\mathbf{D}(\operatorname{Mod} A)$. Following [35] we extend the Nakayama functor to all of $\mathbf{K}(\operatorname{Inj} A)$, which one may think of as a triangulated analogue of the ind-completion of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. This *completion* of the Nakayama functor is also an equivalence:

$$\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj} A)\,.$$

This is proved in Section 5, where we establish also that it restricts to an equivalence on $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. The induced equivalence on the stable category of G-projective modules is precisely the functor S in the statement of Theorem 1.2; see Section 6 where the singularity category of A, in the sense of Buchweitz [13] and Orlov [42] also appears. To make this identification, we need to extend results of Auslander and Buchweitz concerning G-approximations; this is dealt with in Appendix A.

Our debt to Grothendieck is evident. It ought to be clear by now that the work of Auslander and Buchweitz also provides much inspiration for this paper. Whatever new insight we bring is through the systematic use of the homotopy category of injective modules and methods from abstract homotopy theory, especially the Brown representability theorem. To that end we need the structure theory of injectives over finite R-algebras from Gabriel's thesis [20]. Gabriel also introduced the Nakayama functor in representation theory of Artin algebra in his exposition of Auslander–Reiten duality; it is the categorical analogue of the Nakayama automorphism that permutes the isomorphism classes of simple modules over a self-injective algebra [21]. Moreover, it was Gabriel who pointed out the parallel between derived equivalences induced by tilting modules and the duality of Grothendieck and Roos [34].

2. Homotopy category of injectives

In this section, we describe certain functors on homotopy categories attached to Noetherian rings. Our basic references for this material are [32, 35].

Throughout, A will be a ring that is Noetherian on both sides; that is to say, A is Noetherian as a left and as a right A-module. In what follows, A-modules will mean left A-modules, and A^{op}-modules are identified with right A-modules. We write $\operatorname{Mod} A$ for the (abelian) category of A-modules and $\operatorname{mod} A$ for its full subcategory consisting of finitely generated modules. Also, $\operatorname{Inj} A$ and $\operatorname{Proj} A$ are the full subcategories of $\operatorname{Mod} A$ consisting of injective and projective modules, respectively.

For any additive category $\mathcal{A} \subseteq \operatorname{Mod} A$, like the ones in the last paragraph, $\mathbf{K}(\mathcal{A})$ will denote the associated homotopy category, with its natural structure as a triangulated category. Morphisms in this category are denoted $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-,-)$. An object X in $\mathbf{K}(\mathcal{A})$ is *acyclic* if $H^*(X) = 0$, and the full subcategory of acyclic objects in $\mathbf{K}(\mathcal{A})$ is denoted $\mathbf{K}_{\mathrm{ac}}(\mathcal{A})$. A complex $X \in \mathbf{K}(\mathcal{A})$ is said to be *bounded above* if $X^i = 0$ for $i \gg 0$, and *bounded below* if $X^i = 0$ for $i \ll 0$.

In the sequel our focus in mostly on $\mathbf{K}(\operatorname{Inj} A)$, the homotopy category of injective modules, and its various subcategories; the analogous categories of projectives play a more subsidiary role. From work in [33, 35, 41], we know that the triangulated categories $\mathbf{K}(\operatorname{Inj} A)$ and $\mathbf{K}(\operatorname{Proj} A)$ are compactly generated since the ring A is Noetherian on both sides; the compact objects in these categories are described further below. Let $\mathbf{D}(\operatorname{Mod} A)$ denote the (full) derived category of A-modules and $\mathbf{q}\colon \mathbf{K}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ the localization functor; its kernel is $\mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A)$. We write $\mathbf{q}$ also for its restriction to the homotopy categories of injectives and projectives. These functors have adjoints:

$$\mathbf{K}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{q}}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} A) \qquad \text{and} \qquad \mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{q}}{\overset{\mathbf{p}}{\leftrightarrows}} \mathbf{D}(\operatorname{Mod} A)\,.$$

Our convention is to write the left adjoint above the corresponding right one. In what follows, it is convenient to conflate $\mathbf{i}$ and $\mathbf{p}$ with $\mathbf{i}\circ\mathbf{q}$ and $\mathbf{p}\circ\mathbf{q}$, respectively. The images of $\mathbf{i}$ and $\mathbf{p}$ are the K-injectives and K-projectives, respectively. Recall that an object X in $\mathbf{K}(\operatorname{Inj} A)$ is *K-injective* if $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = 0$ for any acyclic complex W in $\mathbf{K}(\operatorname{Mod} A)$. We write $\mathbf{K}_{\mathrm{inj}}(A)$ for the full subcategory of $\mathbf{K}(\operatorname{Inj} A)$ consisting of K-injective complexes. The subcategory $\mathbf{K}_{\mathrm{proj}}(A) \subseteq \mathbf{K}(\operatorname{Proj} A)$ of K-projective complexes is defined similarly.

Compact objects. — Since A is Noetherian $\operatorname{Inj} A$ is closed under arbitrary direct sums, and hence so is the subcategory $\mathbf{K}(\operatorname{Inj} A)$ of $\mathbf{K}(\operatorname{Mod} A)$. As in any triangulated category with arbitrary direct sums, an object X in $\mathbf{K}(\operatorname{Inj} A)$ is *compact* if $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, -)$ commutes with direct sums. The compact objects in $\mathbf{K}(\operatorname{Inj} A)$ form a thick subcategory, denoted $\mathbf{K}^{\mathrm{c}}(\operatorname{Inj} A)$. The adjoint pair $(\mathbf{q}, \mathbf{i})$ above restricts to an equivalence of triangulated categories

$$\mathbf{K}^{\mathrm{c}}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{q}}{\underset{\sim}{\rightleftarrows}}} \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)\,,$$

where $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ denotes the bounded derived category of $\operatorname{mod} A$; see [35, Proposition 2.3] for a proof of this assertion. The corresponding identification of the compact objects in $\mathbf{K}(\operatorname{Proj} A)$ is a bit more involved and is due to Jørgensen [33, Theorem 3.2]. The assignment $M \mapsto \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, A)$ induces an equivalence

$$\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}^{\mathrm{c}}(\operatorname{Proj} A)\,.$$

See also [32], where these two equivalences are related. The formula below for computing morphisms from compacts in $\mathbf{K}(\operatorname{Inj} A)$ is useful in the sequel.

Lemma 2.1. — *For $C, X \in \mathbf{K}(\operatorname{Inj} A)$ with C compact, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X) \cong H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X)\,.$$

Proof. — Since C is compact its K-projective resolution $\mathbf{p}C$ is homotopy equivalent to a complex that is bounded above and consists of finitely generated projective A-modules. For each integer n, let $X(n)$ be the subcomplex $X^{\geqslant -n}$ of X. Since $X(n)$ is K-injective, the quasi-isomorphism $\mathbf{p}C \to C$ induces the one on the left

$$\operatorname{Hom}_A(C, X(n)) \xrightarrow{\sim} \operatorname{Hom}_A(\mathbf{p}C, X(n)) \xleftarrow{\sim} \operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n)\,.$$

The one on the right is the standard one and holds because of the aforementioned properties of $\mathbf{p}C$ and the fact that $X(n)$ is bounded below. One thus gets a canonical isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X(n)) \xrightarrow{\sim} H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n))\,.$$

It is compatible with the inclusions $X(n) \subseteq X(n+1)$, so induces the isomorphism in the bottom row of the following diagram.

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, \operatorname{hocolim}_{n\geqslant 0} X(n)) & \xrightarrow{\sim} & H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A \operatorname{hocolim}_{n\geqslant 0} X(n)) \\
\downarrow\wr & & \downarrow\wr \\
\operatorname{colim}_{n\geqslant 0} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X(n)) & \xrightarrow{\sim} & \operatorname{colim}_{n\geqslant 0} H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n))\,.
\end{array}$$

The isomorphism on the left holds by the compactness of C, while the one on the right holds because $H^0(-)$ commutes with homotopy colimits. It remains to note that $\operatorname{hocolim}_{n\geqslant 0} X(n) = X$ in $\mathbf{K}(\operatorname{Inj} A)$. □

A recollement. — The functors $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \xrightarrow{\mathrm{incl}} \mathbf{K}(\operatorname{Inj} A) \xrightarrow{\mathbf{q}} \mathbf{D}(\operatorname{Mod} A)$ induce a recollement of triangulated categories

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \;\underset{\overset{\longleftarrow}{\mathbf{r}}}{\overset{\overset{\mathbf{s}}{\longleftarrow}}{\xrightarrow{\mathrm{incl}}}}\; \mathbf{K}(\operatorname{Inj} A) \;\underset{\overset{\longleftarrow}{\mathbf{i}}}{\overset{\overset{\mathbf{j}}{\longleftarrow}}{\xrightarrow{\mathbf{q}}}}\; \mathbf{D}(\operatorname{Mod} A)\,. \tag{1}$$

The functor $\mathbf{i}$ is the one discussed above; it embeds $\mathbf{D}(\operatorname{Mod} A)$ as the homotopy category of K-injective complexes. The functor $\mathbf{r}$ thus has a simple description: there is an exact triangle

$$\mathbf{r}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow, \tag{2}$$

where the morphism $X \to \mathbf{i}X$ is the canonical one. Indeed, $\mathbf{r}X$ is evidently acyclic, and if W is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, the induced map $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, \mathbf{r}X) \to \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X)$ is an isomorphism, for one has $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, \mathbf{i}X) = 0$.

The functor $\mathbf{j}\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{K}(\operatorname{Inj} A)$ is fully faithful. The image of $\mathbf{j}$ equals the kernel of $\mathbf{s}$ and identifies with $\operatorname{Loc}(\mathbf{i}A)$, the localizing subcategory of $\mathbf{K}(\operatorname{Inj} A)$ generated by the injective resolution of A; see [35, Theorem 4.2]. One may think of $\mathbf{j}$ as the injective version of taking projective resolutions; see Lemma 2.5. To justify this claim takes preparation.

Lemma 2.2. — *Restricted to the subcategory* $\operatorname{Loc}(\mathbf{i}A)$ *of* $\mathbf{K}(\operatorname{Inj} A)$ *there is a natural isomorphism of functors* $\mathbf{r} \xrightarrow{\sim} \Sigma^{-1}\mathbf{si}$.

Proof. — Consider anew the exact triangle (2), but for X in $\operatorname{Loc}(\mathbf{i}A)$:

$$\mathbf{r}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow \Sigma\mathbf{r}X\,.$$

Apply $\mathbf{s}$ and remember that its kernel is $\operatorname{Loc}(\mathbf{i}A)$. □

Projective algebras. — In the remainder of this section, we assume that the ring A (which hitherto has been Noetherian on both sides) is also projective, as a module, over some central subring R. For the moment, the only role R plays is to allow for constructions of bimodule resolutions with good properties. Set $A^{\mathrm{ev}} := A \otimes_R A^{\mathrm{op}}$, the enveloping algebra of the R-algebra A, and set

$$E := \mathbf{i}_{A^{\mathrm{ev}}} A\,.$$

This is an injective resolution of A as a (left) module over A^{ev}. Since E is a complex of A-bimodules, for any complex X of A-modules, the right action of A on E induces a left A-action on $\operatorname{Hom}_A(E, X)$. The structure map $A \to E$ of bimodules induces a morphism of A-complexes

$$\operatorname{Hom}_A(E, X) \longrightarrow \operatorname{Hom}_A(A, X) \cong X \quad \text{for } X \in \mathbf{K}(\operatorname{Mod} A). \tag{3}$$

The computation below will be used often:

Lemma 2.3. — *The morphism in* (3) *is a quasi-isomorphism for* $X \in \mathbf{K}(\operatorname{Inj} A)$.

Proof. — By considering the mapping cone of $A \to E$, the desired statement reduces to: For any complex $W \in \mathbf{K}(\operatorname{Mod} A)$ that is acyclic and satisfies $W^i = 0$ for $i \ll 0$, one has $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = 0$. Without loss of generality we can assume $W^i = 0$ for $i < 0$. Then one gets the first equality below

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X^{\geqslant -1}) = 0,$$

and the second one holds because $X^{\geqslant -1}$ is K-injective. □

Since A is projective as an R-module, A^{ev} is projective as an A-module both on the left and on the right. The latter condition implies, by adjunction, that as a complex of left A-modules E consists of injectives. In particular, for any projective A-module P, the A-complex $E \otimes_A P$ consists of injective modules. Thus, one has an exact functor

$$E \otimes_A -\colon \mathbf{K}(\operatorname{Proj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A)\,.$$

For each X in $\mathbf{K}(\operatorname{Inj} A)$, one has isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E \otimes_A \mathbf{p}X, X) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{p}X, \operatorname{Hom}_A(E, X))\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{p}X, X)\,.\end{aligned}$$

The second isomorphism is a consequence of Lemma 2.3 and the K-projectivity of $\mathbf{p}X$. Thus, corresponding to the morphism $\mathbf{p}X \to X$, there is natural morphism

$$\pi(X)\colon E \otimes_A \mathbf{p}X \longrightarrow X \tag{4}$$

of complexes of A-modules.

Lemma 2.4. — *The morphism $\pi(X)$ in* (4) *is a quasi-isomorphism for each X.*

Proof. — Let $\eta\colon A \to E$ and $\varepsilon\colon \mathbf{p}X \to X$ denote the structure maps. These fit in the commutative diagram

$$\begin{array}{ccc} A \otimes_A \mathbf{p}X & \xrightarrow{\ \sim\ } & \mathbf{p}X \\ {\scriptstyle \eta\otimes_A \mathbf{p}X}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ E \otimes_A \mathbf{p}X & \xrightarrow{\ \pi(X)\ } & X\,. \end{array}$$

The map $\eta \otimes_A \mathbf{p}X$ is a quasi-isomorphism as η is one and $\mathbf{p}X$ is K-projective. Thus, $\pi(X)$ is a quasi-isomorphism. □

The stabilization functor. — The functor $\mathbf{s}\colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ from (1) admits the following description in terms of its kernel, which uses the natural transformation $\pi\colon E \otimes_A \mathbf{p}(-) \to \mathrm{id}$ of functors on $\mathbf{K}(\operatorname{Inj} A)$ from (4).

Lemma 2.5. — *Each object X in $\mathbf{K}(\operatorname{Inj} A)$ fits into an exact triangle*

$$E \otimes_A \mathbf{p}X \xrightarrow{\pi(X)} X \longrightarrow \mathbf{s}X \longrightarrow\,,$$

and this yields a natural isomorphism $E \otimes_A \mathbf{p}X \xrightarrow{\sim} \mathbf{j}X$.

Proof. — Since $\pi(X)$ is a quasi-isomorphism, by Lemma 2.4, the complex $\mathbf{s}X$ is acyclic. In $\mathbf{K}(\operatorname{Proj} A)$, the complex $\mathbf{p}X$ is in $\operatorname{Loc}(A)$, and hence in $\mathbf{K}(\operatorname{Inj} A)$, the complex $E \otimes_A \mathbf{p}X$ is in $\operatorname{Loc}(E)$. It remains to observe that if $W \in \mathbf{K}(\operatorname{Inj} A)$ is acyclic, then $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, W) = 0$ by Lemma 2.3. □

3. The Nakayama functor and its completion

The Nakayama functor is a standard tool in representation theory of Artin algebras. For instance, the functor interchanges projective and injective modules, thereby providing an efficient method to compute the Auslander–Reiten translate of a finitely generated module [21]. In this section, we discuss the extension of the Nakayama functor from modules to the homotopy category of injectives.

Throughout the rest of this work, we say that a ring A is a *finite R-algebra* if

(1) R is a commutative Noetherian ring;
(2) A is an R-algebra, that is to say, there is a map of rings $R \to A$ whose image is in the center of A;
(3) A is finitely generated as an R-module.

These conditions imply that A is a Noetherian ring, finitely generated as a module over its center, which is thus also Noetherian. Hence, A is a finite algebra over its center. When A is a finite R-algebra, so is the opposite ring A^{op}.

Let A be a finite R-algebra. Following Buchweitz [13, §7.6], which in turn is inspired by the terminology in commutative algebra, we call the A-bimodule

$$\omega_{A/R} := \operatorname{Hom}_R(A, R)$$

the *dualizing bimodule* of the R-algebra A. It is finitely generated as an A-module, on either side. Extending the terminology from the context of finite dimensional algebras over fields we call

$$\mathbf{N}_{A/R} := \operatorname{Hom}_A(\omega_{A/R}, -)\colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} A \tag{5}$$

the *Nakayama functor* of the R-algebra A. Sometimes, this name is used for the functor $\omega_{A/R} \otimes_A -$, which is left adjoint to $\mathbf{N}_{A/R}$, but in this work, the one above plays a more central role, hence our choice of nomenclature. When the algebra in question is clear, we drop the "A/R" from subscripts, to write ω and $\mathbf{N}$. In our applications, A will be projective as an R-module. Then the left adjoint of $\mathbf{N}_{A/R}$ is a Nakayama functor relative to the restriction $\operatorname{Mod} A \to \operatorname{Mod} R$ in the sense of Kvamme [37].

The Nakayama functor can be extended to $\mathbf{D}(\operatorname{Mod} A)$, yielding the *derived Nakayama functor*

$$\operatorname{RHom}_A(\omega, -)\colon \mathbf{D}(\operatorname{Mod} A) \longrightarrow \mathbf{D}(\operatorname{Mod} A)\,.$$

This functor and its left adjoint has been considered by several authors; see [27]. Here, we study the extension to $\mathbf{K}(\operatorname{Inj} A)$, following [35, §6].

The Nakayama functor is evidently additive and, therefore, admits an extension to $\mathbf{K}(\operatorname{Inj} A)$ as follows. Extend $\mathbf{N}$ to $\mathbf{K}(\operatorname{Mod} A)$, by applying it term-wise; denote this functor also $\mathbf{N}$. Brown representability yields a left adjoint to the inclusion $\mathbf{K}(\operatorname{Inj} A) \hookrightarrow \mathbf{K}(\operatorname{Mod} A)$, say λ. Set

$$\widehat{\mathbf{N}}_{A/R} \colon \mathbf{K}(\operatorname{Inj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A) \tag{6}$$

to be the composite of functors

$$\mathbf{K}(\operatorname{Inj} A) \hookrightarrow \mathbf{K}(\operatorname{Mod} A) \xrightarrow{\mathbf{N}} \mathbf{K}(\operatorname{Mod} A) \xrightarrow{\lambda} \mathbf{K}(\operatorname{Inj} A)\,.$$

Our notation is motivated by the fact that $\mathbf{K}(\operatorname{Inj} A)$ can be viewed as a completion of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, as is explained in [35, §2]. The next result is another reason for this choice. Here, $\mathbf{K}^{+}(\operatorname{Inj} A)$ denotes the full subcategory of $\mathbf{K}(\operatorname{Inj} A)$ consisting of complexes W that are bounded below. Note that $\mathbf{K}^{+}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{D}^{+}(\operatorname{Mod} A)$.

Lemma 3.1. — *On the subcategory* $\mathbf{K}^{+}(\operatorname{Inj} A)$*, there is an isomorphism of functors*

$$\widehat{\mathbf{N}}_{A/R} \xrightarrow{\sim} \mathbf{i}\operatorname{Hom}_A(\omega_{A/R}, -)\,.$$

making the following diagram commutative:

$$\begin{array}{ccccc} \operatorname{Mod} A & \xrightarrow{\text{incl}} & \mathbf{D}^{+}(\operatorname{Mod} A) & \xrightarrow{\mathbf{i}} & \mathbf{K}(\operatorname{Inj} A) \\ \downarrow{\scriptstyle \mathbf{N}} & & \downarrow{\scriptstyle \operatorname{RHom}_A(\omega,-)} & & \downarrow{\scriptstyle \widehat{\mathbf{N}}} \\ \operatorname{Mod} A & \xrightarrow{\text{incl}} & \mathbf{D}(\operatorname{Mod} A) & \xrightarrow{\mathbf{i}} & \mathbf{K}(\operatorname{Inj} A)\,. \end{array}$$

The functor $\widehat{\mathbf{N}}_{A/R} \colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}(\operatorname{Inj} A)$ *preserves arbitrary direct sums and on compact objects* $\widehat{\mathbf{N}}$ *identifies with the functor*

$$\operatorname{RHom}_A(\omega_{A/R}, -) \colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \longrightarrow \mathbf{D}(\operatorname{Mod} A)\,.$$

In general, the above square on the right will not be commutative, if one replaces $\mathbf{D}^{+}(\operatorname{Mod} A)$ by $\mathbf{D}(\operatorname{Mod} A)$; compare Theorem 5.1. We examine these functors in greater detail in the next section.

Proof. — Fix $X \in \mathbf{K}^{+}(\operatorname{Mod} A)$. The key observation is the following.

Claim. — $\lambda X \xrightarrow{\sim} \mathbf{i}X$*, the K-injective resolution of* X*.*

Indeed, since X is bounded below one can assume that so is $\mathbf{i}X$, and hence also the mapping cone, say Z, of the morphism $X \to \mathbf{i}X$. Since Z is also acyclic, arguing as in the proof of Lemma 2.3 one gets that $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(Z, Y) = 0$, for any $Y \in \mathbf{K}(\operatorname{Inj} A)$. Thus, the morphism $X \to \mathbf{i}X$ induces an isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{i}X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, Y)\,,$$

and this justifies the claim.

When X is bounded below, so is $\operatorname{Hom}_A(\omega, X)$. Thus, the claim above yields

$$\widehat{\mathbf{N}}(X) = \lambda \operatorname{Hom}_A(\omega, X) \cong \mathbf{i}\operatorname{Hom}_A(\omega, X)\,.$$

Now fix $X \in \mathbf{D}^+(\operatorname{Mod} A)$. Again, one can assume $\mathbf{i}X$ is also bounded below, and, therefore,

$$\widehat{\mathbf{N}}(\mathbf{i}X) = \lambda \operatorname{Hom}_A(\omega, \mathbf{i}X) \cong \mathbf{i} \operatorname{Hom}_A(\omega, \mathbf{i}X) = \mathbf{i} \operatorname{RHom}_A(\omega, X)\,.$$

This yields the commutativity of the right-hand square.

For the second part of the lemma, it remains to note that the functor $\mathbf{N}$ preserves direct sums, as the A-module ω is finitely generated, and λ preserves direct sums, as it is a left adjoint. □

4. Gorenstein algebras and their derived categories

In this section, we introduce Gorenstein algebras and characterize them in terms of the derived Nakayama functor. This generalizes a well-known fact for Artin algebras. In that case, the algebra is Gorenstein if and only if the dualizing module is a tilting module, so that the derived Nakayama functor is an equivalence.

Commutative Gorenstein rings. — A commutative Noetherian ring R is *Gorenstein* if for each prime (equivalently, maximal) ideal $\mathfrak{p}$, the local ring $R_\mathfrak{p}$ has finite injective dimension as a module over itself [4]. When the Krull dimension of R is finite, this condition is equivalent to R itself having finite injective dimension; see [4, Theorem, §1] for details.

Gorenstein algebras. — We say that a ring A is a *Gorenstein* R-algebra if

(1) A is a finite R-algebra;
(2) A is projective as an R-module;
(3) $A_\mathfrak{p}$ is Iwanaga–Gorenstein for each $\mathfrak{p} \in \operatorname{Spec} R$ with $A_\mathfrak{p} \neq 0$.

Condition (3) means $A_\mathfrak{p}$ has finite injective dimension as a module over itself, on the left and on the right; then the injective dimensions coincide; see [45, Lemma A].

The following lemma provides a comparison between A and R with respect to the Gorenstein property.

Lemma 4.1. — *Let A be a Gorenstein R-algebra and $\mathfrak{p} \in \operatorname{Spec} R$. Then the ring $R_\mathfrak{p}$ is Gorenstein whenever $A_\mathfrak{p} \neq 0$.*

Proof. — As the R-module A is projective so is the $R_\mathfrak{p}$-module $A_\mathfrak{p}$, and hence for each finitely generated $R_\mathfrak{p}$-module M one has the isomorphism below

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(M, R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} A_\mathfrak{p} \cong \operatorname{Ext}^i_{A_\mathfrak{p}}(M \otimes_{R_\mathfrak{p}} A_\mathfrak{p}, A_\mathfrak{p}) = 0 \qquad \text{for } i \gg 0.$$

The equality on the right holds because the injective dimension of $A_\mathfrak{p}$ is finite. We deduce from the computation above that $\operatorname{Ext}^i_{R_\mathfrak{p}}(M, R_\mathfrak{p}) = 0$ for $i \gg 0$, since $A_\mathfrak{p} \neq 0$. Hence, $R_\mathfrak{p}$ is Gorenstein; see [12, Proposition 3.1.14]. □

Let A be a finite R-algebra that is projective as an R-module. Then R admits a decomposition $R' \times R''$ such that $A_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R'$, and A is finitely generated over R'. Thus, one may assume that A is faithful as an R-module, and then the Gorenstein property for A implies that R is Gorenstein; see [22] for details.

The preceding result also has a converse, but this plays no role in the sequel, so we discuss this at the end of this section; see Theorem 4.6. The Gorenstein condition is reflected also in the dualizing bimodule of the R-algebra A. To discuss this, we recall some aspects of perfect complexes over finite algebras.

Let A be a finite R-algebra and M a complex of A-modules. Recall that M is *perfect* if it is isomorphic in $\mathbf{D}(\operatorname{Mod} A)$ to a bounded complex of finitely generated projective A-modules; equivalently, M is compact, as an object in the triangulated category $\mathbf{D}(\operatorname{Mod} A)$; equivalently, M is in $\operatorname{Thick}(A)$; see [40, Theorem 2.2].

The following criterion for detecting perfect complexes will be handy.

Lemma 4.2. — *Let A be a finite R-algebra. For $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, the following conditions are equivalent.*

(1) *M is perfect in $\mathbf{D}(\operatorname{Mod} A)$;*
(2) *$M_{\mathfrak{m}}$ is perfect in $\mathbf{D}(\operatorname{Mod} A_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ in R;*
(3) *$\operatorname{Tor}_i^A(L, M) = 0$ for each $L \in \operatorname{mod} A^{\mathrm{op}}$ and $i \gg 0$.*

Proof. — The equivalence of (1) and (2) is due to Bass [5, Proposition III.6.6]. Evidently, (1) implies (3), and the reverse implication can be verified by an argument akin to that for [2, Theorem A.1.2]. □

Remark 4.3. — We say that a complex M of A-bimodules is *perfect on both sides*, if it is perfect both in $\mathbf{D}(\operatorname{Mod} A)$ and in $\mathbf{D}(\operatorname{Mod} A^{\mathrm{op}})$; said otherwise, the restriction of M along either map $A \to A^{\mathrm{ev}} \leftarrow A^{\mathrm{op}}$ is perfect, in the corresponding category.

We note also that when M is a complex of A-bimodules, $\operatorname{RHom}_A(M, A)$ has a left A-action induced by the right A-action on M, and a right action induced by the right A-action on A. In our context A is a projective R-module, so one can realize $\operatorname{RHom}_A(M, A)$ as a complex of bimodules, namely, the complex $\operatorname{Hom}_A(M, \mathbf{i}_{A^{\mathrm{ev}}} A)$.

Lemma 4.4. — *Let A be a finite R-algebra and M a complex of A-bimodules that is perfect on both sides. The following statements hold:*

(1) *There exists a quasi-isomorphism $P \to M$ of A-bimodules where P is bounded, consisting of finitely generated A-bimodules that are projective on both sides.*
(2) *When A is a Gorenstein R-algebra,* $\operatorname{RHom}_A(M, A)$ *is perfect on both sides.*

Proof. — (1) The hypothesis on M implies that the A^{ev}-module $H^*(M)$ is finitely generated. There thus exists a projective A^{ev}-resolution, say $Q \to M$ with each Q_i finitely generated and 0 for $i \ll 0$. Fix an integer

$$i \geq \max\{\operatorname{proj\,dim}_A M, \operatorname{proj\,dim}_{A^{\mathrm{op}}} M\}\,.$$

The morphism $Q \to M$ factors through the quotient complex

$$P := 0 \longrightarrow \operatorname{Coker}(d^Q_{i+1}) \longrightarrow Q_i \longrightarrow Q_{i-1} \longrightarrow \cdots$$

Since A-modules Q_i are projective on both sides, it follows by the choice of i that so is the A-module $\operatorname{Coker}(d^Q_{i+1})$. Thus, P is the complex we seek.

(2) That $\operatorname{RHom}_A(M,A)$ is perfect on the right is clear; for example, it is equivalent to $\operatorname{Hom}_A(P,A)$ with P as above; this does not involve the Gorenstein property.

As for the perfection on the left, by Lemma 4.2 it suffices to check the perfection locally on $\operatorname{Spec} R$. Thus, we can assume that the injective dimension of A is finite. For any finitely generated A^{op}-module L, one has a natural isomorphism

$$L \otimes^{\mathrm{L}}_A \operatorname{RHom}_A(M,A) \xrightarrow{\ \sim\ } \operatorname{RHom}_A(\operatorname{RHom}_{A^{\mathrm{op}}}(L,M),A)\,.$$

Since M is perfect over A^{op}, and A has finite injective dimension (on the right), so does M, and, hence, $H^*(\operatorname{RHom}_{A^{\mathrm{op}}}(L,M))$ is bounded. Then the finiteness of the injective dimension of A on the left implies that

$$H^*(\operatorname{RHom}_A(\operatorname{RHom}_{A^{\mathrm{op}}}(L,M),A))$$

is bounded. It thus follows from the quasi-isomorphism above that

$$\operatorname{Tor}^A_i(L, \operatorname{RHom}_A(M,A)) = 0 \qquad \text{for } |i| \gg 0.$$

This implies $\operatorname{RHom}_A(M,A)$ is perfect on the left; see Lemma 4.2. □

An equivalence of categories. — Let A be a Gorenstein R-algebra, $\omega_{A/R}$ its dualizing module, and $\mathbf{N}_{A/R}$ the Nakayama functor; see (5). As for finite dimensional algebras [28] the derived functor of the Nakayama functor is an auto-equivalence of the bounded derived category. In other words, $\omega_{A/R}$ is a tilting complex for A.

Theorem 4.5. — *Let A be a Gorenstein R-algebra. The A-bimodule $\omega_{A/R}$ is perfect on both sides and induces adjoint equivalences of triangulated categories*

$$\mathbf{D}(\operatorname{Mod} A) \underset{\operatorname{RHom}_A(\omega_{A/R},-)}{\overset{\omega_{A/R}\otimes^{\mathrm{L}}_A -}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} A)\,.$$

Moreover, these restrict to adjoint equivalences on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Proof. — The argument becomes a bit more transparent once we consider the ring $E := \operatorname{End}_A(\omega)$, and its natural left action on ω that is compatible with the left A-module structure. We first verify the following properties of ω:

(1) The natural maps $A \to E^{\mathrm{op}}$ and $A \to \operatorname{End}_E(\omega)$ of rings are isomorphisms.
(2) $\operatorname{Ext}^i_A(\omega,\omega) = 0 = \operatorname{Ext}^i_E(\omega,\omega)$ for $i \geq 1$.
(3) ω is compact both in $\mathbf{D}(\operatorname{Mod} A)$ and in $\mathbf{D}(\operatorname{Mod} E)$.

The first map in (1) is

$$A \longrightarrow \operatorname{End}_A(\omega)^{\mathrm{op}} \quad \text{where } a \mapsto (w \mapsto wa)\,.$$

A routine computation reveals that this is, indeed, a map of rings. Its bijectivity follows from the computation:

$$\begin{aligned} \operatorname{RHom}_A(\omega,\omega) &\cong \operatorname{RHom}_R(\operatorname{Hom}_R(A,R),R) \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(A,R),R) \\ &\cong A\,, \end{aligned}$$

where the first isomorphism is an adjunction, and the others hold because the R-module A is finite and projective. The computation above also establishes that $\operatorname{Ext}^i_A(\omega,\omega) = 0$ for $i \geq 1$. This justifies the first parts of the (1) and (2). Given that $A \xrightarrow{\sim} E^{\mathrm{op}}$, applying the already established part of the result to A^{op} completes the argument for (1) and (2).

It remains to verify (3), and again, given that $E \cong A^{\mathrm{op}}$ as rings, it suffices to check that ω is perfect in $\mathbf{D}(\operatorname{Mod} A)$. Since the A-module ω is finitely generated it suffices to prove that it has finite projective dimension as an A-module. By Lemma 4.2 it suffices to verify that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ has finite projective dimension for each $\mathfrak{p} \in \operatorname{Spec} R$. Since

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong \operatorname{Hom}_R(A,R)_{\mathfrak{p}}$$

as $A_{\mathfrak{p}}$-bimodules, and $A_{\mathfrak{p}}$ is a Gorenstein $R_{\mathfrak{p}}$-algebra, replacing R and A by their localizations at $\mathfrak{p}$ we can assume that $(R,\mathfrak{m},k)$ is a local ring and A is a Gorenstein R-algebra of finite injective dimension; the desired conclusion is that the projective dimension of $\operatorname{Hom}_R(A,R)$ is finite. At this point, one can invoke [13, Proposition 7.6.3(ii)] to complete the proof. The proof of *op. cit.* uses the theory of Cohen–Macaulay approximations. Here is a direct argument:

Since R is Gorenstein, by Lemma 4.1, and local, it has finite injective dimension; choose a finite injective resolution $R \to \mathbf{i}R$. Choose also a finite injective resolution $A \to \mathbf{i}A$. Then $\operatorname{Hom}_R(\mathbf{i}A, \mathbf{i}R)$ is a bounded complex of flat A-modules, quasi-isomorphic to $\operatorname{Hom}_R(A,R)$; thus the A-module $\operatorname{Hom}_R(A,R)$ has finite flat dimension. Since it is also finitely generated, it follows that its projective dimension is finite; see Lemma 4.2.

This completes the proofs of assertions (1)–(3).

Next we verify the stated equivalence of (the full derived) categories. This is a standard argument, given the properties of ω. Here is a sketch. To begin with, given the isomorphism $A \cong E^{\mathrm{op}}$ of rings, the stated adjunction can be factored as

$$\mathbf{D}(\operatorname{Mod} A) \underset{-\otimes^{\mathrm{L}}_{E}\omega}{\overset{\mathrm{RHom}_A(\omega,-)}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} E^{\mathrm{op}}) \xrightarrow{\ \sim\ } \mathbf{D}(\operatorname{Mod} A)\,.$$

It thus suffices to verify that the adjoint pair on the left are quasi-inverses to each other, that is to say that their counit and unit of the adjunction are isomorphisms. The counit is the evaluation map

$$\varepsilon(M)\colon\ \mathrm{RHom}_A(\omega, M)\otimes^{\mathrm{L}}_{E}\omega \longrightarrow M \quad \text{for } M \text{ in } \mathbf{D}(\operatorname{Mod} A).$$

The map above is an isomorphism, for it factors as the composition of isomorphisms

$$\begin{aligned}\mathrm{RHom}_A(\omega, M)\otimes^{\mathrm{L}}_{E}\omega &\xrightarrow{\sim} \mathrm{RHom}_A(\mathrm{RHom}_E(\omega,\omega), M)\\ &\xrightarrow{\sim} \mathrm{RHom}_A(A, M)\\ &\xrightarrow{\sim} M\,,\end{aligned}$$

where the first map is standard and is a quasi-isomorphism because ω is compact in $\mathbf{D}(\operatorname{Mod} E)$, by (3) above, and the second map is induced by the natural map $A \to \mathrm{RHom}_E(\omega,\omega)$ that is a quasi-isomorphism because of properties (1) and (2). Similarly, the unit map

$$N \longrightarrow \mathrm{RHom}_A(\omega, N\otimes^{\mathrm{L}}_{E}\omega)$$

is a quasi-isomorphism, for all N in $\mathbf{D}(\operatorname{Mod} A)$, for it factors as the composition

$$N \xrightarrow{\sim} N\otimes_E \mathrm{RHom}_A(\omega,\omega) \xrightarrow{\sim} \mathrm{RHom}_A(\omega, N\otimes_E\omega)\,,$$

where the first map is induced by the isomorphism $E \xrightarrow{\sim} \mathrm{RHom}_A(\omega,\omega)$, and the second one is standard and is an isomorphism because ω is perfect in $\mathbf{D}(\operatorname{Mod} A)$.

This completes the proof that the stated adjoint pair of functors induce an equivalence on $\mathbf{D}(\operatorname{Mod} A)$. It remains to note that for each M in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, the A-complex $\mathrm{RHom}_A(\omega, M)$ and $\omega\otimes^{\mathrm{L}}_{A} M$ are in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ as well, because ω is compact on both sides. Thus, they restrict to adjoint equivalences on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. □

We can now offer converses to Lemma 4.1; see Goto [23] for a similar statement in commutative algebra. Regarding condition (3), it is noteworthy that the injective dimension of A need not be finite; so there need not be a global bound (independent of M) on the degree i beyond which $\mathrm{Ext}^i_A(M, A)$ is zero. Indeed, there exist even commutative Gorenstein rings R that exhibit this phenomenon; see [39, A1].

THEOREM 4.6. — *Let R be a commutative Noetherian Gorenstein ring, and A a finite, projective, R-algebra. The following conditions are equivalent.*

(1) *The R-algebra A is Gorenstein.*
(2) *The A-bimodule $\omega_{A/R}$ is perfect on both sides.*
(3) *For each $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} A^{\mathrm{op}}$, we have $\operatorname{Ext}^i_A(M, A) = 0$ for $i \gg 0$ and $\operatorname{Ext}^i_{A^{\mathrm{op}}}(N, A) = 0$ for $i \gg 0$.*
(4) *The functors $\operatorname{RHom}_A(-, A)$ and $\operatorname{RHom}_{A^{\mathrm{op}}}(-, A)$ induce triangle equivalences*

$$\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)^{\mathrm{op}} \underset{\operatorname{RHom}_{A^{\mathrm{op}}}(-,A)}{\overset{\operatorname{RHom}_A(-,A)}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}}) .$$

Proof. — The proof that (1)⇒(2) is contained in Lemma 4.1 and Theorem 4.5.

(2)⇒(1) The hypotheses are local with respect to primes in $\operatorname{Spec} R$, as is the conclusion, by definition. We may thus assume that R is local and, hence, of finite injective dimension. Then, since A is a projective R-module, it follows from adjunction that the A-module $\omega = \operatorname{Hom}_R(A, R)$ has finite injective dimension on both sides. For the same reason, one gets that the following natural map is a quasi-isomorphism

$$A \longrightarrow \operatorname{RHom}_{A^{\mathrm{op}}}(\omega, \omega) ;$$

see the proof of Theorem 4.5. As ω is perfect on the right, it is in $\operatorname{Thick}(A)$ in $\mathbf{D}(\operatorname{Mod} A^{\mathrm{op}})$, and the quasi-isomorphism above implies that A is in $\operatorname{Thick}(\omega)$ in $\mathbf{D}(\operatorname{Mod} A)$. In particular, since the injective dimension of ω as a left A-module is finite, so is that of A. Similarly, we deduce that the injective dimension of A is finite also on the right.

(1)⇒(3) Suppose A is a Gorenstein R-algebra and fix an M in $\operatorname{mod} A$. Since A is in $\operatorname{Thick}(\omega)$ in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, it suffices to verify that $\operatorname{Ext}^i_A(M, \omega)$ for $i \gg 0$. Adjunction yields

$$\operatorname{Ext}^i_A(M, \omega) = \operatorname{Ext}^i_A(M, \operatorname{Hom}_R(A, R)) \cong \operatorname{Ext}^i_R(M, R) .$$

As R is Gorenstein, by Lemma 4.1, the problem reduces to the commutative case, where the result is due to Goto [23, Theorem 1]. The same argument gives the result for N in $\operatorname{mod} A^{\mathrm{op}}$.

(3)⇒(1) For each prime $\mathfrak{p}$ in $\operatorname{Spec} R$ and M in $\operatorname{mod} A$, we have an isomorphism

$$\operatorname{Ext}^i_A(M, A)_{\mathfrak{p}} \cong \operatorname{Ext}^i_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, A_{\mathfrak{p}}) \qquad (i \geq 0) .$$

If this vanishes for each M and $i \gg 0$, then $A_{\mathfrak{p}}$ has finite injective dimension as a left $A_{\mathfrak{p}}$-module. Analogously, $A_{\mathfrak{p}}$ has finite injective dimension as a right $A_{\mathfrak{p}}$-module. Thus, A is Gorenstein.

(1)⇒(4) For each $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, the A^{op}-complex $\operatorname{RHom}_A(M, A)$ belongs to $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})$, by the already verified implication (1)⇒(3), so it remains to verify that the natural biduality morphism

$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M,A),A)$$

is an isomorphism. Since $\mathrm{RHom}_A(M,A)$ is in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A^{\mathrm{op}})$ this can be checked locally on $\mathrm{Spec}\,R$, where it holds for the injective dimension of A is locally finite. The same argument gives the result for N in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A^{\mathrm{op}})$.

(4)⇒(3) Clear. □

Remark 4.7. — The argument in the proof of Theorem 4.6 raises the question: When A is a Gorenstein R-algebra, is $\omega_{A/R}$ generated by A in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A^{\mathrm{ev}})$, that is to say, is it in $\mathrm{Thick}_{A^{\mathrm{ev}}}(A)$? By standard arguments, this question is equivalent to: Is

$$\mathrm{RHom}_R(A\otimes^{\mathrm{L}}_{A^{\mathrm{ev}}}A,R) \xrightarrow{\sim} \mathrm{RHom}_{A^{\mathrm{ev}}}(A,\omega_{A/R})$$

perfect as a dg module over $\mathcal{E} := \mathrm{RHom}_{A^{\mathrm{ev}}}(A,A)$, the (derived) Hochschild cohomology algebra? When this condition holds, it would follow from the isomorphism above that if $\mathrm{HH}^i(A/R)=0$ for $i\gg 0$, then also $\mathrm{HH}_i(A/R)=0$ for $i\gg 0$.

This turns out not to be the case when A is finite dimensional and self-injective over a field: Let k be a field, $q\in k$ an element that is nonzero and not a root of unity, and set

$$\Lambda := \frac{k\langle x,y\rangle}{(x^2, xy+qyx, y^2)}\,.$$

Then Buchweitz, Madsen, Green, and Solberg prove that $\mathrm{rank}_k\,\mathrm{HH}^*(A/k)=5$, whereas $\mathrm{HH}_i(A/k)$ is nonzero for each $i\geq 0$ [14].

On the other hand, the question has, trivially, a positive answer when A is a symmetric R-algebra, that is to say, when $\omega_{A/R}\cong A$ as an A-bimodule. So this begs the question: If $\omega_{A/R}$ is in $\mathrm{Thick}_{A^{\mathrm{ev}}}(A)$, is then A a symmetric R-algebra?

5. Gorenstein algebras and their homotopy categories

Let A be a Gorenstein R-algebra. We study in this case the properties of the Nakayama functor for the homotopy category of injectives $\mathbf{K}(\mathrm{Inj}\,A)$.

The Nakayama functor. — As explained in Section 3, the Nakayama functor admits a canonical extension to a functor $\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\mathrm{Inj}\,A)\to\mathbf{K}(\mathrm{Inj}\,A)$. The following result discusses the compatibility of this functor with the recollement for $\mathbf{K}(\mathrm{Inj}\,A)$ introduced in (1) and the equivalence on $\mathbf{D}(\mathrm{Mod}\,A)$ in Theorem 4.5.

THEOREM 5.1. — *Let A be a Gorenstein R-algebra. The functor $\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}(\operatorname{Inj} A)$ is a triangle equivalence making the following square commutative:*

$$\begin{array}{ccc} \mathbf{D}(\operatorname{Mod} A) & \xrightarrow{\ \mathbf{i}\ } & \mathbf{K}(\operatorname{Inj} A) \\ \downarrow{\scriptstyle \operatorname{RHom}_A(\omega,-)} & & \downarrow{\scriptstyle \widehat{\mathbf{N}}} \\ \mathbf{D}(\operatorname{Mod} A) & \xrightarrow{\ \mathbf{i}\ } & \mathbf{K}(\operatorname{Inj} A)\,. \end{array}$$

Moreover, $\widehat{\mathbf{N}}_{A/R}$ restricts to an equivalence $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$.

The key step in the proof of the result is a “concrete" description of $\widehat{\mathbf{N}}$; see Lemma 5.2 below. To that end note that Lemma 4.4 applies to the dualizing bimodule $\omega_{A/R}$; fix a complex P provided by that result and set $\widehat{\omega}_{A/R} := P$. Thus,

$$\widehat{\omega}_{A/R} \longrightarrow \omega_{A/R}$$

is a finite resolution of $\omega_{A/R}$ by finitely generated A-bimodules that are projective on either side. This implies, in particular, that when X is a complex of injective A-modules, so is $\operatorname{Hom}_A(\widehat{\omega}_{A/R}, X)$; this follows from the standard Hom-tensor adjunction and requires only that $\widehat{\omega}_{A/R}$ consists of modules projective on the right. One thus has the induced exact functor

$$\operatorname{Hom}_A(\widehat{\omega}_{A/R}, -)\colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}(\operatorname{Inj} A)\,.$$

Here is the vouched for description of the completion of the Nakayama functor.

LEMMA 5.2. — *The quasi-isomorphism $\widehat{\omega}_{A/R} \to \omega_{A/R}$ induces an isomorphism*

$$\widehat{\mathbf{N}}_{A/R} \xrightarrow{\sim} \operatorname{Hom}_A(\widehat{\omega}_{A/R}, -)$$

of functors on $\mathbf{K}(\operatorname{Inj} A)$.

Proof. — For $X \in \mathbf{K}(\operatorname{Inj} A)$, the morphism $\widehat{\omega} \to \omega$ induces the morphism

$$\operatorname{Hom}_A(\omega, X) \longrightarrow \operatorname{Hom}_A(\widehat{\omega}, X)$$

of complexes of A-modules. Since $\operatorname{Hom}_A(\widehat{\omega}, X)$ consists of injective modules, one gets an induced morphism

$$\widehat{\mathbf{N}}(X) = \lambda \operatorname{Hom}_A(\omega, X) \longrightarrow \operatorname{Hom}_A(\widehat{\omega}, X)\,.$$

This is the natural transformation in question. The functors $\widehat{\mathbf{N}}$ and $\operatorname{Hom}_A(\widehat{\omega}, -)$ preserve arbitrary direct sums, the former by Lemma 3.1 and the latter because $\widehat{\omega}$ is a bounded complex of finitely generated modules, by choice. Thus, it suffices to verify that the morphism above is an isomorphism when X is compact in $\mathbf{K}(\operatorname{Inj} A)$, that is to say, when it is of the form $\mathbf{i}M$, for some $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. In this case, the morphism in question is the composite

$$\widehat{\mathbf{N}}(\mathbf{i}M) \xrightarrow{\sim} \mathbf{i}\operatorname{Hom}_A(\omega, \mathbf{i}M) \to \operatorname{Hom}_A(\widehat{\omega}, \mathbf{i}M)\,,$$

where the isomorphism is taken from Lemma 3.1. The map above is a quasi-isomorphism and its source and target are K-injective; the former by construction and the latter because $\widehat{\omega}$ is a bounded complex of projectives. It remains to observe that a quasi-isomorphism between K-injectives is an isomorphism in $\mathbf{K}(\operatorname{Inj} A)$. □

Proof of Theorem 5.1. — Given Lemma 5.2, a standard dévissage argument shows that $\widehat{\mathbf{N}}$ is a triangle equivalence: the functor preserves arbitrary direct sums and identifies with $\operatorname{RHom}_A(\omega, -)$ when restricted to compacts, by Lemma 3.1. It remains to note that $\operatorname{RHom}_A(\omega, -)$ is an equivalence on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, by Theorem 4.5.

For the commutativity of the square, fix a complex $X \in \mathbf{D}(\operatorname{Mod} A)$. We have already seen in Lemma 3.1 that

$$\widehat{\mathbf{N}}(\mathbf{i}X) \xrightarrow{\sim} \mathbf{i}\operatorname{RHom}_A(\omega, X)\,,$$

when X is bounded below. An arbitrary complex in $\mathbf{D}(\operatorname{Mod} A)$ is quasi-isomorphic to a homotopy limit of complexes that are bounded below. Thus, it remains to observe that both functors preserve homotopy limits.

It remains to verify that $\widehat{\mathbf{N}}$ restricts to an equivalence between acyclic complexes; equivalently that a complex $X \in \mathbf{K}(\operatorname{Inj} A)$ is acyclic if and only if $\widehat{\mathbf{N}}(X)$ is acyclic.

Since $\widehat{\omega}$ is perfect on the left, $\widehat{\mathbf{N}}$ preserves acyclic complexes. On the other hand, since $\operatorname{RHom}_A(\omega, A)$ is in $\operatorname{Thick}(A)$ in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ by Lemma 4.4, it follows that $\widehat{\mathbf{N}}(\mathbf{i}A)$ is in $\operatorname{Thick}(\mathbf{i}A)$. Using the isomorphism

$$H^n(X) \cong \operatorname{Hom}_{\mathbf{K}}(\mathbf{i}A, \Sigma^n X) \cong \operatorname{Hom}_{\mathbf{K}}(\widehat{\mathbf{N}}(\mathbf{i}A), \Sigma^n \widehat{\mathbf{N}}(X))$$

it follows that when $\widehat{\mathbf{N}}(X)$ is acyclic so is X. □

Remark 5.3. — One may turn $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ into a dg category such that $\mathbf{K}(\operatorname{Inj} A)$ identifies with its derived category; see [35, Appendix A]. Then $\widehat{\mathbf{N}}_{A/R}$ identifies with the lift of the Nakayama functor $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Remark 5.4. — If X is a complex of projective A-modules, then so is the A-complex $\widehat{\omega}_{A/R} \otimes_A X$; this is because $\widehat{\omega}$ consists of modules projective on the left. Thus, one gets an exact functor

$$\widehat{\omega}_{A/R} \otimes_A - \colon \mathbf{K}(\operatorname{Proj} A) \longrightarrow \mathbf{K}(\operatorname{Proj} A)\,.$$

Arguing as in the proof of Theorem 5.1 one can verify that this is also an equivalence of categories.

