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ABOUT $\mathcal{C}^\infty$ FOLIATIONS BY HOLOMORPHIC CURVES ON COMPLEX SURFACES

BY OLIVIER THOM

ABSTRACT. — We study those real $\mathcal{C}^\infty$ foliations in complex surfaces whose leaves are holomorphic curves. The main motivation is to try and understand these foliations in neighborhoods of curves: can we expect the space of foliations in a fixed neighborhood to be infinite-dimensional, or are there some contexts under which every such foliation is holomorphic?

We give some restrictions and study in more detail the geometry of foliations whose leaves belong to a holomorphic family of holomorphic curves. In particular, we classify all real-analytic foliations on neighborhoods of curves that are locally diffeomorphic to foliations by lines, under some non-degeneracy hypothesis.

RÉSUMÉ (*Sur les feuilletages $\mathcal{C}^\infty$ par courbes holomorphes dans les surfaces complexes*). — Nous étudions ces feuilletages $\mathcal{C}^\infty$ dans des surfaces complexes dont les feuilles sont des courbes holomorphes. La principale motivation est d'essayer de comprendre ces feuilletages dans des voisinages de courbes: peut-on s'attendre à ce que l'espace des feuilletages dans un voisinage fixé soit de dimension infinie, ou y a-t-il des contextes dans lesquels chacun des ces feuilletages est holomorphe?

Nous donnons quelques restrictions et étudions plus en détail la géométrie des feuilletages dont les feuilles appartiennent à une famille holomorphe de courbes holomorphes. En particulier, nous classifions tous les feuilletages analytiques réels dans des voisinages de courbes qui sont localement difféomorphes à des feuilletages par droites, sous certaines hypothèses de non-dégénéricité.

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1. Introduction

In this paper, we study real $\mathcal{C}^\infty$ foliations in complex surfaces whose leaves are holomorphic curves; we will call such foliations *semiholomorphic* for short in this text.

These foliations appear naturally in many different contexts and were studied from different points of view, sometimes under different names. After giving some definitions, we will briefly review some problems related to semiholomorphic foliations and recall some results that might be of interest.

1.1. Equations. — Let $U \subset \mathbb{C}^2$ be an open set and $\mathcal{F}$ a C^∞ real codimension 2 foliation on U. The foliation $\mathcal{F}$ is called *semiholomorphic* if the subsheaf $T\mathcal{F} \subset TU$ consists of holomorphic directions: $T\mathcal{F} \subset T^{1,0}U$.

Suppose that U is equipped with holomorphic coordinates (x, y) and that the foliation is smooth and nowhere vertical: $\frac{\partial}{\partial y} \notin T_p\mathcal{F}$ for every $p \in U$. Then the foliation can be described by the $(1,0)$-form $\omega = dy - \lambda dx$, where $\lambda \in C^\infty(U, \mathbb{C})$ is the slope. This $(1,0)$-form satisfies the integrability condition

$$\omega \wedge \overline{\omega} \wedge d\omega = 0. \tag{1}$$

In terms of the function λ, this equation writes:

$$\frac{\partial \lambda}{\partial \bar{x}} + \overline{\lambda} \frac{\partial \lambda}{\partial \bar{y}} = 0. \tag{2}$$

Conversely, a field of holomorphic directions written as the kernel of a $(1,0)$-form ω defines a semiholomorphic foliation if and only if the integrability condition (1) is satisfied.

Suppose that $\mathcal{F}$ is smooth on U and that L is a real codimension 2 subvariety of U invariant by $\mathcal{F}$. Then at every point $p \in L$, the tangent space TL is a complex direction. This shows that L is a complex curve, and that a semiholomorphic foliation is exactly a C^∞ foliation by holomorphic curves.

Example 1.1. — Consider the $(1,0)$-form $\mathrm{Im}(x)dy - \mathrm{Im}(y)dx$. It satisfies equation (1) and so defines a semiholomorphic foliation. This foliation has singular set $\mathbb{R} \times \mathbb{R}$. The leaf passing through $(x_0, y_0) \notin \mathbb{R}^2$ has the equation

$$\begin{aligned} y &= y_0 + \frac{\mathrm{Im}(y_0)}{\mathrm{Im}(x_0)}(x - x_0) \\ &= \frac{\mathrm{Im}(y_0)}{\mathrm{Im}(x_0)} x + \left(\mathrm{Re}(y_0) - \frac{\mathrm{Im}(y_0)}{\mathrm{Im}(x_0)} \mathrm{Re}(x_0) \right). \end{aligned}$$

Note that the leaves of this foliation are exactly the complex affine lines with real parameters.

