

PARAMETRIC GEOMETRY OF NUMBERS OVER A NUMBER FIELD AND EXTENSION OF SCALARS

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PARAMETRIC GEOMETRY OF NUMBERS OVER A NUMBER FIELD AND EXTENSION OF SCALARS

BY ANTHONY POËLS & DAMIEN ROY

Dedicated to Jeff Thunder on his 60th birthday.

ABSTRACT. — The parametric geometry of numbers of Schmidt and Summerer deals with rational approximation to points in \mathbb{R}^n . We extend this theory to a number field K and its completion K_w at a place w in order to treat approximation over K to points in K_w^n . As a consequence, we find that exponents of approximation over \mathbb{Q} in \mathbb{R}^n have the same spectrum as their generalizations over K in K_w^n . When w has relative degree 1 over a place ℓ of \mathbb{Q} , we further relate approximation over K to a point $\boldsymbol{\xi}$ in K_w^n , to approximation over \mathbb{Q} to a point Ξ in \mathbb{Q}_ℓ^{nd} obtained from $\boldsymbol{\xi}$ by extension of scalars, where d is the degree of K over \mathbb{Q} . By combination with a result of P. Bel, this allows us to construct algebraic curves in \mathbb{R}^{3d} defined over \mathbb{Q} , of degree 2d, containing points that are very singular with respect to rational approximation.

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RÉSUMÉ (*Géométrie paramétrique des nombres sur un corps de nombres et extension des scalaires*). — La géométrie paramétrique des nombres de Schmidt et Summerer étudie l'approximation rationnelle des points de \mathbb{R}^n . Nous étendons cette théorie à un corps de nombres K et à son complété K_w en une place w pour traiter de l'approximation sur K des points de K_w^n . Nous en déduisons que les exposants d'approximation sur \mathbb{Q} des points de \mathbb{R}^n possèdent le même spectre que leurs généralisations sur K dans K_w^n . Lorsque w est de degré relatif égal à un au-dessus d'une place ℓ de \mathbb{Q} , nous relions aussi l'approximation sur K d'un point $\boldsymbol{\xi}$ de K_w^n à celle sur \mathbb{Q} du point Ξ de \mathbb{Q}_ℓ^{nd} obtenu à partir de $\boldsymbol{\xi}$ par extension des scalaires, où d désigne le degré de K sur \mathbb{Q} . En combinant cette observation à un résultat de P. Bel, nous parvenons ainsi à construire des courbes algébriques dans \mathbb{R}^{3d} définies sur \mathbb{Q} , de degré 2d, contenant des points qui sont très singuliers vis à vis de l'approximation rationnelle.

1. Introduction

In Diophantine approximation, one is interested in measuring how well a given non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ with $n \geq 2$ can be approximated by subspaces of \mathbb{R}^n defined over \mathbb{Q} of a given dimension k. The most important cases are k = 1 and k = n - 1, and each naturally gives rise to a pair of exponents of approximation. For k = 1, they are $\hat{\lambda}(\boldsymbol{\xi})$ (resp. $\lambda(\boldsymbol{\xi})$) defined as the supremum of all real numbers λ for which the inequalities

(1)
$$\|\mathbf{x}\| \le Q \text{ and } \|\mathbf{x} \wedge \boldsymbol{\xi}\| \le Q^{-\lambda}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for each large enough $Q \geq 1$ (resp. for arbitrarily large values of $Q \geq 1$). For k = n - 1, they are $\widehat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) defined as the supremum of all real numbers ω for which the inequalities

(2)
$$\|\mathbf{x}\| \le Q \text{ and } |\mathbf{x} \cdot \boldsymbol{\xi}| \le Q^{-\omega}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for each large enough $Q \geq 1$ (resp. for arbitrarily large values of $Q \geq 1$), where the dot represents the usual scalar product in \mathbb{R}^n . This is independent of the choice of norms in \mathbb{R}^n and in $\bigwedge^2 \mathbb{R}^n$, but for convenience, we use the Euclidean norms. As these exponents depend only on the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(\mathbb{R})$, we may assume that $\|\boldsymbol{\xi}\| = 1$. We refer the reader to the paper by Laurent [9] for generalizations in intermediate dimensions k.

While studying such exponents, it is important to restrict ourselves to points $\boldsymbol{\xi} \in \mathbb{R}^n$ with \mathbb{Q} -linearly independent coordinates, as this yields simpler statements and can be achieved by dropping redundant coordinates if necessary. For such points, a result of Dirichlet gives

$$(n-1)^{-1} \leq \lambda(\boldsymbol{\xi}) \leq \lambda(\boldsymbol{\xi}) \text{ and } n-1 \leq \widehat{\omega}(\boldsymbol{\xi}) \leq \omega(\boldsymbol{\xi}).$$

However, this does not fully describe the *spectrum* of $(\hat{\lambda}, \lambda, \hat{\omega}, \omega)$, namely the set of all quadruples $(\hat{\lambda}(\boldsymbol{\xi}), \lambda(\boldsymbol{\xi}), \hat{\omega}(\boldsymbol{\xi}), \omega(\boldsymbol{\xi}))$ associated with these $\boldsymbol{\xi}$. For n = 2,

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a complete description is simply given by

$$1 = \widehat{\lambda}(\boldsymbol{\xi}) = \widehat{\omega}(\boldsymbol{\xi}) \le \lambda(\boldsymbol{\xi}) = \omega(\boldsymbol{\xi}) \le \infty$$

For n = 3, the description is more complicated and was achieved by Laurent in [8], showing it as a semi-algebraic set. One of the constraints that it involves is the following remarkable identity due to Jarník [7, Satz 1],

(3)
$$\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})} - 1 = \frac{1}{\widehat{\omega}(\boldsymbol{\xi}) - 1},$$

which together with $2 \leq \hat{\omega}(\boldsymbol{\xi}) \leq \infty$ fully describes the spectrum of the pair $(\hat{\lambda}, \hat{\omega})$. For $n \geq 4$, the spectrum of the four exponents is not known, but Marnat [11] has shown that it contains an open subset of \mathbb{R}^4 , and thus obeys no algebraic relation such as (3).

Much of the recent progress, including the breakthrough of Marnat and Moshchevitin [12] who determined the spectra of the pairs $(\hat{\lambda}, \lambda)$ and $(\hat{\omega}, \omega)$ for each $n \geq 3$, use Schmidt's and Summerer's parametric geometry of numbers [22] in a crucial way. In the dual but equivalent setting of [17], this theory attaches to any point $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\|\boldsymbol{\xi}\| = 1$, the family of symmetric convex bodies of \mathbb{R}^n

$$C_{\boldsymbol{\xi}}(q) = \{ \mathbf{x} \in \mathbb{R}^n ; \|\mathbf{x}\| \le 1 \text{ and } |\mathbf{x} \cdot \boldsymbol{\xi}| \le e^{-q} \} \subseteq \mathbb{R}^n$$

parametrized by real numbers $q \ge 0$. For each $j = 1, \ldots, n$, let $L_{\boldsymbol{\xi},j}(q)$ denote the logarithm of the *j*-th minimum of $C_{\boldsymbol{\xi}}(q)$ with respect to \mathbb{Z}^n , namely the smallest real number *t* such that $e^t C_{\boldsymbol{\xi}}(q)$ contains at least *j* linearly independent points of \mathbb{Z}^n . Then, form the map

(4)
$$\mathbf{L}_{\boldsymbol{\xi}} \colon [0,\infty) \longrightarrow \mathbb{R}^n \\ q \longmapsto (L_{\boldsymbol{\xi},1}(q),\dots,L_{\boldsymbol{\xi},n}(q)).$$

Transposed in this setting, the main results of Schmidt and Summerer in [22] can be summarized as follows. Firstly, they note that the standard exponents of approximation to $\boldsymbol{\xi}$, including the four mentioned above, are given by simple formulas in terms of the inferior and superior limits of the ratios $L_{\boldsymbol{\xi},j}(q)/q$ as q goes to infinity. Secondly, they show the existence of a constant $\gamma \geq 0$ and of a continuous piecewise linear map $\mathbf{P} \colon [0, \infty) \to \mathbb{R}^n$ with growth conditions involving γ , such that the difference $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded. Thus, the abovementioned exponents of approximation to $\boldsymbol{\xi}$ can be computed, via the same formulas, in terms of the behaviour of \mathbf{P} at infinity. They call such a map \mathbf{P} an (n, γ) -system, and their set increases as the deformation parameter γ increases. The (n, 0)-systems, whose simpler description is recalled in Section 2, are simply called *n*-systems for brevity.

The main result of [17] provides a converse and shows more precisely that the set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ with $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\|\boldsymbol{\xi}\| = 1$ coincides with the set of *n*systems modulo the additive group of bounded functions from $[0, \infty)$ to \mathbb{R}^n .

Moreover, $\boldsymbol{\xi}$ has \mathbb{Q} -linearly independent coordinates if and only if any corresponding *n*-system $\mathbf{P} = (P_1, \ldots, P_n)$ satisfies $\lim_{q\to\infty} P_1(q) = \infty$. This reduces the determination of the spectrum of a family of exponents of approximation to a combinatorial problem about such *n*-systems.

A similar theory is developed in [19], with \mathbb{Q} replaced by a field of rational functions in one variable F(T) over an arbitrary field F, and \mathbb{R} replaced by the completion F((1/T)) of F(T) for the degree valuation.

The first goal of this paper is to extend the theory to a number field K and its completion K_w at a place w, in order to study approximation over K to an arbitrary non-zero point $\boldsymbol{\xi}$ of K_w^n . In the next section, we show how to attach to such a point a function $\mathbf{L}_{\boldsymbol{\xi}}: [0, \infty) \to \mathbb{R}^n$ from which the four exponents of approximation to $\boldsymbol{\xi}$ can be computed in the same way as in the case where K is \mathbb{Q} , and w is its place at infinity. We will show that this set of maps also coincides with the set of *n*-systems modulo bounded functions. Thus, the spectrum of these exponents remains the same in this new context. In particular, Jarník's identity (3) holds for any point $\boldsymbol{\xi}$ of K_w^3 with linearly independent coordinates over K.

The second goal of this paper deals with extension of scalars from \mathbb{Q} to a number field K. For this, we assume that w is a place of K of relative degree 1 over \mathbb{Q} , so that $K_w = \mathbb{Q}_\ell$ for the place ℓ of \mathbb{Q} induced by w. We also choose a basis $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ of K over \mathbb{Q} and for each point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n$ we define

(5)
$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi}) \in K_w^{nd} = \mathbb{Q}_\ell^{nd}$$

and say that Ξ is obtained from $\boldsymbol{\xi}$ by extending scalars from \mathbb{Q} to K. If $\boldsymbol{\xi}$ has linearly independent coordinates over K, then Ξ has linearly independent coordinates over \mathbb{Q} , and we will show a close relationship between the maps $\mathbf{L}_{\boldsymbol{\xi}}$ and \mathbf{L}_{Ξ} . From this, we will deduce formulas linking the Diophantine exponents of approximation to $\boldsymbol{\xi}$ over K with those of Ξ over \mathbb{Q} . As a consequence, we will see that Jarník's identity (3) yields

(6)
$$\frac{1}{\widehat{\lambda}(\Xi)} - (2d-1) = \frac{d^2}{\widehat{\omega}(\Xi) - (2d-1)},$$

for any $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} \in \mathbb{Q}_{\ell}^{3d}$ constructed from a point $\boldsymbol{\xi} \in K_w^3$ with K-linearly independent coordinates.

Let ℓ be a place of \mathbb{Q} . We say that a point $\boldsymbol{\xi} \in \mathbb{Q}_{\ell}^{n}$ is very singular if it has linearly independent coordinates over \mathbb{Q} and satisfies $\widehat{\lambda}(\boldsymbol{\xi}) > 1/(n-1)$. This requires $n \geq 3$. Moreover, by Schmidt's subspace theorem, such a point is not algebraic, and so it generates a field $\mathbb{Q}(\boldsymbol{\xi})$ of transcendence degree at least 1 over \mathbb{Q} . The third goal, and the initial motivation of this paper, is to provide new examples of very singular points of transcendence degree 1. Up to now, all known examples come from dimension n = 3 and, apart from the constructions of [16], they are all of the form $\boldsymbol{\xi} = (1, \xi, \xi^{2})$. Moreover, the

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supremum of $\widehat{\lambda}(1,\xi,\xi^2)$ for a transcendental number $\xi \in \mathbb{Q}_{\ell}$ is $1/\gamma \simeq 0.618$, where $\gamma = (1 + \sqrt{5})/2$ denotes the golden ratio. For $\mathbb{Q}_{\ell} = \mathbb{R}$, this follows from the constructions of [14, 15] together with the upper bound of [6, Theorem 1a]. For a prime number ℓ , this follows from [25, Chapter 2] or [3] together with [23, Théorème 2]. More generally, in [1], Bel showed that the result extends to any number field K and its completion K_w at a place w. Assuming that w extends ℓ with relative degree 1 and choosing a basis $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ of K over \mathbb{Q} , we will deduce that \mathbb{Q}_{ℓ}^{3d} contains very singular points of the form $(\boldsymbol{\alpha}, \xi \boldsymbol{\alpha}, \xi^2 \boldsymbol{\alpha})$ with $\xi \in \mathbb{Q}_{\ell}$.

2. Notation and main results

Throughout this paper, we fix an algebraic extension K of \mathbb{Q} of finite degree d.

2.1. Absolute values. — We denote by M(K) the set of non-trivial places of K and by $M_{\infty}(K)$ the subset of its archimedean places. For each $v \in M(K)$, we denote by K_v the completion of K at v and by $d_v = [K_v : \mathbb{Q}_v]$ its local degree. When $v \in M_{\infty}(K)$, we normalize the absolute value $| |_v$ on K_v so that it extends the usual absolute value $| |_{\infty}$ on \mathbb{Q} . Then K_v embeds isometrically into \mathbb{C} . We identify it with its image \mathbb{R} or \mathbb{C} and write $v | \infty$. Otherwise, there is a unique prime number p with $|p|_v < 1$, and we ask that $|p|_v = p^{-1}$, so that $| |_v$ extends the usual p-adic absolute value on \mathbb{Q} . We then write v | p. For these normalizations and for each $a \in K^*$, the product formula reads

$$\prod_{v \in M(K)} |a|_v^{d_v/d} = 1$$

2.2. Local norms and heights. — Given a positive integer n and a place v in M(K), we define the norm of a point $\mathbf{x} = (x_1, \ldots, x_n)$ in K_v^n by

$$\|\mathbf{x}\|_{\nu} = \begin{cases} (|x_1|_{\nu}^2 + \dots + |x_n|_{\nu}^2)^{1/2} & \text{if } \nu \mid \infty, \\ \max\{|x_1|_{\nu}, \dots, |x_n|_{\nu}\} & \text{otherwise.} \end{cases}$$

For this choice of local norms, we define the *height* of a non-zero point \mathbf{x} in K^n by

$$H(\mathbf{x}) = \prod_{v \in M(K)} \|\mathbf{x}\|_v^{d_v/d}.$$

By the product formula, it depends only on the class of \mathbf{x} in $\mathbb{P}^{n-1}(K)$ and satisfies $H(\mathbf{x}) \geq 1$.

More generally, for each $k \in \{1, ..., n\}$ and each $v \in M(K)$, we define the norm of a point in $\bigwedge^k K_v^n$ to be the norm of its set of Plücker coordinates in K_v^N , where $N = \binom{n}{k}$. We also define the height of a point in $\bigwedge^k K^n$ to be

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the height of its set of Plücker coordinates in K^N . This is independent of the ordering of these coordinates. Then, we define the height of a k-dimensional subspace V of K^n to be

$$H(V) = H(\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k)$$

independently of the choice of a basis $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of V over K. For the subspace 0 of K^n , we set H(0) = 1.

2.3. The canonical bilinear form. — We endow K^n with the bilinear form given by

(7)
$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n$$

for each $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in K^n . Then we define the orthogonal space to a subspace V of K^n to be

(8)
$$V^{\perp} = \{ \mathbf{y} \in K^n ; \mathbf{x} \cdot \mathbf{y} = 0 \text{ for each } \mathbf{x} \in V \}.$$

According to a result of Schmidt, it has the same height $H(V^{\perp}) = H(V)$ as V.

For each $v \in M(K)$, the same formula (7) provides a bilinear form on K_{v}^{n} , which we denote in the same way. For a subspace V of K_{v}^{n} , we also define V^{\perp} by (8) but allowing **y** to run through K_{v}^{n} .

2.4. Exponents of approximation. — Fix a place $w \in M(K)$ and a non-zero point $\boldsymbol{\xi} \in K_w^n$. For each non-zero $\mathbf{x} \in K^n$, we slightly modify the notation of P. Bel in [1] by setting

$$D_{\boldsymbol{\xi}}^{*}(\mathbf{x}) = \left(\frac{\|\mathbf{x} \wedge \boldsymbol{\xi}\|_{w}}{\|\boldsymbol{\xi}\|_{w}}\right)^{d_{w}/d} \prod_{\nu \neq w} \|\mathbf{x}\|_{\nu}^{d_{\nu}/d},$$
$$D_{\boldsymbol{\xi}}(\mathbf{x}) = \left(\frac{\|\mathbf{x} \cdot \boldsymbol{\xi}\|_{w}}{\|\boldsymbol{\xi}\|_{w}}\right)^{d_{w}/d} \prod_{\nu \neq w} \|\mathbf{x}\|_{\nu}^{d_{\nu}/d}.$$

In view of the product formula, these numbers depend only on the class of \mathbf{x} in $\mathbb{P}^{n-1}(K)$. Clearly, they also depend only on the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(K_w)$. So, in practice, we may always normalize $\boldsymbol{\xi}$ so that $\|\boldsymbol{\xi}\|_w = 1$.

DEFINITION 2.1. — We denote by $\widehat{\lambda}(\boldsymbol{\xi}, K, w)$ (resp. $\lambda(\boldsymbol{\xi}, K, w)$) the supremum of all real numbers λ for which the inequalities

$$H(\mathbf{x}) \le Q$$
 and $D^*_{\boldsymbol{\xi}}(\mathbf{x}) \le Q^{-\lambda}$

admit a non-zero solution $\mathbf{x} \in K^n$, for all sufficiently large (resp. for arbitrarily large) real numbers $Q \ge 1$. We also denote by $\widehat{\omega}(\boldsymbol{\xi}, K, w)$ (resp. $\omega(\boldsymbol{\xi}, K, w)$) the supremum of all real numbers ω for which the inequalities

$$H(\mathbf{x}) \le Q$$
 and $D_{\boldsymbol{\xi}}(\mathbf{x}) \le Q^{-\omega}$

admit a non-zero solution $\mathbf{x} \in K^n$, for all sufficiently large (resp. for arbitrarily large) real numbers $Q \ge 1$.

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By construction, these numbers depend only on the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(K_w)$. Moreover, when $K = \mathbb{Q}$ and $w = \infty$, these are simply the standard exponents of approximation to a non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ from the Introduction. Indeed, each point of $\mathbb{P}^{n-1}(\mathbb{Q})$ is represented by a primitive integer point \mathbf{x} , that is a point of \mathbb{Z}^n with relatively prime coordinates, and we have $H(\mathbf{x}) = \|\mathbf{x}\|$, $D_{\boldsymbol{\xi}}^{\boldsymbol{\xi}}(\mathbf{x}) = \|\mathbf{x} \wedge \boldsymbol{\xi}\|$ and $D_{\boldsymbol{\xi}}(\mathbf{x}) = |\mathbf{x} \cdot \boldsymbol{\xi}|$ when $\|\boldsymbol{\xi}\| = 1$.

We can now state the main result of P. Bel in [1] to which we alluded in the Introduction.

THEOREM 2.2 (Bel, 2013). — Let $w \in M(K)$, and let S denote the set of elements of K_w^3 of the form $\boldsymbol{\xi} = (1, \xi, \xi^2)$ that have linearly independent coordinates over K. Then, the supremum of the numbers $\hat{\lambda}(\boldsymbol{\xi}, K, w)$ with $\boldsymbol{\xi} \in S$ is $1/\gamma \simeq 0.618$, where $\gamma = (1 + \sqrt{5})/2$ stands for the golden ratio.

2.5. Two dual families of minima. — Let w and $\boldsymbol{\xi} \in K_w^n$ be as in Section 2.4. For each $j = 1, \ldots, n$ and each $q \ge 0$, we define $L_{\boldsymbol{\xi},j}(q)$ (resp. $L_{\boldsymbol{\xi},j}^*(q)$) to be the smallest real number $t \ge 0$ for which the conditions

(9)
$$H(\mathbf{x}) \le e^t$$
 and $D_{\boldsymbol{\xi}}(\mathbf{x}) \le e^{t-q}$ (resp. $D^*_{\boldsymbol{\xi}}(\mathbf{x}) \le e^{t-q}$)

admit at least j linearly independent solutions over K in K^n . This minimum exists since, for any number $B \ge 1$, there are only finitely many elements of $\mathbb{P}^{n-1}(K)$ of height at most B. We combine these functions into two maps

$$\mathbf{L}_{\boldsymbol{\xi}} = (L_{\boldsymbol{\xi},1}, \dots, L_{\boldsymbol{\xi},n}) \text{ and } \mathbf{L}_{\boldsymbol{\xi}}^* = (L_{\boldsymbol{\xi},1}^*, \dots, L_{\boldsymbol{\xi},n}^*)$$

from $[0,\infty)$ to \mathbb{R}^n . For $K = \mathbb{Q}$ and $K_w = \mathbb{R}$, the map $\mathbf{L}_{\boldsymbol{\xi}}$ is the same as in the Introduction.

2.6. The *n*-systems. — Let $q_0 \in [0, \infty)$. An *n*-system on $[q_0, \infty)$ is a continuous function $\mathbf{P} = (P_1, \ldots, P_n)$ from $[q_0, \infty)$ to \mathbb{R}^n with the following combinatorial properties.

- (S1) We have $0 \le P_1(q) \le \cdots \le P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$ for each $q \in [q_0, \infty)$.
- (S2) There exist $s \in \{1, 2, ...\} \cup \{\infty\}$ and a strictly increasing sequence $(q_i)_{0 \le i < s}$ in $[q_0, \infty)$, which is unbounded if $s = \infty$, such that, over each subinterval $I_i = [q_{i-1}, q_i]$ with $1 \le i < s$ including $I_s = [q_{s-1}, \infty)$ if $s < \infty$, the union of the graphs of P_1, \ldots, P_n decomposes as the union of some horizontal line segments and one line segment Γ_i of slope 1 (with possible crossings), which all project down onto I_i .
- (S3) For each index i with $1 \le i < s$, the line segment Γ_i ends strictly above the point where Γ_{i+1} starts (on the vertical line with abscissa q_i).

