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ON THE CLASSIFICATION OF CUBIC PLANAR CREMONA MAPS

by Alberto Calabri & Nguyen Thi Ngoc Giao

Abstract. — We give a fine and complete classification of cubic planar Cremona maps, up to automorphisms of the plane. For this purpose, we introduce a new discrete invariant for cubic planar Cremona maps, called enriched weighted proximity graph, which encodes some properties of the base locus of the Cremona map.

Résumé (*Sur la classification des transformations cubiques planes de Cremona*). — Nous donnons une classification complète et fine des transformations cubiques planes de Cremona, modulo l'action des automorphismes du plan. Dans ce but, nous introduisons un nouvel invariant discret pour les transformations cubiques planes de Cremona, appelé graphe de proximité pondéré enrichi, qui décrit certaines propriétés du lieu de base de la transformation de Cremona.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. We denote by $\mathbb{P}^2$ the projective plane and by $\mathrm{Bir}(\mathbb{P}^2)$ the *plane Cremona group*, that is, the group of birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Generators of $\mathrm{Bir}(\mathbb{P}^2)$ are very well known, for over a century, thanks to the classical works of Noether and Castelnuovo, namely

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$\mathrm{Bir}(\mathbb{P}^2)$ is generated by automorphisms of $\mathbb{P}^2$ and a single quadratic transformation, e.g., the so-called elementary quadratic map

$$\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [yz:xz:xy]. \tag{1}$$

Nonetheless, the plane Cremona group is currently still a very active research area. Very recent remarkable results in this subject are, e.g., a new presentation of $\mathrm{Bir}(\mathbb{P}^2)$, due to Urech and Zimmermann in [14], and the study by Blanc and Furter in [3] of the so-called *length* of a plane Cremona map φ, that is, the minimum number of *Jonquières* maps needed to decompose φ; see Definition 5.3 for Jonquières maps.

Since all cubic planar Cremona maps trivially have length 1, let us introduce two refinements of the notion of length, namely the *quadratic length* (or *ordinary quadratic length*) of a plane Cremona map φ, that is, the minimum number of quadratic (or ordinary quadratic) maps needed to decompose φ, where an ordinary quadratic map is such that its three base points are all proper, cf. Section 5 for more details.

Let us say that two plane Cremona maps $\varphi, \varphi'\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are *equivalent* if there exist two automorphisms $\alpha, \alpha' \in \mathrm{Aut}(\mathbb{P}^2)$, such that $\varphi' = \alpha' \circ \varphi \circ \alpha$. It is very well known from the beginning of the study of plane Cremona maps, more than one hundred years ago, that there are exactly three equivalence classes of quadratic planar Cremona maps, which can be classified by the number of proper base points of the map; cf. Remark 5.2.

A classification of equivalence classes of cubic planar Cremona maps was described only few years ago by Cerveau and Déserti in [5]; they found 32 types of cubic planar Cremona maps, namely 27 types that are a single map, four types that are families of maps depending on one parameter, and one type that is a family of maps depending on two parameters. Their classification is based on the detailed analysis of those plane curves that are contracted by a cubic planar Cremona map.

However, it turns out that the classification in [5] is not complete, and it contains some inaccuracies; see Section 8 for a more detailed account.

- We found a map (our type 15 in Table 1.2) that does not occur in their list.
- We found that their type 17, that is, a single map, should be replaced by a one-parameter set of maps (our type 28 in Table 1.2).
- We found that their type 19 is equivalent to a particular case of their type 18.
- We found that their type 31 is equivalent to a particular case of their type 30.

Furthermore, in [5] it is not clear when two maps of the same type, that depend on parameters, are equivalent.

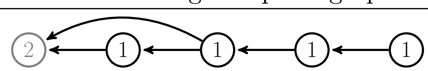
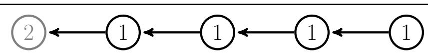
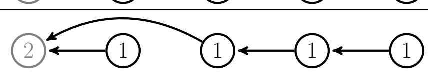
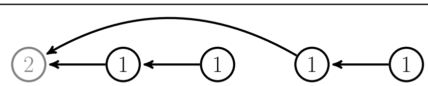
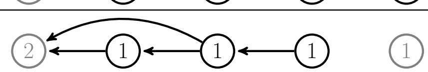
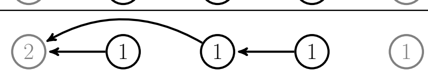
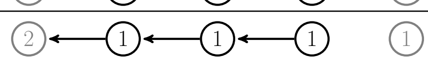
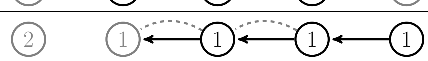
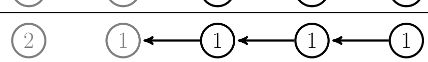
Our main result in this paper is a fine classification of equivalence classes of cubic planar Cremona maps. For this purpose, we introduce a new discrete invariant, which we call *enriched weighted proximity graph* of a cubic planar Cremona map φ, determined by the properties of the base points of the homaloidal net $\mathcal{L}_\varphi$ defining φ. Namely, it is a directed graph with five vertices: one vertex p_1 with weight 2 corresponding to the base point of multiplicity 2 of $\mathcal{L}_\varphi$ and the other four vertices $p_2, \ldots, p_5$ with weight 1 corresponding to the other four (proper or infinitely near) base points of multiplicity 1 of $\mathcal{L}_\varphi$, such that

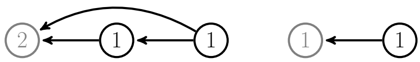
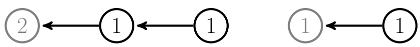
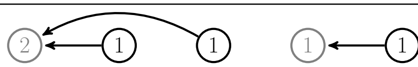
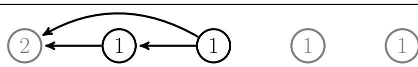
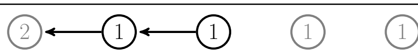
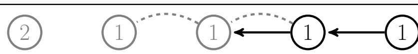
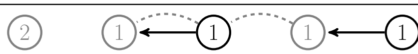
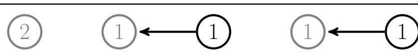
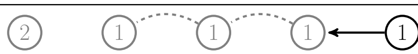
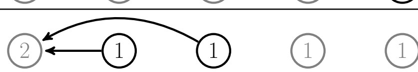
- there is an arrow starting from a vertex p_i towards a vertex p_j if and only if p_i is *proximate* to p_j; cf. Definition 2.6;
- there is a dashed line joining three vertices if and only if the corresponding three base points are aligned.

In other words, the enriched weighted proximity graph takes into account the proximity relations among the base points, their multiplicities and their relative position; cf. Section 6. Our first result is their classification.

Theorem 1.1. — *There are exactly 31 enriched weighted proximity graphs of cubic planar Cremona maps, up to isomorphism, listed in Table 1.1.*

Table 1.1: Enriched weighted proximity graphs and ordinary quadratic lengths of cubic planar Cremona maps.

#	Enriched weighted prox. graph	oql
1		6
2		5
3		5
4		4
5		5
6		4
7		4
8		5
9		4

10		3
11		3
12		3
13		3
14		3
15		3
16		3
17		4
18		3
19		3
20		3
21		2
22		3
23		2
24		3
25		2
26		2
27		2
28		3
29		2
30		2
31		2

Correspondingly, we find 31 types of equivalence classes of cubic planar Cremona maps. Before stating our classification theorem, we need a bit of notation.

Let us set $\mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$ and let us define the following maps:

$$g_1, g_2 \colon \mathbb{C}^{**} \times \mathbb{C}^{**} \to \mathbb{C}^{**} \times \mathbb{C}^{**}, \qquad g_1(a,b) = (b,a), \qquad g_2(a,b) = \left(\frac{1}{a}, \frac{1}{b}\right).$$

Therefore, $g_3 := g_2 \circ g_1 = g_1 \circ g_2$ is the map $(a,b) \mapsto (1/b, 1/a)$. Clearly,

$$G = \{\mathrm{id}, g_1, g_2, g_3\}$$

is a group, under the composition, which is isomorphic to $((\mathbb{Z}/2\mathbb{Z})^2, +)$.

For $a \neq b$ and $a, b \in \mathbb{C}^{**}$, let us denote by S' the following set

$$S' = \left\{(a,b), \left(\frac{a}{a-1}, \frac{a-b}{a-1}\right), \left(\frac{b}{b-1}, \frac{b-a}{b-1}\right), \left(\frac{a-b}{b(a-1)}, \frac{1}{1-a}\right), \left(\frac{b-a}{a(b-1)}, \frac{1}{1-b}\right), \left(\frac{a-1}{b-1}, \frac{b(a-1)}{a(b-1)}\right)\right\}$$

and let us define

$$S = \{g(s) \mid g \in G \text{ and } s \in S'\}. \tag{2}$$

Theorem 1.2. — *Any cubic planar Cremona map is equivalent to one of the maps in Table 1.2, where the first 25 types are single maps, types 26–30 depend on one parameter $\gamma \neq 0,1$ and type 31 depends on two parameters a, b, where $a, b \neq 0, 1$ and $a \neq b$.*

Two cubic planar Cremona maps of two different types are not equivalent.

Concerning the types depending on parameters:

- $\varphi_{26,\gamma}$, *that is type 26 in Table 1.2 with parameter $\gamma \neq 0,1$, is equivalent to $\varphi_{26,\gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = \gamma/(\gamma-1)$;*
- $\varphi_{27,\gamma}$, *that is type 27 in Table 1.2 with parameter $\gamma \neq 0,1$, is equivalent to $\varphi_{27,\gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = 1/\gamma$;*
- *for $n \in \{28, 29, 30\}$, the map $\varphi_{n,\gamma}$, that is type n in Table 1.2 with parameter $\gamma \neq 0,1$, is equivalent to $\varphi_{n,\gamma'}$ if and only if*

$$\gamma' \in \left\{\gamma, \frac{1}{\gamma}, 1-\gamma, \frac{1}{1-\gamma}, \frac{\gamma}{\gamma-1}, \frac{\gamma-1}{\gamma}\right\};$$

- $\varphi_{31,a,b}$, *that is type 31 in Table 1.2 with two parameters $a, b \neq 0, 1$, $a \neq b$, is equivalent to $\varphi_{31,a',b'}$ if and only if $(a', b') \in S$, where S is defined in* (2).

In Table 1.2, the first column lists our type, the second column lists the formula of the map, the third column lists the corresponding type in [5], cf. Section 8, and finally the fourth column lists the type of the inverse map.

As a first application of our classification, we compute the ordinary quadratic length and the quadratic length of all cubic planar Cremona maps.

TABLE 1.2. Types of cubic planar Cremona maps.

#	Map	[5]	Inv.
1	$[xz^2+y^3:yz^2:z^3]$	1	1
2	$[x(x^2+yz):y^3:y(x^2+yz)]$	20	8
3	$[xz^2:x^3+xyz:z^3]$	3	5
4	$[x^2z:x^3+z^3+xyz:xz^2]$	4	4
5	$[x^2z:x^2y+z^3:xz^2]$	5	3
6	$[x^2(x-y):xy(x-y):xyz+y^3]$	12	6
7	$[x(x^2+yz):y(x^2+yz):xy^2]$	24	17
8	$[xyz:yz^2:z^3-x^2y]$	6	2
9	$[y^2z:x(xz+y^2):y(xz+y^2)]$	21	9
10	$[x^3:y^2z:xyz]$	7	10
11	$[x(y^2+xz):y(y^2+xz):xyz]$	22	18
12	$[xz^2:x^2y:z^3]$	2	12
13	$[x(y^2+xz):y(y^2+xz):xy^2]$	23	20
14	$[x^3:x^2y:(x-y)yz]$	11	15
15	$[x^2y:xy^2:(x-y)^2z]$	$(\star)$	14
16	$[x(x^2+yz):y(x^2+yz):xy(x-y)]$	28	24
17	$[xyz:y^2z:x(y^2-xz)]$	10	7
18	$[x^2(y-z):xy(y-z):y^2z]$	8	11
19	$[x(x^2+yz+xz):y(x^2+yz+xz):xyz]$	26	19
20	$[x^2z:xyz:y^2(x-z)]$	9	13
21	$[x(xy+xz+yz):y(xy+xz+yz):xyz]$	25	21
22	$[xz(x+y):yz(x+y):xy^2]$	13	22
23	$[x(x^2+xy+yz):y(x^2+xy+yz):xyz]$	27	25
24	$[xyz:(y-x)yz:x(x-y)(y-z)]$	15	16
25	$[x(x+y)(y+z):y(x+y)(y+z):xyz]$	14	23
26	$[x(\gamma xz-\gamma y^2-xy+y^2):\gamma xy(z-y):\gamma y^2(z-x)]$	29	26
27	$[\gamma x^2y:\gamma xy^2:(x+y)(x+\gamma y)z]$	16	27
28	$[xy(x-y):xz(y-\gamma x):z(y+\gamma x)(y-\gamma x)]$	$17^\dagger$	28
29	$[xy(x-y):x(xy-\gamma xy+\gamma xz-yz):x^2y-\gamma^2x^2y+\gamma^2x^2z-y^2z]$	30	30
30	$[x(xy+\gamma xz-xz-\gamma y^2):\gamma xz(x-y):\gamma z(x-y)(x+y)]$	18	29
31	$[ax(-abxz+aby^2-b^2xy+b^2xz+axy-ay^2)$ $:ax(-abxz+abyz+axy-ayz-bxy+bxz)$ $:-a^2bx^2z+a^2by^2z+a^2x^2y-a^2y^2z-b^2x^2y+b^2x^2z]$	32	31

Theorem 1.3. — *Plane Cremona maps equivalent to type 1 in Table 1.2 have quadratic length 3, while all other cubic planar Cremona maps have quadratic length 2.*

A plane Cremona map equivalent to type n, $1 \leq n \leq 31$*, in Table 1.2 has the respective ordinary quadratic length listed in the third column in Table 1.1.*

Further applications of the classification of cubic planar Cremona maps, like, for example, to the study of their dynamical properties and to the study of algebraic foliations, can be found in [5].

Recall that the set of all plane Cremona maps of fixed degree d has a natural structure of quasi-projective variety, in particular for degree $d = 3$ it is irreducible, rational and of dimension 18; see, e.g., [2].

Finally, we believe that our new approach to cubic planar Cremona maps will led to a better understanding also of cubic Cremona maps in $\mathbb{P}^3$; cf. the recent paper [6].

The results contained in this paper are part of the second author's Ph.D. thesis [12], where there is also a classification of analogous enriched weighted proximity graphs of quartic planar Cremona maps. Note that weighted proximity graphs can be defined similarly for planar Cremona maps of any degree, but it is not clear to us how to enrich these graphs in order to further distinguish their equivalence classes.

Let us briefly describe the contents of this paper.

In Section 2, we fix notation and recall the definition of infinitely near and proximate points. In Section 3, we introduce special coordinates in order to deal with infinitely near points that we call *standard coordinates.* In Section 4, we consider plane conics passing through five proper or infinitely near points. In Section 5, we recall the definition of the length of a plane Cremona map and we introduce the notion of quadratic length and ordinary quadratic length of a plane Cremona map. In Section 6, we define the enriched weighted proximity graph of the base points of the homaloidal net defining a cubic planar Cremona map and we prove Theorem 1.1. In Section 7, we define the height of a base point of a plane Cremona map, which allows us to give a lower bound on the ordinary quadratic length of a plane Cremona map. In Section 8, we compare our classification with that of Cerveau and Déserti in [5]. In Section 9, we prove Theorem 1.2. In Section 10, we compute the quadratic length and ordinary quadratic length of all cubic planar Cremona maps, proving Theorem 1.3.

2. Notation and infinitely near points

Notation 2.1. — Any non-zero complex number z can be written uniquely as follows

$$z = re^{i\theta} = r\left(\cos(\theta) + i\sin(\theta)\right), \qquad \text{with } r > 0, \text{ and } \theta \in [0, 2\pi).$$

Any non-zero complex number $z = r(\cos(\theta) + i\sin(\theta))$ has two square roots, namely

$$z_0 = \sqrt{r}\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right], \qquad z_1 = -z_0.$$

From now on, we denote z_0 by $\sqrt{z}$ and z_1 by $-\sqrt{z}$.

For any $t \in \mathbb{C}$, such that $t^2 \neq 4$, set $t^\bullet = \sqrt{t^2-4}$, $t_+ = (t+t^\bullet)/2$ and $t_- = (t-t^\bullet)/2$, that is, $t_\pm$ are the roots of the equation $x^2 - tx + 1 = 0$. Note that, if $t^2 \neq 4$, then $t_+ \neq t_-$ and $t_+, t_- \neq 0$.

Next, we recall some very well-known facts about the projective plane and plane curves.

Lemma 2.2. — *Let $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ be four points, such that no three are collinear. Then, there exists a unique automorphism of $\mathbb{P}^2$, which maps $e_1 = [1:0:0], e_2 = [0:1:0], e_3 = [0:0:1], e_4 = [1:1:1]$ to p_1, p_2, p_3, p_4, respectively.*

Proof. — See, e.g., §11.2 in [8]. □

Lemma 2.3. — *Let $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}^2$ be five points, such that no three are collinear. Then, there exists a unique irreducible conic passing through $p_1, \dots, p_5$.*

Proof. — See, e.g., §5.2 in [13]. □

Lemma 2.4 (cf. Lemma 1.2.3 in [11]). — *Let C_1 and C_2 be two irreducible conics. Then, there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$.*

Proof. — It suffices to show that there exists an automorphism of $\mathbb{P}^2$ that maps C_1 to the conic $C_0\colon xz - y^2 = 0$. Choose three distinct points p_1, p_2, p_3 of C_1. Let p_4 be the intersection point of the tangent line to C_1 at p_1 with the tangent line to C_1 at p_2. Clearly, no three among p_1, p_2, p_3, p_4 are collinear. Therefore, by Lemma 2.2, there exists an automorphism β of $\mathbb{P}^2$ that maps p_1, p_2, p_3, p_4 to e_1, e_3, e_4, e_2, respectively. Hence, $\beta(C_1) = C_0$. □

The proof of the previous lemma also shows the following:

Lemma 2.5. — *Let $n \in \{1, 2, 3\}$. Let C_1 and C_2 be two irreducible conics. Let $p_1, \dots, p_n$ be distinct points of C_1 and let $q_1, \dots, q_n$ be distinct points of C_2. Then, there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 1, \dots, n$.*

Let us recall the definition of the bubble space of $\mathbb{P}^2$, which is useful to define infinitely near points and related properties.

Definition 2.6 (cf. [7, §7.3.2], [10, Chapter 5, §35]). — Let us denote by $\mathcal{B}(\mathbb{P}^2)$ the so-called *bubble space* of $\mathbb{P}^2$, which is defined as follows. Consider all surfaces X *above* $\mathbb{P}^2$, i.e., all surfaces X, such that there exists a birational morphism $X \to \mathbb{P}^2$. If X_1, X_2 are two surfaces above $\mathbb{P}^2$, say $\pi_1 \colon X_1 \to \mathbb{P}^2$ and $\pi_2 \colon X_2 \to \mathbb{P}^2$ are birational morphisms, one identifies $p_1 \in X_1$ with $p_2 \in X_2$ if the birational map $(\pi_2)^{-1} \circ \pi_1 \colon X_1 \dashrightarrow X_2$ is a local isomorphism at p_1 that sends p_1 to p_2. The bubble space $\mathcal{B}(\mathbb{P}^2)$ is the union of all points of all surfaces above $\mathbb{P}^2$ modulo the equivalence relation generated by these identifications.

For any birational morphism $X \to \mathbb{P}^2$, there is an injective map $X \to \mathcal{B}(\mathbb{P}^2)$, and, therefore, we will identify points of X with their images in $\mathcal{B}(\mathbb{P}^2)$.