Since the Nakayama functor $\widehat{\mathbf{N}}_{A/R}$ is an equivalence, it has a quasi-inverse. This is described below.

A quasi-inverse. — Set $V := \operatorname{Hom}_A(\widehat{\omega}_{A/R}, A)$; this is a bounded complex of A-bimodules where the left action is through the right A-module structure on $\widehat{\omega}_{A/R}$ and the right action is through the right A-module structure of A.

PROPOSITION 5.5. — *The assignment $X \mapsto \operatorname{Hom}_A(V, X)$ induces an exact functor*

$$\operatorname{Hom}_A(V, -) \colon \mathbf{K}(\operatorname{Inj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A)\,.$$

This functor is a quasi-inverse of $\widehat{\mathbf{N}}_{A/R}$, and so an equivalence of categories.

Proof. — The complex $\widehat{\omega}$ consists of modules projective on the left, and the right A-action on $V = \operatorname{Hom}_A(\widehat{\omega}, A)$ is through A, so V consists of modules that are projective on the right. Given this it is easy to verify that $\operatorname{Hom}_A(V, -)$ maps complexes of injectives to complexes of injectives and so induces an exact functor on $\mathbf{K}(\operatorname{Inj} A)$. For $X \in \mathbf{K}(\operatorname{Inj} A)$, the natural morphism of complexes

$$V \otimes_A X = \operatorname{Hom}_A(\widehat{\omega}, A) \otimes_A X \longrightarrow \operatorname{Hom}_A(\widehat{\omega}, X)$$

is an isomorphism because the complex $\widehat{\omega}$ is a bounded complex of modules projective on the left. This justifies the second isomorphism below:

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, \operatorname{Hom}_A(V, \operatorname{Hom}_A(\widehat{\omega}, X))) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(V \otimes_A X, \operatorname{Hom}_A(\widehat{\omega}, X))\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\operatorname{Hom}_A(\widehat{\omega}, X), \operatorname{Hom}_A(\widehat{\omega}, X))\,.\end{aligned}$$

The first one is adjunction. Thus, the identity on $\operatorname{Hom}_A(\widehat{\omega}, X)$ induces a morphism

$$\eta(X) \colon X \longrightarrow \operatorname{Hom}_A(V, \operatorname{Hom}_A(\widehat{\omega}, X))\,,$$

which is natural in X. As functors of X, both the source and the target of η are exact and preserves direct sums; thus, to verify that $\eta(X)$ is an isomorphism for each X it suffices to verify that this is so for compact objects in $\mathbf{K}(\operatorname{Inj} A)$, that is to say, for the induced natural transformation on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. This is the map

$$M \mapsto \operatorname{RHom}_A(\operatorname{RHom}_A(\omega, A), \operatorname{RHom}_A(\omega, M))\,.$$

Since ω and $\operatorname{RHom}_A(\omega, A)$ are perfect as complexes of left A-modules, by Theorem 4.5 and Lemma 4.4, respectively, the map above can be obtained by applying $(-) \otimes^{\mathrm{L}}_A M$ to the natural homothety morphism

$$A \longrightarrow \operatorname{RHom}_A(\operatorname{RHom}_A(\omega, A), \operatorname{RHom}_A(\omega, A))\,.$$

Observe this a morphism in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{ev}})$. It remains to note that the map above is a quasi-isomorphism by, for example, Theorem 4.5. □

Acyclicity versus total acyclicity. — Set $E := \mathbf{i}_{A^{\mathrm{ev}}} A$, the injective resolution of A as an A-bimodule, and consider adjoint functors

$$\mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{f}}{\overset{}{\rightleftarrows}} \mathbf{K}(\operatorname{Flat} A) \underset{\operatorname{Hom}_A(E,-)}{\overset{E\otimes_A -}{\rightleftarrows}} \mathbf{K}(\operatorname{Inj} A)\,,$$

where $\mathbf{f}$ is the right adjoint to the inclusion. It exists because $\mathbf{K}(\operatorname{Proj} A)$ is a compactly generated triangulated category, and its inclusion in $\mathbf{K}(\operatorname{Flat} A)$ is compatible with coproducts; see [32, Proposition 2.4]. One thus gets an adjoint pair

$$\mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{h}}{\overset{\mathbf{t}}{\rightleftarrows}} \mathbf{K}(\operatorname{Inj} A)\,,$$

where $\mathbf{t} := E \otimes_A -$ and $\mathbf{h} := \mathbf{f} \circ \operatorname{Hom}_A(E, -)$.

Let $\mathcal{A}$ be an additive category. A complex $X \in \mathbf{K}(\mathcal{A})$ is called *totally acyclic* if $\operatorname{Hom}(W, X)$ and $\operatorname{Hom}(X, W)$ are acyclic complexes of abelian groups for all $W \in \mathcal{A}$. We denote by $\mathbf{K}_{\mathrm{tac}}(\mathcal{A})$ the full subcategory of totally acyclic complexes.

Theorem 5.6. — *Let A be a Gorenstein R-algebra. The adjoint functors* $(\mathbf{t}, \mathbf{h})$ *above are equivalences of categories, and they restrict to equivalences*

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \underset{\mathbf{h}}{\overset{\mathbf{t}}{\underset{\sim}{\rightleftarrows}}} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,.$$

Moreover, there are equalities

$$\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \quad \textit{and} \quad \mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} A) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,.$$

Proof. — It is clear that the functor $\mathbf{t}$ preserves direct sums. It also preserves compact objects, as we now explain. We may assume that a compact object in $\mathbf{K}(\operatorname{Proj} A)$ is of the form $\operatorname{Hom}_A(\mathbf{p}M, A)$ for some $M \in \operatorname{mod} A^{\mathrm{op}}$. This yields a complex

$$E \otimes_A \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, A) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, E)\,,$$

which is compact in $\mathbf{K}(\operatorname{Inj} A)$ because it is bounded below with

$$H^i \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, E) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(M, A) = 0$$

, for $i \gg 0$, by Theorem 4.6. In fact, the functor $\mathbf{t}$ restricted to compacts identifies with

$$\operatorname{RHom}_{A^{\mathrm{op}}}(-, A)\colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}}) \longrightarrow \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)^{\mathrm{op}}\,,$$

and this is an equivalence, again by Theorem 4.6. Thus, $\mathbf{t}$ is an equivalence of categories. Moreover, since $\mathbf{h}$ is its adjoint, the latter is the quasi-inverse to $\mathbf{t}$.

For $X \in \mathbf{K}(\operatorname{Proj} A)$, the equivalence of categories and Lemma 2.3 yield

$$H^n(X) = \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(A, \Sigma^n X) \cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, \Sigma^n \mathbf{t}X) = H^n(\mathbf{t}X)\,,$$

for each integer n. Thus, X is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ if and only $\mathbf{t}X$ is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. Therefore, $(\mathbf{t}, \mathbf{h})$ induce an equivalence on the subcategory of acyclic complexes.

The key to verifying the remaining assertions is the following.

Claim. — $\operatorname{Inj} A \subset \operatorname{Loc}(E)$, *in* $\mathbf{K}(\operatorname{Inj} A)$.

Indeed, given the already established equivalence, it suffices to verify that $\mathbf{h}I$ is in $\operatorname{Loc}(A)$ for any injective A-module I, since $\mathbf{h}$ identifies E with A. As E is a complex of injective modules that are bounded below, $\operatorname{Hom}_A(E, I)$ is a complex of flat modules that are bounded above, and it is quasi-isomorphic to I, by Lemma 2.3. Therefore, $\mathbf{h}I = \mathbf{f}\operatorname{Hom}_A(E, I)$ is a projective resolution of I; see [32, Theorem 2.7(2)]. Thus, $\mathbf{h}I$ is in $\operatorname{Loc}(A)$, as desired.

Fix $Y \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. Then $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, \Sigma^n Y) = 0$ for each integer n, so the claim yields $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(I, \Sigma^n Y) = 0$, for $I \in \operatorname{Inj} A$ and integers n, that is to say, Y is totally acyclic. Thus, any acyclic complex of injective modules is totally acyclic.

Fix an acyclic complex X in $\mathbf{K}(\operatorname{Proj} A)$. We want to verify that X is totally acyclic, that is to say, $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, -) = 0$ on $\operatorname{Add} A$. Since $\mathbf{t}$ is an equivalence of categories, it suffices to verify that $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{t}X, -) = 0$ on $\operatorname{Add} \mathbf{t}A$, that is to say, on $\operatorname{Add} E$. However, $\mathbf{t}X$ is also acyclic, by the already established part of the result, and any complex in $\operatorname{Add} E$ is bounded below, and hence K-injective. This implies the desired result. □

6. Gorenstein projective modules

Let A be a Gorenstein R-algebra. An A-module M is *Gorenstein projective* (abbreviated to G-projective) if M is a syzygy in a totally acyclic complex of projective modules, that is, $M \cong \operatorname{Coker}(d_X^{-1})$, for some X in $\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$. Given Theorem 5.6, one can "totally" drop from the definition. We write $\operatorname{GProj} A$ for the full subcategory of $\operatorname{Mod} A$ consisting of G-projectives, and $\operatorname{Gproj} A$ for $\operatorname{GProj} A \cap \operatorname{mod} A$.

Starting from Theorem 5.6, and also the results below, one can develop the theory of G-projective modules along the lines in [13], but we shall be content with recording a few observations needed to prove the duality theorems in Section 9. All these are well known when A is Iwanaga–Gorenstein.

Lemma 6.1. — *Let M be a G-projective A-module. The following statements hold.*

(1) *$M_{\mathfrak{p}}$ is G-projective as an $A_{\mathfrak{p}}$-module for $\mathfrak{p} \in \operatorname{Spec} R$.*
(2) $\operatorname{Tor}_i^A(\omega_{A/R}, M) = 0 = \operatorname{Ext}_A^i(\omega_{A/R}, M)$ *for $i \geq 1$.*

Proof. — Evidently, the localization of an acyclic complex is acyclic, so (1) follows.

(2) Since an A-module is zero if it is zero locally on $\operatorname{Spec} R$, given (1) and the finite generation of ω, we can reduce the verification of (2) to the case when R is local and so assume that the injective dimension of A is finite. Let I be the injective hull of the residue field of R and set $J := \operatorname{Hom}_R(\omega, I)$.

Claim. — *The A-module J is a faithful injective and has finite projective dimension.*

Indeed, as I is a faithful injective R-module, it follows by adjunction that the A-module J is faithful and injective. Since R is a Gorenstein local ring it has finite injective dimension, so I has finite projective dimension; that is to say, I is in $\operatorname{Thick}(\operatorname{Add} R)$ in $\mathbf{D}(\operatorname{Mod} R)$. Since ω is a finite projective R-module $\operatorname{Hom}_R(\omega, -)$ is an exact functor on $\mathbf{D}(\operatorname{Mod} A)$, so we deduce that J is in $\operatorname{Thick}(\operatorname{Add} \operatorname{Hom}_R(\omega, R))$ in $\mathbf{D}(\operatorname{Mod} A)$. Finally, observe that $A \cong \operatorname{Hom}_R(\omega, R)$ as A-modules.

The claim and the hypothesis that M is G-projective justify the equality below:

$$\operatorname{Hom}_R(\operatorname{Tor}_i^A(\omega, M), I) \cong \operatorname{Ext}_A^i(M, J) = 0 \quad \text{for } i \geq 1;$$

see also (7). The isomorphism is a standard adjunction. Since I is a faithful injective, it follows that $\operatorname{Tor}_i^A(\omega, M) = 0$ as desired.

A similar argument settles the claim about the vanishing of Ext-modules. □

When M is G-projective and $X \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ is as above, the truncation $X_{\geqslant 0}$ is a projective resolution of M, and the total acyclicity of X implies

$$\operatorname{Ext}_A^i(M, P) = 0 \quad \text{for each projective module } P \text{ and } i \geq 1. \tag{7}$$

Here is a partial converse.

Lemma 6.2. — *A finitely generated A-module M satisfying $\operatorname{Ext}_A^i(M, A) = 0$, for $i \geq 1$, is G-projective. Moreover, such a module is a syzygy in an acyclic complex of finitely generated projective A-modules.*

Proof. — It suffices to verify that $\operatorname{Ext}_{A^{\mathrm{op}}}^i(M^*, A) = 0$, for $i \geq 1$, and that the biduality map $M \to M^{**}$ is bijective; given these, it is straightforward to construct an acyclic complex with M as a syzygy. What is more, using resolutions of M and M^* by finitely generated projective modules, one can get an acyclic complex consisting of finitely generated projective modules. Since M is finitely generated, and A is a finite R-algebra, both the conditions in question can be checked locally on $\operatorname{Spec} R$. We may thus assume that A is Iwananga–Gorenstein, in which case, the desired result is contained in [13, Lemma 4.2.2(iii)]. □

With exact structure inherited from $\operatorname{Mod} A$, the category $\operatorname{GProj} A$ is Frobenius, with projective objects $\operatorname{Proj} A$. Thus, the associated stable category,

$\underline{\mathrm{GProj}}A$, is triangulated. It is also compactly generated, with compact objects $\underline{\mathrm{Gproj}}A$; see, for example, [10, Proposition 2.10]. By the very definition, G-projectives are related to acyclic complexes of projectives. To clarify this connection, we recall from [32, §7.6] that there is an adjoint pair

$$\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A) \underset{\mathbf{a}}{\overset{}{\rightleftarrows}} \mathbf{K}(\mathrm{Proj}\,A)\,,$$

where the left adjoint is the inclusion. The next result is well known and can be readily proved by adapting the argument for [13, Theorem 4.4.1].

Proposition 6.3. — *The composition of functors* $\mathbf{a}\circ\mathbf{p}\colon \mathrm{Mod}\,A \to \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A)$ *induces a triangle equivalence*

$$\mathbf{ap}\colon \underline{\mathrm{GProj}}A \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A)\,,$$

with the quasi-inverse defined by the assignment $X \mapsto \mathrm{Coker}(d_X^{-1})$. □

The singularity category. — Let $\mathbf{D}_{\mathrm{sg}}(A)$ be the *singularity category* of A introduced by Buchweitz [13] as the *stable derived category.* It is $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A)$ modulo the perfect complexes. Any perfect complex is in the kernel of the functor

$$\mathbf{si}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A) \to \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\,A)^{\mathrm{c}}\,,$$

where the functors $\mathbf{s}$ and $\mathbf{i}$ are from (1). Hence, there is an induced exact functor

$$\mathbf{D}_{\mathrm{sg}}(A) \to \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\,A)^{\mathrm{c}}\,,$$

which we also denote $\mathbf{si}$. On the other hand, the embedding $\mathrm{Gproj}\,A \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A)$ induces an exact functor

$$\mathbf{g}\colon \underline{\mathrm{Gproj}}A \longrightarrow \mathbf{D}_{\mathrm{sg}}(A)\,.$$

The result below was proved by Buchweitz [13, Theorem 4.4.1] when A is an Iwanaga–Gorenstein ring.

Theorem 6.4. — *Let A be a Gorenstein R-algebra. The functors* $\mathbf{g}$ *and* $\mathbf{si}$ *are equivalences, up to direct summands, of triangulated categories:*

$$\underline{\mathrm{Gproj}}(A) \xrightarrow[\mathbf{g}]{\sim} \mathbf{D}_{\mathrm{sg}}(A) \xrightarrow[\mathbf{si}]{\sim} \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\,A)^{\mathrm{c}}\,.$$

Proof. — The assertion about $\mathbf{si}$ is by [35, Corollary 5.4].

Let M, N be finitely generated G-projective A-modules. As noted in (7), one has $\mathrm{Ext}^i_A(M, A) = 0$ for $i \geq 1$. Arguing as in the proof of [42, Proposition 1.21] one gets that $\mathbf{g}$ induces a bijection:

$$\underline{\mathrm{Hom}}_A(M,N) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}}(\mathbf{g}M, \mathbf{g}N)\,.$$

Thus, $\mathbf{g}$ is fully faithful. It remains to prove that it is essentially surjective.

Fix X in $\mathbf{D}_{\mathrm{sg}}(A)$; we can assume that X is a bounded-above complex of finitely generated projective A-modules. Suppose $H^i(X) = 0$, for all $i < n$. Truncating at n yields a morphism $X \to \sigma_{\leq n}X$, which is an isomorphism in $\mathbf{D}_{\mathrm{sg}}(A)$ since its cone is perfect. Thus, X is isomorphic to a suspension of $M := \operatorname{Coker}(d_X^{n-1})$ in $\mathbf{D}_{\mathrm{sg}}(A)$. Since $\operatorname{Ext}_A^i(M, A)$, for $i \gg 0$ by Theorem 4.6, some syzygy of M is G-projective by Lemma 6.2, and we conclude that $\mathbf{g}$ is essentially surjective. □

A standard dévissage argument yields the following consequence.

Corollary 6.5. — *The composition of functor* $\mathbf{s} \circ \mathbf{i} \colon \operatorname{Mod} A \to \mathbf{K}(\operatorname{Inj} A)$ *induces a triangle equivalence*

$$\mathbf{si} \colon \underline{\mathrm{GProj}}A \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,.$$

Proof. — The triangulated categories $\underline{\mathrm{GProj}}A$ and $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj})$ are both compactly generated, and the functor $\mathbf{si}$ preserves coproducts. For the compact generation of $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj})$, see [35, Corollary 5.4], and $\mathbf{si}$ preserves coproducts since $\mathbf{s}$ is a left adjoint. Thus, the assertion follows from the fact that $\mathbf{si}$ is a triangle equivalence when restricted to the subcategories of compact objects; see Theorem 6.4. □

The Nakayama functor. — Via the equivalences of categories established above the auto-equivalence of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ given by Nakayama functor induces an auto-equivalence on $\underline{\mathrm{GProj}}A$ and on the singularity category. This is made explicit in the next two results. The functor $\mathrm{GP}(-)$ that appears in the statements is the G-projective approximation whose existence is established in Theorem A.1. When A is a Gorenstein R-algebra, it follows from Theorem 4.5 that functor $\omega_{A/R}$ takes perfect complexes to perfect complexes, and hence induces a functor on the quotient $\mathbf{D}_{\mathrm{sg}}(A)$; we also denote that functor $\omega_{A/R} \otimes^{\mathrm{L}}_A (-)$.

Proposition 6.6. — *Let A be a Gorenstein R-algebra. One has the following diagram of equivalences of categories*

$$\begin{array}{ccccc}
\underline{\mathrm{Gproj}}(A) & \xrightarrow[\mathbf{g}]{\sim} & \mathbf{D}_{\mathrm{sg}}(A) & \xrightarrow[\mathbf{si}]{\sim} & \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}} \\
{\scriptstyle \mathrm{GP}(\omega_{A/R}\otimes_A(-))}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle \omega_{A/R}\otimes^{\mathrm{L}}_A(-)}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle \widehat{\mathbf{N}}^{-1}_{A/R}} \\
\underline{\mathrm{Gproj}}(A) & \xrightarrow[\mathbf{g}]{\sim} & \mathbf{D}_{\mathrm{sg}}(A) & \xrightarrow[\mathbf{si}]{\sim} & \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}}
\end{array},$$

where the squares commute up to an isomorphism of functors.

Proof. — The equivalences in the rows are from Theorem 6.4. We already know that $\widehat{\mathbf{N}}$ is an equivalence, so one has only to verify the commutativity of the diagram.

The commutativity of the square on the left is tantamount to: For each G-projective A-module M there is a natural isomorphism between $\omega \otimes^{\mathrm{L}}_A M$ and

$\mathrm{GP}(\omega \otimes_A M)$, viewed as objects in $\mathbf{D}_{\mathrm{sg}}(A)$. As noted in Lemma 6.2, finitely generated G-projective modules are syzygies in acyclic complexes of finitely generated projective modules. Thus, the proof of Theorem A.1 yields an exact sequence of A-modules

$$0 \longrightarrow P \longrightarrow \mathrm{GP}(\omega \otimes_A M) \longrightarrow \omega \otimes_A M \longrightarrow 0\,,$$

with $\mathrm{GP}(\omega \otimes_A M)$ a G-projective and P a finitely generated projective. This gives the isomorphism on the left

$$\mathrm{GP}(\omega \otimes_A M) \xrightarrow{\ \sim\ } \omega \otimes_A M \xleftarrow{\ \sim\ } \omega \otimes^{\mathrm{L}}_A M$$

in $\mathbf{D}_{\mathrm{sg}}(A)$. The one on the right is by Lemma 6.1(2), for the latter is tantamount to the statement that the natural morphism of complexes $\omega \otimes^{\mathrm{L}}_A M \to (\omega \otimes_A M)$ is an isomorphism in $\mathbf{D}(\mathrm{Mod}\,A)$, and so also in $\mathbf{D}_{\mathrm{sg}}A$.

For $X \in \mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A)$, from Lemma 3.1 and Theorem 4.5 one gets isomorphisms

$$\widehat{\mathbf{N}}\mathbf{i}(\omega \otimes^{\mathrm{L}}_A X) \cong \widehat{\mathbf{N}}(\omega \otimes^{\mathrm{L}}_A \mathbf{i}X) \cong \mathrm{RHom}_A(\omega, \omega \otimes^{\mathrm{L}}_A \mathbf{i}X) \cong \mathbf{i}X\,.$$

Applying $\mathbf{s}$ to the composition and observing that $\widehat{\mathbf{N}}$ commutes with $\mathbf{s}$ by Theorem 5.1, yields the commutativity of the square on the right. □

The commutativity of the outer square in Proposition 6.6 lifts to the corresponding "big" categories.

Proposition 6.7. — *The functor* $\mathrm{GP}(\omega_{A/R} \otimes_A -)\colon \underline{\mathrm{GProj}}A \to \underline{\mathrm{GProj}}A$ *is an equivalence of triangulated categories, with quasi-inverse* $\mathrm{GP}\,\mathrm{Hom}_A(\omega_{A/R}, -)$. *Moreover, the diagram below commutes up to an isomorphism of functors:*

$$\begin{array}{ccc} \underline{\mathrm{GProj}}A & \xrightarrow[\sim]{\mathbf{si}} & \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\,A) \\ {\scriptstyle \mathrm{GP}_A(\omega_{A/R}\otimes_A -)}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle \widehat{\mathbf{N}}^{-1}_{A/R}} \\ \underline{\mathrm{GProj}}A & \xrightarrow[\sim]{\mathbf{si}} & \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\,A)\,. \end{array}$$

Proof. — The crucial observation is that the categories involved are compactly generated, and all the functors involved commute with direct sums. Thus, the desired result is a consequence of Proposition 6.6. □

7. Localization and torsion functors

As before, let A be a finite R-algebra. In what follows, we apply the theory of local cohomology and localization from [7], with respect to the action of the ring R on the homotopy category of injective A-modules. To that end we recall some results concerning the structure of injective A-modules discovered by Gabriel [20]; it extends the (by now well-known) theory for commutative rings.

To begin with, by the *spectrum* of A we mean the collection of two-sided prime ideals of A, denoted $\operatorname{Spec} A$. Since the map $\eta\colon R \to A$ is central and finite, the induced map on spectra

$$\operatorname{Spec} A \longrightarrow \operatorname{Spec} R \quad \text{where } \mathfrak{q} \mapsto \mathfrak{q} \cap R \text{ for } \mathfrak{q} \in \operatorname{Spec} A,$$

is surjective onto $\operatorname{Spec} \eta(R)$, which is a closed subset of $\operatorname{Spec} R$. Moreover, the fibers of the map are *discrete*: if $\mathfrak{q}' \subseteq \mathfrak{q}$ are elements of $\operatorname{Spec} A$ such that $\mathfrak{q}' \cap R = \mathfrak{q} \cap R$, then $\mathfrak{q}' = \mathfrak{q}$; see [20, Proposition V.11].

Torsion. — For each $\mathfrak{p}$ in $\operatorname{Spec} R$, there is a natural A-module structure of $M_{\mathfrak{p}}$ for which the canonical map $M \to M_{\mathfrak{p}}$ is A-linear.

A subset $V \subseteq \operatorname{Spec} R$ is *specialization closed* when it has the following property: If $\mathfrak{p} \subseteq \mathfrak{p}'$ are prime ideals in R and $\mathfrak{p}$ is in V, then $\mathfrak{p}'$ is in V; equivalently, that V contains the closure (in the Zariski) topology of its points. The following specialization closed subsets play a central role: Given an ideal $\mathfrak{a} \subset R$, the subset

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a}\}$$

of $\operatorname{Spec} R$, and given a prime $\mathfrak{p}$ in $\operatorname{Spec} R$, the subset

$$Z(\mathfrak{p}) := \{\mathfrak{p}' \in \operatorname{Spec} R \mid \mathfrak{p}' \not\subseteq \mathfrak{p}\}\,.$$

Observe that $Z(\mathfrak{p})$ equals $\operatorname{Spec} R \setminus \operatorname{Spec} R_{\mathfrak{p}}$.

Give a specialization closed subset V of $\operatorname{Spec} R$, the V*-torsion* submodule of an A-module M is defined by

$$\varGamma_V M := \operatorname{Ker}(M \longrightarrow \prod_{\mathfrak{p} \notin V} M_{\mathfrak{p}})\,.$$

The assignment $M \mapsto \varGamma_V M$ is an additive, left-exact, functor on $\operatorname{Mod} A$. The module M is called V*-torsion* if $\varGamma_V M = M$.

It is easy to verify that when $V := V(r)$ for an element $r \in R$, one has

$$\varGamma_{V(r)} M = \operatorname{Ker}(M \longrightarrow M_r)\,,$$

where M_r is the localization of M at the multiplicatively closed subset $\{r^n\}_{n \geqslant 0}$, and that when $V := Z(\mathfrak{p})$, for some $\mathfrak{p} \in \operatorname{Spec} R$, one gets

$$\varGamma_{Z(\mathfrak{p})} M = \operatorname{Ker}(M \longrightarrow M_{\mathfrak{p}})\,.$$

Injective modules. — Since A is Noetherian, $\operatorname{Inj} A$, the full subcategory of $\operatorname{Mod} A$ consisting of injective modules, is closed under arbitrary direct sums. For a $\mathfrak{q}$ in $\operatorname{Spec} A$ the injective hull of the A-module $A/\mathfrak{q}$ decomposes into a finite direct sum of copies of an indecomposable injective module, which we denote by $I(\mathfrak{q})$. Since A is a finite R-algebra, the assignment $\mathfrak{q} \mapsto I(\mathfrak{q})$ is a bijection between $\operatorname{Spec} A$ and the isomorphism classes of indecomposable injective A-modules, by [20, V.4]. Thus, each injective A-module is a direct sum of copies of $I(\mathfrak{q})$, as $\mathfrak{q}$ varies over $\operatorname{Spec} A$.

LEMMA 7.1. — *Let $V \subseteq \operatorname{Spec} R$ be specialization closed. For each injective A-module I, the module $\Gamma_V I$ is a direct summand of I. Thus, for $\mathfrak{q}$ in $\operatorname{Spec} A$, one has*

$$\Gamma_V I(\mathfrak{q}) = \begin{cases} I(\mathfrak{q}) & \text{when } \mathfrak{q} \cap R \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The functor Γ_V provides a right adjoint for the inclusion of the localizing subcategory of A-modules that are V-torsion. This functor preserves injectivity, since the localizing subcategory is stable under taking injective envelopes, by [20, Proposition V.12]. Thus, $\Gamma_V I$ is a direct summand of I for every injective A-module I. In particular, we have $\Gamma_V I = I$, or $\Gamma_V I = 0$ when I is indecomposable. □

Since Γ_V is an additive functor, it induces a functor on the category of A-complexes. For each complex X of injective A-modules set

$$L_V X := \operatorname{Coker}(\Gamma_V X \longrightarrow X)\,.$$

Thus, one gets an exact sequence of A-complexes

$$0 \longrightarrow \Gamma_V X \longrightarrow X \longrightarrow L_V X \longrightarrow 0\,.$$

By Lemma 7.1 the subcomplex $\Gamma_V X$ consists of injective A-modules so the sequence above is degree-wise split exact, and hence induces in $\mathbf{K}(\operatorname{Inj} A)$ an exact triangle

$$\Gamma_V X \longrightarrow X \longrightarrow L_V X \longrightarrow \Sigma\Gamma_V X\,. \tag{8}$$

The functor L_V has an explicit description in a couple of cases.

EXAMPLE 7.2. — Suppose $V := V(r)$, for some $r \in R$. Then the map $X \to X_r$ is surjective, by Lemma 7.1, so there is an exact sequence

$$0 \longrightarrow \Gamma_{V(r)} X \longrightarrow X \longrightarrow X_r \longrightarrow 0$$

of A-complexes, and hence $L_{V(r)}X = X_r$. By the same token, when $V := Z(\mathfrak{p})$ for some prime $\mathfrak{p}$ in $\operatorname{Spec} R$, one gets an exact sequence

$$0 \longrightarrow \Gamma_{Z(\mathfrak{p})} X \longrightarrow X \longrightarrow X_{\mathfrak{p}} \longrightarrow 0$$

of A-complexes, so that $L_{Z(\mathfrak{p})}X = X_{\mathfrak{p}}$.

Localization and local cohomology. — The ring R acts on $\mathbf{K}(\operatorname{Mod} A)$ and hence on its subcategories discussed above, in the sense of [7]. We focus on $\mathcal{T} = \mathbf{K}(\operatorname{Inj} A)$.

For any localizing subcategory $\mathcal{C} \subseteq \mathcal{T}$ and object $X \in \mathcal{T}$, we call an exact triangle

$$\Gamma X \longrightarrow X \longrightarrow LX \longrightarrow \Sigma\Gamma X$$

a *localization triangle* provided that $\Gamma X \in \mathcal{C}$ and $LX \in \mathcal{C}^{\perp}$, where $\mathcal{C}^{\perp} \subseteq \mathcal{T}$ denotes the colocalizing subcategory consisting of objects Y such that $\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0$, for all $X \in \mathcal{C}$. If such a triangle exists for all objects $X \in \mathcal{T}$, then Γ yields a right adjoint for the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$, and L yields a left adjoint for the inclusion $\mathcal{C}^{\perp} \hookrightarrow \mathcal{T}$.

Given a specialization closed subset $V \subseteq \operatorname{Spec} R$, an object X in $\mathcal{T}$ is *V-torsion* provided that $\operatorname{Hom}_{\mathcal{T}}(C, X)$ is a V-torsion A-module for each compact $C \in \mathcal{T}$.

LEMMA 7.3. — *For a specialization closed subset $V \subseteq \operatorname{Spec} R$, the triangle* (8) *is the localization triangle associated to the localizing subcategory of V-torsion objects in* $\mathbf{K}(\operatorname{Inj} A)$.

Proof. — Fix $X \in \mathbf{K}(\operatorname{Inj} A)$. Then $\Gamma_V X$ is V-torsion by construction. Moreover, for every injective A-module I, it is easy to verify that

$$\operatorname{Hom}(\Gamma_V I, I/\Gamma_V I) = 0\,.$$

Thus, $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X', L_V X) = 0$, for all V-torsion $X' \in \mathbf{K}(\operatorname{Inj} A)$. □

LEMMA 7.4. — *For any $\mathfrak{p}$ in* $\operatorname{Spec} R$ *and X in* $\mathbf{K}(\operatorname{Inj} A)$, *we have $L_{Z(\mathfrak{p})} X \cong X_{\mathfrak{p}}$.*

Proof. — This follows from Example 7.2. □

For an object X in $\mathbf{K}(\operatorname{Inj} A)$, we write $\operatorname{Loc}(X)$ for the smallest localizing subcategory of $\mathbf{K}(\operatorname{Inj} A)$ that contains X.

LEMMA 7.5. — *Let $V \subseteq$* $\operatorname{Spec} R$ *be specialization closed. For any X in* $\mathbf{K}(\operatorname{Inj} A)$, *the A-complexes $\Gamma_V X$ and $L_V X$ are in* $\operatorname{Loc}(X)$.

Proof. — This follows from the local-to-global principle discussed in [8]. More specifically, one combines [8, Theorem 3.1] with [44, Theorem 6.9]. □

LEMMA 7.6. — *Let $V \subseteq$* $\operatorname{Spec} R$ *be specialization closed. If an A-complex X of injective A-modules is acyclic, then so are the complexes $\Gamma_V X$ and $L_V X$.*

Proof. — The subcategory $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ of $\mathbf{K}(\operatorname{Inj} A)$ is localizing, hence when X is acyclic, so are the complexes in $\operatorname{Loc}(X)$. It remains to recall Lemma 7.5. □

For an object X in $\mathbf{K}(\operatorname{Inj} A)$ and $\mathfrak{p} \in \operatorname{Spec} R$ the *local cohomology* at $\mathfrak{p}$ is

$$\Gamma_{\mathfrak{p}} X := \Gamma_{V(\mathfrak{p})}(X_{\mathfrak{p}})\,.$$

The following observation will be useful.

LEMMA 7.7. — *For any X in* $\mathbf{K}(\operatorname{Inj} A)$, *the complex $\Gamma_{\mathfrak{p}} X$ is a subquotient of X. In particular, if $X^i = 0$ for some $i \in \mathbb{Z}$, then $(\Gamma_{\mathfrak{p}} X)^i = 0$ as well.* □

Remark 7.8. — The triangulated category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is compactly generated and R-linear, so has its own localization functors for a specialization closed subset V of $\operatorname{Spec} R$. It follows from Lemma 7.6 that these are just restrictions of the corresponding functors on $\mathbf{K}(\operatorname{Inj} A)$.

The triangulated category $\mathbf{D}(\operatorname{Mod} A)$ is also compactly generated and R-linear. However, the embedding $\mathbf{i}\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{K}(\operatorname{Inj} A)$ is not compatible with the localization functors; in other words, for a K-injective complex X, the complex $\varGamma_V X$ need not be K-injective; see [16]. On the other hand, it is easy to verify that these functors are compatible with the restriction functor $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} R)$.

Remark 7.9. — Fix a $\mathfrak{p}$ in $\operatorname{Spec} R$ and consider the diagram of exact functors.

$$\begin{array}{ccc}
\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) & \underset{\mathrm{incl}}{\overset{\mathbf{s}}{\leftrightarrows}} & \mathbf{K}(\operatorname{Inj} A) \\
{\scriptstyle (-)_{\mathfrak{p}}}\downarrow\uparrow{\scriptstyle \mathrm{res}} & & {\scriptstyle (-)_{\mathfrak{p}}}\downarrow\uparrow{\scriptstyle \mathrm{res}} \\
\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A_{\mathfrak{p}}) & \underset{\mathrm{incl}}{\overset{\mathbf{s}_{\mathfrak{p}}}{\leftrightarrows}} & \mathbf{K}(\operatorname{Inj} A_{\mathfrak{p}})
\end{array}$$

It is clear that the two compositions of right adjoints, from the bottom left to the top right, coincide. It follows that the composition of the corresponding left adjoint functors are isomorphic: $(\mathbf{s}X)_{\mathfrak{p}} \cong \mathbf{s}_{\mathfrak{p}}(X_{\mathfrak{p}})$ for X in $\mathbf{K}(\operatorname{Inj} A)$.

Support. — Let $\mathcal{T}$ be $\mathbf{K}(\operatorname{Inj} A)$ or $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. Specializing the definition from [7] to our context, we introduce the *support* of an object X in $\mathcal{T}$ to be the subset

$$\operatorname{supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} X \neq 0\}\,.$$

It follows from Remark 7.8 that the support an object in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is the same as its support when we view it as an object in $\mathbf{K}(\operatorname{Inj} A)$.

The *support* of $\mathcal{T}$ is the subset of $\operatorname{Spec} R$ defined by

$$\operatorname{supp}_R \mathcal{T} := \bigcup_{X \in \mathcal{T}^{\mathrm{c}}} \operatorname{supp}_R X\,.$$

Here are some alternative characterizations of support for acyclic complexes.

Proposition 7.10. — *Let A be a finite R-algebra, fix $X \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ and $\mathfrak{p}$ in $\operatorname{Spec} R$. The following conditions are equivalent:*

(1) *The prime $\mathfrak{p}$ is not in $\operatorname{supp}_R X$.*
(2) *The complex $\varGamma_{\mathfrak{p}} X$ is contractible.*
(3) *The A-module $\varGamma_{\mathfrak{p}}(\Omega^i(X))$ is injective for each (equivalently, some) integer i.*

Proof. — An acyclic complex of injective modules is zero in $\mathbf{K}(\operatorname{Inj} A)$ if and only if it is contractible, if and only if each, equivalently, one of its syzygy modules is injective. From this, we get that (1)⇔(2) and also that these conditions are equivalent to $\Omega^i(\Gamma_{\mathfrak{p}}X)$ injective for each, equivalently, some, i. It remains to note that since the functor $\Gamma_{\mathfrak{p}}$ is left-exact and preserves acyclicity of complexes in $\mathbf{K}(\operatorname{Inj} A)$, one gets

$$\Omega^i(\Gamma_{\mathfrak{p}}X) \cong \Gamma_{\mathfrak{p}}\Omega^i(X) \quad \text{for each integer } i.$$

This completes the proof. □

The following observation concerning generators for $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is well known.

Lemma 7.11. — *The compact objects in* $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ *are direct summands of objects of the form* $\mathbf{s}C$*, where* C *is a compact object in* $\mathbf{K}(\operatorname{Inj} A)$*.*

Proof. — The functor $\mathbf{s}$ is left adjoint to the inclusion $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \subset \mathbf{K}(\operatorname{Inj} A)$, so it is essentially surjective; it also preserves compactness for the inclusion preserves direct sums. It follows that up to direct summands all compact objects of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ are in the image of $\mathbf{s}$; see [40, Theorem 2.1]. □

A Noetherian ring A is *regular* if each $M \in \operatorname{mod} A$ has finite projective dimension; equivalently, each M in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ is perfect. We say that A is *singular* to mean that it is not regular. When A is a finite R-algebra its *regular locus* will mean the collection of primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $A_{\mathfrak{p}}$ is regular. Its complement in $\operatorname{Spec} R$ is the *singular locus.*

Corollary 7.12. — *The singular locus of* A *equals* $\operatorname{supp}_R \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$*.*

Proof. — By Lemma 7.11 the support of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is the union of the supports of $\mathbf{s}(\mathbf{i}M)$, for $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. For any $\mathfrak{p} \in \operatorname{Spec} R$, one has isomorphisms

$$\mathbf{s}(\mathbf{i}M)_{\mathfrak{p}} \cong \mathbf{s}_{\mathfrak{p}}((\mathbf{i}M)_{\mathfrak{p}}) \cong \mathbf{s}_{\mathfrak{p}}(\mathbf{i}(M_{\mathfrak{p}}))$$

in $\mathbf{K}(\operatorname{Inj} A_{\mathfrak{p}})$, where the first one is by Remark 7.9, and the second one is standard. Thus, $\mathbf{s}(\mathbf{i}M)_{\mathfrak{p}} \cong 0$ if and only if $M_{\mathfrak{p}}$ is perfect in $\mathbf{D}(\operatorname{Mod} A_{\mathfrak{p}})$. Consequently, if $\mathfrak{p}$ is in the regular locus of A, then $\mathbf{s}(\mathbf{i}M)_{\mathfrak{p}} = 0$, for each M in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, and hence $\mathfrak{p}$ is not in the support of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$.

Conversely, if $A_{\mathfrak{p}}$ is not regular, then there exists an $M \in \operatorname{mod} A$ such that $M_{\mathfrak{p}}$ is not perfect; one can choose M to be $V(\mathfrak{p})$-torsion. Then $\Gamma_{\mathfrak{p}}\mathbf{s}(\mathbf{i}M) \cong \mathbf{s}(\mathbf{i}M)_{\mathfrak{p}}$ is nonzero, so $\mathfrak{p}$ is in the support of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. □

8. Matlis duality and Gorenstein categories

This section is about avatars of Matlis duality in various homotopy categories we have been dealing with. To set the stage for the discussion, it helps to consider a general, compactly generated, triangulated category $\mathcal{T}$ with the action of a commutative Noetherian ring R, in the sense of [7]. Fix an injective R-module I. For each compact object C in $\mathcal{T}$, the functor

$$X \longmapsto \operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(C, X), I)\,,$$

from $\mathcal{T}$ to $\operatorname{Mod} R$, is homological and takes coproducts to products. The Brown representability theorem implies that it is representable: There is an object, say $T_I(C)$, in $\mathcal{T}$ and an isomorphism of functors

$$\operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(C, -), I) \cong \operatorname{Hom}_{\mathcal{T}}(-, T_I(C))\,.$$

In this way, the assignment $C \times I \mapsto T_I(C)$ yields a functor

$$T\colon \mathcal{T}^{\mathrm{c}} \times \operatorname{Inj} R \longrightarrow \mathcal{T}\,.$$

Borrowing terminology from [18] we call the functor $T_I(-)$ the *Matlis lift* of I to $\mathcal{T}$. In what follows, for $\mathfrak{p}$ in $\operatorname{Spec} R$, we write $T_{\mathfrak{p}}(-)$ for $T_{I(\mathfrak{p})}(-)$, where $I(\mathfrak{p})$ is the injective hull of the R-module $R/\mathfrak{p}$.

Now, let A be a finite R-algebra as before. The description of the Matlis lifts of injective R-modules to the R-linear category $\mathbf{D}(\operatorname{Mod} A)$ is straightforward.

Proposition 8.1. — *The Matlis lift to $\mathbf{D}(\operatorname{Mod} A)$ of an injective R-module I is given by the functor $C \mapsto \operatorname{RHom}_R(A, I) \otimes^{\mathrm{L}}_A C$.*

Proof. — Given objects $X \in \mathbf{D}(\operatorname{Mod} A)$ and a finitely generated projective A-module P, there are natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_A(P, X), I) &\cong \operatorname{Hom}_R(X, I) \otimes_A P\\ &\cong \operatorname{Hom}_A(X, \operatorname{Hom}_R(A, I)) \otimes_A P\\ &\cong \operatorname{Hom}_A(X, \operatorname{Hom}_R(A, I) \otimes_A P)\,.\end{aligned}$$

It remains to observe that any compact object in $\mathbf{D}(\operatorname{Mod} A)$ is isomorphic to a bounded complex of finitely generated projective A-modules. □

The Matlis lifts of injective R-modules to the R-linear category $\mathbf{K}(\operatorname{Inj} A)$ is described in the next result, which is modeled on [36, Theorem 3.4]; the proof we give is somewhat different.

Theorem 8.2. — *Let A be a finite R-algebra. The Matlis lift to $\mathbf{K}(\operatorname{Inj} A)$ of an injective R-module I is given by*

$$C \longmapsto \operatorname{Hom}_R(A, I) \otimes_A \mathbf{p}C\,.$$

Proof. — Fix objects C, X in $\mathbf{K}(\operatorname{Inj} A)$ with C compact. The key input is Lemma 2.1 that yields the first isomorphism below

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C,X),I) &\cong \operatorname{Hom}_R(H^0(\operatorname{Hom}_A(\mathbf{p}C,A)\otimes_A X),I)\\ &\cong H^0(\operatorname{Hom}_R(\operatorname{Hom}_A(\mathbf{p}C,A)\otimes_A X,I))\\ &\cong H^0(\operatorname{Hom}_A(X,\operatorname{Hom}_R(\operatorname{Hom}_A(\mathbf{p}C,A),I))\\ &\cong H^0(\operatorname{Hom}_A(X,\operatorname{Hom}_R(A,I)\otimes_A \mathbf{p}C))\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,\operatorname{Hom}_R(A,I)\otimes_A \mathbf{p}C)\,.\end{aligned}$$

The second one holds because I is injective. The rest are standard. □

The next result describes Matlis lifts to $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, using the functors from (1). In Lemma 7.11 we described the compact objects in that category.

Corollary 8.3. — *For a compact object in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ of the form $\mathbf{s}C$, given by a compact object C in $\mathbf{K}(\operatorname{Inj} A)$, the Matlis lift of an injective R-module I is the complex*

$$T_I(\mathbf{s}C) \cong \mathbf{r}(T_I C) \cong \mathbf{r}(\operatorname{Hom}_R(A,I)\otimes_A \mathbf{p}C)\,.$$

Proof. — For any acyclic complex X of injective R-modules, one has

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,\mathbf{r}(T_I(C)) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,T_I(C))\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C,X),I)\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{s}C,X),I)\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,T_I(\mathbf{s}C))\,.\end{aligned}$$

This justifies the first isomorphism. For the second one, see Theorem 8.2. □

Remark 8.4. — There is a notion of purity for compactly generated triangulated categories, analogous to the classical concept of purity for module categories; see Crawley-Boevey's survey [17]. It follows from the construction that any Matlis lift is a pure-injective object. In particular, we obtain from a Matlis lift a pure-injective module when an acyclic complex is identified with an A-module.

Gorenstein categories. — Let $\mathcal{T}$ be an R-linear category. Following [9] we say that $\mathcal{T}$ is *Gorenstein* if there is an R-linear triangle equivalence

$$F\colon \mathcal{T}^{\mathrm{c}} \xrightarrow{\sim} \mathcal{T}^{\mathrm{c}}$$

such that for each $\mathfrak{p}$ in $\operatorname{supp}_R \mathcal{T}$, there is an integer $d(\mathfrak{p})$ and a natural isomorphism

$$\varGamma_{\mathfrak{p}} \circ F \cong \Sigma^{-d(\mathfrak{p})} \circ T_{\mathfrak{p}}$$

of functors $\mathcal{T}^{c} \to \mathcal{T}$. The functor F plays the role of a *global Serre functor* because it induces a Serre functor, in the sense of Bondal and Kapranov [11], on the subcategory of compacts objects in $\mathcal{T}_{\mathfrak{p}}$, the $\mathfrak{p}$-local $\mathfrak{p}$-torsion objects in $\mathcal{T}$, for $\mathfrak{p}$ in $\operatorname{Spec} R$. More precisely, localizing with respect to $\mathfrak{p}$ yields a functor $F_{\mathfrak{p}}\colon \mathcal{T}^{c}_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{T}^{c}_{\mathfrak{p}}$ and a natural isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(X,Y), I(\mathfrak{p})) \cong \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{d(\mathfrak{p})} F_{\mathfrak{p}} X)\,,$$

for objects $X, Y \in \mathcal{T}_{\mathfrak{p}}$ such that X is compact and $\operatorname{supp}_R X = \{\mathfrak{p}\}$. This is explained in [9, §7]. In what follows we focus on the following special case.

Proposition 8.5. — *Let $\mathcal{T}$ be a compactly generated R-linear category that is Gorenstein, with global Serre functor F. Fix a maximal ideal $\mathfrak{m}$ in R. For any $X \in \mathcal{T}^{c}$ and $Y \in \mathcal{T}$ with $\operatorname{supp}_R X = \{\mathfrak{m}\}$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(X,Y), I(\mathfrak{m})) \cong \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{d(\mathfrak{m})} F X)\,.$$

In particular, if $\operatorname{supp}_R \mathcal{T} = \{\mathfrak{m}\}$, then $\Sigma^{d(\mathfrak{m})}F$ is a Serre functor on $\mathcal{T}^{c}$.

Proof. — Since $\mathfrak{m}$ is maximal, any object supported on $\mathfrak{m}$ is already $\mathfrak{m}$-local. Thus, the desired isomorphism is a special case of [9, Proposition 7.3]. □

Gorenstein rings. — Let R be a commutative Gorenstein ring. For $\mathfrak{p}$ in $\operatorname{Spec} R$, set $h(\mathfrak{p}) = \dim R_{\mathfrak{p}}$; this is the height of $\mathfrak{p}$. The Gorenstein property for R is equivalent to the condition that the minimal injective resolution I of R satisfies

$$I^n = \bigoplus_{h(\mathfrak{p})=n} I(\mathfrak{p}) \quad \text{for each } n.$$

This translates to the condition that in $\mathbf{K}(\operatorname{Inj} R)$, there are isomorphisms

$$\varGamma_{\mathfrak{p}}(\mathbf{i}R) \cong \Sigma^{-h(\mathfrak{p})} I(\mathfrak{p}) \quad \text{for each } \mathfrak{p} \in \operatorname{Spec} R. \tag{9}$$

This result is due to Grothendieck, cf. [12, Proposition 3.5.4].

Proposition 8.6. — *Let A be a finite R-algebra that is projective as an R-module. The following conditions are equivalent:*

(1) *The R-algebra A is Gorenstein.*
(2) *The R-linear category $\mathbf{D}(\operatorname{Mod} A)$ is Gorenstein.*

When they hold the global Serre functor is $\omega_{A/R} \otimes^{\mathrm{L}}_A -$, and $d(\mathfrak{p}) = \dim R_{\mathfrak{p}}$.

Proof. — (1)⇒(2) As the R-algebra A is Gorenstein, the functor $F := \omega \otimes^{\mathrm{L}}_A -$ is an equivalence on $\mathbf{D}(\operatorname{Mod} A)$ and hence restricts to an equivalence $\mathbf{D}(\operatorname{Mod} A)^{c}$ the subcategory of perfect complexes; see Theorem 4.5. With $d(\mathfrak{p})$ as in the statement, for any perfect complex C, from Proposition 8.1 one gets the equality below

$$\begin{aligned} T_{\mathfrak{p}}(C) &= \mathrm{RHom}_R(A, I(\mathfrak{p})) \otimes^{\mathrm{L}}_A C \\ &\cong I(\mathfrak{p}) \otimes^{\mathrm{L}}_R \mathrm{Hom}_R(A,R) \otimes^{\mathrm{L}}_A C \\ &\cong \Sigma^{d(\mathfrak{p})} \varGamma_{\mathfrak{p}}(\mathbf{i}R) \otimes^{\mathrm{L}}_R FC \\ &\cong \Sigma^{d(\mathfrak{p})} \varGamma_{\mathfrak{p}}(\mathbf{i}R \otimes^{\mathrm{L}}_R FC) \\ &\cong \Sigma^{d(\mathfrak{p})} \varGamma_{\mathfrak{p}} FC\,. \end{aligned}$$

The third isomorphism is from (9), and the rest are standard. Thus, $\mathbf{D}(\operatorname{Mod} A)$ is Gorenstein, with the prescribed global Serre functor and shift $d(\mathfrak{p})$.