The sequence $(q_i)_{0 \le i < s}$ is uniquely determined by **P**. Its elements are called the *switch numbers* of **P**. We say that an *n*-system is *rigid of mesh c*, for a given c > 0, if the *n* coordinates of $\mathbf{P}(q_i)$ are distinct positive multiples of *c* for

each index i with $0 \le i < s$. Then each q_i is also a positive multiple of c by condition (S1). See [17, Figure 1] for a picture showing the combined graph of a rigid 5-system with s = 3.

For each *n*-system $\mathbf{P} = (P_1, \ldots, P_n) \colon [q_0, \infty) \to \mathbb{R}^n$, we define its *dual* to be the map $\mathbf{P}^* \colon [q_0, \infty) \to \mathbb{R}^n$ given by

(10)
$$\mathbf{P}^*(q) = \left(q - P_n(q), q - P_{n-1}(q), \dots, q - P_1(q)\right) \text{ for each } q \ge q_0.$$

Note that \mathbf{P}^* is not an *n*-system unless n = 2, in which case $\mathbf{P}^* = \mathbf{P}$.

2.7. Main results. — With the notation of $\S2.5$, we will show that the main result of parametric geometry of numbers from [17] naturally extends to the present more general setting. We state it below in dual form as well.

THEOREM A. — Let $n \ge 2$ be an integer and let $w \in M(K)$. There are constants c, c' > 0 depending only on K, w and n with the following property. For each non-zero point $\boldsymbol{\xi} \in K_w^n$, there is an n-system **P** on $[0, \infty)$ such that

(11)
$$\sup_{q\geq 0} \|\mathbf{L}_{\boldsymbol{\xi}}(q) - \mathbf{P}(q)\| \leq c \quad and \quad \sup_{q\geq 0} \|\mathbf{L}_{\boldsymbol{\xi}}^*(q) - \mathbf{P}^*(q)\| \leq c.$$

Conversely, for each n-system **P** on $[0, \infty)$, there is a non-zero point $\boldsymbol{\xi} \in K_w^n$ for which one of the two conditions in (11) holds. Then the second condition holds with c replaced by c'.

This means, in particular, that the two conditions in (11) are equivalent up to the value of the constant c. For n = 1, the statement of Theorem A is also true but not interesting because there is a unique 1-system **P** on $[0, \infty)$, and it satisfies $\mathbf{P}(q) = \mathbf{L}_{\boldsymbol{\xi}}(q) = q$ and $\mathbf{P}^*(q) = \mathbf{L}^*_{\boldsymbol{\xi}}(q) = 0$ for any $q \ge 0$ and any non-zero $\boldsymbol{\xi} \in K_w$. The next result deals with extension of scalars from \mathbb{Q} to K.

THEOREM B. — Let $n \geq 2$ be an integer, let $w \in M(K)$ be a place of K of relative degree $d_w = 1$ over a place ℓ of \mathbb{Q} , and let $\alpha \in K^d$ be a basis of K over \mathbb{Q} . There is a constant c'' > 0 with the following property. For each non-zero $\boldsymbol{\xi} \in K_w^n$, the point $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} \in \mathbb{Q}_{\ell}^{nd}$ (defined in (5)) satisfies

(12)
$$\begin{aligned} |L_{\Xi,d(i-1)+j}(dq) - L_{\xi,i}(q)| &\leq c'', \\ |L_{\Xi,d(i-1)+j}^*(dq) - L_{\xi,i}^*(q) - (d-1)q| &\leq c'' \end{aligned}$$

for any choice of $q \ge 0$, $i = 1, \ldots, n$ and $j = 1, \ldots, d$.

Again, the two sets of conditions in (12) in terms of the functions L and L^* are equivalent up to the value of c''. Our last main result provides very singular points on projective algebraic curves of degree 2d defined and irreducible over \mathbb{Q} .

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THEOREM C. — Suppose that K embeds in \mathbb{Q}_{ℓ} for a place ℓ of \mathbb{Q} . Identify K with its image and choose a basis $\boldsymbol{\alpha} \in \mathbb{Q}_{\ell}^d$ of K over \mathbb{Q} . Then we have

$$\sup\left\{\widehat{\lambda}\left((\boldsymbol{\alpha}, \boldsymbol{\xi}\boldsymbol{\alpha}, \boldsymbol{\xi}^{2}\boldsymbol{\alpha}), \mathbb{Q}, \ell\right); \boldsymbol{\xi} \in \mathbb{Q}_{\ell} \text{ and } [K(\boldsymbol{\xi}) : K] > 2\right\} = (d\gamma^{2} - 1)^{-1},$$
$$\sup\left\{\widehat{\omega}\left((\boldsymbol{\alpha}, \boldsymbol{\xi}\boldsymbol{\alpha}, \boldsymbol{\xi}^{2}\boldsymbol{\alpha}), \mathbb{Q}, \ell\right); \boldsymbol{\xi} \in \mathbb{Q}_{\ell} \text{ and } [K(\boldsymbol{\xi}) : K] > 2\right\} = d(\gamma^{2} + 1) - 1$$

where $\gamma = (1 + \sqrt{5})/2$ stands for the golden ratio.

Since $2 < \gamma^2 < 3$, this, indeed, provides very singular points $(\boldsymbol{\alpha}, \boldsymbol{\xi}\boldsymbol{\alpha}, \boldsymbol{\xi}^2\boldsymbol{\alpha}) \in \mathbb{Q}_{\ell}^{3d}$.

2.8. Outline of the paper. — Most of the paper is devoted to the proof of Theorem A. This is done in two steps, which we briefly sketch below.

We first show in Section 7 how to attach an *n*-system to a non-zero point $\boldsymbol{\xi} \in K_w^n$. The general strategy is similar to that of Schmidt and Summerer in [22], but we need the adelic versions of Minkowski's convex body theorem and of Mahler's theory of compound bodies recalled in Section 4. We also need a notion of distance $\lambda(\mathbf{x}, C)$ between a non-zero point \mathbf{x} of K^n and an adelic convex body C, and a related notion of adelic minima for C defined in Section 5. With those tools, we construct a family of adelic convex bodies $C_{\boldsymbol{\xi}}(q)$ whose minima are closely related to the map $\mathbf{L}_{\boldsymbol{\xi}}(q)$, and we obtain information on this map by considering approximate compounds $\mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q)$ of $\mathcal{C}_{\boldsymbol{\xi}}(q)$. The existence of an *n*-system that approximates $\mathbf{L}_{\boldsymbol{\xi}}(q)$ up to a bounded function then follows from a combinatorial result of [17] that is recalled in Section 6.

The combinatorial result of Section 6 is also used in order to attach a point $\boldsymbol{\xi}$ to an *n*-system. It shows that we simply have to do it for a rigid *n*-system **R** with a large mesh. The construction of $\boldsymbol{\xi}$ is done in Section 9. We work over the ring \mathcal{O}_S of S-integers of K, where S consists of w and all archimedean places of K. We recursively construct a sequence of ordered bases $\mathbf{x}^{(i)}$ of \mathcal{O}_S^n over \mathcal{O}_S , one for each of the switch points q_i of **R**. The basis $\mathbf{x}^{(i)}$ will realize, up to bounded factors, the successive minima of the adelic convex body $\mathcal{C}_{\boldsymbol{\xi}}(q)$ in the interval between q_i and q_{i+1} for the point $\pmb{\xi}$ that we want (as illustrated, for example, in [17, Figure 2]). Each basis $\mathbf{x}^{(i)}$, except for the first, is constructed from the preceding $\mathbf{x}^{(i-1)}$ by modifying only one point of it and by moving the new point up in the sequence, according to the behaviour of the map ${f R}$ between q_{i-1} and q_i . This new point is obtained by multiplying the old one by an appropriate S-unit and by adding to this product a linear combination of some other points of $\mathbf{x}^{(i-1)}$ with coefficients in \mathcal{O}_S , in order to keep control of the geometry of $\mathbf{x}^{(i)}$ in K_v^n for each place v of S. For the places v distinct from w, this is done so that the image of $\mathbf{x}^{(i)}$ in K_{v}^{n} remains bounded and almost orthogonal in a sense that is defined in Section 3. Furthermore, at the place w, the norms of the basis elements of $\mathbf{x}^{(i)}$ in K_w^n are governed by the coordinates of $\mathbf{R}(q_i)$, and the subsequence of $\mathbf{x}^{(i)}$ common to $\mathbf{x}^{(i+1)}$ is almost orthogonal. Then the lines of K_w^n that are orthogonal to these subsequences for the dot

product converge to a line whose generator $\boldsymbol{\xi}$ has the required property. The local estimates that are needed are developed in Section 3, and the recursive procedure is presented in Section 8, together with crucial estimates for local norms at the place w expressed in terms of heights only.

With the help of Theorem A, we show in Section 10 that the spectrum of the four exponents $(\omega, \hat{\omega}, \lambda, \hat{\lambda})$ is independent of K and w and that it can be computed in terms of *n*-systems. We also extend the intermediate exponents of Laurent to the number field setting and derive the same conclusion for their spectrum.

Finally, Theorems B and C are proved in Section 12 using a general construction in adelic geometry of numbers from Section 11 that is reminiscent of work of Jeff Thunder in [24].

3. Local metric estimates

For the sake of generality, we fix here an arbitrary local field L, namely a complete field with respect to an absolute value | | which either is archimedean or has a discrete valuation group $|L^*|$ in \mathbb{R}^* . For our applications, this will be K_v for some place v of K. If L is archimedean, we identify it with \mathbb{R} or \mathbb{C} through an isometric field embedding in \mathbb{C} (unique up to composition with complex conjugation). Otherwise, we denote by $\mathcal{O} = \{x \in L; |x| \leq 1\}$ the valuation ring of L. In this section, we define notions of orthogonality and distance and provide several estimates that will be needed in later sections (cf. [17, §4]).

3.1. Norms and orthogonality. — Let k and n be integers with $1 \le k \le n$ and let U be a vector space over L of dimension n. If $L \subseteq \mathbb{C}$, we equip U with the Euclidean norm associated to an inner product on U (real if $L = \mathbb{R}$ and complex if $L = \mathbb{C}$). Then there is a unique inner product on $\bigwedge^k U$ such that, for any orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ of U, the products $\mathbf{u}_{i_1} \land \cdots \land \mathbf{u}_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$ form an orthonormal basis of $\bigwedge^k U$, and we equip this space with the associated Euclidean norm. If L is not archimedean, the ring \mathcal{O} is a principal ideal domain, and we equip U with the maximum norm with respect to some basis of U over L. Then, the unit ball \mathcal{B} for that norm is the free rank n sub- \mathcal{O} -module of U generated by this basis, and, for each $\mathbf{x} \in U$, we have

$$\|\mathbf{x}\| = \min\{|a|; a \in L \text{ and } \mathbf{x} \in a\mathcal{B}\}.$$

Moreover, the sub- \mathcal{O} -module $\bigwedge^k \mathcal{B}$ of $\bigwedge^k U$ generated by the products of k elements of \mathcal{B} is free of rank $N = \binom{n}{k}$, and we equip $\bigwedge^k U$ with the corresponding norm.

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If V is a subspace of U, we endow it with the induced norm. This norm is admissible because, if $L \not\subseteq \mathbb{C}$, it is associated to the sub- \mathcal{O} -module $\mathcal{B} \cap V$ of V, which is free of rank $\dim_L(V)$. We say that subspaces V_1, \ldots, V_m of V are (topologically) orthogonal and, following the notation of [19, §2.2], we write their sum as

$$V_1 \perp_{\mathrm{top}} \cdots \perp_{\mathrm{top}} V_m,$$

if, for any choice of $(\mathbf{x}_1, \ldots, \mathbf{x}_m) \in V_1 \times \cdots \times V_m$, we have

$$\|\mathbf{x}_1 + \dots + \mathbf{x}_m\| = \begin{cases} (\|\mathbf{x}_1\|^2 + \dots + \|\mathbf{x}_m\|^2)^{1/2} & \text{if } L \subseteq \mathbb{C}, \\ \max\{\|\mathbf{x}_1\|, \dots, \|\mathbf{x}_m\|\} & \text{otherwise.} \end{cases}$$

When $L \subseteq \mathbb{C}$, this is the usual notion, and it amounts to asking that V_1, \ldots, V_m be pairwise orthogonal. However, when L is non-archimedean, the latter condition is necessary but not sufficient. We say that a point $\mathbf{x} \in U$ is orthogonal to a subspace V of U if $\langle \mathbf{x} \rangle_L$ and V are orthogonal. We say that an m-tuple of vectors $(\mathbf{x}_1, \ldots, \mathbf{x}_m) \in U^m$ is orthogonal if the subspaces $\langle \mathbf{x}_1 \rangle_L, \ldots, \langle \mathbf{x}_m \rangle_L$ that they span are orthogonal. We say that it is orthonormal if, moreover, they have norm 1. Again, these are the usual notions when $L \subseteq \mathbb{C}$. When $L \not\subseteq \mathbb{C}$, an orthonormal basis of U is simply a basis of \mathcal{B} as an \mathcal{O} -module. In general, an m-tuple of non-zero vectors $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ of U is orthogonal (resp. orthonormal) if and only if it can be extended to an orthogonal (resp. orthonormal) basis $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of U. We will also need the following criterion.

LEMMA 3.1. — With the above notation, let $\mathbf{x}_1, \ldots, \mathbf{x}_m \in U \setminus \{0\}$. Then, we have

$$\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_m\| \le \|\mathbf{x}_1\| \cdots \|\mathbf{x}_m\|,$$

with equality if and only if $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is orthogonal.

On L^n we have the canonical bilinear form or dot product given by (7) for any pair of points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in L^n . If $L \subseteq \mathbb{C}$, this is connected with the inner product

$$(\mathbf{x},\mathbf{y}) := \mathbf{x} \cdot \overline{\mathbf{y}} = x_1 \overline{y}_1 + \dots + x_n \overline{y}_n,$$

where $\overline{\mathbf{y}} = (\overline{y}_1, \ldots, \overline{y}_n)$ denotes the complex conjugate of \mathbf{y} , and we equip L^n with the corresponding Euclidean norm. Otherwise, we equip L^n with the maximum norm, so that its unit ball is \mathcal{O}^n . When $L = K_v$, this agrees with the definitions of Section 2.2 both for the norm on L^n and the corresponding norm on $\bigwedge^k L^n$. We conclude with the following observation.

LEMMA 3.2. — Let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an orthonormal basis of L^n . The dual basis $(\mathbf{u}_1^*, \ldots, \mathbf{u}_n^*)$ of L^n with respect to the dot product is also orthonormal.

Proof. — If $L \subseteq \mathbb{C}$, then \mathbf{u}_j^* is the complex conjugate $\overline{\mathbf{u}}_j$ of \mathbf{u}_j for each index $j = 1, \ldots, n$, thus $(\mathbf{u}_i^*, \mathbf{u}_j^*) = \overline{(\mathbf{u}_i, \mathbf{u}_j)} = \delta_{i,j}$ for each $i, j \in \{1, \ldots, n\}$, and we are done. If $L \not\subseteq \mathbb{C}$, then $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ is a basis of \mathcal{O}^n as an \mathcal{O} -module, thus $(\mathbf{u}_1^*, \ldots, \mathbf{u}_n^*)$ is also a basis of \mathcal{O}^n as needed.

3.2. Distances. — Again, let $1 \le k \le n$ be integers and let U be a vector space over L of dimension n equipped with an admissible norm, as above. By the choice of norm on L^n , a basis $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ of U is orthonormal if and only if the linear map from L^n to U sending a point $(a_1, \ldots, a_n) \in L^n$ to $a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \in U$ is an isometry. We consider the following notions of distance.

DEFINITION 3.3. — The (projective) distance between non-zero points \mathbf{x} and \mathbf{y} of U, or between the lines $\langle \mathbf{x} \rangle_L$ and $\langle \mathbf{y} \rangle_L$ that they generate, is

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) := \operatorname{dist}(\langle \mathbf{x} \rangle_L, \langle \mathbf{y} \rangle_L) := \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|} \in [0, 1]$$

If V_1 , V_2 are subspaces of U of the same dimension k, then $\bigwedge^k V_1$, $\bigwedge^k V_2$ are one-dimensional subspaces of $\bigwedge^k U$, and we define

$$\operatorname{dist}(V_1, V_2) = \operatorname{dist}\left(\bigwedge^k V_1, \bigwedge^k V_2\right).$$

As Schmidt notes in [20, §8], the distance between non-zero points of U satisfies the triangle inequality when $L \subseteq \mathbb{C}$. When L is non-archimedean, we state below a stronger inequality, which we leave to the reader.

LEMMA 3.4. — For any non-zero points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in U$, we have

$$\operatorname{dist}(\mathbf{x}_1, \mathbf{x}_3) \leq \begin{cases} \operatorname{dist}(\mathbf{x}_1, \mathbf{x}_2) + \operatorname{dist}(\mathbf{x}_2, \mathbf{x}_3) & \text{if } L \subseteq \mathbb{C}, \\ \max\{\operatorname{dist}(\mathbf{x}_1, \mathbf{x}_2), \operatorname{dist}(\mathbf{x}_2, \mathbf{x}_3)\} & \text{else.} \end{cases}$$

The same holds if \mathbf{x}_j is replaced by a subspace V_j of U of dimension $k \ge 1$ for j = 1, 2, 3.

The second assertion of the lemma follows from the first when k = 1. The general case where k > 1 follows by considering the lines $\bigwedge^k V_i$ inside $\bigwedge^k U$.

For any subspace V of U, we can write $U = W \perp_{top} V$ for some other subspace W of U. It suffices to choose an orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ for V (empty if V = 0), to complete it to an orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ of U and to take $W = \langle \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n \rangle_L$. So, we may write any $\mathbf{x} \in U$ in the form $\mathbf{x} = \mathbf{w} + \mathbf{v}$ with $\mathbf{w} \in W$ orthogonal to V and $\mathbf{v} \in V$. If $L \subseteq \mathbb{C}$, this decomposition is unique. In general, it is not unique, but the next result shows that $\|\mathbf{w}\|$ is independent of the decomposition (upon noting that $\mathbf{w} = 0$ when $\mathbf{x} = 0$).

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LEMMA 3.5. — Let V be a non-zero subspace of U and let $\mathbf{x} \in U \setminus \{0\}$. Write $\mathbf{x} = \mathbf{w} + \mathbf{v}$ with \mathbf{w} orthogonal to V and $\mathbf{v} \in V$. Then we have

(13)
$$\operatorname{dist}(\mathbf{x}, V) := \min\left\{\operatorname{dist}(\mathbf{x}, \mathbf{y}); \, \mathbf{y} \in V \setminus \{0\}\right\} = \frac{\|\mathbf{w}\|}{\|\mathbf{x}\|}.$$

Proof. — Let $\mathbf{y} \in V \setminus \{0\}$. Since \mathbf{w} is orthogonal to V, it is orthogonal to \mathbf{y} . So, the products $\mathbf{w} \wedge \mathbf{y}$ and $\mathbf{v} \wedge \mathbf{y}$ are orthogonal in $\bigwedge^2 U$. Consequently, we find

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{w} \wedge \mathbf{y} + \mathbf{v} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|} \ge \frac{\|\mathbf{w} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{\|\mathbf{w}\|}{\|\mathbf{x}\|},$$

with equality everywhere, if $\mathbf{v} = 0$ or if $\mathbf{y} = \mathbf{v} \neq 0$.

By Lemma 3.1, non-zero points \mathbf{x} , \mathbf{y} of U are orthogonal if and only if dist $(\mathbf{x}, \mathbf{y}) = 1$. Thus, with the notation of Lemma 3.5, the point \mathbf{x} is orthogonal to V if and only if dist $(\mathbf{x}, V) = 1$. Moreover, we have $\mathbf{x} \in V$ if and only if dist $(\mathbf{x}, V) = 0$. We also note the following alternative formula for dist (\mathbf{x}, V) .

LEMMA 3.6. — Let \mathbf{x} and V be as in Lemma 3.5 and let $(\mathbf{y}_1, \ldots, \mathbf{y}_k)$ be a basis of V. Then we have

(14)
$$\operatorname{dist}(\mathbf{x}, V) = \frac{\|\mathbf{x} \wedge \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\|}{\|\mathbf{x}\| \|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\|}.$$

Proof. — Since the right-hand side of (14) is independent of the choice of $(\mathbf{y}_1, \ldots, \mathbf{y}_k)$, we may assume that this basis is orthogonal. Then, for the decomposition $\mathbf{x} = \mathbf{w} + \mathbf{v}$ of Lemma 3.5, the sequence $(\mathbf{w}, \mathbf{y}_1, \ldots, \mathbf{y}_k)$ is also orthogonal. Thus, using Lemma 3.1, we find

$$\|\mathbf{x} \wedge \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\| = \|\mathbf{w} \wedge \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\| = \|\mathbf{w}\| \|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\|,$$

and so the right-hand side of (14) reduces to $\|\mathbf{w}\| / \|\mathbf{x}\| = \operatorname{dist}(\mathbf{x}, V)$.

For the next crucial lemma, we apply the previous results with $U = L^n$.

