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ON THE GLOBAL DETERMINANT METHOD

BY CHUNHUI LIU

ABSTRACT. — In this paper, we build the global determinant method of Salberger by Arakelov geometry explicitly. As an application, we study the dependence on the degree of the number of rational points of bounded height in plane curves. We will also explain why some constants will be more explicit if we admit the generalized Riemann hypothesis.

RÉSUMÉ (*Autour de la méthode globale de déterminant*). — Dans cet article, on construit la méthode globale de déterminant de Salberger par la géométrie d'Arakelov explicitement. Comme une application, on étudie la dépendance du degré du nombre de points rationnels de hauteur majorée dans courbes planes. On expliquera aussi pourquoi certaines constantes seront plus explicites si on admet l'hypothèse généralisée de Riemann.

1. Introduction

Let $X \hookrightarrow \mathbb{P}_K^n$ be a projective variety over a number field K . For every rational point $\xi \in X(K)$, we denote by $H_K(\xi)$ the height (see (1) for the

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definition) of ξ with respect to the above closed immersion, for example, the classic Weil height (cf. [25, §B.2, Definition]). Let

$$S(X; B) = \{\xi \in X(K) \mid H_K(\xi) \leq B\},$$

where $B \geq 1$ and the embedding morphism is omitted. By Northcott's property, the cardinality $\#S(X; B)$ is finite for a fixed $B \in \mathbb{R}$.

In order to understand the density of the rational points of X , it is an important approach to study the function $\#S(X; B)$ with the variable $B \in \mathbb{R}^+$. For different required properties of $\#S(X; B)$, numerous methods have been applied. In this article, we are interested in the uniform upper bound of $\#S(X; B)$ for all $X \hookrightarrow \mathbb{P}_K^n$ with fixed degree and dimension, and for those satisfying certain common conditions.

1.1. Determinant method. — In order to understand the function $\#S(X; B)$ of the variable $B \in \mathbb{R}^+$, we will introduce the so-called *determinant method* to study the number of rational points with bounded height in arithmetic varieties, which was proposed in [24].

1.1.1. Basic ideas and history. — Traditionally, the determinant method was proposed over the rational number field \mathbb{Q} to avoid some extra technical troubles. In [3] (see also [35]), Bombieri and Pila proposed a method of determinant argument to study plane affine curves. The monomials of a certain degree evaluated on a family of rational points in $S(X; B)$ having the same reduction modulo some prime numbers form a matrix whose determinant is zero by a local estimate. By this method, they proved $\#S(X; B) \ll_{\delta, \epsilon} B^{2/\delta + \epsilon}$ for all $\epsilon > 0$, where $\delta = \deg(X)$.

In [24], Heath-Brown generalized the method of [3] to the higher dimensional case. His idea is to focus on a subset of $S(X; B)$ whose reductions modulo a prime number are a same regular point, and he proved that this subset can be covered by a bounded degree hypersurface, which does not contain the generic point of X . Then he counted the number of regular points over finite fields and controlled the regular reductions. In [5], Broberg generalized it to the case over arbitrary number fields.

In [41, 42], Serre asked whether $\#S(X; B) \ll_X B^{\dim(X)} (\log B)^c$ is verified for all arithmetic varieties X with a particular constant c . In [24], Heath-Brown proposed a uniform version $\#S(X; B) \ll_{d, \delta, \epsilon} B^{d + \epsilon}$ for all $\epsilon > 0$ with $\delta = \deg(X)$ and $d = \dim(X)$, which is called *the dimension growth conjecture*. He proved this conjecture for some special cases. Later, Browning, Heath-Brown and Salberger had some contributions to this subject, see [6, 7, 8] for the improvements of the determinant method and the proofs under certain conditions. In [39], Salberger considered the general reductions, and the multiplicities of rational points were taken into consideration, and he proved the dimension growth conjecture with certain conditions on the subvarieties of X .

1.1.2. *A global version.* — The so-called *global determinant method* was first introduced by Salberger in [40] in order to study the dimension growth conjecture mentioned above. In general, it allows one to use only one auxiliary hypersurface to cover the rational points of bounded height, and one needs to optimize the degree of this hypersurface. By the global version, he proved the dimension growth conjecture for $\deg(X) = \delta \geq 4$ and $\#S(X; B) \ll_{\delta} B^{\frac{2}{\delta}} \log B$, when X is a curve.

In [48], Walsh refined the global determinant method in [40], and he removed the $\log B$ term in [40], when X is a curve.

1.1.3. *The dependence on degree.* — Let $X \hookrightarrow \mathbb{P}_{\mathbb{Q}}^n$ be a geometrically integral variety of degree δ and dimension d . We are also interested in the dependence of the uniform upper bound of $\#S(X; B)$ on δ , in particular when X is a plane curve ($n = 2$ and $d = 1$).

In [47], Walkowiak studied this problem by counting integral points over \mathbb{Z} . In [32, Théorème 2.10], Motte obtained an estimate, which has a better dependence on B but a worse dependence on δ than that in [47, Théorème 1].

In fact, one is able to obtain a better dependence on δ by the global determinant method. In [10], Castryck, Cluckers, Dittmann and Nguyen improved [48] by giving an explicit dependence on δ . As applications, they obtained $\#S(X; B) \ll \delta^4 B^{2/\delta}$, when X is a plane curve, and a better partial result of the dimension growth conjecture than that in [40], and the estimates in [47] and [32] for the case of plane curves.

In [34], the work [10] was generalized over an arbitrary global field. Before [34] was announced, Vermeulen studied the case over $\mathbb{F}_q(t)$ in [46].

Besides the uniform bounds of rational points, [10] also studied the bound of 2-torsion points of the class group of number fields, which improved the work [2, Theorem 1.1] of Bhargava, Shankar, Taniguchi, Thorne, Tsimerman and Y. Zhao.

1.1.4. *Formulation with Arakelov geometry.* — In [12, 13], H. Chen reformulated the work of Salberger [39] by Bost's slope method from Arakelov geometry developed in [4]. In this formulation, H. Chen replaced the matrix of monomials by the evaluation map, which sends a global section of a particular line bundle to its values on a family of rational points. With the slope inequalities, we can control the height of the evaluation map in the slope method, which replaces the role of Siegel's lemma in controlling heights.

There are two advantages of the Arakelov geometry approach. First, Arakelov geometry gives a natural conceptual framework for the determinant method over arbitrary number fields. Next, it is easier to obtain explicit estimates, since the constants obtained from the slope inequalities are given explicitly in general.

1.2. A global version with the formulation of Arakelov geometry. — In this article, we will construct the global determinant method over an arbitrary number field by Arakelov geometry following the strategy of [12, 13]. As a direct application, we will study the problem of counting rational points in plane curves and we consider how these upper bounds depend on the degree. Some of the ideas were inspired by [40, 48, 10].

1.2.1. Main results. — First, we have the control of auxiliary hypersurface below in Corollary 5.5 and Corollary 5.6, which are deduced from Theorem 5.4.

THEOREM 1.1. — *Let X be a geometrically integral hypersurface in \mathbb{P}_K^n of degree δ , and*

$$a_n = \frac{n}{(n-1)\delta^{1/(n-1)}}$$

be a constant depending on n . Then there is a hypersurface of degree ϖ , which covers $S(X; B)$ but does not contain the generic point of X . In addition, we have

$$\varpi \ll_{K,n} \delta^3 B^{a_n}$$

in Corollary 5.5, and

$$\varpi \ll_{K,n} \delta^{3-1/(n-1)} B^{a_n} \max \left\{ \frac{\log B}{[K : \mathbb{Q}]}, 1 \right\}$$

in Corollary 5.6.

If we assume the generalized Riemann hypothesis, the above constants depending on K and n will be given explicitly in Corollary 5.5 and Corollary 5.6.

Since we apply the approach of Arakelov geometry in this article, we no longer use the technique of “change of coordinate” in [48, §3] and [10, §3.4]. Instead, we are able to obtain a uniform estimate directly.

1.2.2. Potential applications. — Similar to the previous applications of the determinant, the above estimates can be applied to study the uniform upper bound of the number of rational points with bounded height, where we will initiate the induction on the dimension as usual and the study of the distribution of the loci of small degree in a variety (see [40, §4] for such an example, which considered the density of conics in a cubic surface). By the above operation, we are able to obtain estimates of general arithmetic varieties from those of hypersurfaces via a suitable linear projection.

Since our method works over an arbitrary number field and gives an explicit estimate (or under some technical conditions), it is possible that further applications will work well under the same conditions and also be explicit.

As a direct application, we have the following results on counting the rational points of the bounded height in plane curves in Theorem 6.1 and Theorem 6.2.

THEOREM 1.2. — *Let X be a geometrically integral plane curve in \mathbb{P}_K^2 of degree δ . Then we have*

$$\#S(X; B) \ll_K \delta^4 B^{2/\delta}$$

in Theorem 6.1 and

$$\#S(X; B) \ll_K \delta^3 B^{2/\delta} \log B$$

in Theorem 6.2.

If we assume the generalized Riemann hypothesis, the above constants depending on K will be given explicitly in Theorem 6.1 and Theorem 6.2.

Theorem 6.1 generalizes [10, Theorem 2] over an arbitrary number field and gives an explicit estimate under the assumption of the generalized Riemann hypothesis. Theorem 6.2 can be viewed as a projective analogue of [10, Theorem 3] over an arbitrary number field and a better partial result of the conjecture of Heath-Brown referred to in Remark 6.3. These two estimates are better than those given in [47, Théorème 1] and [32, Théorème 2.10].

1.2.3. The role of the generalized Riemann hypothesis. — In this work, some explicit estimates of the distribution of prime ideals are applied. If we admit **GRH (the generalized Riemann hypothesis)** of the Dedekind zeta function of the base number field, we are able to obtain more explicit estimates; see [20], for example. Without the assumptions of GRH, it seems to be very difficult to obtain such explicit estimates over an arbitrary number field, since we do not know the zero-free region of the Dedekind zeta function. If we know the zero-free region clearly enough, for example, if we work in the rational number field \mathbb{Q} or totally imaginary fields (see [44] and [22], respectively), or we just want an implicit estimate (see [37]), we do not need to suppose GRH.

1.3. Organization of the article. — This paper is organized as following. In §2, we provide some preliminaries to construct the determinant method. In §3, we formulate the global determinant method by the slope method. In §4, we give some useful estimates on the non-geometrically integral reductions, a count of multiplicities over finite fields, the distributions of some particular prime ideals, and the geometric Hilbert–Samuel function. In §5, we provide an explicit upper bound of the determinant and lower bounds of auxiliary hypersurfaces. In §6, we give two uniform upper bounds of rational points of bounded height in plane curves. In §7, under the assumption of GRH, we give some explicit estimates of the distribution of prime ideals of bounded norm in a ring of integers and explain how to apply these explicit estimates in the global determinant to get more explicit estimates. In Appendix A, we will give an explicit lower bound of a useful function induced by the local Hilbert–Samuel function.

2. Fundamental settings

In this section, we will introduce some preliminaries to help understand the problem of counting rational points of bounded height. In particular, we will provide some basic notions in Arakelov geometry.

2.1. Counting rational points of bounded height. — Let K be a number field and \mathcal{O}_K be its ring of integers. We denote by $M_{K,f}$ the set of finite places of K and by $M_{K,\infty}$ the set of infinite places of K . In addition, we denote by $M_K = M_{K,f} \sqcup M_{K,\infty}$ the set of places of K . For every $v \in M_K$ and $x \in K$, we define the absolute value $|x|_v = |N_{K_v/\mathbb{Q}_v}(x)|_v^{\frac{1}{[K_v:\mathbb{Q}_v]}}$ for each $v \in M_K$, extending the usual absolute values of \mathbb{Q}_p or \mathbb{R} . Here, \mathbb{Q}_v denotes the p -adic field \mathbb{Q}_p , where v is extended from p under the extension K/\mathbb{Q} .

Let $\xi = [\xi_0 : \cdots : \xi_n] \in \mathbb{P}_K^n(K)$. We define the *height* of ξ in \mathbb{P}_K^n as

$$(1) \quad H_K(\xi) = \prod_{v \in M_K} \max_{0 \leq i \leq n} \left\{ |\xi_i|_v^{[K_v:\mathbb{Q}_v]} \right\}.$$

We also define the logarithmic height of ξ as

$$(2) \quad h(\xi) = \frac{1}{[K:\mathbb{Q}]} \log H_K(\xi),$$

which is invariant under the extensions over K (cf. [25, Lemma B.2.1]).

Suppose that X is a closed integral subscheme of \mathbb{P}_K^n , and $\phi: X \hookrightarrow \mathbb{P}_K^n$ is the closed immersion. For $\xi \in X(K)$, we define $H_K(\xi) = H_K(\phi(\xi))$ and, usually, we omit the closed immersion ϕ if there is no confusion. Next, we denote

$$S(X; B) = \{\xi \in X(K) \mid H_K(\xi) \leq B\}, \text{ and } N(X; B) = \#S(X; B).$$

By Northcott's property (cf. [25, Theorem B.2.3]), the cardinality $N(X; B)$ is finite for a fixed $B \geq 1$.

2.2. A function induced by local Hilbert–Samuel functions. — In this part, we will introduce a function induced by the local Hilbert–Samuel function of schemes at a closed point and we will use this function in Proposition 3.4. For the motivation and background, see [39, §2] and [13, §3.2].

Let k be a field and X be a closed subscheme of \mathbb{P}_k^n of pure dimension d , which means all its irreducible components have the same dimension. Let ξ be a closed point of X . We denote by

$$(3) \quad H_\xi(s) = \dim_{\kappa(\xi)} \left(\mathfrak{m}_{X,\xi}^s / \mathfrak{m}_{X,\xi}^{s+1} \right)$$

the local Hilbert–Samuel function of X at the point ξ with the variable $s \in \mathbb{N}$, where $\mathfrak{m}_{X,\xi}$ is the maximal ideal of the local ring $\mathcal{O}_{X,\xi}$, and $\kappa(\xi)$ is the residue

field of the local ring $\mathcal{O}_{X,\xi}$. For this function, we have the polynomial asymptotic

$$(4) \quad H_\xi(s) = \frac{\mu_\xi(X)}{(d-1)!} s^{d-1} + o(s^{d-1}),$$

when $s \rightarrow +\infty$ and we define the positive integer $\mu_\xi(X)$ as the *multiplicity* of point ξ in X .