One says that $p_1 \in \mathcal{B}(\mathbb{P}^2)$ is *infinitely near* $p_2 \in \mathcal{B}(\mathbb{P}^2)$, say $p_1 \in X_1$ and $p_2 \in X_2$, with birational morphisms $\pi_1 \colon X_1 \to \mathbb{P}^2$ and $\pi_2 \colon X_2 \to \mathbb{P}^2$, if the birational map $(\pi_2)^{-1} \circ \pi_1 \colon X_1 \dashrightarrow X_2$ is defined at p_1, sends p_1 to p_2 but is not a local isomorphism at p_1. In such a case, we write that $p_1 \succ p_2$.

Moreover, one says that p_1 is *in the first neighbourhood* of p_2, or that p_1 is *infinitely near* p_2 *of the first order*, if $(\pi_2)^{-1} \circ \pi_1$ corresponds locally to the blow up of p_2. In such a case, we write that $p_1 \succ_1 p_2$.

If $p_1 \succ p_2$, then one can define the *infinitesimal order* of p_1 with respect to p_2 by induction, namely if $p_1 \succ_1 p_3$ and $p_3 \succ_k p_2$ for some k, then p_1 is *infinitely near* p_2 *of order* $k+1$.

If $p_1 \succ p_2$ and $p_1 \in X_1$, then there is a unique irreducible curve $E_2 \subset X_1$, which corresponds to the exceptional curve of the blowing up of $p_2 \in X_2$. One says that p_1 is *proximate* to p_2 if $p_1 \in E_2$. In such a case, we write that $p_1 \dashrightarrow p_2$.

If $p_1 \dashrightarrow p_2$ and $p_1 \succ_k p_2$ with $k > 1$, then we say that p_1 is *satellite* to p_2 and we write $p_1 \odot p_2$. Otherwise, if p_1 is not satellite to p_2, we then write that $p_1 \not\odot p_2$.

One says that a point $p \in \mathbb{P}^2 \subset \mathcal{B}(\mathbb{P}^2)$ is a *proper point of* $\mathbb{P}^2$.

If $p_1 \succ p_2 \in \mathbb{P}^2$, where $p_1 \in X_1$ and $\pi_1 : X_1 \to \mathbb{P}^2$ is a birational morphism, we say that a plane curve C passes through p_1 if C passes through p_2, and the strict transform of C on X_1 via π_1 passes through p_1.

Remark 2.7. — Note that an infinitely near point is proximate to either one or two points; cf., e.g., [4, 5.3].

Proposition 2.8 (Proximity inequality). — *Let $\varphi \colon S \to \mathbb{P}^2$ be a birational morphism, that is, the composition of the blowing ups at single points, namely $\varphi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{r-1} \circ \pi_r$, where $\pi_i \colon S_i \to S_{i-1}$ is the blowing up at a point $p_i \in S_{i-1}$, $i = 1, \ldots, r$, with $\mathbb{P}^2 = S_0$ and $S = S_r$. Let C be a plane curve and let C_i be the strict transform of C in S_i for $i = 1, \ldots, r$. Setting $C_0 = C$ and $m_i = \mathrm{mult}_{p_i}(C_{i-1})$ for $i = 1, \ldots, r$, one has, for each $j = 1, \ldots, r$,*

$$m_j \geqslant \sum_{p_k \dashrightarrow p_j} m_k.$$

Proof. — See §2.2 in [1] or Theorem 3.5.3, Corollary 3.5.4 in [4]. □

3. Standard coordinates of infinitely near points

In this section, we want to give a way to describe infinitely near points that we call *standard coordinates.*

Let $p_1 = [a : b : c] \in \mathbb{P}^2$. Let us consider three cases:

(i) if $c \neq 0$, then $p_1 = \left[\frac{a}{c} : \frac{b}{c} : 1\right] = [\overline{a} : \overline{b} : 1]$;

(ii) if $c = 0$ and $b \neq 0$, then $p_1 = \left[\frac{a}{b} : 1 : 0\right] = [\overline{a} : 1 : 0]$;

(iii) if $c = b = 0$, then $p_1 = [1 : 0 : 0]$.

In case (i), we work on the affine chart $U_2 \simeq \mathbb{C}^2_{\overline{x},\overline{y}}$, so that p_1 corresponds to the point $\overline{p}_1 = (\overline{a}, \overline{b})$, and we define the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\overline{x},\overline{y}} \to \mathbb{C}^2_{x_0,y_0}$ by

$$\alpha_1(\overline{x}, \overline{y}) = (\overline{x} - \overline{a}, \overline{y} - \overline{b}).$$

In case (ii), we work on the affine chart $U_1 \simeq \mathbb{C}^2_{\overline{x},\overline{z}}$, so that p_1 corresponds to the point $\overline{p}_1 = (\overline{a}, 0)$, and we define the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\overline{x},\overline{z}} \to \mathbb{C}^2_{x_0,y_0}$ by

$$\alpha_1(\overline{x}, \overline{z}) = (\overline{x} - \overline{a}, \overline{z}).$$

In case (iii), we work on the affine chart $U_0 \simeq \mathbb{C}^2_{\overline{y},\overline{z}}$, so that p_1 corresponds to the point $\bar{p}_1 = (0, 0)$, and we define the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\overline{y},\overline{z}} \to \mathbb{C}^2_{x_0,y_0}$ by

$$\alpha_1(\overline{y}, \overline{z}) = (\overline{y}, \overline{z}).$$

In all three cases, we defined α_1 in such a way that $\alpha_1(\overline{p}_1) = (0, 0) \in \mathbb{C}^2_{x_0,y_0}$.

We blow up $\mathbb{C}^2_{x_0,y_0}$ at $(0, 0)$ and we consider the first chart $\mathbb{C}^2_{x_1,y_1}$ where the blowing-up map is given in coordinates by $x_0 = x_1, y_0 = x_1 y_1$.

In this chart, the exceptional curve E_1 has local equation $x_1 = 0$; hence, a point $p_2 \succ_1 p_1$ corresponds either to the point $(0, t_2) \in E_1$ with $t_2 \in \mathbb{C}$ or to the point that is the origin of the second chart. In the former case, let us say that p_2 has standard coordinates $p_2 = (p_1, t_2)$, while in the latter case, let us say that p_2 has standard coordinates $p_2 = (p_1, \infty)$. Setting $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, let us denote the standard coordinates of p_2 by $p_2 = (p_1, t_2)$ with $t_2 \in \mathbb{P}^1$.

Remark 3.1. — Recall that a point $p_2 \succ_1 p_1$ corresponds to the direction of a line passing through p_1. More precisely, one can see that the point $p_2 = (p_1, t_2)$, with $p_1 = [a : b : c]$, corresponds to the line defined by the following equations

$$\begin{cases} cy - bz = t_2(cx - az) & \text{when } c \neq 0 \text{ and } t_2 \in \mathbb{C}, \\ cx - az = 0 & \text{when } c \neq 0 \text{ and } t_2 = \infty, \\ bz = t_2(bx - ay) & \text{when } c = 0, b \neq 0 \text{ and } t_2 \in \mathbb{C}, \\ bx = ay & \text{when } c = 0, b \neq 0 \text{ and } t_2 = \infty, \\ z = t_2 y & \text{when } b = c = 0 \text{ and } t_2 \in \mathbb{C}, \\ y = 0 & \text{when } b = c = 0 \text{ and } t_2 = \infty. \end{cases}$$

In other words, the above equations define the unique line passing through p_1 and p_2.

We want to go on by blowing up at $p_2 = (p_1, t_2)$, with $t_2 \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Either $t_2 \in \mathbb{C}$ or $t_2 = \infty$. In the former case, with notation as above, let $\alpha_2 \colon \mathbb{C}^2_{x_1,y_1} \to \mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ be the isomorphism defined by

$$\alpha_2(x_1, y_1) = (x_1, y_1 - t_2).$$

In the latter case, p_2 corresponds to the origin of the second chart of the blowing up of $\mathbb{C}^2_{x_0,y_0}$ at $(0,0)$, which we write as $\mathbb{C}^2_{x_1',y_1'}$, where the blowing-up map is given by $x_0 = x_1'y_1', y_0 = y_1'$. Let $\alpha_2 \colon \mathbb{C}^2_{x_1',y_1'} \to \mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ be the isomorphism

$$\alpha_2(x_1', y_1') = (y_1', x_1').$$

In this way, in both cases, in $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ the exceptional curve E_1 has local equation $\bar{x}_1 = 0$, and the point p_2 corresponds to the origin $(0,0)$.

We blow up $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ at $(0,0)$ and we consider the first chart $\mathbb{C}^2_{x_2,y_2}$ where the blowing-up map is given in coordinates by $\bar{x}_1 = x_2, \bar{y}_1 = x_2y_2$. In this chart, the exceptional curve E_2 has local equation $x_2 = 0$, hence a point $p_3 \succ_1 p_2$ corresponds either to the point $(0, t_3) \in E_2$ with $t_3 \in \mathbb{C}$ or to the point that is the origin of the second chart.

Let us say that p_3 has standard coordinates $p_3 = (p_1, t_2, t_3)$, where either $t_3 \in \mathbb{C}$ in the former case or $t_3 = \infty$ in the latter case.

Note that the strict transform of E_1 can be seen only in the second chart, and it meets E_2 at the origin of the second chart. In other words, the point with standard coordinates (p_1, t_2, ∞) is satellite to p_1.

More generally, let us proceed by induction of the infinitesimal order. Suppose that we have blown up the point p_{r-1} with standard coordinates $p_{r-1} = (p_1, t_2, \dots, t_{r-1})$, with $t_i \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $i = 2, \dots, r-1$. Following the procedure described above, we may assume that p_{r-1} is the origin of a chart $\mathbb{C}^2_{\bar{x}_{r-1},\bar{y}_{r-1}}$ in such a way that the exceptional curve E_{r-1} has local equation $\bar{x}_{r-1} = 0$.

In the first chart of the blowing up of $\mathbb{C}^2_{\bar{x}_{r-1},\bar{y}_{r-1}}$ at $(0,0)$, given in coordinates by $\bar{x}_{r-1} = x_r, \bar{y}_{r-1} = x_ry_r$, the exceptional curve E_r has local equation $x_r = 0$; hence a point $p_r \succ_1 p_{r-1}$ corresponds either to the point $(0, t_r) \in E_r$ with $t_r \in \mathbb{C}$ or to the point that is the origin of the second chart, given in coordinates by $\bar{x}_{r-1} = x_ry_r, \bar{y}_{r-1} = y_r$.

Let us say that p_r has standard coordinates $p_r = (p_1, t_2, \dots, t_r)$, where $t_r \in \mathbb{C}$ in the former case and $t_r = \infty$ in the latter case.

The above discussion proves the following:

Lemma 3.2. — *Let $p_1 \in \mathbb{P}^2$. Then, there is a one-to-one correspondence between points infinitely near p_1 of order r and* $(\mathbb{P}^1)^r = \underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{r \text{ times}}$.

Corollary 3.3. — *There is a one-to-one correspondence between points infinitely near a proper point of order r and $W = \mathbb{P}^2 \times (\mathbb{P}^1)^r$.*

DEFINITION 3.4. — We call *standard coordinates* of an infinitely near point the point of W obtained with the above construction.

4. Conics and infinitely near points

The contents of this section are well known, but we have not found a proper reference for them.

REMARK 4.1. — If $p_1 \in \mathbb{P}^2, p_3 \succ_1 p_2 \succ_1 p_1$ and $p_3 \odot p_1$, i.e., $p_3 \dashrightarrow p_1$, then there is no smooth plane curve passing through p_1, p_2, p_3 because of the proximity inequality at p_1.

For the same reason, if $p_1 \in \mathbb{P}^2$, $p_2 \succ_1 p_1$, $p_3 \succ_1 p_1$ and $p_2 \neq p_3$, then there is no smooth plane curve passing through p_1, p_2, p_3.

LEMMA 4.2. — *If $p_1 \in \mathbb{P}^2, p_3 \succ_1 p_2 \succ_1 p_1$ and p_1, p_2, p_3 are collinear, namely p_3 lies on the strict transform of the line passing through p_1 and p_2, then there is no irreducible conic passing through p_1, p_2, p_3.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0]$, and p_2 has standard coordinates $p_2 = (p_1, 0)$, so p_3 is uniquely determined by p_1, p_2, namely p_3 has standard coordinates $p_3 = (p_1, 0, 0)$.

Suppose that C is an irreducible conic passing through p_1, p_2. Then, C has equation

$$a_2 y^2 + a_3 xz + a_4 yz + a_5 z^2 = 0,$$

where $a_2, a_3, a_4, a_5 \in \mathbb{C}$, and $a_2, a_3 \neq 0$ because C is irreducible.

We work in the affine chart $U_0 \simeq \mathbb{C}^2_{\bar{y},\bar{z}}$ and we consider the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\bar{y},\bar{z}} \to \mathbb{C}^2_{x_0,y_0}$ defined by $\alpha_1(\bar{y}, \bar{z}) = (\bar{y}, \bar{z})$, where the conic C has local equation

$$a_2 x_0^2 + a_3 y_0 + a_4 x_0 y_0 + a_5 y_0^2 = 0.$$

In the first chart of the blowing up of $\mathbb{C}^2_{x_0,y_0}$ at the origin $(0,0)$, where $x_0 = x_1, y_0 = x_1 y_1$, the strict transform of C has local equation

$$a_2 x_1 + a_3 y_1 + a_4 x_1 y_1 + a_5 x_1 y_1^2 = 0.$$

Note that p_2 is just the origin of $\mathbb{C}^2_{x_1,y_1}$.
Then, the strict transform of C via the blowing up of $\mathbb{C}^2_{x_1,y_1}$ at the origin $(0,0)$ has local equation in the first chart, where $x_1 = x_2, y_1 = x_2 y_2$,

$$a_2 + a_3 y_2 + a_4 x_2 y_2 + a_5 x_2 y_2^2 = 0.$$

Note that p_3 is just the origin of $\mathbb{C}^2_{y_2,z_2}$, but the strict transform of C does not pass through $(0,0)$ because $a_2 \neq 0$. □

Remark 4.3. — It is easy to check that if $p_1 \in \mathbb{P}^2, p_3 \succ_1 p_2 \succ_1 p_1$, $p_3 \not\odot p_1$ and p_1, p_2, p_3 are not collinear, then there are irreducible conics passing through p_1, p_2, p_3.

Lemma 4.4. — *Let $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ and $p_5 \succ_1 p_1$, such that no three among $p_1, \dots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \dots, p_5$.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1], p_4 = [1:1:1]$. Then, p_5 has standard coordinates $p_5 = (p_1, t_5)$, namely p_5 is infinitely near p_1 of the first order in the direction of the line $z - t_5 y = 0$, where $t_5 \in \mathbb{C} \setminus \{0, 1\}$; indeed, if $t_5 = 0$, then p_5, p_2, p_1 would be collinear; if $t_5 = 1$, then p_5, p_4, p_1 would be collinear and, finally, if $t_5 = \infty$, then p_5, p_3, p_1 would be collinear. Then, one can check that the conic

$$xz - t_5 xy + (t_5 - 1) yz = 0$$

is the unique irreducible conic passing through $p_1, \dots, p_5$. □

Lemma 4.5. — *Let $p_1, p_2, p_3 \in \mathbb{P}^2$ and $p_5 \succ_1 p_4 \succ_1 p_1$, such that $p_5 \not\odot p_1$ and no three among $p_1, \dots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \dots, p_5$.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1]$ and that p_4 has standard coordinates $p_4 = (p_1, 1)$, namely p_4 is infinitely near p_1 of the first order in the direction of the line $y = z$. Then, p_5 has standard coordinates $p_5 = (p_1, 1, t_5)$, where $t_5 \in \mathbb{C}^*$; indeed, if $t_5 = 0$, then p_5, p_4, p_1 would be collinear, and if $t_5 = \infty$, then $p_5 \odot p_1$. Then, one can check that the conic

$$xz - xy - t_5 yz = 0$$

is the unique irreducible conic passing through $p_1, \dots, p_5$. □

Lemma 4.6. — *Let $p_1, p_2, p_3 \in \mathbb{P}^2$ and $p_4 \succ_1 p_1, p_5 \succ_1 p_2$, such that no three among $p_1, \dots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \dots, p_5$.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1]$ and that the two lines, one through p_1, p_4 and the other one through p_2, p_5, meet at $[1:1:1]$, namely p_4 is infinitely near p_1 of the first order in the direction of the line $y = z$, and p_5 is infinitely near p_2 of the first order in the direction of the line $x = z$. In other words, p_4 has standard coordinates $p_4 = (p_1, 1)$, and p_5 has standard coordinates $p_5 = (p_2, 1)$. Then, it is clear that the conic

$$xy - yz - xz = 0$$

is the unique irreducible conic passing through $p_1, \dots, p_5$. □

Lemma 4.7. — *Let* $p_1, p_2 \in \mathbb{P}^2$ *and* $p_5 \succ_1 p_3 \succ_1 p_1, p_4 \succ_1 p_2$*, such that* $p_5 \not\odot p_1$*, and no three among* $p_1, \ldots, p_5$ *are collinear. Then, there exists a unique irreducible conic passing through* $p_1, \ldots, p_5$*.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0], p_2 = [0:1:0]$, and that the two lines, one through p_1, p_3 and the other one through p_2, p_4, meet at $[0:0:1]$, namely p_3 is infinitely near p_1 of the first order in the direction of the line $y = 0$, and p_4 is infinitely near p_2 of the first order in the direction of the line $x = 0$. In other words, p_3 has standard coordinates $p_3 = (p_1, \infty)$, and p_4 has standard coordinates $p_4 = (p_2, \infty)$. Then, p_5 has standard coordinates $p_5 = (p_1, \infty, t_5)$, where $t_5 \in \mathbb{C}^*$; indeed, if $t_5 = 0$, then p_5, p_3, p_1 would be collinear, and if $t_5 = \infty$, then $p_5 \odot p_1$. One can check that the conic

$$t_5 xy - z^2 = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. □

Remark 4.8. — The previous lemmas explain Remark V.4.2.1 in [9] better.

Lemma 4.9. — *Let* $p_1, p_2 \in \mathbb{P}^2$ *and* $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_1$*, such that* $p_4 \not\odot p_1$*,* $p_5 \not\odot p_3$*, and no three among* $p_1, \ldots, p_4$ *are collinear. Then, there exists a unique irreducible conic passing through* $p_1, \ldots, p_5$*.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, and p_3, p_4 have standard coordinates, respectively, $p_3 = (p_1, \infty)$ and $p_4 = (p_1, \infty, 1)$, according to the proof of the previous lemma. Then, p_5 has standard coordinates $p_5 = (p_1, \infty, 1, t_5)$, where $t_5 \in \mathbb{C}$; indeed, if $p_5 = \infty$, then we would have $p_5 \odot p_3$, contradicting the hypothesis. One can check that the conic

$$xy + t_5 yz - z^2 = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. □

Lemma 4.10. — *Let* $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1 \in \mathbb{P}^2$ *such that* $p_3 \not\odot p_1$*,* $p_4 \not\odot p_2$*,* $p_5 \not\odot p_3$ *and* p_1, p_2, p_3 *are not collinear. Then, there exists a unique irreducible conic passing through* $p_1, \ldots, p_5$*.*

Proof. — Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0]$ and p_2, p_3, p_4 have standard coordinates, respectively, $p_2 = (p_1, \infty)$, $p_3 = (p_1, \infty, 1)$, $p_4 = (p_1, \infty, 1, 0)$, according to the proof of the previous lemma. Then, p_5 has standard coordinates $p_5 = (p_1, \infty, 1, 0, t_5)$, where $t_5 \in \mathbb{C}$; indeed, if $t_5 = \infty$, then we would have $p_5 \odot p_3$, contradicting the hypothesis. One can check that the conic

$$xy - z^2 + t_5 y^2 = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. □

5. Lengths of plane Cremona maps

A plane Cremona map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be written as

$$\varphi([x:y:z]) = [f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)],$$

where $f_i \in \mathbb{C}[x,y,z]$, $i = 0,1,2$, are homogeneous polynomials of the same degree, say d, that is called the *degree* of φ if f_0, f_1, f_2 have no common factor.