LEMMA 3.7. — Suppose that $n \ge m \ge 2$ for an integer m. Let V_1 , V_2 be subspaces of L^n of dimension m - 1 for which $W = V_1 \cap V_2$ has dimension at least m - 2. Then we have

(15)
$$\operatorname{dist}(V_1, V_2) = \max\left\{\operatorname{dist}(\mathbf{x}, V_2) \, ; \, \mathbf{x} \in V_1 \setminus \{0\}\right\}.$$

Moreover, if $V_1 \neq V_2$, if $(\mathbf{w}_1, \ldots, \mathbf{w}_{m-2})$ is a basis of W, and if $\mathbf{v}_i \in V_i \setminus W$ for i = 1, 2, then upon writing $\omega = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{m-2}$ we have

(16)
$$\operatorname{dist}(V_1, V_2) = \frac{\|\omega\| \|\omega \wedge \mathbf{v}_1 \wedge \mathbf{v}_2\|}{\|\omega \wedge \mathbf{v}_1\| \|\omega \wedge \mathbf{v}_2\|}.$$

Proof. — We may assume that $V_1 \neq V_2$ since otherwise both sides of (15) are zero. We also note that the right-hand side of (16) is independent of the choice of $\mathbf{w}_1, \ldots, \mathbf{w}_{m-2}, \mathbf{v}_1, \mathbf{v}_2$. So, we choose for $(\mathbf{w}_1, \ldots, \mathbf{w}_{m-2})$ an orthonormal basis of W and we complete it to an orthonormal basis $(\mathbf{w}_1, \ldots, \mathbf{w}_{m-2}, \mathbf{v}_2, \mathbf{u})$ of $V_1 + V_2$ with $\mathbf{v}_2 \in V_2$. We also choose a unit vector $\mathbf{v}_1 \in V_1$ of the form $\mathbf{v}_1 = a\mathbf{v}_2 + b\mathbf{u}$ with $(a, b) \in L^2$. Then $(\mathbf{w}_1, \ldots, \mathbf{w}_{m-2}, \mathbf{v}_i)$ is an orthonormal basis of V_i for i = 1, 2 and we have ||(a, b)|| = 1. Moreover, the pair $(\omega \wedge \mathbf{v}_2, \omega \wedge \mathbf{u})$ is orthonormal in $\bigwedge^{m-1}L^n$. Since $\omega \wedge \mathbf{v}_1 = a\omega \wedge \mathbf{v}_2 + b\omega \wedge \mathbf{u}$, we deduce that

$$\operatorname{dist}(V_1, V_2) = \operatorname{dist}(\omega \wedge \mathbf{v}_1, \omega \wedge \mathbf{v}_2) = |b|$$

This proves (16) since $\|\omega \wedge \mathbf{v}_1 \wedge \mathbf{v}_2\| = \|b\omega \wedge \mathbf{u} \wedge \mathbf{v}_2\| = |b|$. Finally, let $\mathbf{x} \in V_1 \setminus \{0\}$ and write $\mathbf{x} = \mathbf{w} + t\mathbf{v}_1$ with $\mathbf{w} \in W$ and $t \in L$. Then, $\mathbf{x} = tb\mathbf{u} + \mathbf{v}$, where $\mathbf{v} = \mathbf{w} + ta\mathbf{v}_2 \in V_2$, and \mathbf{u} is orthogonal to V_2 . So, Lemma 3.5 gives

$$\operatorname{dist}(\mathbf{x}, V_2) = \frac{|tb|}{\|\mathbf{x}\|} \le \frac{|tb|}{|t|} = |b|$$

with equality if $\mathbf{x} = \mathbf{v}_1$. This proves (15).

COROLLARY 3.8. — Let V_1 , V_2 be as in Lemma 3.7 and let $\mathbf{x} \in L^n \setminus \{0\}$. Then we have

$$\operatorname{dist}(\mathbf{x}, V_2) \leq \begin{cases} \operatorname{dist}(\mathbf{x}, V_1) + \operatorname{dist}(V_1, V_2) & \text{if } L \subseteq \mathbb{C}, \\ \max\{\operatorname{dist}(\mathbf{x}, V_1), \operatorname{dist}(V_1, V_2)\} & \text{else.} \end{cases}$$

Proof. — Choose $\mathbf{y}_1 \in V_1 \setminus \{0\}$ such that $\operatorname{dist}(\mathbf{x}, \mathbf{y}_1) = \operatorname{dist}(\mathbf{x}, V_1)$, and $\mathbf{y}_2 \in V_2 \setminus \{0\}$ such that $\operatorname{dist}(\mathbf{y}_1, \mathbf{y}_2) = \operatorname{dist}(\mathbf{y}_1, V_2)$. By Lemma 3.7, we have $\operatorname{dist}(\mathbf{y}_1, \mathbf{y}_2) \leq \operatorname{dist}(V_1, V_2)$. As $\operatorname{dist}(\mathbf{x}, V_2) \leq \operatorname{dist}(\mathbf{x}, \mathbf{y}_2)$, the conclusion follows from the triangle inequality of Lemma 3.4 applied to \mathbf{x}, \mathbf{y}_1 and \mathbf{y}_2 .

3.3. Duality. — For each k = 0, ..., n, the dot product on $L^n = \bigwedge^1 L^n$ induces a non-degenerate bilinear map from $\bigwedge^k L^n \times \bigwedge^k L^n$ to L, also denoted by a dot and given on pure products by

$$(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k) \cdot (\mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k) = \det(\mathbf{x}_i \cdot \mathbf{y}_j),$$

with the convention that, for k = 0, the empty wedge product is $1 \in L = \bigwedge^0 L^n$, and the empty determinant is 1 as well.

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ denote the canonical basis of L^n and let $\mathbf{E} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$. For k as above, there is a unique isomorphism $\varphi_k \colon \bigwedge^k L^n \to \bigwedge^{n-k} L^n$ such that

$$(\mathbf{X} \wedge \mathbf{Y}) \cdot \mathbf{E} = \varphi_k(\mathbf{X}) \cdot \mathbf{Y},$$

for any $\mathbf{X} \in \bigwedge^k L^n$ and $\mathbf{Y} \in \bigwedge^{n-k} L^n$. If $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ is any basis of L^n with $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_n = \mathbf{E}$, and if $(\mathbf{u}_1^*, \ldots, \mathbf{u}_n^*)$ denotes the dual basis of L^n for

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the dot product, a short computation shows that, for any k-tuple of integers $\mathbf{i} = (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le n$, we have

(17)
$$\varphi_k(\mathbf{u}_{i_1} \wedge \dots \wedge \mathbf{u}_{i_k}) = \epsilon(\mathbf{i}, \mathbf{j}) \mathbf{u}_{j_1}^* \wedge \dots \wedge \mathbf{u}_{j_{n-k}}^*$$

where $\mathbf{j} = (j_1, \ldots, j_{n-k})$ denotes the complementary increasing sequence of integers for which (\mathbf{i}, \mathbf{j}) is a permutation of $(1, \ldots, n)$, and $\epsilon(\mathbf{i}, \mathbf{j}) \in \{-1, 1\}$ is the signature of this permutation. In particular, if we choose $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ to be the canonical basis of L^n , which is its own dual, this formula shows that φ_k is an isometry. Furthermore, if V is a subspace of L^n of dimension k, we may choose $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ so that $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ is a basis of V. Then $(\mathbf{u}_{k+1}^*, \ldots, \mathbf{u}_n^*)$ is a basis of the subspace

$$V^{\perp} = \{ \mathbf{y} \in L^n \, ; \, \mathbf{x} \cdot \mathbf{y} = 0 \text{ for each } \mathbf{x} \in V \},\$$

and the same formula implies that

$$\varphi_k(\bigwedge^k V) = \bigwedge^{n-k} V^{\perp}.$$

As φ_k is an isometry, it preserves the distance, and so we conclude as follows.

LEMMA 3.9. — For any pair of subspaces V_1 , V_2 of L^n of the same dimension k with 0 < k < n, we have

$$\operatorname{dist}(V_1, V_2) = \operatorname{dist}(V_1^{\perp}, V_2^{\perp}).$$

In particular, if V_1 , V_2 have dimension n-1 > 0, and if $V_i^{\perp} = \langle \mathbf{u}_i \rangle_L$ for i = 1, 2, then $\operatorname{dist}(V_1, V_2) = \operatorname{dist}(\mathbf{u}_1, \mathbf{u}_2)$. When $L = \mathbb{R}$, this observation also follows from [17, Lemma 4.4].

3.4. Almost orthogonal sequences. — We set

(18)
$$\delta = \begin{cases} 1 & \text{if } L \subseteq \mathbb{C}, \\ 0 & \text{otherwise} \end{cases}$$

and say that a non-empty sequence $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ in L^n is almost orthogonal if it is linearly independent over L and satisfies

$$\operatorname{dist}(\mathbf{x}_j, \langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1} \rangle_L) \ge 1 - \delta/2^{j-1} \quad (2 \le j \le m).$$

Thus, almost orthogonal means orthogonal when L is non-archimedean. As in [17, §4], we note that any non-empty subsequence of an almost orthogonal sequence is almost orthogonal. Since $\prod_{j\geq 2}(1-\delta/2^{j-1}) \geq e^{-2\delta}$, we find the following estimate (cf. [17, Lemma 4.6]).

LEMMA 3.10. — For any almost orthogonal sequence $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ in L^n , we have

$$e^{-2\delta} \|\mathbf{x}_1\| \cdots \|\mathbf{x}_m\| \le \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_m\| \le \|\mathbf{x}_1\| \cdots \|\mathbf{x}_m\|.$$

The next crucial result is analogous to [17, Lemma 4.7]. It uses the convention that a hat on an element of a sequence or product means that this element is omitted from the sequence or product.

LEMMA 3.11. — Let k, ℓ , m be integers with $1 \leq k < \ell \leq m \leq n$ and let $\mathbf{y}_1, \ldots, \mathbf{y}_m$ be linearly independent points of L^n . Suppose that each of the sequences $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_\ell}, \ldots, \mathbf{y}_m)$ and $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_k}, \ldots, \mathbf{y}_m)$ is almost orthogonal. Then, the subspaces

$$V_1 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}_\ell}, \dots, \mathbf{y}_m \rangle_L$$
 and $V_2 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}_k}, \dots, \mathbf{y}_m \rangle_L$

that they span in L^n satisfy

$$\operatorname{dist}(V_1, V_2) \le e^{4\delta} \frac{\|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_m\|}{\|\mathbf{y}_1\| \cdots \|\mathbf{y}_m\|}$$

Proof. — Upon setting $\omega = \mathbf{y}_1 \wedge \cdots \wedge \widehat{\mathbf{y}_k} \wedge \cdots \wedge \widehat{\mathbf{y}_\ell} \wedge \cdots \wedge \mathbf{y}_m$, Lemma 3.7 gives

$$\operatorname{dist}(V_1, V_2) = \frac{\|\omega\| \|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_m\|}{\|\omega \wedge \mathbf{y}_\ell\| \|\omega \wedge \mathbf{y}_k\|}$$

The conclusion follows because, by Lemma 3.10,

$$\begin{aligned} \|\boldsymbol{\omega} \wedge \mathbf{y}_{\ell}\| \|\boldsymbol{\omega} \wedge \mathbf{y}_{k}\| &\geq e^{-4\delta} \big(\|\mathbf{y}_{1}\| \cdots \| \mathbf{y}_{k}\| \cdots \|\mathbf{y}_{m}\| \big) \big(\|\mathbf{y}_{1}\| \cdots \| \mathbf{y}_{\ell}\| \cdots \|\mathbf{y}_{m}\| \big) \\ &\geq e^{-4\delta} \|\boldsymbol{\omega}\| \big(\|\mathbf{y}_{1}\| \cdots \| \mathbf{y}_{m}\| \big). \end{aligned}$$

We conclude with a simple estimate.

LEMMA 3.12. — For any unit vectors $\mathbf{u}, \mathbf{u}' \in L^n$ and any $\mathbf{x} \in L^n$, we have

(19)
$$|\mathbf{x} \cdot \mathbf{u}| \le 2^{\delta} \max\{|\mathbf{x} \cdot \mathbf{u}'|, \|\mathbf{x}\| \operatorname{dist}(\mathbf{u}, \mathbf{u}')\}.$$

Proof. — We have $\|(\mathbf{x} \cdot \mathbf{u})\mathbf{u}' - (\mathbf{x} \cdot \mathbf{u}')\mathbf{u}\| \le \|\mathbf{x}\| \|\mathbf{u} \wedge \mathbf{u}'\|$ for any $\mathbf{x}, \mathbf{u}, \mathbf{u}' \in L^n$. \Box

When $\mathbf{x} \cdot \mathbf{u}' = 0$, this simply becomes $|\mathbf{x} \cdot \mathbf{u}| \le 2^{\delta} ||\mathbf{x}|| \operatorname{dist}(\mathbf{u}, \mathbf{u}')$.

4. Adelic geometry of numbers

For each place v of K, we form the compact set

$$\mathcal{O}_{\nu} = \{ x \in K_{\nu} ; |x|_{\nu} \le 1 \}.$$

When $\nu \nmid \infty$, this is the ring of integers of K_{ν} , and we follow MacFeat [13] in normalizing the Haar measure μ_{ν} on K_{ν} so that $\mu_{\nu}(\mathcal{O}_{\nu}) = 1$. When $\nu \mid \infty$, we set μ_{ν} to be the Lebesgue measure on K_{ν} , with $\mu_{\nu}(\mathcal{O}_{\nu}) = 2$ if $K_{\nu} = \mathbb{R}$, and $\mu_{\nu}(\mathcal{O}_{\nu}) = \pi$ if $K_{\nu} = \mathbb{C}$. We also denote by μ_{ν} the product measure on K_{ν}^{n} .

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The ring of adèles of K is the subring $K_{\mathbb{A}}$ of $\prod_{v \in M(K)} K_v$, which consists of the sequences (a_v) with $a_v \in \mathcal{O}_v$ for all but finitely v. It is endowed with the unique topology that extends the product topology on the set

$$\mathcal{O}_{\mathbb{A}} := \prod_{\nu \mid \infty} K_{\nu} \times \prod_{\nu \nmid \infty} \mathcal{O}_{\nu}$$

and makes $K_{\mathbb{A}}$ into a locally compact ring with $\mathcal{O}_{\mathbb{A}}$ as an open subring. Then K embeds in $K_{\mathbb{A}}$ as a discrete subring via the diagonal map. We denote by μ the Haar measure on $K_{\mathbb{A}}$ whose restriction to $\mathcal{O}_{\mathbb{A}}$ is the product of the μ_{ν} , and we use the same notation for the product measure on $K_{\mathbb{A}}^n$.

When $v \mid \infty$, a (Minkowski) convex body of K_v^n is any compact convex neighbourhood C_v of 0 such that $a C_v \subseteq C_v$ for each $a \in \mathcal{O}_v$. When $v \nmid \infty$, this is any finitely generated (thus free and compact) \mathcal{O}_v -submodule \mathcal{C}_v of K_v^n of rank n. Finally, a convex body of $K_{\mathbb{A}}^n$ is any product $\mathcal{C} = \prod_v \mathcal{C}_v$ where \mathcal{C}_v is a convex body of K_v^n for each v, and $\mathcal{C}_v = \mathcal{O}_v^n$ for all but finitely v. Then the induced topology on \mathcal{C} coincides with the usual product topology, and its volume $\mu(\mathcal{C}) = \prod_v \mu_v(\mathcal{C}_v)$ is finite and positive.

For each j = 1, ..., n, we define the *j*-th minimum $\lambda_j(\mathcal{C})$ of a convex body $\mathcal{C} = \prod_{\nu} \mathcal{C}_{\nu}$ of $K^n_{\mathbb{A}}$ to be the smallest $\lambda > 0$ for which the dilated convex body

$$\lambda \, \mathcal{C} = \prod_{\nu \mid \infty} \left(\lambda \, \mathcal{C}_{\nu} \right) \prod_{\nu \nmid \infty} \mathcal{C}_{\nu}$$

contains at least j linearly independent elements of K^n over K. With this notation, the adelic version of Minkowski's theorem reads as follows [13, 2].

THEOREM 4.1 (McFeat, 1971; Bombieri and Vaaler, 1983). — For any convex body C of $K^n_{\mathbb{A}}$, we have

$$(\lambda_1(\mathcal{C})\cdots\lambda_n(\mathcal{C}))^d \mu(\mathcal{C}) \asymp 1,$$

with implicit constants that depend only on K and n.

We refer the reader to [13, Theorems 5 and 6], [2, Theorems 3 and 6] and [24, Corollary of Theorem 1] for explicit lower bounds and upper bounds. In particular, this result implies that, if the volume $\mu(\mathcal{C})$ of \mathcal{C} is large enough, then $\lambda_1(\mathcal{C}) \leq 1$, and so \mathcal{C} contains a non-zero point of K^n .

More generally, fix an integer k with $1 \leq k \leq n$ and set $N = \binom{n}{k}$. The K-linear isomorphism from $\bigwedge^k K^n$ to K^N that sends a point to its Plücker coordinates extends to a K_{ν} -linear topological isomorphism from $\bigwedge^k K^n_{\nu}$ to K^N_{ν} and to a $K_{\mathbb{A}}$ -linear topological isomorphism from $\bigwedge^k K^n_{\mathbb{A}}$ to $K^N_{\mathbb{A}}$. Identifying these pairs of spaces, we obtain a measure μ_{ν} on $\bigwedge^k K^n_{\nu}$ for each $\nu \in M(K)$ and a measure μ on $\bigwedge^k K^n_{\mathbb{A}}$. This also provides the notion of a (Minkowski) convex body \mathcal{K}_{ν} of $\bigwedge^k K^n_{\nu}$ for each $\nu \in M(K)$ and of a convex body $\mathcal{K} = \prod_{\nu} \mathcal{K}_{\nu}$

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of $\bigwedge^k K^n_{\mathbb{A}}$, as well as the notion of the *j*-th minimum $\lambda_j(\mathcal{K})$ of \mathcal{K} with respect to $\bigwedge^k K^n$, for each j = 1, ..., N.

Let $\mathcal{C} = \prod_{\nu} \mathcal{C}_{\nu}$ be a convex body of $K^n_{\mathbb{A}}$. Its k-th compound is the convex body $\bigwedge^k \mathcal{C} = \prod_{\nu} (\bigwedge^k \mathcal{C}_{\nu})$ of $\bigwedge^k K^n_{\mathbb{A}}$ whose component $\bigwedge^k \mathcal{C}_{\nu}$ at a place ν is the smallest Minkowski convex body of $\bigwedge^k K^n_{\nu}$ containing all products $\mathbf{x}_1 \land \cdots \land \mathbf{x}_k$ of k elements $\mathbf{x}_1, \ldots, \mathbf{x}_k$ of \mathcal{C}_{ν} . In particular, we have $\bigwedge^1 \mathcal{C} = \mathcal{C}$. In this context, E. B. Burger extended Mahler's theory of compound bodies in [5]. Leaving out the explicit values of the constants from [5, Theorem 1.2], he showed that the minima of these convex bodies are related as follows.

THEOREM 4.2 (Burger, 1993). — With the above notation, order the N products $\lambda_{i_1}(\mathcal{C}) \cdots \lambda_{i_k}(\mathcal{C})$ with $1 \leq i_1 < \cdots < i_k \leq n$ into a non-decreasing sequence $\Lambda_1 \leq \cdots \leq \Lambda_N$. Then, for each $j = 1, \ldots, N$, we have

$$\lambda_j\left(\bigwedge^k \mathcal{C}\right) \asymp \Lambda_j,$$

with implicit constants depending only on K and n.

We note that $\Lambda_1 = \lambda_1(\mathcal{C}) \cdots \lambda_k(\mathcal{C})$, and that $\Lambda_2 = \lambda_1(\mathcal{C}) \cdots \lambda_{k-1}(\mathcal{C})\lambda_{k+1}(\mathcal{C})$ if k < n. Moreover, if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent elements of K^n which realize the successive minima of \mathcal{C} in the sense that $\mathbf{x}_i \in \lambda_i(\mathcal{C})\mathcal{C}$ for $i = 1, \ldots, n$, then $\mathbf{X} = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ belongs to $\Lambda_1 \bigwedge^k \mathcal{C}$. Thus, by the above theorem, the first minimum of $\bigwedge^k \mathcal{C}$ is realized up to a bounded factor by the pure product \mathbf{X} .

In practice, the compounds of a given convex body are difficult to compute exactly. So, we instead use approximations of them, like in the standard theory (see [21, Chapter IV, §7]).

5. Dilations

The group of idèles of K is the group $K^*_{\mathbb{A}}$ of invertible elements of $K_{\mathbb{A}}$. It contains the multiplicative group K^* of K as a subgroup. We define the module $|\boldsymbol{a}|_{\mathbb{A}}$ of an idèle $\boldsymbol{a} = (a_v) \in K^*_{\mathbb{A}}$ by

$$|\boldsymbol{a}|_{\mathbb{A}} = \prod_{v \in M(K)} |a_v|_v^{d_v/d} \in \mathbb{R}_{>0}$$

and recall that $|\alpha|_{\mathbb{A}} = 1$ for any $\alpha \in K^*$. Then, for each convex body $\mathcal{C} = \prod_{\nu} \mathcal{C}_{\nu}$ of $K^n_{\mathbb{A}}$, the product

$$\boldsymbol{a}\,\mathcal{C}=\prod_{\boldsymbol{v}\in M(K)}(a_{\boldsymbol{v}}\,\mathcal{C}_{\boldsymbol{v}})$$

is a convex body of volume $\mu(\boldsymbol{a} \, \mathcal{C}) = |\boldsymbol{a}|_{\mathbb{A}}^{dn} \mu(\mathcal{C})$. This construction extends the definition of $\lambda \mathcal{C}$ with $\lambda \in \mathbb{R}_{>0}$ by identifying any such λ with the idèle having component λ at each archimedean place $v \mid \infty$ and component 1 at all other places.

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DEFINITION 5.1. — For C as above and for each non-zero $\mathbf{x} \in K^n$, we set (20) $\lambda(\mathbf{x}, C) = \min\{|\boldsymbol{a}|_{\mathbb{A}}; \boldsymbol{a} \in K^*_{\mathbb{A}} \text{ and } \mathbf{x} \in \boldsymbol{a}C\}.$

For each j = 1, ..., n, we also define $\lambda_j^{\mathbb{A}}(\mathcal{C})$ to be the smallest $\lambda > 0$ for which there are at least j linearly independent elements \mathbf{x} of K^n with $\lambda(\mathbf{x}, \mathcal{C}) \leq \lambda$.

The minimum exists in (20) because for each $v \in M(K)$ there is a non-zero $a_v \in K_v$ with $|a_v|_v$ minimal, such that $\mathbf{x} \in a_v \mathcal{C}_v$, and we may choose $a_v = 1$ for all but finitely many v. Moreover, by the product formula, the value $\lambda(\mathbf{x}, \mathcal{C})$, which we view as a sort of distance from \mathbf{x} to \mathcal{C} , depends only on the class of \mathbf{x} in $\mathbb{P}^{n-1}(K)$. In particular, it is independent of \mathbf{x} if n = 1. We also note that, for any given t > 0, the non-zero points \mathbf{x} of K^n with $\lambda(\mathbf{x}, \mathcal{C}) \leq t$ have height at most ct for a constant c > 0 depending only on \mathcal{C} . So these points \mathbf{x} belong to finitely many classes in $\mathbb{P}^{n-1}(K)$, and for them $\lambda(\mathbf{x}, \mathcal{C})$ takes finitely many values in [0, t]. Hence, there is a basis $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of K^n over K such that $\lambda_j^{\mathbb{A}}(\mathcal{C}) = \lambda(\mathbf{x}_j, \mathcal{C})$ for each $j = 1, \ldots, n$. In particular, it is sensible to define each $\lambda_j^{\mathbb{A}}(\mathcal{C})$ as a minimum.