We define the series $\{q_\xi(m)\}_{m \geq 0}$ as the increasing series of non-negative integers such that every integer $s \in \mathbb{N}$ appears exactly $H_\xi(s)$ times in this series. For example, if $H_\xi(0) = 1, H_\xi(1) = 2, H_\xi(2) = 4, H_\xi(3) = 5, \dots$, then the series $\{q_\xi(m)\}_{m \geq 0}$ is

$$\{0, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, \dots\}.$$

Let $\{Q_\xi(m)\}_{m \geq 0}$ be the partial sum of the series $\{q_\xi(m)\}_{m \geq 0}$, which is

$$(5) \quad Q_\xi(m) = q_\xi(0) + q_\xi(1) + \dots + q_\xi(m),$$

for all $m \in \mathbb{N}$.

If X is a hypersurface of \mathbb{P}_k^n , then by [26, Example 2.70 (2)] the local Hilbert–Samuel function of X at the point ξ defined in (3) is

$$H_\xi(s) = \binom{n+s-1}{s} - \binom{n+s-\mu_\xi(X)-1}{s-\mu_\xi(X)}.$$

In this case, we have the following explicit lower bound of $Q_\xi(m)$, which is

$$(6) \quad Q_\xi(m) > \left(\frac{(n-1)!}{\mu_\xi(X)} \right)^{\frac{1}{n-1}} \left(\frac{n-1}{n} \right) m^{\frac{n}{n-1}} - \frac{n^3 + 2n^2 + n - 4}{2n(n+1)} m.$$

This lower bound has the optimal dominant term by the argument in [39, Main Lemma 2.5] and some other subsequent references. In Appendix A, we will provide a detailed proof of this lower bound.

2.3. Normed vector bundles. — A *normed vector bundle* over $\text{Spec } \mathcal{O}_K$ is all the pairings $\overline{E} = (E, (\|\cdot\|_v)_{v \in M_{K,\infty}})$, where

- E is a projective \mathcal{O}_K -module of finite rank;
- $(\|\cdot\|_v)_{v \in M_{K,\infty}}$ is a family of norms, where $\|\cdot\|_v$ is a norm over $E \otimes_{\mathcal{O}_K} \mathbb{C}$, which is invariant under the action of $\text{Gal}(\mathbb{C}/K_v)$. We consider a complex place and its conjugation as two different places.

If for all $v \in M_{K,\infty}$, the norms $(\|\cdot\|_v)_{v \in M_{K,\infty}}$ are Hermitian, we say that \overline{E} is a *Hermitian vector bundle* over $\text{Spec } \mathcal{O}_K$. If $\text{rk}_{\mathcal{O}_K}(E) = 1$, we say that \overline{E} is a *Hermitian line bundle*.

Suppose that F is a sub- \mathcal{O}_K -module of E . We say that F is a *saturated* sub- \mathcal{O}_K -module if E/F is a torsion-free \mathcal{O}_K -module.

Let $\overline{E} = (E, (\|\cdot\|_{E,v})_{v \in M_{K,\infty}})$ and $\overline{F} = (F, (\|\cdot\|_{F,v})_{v \in M_{K,\infty}})$ be two Hermitian vector bundles. If F is a saturated sub- \mathcal{O}_K -module of E and $\|\cdot\|_{F,v}$ is

the restriction of $\|\cdot\|_{E,v}$ over $F \otimes_{\mathcal{O}_K,v} \mathbb{C}$ for every $v \in M_{K,\infty}$, we say that \overline{F} is a *sub-Hermitian vector bundle* of \overline{E} over $\text{Spec } \mathcal{O}_K$.

We say that $\overline{G} = (G, (\|\cdot\|_{G,v})_{v \in M_{K,\infty}})$ is a *quotient Hermitian vector bundle* of \overline{E} over $\text{Spec } \mathcal{O}_K$, if for every $v \in M_{K,\infty}$, the module G is a projective quotient \mathcal{O}_K -module of E , and $\|\cdot\|_{G,v}$ is the induced quotient space norm of $\|\cdot\|_{E,v}$.

For simplicity, we will denote it by $E_K = E \otimes_{\mathcal{O}_K} K$ below.

2.4. Arakelov invariants. — Let \overline{E} be a Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, and $\{s_1, \dots, s_r\}$ be a K -basis of E_K . We will introduce some invariants in Arakelov geometry below.

2.4.1. *Arakelov degree.* — The *Arakelov degree* of \overline{E} is defined as

$$\begin{aligned} \widehat{\deg}(\overline{E}) &= - \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log \|s_1 \wedge \cdots \wedge s_r\|_v \\ &= \log (\# (E/\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r)) \\ &\quad - \frac{1}{2} \sum_{v \in M_{K,\infty}} [K_v : \mathbb{Q}_v] \log \det (\langle s_i, s_j \rangle_{v, 1 \leq i, j \leq r}), \end{aligned}$$

where $\|s_1 \wedge \cdots \wedge s_r\|_v$ follows the definition in [11, 2.1.9] for all $v \in M_{K,\infty}$, and $\langle s_i, s_j \rangle_{v, 1 \leq i, j \leq r}$ is the Gram matrix of the basis $\{s_1, \dots, s_r\}$ with respect to $v \in M_{K,\infty}$. We refer the readers to [17, 2.4.1] for a proof of the equivalence of the above two definitions. The Arakelov degree is independent of the choice of the basis $\{s_1, \dots, s_r\}$ by the product formula (cf. [33, Chap. III, Proposition 1.3]). In addition, we define

$$\widehat{\deg}_n(\overline{E}) = \frac{1}{[K : \mathbb{Q}]} \widehat{\deg}(\overline{E})$$

as the *normalized Arakelov degree* of \overline{E} , which is independent of the choice of K .

2.4.2. *Slope.* — Let \overline{E} be a non-zero Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ and $\text{rk}(E)$ be the rank of E . The *slope* of \overline{E} is defined as

$$\hat{\mu}(\overline{E}) := \frac{1}{\text{rk}(E)} \widehat{\deg}_n(\overline{E}).$$

In addition, we denote by $\hat{\mu}_{\max}(\overline{E})$ the maximal value of slopes of all non-zero Hermitian subbundles and by $\hat{\mu}_{\min}(\overline{E})$ the minimal value of slopes of all non-zero Hermitian quotients bundles of \overline{E} .

2.4.3. *Height of linear maps.* — Let \overline{E} and \overline{F} be two non-zero Hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ and $\phi : E_K \rightarrow F_K$ be a non-zero homomorphism.

The *height* of ϕ is defined as

$$h(\phi) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \|\phi\|_v,$$

where $\|\phi\|_v$ is the operator norm of $\phi_v : E \otimes_K K_v \rightarrow F \otimes_K K_v$ induced by the above linear homomorphism with respect to $v \in M_K$.

We refer the readers to [4, Appendix A] for some equalities and inequalities on Arakelov degrees and corresponding heights of homomorphisms.

2.5. Arithmetic Hilbert–Samuel function. — Let $\bar{\mathcal{E}}$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec } \mathcal{O}_K$ and $\mathbb{P}(\mathcal{E})$ be the projective space that represents the functor from the category of commutative \mathcal{O}_K -algebras to the category of sets mapping all \mathcal{O}_K -algebra A to the set of projective quotient A -module of $\mathcal{E} \otimes_{\mathcal{O}_K} A$ of rank 1. Let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ (or $\mathcal{O}(1)$ if there is no confusion) be the universal bundle, and we denote by $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(D)$ (or $\mathcal{O}(D)$) the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)^{\otimes D}$ for simplicity. The Hermitian metrics on \mathcal{E} induce by quotient of Hermitian metrics (i.e., Fubini–Study metrics) on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, which define a Hermitian line bundle $\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$ on $\mathbb{P}(\mathcal{E})$.

For every $D \in \mathbb{N}^+$, let

$$(7) \quad E_D = H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(D)),$$

and let $r(n, D)$ be its rank over \mathcal{O}_K . In fact, we have

$$(8) \quad r(n, D) = \binom{n+D}{D}.$$

For each $v \in M_{K,\infty}$, we denote by $\|\cdot\|_{v,\text{sup}}$ the norm over $E_{D,v} = E_D \otimes_{\mathcal{O}_K,v} \mathbb{C}$, such that

$$(9) \quad \forall s \in E_{D,v}, \quad \|s\|_{v,\text{sup}} = \sup_{x \in \mathbb{P}(\mathcal{E}_K)_v(\mathbb{C})} \|s(x)\|_{v,\text{FS}},$$

where $\|\cdot\|_{v,\text{FS}}$ is the corresponding Fubini–Study norm.

2.5.1. Metric of John. — Next, we introduce the *metric of John*, see [45] for a systematic introduction of this notion. In general, for a given symmetric convex body C , there exists the unique ellipsoid, called the *ellipsoid of John*, contained in C with the maximal volume.

For the \mathcal{O}_K -module E_D and any place $v \in M_{K,\infty}$, we take the ellipsoid of John of its unit closed ball defined via the norm $\|\cdot\|_{v,\text{sup}}$, and this ellipsoid induces a Hermitian norm, denoted by $\|\cdot\|_{v,J}$. For every section $s \in E_D$, the inequality

$$(10) \quad \|s\|_{v,\text{sup}} \leq \|s\|_{v,J} \leq \sqrt{r(n, D)} \|s\|_{v,\text{sup}}$$

is verified by [45, Theorem 3.3.6].

2.5.2. Evaluation map. — Let X be an integral closed subscheme of $\mathbb{P}(\mathcal{E}_K)$ and \mathcal{X} be the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. We denote by

$$(11) \quad \eta_{X,D} : E_{D,K} = H^0(\mathbb{P}(\mathcal{E}_K), \mathcal{O}(D)) \rightarrow H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_X^{\otimes D})$$

the *evaluation map* over X induced by the closed immersion of X in $\mathbb{P}(\mathcal{E}_K)$. We denote by F_D the largest saturated sub- \mathcal{O}_K -module of $H^0(\mathcal{X}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\mathcal{X}}^{\otimes D})$, such that $F_{D,K} = \text{Im}(\eta_{X,D})$. When the integer D is large enough, the homomorphism $\eta_{X,D}$ is surjective, which means $F_D = H^0(\mathcal{X}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\mathcal{X}}^{\otimes D})$.

The \mathcal{O}_K -module F_D is equipped with the quotient metrics (from \overline{E}_D), such that F_D is a Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$. We denote this Hermitian vector bundle by \overline{F}_D .

DEFINITION 2.1 (Arithmetic Hilbert–Samuel function). — Let \overline{F}_D be the Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ defined above from the map (11). We say that the function that maps the positive integer D to $\widehat{\mu}(\overline{F}_D)$ is the *arithmetic Hilbert–Samuel function* of X with respect to the Hermitian line bundle $\overline{\mathcal{O}(1)}$.

2.6. Height of rational points. — In this part, we will define a height function of rational points by Arakelov geometry.

Let $\overline{\mathcal{E}}$ be a Hermitian vector bundle of rank $n + 1$ over $\text{Spec } \mathcal{O}_K$, $P \in \mathbb{P}(\mathcal{E}_K)(K)$ and $\mathcal{P} \in \mathbb{P}(\mathcal{E})(\mathcal{O}_K)$ be the Zariski closure of P in $\mathbb{P}(\mathcal{E})$. Let $\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$ be the universal bundle equipped with the corresponding Fubini–Study metric at each $v \in M_{K,\infty}$; then $\mathcal{P}^* \overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$ is a Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$. We define the *height* of the rational point P with respect to $\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$ as

$$(12) \quad h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(P) = \widehat{\deg}_n \left(\mathcal{P}^* \overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} \right).$$

We keep all the above notations. We choose

$$(13) \quad \overline{\mathcal{E}} = \left(\mathcal{O}_K^{\oplus(n+1)}, (\|\cdot\|_v)_{v \in M_{K,\infty}} \right),$$

where for every $v \in M_{K,\infty}$, $\|\cdot\|_v$ is the ℓ^2 -norm mapping (t_0, \dots, t_n) to $\sqrt{|v(t_0)|^2 + \dots + |v(t_n)|^2}$. In this case, we use the notations $\mathbb{P}_K^n = \mathbb{P}(\mathcal{E}_K)$ and $\mathbb{P}_{\mathcal{O}_K}^n = \mathbb{P}(\mathcal{E})$ for simplicity. We suppose that P has a K -rational projective coordinate $[x_0 : \dots : x_n]$; then we have (cf. [31, Proposition 9.10])

$$\begin{aligned} h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(P) &= \sum_{v \in M_{K,f}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \left(\max_{1 \leq i \leq n} |x_i|_v \right) \\ &\quad + \frac{1}{2} \sum_{v \in M_{K,\infty}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \left(\sum_{j=0}^n |v(x_j)|^2 \right). \end{aligned}$$

REMARK 2.2. — We compare the logarithmic height $h(\cdot)$ defined in (2) and the height $h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(\cdot)$ defined in (12) by Arakelov geometry, where $\overline{\mathcal{E}}$ is defined in (13). In fact, by an elementary calculation, the inequality

$$\left| h(P) - h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(P) \right| \leq \frac{1}{2} \log(n+1)$$

is uniformly verified for all $P \in \mathbb{P}_K^n(K)$.

Let $\psi : X \hookrightarrow \mathbb{P}_K^n$ be a projective scheme and $P \in X(K)$. We define the height of P as $h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(\psi(P))$. We will just use the notation $h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(P)$ or $h(P)$ if there is no confusion of the morphism ψ and the Hermitian line bundle $\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$.

2.7. Height functions of arithmetic varieties. — In this part, we will introduce several height functions of arithmetic varieties, which evaluate their arithmetic complexities.

2.7.1. *Arakelov height.* — First, we will introduce a kind of height functions of arithmetic varieties defined by the arithmetic intersection theory developed by Gillet and Soulé in [18], which was first introduced by Faltings in [15, Definition 2.5]; see also [43, III.6].