1.2. Holomorphic motion. — Consider an open set $U \subset \mathbb{C}^2$ and a semiholomorphic foliation $\mathcal{F}$ in U. Fix a transverse holomorphic fibration $\{x = cte\}$ on U and a local trivialization (x, y) of this fibration. Write T_x as the fiber above x. Consider also an origin $0 \in U$ and suppose that the curve $\{y = 0\}$ is a leaf of $\mathcal{F}$.

For each x, consider the holonomy transport $\varphi_x : T_0 \to T_x$ obtained by following the leaves of $\mathcal{F}$. This is a family of $\mathcal{C}^\infty$ diffeomorphisms depending holomorphically on the parameter x. Using the trivialization, we can consider φ_x as a family of germs of diffeomorphisms in the variable y. Conversely, given any family φ_x of germs of diffeomorphisms depending holomorphically on the parameter x, we can construct a semiholomorphic foliation transverse to the fibration $\{x = cte\}$ by taking trajectories of points $y_0 \in T_0$.

To find the equivalence with the previous point of view, consider a point $(x, y) \in U$ and write y_0 as the point of intersection between T_0 and the leaf passing through (x, y). This means that $y = \varphi_x(y_0)$, or rather $y_0 = \varphi_x^{-1}(y)$. Then the slope of the foliation is

$$\lambda(x, y) = \frac{\partial \varphi}{\partial x}(x, \varphi_x^{-1}(y)).$$

Example 1.2. — In Example 1.1, the holonomy transport between transversals above $x_0 = i$ and x is written as

$$\varphi_x(y) = \mathrm{Re}(y) + x\mathrm{Im}(y).$$

This point of view appears naturally in holomorphic dynamics; see, for example, [11] and [14]. The main problem studied from this point of view is the extension of the semiholomorphic foliation in the transverse direction; the results show that the transversal behavior is very similar to that of $\mathcal{C}^\infty$ objects (for example, the proof of [14, §2] involves partitions of unity). The most complete theorem obtained in this direction seems to be [13]: in $\mathbb{D} \times \mathbb{C}$, any set of disjoint complex curves $(C_e)_{e \in E}$ transverse to the fibers $\mathbb{C}$ can be extended to a semiholomorphic foliation in $\mathbb{D} \times \mathbb{C}$ transverse to the vertical fibration.

1.3. Levi-flat hypersurfaces and Ueda theory. — An interesting motivation for semiholomorphic foliations comes from Ueda theory and, more generally, the study of Levi-flat hypersurfaces. Recall that a smooth $\mathcal{C}^\infty$ real hypersurface H in a complex surface S is called Levi-flat if the field of complex directions $TH \cap J(TH)$ is integrable on H, where $J : TS \to TS$ denotes the operator given by multiplication by i.

It follows that H is foliated by complex curves. In the $\mathcal{C}^\infty$ category, we cannot say anything a priori about the transverse regularity of this foliation; however, when a Levi-flat hypersurface H is real analytic, then by [5] this foliation can be extended to a holomorphic foliation in a neighborhood of H. Note that if we have a smooth $\mathcal{C}^\infty$ foliation by smooth Levi-flat hypersurfaces $\mathcal{H}$ in S, the field of complex directions $T\mathcal{H} \cap J(T\mathcal{H})$ defines a smooth semiholomorphic foliation.

These kinds of objects appear naturally in the study of neighborhoods of compact complex curves; by a theorem of T. Ueda [16], for a smooth compact curve C in a complex surface S with torsion normal bundle, either C admits a system of strictly pseudo-concave neighborhoods, or C admits a logarithmic 1-form ω with poles along C and purely imaginary periods. The foliation defined by this 1-form is a smooth holomorphic foliation admitting C as a leaf and with unitary holonomy. It follows that there exists in S a foliation by Levi-flat hypersurfaces, each of which is the border of a neighborhood of C, and that the foliation defined by ω is the semiholomorphic foliation tangent to it. Thus, when N_C is torsion, any semiholomorphic foliation defined in a neighborhood of C, tangent to a foliation by Levi-flat hypersurfaces of this kind, is necessarily holomorphic.

This theorem is also true for smooth compact curves C with a generic normal bundle of degree 0, but in non-generic cases, the question is still open (typically, the non-generic cases correspond to neighborhood S of C, which are formally but not analytically linearizable).

An interesting question would then be to understand when a neighborhood S of a compact curve C admits smooth semiholomorphic foliations tangent to C, which are not holomorphic, and, in particular, whether these semiholomorphic foliations are exceptional objects or if any neighborhood S admits one of them.

1.4. Teichmüller theory. — Holomorphic motions, and thus semiholomorphic foliations, appear naturally in the context of Teichmüller theory, giving rise to some interesting examples; see, for example, [3] and [12] for more details.