To compare these minima to those of MacFeat and Bombieri–Vaaler, we need the following special case of the strong approximation theorem from [10, Theorem 3].

LEMMA 5.2. — There exists a constant $c_1 = c_1(K) > 0$ with the following property. For each $\mathbf{a} = (a_v) \in K^*_{\mathbb{A}}$ with $|\mathbf{a}|_{\mathbb{A}} \ge c_1$, there exists $\alpha \in K^*$ such that $|\alpha|_v \le |a_v|_v$ for each $v \in M(K)$.

Note that this also follows from the adelic version of Minkowski's theorem, because, for given $\boldsymbol{a} = (a_v) \in K^*_{\mathbb{A}}$, the set of points $(x_v) \in K_{\mathbb{A}}$ with $|x_v|_v \leq |a_v|_v$ for all v is the convex body $\boldsymbol{a}\mathcal{B}$ of $K_{\mathbb{A}}$ of volume $|\boldsymbol{a}|^d_{\mathbb{A}}\mu(\mathcal{B})$, where $\mathcal{B} = \prod_v \mathcal{O}_v$. So, if $|\boldsymbol{a}|_{\mathbb{A}}$ is large enough, Theorem 4.1 gives $\lambda_1(\boldsymbol{a}\mathcal{B}) \leq 1$, and thus $\boldsymbol{a}\mathcal{B}$ contains some non-zero element of K.

PROPOSITION 5.3. — Let C be a convex body of $K^n_{\mathbb{A}}$ and let $j \in \{1, \ldots, n\}$. Then, we have

(21)
$$c_1^{-1}\lambda_j(\mathcal{C}) \le \lambda_j^{\mathbb{A}}(\mathcal{C}) \le \lambda_j(\mathcal{C}),$$

where c_1 comes from Lemma 5.2. Moreover, for each idèle $\mathbf{a} \in K^*_{\mathbb{A}}$, we also have

(22)
$$\lambda_j^{\mathbb{A}}(\boldsymbol{a}\,\mathcal{C}) = |\boldsymbol{a}|_{\mathbb{A}}^{-1}\lambda_j^{\mathbb{A}}(\mathcal{C}).$$

Proof. — Set $\lambda = \lambda_j^{\mathbb{A}}(\mathcal{C})$ and choose a set F of j linearly independent points \mathbf{x} of K^n with $\lambda(\mathbf{x}, \mathcal{C}) \leq \lambda$. Given $\mathbf{x} \in F$, there exists $\mathbf{a} \in K_{\mathbb{A}}^*$ with $|\mathbf{a}|_{\mathbb{A}} \leq \lambda$ such that $\mathbf{x} \in \mathbf{a}\mathcal{C}$. As the idèle $\mathbf{a}' = (a'_v) := c_1\lambda\mathbf{a}^{-1}$ satisfies $|\mathbf{a}'|_{\mathbb{A}} \geq c_1$, Lemma 5.2 provides $\alpha \in K^*$, such that $|\alpha|_v \leq |a'_v|_v$ for each $v \in M(K)$, and then the point $\alpha \mathbf{x}$ of K^n belongs to $\alpha \mathbf{a}\mathcal{C} \subseteq \mathbf{a}'\mathbf{a}\mathcal{C} = c_1\lambda\mathcal{C}$. Doing this for each $\mathbf{x} \in F$, we obtain

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j linearly independent points of K^n in $c_1 \lambda C$. This means that $\lambda_j(C) \leq c_1 \lambda$, which amounts to the first inequality in (21).

To prove the second inequality in (21), set $\lambda = \lambda_j(\mathcal{C})$. Then $\lambda \mathcal{C}$ contains at least *j* linearly independent elements of K^n . As $|\lambda|_{\mathbb{A}} = \lambda$, this implies that $\lambda_j^{\mathbb{A}}(\mathcal{C}) \leq \lambda$, and we are done.

Finally, (22) follows from the definitions and the multiplicativity of the module on $K^*_{\mathbb{A}}$.

In view of our identifications (see Section 4), the above results and definitions apply with \mathcal{C} replaced by any convex body $\mathcal{K} = \prod_{\nu} \mathcal{K}_{\nu}$ of $\bigwedge^{k} K^{n}_{\mathbb{A}}$ for any integer $1 \leq k \leq n$, provided that n is replaced by $N = \binom{n}{k}$ and that K^{n} is replaced by $\bigwedge^{k} K^{n}$.

6. A combinatorial result

In preparation for the proof of Theorem A in the next sections, we will need the following result from [17]. We refer the reader to Section 2.6 for the definition of an *n*-system.

PROPOSITION 6.1. — Let $c \ge 0$. Suppose that, for each k = 1, ..., n, there are continuous functions $L_k: [0, \infty) \to \mathbb{R}$ and $M_k: [0, \infty) \to \mathbb{R}$ which are piecewise linear with slopes 0 and 1 and which satisfy the following properties:

- (1) $0 \leq L_1(q) \leq \cdots \leq L_n(q) \leq q$ for each $q \geq 0$;
- (2) $|M_k(q) L_1(q) \dots L_k(q)| \le c \text{ for each } k = 1, \dots, n \text{ and each } q \ge 0;$
- (3) $M_n(q) = q$ for each $q \ge 0$;
- (4) if, for some integer k with $1 \le k < n$ and some q > 0, the function M_k changes slope from 1 to 0 at q, then $|L_{k+1}(q) L_k(q)| \le 2c$.

Choose $c' > 24n^3c$ and set

$$t_i = (1 + 2 + \dots + i)c'$$
 for $i = 0, 1, \dots, n$.

Then there exists an n-system $\mathbf{R} = (R_1, \ldots, R_n)$ on $[0, \infty)$, whose restriction to $[t_n, \infty)$ is rigid of mesh c', such that

(5) $\max_{1 \le k \le n} |L_k(q) - R_k(q)| \le 4n^2 c'$ for each $q \ge 0$;

(6) $\mathbf{R}(t_i) = (0, \dots, 0, c', 2c', \dots, ic')$ for each $i = 0, 1, \dots, n$.

Proof. — Define $M_0 = 0$ and $P_k = M_k - M_{k-1}$ for $k = 1, \ldots, n$. Also put $\gamma = 6c$. By adapting the proof of [17, Theorem 2.9], we find that the map $\mathbf{P} = (P_1, \ldots, P_n) \colon [0, \infty) \to \mathbb{R}^n$ is an (n, γ) -system in the sense of [17, Definition 2.8], with $|L_k(q) - P_k(q)| \leq \gamma$ for each $q \geq 0$ and $k = 1, \ldots, n$. Then, arguing as in the proof of [17, Theorem 8.2], we obtain a rigid *n*-system $\mathbf{R} = (R_1, \ldots, R_n) \colon [t_n, \infty) \to \mathbb{R}^n$ of mesh c', which satisfies condition (5) for each $q \geq t_n$. In particular, $\mathbf{R}(t_n)$ is a strictly increasing sequence of positive

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integer multiples of c' with sum t_n , and so $\mathbf{R}(t_n) = (c', 2c', \dots, nc')$. From this it follows that \mathbf{R} extends uniquely to an *n*-system on $[0, \infty)$ satisfying condition (6) (see the proof of [17, Theorem 8.1]). For each $k = 1, \dots, n$ and each $q \ge 0$, we have $0 \le L_k(q), R_k(q) \le q$, and thus $|L_k - R_k|$ is bounded above by $\max\{t_n, 4n^2c'\} = 4n^2c'$ on $[0, \infty)$.

Note that, for c = 0, the hypotheses of Proposition 6.1 amount to asking that the map $\mathbf{L} := (L_1, \ldots, L_n)$ itself be an *n*-system on $[0, \infty)$ (and that $M_k = L_1 + \cdots + L_k$ for each $k = 1, \ldots, n$). In fact, this is how *n*-systems are defined in [17, §2.5] (where they are called (n, 0)-systems). From this, we infer the following result of the approximation.

COROLLARY 6.2. — Let $\mathbf{L} = (L_1, \ldots, L_n)$ be an n-system on $[0, \infty)$, let c' > 0, and let $q_0 = (n^2 - n + 1)c'/2$. Then, there exists an n-system $\mathbf{R} = (R_1, \ldots, R_n)$ on $[0, \infty)$ whose restriction to $[q_0, \infty)$ is rigid of mesh c'/2, which satisfies $\max_{1 \le k \le n} |L_k(q) - R_k(q)| \le 4n^2c'$ for each $q \ge 0$, and for which R_1 has slope 1 on $[q_0, q_0 + c'/2]$.

Proof. — The conditions (1)–(4) of Proposition 6.1 are satisfied for the choice of c = 0 and of $M_k = L_1 + \cdots + L_k$ for each $k = 1, \ldots, n$. So its conclusion applies for the given c'. Consider the resulting *n*-system $\mathbf{R} = (R_1, \ldots, R_n)$ on $[0, \infty)$. On $[t_{n-1}, t_n]$, the union of the graphs of R_1, \ldots, R_n consists of n-1horizontal line segments of ordinates $c', 2c', \ldots, (n-1)c'$ and one line segment of slope 1 joining $(t_{n-1}, 0)$ to (t_n, nc') . Since $q_0 = t_{n-1} + c'/2$, we deduce that $\mathbf{R}(q_0) = (c'/2, c', 2c', \ldots, (n-1)c')$ and that R_1 has slope 1 on $[q_0, q_0 + c'/2]$. Finally, since \mathbf{R} is rigid of mesh c' on $[t_n, \infty)$, it is also rigid of mesh c'/2 on $[q_0, \infty)$. □

This corollary will be useful when it comes to approximate an *n*-system **L** by the map $\mathbf{L}_{\boldsymbol{\xi}}$ attached to a non-zero point $\boldsymbol{\xi} \in K_w^n$, because it reduces the problem to approximating an *n*-system **R** as in the corollary. The property that R_1 has slope 1 to the right of q_0 will simplify the argument.

7. From points to *n*-systems

The goal of this section is to prove the first and last assertions of Theorem A. To this end, we fix an integer $n \ge 2$, a place $w \in M(K)$, and a non-zero point $\boldsymbol{\xi} \in K_w^n$. As $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{L}_{\boldsymbol{\xi}}^*$ depend only on the line $\langle \boldsymbol{\xi} \rangle_{K_w}$ spanned by $\boldsymbol{\xi}$ in K_w^n , we assume, to simplify the computations, that

$$\|\boldsymbol{\xi}\|_{w} = 1.$$

Using the general strategy of Schmidt and Summerer in [22], we will show that the components L_1, \ldots, L_n of $\mathbf{L}_{\boldsymbol{\xi}}$ satisfy the hypotheses of Proposition 6.1 for some choice of functions M_1, \ldots, M_n and some constant $c = c(K, w, n) \geq 1$. This will ensure the existence of an *n*-system $\mathbf{P}: [0, \infty) \to \mathbb{R}^n$ for which the

difference $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded, and we will show that this is equivalent to $\mathbf{L}_{\boldsymbol{\xi}}^* - \mathbf{P}^*$ being bounded. The precise argument given below is adapted from [17, §2]. In all estimates, the implicit constants involved in the symbol \approx depend only on K, w and n. We also use the convention that a hat on an element of a sequence or product means that this element is omitted from the sequence or product.

For each $k \in \{1, \ldots, n-1\}$, there is a unique bilinear map

$$\begin{array}{c} K_w^n \times \bigwedge^k K_w^n \longrightarrow \bigwedge^{k-1} K_w^n \\ (\mathbf{y}, \mathbf{X}) \longmapsto \mathbf{y} \,\lrcorner\, \mathbf{X} \end{array}$$

called *contraction*, which satisfies

(23)
$$\mathbf{y} \lrcorner (\mathbf{x}_1 \land \dots \land \mathbf{x}_k) = \sum_{i=1}^k (-1)^{i-1} (\mathbf{y} \cdot \mathbf{x}_i) \mathbf{x}_1 \land \dots \land \widehat{\mathbf{x}_i} \land \dots \land \mathbf{x}_k,$$

for any $\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_k \in K_w^n$. For k = 1, this is the dot product $\mathbf{y} \perp \mathbf{x} = \mathbf{y} \cdot \mathbf{x}$. We use this to define a map $\mathbf{L}_{\boldsymbol{\xi}}^{(k)}$ as follows.

DEFINITION 7.1. — Let $k \in \{1, ..., n-1\}$ and let $N = \binom{n}{k}$. For each non-zero $\mathbf{X} \in \bigwedge^k K^n$, we set

$$D_{\boldsymbol{\xi}}(\mathbf{X}) = \|\boldsymbol{\xi} \,\lrcorner\, \mathbf{X}\|_{w}^{d_{w}/d} \prod_{v \neq w} \|\mathbf{X}\|_{v}^{d_{v}/d}.$$

As $\|\boldsymbol{\xi}\|_w = 1$, this agrees with the definition of Section 2.4 for k = 1. We also define a map $L_{\boldsymbol{\xi}}(\mathbf{X}, \cdot) \colon [0, \infty) \to \mathbb{R}$ by

(24)
$$L_{\boldsymbol{\xi}}(\mathbf{X},q) = \max\left\{\log H(\mathbf{X}), \, q + \log D_{\boldsymbol{\xi}}(\mathbf{X})\right\} \quad (q \ge 0).$$

For each j = 1, ..., N and $q \ge 0$, we denote by $L_{\boldsymbol{\xi},j}^{(k)}(q)$ the smallest real number $t \ge 0$ for which there exist at least j linearly independent elements \mathbf{X} of $\bigwedge^k K^n$ for which $L_{\boldsymbol{\xi}}(\mathbf{X}, q) \le t$ or equivalently for which

$$H(\mathbf{X}) \le e^t$$
 and $D_{\boldsymbol{\xi}}(\mathbf{X}) \le e^{t-q}$

Finally, we define $\mathbf{L}_{\boldsymbol{\xi}}^{(k)} \colon [0,\infty) \to \mathbb{R}^N$ by $\mathbf{L}_{\boldsymbol{\xi}}^{(k)}(q) = \left(L_{\boldsymbol{\xi},1}^{(k)}(q), \ldots, L_{\boldsymbol{\xi},N}^{(k)}(q)\right)$ for each $q \ge 0$.

Since, for a non-zero $\mathbf{X} \in \bigwedge^k K^n$, the numbers $H(\mathbf{X})$ and $D_{\boldsymbol{\xi}}(\mathbf{X})$ depend only on the class of \mathbf{X} in projective space on $\bigwedge^k K^n$, and since for each $B \ge 1$ there are finitely many classes of height at most B, each number $L_{\boldsymbol{\xi},j}^{(k)}(q)$ can, indeed, be defined as a minimum. For k = 1, we recover $\mathbf{L}_{\boldsymbol{\xi}}^{(1)} = \mathbf{L}_{\boldsymbol{\xi}}$. The first step is to compare these maps with the minima of the following families of adelic convex bodies.

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DEFINITION 7.2. — Let k and N be as in Definition 7.1. For each $q \ge 0$, we denote by $\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)$ the convex body of $\bigwedge^k K^n_{\mathbb{A}}$, which consists of the points $\mathbf{X} = (\mathbf{X}_v)$ satisfying

$$\|\mathbf{X}_{v}\|_{v} \leq 1$$
 for each $v \in M(K)$ and $\|\boldsymbol{\xi} \,\lrcorner\, \mathbf{X}_{w}\|_{w} \leq e^{-qd/d_{w}}$

We also set $C_{\boldsymbol{\xi}}(q) = C_{\boldsymbol{\xi}}^{(1)}(q)$.

Thus, $C_{\boldsymbol{\xi}}(q)$ consists of the points $(\mathbf{x}_{\nu}) \in K^n_{\mathbb{A}}$ satisfying

 $\|\mathbf{x}_{v}\|_{v} \leq 1$ for each $v \in M(K)$ and $\|\mathbf{x}_{w} \cdot \boldsymbol{\xi}\|_{w} \leq e^{-qd/d_{w}}$.

Its volume is $\mu(\mathcal{C}_{\boldsymbol{\xi}}(q)) \simeq e^{-qd}$. Applying Definition 5.1, we first obtain the following estimate.

LEMMA 7.3. — Let $k \in \{1, \ldots, n-1\}$, let $\mathbf{X} \in \bigwedge^k K^n \setminus \{0\}$ and let $q \ge 0$. Then, we have

$$\lambda(\mathbf{X}, \mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q)) \asymp \max\{H(\mathbf{X}), e^q D_{\boldsymbol{\xi}}(\mathbf{X})\} = \exp(L_{\boldsymbol{\xi}}(\mathbf{X}, q)).$$

Proof. — An idèle $\boldsymbol{a} = (a_v)$ of K of smallest module such that $\mathbf{X} \in \boldsymbol{aC}_{\boldsymbol{\xi}}^{(k)}(q)$ has $|a_v|_v = \|\mathbf{X}\|_v$ for each place $v \neq w$, and $|a_w|_w = \max\{\|\mathbf{X}\|_w, r\|\boldsymbol{\xi} \, \lrcorner \, \mathbf{X}\|_w\}$, where r is the smallest element of the valuation group $|K_*^*|_w$ at w with $r \geq e^{qd/d_w}$. The estimate follows since $r \asymp e^{qd/d_w}$ and $\lambda(\mathbf{X}, \mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q)) = |\boldsymbol{a}|_{\mathbb{A}}$ for such \boldsymbol{a} . \Box

LEMMA 7.4. — Let $k \in \{1, \ldots, n-1\}$ and $N = \binom{n}{k}$. For each $q \ge 0$, we have

(25)
$$0 \leq L_{\boldsymbol{\xi},1}^{(k)}(q) \leq \cdots \leq L_{\boldsymbol{\xi},N}^{(k)}(q) \leq q, \\ \exp(L_{\boldsymbol{\xi},j}^{(k)}(q)) \asymp \lambda_j(\mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q)) \quad (1 \leq j \leq N).$$

Moreover, the functions $L_{\boldsymbol{\xi},j}^{(k)}$ are continuous and piecewise linear with slopes 0 and 1 on $[0,\infty)$. Finally, if $L_{\boldsymbol{\xi},1}^{(k)}$ changes slope from 1 to 0 at a point q > 0, then $L_{\boldsymbol{\xi},1}^{(k)}(q) = L_{\boldsymbol{\xi},2}^{(k)}(q)$.

Proof. — For given $q \ge 0$ and $j \in \{1, \ldots, N\}$, the number $L^{(k)}_{\boldsymbol{\xi},j}(q)$ (resp. $\lambda_j^{\mathbb{A}}(\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q))$) is, by definition, the minimum of

(26)
$$\max_{\mathbf{X}\in E} \exp(L_{\boldsymbol{\xi}}(\mathbf{X},q)) \quad \Big(\text{ resp. } \max_{\mathbf{X}\in E} \lambda(\mathbf{X}, \mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q)) \Big),$$

where E runs through all sets of j linearly independent elements of $\bigwedge^k K^n$. Taking for E a set of j products $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the canonical basis of K^n , we deduce that $L_{\boldsymbol{\xi},j}^{(k)}(q) \leq q$ because those products \mathbf{e} have $D_{\boldsymbol{\xi}}(\mathbf{e}) \leq 1 = H(\mathbf{e})$ as $\|\boldsymbol{\xi}\|_w = 1$. This yields the first set of inequalities in (25). Using Lemma 7.3, we also deduce that

 $\exp(L_{\boldsymbol{\xi},j}^{(k)}(q)) \asymp \lambda_j^{\mathbb{A}}(\mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q))$. The second set of estimates in (25) then follows using Proposition 5.3 with $\bigwedge^k K_{\mathbb{A}}^n$ identified to $K_{\mathbb{A}}^N$.

Fix Q > 0. For each $q \in [0, Q]$, we have $L_{\boldsymbol{\xi}, j}^{(k)}(q) \leq q \leq Q$. Thus, in computing $L_{\boldsymbol{\xi}, j}^{(k)}(q)$ on [0, Q] in terms of the projective invariants (26), it suffices to choose E inside a set F of representatives in $\bigwedge^k K^n$ of points of $\mathbb{P}(\bigwedge^k K^n)$ of height at most e^Q . Since F is finite, we deduce that $L_{\boldsymbol{\xi}, j}^{(k)}$ is continuous and piecewise linear with slopes 0 and 1 on [0, Q]. As Q can be taken arbitrarily large, this property extends to $[0, \infty)$. Finally, if $L_{\boldsymbol{\xi}, 1}^{(k)}$ changes slope from 1 to 0 at a point q > 0, there exist $\epsilon > 0$ and $\mathbf{X}, \mathbf{Y} \in \bigwedge^k K^n$, such that

$$L_{\boldsymbol{\xi},1}^{(k)}(t) = \begin{cases} L_{\boldsymbol{\xi}}(\mathbf{X},t) = t + \log D_{\boldsymbol{\xi}}(\mathbf{X}) & \text{for } q - \epsilon \le t \le q, \\ L_{\boldsymbol{\xi}}(\mathbf{Y},t) = \log H(\mathbf{Y}) & \text{for } q \le t \le q + \epsilon. \end{cases}$$

Thus, **X**, **Y** are linearly independent, and so $L_{\boldsymbol{\xi},1}^{(k)}(q) = L_{\boldsymbol{\xi},2}^{(k)}(q)$.

The next lemma compares the convex body $C_{\boldsymbol{\xi}}^{(k)}(q)$ with the k-th compound of $\mathcal{C}_{\boldsymbol{\xi}}(q)$.

LEMMA 7.5. — Let k and N be as in Lemma 7.4. For each $q \ge 0$, we have

(27)
$$\bigwedge^{k} \mathcal{C}_{\boldsymbol{\xi}}(q) \subseteq k \mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q) \quad and \quad \mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q) \subseteq N \bigwedge^{k} \mathcal{C}_{\boldsymbol{\xi}}(q).$$

Proof. — Fix a choice of $q \ge 0$ and, for simplicity, set

$$\mathcal{C} := \mathcal{C}_{\boldsymbol{\xi}}(q) = \prod_{\nu} \mathcal{C}_{\nu} \text{ and } \mathcal{C}^{(k)} := \mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q) = \prod_{\nu} \mathcal{C}^{(k)}_{\nu}.$$

Let $v \in M(K)$ and let $\mathbf{X}_v = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ with $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{C}_v$. We find

$$\|\mathbf{X}_{v}\|_{v} \leq \|\mathbf{x}_{1}\|_{v} \cdots \|\mathbf{x}_{k}\|_{v} \leq 1,$$

thus, $\mathbf{X}_{v} \in \mathcal{C}_{v}^{(k)}$ if $v \neq w$. If v = w, formula (23) also yields $\|\boldsymbol{\xi} \,|\, \mathbf{X}_{w}\|_{w} \leq k^{\delta} e^{-qd/d_{w}}$, where $\delta = 1$ if $w \mid \infty$ and $\delta = 0$ else, thus, $\mathbf{X}_{w} \in k^{\delta} \mathcal{C}_{w}^{(k)}$. This implies the first inclusion in (27).