DEFINITION 2.3 (Arakelov height). — Let K be a number field, \mathcal{O}_K be its ring of integers, $\overline{\mathcal{E}}$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec } \mathcal{O}_K$ and $\overline{\mathcal{L}}$ be a Hermitian line bundle over $\mathbb{P}(\mathcal{E})$. Let X be a pure dimensional closed subscheme of $\mathbb{P}(\mathcal{E}_K)$ of dimension d and \mathcal{X} be the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. The *Arakelov height* of X is defined as the arithmetic intersection number

$$\frac{1}{[K : \mathbb{Q}]} \widehat{\deg} (\widehat{c}_1(\overline{\mathcal{L}})^{d+1} \cdot [\mathcal{X}]),$$

where $\widehat{c}_1(\overline{\mathcal{L}})$ is the arithmetic first Chern class of $\overline{\mathcal{L}}$ (cf. [43, Chap. III.4, Proposition 1] for its definition). This height is noted by $h_{\overline{\mathcal{L}}}(X)$ or $h_{\overline{\mathcal{L}}}(\mathcal{X})$.

REMARK 2.4. — With all the notations in Definition 2.1 and Definition 2.3, by [36, Théorème A], we have

$$h_{\overline{\mathcal{O}(1)}}(X) = \lim_{D \rightarrow +\infty} \frac{\widehat{\deg}_n(\overline{F}_D)}{D^{d+1}/(d+1)!}.$$

2.7.2. *Heights of hypersurfaces.* — Let X be a hypersurface in \mathbb{P}_K^n of degree δ . By [23, Proposition 7.6 (d), Chap. I], X is defined by a homogeneous polynomial of degree δ . We define a height function of hypersurfaces by considering its polynomial of definition.

DEFINITION 2.5 (Naive height). — Let

$$f(T_0, \dots, T_n) = \sum_{(i_0, \dots, i_n) \in \mathbb{N}^{n+1}} a_{i_0, \dots, i_n} T_0^{i_0} \cdots T_n^{i_n} \in K[T_0, \dots, T_n].$$

We define the *naive height* of $f(T_0, \dots, T_n)$ as

$$H_K(f) = \prod_{v \in M_K} \max_{(i_0, \dots, i_n) \in \mathbb{N}^{n+1}} \{ |a_{i_0, \dots, i_n}|_v \}^{[K_v : \mathbb{Q}_v]},$$

and

$$h(f) = \frac{1}{[K : \mathbb{Q}]} \log H_K(f).$$

In addition, if $f(T_0, \dots, T_n)$ is homogeneous and defines the hypersurface $X \hookrightarrow \mathbb{P}_K^n$, we define the *naive height* of X as

$$H_K(X) = H_K(f) \text{ and } h(X) = h(f).$$

2.7.3. *Comparison of height functions.* — In order to compare $h_{\overline{\mathcal{O}(1)}}(\mathcal{X})$ and $h(X)$ for a hypersurface X , we refer to the following result in [27].

PROPOSITION 2.6. — *Let X be a hypersurface in \mathbb{P}_K^n of degree δ . With all the notations above, we have*

$$\begin{aligned} -\delta \left(\frac{1}{2\delta} \log((n+1)(\delta+1)) + \frac{1}{2} \mathcal{H}_n \right) &\leq h(X) - h_{\overline{\mathcal{O}(1)}}(\mathcal{X}) \\ &\leq \delta \left(\log 2 + 5 \log(n+1) - \frac{1}{2} \mathcal{H}_n \right), \end{aligned}$$

where $\mathcal{H}_n = 1 + \cdots + \frac{1}{n}$.

Proof. — Since X is a hypersurface, the Chow variety of X is just X itself. Then we have the result from [27, Proposition 3.7] directly after some elementary calculations. \square

3. Global determinant method for hypersurfaces

In the rest of this article, unless especially mentioned, we suppose that X is an integral hypersurface in \mathbb{P}_K^n , and \mathcal{X} is its Zariski closure in $\mathbb{P}_{\mathcal{O}_K}^n$. In fact, $X \rightarrow \operatorname{Spec} K$ is the generic fiber of $\mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}_K$. When we consider the height $h(P)$ of a rational point $P \in X(K)$ embedded into \mathbb{P}_K^n , we use the definition in (12) by Arakelov geometry. Let \mathfrak{p} be a maximal ideal of \mathcal{O}_K ; we denote by $\mathcal{X}_{\mathfrak{p}} = \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathbb{F}_{\mathfrak{p}} \rightarrow \mathbb{F}_{\mathfrak{p}}$ the fiber at \mathfrak{p} .

Let $r_1(n, D)$ be the rank of F_D over \mathcal{O}_K , where F_D is defined in §2.5. For the case where X is a hypersurface of degree δ in \mathbb{P}_K^n , we have

$$r_1(n, D) = \binom{n+D}{n} - \binom{n+D-\delta}{n}.$$

Our main target of this section is to prove the following result.

THEOREM 3.1. — *We keep all the notations in §2.5 and this section. Let X be a closed integral subscheme in \mathbb{P}_K^n and \mathcal{X} be its Zariski closure in $\mathbb{P}_{\mathcal{O}_K}^n$. Let $\{\mathfrak{p}_j\}_{j \in J}$ be a finite family of maximal ideals of \mathcal{O}_K , and $\{P_i\}_{i \in I}$ be a family of rational points of X . For a fixed prime ideal \mathfrak{p} of \mathcal{O}_K , let $\mu_\xi(\mathcal{X}_{\mathfrak{p}})$ be the multiplicity of the point ξ in $\mathcal{X}_{\mathfrak{p}}$; we denote $n(\mathcal{X}_{\mathfrak{p}}) = \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \mu_\xi(\mathcal{X}_{\mathfrak{p}})$. If the inequality*

(14)

$$\sup_{i \in I} h(P_i) < \frac{\widehat{\mu}(\overline{F}_D)}{D} - \frac{\log r_1(n, D)}{2D} \\ + \frac{1}{[K:\mathbb{Q}]} \sum_{j \in J} \left(\frac{(n-1)!^{\frac{1}{n-1}} (n-1) r_1(n, D)^{\frac{1}{n-1}}}{n D n(\mathcal{X}_{\mathfrak{p}_j})^{\frac{1}{n-1}}} - \frac{n^3 + 2n^2 + n - 4}{2Dn(n+1)} \right) \log N(\mathfrak{p}_j)$$

is verified, then there exists a section $s \in E_{D,K}$, which contains $\{P_i\}_{i \in I}$ but does not contain the generic point of X . In other words, $\{P_i\}_{i \in I}$ can be covered by a hypersurface of degree D that does not contain the generic point of X .

3.1. Auxiliary results. — We refer to some results in [12, 13], which are used in the reformulation of the determinant method by Arakelov geometry. We will also prove a new auxiliary lemma.

PROPOSITION 3.2 ([12], Proposition 2.2). — *Let \overline{E} be a Hermitian vector bundle of rank $r > 0$ over $\text{Spec } \mathcal{O}_K$ and $\{\overline{L}_i\}_{i \in I}$ be a family of Hermitian line bundles over $\text{Spec } \mathcal{O}_K$. If*

$$\phi: E_K \rightarrow \bigoplus_{i \in I} L_{i,K}$$

is an injective homomorphism, then there exists a subset I_0 of I whose cardinality is r such that the following equality

$$\widehat{\mu}(\overline{E}) = \frac{1}{r} \left(\sum_{i \in I_0} \widehat{\mu}(\overline{L}_i) + h(\wedge^r (\text{pr}_{I_0} \circ \phi)) \right)$$

is verified, where $\text{pr}_{I_0}: \bigoplus_{i \in I} L_{i,K} \rightarrow \bigoplus_{i \in I_0} L_{i,K}$ is the canonical projection.

In order to benefit the readers, we will provide the details of the construction of certain local homomorphisms, which are introduced in [39, Lemma 2.4]; see also [13, §3.2].

Let X be an integral closed subscheme of \mathbb{P}_K^n and \mathcal{X} be the Zariski closure of X in $\mathbb{P}_{\mathcal{O}_K}^n$. Let \mathfrak{p} be a maximal ideal of \mathcal{O}_K and $\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})$. In this case, $\mathcal{O}_{\mathcal{X},\xi}$ is a local algebra over $\mathcal{O}_{K,\mathfrak{p}}$. Let $(f_i)_{1 \leq i \leq m}$ be a family of local homomorphisms of $\mathcal{O}_{K,\mathfrak{p}}$ -algebras from $\mathcal{O}_{\mathcal{X},\xi}$ to $\mathcal{O}_{K,\mathfrak{p}}$.

Let E be a free sub- $\mathcal{O}_{K,\mathfrak{p}}$ -module of finite type of $\mathcal{O}_{\mathcal{X},\xi}$ and let f be the $\mathcal{O}_{K,\mathfrak{p}}$ -linear homomorphism

$$(f_i|_E)_{1 \leq i \leq m} : E \rightarrow \mathcal{O}_{K,\mathfrak{p}}^{\oplus m}.$$

Since f_1 is a local homomorphism of $\mathcal{O}_{K,\mathfrak{p}}$ -algebras, it must be surjective. Let \mathfrak{a} be the kernel of f_1 , then we have $\mathcal{O}_{\mathcal{X},\xi}/\mathfrak{a} \cong \mathcal{O}_{K,\mathfrak{p}}$. Furthermore, since $\mathcal{O}_{\mathcal{X},\xi}$ is a local ring, and we suppose that \mathfrak{m}_ξ is its maximal ideal, we have $\mathfrak{m}_\xi \supseteq \mathfrak{a}$. Moreover, since f_1 is a local homomorphism, we have $\mathfrak{a} + \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi} = \mathfrak{m}_\xi$. For each $j \in \mathbb{N}$, $\mathfrak{a}^j/\mathfrak{a}^{j+1}$ is an $\mathcal{O}_{\mathcal{X},\xi}/\mathfrak{a} \cong \mathcal{O}_{K,\mathfrak{p}}$ -module of finite type.

In order to estimate its rank, we need the following result.

LEMMA 3.3. — *With all the above notations and constructions, we have*

$$\begin{aligned} \mathbb{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{K,\mathfrak{p}}} (\mathfrak{a}^j/\mathfrak{a}^{j+1}) &\cong (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^j / (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^{j+1} \\ &\cong (\mathfrak{m}_\xi/\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^j / (\mathfrak{m}_\xi/\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^{j+1}. \end{aligned}$$

Proof. — By definition, we have

$$\mathbb{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{K,\mathfrak{p}}} (\mathfrak{a}^j/\mathfrak{a}^{j+1}) \cong (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^j / (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^{j+1}.$$

Next, from the facts $\mathfrak{a} + \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi} = \mathfrak{m}_\xi$ and $\mathfrak{a} \subseteq \mathfrak{m}_\xi$, we claim that

$$\mathfrak{a} + \mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi} = \mathfrak{m}_\xi \cap (\mathfrak{a} + \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi}) = \mathfrak{m}_\xi$$

is verified. In fact, for every $x \in \mathfrak{m}_\xi = \mathfrak{m}_\xi \cap (\mathfrak{a} + \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})$, there exist $v \in \mathfrak{a}$ and $w \in \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi}$, such that $x = v + w$. Then $w = x - v \in \mathfrak{m}_\xi$. Since $v, x \in \mathfrak{m}_\xi$, then $w \in \mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi}$. So we have $x \in \mathfrak{a} + \mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi}$. Conversely, since \mathfrak{m}_ξ is the maximal ideal, $\mathfrak{a} + \mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi} \subseteq \mathfrak{m}_\xi$.

By the above fact, we have

$$\begin{aligned} \mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi} &\cong \mathfrak{a}/\mathfrak{a} \cap (\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi}) \\ &\cong (\mathfrak{a} + \mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})/\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi} \cong \mathfrak{m}_\xi/\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi}, \end{aligned}$$

which terminates the proof. \square

With Nakayama's lemma (cf. [30, Theorem 2.2]) we deduce that the rank of $\mathfrak{a}^j/\mathfrak{a}^{j+1}$ over $\mathcal{O}_{K,\mathfrak{p}}$ is equal to the rank of $(\mathfrak{m}_\xi/\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^j/(\mathfrak{m}_\xi/\mathfrak{m}_\xi \cap \mathfrak{p}\mathcal{O}_{\mathcal{X},\xi})^{j+1}$ over $\mathbb{F}_{\mathfrak{p}}$ from the isomorphism in Lemma 3.3, which is the value of the local Hilbert–Samuel function $H_\xi(j)$ defined in (3).

With this fact, we consider the filtration

$$\mathcal{O}_{\mathcal{X},\xi} = \mathfrak{a}^0 \supseteq \mathfrak{a} \supseteq \mathfrak{a}^2 \supseteq \cdots \supseteq \mathfrak{a}^j \supseteq \mathfrak{a}^{j+1} \supseteq \cdots$$

of $\mathcal{O}_{\mathcal{X},\xi}$, which induces the filtration

$$(15) \quad \mathcal{F} : E = E \cap \mathfrak{a}^0 \supseteq E \cap \mathfrak{a} \supseteq E \cap \mathfrak{a}^2 \supseteq \cdots \supseteq E \cap \mathfrak{a}^j \supseteq E \cap \mathfrak{a}^{j+1} \supseteq \cdots$$

of E , whose j -th sub-quotient $E \cap \mathfrak{a}^j / E \cap \mathfrak{a}^{j+1}$ is a free $\mathcal{O}_{K,\mathfrak{p}}$ -module of rank smaller than or equal to $H_\xi(j)$.

We suppose that the reductions of all the above local homomorphisms f_1, \dots, f_m modulo \mathfrak{p} are same, which means that all the composed homomorphisms

$$\mathcal{O}_{\mathcal{X}, \xi} \xrightarrow{f_i} \mathcal{O}_{K, \mathfrak{p}} \rightarrow \mathbb{F}_{\mathfrak{p}}$$

are the same for all $i = 1, \dots, m$, where the last arrow is the canonical reduction morphism modulo \mathfrak{p} . We note that $N(\mathfrak{p}) = \#\mathbb{F}_{\mathfrak{p}}$. In this case, the restriction of f on $E \cap \mathfrak{a}^j$ has its norm smaller than $N(\mathfrak{p})^{-j}$. In fact, for any $1 \leq i \leq m$, we have $f_i(\mathfrak{a}) \subseteq \mathfrak{p}\mathcal{O}_{K, \mathfrak{p}}$ and, hence, we have $f_i(\mathfrak{a}^j) \subseteq \mathfrak{p}^j\mathcal{O}_{K, \mathfrak{p}}$.