Plane Cremona maps of degree 1 are automorphisms of $\mathbb{P}^2$, i.e., elements of $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$. Plane Cremona maps of degree 2 (and 3) are called *quadratic* (and *cubic*).

Definition 5.1. — Let us say that two plane Cremona maps $\varphi, \varphi'\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are *equivalent* if there exist two automorphisms $\alpha, \alpha' \in \mathrm{Aut}(\mathbb{P}^2)$, such that

$$\varphi' = \alpha' \circ \varphi \circ \alpha.$$

Remark 5.2. — It is very well known that quadratic planar Cremona maps are equivalent to one and only one of the following three maps: the elementary quadratic transformation σ;

$$\rho([x:y:z]) = [xy : z^2 : yz]; \qquad \tau([x:y:z]) = [x^2 : xy : y^2 - xz]. \tag{3}$$

Recall that a quadratic planar Cremona map is called *ordinary* (or *of the first type*) if it has three proper base points, i.e., if it is equivalent to σ. Furthermore, a quadratic planar Cremona map is called *of the second type* if it has two proper base points, i.e., if it is equivalent to ρ, and, finally, a quadratic planar Cremona map is called *of the third type* if it has only one proper base point, i.e., if it is equivalent to τ.

Recall that the linear system defining a plane Cremona map φ of degree d has base points $p_1, \dots, p_r \in \mathcal{B}(\mathbb{P}^2)$ of respective multiplicity $m_1, \dots, m_r$ that satisfy the following equations:

$$d^2 - 1 = \sum_{i=1}^{r} m_i^2, \qquad 3(d-1) = \sum_{i=1}^{r} m_i. \tag{4}$$

Let us say that a base point p_i is *simple* if its multiplicity m_i is 1.

Definition 5.3. — A plane Cremona map is called *de Jonquières* if it has degree d and a base point of multiplicity $d-1$.

Equations (4) imply that plane Cremona maps of degree 2 and 3 are de Jonquières.

According to the Noether-Castelnuovo Theorem, any plane Cremona map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be written as

$$\varphi = \alpha_n \circ \sigma \circ \alpha_{n-1} \circ \sigma \circ \cdots \circ \alpha_1 \circ \sigma \circ \alpha_0, \tag{5}$$

where $\alpha_i \in \mathrm{Aut}(\mathbb{P}^2)$ for any $i = 0, \dots, n$, for some integer n.

Definition 5.4. — Let us call (5) a *decomposition* of φ. Let us say that a decomposition (5) is *minimal* if n is minimal among all decompositions of φ. Let us call such n the *ordinary quadratic length* of φ and let us denote it by $\mathrm{oql}(\varphi)$.

Therefore, the ordinary quadratic length of a plane Cremona map φ of degree ≥ 2 is the minimum n such that there exist ordinary quadratic maps $\psi_1, \psi_2, \ldots, \psi_n$ with

$$\varphi = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_2 \circ \psi_1. \tag{6}$$

Definition 5.5. — Let us call the *quadratic length* of plane Cremona map φ the minimum n such that there exists a decomposition (6) where ψ_i is a (not necessarily ordinary) quadratic map, for each $i = 1, \ldots, n$, and denote it by $\mathrm{ql}(\varphi)$.

Recall that Blanc and Furter in [3] defined the *length* of a plane Cremona map φ as the minimum n, such that there exists a decomposition (6) where ψ_i is a *de Jonquières* map, for each $i = 1, \ldots, n$, and denoted it by $\mathrm{lgth}(\varphi)$. Clearly, one has that

$$\mathrm{lgth}(\varphi) \leq \mathrm{ql}(\varphi) \leq \mathrm{oql}(\varphi).$$

Definition 5.6. — A plane Cremona map φ is called *involutory*, or *an involution*, if $\varphi^{-1} = \varphi$.

Example 5.7. — If $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$, then $\alpha^{-1} \circ \sigma \circ \alpha$ is an involutory ordinary quadratic map, which we call *standard involution.*

Remark 5.8. — In order to compute the ordinary quadratic length of plane Cremona maps, it suffices to work with standard involutions, cf. Théorème 5.24 in [5].

Remark 5.9. — Two equivalent plane Cremona maps clearly have the same length, quadratic length and ordinary quadratic length.

The following lemma is a straightforward application of the definitions.

Lemma 5.10. — *Let* $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a plane Cremona map. Then,*

$$\mathrm{oql}(\varphi) = 0 \textit{ if and only if } \varphi \in \mathrm{Aut}(\mathbb{P}^2).$$

Moreover, one has

- $\mathrm{oql}(\varphi) = 1$ *if and only if* φ *is an ordinary quadratic map;*
- $\mathrm{ql}(\varphi) = 1$ *if and only if* φ *is a quadratic map;*
- $\mathrm{lgth}(\varphi) = 1$ *if and only if* φ *is a de Jonquières map.*

Corollary 5.11. — *Let* $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a plane Cremona map of degree* $d \geq 3$. *Then,*

$$\operatorname{oql}(\varphi) \geq \operatorname{ql}(\varphi) \geq 2.$$

Example 5.12. — Let ρ be the quadratic map defined in (3). It is classically very well known that $\operatorname{oql}(\rho) = 2$. A minimal decomposition of ρ is:

$$\rho = [x : z - y : z] \circ \sigma \circ [x : y + z : z] \circ \sigma \circ [x : y - z : z]. \tag{7}$$

Example 5.13. — Let τ be the quadratic map defined in (3). It is classically well known that τ is the composition of two quadratic maps of the second type and, therefore, the composition of four ordinary quadratic maps. A decomposition of τ, given in [5], is:

$$\begin{aligned}\tau = [y - x : 2y - x : x - y + z] \circ \sigma \circ [x + z : x : y] \circ \sigma \circ [-y : x - 3y + z : x] \circ \\ \circ \sigma \circ [x + z : x : y] \circ \sigma \circ [y - x : -2x + z : 2x - y].\end{aligned} \tag{8}$$

However, we found no reference with a proof that $\operatorname{oql}(\tau) = 4$, hence that the above decomposition is minimal, even if we believe that it was classically known. This fact can be shown as a consequence of our Theorem 1.3; see Corollary 10.12 later.

Corollary 5.14. — *Let* $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a plane Cremona map of degree* $d \geq 5$. *Then,*

$$\operatorname{oql}(\varphi) \geq \operatorname{ql}(\varphi) \geq 3.$$

Proof. — We claim that if $\operatorname{ql}(\varphi) \leq 2$, then $\deg(\varphi) \leq 4$. This is trivial if $\operatorname{ql}(\varphi) \leq 1$. Suppose that $\operatorname{ql}(\varphi) = 2$, namely $\varphi = \rho_2 \circ \rho_1$, where ρ_1, ρ_2 are quadratic maps. Let p_1, p_2, p_3 be the base points of ρ_2. If m_1, m_2, m_3 are the multiplicities of ρ_1^{-1} at p_1, p_2, p_3, respectively, then

$$\deg(\varphi) = \deg(\rho_2 \circ \rho_1) = 4 - m_1 - m_2 - m_3 \leq 4,$$

which is our claim. □

Lemma 5.15. — *Let* $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a plane de Jonquières map of degree* $d \leq 5$. *Then,*

$$\operatorname{oql}(\varphi) \geq \operatorname{ql}(\varphi) \geq d - 1.$$

Proof. — It is trivial if $d \leq 3$. Let us first consider the case $d = 4$.

By contradiction, suppose that $\operatorname{ql}(\varphi) \leq 2$. Clearly, $\operatorname{ql}(\varphi)$ cannot be less than 2, so we can write $\varphi = \varrho_2 \circ \varrho_1$, where ϱ_1, ϱ_2 are two quadratic planar Cremona maps. In other words, one has that $\varphi \circ \varrho_1^{-1}$ is the quadratic map ϱ_2. We claim that the composition $\varphi \circ \varrho_1^{-1}$ always has degree ≥ 3, which is a contradiction.

We now prove our claim. Suppose that $p_0, p_1, \ldots, p_6$ are the base points of φ, where p_0 is the triple base point, and $p_1, \ldots, p_6$ are simple base points.

We distinguish four possibilities:

- if ϱ_1 has base points p_0, p_i, p_j with $0 < i < j \leq 6$, then $\varphi \circ \varrho_1^{-1}$ has degree 3;
- if ϱ_1 has base points p_0, p_i with $0 < i \leq 6$ and p_j is not a base point of ϱ_1 for any j, such that $0 \leq j \leq 6$ and $j \neq 0, i$, then $\varphi \circ \varrho_1^{-1}$ has degree 4;
- if ϱ_1 has base point p_0 and $p_1, \ldots, p_6$ are not base points of ϱ_1, then $\varphi \circ \varrho_1^{-1}$ has degree 5;
- if p_0 is not a base point of ϱ_1, then $5 \leq \deg(\varphi \circ \varrho_1^{-1}) \leq 8$.

Our claim is proved.

We are left with the case $d = 5$.

By contradiction, suppose that $\mathrm{ql}(\varphi) \leq 3$, hence, $\mathrm{ql}(\varphi) = 3$ by Corollary 5.14, and we can write $\varphi = \varrho_3 \circ \varrho_2 \circ \varrho_1$, where $\varrho_1, \varrho_2, \varrho_3$ are quadratic planar Cremona maps. In other words, one has that $\varphi \circ \varrho_1^{-1} = \varrho_3 \circ \varrho_2$ has quadratic length 2.

Let p_0 be the base point of multiplicity 4 of φ. There are two cases: either p_0 is a base point of ϱ_1 or p_0 is not a base point of ϱ_1.

In the former case, the map $\varphi \circ \varrho_1^{-1} = \varrho_3 \circ \varrho_2$ is a de Jonquières map of degree d' with $4 \leq d' \leq 6$. If $d' = 5, 6$, then Corollary 5.14 gives a contradiction. Otherwise $d' = 4$, which is another contradiction with the first part of this proof.

In the latter case, the map $\varphi \circ \varrho_1^{-1} = \varrho_3 \circ \varrho_2$ has degree d'' with $7 \leq d'' \leq 10$, and we again have a contradiction with Corollary 5.14. □

6. Enriched weighted proximity graphs

Definition 6.1. — Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map. Let us associate to φ a *weighted digraph* G_φ, called the *weighted proximity graph of (the base points of)* φ, defined as follows:

- the vertices of G_φ are the base points $p_1, \ldots, p_r \in \mathcal{B}(\mathbb{P}^2)$ of φ;
- there is an arrow $p_i \to p_j$, if and only if p_i is proximate to p_j;
- each vertex p_i is weighted with the multiplicity of φ at p_i.

It is easy to check that (weighted) proximity graphs have the property of being *admissible*, according to the following definition.

Definition 6.2. — Let us say that a digraph is *admissible* if it is acyclic and satisfies the following three properties:

(i) each vertex has outdegree at most 2;
(ii) if a vertex u has outdegree 2, say $u \to v$ and $u \to w$, then either $v \to w$ or $w \to v$;
(iii) fixing two vertices v and w, there exists at most one vertex u, such that $u \to v$ and $u \to w$.

Remark 6.3. — Property (i) for a proximity graph follows from Remark 2.7.

Example 6.4. — A de Jonquières map of degree d has weighted proximity graph with $2d-1$ vertices, one with weight $d-1$ and the other $2d-2$ vertices with weight 1.

Remark 6.5. — Note that the number of connected components of G_φ equals the number of proper base points in $\mathbb{P}^2$ among the base points $p_1, \ldots, p_r \in \mathcal{B}(\mathbb{P}^2)$ of φ.

Remark 6.6. — Clearly, two equivalent plane Cremona maps have the same weighted proximity graph. The converse is true for quadratic maps, but it is not true in general.

Example 6.7. — We will see later that the two cubic planar Cremona maps

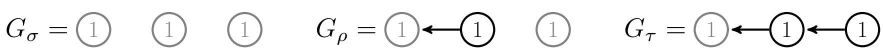
$$\varphi_{10} = [x^3 : y^2 z : xyz] \quad \text{and} \quad \varphi_{11} = [x(y^2+xz) : y(y^2+xz) : xyz]$$

have the same weighted proximity graph

but they are not equivalent.

Notation 6.8. — When we draw the weighted proximity graph of a plane Cremona map φ, for the readers' convenience we write *proper base points* in gray and *infinitely near points* in black.

Example 6.9. — Let σ, ρ and τ be the quadratic maps defined in (1) and (3). Their respective proximity graphs G_σ, G_ρ and G_τ are:



Remark 6.10. — Let

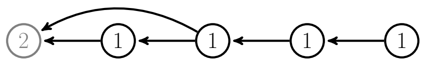
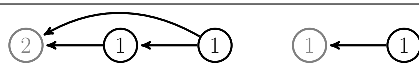
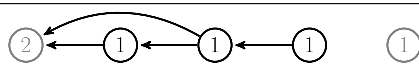
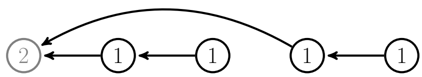
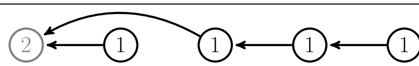
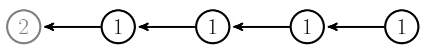
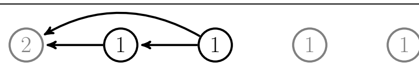
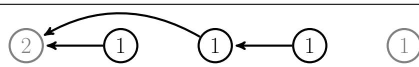
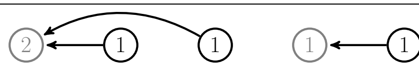
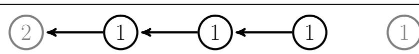
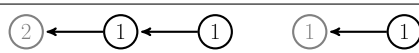
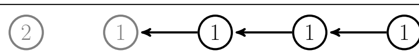
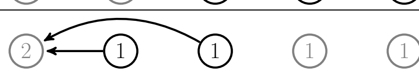
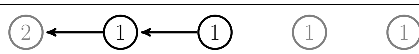
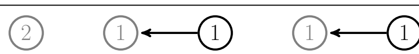
Then, there is no plane Cremona map with G or G' as a weighted proximity graph; cf. the proximity inequality 2.8 and Remark 4.1.

We now classify weighted proximity graphs of cubic planar Cremona maps.

Theorem 6.11. — *There are exactly 21 weighted proximity graphs of cubic planar Cremona maps, up to isomorphism, which are listed in Table 6.1.*

Proof of Theorem 6.11. — Weighted proximity graphs of cubic planar Cremona maps have five vertices, one with weight 2 and the other four with weight 1. Moreover, the proximity inequalities imply that only the double point may have satellite points, and there can be at most one of them. For the same reason, a simple base point may have at most one proximate point, while the double point may have at most two proximate points. We claim that these

Table 6.1. Weighted proximity graphs of cubic planar Cremona maps.

n°	Weighted proximity graph
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	

conditions are enough to find the 21 weighted proximity graphs of cubic planar Cremona maps, which are listed in Table 6.1.

Indeed, we may start from the weighted graph with no arrow, which is number 21 in the list of Table 6.1. We then add one arrow at each time in such a way that the graph is still admissible, and the weights satisfy the proximity inequalities for all vertices. For example, if we add one arrow to graph 21, then we find exactly two non-isomorphic weighted proximity graphs, which are numbers 19 and 20. If we add a second arrow, then we find other 5 graphs, which are numbers 14–18. And so on; in the following step we find the graphs with three arrows, which are numbers 7–13. In the next step, we find numbers 2–6 with four arrows, and finally there is only one graph, number 1, with five arrows. □

Definition 6.12. — Let us add to the weighted proximity graph G_φ of a *cubic* planar Cremona map φ the list of lines passing through *three* base points of φ. Let us call this object the *enriched weighted proximity graph* of φ.

Remark 6.13. — These lines are *unexpected*, in the sense that three points in general position are not aligned.

A line through three base points of a cubic planar Cremona map φ cannot pass through the (proper) base point of multiplicity 2, otherwise the linear system defining the map would be reducible by the Bézout theorem. For the same reason, a line cannot pass through all four simple base points of φ. Furthermore, there cannot be two different such lines, because they should have two points in common.

Notation 6.14. — The line passing through three base points of a cubic planar Cremona map are indicated as a gray dashed curve in the pictures of weighted proximity graphs.

Theorem 6.15. — *There are exactly 31 enriched weighted proximity graphs of cubic planar Cremona maps, up to isomorphism, listed in Table 1.1.*

Proof. — Recall that a line ℓ passes through an infinitely near point p only if ℓ passes through the proper point $q \in \mathbb{P}^2$, such that $p \succ q$, and the strict transform of ℓ passes through p. Therefore, the enriched weighted proximity graph cannot include a line passing through a base point infinitely near the base point of multiplicity 2, by Remark 6.13.

Hence, there is no line through three base points in the weighted proximity graphs 1–11, 14 and 15 in Table 6.1.

Let us denote by p_1 the base point of multiplicity 2 and by $p_2, \ldots, p_5$ the other simple base points going from left to right in the pictures of the weighted proximity graphs in Table 6.1.

The weighted proximity graph 12 in Table 6.1 may have a line through the proper simple base point p_3 and both of its infinitely near base points, which are p_4 and p_5. Accordingly, we find the two enriched weighted proximity graphs 10 and 11 in Table 1.1.

Similarly, the weighted proximity graph 13 in Table 6.1 may have a line through p_2, p_3, p_4, and we find the two enriched weighted proximity graphs 8 and 9 in Table 1.1.

Then, the weighted proximity graph 16 in Table 6.1 may have a line through p_3, p_4, p_5, and we find the two enriched weighted proximity graphs 22 and 23 in Table 1.1.

The weighted proximity graph 17 in Table 6.1 may have either a line through p_2, p_4, p_5 or a line through p_2, p_3, p_4, which, however, give two isomorphic enriched weighted proximity graphs; hence, we find the two enriched weighted proximity graphs 20 and 21 in Table 1.1.

The weighted proximity graph 18 in Table 6.1 may have either a line through p_3, p_4, p_5 or a line through p_2, p_3, p_4. Accordingly, we find the three enriched weighted proximity graphs 17, 18 and 19 in Table 1.1.

The weighted proximity graph 19 in Table 6.1 may have a line through p_3, p_4, p_5, and we find the two enriched weighted proximity graphs 28 and 29 in Table 1.1.

The weighted proximity graph 20 in Table 6.1 may have either a line through p_2, p_3, p_4 or a line through p_2, p_4, p_5. (There could also be a line through p_3, p_4, p_5, but the resulting enriched weighted proximity graph would be isomorphic to a previous one.) Accordingly, we find the three enriched weighted proximity graphs 24, 25 and 26 in Table 1.1.

Finally, the weighted proximity graph 21 in Table 6.1 may have four different lines, which, however, give four isomorphic enriched weighted proximity graphs. Hence, we find the two enriched weighted proximity graphs 30 and 31 in Table 1.1. □

7. Height of points with respect to a plane Cremona map

DEFINITION 7.1. — Let φ be a plane Cremona map. Let us define the *height $h_\varphi(p)$ of a point $p \in \mathcal{B}(\mathbb{P}^2)$ with respect to φ* as follows:

$$\mathrm{h}_\varphi(p) = \begin{cases} 0 & \text{if } p \text{ is not a base point of } \varphi, \\ 1 & \text{if } p \text{ is a proper base point of } \varphi, \\ n+1 & \text{if } p \text{ is a base point of } \varphi \text{ and } p \succ_n p' \in \mathbb{P}^2. \end{cases}$$

DEFINITION 7.2. — Let φ be a plane Cremona map. Let us also define the *load* of a proper base point p with respect to φ as follows:

$$\mathrm{load}_\varphi(p) = \sharp\big\{q \text{ is a base point of } \varphi \;\big|\; q \succ p\big\} + 1,$$

which is the number of base points of φ that are infinitely near p increased by 1.