To prove the second inclusion, it suffices to show that each $\mathbf{X}_{\nu} \in C_{\nu}^{(k)}$ can be written as a sum of N products of k elements of C_{ν} . For $\nu \neq w$, this is immediate since C_{ν} contains $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and since the products $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$ form an orthonormal basis of K_{ν}^n . As $\|\mathbf{X}_{\nu}\|_{\nu} \leq 1$, the N coordinates of \mathbf{X}_{ν} in this basis have absolute values of at most 1, and we are done. For $\nu = w$, we complete $\mathbf{u}_1 = \boldsymbol{\xi}$ into an orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ of K_w^n and we form its dual basis $(\mathbf{u}_1^*, \ldots, \mathbf{u}_n^*)$ with respect to the dot product. By Lemma 3.2, this new basis is also orthonormal. Moreover, we have $\boldsymbol{\xi} \cdot \mathbf{u}_1^* = 1$

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and $\boldsymbol{\xi} \cdot \mathbf{u}_i^* = 0$ for i = 2, ..., n. Thus, \mathcal{C}_w contains $\mathbf{u}_2^*, ..., \mathbf{u}_n^*$, as well as $c\mathbf{u}_1^*$ for any $c \in K_w$ with $|c|_w \leq e^{-qd/d_w}$. Upon writing

$$\mathbf{X}_{w} = \sum_{1 \leq i_{1} < \cdots < i_{k} \leq n} c_{i_{1}, \dots, i_{k}} \mathbf{u}_{i_{1}}^{*} \wedge \cdots \wedge \mathbf{u}_{i_{k}}^{*},$$

we find

$$\boldsymbol{\xi} \,\lrcorner\, \mathbf{X}_w = \sum_{1 < i_2 < \cdots < i_k \leq n} c_{1,i_2,\dots,i_k} \mathbf{u}_{i_2}^* \wedge \cdots \wedge \mathbf{u}_{i_k}^*.$$

As $\|\mathbf{X}_w\|_w \leq 1$ and $\|\boldsymbol{\xi} \,\lrcorner\, \mathbf{X}_w\|_w \leq e^{-qd/d_w}$, we deduce that $|c_{i_1,\ldots,i_k}|_w$ is bounded above by e^{-qd/d_w} if $i_1 = 1$, and by 1 otherwise, so we are done.

We can now prove the following part of Theorem A.

PROPOSITION 7.6. — There is an n-system $\mathbf{P} \colon [0, \infty) \to \mathbb{R}^n$ such that $\|\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}\|$ is uniformly bounded by a constant depending only on K, w and n.

Proof. — Set $M_k = L_{\boldsymbol{\xi},1}^{(k)}$ for $k = 1, \ldots, n-1$ and define $M_n(q) = q$ for each $q \geq 0$. It suffices to show that these functions and the functions $L_k = L_{\boldsymbol{\xi},k}$ satisfy all the hypotheses of Proposition 6.1 for some constant c = c(K, w, n). By Lemma 7.4, we only have to verify conditions (2) and (4) of that proposition. Theorem 4.1 gives

(28)
$$\lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q)) \cdots \lambda_n(\mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp \mu(\mathcal{C}_{\boldsymbol{\xi}}(q))^{-1/d} \asymp e^q,$$

while for each k = 1, ..., n - 1, Theorem 4.2 provides

$$\lambda_1(\bigwedge^k \mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp \lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q)) \cdots \lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q)),$$

$$\lambda_2(\bigwedge^k \mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp \lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q)) \cdots \lambda_{k-1}(\mathcal{C}_{\boldsymbol{\xi}}(q))\lambda_{k+1}(\mathcal{C}_{\boldsymbol{\xi}}(q)).$$

The inclusions of Lemma 7.5 combined with (25) also imply that

$$\lambda_j \left(\bigwedge^k \mathcal{C}_{\boldsymbol{\xi}}(q) \right) \asymp \lambda_j \left(\mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q) \right) \asymp \exp(L_{\boldsymbol{\xi},j}^{(k)}(q)) \quad \text{for } j = 1, \dots, \binom{n}{k}.$$

Taking logarithms, we obtain the inequalities of Proposition 6.1(2), as well as

$$|L_{\boldsymbol{\xi},2}^{(k)}(q) - L_1(q) - \dots - L_{k-1}(q) - L_{k+1}(q)| \le c \quad (1 \le k < n, \ q \ge 0),$$

for some constant c = c(K, w, n). Finally, if, for some k < n, the function M_k changes slope from 1 to 0 at a point q > 0, Lemma 7.4 gives

$$L_{\boldsymbol{\xi},2}^{(k)}(q) = L_{\boldsymbol{\xi},1}^{(k)}(q) = M_k(q).$$

Then comparing the last estimate with that of Proposition 6.1 (2), we obtain $|L_{k+1}(q) - L_k(q)| \leq 2c$. Thus, condition (4) of Proposition 6.1 holds as well. \Box

To compare approximation to $\mathbf{L}_{\boldsymbol{\xi}}$ by an *n*-system **P** and approximation to $\mathbf{L}_{\boldsymbol{\xi}}^*$ by the dual map \mathbf{P}^* defined by (10), we first note the following equality.

LEMMA 7.7. — We have $\mathbf{L}_{\boldsymbol{\xi}}^{(n-1)} = \mathbf{L}_{\boldsymbol{\xi}}^*$.

Proof. — Consider the K-linear map $\varphi \colon K^n \to \bigwedge^{n-1} K^n$ given by $\varphi(\mathbf{x}) = \mathbf{x} \sqcup (\mathbf{e}_1 \land \dots \land \mathbf{e}_n),$

for each $\mathbf{x} \in K^n$. Writing $\mathbf{x} = (x_1, \ldots, x_n)$, we find that

$$\varphi(\mathbf{x}) = \sum_{i=1}^{n} (-1)^{i-1} x_i \mathbf{e}_1 \wedge \cdots \wedge \widehat{\mathbf{e}_i} \wedge \cdots \wedge \mathbf{e}_n.$$

Thus, φ is an isomorphism and, for each place v of K, it extends to a K_v -linear isometry $\varphi_v \colon K_v^n \to \bigwedge^{n-1} K_v^n$ given by the same formulas. Moreover, a short computation shows that

$$\|\boldsymbol{\xi} \,\lrcorner\, \varphi_w(\mathbf{x})\|_w = \|\mathbf{x} \wedge \boldsymbol{\xi}\|_w$$

for each $\mathbf{x} \in K_w^n$. Thus, for each non-zero $\mathbf{x} \in K^n$, we have

$$H(\varphi(\mathbf{x})) = H(\mathbf{x}) \quad \text{and} \quad D_{\boldsymbol{\xi}}(\varphi(\mathbf{x})) = D_{\boldsymbol{\xi}}^*(\mathbf{x}),$$

and the conclusion follows since φ is an isomorphism.

LEMMA 7.8. — There is a constant $c^* = c^*(K, w, n)$ such that

$$|L^*_{\xi,j}(q) + L_{\xi,k}(q) - q| \le c^*$$

for each $q \ge 0$ and each $j, k \in \{1, \ldots, n\}$ with j + k = n + 1.

Proof. — For q, j and k as above, Theorem 4.2 combined with (28) provides

$$\lambda_j \left(\bigwedge^{n-1} \mathcal{C}_{\boldsymbol{\xi}}(q) \right) \asymp \lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q)) \cdots \widehat{\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q))} \cdots \lambda_n(\mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp \frac{e^q}{\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q))}.$$

Using Lemmas 7.5, 7.4 and 7.7 in this order, we also find

$$\lambda_j \left(\bigwedge^{n-1} \mathcal{C}_{\boldsymbol{\xi}}(q) \right) \asymp \lambda_j \left(\mathcal{C}_{\boldsymbol{\xi}}^{(n-1)}(q) \right) \asymp \exp(L_{\boldsymbol{\xi},j}^{(n-1)}(q)) = \exp(L_{\boldsymbol{\xi},j}^*(q)),$$

while $\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q)) \simeq \exp(L_{\boldsymbol{\xi},k}(q))$. The conclusion follows by taking logarithms.

We deduce the following complement to Proposition 7.6.

PROPOSITION 7.9. — Let $\mathbf{P} = (P_1, \ldots, P_n)$ be an n-system on $[0, \infty)$. If one of the conditions (11) from Theorem A holds with a constant c, then the other holds with c replaced by $c + c^* \sqrt{n}$, where c^* comes from Lemma 7.8.

Proof. — With q, j and k as in Lemma 7.8, the definition of \mathbf{P}^* in (10) gives $P_i^*(q) + P_k(q) = q$. Hence, the inequality of the lemma may be restated as

$$\left| (L^*_{\boldsymbol{\xi},j}(q) - P^*_j(q)) + (L_{\boldsymbol{\xi},k}(q) - P_k(q)) \right| \le c^*,$$

and the result follows.

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We conclude this section with the following observation.

LEMMA 7.10. — For $q \ge 0$, define $\mathcal{C}^*_{\boldsymbol{\xi}}(q)$ to be the set of points $(\mathbf{x}_v) \in K^n_{\mathbb{A}}$ satisfying

$$\|\mathbf{x}_{v}\|_{v} \leq 1$$
 for each $v \in M(K)$ and $\|\mathbf{x}_{w} \wedge \boldsymbol{\xi}\|_{w} \leq e^{-qd/d_{w}}$.

Then $\lambda_j(\mathcal{C}^*_{\boldsymbol{\xi}}(q)) = \lambda_j(\mathcal{C}^{(n-1)}_{\boldsymbol{\xi}}(q)) \approx \exp(L^{(n-1)}_{\boldsymbol{\xi},j}(q)) = \exp(L^*_{\boldsymbol{\xi},j}(q))$, for each $j = 1, \ldots, n$.

Proof. — Going back to the proof of Lemma 7.7, we find that the $K_{\mathbb{A}}$ -linear isomorphism $\varphi_{\mathbb{A}}: K_{\mathbb{A}}^n \to \bigwedge^{n-1} K_{\mathbb{A}}^n$, which extends φ , maps $\mathcal{C}^*_{\boldsymbol{\xi}}(q)$ to $\mathcal{C}^{(n-1)}_{\boldsymbol{\xi}}(q)$ for each $q \geq 0$. Thus, these convex bodies have the same minima. The remaining estimates follow from (25) in Lemma 7.4 and from Lemma 7.7.

8. Construction of bases

As in the preceding section, we assume $n \ge 2$ and we fix a place $w \in M(K)$. We also set

$$S = M_{\infty}(K) \cup \{w\}$$

and denote by $\mathcal{O}_S = \bigcap_{v \notin S} (K \cap \mathcal{O}_v)$ the ring of S-integers of K. The goal of this section is to provide a general recursive construction of bases of \mathcal{O}_S^n as an \mathcal{O}_S -module. In the next section, we will use it to complete the proof of Theorem A. The general strategy is similar to that of [17, §5] but complicated by the fact that we need these bases to obey several properties at each place of S. In particular, we will need them to be almost orthogonal in K_v^n for each $v \in S \setminus \{w\}$, in the sense of Section 3.4 for $L = K_v$. We start by recalling two general results of approximation by elements of \mathcal{O}_S within $\prod_{v \in S} K_v$.

The group of S-units of K is the group \mathcal{O}_S^* of invertible elements of \mathcal{O}_S . It is well-known that its image under the logarithmic embedding, namely the set of points $(\log |\varepsilon|_v)_{v \in S}$ with $\varepsilon \in \mathcal{O}_S^*$, forms a lattice within the hyperplane of \mathbb{R}^S of points $(x_v)_{v \in S}$ with $\sum_{v \in S} d_v x_v = 0$. Thus, there is a constant $c_2 = c_2(K, S) \ge 1$ with the following property.

LEMMA 8.1. — For any choice of positive real numbers $(r_v)_{v\in S}$ with $\prod_{v\in S} r_v^{d_v} = 1$, there exists an S-unit $\varepsilon \in \mathcal{O}_S^*$ which satisfies $c_2^{-1}r_v \leq |\varepsilon|_v \leq c_2r_v$ for each $v \in S$.

From [10, Theorem 3] of Mahler, there is also a constant $c_3 = c_3(K, S) \ge 1$ with the following property.

LEMMA 8.2. — For any $(a_v)_{v \in S} \in \prod_{v \in S} K_v$ and any family of positive real numbers $(t_v)_{v \in S}$ with $\prod_{v \in S} t_v^{d_v/d} \ge c_3$, there exists an S-integer $\alpha \in \mathcal{O}_S$ which satisfies $|\alpha - a_v|_v \le t_v$ for each $v \in S$.

In fact, the result of Mahler shows this with a constant c_3 that depends only on K when each t_v belongs to the valuation group of K_v . As we do not require this, our constant c_3 depends on S as well, but in a weak form.

From now on, we fix a constant C in the valuation group of K_w , with

(29)
$$C \ge n2^{n+1}(c_3c_4)^{d/d_w}$$
 where $c_4 = n2^n(2ec_2)^2$

In agreement with the notation of Section 3.4 for $L = K_w$, we set

$$\delta = \begin{cases} 1 & \text{if } w \mid \infty, \\ 0 & \text{otherwise.} \end{cases}$$

For the other places $v \in S \setminus \{w\}$, no special notation is needed since they are all archimedean, and so the results of that section apply to $L = K_v$ with $\delta = 1$. For convenience, we also define

(30)
$$\Delta = \{(a_1, \dots, a_n) \in \mathbb{Z}^n ; 0 \le a_1 \le \dots \le a_n\}.$$

We are interested in bases of \mathcal{O}_S^n with three kinds of properties.

DEFINITION 8.3. — Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ be a basis of \mathcal{O}_S^n over \mathcal{O}_S . We say that

- **x** is admissible if, for any $v \in S \setminus \{w\}$, it is almost orthogonal in K_v^n and satisfies $1 \leq ||\mathbf{x}_j||_v \leq (2ec_2)^2$ for j = 1, ..., n;
- **x** has size $\boldsymbol{a} = (a_1, \ldots, a_n) \in \Delta$ if $C^{a_j} \leq \|\mathbf{x}_j\|_w \leq (1+\delta)C^{a_j}$ for $j = 1, \ldots, n;$
- **x** has type (k, ℓ) for integers $1 \le k < \ell \le n$ if

$$\operatorname{dist}_{w}\left(\mathbf{x}_{\ell}, \langle \mathbf{x}_{1}, \ldots, \widehat{\mathbf{x}_{k}}, \ldots, \mathbf{x}_{\ell-1} \rangle_{K_{w}}\right) \geq 1 - \frac{1}{2^{\ell-1}}.$$

We start with two quick consequences.

LEMMA 8.4. — Suppose that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is an admissible basis of \mathcal{O}_S^n over \mathcal{O}_S of size $\mathbf{a} = (a_1, \ldots, a_n) \in \Delta$. Then, for each index $j = 1, \ldots, \ell$, we have $C^{a_j d_w/d} \leq H(\mathbf{x}_j) \leq c_5 C^{a_j d_w/d}$, where $c_5 = (2ec_2)^2$.

Proof. — Let $j \in \{1, ..., n\}$. For each place v of K not in S, the *n*-tuple \mathbf{x} is a basis of \mathcal{O}_{v}^{n} over \mathcal{O}_{v} , hence $\|\mathbf{x}_{j}\|_{v} = 1$. This yields $H(\mathbf{x}_{j}) = \prod_{v \in S} \|\mathbf{x}_{j}\|_{v}^{d_{v}/d}$, and the conclusion follows.

LEMMA 8.5. — Let k, ℓ , m be integers with $1 \leq k < \ell \leq m \leq n$ and let $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ be an admissible basis of \mathcal{O}_S^n over \mathcal{O}_S . Suppose that the subsequences $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_\ell}, \ldots, \mathbf{y}_m)$ and $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_k}, \ldots, \mathbf{y}_m)$ are both almost orthogonal in K_w^n . Then, the subspaces

$$V_1 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}_\ell}, \dots, \mathbf{y}_m \rangle_{K_w} \quad and \quad V_2 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}_k}, \dots, \mathbf{y}_m \rangle_{K_w}$$

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that they span in K_w^n satisfy

$$\operatorname{dist}_{w}(V_{1}, V_{2})^{d_{w}/d} \leq e^{4} \frac{H(\langle \mathbf{y}_{1}, \dots, \mathbf{y}_{m} \rangle_{K})}{H(\mathbf{y}_{1}) \cdots H(\mathbf{y}_{m})}.$$

Proof. — Using Lemma 3.11 with $L = K_w$, we find

$$\operatorname{dist}_{w}(V_{1}, V_{2})^{d_{w}/d} \leq \left(e^{4\delta} \frac{\|\mathbf{y}_{1} \wedge \dots \wedge \mathbf{y}_{m}\|_{w}}{\|\mathbf{y}_{1}\|_{w} \cdots \|\mathbf{y}_{m}\|_{w}}\right)^{d_{w}/d} = A \frac{H(\langle \mathbf{y}_{1}, \dots, \mathbf{y}_{m} \rangle_{K})}{H(\mathbf{y}_{1}) \cdots H(\mathbf{y}_{m})},$$

where

$$A = e^{4\delta d_w/d} \prod_{\nu \in S \setminus \{w\}} \left(\frac{\|\mathbf{y}_1\|_{\nu} \cdots \|\mathbf{y}_m\|_{\nu}}{\|\mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_m\|_{\nu}} \right)^{d_\nu/d}.$$

Since \mathbf{y} is an admissible basis of \mathcal{O}_S^n , the sequence $(\mathbf{y}_1, \ldots, \mathbf{y}_m)$ is almost orthogonal in K_v^n for each place $v \in S$ other than w. As such a place v is archimedean, Lemma 3.10 shows that the corresponding factor of A is bounded above by $e^{2d_v/d}$. This gives $A \leq e^4$.

The main result of this section is the following construction which, in essence, generalizes [17, Lemma 5.1]. The crucial novelty is the introduction of an S-unit ε in condition (3) below (in the context of [17] where $K = \mathbb{Q}$ and $S = \{\infty\}$, it would simply be $\varepsilon = \pm 1$).

LEMMA 8.6. — Let h, k, ℓ be integers with

 $1 \le k < \ell \le n$ and $1 \le h \le \ell \le n$.

Suppose that elements $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ of Δ satisfy

(1) $b_{\ell} > a_{\ell}$ and $(b_1, \ldots, \widehat{b_{\ell}}, \ldots, b_n) = (a_1, \ldots, \widehat{a_h}, \ldots, a_n).$

Suppose, moreover, that $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an admissible basis of \mathcal{O}_S^n over \mathcal{O}_S of size \mathbf{a} . Then there exists an admissible basis $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$ of \mathcal{O}_S^n over \mathcal{O}_S of size \mathbf{b} and type (k, ℓ) such that

- (2) $(\mathbf{y}_1,\ldots,\widehat{\mathbf{y}_\ell},\ldots,\mathbf{y}_n) = (\mathbf{x}_1,\ldots,\widehat{\mathbf{x}_h},\ldots,\mathbf{x}_n),$
- (3) $\mathbf{y}_{\ell} \in \varepsilon \mathbf{x}_h + \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}_h}, \dots, \mathbf{x}_{\ell} \rangle_{\mathcal{O}_S}$ for some $\varepsilon \in \mathcal{O}_S^*$.

Proof. — We use (2) as a definition of $\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_\ell}, \ldots, \mathbf{y}_n$. Then, the *n*-tuple $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ is a basis of \mathcal{O}_S^n over \mathcal{O}_S , for any choice of \mathbf{y}_ℓ satisfying (3).

We set $r_v = 2e^2c_2 \|\mathbf{x}_h\|_v^{-1}$ for each $v \in S \setminus \{w\}$, and define r_w by the condition $\prod_{v \in S} r_v^{d_v} = 1$. Since \mathbf{x} is admissible, we have $r_v \geq (2c_2)^{-1}$ for $v \neq w$, so $r_w^{d_w} \leq (2c_2)^d$. Then Lemma 8.1 provides an S-unit $\varepsilon \in \mathcal{O}_S^*$ with

(31)
$$2e^2 \le \|\varepsilon \mathbf{x}_h\|_v \le 2e^2c_2^2 \text{ for each } v \in S \setminus \{w\},$$

(32) $|\varepsilon|_{w}^{d_{w}} \leq (c_{2}r_{w})^{d_{w}} \leq (2c_{2})^{2d}.$

For each $v \in S$, we define

$$U_{\nu} = \langle \mathbf{x}_{1}, \dots, \mathbf{x}_{\ell} \rangle_{K_{\nu}},$$

$$V_{\nu} = \langle \mathbf{x}_{1}, \dots, \widehat{\mathbf{x}_{h}}, \dots, \mathbf{x}_{\ell} \rangle_{K_{\nu}} = \langle \mathbf{y}_{1}, \dots, \mathbf{y}_{\ell-1} \rangle_{K_{\nu}}$$

and we choose a unit vector $\mathbf{u}_{v} \in U_{v}$ such that

$$U_{\nu} = \langle \mathbf{u}_{\nu} \rangle_{K_{\nu}} \perp_{\mathrm{top}} V_{\nu}$$

If $v \neq w$, we write

(33)
$$\varepsilon \mathbf{x}_h = c_v \mathbf{u}_v + \sum_{j=1}^{\ell-1} c_{v,j} \mathbf{y}_j,$$

with coefficients c_v and $c_{v,j}$ in K_v . For v = w, we also define

$$W_w = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}_k}, \dots, \mathbf{y}_{\ell-1} \rangle_{K_w}$$

and choose a unit vector $\mathbf{v}_w \in V_w$ such that

$$V_w = \langle \mathbf{v}_w \rangle_{K_w} \perp_{\mathrm{top}} W_w.$$

This provides a decomposition $U_w = \langle \mathbf{u}_w \rangle_{K_w} \perp_{\text{top}} \langle \mathbf{v}_w \rangle_{K_w} \perp_{\text{top}} W_w$. We choose $B \in K_w$ with

(34)
$$|B|_w = (1 + \delta/2)C^{b_\ell}.$$

This is possible because if $w \nmid \infty$, then $\delta = 0$ and C belongs to the valuation group of K_w . We then write

(35)
$$\varepsilon \mathbf{x}_h = c_w \mathbf{u}_w + B \mathbf{v}_w + \sum_{j=1}^{\ell-1} c_{w,j} \mathbf{y}_j,$$

with c_w and $c_{w,j}$ in K_w . The approximation lemma 8.2 provides, for each $j = 1, \ldots, \ell - 1$, an S-integer $\alpha_j \in \mathcal{O}_S$ such that

(36)
$$\begin{aligned} |\alpha_j - c_{\nu,j}|_\nu &\leq c_4^{-1} \quad \text{for all } \nu \in S \setminus \{w\},\\ |\alpha_j - c_{w,j}|_w &\leq (c_3 c_4)^{d/d_w}, \end{aligned}$$

for the constant $c_4 = n2^n (2ec_2)^2$ defined in (29). Then the point

(37)
$$\mathbf{y}_{\ell} = \varepsilon \mathbf{x}_h - \sum_{j=1}^{\ell-1} \alpha_j \mathbf{y}_j$$

fulfills condition (3), and so $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$ is an \mathcal{O}_S -basis of \mathcal{O}_S^n .