From the above argument, we have the following result from [13, Lemma 3.2, Lemma 3.3].

PROPOSITION 3.4 ([13], Proposition 3.4). — *Let \mathfrak{p} be a maximal ideal of \mathcal{O}_K and $\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})$. Suppose that $\{f_i\}_{1 \leq i \leq m}$ is a family of local $\mathcal{O}_{K, \mathfrak{p}}$ -linear homomorphisms from $\mathcal{O}_{\mathcal{X}, \xi}$ to $\mathcal{O}_{K, \mathfrak{p}}$, whose reduction modulo \mathfrak{p} is the same. Let E be a free sub- $\mathcal{O}_{K, \mathfrak{p}}$ -module of finite type of $\mathcal{O}_{\mathcal{X}, \xi}$ and $f = (f_i|_E)_{1 \leq i \leq m}$. Then for any integer $r \geq 1$, we have*

$$(16) \quad \|\wedge^r f_K\| \leq N(\mathfrak{p})^{-Q_{\xi}(r)},$$

where $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$, and $Q_{\xi}(r)$ is defined in (5).

The following lemma will be used in the global determinant estimate.

LEMMA 3.5. — *Let $(K, |\cdot|)$ be a normed field, E_1, E_2, F_1, F_2 be four normed vector spaces over K , $f_1 : E_1 \rightarrow F_1$ and $f_2 : E_2 \rightarrow F_2$ be two K -linear isomorphisms. Suppose $\dim_K(E_1) = \dim_K(F_1) = r_1$ and $\dim_K(E_2) = \dim_K(F_2) = r_2$. We equipped*

$$f_1 \oplus f_2 : E_1 \oplus E_2 \rightarrow F_2 \oplus F_2$$

with the corresponding maximal value norms. Then we have

$$\|\wedge^{r_1+r_2} (f_1 \oplus f_2)\| = \|\wedge^{r_1} f_1\| \cdot \|\wedge^{r_2} f_2\|,$$

where the above $\|\cdot\|$ is the norm of operators.

Proof. — By definition, the linear maps $\wedge^{r_1} f_1$ and $\wedge^{r_2} f_2$ are both scalar products by the corresponding determinants, and $\wedge^{r_1+r_2} (f_1 \oplus f_2)$ is the scalar product of the above two determinants. Then we have the result directly by definition. \square

3.2. Proof of Theorem 3.1. — We will prove Theorem 3.1; some ideas of the proof below were inspired by [13, §3].

Proof of Theorem 3.1. — Let D be an integer larger than 1. We suppose that the global section predicted by Theorem 3.1 does not exist. Then the evaluation map

$$f : F_{D,K} \rightarrow \bigoplus_{i \in I} P_i^* \mathcal{O}_{\mathbb{P}_K^n}(D)$$

is injective. We can replace I by one of its subsets, such that f is an isomorphism. From now on, we suppose f is isomorphic, which means $\#I = r_1(n, D) = \text{rk } F_{D,K}$. Then by Proposition 3.2 we have

$$\widehat{\mu}(\overline{F}_D) = \frac{1}{r_1(n, D)} \left(D \sum_{i \in I} h(P_i) + h\left(\wedge^{r_1(n, D)} f\right) \right),$$

which implies

$$\frac{\widehat{\mu}(\overline{F}_D)}{D} \leq \sup_{i \in I} h(P_i) + \frac{1}{Dr_1(n, D)} h\left(\wedge^{r_1(n, D)} f\right).$$

We now estimate the height of $\wedge^{r_1(n, D)} f$. For every $v \in M_{K, \infty}$, we have

$$\frac{1}{r_1(n, D)} \log \|\wedge^{r_1(n, D)} f\|_v \leq \log \|f\|_v \leq \log \sqrt{r_1(n, D)},$$

where the second inequality comes from the definition of metrics of John in §2.5.1.

We now consider the case of $v \in M_{K, f}$. The homomorphism f is induced by a homomorphism of \mathcal{O}_K -module

$$F_D \rightarrow \bigoplus_{i \in I} \mathcal{P}_i^* \mathcal{O}_{\mathbb{P}_{\mathcal{O}_K}^n}(D),$$

where \mathcal{P}_i is the Zariski closure of P_i in \mathcal{X} for each $i \in I$. Then for every $v \in M_{K, f}$, we have $\log \|\wedge^{r_1(n, D)} f\|_v \leq 0$.

We fix a maximal ideal \mathfrak{p} of \mathcal{O}_K corresponding to $v \in M_{K, f}$ and decompose the set $\{\mathcal{P}_i\}_{i \in I}$ as the disjoint union

$$\{\mathcal{P}_i\}_{i \in I} = \bigcup_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \{\mathcal{P}_{l, \xi}\}_{l=1}^{m_{\xi}},$$

where all elements in $\{\mathcal{P}_{l, \xi}\}_{l=1}^{m_{\xi}}$ modulo \mathfrak{p} are the same point $\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})$. If $\{\mathcal{P}_{l, \xi}\}_{l=1}^{m_{\xi}}$ is empty for some $\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})$, we define $m_{\xi} = 0$ for simplicity. With the above notations, let

$$\bigcup_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \{s_{l, \xi}\}_{l=1}^{m_{\xi}}$$

be an $\mathcal{O}_{K, \mathfrak{p}}$ -basis of $F_{D, \mathfrak{p}}$, such that $f(s_{l, \xi})$ generates $\mathcal{P}_{l, \xi}^* \mathcal{O}_{\mathbb{P}_{\mathcal{O}_K}^n}(D)$ for all $l = 1, \dots, m_{\xi}$ and $\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})$. Since $\mathcal{O}_{K, \mathfrak{p}}$ is a local ring, the $\mathcal{O}_{K, \mathfrak{p}}$ -module $F_{D, \mathfrak{p}}$ is

free, and then there exists such a basis for a fixed maximal ideal \mathfrak{p} . We denote by $F_{D,\xi}$ the sub- $\mathcal{O}_{K,\mathfrak{p}}$ -module of $F_{D,\mathfrak{p}}$ generated by $\{s_{l,\xi}\}_{l=1}^{m_\xi}$.

By Proposition 3.4, we have

$$\log \left\| \wedge^{\mathrm{rk}(F_{D,\xi})} f|_{F_{D,\xi}} \right\|_{\mathfrak{p}} \leq -Q_\xi(\mathrm{rk}(F_{D,\xi})) \log N(\mathfrak{p}).$$

By definition, we have

$$F_{D,\mathfrak{p}} = \bigoplus_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} F_{D,\xi},$$

and

$$(17) \quad r_1(n, D) = \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \mathrm{rk}(F_{D,\xi}).$$

Then from the above construction, by applying Lemma 3.5 and Proposition 3.4, respectively, we obtain

$$\begin{aligned} \log \left\| \wedge^{r_1(n,D)} f \right\|_{\mathfrak{p}} &= \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \log \left\| \wedge^{\mathrm{rk}(F_{D,\xi})} f|_{F_{D,\xi}} \right\|_{\mathfrak{p}} \\ &\leq - \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} Q_\xi(\mathrm{rk}(F_{D,\xi})) \log N(\mathfrak{p}). \end{aligned}$$

In order to estimate the term

$$\frac{1}{r_1(n, D)} \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} Q_\xi(\mathrm{rk}(F_{D,\xi})),$$

by (6), we have

$$\begin{aligned} &\sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} Q_\xi(\mathrm{rk}(F_{D,\xi})) \\ &> \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \left(\left(\frac{(n-1)!}{\mu_\xi(\mathcal{X}_{\mathfrak{p}})} \right)^{\frac{1}{n-1}} \left(\frac{n-1}{n} \right) \mathrm{rk}(F_{D,\xi})^{\frac{n}{n-1}} - \frac{n^3 + 2n^2 + n - 4}{2n(n+1)} \mathrm{rk}(F_{D,\xi}) \right) \\ &= \frac{(n-1)!^{\frac{1}{n-1}} (n-1)}{n} \sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \frac{\mathrm{rk}(F_{D,\xi})^{\frac{n}{n-1}}}{\mu_\xi(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}} - \frac{n^3 + 2n^2 + n - 4}{2n(n+1)} r_1(n, D). \end{aligned}$$

By (17) and Hölder's inequality, we have

$$\sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \frac{\mathrm{rk}(F_{D,\xi})^{\frac{n}{n-1}}}{\mu_\xi(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}} \geq \frac{\left(\sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \mathrm{rk}(F_{D,\xi}) \right)^{\frac{n}{n-1}}}{\left(\sum_{\xi \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \mu_\xi(\mathcal{X}_{\mathfrak{p}}) \right)^{\frac{1}{n-1}}} = \frac{r_1(n, D)^{\frac{n}{n-1}}}{n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}},$$

where $n(\mathcal{X}_{\mathfrak{p}})$ is defined in the statement of Theorem 3.1. Then we obtain the inequality

$$\frac{\widehat{\mu}(\overline{F}_D)}{D} \leq \sup_{i \in I} h(P_i) + \frac{\log r_1(n, D)}{2D} - \frac{1}{[K : \mathbb{Q}]} \sum_{j \in J} \left(\frac{(n-1)!^{\frac{1}{n-1}} (n-1) r_1(n, D)^{\frac{1}{n-1}}}{n D n(\mathcal{X}_{\mathfrak{p}_j})^{\frac{1}{n-1}}} - \frac{n^3 + 2n^2 + n - 4}{2Dn(n+1)} \right) \log N(\mathfrak{p}_j),$$

which leads to a contradiction. \square

4. Some quantitative estimates

In order to apply the global determinant method introduced in Theorem 3.1, we need to gather enough information on the term $n(\mathcal{X}_{\mathfrak{p}})$ in it. For this target, we need to have a control of the reduction type of $\mathcal{X} \hookrightarrow \mathbb{P}_{\mathcal{O}_K}^n \rightarrow \text{Spec } \mathcal{O}_K$, an upper bound of $n(\mathcal{X}_{\mathfrak{p}})$ when $\mathcal{X}_{\mathfrak{p}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}}$ is geometrically integral, and a distribution of certain prime ideals of \mathcal{O}_K . We will also provide an explicit estimate of the geometric Hilbert–Samuel function of hypersurfaces.

4.1. Control of the non-geometrically integral reductions. — Let $X \hookrightarrow \mathbb{P}_K^n$ be a geometrically integral hypersurface of degree δ , $\mathcal{X} \hookrightarrow \mathbb{P}_{\mathcal{O}_K}^n \rightarrow \text{Spec } \mathcal{O}_K$ be its Zariski closure, and $\mathcal{X}_{\mathbb{F}_{\mathfrak{p}}} = \mathcal{X} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathbb{F}_{\mathfrak{p}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spm } \mathcal{O}_K$. By [21, Théorème 9.7.7], the set

$$(18) \quad \mathcal{Q}(\mathcal{X}) = \{\mathfrak{p} \in \text{Spm } \mathcal{O}_K \mid \mathcal{X}_{\mathbb{F}_{\mathfrak{p}}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}} \text{ is not geometrically integral}\}$$

is finite.

Next, we introduce a numerical description of the set $\mathcal{Q}(\mathcal{X})$. In fact, there are fruitful results on this subject, but most of them are over rational number field \mathbb{Q} . In [28], the estimate [38, Satz 4] was generalized over arbitrary number fields by using a height function in an adelic sense by the approach of [27, §3.4]. By [28, Proposition 4.1], we have

$$(19) \quad \frac{1}{[K : \mathbb{Q}]} \sum_{\mathfrak{p} \in \mathcal{Q}(\mathcal{X})} \log N(\mathfrak{p}) \leq (\delta^2 - 1)h(X) + C(n, \delta),$$

where $h(X)$ is the naive height of X defined in Definition 2.5, $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$, and the constant

$$C(n, \delta) = (\delta^2 - 1) \left(3 \log \delta + \delta \log 3 + \log \binom{n + \delta}{\delta} \right).$$

In fact, we have $C(n, \delta) \ll_n \delta^3$.

4.2. Quantitative estimates over finite fields. — In this subsection, we give an upper bound of the term $n(\mathcal{X}_{\mathfrak{p}})$ for an arbitrary maximal ideal \mathfrak{p} of \mathcal{O}_K , where $n(\mathcal{X}_{\mathfrak{p}})$ is defined in the statement of Theorem 3.1. In this part, we consider this problem over arbitrary finite fields.

Let \mathbb{F}_q be the finite field with q elements, X be a geometrically integral hypersurface in $\mathbb{P}_{\mathbb{F}_q}^n$ of degree δ , and $n(X_{\mathbb{F}_q}) = \sum_{\xi \in X(\mathbb{F}_q)} \mu_{\xi}(X)$, where $\mu_{\xi}(X)$ is the multiplicity of ξ in X defined via the local Hilbert–Samuel function in (4). Then we have

$$n(X_{\mathbb{F}_q}) = \#X(\mathbb{F}_q) + \sum_{\xi \in X(\mathbb{F}_q)} (\mu_{\xi}(X) - 1).$$

In order to estimate $n(X_{\mathbb{F}_q})$, we will consider the terms $\#X(\mathbb{F}_q)$ and $\sum_{\xi \in X(\mathbb{F}_q)} (\mu_{\xi}(X) - 1)$ separately.

4.2.1. For the estimate of $\#X(\mathbb{F}_q)$, there are fruitful results. For our application, we have the following result deduced from [9, Corollary 5.6].

PROPOSITION 4.1. — *Let $X \hookrightarrow \mathbb{P}_{\mathbb{F}_q}^n$ be a geometrically integral hypersurface of degree δ over the finite field \mathbb{F}_q . When $q \leq \delta^2$ or $q \geq 27\delta^4$, we have*

$$\#X(\mathbb{F}_q) - q^{n-1} \leq n\delta^2 q^{n-\frac{3}{2}}.$$

Proof. — We consider this estimate case by case as follows.