(2)⇒(1) It suffices to verify that the injective dimension of $A_{\mathfrak{m}}$ is finite for any maximal ideal $\mathfrak{m}$ in R. For this, it suffices to verify that $M \in \operatorname{mod}(A/\mathfrak{m}A)$ satisfy

$$\operatorname{Ext}^i_A(M, A) = 0 \quad \text{for } i \gg 0.$$

For then an argument along the lines of the proof of [2, Proposition A.1.5] yields that $A_{\mathfrak{m}}$ has finite injective dimension over itself.

Let $F\colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)^{\mathrm{c}} \to \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)^{\mathrm{c}}$ be a global Serre functor and F^{-1} its quasi-inverse. Since M is $\mathfrak{m}$-torsion from Proposition 8.5 we get the isomorphism below

$$\mathrm{Hom}_{\mathbf{D}(A)}(M, \Sigma^i A) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{D}(A)}(F^{-1}A, \Sigma^{d(\mathfrak{m})-i} M), I(\mathfrak{m}))\,.$$

It remains to note that since $F^{-1}A$ is perfect one has

$$\mathrm{Hom}_{\mathbf{D}(A)}(F^{-1}A, \Sigma^j(-)) = 0 \quad \text{on } \operatorname{Mod} A,$$

for all $|j| \gg 0$. This implies the desired result. □

Here is the analogue of the preceding result dealing with homotopy categories.

Proposition 8.7. — *Let A be a finite R-algebra that is projective as an R-module. The R-linear category $\mathbf{K}(\operatorname{Inj} A)$ is Gorenstein if and only if A is regular.*

Proof. — When A is regular, the canonical functor $\mathbf{K}(\operatorname{Inj} A) \to \mathbf{D}(\operatorname{Mod} A)$ is an equivalence and $\mathbf{D}(\operatorname{Mod} A)$ is Gorenstein, by Proposition 8.6. As to the converse, it suffices check that $A_{\mathfrak{m}}$ is regular for each maximal ideal $\mathfrak{m}$ in R.

Arguing as in the proof of (2)⇒(1) in Proposition 8.6 one deduces that for each $M \in \operatorname{mod}(A/\mathfrak{m}A)$ and $N \in \operatorname{mod} A$, one has

$$\operatorname{Ext}^i_A(M, N) = 0 \quad \text{for } i \gg 0.$$

This implies that $A_{\mathfrak{m}}$ is regular. □

The preceding results concern the Gorenstein property for the derived category and the homotopy category of injectives for two of the three categories that appear in the recollement (1). That of the last one is dealt with in the next section.

9. Grothendieck duality for $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, A)$

This section is dedicated to the proof of the following result. As explained in the Introduction, this has been the guiding light for the results presented in this work.

THEOREM 9.1. — *Let A be a Gorenstein R-algebra. For each compact object X in $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, A)$ and $\mathfrak{p}$ in the singular locus of A, there is a natural isomorphism*

$$\varGamma_{\mathfrak{p}} X \cong \Sigma^{-d(\mathfrak{p})} T_{\mathfrak{p}}(\widehat{\mathbf{N}}_{A/R} X)\,,$$

where $d(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) - 1$. In particular, the R-linear category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, A)$ is Gorenstein, with the global Serre functor the quasi-inverse of $\widehat{\mathbf{N}}_{A/R}$.

The proof is given further below. Theorem 1.2 from the Introduction is an immediate consequence.

COROLLARY 9.2. — *Let A be a Gorenstein R-algebra and let M, N be G-projective A-modules with M finitely generated. For each $\mathfrak{p} \in \operatorname{Spec} R$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\widehat{\operatorname{Ext}}^i_A(M,N), I(\mathfrak{p})) \cong \widehat{\operatorname{Ext}}^{d(\mathfrak{p})-i}_A(N, \varGamma_{\mathfrak{p}} S(M)).$$

Proof. — The assertion is a direct translation of Theorem 9.1, given the equivalence $\underline{\mathrm{GProj}}A \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, A)$ from Proposition 6.7. □

We continue with a consequence concerning duality for the category of compact objects. The statements are simpler, and perhaps more striking, when specialized to the case of local isolated singularities, and that is what we do.

Isolated singularities. — Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring and A a finite projective R-algebra. We say that A has an *isolated singularity* if its singular locus is $\{\mathfrak{m}\}$; that is to say, if the ring $A_{\mathfrak{p}}$ is regular for each non-maximal ideal $\mathfrak{p}$ in $\operatorname{Spec} R$; see the discussion around Corollary 7.12.

COROLLARY 9.3. — *Let R be a commutative Noetherian local ring of Krull dimension d. If A is a Gorenstein R-algebra with an isolated singularity, then the assignment*

$$X \mapsto \Sigma^{d-1} \widehat{\mathbf{N}}^{-1}(X)$$

is a Serre functor on the R-linear category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, A)^{\mathrm{c}}$.

Proof. — Since A has an isolated singularity, the R-linear category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is supported at $\mathfrak{m}$, the maximal ideal of R; see Corollary 7.12. Thus, Theorem 9.1 and Proposition 8.5 yield the desired result. □

Given the equivalences in Theorem 6.4 one can recast the duality statement above in terms of the singularity category and the stable category of Gorenstein projective modules. Here, too, we are following Buchweitz's footsteps [13], except that he does not require A to be projective over a central subalgebra; on the other hand, he considers only rings of finite injective dimension. We can get away with local finiteness of injective dimension, thanks to Theorem 4.6.

Corollary 9.4. — *For R and A as in Corollary 9.3, the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ has Serre duality, with Serre functor $\Sigma^{d-1}\omega_{A/R}\otimes^{\mathrm{L}}_A(-)$.*

Proof. — This is a direct translation of Corollary 9.3, made using Theorem 6.6. □

Moreover, here is Corollary 9.3 transported to the world of G-projective modules.

Corollary 9.5. — *For R and A as in Corollary 9.3, the functor*

$$M\mapsto \Omega^{1-d}\operatorname{GP}(\omega_{A/R}\otimes_A M)$$

is a Serre functor on the triangulated category $\underline{\operatorname{Gproj}}A$. □

Remark 9.6. — Set $S:=\Omega^{1-d}\operatorname{GP}(\omega\otimes_A(-))$; the Serre functor on $\underline{\operatorname{Gproj}}(A)$. Theorem 9.5 translates to the statement that there is an R-linear *trace* map

$$\underline{\operatorname{Hom}}_A(M,SM)\xrightarrow{\tau} I(\mathfrak{m})$$

such that the bilinear pairing

$$\underline{\operatorname{Hom}}_A(N,SM)\times\underline{\operatorname{Hom}}_A(M,N)\xrightarrow{-\circ-}\underline{\operatorname{Hom}}_A(M,SM))\xrightarrow{\tau} I(\mathfrak{m})\,,$$

where $-\circ-$ is the obvious composition, is non-degenerate. Murfet [38] describes the trace map in the case when $A=R$, that is to say, in the case of commutative rings; this involves the theory of residues and differentials forms. It would be interesting to extend his work to the present context.

We now prepare for the proof of Theorem 9.1.

Lemma 9.7. — *Let A be a finite R-algebra. For each X in $\operatorname{Loc}(\mathbf{i}A)$ for which $\mathbf{i}X$ is in $\mathbf{K}^+(\operatorname{Inj}A)$, the isomorphism* (2.2) *induces isomorphisms*

$$\Sigma^{-1}\varGamma_{\mathfrak{p}}\mathbf{s}(\mathbf{i}X)\xrightarrow{\sim}\mathbf{r}\varGamma_{\mathfrak{p}}X\quad \textit{for each } \mathfrak{p}\in\operatorname{Spec}R.$$

Proof. — Since X is in $\operatorname{Loc}(\mathbf{i}A)$, from (2) and Lemma 2.2, we get an exact triangle

$$\Sigma^{-1}\mathbf{s}(\mathbf{i}X) \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow$$

Applying $\varGamma_{\mathfrak{p}}$ to this yields the exact triangle

$$\Sigma^{-1}\varGamma_{\mathfrak{p}}(\mathbf{s}(\mathbf{i}X)) \longrightarrow \varGamma_{\mathfrak{p}}X \longrightarrow \varGamma_{\mathfrak{p}}(\mathbf{i}X) \longrightarrow .$$

Since $\mathbf{s}(\mathbf{i}X)$ is acyclic so is the complex $\varGamma_{\mathfrak{p}}(\Sigma^{-1}\mathbf{s}(\mathbf{i}X))$, by Lemma 7.6. Hence, $\mathbf{r}(-)$ is (isomorphic to) the identity on this complex. On the other hand, since $\mathbf{i}X$ is bounded below, so is $\varGamma_{\mathfrak{p}}(\mathbf{i}X)$, by Lemma 7.7, and hence $\mathbf{r}(-)$ vanishes on this complex. Keeping these observations in mind and applying the functor $\mathbf{r}$ to the exact triangle above yields the stated isomorphism. □

Proof of Theorem 9.1. — It suffices to establish the result for objects of the form $\mathbf{s}(C)$, where $C \in \mathbf{K}(\operatorname{Inj} A)$ is a compact object; we can assume that C is bounded below. Set $D := \operatorname{Hom}_R(A, \mathbf{i}R)$. We shall be interested in the complex of injective A-modules

$$X := D \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C)\,.$$

We claim that this complex satisfies the hypotheses of Lemma 9.7.

Claim. — *X is in $\operatorname{Loc}(\mathbf{i}A)$ and $\mathbf{i}X \xrightarrow{\sim} C$ and, in particular, it is bounded below.*

Indeed, the complex D consists of A-bimodules that are injective on either side, and the map $R \to \mathbf{i}R$ induces a quasi-isomorphism

$$\omega = \operatorname{Hom}_R(A, R) \longrightarrow \operatorname{Hom}_R(A, \mathbf{i}R) = D$$

of A-bimodules. Thus, D is an injective resolution of ω on both sides. It follows that in $\mathbf{D}(\operatorname{Mod} A)$ there are natural isomorphisms

$$D \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C) \cong \omega \otimes^{\mathrm{L}}_A \operatorname{RHom}_A(\omega, C) \cong C\,,$$

where the second one is by Theorem 4.5. Therefore, in $\mathbf{K}(\operatorname{Inj} A)$, one gets that

$$\mathbf{i}X = \mathbf{i}(D \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C)) \xrightarrow{\ \sim\ } C.$$

As to the first part of the claim, $\mathbf{p}(\widehat{\mathbf{N}}C)$ is in $\operatorname{Loc}(A) \subseteq \mathbf{K}(\operatorname{Proj} A)$, hence X is in $\operatorname{Loc}(D)$ in $\mathbf{K}(\operatorname{Inj} A)$. However, D is an injective resolution of ω, and the latter is perfect, as an object of $\mathbf{D}(\operatorname{Mod} A)$, so D is in $\operatorname{Thick}(\mathbf{i}A)$. It follows that X is in $\operatorname{Loc}(\mathbf{i}A)$, as claimed.

From the claim and Lemma 9.7, we deduce that

$$\Sigma^{-1}\varGamma_{\mathfrak{p}}\mathbf{s}(\mathbf{i}X) \cong \mathbf{r}\varGamma_{\mathfrak{p}}X \quad \text{for each } \mathfrak{p} \in \operatorname{Spec} R.$$

This justifies the penultimate isomorphism below, where h stands for $\dim R_{\mathfrak{p}}$:

$$\begin{aligned} T_{\mathfrak{p}}(\widehat{\mathbf{N}}(\mathbf{s}C)) &\cong T_{\mathfrak{p}}(\mathbf{s}(\widehat{\mathbf{N}}C)) \\ &\cong \mathbf{r}(\operatorname{Hom}_R(A, I(\mathfrak{p})) \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C))) \\ &\cong \mathbf{r}(\operatorname{Hom}_R(A, \Sigma^h \mathit{\Gamma}_{\mathfrak{p}} \mathbf{i}R) \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C))) \\ &\cong \Sigma^h \mathbf{r} \mathit{\Gamma}_{\mathfrak{p}}(\operatorname{Hom}_R(A, \mathbf{i}R) \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C)) \\ &= \Sigma^h \mathbf{r} \mathit{\Gamma}_{\mathfrak{p}}(X) \\ &\cong \Sigma^{h-1} \mathit{\Gamma}_{\mathfrak{p}} \mathbf{s}(\mathbf{i}X) \\ &\cong \Sigma^{h-1} \mathit{\Gamma}_{\mathfrak{p}} \mathbf{s}(C)\,. \end{aligned}$$

The first isomorphism is by Theorem 5.1; the second is by Corollary 8.3; the third is from (9), which applies as R is Gorenstein, by Lemma 4.1. The last isomorphism is again by the claim above. This finishes the proof. □

In contrast with Proposition 8.6 and Proposition 8.7, we do not know if the Gorenstein property of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ characterizes Gorenstein algebras; except when A is commutative.

Theorem 9.8. — *Let R be a commutative Noetherian ring. The R-linear category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ is Gorenstein if and only if the ring R is Gorenstein.*

Proof. — The reverse implication is contained in Theorem 9.1.

Suppose $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ is Gorenstein as an R-linear category, with global Serre functor F. Let $\mathfrak{m}$ be a maximal ideal of R, and $k := R/\mathfrak{m}$ its residue field. The object $\mathbf{si}k$ in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is compact and $\mathfrak{m}$-torsion so Proposition 8.5 yields

$$\begin{aligned} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{si}k, \mathbf{si}k)^{\vee\vee} &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{si}k, \Sigma^{d(\mathfrak{m})} F(\mathbf{si}k))^{\vee} \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\Sigma^{d(\mathfrak{m})} F(\mathbf{si}k), \Sigma^{d(\mathfrak{m})} F(\mathbf{si}k)) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{si}k, \mathbf{si}k)\,. \end{aligned}$$

Thus, one gets an isomorphism of Tate cohomology modules

$$\widehat{\operatorname{Ext}}^0_R(k,k) \cong \widehat{\operatorname{Ext}}^0_R(k,k)^{\vee\vee} \quad \text{for each } i \in \mathbb{N}.$$

These modules are annihilated by $\mathfrak{m}$, and so are k-vector spaces. The isomorphism above implies that each of them has finite rank over k. It remains to recall the result of Avramov and Veliche [3, Theorem 6.4] that the finiteness of the rank of $\widehat{\operatorname{Ext}}^i_R(k,k)$ for *some* i already implies that $R_{\mathfrak{m}}$ is Gorenstein. □

The proof of the preceding result does not go through for non-commutative rings, for there exist finite dimensional algebras A over a field k that are not Gorenstein, and yet $\widehat{\operatorname{Ext}}^i_A(M,N)$ is finite dimensional over k for each i, and finite dimensional A-modules M, N; see, for example, [15, Example 4.3, (1), (2)].

Appendix A. Gorenstein approximations

Let $\mathcal{A}$ be an additive category. Recall that a complex $X \in \mathbf{K}(\mathcal{A})$ is called *totally acyclic* if the complexes of abelian groups $\operatorname{Hom}(W, X)$ and $\operatorname{Hom}(X, W)$ are acyclic for all $W \in \mathcal{A}$. When $\mathcal{A}$ is abelian, and $\mathcal{C} \subseteq \mathcal{A}$ is a class of objects, we set

$$\begin{aligned} {}^{\perp}\mathcal{C} &= \{X \in \mathcal{A} \mid \operatorname{Ext}^n(X, Y) = 0 \text{ for all } Y \in \mathcal{C},\ n > 0\} \\ \mathcal{C}^{\perp} &= \{Y \in \mathcal{A} \mid \operatorname{Ext}^n(X, Y) = 0 \text{ for all } X \in \mathcal{C},\ n > 0\}. \end{aligned}$$

A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{A}$ is a (hereditary and complete) *cotorsion pair* for $\mathcal{A}$ if

$$\mathcal{X}^{\perp} = \mathcal{Y} \qquad \text{and} \qquad \mathcal{X} = {}^{\perp}\mathcal{Y}\,,$$

and every object $M \in \mathcal{A}$ fits into exact *approximation sequences*

$$0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0\,,$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

Gorenstein algebras. — Fix a ring A. Recall that an A-module is *G-projective*, if it is of the form

$$C^0(X) := \operatorname{Coker}(X^{-1} \xrightarrow{d^{-1}} X^0)\,,$$

for a totally acyclic $X \in \mathbf{K}(\operatorname{Proj} A)$. The *G-injective* modules are those of the form

$$Z^0(X) := \operatorname{Ker}(X^0 \xrightarrow{d^0} X^1)\,,$$

for some totally acyclic $X \in \mathbf{K}(\operatorname{Inj} A)$. We write $\operatorname{GProj} A$ for the full subcategory of all G-projective modules and $\operatorname{GInj} A$ for the full subcategory of all G-injective modules. The theorem below provides Gorenstein approximations for all modules over a Gorenstein algebra.

Let $\operatorname{Fin} A$ be the full subcategory of A-modules having finite projective and finite injective dimension. When A is a finite R-algebra, we consider the category

$$\operatorname{Fin}(A/R) := \{M \in \operatorname{Mod} A \mid M_{\mathfrak{p}} \in \operatorname{Fin}(A_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}.$$

Observe that when the R-algebra A is Gorenstein, $\operatorname{Fin}(A_{\mathfrak{p}}/R)$ is the category of $A_{\mathfrak{p}}$-modules of finite projective—equivalently, finite injective—dimension. One of the consequences of the result below is that, at least for Gorenstein algebras, $\operatorname{Fin}(A/R)$ is independent of the ring R.

Theorem A.1. — *Let A be a Gorenstein R-algebra. Then there are equalities*

$$(\operatorname{GProj} A)^{\perp} = \operatorname{Fin}(A/R) = {}^{\perp}(\operatorname{GInj} A)\,.$$

Also, $(\operatorname{GProj} A, \operatorname{Fin}(A/R))$ *and* $(\operatorname{Fin}(A/R), \operatorname{GInj} A)$ *are cotorsion pairs for* $\operatorname{Mod} A$*.*

The map $X_M \to M$ for $\mathcal{X} = \operatorname{GProj} A$ is called the *G-projective approximation*; we set $\operatorname{GP}(M) = X_M$. This module is unique up to morphisms that factor through a projective module. Analogously, the map $M \to Y^M$ for $\mathcal{Y} = \operatorname{GInj} A$ is called *G-injective approximation*, and we set $\operatorname{GI}(M) = Y^M$; it is unique up to morphisms that factor through an injective module.

Proof. — First, observe that any acyclic complex of projective or injective A-modules is totally acyclic by Theorem 5.6. This means that G-projective and G-injective modules are obtained from acyclic complexes.

We begin with the construction of G-injective approximations, using the recollement (1) as follows. Set $\mathcal{Y} = \operatorname{GInj} A$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Fix an A-module M. Then an injective resolution $\mathbf{i}M$ fits into an exact triangle

$$\mathbf{jq}(\mathbf{i}M) \longrightarrow \mathbf{i}M \longrightarrow \mathbf{s}(\mathbf{i}M) \longrightarrow$$

given by an exact sequence of complexes

$$0 \longrightarrow \mathbf{i}M \longrightarrow \mathbf{s}(\mathbf{i}M) \longrightarrow \Sigma(\mathbf{jq}(\mathbf{i}M)) \longrightarrow 0\,,$$

which is split-exact in each degree. Thus $Z^0(-)$ gives an exact sequence

$$0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0\,,$$

with $X^M \in \mathcal{X}$ and $Y^M \in \mathcal{Y}$. The other sequence $0 \to Y_M \to X_M \to M \to 0$ is obtained by rotating this triangle. This justifies the claim that $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair; see [35, Theorem 7.12] for details.

It remains to identify $\mathcal{X}$, the left orthogonal to $\operatorname{GInj} A$. A standard argument yields the equality $\mathcal{X} = \operatorname{Fin} A$ when A is Iwanaga–Gorenstein. For a Gorenstein algebra A, the equality $\mathcal{X} = \operatorname{Fin}(A/R)$ follows once we can show that for each $\mathfrak{p} \in \operatorname{Spec} R$, the $\mathfrak{p}$-localization of an approximation sequence for $M \in \operatorname{Mod} A$ yields an approximation sequence for $M_{\mathfrak{p}}$ in $\operatorname{Mod} A_{\mathfrak{p}}$. It follows from the discussion in Remark 7.9 that for any A-module M, one has isomorphisms

$$(\mathbf{s}(\mathbf{i}M))_{\mathfrak{p}} \cong \mathbf{s}_{\mathfrak{p}}((\mathbf{i}M)_{\mathfrak{p}}) \cong \mathbf{s}_{\mathfrak{p}}(\mathbf{i}_{\mathfrak{p}}M_{\mathfrak{p}})\,.$$

This implies $(X^M)_{\mathfrak{p}} \cong X^{M_{\mathfrak{p}}}$ and $(X_M)_{\mathfrak{p}} \cong X_{M_{\mathfrak{p}}}$. Thus, both modules have finite projective and finite injective dimension. We conclude that $\mathcal{X} = \operatorname{Fin}(A/R)$.

Next, we consider G-projective approximations using the analogue of the recollement (1) for $\mathbf{K}(\operatorname{Proj} A)$. The proof that $(\operatorname{GProj} A, (\operatorname{GProj} A)^{\perp})$ is a cotorsion pair is similar to that for $\operatorname{GInj} A$, for it uses the right adjoint of the inclusion $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \hookrightarrow \mathbf{K}(\operatorname{Proj} A)$; we omit the details. The equality $(\operatorname{GProj} A)^{\perp} = \operatorname{Fin}(A/R)$ can be verified as follows. Recall from Theorem 5.6 that there is an adjoint pair of triangle equivalences

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \underset{\mathbf{h}}{\overset{E\otimes_A -}{\underset{\sim}{\rightleftarrows}}} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\quad.$$

Consider the exact triangle

$$E \otimes_A \mathbf{p}M \longrightarrow \mathbf{i}M \longrightarrow \mathbf{s}(\mathbf{i}M) \longrightarrow$$

from Lemma 2.5, which we used for constructing a G-injective approximation of M. Applying the equivalence $\mathbf{h}$ and rotating yields an exact triangle

$$\Sigma^{-1}\mathbf{hs}(\mathbf{i}M) \longrightarrow \mathbf{p}M \longrightarrow \mathbf{h}(\mathbf{i}M) \longrightarrow$$

which provides us with the G-projective approximation of M. We claim that for each $\mathfrak{p} \in \operatorname{Spec} R$, the $\mathfrak{p}$-localization of this triangle yields the Gorenstein-projective approximation of $M_\mathfrak{p}$. To this end consider the following diagram of exact functors.

$$\begin{array}{ccc}
\mathbf{K}(\operatorname{Proj} A) & \underset{\mathbf{h}}{\overset{E\otimes_A -}{\rightleftarrows}} & \mathbf{K}(\operatorname{Inj} A) \\
{\scriptstyle (-)_\mathfrak{p}}\big\downarrow\big\uparrow{\scriptstyle \operatorname{res}} & & {\scriptstyle (-)_\mathfrak{p}}\big\downarrow\big\uparrow{\scriptstyle \operatorname{res}} \\
\mathbf{K}(\operatorname{Proj} A_\mathfrak{p}) & \underset{\mathbf{h}_\mathfrak{p}}{\overset{E_\mathfrak{p}\otimes_{A_\mathfrak{p}} -}{\rightleftarrows}} & \mathbf{K}(\operatorname{Inj} A_\mathfrak{p})
\end{array}$$

It is easily checked that for each A-module M, one has isomorphisms

$$(\mathbf{hs}(\mathbf{i}M))_\mathfrak{p} \cong \mathbf{h}_\mathfrak{p}((\mathbf{s}(\mathbf{i}M))_\mathfrak{p}) \cong \mathbf{h}_\mathfrak{p}\mathbf{s}_\mathfrak{p}(\mathbf{i}_\mathfrak{p}M_\mathfrak{p})\,.$$

This implies that $(Y^M)_\mathfrak{p} \cong Y^{M_\mathfrak{p}}$ and $(Y_M)_\mathfrak{p} \cong Y_{M_\mathfrak{p}}$. Thus, both modules have finite projective and finite injective dimension, so $(\operatorname{GProj} A)^\perp = \operatorname{Fin}(A/R)$. □

Remark A.2. — The above theorem shows that Gorenstein algebras are virtually Gorenstein in the sense of Beligiannis and Reiten [6], which means that the classes $(\operatorname{GProj} A)^\perp$ and ${}^\perp(\operatorname{GInj} A)$ coincide.

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SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_n(F)$ DISTINGUISHED BY A UNITARY INVOLUTION

BY JIANDI ZOU

ABSTRACT. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residue characteristic $p \neq 2$. Let R be an algebraically closed field of characteristic different from p. For π a supercuspidal representation of $G = \mathrm{GL}_n(F)$ over R and G^τ a unitary subgroup of G with respect to F/F_0, we prove that π is distinguished by G^τ, if and only if π is Galois invariant. When $R = \mathbb{C}$ and F is a p-adic field, this result was first a conjecture proposed by Jacquet and was proved in the 2010s by Feigon–Lapid–Offen by using global methods. Our proof is local and works for both complex representations and l-modular representations with $l \neq p$. We further study the dimension of $\mathrm{Hom}_{G^\tau}(\pi, 1)$ and show that it is at most 1.

RÉSUMÉ (*Représentations supercuspidales de* $\mathrm{GL}_n(F)$ *distinguées par une involution unitaire*). — Soit F/F_0 une extension quadratique de corps localement compacts non archimédiens de caractéristique résiduelle $p \neq 2$. Soit R un corps algébriquement clos de caractéristique différente de p. Pour π une représentation supercuspidale de $G = \mathrm{GL}_n(F)$ sur R et G^τ un sous-groupe unitaire de G par rapport à F/F_0, on montre que π est distinguée par G^τ si et seulement si π est invariante galoisienne. Lorsque $R = \mathbb{C}$ et F est un corps p-adique, ce résultat d'abord sous la forme d'une conjecture proposée par Jacquet a été prouvé dans les années 2010 par Feigon-Lapid-Offen en utilisant des méthodes globales. Notre preuve est locale et fonctionne à la fois pour les représentations complexes et les représentations l-modulaires avec $l \neq p$. Nous étudions plus en détail la dimension de $\mathrm{Hom}_{G^\tau}(\pi, 1)$ et montrons qu'elle est au plus un.

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1. Introduction

Let F/F_0 be a quadratic extension of p-adic fields of residue characteristic p and let σ denote its non-trivial automorphism. For $G = \mathrm{GL}_n(F)$, let ε be a *hermitian matrix* in G, that is, $\sigma({}^t\varepsilon) = \varepsilon$ with ^{t} denoting the transpose of matrices. We define

$$\tau_\varepsilon(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1},$$

for any $x \in G$, called a *unitary involution* on G. We fix $\tau = \tau_\varepsilon$ and we denote by G^τ the subgroup of G consisting of the elements fixed by τ, called the *unitary subgroup* of G with respect to τ. For π an irreducible smooth representation of G over $\mathbb{C}$, Jacquet proposed to study the space of G^τ-invariant linear forms on π, that is, the space

$$\mathrm{Hom}_{G^\tau}(\pi, 1).$$

When the space is non-zero, he called π *distinguished by* G^τ. For $n = 3$ and π supercuspidal, he proved in [26] by using global argument that π is distinguished by G^τ, if and only if π is σ-invariant, that is, $\pi^\sigma \cong \pi$, where $\pi^\sigma := \pi \circ \sigma$. Moreover, he showed that this space is of dimension 1 as a complex vector space when the condition above is satisfied. Moreover, in *ibid.*, he also sketched a similar proof when $n = 2$ and π is supercuspidal to give the same criterion of being distinguished and the same dimension 1 theorem. Based on these results, he conjectured that, in general, π is distinguished by G^τ, if and only if π is σ-invariant. Moreover, it is also interesting to determine the dimension of the space of G^τ-invariant linear forms that is not necessarily 1 in general. Under the assumption that π is σ-invariant and supercuspidal, Jacquet further conjectured that the dimension is 1.

In addition, an irreducible representation π of G is contained in the image of quadratic base change with respect to F/F_0, if and only if it is σ-invariant ([3]). Thus, for irreducible representations, the conjecture of Jacquet gives a connection between quadratic base change and G^τ-distinction.

Besides the special case mentioned above, the following two evidences also support the conjecture. First, we consider the analogue of the conjecture in the finite field case. For $\overline{\rho}$ an irreducible complex representation of $\mathrm{GL}_n(\mathbb{F}_{q^2})$, Gow [16] proved that $\overline{\rho}$ is distinguished by the unitary subgroup $\mathrm{U}_n(\mathbb{F}_q)$, if and only if $\overline{\rho}$ is isomorphic to its twist under the non-trivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. Under this condition, he also showed that the space of $\mathrm{U}_n(\mathbb{F}_q)$-invariant linear forms is of dimension 1 as a complex vector space. In addition, Shintani [41] showed that there is a one-to-one correspondence between the set of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and that of Galois-invariant irreducible representations of $\mathrm{GL}_n(\mathbb{F}_{q^2})$, where the correspondence, called the *base change map*, is characterized by a trace identity. Thus, these two results relate the $\mathrm{U}_n(\mathbb{F}_q)$-distinction to the base change map. Finally, when $\overline{\rho}$ is generic and Galois-invariant, Anandavardhanan and Matringe [2] recently showed that the $\mathrm{U}_n(\mathbb{F}_q)$-average of the Bessel function of $\overline{\rho}$ on the Whittaker model as a $\mathrm{U}_n(\mathbb{F}_q)$-invariant linear form is non-zero. Since the space of $\mathrm{U}_n(\mathbb{F}_q)$-invariant linear forms is of dimension 1, their result gives us a concrete characterization of the space.

The other evidence for the Jacquet conjecture is its global analogue. We assume $\mathcal{K}/\mathcal{K}_0$ to be a quadratic extension of number fields and we denote by σ its non-trivial automorphism. We choose τ to be a unitary involution on $\mathrm{GL}_n(\mathcal{K})$, which also gives us an involution on $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$, still denoted by τ by abuse of notation, where $\mathbb{A}_{\mathcal{K}}$ denotes the ring of adèles of $\mathcal{K}$. We denote by $\mathrm{GL}_n(\mathcal{K})^\tau$ (or $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$) the unitary subgroup of $\mathrm{GL}_n(\mathcal{K})$ (or $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$) with respect to τ. For ϕ a cusp form of $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$, we define

$$\mathcal{P}_\tau(\phi) = \int_{\mathrm{GL}_n(\mathcal{K})^\tau \backslash \mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau} \phi(h)dh$$

to be the *unitary period integral* of ϕ with respect to τ. We say that a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ is $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$-distinguished if there exists a cusp form in the space of Π such that $\mathcal{P}_\tau(\phi) \neq 0$. In the 1990s, Jacquet and Ye began to study the relation between $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$-distinction and global base change (see, for example, [28] when $n = 3$). For general n, Jacquet [27] showed that Π is contained in the image of the quadratic base change map (or equivalently Π is σ-invariant [3]) with respect to $\mathcal{K}/\mathcal{K}_0$, if and only if there exists a unitary involution τ such that Π is G^τ-distinguished. This result may be viewed as the global version of the Jacquet conjecture for supercuspidal representations.

In fact, for the special case of the Jacquet conjecture in [26], Jacquet used the global analogue of the same conjecture and the relative trace formula to finish the proof. To say it simply, he first proved the global analogue of the conjecture. Then he used the relative trace formula to write a non-zero unitary period integral as the product of its local components at each place of $\mathcal{K}_0$, where each local component characterizes the distinction of the local component of Π with respect to the corresponding unitary group over local fields. When π is σ-invariant, he chose Π to be a σ-invariant cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ and v_0 to be a non-Archimedean place of $\mathcal{K}_0$, such that $(G^\tau, \pi) = (\mathrm{GL}_n(\mathcal{K}_{v_0})^\tau, \Pi_{v_0})$. Then the product decomposition leads to the proof of the "if" part of the conjecture. The "only if" part of the conjecture, which will be discussed in Section 4, requires the application of a globalization theorem. His method was generalized by Feigon–Lapid–Offen in [14] to general n and more general families of representations. They showed that the Jacquet conjecture works for generic representations of G. Moreover, they were able to give a lower bound for the dimension of $\mathrm{Hom}_{G^\tau}(\pi, 1)$ and they further conjectured that the inequality they gave is actually an equality. Finally, Beuzart-Plessis recently verified the equality conjectured above [5]. Thus, for generic representations of G, the Jacquet conjecture was settled.

Instead of using global methods, there are other methods to study this conjecture, which are local and algebraic. Hakim–Mao [19] verified the conjecture when $p \neq 2$ and π is supercuspidal of level zero, that is, π is supercuspidal such that $\pi^{1+\mathfrak{p}_F \mathrm{M}_n(\mathfrak{o}_F)} \neq 0$, where $\mathfrak{o}_F$ denotes the ring of integers of F and $\mathfrak{p}_F$ denotes its maximal ideal. When π is supercuspidal and F/F_0 is unramified, Prasad [34] proved the conjecture by applying the simple type theory developed by Bushnell-Kutzko in [9]. When $p \neq 2$ and π is tame supercuspidal, that is, π is a supercuspidal representation arising from the construction of Howe [24], Hakim-Murnaghan [21] verified the conjecture.

The discussion above leaves us an open question: *Is there any local and algebraic method that leads to a proof of the Jacquet conjecture that works for all supercuspidal representations of G?* First, this will lead to a new proof of the results of Hakim–Mao, Prasad and Hakim–Murnaghan, which we mentioned in the last paragraph. Secondly, instead of considering complex representations, we are also willing to study l-modular representations with $l \neq p$. One hopes to prove an analogue of the Jacquet conjecture for l-modular supercuspidal representations, which will generalize the result of Feigon–Lapid–Offen for supercuspidal representations. Noting that they use global methods in their proof, which strongly relies on the assumption that all the representations are complex. Thus, their method does no longer works for l-modular representations. Finally, we are willing to consider F/F_0 to be a quadratic extension of non-Archimedean locally compact fields instead of p-adic fields. Since the result of Feigon–Lapid–Offen heavily relies on the fact that the characteristic of F equals 0, their method fails when considering non-Archimedean locally compact fields of positive characteristic. The aim of this paper is to answer this question.

We will say a bit more about l-modular representations. The study of smooth l-modular representations of $G = \mathrm{GL}_n(F)$ was initiated by Vignéras [43], [44] to extend the local Langlands program to l-modular representations. In this spirit, many classical results related to smooth complex representations of p-adic groups have been generalized to l-modular representations. For example, the local Jacquet–Langlands correspondence related to l-modular representations has been studied in detail in [11], [33] and [37]. Thus, it is also natural to consider the l-modular version of the Jacquet conjecture, which hopes to build up the relation between distinction and an expected l-modular version of quadratic base change. This paper is the starting point of the whole project.

To begin with, from now on we assume F/F_0 to be a quadratic extension of non-Archimedean locally compact fields of residue characteristic $p \neq 2$ instead of p-adic fields. We fix R an algebraically closed field of characteristic $l \neq p$, allowing that $l = 0$. When $l > 0$, we say that we are in the l-modular case (or modular case for short). Later on, we always consider smooth representations over R and we assume π to be a supercuspidal representation of G over R. Be aware that when $l \neq 0$, a supercuspidal representation is not the same as a cuspidal representation of G, although they are the same when $l = 0$ (see, for example, Vignéras [43], chapitre II, section 2). Now we state our first main theorem:

THEOREM 1.1. — *For π a supercuspidal representation of $G = \mathrm{GL}_n(F)$ and τ a unitary involution, π is distinguished by G^τ if and only if $\pi^\sigma \cong \pi$.*

Moreover, we may also calculate the dimension of the space of G^τ-invariant linear forms:

THEOREM 1.2. — *For π a σ-invariant supercuspidal representation of G, we have*

$$\dim_R \mathrm{Hom}_{G^\tau}(\pi, 1) = 1.$$

One important corollary of Theorem 1.1 relates to the $\overline{\mathbb{Q}_l}$-lift of a σ-invariant supercuspidal representation of G over $\overline{\mathbb{F}_l}$ when $l > 0$, where we denote by $\overline{\mathbb{Q}_l}$, $\overline{\mathbb{Z}_l}$ and $\overline{\mathbb{F}_l}$ the algebraic closure of an l-adic field, its ring of integers and the algebraic closure of the finite field of l elements, respectively. For $(\widetilde{\pi}, V)$ a smooth irreducible representation of G over $\overline{\mathbb{Q}_l}$, we call it *integral* if it admits an integral structure, that is, a $\overline{\mathbb{Z}_l}[G]$-submodule L_V of V generated by a $\overline{\mathbb{Q}_l}$-basis of V. For such a representation, the semi-simplification of $L_V \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}$ does not depend on the choice of L_V, which we denote by $r_l(\widetilde{\pi})$ as a representation of G over $\overline{\mathbb{F}_l}$, called the *reduction modulo l* of π (see [43] for more details). The following theorem, which will be proved at the end of Section 8, says that it is always possible to find a σ-invariant $\overline{\mathbb{Q}_l}$-lift for a σ-invariant supercuspidal representation of G over $\overline{\mathbb{F}_l}$.

THEOREM 1.3. — *For π a σ-invariant supercuspidal representation of G over $\overline{\mathbb{F}_l}$, there exists an integral σ-invariant supercuspidal representation $\widetilde{\pi}$ of G over $\overline{\mathbb{Q}_l}$, such that $r_l(\widetilde{\pi}) = \pi$.*

Let us outline the contents of this paper by introducing the strategy of our proof for Theorem 1.1 and Theorem 1.2. In Section 2, we introduce our settings and basic knowledge about hermitian matrices and unitary subgroups. Our main tool to prove the theorems will be the simple type theory developed by Bushnell–Kutzko in [9] and further generalized by Vignéras [43] and Mínguez-Sécherre [32] to the l-modular case. In Section 3, we will give a detailed introduction to this theory, but here we also recall a little bit for convenience. The idea of simple type theory is to realize any cuspidal representation π of G as the compact induction of a finite dimensional representation Λ of $\boldsymbol{J}$, which is an open subgroup of G compact modulo its centre. Such a pair $(\boldsymbol{J}, \Lambda)$, constructed as in [9], is called an *extended maximal simple type*, which we will abbreviate to *simple type* for simplicity. We also mention the following main properties of $(\boldsymbol{J}, \Lambda)$:

(1) The group $\boldsymbol{J}$ contains a unique maximal open compact subgroup J, which contains a unique maximal normal pro-p-subgroup J^1;
(2) We have $J/J^1 \cong \mathrm{GL}_m(\boldsymbol{l})$, where E/F is a certain field extension of degree d with $\boldsymbol{l}$ denoting the residue field of E and $n = md$;
(3) We may write $\Lambda = \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$, where $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$ are irreducible representations of $\boldsymbol{J}$, such that the restriction $\boldsymbol{\kappa}|_{J^1} = \eta$ is an irreducible representation of J^1, called a *Heisenberg representation*, and $\boldsymbol{\rho}|_J$ is the inflation of a cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l}) \cong J/J^1$.

For a given supercuspidal representation π of G, our starting point is to prove the "only if" part of Theorem 1.1. When $R = \mathbb{C}$ and $\mathrm{char}(F) = 0$, it is a standard result by using global argument, especially the globalization theorem ([20], Theorem 1). When $\mathrm{char}(F) = p > 0$, we may keep the original proof except that we need a characteristic p version of the globalization theorem. Fortunately, we can use a more general result due to Gan–Lomelí [15] to get the result we need. Since any supercuspidal representation of G over a characteristic 0 algebraically closed field can be realized as a representation over $\overline{\mathbb{Q}}$ up to twisting by an unramified character, we finish the proof when $\mathrm{char}(R) = 0$. When $R = \overline{\mathbb{F}_l}$, we consider the projective envelope $P_{\Lambda|_J}$ of $\Lambda|_J$ and we use the results in [43] to study its irreducible components and the irreducible components of its $\overline{\mathbb{Q}_l}$-lift. Finally, we will show that there exists a $\overline{\mathbb{Q}_l}$-lift of π, which is supercuspidal and G^τ-distinguished. Thus, by using the characteristic 0 case, we finish the proof for the "only if" part, for any R under our settings. The details will be presented in Section 4.

In Section 5, we prove the *τ-self-dual type theorem*, which says that for a unitary involution τ and a σ-invariant cuspidal representation π of G with a technical condition, we may find a simple type $(\boldsymbol{J}, \Lambda)$ contained in π, such that $\tau(\boldsymbol{J}) = \boldsymbol{J}$ and $\Lambda^\tau \cong \Lambda^\vee$, where $^\vee$ denotes the smooth contragradient. In other words, we find a "symmetric" simple type contained in π with respect to τ. Our strategy follows from [1], section 4. First, we consider the case where E/F is totally wildly ramified and $n = d$. Then for E/F in general with $n = d$, we make use of the techniques about endo-class and tame lifting developed in [6] to prove the theorem by reducing it to the former case. Finally, by using the $n = d$ case, we prove the general theorem.

In Section 6, for τ, π as in Section 5 satisfying the technical condition, we first choose a τ-self-dual simple type $(\boldsymbol{J}, \Lambda)$ contained in π. The main result of Section 6, which we call the *distinguished type theorem*, says that π is distinguished by G^τ if and only if there exists a τ-self-dual and distinguished simple type of π. More specifically, by Frobenius reciprocity and the Mackey formula, we have

$$\mathrm{Hom}_{G^\tau}(\pi,1)\cong\prod_{g\in\boldsymbol{J}\backslash G/G^\tau}\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g,1).$$

We concentrate on those g in the double coset such that $\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g,1)\neq 0$. The proof of the distinguished type theorem also shows that there are at most two such double cosets, which can be written down explicitly. Moreover, for those g, we have

$$\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g,1)\cong\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\kappa}^g,\chi^{-1})\otimes_R\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\rho}^g,\chi),$$

where $\boldsymbol{\kappa}$ is well chosen, such that $\boldsymbol{\kappa}^\tau\cong\boldsymbol{\kappa}^\vee$, and χ is a quadratic character of $\boldsymbol{J}^g\cap G^\tau$, which is trivial on $J^{1g}\cap G^\tau$. In the tensor product, the first term $\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\kappa}^g,\chi^{-1})$ is of dimension 1 as an R-vector space. So essentially we only need to study the second term. If we denote by $\overline{\rho^g}$ the cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})\cong J^g/J^{1g}$, whose inflation equals $\boldsymbol{\rho}^g|_{J^g}$ and by $\overline{\chi}$ the character of $H:=J^g\cap G^\tau/J^{1g}\cap G^\tau$, whose inflation equals $\chi|_{J^g\cap G^\tau}$, then we further have

$$\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\rho}^g,\chi)\cong\mathrm{Hom}_H(\overline{\rho^g},\overline{\chi}).$$

Here, H could be a unitary subgroup, an orthogonal subgroup or a symplectic subgroup of $\mathrm{GL}_m(\boldsymbol{l})$. When π is supercuspidal, the technical condition in the τ-self-dual type theorem is always satisfied, and we reduce our problem to studying the H-distinction of a supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$.

Moreover, at the beginning of Section 6, we use the result in Section 5 to extend σ to a non-trivial involution on E. We write $E_0=E^\sigma$ and we deduce that E/E_0 is a quadratic extension. When E/E_0 is unramified, H is a unitary subgroup. We first use the result of Gow [16] to deal with the characteristic 0 case. For $\mathrm{char}(R)>0$, we consider the projective envelope as in Section 4. When E/E_0 is ramified, H is either an orthogonal subgroup or a symplectic subgroup. When H is orthogonal, we use Deligne–Lusztig theory [12], precisely a formula given by Hakim–Lansky [18] to calculate the dimension of $\mathrm{Hom}_H(\overline{\rho^g},\overline{\chi})$, when $\mathrm{char}(R)=0$. For $\mathrm{char}(R)>0$, we again use the same method as in Section 4 to finish the proof. When H is symplectic, by [31], the space is always $\{0\}$. These two cases will be studied in Section 7 and Section 8 separately. Finally, in Section 9, we give a purely local proof of the main theorem of Section 4.

It is worth mentioning that in [35], Sécherre studied the σ-self-dual supercuspidal representations of G over R, with the same notation unchanged as before. He proved the following *Dichotomy Theorem* and *Disjunction Theorem*: For π a supercuspidal representation of G, it is σ-self-dual (that is, $\pi^\sigma\cong\pi^\vee$), if and only if it is either distinguished by $\mathrm{GL}_n(F_0)$ or ω-distinguished, where ω denotes the unique non-trivial character of $F_0^\times$ trivial on $\mathrm{N}_{F/F_0}(F^\times)$. The method that we use in this paper is the same as that developed in *ibid.* For example, our Section 5 corresponds to section 4 of [1], and our Section 6 corresponds to section 6 of [35], etc.

We point out the main differences in our case to end this Introduction. First, in Section 5, we will find that in a certain case, it is even impossible to find a hereditary order $\mathfrak{a}$, such that $\tau(\mathfrak{a}) = \mathfrak{a}$, which is not a problem in section 4 of [1]. That is why we need to add a technical condition in the main theorem of Section 5 and finally verify it for supercuspidal representations. Precisely, for a σ-invariant supercuspidal representation, we first consider the unitary involution $\tau = \tau_1$ corresponding to the identity hermitian matrix I_n. In this case, we may use our discussion in Section 5 to find a τ-self-dual type contained in π and we may further use our discussion in Section 6 and Section 7 to show that m is odd when E/E_0 is unramified. This exactly affirms the condition we need, and we may repeat the procedure of Section 5 and Section 6 for general unitary involutions. This detouring argument also indicates that a σ-invariant cuspidal not supercuspidal representation does not always contain a τ-self-dual simple type, which justifies that our supercuspidal (instead of cuspidal) assumption is somehow important.

Furthermore, in Section 8, it is unclear whether or not the character χ mentioned above can be realized as a character of $\boldsymbol{J}$ and thus cannot be assumed to be trivial a priori as in [35]. This means that we need to consider a supercuspidal representation of the general linear group over a finite field distinguished by a non-trivial character of an orthogonal subgroup instead of the trivial one. This is why the result of Hakim–Lansky ([18] Theorem 3.11) shows up.

Last but not least, in Section 6, a large part of our results is stated and proved for a general involution instead of a unitary one. This provides the possibility of using the same method to study the distinction of supercuspidal representations of G by other involutions. For instance, a similar problem for orthogonal subgroups was also considered by the author [45].

2. Notation and basic definitions

2.1. Notation. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields with residue characteristic $p \neq 2$ and let σ be the unique non-trivial involution in the Galois group. Write $\mathfrak{o}_F$ (or $\mathfrak{o}_{F_0}$) for the ring of integers of F (or F_0) and $\boldsymbol{k}$ (or $\boldsymbol{k}_0$) for the residue field of F (or F_0). The involution σ induces a $\boldsymbol{k}_0$-automorphism of $\boldsymbol{k}$ generating $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{k}_0)$, still denoted by σ.

Let R be an algebraically closed field of characteristic $l \geq 0$ different from p. If $l > 0$, then we are in the "modular case".

We fix a character

$$\psi_0 : F_0 \to R^\times$$

trivial on the maximal ideal of $\mathfrak{o}_{F_0}$ but not on $\mathfrak{o}_{F_0}$, and we define $\psi = \psi_0 \circ \mathrm{tr}_{F/F_0}$.

Let G be the locally profinite group $\mathrm{GL}_n(F)$ with $n \geq 1$, equipped with the involution σ acting componentwise. Let ε be a *hermitian matrix* in G, which means that $\varepsilon^* = \varepsilon$. Here, $x^* := \sigma({}^t x)$, for any $x \in \mathrm{M}_n(F)$, with ^{t} denoting the transpose operator. Sometimes, we write $\sigma_t(x) := x^*$, for any $x \in \mathrm{M}_n(F)$, to emphasize that σ_t is an anti-involution on $\mathrm{M}_n(F)$ extending σ. For ε hermitian and $g \in G$, we define $\tau_\varepsilon(g) = \varepsilon\sigma({}^t g^{-1})\varepsilon^{-1}$, called the *unitary involution* corresponding to ε. For $\tau = \tau_\varepsilon$ a fixed unitary involution, we denote by G^τ the corresponding *unitary subgroup*, which consists of the elements of G fixed by τ.

By *representations* of a locally profinite group, we always mean smooth representations on an R-module. Given a representation π of a closed subgroup H of G, we write $\pi^\vee$ for the smooth contragradient of π. We write π^σ and π^τ for the representations $\pi \circ \sigma$ and $\pi \circ \tau$ of groups $\sigma(H)$ and $\tau(H)$, respectively. We say that π is *τ-self,dual* if H is τ-stable, and π^τ is isomorphic to $\pi^\vee$. We say that π is *σ-invariant*, if H is σ-stable, and π^σ is isomorphic to π. For $g \in G$, we write $H^g = \{g^{-1}hg | h \in H\}$ a closed subgroup and we write $\pi^g : x \mapsto \pi(gxg^{-1})$ a representation of H^g.

For $\mathfrak{a}$ an $\mathfrak{o}_F$-subalgebra of $\mathrm{M}_n(F)$ and $\tau = \tau_\varepsilon$ a unitary involution, we denote by

$$\tau(\mathfrak{a}) := \sigma_\varepsilon(\mathfrak{a}) = \{\sigma_\varepsilon(x) | x \in \mathfrak{a}\}$$

an $\mathfrak{o}_F$-subalgebra of $\mathrm{M}_n(F)$, where $\sigma_\varepsilon(x) := \varepsilon\sigma({}^t x)\varepsilon^{-1}$ is an anti-involution for any $x \in \mathrm{M}_n(F)$. We say that $\mathfrak{a}$ is *τ-stable* if $\tau(\mathfrak{a}) = \mathfrak{a}$. Moreover, for $g \in G$, we obtain

$$\tau(\mathfrak{a})^g = g^{-1}\sigma_\varepsilon(\mathfrak{a})g = \sigma_\varepsilon(\sigma_\varepsilon(g)\mathfrak{a}\sigma_\varepsilon(g^{-1})) = \sigma_\varepsilon(\tau(g)^{-1}\mathfrak{a}\tau(g)) = \tau(\mathfrak{a}^{\tau(g)}).$$

In other words, the notation $\tau(\mathfrak{a})$ is compatible with G-conjugacy.