1° To show that **y** is admissible, we fix a place $v \in S \setminus \{w\} \subseteq M_{\infty}(K)$. Since **x** is admissible, we directly obtain $1 \leq ||\mathbf{y}_j||_v \leq (2ec_2)^2$ for each $j \neq \ell$ because of equality (2). As **x** is almost orthogonal in K_v^n , its subsequence

$$(\mathbf{x}_1,\ldots,\widehat{\mathbf{x}_h},\ldots,\mathbf{x}_\ell)=(\mathbf{y}_1,\ldots,\mathbf{y}_{\ell-1})$$

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is also almost orthogonal. Moreover, for each integer j with $\ell + 1 \leq j \leq n$, we have $\mathbf{x}_j = \mathbf{y}_j$ and $\langle \mathbf{x}_1, \ldots, \mathbf{x}_{j-1} \rangle_{K_\nu} = \langle \mathbf{y}_1, \ldots, \mathbf{y}_{j-1} \rangle_{K_\nu}$, and thus

$$\operatorname{dist}_{\nu}\left(\mathbf{y}_{j}, \langle \mathbf{y}_{1}, \dots, \mathbf{y}_{j-1} \rangle_{K_{\nu}}\right) = \operatorname{dist}_{\nu}\left(\mathbf{x}_{j}, \langle \mathbf{x}_{1}, \dots, \mathbf{x}_{j-1} \rangle_{K_{\nu}}\right) \geq 1 - (1/2)^{j-1}$$

Since $\langle \mathbf{y}_1, \ldots, \mathbf{y}_{\ell-1} \rangle_{K_v} = V_v$, it only remains to show that

(38)
$$1 \leq \|\mathbf{y}_{\ell}\|_{\nu} \leq (2ec_2)^2$$
 and $\operatorname{dist}_{\nu}(\mathbf{y}_{\ell}, V_{\nu}) \geq 1 - (1/2)^{\ell-1}$.

To this end, we first note that, since $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is almost orthogonal in K_{ν}^n , Lemmas 3.6 and 3.10 yield

$$1 \ge \operatorname{dist}_{\nu}(\mathbf{x}_{h}, V_{\nu}) = \frac{\|\mathbf{x}_{1} \wedge \cdots \wedge \mathbf{x}_{\ell}\|_{\nu}}{\|\mathbf{x}_{h}\|_{\nu} \|\mathbf{x}_{1} \wedge \cdots \wedge \widehat{\mathbf{x}_{h}} \wedge \cdots \wedge \mathbf{x}_{\ell}\|_{\nu}} \ge \frac{\|\mathbf{x}_{1} \wedge \cdots \wedge \mathbf{x}_{\ell}\|_{\nu}}{\|\mathbf{x}_{1}\|_{\nu} \cdots \|\mathbf{x}_{\ell}\|_{\nu}} \ge e^{-2}.$$

On the other hand, Lemma 3.5 applied to the decomposition of $\epsilon \mathbf{x}_h$ in (33) gives

$$\operatorname{dist}_{\nu}(\mathbf{x}_{h}, V_{\nu}) = \operatorname{dist}_{\nu}(\varepsilon \mathbf{x}_{h}, V_{\nu}) = \frac{|c_{\nu}|_{\nu}}{\|\varepsilon \mathbf{x}_{h}\|_{\nu}}$$

Using the estimates of (31) for $\|\varepsilon \mathbf{x}_h\|_{\nu}$, we deduce that

(39)
$$2 \le |c_{\nu}|_{\nu} \le 2e^2 c_2^2.$$

Combining (33) and (37), we obtain

$$\mathbf{y}_{\ell} = c_{\nu} \mathbf{u}_{\nu} + \sum_{j=1}^{\ell-1} (c_{\nu,j} - \alpha_j) \mathbf{y}_j.$$

Using (36), this decomposition of \mathbf{y}_{ℓ} implies

$$\|\mathbf{y}_{\ell} - c_{\nu}\mathbf{u}_{\nu}\|_{\nu} \leq \sum_{j=1}^{\ell-1} c_4^{-1} \|\mathbf{y}_j\|_{\nu} \leq nc_4^{-1} (2ec_2)^2 = 2^{-n},$$

and, thus, $1 \leq \|\mathbf{y}_{\ell}\|_{\nu} \leq (2ec_2)^2$ by (39). Finally, Lemma 3.5 gives

$$\operatorname{dist}_{\nu}(\mathbf{y}_{\ell}, V_{\nu}) = \frac{|c_{\nu}|_{\nu}}{\|\mathbf{y}_{\ell}\|_{\nu}} \ge \frac{|c_{\nu}|_{\nu}}{|c_{\nu}|_{\nu} + 2^{-n}} \ge \frac{2}{2 + 2^{-n}} \ge 1 - \frac{1}{2^{n+1}},$$

which completes the proof of (38). Thus, **y** is admissible.

2° We now show that **y** has size **b**. Since **x** has size **a**, relations (1) and (2) reduce the problem to showing that

(40)
$$C^{b_{\ell}} \le \|\mathbf{y}_{\ell}\|_{w} \le (1+\delta)C^{b_{\ell}}$$

To prove this, we first combine (35) and (37) to obtain

$$\mathbf{y}_{\ell} = c_{w}\mathbf{u}_{w} + B\mathbf{v}_{w} + \sum_{j=1}^{\ell-1} (c_{w,j} - \alpha_{j})\mathbf{y}_{j},$$

and note that the decomposition of $\epsilon \mathbf{x}_h$ in (35) implies

$$c_w|_w = \|c_w \mathbf{u}_w\|_w \le \|\varepsilon \mathbf{x}_h\|_w \le (2c_2)^{2d/d_w} \|\mathbf{x}_h\|_w,$$

where the last estimation comes from (32). Using (36), we deduce that

$$\|\mathbf{y}_{\ell} - B\mathbf{v}_{w}\|_{w} \le |c_{w}|_{w} + \sum_{j=1}^{\ell-1} (c_{3}c_{4})^{d/d_{w}} \|\mathbf{y}_{j}\|_{w} \le (c_{3}c_{4})^{d/d_{w}} \sum_{j=1}^{\ell} \|\mathbf{x}_{j}\|_{w}.$$

We also note that $\|\mathbf{x}_j\|_w \leq 2C^{a_j} \leq 2C^{b_\ell-1}$ for $j = 1, \ldots, \ell$ because \mathbf{x} has size a, and hypothesis (1) gives $a_1 \leq \cdots \leq a_\ell < b_\ell$. Using hypothesis (29) on C, we conclude that

(41)
$$\|\mathbf{y}_{\ell} - B\mathbf{v}_{w}\|_{w} \leq 2n(c_{3}c_{4})^{d/d_{w}}C^{b_{\ell}-1} \leq 2^{-n}C^{b_{\ell}}.$$

Using the value for $|B|_w$ in (34), this yields (40). So **y** has size **b**.

3° It remains to show that **y** has type (k, ℓ) .

By (41), we have a decomposition $\mathbf{y}_{\ell} = B\mathbf{v}_w + \mathbf{z}$ with $\|\mathbf{z}\|_w \leq 2^{-n}|B|_w$. Thus,

$$dist_{w}(\mathbf{y}_{\ell}, W_{w}) \geq \frac{|B|_{w} - \|\mathbf{z}\|_{w}}{\|\mathbf{y}_{\ell}\|_{w}} \geq \frac{1 - 2^{-n}}{1 + 2^{-n}} \geq 1 - \frac{1}{2^{n-1}} \quad \text{if } w \mid \infty,$$

$$dist_{w}(\mathbf{y}_{\ell}, W_{w}) \geq \frac{|B|_{w}}{\|\mathbf{y}_{\ell}\|_{w}} = 1 \quad \text{otherwise}$$

In both cases, this yields $\operatorname{dist}_{w}(\mathbf{y}_{\ell}, W_{w}) \geq 1 - \delta/2^{\ell-1}$. So **y** has type (k, ℓ) . \Box

We will also need the following complementary result.

LEMMA 8.7. — With the same hypotheses and notation, let m be an integer with $\ell \leq m \leq n$.

- (i) We have $\langle \mathbf{x}_1, \ldots, \mathbf{x}_m \rangle_K = \langle \mathbf{y}_1, \ldots, \mathbf{y}_m \rangle_K$. If m < n, we also have $\mathbf{x}_{m+1} = \mathbf{y}_{m+1}$.
- (ii) If the sequence (x₁,..., x_ℓ) is almost orthogonal in Kⁿ_w, then the sequence (y₁,..., y_ℓ) is also almost orthogonal in Kⁿ_w.
- (iii) If both $(\mathbf{x}_1, \ldots, \widehat{\mathbf{x}_h}, \ldots, \mathbf{x}_m)$ and $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_k}, \ldots, \mathbf{y}_m)$ are almost orthogonal in K_w^n , then the subspaces of K_w^n that they span,

$$V_1 = \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}_h}, \dots, \mathbf{x}_m \rangle_{K_w} \quad and \quad V_2 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}_k}, \dots, \mathbf{y}_m \rangle_{K_w}$$

satisfy

(42)
$$\operatorname{dist}_{w}(V_{1}, V_{2})^{d_{w}/d} \leq e^{4} \frac{H(\langle \mathbf{y}_{1}, \dots, \mathbf{y}_{m} \rangle_{K})}{H(\mathbf{y}_{1}) \cdots H(\mathbf{y}_{m})} \leq e^{4} \frac{c_{5}}{C^{d_{w}/d}} \frac{H(\langle \mathbf{x}_{1}, \dots, \mathbf{x}_{m} \rangle_{K})}{H(\mathbf{x}_{1}) \cdots H(\mathbf{x}_{m})}.$$

When (iii) holds with m < n, estimates (42) together with Corollary 3.8 allow us to connect the distances from $\mathbf{x}_{m+1} = \mathbf{y}_{m+1}$ to V_1 and V_2 in terms of heights only.

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Proof. — Part (i) follows directly from conditions (2) and (3) of Lemma 8.6.

To prove (ii) we note that $(\mathbf{x}_1, \ldots, \widehat{\mathbf{x}_h}, \ldots, \mathbf{x}_\ell) = (\mathbf{y}_1, \ldots, \mathbf{y}_{\ell-1})$. Thus, if this sequence is almost orthogonal in K_w^n , so is its subsequence $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_k}, \ldots, \mathbf{y}_{\ell-1})$. Since \mathbf{y} is of type (k, ℓ) , we then conclude that $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_k}, \ldots, \mathbf{y}_\ell)$ is almost orthogonal in K_w^n .

Under the hypotheses of (iii), the first inequality in (42) follows from Lemma 8.5 because $(\mathbf{x}_1, \ldots, \widehat{\mathbf{x}_h}, \ldots, \mathbf{x}_m)$ coincides with $(\mathbf{y}_1, \ldots, \widehat{\mathbf{y}_\ell}, \ldots, \mathbf{y}_m)$. To prove the second inequality, we use the fact that \mathbf{x} and \mathbf{y} have respective sizes \boldsymbol{a} and \boldsymbol{b} with $a_h \leq a_\ell < b_\ell$. By Lemma 8.4, we obtain

$$\frac{H(\mathbf{x}_1)\cdots H(\mathbf{x}_m)}{H(\mathbf{y}_1)\cdots H(\mathbf{y}_m)} = \frac{H(\mathbf{x}_h)}{H(\mathbf{y}_\ell)} \le c_5 C^{(a_h - b_\ell)d_w/d} \le \frac{c_5}{C^{d_w/d}}$$

The required inequality follows since $H(\langle \mathbf{y}_1, \dots, \mathbf{y}_m \rangle_K) = H(\langle \mathbf{x}_1, \dots, \mathbf{x}_m \rangle_K)$ by (i).

We conclude with the following existence result, which for us replaces [17, Lemma 5.2].

LEMMA 8.8. — Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $0 \leq a_1 < \cdots < a_n$. There exists an admissible basis $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of \mathcal{O}_S^n over \mathcal{O}_S of size \mathbf{a} and type (1, n) such that $(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$ is almost orthogonal in K_w^n .

Proof. — Starting from the canonical basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of K^n , Lemma 8.6 applied recursively n times with h = k = 1 and $\ell = n$ provides points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of \mathcal{O}_S^n such that, for each $j = 0, \ldots, n$, the n-tuple $(\mathbf{e}_{j+1}, \ldots, \mathbf{e}_n, \mathbf{x}_1, \ldots, \mathbf{x}_j)$ is an admissible basis of \mathcal{O}_S^n of size $(0, \ldots, 0, a_1, \ldots, a_j)$ and type (1, n). For each j with $2 \leq j \leq n-1$, we find

$$\operatorname{dist}_{w}\left(\mathbf{x}_{j}, \langle \mathbf{x}_{1}, \dots, \mathbf{x}_{j-1} \rangle_{K_{w}}\right) \geq \operatorname{dist}_{w}\left(\mathbf{x}_{j}, \langle \mathbf{e}_{j+2}, \dots, \mathbf{e}_{n}, \mathbf{x}_{1}, \dots, \mathbf{x}_{j-1} \rangle_{K_{w}}\right)$$
$$\geq 1 - 1/2^{n-1}.$$

Thus, $(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$ is almost orthogonal in K_w^n .

9. From *n*-systems to points

Let $\mathbf{L}: [0, \infty) \to \mathbb{R}^n$ be an arbitrary *n*-system. To complete the proof of Theorem A, it remains to show the existence of a non-zero point $\boldsymbol{\xi} \in K_w^n$ for which $\|\mathbf{L} - \mathbf{L}_{\boldsymbol{\xi}}\|$ is bounded above by a constant depending only on K, w and n. To this end, consider the *n*-system $\mathbf{R} = (R_1, \ldots, R_n)$ provided by Corollary 6.2 for the choice of c' = 2c, where

$$c = \frac{d_w}{d} \log(C),$$

for the constant C = C(K, w, n) of the preceding section, satisfying (29). Since $\|\mathbf{L} - \mathbf{R}\|$ is bounded above by a constant depending only on K, n and w,

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it suffices to construct a non-zero point $\boldsymbol{\xi} \in K_w^n$ for which $\|\mathbf{R} - \mathbf{L}_{\boldsymbol{\xi}}\|$ is also bounded above by such a constant.

Let $q_0 = (n^2 - n + 1)c$, so that the restriction of **R** to $[q_0, \infty)$ is rigid of mesh c. We denote by $(q_i)_{0 \le i < s}$ the finite or infinite sequence of switch points of **R** on that interval, with cardinality $s \in \{1, 2, ...\} \cup \{\infty\}$. For each integer i with $0 \le i < s$, we set

$$a^{(i)} = c^{-1}\mathbf{R}(q_i) = (a_1^{(i)}, \dots, a_n^{(i)}) \in \Delta,$$

where $\Delta \subset \mathbb{Z}^n$ is defined by (30). We have $R_j(q_i) = ca_j^{(i)}$ for $j = 1, \ldots, n$ and

(43)
$$q_i = ca_1^{(i)} + \dots + ca_n^{(i)} \quad (0 \le i < s).$$

We also denote by k_i the index j for which the right derivative of R_j at q_i is 1 and, when i > 0, we denote by ℓ_i the index j for which the left derivative of R_j at q_i is 1. By the choice of **R**, we have $k_0 = 1$. Finally, we set $\ell_0 = n$. Then, for each integer i with $1 \le i < s$, we have

(P1) $1 = k_0 < \ell_0 = n$ and $1 \le k_i < \ell_i \le n$,

(P2) $\ell_i \ge k_{i-1}$ and $a_{\ell_i}^{(i)} > a_{\ell_i}^{(i-1)}$,

(P3) $(a_1^{(i)}, \dots, \widehat{a_{\ell_i}^{(i)}}, \dots, a_n^{(i)}) = (a_1^{(i-1)}, \dots, \widehat{a_{k_{i-1}}^{(i-1)}}, \dots, a_n^{(i-1)}).$

From these data, it is a simple matter to reconstruct the function \mathbf{R} . Let

 $\Phi \colon \mathbb{R}^n \to \{ (x_1, \dots, x_n) \in \mathbb{R}^n \, ; \, x_1 \le x_2 \le \dots \le x_n \}$

denote the continuous function that reorders the coordinates of a point as a non-decreasing sequence and set $q_s = \infty$ if $s < \infty$. Then, for each $q \in [q_i, q_{i+1})$, we have

(44)
$$\mathbf{R}(q) = \Phi(\mathbf{R}(q_i) + (q - q_i)\mathbf{e}_{k_i})$$
$$= \Phi(R_1(q_i), \dots, R_{k_i}(q_i) + q - q_i, \dots, R_n(q_i)).$$

The formula also extends to $q = q_{i+1}$, if i + 1 < s.

We first apply the results of the preceding section to construct a specific sequence of bases of \mathcal{O}_S^n . Its relevance to our problem will become clear in the corollaries that we derive afterwards.

PROPOSITION 9.1. — There exists a sequence $(\mathbf{x}^{(i)})_{0 \leq i < s}$ of bases of \mathcal{O}_S^n over \mathcal{O}_S such that, for each integer i with $0 \leq i < s$, the basis $\mathbf{x}^{(i)} = (\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ is admissible of size $\mathbf{a}^{(i)}$ and type (k_i, ℓ_i) with the additional property that, when $i \geq 1$,

(1)
$$(\mathbf{x}_{1}^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_{i}}^{(i)}}, \dots, \mathbf{x}_{n}^{(i)}) = (\mathbf{x}_{1}^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_{n}^{(i-1)}),$$

(2) $\mathbf{x}_{\ell_{i}}^{(i)} \in \varepsilon_{i} \mathbf{x}_{k_{i-1}}^{(i-1)} + \langle \mathbf{x}_{1}^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_{\ell_{i}}^{(i-1)} \rangle_{\mathcal{O}_{S}} \text{ for some } \varepsilon_{i} \in \mathcal{O}_{S}^{*}.$

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We may further require that the sequence $\widehat{\mathbf{x}}^{(-1)} := (\mathbf{x}_1^{(0)}, \ldots, \mathbf{x}_{n-1}^{(0)})$ is almost orthogonal in K_w^n . Then, for each integer i with $0 \leq i < s$, the sequence $\widehat{\mathbf{x}}^{(i)} := (\mathbf{x}_1^{(i)}, \ldots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \ldots, \mathbf{x}_n^{(i)})$ is also almost orthogonal in K_w^n . Finally, for each i with $-1 \leq i < s$, choose a unit vector \mathbf{u}_i of K_w^n which is orthogonal to each vector of $\widehat{\mathbf{x}}^{(i)}$ with respect to the dot product. Then we further have

(45)
$$\operatorname{dist}_{w}(\mathbf{u}_{i}, \mathbf{u}_{j}) \leq 2^{\delta} \exp((4 - q_{i+1})d/d_{w}) \quad (-1 \leq i < j < s).$$

Proof. — Lemma 8.6 provides such a sequence of bases recursively, starting from any admissible basis $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_n^{(0)})$ of size $\mathbf{a}^{(0)}$ and type (k_0, ℓ_0) . To build $\mathbf{x}^{(i)}$ from $\mathbf{x}^{(i-1)}$ for an integer i with $1 \le i < s$, we apply this lemma with $h = k_{i-1}, (k, \ell) = (k_i, \ell_i), \mathbf{a} = \mathbf{a}^{(i-1)}$ and $\mathbf{b} = \mathbf{a}^{(i)}$. The hypotheses of the lemma are fulfilled by virtue of conditions (P1)–(P3).

By Lemma 8.8 we may choose the initial basis $\mathbf{x}^{(0)}$ so that $\hat{\mathbf{x}}^{(-1)}$ is almost orthogonal in K_w^n . Assuming this, we now prove by induction that $\hat{\mathbf{x}}^{(i)}$ is almost orthogonal in K_w^n for each *i* with $0 \le i < s$.

We first note that $\widehat{\mathbf{x}}^{(0)} = (\mathbf{x}_2^{(0)}, \dots, \mathbf{x}_n^{(0)})$ is almost orthogonal in K_w^n because $\mathbf{x}^{(0)}$ has type (1, n) and the sequence $(\mathbf{x}_2^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)})$ is almost orthogonal in K_w^n , as a subsequence of the almost orthogonal sequence $\widehat{\mathbf{x}}^{(-1)}$.

Suppose now that $\widehat{\mathbf{x}}^{(i)}$ is almost orthogonal in K_w^n for each $i = 0, \ldots, t-1$, where t is an integer with $1 \leq t < s$. To complete the induction step, we will show, by induction on m, that $(\mathbf{x}_1^{(t)}, \ldots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \ldots, \mathbf{x}_m^{(t)})$ is almost orthogonal in K_w^n , for each $m = \ell_t, \ldots, n$. For $m = \ell_t$, this follows from Lemma 8.7 (ii) since $(\mathbf{x}_1^{(t-1)}, \ldots, \widehat{\mathbf{x}_{k_{t-1}}^{(t-1)}}, \ldots, \mathbf{x}_{\ell_t}^{(t-1)})$ is almost orthogonal in K_w^n . If $\ell_t = n$, we are done. Otherwise, let m be an integer with $\ell_t \leq m < n$ for which $(\mathbf{x}_1^{(t)}, \ldots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \ldots, \mathbf{x}_m^{(t)})$ is almost orthogonal in K_w^n . Since $\ell_0 = n > m$, there is a largest integer r with $0 \leq r < t$ such that $\ell_r > m$. This means that $\ell_{r+1}, \ldots, \ell_t \leq m$, and so $k_r, \ldots, k_t \leq m$ by (P1) and (P2). Moreover, we have $\mathbf{x}_{m+1}^{(r)} = \cdots = \mathbf{x}_{m+1}^{(t)}$ by Lemma 8.7 (i). Define

(46)
$$U^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_m^{(i)} \rangle_{K_w}$$
 and $V^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_m^{(i)} \rangle_{K_w}$

for $i = r, \ldots, t$. We claim that

(47)
$$\operatorname{dist}_{w}(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) \ge 1 - \frac{\delta}{2^{m}} \quad \text{and} \quad \operatorname{dist}_{w}(V^{(r)}, V^{(t)}) \le \frac{1}{2^{m}}.$$

If we take this for granted, Corollary 3.8 gives $\operatorname{dist}_{w}(\mathbf{x}_{m+1}^{(t)}, V^{(t)}) \geq 1 - \delta/2^{m-1}$, which is exactly what we need to complete the induction on m and, thus, to complete the main induction as well.