1. If $q \leq \delta$, we have $\#X(\mathbb{F}_q) \leq \#\mathbb{P}^n(\mathbb{F}_q) = q^n + \cdots + 1$. Then

$$\#X(\mathbb{F}_q) - q^{n-1} \leq nq^n \leq n\delta^2 q^{n-\frac{3}{2}}.$$

2. If $\delta + 1 \leq q \leq \delta^2$, we have $\#X(\mathbb{F}_q) \leq \delta \#\mathbb{P}^{n-1}(\mathbb{F}_q) = \delta(q^{n-1} + \cdots + 1)$. Then

$$\#X(\mathbb{F}_q) - q^{n-1} \leq (\delta - 1)q^{n-1} + \delta(q^{n-2} + \cdots + 1) \leq n\delta^2 q^{n-\frac{3}{2}}.$$

3. If $q \geq 27\delta^4$, by [9, Corollary 5.6], we have

$$\#X(\mathbb{F}_q) - q^{n-1} \leq (\delta - 1)(\delta - 2)q^{n-\frac{3}{2}} + (5\delta^2 + \delta + 1)q^{n-2} \leq n\delta^2 q^{n-\frac{3}{2}}. \quad \square$$

REMARK 4.2. — With all the notations in Proposition 4.1, when $\delta^2 \ll_n q \ll_n \delta^4$, by [9, Corollary 5.6], we have

$$\#X(\mathbb{F}_q) - q^{n-1} \leq (\delta - 1)(\delta - 2)q^{n-\frac{3}{2}} + B(n, \delta)q^{n-2},$$

where the constant satisfies $B(n, \delta) \ll_n \delta^4$. It seems that the constant $B(n, \delta)$ could have a better dependence on δ , but to the author's knowledge, we do not know the answer.

4.2.2. For the term $\sum_{\xi \in X(\mathbb{F}_q)} (\mu_\xi(X) - 1)$, by [29, Theorem 5.1], we have

$$(20) \quad \sum_{\xi \in X(\mathbb{F}_q)} (\mu_\xi(X) - 1) \leq \frac{1}{2} \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X) (\mu_\xi(X) - 1) \\ \leq \frac{(n-1)^2}{2} \delta(\delta-1) \max\{\delta-1, q\}^{n-2},$$

which has the optimal dependances on δ and $\max\{\delta-1, q\}$, when $q \geq \delta-1$.

4.2.3. We combine Proposition 4.1 and the estimate (20). When $q \leq \delta^2$ or $q \geq 27\delta^4$, we have

$$n(X_{\mathbb{F}_q}) \leq q^{n-1} + n^2 \delta^2 \max\{q, \delta-1\}^{n-\frac{3}{2}}$$

by an elementary calculation. In addition, we have

$$(21) \quad \frac{1}{n(X_{\mathbb{F}_q})^{\frac{1}{n-1}}} \geq \frac{1}{q} - \frac{n^2 \delta^2}{\max\{q, \delta-1\}^{\frac{3}{2}}}$$

under the same assumption of q and δ as above.

4.3. Distribution of certain prime ideals. — In this part, we will consider some distributions of prime ideals of the ring of integers of number fields.

4.3.1. *Distribution of prime ideals containing a fixed ideal.* — In this part, we will consider the distribution of certain maximal ideals of \mathcal{O}_K . First, we generalize [40, Lemma 1.10] over an arbitrary number field, where the former result works over \mathbb{Z} only.

LEMMA 4.3. — *Let \mathfrak{a} be a proper ideal of \mathcal{O}_K , \mathfrak{p} be a prime ideal of \mathcal{O}_K , and $N(\mathfrak{a}) = \#(\mathcal{O}_K/\mathfrak{a})$. Then we have*

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\mathfrak{p} \supseteq \mathfrak{a}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \leq \log \log (N(\mathfrak{a})) + 2,$$

where the above sum runs over all prime ideals contained in \mathfrak{a} of \mathcal{O}_K .

Proof. — We will prove the inequality for the case of $K = \mathbb{Q}$ at first and then show the general case with it.

Case of $K = \mathbb{Q}$. — In this case, we will repeat the proof of [40, Lemma 1.10] by Salberger, since this preprint is not easily available. Suppose that \mathfrak{a} is generated by the positive square-free integer π and let m be a positive integer, such that $m \leq \pi$. For the prime p , let $v_p(m)$ be the largest integer, such that $p^{v_p(m)} \mid m$. By [44, Tome I, Corollaire 1.7] and [44, Tome I, Théorème 1.8], we have

$$m \sum_{p|\pi} \frac{\log p}{p} - \sum_{p|\pi} \log p \leq \sum_{p|\pi} v_p(m!) \log p \leq \sum_{p \leq \pi} v_p(m!) \log p = \log m! \leq m \log m,$$

and then we obtain

$$\sum_{p|\pi} \frac{\log p}{p} \leq \log m + \frac{1}{m} \sum_{p|\pi} \log p \leq \log m + \frac{1}{m} \log \pi.$$

Let $m = [\log \pi]$ for $\pi \geq 2$; then we accomplish the proof for $K = \mathbb{Q}$.

Case of arbitrary number fields. — Let

$$\mathfrak{a} = \mathfrak{p}_1^{v_{\mathfrak{p}_1}(\mathfrak{a})} \cdots \mathfrak{p}_k^{v_{\mathfrak{p}_k}(\mathfrak{a})},$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are distinct prime ideals of \mathcal{O}_K , and $v_{\mathfrak{p}_i}(\mathfrak{a}) \in \mathbb{N}^+$ for all $i = 1, \dots, k$. Let the prime p_i be the characteristic of the prime ideal \mathfrak{p}_i , where $i = 1, \dots, k$ as above. For a fixed prime p , there are at most $[K : \mathbb{Q}]$ prime ideals of characteristic p in \mathcal{O}_K . For all prime p and $f \in \mathbb{N}^+$, we have

$$\frac{\log p^f}{p^f} \leq \frac{\log p}{p}.$$

Let $P(\mathfrak{a})$ be the product of all the different characteristics of $\mathfrak{p}_1, \dots, \mathfrak{p}_k$; we have $P(\mathfrak{a}) \leq N(\mathfrak{a})$ by definition directly. Then by the above facts, we obtain

$$\sum_{\mathfrak{p} \supseteq \mathfrak{a}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \leq [K : \mathbb{Q}] \sum_{p|P(\mathfrak{a})} \frac{\log p}{p}.$$

With the case $K = \mathbb{Q}$, we have

$$\sum_{p|P(\mathfrak{a})} \frac{\log p}{p} \leq \log \log P(\mathfrak{a}) + 2 \leq \log \log N(\mathfrak{a}) + 2,$$

which proves the assertion. \square

4.3.2. Distribution of prime ideals with bounded norm. — Let $x \in \mathbb{R}^+$, $\mathfrak{p} \in \text{Spm } \mathcal{O}_K$ and $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$. In this part, we consider

$$(22) \quad \theta_K(x) = \sum_{N(\mathfrak{p}) \leq x} \log N(\mathfrak{p}), \quad \psi_K(x) = \sum_{N(\mathfrak{p}) \leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})}, \quad \phi_K(x) = \sum_{N(\mathfrak{p}) \leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{3}{2}}}.$$

When $K = \mathbb{Q}$, these are classic estimates of Chebyshev function (cf. [44, Tome I, Théorème 2.11]) and Mertens' first theorem (cf. [44, Tome I, Théorème 1.8]). For the case of arbitrary number fields, a generalization of [44, Tome I, Théorème 2.11] is Landau's prime ideals theorem (cf. [37, Theorem 2.2]), and a generalization of Mertens' first theorem was deduced from this in [37, Lemma 2.3].

In this part, we will give a more explicit version of some results [37] that will be used in the application of Theorem 3.1.

By Landau's prime ideal theorem (cf. [37, Theorem 2.2]), we have

$$\theta_k(x) = x + O_K(xe^{-c\sqrt{\log x}}),$$

where c is a constant depending on K . Then there exists a function $\epsilon_1(K, x)$ of the number field K and $x \in \mathbb{R}^+$, such that

$$(23) \quad |\theta_K(x) - x| \leq \epsilon_1(K, x),$$

where $\epsilon_1(K, x) = O_K(xe^{-c\sqrt{\log x}})$ for all $x \in \mathbb{R}^+$, and c depends on K only.

By [37, Lemma 2.3], we have

$$\psi_K(x) = \log x + O_K(1),$$

which is obtained by Abel's summation formula applied in [37, Lemma 2.1]. Then there exists a function $\epsilon_2(K)$ of the number field K , such that

$$(24) \quad |\psi_K(x) - \log x| \leq \epsilon_2(K).$$

Similarly to the application of [37, Lemma 2.1] to the proof of [37, Lemma 2.3], we have

$$\phi_K(x) = \frac{\theta_K(x)}{x^{\frac{3}{2}}} + \frac{3}{2} \int_2^x \frac{\theta_K(t)}{t^{\frac{5}{2}}} dt$$

by [37, Lemma 2.1]. Then by (23), we have

$$\left| \phi_K(x) - \frac{1}{\sqrt{x}} - \frac{3}{2} \int_2^x \frac{1}{t^{\frac{3}{2}}} dt \right| \leq \frac{\epsilon_1(K, x)}{x^{\frac{3}{2}}} + \frac{3}{2} \int_2^x \frac{\epsilon_1(K, t)}{t^{\frac{5}{2}}} dt.$$

Then by an elementary calculation, there exists a function $\epsilon_3(K, x)$ of the number field K and $x \in \mathbb{R}^+$, such that

$$(25) \quad \left| \phi_K(x) - \frac{3}{2}\sqrt{2} + \frac{2}{\sqrt{x}} \right| \leq \epsilon_3(K, x),$$

where

$$(26) \quad \epsilon_3(K, x) = \frac{\epsilon_1(K, x)}{x^{\frac{3}{2}}} + \frac{3}{2} \int_2^x \frac{\epsilon_1(K, t)}{t^{\frac{5}{2}}} dt.$$

4.3.3. Distribution of non-geometrically integral reductions. — In this part, we consider a distribution of the prime ideals modulo, of which the reductions are not geometrically integral. In the estimate below, the estimates of (22) will be involved.

PROPOSITION 4.4. — *Let X be a geometrically integral hypersurface of degree δ in \mathbb{P}_K^n and \mathcal{X} be its Zariski closure in $\mathbb{P}_{\mathcal{O}_K}^n$. Let $\mathfrak{p} \in \text{Spm } \mathcal{O}_K$, $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$,*

$$\mathcal{Q}'(\mathcal{X}) = \{\mathfrak{p} \in \text{Spm } \mathcal{O}_K \mid N(\mathfrak{p}) > 27\delta^4 \text{ and } \mathcal{X}_{\mathfrak{p}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}} \text{ not geometrically integral}\},$$

and

$$b'(\mathcal{X}) = \prod_{\mathfrak{p} \in \mathcal{Q}'(\mathcal{X})} \exp \left(\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \right).$$

Then we have

$$b'(\mathcal{X}) \leq \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}]) \cdot (\delta^{-2} - \delta^{-4}) \left(h(X) + \left(3\log \delta + \delta \log 3 + \log \binom{n+\delta}{\delta} \right) \right),$$

where $h(X)$ is defined in Definition 2.5, and $\epsilon_2(K)$ is defined in (24).

Proof. — We denote by $\mathcal{P}'(\mathcal{X})$ the product of all maximal ideals in $\mathcal{Q}'(\mathcal{X})$, and

$$c'(\mathcal{X}) = (\delta^2 - 1) \left(h(X) + \left(3\log \delta + \delta \log 3 + \log \binom{n+\delta}{\delta} \right) \right).$$

Then by Lemma 4.3 and (19), we have

$$\begin{aligned} \frac{1}{[K : \mathbb{Q}]} \log b'(\mathcal{X}) &= \frac{1}{[K : \mathbb{Q}]} \sum_{\mathfrak{p} \in \mathcal{Q}'(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \\ &\leq \frac{1}{[K : \mathbb{Q}]} \sum_{27\delta^4 < N(\mathfrak{p}) \leq c'(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} + \frac{1}{[K : \mathbb{Q}]} \sum_{\substack{\mathfrak{p} \in \mathcal{Q}'(\mathcal{X}) \\ N(\mathfrak{p}) > c'(\mathcal{X})}} \frac{\log N(\mathfrak{p})}{c'(\mathcal{X})}. \end{aligned}$$

By (24), we have

$$\begin{aligned} &\frac{1}{[K : \mathbb{Q}]} \sum_{27\delta^4 < N(\mathfrak{p}) \leq c'(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \\ &= \frac{1}{[K : \mathbb{Q}]} \sum_{N(\mathfrak{p}) \leq c'(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} - \frac{1}{[K : \mathbb{Q}]} \sum_{N(\mathfrak{p}) \leq 27\delta^4} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \\ &\leq \frac{1}{[K : \mathbb{Q}]} (\log c'(\mathcal{X}) - 4\log \delta + 2\epsilon_2(K) - 3\log 3), \end{aligned}$$

where $\epsilon_2(K)$ is defined in (24) depending on K only.

Let

$$\mathcal{Q}(\mathcal{X}) = \{ \mathfrak{p} \in \text{Spm } \mathcal{O}_K \mid \mathcal{X}_{\mathbb{F}_{\mathfrak{p}}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}} \text{ is not geometrically integral} \}.$$

Since $\mathcal{Q}'(\mathcal{X}) \subseteq \mathcal{Q}(\mathcal{X})$, we have

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\substack{\mathfrak{p} \in \mathcal{Q}'(\mathcal{X}) \\ N(\mathfrak{p}) > c'(\mathcal{X})}} \frac{\log N(\mathfrak{p})}{c'(\mathcal{X})} \leq \frac{1}{[K : \mathbb{Q}]c'(\mathcal{X})} \sum_{\mathfrak{p} \in \mathcal{Q}(\mathcal{X})} \log N(\mathfrak{p}) \leq 1,$$

where the last inequality is from (19). By combining the above two estimates, we terminate the proof. \square

REMARK 4.5. — With all the notations and assumptions in Proposition 4.4, we have

$$b'(\mathcal{X}) \ll_{n,K} \max \{ \delta^{-2} h(X), \delta^{-1} \}.$$

4.4. An explicit estimate of the geometric Hilbert–Samuel function. — In this part, we will provide an explicit lower bound of the geometric Hilbert–Samuel function of a projective hypersurface, which will be used in the application of the determinant method. The inequality

$$\frac{(N - m + 1)^m}{m!} \leq \binom{N}{m} \leq \frac{(N - (m - 1)/2)^m}{m!}$$

will be helpful in the calculation below.