Remarks 7.3. — (i) If p is a simple proper base point of φ, then $\mathrm{load}_\varphi(p)$ is equal to the maximum height of base points that are infinitely near p; indeed, the proximity inequality implies that base points that are infinitely near p cannot be satellite; in other words, there is a sequence $p_n \succ_1 p_{n-1} \succ_1 \cdots \succ_1 p_1 \succ_1 p$, where p_i is a base point infinitely near p of order i, $i = 1, \dots, n$.

(ii) If φ is a de Jonquières map of degree d, and it has a unique proper base point p, then $\mathrm{load}_\varphi(p) = 2d - 1$.

Notation 7.4. — Let $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$ and set $\varrho = \alpha^{-1} \circ \sigma \circ \alpha$, which is an involutory ordinary quadratic map based at three points $p_1, p_2, p_3 \in \mathbb{P}^2$. Denote by ℓ_1 (or ℓ_2, ℓ_3) the line passing through p_2 and p_3 (or p_1 and p_3, p_1 and p_2) and set $T = \ell_1 \cup \ell_2 \cup \ell_3$.

Setting $\varphi\colon S \to \mathbb{P}^2$ the blowing up of $\mathbb{P}^2$ at p_1, p_2, p_3, recall that $\tilde{\varrho} = \varphi^{-1} \circ \varrho \circ \varphi \in \mathrm{Aut}(S)$ and, therefore, it induces a bijection $\bar{\varrho}\colon \mathcal{B}(\mathbb{P}^2) \to \mathcal{B}(\mathbb{P}^2)$, such that:

- if $p = p_i$, $i = 1, 2, 3$, then $\bar{\varrho}(p) = p$;
- if $p \in \mathbb{P}^2 \setminus T$, then $\bar{\varrho}(p) = \varrho(p)$;
- if $p \in \ell_i \setminus \{p_j, p_k\}$, $\{i, j, k\} = \{1, 2, 3\}$, then $\bar{\varrho}(p) \succ_1 p_i$ in the direction of the line passing through p_i and p;
- if $p \succ_1 p_i$ in the direction of the line ℓ_j, where $\{i, j\} \subset \{1, 2, 3\}$, then $\bar{\varrho}(p) \succ_1 p_j$ in the direction of the line ℓ_i;
- if $p \succ_1 p_i$ (not lying on ℓ_j, ℓ_k, $\{i, j, k\} = \{1, 2, 3\}$), then $\bar{\varrho}(p)$ is a proper point of ℓ_i.

Let us say that $\bar{\varrho}(p) \in \mathcal{B}(\mathbb{P}^2)$ is the point corresponding to $p \in \mathcal{B}(\mathbb{P}^2)$ via ϱ.

Proposition 7.5. — *Let p_1, p_2, p_3 be the base points of an involutory ordinary quadratic planar Cremona map $\varrho = \alpha^{-1} \circ \sigma \circ \alpha$, where $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$. Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree $d > 1$ with base points $p_4, \dots, p_r$ and possibly p_1, p_2, p_3. Denote by m_i the multiplicity of φ at p_i, $i = 1, \dots, r$ (that is $m_i = 0$ if p_i is not a base point of φ, $i = 1, 2, 3$). Denote by $\bar{\varrho}(p)$ the point corresponding to p via ϱ as in Notation 7.4.*

Then, the composite map $\varphi \circ \varrho^{-1} = \varphi \circ \varrho$ has degree $d - \varepsilon$, where

$$\varepsilon = m_1 + m_2 + m_3 - d,$$

and it has $\bar{\varrho}(p_i)$, $i = 4, \dots, r$, as the base point of multiplicity m_i. Furthermore, it has multiplicity $m_i - \varepsilon \geq 0$ at p_i, $i = 1, 2, 3$ (that is, p_i is not a base point of $\varphi \circ \varrho$ when $\varepsilon = m_i$).

Proof. — Cf. Proposition 4.2.5 in [1]. □

Lemma 7.6. — *Let φ be a plane Cremona map and $\varrho = \alpha^{-1} \circ \sigma \circ \alpha$ an involutory ordinary quadratic map, where $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$. If $p \in \mathcal{B}(\mathbb{P}^2)$ and $\bar{p} = \bar{\varrho}(p) \in \mathcal{B}(\mathbb{P}^2)$ as in Notation 7.4, then*

$$-1 \leq \mathrm{h}_\varphi(p) - \mathrm{h}_{\varphi \circ \varrho}(\bar{p}) \leq 1.$$

Proof. — Set $\varphi' = \varphi \circ \varrho$. Let us see the possible cases.

- If p is not a base point of φ, that is, $\mathrm{h}_\varphi(p) = 0$, then either $\bar{p}$ is not a base point of φ' or $\bar{p}$ is a proper base point of φ' by Proposition 7.5 and Notation 7.4. In the former case, one has $\mathrm{h}_{\varphi'}(\bar{p}) = 0$, whereas in the latter case, one has $\mathrm{h}_{\varphi'}(\bar{p}) = 1$, and the assertion follows.
- If p is a proper base point of φ, that is, $\mathrm{h}_\varphi(p) = 1$, then Proposition 7.5 and Notation 7.4 imply that three cases may occur:
  1. $\bar{p}$ is not a base point of φ',
  2. $\bar{p}$ is still a proper base point of φ',
  3. $\bar{p}$ is a base point of φ' that is infinitely near (of order 1) a proper base point;

  accordingly, one has $\mathrm{h}_{\varphi'}(\bar{p}) = 0$, $\mathrm{h}_{\varphi'}(\bar{p}) = 1$, $\mathrm{h}_{\varphi'}(\bar{p}) = 2$, and the assertion follows.
- If p is a base point of φ, and p is infinitely near p' of order n, where p' is a proper base point of φ, that is, $\mathrm{h}_\varphi(p) = n + 1$ and $\mathrm{h}_\varphi(p') = 1$, then the previous analysis shows that $0 \leq \mathrm{h}_{\varphi'}(\bar{p}') \leq 2$ and accordingly $n \leq \mathrm{h}_{\varphi'}(\bar{p}) \leq n + 2$, which is the assertion.

We conclude that the assertion holds in any case. □

Proposition 7.7. — *Let φ be a plane Cremona map. Then*

$$\mathrm{oql}(\varphi) \geq \max\{\mathrm{h}_\varphi(p) \mid p \in \mathcal{B}(\mathbb{P}^2)\}.$$

Proof. — Let us set $n = \mathrm{oql}(\varphi)$ and let

$$\varphi = \alpha \circ \varrho_n \circ \varrho_{n-1} \circ \cdots \circ \varrho_2 \circ \varrho_1$$

be a minimal decomposition of φ, where ϱ_i, $i = 1, \ldots, n$, is an involutory ordinary quadratic map, and α is an automorphism of $\mathbb{P}^2$. We proceed by induction on n. Let us set

$$m(\varphi) = \max\{\mathrm{h}_\varphi(p) \mid p \in \mathcal{B}(\mathbb{P}^2)\}.$$

The assertion is clearly true for $n = 0, 1$ because an automorphism has no base point, and an ordinary quadratic map has exactly three points of height 1.

We then suppose that $n \geq 2$, and we denote $\varphi \circ \varrho_1$ by φ', so that $\mathrm{oql}(\varphi') = n-1$ and by induction hypothesis $n-1 \geq m(\varphi')$. Now Lemma 7.6 implies that

$$\mathrm{h}_{\varphi'}(\bar{\varrho}_1(p)) \geq \mathrm{h}_\varphi(p) - 1,$$

for any $p \in \mathcal{B}(\mathbb{P}^2)$, hence $m(\varphi') \geq m(\varphi) - 1$. Therefore, we conclude that

$$n = \mathrm{oql}(\varphi) = (n-1) + 1 \geq m(\varphi') + 1 \geq m(\varphi),$$

which is the assertion. □

8. Comparison with the classification in [5]

In this section, we compare our classification with the one in [5].
We will freely use Notation 2.1.

REMARKS 8.1. — The classification in [5] is not complete.

- Our type 15 in Table 1.2 does not occur in their list, even if it is equivalent to the inverse of their type 11.
- Type 17 in [5], which we denote by ψ_{17}, is equivalent to our type 28 in Table 1.2 with $\gamma_0 = -1$, which we denote by $\varphi_{28,-1}$, because

$$[y : y + z : x] \circ \varphi_{28,-1} = \psi_{17}.$$

However, our type 28 with $\gamma \neq -1$ does not occur in the list in [5]. This explains why we added † at type 17 in the third column of Table 1.2.

REMARKS 8.2. —

- In [5], their type 19, which we denote by ψ_{19}, is equivalent to their type 18 with parameter $\gamma_0 = -3/\sqrt{2}$, which we denote by ψ_{18,γ_0}. Indeed, one has

$$\psi_{19} = [\sqrt{2}y + z : -z : x - \sqrt{2}y] \circ \psi_{18,\gamma_0} \circ [\sqrt{2}x : x - y : y - z].$$

- Similarly, in [5], their type 31, which we denote by ψ_{31}, is equivalent to their type 30 with parameter $\gamma_0 = 3/\sqrt{2}$, which we denote by ψ_{30,γ_0}. Indeed, one has

$$\psi_{31} = [y + \sqrt{2}x : -y : 2(z - y)] \circ \psi_{30,\gamma_0} \circ [x + y : -\sqrt{2}y : x + 2z].$$

This explains why types 19 and 31 in [5] do not appear in the third column of Table 1.2.

REMARKS 8.3. —

- Let φ_{24} be the map 24 in Table 1.2. Then, φ_{24} is equivalent to type 15 in [5], which we denote by ψ_{15}. Indeed, one has

$$\varphi_{24} \circ [x : x + y : x + y + z] = \psi_{15}.$$

- Let $\varphi_{26,\gamma}$ be the map 26 with parameter $\gamma \neq 0, 1$ in Table 1.2. Let $\psi_{29,t}$ be the map of type 29 with parameter $t \neq 0, 1$ in [5]. Then, one has that

$$[ty - x : t(y - tz) : ty] \circ \varphi_{26,\gamma_0} \circ [-tx : y : y + z] = \psi_{29,t},$$

where $\gamma_0 = 1/(1 - t)$, which shows that φ_{26,γ_0} is equivalent to $\psi_{29,t}$.
- Let $\varphi_{27,\gamma}$ be the map 27 with parameter $\gamma \neq 0, 1$ in Table 1.2. Let $\psi_{16,t}$ be the map of type 16 with parameter t such that $t^2 \neq 4$ in [5]. Then, one has that $\psi_{16,t}$ and

$$\varphi_{27,\gamma_0} \circ [-(t_- x + y) : t_+ x + y : z],$$

where $\gamma_0 = t_-/t_+$ are defined by the same homaloidal net, and, therefore, φ_{27,γ_0} is equivalent to $\psi_{16,t}$.

- Let $\varphi_{29,\gamma}$ be the map 29 with parameter $\gamma \neq 0, 1$ in Table 1.2. Let $\psi_{30,t}$ be the map of type 30 with parameter t, such that $t^2 \neq 4$ in [5]. Then, one has that $\psi_{30,t}$ and
$$\varphi_{29,\gamma_0} \circ [t^\bullet y : t_+(y + t_-x) : t_+y + x + z)],$$
where $\gamma_0 = 1/2 + t/(2t^\bullet)$ are defined by the same homaloidal net, and, therefore, φ_{29,γ_0} is equivalent to $\psi_{30,t}$.
- Let $\varphi_{30,\gamma}$ be the map 30 with parameter $\gamma \neq 0, 1$ in Table 1.2. Let $\psi_{18,t}$ be the map of type 18 with parameter t such that $t^2 \neq 4$ in [5]. Then, one has that φ_{30,γ_0}, where $\gamma_0 = tt_+ - 1$, and
$$\psi_{18,t} \circ [x : -t_+y : t_+y - t_-x - t^\bullet z]$$
are defined by the same homaloidal net, and, therefore, φ_{30,γ_0} is equivalent to $\psi_{18,t}$.
- Let $\varphi_{31,a,b}$ be the map 31 with two parameters a, b, such that $a \neq b$ and $a, b \neq 0, 1$ in Table 1.2. Let $\psi_{32,t,h}$ be the map of type 32 with two parameters t, h, such that $t^2 \neq 4$ and $h \neq t_\pm$ in [5]. Then, one has that $\psi_{32,t,h}$, and
$$\varphi_{31,a_0,b_0} \circ [t^\bullet x : -t_-x - y : -t_-x - y - z],$$
$$(a_0, b_0) = \left(\frac{(2 - tt_+)h}{h - t_+}, \frac{h}{h - t_+}\right)$$
are defined by the same homaloidal net, and, therefore, φ_{31,a_0,b_0} is equivalent to $\psi_{32,t,h}$.

Remarks 8.4. —

- Type 19 in [5], which we denote by ψ_{19}, is equivalent to $\varphi_{30,-1}$, which is type 30 in Table 1.2 with parameter $\gamma = -1$, because
$$[y - x + z : x - y : y - z] \circ \varphi_{30,-1} \circ [-x : y : z] = \psi_{19}.$$
- Type 31 in [5], which we denote by ψ_{31}, is equivalent to $\varphi_{29,-1}$, which is type 29 in Table 1.2 with parameter $\gamma = -1$, because
$$[-y : 2x - y - z : 2x] \circ \varphi_{29,-1} \circ [x : y : 2x + 2z] = \psi_{31}.$$

Remark 8.5. — In Section 6.4, Théorème 6.39 in [5], there is a list of decompositions in quadratic maps of their 32 types of cubic planar Cremona maps.

Note that the decompositions of their types 25 and 26 are exchanged, and the decomposition of their type 24 is incorrect.

In Table 8.1 we list decompositions into quadratic maps of our 31 types of cubic planar Cremona maps, cf. Table 1.2. These decompositions into quadratic maps turn out to be minimal, according to Theorem 1.3.

If one replaces the maps ρ and τ with their respective decompositions (7) and (8) into ordinary quadratic maps, one then gets decompositions of our 31 types of cubic planar Cremona maps into ordinary quadratic maps, which are not necessarily minimal.

TABLE 8.1. Decompositions into quadratic maps of the 31 maps in Table 1.2.

#	A decomposition into quadratic maps
1	$[x:z:y]\circ\rho\circ[z:y:x]\circ\tau\circ[z:y:-x]\circ\rho\circ[x:z:y]$
2	$[x+z:y:z]\circ\rho\circ[y-x:z:x]\circ\tau\circ[y:x:-z]$
3	$[z:x:y]\circ\rho\circ[z:y:x]\circ\tau\circ[z:x:-y]$
4	$[y:z:x]\circ\tau\circ[y:x:-z]\circ\tau\circ[x:z:-y]$
5	$[x:z:y]\circ\tau\circ[-z:-y:x+z]\circ\rho\circ[y-z:x:z]$
6	$[-y:z:x]\circ\rho\circ[x+y+2z:y+z:-z]\circ\rho\circ[x+z:x:y-x]$
7	$[-x-z:z:y]\circ\rho\circ[-z-y:x+y+z:z]\circ\rho\circ[z-x:y:x]$
8	$[y:x:-z]\circ\tau\circ[z:x+z:y]\circ\rho\circ[x-z:y:z]$
9	$[y:-x-z:z]\circ\rho\circ[-2z-x:x+y+z:z]\circ\rho\circ[x-y:z:y]$
10	$[y:x+z:z]\circ\rho\circ[z-y:x+z:y]\circ\rho\circ[z-x:y:x]$
11	$[z:x:y]\circ\sigma\circ[x:x+y:z]\circ\rho\circ[x:z:y]$
12	$[z:x:y]\circ\rho\circ[x:z:y]\circ\rho\circ[y:x:z]$
13	$[z:y:x]\circ\sigma\circ[y+x+z:z:y]\circ\rho\circ[z-y:x:y]$
14	$[x-z:-z:y]\circ\sigma\circ[x:y:x-z]\circ\rho\circ[y:z:x]$
15	$[x:z+x:y]\circ\rho\circ[y:z:x-y]\circ\sigma$
16	$[x:z:y+z]\circ\rho\circ[y:x-z-y:y+z]\circ\sigma\circ[x+z:y-x:x]$
17	$[z:y:x]\circ\rho\circ[y-x:x:z]\circ\rho\circ[x:z:y]$
18	$[x+z:z:-y]\circ\rho\circ[y-x:y-z:x]\circ\sigma$
19	$[z:x:y+z]\circ\rho\circ[z:x-y-z:y]\circ\sigma\circ[x+z:y:x]$
20	$[y:z:x+z]\circ\rho\circ[z-x-y:x:y]\circ\sigma$
21	$\sigma\circ[x:y:x+y+z]\circ\sigma$
22	$[y-z:z:x+z]\circ\rho\circ[z-x-y:x:x+y]\circ\sigma$
23	$[x:-y:z]\circ\sigma\circ[y+z:z:x+y+z]\circ\sigma\circ[z:x:-x-y]$
24	$[y+z:-z:x-z]\circ\rho\circ[x-y+z:y-x:x]\circ\sigma$
25	$[-x:z:y]\circ\sigma\circ[z:x+y:y+z]\circ\sigma\circ[z:y:-x-y]$
26	$[\gamma(\gamma x-2x+y)+x+z:\gamma(\gamma x-x+y):\gamma(\gamma x+y)]\circ\sigma\circ$ $\circ[\gamma((\gamma-1)x-\gamma y+z):\gamma(y-x):\gamma x]\circ\sigma$
27	$[\gamma(\gamma x+y):-\gamma(x+y):(\gamma-1)^2z]\circ\sigma\circ[\gamma x+y:-x-y:(\gamma-1)z]\circ\sigma$
28	$[x:z-y:2\gamma z-(1+\gamma)y]\circ\rho\circ[(1-\gamma)z:x-y:x-\gamma y]\circ\sigma$
29	$[y+z:y-x:\gamma(y-x-z)-x+y+z]\circ\sigma\circ$ $\circ[x-y:x-\gamma y:(1-\gamma)z-x+\gamma y]\circ\sigma$
30	$[\gamma^2x+(1-\gamma)y-z:\gamma(\gamma x-y):\gamma((\gamma+1)x-y)]\circ\sigma\circ$ $\circ[\gamma(y+z)-y:y+z:x+z]\circ\sigma\circ[z-x:y-x:x]$
31	$[a(a(x+(b-1)^2z)+by):a(ax+y):by-((b-1)z-x)a^2-(b((1-b)z-$ $-x)-y)a]\circ\sigma\circ[ax-by:y-x:(b-1)ax-b(a-1)y+(a-b)z]\circ\sigma$

Table 8.2. Decompositions into ordinary quadratic maps of some cubic planar Cremona maps.