For τ a unitary involution and π a representation of H as above, we say that π is *$H \cap G^\tau$-distinguished*, or just *distinguished*, if the space $\mathrm{Hom}_{H \cap G^\tau}(\pi, 1)$ is non-zero.

An irreducible representation of G is called *cuspidal* (or *supercuspidal*) if it does not occur as a sub-representation (or subquotient) of a parabolically induced representation with respect to a proper parabolic subgroup of G.

2.2. Hermitian matrices and unitary groups. — We make use of this subsection to introduce basic knowledge of hermitian matrices and unitary groups. The references will be [19] and [25].

Let E/E_0 be a quadratic extension of non-Archimedean locally compact fields, which are algebraic extensions of F and F_0, respectively. Write $\mathfrak{o}_E$ for the ring of integers of E and $\mathfrak{o}_{E_0}$ for that of E_0. Let $\sigma' \in \mathrm{Gal}(E/E_0)$ be the unique non-trivial involution in the Galois group. For $\varepsilon' \in \mathrm{GL}_m(E)$, just as in the last subsection, we say that ε' is a *hermitian matrix* if $(\varepsilon')^* = \varepsilon'$, where we consider $(\cdot)^*$ as before with n, F, F_0, σ replaced by m, E, E_0, σ', respectively. Write ϖ_E for a uniformizer of E such that

$$\sigma'(\varpi_E) = \begin{cases} \varpi_E & \text{if } E/E_0 \text{ is unramified,} \\ -\varpi_E & \text{if } E/E_0 \text{ is ramified.} \end{cases}$$

Let $\mathcal{X}$ denote the set of all the hermitian matrices in $\mathrm{GL}_m(E)$ for E/E_0. The group $\mathrm{GL}_m(E)$ acts on $\mathcal{X}$ by $g \cdot x = gxg^*$.

Proposition 2.1 ([25], Theorem 3.1). — *There are exactly two* $\mathrm{GL}_m(E)$-*orbits of* $\mathcal{X}$ *with respect to the action given above. Furthermore, the elements in each orbit are exactly determined by the classes of their determinants in* $E_0^\times/\mathrm{N}_{E/E_0}(E^\times)$.

We also consider the $\mathrm{GL}_m(\mathfrak{o}_E)$-orbits of $\mathcal{X}$. We consider sequences $\alpha = (\alpha_1, \ldots, \alpha_r)$ of certain triples $\alpha_i = (a_i, m_i, \delta_i)$, such that $a_1 > \ldots > a_r$ is a decreasing sequence of integers, and $m_1 + \ldots + m_r = m$ is a partition of m by positive integers, and $\delta_1, \ldots, \delta_r$ are elements of E, such that:

(1) If E/E_0 is unramified, then $\delta_i = 1$.
(2) If E/E_0 is ramified and a_i is odd, then $\delta_i = 1$ and m_i is even.
(3) If E/E_0 is ramified and a_i is even, then δ_i is either 1 or ϵ, with $\epsilon \in \mathfrak{o}_{E_0}^\times - \mathrm{N}_{E/E_0}(\mathfrak{o}_E^\times)$ fixed.

For each $\alpha = (\alpha_1, \ldots, \alpha_r)$ as above, we introduce a hermitian matrix $\varpi_E^\alpha = \varpi_E^{\alpha_1} \oplus \ldots \oplus \varpi_E^{\alpha_r}$, where $\varpi_E^{\alpha_i} \in \mathrm{GL}_{m_i}(E)$ is a hermitian matrix, such that:

(i) In the case (1), $\varpi_E^{\alpha_i} = \varpi_E^{a_i} I_{m_i}$.
(ii) In the case (2), $\varpi_E^{\alpha_i} = \varpi_E^{a_i} J_{m_i/2}$, where $J_{m_i/2} = \begin{pmatrix} 0 & I_{m_i/2} \\ -I_{m_i/2} & 0 \end{pmatrix}$;
(iii) In the case (3), $\varpi_E^{\alpha_i} = \varpi_E^{a_i} \mathrm{diag}(1, \ldots, 1, \delta_i)$, where $\mathrm{diag}(*, \ldots, *)$ denotes the diagonal matrix with corresponding diagonal elements.

We state the following proposition, which classifies all the $\mathrm{GL}_m(\mathfrak{o}_E)$-orbits of $\mathcal{X}$.

Proposition 2.2 ([25], Theorem 7.1, Theorem 8.2). — *Each class of the* $\mathrm{GL}_m(\mathfrak{o}_E)$-*orbits of* $\mathcal{X}$ *contains a unique representative of the form* ϖ_E^α *for a certain* α *as above.*

Now we study unitary groups. For $\varepsilon' \in \mathcal{X}$, we denote by $\mathrm{U}_m(\varepsilon')$ the unitary group consisting of those $g \in \mathrm{GL}_m(E)$ such that $g\varepsilon' g^* = \varepsilon'$. We say that two unitary groups are *equivalent*, if and only if they are conjugate by some $g \in \mathrm{GL}_m(E)$. Since it is easy to check that $g\mathrm{U}_m(\varepsilon')g^{-1} = \mathrm{U}_m(g\varepsilon' g^*)$, by Proposition 2.1, there are at most two equivalence classes of unitary groups, which are represented by $\mathrm{U}_m(E/E_0) := \mathrm{U}_m(I_m)$ and $\mathrm{U}'_m(E/E_0) := \mathrm{U}_m(\varepsilon)$ for $\varepsilon = \mathrm{diag}(1, \ldots, 1, \epsilon)$, where $\epsilon \in E_0^\times - \mathrm{N}_{E/E_0}(E^\times)$ is fixed.

Remark 2.3. — While we will not use it, we list the following result for completeness: $\mathrm{U}_m(E/E_0)$ is equivalent to $\mathrm{U}'_m(E/E_0)$ if and only if m is odd.

REMARK 2.4. — In the future, we only consider the following two cases. First, we consider $E = F$, $E_0 = F_0$, $m = n$ and $\sigma' = \sigma$. For any two unitary involutions with the corresponding hermitian matrices in the same $\mathrm{GL}_n(F)$-orbit, we have already shown that the corresponding two unitary groups are equivalent. Since distinction is a property invariant up to equivalence of unitary groups, we may choose a hermitian matrix in its $\mathrm{GL}_n(F)$-orbit such that the corresponding unitary involution τ is simple enough to simplify the problem. Secondly, we consider E as a finite field extension of F determined by a cuspidal representation π such that $n = m[E : F]$. We will find out that if $\pi^\sigma \cong \pi$, then we may find an involution σ' on E such that $E_0 = E^{\sigma'}$ and $\sigma'|_F = \sigma$. So we may make use of the propositions in this subsection to study hermitian matrices and unitary groups of $\mathrm{GL}_m(E)$.

3. Preliminaries on simple types

In this section, we recall the main results that we will need on simple strata, characters and types [6], [8], [9], [32], [43]. We mainly follow the structure of [1] and [35].

3.1. Simple strata and characters. — Let $[\mathfrak{a}, \beta]$ be a simple stratum in $\mathrm{M}_n(F)$ for a certain $n \geq 1$. Recall that $\mathfrak{a}$ is a hereditary order in $\mathrm{M}_n(F)$, and β is in $G = \mathrm{GL}_n(F)$, such that:

(1) the F-algebra $E = F[\beta]$ is a field with degree d over F;
(2) $E^\times$ normalizes $\mathfrak{a}^\times$.

The centralizer of E in $\mathrm{M}_n(F)$, denoted by B, is an E-algebra isomorphic to $\mathrm{M}_m(E)$ with $n = md$. The intersection $\mathfrak{b} := \mathfrak{a} \cap B$ is a hereditary order in B.

We denote by $\mathfrak{p}_\mathfrak{a}$ the Jacobson radical of $\mathfrak{a}$, and $U^1(\mathfrak{a})$ the compact open pro-p-subgroup $1 + \mathfrak{p}_\mathfrak{a}$ of G. Similarly, we denote by $\mathfrak{p}_\mathfrak{b}$ the Jacobson radical of $\mathfrak{b}$ and $U^1(\mathfrak{b})$ the compact open pro-p-subgroup $1 + \mathfrak{p}_\mathfrak{b}$ of $B^\times$. For any $x \in B^\times$, we have ([9], Theorem 1.6.1)

(1) $$U^1(\mathfrak{a})xU^1(\mathfrak{a}) \cap B^\times = U^1(\mathfrak{b})xU^1(\mathfrak{b}).$$

Associated with $[\mathfrak{a}, \beta]$, there are open compact subgroups

$$H^1(\mathfrak{a}, \beta) \subset J^1(\mathfrak{a}, \beta) \subset J(\mathfrak{a}, \beta)$$

of $\mathfrak{a}^\times$ and a finite set $\mathcal{C}(\mathfrak{a}, \beta)$ of *simple characters* of $H^1(\mathfrak{a}, \beta)$ depending on the choice of ψ. We denote by $\boldsymbol{J}(\mathfrak{a}, \beta)$ the subgroup of G generated by $J(\mathfrak{a}, \beta)$ and the normalizer of $\mathfrak{b}^\times$ in $B^\times$.

PROPOSITION 3.1 ([9], section 3). — *We have the following properties:*

(1) *The group $J(\mathfrak{a}, \beta)$ is the unique maximal compact subgroup of $\boldsymbol{J}(\mathfrak{a}, \beta)$.*
(2) *The group $J^1(\mathfrak{a}, \beta)$ is the unique maximal normal pro-p-subgroup of $J(\mathfrak{a}, \beta)$.*

(3) *The group* $J(\mathfrak{a},\beta)$ *is generated by* $J^1(\mathfrak{a},\beta)$ *and* $\mathfrak{b}^\times$*, and we have*

$$J(\mathfrak{a},\beta)\cap B^\times=\mathfrak{b}^\times, J^1(\mathfrak{a},\beta)\cap B^\times=U^1(\mathfrak{b}). \tag{2}$$

(4) *The normalizer of any simple character* $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ *in* G *is equal to* $\boldsymbol{J}(\mathfrak{a},\beta)$*.*

(5) *The intertwining set of any* $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ *in* G*, which we denote by* $I_G(\theta)$*, is equal to* $J^1(\mathfrak{a},\beta)B^\times J^1(\mathfrak{a},\beta)=J(\mathfrak{a},\beta)B^\times J(\mathfrak{a},\beta)$*.*

REMARK 3.2. — For short, we write J, J^1, H^1 for $J(\mathfrak{a},\beta)$, $J^1(\mathfrak{a},\beta)$, $H^1(\mathfrak{a},\beta)$, respectively, if $\mathfrak{a}$ and β are clear to us.

When $\mathfrak{b}$ is a maximal order in B, we call the simple stratum $[\mathfrak{a},\beta]$ and the simple characters in $\mathcal{C}(\mathfrak{a},\beta)$ *maximal*. In this case, we may find an isomorphism of E-algebras $B\cong \mathrm{M}_m(E)$, which identifies $\mathfrak{b}$ with the standard maximal order, and, moreover, we have group isomorphisms

$$J(\mathfrak{a},\beta)/J^1(\mathfrak{a},\beta)\cong\mathfrak{b}^\times/U^1(\mathfrak{b})\cong \mathrm{GL}_m(\boldsymbol{l}), \tag{3}$$

where $\boldsymbol{l}$ denotes the residue field of E.

3.2. Simple types and cuspidal representations. — A pair $(\boldsymbol{J},\Lambda)$, called an *extended maximal simple type* in G (we always write *simple type* for short) and constructed in [9] in the characteristic 0 case and in [43], [32] in the modular case, is made of a subgroup $\boldsymbol{J}$ of G, which is open and compact modulo its centre, and an irreducible representation Λ of $\boldsymbol{J}$.

Given a simple type $(\boldsymbol{J},\Lambda)$ in G, there are a maximal simple stratum $[\mathfrak{a},\beta]$ in $\mathrm{M}_n(F)$ and a maximal simple character $\theta\in\mathcal{C}(\mathfrak{a},\beta)$, such that $\boldsymbol{J}(\mathfrak{a},\beta)=\boldsymbol{J}$, and θ is contained in the restriction of Λ to $H^1(\mathfrak{a},\beta)$. Such a character θ is said to be *attached to* Λ. By [9], Proposition 5.1.1 (or [32], Proposition 2.1 in the modular case), the group $J^1(\mathfrak{a},\beta)$ has, up to isomorphism, a unique irreducible representation η whose restriction to $H^1(\mathfrak{a},\beta)$ contains θ. Such a representation η, called the *Heisenberg representation* associated to θ, has the following properties:

(1) The restriction of η to $H^1(\mathfrak{a},\beta)$ is made of $(J^1(\mathfrak{a},\beta):H^1(\mathfrak{a},\beta))^{1/2}$ copies of θ. Here, $(J^1(\mathfrak{a},\beta):H^1(\mathfrak{a},\beta))^{1/2}$ is a power of p.

(2) The direct sum of $(J^1(\mathfrak{a},\beta):H^1(\mathfrak{a},\beta))^{1/2}$ copies of η, which we denote by $\eta^{(J^1(\mathfrak{a},\beta):H^1(\mathfrak{a},\beta))^{1/2}}$, is isomorphic to $\mathrm{Ind}_{H^1}^{J^1}\theta$.

(3) The representation η extends to $\boldsymbol{J}$.

(4) The intertwining set of η, which we denote by $I_G(\eta)$, equals $I_G(\theta)$.

(5) For $h\in I_G(\eta)$, we have $\dim_R(\mathrm{Hom}_{J^1\cap J^{1h}}(\eta^h,\eta))=1$.

For any representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}$ extending η, there exists a unique irreducible representation $\boldsymbol{\rho}$ of $\boldsymbol{J}$ trivial on $J^1(\mathfrak{a},\beta)$, such that $\Lambda\cong\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Through (3), the restriction of $\boldsymbol{\rho}$ to $J=J(\mathfrak{a},\beta)$ is identified with the inflation of a cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$.

Remark 3.3. — Recall that in [9], Bushnell and Kutzko also assume $\kappa^0 = \boldsymbol{\kappa}|_{J(\mathfrak{a},\beta)}$ to be a so-called *beta-extension*, which means that:

(1) κ^0 is an extension of η;
(2) if we denote by $I_G(\kappa^0)$ the intertwining set of κ^0, then $I_G(\kappa^0) = I_G(\eta) = I_G(\theta)$.

However, in our case, since $\mathrm{GL}_m(\boldsymbol{l})$ is not isomorphic to $\mathrm{GL}_2(\mathbb{F}_2)$ $(p \neq 2)$, any character of $\mathrm{GL}_m(\boldsymbol{l})$ factors through the determinant. It follows that any representation of J extending η is a beta-extension. So, finally, our consideration of κ^0 coincides with the original assumption of Bushnell and Kutzko.

We now give the classification of irreducible cuspidal representations of G in terms of simple types (see [9], §6.2, §8.4 and [32], section 3 in the modular case).

Proposition 3.4 ([9],[32]). — *Let π be a cuspidal representation of G.*

(1) *There is a simple type $(\boldsymbol{J}, \Lambda)$ such that Λ is a sub-representation of the restriction of π to $\boldsymbol{J}$. It is unique up to G-conjugacy.*
(2) *Compact induction c-$\mathrm{Ind}_{\boldsymbol{J}}^G$ gives a bijection between the G-conjugacy classes of simple types and the isomorphism classes of cuspidal representations of G.*

3.3. Endo-classes, tame parameter fields and tame lifting. — In this subsection, we introduce the concepts of endo-classes, tame parameter fields and tame lifting. The main references will be [9], [6] and [8].

For $[\mathfrak{a},\beta]$, a simple stratum in $\mathrm{M}_n(F)$ and $[\mathfrak{a}',\beta']$ a simple stratum in $\mathrm{M}_{n'}(F)$ with $n, n' \geq 1$, if we have an isomorphism of F-algebras $\phi : F[\beta] \to F[\beta']$, such that $\phi(\beta) = \beta'$, then there exists a canonical bijection

$$t_{\mathfrak{a},\mathfrak{a}'}^{\beta,\beta'} : \mathcal{C}(\mathfrak{a},\beta) \to \mathcal{C}(\mathfrak{a}',\beta'),$$

called the *transfer map* (see [9], Theorem 3.6.14).

Now let $[\mathfrak{a}_1,\beta_1]$ and $[\mathfrak{a}_2,\beta_2]$ be simple strata in $\mathrm{M}_{n_1}(F)$ and $\mathrm{M}_{n_2}(F)$, respectively, with $n_1, n_2 \geq 1$. We call two simple characters $\theta_1 \in \mathcal{C}(\mathfrak{a}_1,\beta_1)$ and $\theta_2 \in \mathcal{C}(\mathfrak{a}_2,\beta_2)$ *endo-equivalent*, if there are simple strata $[\mathfrak{a}',\beta_1']$ and $[\mathfrak{a}',\beta_2']$ in $\mathrm{M}_{n'}(F)$, for some $n' \geq 1$ such that θ_1 and θ_2 transfer to two simple characters $\theta_1' \in \mathcal{C}(\mathfrak{a}',\beta_1')$ and $\theta_2' \in \mathcal{C}(\mathfrak{a}',\beta_2')$, respectively, which intertwine (or by [9], Theorem 3.5.11, which are $\mathrm{GL}_{n'}(F)$-conjugate). This defines an equivalence relation on

$$\bigcup_{[\mathfrak{a},\beta]} \mathcal{C}(\mathfrak{a},\beta),$$

where the union runs over all simple strata of $\mathrm{M}_n(F)$, for all $n \geq 1$ (see [6], section 8). An equivalence class for this equivalence relation is called an *endo-class.*

For π a cuspidal representation of $G = \mathrm{GL}_n(F)$, there exist a simple stratum $[\mathfrak{a}, \beta]$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ contained in π. The set of simple characters θ contained in π constitutes a G-conjugacy class, thus those simple characters are endo-equivalent. So we may denote by Θ_π the endo-class of π, which is the endo-class determined by any θ contained in π.

Given $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, the degree of E/F, its ramification index and its residue degree depend only on the endo-class of θ. They are called the degree, ramification index and residue degree of this endo-class. Although the field extension E/F is not uniquely determined, its maximal tamely ramified sub-extension is uniquely determined by the endo-class of θ up to F-isomorphism. This field is called a *tame parameter field* of the endo-class (see [8], §2.2, §2.4).

We denote by $\mathcal{E}(F)$ the set of endo-classes of simple characters over F. Given a finite tamely ramified extension T of F, we have a surjection

$$\mathcal{E}(T) \to \mathcal{E}(F)$$

with finite fibers, which is called a *restriction map* (see [8], §2.3). Given $\Theta \in \mathcal{E}(F)$, the endo-classes $\Psi \in \mathcal{E}(T)$ restricting to Θ are called the T/F-lifts of Θ. If Θ has a tame parameter field T, then $\mathrm{Aut}_F(T)$ acts faithfully and transitively on the set of T/F-lifts of Θ (see [8], §2.3, §2.4).

Let $[\mathfrak{a}, \beta]$ be a simple stratum and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character, let T be the maximal tamely ramified extension of F in E, and let Θ be the endo-class of θ, then T is a tame parameter field for Θ. Let $C \cong \mathrm{M}_{n/t}(T)$ denote the centralizer of T in $\mathrm{M}_n(F)$, where $t = [T : F]$. The intersection $\mathfrak{c} = \mathfrak{a} \cap C$ is an order in C, which gives rise to a simple stratum $[\mathfrak{c}, \beta]$. The restriction of θ to $H^1(\mathfrak{c}, \beta)$, denoted by θ_T, is a simple character associated to this simple stratum, called the *interior T/F-lift* of θ. Its endo-class, denoted by Ψ, is a T/F-lift of Θ. For the origin and details of the construction of Ψ, see [6].

For $T \subset \mathrm{M}_n(F)$ a tamely ramified sub-extension over F, the map

$$\mathfrak{a} \mapsto \mathfrak{a} \cap C$$

is injective from the set of hereditary orders of $\mathrm{M}_n(F)$ normalized by $T^\times$ to the set of hereditary orders of C (see [6], Section 2), where we still denote by C the centralizer of T in $\mathrm{M}_n(F)$. For $[\mathfrak{a}, \beta_1]$, $[\mathfrak{a}_2, \theta_2]$ two simple strata, and $\theta_1 \in \mathcal{C}(\mathfrak{a}, \beta_1)$, $\theta_2 \in \mathcal{C}(\mathfrak{a}, \beta_2)$ two simple characters, such that θ_1 and θ_2 have the same tame parameter field T, if

$$\mathcal{C}(\mathfrak{c}, \beta_1) = \mathcal{C}(\mathfrak{c}, \beta_2) \quad \text{and} \quad (\theta_1)_T = (\theta_2)_T,$$

then (see [BH96], Theorem 7.10, Theorem 7.15)

$$\mathcal{C}(\mathfrak{a}, \beta_1) = \mathcal{C}(\mathfrak{a}, \beta_2) \quad \text{and} \quad \theta_1 = \theta_2.$$

In particular, when $\beta_1 = \beta_2 = \beta$, the interior T/F-lift is injective from $\mathcal{C}(\mathfrak{a}, \beta)$ to $\mathcal{C}(\mathfrak{c}, \beta)$.

3.4. Supercuspidal representations. — Let π be a cuspidal representation of G. By Proposition 3.4, it contains a simple type $(\boldsymbol{J}, \Lambda)$. Fix a maximal simple stratum $[\mathfrak{a}, \beta]$, such that $\boldsymbol{J} = \boldsymbol{J}(\mathfrak{a}, \beta)$, and write $\Lambda = \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ as in §3.2. Let $\overline{\rho}$ be the cuspidal representation of $J/J^1 \cong \mathrm{GL}_m(\boldsymbol{l})$ whose inflation equals $\boldsymbol{\rho}|_J$. We have the following proposition:

Proposition 3.5 ([43], Chapitre III, 5.14). — *The representation π is supercuspidal if and only if $\overline{\rho}$ is supercuspidal.*

4. Distinction implies Galois invariance for a supercuspidal representation

Let $G = \mathrm{GL}_n(F)$ and let G^τ be the unitary group corresponding to a unitary involution τ. We state the following theorem, which is well-known when $R = \mathbb{C}$ and $\mathrm{char}(F) = 0$ (see, for example, [20], section 4, corollary or the earlier paper [23] which illustrates the idea).

Theorem 4.1. — *Let π be a supercuspidal representation of G. If π is distinguished by G^τ, then π is σ-invariant.*

Before proving Theorem 4.1, we state a useful lemma, which will be used not only in the proof of the theorem but also in the latter sections.

Lemma 4.2. — *For δ a unitary involution on G and for $(\boldsymbol{J}, \Lambda)$ a simple type in G, we have $\boldsymbol{J} \cap G^\delta = J \cap G^\delta$.*

Proof. — For $x \in \boldsymbol{J} \cap G^\delta$, we have $\delta(x) = x$, which implies that $\sigma(\det(x))\det(x) = 1$, where we denote by $\det(\cdot)$ the determinant function defined on G. Thus, $\det(x) \in \mathfrak{o}_F^\times$. Since $\boldsymbol{J} = E^\times J$, we get $x \in \mathfrak{o}_E^\times J \cap G^\delta = J \cap G^\delta$. □

Moreover, we need the following lemma, which says that the properties of distinction and σ-invariance are maintained up to change of base fields.

Lemma 4.3. — *Let $R_1 \hookrightarrow R_2$ be a fixed embedding of two algebraically closed fields of characteristic $l \geq 0$. Let π_0 be a supercuspidal representation of G over R_1. Let $\pi = \pi_0 \otimes_{R_1} R_2$ be the corresponding representation of G over R_2. Then:*

(1) *π_0 is distinguished by G^τ if and only if π is distinguished by G^τ.*
(2) *$\pi_0^\sigma \cong \pi_0$ if and only if $\pi^\sigma \cong \pi$.*

Proof. — For (1), let $(\boldsymbol{J}, \Lambda_0)$ be a simple type of π_0. Then $(\boldsymbol{J}, \Lambda) := (\boldsymbol{J}, \Lambda_0 \otimes_{R_1} R_2)$ is a simple type of π, and thus π is also supercuspidal. Using Frobenius reciprocity and the Mackey formula[1],

$$\mathrm{Hom}_{R_1[G^\tau]}(\pi_0,1)\neq 0 \Longleftrightarrow \begin{array}{l}\text{There exists } \ g\in G \text{ such that}\\ \mathrm{Hom}_{R_1[\boldsymbol{J}^g\cap G^\tau]}(\Lambda_0^g,1)\neq 0\end{array}$$

and

$$\mathrm{Hom}_{R_2[G^\tau]}(\pi,1)\neq 0 \Longleftrightarrow \begin{array}{l}\text{There exists } \ g\in G \text{ such that}\\ \mathrm{Hom}_{R_2[\boldsymbol{J}^g\cap G^\tau]}(\Lambda^g,1)\neq 0.\end{array}$$

By Lemma 4.2, $\boldsymbol{J}^g\cap G^\tau = J^g\cap G^\tau$ is a compact group, and Λ_0^g is a representation of finite dimension. Thus,

$$\mathrm{Hom}_{R_1[\boldsymbol{J}^g\cap G^\tau]}(\Lambda_0^g,1)\otimes_{R_1}R_2\cong \mathrm{Hom}_{R_2[\boldsymbol{J}^g\cap G^\tau]}(\Lambda^g,1),$$

which finishes the proof of (1). For (2), from [43], Chapitre I, 6.13, we know that π_0 is isomorphic to π_0^σ if and only if their trace characters are equal up to a scalar in $R_1^\times$, which works similarly for π and π^σ. Since the trace characters of π_0 and π are equal up to the change of scalars, which works similarly for π_0^σ and π^σ, we finish the proof of (2). [2] □

Proof of Theorem 4.1. — First we consider $R=\mathbb{C}$. If $\mathrm{char}(F)=0$, it is a standard result proved by using a global method ([20], section 4, corollary). Especially, their result is based on the globalization theorem, saying a distinguished π under our settings can be realized as a local component of a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$, which is distinguished by a unitary subgroup of $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$ with respect to a quadratic extension of number fields $\mathcal{K}/\mathcal{K}_0$ (see *ibid.*, Theorem 1). If $\mathrm{char}(F)>0$, in order to use the proof of Hakim–Murnaghan, we only need a variant of globalization theorem for the characteristic positive case. Fortunately, Gan–Lomelí already built up the globalization theorem for general reductive groups over function fields and locally compact fields of positive characteristic (see [15], Theorem 1.3). Following their notations, we choose the reductive group H to be $\mathrm{R}_{\mathcal{K}/\mathcal{K}_0}(\mathrm{GL}_n(\mathcal{K}))$, where $\mathcal{K}/\mathcal{K}_0$ is a quadratic extension of function fields, and $\mathrm{R}_{\mathcal{K}/\mathcal{K}_0}$ is the Weil restriction. We choose V to be $\mathrm{M}_n(\mathcal{K})$ as a $\mathcal{K}_0$-vector space and $\iota: H\rightarrow \mathrm{GL}(V)$ to be a representation over $\mathcal{K}_0$ defined by

$$\iota(h)x=hx\sigma({}^th),\quad x\in V,\ h\in H,$$

where σ denotes the non-trivial involution in $\mathrm{Gal}(\mathcal{K}/\mathcal{K}_0)$. If we choose $x_0\in V$ to be a hermitian matrix in $\mathrm{M}_n(\mathcal{K})$ and H^{x_0} to be the stabilizer of x_0, then H^{x_0} becomes a unitary subgroup of H, which satisfies the condition of *loc. cit.* In order to use their result, we only need to verify the conditions (a) and (b) in their theorem. For condition (a), ι is semi-simple since it is the direct

1. This argument will occur several times in this section, so for more details we refer the reader to the proof of Theorem 4.1.

2. Note that if the trace characters of π_0^σ and π_0 are equal up to a scalar in $R_2^\times$, then that scalar is in $R_1^\times$ since the trace of π_0 and π_0^σ take values in R_1.

sum of two irreducible sub-representations, composed of hermitian matrices and anti-hermitian matrices, respectively [3]. For condition (b), since we only care about the case where $\chi = 1$, it is automatically satisfied. Thus, if we use [15], Theorem 1.3 to replace [20], Theorem 1 and follow the proof in [20], then we finish the proof when $R = \mathbb{C}$, and F/F_0 is a quadratic extension of locally compact fields of characteristic p.

For $\mathrm{char}(R) = 0$ in general, a supercuspidal representation of G can be realized as a representation over $\overline{\mathbb{Q}}$ up to twisting by an unramified character, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. More precisely, there exists a character $\chi : F^\times \to R^\times$ such that $\chi|_{\mathfrak{o}_F^\times} = 1$ and $\pi \cdot \chi \circ \det$ can be realized as a representation over $\overline{\mathbb{Q}}$. Since $\det(G^\tau) \subset \mathfrak{o}_F^\times$ and $\chi \circ \det|_{G^\tau}$ is trivial, π is G^τ-distinguished if and only if $\pi \cdot \chi \circ \det$ is, as a representation over R, and also as a representation over $\overline{\mathbb{Q}}$ or $\mathbb{C}$ by Lemma 4.3.(1). Using the complex case, $\pi \cdot \chi \circ \det$ is σ-invariant as a representation over $\mathbb{C}$, and also as a representation over $\overline{\mathbb{Q}}$ or R by Lemma 4.3.(2). By definition, χ is σ-invariant, and thus π is also σ-invariant.

For $R = \overline{\mathbb{F}_l}$, we write $\pi \cong \text{c-Ind}_{\boldsymbol{J}}^G \Lambda$ for a simple type $(\boldsymbol{J}, \Lambda)$. Using the Mackey formula and Frobenius reciprocity, we have

$$0 \neq \mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \boldsymbol{J} \backslash G / G^\tau} \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1).$$

Thus, π is distinguished if and only if there exists $g \in G$ such that $\mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$. Let $\gamma = \tau(g) g^{-1}$ and let $\delta(x) = \gamma^{-1} \tau(x) \gamma$ for $x \in G$, which is also a unitary involution; then we have

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1) &\cong \mathrm{Hom}_{\boldsymbol{J} \cap G^\delta}(\Lambda, 1) = \mathrm{Hom}_{J \cap G^\delta}(\Lambda^0, 1) \\ &\cong \mathrm{Hom}_J(\Lambda^0, \mathrm{Ind}_{J \cap G^\delta}^J \overline{\mathbb{F}_l}), \end{aligned}$$

where $\Lambda^0 = \Lambda|_J$, and we use the fact that $\boldsymbol{J} \cap G^\delta = J \cap G^\delta$ by Lemma 4.2.

We consider P_{Λ^0} to be the projective envelope of Λ^0 as a $\overline{\mathbb{Z}_l}[J]$-module, where we denote by $\overline{\mathbb{Z}_l}$ the ring of integers of $\overline{\mathbb{Q}_l}$; then we have ([43], Chapitre III, 4.28 and [39], Proposition 42 for finite group case. Since Λ^0 is a smooth representation of the compact group J of finite dimension, it can be regarded as a representation of a finite group.):

(1) $P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}$ is the projective envelope of Λ^0 as a $\overline{\mathbb{F}_l}[J]$-module, which is indecomposable of finite length, with each irreducible component isomorphic to Λ^0. Thus, $\mathrm{Hom}_{\overline{\mathbb{F}_l}[J]}(P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}, \mathrm{Ind}_{J \cap G^\delta}^J \overline{\mathbb{F}_l}) \neq 0$.

(2) For $\widetilde{P_{\Lambda^0}} = P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{Q}_l}$ the $\overline{\mathbb{Q}_l}$-lift of P_{Λ^0}, we have $\widetilde{P_{\Lambda^0}} \cong \bigoplus \widetilde{\Lambda^0}$, where $\widetilde{\Lambda^0}$ in the direct sum are $\overline{\mathbb{Q}_l}$-lifts of Λ^0 of multiplicity 1 (the multiplicity 1 statement is derived from counting the length of $P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}$ and the number of different $\widetilde{\Lambda^0}$ in $\widetilde{P_{\Lambda^0}}$, and then showing that they are equal.

3. Here we need the assumption $p \neq 2$.

The argument is indicated in the proof of [43], Chapitre III, 4.28, or more precisely, *ibid.*, Chapitre III, Théorème 2.2 and Théorème 2.9).

(3) In (2), each $(J, \widetilde{\Lambda^0})$ can be extended to a simple type $(\boldsymbol{J}, \widetilde{\Lambda})$ of G as a $\overline{\mathbb{Q}_l}$-lift of $(\boldsymbol{J}, \Lambda)$ ([43], Chapitre III, 4.29).

Using (1), $\mathrm{Hom}_{\overline{\mathbb{F}_l}[J]}(P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{F}_l}) \neq 0$. Since P_{Λ^0} is a projective $\overline{\mathbb{Z}_l}[J]$-module, it is a free $\overline{\mathbb{Z}_l}$-module. Since $\mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Z}_l}$ is a free $\overline{\mathbb{Z}_l}$-module,

$$\mathrm{Hom}_{\overline{\mathbb{Z}_l}[J]}(P_{\Lambda^0}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Z}_l})$$

is a free $\overline{\mathbb{Z}_l}$-module. As a result,

$$\mathrm{Hom}_{\overline{\mathbb{F}_l}[J]}(P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{F}_l}) \cong \mathrm{Hom}_{\overline{\mathbb{Z}_l}[J]}(P_{\Lambda^0}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Z}_l}) \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l} \neq 0$$

if and only if

$$\mathrm{Hom}_{\overline{\mathbb{Z}_l}[J]}(P_{\Lambda^0}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Z}_l}) \neq 0$$

if and only if

$$\mathrm{Hom}_{\overline{\mathbb{Q}_l}[J]}(\widetilde{P_{\Lambda^0}}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Q}_l}) \cong \mathrm{Hom}_{\overline{\mathbb{Z}_l}[J]}(P_{\Lambda^0}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Z}_l}) \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{Q}_l} \neq 0.$$

So there exists $\widetilde{\Lambda^0}$ as in condition (2), such that $\mathrm{Hom}_{\overline{\mathbb{Q}_l}[J]}(\widetilde{\Lambda^0}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Q}_l}) \neq 0$. Using (3) we may choose $(\boldsymbol{J}, \widetilde{\Lambda})$ as an extension of $(J, \widetilde{\Lambda^0})$. We write $\widetilde{\pi} = c\text{-}\mathrm{Ind}_{\boldsymbol{J}}^G \widetilde{\Lambda}$, which is a supercuspidal representation of G over $\overline{\mathbb{Q}_l}$. By using

$$\begin{aligned}\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\widetilde{\Lambda}^g, 1) &\cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\delta}(\widetilde{\Lambda}, 1) = \mathrm{Hom}_{J\cap G^\delta}(\widetilde{\Lambda^0}, 1)\\ &\cong \mathrm{Hom}_J(\widetilde{\Lambda^0}, \mathrm{Ind}_{J\cap G^\delta}^J \overline{\mathbb{Q}_l}) \neq 0\end{aligned}$$

and by the Mackey formula and Frobenius reciprocity as before, $\widetilde{\pi}$ is G^τ-distinguished. Using the result of the characteristic 0 case, we have $\widetilde{\pi}^\sigma \cong \widetilde{\pi}$. By (3), $\widetilde{\Lambda}$ is a $\overline{\mathbb{Q}_l}$-lift of Λ. So $\widetilde{\pi}$ is a $\overline{\mathbb{Q}_l}$-lift of π, and we have $\pi^\sigma \cong \pi$.

For $\mathrm{char}(R) = l > 0$ in general, as in the characteristic zero case, there exists a character $\chi : F^\times \to R^\times$ such that $\chi|_{\mathfrak{o}_F^\times} = 1$ and $\pi \cdot \chi \circ \det$ can be realized as a representation over $\overline{\mathbb{F}_l}$. Similarly, we deduce that π is G^τ-distinguished if and only if $\pi \cdot \chi \circ \det$ is, as a representation over R, and also as a representation over $\overline{\mathbb{F}}_l$ by Lemma 4.3.(1). Using the case above, $\pi \cdot \chi \circ \det$ is σ-invariant, as a representation over $\overline{\mathbb{F}}_l$, and also as a representation over R by Lemma 4.3.(2). By definition, χ is σ-invariant, and thus π is also σ-invariant. □

Remark 4.4. — It is also possible to give a purely local proof (without using the result of the complex supercuspidal case) for this theorem, which also works for cuspidal representations. Since our proof relies on the refinement of the results and the arguments in Section 5 to Section 8, we leave it to the last section to avoid breaking up the structure of the paper.

5. The τ-self-dual type theorem

Let $G = \mathrm{GL}_n(F)$ and let τ be the unitary involution of G corresponding to a hermitian matrix ε. Let π be a cuspidal representation of G. We choose a maximal simple stratum $[\mathfrak{a}, \beta]$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ contained in π.

Lemma 5.1. — *If π is σ-invariant, then we may choose the simple stratum above such that $\sigma({}^t\beta) = \beta$. As a result, σ_t (see Section 2) is an involution defined on E whose restriction to F is σ.*

Let $E_0 = E^{\sigma_t}$, where $E = F[\beta]$ and β is chosen as in Lemma 5.1.

Theorem 5.2. — *Let π be a σ-invariant cuspidal representation of G and let τ be a unitary involution. We also assume the following additional condition:*

If the hermitian matrix corresponding to τ is not in the same G-class as I_n in $\mathcal{X}$ and if there exists a maximal simple stratum $[\mathfrak{a}, \beta]$ as in Lemma 5.1 with a $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ contained in π, such that the corresponding E/E_0 is unramified, then m is odd.

Then there exist a maximal simple stratum $[\mathfrak{a}', \beta']$ and a simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ contained in π, such that:

(1) $\tau(\beta') = \beta'^{-1}$.

(2) $\tau(\mathfrak{a}') = \mathfrak{a}'$ *and*[4] $\tau(H^1(\mathfrak{a}', \beta')) = H^1(\mathfrak{a}', \beta')$.

(3) $\theta' \circ \tau = \theta'^{-1}$.

As a corollary of Theorem 5.2, we state the main theorem of this section:

Theorem 5.3 (The τ-self-dual type theorem). — *Under the same conditions as Theorem 5.2, there exists a simple type $(\boldsymbol{J}, \Lambda)$ contained in π such that $\tau(\boldsymbol{J}) = \boldsymbol{J}$ and $\Lambda^\tau \cong \Lambda^\vee$.*

In the following subsections, we will focus on the proof of the results stated.

5.1. Endo-class version of main results. — To prove Theorem 5.2 and Theorem 5.3, we consider their corresponding analogues for the endo-class. Let Θ be an endo-class over F. As mentioned in Section 3, we write $d = \deg(\Theta)$. Moreover, its tame parameter field T as a tamely ramified extension over F is unique up to F-isomorphism.

From the definition of the endo-class, we may choose a maximal simple stratum $[\mathfrak{a}, \beta]$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ such that $\theta \in \Theta$. We denote by Θ^σ the endo-class of θ^σ, which does not depend on the choice of θ. We denote by n the size of $\mathfrak{a}$, that is, $\mathfrak{a} \hookrightarrow \mathrm{M}_n(F)$ as a hereditary order. We write $n = md$ with m a positive integer. First of all, we have the following lemma as an endo-class version of Lemma 5.1, which will be proved in §5.4.

4. For the definition of $\tau(\mathfrak{a}')$, see §2.1. We will use the same notation for Theorem 5.5 and further proofs.

Lemma 5.4. — *If $\Theta^\sigma = \Theta$, then we may choose the simple stratum above such that $\sigma({}^t\beta) = \beta$. As a result, σ_t is an involution defined on E whose restriction to F is σ.*

Let $E_0 = E^{\sigma_t}$, where $E = F[\beta]$ and β is chosen as in Lemma 5.4. The following theorem as an endo-class version of Theorem 5.2 says that we may adjust our choice of the simple stratum and simple character such that they are τ-self-dual with respect to a unitary involution τ:

Theorem 5.5. — *Let $\Theta \in \mathcal{E}(F)$ be an endo-class over F such that $\Theta^\sigma = \Theta$. Let τ be a unitary involution of G. We also assume the following additional condition:*

If the hermitian matrix corresponding to τ is not in the same G-class as I_n in $\mathcal{X}$, and if there exists a maximal simple stratum $[\mathfrak{a}, \beta]$ as in Lemma 5.4 with a $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ contained in Θ, such that the corresponding E/E_0 is unramified, then $m = n/d$ is odd.

Then there exist a maximal simple stratum $[\mathfrak{a}', \beta']$ in $\mathrm{M}_n(F)$ and a simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ such that:

(1) $\tau(\beta') = \beta'^{-1}$.
(2) $\tau(\mathfrak{a}') = \mathfrak{a}'$ *and* $\tau(H^1(\mathfrak{a}', \beta')) = H^1(\mathfrak{a}', \beta')$.
(3) $\theta' \in \Theta$ *and* $\theta' \circ \tau = \theta'^{-1}$.

Later we will focus on the proof of Lemma 5.4 and Theorem 5.5. So before we begin our proof, we illustrate how this theorem implies Lemma 5.1, Theorem 5.2 and Theorem 5.3. First, we have the following important result due to Gelfand and Kazhdan (see [4], Theorem 7.3 for the complex case and [38], Proposition 8.4 for the l-modular case):

Proposition 5.6. — *For π an irreducible representation of $\mathrm{GL}_n(F)$, the representation defined by $g \mapsto \pi({}^t g^{-1})$ is isomorphic to $\pi^\vee$.*

For π given as in Lemma 5.1, if we denote by Θ_π the endo-class of π, then we get $\Theta_\pi^\sigma = \Theta_\pi$. So we may use Lemma 5.4 to get Lemma 5.1 and use Theorem 5.5 to get Theorem 5.2.

Now we show that Theorem 5.2 implies Theorem 5.3. Using Proposition 5.6, we have $\pi^{\tau\vee} \cong \pi^\sigma \cong \pi$. Let $(\boldsymbol{J}, \Lambda)$ be a simple type of π containing θ', where θ' is obtained from Theorem 5.2 such that $\theta' \circ \tau = \theta'^{-1}$. Thus $\tau(\boldsymbol{J}) = \boldsymbol{J}$ since they are the G-normalizers of $\theta' \circ \tau$ and θ'^{-1}, respectively. Since $\pi^{\tau\vee} \cong \pi$, it contains both $(\boldsymbol{J}, \Lambda)$ and $(\boldsymbol{J}, \Lambda^{\tau\vee})$. By Proposition 3.4, there exists $g \in G$ such that $(\boldsymbol{J}, \Lambda^{\tau\vee}) = (\boldsymbol{J}^g, \Lambda^g)$. Since $\Lambda^{\tau\vee} \cong \Lambda^g$ contains both $(\theta' \circ \tau)^{-1} = \theta'$ and θ'^g as simple characters, the restriction of Λ^g to the intersection

$$H^1(\mathfrak{a}', \beta') \cap H^1(\mathfrak{a}', \beta')^g, \tag{4}$$

which is a direct sum of copies of θ'^g restricting to (4), contains the restriction of θ' to (4). It follows that g intertwines θ'. By Proposition 3.1.(5), $g \in$

$J(\mathfrak{a}',\beta')B'^{\times}J(\mathfrak{a}',\beta')$ with B' the centralizer of E' in $\mathrm{M}_n(F)$. Thus we may assume $g\in B'^{\times}$. From the uniqueness of the maximal compact subgroup in $\boldsymbol{J}$, we deduce that $\boldsymbol{J}^g=\boldsymbol{J}$ implies $J(\mathfrak{a}',\beta')^g=J(\mathfrak{a}',\beta')$. Intersecting it with $B'^{\times}$ implies that $\mathfrak{b}'^{\times g}=\mathfrak{b}'^{\times}$. Since $\mathfrak{b}'^{\times}$ is a maximal compact subgroup of $B'^{\times}\cong\mathrm{GL}_m(E')$ and $g\in B'^{\times}$, we deduce that $g\in E'^{\times}\mathfrak{b}'^{\times}\subset J(\mathfrak{a}',\beta')$. Thus, $(\boldsymbol{J}^g,\Lambda^g)=(\boldsymbol{J},\Lambda)$, which finishes the proof of Theorem 5.3.

Finally, we state the following two lemmas, which will be useful in our further proof:

Lemma 5.7. — *Let $[\mathfrak{a},\beta]$ be a maximal simple stratum in $\mathrm{M}_n(F)$ and let Θ be a σ-invariant endo-class over F, such that there exists $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ a simple character in Θ. Then $\theta\circ\tau$ and θ^{-1} are in the same endo-class. In particular, if the hereditary order $\mathfrak{a}$ is τ-invariant, then $\theta\circ\tau$ conjugates to θ^{-1} by an element in $U(\mathfrak{a})$.*

Proof. — We choose π a cuspidal representation of G containing θ. Thus by definition, we have $\Theta_\pi=\Theta$. Using Proposition 5.6, we have $\pi^\tau\cong\pi^{\sigma\vee}$. So $\theta\circ\tau\in\Theta_{\pi^\tau}=\Theta_{\pi^{\sigma\vee}}=\Theta_{\pi^\vee}^\sigma$ and $\theta^{-1}\in\Theta_{\pi^\vee}$. Since $\Theta^\sigma=\Theta$, we have $\Theta_{\pi^\vee}^\sigma=\Theta_{\pi^\vee}$, which means that $\theta\circ\tau$ and θ^{-1} are in the same endo-class. If $\tau(\mathfrak{a})=\mathfrak{a}$, then by definition of the endo-equivalence ([6], Theorem 8.7), $\theta\circ\tau$ intertwines with θ^{-1}. By [9], Theorem 3.5.11, $\theta\circ\tau$ conjugates to θ^{-1} by an element in $U(\mathfrak{a})$. □

The following lemma will be used to change the choice of a unitary involution up to G-action on its corresponding hermitian matrix.

Lemma 5.8. — *Let $\tau=\tau_\varepsilon$ be the unitary involution on G corresponding to a hermitian matrix ε, and let $[\mathfrak{a},\beta]$ be a maximal simple stratum in $\mathrm{M}_n(F)$ and let $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ be a simple character, such that*

$$\tau(\mathfrak{a})=\mathfrak{a},\quad \theta\circ\tau=\theta^{-1}\quad (\textit{and } \tau(\beta)=\beta^{-1}).$$

Then for $\tau'=\tau_{\varepsilon'}$ the unitary involution corresponding to a hermitian matrix $\varepsilon'=g^{-1}\varepsilon\sigma({}^tg^{-1})$, we have

$$\tau'(\mathfrak{a}^g)=\mathfrak{a}^g,\quad \theta^g\circ\tau'=(\theta^g)^{-1}\quad (\textit{and } \tau'(\beta^g)=(\beta^g)^{-1}).$$

Proof. — The proof is just a simple calculation. We have

$$\tau'(\mathfrak{a}^g)=\tau'(g^{-1})\tau'(\mathfrak{a})\tau'(g)=\tau'(g^{-1})\varepsilon'\varepsilon^{-1}\tau(\mathfrak{a})(\varepsilon'\varepsilon^{-1})^{-1}\tau'(g)=g^{-1}\tau(\mathfrak{a})g,$$

where in the last step we use

$$(\varepsilon'\varepsilon^{-1})^{-1}\tau'(g)=\varepsilon\sigma({}^tg^{-1})\varepsilon'^{-1}=g.$$

Since $\tau(\mathfrak{a})=\mathfrak{a}$, we get $\tau'(\mathfrak{a}^g)=\mathfrak{a}^g$. The other two equations can be proved in a similar way. □

5.2. The maximal and totally wildly ramified case. — Now we focus on the proof of Theorem 5.5. We imitate the strategy in [1], section 4, which first considered a special case, and the used tame lifting developed by Bushnell and Henniart [6] and other tools developed by Bushnell and Kutzko [9] to generalize their result. In this subsection, we prove the following proposition as a special case of (2) and (3) of Theorem 5.5:

PROPOSITION 5.9. — *Let* $[\mathfrak{a},\beta]$ *be a simple stratum in* $\mathrm{M}_n(F)$ *and let* $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ *such that* $\theta\in\Theta$ *with* Θ *a* σ*-invariant endo-class. Let* E/F *be totally wildly ramified of degree* n*. Let* $\tau=\tau_1$ *with* $\tau_1(x):=\sigma({}^tx^{-1})$ *for any* $x\in G$*. Then there exist a simple stratum* $[\mathfrak{a}'',\beta'']$ *and a simple character* $\theta''\in\mathcal{C}(\mathfrak{a}'',\beta'')$ *such that* $(\mathfrak{a}'',\theta'')$ *is* G*-conjugate to* $(\mathfrak{a},\theta)$ *with the property* $\tau(\mathfrak{a}'')=\mathfrak{a}''$ *and* $\theta''\circ\tau=\theta''^{-1}$*.*

REMARK 5.10. — In Proposition 5.9 we have $[E:F]=d=n$, which is a power of p as an odd number.

Up to G-conjugacy, we may and will assume $\mathfrak{a}$ to be standard (that is, $\mathfrak{a}$ is made of matrices with upper triangular elements in $\mathfrak{o}_F$ and other elements in $\mathfrak{p}_F$.).

LEMMA 5.11. — *There exist* $g_1\in G$ *and* $a_1,\dots,a_n\in\mathfrak{o}_F^\times$*, such that*

$$\tau(g_1)g_1^{-1}=A:=\begin{pmatrix}0&0&\dots&0&a_1\\0&\ddots&\ddots&a_2&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&a_{n-1}&\ddots&\ddots&0\\a_n&0&\dots&0&0\end{pmatrix}.$$

Moreover, if we define $\mathfrak{a}':=\mathfrak{a}^{g_1}$*, then we have* $\tau(\mathfrak{a}')=\mathfrak{a}'$*.*

Proof. — First we claim that we may choose $a_i\in\mathfrak{o}_F^\times$ such that A is a hermitian matrix and $\det(A)\in\mathrm{N}_{F/F_0}(F^\times)$. To do this, noting that $A^*=A$ if and only if $a_i=\sigma(a_{n+1-i})$ for $i=1,2,\dots,n$, we choose $a_i=\sigma(a_{n+1-i})$ for $i=1,2,\dots,(n-1)/2$ randomly but only to make sure that they are in $\mathfrak{o}_F^\times$ and we choose $a_{(n+1)/2}\in\mathfrak{o}_{F_0}^\times$ to make sure that $\det(A)\in\mathrm{N}_{F/F_0}(F^\times)$.