LEMMA 4.6. — *Let X be a hypersurface of degree δ in \mathbb{P}_K^n . We denote by $r_1(n, D)$ its geometric Hilbert–Samuel function with the variable D . When $D \geq \delta + 1$, we have*

$$r_1(n, D)^{\frac{1}{n-1}} \geq {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}} D - (\delta - 2) {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}},$$

and

$$r_1(n, D)^{\frac{1}{n-1}} \leq {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}} D + \frac{n}{2} {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}}.$$

Proof. — In fact, we have

$$r_1(n, D) = \binom{n+D}{n} - \binom{n+D-\delta}{n}$$

when $D \geq \delta + 1$.

In order to obtain the lower bound, we have

$$\begin{aligned} (27) \quad r_1(n, D) &= \sum_{j=1}^{\delta} \binom{D - \delta + n - 1 + j}{n-1} \geq \frac{\delta(D - \delta + 2)^{n-1}}{(n-1)!} \\ &\geq \frac{\delta}{(n-1)!} D^{n-1} - \frac{\delta(\delta-2)}{(n-1)!} D^{n-2}. \end{aligned}$$

Then we obtain

$$\begin{aligned} r_1(n, D)^{\frac{1}{n-1}} &\geq {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}} D \left(1 - \frac{\delta-2}{D} \right)^{\frac{1}{n-1}} \\ &\geq {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}} D - (\delta-2) {}^{n-1}\sqrt{\frac{\delta}{(n-1)!}} \end{aligned}$$

when $D \geq \delta + 1$.

On the other hand, we have

$$(28) \quad r_1(n, D) = \sum_{j=1}^{\delta} \binom{D - \delta + n - 1 + j}{n-1} \\ \leq \frac{\delta(D + \frac{n}{2})^{n-1}}{(n-1)!} = \frac{\delta D^{n-1}}{(n-1)!} \left(1 + \frac{n}{2D}\right)^{n-1},$$

which terminates the proof by an elementary calculation. \square

5. An explicit estimate of the determinant

In this section, we will give an upper bound of the degree of the auxiliary hypersurface determined by Theorem 3.1.

5.1. Uniform lower bound of arithmetic Hilbert–Samuel functions. — Firstly, we refer to a result in [12], which is an application of the uniform lower bound of the arithmetic Hilbert–Samuel functions to the determinant method.

PROPOSITION 5.1 ([12], Proposition 2.12). — *We keep all the notations in §2.5. Let X be a closed integral subscheme of \mathbb{P}_K^n , $Z = (P_i)_{i \in I}$ be a family of rational points and*

$$\phi_{Z,D} : F_{D,K} \rightarrow \bigoplus_{i \in I} P_i^* \mathcal{O}_{\mathbb{P}_K^n}(D)$$

be the evaluation map. If we have the inequality

$$\sup_{i \in I} h_{\overline{\mathcal{O}(1)}}(P_i) < \frac{\widehat{\mu}_{\max}(\overline{F}_D)}{D} - \frac{1}{2D} \log r_1(n, D),$$

where $r_1(n, D) = \text{rk}(F_D)$, and the height function $h_{\overline{\mathcal{O}(1)}}(\cdot)$ is defined in (12), then the homomorphism $\phi_{Z,D}$ is not able to be injective.

The uniform lower bound of $\widehat{\mu}(\overline{F}_D)$ for all $D \geq 1$ will play a significant role in the construction of auxiliary hypersurfaces if we want to apply Proposition 5.1. In [14], David and Philippon give an explicit uniform lower bound of $\widehat{\mu}(\overline{F}_D)$. This result is reformulated by H. Chen in [12, Theorem 4.8] with the language of the slope method. In fact, let X be a closed integral subscheme of \mathbb{P}_K^n of dimension d and degree δ , and \mathcal{X} be its Zariski closure in $\mathbb{P}_{\mathcal{O}_K}^n$. The inequality

$$(29) \quad \frac{\widehat{\mu}(\overline{F}_D)}{D} \geq \frac{d!}{\delta(2d+2)^{d+1}} h_{\overline{\mathcal{O}(1)}}(\mathcal{X}) - \log(n+1) - 2^d$$

is uniformly verified for all $D \geq 2(n-d)(\delta-1) + d+2$ (see also [12, Remark 4.9] for some minor modifications), where $h_{\overline{\mathcal{O}(1)}}(\mathcal{X})$ follows the definition in Definition 2.3.

By Proposition 5.1, all the rational points with small heights in a projective variety can be covered by one hypersurface that does not contain the generic point of the original variety. The following result gives a numerical description of this observation.

PROPOSITION 5.2. — *Let X be an integral hypersurface of degree δ in \mathbb{P}_K^n , the constant $\mathcal{H}_n = 1 + \cdots + \frac{1}{n}$ and the constant*

$$(30) \quad C_1(n) = -\frac{(2n)^n}{(n-1)!} \left(\log 2 + 5 \log(n+1) - \frac{1}{2} \mathcal{H}_n \right) - \frac{3}{2} \log(n+1) - 2^{n-1}.$$

If

$$\frac{\log B}{[K : \mathbb{Q}]} < \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n),$$

then there exists a hypersurface of degree smaller than $2\delta + n - 1$, which contains all rational points in $S(X; B)$ but does not contain the generic point of X , where we use the height function defined in (12).

Proof. — If there does not exist such a hypersurface, the evaluation map $\phi_{Z,D}$ in Proposition 5.1 is injective. On the other hand, by (29), Proposition 2.6 and the fact that

$$r_1(n, D) \leq \binom{n+D}{n} \leq (n+1)^D$$

is uniformly verified for all $n, D \geq 1$, we obtain

$$\begin{aligned} \frac{\log B}{[K : \mathbb{Q}]} &< \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n) \\ &\leq \frac{(n-1)!}{\delta(2n)^n} h_{\mathcal{O}(1)}(\mathcal{X}) - \frac{3}{2} \log(n+1) - 2^{n-1} \\ &\leq \frac{\hat{\mu}_{\max}(\overline{F}_D)}{D} - \frac{1}{2D} \log r_1(n, D), \end{aligned}$$

which contradicts Proposition 5.1. \square

REMARK 5.3. — With all the notations in Proposition 5.2; with the arithmetic Hilbert–Samuel Theorem of arithmetic ample line bundles (cf. [19, Theorem 8], [49, Theorem 1.4] and [1, Théorème principal]), we have

$$\hat{\mu}(\overline{F}_D) = \frac{h_{\mathcal{O}(1)}(\mathcal{X})}{n\delta} D + o(D)$$

for D tends into infinity. So it is expected that we can obtain a better uniform lower bound of $\hat{\mu}(\overline{F}_D)$ than that in (29). If we have a better explicit lower bound, we can improve the bound given in Proposition 5.2.

5.2. Estimate of the determinant. — In the global determinant method, for each geometrically integral hypersurface, we allow only one auxiliary hypersurface to cover its rational points with bounded height not containing the generic point of the original hypersurface and we optimize the degree of this auxiliary hypersurface.

In this part, we will give an upper bound of the degree of the auxiliary hypersurface determined in Theorem 3.1, where the size of non-geometrically integral reductions and the height of the original hypersurface will be involved.

Before the statement of the main theorem in this paragraph, we will introduce two constants depending on the number field K and the positive integer $n \geq 2$, which will be used in the estimate of the determinant.

Let K be a number field; we denote

$$(31) \quad \kappa_1(K) = \sup_{x \in \mathbb{R}^+} \frac{\epsilon_1(K, x)}{x},$$

where $\epsilon_1(K, x)$ is introduced in (23). By [37, Theorem 2.2], the above supremum exists, and $\kappa_1(K)$ depends on K only.

Let $\delta \geq 1$ and $n \geq 2$ be two integers; we denote

$$(32) \quad \kappa_2(K, n) = \sup_{\delta \geq 1} \left\{ -3 \log 3 + \frac{2n^2}{3\sqrt{3}} + 2n^2 \delta^2 \epsilon_3(K, 27\delta^4) + 2\epsilon_2(K) \right\},$$

where $\epsilon_2(K)$ is defined in (24), and $\epsilon_3(K, x)$ is defined in (26). By taking [37, Theorem 2.2] into the estimate of (26), the above supremum exists, and $\kappa_2(K, n)$ depends on K and n only.

THEOREM 5.4. — *Let K be a number field. Let X be a geometrically integral hypersurface in \mathbb{P}_K^n of degree $\delta \geq 2$ and $S(X; B)$ be the set of rational points in X whose height is smaller than B with respect to the above closed immersion; see (12) for the definition of the height function used above. Then there exists a hypersurface in \mathbb{P}_K^n of degree smaller than*

$$e^{C_2(n, K)} B^{n/((n-1)\delta^{1/(n-1)})} \delta^{4-1/(n-1)} \frac{b'(\mathcal{X})}{H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1-1/(n-1)}}},$$

which covers $S(X; B)$ but does not contain the generic point of X , where the constant

$$\begin{aligned} C_2(n, K) = & \frac{nC_1(n)[K : \mathbb{Q}]}{(n-1)^{n-1}\sqrt[n]{2}} + \kappa_2(K, n) + \frac{\log(n-1)!}{n-1} \\ & + 3 + \frac{n^3 + 2n^2 + n - 4}{2(n^2 - 1)^{n-1}\sqrt[n]{(n-1)!}} \left(1 + \frac{n}{4}\right) (1 + \kappa_1(K)), \end{aligned}$$

the constant $C_1(n)$ is defined in (30), $b'(\mathcal{X})$ is defined in Proposition 4.4, $\kappa_1(K)$ is defined in (31), $\kappa_2(K, n)$ is defined in (32), and the height $H_K(X)$ of X is defined in Definition 2.5.

Proof. — By Proposition 5.2, we divide the proof into two parts.

I. The case of large height varieties. — If

$$\frac{\log B}{[K : \mathbb{Q}]} < \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n),$$

where the constant $C_1(n)$ is defined in (30) and $h(X)$ is defined in Definition 2.5, then by Proposition 5.2, $S(X; B)$ can be covered by a hypersurface of degree no more than $2\delta + n - 1$, which does not contain the generic point of X . By an elementary calculation, we obtain that $2\delta + n - 1$ is smaller than the bound provided in the statement of the theorem, for $n \geq 2$, $\delta \geq 1$, $b'(\mathcal{X}) \geq 1$ and $H_K(X) \geq 1$.

II. The case of small height varieties. — For the case of

$$\frac{\log B}{[K : \mathbb{Q}]} \geq \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n),$$

which is equivalent to

$$h(X) \leq \frac{\delta(2n)^n}{(n-1)!} \cdot \left(\frac{\log B}{[K : \mathbb{Q}]} - C_1(n) \right),$$

we will treat it as follows. We keep all the notations in Theorem 3.1 and we suppose $D \geq 3\delta \log \delta + n - 1 \geq 2\delta + n - 1$ from now on. We denote the set

$$\mathcal{R}(\mathcal{X}) = \{\mathfrak{p} \in \text{Spm } \mathcal{O}_K \mid 27\delta^4 \leq N(\mathfrak{p}) \leq r_1(n, D)^{\frac{1}{n-1}}, \\ \mathcal{X}_{\mathfrak{p}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}} \text{ is geometrically integral}\}$$

and we apply Theorem 3.1 to the reductions at $\mathcal{R}(\mathcal{X})$. If there does not exist such a hypersurface, then by Theorem 3.1 applied in the above sense, we have

$$\frac{\log B}{[K : \mathbb{Q}]} \geq \frac{\widehat{\mu}(\overline{F}_D)}{D} - \frac{\log r_1(n, D)}{2D} \\ + \frac{1}{[K : \mathbb{Q}]} \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \left(\frac{(n-1)!^{\frac{1}{n-1}} (n-1) r_1(n, D)^{\frac{1}{n-1}}}{n D n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}} - \frac{n^3 + 2n^2 + n - 4}{2Dn(n+1)} \right) \log N(\mathfrak{p}).$$

From the explicit lower bound of $\widehat{\mu}(F_D)$ provided at (29) and Proposition 5.2, we deduce

$$(33) \quad \frac{\log B}{[K : \mathbb{Q}]} - \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n) \\ \geq \frac{(n-1)!^{\frac{1}{n-1}} (n-1) r_1(n, D)^{\frac{1}{n-1}}}{n D [K : \mathbb{Q}]} \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}} \\ - \frac{n^3 + 2n^2 + n - 4}{2nD(n+1)[K : \mathbb{Q}]} \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \log N(\mathfrak{p}).$$

II-1. Estimate of $\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}}$. — In order to estimate the sum $\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}}$ in (33), by (21), we have

$$\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}} \geq \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} - n^2 \delta^2 \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{3}{2}}}.$$

For the estimate of $\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})}$, we denote

$$\mathcal{Q}'(\mathcal{X}) = \left\{ \mathfrak{p} \in \text{Spm } \mathcal{O}_K \mid 27\delta^4 \leq N(\mathfrak{p}) \leq r_1(n, D)^{\frac{1}{n-1}}, \right. \\ \left. \mathcal{X}_{\mathbb{F}_{\mathfrak{p}}} \rightarrow \text{Spec } \mathbb{F}_{\mathfrak{p}} \text{ is not geometrically integral} \right\}.$$

Then by (24), we have

$$\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \sum_{27\delta^4 \leq N(\mathfrak{p}) \leq r_1(n, D)^{\frac{1}{n-1}}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} - \sum_{\mathfrak{p} \in \mathcal{Q}'(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} \\ \geq \frac{1}{n-1} \log r_1(n, D) - 3 \log 3 - 4 \log \delta - 2\epsilon_2(K) - \log(b'(\mathcal{X})),$$

where the notation $b'(\mathcal{X})$ is introduced in Proposition 4.4, and $\epsilon_2(K)$ is defined in (24).