#	A decomposition $\alpha_n \circ \sigma \circ \alpha_{n-1} \circ \sigma \circ \cdots \circ \alpha_1 \circ \sigma \circ \alpha_0$
1	$\alpha_6 = [27y + 225z : 12y : 8x - 8y]$, $\alpha_5 = [2x + 5y : 5y - x : 15x + 15z]$, $\alpha_4 = [2x + 2z : 5x : 3x + 10y - 2z]$, $\alpha_3 = [x - y : z + 2y - x : 2y]$, $\alpha_2 = [z : z - 2x : 2x + 2y - z]$, $\alpha_1 = [x - y : z - x + y : 2x - y]$, $\alpha_0 = [y : y + z : x]$
2	$\alpha_5 = [8y - 8x : x + z : 4x]$, $\alpha_4 = [x + y : y : z - x]$, $\alpha_3 = [2x : -y - 2x : y + 2x - 2z]$, $\alpha_2 = [y - x : x : x + z - y]$, $\alpha_1 = [x : z - x : y]$, $\alpha_0 = [x : z : x + y]$
3	$\alpha_5 = [4y : 4y + 3x : 4y + 4z]$, $\alpha_4 = [3x - z : z - y : y]$, $\alpha_3 = [9z + 3x : y : 3z - y]$, $\alpha_2 = [3y + 4z - x : x - z : 3x]$, $\alpha_1 = [y + z : x - y + z : y - z]$, $\alpha_0 = [2y : x + z : x - z]$
4	$\alpha_4 = [y + z : x + 2z : z - y]$, $\alpha_3 = [2x : y - z : y + z]$, $\alpha_2 = [y - 4x - 4z : x : z]$, $\alpha_1 = [y + z : x : y - z]$, $\alpha_0 = [2y : x + z : x - z]$
5	$\alpha_5 = [4y + 4z : 12z + x + 9y : 6y + 8z]$, $\alpha_4 = [2y + z : -2x - z : 2x + 2z]$, $\alpha_3 = [2y : 2y - z + x : z - y]$, $\alpha_2 = [2z + 2x - y : 2z - y : y]$, $\alpha_1 = [y - x - z : 2z + x : x + z]$, $\alpha_0 = [x - z : y : z]$
8	$\alpha_5 = [2y : -4x - 4y : 8x + 9y + z]$, $\alpha_4 = [2x - 2y : 2y - x : x + 2z]$, $\alpha_3 = [x + y : x : z - y - 2x]$, $\alpha_2 = [x + y : -2x : 2x + y + z]$, $\alpha_1 = [2x + y : y : 2x - y + 2z]$, $\alpha_0 = [-z : x + z : y + z]$
10	$\alpha_3 = [x + y : -y - z : y]$, $\alpha_2 = [z - x : x + y : -y]$, $\alpha_1 = [z : x - z : y + z - x]$, $\alpha_0 = [x : x + z : x + y]$
12	$\alpha_3 = [-x : x - z : x + y]$, $\alpha_2 = [y - x : x : y + z]$, $\alpha_1 = [y : x + y : z - x - y]$, $\alpha_0 = [-z : x + z : y - z]$

More precisely, one may check that such decompositions into ordinary quadratic maps are still minimal for types 6, 7, 9, 11, 13–31, according to Theorem 1.3.

Therefore, in Table 8.2, we give other decompositions into ordinary quadratic maps of types 1, 2, 3, 4, 5, 8, 10, 12, which turn out to be minimal, according to Theorem 1.3; cf. Table 1.1.

9. Proof of the classification Theorem 1.2

According to Theorem 6.15, there are 31 enriched weighted proximity graphs of cubic planar Cremona maps; they are listed in Table 1.1. We will show that a cubic planar Cremona map with enriched weighted proximity graph of type n, $1 \leq n \leq 31$, in Table 1.1 is equivalent to the map of type n in Table 1.2.

LEMMA 9.1. — *Let φ_1 be the map 1 in Table 1.2 and let ψ_1 be a map with enriched weighted proximity graph 1 in Table 1.1. Then, ψ_1 is equivalent to φ_1.*

Proof. — The base points of φ_1 are $p_0 = [1 : 0 : 0]$ of multiplicity 2 and $p_1, \ldots, p_4$ with $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1 \succ_1 p_0$ and $p_2 \odot p_0$, whose standard coordinates are $p_1 = (p_0, 0)$, $p_2 = (p_0, 0, \infty)$, $p_3 = (p_0, 0, \infty, -1)$, $p_4 = (p_0, 0, \infty, -1, 0)$.

The base points of ψ_1 are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ with $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_1 \succ_1 q_0$ and $q_2 \odot q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_0) = q_0$ and $\alpha_1(p_1) = q_1$, so that also $\alpha_1(p_2) = q_2$.

The base points of $\psi_1 \circ \alpha_1$ are then $p_0, p_1, p_2, q_3', q_4'$, where q_3' has standard coordinates $q_3' = (p_0, 0, \infty, u_3)$ for some $u_3 \in \mathbb{C}^*$ because, if u_3 were 0, then q_3' would be proximate to p_0, which is a contradiction, and, if u_3 were ∞, then q_3' would be proximate to p_1, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2 and that maps p_3 to q_3' is

$$\alpha_2([x : y : z]) = [-x : u_3 y : u_3 z].$$

The base points of $\psi_1 \circ \alpha_1 \circ \alpha_2$ are then $p_0, p_1, p_2, p_3, q_4''$, where q_4'' has standard coordinates $q_4'' = (p_0, 0, \infty, -1, u_4)$ for some $u_4 \in \mathbb{C}$ because, if u_4 were ∞, then q_4'' would be proximate to p_2, which is a contradiction.

An automorphism α_3 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps p_4 to q_4'' is

$$\alpha_3([x : y : z]) = [3x : 3y - u_4 z : 3z].$$

Therefore, the maps φ_1 and $\psi_1 \circ \alpha_1 \circ \alpha_2 \circ \alpha_3$ are defined by the same homaloidal net and, hence, φ_1 and ψ_1 are equivalent. □

LEMMA 9.2. — *Let φ_2 be the map 2 in Table 1.2 and let ψ_2 be a map with enriched weighted proximity graph 2 in Table 1.1. Then, ψ_2 is equivalent to φ_2.*

Proof. — The base points of φ_2 are $p_0 = [0 : 0 : 1]$ of multiplicity 2 and $p_1, \ldots, p_4$ with standard coordinates $p_1 = (p_0, 0)$, $p_2 = (p_0, 0, -1)$, $p_3 = (p_0, 0, -1, 0)$, $p_4 = (p_0, 0, -1, 0, 0)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, which is $C_1 \colon x^2 + yz = 0$. Let q_0 be the double base point of ψ_2 and let $q_1, \ldots, q_4$ be the simple base points of ψ_2. According to Lemma 4.10, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(q_0) = p_0$ and $\alpha(C_2) = C_1$. This forces $\alpha(q_i) = p_i$, $i = 1, 2, 3, 4$. Therefore, ψ_2 is equivalent to φ_2. □

LEMMA 9.3. — *Let φ_3 be the map 3 in Table 1.2 and let ψ_3 be a map with enriched weighted proximity graph 3 in Table 1.1. Then, ψ_3 is equivalent to φ_3.*

Proof. — The base points of φ_3 are $p_0 = [0 : 1 : 0]$ of multiplicity 2 and p_1, p_2, p_3, p_4, where $p_1 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_0$ with standard coordinates $p_1 = (p_0, \infty), p_2 = (p_0, 0), p_3 = (p_0, 0, -1)$ and $p_4 = (p_0, 0, -1, 0)$.

The base points of ψ_3 are q_0 of multiplicity 2 and $q_1, \dots, q_4$, where $q_1 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(q_i) = p_i$ for $i = 0, 1, 2$.

The base points of $\psi_3 \circ \alpha_1$ are then $p_0, p_1, p_2, q'_3, q'_4$, where q'_3 has standard coordinates $q'_3 = (p_0, 0, u_3)$ for some $u_3 \in \mathbb{C}^*$ because, if u_3 were 0, then q'_3 would be aligned with p_0 and p_2, which is a contradiction, and, if u_3 were ∞, then q'_3 would be proximate to p_0, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2 and that maps $p_3 = (p_0, 0, -1)$ to $q'_3 = (p_0, 0, u_3)$ is

$$\alpha_2([x : y : z]) = [x : -u_3 y : z].$$

The base points of $\psi_3 \circ \alpha_1 \circ \alpha_2$ are then $p_0, p_1, p_2, p_3, q''_4$ where q''_4 has standard coordinates $q''_4 = (p_0, 0, -1, u_4)$ for some $u_4 \in \mathbb{C}$ because, if u_4 were ∞, then q''_4 would be proximate to p_2, a contradiction.

An automorphism α_3 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps p_4 to q''_4 is

$$\alpha_3([x : y : z]) = [x : y - u_4 x : z].$$

Therefore, the maps φ_3 and $\psi_3 \circ \alpha_1 \circ \alpha_2 \circ \alpha_3$ are defined by the same homaloidal net and, hence, φ_3 and ψ_3 are equivalent. □

LEMMA 9.4. — *Let φ_4 be the map 4 in Table 1.2 and let ψ_4 be a map with enriched weighted proximity graph 4 in Table 1.1. Then, ψ_4 is equivalent to φ_4.*

Proof. — The base points of φ_4 are $p_0 = [0 : 1 : 0]$ of multiplicity 2 and $p_1, \dots, p_4$ where $p_3 \succ_1 p_1 \succ_1 p_0$ and $p_4 \succ_1 p_2 \succ_1 p_0$, with standard coordinates $p_1 = (p_0, \infty)$, $p_3 = (p_0, \infty, -1)$, $p_2 = (p_0, 0)$ and $p_4 = (p_0, 0, -1)$.

The base points of ψ_4 are q_0 of multiplicity 2 and $q_1, \dots, q_4$, where $q_3 \succ_1 q_1 \succ_1 q_0$ and $q_4 \succ_1 q_2 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$.

The base points of $\psi_4 \circ \alpha_1$ are then $p_0, p_1, p_2, q'_3, q'_4$, where q'_3 has standard coordinates $q'_3 = (p_0, \infty, u_3)$ for some $u_3 \in \mathbb{C}^*$, because if u_3 were 0, then q'_3 would be aligned with p_0 and p_1, which is a contradiction, and, if u_3 were ∞, then q'_3 would be proximate to p_0, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2 and that maps $p_3 = (p_0, \infty, -1)$ to $q'_3 = (p_0, \infty, u_3)$ is

$$\alpha_2([x : y : z]) = [-u_3 x : y : z].$$

The base points of $\psi_4 \circ \alpha_1 \circ \alpha_2$ are then $p_0, p_1, p_2, p_3, q''_4$, where q''_4 has standard coordinates $q''_4 = (p_0, 0, u_4)$ for some $u_4 \in \mathbb{C}^*$, because if u_4 were 0, then q'_4 would be aligned with p_0 and p_2, which is a contradiction, and if u_4 were ∞, then q''_4 would be proximate to p_0, which is a contradiction.

An automorphism α_3 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_0, 0, -1)$ to $q''_4 = (p_0, 0, u_4)$ is

$$\alpha_3([x : y : z]) = [x : \delta^2 y : \delta z],$$

where δ is a cube root of $-u_4$, i.e., $\delta^3 = -u_4$. Therefore, the maps φ_4 and $\psi_4 \circ \alpha_1 \circ \alpha_2 \circ \alpha_3$ are defined by the same homaloidal net and, hence, φ_4 and ψ_4 are equivalent. □

Lemma 9.5. — *Let φ_5 be the map 5 in Table 1.2 and let ψ_5 be a map with enriched weighted proximity graph 5 in Table 1.1. Then, ψ_5 is equivalent to φ_5.*

Proof. — The base points of φ_5 are $p_0 = [0 : 1 : 0]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_0$ and $p_3 \odot p_0$, with standard coordinates $p_2 = (p_0, \infty)$, $p_3 = (p_0, \infty, \infty)$ and $p_4 = (p_0, \infty, \infty, -1)$.

The base points of ψ_5 are $q_0 \in \mathbb{P}^2$ of multiplicity 2, $q_1 \in \mathbb{P}^2$, and q_2, q_3, q_4, where $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_0$ and $q_3 \odot q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$. It follows that also $\alpha_1(p_3) = q_3$.

The base points of $\psi_5 \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q'_4, where q'_4 has standard coordinates $q'_4 = (p_0, \infty, \infty, u_4)$ for some $u_4 \in \mathbb{C}^*$ because if u_4 were 0, then q'_4 would be proximate to p_0, which is a contradiction, and, if u_4 were ∞, then q'_4 would be proximate to p_2, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_0, \infty, \infty, -1)$ to $q'_4 = (p_0, \infty, \infty, u_4)$ is

$$\alpha_2([x : y : z]) = [u_4 x : -y : u_4 z].$$

Therefore, the maps φ_5 and $\psi_5 \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_5 and ψ_5 are equivalent. □

Lemma 9.6. — *Let φ_6 be the map 6 in Table 1.2 and let ψ_6 be a map with enriched weighted proximity graph 6 in Table 1.1. Then, ψ_6 is equivalent to φ_6.*

Proof. — The base points of φ_6 are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 1 : -1]$ and p_2, p_3, p_4, where $p_2 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_0$, with standard coordinates $p_2 = (p_0, 0)$, $p_3 = (p_0, \infty)$ and $p_4 = (p_0, \infty, -1)$.

The base points of ψ_6 are $q_0 \in \mathbb{P}^2$ of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4, where $q_2 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_6 \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q'_4, where q'_4 has standard coordinates $q'_4 = (p_0, \infty, u_4)$ for some $u_4 \in \mathbb{C}^*$ because if u_4 were 0, then q'_4 would be aligned with p_0 and p_3, which is a contradiction, and, if u_4 were ∞, then q'_4 would be proximate to p_0, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_0, \infty, -1)$ to $q'_4 = (p_0, \infty, u_4)$ is

$$\alpha_2([x : y : z]) = [u_4 x : u_4 y : -(u_4 + 1)x - z].$$

Therefore, the maps φ_6 and $\psi_6 \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_6 and ψ_6 are equivalent. □

LEMMA 9.7. — *Let φ_7 be the map 7 in Table 1.2 and let ψ_7 be a map with enriched weighted proximity graph 7 in Table 1.1. Then, ψ_7 is equivalent to φ_7.*

Proof. — The base points of φ_7 are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$ and p_2, p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_0$ with standard coordinates $p_2 = (p_0, 0), p_3 = (p_0, 0, -1), p_4 = (p_0, 0, -1, 0)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, which is $C_1 \colon x^2+yz = 0$. The base points of ψ_7 are q_0 of multiplicity 2 and $q_1, \ldots, q_4$, where $q_1 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_0$. According to Lemma 4.9, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_7 is equivalent to φ_7. □

LEMMA 9.8. — *Let φ_8 be the map 8 in Table 1.2 and let ψ_8 be a map with enriched weighted proximity graph 8 in Table 1.1. Then, ψ_8 is equivalent to φ_8.*

Proof. — The base points of φ_8 are $p_0 = [0 : 1 : 0]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with standard coordinates $p_2 = (p_1, \infty), p_3 = (p_1, \infty, 0)$ and $p_4 = (p_1, \infty, 0, 1)$.

The base points of ψ_8 are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4, where $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_1$, and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$. It follows that also $\alpha_1(p_3) = q_3$.

The base points of $\psi_8 \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4', where q_4' has standard coordinates $q_4' = (p_1, \infty, 0, u_4)$ for some $u_4 \in \mathbb{C}^*$ because if u_4 were 0, then q_4' would be aligned with p_1, p_2, p_3, which is a contradiction, and, if u_4 were ∞, then q_4' would be proximate to p_2, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_1, \infty, 0, 1)$ to $q_4' = (p_0, \infty, 0, u_4)$ is

$$\alpha_2([x : y : z]) = [x : u_4 y : z].$$

Therefore, the maps φ_8 and $\psi_8 \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_8 and ψ_8 are equivalent. □

LEMMA 9.9. — *Let φ_9 be the map 9 in Table 1.2 and let ψ_9 be a map with enriched weighted proximity graph 9 in Table 1.1. Then, ψ_9 is equivalent to φ_9.*

Proof. — The base points of φ_9 are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with standard coordinates $p_2 = (p_1, 0), p_3 = (p_1, 0, -1), p_4 = (p_1, 0, -1, 0)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, which is $C_1 \colon xz+y^2 = 0$. The base points of ψ_9 are q_0 of multiplicity 2 and $q_1, \ldots, q_4$, where $q_1 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_1$. According to Lemma 4.9, there is a unique irreducible conic C_2 passing through

$q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_9 is equivalent to φ_9. □

Lemma 9.10. — *Let φ_{10} be the map 10 in Table 1.2 and let ψ_{10} be a map with enriched weighted proximity graph 10 in Table 1.1. Then, ψ_{10} is equivalent to φ_{10}.*

Proof. — The base points of φ_{10} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$ and p_2, p_3, p_4, where $p_2 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_2 = (p_0, 0), p_3 = (p_1, 0)$ and $p_4 = (p_1, 0, 0)$. The base points of ψ_{10} are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4, where $q_2 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_1$, and q_4 is aligned with q_1 and q_3. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha(p_4) = q_4$, so the maps φ_{10} and $\psi_{10} \circ \alpha$ are defined by the same homaloidal net, and, therefore, φ_{10} and ψ_{10} are equivalent. □

Lemma 9.11. — *Let φ_{11} be the map 11 in Table 1.2 and let ψ_{11} be a map with enriched weighted proximity graph 11 in Table 1.1. Then, ψ_{11} is equivalent to φ_{11}.*

Proof. — The base points of φ_{11} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4, where $p_2 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_2 = (p_0, \infty)$, $p_3 = (p_1, 0)$, $p_4 = (p_1, 0, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, which is $C_1 : xz + y^2 = 0$. The base points of ψ_{11} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$, where $q_1 \in \mathbb{P}^2$ and $q_2 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_1$. According to Lemma 4.7, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_{11} is equivalent to φ_{11}. □

Lemma 9.12. — *Let φ_{12} be the map 12 in Table 1.2 and let ψ_{12} be a map with enriched weighted proximity graph 12 in Table 1.1. Then, ψ_{12} is equivalent to φ_{12}.*

Proof. — The base points of φ_{12} are $p_0 = [0 : 1 : 0]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4, where $p_3 \succ_1 p_1$, $p_4 \succ_1 p_2 \succ_1 p_0$ and $p_4 \odot p_0$ with standard coordinates $p_2 = (p_0, \infty), p_3 = (p_1, \infty)$ and $p_4 = (p_0, \infty, \infty)$. The base points of ψ_{12} are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4, where $q_3 \succ_1 q_1$, $q_4 \succ_1 q_2 \succ_1 q_0$ and $q_4 \odot q_0$. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha(p_4) = q_4$, so the maps φ_{12} and $\psi_{12} \circ \alpha$ are defined by the same homaloidal net, and, therefore, φ_{12} and ψ_{12} are equivalent. □

Lemma 9.13. — *Let φ_{13} be the map 13 in Table 1.2 and let ψ_{13} be a map with enriched weighted proximity graph 13 in Table 1.1. Then, ψ_{13} is equivalent to φ_{13}.*

Proof. — The base points of φ_{13} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4, where $p_2 \succ_1 p_1$ and $p_4 \succ_1 p_3 \succ_1 p_0$ with standard coordinates $p_2 = (p_1, 0)$, $p_3 = (p_0, \infty)$, $p_4 = (p_0, \infty, -1)$. So there is a unique irreducible conic passing through $p_0, \dots, p_4$, that is $C_1\colon xz + y^2 = 0$. The base points of ψ_{13} are q_0 of multiplicity 2 and $q_1, \dots, q_4$, where $q_1 \in \mathbb{P}^2$ and $q_2 \succ_1 q_1$ and $q_4 \succ_1 q_3 \succ_1 q_0$. According to Lemma 4.7, there is a unique irreducible conic C_2 passing through $q_0, \dots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_{13} is equivalent to φ_{13}. □

Lemma 9.14. — *Let φ_{14} be the map 14 in Table 1.2 and let ψ_{14} be a map with enriched weighted proximity graph 14 in Table 1.1. Then, ψ_{14} is equivalent to φ_{14}.*

Proof. — The base points of φ_{14} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$ and p_2, p_3, p_4, where $p_2 \succ_1 p_0$, $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_1$ with standard coordinates $p_2 = (p_0, 0), p_3 = (p_0, 1)$ and $p_4 = (p_1, 0)$. The base points of ψ_{14} are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4, where $q_2 \succ_1 q_0$, $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_1$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 4$.