If $r \geq 1$ and $m + 1 < \ell_r$, the (m + 1)-tuple $(\mathbf{x}_1^{(r)}, \ldots, \mathbf{x}_{m+1}^{(r)})$ is almost orthogonal in K_w^n as a subsequence of

$$\left(\mathbf{x}_{1}^{(r)},\ldots,\widehat{\mathbf{x}_{\ell_{r}}^{(r)}},\ldots,\mathbf{x}_{n}^{(r)}\right)=\left(\mathbf{x}_{1}^{(r-1)},\ldots,\widehat{\mathbf{x}_{k_{r-1}}^{(r-1)}},\ldots,\mathbf{x}_{n}^{(r-1)}\right).$$

If r = 0 and $m+1 < \ell_0 = n$, it is also almost orthogonal in K_w^n , as a subsequence of $\widehat{\mathbf{x}}^{(-1)}$. So, independently of r, if $m+1 < \ell_r$, we obtain

$$\operatorname{dist}_{w}(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) \ge \operatorname{dist}_{w}(\mathbf{x}_{m+1}^{(r)}, U^{(r)}) \ge 1 - \frac{\delta}{2^{m}},$$

which gives the first inequality in (47). If $m + 1 = \ell_r$, the latter inequality holds by construction, since $\mathbf{x}^{(r)}$ has type (k_r, ℓ_r) .

To prove the second inequality in (47), we apply Lemma 8.7 (iii). For each $i = r + 1, \ldots, t$, it gives

$$dist_{w}(V^{(i-1)}, V^{(i)})^{d_{w}/d} \leq e^{4} \frac{H(\langle \mathbf{x}_{1}^{(i)}, \dots, \mathbf{x}_{m}^{(i)} \rangle_{K})}{H(\mathbf{x}_{1}^{(i)}) \cdots H(\mathbf{x}_{m}^{(i)})} \\ \leq e^{4} \frac{c_{5}}{C^{d_{w}/d}} \frac{H(\langle \mathbf{x}_{1}^{(i-1)}, \dots, \mathbf{x}_{m}^{(i-1)} \rangle_{K})}{H(\mathbf{x}_{1}^{(i-1)}) \cdots H(\mathbf{x}_{m}^{(i-1)})}$$

and so, for the same values of i, we obtain

$$dist_{w}(V^{(i-1)}, V^{(i)})^{d_{w}/d} \leq e^{4} \left(\frac{c_{5}}{C^{d_{w}/d}}\right)^{i-r} \frac{H(\langle \mathbf{x}_{1}^{(r)}, \dots, \mathbf{x}_{m}^{(r)} \rangle_{K})}{H(\mathbf{x}_{1}^{(r)}) \cdots H(\mathbf{x}_{m}^{(r)})} \leq e^{4} \left(\frac{c_{5}}{C^{d_{w}/d}}\right)^{i-r}.$$

Since $C \geq 2^n (e^4 c_5)^{d/d_w}$, this yields $\operatorname{dist}_w(V^{(i-1)}, V^{(i)}) \leq 2^{-(i-r)n}$ for each integer $i = r + 1, \ldots, t$, and so by the triangle inequality of Lemma 3.4 we obtain

$$\operatorname{dist}_{w}(V^{(r)}, V^{(t)}) \leq \sum_{i=r+1}^{t} \operatorname{dist}_{w}(V^{(i-1)}, V^{(i)}) \leq \frac{1}{2^{n-1}} \leq \frac{1}{2^{m}}$$

which completes the proof of (47).

By the above, the sequence $\widehat{\mathbf{x}}^{(i)}$ is almost orthogonal in K_w^n for each i with $-1 \leq i < s$. For those i, take $V^{(i)}$ to be the subspace of K_w^n spanned by $\widehat{\mathbf{x}}^{(i)}$, so that $(V^{(i)})^{\perp} = \langle \mathbf{u}_i \rangle_{K_w}$. When $0 \leq i < s$, both $\widehat{\mathbf{x}}^{(i-1)}$ and $\widehat{\mathbf{x}}^{(i)}$ are almost orthogonal subsequences of $\mathbf{x}^{(i)}$. Then Lemmas 3.9 and 8.5 yield

$$\operatorname{dist}_{w}(\mathbf{u}_{i-1}, \mathbf{u}_{i})^{d_{w}/d} = \operatorname{dist}_{w}(V^{(i-1)}, V^{(i)})^{d_{w}/d} \le \frac{e^{4}}{H(\mathbf{x}_{1}^{(i)}) \cdots H(\mathbf{x}_{n}^{(i)})} \quad (0 \le i < s)$$

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because $\langle \mathbf{x}_1^{(i)}, \ldots, \mathbf{x}_n^{(i)} \rangle_K = K^n$ has height 1. Since the basis $\mathbf{x}^{(i)}$ is admissible of size $a^{(i)}$, Lemma 8.4 further gives

$$\log H(\mathbf{x}_{j}^{(i)}) \ge a_{j}^{(i)}(d_{w}/d)\log(C) = ca_{j}^{(i)}$$
 for $j = 1, \dots, n$.

Using (43), we conclude that

$$\operatorname{dist}_{w}(\mathbf{u}_{i-1}, \mathbf{u}_{i})^{d_{w}/d} \leq \exp\left(4 - \sum_{j=1}^{n} c a_{j}^{(i)}\right) = \exp(4 - q_{i}) \quad (0 \leq i < s).$$

Since $(q_i)_{0 \le i < s}$ is a strictly increasing sequence of multiples of c and since $cd/d_w = \log(C) \ge \log(2)$, we deduce that

$$dist_w(\mathbf{u}_{j-1}, \mathbf{u}_j) \le \exp((4 - q_j)d/d_w) \le (1/2)^{j-i}\exp((4 - q_i)d/d_w),$$

for each pair of integers $0 \le i < j < s$. Then, (45) follows from the triangle inequality of Lemma 3.4.

For the corollary below, we recall our convention that $q_s = \infty$ when $s < \infty$. As a special case of (24), we also recall that, for each $\mathbf{x} \in K^n$, each non-zero $\boldsymbol{\xi} \in K_w^n$ and each $q \ge 0$, we have by definition

$$L_{\boldsymbol{\xi}}(\mathbf{x}, q) = \max\{\log H(\mathbf{x}), q + \log D_{\boldsymbol{\xi}}(\mathbf{x})\}.$$

COROLLARY 9.2. — Under the hypotheses of Proposition 9.1, there is a unit vector $\boldsymbol{\xi} \in K_w^n$ such that, for each integer i with $0 \leq i < s$ and each $q \in [q_i, q_{i+1})$, we have

$$L_{\boldsymbol{\xi}}(\mathbf{x}_{j}^{(i)}, q) \leq c_{6} + R_{j}(q_{i}) + \begin{cases} q - q_{i} & \text{if } j = k_{i}, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_6 = 6 + \log(c_5)$.

Proof. — Consider the sequence of unit vectors $(\mathbf{u}_i)_{-1 \leq i < s}$ given by the proposition. If $s = \infty$, it follows from (45) that its image in $\mathbb{P}^{n-1}(K_w)$ converges to the class of a unit vector $\boldsymbol{\xi} \in K_w^n$, such that

(48)
$$\operatorname{dist}_{w}(\mathbf{u}_{i}, \boldsymbol{\xi}) \leq 2^{\delta} \exp((4 - q_{i+1})d/d_{w}) \quad (-1 \leq i < s).$$

When $s < \infty$, we have $q_s = \infty$ by convention, and (48) holds with $\boldsymbol{\xi} = \mathbf{u}_{s-1}$. We now fix *i* and *j* with $0 \le i < s$ and $1 \le j \le n$. For each $q \ge 0$, we have

(49)
$$L_{\boldsymbol{\xi}}(\mathbf{x}_{j}^{(i)}, q) = \log H(\mathbf{x}_{j}^{(i)}) + \max\left\{0, \ q + \frac{d_{w}}{d}\log\frac{|\mathbf{x}_{j}^{(i)} \cdot \boldsymbol{\xi}|_{w}}{\|\mathbf{x}_{j}^{(i)}\|_{w}}\right\}$$

We also have $\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}_{i-1} = 0$ because $\mathbf{x}_{k_i}^{(i)}$ belongs to the sequence $\hat{\mathbf{x}}^{(i-1)}$. So (19) yields

$$\|\mathbf{x}_{k_i}^{(i)} \cdot \boldsymbol{\xi}\|_{w} \leq 2^{\delta} \|\mathbf{x}_{k_i}^{(i)}\|_{w} \operatorname{dist}_{w}(\mathbf{u}_{i-1}, \boldsymbol{\xi}).$$

If $j \neq k_i$, we instead have $\mathbf{x}_j^{(i)} \cdot \mathbf{u}_i = 0$ because $\mathbf{x}_j^{(i)}$ belongs to $\hat{\mathbf{x}}^{(i)}$, and so (19) yields

$$\mathbf{x}_{j}^{(i)} \cdot \boldsymbol{\xi}|_{w} \leq 2^{\delta} \|\mathbf{x}_{j}^{(i)}\|_{w} \operatorname{dist}_{w}(\mathbf{u}_{i}, \boldsymbol{\xi}).$$

Using (48) and noting that $4^{\delta} \leq \exp(2d/d_w)$, we deduce that

$$\frac{d_w}{d} \log \frac{|\mathbf{x}_j^{(i)} \cdot \boldsymbol{\xi}|_w}{\|\mathbf{x}_j^{(i)}\|_w} \le \begin{cases} 6 - q_i & \text{if } j = k_i, \\ 6 - q_{i+1} & \text{otherwise.} \end{cases}$$

Moreover, Lemma 8.4 gives $\log H(\mathbf{x}_j^{(i)}) \leq \log(c_5) + ca_j^{(i)} = c_6 - 6 + R_j(q_i)$ since $\mathbf{x}^{(i)}$ has size \mathbf{a} . The conclusion follows by substituting the last two estimates into (49).

We can now complete the proof of Theorem A as follows.

COROLLARY 9.3. — Let $\boldsymbol{\xi}$ be as in Corollary 9.2. Then, there is a constant $c_7 = c_7(K, w, n)$ such that

(50)
$$\max_{1 \le j \le n} |L_{\xi,j}(q) - R_j(q)| \le c_7,$$

for each $q \geq 0$.

Proof. — Fix an integer i with $0 \le i < s$ and a point $q \in [q_i, q_{i+1})$. Denote by

 $(r_1,\ldots,r_n) = \Phi\left(L_{\boldsymbol{\xi}}(\mathbf{x}_1^{(i)},q),\ldots,L_{\boldsymbol{\xi}}(\mathbf{x}_n^{(i)},q)\right)$

the numbers $L_{\boldsymbol{\xi}}(\mathbf{x}_j^{(i)}, q)$ with $1 \leq j \leq n$ written in non-decreasing order. Since $\mathbf{x}^{(i)}$ is a basis of \mathcal{O}_S^n over \mathcal{O}_S , it is also a basis of K^n over K, and so by definition we have

 $L_{\boldsymbol{\xi},j}(q) \le r_j \quad (1 \le j \le n).$

By Corollary 9.2 and formula (44), we also have

$$r_j \le c_6 + R_j(q) \quad (1 \le j \le n)$$

By definition of an *n*-system, we further have $\sum_{j=1}^{n} R_j(q) = q$. Thus, by Lemma 7.4 (for k = 1) and the estimates of (28), we find

$$\sum_{j=1}^{n} \left(c_6 + R_j(q) - L_{\boldsymbol{\xi},j}(q) \right) = nc_6 + q - \sum_{j=1}^{n} L_{\boldsymbol{\xi},j}(q) \le c_6 + c_7,$$

for a constant $c_7 = c_7(K, w, n) \ge \max\{c_6, q_0\}$. This yields (50) because each summand in the main sum on the left-hand side is non-negative. Finally, (50) also holds for each $q \in [0, q_0)$, because for such q, the numbers $R_j(q)$ and $L_{\boldsymbol{\xi},j}(q)$ belong to $[0, q_0) \subseteq [0, c_7)$.

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10. Spectra of exponents of approximation

We fix a place w of K and an integer $n \geq 2$. For each non-zero $\boldsymbol{\xi} \in K_w^n$, we write $\widehat{\omega}(\boldsymbol{\xi})$ for $\widehat{\omega}(\boldsymbol{\xi}, K, w)$ and similarly for the three other exponents introduced in Section 2.4. Our goal is to show that their spectrum is independent of the choice of K and w and, more precisely, that it can be expressed in terms of n-systems, as mentioned in the Introduction. We will also generalize to the present setting the exponents of Laurent from [9] and show that the same applies to their spectrum. We start with the following observation.

LEMMA 10.1. — For each non-zero $\boldsymbol{\xi} \in K_w^n$, we have

$$\begin{split} & \liminf_{q \to \infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\omega(\boldsymbol{\xi}) + 1}, & \qquad \limsup_{q \to \infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\widehat{\omega}(\boldsymbol{\xi}) + 1}, \\ & \liminf_{q \to \infty} \frac{L_{\boldsymbol{\xi},1}^*(q)}{q} = \frac{1}{\lambda(\boldsymbol{\xi}) + 1}, & \qquad \limsup_{q \to \infty} \frac{L_{\boldsymbol{\xi},1}^*(q)}{q} = \frac{1}{\widehat{\lambda}(\boldsymbol{\xi}) + 1}. \end{split}$$

Proof. — By Definition 2.1, the number $\hat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) is the supremum of all $\omega \geq 0$ such that, for each sufficiently large t > 0 (resp. for arbitrarily large t > 0), there is a non-zero point $\mathbf{x} \in K^n$ with $H(\mathbf{x}) \leq e^t$ and $D_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{-\omega t}$. By definition of $L_{\boldsymbol{\xi},1}$ in Section 2.5, the existence of such \mathbf{x} translates into $L_{\boldsymbol{\xi},1}((\omega+1)t) \leq t$. Using the change of variables $q = (\omega+1)t$, we deduce that $\hat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) is the supremum of all $\omega \geq 0$ for which $q^{-1}L_{\boldsymbol{\xi},1}(q) \leq 1/(\omega+1)$ for each sufficiently large q > 0 (resp. for arbitrarily large q > 0). This yields the first row of formulas. The second one is proved in the same way. □

For any function $\mathbf{P} = (P_1, \ldots, P_n) \colon [0, \infty) \to \mathbb{R}^n$ and any $j = 1, \ldots, n$, we define

$$\varphi_j(\mathbf{P}) = \liminf_{q \to \infty} \frac{P_j(q)}{q} \quad \text{and} \quad \bar{\varphi}_j(\mathbf{P}) = \limsup_{q \to \infty} \frac{P_j(q)}{q}$$

The following generalization of [17, Corollary 1.4] characterizes the spectrum of $(\omega, \hat{\omega}, \lambda, \hat{\lambda})$.

PROPOSITION 10.2. — The set S of quadruples

$$\left(\frac{1}{\omega(\boldsymbol{\xi})+1}, \frac{1}{\widehat{\omega}(\boldsymbol{\xi})+1}, \frac{1}{\lambda(\boldsymbol{\xi})+1}, \frac{1}{\widehat{\lambda}(\boldsymbol{\xi})+1}\right) \in [0,1]^4$$

where $\boldsymbol{\xi} \in K_w^n$ has K-linearly independent coordinates coincides with the set of quadruples

$$\left(\varphi_1(\mathbf{P}), \bar{\varphi}_1(\mathbf{P}), 1 - \bar{\varphi}_n(\mathbf{P}), 1 - \varphi_n(\mathbf{P})\right) \in [0, 1]^4$$

where $\mathbf{P} = (P_1, \ldots, P_n)$ is an n-system with first component P_1 unbounded.

Proof. — By the lemma, \mathcal{S} consists of the points

$$(\varphi_1(\mathbf{L}_{\boldsymbol{\xi}}), \bar{\varphi}_1(\mathbf{L}_{\boldsymbol{\xi}}), \varphi_1(\mathbf{L}_{\boldsymbol{\xi}}^*), \bar{\varphi}_1(\mathbf{L}_{\boldsymbol{\xi}}^*)))$$

for all $\boldsymbol{\xi} \in K_{w}^{n}$ with K-linearly independent coordinates. By definition of $L_{\boldsymbol{\xi},1}$ in Section 2.5, that condition on $\boldsymbol{\xi}$ is equivalent to asking that $L_{\boldsymbol{\xi},1}$ is unbounded. Thus, by Theorem A, the set \mathcal{S} consists of the points

$$(\underline{\varphi}_1(\mathbf{P}), \overline{\varphi}_1(\mathbf{P}), \underline{\varphi}_1(\mathbf{P}^*), \overline{\varphi}_1(\mathbf{P}^*))$$

where $\mathbf{P} = (P_1, \dots, P_n)$ is an *n*-system whose first component P_1 is unbounded. The conclusion follows since, for any *n*-system \mathbf{P} , one has $\varphi_1(\mathbf{P}^*) = 1 - \overline{\varphi}_n(\mathbf{P})$ and $\overline{\varphi}_1(\mathbf{P}^*) = 1 - \underline{\varphi}_n(\mathbf{P})$.

COROLLARY 10.3. — The set S is independent of the choice of K and w. In particular, since Jarník's identity (3) holds for any point $\boldsymbol{\xi}$ of $\mathbb{Q}^3_{\infty} = \mathbb{R}^3$ with \mathbb{Q} -linearly independent coordinates, it also holds for any point $\boldsymbol{\xi}$ of K^3_w with K-linearly independent coordinates.

To generalize the exponents of Laurent from [9], we fix an integer k with $1 \le k \le n-1$, and we assume for simplicity that $\|\boldsymbol{\xi}\|_w = 1$. For each non-zero $\mathbf{X} \in \bigwedge^k K^n$, we define

$$D_{\boldsymbol{\xi}}^*(\mathbf{X}) = \| \boldsymbol{\xi} \wedge \mathbf{X} \|_w^{d_w/d} \prod_{v \neq w} \| \mathbf{X} \|_v^{d_v/d}$$

in addition to the quantity $D_{\boldsymbol{\xi}}(\mathbf{X})$ from Definition 7.1. For k = 1, this agrees with the notation of Section 2.4 since $\|\boldsymbol{\xi}\|_w = 1$. We note the following fact.

LEMMA 10.4. — For given $\omega \geq 0$ and $Q \geq 1$, the following conditions are equivalent:

- (i) there is a non-zero $\mathbf{X} \in \bigwedge^k K^n$ with $H(\mathbf{X}) \leq Q$ and $D^*_{\boldsymbol{\xi}}(\mathbf{X}) \leq Q^{-\omega}$;
- (ii) there is a non-zero $\mathbf{Y} \in \bigwedge^{n-k} K^n$ with $H(\mathbf{Y}) \leq Q$ and $D_{\boldsymbol{\xi}}(\mathbf{Y}) \leq Q^{-\omega}$.

Proof. — Consider the K_{ν} -linear isometry $\varphi_{k,\nu} \colon \bigwedge^k K_{\nu}^n \to \bigwedge^{n-k} K_{\nu}^n$ from Section 3.3 for each $\nu \in M(K)$. They all restrict to a single K-linear isomorphism $\varphi_k \colon \bigwedge^k K^n \to \bigwedge^{n-k} K^n$. Moreover, for each $\mathbf{X} \in \bigwedge^k K_w^n$, we have

$$\boldsymbol{\xi} \,\lrcorner\, \varphi_{k,w}(\mathbf{X}) = \varphi_{k+1,w}(\mathbf{X} \wedge \boldsymbol{\xi})$$

(cf. [4, §3, Lemma 2]) and thus $\|\boldsymbol{\xi} \,\sqcup\, \varphi_{k,w}(\mathbf{X})\|_{w} = \|\mathbf{X} \wedge \boldsymbol{\xi}\|_{w}$. We conclude that, if $\mathbf{X} \in \bigwedge^{k} K^{n}$ is non-zero, then the point $\mathbf{Y} = \varphi_{k}(\mathbf{X}) \in \bigwedge^{n-k} K^{n}$ is non-zero with $H(\mathbf{Y}) = H(\mathbf{X})$ and $D_{\boldsymbol{\xi}}(\mathbf{Y}) = D^{*}_{\boldsymbol{\xi}}(\mathbf{X})$. Thus, the two conditions are equivalent.

DEFINITION 10.5. — We denote by $\omega_{k-1}(\boldsymbol{\xi})$ (resp. $\widehat{\omega}_{k-1}(\boldsymbol{\xi})$) the supremum of all $\omega \geq 0$ for which the equivalent conditions of Lemma 10.4 are fulfilled for arbitrarily large values of Q (resp. for all sufficiently large values of Q).

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For k = 1, applying condition (i) shows that $\omega_0(\boldsymbol{\xi}) = \lambda(\boldsymbol{\xi})$ and $\widehat{\omega}_0(\boldsymbol{\xi}) = \widehat{\lambda}(\boldsymbol{\xi})$. For k = n - 1, applying condition (ii) instead shows that $\omega_{n-2}(\boldsymbol{\xi}) = \omega(\boldsymbol{\xi})$ and $\widehat{\omega}_{n-2}(\boldsymbol{\xi}) = \widehat{\omega}(\boldsymbol{\xi})$. Moreover, for $K = \mathbb{Q}$, $w = \infty$ and any choice of k, the number $\omega_{k-1}(\boldsymbol{\xi})$ defined above coincides with the exponent of Laurent from [9], in view of condition (i): see [9, §2, Remark] or [4, §4, Proposition].

Arguing as in Lemma 10.1, using condition (ii) and the definition of $L_{\boldsymbol{\xi},1}^{(k-1)}$, we deduce that

$$\frac{1}{\omega_{k-1}(\boldsymbol{\xi})+1} = \liminf_{q \to \infty} \frac{L_{\boldsymbol{\xi},1}^{(n-k)}(q)}{q} \quad \text{and} \quad \frac{1}{\widehat{\omega}_{k-1}(\boldsymbol{\xi})+1} = \limsup_{q \to \infty} \frac{L_{\boldsymbol{\xi},1}^{(n-k)}(q)}{q}.$$

On the other hand, the proof of Proposition 7.6 shows that $L_{\boldsymbol{\xi},1}^{(j)}$ differs by a bounded function from $L_{\boldsymbol{\xi},1} + \cdots + L_{\boldsymbol{\xi},j}$, for each $j = 1, \ldots, n-1$. We conclude that the spectrum of these 2n-2 exponents is independent of K and w, and characterized as follows (cf. [18, Proposition 3.1]).