For the term $\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{3}{2}}}$, it is equal to zero when $r_1(n, D)^{\frac{1}{n-1}} \leq 27\delta^4$.

When $r_1(n, D)^{\frac{1}{n-1}} > 27\delta^4$, by (25), we have

$$\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{3}{2}}} \leq \sum_{27\delta^4 \leq N(\mathfrak{p}) \leq r_1(n, D)^{\frac{1}{n-1}}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{3}{2}}} \\ \leq \frac{2}{3\sqrt{3}\delta^2} - 2r_1(n, D)^{-\frac{1}{2(n-1)}} + 2\epsilon_3(K, 27\delta^4).$$

By the above two estimates, we obtain

$$(34) \quad \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \frac{\log N(\mathfrak{p})}{n(\mathcal{X}_{\mathfrak{p}})^{\frac{1}{n-1}}} \\ \geq \frac{1}{n-1} \log r_1(n, D) - 3 \log 3 - 4 \log \delta - \log(b'(\mathcal{X})) - 2\epsilon_2(K) \\ - n^2 \delta^2 \left(\frac{2}{3\sqrt{3}\delta^2} - 2r_1(n, D)^{-\frac{1}{2(n-1)}} + 2\epsilon_3(K, 27\delta^4) \right)$$

by combining the above two inequalities.

II-2. Estimate of $\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \log N(\mathfrak{p})$. — For the estimate of $\sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \log N(\mathfrak{p})$, by (23), we have

$$(35) \quad \frac{1}{D} \sum_{\mathfrak{p} \in \mathcal{R}(\mathcal{X})} \log N(\mathfrak{p}) \leq \frac{1}{D} \sum_{N(\mathfrak{p}) \leq r_1(n, D)^{\frac{1}{n-1}}} \log N(\mathfrak{p}) \\ \leq \frac{1}{D} \left(r_1(n, D)^{\frac{1}{n-1}} + \epsilon_1 \left(K, r_1(n, D)^{\frac{1}{n-1}} \right) \right),$$

where $\epsilon_1(K, x)$ is defined in (23).

II-3. Deducing the contradiction. — We take (34) and (35) into (33), and we do some elementary calculations. Then the inequality

$$(36) \quad \frac{\log B}{[K : \mathbb{Q}]} - \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n) \\ \geq \frac{(n-1)!^{\frac{1}{n-1}} (n-1)}{n[K : \mathbb{Q}]} \cdot \frac{r_1(n, D)^{\frac{1}{n-1}}}{D} \left(\frac{1}{n-1} \log r_1(n, D) - \log(b'(\mathcal{X})) \right. \\ \left. - 4 \log \delta - 3 \log 3 - \frac{2n^2}{3\sqrt{3}} + \frac{2n^2\delta^2}{r_1(n, D)^{\frac{1}{2(n-1)}}} - 2n^2\delta^2\epsilon_3(K, 27\delta^4) - 2\epsilon_2(K) \right) \\ - \frac{n^3 + 2n^2 + n - 4}{2n(n+1)[K : \mathbb{Q}]} \cdot \frac{r_1(n, D)^{\frac{1}{n-1}}}{D} \left(1 + \frac{\epsilon_1 \left(K, r_1(n, D)^{\frac{1}{n-1}} \right)}{r_1(n, D)^{\frac{1}{n-1}}} \right)$$

is uniformly verified for all $D \geq 3\delta \log \delta + n - 1 \geq 2\delta + n - 1$.

From (27) in Lemma 4.6, we have

$$(37) \quad \frac{1}{n-1} \log r_1(n, D) \geq \log D + \frac{1}{n-1} \log \delta - \frac{1}{n-1} \log(n-1)!$$

when $D \geq 2\delta + n - 1$.

We take Lemma 4.6 and (37) into (36), and by the fact

$$\frac{\epsilon_1 \left(K, r_1(n, D)^{\frac{1}{n-1}} \right)}{r_1(n, D)^{\frac{1}{n-1}}} \leq \kappa_1(K)$$

and

$$3 \log 3 - \frac{2n^2}{3\sqrt{3}} + \frac{2n^2\delta^2}{r_1(n, D)^{\frac{1}{2(n-1)}}} - 2n^2\delta^2\epsilon_3(K, 27\delta^4) - 2\epsilon_2(K) \geq -\kappa_2(n, K),$$

we obtain

$$\begin{aligned}
 (38) \quad & \frac{n}{(n-1)^{n-1}\sqrt[n]{\delta}} \left(\log B - \frac{[K:\mathbb{Q}](n-1)!}{\delta(2n)^n} h(X) + C_1(n)[K:\mathbb{Q}] \right) \\
 & \geq \left(1 - \frac{\delta-2}{D} \right) \left(\log D - \left(4 - \frac{1}{n-1} \right) \log \delta \right. \\
 & \quad \left. - \log(b'(\mathcal{X})) - \kappa_2(n, K) - \frac{\log(n-1)!}{n-1} \right) \\
 & \quad - \frac{n^3 + 2n^2 + n - 4}{2(n^2 - 1)^{n-1}\sqrt[n-1]{(n-1)!}} \cdot \left(1 + \frac{n}{4} \right) (1 + \kappa_1(K)).
 \end{aligned}$$

When $D \geq 3\delta \log \delta + n - 1 \geq 2\delta + n - 1$ and $\delta \geq 2$, we have

$$\begin{aligned}
 & \frac{\delta-2}{D} \left(\log D - \left(4 - \frac{1}{n-1} \right) \log \delta - \frac{\log(n-1)!}{n-1} - \log(b'(\mathcal{X})) - \kappa_2(K, n) \right) \\
 & \leq \frac{\delta-2}{D} \log D \leq 3
 \end{aligned}$$

by an elementary calculation. We take the above inequality into (38) and then we obtain

$$\begin{aligned}
 & \frac{n}{(n-1)^{n-1}\sqrt[n]{\delta}} \left(\log B - \frac{[K:\mathbb{Q}](n-1)!}{\delta(2n)^n} h(X) + C_1(n)[K:\mathbb{Q}] \right) \\
 & \geq \log D - \left(4 - \frac{1}{n-1} \right) \log \delta - \log(b'(\mathcal{X})) - \kappa_2(K, n) - 3 - \frac{\log(n-1)!}{n-1} \\
 & \quad - \frac{n^3 + 2n^2 + n - 4}{2(n^2 - 1)^{n-1}\sqrt[n-1]{(n-1)!}} \cdot \left(1 + \frac{n}{4} \right) (1 + \kappa_1(K)),
 \end{aligned}$$

which deduces

$$\begin{aligned}
 \log D & \leq \frac{n \log B}{(n-1)^{n-1}\sqrt[n]{\delta}} - \frac{[K:\mathbb{Q}]n!}{\delta^{1+\frac{1}{n-1}}(n-1)(2n)^n} h(X) + \log(b'(\mathcal{X})) \\
 & \quad + \left(4 - \frac{1}{n-1} \right) \log \delta + C_2(n, K),
 \end{aligned}$$

with the constant $C_2(n, K)$ in the statement of this theorem, and this leads to the contradiction. \square

5.3. Control of auxiliary hypersurfaces. — The following two upper bounds of the degree of the auxiliary hypersurface are deduced from Theorem 5.4 directly.

COROLLARY 5.5. — *We keep all the notations and conditions in Theorem 5.4. Then there exists a hypersurface of degree smaller than*

$$C_3(n, K)\delta^3 B^{n/((n-1)\delta^{1/(n-1)})},$$

which covers $S(X; B)$ but does not contain the generic point of X , where the constant

$$C_3(n, K) = e^{C_2(n, K)} \frac{(n+6)(n-1)(2n)^n \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}])}{n!},$$

and $C_2(n, K)$ is defined in Theorem 5.4.

Proof. — By the upper bound of $b'(\mathcal{X})$ given in Proposition 4.4, we have

$$\begin{aligned} b'(\mathcal{X}) &\leq \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}]) \\ &\quad \cdot (\delta^{-2} - \delta^{-4}) \left(h(X) + \left(3\log \delta + \delta \log 3 + \log \left(\frac{n+\delta}{\delta} \right) \right) \right) \\ &\leq \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}]) \delta^{-2} (h(X) + (3\delta + 2\delta + \delta \log(n+1))) \\ &\leq (n+6) \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}]) \max\{\delta^{-2}h(X), \delta^{-1}\}. \end{aligned}$$

We denote by $G_K(X) = H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1-\frac{1}{n-1}}} \geq 1$ for simplicity, where the last inequality is obtained by definition directly. Then by an elementary calculation, we have

$$\begin{aligned} &\frac{b'(\mathcal{X})}{H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1-\frac{1}{n-1}}}} \\ &\leq (n+6) \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}]) \\ &\quad \cdot \frac{\max\left\{ \frac{\delta^{-1+\frac{1}{n-1}}(n-1)(2n)^n}{n![K:\mathbb{Q}]} \log G_K(X), \delta^{-1} \right\}}{G_K(X)} \\ &\leq \frac{(n+6)(n-1)(2n)^n \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}])}{n!} \delta^{-1+\frac{1}{n-1}}. \end{aligned}$$

We have the assertion by taking the above estimate into Theorem 5.4. \square

Compared with Corollary 5.5, the result below has a better dependence on the degree of the original hypersurface but a little worse dependence on the bound of heights.

COROLLARY 5.6. — *We keep all the notations and conditions in Theorem 5.4. Then there exists a hypersurface of degree smaller than*

$$\frac{C'_3(n, K) \delta^{3-\frac{1}{n-1}}}{H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1-\frac{1}{n-1}}}} B^{n/((n-1)\delta^{1/(n-1)})} \max\left\{ \frac{\log B}{[K : \mathbb{Q}]}, 1 \right\},$$

which covers $S(X; B)$ but does not contain the generic point of X , where the constant

$$C'_3(n, K) = -e^{C_2(n, K)} \frac{C_1(n)(n+6)(2n)^n \exp(2\epsilon_2(K) - 3\log 3 + [K : \mathbb{Q}])}{(n-1)!},$$

$C_1(n)$ is defined in (30), $C_2(n, K)$ is defined in Theorem 5.4 and $H_K(X)$ is defined in Definition 2.5.

Proof. — If

$$\frac{\log B}{[K : \mathbb{Q}]} < \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n),$$

then by Proposition 5.2, $S(X; B)$ can be covered by a hypersurface of degree no more than $2\delta + n - 1$, which does not contain the generic point of X . The upper bound of the degree satisfies the bound provided in the statement.

If

$$\frac{\log B}{[K : \mathbb{Q}]} \geq \frac{(n-1)!}{\delta(2n)^n} h(X) + C_1(n),$$

which is equivalent to

$$h(X) \leq \frac{\delta(2n)^n}{(n-1)!} \cdot \frac{\log B}{[K : \mathbb{Q}]} - \frac{\delta(2n)^n}{(n-1)!} C_1(n),$$

then we deal with it as follows. Similarly to the proof of Corollary 5.5, we have

$$b'(\mathcal{X}) \leq (n+6) \exp(2\epsilon_2(K) - 3 \log 3 + [K : \mathbb{Q}]) \max\{\delta^{-2} h(X), \delta^{-1}\},$$

where $b'(\mathcal{X})$ is the same as in Proposition 4.4 and Theorem 5.4. Then we have

$$\begin{aligned} & \frac{b'(\mathcal{X})}{H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1 - \frac{1}{n-1}}}} \\ & \leq \frac{(n+6) \exp(2\epsilon_2(K) - 3 \log 3 + [K : \mathbb{Q}]) \delta^{-1}}{H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1 - \frac{1}{n-1}}}} \\ & \quad \cdot \max \left\{ \frac{(2n)^n}{(n-1)!} \left(\frac{\log B}{[K : \mathbb{Q}]} - C_1(n) \right), 1 \right\} \\ & \leq \frac{-C_1(n)(n+6)(2n)^n \exp(2\epsilon_2(K) - 3 \log 3 + [K : \mathbb{Q}]) \delta^{-1} \max \left\{ \frac{\log B}{[K : \mathbb{Q}]}, 1 \right\}}{(n-1)! H_K(X)^{\frac{n!}{(n-1)(2n)^n} \delta^{-1 - \frac{1}{n-1}}}}, \end{aligned}$$

and we obtain the assertion by taking the above inequality into Theorem 5.4. \square

6. Counting rational points in plane curves

As applications of Corollary 5.5 and Corollary 5.6 we have the following uniform upper bounds of the number of rational points with bounded height in plane curves.

6.1. A generalization over an arbitrary number field. — The following result generalizes [10, Theorem 2] over an arbitrary number field, which has the optimal dependence on the bound of heights.

THEOREM 6.1. — *Let X be a geometrically integral curves in \mathbb{P}_K^2 of degree δ . Then we have*

$$\#S(X; B) \leq C_3(2, K) \delta^4 B^{2/\delta},$$

where the constant $C_3(2, K)$ is defined in Corollary 5.5. In addition, we have

$$\#S(X; B) \ll_K \delta^4 B^{2/\delta}.$$

Proof. — We apply the Bézout theorem in the intersection theory (cf. [16, Proposition 8.4]) to X and the auxiliary hypersurface determined in Corollary 5.5 for the case of $n = 2$, and then we obtain the result. \square

6.2. A better dependence on the degree. — In this part, we will provide another uniform upper bound of rational points with bounded height in plane curves. This result has a better dependence on the degree than that of Theorem 6.1, but a bit worse dependence on the bound of heights.

THEOREM 6.2. — *Let X be a geometrically integral curves in \mathbb{P}_K^2 of degree δ . Then we have*

$$\#S(X; B) \leq C'_3(2, K) \delta^3 B^{2/\delta} \max \left\{ \frac{\log B}{[K : \mathbb{Q}]}, 1 \right\},$$

where the constant $C'_3(2, K)$ is defined in Corollary 5.6. In addition, we have

$$\#S(X; B) \ll_K \delta^3 B^{2/\delta} \log B,$$

when $B \geq \exp([K : \mathbb{Q}])$.