The base points of $\psi_{14} \circ \alpha_1$ are then p_0, p_1, p_2, q_3', p_4, where q_3' has standard coordinates $q_3' = (p_0, u_3)$ for some $u_3 \in \mathbb{C}^*$ because if u_3 were 0, then q_3' would be equal to p_2, which is a contradiction, and, if u_3 were ∞, then q_3' would be aligned with p_0 and p_1, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_4 and that maps $p_3 = (p_0, 1)$ to $q_3' = (p_0, u_3)$ is

$$\alpha_2([x : y : z]) = [x : u_3 y : z].$$

Therefore, the maps φ_{14} and $\psi_{14} \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net, and, hence, φ_{14} and ψ_{14} are equivalent. □

Lemma 9.15. — *Let φ_{15} be the map 15 in Table 1.2 and let ψ_{15} be a map with enriched weighted proximity graph 15 in Table 1.1. Then, ψ_{15} is equivalent to φ_{15}.*

Proof. — The base points of φ_{15} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$ and p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_0$ and $p_4 \odot p_0$ with standard coordinates $p_3 = (p_0, 1)$ and $p_4 = (p_0, 1, \infty)$. The base points of ψ_{15} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4, where $q_4 \succ_1 q_3 \succ_1 q_0$ and $q_4 \odot q_0$. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha(p_4) = q_4$, so the maps φ_{15} and $\psi_{15} \circ \alpha$ are defined by the same homaloidal net, and, therefore, φ_{15} and ψ_{15} are equivalent. □

LEMMA 9.16. — *Let φ_{16} be the map 16 in Table 1.2 and let ψ_{16} be a map with enriched weighted proximity graph 16 in Table 1.1. Then, ψ_{16} is equivalent to φ_{16}.*

Proof. — The base points of φ_{16} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [0:1:0]$, $p_2 = [1:1:-1]$ and p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_0$ with standard coordinates $p_3 = (p_0, 0)$, $p_4 = (p_0, 0, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, which is $C_1 \colon x^2 + yz = 0$. The base points of ψ_{16} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$, where $q_1, q_2 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_0$. According to Lemma 4.5, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{16} is equivalent to φ_{16}. □

LEMMA 9.17. — *Let φ_{17} be the map 17 in Table 1.2 and let ψ_{17} be a map with enriched weighted proximity graph 17 in Table 1.1. Then, ψ_{17} is equivalent to φ_{17}.*

Proof. — The base points of φ_{17} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$, $p_2 = [0:1:0]$ and p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_3 = (p_1, 0)$ and $p_4 = (p_1, 0, 1)$. The base points of ψ_{17} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4, where $q_4 \succ_1 q_3 \succ_1 q_1$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$. It follows that also $\alpha_1(p_3) = q_3$.

The base points of $\psi_{17} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4', where q_4' has standard coordinates $q_4' = (p_1, 0, u_4)$ for some $u_4 \in \mathbb{C}^*$ because if u_4 were 0, then q_4' would be aligned with p_1, p_2 and p_3, which is a contradiction, and if u_4 were ∞, then q_4' would be satellite to p_1, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and maps $p_4 = (p_1, 0, 1)$ to $q_4' = (p_1, 0, u_4)$ is

$$\alpha_2([x:y:z]) = [u_4 x : y : z].$$

Therefore, the maps φ_{17} and $\psi_{17} \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_{17} and ψ_{17} are equivalent. □

LEMMA 9.18. — *Let φ_{18} be the map 18 in Table 1.2 and let ψ_{18} be a map with enriched weighted proximity graph 18 in Table 1.1. Then, ψ_{18} is equivalent to φ_{18}.*

Proof. — The base points of φ_{18} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$, $p_2 = [0:1:0]$ and p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_3 = (p_1, 1)$ and $p_4 = (p_1, 1, 0)$. The base points of ψ_{18} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4, where $q_4 \succ_1 q_3 \succ_1 q_1$ and q_4 is aligned with q_1 and q_3. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{18} and $\psi_{18} \circ \alpha_1$ are defined by the same homaloidal net, and, therefore, φ_{18} and ψ_{18} are equivalent. □

LEMMA 9.19. — *Let φ_{19} be the map 19 in Table 1.2 and let ψ_{19} be a map with enriched weighted proximity graph 19 in Table 1.1. Then, ψ_{19} is equivalent to φ_{19}.*

Proof. — The base points of φ_{19} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : -1]$ and p_3, p_4, where $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_3 = (p_1, 0)$, $p_4 = (p_1, 0, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, which is $C_1 \colon x^2 + xz + yz = 0$. The base points of ψ_{19} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$, where $q_1, q_2 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_1$. According to Lemma 4.5, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{19} is equivalent to φ_{19}. □

LEMMA 9.20. — *Let φ_{20} be the map 20 in Table 1.2 and let ψ_{20} be a map with enriched weighted proximity graph 20 in Table 1.1. Then, ψ_{20} is equivalent to φ_{20}.*

Proof. — The base points of φ_{20} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4, where $p_3 \succ_1 p_1$ and $p_4 \succ_1 p_2$ with standard coordinates $p_3 = (p_1, 0)$ and $p_4 = (p_2, 1)$. The base points of ψ_{20} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4, where $q_3 \succ_1 q_1$ and $q_4 \succ_1 q_2$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 4$. It follows that also $\alpha_1(p_3) = q_3$, so the maps φ_{20} and $\psi_{20} \circ \alpha_1$ are defined by the same homaloidal net, and, therefore, φ_{20} and ψ_{20} are equivalent. □

LEMMA 9.21. — *Let φ_{21} be the map 21 in Table 1.2 and let ψ_{21} be a map with enriched weighted proximity graph 21 in Table 1.1. Then, ψ_{21} is equivalent to φ_{21}.*

Proof. — The base points of φ_{21} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4, where $p_3 \succ_1 p_1$ and $p_4 \succ_1 p_2$ with standard coordinates $p_3 = (p_1, -1)$, $p_4 = (p_2, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon xy + xz + yz = 0$. The base points of ψ_{21} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1, q_2 \in \mathbb{P}^2$, $q_3 \succ_1 q_1$ and $q_4 \succ_1 q_2$. According to Lemma 4.6, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{21} is equivalent to φ_{21}. □

LEMMA 9.22. — *Let φ_{22} be the map 22 in Table 1.2 and let ψ_{22} be a map with enriched weighted proximity graph 22 in Table 1.1. Then, ψ_{22} is equivalent to φ_{22}.*

Proof. — The base points of φ_{22} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$, $p_2 = [0:1:0]$ and p_3, p_4, where $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_1$ with standard coordinates $p_3 = (p_0, -1)$ and $p_4 = (p_1, 0)$. The base points of ψ_{22} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4, where $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_1$ and q_4 is aligned with q_1 and q_2. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha(p_4) = q_4$, so the maps φ_{22} and $\psi_{22} \circ \alpha$ are defined by the same homaloidal net, and, therefore, φ_{22} and ψ_{22} are equivalent. □

LEMMA 9.23. — *Let φ_{23} be the map 23 in Table 1.2 and let ψ_{23} be a map with enriched weighted proximity graph 23 in Table 1.1. Then, ψ_{23} is equivalent to φ_{23}.*

Proof. — The base points of φ_{23} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [0:1:0]$, $p_2 = [1:-1:0]$ and p_3, p_4, where $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_1$ with standard coordinates $p_3 = (p_0, 0)$, $p_4 = (p_1, -1)$. So there is a unique irreducible conic passing through $p_0, \dots, p_4$, that is $C_1 \colon x^2 + xy + yz = 0$. The base points of ψ_{23} are q_0 of multiplicity 2 and $q_1, \dots, q_4$ where $q_1, q_2 \in \mathbb{P}^2$, $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_1$. According to Lemma 4.6, there is a unique irreducible conic C_2 passing through $q_0, \dots, q_4$. Moreover, Lemma 2.5 implies that there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{23} is equivalent to φ_{23}. □

LEMMA 9.24. — *Let φ_{24} be the map 24 in Table 1.2 and let ψ_{24} be a map with enriched weighted proximity graph 24 in Table 1.1. Then, ψ_{24} is equivalent to φ_{24}.*

Proof. — The base points of φ_{24} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [1:1:0]$ and p_4, where $p_4 \succ_1 p_1$ with standard coordinates $p_4 = (p_1, 1)$. The base points of ψ_{24} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4, where $q_4 \succ_1 q_1$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$, such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{24} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4', where q_4' has standard coordinates $q_4' = (p_1, u_4)$ for some $u_4 \in \mathbb{C}^*$ because if u_4 were 0, then q_4' would be aligned with p_1, p_2 and p_3, which is a contradiction, and, if u_4 were ∞, then q_4' would be aligned with p_0 and p_1, which is again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_1, 1)$ to $q_4' = (p_1, u_4)$ is

$$\alpha_2([x:y:z]) = [x:y:u_4 z].$$

Therefore, the maps φ_{24} and $\psi_{24} \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net, and, hence, φ_{24} and ψ_{24} are equivalent. □

LEMMA 9.25. — *Let φ_{25} be the map 25 in Table 1.2 and let ψ_{25} be a map with enriched weighted proximity graph 25 in Table 1.1. Then, ψ_{25} is equivalent to φ_{25}.*

Proof. — The base points of φ_{25} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$, $p_2 = [0:1:-1]$, $p_3 = [1:-1:0]$ and p_4, where $p_4 \succ_1 p_1$ with standard coordinates $p_4 = (p_1, -1)$. The base points of ψ_{25} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4, where $q_4 \succ_1 q_1$ and q_4 is aligned with q_1 and q_2. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0,1,2,3$. It follows that also $\alpha(p_4) = q_4$, so the maps φ_{25} and $\psi_{25} \circ \alpha$ are defined by the same homaloidal net, and, therefore, φ_{25} and ψ_{25} are equivalent. □

LEMMA 9.26. — *Let $\varphi_{26,\gamma}$ be the map 26 in Table 1.2 with parameter γ and let ψ_{26} be a map with enriched weighted proximity graph 26 in Table 1.1. Then, ψ_{26} is equivalent to $\varphi_{26,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. — The base points of $\varphi_{26,\gamma}$ are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [1:1:1]$ and p_4, where $p_4 \succ_1 p_1$ with standard coordinates $p_4 = (p_1, 1/\gamma)$. The base points of ψ_{26} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4, where $q_4 \succ_1 q_1$. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0,1,2,3$.

The base points of $\psi_{26} \circ \alpha$ are then p_0, p_1, p_2, p_3, q_4', where q_4' has standard coordinates $q_4' = (p_1, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because if u_4 were 0, then q_4' would be aligned with p_1 and p_2, which is a contradiction; if u_4 were ∞, then q_4' would be aligned with p_0 and p_1, which is again a contradiction, and, if u_4 were 1, then q_4' would be aligned with p_1 and p_3, which is still a contradiction. Setting $\gamma = 1/u_4$, the maps $\varphi_{26,\gamma}$ and $\psi_{26} \circ \alpha$ are defined by the same homaloidal net, and, therefore, $\varphi_{26,\gamma}$ and ψ_{26} are equivalent. □

LEMMA 9.27. — *Let $\varphi_{27,\gamma}$ be the map 27 in Table 1.2 with parameter γ and let ψ_{27} be a map with enriched weighted proximity graph 27 in Table 1.1. Then, ψ_{27} is equivalent to $\varphi_{27,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. — The base points of $\varphi_{27,\gamma}$ are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [0:1:0]$, $p_2 = [1:0:0]$ and p_3, p_4, where $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_0$ with standard coordinates $p_3 = (p_0, -1)$ and $p_4 = (p_0, -1/\gamma)$. The base points of ψ_{27} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4, where $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_0$. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0,1,2,3$.

The base points of $\psi_{27} \circ \alpha$ are then p_0, p_1, p_2, p_3, q_4', where q_4' has standard coordinates $q_4' = (p_0, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because if u_4 were 0, then q_4' would be aligned with p_0 and p_2, which is a contradiction; if u_4 were ∞, then q_4' would be aligned with p_0 and p_1, which is again a contradiction, and, if u_4

were 1, then q'_4 would be equal to p_3, which is still a contradiction. Setting $\gamma = -1/u_4$, the maps $\varphi_{27,\gamma}$ and $\psi_{27} \circ \alpha$ are defined by the same homaloidal net, and, therefore, $\varphi_{27,\gamma}$ and ψ_{27} are equivalent. □

Lemma 9.28. — *Let $\varphi_{28,\gamma}$ be the map 28 in Table 1.2 with parameter γ and let ψ_{28} be a map with enriched weighted proximity graph 28 in Table 1.1. Then, ψ_{28} is equivalent to $\varphi_{28,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. — The base points of $\varphi_{28,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 0]$ and p_4, where $p_4 \succ_1 p_0$ with standard coordinates $p_4 = (p_0, \gamma)$. The base points of ψ_{28} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4, where $q_4 \succ_1 q_0$ and q_1, q_2, q_3 are collinear. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{28} \circ \alpha$ are then p_0, p_1, p_2, p_3, q'_4, where $q'_4 = (p_0, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because if u_4 were 0, then q'_4 would be aligned with p_0 and p_2, which is a contradiction; if u_4 were ∞, then q'_4 would be aligned with p_0 and p_1, which is again a contradiction, and, if u_4 were 1, then q'_4 would be aligned with p_0 and p_3, which is still a contradiction. Setting $\gamma = u_4$, the maps $\varphi_{28,\gamma}$ and $\psi_{28} \circ \alpha$ are defined by the same homaloidal net, and, therefore, $\varphi_{28,\gamma}$ and ψ_{28} are equivalent. □

Lemma 9.29. — *Let $\varphi_{29,\gamma}$ be the map 29 in Table 1.2 with parameter γ and let ψ_{29} be a map with enriched weighted proximity graph 29 in Table 1.1. Then, ψ_{29} is equivalent to $\varphi_{29,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. — The base points of $\varphi_{29,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 1]$ and p_4, where $p_4 \succ_1 p_0$ with standard coordinates $p_4 = (p_0, \gamma)$. The base points of ψ_{29} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4, where $q_4 \succ_1 q_0$. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{29} \circ \alpha$ are then p_0, p_1, p_2, p_3, q'_4, where $q'_4 = (p_0, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because if u_4 were 0, then q'_4 would be aligned with p_0 and p_2, which is a contradiction; if u_4 were ∞, then q'_4 would be aligned with p_0 and p_1, which is again a contradiction, and, if u_4 were 1, then q'_4 would be aligned with p_0 and p_3, which is still a contradiction. Setting $\gamma = u_4$, the maps $\varphi_{29,\gamma}$ and $\psi_{29} \circ \alpha$ are defined by the same homaloidal net,and, therefore, $\varphi_{29,\gamma}$ and ψ_{29} are equivalent. □

Lemma 9.30. — *Let $\varphi_{30,\gamma}$ be the map 30 in Table 1.2 with parameter γ and let ψ_{30} be a map with enriched weighted proximity graph 30 in Table 1.1. Then, ψ_{30} is equivalent to $\varphi_{30,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. — The base points of $\varphi_{30,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [\gamma : 1 : 0]$ and $p_4 = [1 : 1 : 1]$. The base points of ψ_{30}

are q_0 of multiplicity 2, $q_1, q_2, q_3, q_4 \in \mathbb{P}^2$, where q_1, q_2, q_3 are collinear. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 4$.

The base points of $\psi_{30} \circ \alpha$ are then p_0, p_1, p_2, q_3', p_4, where $q_3' = [u_3 : 1 : 0]$ for some $u_3 \in \mathbb{C}^{**}$ because if u_3 were 0, then q_3' would be equal to p_1, which is a contradiction, and, if u_3 were 1, then q_3' would be aligned with p_0 and p_4, which is again a contradiction. Setting $\gamma = u_3$, the maps $\varphi_{30,\gamma}$ and $\psi_{30} \circ \alpha$ are defined by the same homaloidal net, and, therefore, $\varphi_{30,\gamma}$ and ψ_{30} are equivalent. □

Lemma 9.31. — *Let $\varphi_{31,a,b}$ be the map 31 in Table 1.2 with parameters a, b and let ψ_{31} be a map with enriched weighted proximity graph 31 in Table 1.1. Then, ψ_{31} is equivalent to $\varphi_{31,\gamma}$ for some $a, b \neq 0, 1$, $a \neq b$.*

Proof. — The base points of $\varphi_{31,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 1]$ and $p_4 = [a : b : 1]$. The base points of ψ_{31} are q_0 of multiplicity 2 and $q_1, q_2, q_3, q_4 \in \mathbb{P}^2$. Clearly, there exists an automorphism α of $\mathbb{P}^2$, such that $\alpha(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{31} \circ \alpha$ are then p_0, p_1, p_2, p_3, q_4', where $q_4' = [t_4 : u_4 : v_4]$ with $t_4, u_4, v_4 \in \mathbb{C}^*$; indeed,

- $v_4 \neq 0$ because otherwise q_4' would be aligned with p_1 and p_2;
- $u_4 \neq 0$ because otherwise q_4' would be aligned with p_0 and p_1;
- $t_4 \neq 0$ because otherwise q_4' would be aligned with p_0 and p_2.

Moreover, t_4/v_4 and u_4/v_4 satisfy the following conditions:

- $t_4/v_4 \neq 1$ because otherwise q_4' would be aligned with p_1 and p_3;
- $u_4/v_4 \neq 1$ because otherwise q_4' would be aligned with p_2 and p_3;
- $t_4/v_4 \neq u_4/v_4$ because otherwise q_4' would be aligned with p_0 and p_3.

Setting $a = t_4/v_4$ and $b = u_4/v_4$, it follows that $a, b \in \mathbb{C}^{**}$ and $a \neq b$, the maps $\varphi_{31,a,b}$ and $\psi_{31} \circ \alpha$ are defined by the same homaloidal net, and, therefore, $\varphi_{31,a,b}$ and ψ_{31} are equivalent. □

Lemma 9.32. — *Set $\varphi_{26,\gamma}$ the map of type 26 in Table 1.2 with parameter γ, where $\gamma \neq 0, 1$. Then, $\varphi_{26,\gamma}$ is equivalent to $\varphi_{26,\gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = \gamma/(\gamma - 1)$.*

Proof. — Let $p_0, p_1, \ldots, p_4$ be the base points of $\varphi_{26,\gamma}$ as in the proof of Lemma 9.26.

An automorphism α of $\mathbb{P}^2$ that fixes the homaloidal net defining $\varphi_{26,\gamma}$, and this is different from the identity, is such that $\alpha(p_i) = p_i$, $i = 0, 1$, $\alpha(p_2) = p_3$ and $\alpha(p_3) = p_2$. Therefore, α is unique, and it is defined by

$$\alpha([x : y : z]) = [y - x : y : y - z],$$

so $\alpha(p_4)$ has standard coordinates $(p_1, (\gamma - 1)/\gamma)$, and, hence, $\varphi_{26,\gamma/(\gamma-1)}$ is equivalent to $\varphi_{26,\gamma}$. □

Lemma 9.33. — *Set* $\varphi_{27,\gamma}$ *the map of type* 27 *in Table 1.2 with parameter* γ, *where* $\gamma \neq 0, 1$. *Then,* $\varphi_{27,\gamma}$ *is equivalent to* $\varphi_{27,\gamma'}$ *if and only if either* $\gamma' = \gamma$ *or* $\gamma' = 1/\gamma$.

Proof. — Let p_0, p_1, p_2, p_3, p_4 be the base points of $\varphi_{27,\gamma}$ as in the proof of Lemma 9.27.