Since A is a hermitian matrix that is in the same G-orbit as I_n by considering the determinant, using Proposition 2.1, there exists an element $g_1\in G$ such that $(g_1^{-1})^*g_1^{-1}=A$, which means that $\tau(g_1)g_1^{-1}=A$. By definition $\tau(\mathfrak{a}')=\mathfrak{a}'$ if and only if $\tau(g_1^{-1})\tau(\mathfrak{a})\tau(g_1)=g_1^{-1}\mathfrak{a}g_1$. Since $\mathfrak{a}^*={}^t\mathfrak{a}$, we deduce that $\tau(\mathfrak{a}')=\mathfrak{a}'$ if and only if $A^{-1}\,{}^t\mathfrak{a}A=(\tau(g_1)g_1^{-1})^{-1}\,{}^t\mathfrak{a}\tau(g_1)g_1^{-1}=\mathfrak{a}$. From our choice of A and the definition of $\mathfrak{a}$, this can be verified directly. □

Now fix g_1 as in Lemma 5.11. We write $\theta' = \theta^{g_1}$ and $\beta' = \beta^{g_1}$. Since $\mathfrak{a}' = \mathfrak{a}^{g_1}$, we also have:

(1) $U'^i := U^i(\mathfrak{a}') = U^i(\mathfrak{a})^{g_1}$, where $U^i(\mathfrak{a}) := 1 + \mathfrak{p}_{\mathfrak{a}}^i$ for $i \geq 1$.
(2) $J' := J(\mathfrak{a}', \beta') = J(\mathfrak{a}, \beta)^{g_1}$.
(3) $J'^1 := J^1(\mathfrak{a}', \beta') = J^1(\mathfrak{a}, \beta)^{g_1}$.
(4) $\boldsymbol{J}' := \boldsymbol{J}(\mathfrak{a}', \beta') = \boldsymbol{J}(\mathfrak{a}, \beta)^{g_1}$.
(5) $H'^1 := H^1(\mathfrak{a}', \beta') = H^1(\mathfrak{a}, \beta)^{g_1}$.
(6) $M' := M^{g_1}$, where $M = \mathfrak{o}_F^\times \times \ldots \times \mathfrak{o}_F^\times$ is the subgroup of diagonal matrices contained in $\mathfrak{a}$.

Since $\mathfrak{a}'$ is τ-stable and $\Theta^\sigma = \Theta$, using Lemma 5.7, there exists $u' \in U(\mathfrak{a}')$ such that $\theta' \circ \tau = (\theta'^{-1})^{u'}$. Since $\theta' = \theta' \circ \tau \circ \tau = (\theta'^{-1})^{u'} \circ \tau = \theta'^{u'\tau(u')}$, we deduce that $u'\tau(u')$ normalizes θ', which means that $u'\tau(u') \in \boldsymbol{J}' \cap U(\mathfrak{a}') = J'$ by using Proposition 3.1.(4). To prove Proposition 5.9, we only need to find $x' \in G$ such that $\mathfrak{a}'' := \mathfrak{a}'^{x'}$ and $\theta'' := \theta'^{x'}$ have the desired property. By direct calculation, this means that $\tau(x')x'^{-1}$ normalizes $\mathfrak{a}'$ and $u'\tau(x')x'^{-1}$ normalizes θ', so using Proposition 3.1.(4) and the fact that $u'^{-1}\boldsymbol{J}'$ is contained in the normalizer of $\mathfrak{a}'$, it suffices to choose x' such that $u'\tau(x')x'^{-1} \in \boldsymbol{J}'$.

Lemma 5.12. — *There exists $y' \in M'$ such that $u'\tau(y')y'^{-1} \in J(\mathfrak{a}', \beta')U^1(\mathfrak{a}') = \mathfrak{o}_F^\times U^1(\mathfrak{a}')$.*

Proof. — First we write $u' = g_1^{-1} u g_1$ for a certain $u \in U(\mathfrak{a})$. Then $u'\tau(u') \in J(\mathfrak{a}', \beta')$ implies that $uA^{-1}(u^{-1})^* A \in J(\mathfrak{a}, \beta) \subset \mathfrak{o}_F^\times U^1(\mathfrak{a})$ by direct calculation, where A is defined as in Lemma 5.11.

We choose $y' = g_1^{-1} y g_1$ with $y = \mathrm{diag}(y_1, \ldots, y_n) \in M = \mathfrak{o}_F^\times \times \ldots \times \mathfrak{o}_F^\times$ to be determined. By direct calculation, $u'\tau(y')y'^{-1} \in J(\mathfrak{a}', \beta')U^1(\mathfrak{a}')$ if and only if $uA^{-1}(y^{-1})^* A y^{-1} \in J(\mathfrak{a}, \beta)U^1(\mathfrak{a}) = \mathfrak{o}_F^\times U^1(\mathfrak{a})$. We use $\overline{u_i}$, $\overline{a}$, $\overline{y_i}$ and $\overline{b}$ to denote the image of u_i, a, y_i, b in $k_F \cong \mathfrak{o}_F/\mathfrak{p}_F$, respectively, where $u_i, a, b \in \mathfrak{o}_F$ will be defined in the following two paragraphs.

We write $$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & a_1 \\ 0 & \ddots & \ddots & a_2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_{n-1} & \ddots & \ddots & 0 \\ a_n & 0 & \ldots & 0 & 0 \end{pmatrix} \text{ and } u = \begin{pmatrix} u_1 & *_{\mathfrak{o}_F} & \ldots & \ldots & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & u_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1} & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & \ldots & \ldots & *_{\mathfrak{p}_F} & u_n \end{pmatrix},$$
where $*_{\mathfrak{o}_F}$ and $*_{\mathfrak{p}_F}$ represent elements in $\mathfrak{o}_F$ and $\mathfrak{p}_F$, respectively. By direct calculation, we have

$$uA^{-1}(u^{-1})^* A = \begin{pmatrix} u_1\sigma(u_n^{-1}) & *_{\mathfrak{o}_F} & \ldots & \ldots & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & u_2\sigma(u_{n-1}^{-1}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1}\sigma(u_2^{-1}) & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & \ldots & \ldots & *_{\mathfrak{p}_F} & u_n\sigma(u_1^{-1}) \end{pmatrix} \in \mathfrak{o}_F^\times U^1(\mathfrak{a}),$$

which means that there exists $a \in \mathfrak{o}_F^\times$ such that

$$u_1\sigma(u_n^{-1}), u_2\sigma(u_{n-1}^{-1}), \ldots, u_n\sigma(u_1^{-1}) \in a(1+\mathfrak{p}_F). \tag{5}$$

Also by direct calculation, we have

$$uA^{-1}(y^{-1})^*Ay^{-1} = \begin{pmatrix} u_1y_1^{-1}\sigma(y_n^{-1}) & *_{\mathfrak{o}_F} & \cdots & \cdots & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & u_2y_2^{-1}\sigma(y_{n-1}^{-1}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1}y_{n-1}^{-1}\sigma(y_2^{-1}) & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & \cdots & \cdots & *_{\mathfrak{p}_F} & u_ny_n^{-1}\sigma(y_1^{-1}) \end{pmatrix},$$

which means that the lemma is true if and only if there exists $b \in \mathfrak{o}_F^\times$ such that

$$u_1y_1^{-1}\sigma(y_n^{-1}), u_2y_2^{-1}\sigma(y_{n-1}^{-1}), \ldots, u_ny_n^{-1}\sigma(y_1^{-1}) \in b(1+\mathfrak{p}_F). \tag{6}$$

If we consider modulo $\mathfrak{p}_F$, then the condition (5) becomes

$$\overline{u_1}\sigma(\overline{u_n}^{-1}) = \overline{u_2}\sigma(\overline{u_{n-1}}^{-1}) = \ldots = \overline{u_n}\sigma(\overline{u_1}^{-1}) = \overline{a}. \tag{7}$$

Moreover, if we consider modulo $U^1(\mathfrak{a})$, then $uA^{-1}(y^{-1})^*Ay^{-1} \in \mathfrak{o}_F^\times U^1(\mathfrak{a})$ if and only if there exist $y_i \in \mathfrak{o}_F^\times$ such that there exists $b \in \mathfrak{o}_F^\times$ in the condition (6) such that

$$\overline{u_1y_1}^{-1}\sigma(\overline{y_n}^{-1}) = \overline{u_2y_2}^{-1}\sigma(\overline{y_{n-1}}^{-1}) = \ldots = \overline{u_ny_n}^{-1}\sigma(\overline{y_1}^{-1}) = \overline{b}. \tag{8}$$

We choose $b = u_{(n+1)/2}$, and then $\overline{b}\sigma(\overline{b}^{-1}) = \overline{a}$. Furthermore, we choose $y_i = b^{-1}u_i$ when $i = 1, 2, \ldots, (n-1)/2$ and $y_i = 1$ when $i = (n+1)/2, \ldots, n$. Combining this with the equation (7), the equation (8) is satisfied. □

Let us write $z'u'\tau(y')y'^{-1} \in U'^1$ for some $y' \in M'$ and $z' \in \mathfrak{o}_F^\times$ given by Lemma 5.12. By replacing the simple stratum $[\mathfrak{a}', \beta']$ with $[\mathfrak{a}'^{y'}, \beta'^{y'}]$, the simple character θ' with $\theta'^{y'}$ and u' with $y'^{-1}z'u'\tau(y')$, which does not affect the fact that the order is τ-stable, we can and will assume that $u' \in U'^1$. We write $J'^i = J' \cap U'^i$ for $i \geq 1$. We state the following two lemmas, which correspond to Lemma 4.16 and Lemma 4.17 in [1]. Actually, the same proofs work when one replaces the Galois involution σ in the original lemmas with any involution τ on G.

Lemma 5.13. — *Let $v' \in U'^i$ for some $i \geq 1$ and assume that $v'\tau(v') \in J'^i$. Then there exist $j' \in J'^i$ and $x' \in U'^i$ such that $j'v'\tau(x')x'^{-1} \in U'^{i+1}$.*

Using Lemma 5.13 to replace Lemma 4.16 in [1], we may prove the following lemma:

LEMMA 5.14. — *There exists a sequence of $(x'_i, j'_i, v'_i) \in U'^i \times J'^i \times U'^{i+1}$ for $i \geq 0$, satisfying the following conditions:*

(1) $(x'_0, j'_0, v'_0) = (1, 1, u')$.
(2) *For all $i \geq 0$, if we set $y'_i = x'_0 x'_1 \ldots x'_i \in U'^1$, then the simple character $\theta'_i = \theta'^{y'_i} \in \mathcal{C}(\mathfrak{a}', \beta'^{y'_i})$ satisfies $\theta'_i \circ \tau = (\theta'^{-1}_i)^{v'_i}$.*
(3) *For all $i \geq 1$, we have $y'_i v'_i = j'_i y'_{i-1} v'_{i-1} \tau(x'_i)$.*

Let $x' \in U'^1$ be the limit of $y'_i = x'_0 x'_1 \ldots x'_i$ and let $h' \in J'^1$ be that of $j'_i \ldots j'_1 j'_0$ when i tends to infinity. By Lemma 5.14.(3), we have

$$y'_i v'_i \tau(y'^{-1}_i) = j'_i y'_{i-1} v'_{i-1} \tau(y'^{-1}_{i-1}) = \ldots = j'_i \ldots j'_1 j'_0 u'.$$

Passing to the limit, we get $x'\tau(x')^{-1} = h'u'$, which implies that $u'\tau(x')x'^{-1} = h'^{-1} \in J'$. Let $(\mathfrak{a}'', \theta'') = (\mathfrak{a}'^{x'}, \theta'^{x'})$, which finishes the proof of Proposition 5.9.

5.3. The maximal case. — In this subsection, we generalize Proposition 5.9 to the following situation:

PROPOSITION 5.15. — *Let $[\mathfrak{a}, \beta]$ be a simple stratum in $\mathrm{M}_n(F)$ such that $[E : F] = n$, let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ such that $\theta \in \Theta$ with Θ a σ-invariant endo-class and let τ be a given unitary involution. Then there exist a simple stratum $[\mathfrak{a}'', \beta'']$ and a simple character $\theta'' \in \mathcal{C}(\mathfrak{a}'', \beta'')$ such that $(\mathfrak{a}'', \theta'')$ is G-conjugate to $(\mathfrak{a}, \theta)$ with the property $\tau(\mathfrak{a}'') = \mathfrak{a}''$ and $\theta'' \circ \tau = \theta''^{-1}$.*

To prove the proposition, we first study an endo-class Θ over F being σ-invariant, that is, $\Theta^\sigma = \Theta$. Let T be a tame parameter field of Θ.

LEMMA 5.16. — *Let Θ be a σ-invariant endo-class and let T/F be its tame parameter field. Then given a T/F-lift Ψ of Θ, there is a unique involution α of T extending σ such that $\Psi^\alpha = \Psi$.*

Proof. — The proof of Lemma 4.8 in [1] can be used almost unchanged for our lemma. We only need to consider Θ instead of $\Theta^\vee$ and Ψ instead of $\Psi^\vee$. □

Let α be the involution of T given by Lemma 5.16 and let T_0 be the sub-field of T fixed by α. Then $T_0 \cap F = F_0$. We write $t = [T : F] = [T_0 : F_0]$. We need the following proposition due to Hakim and Murnaghan:

PROPOSITION 5.17 ([21], Proposition 2.1). — *There exists an embedding $\iota : T \hookrightarrow \mathrm{M}_t(F)$ of F-algebras such that, for $x \in T$, we have $\iota(\alpha(x)) = \iota(x)^* := \sigma({}^t\iota(x))$.*

Proof of Proposition 5.15. — Let $E = F[\beta]$ and let T be the maximal tamely ramified extension of F in E. It is a tame parameter field of the endo-class Θ. The simple character θ gives Ψ, the endo-class of the interior T/F-lift of Θ, as we introduced in §3.3. Let α be defined as in Lemma 5.16 and let ι be defined as in Proposition 5.17. By abuse of notation, we define

$$\iota : \mathrm{M}_{n/t}(T) \hookrightarrow \mathrm{M}_{n/t}(\mathrm{M}_t(F)) = \mathrm{M}_n(F)$$

with each block defined by the original ι. First we consider $\tau(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1}$, for any $x \in G$ with $\varepsilon = I_n$ or $\mathrm{diag}(\iota(\epsilon), \ldots, \iota(\epsilon), \iota(\epsilon))$, where $\epsilon \in T_0^\times - \mathrm{N}_{T/T_0}(T^\times)$. The determinant of the latter matrix is $\mathrm{N}_{T_0/F_0}(\epsilon)^{n/t}$. Since

$$\mathrm{N}_{T_0/F_0} : T_0^\times \to F_0^\times$$

is a homomorphism that maps $\mathrm{N}_{T/T_0}(T^\times)$ to $\mathrm{N}_{F/F_0}(F^\times)$, it leads to a group homomorphism

$$\mathrm{N}_{T_0/F_0} : T_0^\times/\mathrm{N}_{T/T_0}(T^\times) \to F_0^\times/\mathrm{N}_{F/F_0}(F^\times)$$

between two groups of order 2. We state and prove the following lemma in general:

Lemma 5.18. — *Let F, F_0 be defined as before. Let L_0/F_0 be a finite extension such that $L = L_0F$ is a field with $[L : L_0] = 2$ and $F_0 = L_0 \cap F$. Then the group homomorphism*

$$\mathrm{N}_{L_0/F_0} : L_0^\times \to F_0^\times$$

induces an isomorphism

$$\mathrm{N}_{L_0/F_0} : L_0^\times/\mathrm{N}_{L/L_0}(L^\times) \to F_0^\times/\mathrm{N}_{F/F_0}(F^\times)$$

of groups of order 2.

Proof. — We first consider the case where L_0/F_0 is abelian. If, on the contrary, the induced homomorphism is not an isomorphism, then $\mathrm{N}_{L_0/F_0}(L_0^\times) \subset \mathrm{N}_{F/F_0}(F^\times)$, which means that F is contained in L_0 by the local class field theory ([40], Chapter 14, Theorem 1), which is absurd.

When L_0/F_0 is Galois, we may write $F_0 = L_0^0 \subsetneq \ldots \subsetneq L_0^r = L_0$, such that L_0^{i+1}/L_0^i is abelian for $i = 0, \ldots, r-1$ ([40], Chapter 4, Proposition 7). We write $L^i = L_0^i F$. Thus it is easy to show that L^i/L_0^i is quadratic, $L_0^i = L_0^{i+1} \cap L^i$ and $L_0^{i+1}L^i = L^{i+1}$ for $i = 0, \ldots, r-1$. Using the abelian case,

$$\mathrm{N}_{L_0^{i+1}/L_0^i} : L_0^{i+1\times}/\mathrm{N}_{L^{i+1}/L_0^{i+1}}(L^{i+1\times}) \to L_0^{i\times}/\mathrm{N}_{L^i/L_0^i}(L^{i\times})$$

is an isomorphism for $i = 0, 1, \ldots, r-1$. Composing them together, we finish the proof.

When L_0/F_0 is separable, we write L_0' as the normal closure of L_0 over F_0. Thus, L_0' contains L_0, and L_0'/F_0 is a finite Galois extension. We write $L' = L_0'F$. Using the Galois case,

$$\mathrm{N}_{L_0'/F_0} : L_0'^{\times}/\mathrm{N}_{L'/L_0'}(L'^{\times}) \to F_0^{\times}/\mathrm{N}_{F/F_0}(F^{\times})$$

is an isomorphism. Since $\mathrm{N}_{L_0'/F_0}(L_0'^{\times}) \subset \mathrm{N}_{L_0/F_0}(L_0^{\times})$,

$$\mathrm{N}_{L_0/F_0} : L_0^{\times}/\mathrm{N}_{L/L_0}(L^{\times}) \to F_0^{\times}/\mathrm{N}_{F/F_0}(F^{\times})$$

is also an isomorphism.

In the characteristic p case in general, we write L_0^{sep} the maximal separable sub-extension of F_0 contained in L_0, and thus L_0/L_0^{sep} is purely inseparable. Thus $\mathrm{N}_{L_0/L_0^{sep}}(x) = x^{p^{[L_0:L_0^{sep}]}}$, for any $x \in L_0^{\times}$. Since $p \neq 2$ and $L_0^{\times}/\mathrm{N}_{L/L_0}(L^{\times})$ is of order 2,

$$\mathrm{N}_{L_0/L_0^{sep}} : L_0^{\times}/\mathrm{N}_{L/L_0}(L^{\times}) \to L_0^{sep\times}/\mathrm{N}_{L^{sep}/L_0^{sep}}(L^{sep\times})$$

is an isomorphism, where $L^{sep} := LL_0^{sep}$. So we come back to the separable case, which finishes the proof. $\square$

Using Lemma 5.18, for $L_0 = T_0$, the homomorphism above is actually an isomorphism. Since n/t is odd, and $\epsilon \in T_0^{\times} - \mathrm{N}_{T/T_0}(T^{\times})$, we have $\det(\varepsilon) = \mathrm{N}_{T_0/F_0}(\epsilon)^{n/t} \in F_0^{\times} - \mathrm{N}_{F/F_0}(F^{\times})$. So, indeed, these two involutions represent both of the G-classes of hermitian matrices. Thus, using Lemma 5.8, we may from now on assume τ to be the two unitary involutions we mentioned above. Furthermore, $\iota(T)^{\times}$ is normalized by τ from the exact construction of τ and Proposition 5.17, where we regard T as an F-subalgebra of $\mathrm{M}_{n/t}(T)$ given by the diagonal embedding.

Since T and $\iota(T)$ are isomorphic as F-subalgebras contained in $\mathrm{M}_n(F)$, by the Skolem–Noether theorem, there exists $g \in G$ such that $\iota(T) = T^g$. Thus, if we write $[\mathfrak{a}', \beta'] = [\mathfrak{a}^g, \beta^g]$, $\theta' = \theta^g$ and $E' = F[\beta']$, then $\theta' \in \Theta$ such that its tame parameter field equals $\iota(T)$. Since τ normalizes $\iota(T)^{\times}$, we deduce that $\theta' \circ \tau$ and θ'^{-1} have the same parameter field $\iota(T)$. If we write Ψ' the endo-class of the interior $\iota(T)/F$-lift corresponding to θ', and if we choose $\alpha' = \iota|_T \circ \alpha \circ \iota|_{\iota(T)}^{-1}$, then we have $\Psi'^{\alpha'} = \Psi'$.

Let $C' = \mathrm{M}_{n/t}(\iota(T))$ denote the centralizer of $\iota(T)$ in $\mathrm{M}_n(F)$. For $c \in \mathrm{M}_{n/t}(T)$, we have

$$\begin{aligned}\tau(\iota(c)) &= \varepsilon\sigma({}^t\iota(c)^{-1})\varepsilon^{-1} = \varepsilon({}^{t_{C'}}\iota(\alpha(c))^{-1})\varepsilon^{-1} \\ &= \varepsilon(\alpha'({}^{t_{C'}}\iota(c))^{-1})\varepsilon^{-1} = \tau'(\iota(c)),\end{aligned}$$

where we denote by $t_{C'}$ the transpose on $C' = \mathrm{M}_{n/t}(\iota(T))$ and $\tau'(c') = \varepsilon(\alpha'({}^{t_{C'}}c'^{-1}))\varepsilon^{-1}$, for any $c' \in C'^{\times}$. Thus, τ', the restriction of τ to $C'^{\times}$, is the unitary involution τ_1 on $C'^{\times} = \mathrm{GL}_{n/t}(\iota(T))$ with respect to the Galois involution $\alpha' \in \mathrm{Gal}(\iota(T)/F)$. The intersection $\mathfrak{c}' = \mathfrak{a}' \cap C'$ gives rise to a simple stratum $[\mathfrak{c}', \beta']$. The restriction of θ' to $H^1(\mathfrak{c}', \beta')$, denoted by $\theta'_{\iota(T)}$, is a simple character associated to this simple stratum with endo-class Ψ'. Since $E'/\iota(T)$ is totally wildly ramified, using Proposition 5.9 with G, θ, Θ, σ and τ replaced by $C'^{\times}$, $\theta'_{\iota(T)}$, Ψ', α' and τ', respectively, there exists $c' \in C'^{\times}$, such that $\tau'(\mathfrak{c}'^{c'}) = \mathfrak{c}'^{c'}$ and $\theta'^{c'}_{\iota(T)} \circ \tau' = (\theta'^{c'}_{\iota(T)})^{-1}$.

By the injectivity of $\mathfrak{a} \mapsto \mathfrak{a} \cap C'$ between sets of hereditary orders as mentioned in §3.3, $\mathfrak{a}'' := \mathfrak{a}'^{c'}$ is τ-stable. Moreover, if we write $\theta'' = \theta'^{c'}$, then from our construction of τ and the definition of $\iota(T)/F$-lift, the simple characters

$$(\theta'' \circ \tau)_{\iota(T)} = \theta'' \circ \tau|_{H^1(\tau(\mathfrak{c}'),\tau(\beta'))} = \theta'' \circ \tau'|_{H^1(\tau(\mathfrak{c}'),\tau(\beta'))} = \theta''_{\iota(T)} \circ \tau'$$

and

$$(\theta''^{-1})_{\iota(T)} = \theta''^{-1}_{\iota(T)}$$

are equal. By the last paragraph of §3.3, the simple character θ'' satisfies the property $\theta'' \circ \tau = \theta''^{-1}$. □

5.4. The general case. — In this subsection, we finish the proof of Lemma 5.4 and Theorem 5.5. First of all, we recall the following result of Stevens:

PROPOSITION 5.19 ([42], Theorem 6.3). — *Let $[\mathfrak{a}, \beta]$ be a simple stratum in $\mathrm{M}_n(F)$ with $\sigma_t(\mathfrak{a}) = \mathfrak{a}$. Suppose that there exists a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, such that $H^1(\mathfrak{a}, \beta)$ is σ_t-stable and $\theta \circ \sigma_t = \theta$. Then there exists a simple stratum $[\mathfrak{a}, \gamma]$, such that $\theta \in \mathcal{C}(\mathfrak{a}, \gamma)$ and $\sigma_t(\gamma) = \gamma$.*

Proof. — The original proof of [42], Theorem 6.3 can be modified as follows. For any $x \in \mathrm{M}_n(F)$, we use $-\sigma_t(x)$ to replace $\overline{x}$; we use σ_t to replace σ; for $[\mathfrak{a}, \beta]$ a simple stratum, we say that it is σ_t-*invariant* if $\sigma_t(\mathfrak{a}) = \mathfrak{a}$, and $\sigma_t(\beta) = \beta$, and we use this concept to replace the concept *skew simple stratum* in the original proof. With these replacements, the original proof can be used in our case without difficulty (see also the last paragraph of *ibid.*). □

We choose $[\mathfrak{a}_0, \beta_0]$ to be a maximal simple stratum in $\mathrm{M}_d(F)$ and $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \beta_0)$ such that $\theta_0 \in \Theta$. By Proposition 5.15, there are a maximal simple stratum $[\mathfrak{a}'_0, \beta'_0]$ and a simple character $\theta'_0 \in \mathcal{C}(\mathfrak{a}'_0, \beta'_0)$, which is $\mathrm{GL}_d(F)$-conjugate to θ_0, such that:

(1) The order $\mathfrak{a}'_0$ is τ_1-stable.

(2) The group $H^1(\mathfrak{a}'_0, \beta'_0)$ is τ_1-stable, and $\theta'_0 \circ \tau_1 = \theta'^{-1}_0$.

Furthermore, using Proposition 5.19 we may assume that:

(3) $\sigma_t(\beta'_0) = \beta'_0$.

We embed $\mathrm{M}_d(F)$ diagonally into the F-algebra $\mathrm{M}_n(F)$. This gives an F-algebra homomorphism $\iota' : F[\beta'_0] \hookrightarrow \mathrm{M}_n(F)$. Write $\beta' = \iota'(\beta'_0) = \beta'_0 \otimes \ldots \otimes \beta'_0$ and $E' = F[\beta']$. The centralizer B' of E' in $\mathrm{M}_n(F)$ is naturally identified with $\mathrm{M}_m(E')$. We regard σ_t as an involution on E' extending σ, and we write $E'_0 = E'^{\sigma_t}$. Let $\mathfrak{b}'$ be a maximal standard hereditary order in B', which may be identified with $\mathrm{M}_m(\mathfrak{o}_{E'})$, and let $\mathfrak{a}' = \mathrm{M}_m(\mathfrak{a}'_0)$ be the unique hereditary order in $\mathrm{M}_n(F)$ normalized by $E'^{\times}$, such that $\mathfrak{a}' \cap B' = \mathfrak{b}'$. Then the simple stratum $[\mathfrak{a}', \beta']$ satisfies the requirement of Lemma 5.4, finishing its proof.

Now we focus on the proof of Theorem 5.5. By Lemma 5.8, we may change τ up to G-action on its corresponding hermitian matrix, which does not change the content of the theorem. So if ε is in the same G-class as I_n, we may simply choose $\tau = \tau_1$, where $\tau_1(x) = \sigma({}^t x^{-1})$, for any $x \in G$. If not, we fix an $\epsilon \in E_0'^{\times} - \mathrm{N}_{E'/E'_0}(E'^{\times})$. Regarding ϵ as an element in $\mathrm{M}_d(F)$, we have $\det(\epsilon) = \mathrm{N}_{E'_0/F_0}(\epsilon)$. Since

$$\mathrm{N}_{E'_0/F_0} : E_0'^{\times} \to F_0^{\times}$$

is a homomorphism that maps $\mathrm{N}_{E'/E'_0}(E'^{\times})$ to $\mathrm{N}_{F/F_0}(F^{\times})$, by Lemma 5.18 with $L_0 = E'_0$, it leads to an isomorphism

$$\mathrm{N}_{E'_0/F_0} : E_0'^{\times}/\mathrm{N}_{E'/E'_0}(E'^{\times}) \to F_0^{\times}/\mathrm{N}_{F/F_0}(F^{\times})$$

of the two groups of order 2. Thus, $\mathrm{N}_{E'_0/F_0}(\epsilon) \in F_0^{\times} - \mathrm{N}_{F/F_0}(F^{\times})$. If E'/E'_0 is unramified, we write $\varepsilon = \mathrm{diag}(\epsilon, \dots, \epsilon)$. Then $\det(\varepsilon) = \mathrm{N}_{E'_0/F_0}(\epsilon)^m \in F_0^{\times} - \mathrm{N}_{F/F_0}(F^{\times})$, since $F_0^{\times}/\mathrm{N}_{F/F_0}(F^{\times})$ is a group of order 2, and m is odd from the condition of the theorem. If E'/E'_0 is ramified, we may assume further that $\epsilon \in \mathfrak{o}_{E'_0}^{\times}$. We write $\varepsilon = \mathrm{diag}(I_d, \dots, I_d, \epsilon)$ and we have $\det(\varepsilon) = \mathrm{N}_{E'_0/F_0}(\epsilon) \in F_0^{\times} - \mathrm{N}_{F/F_0}(F^{\times})$. For both cases, τ_ε is a unitary involution whose corresponding hermitian matrix is not in the same G-class as I_n. So from now on, we only consider the three unitary involutions above. From our assumption of τ, the restriction of τ to $\mathrm{GL}_m(E')$ is also a unitary involution $\tau' = \tau_1$ or τ_ε with $\varepsilon = \mathrm{diag}(1, \dots, 1, \epsilon)$. In particular, since ϵ is an element in E', we know that ε commutes with elements in E' and we have $\tau(\beta') = \beta'^{-1}$.

Since $\mathfrak{a}'_0$ is τ_1-stable and $\mathfrak{b}'$ is τ'-stable, from our assumption of τ we deduce that $\mathfrak{a}'$ is τ-stable, or by definition $\varepsilon\sigma_t(\mathfrak{a}')\varepsilon^{-1} = \mathfrak{a}'$. Since $\sigma_t(\beta') = \beta'$, by direct calculation we have

$$\begin{aligned}\tau(H^1(\mathfrak{a}', \beta')) &= \varepsilon H^1(\sigma_t(\mathfrak{a}'), \sigma_t(\beta'))^{-1}\varepsilon^{-1} \\ &= H^1(\sigma_t(\mathfrak{a}')^{\varepsilon^{-1}}, \beta'^{\varepsilon^{-1}}) = H^1(\mathfrak{a}', \beta'^{\varepsilon^{-1}}) = H^1(\mathfrak{a}', \beta').\end{aligned}$$

Let M be the standard Levi subgroup of G isomorphic to $\mathrm{GL}_d(F) \times \dots \times \mathrm{GL}_d(F)$, let P be the standard parabolic subgroup of G generated by M and upper triangular matrices, and let N be its unipotent radical. Let N^- be the unipotent radical of the parabolic subgroup opposite to P with respect to M.

By [36], Théorème 2.17, we have

$$H^1(\mathfrak{a}',\beta') = (H^1(\mathfrak{a}',\beta')\cap N^-)\cdot(H^1(\mathfrak{a}',\beta')\cap M)\cdot(H^1(\mathfrak{a}',\beta')\cap N), \tag{9}$$

$$H^1(\mathfrak{a}',\beta')\cap M = H^1(\mathfrak{a}'_0,\beta'_0)\times\ldots\times H^1(\mathfrak{a}'_0,\beta'_0). \tag{10}$$

Let $\theta'\in\mathcal{C}(\mathfrak{a}',\beta')$ be the transfer of θ'_0. By *loc. cit.*, the character θ' is trivial on $H^1(\mathfrak{a}',\beta')\cap N^-$ and $H^1(\mathfrak{a}',\beta')\cap N$, and the restriction of θ' to $H^1(\mathfrak{a}',\beta')\cap M$ equals $\theta'_0\otimes\ldots\otimes\theta'_0$. We have

$$\begin{aligned}\theta'\circ\tau|_{H^1(\mathfrak{a}',\beta')\cap N^-} &= \theta'\circ\tau|_{H^1(\mathfrak{a}',\beta')\cap N}\\ &= \theta'^{-1}|_{H^1(\mathfrak{a}',\beta')\cap N^-} = \theta'^{-1}|_{H^1(\mathfrak{a}',\beta')\cap N} = 1\end{aligned}$$

and

$$\begin{aligned}\theta'\circ\tau|_{H^1(\mathfrak{a}',\beta')\cap M} &= \theta'_0\circ\tau_1\otimes\ldots\otimes\theta'_0\circ\tau_1\\ &= \theta_0'^{-1}\otimes\ldots\otimes\theta_0'^{-1} = \theta'^{-1}|_{H^1(\mathfrak{a}',\beta')\cap M}\end{aligned}$$

for $\tau=\tau_1$ or τ_ε with $\varepsilon=\mathrm{diag}(\epsilon,\ldots,\epsilon)$ or $\mathrm{diag}(1,\ldots,1,\epsilon)$, since $\epsilon\in F[\beta'_0]^\times$ normalizes θ'_0. Thus by equation (9), we have $\theta'\circ\tau=\theta'^{-1}$.

Remark 5.20. — From the proof of Theorem 5.5, we observe that if τ is chosen as one of the three unitary involutions mentioned in the proof, then we may choose the same simple stratum and simple character satisfying the conclusion of the theorem.

Remark 5.21. — We give a counter-example to show that the condition in Theorem 5.5 is necessary. Let $n=2$, let F/F_0 be unramified, let Θ be trivial and let $\varepsilon=\mathrm{diag}(1,\varpi_{F_0})$. Then $d=1$, $m=n=2$, $E=F$ and $E_0=F_0$. If the theorem is true, then $\mathfrak{a}=\mathrm{M}_2(\mathfrak{o}_F)^g$ for some $g\in\mathrm{GL}_2(F)$ and $\tau(\mathfrak{a})=\mathfrak{a}$. By direct calculation $\sigma({}^tg^{-1})\varepsilon^{-1}g^{-1}$ normalizes $\mathrm{M}_2(\mathfrak{o}_F)$, which means that $\sigma({}^tg^{-1})\varepsilon^{-1}g^{-1}\in F^\times\mathrm{GL}_2(\mathfrak{o}_F)$. It is impossible since $\det(\sigma({}^tg^{-1})\varepsilon^{-1}g^{-1})\in\varpi_{F_0}\mathrm{N}_{F/F_0}(F^\times)$, while $\det(F^\times\mathrm{GL}_2(\mathfrak{o}_F))\subset\mathrm{N}_{F/F_0}(F^\times)$.

6. The distinguished type theorem

Let π be a cuspidal representation of G such that $\pi^\sigma\cong\pi$. From the statements and proofs of Theorem 5.2, 5.3 and 5.5, we may assume the following conditions:

Remark 6.1. —
(1) For $\tau=\tau_1$, there exist a simple stratum $[\mathfrak{a},\beta]$ and a simple character $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ contained in π, such that $\tau(\mathfrak{a})=\mathfrak{a}$, $\tau(H^1(\mathfrak{a},\beta))=H^1(\mathfrak{a},\beta)$, $\theta\circ\tau=\theta^{-1}$ and $\tau(\beta)=\beta^{-1}$, where $\tau_1(x):=\sigma({}^tx^{-1})$ for any $x\in G$.
(2) For $\tau=\tau_1$, there exists a simple type $(\boldsymbol{J},\Lambda)$ containing θ and contained in π, such that $\tau(\boldsymbol{J})=\boldsymbol{J}$ and $\Lambda^\tau\cong\Lambda^\vee$.

(3) σ_t is an involution on $E = F[\beta]$, whose restriction to F equals σ. So by abuse of notation, we identify σ with σ_t. Let $E_0 = E^\sigma$. **We assume further in this section that if E/E_0 is unramified, then m is odd**[5]**.**

(4) Write $\tau(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1}$ for any $x \in G$ such that: when E/E_0 is unramified, we assume $\varepsilon = I_n$ or $\mathrm{diag}(\varpi_E, \ldots, \varpi_E) \in \mathrm{GL}_m(E) \hookrightarrow G$; when E/E_0 is ramified, we assume $\varepsilon = I_n$ or $\mathrm{diag}(1, \ldots, 1, \epsilon) \in \mathrm{GL}_m(E) \hookrightarrow G$ with $\epsilon \in \mathfrak{o}_{E_0}^\times - \mathrm{N}_{E/E_0}(\mathfrak{o}_E^\times)$. By Remark 5.20, we assume further that for these three unitary involutions, conditions (1) and (2) are also satisfied. **From now on until the end of this section, we assume ε to be one of these three hermitian matrices and τ to be one of these three corresponding involutions.**

(5) the element β has the block diagonal form:

$$\beta = \mathrm{diag}(\beta_0, \ldots, \beta_0) \in \mathrm{M}_m(\mathrm{M}_d(F)) = \mathrm{M}_n(F),$$

for some $\beta_0 \in \mathrm{M}_d(F)$, where d is the degree of β over F and $n = md$. The centralizer B of E in $\mathrm{M}_n(F)$ is identified with $\mathrm{M}_m(E)$. If we regard τ as the restriction of the original involution to $B^\times$, then it is a unitary involution with respect to $B^\times = \mathrm{GL}_m(E)$, E/E_0 and $\sigma \in \mathrm{Gal}(E/E_0)$.

(6) The order $\mathfrak{b} = \mathfrak{a} \cap B$ is the standard maximal order $\mathrm{M}_m(\mathfrak{o}_E)$ of $\mathrm{M}_m(E)$. Thus, if we write $\mathfrak{a}_0$ as the hereditary order of $\mathrm{M}_d(F)$ normalized by E, then $\mathfrak{a}$ is identified with $\mathrm{M}_m(\mathfrak{a}_0)$.

(7) ϖ_E is a uniformizer of E such that:

$$\sigma(\varpi_E) = \begin{cases} \varpi_E & \text{if } E \text{ is unramified over } E_0; \\ -\varpi_E & \text{if } E \text{ is ramified over } E_0. \end{cases}$$

Now we state the main theorem of this section:

Theorem 6.2 (distinguished type theorem). — *For π a σ-invariant cuspidal representation, it is G^τ-distinguished if and only if it contains a τ-self-dual simple type $(\boldsymbol{J}, \Lambda)$, such that* $\mathrm{Hom}_{\boldsymbol{J} \cap G^\tau}(\Lambda, 1) \neq 0$.

Remark 6.3. — Since every hermitian matrix is equivalent to one of the hermitian matrices mentioned in Remark 6.1.(4) up to G-action, and the property of distinction is invariant up to equivalence of unitary group, the theorem works for every unitary involution, although we only consider those occurring in *loc. cit.*

5. Although this condition seems a little bit annoying, finally in Section 7, we find out that this condition is automatically satisfied for π a σ-invariant supercuspidal representation.

Choose $(\boldsymbol{J}, \Lambda)$ as in Remark 6.1, using the Mackey formula and Frobenius reciprocity, we have

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_g \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1),$$

where g ranges over a set of representatives of $(\boldsymbol{J}, G^\tau)$-double cosets in G. So π is G^τ-distinguished if and only if there exists g as a representative of a $(\boldsymbol{J}, G^\tau)$-double coset, such that $\mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$. We will study such g and will show that $(\boldsymbol{J}^g, \Lambda^g)$ is actually τ-self-dual. So $(\boldsymbol{J}^g, \Lambda^g)$ is a distinguished and τ-self-dual simple type that we are looking for, finishing the proof of the theorem.

6.1. Double cosets contributing to the distinction of $\boldsymbol{\theta}$. —

PROPOSITION 6.4. — *For $g \in G$, the character θ^g is trivial on $H^{1g} \cap G^\tau$ if and only if $\tau(g)g^{-1} \in JB^\times J$.*

Proof. — We only need to use the same proof of [35], Proposition 6.6, with σ replaced by τ. □

As a result, since $\mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ implies that $\mathrm{Hom}_{H^{1g} \cap G^\tau}(\theta^g, 1) \neq 0$, using Proposition 6.4 we have $\gamma := \tau(g)g^{-1} \in JB^\times J$.

6.2. The double coset lemma. — The next step is to prove the following double coset lemma:

LEMMA 6.5. — *Let $g \in G$. Then $\gamma = \tau(g)g^{-1} \in JB^\times J$ if and only if $g \in JB^\times G^\tau$.*

Proof. — If $g \in JB^\times G^\tau$, one verifies immediately that $\gamma \in JB^\times J$. Conversely, supposing that $\gamma \in JB^\times J$, first we need the following lemma:

LEMMA 6.6. — *There exists an element $b \in B^\times$ such that $\gamma \in JbJ$ and $b\tau(b) = 1$.*

Proof. — Since $B^\times \cap J = \mathfrak{b}^\times$ is a maximal compact subgroup of $B^\times$, using the Cartan decomposition over $B^\times \cong \mathrm{GL}_m(E)$, we write $\gamma = xcy$ with $x, y \in J$ and $c = \mathrm{diag}(\varpi_E^{a_1} I_{m_1}, \ldots, \varpi_E^{a_r} I_{m_r})$, where $a_1 > \ldots > a_r$ are integers, and $m_1 + \ldots + m_r = m$.

If E/E_0 is unramified, then by definition $c^* = c$. So if we choose $b = c\varepsilon^{-1}$, then $b\varepsilon(b^*)^{-1}\varepsilon^{-1} = c(c^*)^{-1} = 1$, that is, $b\tau(b) = 1$.

If E/E_0 is ramified, since $\tau(\gamma)\gamma = 1$, we know that $xcy = \varepsilon y^* c^* x^* \varepsilon^{-1}$, which is equivalent to $(y^*)^{-1}\varepsilon^{-1}xc = c^* x^* \varepsilon^{-1} y^{-1}$. Let $z = x^*\varepsilon^{-1}y^{-1} \in J$; then we have $z^* c = c^* z$. We regard z and c as matrices in $\mathrm{M}_m(\mathrm{M}_d(F))$. Denote by $z^{(j)} \in \mathrm{M}_{m_j}(\mathrm{M}_d(F))$ the block matrix in z, which is in the same place as $\varpi_E^{a_j} I_{m_j}$ in c. Since $z^* c = c^* z$, by direct calculation

$$(z^{(j)})^* \varpi_E^{a_j} = (-1)^{a_j} \varpi_E^{a_j} z^{(j)} \quad \text{for } j = 1, \ldots, r. \tag{11}$$

By considering the following embedding

$$\begin{aligned}\mathrm{M}_{m_j}(\mathrm{M}_d(F)) &\hookrightarrow \mathrm{M}_m(\mathrm{M}_d(F))\\ h &\mapsto \mathrm{diag}(0_{m_1 d},\dots,0_{m_{j-1}d},h,0_{m_{j+1}d},\dots,0_{m_r d}),\end{aligned}$$

we regard $\mathrm{M}_{m_j d}(F)$ as a subalgebra of $\mathrm{M}_{md}(F)$ denoted by $A^{(j)}$, where $0_{m_j d}$ represents the zero matrix of size $m_j d \times m_j d$. We write $\mathfrak{a}^{(j)} = \mathfrak{a} \cap A^{(j)}$. By abuse of notation, we identify the element $\beta_0 \otimes \ldots \otimes \beta_0$, which consists of m_j copies of β_0 and is contained in $\mathrm{M}_{m_j}(\mathrm{M}_d(F))$, with β. By [36], Théorème 2.17, since $z \in J(\mathfrak{a},\beta)$, we get $z^{(j)} \in J(\mathfrak{a}^{(j)},\beta)$ for $j = 1,\dots,r$. By *loc. cit.*, if we denote by

$$M = \mathrm{GL}_{m_1 d}(F) \times \ldots \times \mathrm{GL}_{m_r d}(F)$$

the Levi subgroup of G corresponding to the partition $n = m_1 d + \ldots + m_r d$, and then

$$M \cap J = J(\mathfrak{a}^{(1)},\beta) \times \ldots \times J(\mathfrak{a}^{(r)},\beta)$$

and

$$M \cap J^1 = J^1(\mathfrak{a}^{(1)},\beta) \times \ldots \times J^1(\mathfrak{a}^{(r)},\beta).$$

Thus we get $\mathrm{diag}(z^{(1)},\dots,z^{(r)}) \in M \cap J$. Furthermore, we have

$$\begin{aligned}M \cap J/M \cap J^1 &\cong J(\mathfrak{a}^{(1)},\beta)/J^1(\mathfrak{a}^{(1)},\beta) \times \ldots \times J(\mathfrak{a}^{(r)},\beta)/J^1(\mathfrak{a}^{(r)},\beta)\\ &\cong \mathrm{GL}_{m_1}(\boldsymbol{l}) \times \ldots \times \mathrm{GL}_{m_r}(\boldsymbol{l}).\end{aligned}$$

Since $(\cdot)^*$ fixes $M \cap J$ and $M \cap J^1$, it induces a map

$$\begin{aligned}M \cap J/M \cap J^1 &\cong \mathrm{GL}_{m_1}(\boldsymbol{l}) \times \ldots \times \mathrm{GL}_{m_r}(\boldsymbol{l})\\ &\to \mathrm{GL}_{m_1}(\boldsymbol{l}) \times \ldots \times \mathrm{GL}_{m_r}(\boldsymbol{l}) \cong M \cap J/M \cap J^1,\\ (\overline{z^{(1)}},\dots,\overline{z^{(r)}}) &\mapsto (\overline{(z^{(1)})^*},\dots,\overline{(z^{(r)})^*}),\end{aligned}$$

where $\boldsymbol{l}$ is the residue field of E and E_0, and $\overline{z^{(j)}} \in J(\mathfrak{a}^{(j)},\beta)/J^1(\mathfrak{a}^{(j)},\beta) \cong \mathrm{GL}_{m_j}(\boldsymbol{l})$ is the image of $z^{(j)}$.

We show that for any i such that $2 \nmid a_i$, we have $2 \mid m_i$. Considering $j = i$ in equation (11), we get $(z^{(i)})^* = -\varpi_E^{a_i} z^{(i)} \varpi_E^{-a_i}$. Since $J/J^1 \cong U(\mathfrak{b})/U^1(\mathfrak{b})$ on which $E^\times$ acts trivially by conjugation, we get $\overline{z^{(i)}} = \overline{\varpi_E^{a_i} z^{(i)} \varpi_E^{-a_i}} = \overline{-(z^{(i)})^*} = -{}^t\overline{z^{(i)}}$. Since there does not exist any anti-symmetric invertible matrix of odd dimension, we must have $2|m_i$. Now for $\alpha_j = (a_j, m_j)$, define

$$\varpi_E^{\alpha_j} = \begin{cases}\varpi_E^{a_j} I_{m_j} & \text{if } 2|a_j;\\ \varpi_E^{a_j} J_{m_j/2} & \text{if } 2 \nmid a_j,\end{cases}$$

and $c' = \operatorname{diag}(\varpi_E^{\alpha_1}, \ldots, \varpi_E^{\alpha_r})$, where $J_{m_j/2} := \begin{pmatrix} 0 & I_{m_j/2} \\ -I_{m_j/2} & 0 \end{pmatrix}$. We have $c' = c'^*$ and c' is in the same J-J double coset as c. Letting $b = c'\varepsilon^{-1}$, we get $b\tau(b) = 1$. □

Now we write $\gamma = x'bx$ with $x, x' \in J$ and $b \in B^\times$ as in Lemma 6.6. Replacing g by $\tau(x')^{-1}g$ does not change the double coset JgG^τ but changes γ into $bx\tau(x')$. So from now on, we assume that

(12)
$$\gamma = bx, \quad b\tau(b) = 1,\ x \in J, \quad b \text{ is of the form in the proof of Lemma 6.6.}$$

Write K for the group $J \cap b^{-1}Jb$. Since $\tau(b) = b^{-1}$, and J is τ-stable, we have $x \in K$. The following corollary of Lemma 6.6 is obvious.

Corollary 6.7. — *The map $\delta_b : k \mapsto b^{-1}\tau(k)b$ is an involution on K.*

Now for $a_1 > \ldots > a_r$ as in the proof of Lemma 6.6, and $M = \mathrm{GL}_{m_1 d}(F) \times \ldots \times \mathrm{GL}_{m_r d}(F) \subseteq G$, we write P for the standard parabolic subgroup of G generated by M and upper triangular matrices, N for the unipotent radical of P and N^- for the opposite of N as a unipotent sub-group. By definition, b normalizes M, and we have

$$K = (K \cap N^-) \cdot (K \cap M) \cdot (K \cap N).$$

For $V = K \cap B^\times = U \cap b^{-1}Ub$ a subgroup of $B^\times$, similarly we have

$$V = (V \cap N^-) \cdot (V \cap M) \cdot (V \cap N),$$

where $U = U(\mathfrak{b})$ and $U^1 = J^1 \cap B^\times = U^1(\mathfrak{b})$. By definition, V is also fixed by δ_b.

Lemma 6.8. — *The subset*

$$K^1 = (K \cap N^-) \cdot (J^1 \cap M) \cdot (K \cap N)$$

is a δ_b-stable normal pro-p-subgroup of K, and we have $K = VK^1$.