Proof. — This is the same application of the Bézout theorem in the intersection theory (cf. [16, Proposition 8.4]) to Corollary 5.6 as that of Corollary 6.1 when $n = 2$, where we take $H_K(X) \geq 1$ defined in Definition 2.5 into consideration. \square

REMARK 6.3. — It seems that the upper bound given in Theorem 6.1 and Theorem 6.2 are not optimal. Actually, for a geometrically integral plane curve $X \hookrightarrow \mathbb{P}_{\mathbb{Q}}^2$ of degree δ , Heath-Brown conjectured the uniform upper bound

$$\#S(X; B) \ll \delta^2 B^{2/\delta}.$$

By the examples given in [10, §6], the exponent 2 of δ in the above conjecture would be optimal.

7. Explicit estimates under the assumption of GRH

Under the assumption of **GRH** (the generalized Riemann hypothesis) for the Dedekind zeta function of the number field K , we have more explicit estimates of

$$\begin{aligned}\theta_K(x) &= \sum_{N(\mathfrak{p}) \leq x} \log N(\mathfrak{p}), \quad \psi_K(x) = \sum_{N(\mathfrak{p}) \leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})}, \quad \text{and} \\ \phi_K(x) &= \sum_{N(\mathfrak{p}) \leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{3}{2}}},\end{aligned}$$

where $x \in \mathbb{R}^+$, $\mathfrak{p} \in \text{Spm } \mathcal{O}_K$, and $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$. With these results, we are able to obtain more explicit estimates of in the global determinant method and more explicit estimates in the densities of rational points. In this section, we will give explicit estimates of the remainders of the above $\theta_K(x)$, $\psi_K(x)$ and $\phi_K(x)$.

7.1. Explicit estimates of the distribution of prime ideals with bounded norms.

— In order to obtain explicit estimates, first we refer to a result in [20] under the assumption of GRH. Then by the same technique as in [37], we get an explicit generalization of Mertens' first theorem. If we do not need the explicit version, it is not necessary to assume GRH.

Let Δ_K be the discriminant of the number field K . By [20, Corollary 1.3], if $x \geq 3$, we have

$$\begin{aligned}|\theta_K(x) - x| &\leq \sqrt{x} \left(\left(\frac{1}{2\pi} \log \left(\frac{18.8x}{\log^2 x} \right) + 2.3 \right) \log \Delta_K \right. \\ &\quad \left. + \left(\frac{1}{8\pi} \log^2 \left(\frac{18.8x}{\log^2 x} \right) + 1.3 \right) [K : \mathbb{Q}] + 0.3 \log x + 14.6 \right) \\ &\leq \sqrt{x} \log^2 x (\log \Delta_K + [K : \mathbb{Q}]) \left(\frac{1}{2\pi(\log^2 x)} \log \left(\frac{18.8x}{\log^2 x} \right) + 3.6 \right. \\ &\quad \left. + \frac{1}{8\pi(\log^2 x)} \log^2 \left(\frac{18.8x}{\log^2 x} \right) + 0.3 \frac{1}{\log x} + 14.6 \frac{1}{\log^2 x} \right)\end{aligned}$$

under the assumption of GRH. Since $x \geq 3$, we obtain

$$(39) \quad |\theta_K(x) - x| \leq 528\sqrt{x} \log^2 x (\log \Delta_K + [K : \mathbb{Q}])$$

by Mathematica under the same assumption.

Similarly to the application of [37, Lemma 2.1] to the proof of [37, Lemma 2.3], by Abel's summation formula, we have

$$\psi_K(x) = \frac{\theta_K(x)}{x} + \int_2^x \frac{\theta_K(t)}{t^2} dt.$$

From (39), we have

$$\left| \psi_K(x) - 1 - \int_2^x \frac{1}{t} dt \right| \leq 528 (\log \Delta_K + [K : \mathbb{Q}]) \left(\frac{\log^2 x}{\sqrt{x}} + \int_2^x \frac{\log^2 t}{t^{\frac{3}{2}}} dt \right).$$

Then by an elementary calculation executed by Mathematica, we obtain

$$(40) \quad |\psi_K(x) - \log x| \leq 9550 (\log \Delta_K + [K : \mathbb{Q}])$$

for $x \geq 3$ under the assumption of GRH.

As above, we have

$$\phi_K(x) = \frac{\theta_K(x)}{x^{\frac{3}{2}}} + \frac{3}{2} \int_2^x \frac{\theta_K(t)}{t^{\frac{5}{2}}} dt.$$

Then from (39), we obtain

$$\left| \phi_K(x) - \frac{1}{\sqrt{x}} - \frac{3}{2} \int_2^x \frac{1}{t^{\frac{3}{2}}} dt \right| \leq 792 (\log \Delta_K + [K : \mathbb{Q}]) \left(\frac{\log^2 x}{x} + \int_2^x \frac{\log^2 t}{t^2} dt \right).$$

Then after an elementary calculation executed by Mathematica, we have

$$(41) \quad \left| \phi_K(x) - \frac{3}{2} \sqrt{2} + \frac{2}{x^{\frac{1}{2}}} \right| \leq 2516 \frac{\log^2 x}{x} (\log \Delta_K + [K : \mathbb{Q}])$$

uniformly for all $x \in \mathbb{R}^+$ under the assumption of GRH.

7.2. Estimates of remainders. — We compare (23) with (39), (24) with (40) and (25) with (41). Then we can suppose

$$\epsilon_1(K, x) = 528 \sqrt{x} \log^2 x (\log \Delta_K + [K : \mathbb{Q}])$$

in (23),

$$\epsilon_2(K) = 9550 (\log \Delta_K + [K : \mathbb{Q}])$$

in (24) and

$$\epsilon_3(K, x) = 2516 \frac{\log^2 x}{x} (\log \Delta_K + [K : \mathbb{Q}])$$

in (25) under the assumption of GRH. We take the above estimates of $\epsilon_1(K, x)$, $\epsilon_2(K)$ and $\epsilon_3(K, x)$ into Proposition 4.4, (31) and (32), and then we obtain a more explicit estimate in Theorem 5.4. Hence, the estimates in Corollary 5.5, Corollary 5.6, Theorem 6.1 and Theorem 6.2 are more explicit under the assumption of GRH.

REMARK 7.1. — In addition to the case of $K = \mathbb{Q}$, if we work over some other particular number fields, the assumption of the generalized Riemann hypothesis may not be obligatory. For example, in [22, Theorem 2], we are able to do it over totally imaginary fields. It depends on the understanding of the zero-free region of the Dedekind zeta function of the number field K .

Appendix A. An explicit lower bound of $Q_\xi(r)$

In this Appendix, we give an explicit lower bound of the function $Q_\xi(r)$ defined in (5) for the case of hypersurfaces.

In the following proof of Proposition A.1, the inequality

$$\frac{(N - m + 1)^m}{m!} \leq \binom{N}{m} \leq \frac{(N - (m - 1)/2)^m}{m!}$$

will be very useful, where N and m are two positive integers, and $N \geq m \geq 1$.

PROPOSITION A.1. — *Let X be a hypersurface of \mathbb{P}_k^n , ξ be a closed point in X and μ_ξ be the multiplicity of ξ in X induced by its local Hilbert–Samuel function. The function $Q_\xi(r)$ is defined in the equality (5). Then we have*

$$Q_\xi(r) > \left(\frac{(n-1)!}{\mu_\xi} \right)^{\frac{1}{n-1}} \left(\frac{n-1}{n} \right) r^{\frac{n}{n-1}} - \frac{n^3 + 2n^2 + n - 4}{2n(n+1)} r.$$

Proof. — For the case of hypersurfaces, by [26, Example 2.70 (2)], we have

$$H_\xi(s) = \binom{n+s-1}{s} - \binom{n+s-\mu_\xi-1}{s-\mu_\xi}.$$

We define the function $U_\xi(k) = H_\xi(0) + \cdots + H_\xi(k)$, and then we have

$$\begin{aligned} U_\xi(k) &= \sum_{j=0}^k \binom{n+j-1}{j} - \sum_{j=0}^k \binom{n+j-\mu_\xi-1}{j-\mu_\xi} \\ &= \binom{n+k}{n} - \binom{n+k-\mu_\xi}{n}. \end{aligned}$$

Then we obtain

$$\begin{aligned} Q_\xi(U_\xi(k)) &= \sum_{j=0}^n j H_\xi(j) \\ &= \sum_{j=0}^k j \binom{j+n-1}{n-1} - \sum_{j=0}^k j \binom{n+j-\mu_\xi-1}{n-1} \\ &= n \binom{k+n}{n+1} - n \binom{k-\mu_\xi+n}{n+1} - \mu_\xi \binom{n+k-\mu_\xi}{n}. \end{aligned}$$

Let $r \in]U_\xi(k-1), U_\xi(k)]$. By the definition of $Q_\xi(r)$ in the equality (5), we have the inequality

$$Q_\xi(U_\xi(k-1)) \leq Q_\xi(r) \leq Q_\xi(U_\xi(k)).$$

So, we have

$$\begin{aligned} Q_\xi(r) &= Q_\xi(U_\xi(k-1)) + k(r - U_\xi(k-1)) \\ &= n \binom{n+k-1}{n+1} - n \binom{n+k-\mu_\xi-1}{n+1} - \mu_\xi \binom{n+k-\mu_\xi-1}{n} \\ &\quad + kr - k \binom{n+k-1}{n} + k \binom{n+k-\mu_\xi-1}{n} \\ (42) \quad &= kr + \binom{n+k-\mu_\xi}{n+1} - \binom{n+k}{n+1}. \end{aligned}$$

In order to get a lower bound of $Q_\xi(r)$, we need to estimate the term

$$\binom{n+k-\mu_\xi}{n+1} - \binom{n+k}{n+1}.$$

In fact, we have the estimate

$$\begin{aligned} &\left[\binom{n+k}{n+1} - \binom{n+k-\mu_\xi}{n+1} \right] / U_\xi(k-1) \\ &= \left[\binom{n+k}{n+1} - \binom{n+k-\mu_\xi}{n+1} \right] / \left[\binom{n+k-1}{n} - \binom{n+k-\mu_\xi-1}{n} \right] \\ &= \frac{(n+k)(n+k-1) \cdots k - (n+k-\mu_\xi) \cdots (k-\mu_\xi)}{(n+1)[(n+k-1) \cdots k - (n+k-\mu_\xi-1) \cdots (k-\mu_\xi)]} \\ &= \frac{\frac{(n+k)(n+k-1)}{k-1} \frac{(n+k-2) \cdots (k-1)}{(n+k-\mu_\xi-1) \cdots (k-\mu_\xi)} - (n+k-\mu_\xi)}{\left[\frac{n+k-1}{k-1} \left[\frac{(n+k-2) \cdots (k-1)}{(n+k-\mu_\xi-1) \cdots (k-\mu_\xi)} \right] - 1 \right] (n+1)} \\ &= \frac{1}{n+1} \left[n+k + \frac{\mu_\xi}{\frac{n+k-1}{k-1} \cdot \frac{(n+k-2) \cdots (k-1)}{(n+k-\mu_\xi-1) \cdots (k-\mu_\xi)} - 1} \right] \\ &\leq \frac{1}{n+1} \left[n+k + \frac{\mu_\xi}{\left(\frac{n+k-1}{n+k-\mu_\xi-1} \right)^n - 1} \right] \\ &= \frac{1}{n+1} \left[n+k + \frac{(n+k-\mu_\xi-1)^n}{(n+k-1)^{n-1} + \cdots + (n+k-\mu_\xi-1)^{n-1}} \right] \\ &\leq \frac{1}{n+1} \left(n+k + \frac{n+k-\mu_\xi-1}{n} \right) \\ &= \frac{(n+1)k + n^2 + n - \mu_\xi - 1}{n(n+1)}. \end{aligned}$$

By the equality (42), we obtain

$$\begin{aligned}
 (43) \quad Q_{\xi}(r) &= kr + \binom{n+k-\mu_{\xi}}{n+1} - \binom{n+k}{n+1} \\
 &\geq kr - \frac{(n+1)k + n^2 + n - \mu_{\xi} - 1}{n(n+1)} U_{\xi}(k-1) \\
 &> kr - \frac{(n+1)k + n^2 + n - \mu_{\xi} - 1}{n(n+1)} r \\
 &= \frac{(n^2-1)k - n^2 - n + \mu_{\xi} + 1}{n(n+1)} r \\
 &= \left(\frac{n-1}{n} k - \frac{n^2 + n - \mu_{\xi} - 1}{n(n+1)} \right) r,
 \end{aligned}$$

where we use the estimate $U_{\xi}(k-1) < r$ in the inequality (43). In addition, we obtain the inequality

$$\begin{aligned}
 r &\leq U_{\xi}(k) = \binom{n+k}{n} - \binom{n+k-\mu_{\xi}}{n} = \sum_{j=1}^{\mu_{\xi}} \binom{n+k-j}{n-1} \\
 &\leq \frac{1}{(n-1)!} \sum_{j=1}^{\mu_{\xi}} \left(k + \frac{n}{2} - j + 1\right)^{n-1}.
 \end{aligned}$$

In addition, we have

$$\sum_{j=1}^{\mu_{\xi}} \left(k + \frac{n}{2} - j + 1\right)^{n-1} \leq \mu_{\xi} \left(k + \frac{n}{2}\right)^{n-1}.$$

Then

$$k \geq \frac{1}{n^{-1}\sqrt{\mu_{\xi}}} n^{-1}\sqrt{(n-1)!}r - \frac{n}{2}.$$

Finally, we have

$$\begin{aligned}
 Q_{\xi}(r) &> \left(\frac{n-1}{n} \left(\frac{1}{n^{-1}\sqrt{\mu_{\xi}}} n^{-1}\sqrt{(n-1)!}r - \frac{n}{2} \right) - \frac{n^2 + n - \mu_{\xi} - 1}{n(n+1)} \right) r \\
 &\geq \left(\frac{(n-1)!}{\mu_{\xi}} \right)^{\frac{1}{n-1}} \left(\frac{n-1}{n} \right) r^{\frac{n}{n-1}} - \frac{n^3 + 2n^2 + n - 4}{2n(n+1)} r,
 \end{aligned}$$

for $\mu_{\xi} \geq 1$. Then we obtain the result. \square

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