PROPOSITION 10.6. — The set of points

$$((\omega_0(\boldsymbol{\xi})+1)^{-1},\ldots,(\omega_{n-2}(\boldsymbol{\xi})+1)^{-1},(\widehat{\omega}_0(\boldsymbol{\xi})+1)^{-1},\ldots,(\widehat{\omega}_{n-2}(\boldsymbol{\xi})+1)^{-1}))$$

where $\boldsymbol{\xi} \in K_w^n$ has K-linearly independent coordinates coincides with the set of points

$$\left(\underbrace{\psi}_{n-1}(\mathbf{P}), \ldots, \underbrace{\psi}_{1}(\mathbf{P}), \overline{\psi}_{n-1}(\mathbf{P}), \ldots, \overline{\psi}_{1}(\mathbf{P}) \right),$$

where $\mathbf{P} = (P_1, \ldots, P_n)$ is an n-system with first component P_1 unbounded, and

$$\psi_j(\mathbf{P}) = \liminf_{q \to \infty} \frac{P_1(q) + \dots + P_j(q)}{q},$$
$$\bar{\psi}_j(\mathbf{P}) = \limsup_{q \to \infty} \frac{P_1(q) + \dots + P_j(q)}{q}$$

for $j = 1, \ldots, n - 1$.

As in [9, Definition 2], the exponent $\omega_{k-1}(\boldsymbol{\xi})$ can be described geometrically as the supremum of all $\omega \geq 0$ for which there are infinitely many subspaces V of K^n of dimension k that satisfy

$$\inf\{\operatorname{dist}_{W}(\boldsymbol{\xi}, \mathbf{x}) ; \mathbf{x} \in V\} \leq H(V)^{-(\omega+1)d/d_{W}}$$

We simply sketch the proof, similar to that of [4, §4, Proposition], using the results of Section 7. We first observe that, in defining $\omega_{k-1}(\boldsymbol{\xi})$ through condition (ii) of Lemma 10.4, we may take for \mathbf{Y} a point of $\bigwedge^{n-k} K^n$ which realizes the first minimum of $\mathcal{C}_{\boldsymbol{\xi}}^{(n-k)}(t)$, where $t = (\omega + 1)\log(Q)$. Since that convex body is comparable to $\bigwedge^{n-k} \mathcal{C}_{\boldsymbol{\xi}}(t)$, we may even choose \mathbf{Y} to be the wedge product of the first n-k points of a basis of K^n which realizes the minima of $\mathcal{C}_{\boldsymbol{\xi}}(t)$ (see the comments after Theorem 4.2). Let W denote the subspace of K^n spanned by these points, so that \mathbf{Y} spans $\bigwedge^{n-k} W$, and let $V = W^{\perp}$.

Then, going back to the proof of the lemma, we find that $\varphi_k(\bigwedge^k V) = \bigwedge^{n-k} W$ and so $\mathbf{Y} = \varphi_k(\mathbf{X})$ for some generator \mathbf{X} of $\bigwedge^k V$. In defining $\omega_{k-1}(\boldsymbol{\xi})$ through condition (i), we may thus assume that \mathbf{X} has this form. For such a point, Lemma 3.6 yields $D_{\boldsymbol{\xi}}^*(\mathbf{X}) = \operatorname{dist}_w(\boldsymbol{\xi}, \overline{V})^{d_w/d} H(V)$, where $\overline{V} = \langle V \rangle_{K_w}$ is the topological closure of V in K_w^n , and the claim follows.

11. The principle of Thunder

In his alternative proof of the adelic Minkowski theorem [24], Jeff Thunder relates the successive minima of a convex body of $K^n_{\mathbb{A}}$ to those of an appropriate convex body of $\mathbb{Q}^{dn}_{\mathbb{A}}$. We formulate his idea below as a general principle.

To this end, we use the following construction, where \mathbb{Q}^{dn} and K^n are viewed, respectively, as subsets of $\mathbb{Q}^{dn}_{\mathbb{A}}$ and $K^n_{\mathbb{A}}$ under the diagonal embedding.

LEMMA 11.1. — Let $T: \mathbb{Q}^{dn} \to K^n$ be a \mathbb{Q} -linear isomorphism. For each place u of \mathbb{Q} and each place v of K above u, we denote by $T_v: \mathbb{Q}_u^{dn} \to K_v$ the \mathbb{Q}_u -linear map that extends T. Then

$$\begin{array}{ccc} T_u : \mathbb{Q}_u^{dn} \longrightarrow \prod_{v|u} K_v^n \\ \mathbf{x} \longmapsto (T_v(\mathbf{x}))_{v|u} \end{array}$$

is a \mathbb{Q}_u -linear isomorphism. Moreover, the map

$$T_{\mathbb{A}} : \mathbb{Q}^{dn}_{\mathbb{A}} \longrightarrow K^{n}_{\mathbb{A}} \\ (\mathbf{x}_{u})_{u \in M(\mathbb{Q})} \longmapsto ((T_{v}(\mathbf{x}_{u}))_{v|u})_{u \in M(\mathbb{Q})}$$

is the unique $\mathbb{Q}_{\mathbb{A}}$ -linear map that extends T. It is also an isomorphism.

Proof. — By construction the map T_u is \mathbb{Q}_u -linear with domain and codomain of the same dimension $dn = \sum_{v|u} d_v n$ as vector spaces over \mathbb{Q}_u . Moreover, $T_u(\mathbb{Q}^{dn})$ is the image of K^n in $\prod_{v|u} K_v^n$ under the diagonal embedding, which is dense in this product. So T_u is surjective and, therefore, it is an isomorphism. This proves the first assertion, and the others follow from it.

PROPOSITION 11.2 (Thunder's principle). — With the notation of Lemma 11.1, let \mathcal{K} be a convex body of $K^n_{\mathbb{A}}$. Then, $\mathcal{C} := T^{-1}_{\mathbb{A}}(\mathcal{K})$ is a convex body of $\mathbb{Q}^{dn}_{\mathbb{A}}$, and we have

$$\lambda_{d(i-1)+j}(\mathcal{C}) \asymp \lambda_i(\mathcal{K}) \quad (1 \le i \le n, \ 1 \le j \le d),$$

with implied constants that depend only on K, n and T.

Proof. — Writing $\mathcal{K} = \prod_{v \in M(K)} \mathcal{K}_v$, we find that $\mathcal{C} = \prod_{u \in M(\mathbb{Q})} \mathcal{C}_u$, where

$$\mathcal{C}_u = \bigcap_{v|u} T_v^{-1}(\mathcal{K}_v)$$

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is a convex body of \mathbb{Q}_{u}^{dn} for each $u \in M(\mathbb{Q})$. For all but finitely many places $u \neq \infty$, we also have that $T_{u}(\mathbb{Z}_{u}^{dn}) = \prod_{v|u} \mathcal{O}_{v}^{n} = \prod_{v|u} \mathcal{K}_{v}$ and so $\mathcal{C}_{u} = \mathbb{Z}_{u}^{dn}$. Thus, \mathcal{C} is a convex body of $\mathbb{Q}_{\mathbb{A}}^{dn}$.

Choose linearly independent elements $\mathbf{y}_1, \ldots, \mathbf{y}_n$ of K^n over K which realize the successive minima of \mathcal{K} , choose a basis $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ of the ring of integers \mathcal{O}_K of K as a \mathbb{Z} -module and let $c = \max\{|\boldsymbol{\omega}_j|_v; 1 \leq j \leq d \text{ and } v \mid \infty\}$. Then the points

$$\mathbf{x}_{i,j} := T^{-1}(\omega_j \mathbf{y}_i) \in \mathbb{Q}^{dn} \quad (1 \le i \le n, \ 1 \le j \le d)$$

are linearly independent over \mathbb{Q} . For each indexing pair (i, j), we also have $\mathbf{y}_i \in \lambda_i(\mathcal{K})\mathcal{K}$ and $\omega_j\mathcal{K} \subseteq c\mathcal{K}$, thus $\omega_j\mathbf{y}_i \in c\lambda_i(\mathcal{K})\mathcal{K}$. As T_{∞} is linear over $\mathbb{Q}_{\infty} = \mathbb{R}$, this means that

$$\mathbf{x}_{i,j} \in T^{-1}_{\mathbb{A}}(c\lambda_i(\mathcal{K})\mathcal{K}) = c\lambda_i(\mathcal{K})\mathcal{C}.$$

In view of the linear independence of the points $\mathbf{x}_{i,j}$ and the fact that we have $\lambda_1(\mathcal{K}) \leq \cdots \leq \lambda_n(\mathcal{K})$, we conclude that

$$\lambda_{d(i-1)+j}(\mathcal{C}) \le c\lambda_i(\mathcal{K}) \quad (1 \le i \le n, \ 1 \le j \le d).$$

On the other hand, since $T_{\mathbb{A}}$ is $\mathbb{Q}_{\mathbb{A}}$ -linear and invertible, we have $\mu(\mathcal{C}) = c'\mu(\mathcal{K})$ for some constant c' > 0 which depends only on $T_{\mathbb{A}}$. So, the adelic Minkowski's theorem 4.1, applied to \mathcal{C} and \mathcal{K} separately, yields

$$\lambda_1(\mathcal{C})\cdots\lambda_{dn}(\mathcal{C}) \asymp \mu(\mathcal{C})^{-1} \asymp \mu(\mathcal{K})^{-1} \asymp (\lambda_1(\mathcal{K})\cdots\lambda_n(\mathcal{K}))^d.$$

Jusion follows.

The conclusion follows.

12. Proofs of Theorems B and C

We fix a basis $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ of K over \mathbb{Q} , a place w of K of local degree $d_w = 1$ above a place ℓ of \mathbb{Q} , so that $K_w = \mathbb{Q}_\ell$, and a non-zero point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{Q}_\ell^n$, assuming $n \geq 2$. We form

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi}) \in (K_w^n)^d = (\mathbb{Q}_\ell^n)^d,$$

and note that

(51)
$$\|\Xi\|_{\ell} = \|\boldsymbol{\alpha}\|_{w} \|\boldsymbol{\xi}\|_{w}.$$

In order to apply the results of Section 7, we need to adjust Definition 7.2 so that the family of convex bodies in $\mathbb{Q}^{dn}_{\mathbb{A}}$ attached to Ξ , and the family of convex bodies in $K^n_{\mathbb{A}}$ attached to $\boldsymbol{\xi}$ do not depend on the norms of those points. Thus, for each $q \geq 0$, we define

$$\mathcal{C}_{\Xi}(q) = \left\{ (\mathbf{x}_u) \in \mathbb{Q}_{\mathbb{A}}^{dn}; \|\Xi\|_{\ell}^{-1} |\mathbf{x}_{\ell} \cdot \Xi|_{\ell} \leq e^{-q} \text{ and } \|\mathbf{x}_u\|_u \leq 1 \text{ for each } u \in M(\mathbb{Q}) \right\},\$$
$$\mathcal{C}_{\xi}(q) = \left\{ (\mathbf{y}_v) \in K_{\mathbb{A}}^n; \|\boldsymbol{\xi}\|_w^{-1} |\mathbf{y}_w \cdot \boldsymbol{\xi}|_w \leq e^{-qd} \text{ and } \|\mathbf{y}_v\|_v \leq 1 \text{ for each } v \in M(K) \right\},\$$
since $d_w = 1$. In order to relate the minima of these convex bodies and to deduce relationships between the standard four exponents of approximation attached

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to the triples (Ξ, \mathbb{Q}, ℓ) and $(\boldsymbol{\xi}, K, w)$, we consider the \mathbb{Q} -linear isomorphism $T: (\mathbb{Q}^n)^d \to K^n$ given by

$$T(\mathbf{x}_1,\ldots,\mathbf{x}_d) = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_d \mathbf{x}_d,$$

for any $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{Q}^n$. For each place u of \mathbb{Q} and each place v of K above u, it extends by continuity to a \mathbb{Q}_u -linear map T_v from $(\mathbb{Q}_u^n)^d$ to K_v^n given by

$$T_{\nu}(\mathbf{x}_1,\ldots,\mathbf{x}_d)=\alpha_1\mathbf{x}_1+\cdots+\alpha_d\mathbf{x}_d,$$

for any $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{Q}_u^n$. Following Lemma 11.1, this yields a \mathbb{Q}_u -linear isomorphism T_u from \mathbb{Q}_u^{nd} to $\prod_{\nu|u} K_{\nu}^n$, as well as a $\mathbb{Q}_{\mathbb{A}}$ -linear isomorphism $T_{\mathbb{A}}$ from $\mathbb{Q}_{\mathbb{A}}^{nd}$ to $K_{\mathbb{A}}^n$.

PROPOSITION 12.1. — There is an idèle $\mathbf{a} \in K^*_{\mathbb{A}}$ that depends only on K, w, $\mathbf{\alpha}$ and n such that, for each $q \geq 0$,

$$a^{-1}\mathcal{C}_{\Xi}(dq) \subseteq T^{-1}_{\mathbb{A}}(\mathcal{C}_{\xi}(q)) \subseteq a \mathcal{C}_{\Xi}(dq).$$

Proof. — Fix a place u of \mathbb{Q} . Since T_u is a \mathbb{Q}_u -linear isomorphism, there exists a constant $c_u \geq 1$, such that

$$c_u^{-1} \|\mathbf{x}\|_u \leq \max_{v|u} \|T_v(\mathbf{x})\|_v \leq c_u \|\mathbf{x}\|_u,$$

for all $\mathbf{x} \in \mathbb{Q}_u^{nd}$. Since $T_u(\mathbb{Z}_u^{nd}) = \prod_{v|u} \mathcal{O}_v^n$ for all but finitely many $u \neq \infty$, we may take $c_u = 1$ for those u. When $u = \ell$ and $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_d) \in (\mathbb{Q}_\ell^n)^d$, we also find

$$\mathbf{x} \cdot \Xi = \sum_{i=1}^{d} \alpha_i \mathbf{x}_i \cdot \boldsymbol{\xi} = T_w(\mathbf{x}) \cdot \boldsymbol{\xi},$$

thus $|\mathbf{x} \cdot \Xi|_{\ell} = |T_w(\mathbf{x}) \cdot \boldsymbol{\xi}|_w$. Then, using (51) and assuming that we have $c_{\ell} \geq \max\{\|\boldsymbol{\alpha}\|_w, \|\boldsymbol{\alpha}\|_w^{-1}\}$, we deduce that

$$c_{\ell}^{-1} \frac{|\mathbf{x} \cdot \Xi|_{\ell}}{\|\Xi\|_{\ell}} \le \frac{|T_w(\mathbf{x}) \cdot \boldsymbol{\xi}|_w}{\|\boldsymbol{\xi}\|_w} \le c_{\ell} \frac{|\mathbf{x} \cdot \Xi|_{\ell}}{\|\Xi\|_{\ell}}.$$

Choose an idèle $\mathbf{a} = (a_u) \in \mathbb{Q}^*_{\mathbb{A}}$ such that $|a_u|_u \geq c_u$ for each $u \in M(\mathbb{Q})$. Then, if $\mathbf{x} = (\mathbf{x}_u) \in \mathbb{Q}^{nd}_{\mathbb{A}}$ and $\mathbf{y} = (\mathbf{y}_v) \in K^n_{\mathbb{A}}$ are related by $\mathbf{y} = T_{\mathbb{A}}(\mathbf{x})$, the above estimates yield

$$\begin{aligned} \|a_u^{-1}\mathbf{x}_u\|_u &\leq \max_{\nu|u} \|\mathbf{y}_\nu\|_\nu \leq \|a_u\mathbf{x}_u\|_u, \\ \frac{|a_\ell^{-1}\mathbf{x}_\ell \cdot \Xi|_\ell}{\|\Xi\|_\ell} &\leq \frac{|\mathbf{y}_w \cdot \boldsymbol{\xi}|_w}{\|\boldsymbol{\xi}\|_w} \leq \frac{|a_\ell\mathbf{x}_\ell \cdot \Xi|_\ell}{\|\Xi\|_\ell} \end{aligned}$$

So, if $\mathbf{y} \in \mathcal{C}_{\boldsymbol{\xi}}(q)$ (resp. $a\mathbf{x} \in \mathcal{C}_{\Xi}(dq)$) for some $q \geq 0$, then $a^{-1}\mathbf{x} \in \mathcal{C}_{\Xi}(dq)$ (resp. $\mathbf{y} \in \mathcal{C}_{\boldsymbol{\xi}}(q)$). This means that we have $a^{-1}T_{\mathbb{A}}^{-1}(\mathcal{C}_{\boldsymbol{\xi}}(q)) \subseteq \mathcal{C}_{\Xi}(dq)$ (resp. $a^{-1}\mathcal{C}_{\Xi}(dq) \subseteq T_{\mathbb{A}}^{-1}(\mathcal{C}_{\boldsymbol{\xi}}(q))$), as needed. \Box

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By Proposition 5.3 and Thunder's principle (Proposition 11.2), the above result yields the following estimates.

COROLLARY 12.2. — For each i = 1, ..., n, each j = 1, ..., d and each $q \ge 0$, we have

(52)
$$\lambda_{d(i-1)+j}(\mathcal{C}_{\Xi}(dq)) \asymp \lambda_i(\mathcal{C}_{\xi}(q)),$$

with implicit constants that depend only on K, α , n and w.

Proof of Theorem B. — Fix i, j and q as above. Taking logarithms on both sides of (52) and using Lemma 7.4, we obtain that the absolute value of the difference

(53)
$$L_{\Xi,d(i-1)+j}(dq) - L_{\boldsymbol{\xi},i}(q)$$

is bounded above by a constant that depends only on K, α , n and w. Letting i' = n + 1 - i and j' = d + 1 - j, Lemma 7.8 shows that the same applies to

$$L_{\Xi,d(i-1)+j}(dq) + L^*_{\Xi,d(i'-1)+j'}(dq) - dq \quad \text{and} \quad L_{\xi,i}(q) + L^*_{\xi,i'}(q) - q.$$

Subtracting from the first number the sum of the second and of (53), we obtain that

$$L^*_{\Xi,d(i'-1)+j'}(dq) - L^*_{\xi,i'}(q) - (d-1)q$$

also has absolute value bounded above by such a constant. As i' runs from 1 to n with i, and as j' runs from 1 to d with j, this proves the two inequalities of Theorem B.

COROLLARY 12.3. — Upon writing $\widehat{\omega}(\boldsymbol{\xi})$ for $\widehat{\omega}(\boldsymbol{\xi}, K, w)$, $\widehat{\omega}(\Xi)$ for $\widehat{\omega}(\Xi, \mathbb{Q}, \ell)$, and similarly for the other exponents, we have

$$d(\widehat{\omega}(\boldsymbol{\xi})+1) = \widehat{\omega}(\Xi)+1, \qquad \qquad d(\omega(\boldsymbol{\xi})+1) = \omega(\Xi)+1, \\ d\left(\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})}+1\right) = \frac{1}{\widehat{\lambda}(\Xi)}+1, \qquad \qquad d\left(\frac{1}{\lambda(\boldsymbol{\xi})}+1\right) = \frac{1}{\lambda(\Xi)}+1.$$

Proof. — By Theorem B, the ratios $L_{\Xi,1}(dq)/q$ and $L_{\xi,1}(q)/q$ have the same limit points as q goes to infinity. In particular, they have the same superior limit and the same inferior limit. Applying Lemma 10.1 separately to (Ξ, \mathbb{Q}, ℓ) and (ξ, K, w) to compute these limits and comparing the results, we get the first row of equalities.

By Theorem B, the quantities $L^*_{\Xi,1}(dq)/q - d$ and $L^*_{\boldsymbol{\xi},1}(q)/q - 1$ also have the same limit points as q goes to infinity, and the same lemma yields the second row of equalities.

As an application, suppose that n = 3 and that $\boldsymbol{\xi} \in K_w^3$ has linearly independent coordinates over K. Then, $\boldsymbol{\xi}$ satisfies Jarník's identity (3) by Corollary 10.3. Using the formulas of the above corollary, we deduce the identity (6) relating $\widehat{\lambda}(\Xi)$ and $\widehat{\omega}(\Xi)$.

Proof of Theorem C. — Let S denote the subset of K_w^3 from Theorem 2.2 of Bel. Since the supremum of $\hat{\lambda}(\boldsymbol{\xi}, K, w)$ is $1/\gamma$ as $\boldsymbol{\xi}$ runs through S, the formulas of Corollary 12.3 imply that for the corresponding points $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi}$, the supremum of $\hat{\lambda}(\Xi, \mathbb{Q}, \ell)$ is $1/(d\gamma^2 - 1)$. Since the points $\boldsymbol{\xi}$ of S satisfy Jarník's identity (3), the supremum of $\hat{\omega}(\boldsymbol{\xi}, K, w)$ is γ^2 as $\boldsymbol{\xi}$ runs through S. So, for the corresponding points $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi}$, we find similarly that the supremum of $\hat{\omega}(\Xi, \mathbb{Q}, \ell)$ is $d(\gamma^2 + 1) - 1$. This proves Theorem C because, for $\boldsymbol{\xi} = (1, \xi, \xi^2) \in S$, the 3d coordinates of $(\boldsymbol{\alpha}, \boldsymbol{\xi} \boldsymbol{\alpha}, \xi^2 \boldsymbol{\alpha})$ form a permutation of those of $\Xi = (\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi})$, and so these two points have the same exponents of approximation.

As a final remark, suppose that $\mathbf{P} = (P_1, \ldots, P_n)$ is an *n*-system on $[0, \infty)$ for which $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded. Then the difference $\mathbf{L}_{\boldsymbol{\Xi}} - \mathbf{R}$ is bounded for the function $\mathbf{R} = (R_1, \ldots, R_{nd})$ from $[0, \infty)$ to \mathbb{R}^{nd} given by

$$R_{d(i-1)+j}(q) = P_i(q/d) \quad (1 \le i \le n, \ 1 \le j \le d, \ q \ge 0).$$

If d > 1, this is not an *nd*-system because its components are piecewise linear with slopes 0 and 1/d. However, it is a generalized *nd*-system in the sense of [18, Definition 4.5], and so it can easily be approximated uniformly by an *nd*-system, as explained in [18, Section 4].

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