Proof. — The proof is the same as that in [35], Lemma 6.10. □

Lemma 6.9. — *Letting $x \in K$ such that $x\delta_b(x) = 1$, then there are $k \in K$ and $v \in V$ such that:*

1. *The element v is in $\mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \ldots \times \mathrm{GL}_{m_r}(\mathfrak{o}_E) \subseteq B^\times$ such that $v\delta_b(v) = 1$.*
2. *One has $\delta_b(k)xk^{-1} \in vK^1$.*

Proof. — Let $V^1 = V \cap K^1$. We have

$$V^1 = (V \cap N^-) \cdot (U^1 \cap M) \cdot (V \cap N).$$

Thus we have canonical δ_b-equivariant group isomorphisms

$$K/K^1 \cong V/V^1 \cong (U\cap M)/(U^1\cap M). \tag{13}$$

Since $B^\times \cap M = \mathrm{GL}_{m_1}(E)\times\ldots\times \mathrm{GL}_{m_r}(E)$, the right-hand side of (13) is identified with $\mathcal{M} = \mathrm{GL}_{m_1}(\boldsymbol{l})\times\ldots\times\mathrm{GL}_{m_r}(\boldsymbol{l})$, where $\boldsymbol{l}$ denotes the residue field of E. As in the proof of Lemma 6.6, we may write $\varepsilon^{-1}b = \mathrm{diag}(\varpi_E^{a_1}c_1,\ldots,\varpi_E^{a_r}c_r)$ with $c_j\in\mathrm{GL}_{m_j}(\mathfrak{o}_E)$. Moreover, the involution δ_b acts on $\mathcal{M}$ by

$$(g_1,\ldots,g_r)\mapsto(\overline{c_1}^{-1}\sigma({}^tg_1^{-1})\overline{c_1},\ldots,\overline{c_r}^{-1}\sigma({}^tg_r^{-1})\overline{c_r}),$$

where we denote by $\overline{c_j}$ the image of c_j in $\mathrm{GL}_{m_j}(\boldsymbol{l})$. We denote by $(g_1,\ldots,g_r)$ the image of x in $\mathcal{M}=\mathrm{GL}_{m_1}(\boldsymbol{l})\times\ldots\times\mathrm{GL}_{m_r}(\boldsymbol{l})$.

When E/E_0 is unramified, we denote by $\boldsymbol{l_0}$ the residue field of E_0. So $\boldsymbol{l}/\boldsymbol{l_0}$ is quadratic, and the restriction of σ to $\boldsymbol{l}$ is the non-trivial involution in $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l_0})$. Since $(b^{-1}\varepsilon)^* = \varepsilon(b^*)^{-1}\varepsilon^{-1}\varepsilon = \tau(b)\varepsilon = b^{-1}\varepsilon$, we get $\overline{c_j}^* = \overline{c_j}$. If $x\delta_b(x)=1$, and then $(\overline{c_j}g_j)^* = g_j^*\overline{c_j} = \overline{c_j}g_j$.

Lemma 6.10 ([30], Proposition 2.3.1). — *For $\overline{x}=\overline{x}^*$ in $\mathrm{GL}_s(\boldsymbol{l})$, there exists $A\in\mathrm{GL}_s(\boldsymbol{l})$ such that $A\overline{x}A^*=I_s$.*

Using Lemma 6.10, we may choose $k_j\in\mathrm{GL}_{m_j}(\mathfrak{o}_E)$ such that its image $\overline{k_j}$ in $\mathrm{GL}_{m_j}(\boldsymbol{l})$ satisfies $(\overline{k_j}^*)^{-1}\overline{c_j}g_j\overline{k_j}^{-1} = I_{m_j}$. Choosing $k=\mathrm{diag}(k_1,\ldots,k_r)$ and $v=\mathrm{diag}(v_1,\ldots,v_r)=\mathrm{diag}(c_1^{-1},\ldots,c_r^{-1})$, we get $\delta_b(k)xk^{-1}\in vV^1$ and $\delta_b(v)v=\mathrm{diag}(c_1^{-1}c_1^*c_1c_1^{-1},\ldots,c_r^{-1}c_r^*c_rc_r^{-1})=1$.

When E/E_0 is ramified, the restriction of σ to $\boldsymbol{l}$ is trivial. Since $(b^{-1}\varepsilon)^* = b^{-1}\varepsilon$, we get $c_j^*=(-1)^{a_j}c_j$ and ${}^t\overline{c_j}=(-1)^{a_j}\overline{c_j}$.

Lemma 6.11 ([30], Proposition 2.5.4). — *For $\overline{x}={}^t\overline{x}$ in $\mathrm{GL}_s(\boldsymbol{l})$, there exists $A\in\mathrm{GL}_s(\boldsymbol{l})$ such that $A\overline{x}\,{}^tA$ is either I_s or $\overline{\varepsilon_s}=\mathrm{diag}(1,\ldots,1,\overline{\epsilon})$, where $\overline{\epsilon}\in\boldsymbol{l}^\times-\boldsymbol{l}^{\times2}$ with $\boldsymbol{l}^{\times2}$ denoting the group of square elements of $\boldsymbol{l}^\times$.*

Lemma 6.12 ([30], Proposition 2.4.1). — *For $\overline{x}=-{}^t\overline{x}$ in $\mathrm{GL}_s(\boldsymbol{l})$ and $2\mid s$, there exists $A\in\mathrm{GL}_s(\boldsymbol{l})$ such that $A\overline{x}\,{}^tA=J_{s/2}$.*

When a_j is even, using Lemma 6.11 we may choose $k_j\in\mathrm{GL}_{m_j}(\mathfrak{o}_E)$ such that its image $\overline{k_j}$ in $\mathrm{GL}_{m_j}(\boldsymbol{l})$ satisfies that $({}^t\overline{k_j})^{-1}\overline{c_j}g_j\overline{k_j}^{-1}$ equals either I_{m_j} or $\overline{\varepsilon_{m_j}}$, where we choose $\varepsilon_{m_j}=\mathrm{diag}(1,\ldots,1,\epsilon)\in\mathrm{GL}_{m_j}(\mathfrak{o}_E)$ such that its image $\overline{\varepsilon_{m_j}}$ in $\mathrm{GL}_{m_j}(\boldsymbol{l})$ is $\mathrm{diag}(1,\ldots,1,\overline{\epsilon})$ as in Lemma 6.11. Let v_j be c_j^{-1} or $c_j^{-1}\varepsilon_{m_j}$ in the two cases, respectively.

When a_j is odd we deduce that m_j is even from the proof of Lemma 6.6. Using Lemma 6.12, we may choose $k_j\in\mathrm{GL}_{m_j}(\mathfrak{o}_E)$ such that its image $\overline{k_j}$ in $\mathrm{GL}_{m_j}(\boldsymbol{l})$ satisfies $({}^t\overline{k_j})^{-1}\overline{c_j}g_j\overline{k_j}^{-1}=J_{m_j/2}$. We choose $v_j=c_j^{-1}J_{m_j/2}$.

Choosing $k=\mathrm{diag}(k_1,\ldots,k_r)$ and $v=\mathrm{diag}(v_1,\ldots,v_r)$, we know that

$$\delta_b(k)xk^{-1}\in vV^1$$

and

$$\delta_b(v)v = \mathrm{diag}(c_1^{-1}(v_1^*)^{-1}c_1v_1, \ldots, c_r^{-1}(v_r^*)^{-1}c_rv_r) = 1$$

by direct calculation in the two cases, respectively. So no matter whether or not E/E_0 is ramified, we finish the proof. □

Now we finish the proof of Lemma 6.5. Using Lemma 6.9, we choose $k \in K$ and $v \in V$ such that $bv\tau(bv) = 1$ and $\delta_b(k)xk^{-1} \in vK^1$. Thus we have $\tau(k)\gamma k^{-1} \in bvK^1$. Therefore, replacing g by kg and b by bv, we may assume

(14)

$$\gamma = bx, \quad b\tau(b) = 1, \quad x \in K^1, \quad b \in \varpi_E^{a_1}\mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \ldots \times \varpi_E^{a_r}\mathrm{GL}_{m_r}(\mathfrak{o}_E).$$

Furthermore, we have $\delta_b(x)x = 1$.

Since K^1 is a δ_b-stable pro-p-group, and p is odd, the first cohomology set of δ_b on K^1 is trivial. Thus, $x = \delta_b(y)y^{-1}$ for some $y \in K^1$, and hence $\gamma = \tau(y)by^{-1}$. Considering the determinant of this equation, we have $\det(b) \in \mathrm{N}_{F/F_0}(F^\times)$. If we denote by $\det_B$ the determinant function defined on $B^\times = \mathrm{GL}_m(E)$, then $\det(b) = \mathrm{N}_{E/F}(\det_B(b))$. Using Lemma 5.18 for $L = E$, we get $\det_B(b) \in \mathrm{N}_{E/E_0}(E^\times)$ and $\det_B(\varepsilon^{-1}b) \in \det_B(\varepsilon^{-1})\mathrm{N}_{E/E_0}(E^\times)$. Since $\tau(b)b = 1$, we have $(\varepsilon^{-1}b)^* = \varepsilon^{-1}b$. Using Proposition 2.1, there exists $h \in B^\times$, such that $\varepsilon^{-1}b = (h^*)^{-1}\varepsilon^{-1}h^{-1}$. So we have $b = \tau(h)h^{-1}$. Thus, $g \in yhG^\tau \subseteq JB^\times G^\tau$, which finishes the proof of Lemma 6.5. □

6.3. Distinction of the Heisenberg representation. — Now let η be the Heisenberg representation of J^1 associated to θ. We have the following result similar to [35], Proposition 6.12, by replacing σ with τ:

Proposition 6.13. — *Given $g \in G$, we have:*

$$\dim_R \mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g, 1) = \begin{cases} 1 & \textit{if } g \in JB^\times G^\tau, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. — It is useful to recall some details of the proof of this proposition, which will be used in the next subsection. We write $\delta(x) := \gamma^{-1}\tau(x)\gamma$ for any $x \in G$, which is an involution on G. And for any subgroup $H \subset G$, we have $H^g \cap G^\tau = (H \cap G^\delta)^g$.

When $g \notin JB^\times G^\tau$, restricting η^g to H^{1g} and using Proposition 6.4 and Lemma 6.5, the dimension equals 0. When $g \in JB^\times G^\tau$, we need to prove that $\mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g, 1) = \mathrm{Hom}_{J^1\cap G^\delta}(\eta, 1)$ is of dimension 1. We state the following general proposition, which works for a general involution on G:

Proposition 6.14. — *Let δ be an involution on G such that $\delta(H^1) = H^{1\gamma}$ and $\theta \circ \delta = \theta^{-1\gamma}$, where $\gamma \in B^\times$ such that $\delta(\gamma)\gamma = 1$. Then we have*

$$\dim_R \mathrm{Hom}_{J^1\cap G^\delta}(\eta, 1) = 1.$$

Since Proposition 6.14 in our special case implies Proposition 6.13, we only need to focus on the proof of this proposition. We only need to prove that the space

$$\mathrm{Hom}_{J^1\cap G^\delta}(\eta^{(J^1:H^1)^{1/2}},1)\cong \mathrm{Hom}_{J^1\cap G^\delta}(\mathrm{Ind}_{H^1}^{J^1}(\theta),1)$$

is of dimension $(J^1:H^1)^{1/2}$.

LEMMA 6.15. — *For H a subgroup of G such that $\delta(H)=H^\gamma$ with δ and γ as in Proposition 6.14, we have*

$$H\cap G^\delta = H^\gamma\cap G^\delta = H\cap H^\gamma\cap G^\delta.$$

Proof. — We have $H\cap G^\delta=\delta(H\cap G^\delta)=\delta(H)\cap\delta(G^\delta)=H^\gamma\cap G^\delta$, which proves the lemma. □

LEMMA 6.16. — *Let δ and γ be as in Proposition 6.14; then we have the following isomorphisms of finite dimensional representations:*

(1) $\mathrm{Ind}_{H^1}^{J^1}\theta|_{J^1\cap J^{1\gamma}}\cong\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}}\mathrm{Ind}_{H^1\cap J^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta.$

(2) $\mathrm{Ind}_{H^{1\gamma}}^{J^{1\gamma}}\theta^\gamma|_{J^1\cap J^{1\gamma}}\cong\bigoplus_{H^{1\gamma}\backslash J^{1\gamma}/J^1\cap J^{1\gamma}}\mathrm{Ind}_{J^1\cap H^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta^\gamma.$

(3) $\mathrm{Ind}_{H^1}^{J^1}\theta|_{J^1\cap G^\delta}\cong\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}}\bigoplus_{H^1\cap J^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^1\cap G^\delta}\mathrm{Ind}_{H^1\cap G^\delta}^{J^1\cap G^\delta}\theta.$

(4) $\mathrm{Ind}_{H^{1\gamma}}^{J^{1\gamma}}\theta^\gamma|_{J^{1\gamma}\cap G^\delta}\cong\bigoplus_{H^{1\gamma}\backslash J^{1\gamma}/J^1\cap J^{1\gamma}}\bigoplus_{J^1\cap H^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^{1\gamma}\cap G^\delta}\mathrm{Ind}_{H^{1\gamma}\cap G^\delta}^{J^{1\gamma}\cap G^\delta}\theta.$

Proof. — We only prove (1) and (3), since the proofs of (2) and (4) are similar to the proofs of (1) and (3), respectively.

For (1), using the Mackey formula, we have

$$\begin{aligned}\mathrm{Ind}_{H^1}^{J^1}\theta|_{J^1\cap J^{1\gamma}} &\cong \bigoplus_{x\in H^1\backslash J^1/J^1\cap J^{1\gamma}}\mathrm{Ind}_{H^{1x}\cap(J^1\cap J^{1\gamma})}^{J^1\cap J^{1\gamma}}\theta^x\\ &\cong\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}}\mathrm{Ind}_{H^1\cap J^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta.\end{aligned}$$

The last step is because $x\in J^1$ normalizes H^1 and θ.

For (3), using the Mackey formula again, we have

$$\begin{aligned}\mathrm{Ind}_{H^1}^{J^1}\theta|_{J^1\cap G^\delta} &\cong\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}}\mathrm{Ind}_{H^1\cap J^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta|_{J^1\cap G^\delta}\\ &\cong\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}}\ \bigoplus_{y\in H^1\cap J^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^1\cap G^\delta}\mathrm{Ind}_{(H^1\cap J^{1\gamma})^y\cap(J^1\cap G^\delta)}^{J^1\cap G^\delta}\theta^y\\ &\cong\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}}\ \bigoplus_{H^1\cap J^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^1\cap G^\delta}\mathrm{Ind}_{H^1\cap G^\delta}^{J^1\cap G^\delta}\theta.\end{aligned}$$

The last step is because $y\in J^1\cap J^{1\gamma}$ normalizes $H^1\cap J^{1\gamma}$ and θ, and $H^1\cap J^{1\gamma}\cap J^1\cap G^\delta=H^1\cap G^\delta$ by Lemma 6.15.(2) for $H=J^1$. So we finish the proof. □

LEMMA 6.17. — *Let δ and γ be as in Proposition 6.14; then we have:*

(1) $|H^1 \setminus J^1/J^1 \cap J^{1\gamma}| \cdot |H^1 \cap J^{1\gamma} \setminus J^1 \cap J^{1\gamma}/J^1 \cap G^\delta| = (J^1 : H^1)^{1/2}$.
(2) $|H^{1\gamma} \setminus J^{1\gamma}/J^1 \cap J^{1\gamma}| \cdot |J^1 \cap H^{1\gamma} \setminus J^1 \cap J^{1\gamma}/J^{1\gamma} \cap G^\delta| = (J^{1\gamma} : H^{1\gamma})^{1/2}$.
(3) $(J^1 : H^1)^{1/2} = (J^{1\gamma} : H^{1\gamma})^{1/2} = (J^1 \cap G^\delta : H^1 \cap G^\delta)$.

Proof. — For (3), we refer to [35] §6.3 for a proof, by noting that all the results and proofs from Lemma 6.14 to the end of §6.3 in *ibid.* can be generalized to a general involution δ on G, with τ in *loc. cit.* replaced by δ in our settings. For (1), since J^1 normalizes H^1, and $J^1 \cap J^{1\gamma}$ normalizes $H^1 \cap J^{1\gamma}$, we have

$$\begin{aligned}\text{left hand side of (1)} &= (J^1 : H^1(J^1 \cap J^{1\gamma})) \cdot (J^1 \cap J^{1\gamma} : (H^1 \cap J^{1\gamma})(J^1 \cap G^\delta)) \\ &= (J^1 : H^1) \cdot (J^1 \cap J^{1\gamma} : H^1 \cap J^{1\gamma})^{-1} \\ &\quad \cdot (J^1 \cap J^{1\gamma} : H^1 \cap J^{1\gamma}) \cdot (J^1 \cap G^\delta : H^1 \cap J^{1\gamma} \cap G^\delta)^{-1} \\ &= (J^1 : H^1) \cdot (J^1 \cap G^\delta : H^1 \cap G^\delta)^{-1} \\ &= (J^1 : H^1)^{1/2},\end{aligned}$$

where we use Lemma 6.15 for $H = J^{1\gamma}$ and (3) in the last two equations. So we finish the proof of (1), and the proof of (2) is similar. □

Combining Lemma 6.16.(3) with Lemma 6.17.(1),(3), we have

$$\begin{aligned}&\dim_R \mathrm{Hom}_{J^1 \cap G^\delta}(\mathrm{Ind}_{H^1}^{J^1}\theta, 1) \\ &= \dim_R \bigoplus_{H^1 \setminus J^1/J^1 \cap J^{1\gamma}} \bigoplus_{H^1 \cap J^{1\gamma} \setminus J^1 \cap J^{1\gamma}/J^1 \cap G^\delta} \mathrm{Hom}_{J^1 \cap G^\delta}(\mathrm{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta}\theta, 1) \\ &= (J^1 : H^1)^{1/2} \dim_R \mathrm{Hom}_{H^1 \cap G^\delta}(\theta|_{H^1 \cap G^\delta}, 1) \\ &= (J^1 : H^1)^{1/2}.\end{aligned}$$

For the last step, since γ intertwines θ^{-1} and $\theta \circ \delta = \theta^{-1\gamma}$, we know that θ is trivial on

$$\{y\delta(y) | y \in H^1 \cap H^{1\gamma}\}.$$

This set equals $H^1 \cap G^\delta$ since the first cohomology group of δ^{-1}-action on $H^1 \cap H^{1\gamma}$ is trivial. Thus, $\theta|_{H^1 \cap G^\delta}$ is the trivial character. □

6.4. Distinction of extensions of the Heisenberg representation. — Let $\boldsymbol{\kappa}$ be an irreducible representation of $\boldsymbol{J}$ extending η. There is a unique irreducible representation $\boldsymbol{\rho}$ of $\boldsymbol{J}$, which is trivial on J^1 satisfying $\Lambda \cong \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$.

LEMMA 6.18. — *Let $g \in JB^\times G^\tau$.*

(1) *There is a unique character χ of $\boldsymbol{J}^g \cap G^\tau$ trivial on $J^{1g} \cap G^\tau$, such that*

$$\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\boldsymbol{\kappa}^g, \chi^{-1}).$$

(2) *The canonical linear map*

$$\mathrm{Hom}_{J^{1g}\cap G^{\tau}}(\eta^{g},1)\otimes \mathrm{Hom}_{\boldsymbol{J}^{g}\cap G^{\tau}}(\boldsymbol{\rho}^{g},\chi)\to \mathrm{Hom}_{\boldsymbol{J}^{g}\cap G^{\tau}}(\Lambda^{g},1)$$

is an isomorphism.

Proof. — The proof is the same as that in [35], Lemma 6.20. □

For $g\in JB^{\times}G^{\tau}$, we have $\tau(g)\in\tau(JB^{\times}G^{\tau})=JB^{\times}G^{\tau}$, which means that we may consider a similar thing for $\tau(g)$ to that for g in Lemma 6.18. Thus, there exists a unique character χ' of $\boldsymbol{J}^{\tau(g)}\cap G^{\tau}$ trivial on $J^{1\tau(g)}\cap G^{\tau}$, such that

$$\mathrm{Hom}_{J^{1\tau(g)}\cap G^{\tau}}(\eta^{\tau(g)},1)\cong \mathrm{Hom}_{\boldsymbol{J}^{\tau(g)}\cap G^{\tau}}(\boldsymbol{\kappa}^{\tau(g)},\chi'^{-1}).$$

Moreover, $\tau(\boldsymbol{J})=\boldsymbol{J}$, $\tau(J)=J$, $\tau(J^1)=J^1$, and $\tau(H^1)=H^1$, thus using Lemma 4.2 and Lemma 6.15 we have $\boldsymbol{J}^{g}\cap G^{\tau}=\boldsymbol{J}^{\tau(g)}\cap G^{\tau}=J^{g}\cap G^{\tau}=J^{\tau(g)}\cap G^{\tau}$, $J^{1g}\cap G^{\tau}=J^{1\tau(g)}\cap G^{\tau}$ and $H^{1g}\cap G^{\tau}=H^{1\tau(g)}\cap G^{\tau}$. As a result, χ and χ' are characters defined on the same group $\boldsymbol{J}^{g}\cap G^{\tau}=\boldsymbol{J}^{\tau(g)}\cap G^{\tau}$. A natural idea is to compare them. For the rest of this subsection, we focus on the proof of the following proposition:

Proposition 6.19. — *For χ and χ' defined above as characters of $\boldsymbol{J}^{g}\cap G^{\tau}=\boldsymbol{J}^{\tau(g)}\cap G^{\tau}$, we have $\chi=\chi'$.*

We write $\delta(x)=\gamma^{-1}\tau(x)\gamma$ for any $x\in G$ with $\gamma=\tau(g)g^{-1}$. From §3.1, we have $\gamma\in I_G(\eta)=I_G(\kappa^0)$, where $\kappa^0=\boldsymbol{\kappa}|_J$. Moreover, we have

$$\dim_R(\mathrm{Hom}_{J\cap J^{\gamma}}(\kappa^{0\gamma},\kappa^0))=\dim_R(\mathrm{Hom}_{J^1\cap J^{1\gamma}}(\eta^{\gamma},\eta))=1.$$

Using Lemma 6.15, we have $J^1\cap G^{\delta}=J^{1\gamma}\cap G^{\delta}$ as a subgroup of $J^1\cap J^{1\gamma}$ and $H^1\cap G^{\delta}=H^{1\gamma}\cap G^{\delta}$. We claim the following proposition, which works for general γ and δ:

Proposition 6.20. — *Let δ and γ be as in Proposition 6.14, then for a non-zero homomorphism $\varphi\in\mathrm{Hom}_{J^1\cap J^{1\gamma}}(\eta^{\gamma},\eta)=\mathrm{Hom}_{J\cap J^{\gamma}}(\kappa^{0\gamma},\kappa^0)$, it naturally induces an R-vector space isomorphism:*

$$\begin{aligned} f_{\varphi}:\mathrm{Hom}_{J^1\cap G^{\delta}}(\eta,1)&\to\mathrm{Hom}_{J^{1\gamma}\cap G^{\delta}}(\eta^{\gamma},1),\\ \lambda\quad&\mapsto\quad\lambda\circ\varphi.\end{aligned}$$

First, we show that how Proposition 6.20 implies Proposition 6.19. Using Proposition 6.13 for g and $\tau(g)$, respectively, we have $\dim_R\mathrm{Hom}_{J^{1g}\cap G^{\tau}}(\eta^{g},1)=\dim_R\mathrm{Hom}_{J^{1\tau(g)}\cap G^{\tau}}(\eta^{\tau(g)},1)=1$. By Proposition 6.20,

$$\begin{aligned} f_{\varphi}:\mathrm{Hom}_{J^{1g}\cap G^{\tau}}(\eta^{g},1)&\to\mathrm{Hom}_{J^{1\tau(g)}\cap G^{\tau}}(\eta^{\tau(g)},1),\\ \lambda\quad&\mapsto\quad\lambda\circ\varphi\end{aligned}$$

is bijective. If we choose

$$0 \neq \lambda \in \mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g, 1) \quad \text{and}$$
$$0 \neq \lambda' := f_\varphi(\lambda) = \lambda \circ \varphi \in \mathrm{Hom}_{J^{1\tau(g)}\cap G^\tau}(\eta^{\tau(g)}, 1),$$

then for any v in the representation space of η and any $x \in J^g \cap G^\tau = J^{\tau(g)} \cap G^\tau$, we have

$$\begin{aligned} \chi'(x)^{-1}\lambda'(v) &= \lambda'(\kappa^{0\tau(g)}(x)v) && \text{(by Lemma 6.18.(1))} \\ &= \lambda(\varphi(\kappa^{0\tau(g)}(x)v)) && \text{(by definition of } \lambda'\text{)} \\ &= \lambda(\kappa^{0g}(x)\varphi(v)) && \text{(since } \varphi \in \mathrm{Hom}_{J^g\cap J^{\tau(g)}}(\kappa^{0\tau(g)}, \kappa^{0g})\text{)} \\ &= \chi(x)^{-1}\lambda(\varphi(v)) && \text{(by Lemma 6.18.(1))} \\ &= \chi(x)^{-1}\lambda'(v) && \text{(by definition of } \lambda'\text{).} \end{aligned} \tag{15}$$

Since v and $x \in J^g \cap G^\tau = J^{\tau(g)} \cap G^\tau$ are arbitrary, we have $\chi'|_{J^{\tau(g)}\cap G^\tau} = \chi|_{J^g\cap G^\tau}$, which is Proposition 6.19.

So we only need to focus on the proof of Proposition 6.20.

LEMMA 6.21. — *Let δ and γ be as in Proposition 6.14; then there exist an $R[J^1 \cap J^{1\gamma}]$-module homomorphism*

$$\Phi : \eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}} \cong \mathrm{Ind}_{H^{1\gamma}}^{J^{1\gamma}}\theta^\gamma|_{J^1\cap J^{1\gamma}}$$
$$\to \mathrm{Ind}_{H^1}^{J^1}\theta|_{J^1\cap J^{1\gamma}} \cong \eta^{(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}}$$

and a linear form $\widetilde{L_0} \in \mathrm{Hom}_{J^1\cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1)$, such that

$$0 \neq \widetilde{L_0} \circ \Phi \in \mathrm{Hom}_{J^1\cap G^\delta}(\eta^{\gamma(J^{1\gamma}:H^{1\gamma})^{1/2}}, 1).$$

Proof. — We prove this lemma by giving a direct construction of Φ and $\widetilde{L_0}$. First, we choose our $\widetilde{L_0}$. We choose $\lambda_0 \in \mathrm{Hom}_{J^1\cap G^\delta}(\mathrm{Ind}_{H^1\cap G^\delta}^{J^1\cap G^\delta} 1, 1) \cong R$ with the isomorphism given by Frobenius reciprocity, such that its corresponding image in R equals 1. Then we choose $\widetilde{L_0} = (\lambda_0, \ldots, \lambda_0)$ as an element in

$$\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}} \ \bigoplus_{H^1\cap J^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^1\cap G^\delta} \mathrm{Hom}_{J^1\cap G^\delta}(\mathrm{Ind}_{H^1\cap G^\delta}^{J^1\cap G^\delta} 1, 1)$$
$$\cong \mathrm{Hom}_{J^1\cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1),$$

where the isomorphism is determined by Lemma 6.16.(3), and by Lemma 6.17 the number of copies equals $(J^1 : H^1)^{1/2}$.

Now we focus on the construction of Φ. We define

$$f_0(g) := \begin{cases} \theta^\gamma(g_1)\theta(g_2) & \text{if } g = g_1g_2 \in (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) \\ 0 & \text{if } g \in J^1 \cap J^{1\gamma} - (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) \end{cases} \tag{16}$$

as a continuous function defined on $J^1 \cap J^{1\gamma}$ with values in R. Since $(J^1 \cap H^{1\gamma}) \cap (H^1 \cap J^{1\gamma}) = H^1 \cap H^{1\gamma}$ and $\theta^\gamma = \theta$ on $H^1 \cap H^{1\gamma}$, we know that f_0 is well defined.

We want to verify that $f_0 \in \mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$ and $f_0 \in \mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$. Since J^1 normalizes H^1, and $J^{1\gamma}$ normalizes $H^{1\gamma}$, by direct calculation $J^1 \cap J^{1\gamma}$ normalizes $J^1 \cap H^{1\gamma}$ and $H^1 \cap J^{1\gamma}$. In particular, we have $(J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) = (H^1 \cap J^{1\gamma})(J^1 \cap H^{1\gamma})$. Moreover, since J^1 and $J^{1\gamma}$ normalize θ and θ^γ, respectively, $(J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) = (H^1 \cap J^{1\gamma})(J^1 \cap H^{1\gamma})$ normalizes θ and θ^γ.

For $g_1' \in J^1 \cap H^{1\gamma}$, $g_2' \in H^1 \cap J^{1\gamma}$ and $g \in J^1 \cap J^{1\gamma}$, if $g \notin (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma})$, then $g_1'g, g_2'g \notin (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma})$, and thus

$$f_0(g_1'g) = f_0(g_2'g) = 0;$$

if $g = g_1 g_2 \in (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma})$, then

$$f_0(g_1'g) = \theta^\gamma(g_1')\theta^\gamma(g_1)\theta(g_2) = \theta^\gamma(g_1')f_0(g)$$

and

$$\begin{aligned} f_0(g_2'g) &= f_0(g_2'g_1g_2'^{-1}g_2'g_2) \\ &= \theta^\gamma(g_2'g_1g_2'^{-1})\theta(g_2')\theta(g_2) = \theta(g_2')\theta^\gamma(g_1)\theta(g_2) = \theta(g_2')f_0(g). \end{aligned}$$

Considering these facts, we have $f_0 \in \mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$ and $f_0 \in \mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$.

We consider $J^1 \cap J^{1\gamma}$-action on f_0 given by the right translation and we let $\langle f_0 \rangle$ be the $R[J^1 \cap J^{1\gamma}]$-subspace of both $\mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$ and $\mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$ generated by f_0. We choose V_{f_0} to be an $R[J^1 \cap J^{1\gamma}]$-invariant subspace of $\mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$, such that $\mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma = \langle f_0 \rangle \oplus V_{f_0}$.

We define the $R[J^1 \cap J^{1\gamma}]$-module homomorphism

$$\Phi_1 : \mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma \to \mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta,$$

such that $\Phi_1(f_0) = f_0$ and $\Phi_1|_{V_{f_0}} = 0$. Moreover, we define

$$\Phi : \bigoplus_{H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}} \mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma \to \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$$

given by

$$\Phi = \mathrm{diag}(\Phi_1, 0, \ldots, 0) \in \mathrm{M}_{N_1}(\mathrm{Hom}_{R[J^1 \cap J^{1\gamma}]}(\mathrm{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma, \mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta)),$$

where the coordinates are indexed by $N_1 := |H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}| = |H^1 \backslash J^1 / J^1 \cap J^{1\gamma}|$. In particular, we let the first coordinate correspond to the trivial double cosets $H^{1\gamma}(J^1 \cap J^{1\gamma})$ and $H^1(J^1 \cap J^{1\gamma})$, respectively. As a result, Φ gives an $R[J^1 \cap J^{1\gamma}]$-module homomorphism. By Lemma 6.16 we have

$$\eta^{(J^1:H^1)^{1/2}} \cong \mathrm{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap J^{1\gamma}} \cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \mathrm{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta \tag{17}$$

and

$$\eta^{\gamma(J^1:H^1)^{1/2}} \cong \mathrm{Ind}_{H^{1\gamma}}^{J^{1\gamma}}\theta^\gamma|_{J^1\cap J^{1\gamma}} \cong \bigoplus_{H^{1\gamma}\backslash J^{1\gamma}/J^1\cap J^{1\gamma}} \mathrm{Ind}_{J^1\cap H^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta^\gamma. \tag{18}$$

With these two isomorphisms, we may regard Φ as a homomorphism from $\eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}}$ to $\eta^{(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}}$.

Finally, we study $\widetilde{L_0}\circ\Phi$. First, we calculate

$$\Phi_1 : \mathrm{Ind}_{J^1\cap H^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta^\gamma|_{J^1\cap G^\delta} \to \mathrm{Ind}_{H^1\cap J^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta|_{J^1\cap G^\delta}.$$

We have the following isomorphism

$$\mathrm{Ind}_{H^1\cap J^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta|_{J^1\cap G^\delta} \cong \bigoplus_{H^1\cap J^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^1\cap G^\delta} \mathrm{Ind}_{H^1\cap G^\delta}^{J^1\cap G^\delta}1. \tag{19}$$

By definition of Φ_1 and (16),(19), $\Phi_1(f_0|_{J^1\cap G^\delta}) = f_0|_{J^1\cap G^\delta}$ equals

$$(\mathbf{1}_{H^1\cap G^\delta},\dots,\mathbf{1}_{H^1\cap G^\delta},0,\dots,0) \in \bigoplus_{H^1\cap J^{1\gamma}\backslash J^1\cap J^{1\gamma}/J^1\cap G^\delta} \mathrm{Ind}_{H^1\cap G^\delta}^{J^1\cap G^\delta}1, \tag{20}$$

where the coordinates are indexed by the double coset $H^1\cap J^{1\gamma}\setminus J^1\cap J^{1\gamma}/J^1\cap G^\delta$, and those coordinates that equal the characteristic function $\mathbf{1}_{H^1\cap G^\delta}$ are exactly indexed by the subset $H^1\cap J^{1\gamma}\setminus (J^1\cap H^{1\gamma})(J^1\cap H^{1\gamma})/J^1\cap G^\delta$.

We define $v_0 = (f_0|_{J^1\cap G^\delta},0,\dots,0)$ as an element in both

$$\bigoplus_{H^1\backslash J^1/J^1\cap J^{1\gamma}} \mathrm{Ind}_{J^1\cap H^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta^\gamma|_{J^1\cap G^\delta}$$

and

$$\bigoplus_{H^{1\gamma}\backslash J^{1\gamma}/J^1\cap J^{1\gamma}} \mathrm{Ind}_{H^1\cap J^{1\gamma}}^{J^1\cap J^{1\gamma}}\theta|_{J^1\cap G^\delta},$$

where the first coordinate corresponds to the trivial double cosets $H^1(J^1\cap J^{1\gamma})$ and $H^{1\gamma}(J^1\cap J^{1\gamma})$, respectively, as in our definition of Φ. Thus, we have

$$\begin{aligned}(\widetilde{L_0}\circ\Phi)(v_0) &= \widetilde{L_0}((\Phi_1(f_0|_{J^1\cap G^\delta}),0,\dots,0)) = \widetilde{L_0}((f_0|_{J^1\cap G^\delta},0,\dots,0)) \\ &= |H^1\cap J^{1\gamma}\setminus (H^1\cap J^{1\gamma})(J^1\cap H^{1\gamma})/J^1\cap G^\delta|\cdot\lambda_0(\mathbf{1}_{H^1\cap G^\delta}) \neq 0,\end{aligned}$$

where we use the definition of $\widetilde{L_0}$ and (20) for the last equation. Thus, we get $\widetilde{L_0}\circ\Phi\neq 0$, which finishes the proof. □

Lemma 6.22. — *We keep the same notations as in Proposition 6.20 and we fix*

$$0\neq\lambda_0'\in\mathrm{Hom}_{J^1\cap G^\delta}(\eta,1) \quad \textit{and} \quad 0\neq\lambda_0''\in\mathrm{Hom}_{J^1\cap G^\delta}(\eta^\gamma,1).$$

Then:

(1) *For any* $\widetilde{L} \in \mathrm{Hom}_{J^1\cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1)$, *there exists an* $R[J^1 \cap J^{1\gamma}]$-*homomorphism*

$$\mathrm{Pr} : \eta^{(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}} \to \eta|_{J^1\cap J^{1\gamma}}$$

such that $\widetilde{L} = \lambda'_0 \circ \mathrm{Pr}$;

(2) *For any* $\widetilde{L} \in \mathrm{Hom}_{J^1\cap G^\delta}(\eta^{\gamma(J^1:H^1)^{1/2}}, 1)$, *there exists an* $R[J^1 \cap J^{1\gamma}]$-*homomorphism*

$$s : \eta^{\gamma}|_{J^1\cap J^{1\gamma}} \to \eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}}$$

such that $\lambda''_0 = \widetilde{L} \circ s$.

Proof. — The proof is just a simple application of linear algebra. We write $N = (J^1 : H^1)^{1/2}$. For (1), we define $\mathrm{pr}_i : \eta^{(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}} \to \eta|_{J^1\cap J^{1\gamma}}$ as the projection with respect to the i-th coordinate. Since $\lambda'_0 \circ \mathrm{pr}_1, \ldots, \lambda'_0 \circ \mathrm{pr}_N$ are linearly independent, and $\dim_R \mathrm{Hom}_{J^1\cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1) = N$ by Proposition 6.13, $\lambda'_0 \circ \mathrm{pr}_1, \ldots, \lambda'_0 \circ \mathrm{pr}_N$ generate $\mathrm{Hom}_{J^1\cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1)$. So we may choose Pr to be a linear combination of pr_j, which proves (1). The proof of (2) is similar. □

Now we finish the proof of Proposition 6.20. Using Lemma 6.22.(1) we choose Pr such that $\widetilde{L_0} = \lambda'_0 \circ \mathrm{Pr}$, where $\widetilde{L_0}$ is defined as in the statement of Lemma 6.21. Using Lemma 6.21, there exists Φ such that $\widetilde{L_0} \circ \Phi \neq 0$. Using Lemma 6.22.(2) we choose s such that $\widetilde{L_0} \circ \Phi \circ s = \lambda''_0 \neq 0$. We define $\varphi' = \mathrm{Pr} \circ \Phi \circ s$ and we have the following commutative diagram

$$\begin{array}{ccc} \eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}} & \xrightarrow{\Phi} & \eta^{(J^1:H^1)^{1/2}}|_{J^1\cap J^{1\gamma}} \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle \mathrm{Pr}} \\ \eta^{\gamma}|_{J^1\cap J^{1\gamma}} & \xrightarrow{\varphi'} & \eta|_{J^1\cap J^{1\gamma}} \end{array}$$

By definition we have $\lambda'_0 \circ \varphi' = \lambda'_0 \circ \mathrm{Pr} \circ \Phi \circ s = \lambda''_0 \neq 0$, which means that $\varphi' \neq 0$. Since $\mathrm{Hom}_{J^1\cap J^{1\gamma}}(\eta^\gamma, \eta)$ is of dimension 1, we deduce that φ equals φ' multiplying with a non-zero scalar, which means that $\lambda'_0 \circ \varphi \neq 0$. Since $\mathrm{Hom}_{J^1\cap G^\delta}(\eta, 1)$ and $\mathrm{Hom}_{J^1\cap G^\delta}(\eta^\gamma, 1)$ are of dimension 1, we know that f_φ is an R-vector space isomorphism, which proves Proposition 6.20.

6.5. Existence of a $\boldsymbol{\tau}$-self-dual extension of $\boldsymbol{\eta}$. — Now our aim is to choose a simple $\boldsymbol{\kappa}$ as an extension of η. Specifically, under the condition of Remark 6.1, we show that we may assume $\boldsymbol{\kappa}$ to be τ-self-dual, which means that $\boldsymbol{\kappa}^\tau \cong \boldsymbol{\kappa}^\vee$. First of all, we have the following lemma, whose proof is the same as that in [35], Lemma 5.21:

LEMMA 6.23. — *There exists a unique character* μ *of* $\boldsymbol{J}$ *trivial on* J^1 *such that* $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}\mu$. *It satisfies the identity* $\mu \circ \tau = \mu$.

PROPOSITION 6.24. — *When* $\mathrm{char}(R) = 0$, *there exists a character* ϕ *of* $\boldsymbol{J}$ *trivial on* J^1 *such that* $\mu = \phi(\phi \circ \tau)$. *Moreover, for any* R, *we may choose* $\boldsymbol{\kappa}$ *to be an extension of* η *such that* $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}$.

Proof. — First, we consider the case where $\mathrm{char}(R) = 0$. We need the following elementary lemma:

LEMMA 6.25. — *Assume* $\mathrm{char}(R) = 0$. *For* N *odd and* $A \in \mathrm{GL}_N(R)$ *such that* $A^{2^s} = cI_N$ *for* $s \in \mathbb{N}$ *and* $c \in R^\times$, *we have* $\mathrm{Tr}(A) \neq 0$.

Proof. — Because $s = 0$ is trivial, from now on we assume $s \geq 1$. Let ζ_{2^s} be a primitive 2^s-th root of 1 in R and let $c^{1/2^s}$ be a 2^s-th root of c in R; then we get $\mathrm{Tr}(A) = c^{1/2^s} \sum_{i=1}^N \zeta_{2^s}^{n_i}$ with $n_i \in \{0, 1, 2, \dots, 2^s - 1\}$. We know that $P(x) = x^{2^{s-1}} + 1$ is the minimal polynomial of ζ_{2^s} in $\mathbb{Q}[x]$. If $\mathrm{Tr}(A) = 0$, then for $Q(x) = \sum_{i=1}^N x^{n_i}$, we have $Q(\zeta_{2^s}) = 0$. As a result, $P(x)|Q(x)$ in $\mathbb{Q}[x]$ and, thus, in $\mathbb{Z}[x]$ by the Gauss lemma. However, the sum of all the coefficients of $P(x)$ is even, and the sum of all the coefficients of $Q(x)$ equals N, which is odd. We get a contradiction. So $\mathrm{Tr}(A) \neq 0$. □

Let us come back to our proof. We choose $\boldsymbol{\kappa}$ to be an extension of η; thus as in Lemma 6.23, there exists a character μ of $\boldsymbol{J}$ such that $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}\mu$. If E/E_0 is unramified, we let

$$\overline{\mu} : \mathrm{GL}_m(\boldsymbol{l}) \cong J/J^1 \to R^\times$$

be the character whose inflation is $\mu|_J$. There exists a character $\varphi : \boldsymbol{l}^\times \to R^\times$ such that $\overline{\mu} = \varphi \circ \det$. Since $\overline{\mu} \circ \tau = \overline{\mu}$, we get $(\varphi \circ \sigma)\varphi = 1$, or equivalently $\varphi|_{\boldsymbol{l}_0^\times} = 1$, where $\boldsymbol{l}_0$ is the residue field of E_0, and σ acts on $\boldsymbol{l}$ as the Frobenius map corresponding to $\boldsymbol{l}_0$. Let Q be the cardinality of $\boldsymbol{l}_0$; then the cardinality of $\boldsymbol{l}$ is Q^2. If we fix $\zeta_{\boldsymbol{l}}$ a generator of $\boldsymbol{l}^\times$, then $\zeta_{\boldsymbol{l}}^{Q+1}$ is a generator of $\boldsymbol{l}_0^\times$. So we have $\varphi(\zeta_l)^{Q+1} = 1$. Choose $\alpha : \boldsymbol{l}^\times \to R^\times$ a character such that

$$\alpha(\zeta_{\boldsymbol{l}}^m)^{Q-1} = \varphi(\zeta_{\boldsymbol{l}})^{-m} \text{ for } m \in \mathbb{Z}.$$

Since

$$\alpha(\zeta_{\boldsymbol{l}})^{Q^2-1} = \varphi(\zeta_{\boldsymbol{l}})^{-Q-1} = 1,$$

we know that α is well defined as a character of $\boldsymbol{l}^\times$. Moreover, we get $\varphi = \alpha(\alpha \circ \sigma)^{-1}$. Choosing $\phi^0 : J \to R^\times$ as the inflation of $\alpha \circ \det$, we get $\mu|_J = \phi^0(\phi^0 \circ \tau)$.

Since ϖ_E and J generate $\boldsymbol{J}$, to choose ϕ as a character of $\boldsymbol{J}$ extending ϕ^0, it suffices to show that $\mu(\varpi_E) = 1$. Since $\mu = \mu \circ \tau$, we get

$$\mu(\varpi_E) = \mu(\tau(\varpi_E)) = \mu(\varpi_E)^{-1}, \text{ thus } \mu(\varpi_E) \in \{1, -1\}.$$

Let e be the ramification index of E/F, and let $\varpi_E^e = a_0\varpi_F$ for a certain $a_0 \in \mathfrak{o}_E^\times$. We have

$$\varpi_E^{e(Q-1)} = a_0^{Q-1}\varpi_F^{Q-1} \text{ with } a_0^{Q-1} \in 1+\mathfrak{p}_E \subset H^1(\mathfrak{a},\beta).$$

We write $e(Q-1) = 2^s u$ for $2 \nmid u$ and $s \in \mathbb{N}$. For $A = \boldsymbol{\kappa}(\varpi_E^u)$, we have

$$A^{2^s} = \boldsymbol{\kappa}(a_0^{Q-1}\varpi_F^{Q-1}) = \theta(a_0^{Q-1})\omega_{\boldsymbol{\kappa}}(\varpi_F^{Q-1})I_N,$$

where we use the fact that the restriction of $\boldsymbol{\kappa}$ to $H^1(\mathfrak{a},\beta)$ equals N-copies of θ with $N = (J^1 : H^1)^{1/2}$, and $\omega_{\boldsymbol{\kappa}}$ is the central character of $\boldsymbol{\kappa}$. Using Lemma 6.25 with A and $c = \theta(a_0^{Q-1})\omega_{\boldsymbol{\kappa}}(\varpi_F^{Q-1})$, we get $\mathrm{Tr}(\boldsymbol{\kappa}(\varpi_E^u)) \neq 0$. Since $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}\mu$, considering the trace of both sides at ϖ_E^u, we get

$$\mathrm{Tr}(\boldsymbol{\kappa}(\varpi_E^u)) = \mathrm{Tr}(\boldsymbol{\kappa}(\varpi_E^u))\mu(\varpi_E^u),$$

thus $\mu(\varpi_E^u) = 1$. Since u is odd, and $\mu(\varpi_E)$ equals either 1 or -1, we get $\mu(\varpi_E) = 1$, which finishes the proof of this case.

If E/E_0 is ramified, first we show that $\mu|_{\boldsymbol{l}^\times} = 1$, where we consider the embedding $\boldsymbol{l}^\times \hookrightarrow E^\times$. Let Q be the cardinality of $\boldsymbol{l} = \boldsymbol{l}_0$ and let $\zeta_{\boldsymbol{l}}$ be a generator of $\boldsymbol{l}^\times$; then we want to show that $\mu(\zeta_{\boldsymbol{l}}) = 1$. Writing $Q-1 = 2^s u$ with $2 \nmid u$ and using Lemma 6.25 with $A = \boldsymbol{\kappa}(\zeta_{\boldsymbol{l}}^u)$ and $c = 1$, we get $\mathrm{Tr}(\boldsymbol{\kappa}(\zeta_{\boldsymbol{l}}^u)) \neq 0$. Since $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}\mu$, we get

$$\mathrm{Tr}(\boldsymbol{\kappa}(\zeta_{\boldsymbol{l}}^u)) = \mathrm{Tr}(\boldsymbol{\kappa}(\zeta_{\boldsymbol{l}}^u))\mu(\zeta_{\boldsymbol{l}}^u)$$

after considering the trace. Thus, $\mu(\zeta_{\boldsymbol{l}}^u) = 1$. Since $\mu(\zeta_{\boldsymbol{l}})$ equals either 1 or -1, which can be proved as the former case, and u is odd, we get $\mu(\zeta_{\boldsymbol{l}}) = 1$. Thus, $\mu|_J = 1$.

To finish the definition of $\phi : \boldsymbol{J} \to R^\times$ such that $\mu = \phi(\phi\circ\tau)$, we only need to verify the equation

$$\mu(\varpi_E) = \phi(\varpi_E)\phi(\tau(\varpi_E)) = \phi(\varpi_E)\phi(-\varpi_E)^{-1} = \phi(-1)^{-1}.$$

Since we have already showed that $\mu(-1) = 1$, using the relation $\mu = \mu\circ\tau$, we get $\mu(\varpi_E^2) = \mu(-\varpi_E^2) = \mu(\varpi_E)\mu(\tau(\varpi_E))^{-1} = 1$, so we deduce that $\mu(\varpi_E)$ equals either 1 or -1. Choose $\phi(-1) = \mu(\varpi_E)$, which is well defined, we finish the definition of ϕ such that $\mu = \phi(\phi\circ\tau)$. Let $\boldsymbol{\kappa}' = \boldsymbol{\kappa}\phi$, then $\boldsymbol{\kappa}'$ is τ-self-dual.

Now we suppose $R = \overline{\mathbb{F}_l}$. Let $\widetilde{\theta}$ be the lift of θ to $\overline{\mathbb{Q}_l}$ given by the canonical embedding $\overline{\mathbb{F}_l}^\times \hookrightarrow \overline{\mathbb{Q}_l}^\times$, then $\widetilde{\theta}$ is a simple character, and $\widetilde{\theta}\circ\tau = \widetilde{\theta}^{-1}$. There is a τ-self-dual representation $\widetilde{\boldsymbol{\kappa}}$ of $\boldsymbol{J}$ extending the Heisenberg representation $\widetilde{\eta}$ of J^1 corresponding to $\widetilde{\theta}$. Moreover, we can further choose $\widetilde{\boldsymbol{\kappa}}$ such that the central character of $\widetilde{\boldsymbol{\kappa}}$ is integral. To do this, first we choose $\widetilde{\boldsymbol{\kappa}^0}$ to be a representation of J extending η. We extend $\widetilde{\boldsymbol{\kappa}^0}$ to a representation of $F^\times J$. This requires us to choose a quasi-character $\widetilde{\omega} : F^\times \to \overline{\mathbb{Q}_l}^\times$ extending $\omega_{\widetilde{\boldsymbol{\kappa}^0}}$. We choose $\widetilde{\omega}$ such that it is integral. If we further extend this representation to $\widetilde{\boldsymbol{\kappa}}$ as a representation of $J = E^\times J$, then $\widetilde{\boldsymbol{\kappa}}$ is also integral. From the proof of the characteristic 0 case, we may further assume $\widetilde{\boldsymbol{\kappa}}^{\tau\vee} \cong \widetilde{\boldsymbol{\kappa}}$ without losing the property that $\widetilde{\boldsymbol{\kappa}}$ is integral. By [32], §2.11, the reduction of $\widetilde{\boldsymbol{\kappa}}$ to R, denoted by $\boldsymbol{\kappa}$, is thus a τ-self-dual representation of $\boldsymbol{J}$ extending η.