The base points of $\varphi_{27,\gamma'}$ are $q_i = p_i$, $i = 0, 1, 2, 3$, and $q_4 = (q_0, -1/\gamma')$.

Suppose that $\varphi_{27,\gamma'}$ is equivalent to $\varphi_{27,\gamma}$. This implies that there exist automorphisms $\alpha 1, \ldots, \alpha_4$ of $\mathbb{P}^2$ with the following properties:

(1) α_1 is such that $\alpha_1(p_i) = q_i$, $i = 0, 1, 2, 3, 4$;
(2) α_2 is such that $\alpha_2(p_i) = q_i$, $i = 0, 1, 2$, $\alpha_2(p_3) = q_4$ and $\alpha_2(p_4) = q_3$;
(3) α_3 is such that $\alpha_3(p_i) = q_i$, $i = 0, 3, 4$, $\alpha_3(p_1) = q_2$ and $\alpha_3(p_2) = q_1$;
(4) α_4 is such that $\alpha_4(p_0) = q_0$, $\alpha_4(p_1) = q_2$, $\alpha_4(p_2) = q_1$, $\alpha_4(p_3) = q_4$ and $\alpha_4(p_4) = q_3$.

Then, Case (1) occurs only if $\gamma' = \gamma$ and α_1 is the identity. Case (2) occurs only if $\gamma' = 1/\gamma$ and $\alpha_2([x : y : z]) = [x : \gamma y : -\gamma z]$. Case (3) occurs only if $\gamma' = 1/\gamma$ and $\alpha_3([x : y : z]) = [y : x : -z]$. Case (4) occurs only if $\gamma' = \gamma$ and $\alpha_4([x : y : z]) = [\gamma y : x : z]$. □

Let us now recall some definitions of permutations with cycle notation.

Definition 9.34. — Let $\mathfrak{S}_n$ denote the group of permutations of $\{1, 2, \ldots, n\}$. Every permutation can be written as a cycle or a product of disjoint cycles. For $n = 3$, the group $\mathfrak{S}_3$ has six elements:

$$\mathfrak{s}_1 = \mathrm{id}, \quad \mathfrak{s}_2 = (23), \quad \mathfrak{s}_3 = (12), \quad \mathfrak{s}_4 = (123), \quad \mathfrak{s}_5 = (13), \quad \mathfrak{s}_6 = (132).$$

For $n = 4$, the group $\mathfrak{S}_4$ has 24 elements:

$$\begin{array}{llll}
\mathfrak{s}_1 = \mathrm{id}, & \mathfrak{s}_2 = (12), & \mathfrak{s}_3 = (34), & \mathfrak{s}_4 = (12)(34), \\
\mathfrak{s}_5 = (23), & \mathfrak{s}_6 = (123), & \mathfrak{s}_7 = (243), & \mathfrak{s}_8 = (1243), \\
\mathfrak{s}_9 = (132), & \mathfrak{s}_{10} = (13), & \mathfrak{s}_{11} = (1432), & \mathfrak{s}_{12} = (143), \\
\mathfrak{s}_{13} = (1234), & \mathfrak{s}_{14} = (234), & \mathfrak{s}_{15} = (124), & \mathfrak{s}_{16} = (24), \\
\mathfrak{s}_{17} = (134), & \mathfrak{s}_{18} = (1342), & \mathfrak{s}_{19} = (14), & \mathfrak{s}_{20} = (142), \\
\mathfrak{s}_{21} = (13)(24), & \mathfrak{s}_{22} = (1324), & \mathfrak{s}_{23} = (1423), & \mathfrak{s}_{24} = (14)(23).
\end{array}$$

Lemma 9.35. — *For* $n \in \{28, 29, 30\}$, *set* $\varphi_{n,\gamma}$ *the map of type* n *in Table 1.2 with parameter* γ, *where* $\gamma \neq 0, 1$. *Then,* $\varphi_{n,\gamma'}$ *is equivalent to* $\varphi_{n,\gamma}$ *if and only if*

$$\gamma' \in \left\{ \gamma, \frac{1}{\gamma}, 1 - \gamma, \frac{1}{1 - \gamma}, \frac{\gamma}{\gamma - 1}, \frac{\gamma - 1}{\gamma} \right\}.$$

Proof. — We first consider the case $n = 28$.

The map $\varphi_{28,\gamma}$ has base points $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 0]$ and p_4, where $p_4 \succ_1 p_0$ with standard coordinates $p_4 = (p_0, \gamma)$.

The base points of $\varphi_{28,\gamma'}$ are $q_0, \dots, q_4$, where $q_i = p_i$, $i = 0, 1, 2, 3$ and $q_4 = (p_0, \gamma')$.

Suppose that $\varphi_{28,\gamma'}$ is equivalent to $\varphi_{28,\gamma}$. This implies that there exist automorphisms $\alpha_1, \dots, \alpha_6$ of $\mathbb{P}^2$, such that, for $i = 1, \dots, 6$, one has $\alpha_i(p_j) = q_j$, $j = 0, 4$, and

$$\alpha_i(p_j) = q_{\mathfrak{s}_i(j)} \quad \text{for } j = 1, 2, 3,$$

where $\mathfrak{s}_1, \dots, \mathfrak{s}_6$ are the six elements of $\mathfrak{S}_3$ given in Definition 9.34.

- Case $i = 1$ occurs only if $\gamma' = \gamma$ and α_1 is the identity.
- Case $i = 2$ occurs only if $\gamma' = 1 - \gamma$ and $\alpha_2 = [x : x - y : z]$.
- Case $i = 3$ occurs only if $\gamma' = 1/\gamma$ and $\alpha_3 = [y : x : z]$.
- Case $i = 4$ occurs only if $\gamma' = 1/(1 - \gamma)$ and $\alpha_4 = [x - y : x : z]$.
- Case $i = 5$ occurs only if $\gamma' = \gamma/(\gamma - 1)$ and $\alpha_5 = [x - y : -y : z]$.
- Case $i = 6$ occurs only if $\gamma' = \gamma/(\gamma - 1)$ and $\alpha_6 = [y : y - x : z]$.

We proceed similarly for $n = 29$. The map $\varphi_{29,\gamma}$ has the same base points p_i, $i = 0, 1, 2, 4$, of $\varphi_{28,\gamma}$ but $p_3 = [1 : 1 : 1]$. The base points of $\varphi_{29,\gamma'}$ are $q_0, \dots, q_4$, where $q_i = p_i$, $i = 0, 1, 2, 3$ and $q_4 = (q_0, \gamma')$.

If $\varphi_{29,\gamma'}$ is equivalent to $\varphi_{29,\gamma}$, then there exist automorphisms $\alpha_1, \dots, \alpha_6$ of $\mathbb{P}^2$ with the same above properties that occur exactly when γ' is as above, and α_1 is the identity,

$$\begin{aligned} &\alpha_2 = [x : x - y : x - z], \quad \alpha_3 = [y : x : z], \qquad \alpha_4 = [x - y : x : x - z], \\ &\alpha_5 = [y - x : y : y - z], \quad \alpha_6 = [y : y - x : y - z]. \end{aligned}$$

Finally, for $n = 30$, the map $\varphi_{30,\gamma}$ has the same base points p_i, $i = 0, 1, 2$, of $\varphi_{28,\gamma}$ but $p_3 = [\gamma : 1 : 0]$ and $p_4 = [1 : 1 : 1]$. The base points of $\varphi_{30,\gamma'}$ are $q_0, \dots, q_4$ where $q_i = p_i$, $i = 0, 1, 2, 4$ and $q_3 = [\gamma' : 1 : 0]$.

If $\varphi_{30,\gamma'}$ is equivalent to $\varphi_{30,\gamma}$, then there exist automorphisms $\alpha_1, \dots, \alpha_6$ of $\mathbb{P}^2$ with the same above properties that occur exactly when γ' is as above, and α_1 is the identity,

$$\begin{aligned} &\alpha_2 = [(\gamma - 1)x : \gamma y - x : (\gamma - 1)z], \quad \alpha_3 = [y : x : z], \\ &\alpha_4 = [\gamma y - x : (\gamma - 1)x : (\gamma - 1)z], \quad \alpha_5 = [\gamma y - x : (\gamma - 1)y : (\gamma - 1)z], \\ &\alpha_6 = [(\gamma - 1)y : \gamma y - x : (\gamma - 1)z]. \end{aligned}$$ □

REMARK 9.36. — One may check that the numbers in the set of Lemma 9.35 are all different if and only if

$$\gamma \notin \left\{ -1, 2, \frac{1}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} + i\frac{\sqrt{3}}{2} \right\}.$$

Lemma 9.37. — *Set $\varphi_{31,a,b}$ the map of type* 31 *in Table 1.2 with two parameters a, b, where $a \neq b$ and $a, b \neq 0, 1$. Then, $\varphi_{31,a',b'}$ is equivalent to $\varphi_{31,a,b}$ if and only if $(a', b') \in S$, where S is defined in* (2).

Proof. — The base points of $\varphi_{31,a,b}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2 and four simple base points $p_1 = [0 : 1 : 0], p_2 = [1 : 0 : 0], p_3 = [1 : 1 : 1], p_4 = [a : b : 1]$. Similarly, the base points of $\varphi_{31,a',b'}$ are $q_0, \ldots, q_4$, where $q_i = p_i$, $i = 0, 1, 2, 3$ and $q_4 = [a' : b' : 1]$.

Suppose that $\varphi_{31,a',b'}$ is equivalent to $\varphi_{31,a,b}$. Then, there exists an automorphism, says γ, of $\mathbb{P}^2$, such that $\gamma(p_0) = q_0$ and γ maps $p_1, \ldots, p_4$ to a permutation of q_1, q_2, q_3, q_4. Therefore, for each element $\mathfrak{s}_i$, $i = 1, \ldots, 24$, of $\mathfrak{S}_4$ there is an automorphism γ_i, $i = 1, \ldots, 24$, of $\mathbb{P}^2$, such that

$$\gamma_i(p_j) = q_{\mathfrak{s}_i(j)} \quad \text{for } j = 1, \ldots, 4,$$

and, accordingly, we find the values of (a', b') for each one of the 24 cases. In Table 9.1, we list the automorphisms γ_i, $i = 1, \ldots, 24$ and their corresponding values of (a', b'). □

Table 9.1: Automorphisms $\gamma_1, \ldots, \gamma_{24}$ of $\mathbb{P}^2$ and their corresponding values of (a', b').

i	$\gamma_i([x : y : z])$	(a', b')
1	$[x : y : z]$	(a, b)
2	$[y : x : z]$	(b, a)
3	$[bx : ay : abz]$	$\left(\frac{1}{a}, \frac{1}{b}\right)$
4	$[ay : bx : abz]$	$\left(\frac{1}{b}, \frac{1}{a}\right)$
5	$[x : x - y : x - z]$	$\left(\frac{a}{a-1}, \frac{a-b}{a-1}\right)$
6	$[x - y : x : x - z]$	$\left(\frac{a-b}{a-1}, \frac{a}{a-1}\right)$
7	$\left[\frac{x}{a} : \frac{x-y}{a-b} : \frac{x-z}{a-1}\right]$	$\left(\frac{a-1}{a}, \frac{a-1}{a-b}\right)$
8	$\left[\frac{x-y}{a-b} : \frac{x}{a} : \frac{x-z}{a-1}\right]$	$\left(\frac{a-1}{a-b}, \frac{a-1}{a}\right)$
9	$[y : y - x : y - z]$	$\left(\frac{b}{b-1}, \frac{b-a}{b-1}\right)$

10	$[y-x:y:y-z]$	$\left(\frac{b-a}{b-1},\frac{b}{b-1}\right)$
11	$\left[\frac{y}{b}:\frac{x-y}{a-b}:\frac{y-z}{b-1}\right]$	$\left(\frac{b-1}{b},\frac{b-1}{b-a}\right)$
12	$\left[\frac{x-y}{a-b}:\frac{y}{b}:\frac{y-z}{b-1}\right]$	$\left(\frac{b-1}{b-a},\frac{b-1}{b}\right)$
13	$[bx-ay:bx:b(x-az)]$	$\left(\frac{b-a}{b(1-a)},\frac{1}{1-a}\right)$
14	$[bx:bx-ay:b(x-az)]$	$\left(\frac{1}{1-a},\frac{b-a}{b(1-a)}\right)$
15	$\left[\frac{ay-bx}{a-b}:x:\frac{az-x}{a-1}\right]$	$\left(\frac{b(a-1)}{a-b},1-a\right)$
16	$\left[x:\frac{ay-bx}{a-b}:\frac{az-x}{a-1}\right]$	$\left(1-a,\frac{b(a-1)}{a-b}\right)$
17	$[ay-bx:ay:a(y-bz)]$	$\left(\frac{a-b}{a(1-b)},\frac{1}{1-b}\right)$
18	$[ay:ay-bx:a(y-bz)]$	$\left(\frac{1}{1-b},\frac{a-b}{a(1-b)}\right)$
19	$\left[\frac{ay-bx}{a-b}:y:\frac{bz-y}{b-1}\right]$	$\left(\frac{a(1-b)}{a-b},1-b\right)$
20	$\left[y:\frac{ay-bx}{a-b}:\frac{bz-y}{b-1}\right]$	$\left(1-b,\frac{a(1-b)}{a-b}\right)$
21	$\left[y-x:\frac{ay-bx}{a}:\frac{(1-b)x}{a-1}+\frac{(b-a)z}{a-1}+y\right]$	$\left(\frac{a-1}{b-1},\frac{b(a-1)}{a(b-1)}\right)$
22	$\left[\frac{ay-bx}{a}:y-x:\frac{(1-b)x}{a-1}+\frac{(b-a)z}{a-1}+y\right]$	$\left(\frac{b(a-1)}{a(b-1)},\frac{a-1}{b-1}\right)$
23	$\left[y-x:\frac{ay-bx}{b}:\frac{(a-1)y}{b-1}+\frac{(b-a)z}{b-1}-x\right]$	$\left(\frac{b-1}{a-1},\frac{a(b-1)}{b(a-1)}\right)$
24	$\left[\frac{ay-bx}{b}:y-x:\frac{(a-1)y}{b-1}+\frac{(b-a)z}{b-1}-x\right]$	$\left(\frac{a(b-1)}{b(a-1)},\frac{b-1}{a-1}\right)$

REMARK 9.38. — One may check that the pairs in S are all different if and only if (a,b) does not belong to the following set:

$$\left\{\left(a,\frac{1}{a}\right)\middle| a\neq -1\right\}\cup\left\{\left(a,\frac{2a-1}{a}\right)\middle| a\neq\frac{1}{2}\right\}\cup\left\{\left(\frac{2b-1}{b},b\right)\middle| b\neq\frac{1}{2}\right\}$$
$$\cup\left\{(a,b)\middle| a=\frac{3}{2}\pm\frac{i\sqrt{3}}{6}, b=-\frac{1}{2}\pm\frac{i\sqrt{3}}{6}\right\}\cup\left\{(a,\overline{a})\middle| a\in\left\{\frac{1}{2}\pm\frac{i\sqrt{3}}{6},\frac{3}{2}\pm\frac{i\sqrt{3}}{2}\right\}\right\}$$
$$\cup\left\{(a,-\overline{a}),(a,\overline{-\overline{a}})\middle| a\in\left\{-\frac{1}{2}\pm\frac{i\sqrt{3}}{2},\frac{1}{2}\pm\frac{i\sqrt{3}}{2},-\frac{1}{2}\pm\frac{i\sqrt{3}}{6}\right\}\right\}.$$

10. Ordinary quadratic length of cubic planar Cremona maps

In this section, we prove Theorem 1.3. Theorem 1.2 implies that it suffices to compute the lengths of the cubic planar Cremona maps listed in Table 1.2.

Recall that the quadratic length, and, hence, the ordinary quadratic length, of cubic planar Cremona maps is at least 2 (Corollary 5.11). On the other hand, in Table 8.1 there are decompositions of all types of plane cubic maps, but type 1, in exactly two quadratic maps. So, in order to complete the proof of the first assertion of Theorem 1.3, it remains to prove the following lemma.

LEMMA 10.1. — *Let $\varphi_1\in\mathrm{Cr}(\mathbb{P}^2)$ be the map 1 in Table 1.2. Then, φ_1 has quadratic length* 3.

Proof. — Let p_1 be the double base point of φ_1 and let $p_2,\dots,p_5$ be its simple base points, which are all infinitely near p_1, namely $p_5\succ_1 p_4\succ_1 p_3\succ_1 p_2\succ_1 p_1$, where $p_3\odot p_1$. Hence, a quadratic map can be based at p_1 and at p_2, but not at p_3; cf. Remark 6.10.

The decomposition in Table 8.1 implies that $q(\varphi_1)\leq 3$. By contradiction, suppose that $\mathrm{ql}(\varphi_1)=2$. Then, there should exist a quadratic map ρ, such that $\mathrm{ql}(\varphi_1\circ\rho^{-1})=1$, so $\varphi_1\circ\rho^{-1}$ should be a quadratic map by Lemma 5.10. However,

- if ρ is not based at p_1, then $\varphi_1\circ\rho^{-1}$ has degree 6, which is a contradiction;
- if ρ is based at p_1, but not at p_2, then $\varphi_1\circ\rho^{-1}$ has degree 4, which is again a contradiction;
- finally, if ρ is based at p_1 and p_2, then $\varphi_1\circ\rho^{-1}$ has degree 3, which is a contradiction.

Hence, we conclude that $\mathrm{ql}(\varphi_1)=3$. □

We now prove the second assertion of Theorem 1.3, which is that the cubic planar Cremona map of type n, $1\leq n\leq 31$, in Table 1.2 has the respective ordinary quadratic length listed in the third column in Table 1.1.

The decompositions in Table 8.1 show that the maps of types 21, 23, 25, 26, 27, 29, 30, 31 have the ordinary quadratic length exactly 2.

Recall that Proposition 7.7 says the ordinary quadratic length of a plane Cremona map is at least the maximum height of its base points. In particular, the maps φ_n, $n = 10, 11, 12, 13, 15, 16, 18, 19$ have $\mathrm{oql}(\varphi_n) \geq 3$, and the decompositions in Tables 8.1 and 8.2 show that, indeed, $\mathrm{oql}(\varphi_n) = 3$. Similarly, the maps φ_n, $n = 2, 7, 9$ have $\mathrm{oql}(\varphi_n) \geq 4$, and the decompositions in Tables 8.1 and 8.2 show that $\mathrm{oql}(\varphi_n) = 4$.

We now consider the maps of the remaining types, going backwards from the last types to the first ones.

Lemma 10.2. — *Let φ_{28} be the map 28 in Table 1.2. Then,* $\mathrm{oql}(\varphi_{28}) = 3$.

Proof. — Let p_1 be the double base point of φ_{28} and p_3, p_4, p_5 the proper simple base points of φ_{28}, which are collinear. The decomposition of φ_{28} in Table 8.2 shows that $\mathrm{oql}(\varphi_{28}) \leq 3$. Suppose by contradiction that $\mathrm{oql}(\varphi_{28}) = 2$. Therefore, there should exist an ordinary quadratic map ρ, such that $\mathrm{oql}(\varphi_{28} \circ \rho^{-1}) = 1$, i.e., the map $\varphi_{28} \circ \rho^{-1}$ should be an ordinary quadratic map. Since $\varphi_{28} \circ \rho^{-1}$ should have degree 2, the map ρ must be based at p_1 and two proper simple base points of φ_{28}, say p_3, p_4. However, in that case, the quadratic map $\varphi_{28} \circ \rho^{-1}$ is not ordinary, because p_5 would correspond to an infinitely near base point of $\varphi_{28} \circ \rho^{-1}$, which is a contradiction. □

Remark 10.3. — The same argument used in the proof of Lemma 10.2 shows that the maps 20, 22, 24 in Table 1.2 have ordinary quadratic length exactly 3.