For $\mathrm{char}(R) = l > 0$ in general, we fix $\iota : \overline{\mathbb{F}_l} \hookrightarrow R$ an embedding. For θ a simple character over R as before, which is of finite image, there exists a simple character θ_0 over $\overline{\mathbb{F}_l}$ corresponding to the same simple stratum $[\mathfrak{a}, \beta]$, such that $\theta = \iota \circ \theta_0$ and $\theta_0 \circ \tau = \theta_0^{-1}$. Let η_0 be the Heisenberg representation of θ_0 and choose $\boldsymbol{\kappa}_0$ to be a τ-self-dual extension of η_0 by the former case. Then $\boldsymbol{\kappa} = \boldsymbol{\kappa}_0 \otimes_{\overline{\mathbb{F}_l}} R$ is what we want. □

6.6. Proof of Theorem 6.2. — Using Proposition 6.24, we may assume that $\boldsymbol{\kappa}$ is τ-self-dual, which means that $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}$. From its proof, when $R = \overline{\mathbb{F}_l}$, we assume further that $\boldsymbol{\kappa}$ is the reduction of a τ-self-dual representation $\widetilde{\boldsymbol{\kappa}}$ of $\boldsymbol{J}$ over $\overline{\mathbb{Q}_l}$, and when $\mathrm{char}(R) = l > 0$ in general, we assume $\boldsymbol{\kappa}$ to be realized as a $\overline{\mathbb{F}_l}$-representation via a certain field embedding $\overline{\mathbb{F}_l} \hookrightarrow R$.

Proposition 6.26. — *The character χ defined by Lemma 6.18.(1) is quadratic over $J^g \cap G^\tau$, that is, $\chi^2|_{J^g \cap G^\tau} = 1$.*

Proof. — First, we assume that $\mathrm{char}(R) = 0$. We have the following isomorphisms

$$
\begin{aligned}
(21)\quad & \mathrm{Hom}_{J^{1\tau(g)} \cap G^\tau}(\eta^{\tau(g)}, 1) \\
&\cong \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \\
&\cong \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\boldsymbol{\kappa}^g, \chi^{-1}) \\
&\cong \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\chi, \boldsymbol{\kappa}^{g\vee}) && \text{(by the duality of contragradient)} \\
&\cong \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\boldsymbol{\kappa}^{g\vee}, \chi) && \text{(since } \mathrm{char}(R) = 0) \\
&\cong \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\boldsymbol{\kappa}^{g\vee} \circ \tau, \chi \circ \tau) \\
&\cong \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}((\boldsymbol{\kappa}^{\tau\vee})^{\tau(g)}, \chi \circ \tau) \\
&\cong \mathrm{Hom}_{\boldsymbol{J}^{\tau(g)} \cap G^\tau}(\boldsymbol{\kappa}^{\tau(g)}, \chi \circ \tau) && \text{(since } \boldsymbol{\kappa} \text{ is } \tau\text{-self-dual).}
\end{aligned}
$$

Using Proposition 6.19 and the uniqueness of χ' in the *loc. cit.*, we have $\chi \circ \tau = \chi^{-1}$. Since χ is defined on $\boldsymbol{J}^g \cap G^\tau$, which is τ-invariant, we have $\chi \circ \tau = \chi$. Thus, $\chi^2 = \chi(\chi \circ \tau) = 1$.

If $R = \overline{\mathbb{F}_l}$, we denote by $\widetilde{\boldsymbol{\kappa}}$ a τ-self-dual $\overline{\mathbb{Q}_l}$-lift of $\boldsymbol{\kappa}$ and we denote by $\widetilde{\chi}$ the character defined by Lemma 6.18.(1) with respect to $\widetilde{\boldsymbol{\kappa}}$ and $\widetilde{\eta}$, where $\widetilde{\eta}$ is a $J^1 \cap G^\tau$-distinguished $\overline{\mathbb{Q}_l}$-lift of η. Using this proposition for $\overline{\mathbb{Q}_l}$-representations, we get $\widetilde{\chi}^2 = 1$. From the uniqueness of χ, we know that $\widetilde{\chi}$ is a $\overline{\mathbb{Q}_l}$-lift of χ. As a result, we get $\chi^2 = 1$.

If $\mathrm{char}(R) = l > 0$ in general, from the assumption of $\boldsymbol{\kappa}$ mentioned at the beginning of this subsection, via a field embedding $\overline{\mathbb{F}_l} \hookrightarrow R$ we may realize all the representations mentioned in this proposition as representations over $\overline{\mathbb{F}_l}$, so we finish the proof by using the former case. $\square$

As in the proof of Lemma 6.5, we assume $g \in B^\times$ and

(22)
$$\gamma = bx, \quad b\tau(b) = 1, \quad x \in K^1, \quad b \in \varpi_E^{a_1}\mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \ldots \times \varpi_E^{a_r}\mathrm{GL}_{m_r}(\mathfrak{o}_E).$$

There exists a unique standard hereditary order $\mathfrak{b}_m \subseteq \mathfrak{b}$ such that

$$U^1(\mathfrak{b}_m) = (U \cap \delta(U^1))U^1 = (U \cap U^{1\gamma})U^1,$$

where we define $\delta(y) = \gamma^{-1}\tau(y)\gamma$, for any $y \in G$ as an involution on G. We have the following lemma whose proof is the same as that in [35], Lemma 6.22, inspired by [22], Proposition 5.20:

Lemma 6.27. — *We have* $U^1(\mathfrak{b}_m) = (U^1(\mathfrak{b}_m) \cap G^\delta)U^1$.

Theorem 6.28. — *Let* $g \in G$ *and suppose* $\mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1)$ *is non-zero. Then* $\tau(g)g^{-1} \in \boldsymbol{J}$.

Proof. — It is enough to show that $r = 1$ in (22). If not, $\mathfrak{b}_m$ by definition is a proper suborder of $\mathfrak{b}$. Furthermore, $\overline{U^1(\mathfrak{b}_m)} := U^1(\mathfrak{b}_m)/U^1$ is a non-trivial unipotent subgroup of $U/U^1 \cong \mathrm{GL}_m(\boldsymbol{l})$. Using Lemma 6.18.(2), we have

$$\mathrm{Hom}_{\boldsymbol{J} \cap G^\delta}(\boldsymbol{\rho}, \chi^{g^{-1}}) \cong \mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\boldsymbol{\rho}^g, \chi) \neq 0.$$

Restricting ourselves to $U^1(\mathfrak{b}_m) \cap G^\delta$, we have

$$\mathrm{Hom}_{U^1(\mathfrak{b}_m) \cap G^\delta}(\boldsymbol{\rho}, \chi^{g^{-1}}) \neq 0. \tag{23}$$

Using Lemma 6.27, we have the isomorphism

$$(U^1(\mathfrak{b}_m) \cap G^\delta)U^1/U^1 \cong U^1(\mathfrak{b}_m)/U^1.$$

We denote by $\overline{\rho}$ the cuspidal representation of $U^0/U^1 \cong \mathrm{GL}_m(\boldsymbol{l})$ whose inflation is $\boldsymbol{\rho}|_{U^0}$, and by $\overline{\chi^{g^{-1}}}$ the character of $\overline{U^1(\mathfrak{b}_m)}$ whose inflation is $\chi^{g^{-1}}$. So if we consider the equation (23) modulo U^1, then we get

$$\mathrm{Hom}_{\overline{U^1(\mathfrak{b}_m)}}(\overline{\rho}, \overline{\chi^{g^{-1}}}) \neq 0.$$

Since $\chi^{g^{-1}}|_{J \cap G^\delta}$ is quadratic, and $\overline{U^1(\mathfrak{b}_m)}$ is a p-group with $p \neq 2$, we get $\overline{\chi^{g^{-1}}} = 1$, and thus

$$\mathrm{Hom}_{\overline{U^1(\mathfrak{b}_m)}}(\overline{\rho}, 1) \neq 0,$$

which contradicts to the fact that $\overline{\rho}$ is cuspidal. $\square$

Proof of Theorem 6.2. — If there exists a τ-self-dual simple type $(\boldsymbol{J}, \Lambda)$ in π such that $\mathrm{Hom}_{\boldsymbol{J} \cap G^\tau}(\Lambda, 1)$ is non-zero, then π is G^τ-distinguished. Conversely, there exists $g \in G$ such that $\mathrm{Hom}_{\boldsymbol{J}^g \cap G^\tau}(\Lambda^g, 1)$ is non-zero. Using Theorem 6.28 we conclude that $(\boldsymbol{J}^g, \Lambda^g)$ is a τ-self-dual simple type. $\square$

Finally, we state the following corollary of Theorem 6.28 as the end of this section:

COROLLARY 6.29. — *Under the assumption of Theorem 6.28, we have* $g \in \boldsymbol{J}G^\tau$ *or* $g \in \boldsymbol{J}g_1G^\tau$*, where the latter case exists only if* m *is even, and* $g_1 \in B^\times$ *is fixed such that*

$$\tau(g_1)g_1^{-1} = \begin{cases} \varpi_E I_m & \text{if } E/E_0 \text{ is unramified.} \\ \varpi_E J_{m/2} & \text{if } E/E_0 \text{ is ramified.} \end{cases}$$

As a result,

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) \oplus \mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\Lambda^{g_1}, 1).$$

Proof. — Recall that we have already assumed that $g \in B^\times$. Since $\tau(g)g^{-1} \in \boldsymbol{J}\cap B^\times = E^\times\mathfrak{b}^\times$, changing g up to multiplying by an element in $E^\times$, which does not change the double coset it represents, we may assume $(g^*)^{-1}\varepsilon^{-1}g^{-1} \in \mathfrak{b}^\times$ or $\varpi_E\mathfrak{b}^\times$, where ε equals I_m for E/E_0 unramified[6] and ε equals I_m or $\mathrm{diag}(1,\dots,1,\epsilon)$ with $\epsilon \in \mathfrak{o}_{E_0}^\times - \mathrm{N}_{E/E_0}(\mathfrak{o}_E^\times)$ for E/E_0 ramified. Using Proposition 2.2, we may change g^{-1} up to multiplying by an element in $\mathfrak{b}^\times$ on the right, and thus we may write $(g^*)^{-1}\varepsilon^{-1}g^{-1} = \varpi_E^\alpha$, where ϖ_E^α is defined as in §2.2. Thus, we get $\det_B(\varpi_E^\alpha)/\det_B(\varepsilon^{-1}) \in \mathrm{N}_{E/E_0}(E^\times)$.

If $(g^*)^{-1}\varepsilon^{-1}g^{-1} \in \mathfrak{b}^\times$, from the definition and the uniqueness of ϖ_E^α in Proposition 2.2, we get $\varpi_E^\alpha = \varepsilon$. We may further change g^{-1} up to multiplying by an element in $\mathfrak{b}^\times$ on the right, such that $(g^*)^{-1}\varepsilon^{-1}g^{-1} = \varepsilon^{-1}$. Thus, we get $\tau(g) = \varepsilon(g^*)^{-1}\varepsilon^{-1} = g$, which means that $g \in G^\tau$.

If $(g^*)^{-1}\varepsilon^{-1}g^{-1} \in \varpi_E\mathfrak{b}^\times$. Considering the determinant we deduce that $\det_B((g^*)^{-1}\varepsilon^{-1}g^{-1}) \in E^\times$ is of even order with respect to the discrete valuation of E. Since the determinant of elements in $\varpi_E\mathfrak{b}^\times$ is of order m, we know that m is even. Thus, from the definition and the uniqueness of ϖ_E^α in Proposition 2.2, we get $\varpi_E^\alpha = \varpi_E\varepsilon$ when E/E_0 is unramified and $\varpi_E^\alpha = \varpi_E J_{m/2}$ when E/E_0 is ramified. For the former case, we have $\varepsilon = I_m$. Using Proposition 2.1, we may choose $g_1 \in B^\times$ such that $(g_1^*)^{-1}g_1^{-1} = \varpi_E I_m = (g^*)^{-1}g^{-1}$. Thus, $g \in g_1G^\tau$. For the latter case, considering the determinant we must have $\det_B(\varepsilon) \in \mathrm{N}_{E/E_0}(E^\times)$, thus $\varepsilon = I_m$. Using Proposition 2.1, we may choose $g_1 \in B^\times$ such that $(g_1^*)^{-1}g_1^{-1} = \varpi_E J_{m/2} = (g^*)^{-1}g^{-1}$, thus $g \in g_1G^\tau$. □

7. The supercuspidal unramified case

In this section, we study the distinction of σ-invariant supercuspidal representations of G in the case where E/E_0 is unramified.

6. It is also possible in the unramified case that $\varepsilon = \mathrm{diag}(\varpi_E,\dots,\varpi_E)$. However, in this case, $\varepsilon \in E^\times$, which commutes with $B^\times$, thus this case can be combined into the case where $\varepsilon = I_m$.

7.1. The finite field case. — In this subsection, we assume $\boldsymbol{l}/\boldsymbol{l}_0$ to be a quadratic extension of finite fields with characteristic $p \neq 2$. Let $|\boldsymbol{l}_0| = Q$; then $|\boldsymbol{l}| = Q^2$. Let σ be the non-trivial involution in $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$.

Let m be a positive integer and let $\boldsymbol{t}$ be an extension of degree m over $\boldsymbol{l}$. We identify $\boldsymbol{t}^\times$ with a maximal torus of $\mathrm{GL}_m(\boldsymbol{l})$. We call a character $\xi : \boldsymbol{t}^\times \to R^\times$ $\boldsymbol{l}$-regular (or regular for short) if for any $i = 1, \ldots, m-1$, we have $\xi^{|\boldsymbol{l}|^i} \neq \xi$. By Green [17] when $\mathrm{char}(R) = 0$ and James [29] when $\mathrm{char}(R) = l > 0$ prime to p, there is a surjective map

$$\xi \mapsto \overline{\rho_\xi}$$

between $\boldsymbol{l}$-regular characters of $\boldsymbol{t}^\times$ and isomorphism classes of supercuspidal representations of $\mathrm{GL}_m(\boldsymbol{l})$, whose fibers are $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{l})$-orbits.

Lemma 7.1. — (1) *If there exists a σ-invariant supercuspidal representation of* $\mathrm{GL}_m(\boldsymbol{l})$, *then m is odd.*
(2) *When* $\mathrm{char}(R) = 0$, *the converse of (1) is true.*

Proof. — We may follow the same proof of [35], Lemma 2.3, with the concept σ-self-dual in *loc. cit.* replaced by σ-invariant and the corresponding contragradient (or inverse) replaced by the identity. □

Let $H = \mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0) := \mathrm{U}_m(I_m)$ be the unitary subgroup of $\mathrm{GL}_m(\boldsymbol{l})$ corresponding to the hermitian matrix I_m with respect to $\boldsymbol{l}/\boldsymbol{l_0}$. Note that there is only one conjugacy class of unitary subgroup of $\overline{G}$, which is isomorphic to H.

Lemma 7.2. — *Suppose m to be odd and let $\overline{\rho}$ be a supercuspidal representation of* $\mathrm{GL}_m(\boldsymbol{l})$. *The following assertions are equivalent:*

(1) *The representation $\overline{\rho}$ is σ-invariant.*
(2) *The representation $\overline{\rho}$ is H-distinguished.*
(3) *The R-vector space* $\mathrm{Hom}_H(\overline{\rho}, 1)$ *has dimension 1.*

Proof. — When R has characteristic 0, this is [16], Theorem 2.1 and Theorem 2.4. Suppose now that $R = \overline{\mathbb{F}_l}$. First we prove that (1) is equivalent to (2).

For $\overline{\rho}$ a supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$, we denote by $P_{\overline{\rho}}$ the projective envelope of $\overline{\rho}$ as a $\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module, where $\overline{\mathbb{Z}_l}$ is the ring of integers of $\overline{\mathbb{Q}_l}$. Using [43], Chapitre III, Théorème 2.9 and [39], Proposition 42, we have:

(1) $P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}$ is the projective envelope of $\overline{\rho}$ as a $\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module, which is indecomposable of finite length with each irreducible component isomorphic to $\overline{\rho}$.
(2) For $\widetilde{P_{\overline{\rho}}} = P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{Q}_l}$, we have $\widetilde{P_{\overline{\rho}}} \cong \bigoplus \widetilde{\overline{\rho}}$, where $\widetilde{\overline{\rho}}$ in the direct sum ranges over all the supercuspidal $\overline{\mathbb{Q}_l}$-lifts of $\overline{\rho}$ and appears with multiplicity 1.

We have

$$\begin{aligned}
&\mathrm{Hom}_H(\overline{\rho},1)\neq 0\\
&\quad\Longleftrightarrow \mathrm{Hom}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\rho},\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})])\neq 0\\
&\quad\Longleftrightarrow \mathrm{Hom}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(P_{\overline{\rho}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{F}_l},\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})])\neq 0\\
&\quad\Longleftrightarrow \mathrm{Hom}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(P_{\overline{\rho}},\overline{\mathbb{Z}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})])\neq 0\\
&\quad\Longleftrightarrow \mathrm{Hom}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{P_{\overline{\rho}}},\overline{\mathbb{Q}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})])\neq 0\\
&\quad\Longleftrightarrow \text{There exists } \widetilde{\overline{\rho}} \text{ as above such that}\\
&\qquad\quad \mathrm{Hom}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{\overline{\rho}},\overline{\mathbb{Q}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})])\neq 0\\
&\quad\Longleftrightarrow \text{There exists } \widetilde{\overline{\rho}} \text{ as above such that } \widetilde{\overline{\rho}}^{\sigma}=\widetilde{\overline{\rho}}\\
&\quad\Longleftrightarrow \overline{\rho}^{\sigma}=\overline{\rho}.
\end{aligned}$$

The former five equivalences are direct, by noting that a projective $\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module is a free $\overline{\mathbb{Z}_l}$-module. For the second last equivalence, we use the result for the characteristic 0 case. For the last equivalence from the construction of supercuspidal representation given by Green and James, since it is always possible to lift a σ-invariant regular character over $\overline{\mathbb{F}_l}$ to a σ-invariant regular character over $\overline{\mathbb{Q}_l}$, it is always possible to find a σ-invariant $\overline{\mathbb{Q}_l}$-lift $\widetilde{\overline{\rho}}$ for a σ-invariant supercuspidal representation $\overline{\rho}$.

Since (3) implies (2) by definition, we only need to prove that (2) implies (3). We sum up the proof occurring in [35], Lemma 2.19. We have the following $\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module decomposition

$$\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]=V_{\overline{\rho}}\oplus V',$$

where $V_{\overline{\rho}}$ is composed of irreducible components isomorphic to $\overline{\rho}$, and V' has no irreducible component isomorphic to $\overline{\rho}$. First, we verify that $\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}})$ is commutative. By [16], Theorem 2.1, the convolution algebra $\overline{\mathbb{Z}_l}[H\setminus\mathrm{GL}_m(\boldsymbol{l})/H]$ is commutative. Modulo l we deduce that

$$\begin{aligned}
\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})/H]&\cong \mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})])\\
&\cong \mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}})\oplus \mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V')
\end{aligned}$$

is commutative, thus $\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}})$ is commutative.

By [43], Chapitre III, Théorème 2.9, $P=P_{\overline{\rho}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{F}_l}$ is indecomposable with each irreducible subquotient isomorphic to $\overline{\rho}$. By [10], Proposition B.1.2, there exists a nilpotent endomorphism $N\in\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}[P]$ such that

$\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}[P] = \overline{\mathbb{F}_l}[N]$, and there exist $r \geq 1$ and $n_1, \ldots, n_r$ positive integers such that

$$V_{\overline{\rho}} \cong \bigoplus_{i=1}^{r} P/N^{n_i}P.$$

Since $\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}})$ is commutative, we have $r = 1$ and $V_{\overline{\rho}} = P/N^{n_1}P$. Thus,

$$\mathrm{Hom}_H(\overline{\rho}, 1) \cong \mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l})}(\overline{\rho}, V_{\overline{\rho}}) = \mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l})}(\overline{\rho}, P/N^{n_1}P) \cong \overline{\mathbb{F}_l}.$$

For $\mathrm{char}(R) = l > 0$ in general, we fix an embedding $\overline{\mathbb{F}_l} \hookrightarrow R$ and write $\overline{\rho} = \overline{\rho}_0 \otimes_{\overline{\mathbb{F}_l}} R$, where $\overline{\rho}_0$ is a supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ over $\overline{\mathbb{F}_l}$. By considering the Brauer characters, we have

$$\overline{\rho}^{\sigma} \cong \overline{\rho} \quad \text{if and only if} \quad \overline{\rho}_0^{\sigma} \cong \overline{\rho}_0.$$

Moreover,

$$\mathrm{Hom}_{R[H]}(\overline{\rho}, R) \cong \mathrm{Hom}_{\overline{\mathbb{F}_l}[H]}(\overline{\rho}_0, \overline{\mathbb{F}_l}) \otimes_{\overline{\mathbb{F}_l}} R.$$

Thus, we come back to the former case. □

Remark 7.3. — We give an example of a σ-invariant cuspidal non-supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ over $\overline{\mathbb{F}_l}$, which is not distinguished by H. Assume $m = 2$ and $l \neq 2$ such that $l | Q^2 + 1$. Let B be the subgroup of $\mathrm{GL}_2(\boldsymbol{l})$ consisting of upper triangular matrices. For $\mathrm{Ind}_B^{\mathrm{GL}_2(\boldsymbol{l})}\overline{\mathbb{F}_l}$, it is a representation of length 3 with irreducible components of dimension $1, Q^2 - 1, 1$ respectively. Denote by $\overline{\rho}$ the irreducible subquotient of $\mathrm{Ind}_B^{\mathrm{GL}_2(\boldsymbol{l})}\overline{\mathbb{F}_l}$ of dimension $Q^2 - 1$. It is thus cuspidal (not supercuspidal) and σ-invariant. Let $\widetilde{\overline{\rho}}$ be a $\overline{\mathbb{Q}_l}$-lift of $\overline{\rho}$, which is an irreducible cuspidal representation. We write $\widetilde{\overline{\rho}}|_H = V_1 \oplus \ldots \oplus V_r$ its decomposition of irreducible components. Since $|H| = Q(Q+1)(Q^2-1)$ is prime to l, reduction modulo l preserves irreducibility. So $\overline{\rho}|_H$ decomposes as $W_1 \oplus \ldots \oplus W_r$, where the irreducible representation W_i is the reduction of V_i modulo l for each $i = 1, \ldots, r$. Suppose that $\overline{\rho}$ is distinguished. Then $W_i = \overline{\mathbb{F}_l}$ for some i. Thus, V_i is a character that must be trivial, which implies that $\widetilde{\overline{\rho}}$ is distinguished. This is impossible by Lemma 7.1 and Lemma 7.2, since $m = 2$ is even. See [35], Remark 2.8. for the Galois self-dual case.

Finally, we need the following finite group version of Proposition 5.6, which is well known:

Proposition 7.4. — *For $\overline{\rho}$ an irreducible representation of $\mathrm{GL}_m(\boldsymbol{l})$, we have $\overline{\rho}^{\vee} \cong \overline{\rho}({}^{t}\cdot^{-1})$, where $\overline{\rho}({}^{t}\cdot^{-1}) : x \mapsto \overline{\rho}({}^{t}x^{-1})$, for any $x \in \mathrm{GL}_m(\boldsymbol{l})$.*

Proof. — By definition, the Brauer characters of $\overline{\rho}^{\vee}$ and $\overline{\rho}({}^{t}\cdot^{-1})$ are the same. □

7.2. Distinction criterion in the unramified case. — Let π be a σ-invariant supercuspidal representation of G. In this subsection, we prove Theorem 1.1 and Theorem 1.2 in the case where E/E_0 is unramified. To prove Theorem 1.1, it remains to show that π is distinguished by any unitary subgroup G^τ with the aid of Theorem 4.1. Since changing τ up to a G-action does not change the content of the theorem, we only need to consider the two special unitary involutions mentioned in Remark 6.1.(4). To justify the assumption in Remark 6.1.(3), first we prove the following lemma:

Lemma 7.5. — *For any σ-invariant supercuspidal representation π with E/E_0 unramified, m is odd.*

Proof. — We consider $\tau = \tau_1$, where $\tau_1(x) = \sigma({}^t x^{-1})$, for any $x \in G$. We follow the settings of Remark 6.1. For $(\boldsymbol{J}, \Lambda)$ a simple type as in Remark 6.1.(2), we may write $\Lambda \cong \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ as before. Using Proposition 6.24, we may further assume $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}$. Since Λ and $\boldsymbol{\kappa}$ are τ-self-dual, $\boldsymbol{\rho}$ is τ-self-dual. Let $\overline{\rho}$ be the supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l}) \cong J/J^1$ whose inflation equals $\boldsymbol{\rho}|_J$, then $\overline{\rho}^{\tau\vee} \cong \overline{\rho}$ when regarding τ as a unitary involution on $\mathrm{GL}_m(\boldsymbol{l})$. Using Proposition 7.4, we have $\overline{\rho} \circ \sigma \cong \overline{\rho}$. Using Lemma 7.1, we conclude that m is odd. □

With the aid of Lemma 7.5, we may assume as in Remark 6.1.(4) that $\tau(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1}$ for any $x \in G$ with ε equal to I_n or $\mathrm{diag}(\varpi_E, \dots, \varpi_E)$, representing the two classes of unitary involutions. For $(\boldsymbol{J}, \Lambda)$, a simple type as in Remark 6.1.(2), we may write $\Lambda \cong \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ as before. Using Proposition 6.24, we may further assume $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}$. Using Lemma 6.18 with $g = 1$, there exists a quadratic character $\chi : \boldsymbol{J} \cap G^\tau \to R^\times$ such that

$$\dim_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, \chi^{-1}) = 1$$

and

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, \chi^{-1}) \otimes_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\rho}, \chi).$$

We want to show that $\chi = 1$. First, we need the following lemma:

Lemma 7.6. — *The character χ can be extended to a character χ' of $\boldsymbol{J}$.*

Proof. — Using Lemma 4.2, we have $\boldsymbol{J} \cap G^\tau = J \cap G^\tau$. Write $\overline{\chi}$ the character of $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0) \cong J \cap G^\tau / J^1 \cap G^\tau$, whose inflation equals χ. Since it is well known that the derived subgroup of $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$ is $\mathrm{SU}_m(\boldsymbol{l}/\boldsymbol{l}_0) := \{g \in \mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0) | \det(g) = 1\}$ (see [13], II. §5), there exists $\overline{\phi}$ as a quadratic character of $\det(\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)) = \{x \in \boldsymbol{l}^\times | x\sigma(x) = x^{Q+1} = 1\}$, such that $\overline{\chi} = \overline{\phi} \circ \det|_{\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)}$. We extend $\overline{\phi}$ to a character of $\boldsymbol{l}^\times$ and we write $\overline{\chi}' = \overline{\phi} \circ \det$, which is a character of $\mathrm{GL}_m(\boldsymbol{l})$ extending $\overline{\chi}$. Write χ'^0 the inflation of $\overline{\chi}'$ with respect to the isomorphism $\mathrm{GL}_m(\boldsymbol{l}) \cong J/J^1$. Finally, we choose χ' to be a character of $\boldsymbol{J}$ extending χ'^0 by choosing $\chi'(\varpi_E) \neq 0$ randomly. By construction, $\chi'|_{\boldsymbol{J}\cap G^\tau} = \chi$. □

PROPOSITION 7.7. — (1) *When* $\mathrm{char}(R) = 0$, *for any* χ' *extending* χ *to* $\boldsymbol{J}$, *we have* $\chi'(\chi' \circ \tau) = 1$.

(2) *Furthermore, for any* R, *we have* $\chi = 1$.

Proof. — First, we consider $\mathrm{char}(R) = 0$. Since m is odd, Lemma 7.1 implies that $\mathrm{GL}_m(\boldsymbol{l})$ possesses a σ-invariant supercuspidal representation $\overline{\rho'}$. Using Proposition 7.4 we get $\overline{\rho'}^{\tau\vee} \cong \overline{\rho'}$. We denote by $\boldsymbol{\rho}'$ a representation of $\boldsymbol{J}$ trivial on J^1, such that its restriction to J is the inflation of $\overline{\rho'}$. Since $\sigma(\varpi_E) = \varpi_E$, we have $\boldsymbol{\rho}'(\tau(\varpi_E)) = \boldsymbol{\rho}'(\varpi_E)^{-1}$ which means that $\boldsymbol{\rho}'$ is τ-self-dual. By Lemma 7.2 it is also distinguished.

Let Λ' denote the τ-self-dual simple type $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}'$. The natural isomorphism

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda', \chi^{-1}) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, \chi^{-1}) \otimes_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\rho}', 1)$$

shows that Λ' is χ^{-1}-distinguished.

By Lemma 7.6, there exists a character χ' extending χ. The representation $\Lambda'' = \Lambda'\chi'$ is thus a distinguished simple type. Let π'' be the supercuspidal representation of G compactly induced by $(\boldsymbol{J}, \Lambda'')$. It is distinguished, thus τ-self-dual by Theorem 4.1 and Proposition 5.6. Since Λ'' and $\Lambda''^{\tau\vee} \cong \Lambda''\chi'^{-1}(\chi'^{-1} \circ \tau)$ are both contained in π'', it follows that $\chi'(\chi' \circ \tau)$ is trivial.

We write $\overline{\chi} = \overline{\phi} \circ \det$ as in the proof of Lemma 7.6. Since $\chi'(\chi' \circ \tau) = 1$, we get $\overline{\phi}(\overline{\phi \circ \sigma})^{-1} = \overline{\phi}^{1-Q} = 1$. Choose $\zeta_{\boldsymbol{l}}$ to be a primitive root of $\boldsymbol{l}^\times$; then $\zeta_{\boldsymbol{l}}^{Q-1}$ generates the group $\det(\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)) = \{x \in \boldsymbol{l}^\times | x\sigma(x) = x^{Q+1} = 1\}$. Since $\overline{\phi}(\zeta_{\boldsymbol{l}}^{1-Q}) = 1$, we deduce that $\overline{\phi}|_{\det(\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0))}$ is trivial, which means that $\overline{\chi}$ is trivial. Thus, χ as the inflation of $\overline{\chi}$ is also trivial.

Now we consider $R = \overline{\mathbb{F}_l}$. As already mentioned in the proof of Proposition 6.26, if we denote by $\widetilde{\boldsymbol{\kappa}}$ the $\overline{\mathbb{Q}_l}$-lift of $\boldsymbol{\kappa}$ and if we denote by $\widetilde{\chi}$ the character defined by Lemma 6.18.(1) with respect to $\widetilde{\boldsymbol{\kappa}}$ and $\widetilde{\eta}$, then $\widetilde{\chi}$ is a $\overline{\mathbb{Q}_l}$-lift of χ. Using the characteristic 0 case that we already proved, we get $\widetilde{\chi} = 1$, which implies that $\chi = 1$.

When R $= l > 0$ in general, we follow the same logic as in the proof of Proposition 6.26. □

REMARK 7.8. — In fact, in Proposition 7.7, we proved that when m is odd, and E/E_0 is unramified, any τ-self-dual $\boldsymbol{\kappa}$ constructed in Proposition 6.24 as an extension of a $J^1 \cap G^\tau$-distinguished Heisenberg representation η is $\boldsymbol{J} \cap G^\tau$-distinguished.

Now we come back to the proof of our main theorem. We have

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, 1) \otimes_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\rho}, 1),$$

where $\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, 1)$ is of dimension 1, and $\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\rho}, 1) \cong \mathrm{Hom}_{\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)}(\overline{\rho}, 1)$ is also of dimension 1 by Lemma 4.2, Lemma 7.2 and Proposition 7.4. So, $\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1)$ is of dimension 1, which implies that π is G^τ-distinguished.

Thus, we finish the proof of Theorem 1.1 when E/E_0 is unramified. Using Corollary 6.29 and the fact that m is odd we deduce that $\mathrm{Hom}_{G^\tau}(\pi, 1)$ is of dimension 1, which finishes the proof of Theorem 1.2 when E/E_0 is unramified.

8. The supercuspidal ramified case

In this section, we study the distinction of σ-invariant supercuspidal representations of G in the case where E/E_0 is ramified. This finishes the proof of our main theorem.

8.1. The finite field case. — Let $\boldsymbol{l}$ be a finite field of characteristic $p \neq 2$ and let $|\boldsymbol{l}| = Q$. For m a positive integer, we denote by $\mathbf{G}$ the reductive group GL_m over $\boldsymbol{l}$. Thus, by definition, $\mathbf{G}(\boldsymbol{l}) = \mathrm{GL}_m(\boldsymbol{l})$. For $\overline{\varepsilon}$ a matrix in $\mathbf{G}(\boldsymbol{l})$ such that ${}^t\overline{\varepsilon} = \overline{\varepsilon}$, the automorphism defined by $\tau(x) = \overline{\varepsilon}\,{}^t x^{-1} \overline{\varepsilon}^{-1}$, for any $x \in \mathrm{GL}_m(\boldsymbol{l})$, gives an involution on $\mathrm{GL}_m(\boldsymbol{l})$, which induces an involution on $\mathbf{G}$. Thus, $\mathbf{G}^\tau$ is the orthogonal group corresponding to τ, which is a reductive group over $\boldsymbol{l}$, and $\mathbf{G}^\tau(\boldsymbol{l}) = \mathrm{GL}_m(\boldsymbol{l})^\tau$, which is a subgroup of $\mathrm{GL}_m(\boldsymbol{l})$. In this subsection, for $\overline{\rho}$ a supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ and $\overline{\chi}$ a character of $\mathrm{GL}_m(\boldsymbol{l})^\tau$, we state the result mentioned in [18], which gives a criterion for $\overline{\rho}$ distinguished by $\overline{\chi}$.

First of all, we assume $R = \overline{\mathbb{Q}_l}$. We recall a little bit of Deligne–Lusztig theory (see [12]). Let $\mathbf{T}$ be an elliptic maximal $\boldsymbol{l}$-torus in $\mathbf{G}$, where ellipticity means that $\mathbf{T}(\boldsymbol{l}) = \boldsymbol{t}^\times$ and $\boldsymbol{t}/\boldsymbol{l}$ is the field extension of degree m. Let ξ be a regular character of $\mathbf{T}(\boldsymbol{l})$, where regularity means the same as in the construction of Green and James in §7.1. Using [12], Theorem 8.3, there is a virtual character $R_{\mathbf{T},\xi}$ as the character of a cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$. Moreover, if we fix $\mathbf{T}$, we know that $\xi \mapsto R_{\mathbf{T},\xi}$ gives a bijection from the set of Galois orbits of regular characters of $\mathbf{T}$ to the set of cuspidal representations of $\mathrm{GL}_m(\boldsymbol{l})$. So we may choose ξ such that $\mathrm{Trace}(\overline{\rho}) = R_{\mathbf{T},\xi}$. Moreover, using [12], Theorem 4.2, we get $R_{\mathbf{T},\xi}(-1) = \dim(\overline{\rho})\xi(-1)$ with $\dim(\overline{\rho}) = (Q-1)(Q^2-1)\dots(Q^{m-1}-1)$. So if we denote by $\omega_{\overline{\rho}}$ the central character of $\overline{\rho}$, we get $\omega_{\overline{\rho}}(-1) = \xi(-1)$.

Proposition 8.1 ([18], Proposition 6.7). — *For τ, $\overline{\rho}$, $\boldsymbol{T}$ and ξ above, we have:*

$$\dim_R(\mathrm{Hom}_{\mathbf{G}^\tau(\boldsymbol{l})}(\overline{\rho}, \overline{\chi})) = \begin{cases} 1 & \text{if } \omega_{\overline{\rho}}(-1) = \xi(-1) = \overline{\chi}(-1), \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the l-modular case and assume $\mathrm{char}(R) = l > 0$.

Proposition 8.2. — *For τ above and $\overline{\rho}$ a supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ over R, the space $\mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l})^\tau}(\overline{\rho}, \overline{\chi}) \neq 0$ if and only if $\omega_{\overline{\rho}}(-1) = \overline{\chi}(-1)$. Moreover, if the condition is satisfied, then we have $\dim_R(\mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l})^\tau}(\overline{\rho}, \overline{\chi})) = 1$.*

Proof. — First, we assume $R = \overline{\mathbb{F}_l}$. We use a similar proof to that in Lemma 7.2. Let $H = \mathrm{GL}_m(\boldsymbol{l})^\tau$. We choose $\widetilde{\overline{\chi}}$ to be a character of H lifting $\overline{\chi}$, which is defined over $\overline{\mathbb{Z}_l}$ or $\overline{\mathbb{Q}_l}$ by abuse of notation. For $S = \overline{\mathbb{Z}_l}, \overline{\mathbb{Q}_l}$, we define

$$S[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}} := \big\{ f \mid f : \mathrm{GL}_m(\boldsymbol{l}) \to S, \\ f(hg) = \widetilde{\overline{\chi}}(h) f(g) \text{ for any } h \in H, g \in \mathrm{GL}_m(\boldsymbol{l}) \big\}.$$

Especially,

$$\overline{\mathbb{Q}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}} = \mathrm{Ind}_H^{\mathrm{GL}_m(\boldsymbol{l})} \widetilde{\overline{\chi}}$$

as a representation of $\mathrm{GL}_m(\boldsymbol{l})$ over $\overline{\mathbb{Q}_l}$, and $\overline{\mathbb{Z}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}$ is a free $\overline{\mathbb{Z}_l}$-module. If we further define

$$\overline{\mathbb{F}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\overline{\chi}} = \mathrm{Ind}_H^{\mathrm{GL}_m(\boldsymbol{l})} \overline{\chi},$$

then we have

$$\overline{\mathbb{Z}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l} = \overline{\mathbb{F}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\overline{\chi}}$$

and

$$\overline{\mathbb{Z}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{Q}_l} = \overline{\mathbb{Q}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}.$$

We deduce that

$$\begin{aligned} &\mathrm{Hom}_H(\overline{\rho}, \overline{\chi}) \neq 0 \\ &\quad \Longleftrightarrow \text{There exists } \widetilde{\overline{\rho}} \text{ lifting } \overline{\rho} \text{ such that} \\ &\quad\qquad \mathrm{Hom}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{\overline{\rho}}, \overline{\mathbb{Q}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}) \neq 0 \\ &\quad \Longleftrightarrow \text{There exists } \widetilde{\overline{\rho}} \text{ lifting } \overline{\rho} \text{ such that } \omega_{\widetilde{\overline{\rho}}}(-1) = \widetilde{\overline{\chi}}(-1) \\ &\quad \Longleftrightarrow \omega_{\overline{\rho}}(-1) = \overline{\chi}(-1). \end{aligned}$$

The first equivalence is of the same reason as in the proof of Lemma 7.2, and we use Proposition 8.1 for the second equivalence. For the last equivalence, the "$\Rightarrow$" direction is trivial. For the other direction, when $l \neq 2$, we choose $\widetilde{\overline{\rho}}$ to be any supercuspidal $\overline{\mathbb{Q}_l}$-lift of $\overline{\rho}$. Thus, we have $\omega_{\widetilde{\overline{\rho}}}(-1) = \omega_{\overline{\rho}}(-1) = \overline{\chi}(-1) = \widetilde{\overline{\chi}}(-1)$. When $l = 2$, using the construction of Green and James, for ξ a regular character over $\overline{\mathbb{F}_l}$ corresponding to $\overline{\rho}$, we may always find a $\overline{\mathbb{Q}_l}$-lift $\widetilde{\xi}$ that is regular and satisfies $\widetilde{\xi}(-1) = \widetilde{\overline{\chi}}(-1)$. Thus, the supercuspidal representation $\widetilde{\overline{\rho}}$ corresponding to $\widetilde{\xi}$ as a lift of $\overline{\rho}$ satisfies $\omega_{\widetilde{\overline{\rho}}}(-1) = \widetilde{\overline{\chi}}(-1)$. So we finish the proof of the first part.

To calculate the dimension, as in the proof of Lemma 7.2 if we write

$$\overline{\mathbb{F}_l}[H \setminus \mathrm{GL}_m(\boldsymbol{l})]_{\overline{\chi}} = V_{\overline{\rho}} \oplus V',$$

where $V_{\overline{\rho}}$ is composed of irreducible components isomorphic to $\overline{\rho}$, and V' has no irreducible component isomorphic to $\overline{\rho}$, then we only need to show that $\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}})$ is commutative. We consider the following $\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module decomposition

$$\overline{\mathbb{Z}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}=\widetilde{V_{\overline{\rho}}}\oplus \widetilde{V'},$$

where $\widetilde{V_{\overline{\rho}}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l}=\bigoplus_{\widetilde{\overline{\rho}}}\widetilde{\overline{\rho}}$ with the direct sum ranges over all the irreducible representations $\widetilde{\overline{\rho}}$ over $\overline{\mathbb{Q}_l}$ occurring in $\widetilde{P_{\overline{\rho}}}$ counting the multiplicity, and $\widetilde{V'}$ denotes a $\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-complement of $\widetilde{V_{\overline{\rho}}}$, such that $\widetilde{V'}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l}$ contains no irreducible component of $\widetilde{\overline{\rho}}$. Using Proposition 8.1, $\widetilde{V_{\overline{\rho}}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l}$ is multiplicity free, which means that $\mathrm{End}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{V_{\overline{\rho}}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l})$ is commutative. The canonical embedding from $\overline{\mathbb{Z}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}$ to $\overline{\mathbb{Q}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}$ induces the following ring monomorphism

$$\mathrm{End}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\mathbb{Z}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}})\hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\mathbb{Q}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\overline{\chi}})$$

given by tensoring $\overline{\mathbb{Q}_l}$, which leads to the ring monomorphism

$$\mathrm{End}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{V_{\overline{\rho}}})\hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{V_{\overline{\rho}}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l}).$$

Thus $\mathrm{End}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{V_{\overline{\rho}}})$ is also commutative.

The modulo l map from $\overline{\mathbb{Z}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}}$ to $\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\overline{\chi}}$ induces the following ring epimorphism

$$\mathrm{End}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\mathbb{Z}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\widetilde{\overline{\chi}}})\twoheadrightarrow \mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\mathbb{F}_l}[H\setminus \mathrm{GL}_m(\boldsymbol{l})]_{\overline{\chi}}),$$

which leads to the ring epimorphism

$$\mathrm{End}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{V_{\overline{\rho}}})\twoheadrightarrow \mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}}).$$

Since $\mathrm{End}_{\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{V_{\overline{\rho}}})$ is commutative, $\mathrm{End}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(V_{\overline{\rho}})$ is also commutative. Thus, we may use the same proof as in Lemma 7.2 to show that

$$\dim_{\overline{\mathbb{F}_l}}(\mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l})^\tau}(\overline{\rho},\overline{\chi}))=1.$$

Finally, for $\mathrm{char}(R)=l>0$ in general, we follow the corresponding proof in Lemma 7.2. □

Remark 8.3. — For $\mathbf{G}^\tau(\boldsymbol{l})$ an orthogonal group with $m\geq 2$, it is well known that its derived group is always a subgroup of $\mathbf{G}^{\tau 0}(\boldsymbol{l})$ of index 2 (see [13], II. §8), which means that there exists a character of $\mathbf{G}^\tau(\boldsymbol{l})$ that is not trivial on $\mathbf{G}^{\tau 0}(\boldsymbol{l})$. This means that we cannot expect $\overline{\chi}$ to be trivial on $\mathbf{G}^{\tau 0}(\boldsymbol{l})$ in general. However, for those $\overline{\chi}$ occurring in the next subsection, it is highly possible that $\overline{\chi}$ is trivial on $\mathbf{G}^{\tau 0}(\boldsymbol{l})$. For example, [18], Proposition 6.4 gives evidence for this in the case where π is tame supercuspidal. However, the author does not know how to prove it.

Now we assume that m is even. We write $J_{m/2} = \begin{pmatrix} 0 & I_{m/2} \\ -I_{m/2} & 0 \end{pmatrix}$ and we denote by

$$\mathrm{Sp}_m(\boldsymbol{l}) = \{x \in \mathrm{GL}_m(\boldsymbol{l}) |\, {}^t x J_{m/2} x = J_{m/2}\}$$

the symplectic subgroup of $\mathrm{GL}_m(\boldsymbol{l})$.

Proposition 8.4. — *For $\overline{\rho}$, a cuspidal representation of* $\mathrm{GL}_m(\boldsymbol{l})$*, we have* $\mathrm{Hom}_{\mathrm{Sp}_m(\boldsymbol{l})}(\overline{\rho}, 1) = 0$.

Proof. — Using [31], Corollary 1.4., whose proof also works for the l-modular case, we know that an irreducible generic representation cannot be distinguished by a symplectic subgroup. Since a cuspidal representation is generic, we finish the proof. □

8.2. Distinction criterion in the ramified case. — Still let π be a σ-invariant supercuspidal representation of G. In this subsection, we prove Theorem 1.1 and Theorem 1.2 in the case where E/E_0 is ramified. Using Theorem 4.1, we only need to show that π is distinguished by any unitary subgroup G^τ to finish the proof of Theorem 1.1. We may change τ up to a G-action, which does not change the property of being distinguished. Thus, using Remark 6.1.(4), we may assume $\tau(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1}$, for any $x \in G$, where ε equals I_n or $\mathrm{diag}(I_d, \ldots, I_d, \epsilon)$ with $\epsilon \in \mathfrak{o}_{E_0}^\times - \mathrm{N}_{E/E_0}(\mathfrak{o}_E^\times)$, representing the two classes of unitary involutions. We denote by $\overline{\varepsilon}$ the image of ε in $\mathrm{GL}_m(\boldsymbol{l})$.

For $(\boldsymbol{J}, \Lambda)$ a simple type in Remark 6.1.(2), we write $\Lambda \cong \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. Using Proposition 6.24, we may further assume $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}$. Using Lemma 6.18 with $g = 1$, there exists a quadratic character $\chi : \boldsymbol{J} \cap G^\tau \to R^\times$ such that

$$\dim_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, \chi^{-1}) = 1 \tag{24}$$

and

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\kappa}, \chi^{-1}) \otimes_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\rho}, \chi). \tag{25}$$

If we denote by $\omega_{\boldsymbol{\kappa}}$ the central character of $\boldsymbol{\kappa}$ defined on $F^\times$, using (24), we get $\omega_{\boldsymbol{\kappa}} = \chi^{-1}$ as characters of $F^\times \cap (\boldsymbol{J} \cap G^\tau)$. In particular, $\omega_{\boldsymbol{\kappa}}(-1) = \chi^{-1}(-1)$. Since $\boldsymbol{\kappa}^{\tau\vee} \cong \boldsymbol{\kappa}$, we get $\omega_{\boldsymbol{\kappa}} \circ \tau = \omega_{\boldsymbol{\kappa}}^{-1}$. In particular, we have

$$\omega_{\boldsymbol{\kappa}}(\varpi_F)^{-1} = \omega_{\boldsymbol{\kappa}}(\tau(\varpi_F)) = \omega_{\boldsymbol{\kappa}}(\varpi_F)^{-1}\omega_{\boldsymbol{\kappa}}(-1)^{-1},$$

where we use the fact that $\sigma(\varpi_F) = -\varpi_F$. Thus, we get $\omega_{\boldsymbol{\kappa}}(-1) = \chi(-1) = 1$.

Since Λ and $\boldsymbol{\kappa}$ are τ-self-dual, $\boldsymbol{\rho}$ is τ-self-dual. Using the same proof as that for $\boldsymbol{\kappa}$, we get $\omega_{\boldsymbol{\rho}}(-1) = 1$. Let $\overline{\rho}$ be the supercuspidal representation of $\mathrm{GL_m}(\boldsymbol{l}) \cong J/J^1$ whose inflation equals $\boldsymbol{\rho}|_J$ and let $\overline{\chi}$ be the character of

$$\mathbf{G}^\tau(\boldsymbol{l}) \cong J \cap G^\tau / J^1 \cap G^\tau$$

whose inflation equals χ, where τ naturally induces an orthogonal involution on $\mathbf{G}$ with respect to a symmetric matrix $\overline{\varepsilon} \in \mathrm{GL}_m(\boldsymbol{l})$. By definition and Lemma 4.2 we get

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\boldsymbol{\rho}, \chi) \cong \mathrm{Hom}_{\mathbf{G}^\tau(\boldsymbol{l})}(\overline{\rho}, \overline{\chi}).$$

Since $\omega_{\overline{\rho}}(-1) = \overline{\chi}(-1) = 1$, using Proposition 8.1 and Proposition 8.2 the space above is non-zero. Thus, by (25) we have

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) \neq 0,$$

which means that π is distinguished by G^τ, finishing the proof of Theorem 1.1. Moreover, using Proposition 8.1, Proposition 8.2, (24) and (25), we get

$$\dim_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) = 1.$$

Now, if m is even, and $\varepsilon = I_m$, we also need to study the space $\mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\Lambda^{g_1}, 1)$, where g_1 is defined in Corollary 6.29, such that $\tau(g_1)g_1^{-1} = \varpi_E J_{m/2} \in B^\times$. Using Lemma 6.18, there exists a quadratic character $\chi_1 : \boldsymbol{J}^{g_1} \cap G^\tau \to R^\times$ such that

$$\dim_R \mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\boldsymbol{\kappa}^{g_1}, \chi_1^{-1}) = 1 \tag{26}$$

and

$$\mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\Lambda^{g_1}, 1) \cong \mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\boldsymbol{\kappa}^{g_1}, \chi_1^{-1}) \otimes_R \mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\boldsymbol{\rho}^{g_1}, \chi_1). \tag{27}$$

So we only need to study the space $\mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\boldsymbol{\rho}^{g_1}, \chi_1) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^{\delta_{g_1}}}(\boldsymbol{\rho}, \chi_1^{g_1^{-1}})$, where

$$\delta_{g_1}(x) := (\tau(g_1)g_1^{-1})^{-1}\tau(x)(\tau(g_1)g_1^{-1}) = (\varpi_E J_{m/2})^{-1}\tau(x)\varpi_E J_{m/2},$$

for any $x \in G$ as an involution on G.

Let $\overline{\rho}$ be the supercuspidal representation of $\mathrm{GL_m}(\boldsymbol{l}) \cong J/J^1$ whose inflation equals $\boldsymbol{\rho}|_J$ and let $\overline{\chi_1^{g_1^{-1}}}$ be the character of

$$\mathrm{Sp}_m(\boldsymbol{l}) \cong J \cap G^{\delta_{g_1}} / J^1 \cap G^{\delta_{g_1}}$$

whose inflation equals $\chi_1^{g_1^{-1}}$; then we get

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^{\delta_{g_1}}}(\boldsymbol{\rho}, \chi_1^{g_1^{-1}}) \cong \mathrm{Hom}_{\mathrm{Sp}_m(\boldsymbol{l})}(\overline{\rho}, \overline{\chi_1^{g_1^{-1}}}) = \mathrm{Hom}_{\mathrm{Sp}_m(\boldsymbol{l})}(\overline{\rho}, 1),$$

where the last equation is because of the well-known fact that $\mathrm{Sp}_m(\boldsymbol{l})$ equals its derived group ([13], II. §8), thus $\overline{\chi_1^{g_1^{-1}}}|_{\mathrm{Sp}_m(\boldsymbol{l})}$ is trivial. Using Proposition 8.4, we get $\mathrm{Hom}_{\mathrm{Sp}_m(\boldsymbol{l})}(\overline{\rho}, 1) = 0$. Thus, $\mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\Lambda^{g_1}, 1) = 0$.

Using Corollary 6.29, we get

$$\dim_R \mathrm{Hom}_{G^\tau}(\pi, 1) = \dim_R \mathrm{Hom}_{\boldsymbol{J}\cap G^\tau}(\Lambda, 1) + \dim_R \mathrm{Hom}_{\boldsymbol{J}^{g_1}\cap G^\tau}(\Lambda^{g_1}, 1) = 1,$$

which finishes the proof of Theorem 1.2 when E/E_0 is ramified.

8.3. Proof of Theorem 1.3. — We finish the proof of Theorem 1.3. Let π be a σ-invariant supercuspidal representation of G over $\overline{\mathbb{F}_l}$. For τ a unitary involution, by Theorem 1.1, π is distinguished by G^τ. From the proof of Theorem 4.1, there exists a distinguished integral σ-invariant supercuspidal representation $\widetilde{\pi}$ of G over $\overline{\mathbb{Q}_l}$, which lifts π.

9. A purely local proof of Theorem 4.1

In this section, we generalize Theorem 4.1 to irreducible cuspidal representations, meanwhile also giving another proof of the original theorem, which is purely local. Precisely, we prove the following theorem:

THEOREM 9.1. — *Let π be an irreducible cuspidal representation of G over R. If π is distinguished by G^τ, then π is σ-invariant.*

9.1. The finite analogue. —

PROPOSITION 9.2. — *Let $\boldsymbol{l}/\boldsymbol{l}_0$ be a quadratic extension of finite fields of characteristic p and let $\overline{\rho}$ be an irreducible generic representation of $\mathrm{GL}_m(\boldsymbol{l})$ over R. If $\overline{\rho}$ is distinguished by the unitary subgroup H of $\mathrm{GL}_m(\boldsymbol{l})$ with respect to $\boldsymbol{l}/\boldsymbol{l_0}$, then it is σ-invariant.*

Proof. — When $\mathrm{char}(R)=0$, the proposition was proved by Gow [16] for any irreducible representations. So we only consider the l-modular case and without loss of generality we assume $R=\overline{\mathbb{F}_l}$. We write $P_{\overline{\rho}}$ for the projective envelope of $\overline{\rho}$ as a $\overline{\mathbb{Z}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module. Thus, $P_{\overline{\rho}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{F}_l}$ is a projective $\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]$-module, and moreover,

$$\mathrm{Hom}_{\overline{\mathbb{F}_l}[H]}(\overline{\rho},\overline{\mathbb{F}_l})\cong\mathrm{Hom}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\overline{\rho},\overline{\mathbb{F}_l}[H\setminus\mathrm{GL}_m(\boldsymbol{l})])\neq 0$$

implies that

$$\mathrm{Hom}_{\overline{\mathbb{F}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(P_{\overline{\rho}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{F}_l},\overline{\mathbb{F}_l}[H\setminus\mathrm{GL}_m(\boldsymbol{l})])\neq 0.$$

Using the same argument as that in Lemma 7.2, we have

$$\mathrm{Hom}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(P_{\overline{\rho}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l},\overline{\mathbb{Q}_l}[H\setminus\mathrm{GL}_m(\boldsymbol{l})])\neq 0,$$

and, thus, there exists an irreducible constituent $\widetilde{\overline{\rho}}$ of $P_{\overline{\rho}}\otimes_{\overline{\mathbb{Z}_l}}\overline{\mathbb{Q}_l}$ such that

$$\mathrm{Hom}_{\overline{\mathbb{Q}_l}[\mathrm{GL}_m(\boldsymbol{l})]}(\widetilde{\overline{\rho}},\overline{\mathbb{Q}_l}[H\setminus\mathrm{GL}_m(\boldsymbol{l})])\neq 0.$$

By [39], §14.5, §15.4, $\overline{\rho}$ is a constituent of $r_l(\widetilde{\overline{\rho}})$. Since $\widetilde{\overline{\rho}}$ is H-distinguished, it is σ-invariant and so is $r_l(\widetilde{\overline{\rho}})$. For $i=1,\ldots,k$, we choose $\widetilde{\overline{\rho}}_i$ to be a cuspidal representation of $\mathrm{GL}_{m_i}(\boldsymbol{l})$ over $\overline{\mathbb{Q}_l}$, such that $\widetilde{\overline{\rho}}$ is a sub-representation of the parabolic induction $\widetilde{\overline{\rho}}_1\times\ldots\times\widetilde{\overline{\rho}}_k$, where $m_1+\ldots+m_k=m$. For each i, we write $\overline{\rho}_i=r_l(\widetilde{\overline{\rho}}_i)$, which is a cuspidal representation of $\mathrm{GL}_{m_i}(\boldsymbol{l})$ over $\overline{\mathbb{F}_l}$, and then all the irreducible constituents of $r_l(\widetilde{\overline{\rho}})$ are subquotients of $\overline{\rho}_1\times\ldots\times\overline{\rho}_k$, and in particular so is $\overline{\rho}$. Since $\overline{\rho}$ is generic (or non-degenerate), by [43], Chapitre III, 1.10, it is the unique non-degenerate subquotient contained in $\overline{\rho}_1\times\ldots\times\overline{\rho}_k$, thus, it is the unique non-degenerate constituent in $r_l(\widetilde{\overline{\rho}})$. Thus, it is σ-invariant. □

9.2. The cuspidal case. — In this subsection, we prove Theorem 9.1. We choose $(\boldsymbol{J},\Lambda)$ to be a simple type of π, and then by Frobenius reciprocity and the Mackey formula, there exists $g\in G$ such that

$$\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g,1)\neq 0. \tag{28}$$

Let H^1 be the corresponding subgroup of $\boldsymbol{J}$, let θ be the simple character of H^1 contained in Λ and let η be the Heisenberg representation of θ. Restricting (28) to $H^{1g}\cap G^\tau$ we get $\theta^g|_{H^{1g}\cap G^\tau}=1$. Following the proof of [35], Lemma 6.5, we have

$$(\theta\circ\tau)^{\tau(g)}|_{\tau(H^{1g})\cap H^{1g}}=\theta^g\circ\tau|_{\tau(H^{1g})\cap H^{1g}}=(\theta^g)^{-1}|_{\tau(H^{1g})\cap H^{1g}}, \tag{29}$$

or in other words, $\theta\circ\tau$ intertwines with θ^{-1}. Using the intertwining theorem (*cf.* [7]), $\theta\circ\tau$ and θ^{-1} are endo-equivalent, which, from the argument of Lemma 5.7, is equivalent to $\Theta^\sigma=\Theta$, where Θ denotes the endo-class of θ.

We let τ_1 be the unitary involution corresponding to I_n, which in particular satisfies the condition of Theorem 5.5. Since $\Theta^\sigma=\Theta$, by *loc. cit.*, we may choose a simple stratum $[\mathfrak{a},\beta]$ and $\theta'\in\mathcal{C}(\mathfrak{a},\beta)$ with $\theta'\in\Theta$, such that

$$\tau_1(\beta)=\beta^{-1},\quad \tau_1(\mathfrak{a})=\mathfrak{a}\quad\text{and}\quad \theta'\circ\tau_1=\theta'^{-1}.$$

Up to G-conjugacy, we may and will assume that $\boldsymbol{J}=\boldsymbol{J}(\mathfrak{a},\beta)$ and $\theta'=\theta$. We write $E=F[\beta]$ and $B\cong\mathrm{M}_m(E)$ for the centralizer of E in $\mathrm{M}_n(F)$. Using Proposition 6.24, we write $\Lambda=\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ with $\boldsymbol{\kappa}$ an extension of the Heisenberg representation η such that $\boldsymbol{\kappa}^{\tau_1}\cong\boldsymbol{\kappa}^\vee$. Let ε be a hermitian matrix such that $\tau(x)=\varepsilon\sigma({}^tx^{-1})\varepsilon^{-1}=\tau_1(x)^{\varepsilon^{-1}}$ for any $x\in G$. For a fixed $g\in G$, we define $\gamma=\varepsilon^{-1}\tau(g)g^{-1}=\tau_1(g)\varepsilon^{-1}g^{-1}$ and by direct calculation we have $\tau_1(\gamma)=\gamma$.

Proposition 9.3. — *Let $g\in G$ such that* $\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g,1)\neq 0$.

(1) *Changing g by another representative in the same $\boldsymbol{J}$-G^τ double coset, we may assume $\gamma\in B^\times$.*
(2) *The dimension* $\dim_R\mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1)=1$*;*
(3) *There is a unique quadratic character χ of $\boldsymbol{J}^g\cap G^\tau$ trivial on $J^{1g}\cap G^\tau$, such that*

$$\mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1)\cong\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\kappa}^g,\chi^{-1})\cong R.$$

Moreover,

$$\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g,1)\cong\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\kappa}^g,\chi^{-1})\otimes\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\rho}^g,\chi).$$

(4) *The element $\gamma\in\boldsymbol{J}$, thus under the assumption of (1), $\gamma\in B^\times\cap\boldsymbol{J}=E^\times\mathfrak{b}^\times$.*

Proof. — We sketch the proof that follows from that of Theorem 6.2 (actually, we have the same theorem if $\tau = \tau_1$). Using (29) and the fact that $\tau(H^{1g}) = \tau_1(H^1)^{\varepsilon^{-1}\tau(g)} = H^{1\varepsilon^{-1}\tau(g)}$ and $(\theta\circ\tau)^{\tau(g)} = (\theta\circ\tau_1)^{\varepsilon^{-1}\tau(g)} = (\theta^{-1})^{\varepsilon^{-1}\tau(g)}$ we have

$$\begin{aligned}(\theta^{\varepsilon^{-1}\tau(g)})^{-1}|_{H^{1\varepsilon^{-1}\tau(g)}\cap H^{1g}} &= (\theta\circ\tau)^{\tau(g)}|_{\tau(H^{1g})\cap H^{1g}}\\ &= \theta^g\circ\tau|_{\tau(H^{1g})\cap H^{1g}} = (\theta^g)^{-1}|_{H^{1\varepsilon^{-1}\tau(g)}\cap H^{1g}},\end{aligned}$$

which means that γ intertwines θ, or in other words, $\gamma\in JB^\times J$. The following lemma follows from the same proof of Lemma 6.5, once we replace γ there with our γ here and τ there with τ_1.

Lemma 9.4. — *There exist $y\in J = J(\mathfrak{a},\beta)$ and $b\in B^\times$, such that $\gamma = \tau_1(y)by$.*

Thus, we change g by $y^{-1}g$ and then the corresponding $\gamma = b\in B^\times$, which proves (1). For (2), we write

$$\delta(x) := (\tau(g)g^{-1})^{-1}\tau(x)\tau(g)g^{-1} = \gamma^{-1}\tau_1(x)\gamma \quad \text{for any } x\in G$$

an involution on G, and then by definition we have

$$\mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1)\cong\mathrm{Hom}_{J^1\cap G^\delta}(\eta,1),$$

and

$$\gamma\delta(\gamma) = \gamma\gamma^{-1}\tau_1(\gamma)\gamma = 1.$$

Moreover, by direct calculation we have

$$\begin{aligned}\delta(H^1) &= (\tau(g)g^{-1})^{-1}H^{1\varepsilon^{-1}}\tau(g)g^{-1} = H^{1\gamma} \quad \text{and}\\ \theta\circ\delta &= (\theta^{-1})^{\varepsilon^{-1}\tau(g)g^{-1}} = (\theta^{-1})^\gamma.\end{aligned}$$

So using Proposition 6.14, we finish the proof of (2).

Using (2) and the same argument of Proposition 6.18 we get the statement (3), except the part χ being quadratic. To finish that part, since

$$\tau_1(\tau_1(g)\varepsilon^{-1})\varepsilon^{-1}(\tau_1(g)\varepsilon^{-1})^{-1} = g\varepsilon\tau_1(g)^{-1} = (\tau_1(g)\varepsilon^{-1}g^{-1})^{-1} = \gamma^{-1}\in B^\times,$$

we may replace g with $\varepsilon^{-1}\tau(g) = \tau_1(g)\varepsilon^{-1}$ in the statement (3) to get a unique character χ' of $\boldsymbol{J}^{\varepsilon^{-1}\tau(g)}\cap G^\tau$ trivial on $J^{1\varepsilon^{-1}\tau(g)}\cap G^\tau$. Moreover, using the facts $\tau(\boldsymbol{J}) = \boldsymbol{J}^{\varepsilon^{-1}}$, $\tau(J) = J^{\varepsilon^{-1}}$, $\tau(J^1) = J^{1\varepsilon^{-1}}$ and $\tau(H^1) = H^{1\varepsilon^{-1}}$ and Lemma 4.2 it is easy to show that

$$\boldsymbol{J}^g\cap G^\tau = \boldsymbol{J}^{\varepsilon^{-1}\tau(g)}\cap G^\tau = J^g\cap G^\tau = J^{\varepsilon^{-1}\tau(g)}\cap G^\tau. \tag{30}$$

As a result, χ and χ' are characters defined on the same group $\boldsymbol{J}^g\cap G^\tau = \boldsymbol{J}^{\varepsilon^{-1}\tau(g)}\cap G^\tau$. We have the following lemma similar to Proposition 6.19:

Lemma 9.5. — *We have $\chi = \chi'$.*

Proof. — We write δ for the involution defined as above. By §3.2, we have $\gamma \in I_G(\eta) = I_G(\kappa^0)$ and

$$\dim_R(\mathrm{Hom}_{J\cap J^\gamma}(\kappa^{0\gamma},\kappa^0)) = \dim_R(\mathrm{Hom}_{J^1\cap J^{1\gamma}}(\eta^\gamma,\eta)) = 1,$$

where $\kappa^0 = \boldsymbol{\kappa}|_J$. By direct calculation, we have $J^1 \cap G^\delta = J^{1\gamma} \cap G^\delta$ as a subgroup of $J^1 \cap J^{1\gamma}$ and $H^1 \cap G^\delta = H^{1\gamma} \cap G^\delta$. Using statement (2) for g and $\varepsilon^{-1}\tau(g)$, respectively, we get

$$\dim_R \mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1) = \dim_R \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)}\cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)},1) = 1.$$

By Proposition 6.20, for

$$0 \neq \varphi \in \mathrm{Hom}_{J^1\cap J^{1\gamma}}(\eta^\gamma,\eta) = \mathrm{Hom}_{J^{1g}\cap J^{1\varepsilon^{-1}\tau(g)}}(\eta^{\varepsilon^{-1}\tau(g)},\eta^g),$$

the map

$$\begin{aligned} f_\varphi : \mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1) &\to \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)}\cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)},1), \\ \lambda &\mapsto \lambda\circ\varphi \end{aligned}$$

is bijective[7]. If we choose

$$\begin{aligned} &0 \neq \lambda \in \mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1) \quad \text{and} \\ &0 \neq \lambda' := f_\varphi(\lambda) = \lambda\circ\varphi \in \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)}\cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)},1), \end{aligned}$$

then for any v in the representation space of η and any $x \in J^g \cap G^\tau = J^{\varepsilon^{-1}\tau(g)} \cap G^\tau$, using a similar argument to (15) we have

$$\chi'(x)^{-1}\lambda'(v) = \chi(x)^{-1}\lambda'(v).$$

Since v and $x \in J^g \cap G^\tau = J^{\varepsilon^{-1}\tau(g)} \cap G^\tau$ are arbitrary, we have $\chi'|_{J^{\varepsilon^{-1}\tau(g)}\cap G^\tau} = \chi|_{J^g\cap G^\tau}$. Combining this with (30) we finish the proof of the lemma. □

To prove that χ is quadratic, we first assume that $\mathrm{char}(R) = 0$. Using a similar argument to (21) we have the following isomorphism

$$\mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)}\cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)},1) \cong \mathrm{Hom}_{\boldsymbol{J}^{\varepsilon^{-1}\tau(g)}\cap G^\tau}(\boldsymbol{\kappa}^{\varepsilon^{-1}\tau(g)},\chi\circ\tau).$$

Using the above lemma and the uniqueness of χ', we have $\chi\circ\tau = \chi^{-1}$. Since χ is defined on $\boldsymbol{J}^g \cap G^\tau = J^g \cap G^\tau$, which is τ-invariant, we have $\chi\circ\tau = \chi$, and thus $\chi^2 = \chi(\chi\circ\tau) = 1$. When $\mathrm{char}(R) = l > 0$, the same argument in Proposition 6.26 can be used directly.

Finally, using (3) and the distinction of the simple type, we have

$$\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\rho}^g,\chi) \neq 0.$$

7. Noting that $J^{1g} \cap G^\tau = (J^1 \cap G^\delta)^g$ and $J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau = (J^{1\gamma} \cap G^\delta)^g$, thus

$$\mathrm{Hom}_{J^{1g}\cap G^\tau}(\eta^g,1) = \mathrm{Hom}_{J^1\cap G^\delta}(\eta,1) \quad \text{and}$$

$$\mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)}\cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)},1) = \mathrm{Hom}_{J^{1\gamma}\cap G^\delta}(\eta^\gamma,1).$$

Then the proof of (4) is the same as that in §6.6, once we replace γ there with our γ here. □

Corollary 9.6. — *For $g \in G$ such that $\mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\Lambda^g, 1) \neq 0$, we may change g by another representative in the same $\boldsymbol{J}$-G^τ double coset, such that*

$$\gamma = \begin{cases} I_m \text{ or } \varpi_E I_m & \text{if } E/E_0 \text{ is unramified;} \\ I_m \text{ or } \mathrm{diag}(1, \ldots, 1, \epsilon) \text{ or } \varpi_E J_{m/2} & \text{if } E/E_0 \text{ is ramified,} \end{cases}$$

as an element in $\mathrm{GL}_m(E) \cong B^\times \hookrightarrow G$, where $\epsilon \in \mathfrak{o}_{E_0}^\times - \mathrm{N}_{E/E_0}(\mathfrak{o}_E^\times)$

Proof. — We have proved that $\gamma = \tau_1(g)\varepsilon^{-1}g^{-1} \in B^\times \cap \boldsymbol{J} = E^\times\mathfrak{b}^\times$. Changing g up to multiplying by an element in $E^\times$, which does not change the double coset it represents, we may assume $\gamma \in \mathfrak{b}^\times$ or $\varpi_E\mathfrak{b}^\times$. Using Proposition 2.2 and changing g up to multiplying by an element in $\mathfrak{b}^\times$ on the left, we may assume that $\gamma = \varpi_E^\alpha$, and from the uniqueness we must have $\varpi_E^\alpha = I_m$ or $\varpi_E I_m$ when E/E_0 is unramified, and $\varpi_E^\alpha = I_m$, or $\mathrm{diag}(1, \ldots, 1, \epsilon)$ or $\varpi_E J_{m/2}$ when E/E_0 is totally ramified. □

Thus, for $g \in G$ as above, we get

$$\mathrm{Hom}_{J\cap G^\delta}(\boldsymbol{\rho}|_J, \chi^{g^{-1}}) \cong \mathrm{Hom}_{\boldsymbol{J}\cap G^\delta}(\boldsymbol{\rho}, \chi^{g^{-1}}) \cong \mathrm{Hom}_{\boldsymbol{J}^g\cap G^\tau}(\boldsymbol{\rho}^g, \chi) \neq 0.$$

Ww write $H = J \cap G^\delta / J^1 \cap G^\delta$ for the subgroup of $\mathrm{GL}_m(\boldsymbol{l}) \cong J/J^1$, which, from the expression of γ in Corollary 9.6, is either a unitary subgroup, or an orthogonal subgroup, or a symplectic subgroup of $\mathrm{GL}_m(\boldsymbol{l})$. Moreover, we have

$$\mathrm{Hom}_H(\overline{\rho}, \overline{\chi'}) \neq 0,$$

where $\overline{\rho}$ is a cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ whose inflation is $\boldsymbol{\rho}|_J$ and $\overline{\chi'}$ is a quadratic character of H whose inflation is $\chi^{g^{-1}}|_{J\cap G^\delta}$.

When H is unitary, which also means that E/E_0 is unramified, by Lemma 7.6 (or more precisely its argument) $\overline{\chi'}$ can be extended to a quadratic character of $\mathrm{GL}_m(\boldsymbol{l})$. Thus, $\overline{\rho}\overline{\chi'}^{-1}$ as a cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ is distinguished by H, and thus it is σ-invariant by Proposition 9.2. The quadratic character $\overline{\chi'}$ must be σ-invariant, thus $\overline{\rho}$ is also σ-invariant, or by Proposition 7.4, $\overline{\rho}^{\tau_1} \cong \overline{\rho}^\vee$. Thus ,both $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$ are τ_1-self-dual, which means that Λ and π are τ_1-self-dual. By Proposition 5.6, π is σ-invariant.

When H is orthogonal, which also means that E/E_0 is totally ramified, comparing the central character as in §8.2 we have $\overline{\rho}(-I_m) = \mathrm{id}$. Thus, $\boldsymbol{\rho}^{\tau_1}|_J = \boldsymbol{\rho}({}^t\cdot^{-1})|_J \cong \boldsymbol{\rho}|_J$ by Proposition 7.4, and $\boldsymbol{\rho}(\tau_1(\varpi_E)) = \boldsymbol{\rho}(-\varpi_E) = \boldsymbol{\rho}(\varpi_E)$, which means that $\boldsymbol{\rho}$ is τ_1-self-dual, finishing the proof as above.

Finally, by Proposition 8.4 and the fact that $\mathrm{Sp}_m(\boldsymbol{l})$ equals its derived subgroup, the case where H is symplectic never occurs, which ends the proof of Theorem 9.1.

Remark 9.7. — Combining Theorem 9.1 with the argument in [14], section 6, we may further prove that an irreducible *generic* representation π of G distinguished by a unitary subgroup G^τ is σ-invariant.

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CENTRAL POINTS OF THE DOUBLE HEPTAGON TRANSLATION SURFACE ARE NOT CONNECTION POINTS

BY JULIEN BOULANGER

ABSTRACT. — We consider flow directions on the translation surfaces formed from double $(2n+1)$-gons and give a sufficient condition in terms of a natural continued fractions algorithm for a direction to be hyperbolic in the sense that it is a fixed direction for some hyperbolic element of the Veech group of the surface. In particular, we give explicit points with coordinates in the trace field of the double heptagon translation surface, that are not so-called connection points. Among these are the central points of the heptagons, giving a negative answer to a question by P. Hubert and T. Schmidt [1].

RÉSUMÉ (*Les points centraux du double heptagone ne sont pas des points de connexion*). — On s'intéresse au flot directionnel sur les surfaces de translation obtenues à partir de deux $(2n+1)$-gones dont on a recollé les côtés parallèles, et on donne une condition suffisante pour qu'une direction soit hyperbolique, c'est à dire fixée par une direction hyperbolique du groupe de Veech, en termes d'un algorithme de fractions continues naturel sur les directions de la surface. En particulier, cela nous permet d'exhiber des points sur le double heptagone à coordonnées dans le corps de trace qui ne sont pas des points de connexion. Parmi ces points on peut notamment trouver les points centraux des heptagones, ce qui donne une réponse négative à une question de P. Hubert et T. Schmidt [1].

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1. Introduction and statement of the results

A translation surface is a genus g topological surface with an atlas of charts on the surface minus a finite set of points such that all transition functions are translations. These surfaces can also be described as the surfaces obtained by gluing pairs of opposite parallel sides of a collection of Euclidean polygons by translations. Such surfaces arise naturally in the study of billiard table dynamics: the Katok–Zemlyakov unfolding procedure, which consists in reflecting the billiard every time the trajectory hits an edge instead of reflecting the trajectory, replaces the billiard flow on a polygon by a directional flow on isometric translation surfaces. The study of translation surfaces has been flourishing, with major recent advances such as the results in [12], [10], or [11], but there still remains various open questions, for instance in the area of Veech groups. One of these questions is to characterize so-called connection points, for which little is known for translation surfaces whose trace field is of degree 3 or more over $\mathbb{Q}$. In this paper, we look at two particular points of the double heptagon surface, whose trace field is cubic over $\mathbb{Q}$, and show that they are not connection points. For surveys about translation surfaces, see [25] and [24], and for Veech groups, see [16].

Before looking at connection points, one needs to understand better parabolic (or hyperbolic) directions; that is, directions fixed by a parabolic (or hyperbolic) element of the Veech group. For Veech surfaces, periodic directions, saddle connection directions and directions fixed by parabolic elements of the Veech group coincide. For these terms, see the background and [16]. For translation surfaces whose trace field is quadratic or $\mathbb{Q}$, C. McMullen showed in [18] that (after a natural normalization) the periodic directions are exactly those with slopes in the trace field. When the trace field is of higher degree, it is no longer true, and the periodic directions in general form a proper subset of the directions whose slope belong to the trace field. D. Davis and S. Lelièvre [8] characterized the parabolic directions for the double pentagon surface using a continued fractions algorithm. Their results can be directly extended to the $(2n+1)$-gon, which has a trace field of degree n over $\mathbb{Q}$.

In this paper, we use the algorithm to characterize hyperbolic directions whose slopes belong to the trace field for each double $(2n+1)$-gon surface, which are made of two copies of a $(2n+1)$-gon with parallel opposite sides glued together. We find explicit examples of such directions for the double heptagon. This allows us to prove that central points of the double heptagon are not connection points, see Theorem 1.3. This answers negatively a question of P. Hubert and T. Schmidt. Recall that the central points of the double heptagon are the centers of the heptagons. A nonsingular point of a translation surface is called a connection point if every separatrix passing through this point can be extended to a saddle connection. In fact, the author does not know any example of a nonperiodic connection point[1] for a translation surface whose trace field is of degree 3 over $\mathbb{Q}$ or higher.

Theorem 1.1. — *Let $n \geq 2$, for the double $(2n+1)$-gon surface, the directions that end in a periodic sequence (of period ≥ 2) for the continued fractions algorithm are hyperbolic directions.*

Proposition 1.2 (Double heptagon case). — *For the double heptagon surface, there are hyperbolic directions in the trace field.*

This proposition is already known from [2] and [13], where a different method is used. Our method provides an answer to the question of central points as connection points, which was not known.

Theorem 1.3. — *Central points of the double heptagon are not connection points.*

Moreover, one can look at double $(2n+1)$-gons with more sides. For example, the same result holds for the double nonagon:

Theorem 1.4. — *Central points of the double nonagon are not connection points.*

Moreover, various tests that we conducted suggest the following conjecture, which is not new since we found the same ideas in [13].

Conjecture 1.5. — *For the double heptagon and the double nonagon, all the directions in the trace field are either parabolic or hyperbolic.*

What is interesting is that these results do not seem to generalize to the double hendecagon, for example. In fact, for the double hendecagon, we were not able to find any direction in the trace field that ends in a periodic sequence. These issues will be discussed in Section 5.

2. Background

A *translation surface* (X, ω) is a real compact genus g surface X with an atlas ω such that all transition functions are translations except on a finite set of singularities Σ, along with a distinguished direction. Alternatively, it can be seen as a surface obtained from a finite collection of polygons embedded in $\mathbb{C}$ by gluing pairs of parallel opposite sides by translation. We get a surface X with a flat metric and a finite number of singularities. We define $X' = X - \Sigma$, which inherits the translation structure of X and defines a Riemannian structure on X'. Therefore, we have notions of geodesics, length, angle, and geodesic

1. A point is *periodic* if its orbit under the action of the affine group is finite, otherwise it is nonperiodic, see [15].

flow (called directional flow). This allows us make the following definitions, which will be useful in Section 4.

Definitions 2.1. — (i) A *separatrix* is a geodesic line emanating from a singularity.

(ii) A *saddle connection* is a separatrix connecting singularities without any singularities on its interior.

(iii) A nonsingular point of the translation surface is called a *connection point*, if every separatrix passing through this point can be extended to a saddle connection.

The action of $GL_2^+(\mathbb{R})$ on polygons induces an action on the moduli space of translation surfaces (see, for example, [25]). Two surfaces are affinely equivalent, if they lie in the same orbit. The stabilizer of a given translation surface X is called the *Veech group* of X and is denoted by $SL(X)$. In particular, affinely equivalent surfaces have a conjugated Veech group. As well as introducing the notion (although not the name) W.A. Veech showed in [23] that they are discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. Hence, we can classify elements of the Veech group into three types: elliptic ($|\mathrm{tr}(M)| < 2$), parabolic ($|\mathrm{tr}(M)| = 2$), and hyperbolic ($|\mathrm{tr}(M)| > 2$). Any element of the Veech group induces a diffeomorphism of the surface. Such diffeomorphisms are called *affine diffeomorphisms*.

Trace field. — The *trace field* of a group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is the subfield of $\mathbb{R}$ generated over $\mathbb{Q}$ by $\{\mathrm{tr}(M), M \in \Gamma\}$. One defines the trace field of a translation surface to be the trace field of its Veech group.

Let X be a genus g translation surface. We have the following theorems:

Theorem 2.2 (see [17]). — *The trace field of X has degree at most g over $\mathbb{Q}$.*

Assume the Veech group of X contains a hyperbolic element M. Then the trace field is exactly $\mathbb{Q}[tr(M)]$.

It is a classical result (see, for instance, [22]) that after a normalization, there exists an atlas such that every parabolic direction has its slope in the trace field, and every connection point has coordinates in the trace field. Specifically in the quadratic case, we have the following result:

Theorem 2.3 ([18], Theorem 5.1, see also [3]). — *If the trace field is quadratic over $\mathbb{Q}$, then every direction whose slope lies in the trace field is parabolic.*

3. Hyperbolic directions for the double $(2n + 1)$-gon

I. Bouw and M. Möller in [4] gave a large class of Veech surfaces. W.P. Hooper gave a geometric interpretation of these surfaces in [14] and proved in particular that the double $(2n+1)$-gon is affinely equivalent to a staircase polygonal model. See also [6], [9], and [20]. See Figure 3.1 for the double heptagon's staircase model. We will use this model to construct the continued fractions algorithm at the heart of this paper, which is a direct generalization of that described in [8] in the setting of the double pentagon. For more results on the double pentagon, see also [7].

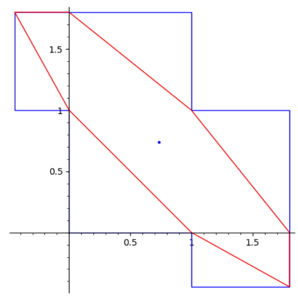

FIGURE 3.1. The staircase model for the double heptagon (in red we show one of the two heptagons).

The staircase model can be constructed as follows : Let each $R_i, i = 1, \ldots, 2n-1$ be the rectangle of side $\sin(\frac{i\pi}{2n+1})$ and $\sin(\frac{(i+1)\pi}{2n+1})$. Glue R_i and R_{i+1} such that edges of the same size are glued together, each side being glued to the opposite side of the other rectangle as shown in Figure 3.2. Parallel edges of R_1 (or R_{2n-1}) that are not glued to an edge of another rectangle are glued together.

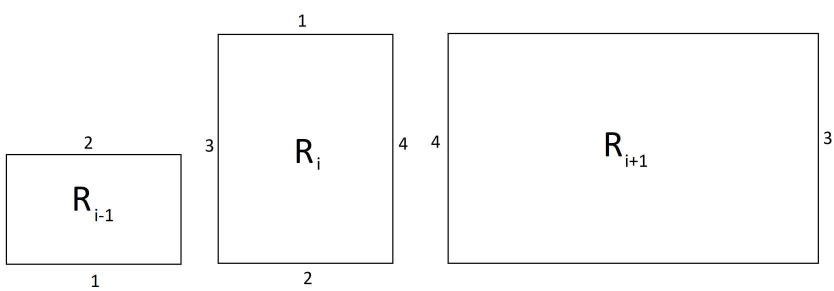

FIGURE 3.2. How to glue the rectangles R_i. Each edge of R_i is glued to the one with the same number in R_{i-1} or R_{i+1}.

It is then an easy calculation to establish the following lemma, which, in fact, is a particular case of Lemma 6.6 from [6] (see also [23]).

LEMMA 3.1. — *Let $n \geq 2$ be an integer. Then in the staircase model for the double $(2n+1)$-gon translation surface, there is a horizontal (or vertical)* *decomposition into cylinders such that all cylinders have modulus equal to* $a_n = 2\cos(\frac{\pi}{2n+1})$.

In fact, for computational reasons, it will be more convenient to rescale the staircase by a factor $\frac{1}{sin(\frac{n\pi}{2n+1})}$, so that each side can be expressed in the trace field, and the longer side has length 1.

Let us now look at the short diagonals of the staircase. We get $2n-1$ short diagonal vectors denoted by $D_i, i \in [\![1, 2n-1]\!]$. We set D_0 to be the shortest horizontal vector and D_{2n} the shortest vertical vector. We rescale such that D_0 and D_{2n} are length 1 vectors. We drew the diagonals in a graph as shown in Figure 3.3 for the double heptagon ($n = 3$). All the D_i's have a Euclidean norm bigger than 1 (except D_0 and D_{2n} with norm equal to 1).

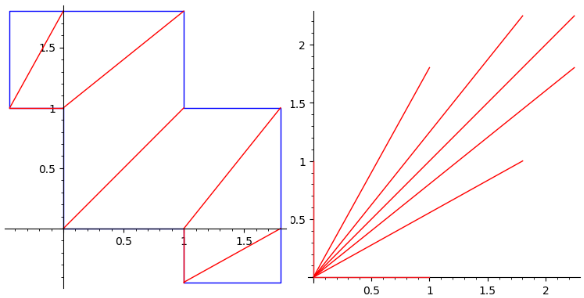

FIGURE 3.3. The diagonals of the double heptagon staircase divide the positive cone into six subcones. The diagonals are rescaled so that D_0 and D_{2n} are length 1 vectors. We have $D_0 = (1,0)$, $D_1 = (a_3, 1)$, $D_2 = (a_3^2 - 1, a_3)$, $D_3 = (a_3^2 - 1, a_3^2 - 1)$, and the other diagonals are symmetrical about the first bisector.

Let M_i, $i \in [\![0, 2n-1]\!]$ be the matrix that maps $D_0 = (1,0)$ to D_i and $D_{2n} = (0,1)$ to D_{i+1}. Let Σ denote the first quadrant, and Σ_i its image under M_i (we include D_i in M_i). The matrix M_i is in the Veech group of the staircase and is associated to an affine homeomorphism of the staircase surface, which we still denote by M_i. This homeomorphism sends parabolic (or hyperbolic) directions[2] to parabolic (or hyperbolic) directions that are in the i^{th} cone. In fact, these matrices M_i already appear in [21]. Iterating this process, we obtain a way to construct new parabolic (or hyperbolic) directions once we have found one. Conversely, we have a continued fractions algorithm given by the following definition.

2. Here and throughout, by direction we mean an element of the projective line $\mathbb{P}(\mathbb{R}^2)$.

Definition 3.2 (continued fractions algorithm for the staircase model). — Given a direction in the first quadrant as the entry, apply the following procedure:

1) If the direction lies in the i^{th} cone, apply M_i^{-1}.
2) If the direction is neither horizontal nor vertical, go back to step 1.

The following theorem is due to D. Davis and S. Lelièvre. It is stated in [8] in the case of the double pentagon, but the same arguments can be directly extended to the double $(2n+1)$-gon.

Theorem 3.3 ([8]). — *A direction on the double $(2n+1)$-gon is parabolic if and only if the continued fractions algorithm terminates at the horizontal direction.*

This theorem gives the first possibility for this algorithm to end. The other possibility would be an eventually periodic ending, i.e., if we apply the algorithm a certain number of times, the direction we get is a direction that we already got in a previous step. Here, we characterize these directions in the trace field and we prove Theorem 1.1, which can be stated more formally in the following way:

Theorem 3.4. — *The continued fractions algorithm is eventually periodic for a direction θ (which is neither horizontal nor vertical) in the trace field if and only if θ is the image by a matrix $M_{i_k}\dots M_{i_1}$ of an eigendirection for a hyperbolic matrix of the form $M_{j_1}\dots M_{j_l}$. In particular, every eventually periodic direction for the continued fractions algorithm is an eigendirection for a hyperbolic matrix of the Veech group.*

Proof. — If θ is eventually periodic for the algorithm, let k denote the length of the preperiod of θ. Then, we have matrices $M_{i_1},\dots,M_{i_k}$, such that $\theta' = (M_{i_k}\dots M_{i_1})^{-1}(\theta)$ is periodic for the algorithm. That is, there exist $M_{j_1},\dots,M_{j_l}$ such that $M_{j_1}\dots M_{j_l}(\theta') = \theta'$. Then $M = M_{j_1}\dots M_{j_l}$ is, indeed, a hyperbolic matrix since all M_js dilate lengths in the first quadrant, which means that the eigenvalue of $M_{j_1}\dots M_{j_l}$ for the direction θ' has to be strictly bigger than 1. Moreover, M belongs to the Veech group, being a product of elements of the Veech group.

Conversely, let us suppose that there are $i_1,\dots,i_k,j_1,\dots,j_l$ such that $M_{j_1}\dots M_{j_l}(\theta')=\theta'$, where $M=M_{j_1}\dots M_{j_l}$ is hyperbolic and $\theta=M_{i_k}\dots M_{i_1}(\theta')$. First, it is clear that θ' belongs to the first quadrant by the Perron–Frobenius theorem since all the matrices M_i have positive entries, and that the only sequences $j_1,\dots,j_l$ such that $M = M_{j_1}\dots M_{j_l}$ have possible zero entries are if $j_1 = \dots = j_l = 0$ or $j_1 = \dots = j_l = 2n$, which gives a matrix M that is parabolic and not hyperbolic. Thus, θ belongs to the first quadrant as well because the M_i's are contractions of the first quadrant. Moreover, at every step q, $M_{i_q}\dots M_{i_1}(\theta')$ belongs to the first quadrant. By construction of the algorithm, it follows that applying the algorithm to the direction θ leads to θ'

after k steps. By the same argument, since $M_{j_1} \dots M_{j_l}(\theta') = \theta'$ and θ' belongs to the first quadrant, we conclude that the sequence $j_l, \dots, j_1$ is exactly the sequence of indices we would have got if we had applied the algorithm to θ', and that θ' is a periodic direction for the algorithm. Hence, θ is an eventually periodic direction for the algorithm. □

Remark 3.5. — A point worth noting is that the sequence of sectors along the algorithm allows us to construct the matrix M, which stabilizes the original direction. This will allow us, for the double heptagon, to find a separatrix whose direction is eventually periodic for the algorithm and, hence, is not parabolic, which means that the separatrix does not extend to a saddle connection.

Example 3.6. — For the continued fractions algorithm on the double heptagon:

- The direction of slope $a_3^2 - 1$ is 2-periodic and fixed by the hyperbolic matrix $M_5 M_0$.
- The direction of slope $\frac{39}{7}a_3^2 + \frac{30}{7}a_3 - \frac{19}{7}$ is 28-periodic and fixed by the hyperbolic matrix $M_5^{12} M_4^2 M_0^{12} M_2 M_0$.

4. Connection points

In this section, we finally show that central points of the double heptagon are not connection points. We first give some motivation to their study.

Connection points have been studied in [15] by P. Hubert and T. Schmidt, who gave a construction of translation surfaces with infinitely generated Veech groups as branched covers over nonperiodic connection points. C. McMullen proved the existence of these points in [19] in the case of a quadratic trace field and implicitly showed that the connection points are exactly the points with coordinates in the trace field. However, in a higher degree there is no such result, neither concerning connection points nor about infinitely generated Veech groups. One of the easiest nonquadratic surfaces is the double heptagon, whose trace field is of degree 3 over $\mathbb{Q}$. P. Arnoux and T. Schmidt implicitly showed (see [2]) that for the double heptagon surface there are points with coordinates in the trace field that are not connection points. Still, it was not known whether or not central points of the double heptagon were connection points. Here, we provide a negative answer to this question.

By definition, for proving that a point is not a connection point, it suffices to find a separatrix passing through it, which cannot be extended to a saddle connection, for instance because the separatrix lies in a hyperbolic direction. We managed to find such a separatrix for a central point, which is drawn in Figure 4.1. Of course, both central points play a symmetric role, so it suffices to consider either one of them.

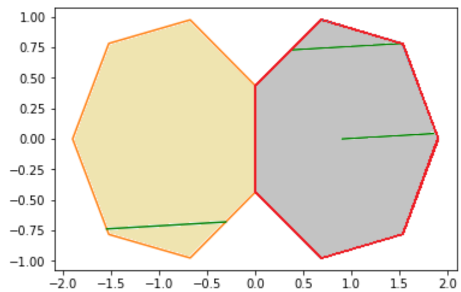

FIGURE 4.1. The green separatrix, passing through one of the central points with slope $\sin(\frac{\pi}{7})(-\frac{8}{3}\cos(\frac{\pi}{7})^2 + 4\cos(\frac{\pi}{7}) - \frac{4}{3})$, does not extend to a saddle connection.

We are now able to prove Proposition 1.2. More precisely:

PROPOSITION 4.1. — *The green separatrix in Figure 4.1 has a hyperbolic direction.*

Proof. — Let us work with the staircase model. Recall that it is affinely equivalent to the double heptagon model. The transition matrix is given by

$$T = \begin{pmatrix} \cos(\frac{\pi}{7}) + 1 & \cos(\frac{\pi}{7}) + 1 \\ -\sin(\frac{\pi}{7}) & \sin(\frac{\pi}{7}) \end{pmatrix}.$$

In this setting, we get Figure 4.2, and the slope of the new green direction is $\frac{3}{13}a^2 + \frac{6}{13}a - \frac{1}{13}$, where $a = a_3 = 2\cos(\frac{\pi}{7})$.

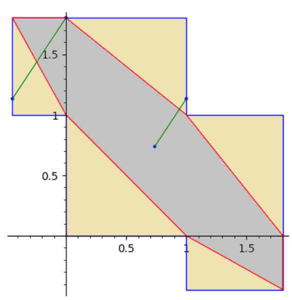

FIGURE 4.2. The same green separatrix in the staircase model does not extend to a saddle connection.

We apply the continued fractions algorithm to the green direction and notice that it ends in a periodic sequence of directions, which means that the green direction is fixed by a hyperbolic matrix of the Veech group, namely,

$$M = M_4^2 M_5 M_0 (M_4^{-1})^2 = \begin{pmatrix} -34a^2 - 26a + 19 & 22a^2 + 21a - 14 \\ -50a^2 - 41a + 28 & 35a^2 + 26a - 17 \end{pmatrix}.$$

It follows that M is hyperbolic (of trace $2+a^2$) and belongs to the Veech group. Explicitly,

$$M = \begin{pmatrix} a & 1 \\ a^2-1 & a \end{pmatrix} \begin{pmatrix} a & 1 \\ a^2-1 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & -1 \\ -a^2+1 & a \end{pmatrix} \begin{pmatrix} a & -1 \\ -a^2+1 & a \end{pmatrix}.$$

Finally, going back to the Veech group of the double heptagon model we get that TMT^{-1} fixes the green direction of Figure 4.1, which is then a hyperbolic direction. □

It follows from this proof that the central points are not connection points, since the green separatrix of Figure 4.1, having a hyperbolic direction, cannot be extended to a saddle connection. This proves Theorem 1.3.

Remark 4.2. — The green separatrix used for the proof is not the only separatrix passing through one of the central points whose direction is hyperbolic. For example, one could have taken the separatrix of Figure 4.3, which is hyperbolic and fixed (in the staircase model) by the matrix $SM_5^3 M_0 M_5^{-2} S^{-1}$. Here, S is the quarter-turn $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the Veech group.

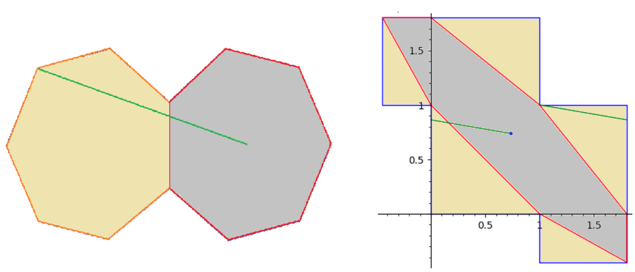

Figure 4.3. Another example of a separatrix whose direction is hyperbolic and in the trace field.

5. Further directions

In the previous sections, we looked at an algorithm defined for all $(2n+1)$-gons and used it for the case of the double heptagon to show that the central points are not connection points. One can ask what happens if we look at double $(2n+1)$-gons with more sides. It appears that the same result holds for the double nonagon. More precisely:

Proposition 5.1. — *The green direction of Figure 5.1 is hyperbolic. Hence, the central points of the double nonagon are not connection points.*

Proof. — The proof is similar to the case of the double heptagon. We work with the staircase model and use the continued fractions algorithm to find a separatrix passing through one of the central points whose direction is hyperbolic. It appears that the green direction of Figure 5.1, starting at a singularity with slope $a_4^2 + 2a_4 + 1$ and reaching one of the central point is hyperbolic and fixed by the matrix

$$M = M_0^4 M_5 M_7^2 = \begin{pmatrix} 23a_4^2 + 12a_4 - 1 & 9a_4 + 4 \\ 5a_4 + 3 & a_4^2 - 1 \end{pmatrix},$$

where $a_4 = 2\cos(\frac{\pi}{9})$, and the M_i's correspond to the matrices of the algorithm for the double nonagon staircase. Namely:

$$M_0 = \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}, \quad M_5 = \begin{pmatrix} a_4^2 - 1 & a_4 \\ a_4 + 1 & a_4^2 - 1 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 1 & 0 \\ a_4 & 1 \end{pmatrix}. \qquad \square$$

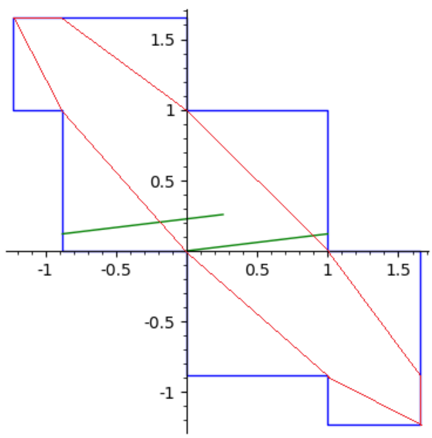

Figure 5.1. The green separatrix in the staircase model for the double nonagon does not extend to a saddle connection.

Conversely, we conducted tests for the double hendecagon but found no directions with periodic ending. This is closely related to Remark 9 of [13] made in the setting of λ-continued fractions for Hecke groups, saying that the authors did not find any hyperbolic direction in the trace field for $11 \leq 2n+1 \leq 29$. The interpretation in our setting relies on Veech having shown in [23] that the Veech group of the double $(2n+1)$-gon is conjugated to the Hecke group H_{2n+1}[3, 4]. In fact, other methods still allow to prove that central points of the double hendecagon are not connection points, this will be shown in a forthcoming work. See also [2] and [5] for related results.

Moreover, the study of directions in the double heptagon and the double nonagon has shown that there are either parabolic or hyperbolic directions in the trace field. However, could there be something else? It is *a priori* possible that the algorithm does not terminate for a given direction. In fact, our tests suggest that this does not happen in those cases, which leads to a precise version of Conjecture 1.5:

Conjecture 5.2. — *For the double heptagon and the double nonagon, every direction in the trace field terminates for the continued fractions algorithm. In particular, every direction in the trace field would be either parabolic or hyperbolic.*

In fact, this conjecture is also related to a conjecture in [13] about the possible orbits on $\mathbb{Q}(2\cos(\frac{\pi}{2n+1})) \cup \{\infty\}$ under the projective action of the Hecke triangle group H_{2n+1}. Once again, the behavior appears to be very different for the double hendecagon: there seems to be directions in the trace field that never terminate for the continued fractions algorithm.

Another interesting corollary of this result is related to billiard trajectories and was suggested to the author by C. McMullen. Recall that the double heptagon surface arises from the unfolding of the triangular billiard with angles $(\frac{\pi}{2}, \frac{\pi}{7}, \frac{5\pi}{14})$. The green separatrix in the proof of Proposition 4.1 is the lift of a vertex-to-vertex trajectory, drawn in Figure 5.2. In particular, there exists vertex-to-vertex trajectories whose directions are not parabolic (which means that there also exists a billiard trajectory in this direction that equidistributes).

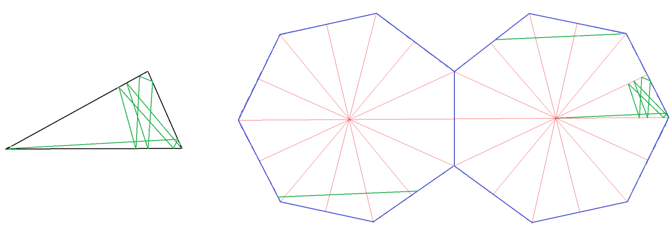

Figure 5.2. The green vertex-to-vertex trajectory on the triangular billiard unfolds to a directional trajectory whose direction is hyperbolic according to Section 4.

3. For $k \geq 3$, $H_k = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & \lambda_k \\ 0 & 1 \end{pmatrix} \right\rangle$, where $\lambda_k = 2\cos(\frac{\pi}{k})$

4. While the Veech group of the $2n$-gon is conjugated to a subgroup of order 2 of the Hecke group H_{2n}.

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