Lemma 10.4. — *Let φ_{17} be the map 17 in Table 1.2. Then,* $\mathrm{oql}(\varphi_{17}) = 4$.

Proof. — The enriched weighted proximity graph of φ_{17} is listed in Table 1.1. Let p_1 be the double base point, p_2, p_3 the two proper simple base points and p_4, p_5, such that $p_5 \succ_1 p_4 \succ_1 p_3$, where p_2, p_3, p_4 are aligned. Then,

$$3 \leqslant \mathrm{oql}(\varphi_{17}) \leqslant 4$$

because of the decomposition of φ_{17} in Table 8.2 and the fact that the height of p_5 with respect to φ_{17} is 3; cf. Proposition 7.7.

Suppose by contradiction that $\mathrm{oql}(\varphi_{17}) = 3$. Then, there should exist an ordinary quadratic map ρ, such that $\mathrm{oql}(\varphi_{17} \circ \rho^{-1}) = 2$. In particular, ρ must be based at p_3, otherwise, the maximum height of the base points of the map $\varphi_{17} \circ \rho^{-1}$ would be still 3, and Proposition 7.7 would give a contradiction.

If ρ is based also at p_2 (or at another point on the line passing through p_3 and p_2), then p_4 would correspond to an infinitely near base point of $\varphi_{17} \circ \rho^{-1}$, and the maximum height of the base points of $\varphi_{17} \circ \rho^{-1}$ would again be 3, which is a contradiction.

There are now two cases: either p_1 is a base point of ρ or p_1 is not a base point of ρ.

In the former case, the map $\varphi_{17} \circ \rho^{-1}$ would have the enriched weighted proximity graph 24 in Table 1.1, and, therefore, would have ordinary quadratic length 3, as we noted in Remark 10.3, which is a contradiction.

In the latter case, the map $\varphi_{17} \circ \rho^{-1}$ would have degree 5, and, therefore, its ordinary quadratic length cannot be 2 by Corollary 5.14, which is a contradiction.

Hence, we conclude that $\operatorname{oql}(\varphi_{17}) = 4$. □

Lemma 10.5. — *Let φ_{14} be the map 14 in Table 1.2. Then,* $\operatorname{oql}(\varphi_{14}) = 3$.

Proof. — The decomposition of φ_{14} in Table 8.2 shows that $\operatorname{oql}(\varphi_{14}) \leq 3$. Suppose by contradiction that $\operatorname{oql}(\varphi_{14}) = 2$. Therefore, there should exist an ordinary quadratic map ρ, such that $\operatorname{oql}(\varphi_{14} \circ \rho^{-1}) = 1$, i.e., the map $\varphi_{14} \circ \rho^{-1}$ should be an ordinary quadratic map. In other words, ρ should be based at the double base point of φ_{14} and other two proper simple base points of φ_{14}; these, however, do not exist. □

Lemma 10.6. — *Let φ_8 be the map 8 in Table 1.2. Then,* $\operatorname{oql}(\varphi_8) = 5$.

Proof. — The enriched weighted proximity graph of φ_8 is listed in Table 1.1. Let p_1 be the double base point, p_2 the proper simple base point and p_3, p_4, p_5 the other infinitely near base points, such that $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2$, where p_2, p_3, p_4 are aligned. Then,

$$4 \leqslant \operatorname{oql}(\varphi_8) \leqslant 5$$

because of the decomposition of φ_8 in Table 8.2 and the fact that the height of p_5 with respect to φ_8 is 4; cf. Proposition 7.7.

Suppose by contradiction that $\operatorname{oql}(\varphi_8) = 4$. Then, there should exist an ordinary quadratic map ρ_1, such that $\operatorname{oql}(\varphi_8 \circ \rho_1^{-1}) = 3$. In particular, ρ_1 must be based at p_2, otherwise, the maximum height of the base points of the map $\varphi_8 \circ \rho_1^{-1}$ would be still 4, and Proposition 7.7 would give a contradiction. For the same reason, ρ_1 cannot be based at p_2 and also at a point on the line passing through p_2 and p_3.

There are now two cases: either p_1 is a base point of ρ_1 or p_1 is not a base point of ρ_1.

In the former case, the map $\varphi_8 \circ \rho^{-1}$ would have the enriched weighted proximity graph 17 in Table 1.1, and, therefore, it would have ordinary quadratic length 4, as we proved in Lemma 10.4, which is a contradiction.

In the latter case, the map $\varphi_8 \circ \rho^{-1}$ would have degree 5 and the following weighted proximity graph:

.

where p'_0, p'_4, p'_5 are aligned. Furthermore, there should exist an ordinary quadratic map ρ_2, such that $\operatorname{oql}(\varphi_8 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 2$. In particular, ρ_2 must be based at p'_4, otherwise the maximum height of the base points of the map $\varphi_8 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be still 3, and Proposition 7.7 would give a contradiction. For the same reason, ρ_2 cannot be based at p'_4 and also at p'_0 or at another point on the line passing through p'_4 and p'_5. Therefore, ρ_2 is based at p'_4 and other two points where $\varphi_8 \circ \rho^{-1}$ has multiplicity ≤ 2, and, hence, the map $\varphi_8 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would have degree ≥ 5, and we get a contraction with Corollary 5.14.

We conclude that $\mathrm{oql}(\varphi_8) = 5$. □

LEMMA 10.7. — *Let φ_6 be the map 6 in Table 1.2. Then,* $\mathrm{oql}(\varphi_6) = 4$.

Proof. — The enriched weighted proximity graph of φ_6 is listed in Table 1.1. Let p_1 be the double base point, p_5 the proper simple base point and p_2, p_3, p_4 the other infinitely near base points, such that $p_2 \succ_1 p_1$ and $p_4 \succ_1 p_3 \succ_1 p_1$. Then,

$$3 \leqslant \mathrm{oql}(\varphi_6) \leqslant 4$$

because of the decomposition of φ_6 in Table 8.2 and the fact that the height of p_4 with respect to φ_6 is 3; cf. Proposition 7.7.

Suppose by contradiction that $\mathrm{oql}(\varphi_6) = 3$. Then, there should exist an ordinary quadratic map ρ, such that $\mathrm{oql}(\varphi_6 \circ \rho^{-1}) = 2$. In particular, ρ must be based at p_1, otherwise the maximum height of the base points of the map $\varphi_6 \circ \rho^{-1}$ would be still 3, and Proposition 7.7 would give a contradiction. For the same reason, ρ_1 cannot be based at p_1 and also at a point on the line passing through p_1 and p_3.

There are now two cases: either ρ is based at p_5 or ρ is not based at p_5.

In the former case, the map $\varphi_6 \circ \rho^{-1}$ would have the enriched weighted proximity graph 24 in Table 1.1, and, therefore, it would have ordinary quadratic length 3 (cf. Remark 10.3), which is a contradiction.

In the latter case, the map $\varphi_6 \circ \rho^{-1}$ would be a de Jonquières map of degree 4, which is a contradiction with Lemma 5.15.

Therefore, we conclude that $\mathrm{oql}(\varphi_6) = 4$. □

LEMMA 10.8. — *Let φ_5 be the map 5 in Table 1.2. Then,* $\mathrm{oql}(\varphi_5) = 5$.

Proof. — The enriched weighted proximity graph of φ_5 is listed in Table 1.1. Let p_1 be the double base point, p_5 the proper simple base point and p_2, p_3, p_4 the other infinitely near base points, such that $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with $p_3 \odot p_1$. Then,

$$4 \leqslant \mathrm{oql}(\varphi_5) \leqslant 5$$

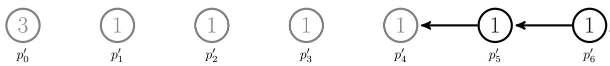
because of the decomposition of φ_5 in Table 8.2 and the fact that the height of p_4 with respect to φ_5 is 4; cf. Proposition 7.7.

Suppose by contradiction that $\mathrm{oql}(\varphi_5) = 4$. Then, there should exist an ordinary quadratic map ρ_1, such that $\mathrm{oql}(\varphi \circ \rho_1^{-1}) = 3$. This implies that ρ_1 must be based at p_1, otherwise the maximum height of the base points of the map $\varphi_5 \circ \rho_1^{-1}$ would be still 4, and Proposition 7.7 would give a contradiction. For the same reason, ρ_1 cannot be based at p_1 and at a point on the line passing through p_1 and p_2.

There are now two cases: either p_5 is a base point of ρ_1 or p_5 is not a base point of ρ_1.

In the former case, the map $\varphi_5 \circ \rho_1^{-1}$ would have enriched weighted proximity graph of type 17 in Table 1.1, and, therefore, it would have ordinary quadratic length 4 by Lemma 10.4, which is a contradiction.

In the latter case, the map $\varphi_5 \circ \rho_1^{-1}$ would be a de Jonquières map of degree 4, and its weighted proximity graph would be



where p'_2, p'_3, p'_4, p'_5 are aligned.

Then, there should exist an ordinary quadratic map ρ_2, such that $\mathrm{oql}(\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 2$. The map ρ_2 must be based at p'_4, and not at p'_2, p'_3, otherwise the maximum height of the base points of the map $\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be at least 3, which is a contradiction with Proposition 7.7. If ρ_2 is not based at p'_0, then $\deg(\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}) \geq 6$, and we get a contradiction with Corollary 5.14. Otherwise ρ_2 is based at p'_0 and, furthermore, either p'_1 is a base point of ρ_2 or p'_1 is not a base point of ρ_2.

In the latter case, the map $\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be a de Jonquières map of degree 4, and we get a contradiction with Lemma 5.15.

In the former case, the map $\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would have the enriched weighted proximity graph of type 24 in Table 1.1, and its ordinary quadratic length would be 3, which is a contradiction.

Hence, we conclude that $\mathrm{oql}(\varphi_5) = 5$. □

LEMMA 10.9. — *Let φ_4 be the map 4 in Table 1.2. Then,* $\mathrm{oql}(\varphi_4) = 4$.

Proof. — The enriched weighted proximity graph of φ_4 is listed in Table 1.1. Let p_1 be the double base point, p_2, p_3, p_4, p_5 the infinitely near simple base points, such that $p_3 \succ_1 p_2 \succ_1 p_1$ and $p_5 \succ_1 p_4 \succ_1 p_1$. Then,

$$3 \leqslant \mathrm{oql}(\varphi_4) \leqslant 4$$

because of the decomposition of φ_4 in Table 8.2 and the fact that the heights of p_3 and of p_5 with respect to φ_4 are 3; cf. Proposition 7.7.

Suppose by contradiction that $\mathrm{oql}(\varphi_4) = 3$. Then, there should exist an ordinary quadratic map ρ, such that $\mathrm{oql}(\varphi \circ \rho^{-1}) = 2$. In particular, ρ must be based at p_1. Then, the map $\varphi \circ \rho^{-1}$ is a de Jonquières map of degree 4, and we get a contradiction with Lemma 5.15. □

LEMMA 10.10. — *Let φ_3 be the map 3 in Table 1.2. Then,* $\mathrm{oql}(\varphi_3) = 5$.

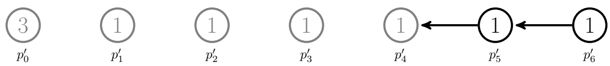
Proof. — The enriched weighted proximity graph of φ_3 is listed in Table 1.1. Let p_1 be the double base point, p_2, p_3, p_4, p_5 the infinitely near simple base points, such that $p_2 \succ_1 p_1$ and $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_1$. Then,

$$4 \leqslant \mathrm{oql}(\varphi_3) \leqslant 5$$

because of the decomposition of φ_3 in Table 8.2 and the fact that the height of p_4 with respect to φ_3 is 4; cf. Proposition 7.7.

Suppose by contradiction that $\mathrm{oql}(\varphi_3) = 4$. Then, there should exist an ordinary quadratic map ρ_1, such that $\mathrm{oql}(\varphi_3 \circ \rho_1^{-1}) = 3$. In particular, ρ_1 must be based at p_1 and not at a point lying on the line passing through p_1 and p_3, otherwise the maximum height of the base points with respect to $\varphi_3 \circ \rho_1^{-1}$ would be still 4. Then, $\varphi_3 \circ \rho_1^{-1}$ is a de Jonquières map of degree 4 and its weighted proximity graph is:

(9)



where p'_1, p'_2, p'_3, p'_4 are aligned.

Then, there should exist an ordinary quadratic map ρ_2, such that $\mathrm{oql}(\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 2$. The map ρ_2 must be based at p'_4, otherwise the maximum height of the base points of the map $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be at least 3, which is a contradiction with Proposition 7.7. Furthermore, the map ρ_2 must be based also at p'_0, otherwise the degree of $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be larger than 4, which is a contradiction with Corollary 5.14.

There are now two cases: either ρ_2 is based at p'_i, for some $i \in \{1, 2, 3\}$, or ρ_2 is not based at $p'_1 .p'_2, p'_3$.

In the former case, the map $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would have the enriched weighted proximity graph of type 14 in Table 1.1, which is a contradiction with Lemma 10.5.

In the latter case, the map $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ is a de Jonquières map of degree 4, which is a contradiction with Lemma 5.15.

Hence, we conclude that $\mathrm{oql}(\varphi_3) = 5$. □

LEMMA 10.11. — *Let φ_1 be the map 1 in Table 1.2. Then,* $\mathrm{oql}(\varphi_1) = 6$.

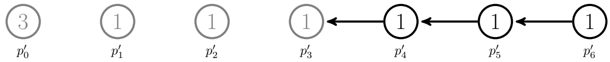
Proof. — The enriched weighted proximity graph of φ_1 is listed in Table 1.1. Let p_1 be the double base point, p_2, p_3, p_4, p_5 the infinitely near simple base points, such that $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with $p_3 \odot p_1$. Then,

$$5 \leqslant \mathrm{oql}(\varphi_1) \leqslant 6$$

because of the decomposition of φ_1 in Table 8.2 and the fact that the height of p_5 with respect to φ_1 is 5; cf. Proposition 7.7.

Suppose by contradiction that $\mathrm{oql}(\varphi_1) = 5$. Then, there should exist an ordinary quadratic map ρ_1, such that $\mathrm{oql}(\varphi_1 \circ \rho_1^{-1}) = 4$. In particular, ρ_1 must be based at p_1 and not at a point lying on the line passing through p_1 and

p_2, otherwise the maximum height of the base points with respect to $\varphi_1 \circ \rho_1^{-1}$ would be still 5. So the map $\varphi_1 \circ \rho_1^{-1}$ is a de Jonquières map of degree 4 and its weighted proximity graph is:



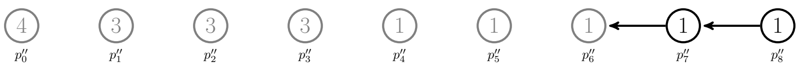
where p'_1, p'_2, p'_3, p'_4 are aligned.

Then, there should exist an ordinary quadratic map ρ_2, such that $\mathrm{oql}(\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 3$. In particular, the map ρ_2 must be based at p'_3 and not at p'_1, p'_2 (or at another point lying on the line passing through p'_3 and p'_4), otherwise the maximum height of the base points of the map $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}$ is 4, which is a contradiction with Proposition 7.7.

There are now two cases: either ρ_2 is based at p'_0 or ρ_2 is not based at p'_0.

In the former case, the map $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}$ is a de Jonquières map of degree 4, and its enriched weighted proximity graph is (9), and we reach a contradiction as in the proof of Lemma 10.10.

In the latter case, the map $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}$ has degree 7, and its weighted proximity graph is:



where p''_2, p''_3, p''_6 are aligned and also $p''_0, p''_4, p''_5, p''_6$ are collinear.

Then, there should exist an ordinary quadratic map ρ_3, such that $\mathrm{oql}(\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1} \circ \rho_3^{-1}) = 2$. Thus, ρ_3 must be based at p''_6, otherwise the maximum height of the base points of $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1} \circ \rho_3^{-1}$ is 3, which is a contradiction with Proposition 7.7. This implies that $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1} \circ \rho_3^{-1}$ would have degree ≥ 6, which is a contradiction with Corollary 5.14.

Hence, we conclude that $\mathrm{oql}(\varphi) = 6$. □

Corollary 10.12. — *Let τ be the quadratic map defined in* (3). *Then,* $\mathrm{oql}(\tau) = 4$, *and the decomposition* (8) *of τ is minimal.*

Proof. — Let p_1, p_2, p_3 be the base points of τ, where $p_3 \succ_1 p_2 \succ_1 p_1 \in \mathbb{P}^2$ and let ℓ be the line through p_1 and p_2. Proposition 7.7 implies that $\mathrm{oql}(\tau) \geq 3$, and the decomposition (8) says that $\mathrm{oql}(\tau) \leq 4$. Suppose by contradiction that $\mathrm{oql}(\tau) = 3$. Then, there exists an involutory ordinary quadratic map ψ, such that $\mathrm{oql}(\tau \circ \psi) = 2$.

There are now two cases: either p_1 is a base point of ψ or p_1 is not a base point of ψ.

In the latter case, $\tau \circ \psi$ has a base point of height 3, and, hence, Proposition 7.7 implies $\mathrm{oql}(\tau \circ \psi) \geq 3$, which is a contradiction.

In the former case, if one of the other two base points of ψ lies on the line ℓ, the map $\tau \circ \psi$ has still a base point of height 3, and we again get the same contradiction. Otherwise, the map $\tau \circ \psi$ has the proximity graph of type 24 in

Table 1.1, which has ordinary quadratic length 3, according to Remark 10.3, which is a contradiction. □

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BIBLIOGRAPHY

[1] M. Alberich-Carramiñana – *Geometry of the plane Cremona maps*, Lecture Notes in Mathematics, no. 1769, Springer-Verlag, 2002.

[2] C. Bisi, A. Calabri & M. Mella – "On plane Cremona transformations of fixed degree", *Journal of Geometric Analysis* **25** (2015), p. 1108–1131.

[3] J. Blanc & J.-P. Furter – "Length in the Cremona group", *Annales Henri Lebesgue* **2** (2019), p. 187–257.

[4] E. Casas-Alvero – *Singularities of plane curves*, London Mathematical Society Student Texts, no. 23, Cambridge University Press, 2000.

[5] D. Cerveau & J. Déserti – *Transformations birationnelles de petit degré*, Cours Spécialisés, no. 19, Société Mathématique de France, 2013.

[6] J. Déserti & F. Han – "On cubic birational maps of $\mathbb{P}^3_{\mathbb{C}}$", *Bulletin de la Société Mathématique de France* **144** (2016), p. 217–249.

[7] I. Dolgachev – *Classical algebraic geometry: A modern view*, Cambridge University Press, 2012.

[8] C. G. Gibson – *Elementary geometry of algebraic curves: An undergraduate introduction*, Cambridge University Press, 1998.

[9] R. Hartshorne – *Algebraic geometry*, Graduate Tests in Mathematics, no. 52, Springer-Verlag New York, 1977.

[10] Y. I. Manin – *Cubic forms. Algebra, geometry, arithmetic*, North-Holland Mathematical Library, no. 4, North-Holland Publishing Co., 1986.

[11] M. Namba – *Geometry of projective algebraic curves*, Pure and Applied Mathematics – A Series of Monographs and Textbooks, Marcel Dekker, 1984.

[12] G. Nguyen Thi Ngoc – "On plane Cremona maps of small degree and their quadratic lengths", Ph.D. thesis, University of Modena and Reggio Emilia, 2020.

[13] K. E. Smith, L. Kahanpää, P. Kekäläinen & W. Traves – *An invitation to algebraic geometry*, Universitext, Springer-Verlag New York, 2006.

[14] C. Urech & S. Zimmermann – "A new presentation of the plane Cremona group", *Proceedings of the American Mathematical Society* **147** (2019), no. 7, p. 2741–2755.