Let us just give one example to explain the link between the two. Consider the plane $\mathbb{C}$ of the variable y, and a polygon P_i inside it equipped with identifications of opposite parallel sides, giving rise to a translation surface C_i. Suppose that one side is included in the real axis and that P_i is contained in the upper half-plane $\mathbb{H}_y$. Now, for every $x \in \mathbb{H}$, consider the linear application $\varphi_x \in \mathrm{GL}_2(\mathbb{R})$ fixing the real axis and sending i to x. The application φ_x sends the polygon P_i to some polygon P_x defining a translation surface C_x.

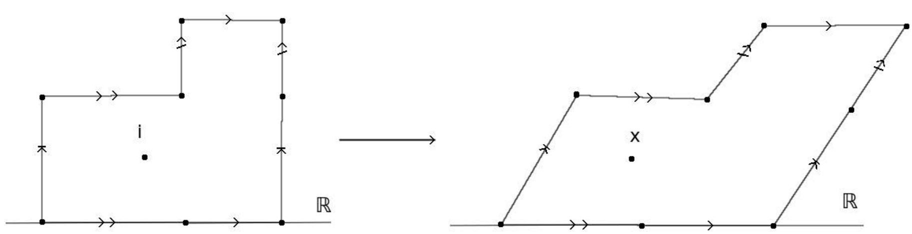

FIGURE 1.1. A deformation of translation surfaces

We can consider the space of pairs $(x, y) \in \mathbb{H} \times \mathbb{C}$ as a complex surface equipped with a holomorphic motion (φ_x); the set of points (x, y) with $y \in P_x$ is stable by the motion, and we can see the union of the translation surfaces $S = \cup_x C_x$ as a quotient of it. The semiholomorphic foliation $\mathcal{F}$ defined by (φ_x) descends to a semiholomorphic foliation on the bundle of translation surfaces S. Note that the application φ_x writes as $\varphi_x(y) = \mathrm{Re}(y) + x\mathrm{Im}(y)$, so that locally the foliation $\mathcal{F}$ is, in fact, that of Example 1.1.

When the translation surfaces admit a lattice of symmetries L, the foliation descends to a foliation on the surface S/L. However, by standard arguments, the lattice L cannot be cocompact. Indeed, if S/L were compact, the length of the shortest loop in C_x, being a continuous function of x, should be bounded by below. However, this length tends to zero for $C_{(ti)}$, when $\mathbb{R} \ni t \to 0$.

1.5. Monge–Ampère foliations. — A large class of examples is also given by Monge–Ampère foliations, in the sense of [2]: given a real plurisubharmonic function $f \in \mathcal{C}^\infty(U)$ in some open set $U \subset \mathbb{C}^2$, introduce the complex hessian $\Omega = \partial\overline{\partial} f$: it is a $(1,1)$-form, and whenever $\Omega \wedge \Omega = 0$, the field of complex directions X defined by the equation $i_X \Omega = 0$ is a semiholomorphic foliation. In dimension 2, each foliation can be obtained this way, but for foliations of higher codimension, the two notions are no more equivalent (see [2] and [7] for more details).

In the article [7], the authors also studied general semiholomorphic foliations and obtained some interesting results. More precisely, following the ideas of [1] for Monge–Ampère foliations, they studied Bott's partial connection of a semiholomorphic foliation $\mathcal{F}$ which is not holomorphic and showed that it induces a connection of negative curvature on the normal bundle $N^{1,0}\mathcal{F}$. This allows us to define an intrinsic metric of curvature -4 on the leaves of $\mathcal{F}$.

These two facts have interesting consequences, for example that a semiholomorphic foliation whose leaves are compact is always holomorphic [7, Example 6.6] (see also [17] for another proof of this fact using different tools), or that a semiholomorphic foliation whose leaves are parabolic is necessarily holomorphic [10].

1.6. Summary of this article. — We begin in Section 2.1 by recalling how the antiholomorphic part $\eta_\mathcal{F}$ of Bott's partial connection gives a hyperbolic singular metric $|\eta_\mathcal{F}|^2$ on the leaves of $\mathcal{F}$, as was already noted in [7]. We then carry on studying the metric $|\eta_\mathcal{F}|^2$ to obtain finer results.

It follows from [7] that if a smooth semiholomorphic foliation $\mathcal{F}$ on a surface S has a compact curve C, then $C \cdot C < 0$. In Theorem 2.8, we show also that $C \cdot C \geq 1 - g$, with equality when the form $i_C^* \eta_\mathcal{F}$ has no zeroes (remember that the metric $|\eta_\mathcal{F}|^2$ is well defined, so that even if $\eta_\mathcal{F}$ is not a well-defined 1-form on C, its zeroes are well defined).

Also, we note in Theorem 2.10 that under some reasonable hypotheses, the leaves of a non-holomorphic semiholomorphic foliation should be complete for the metric $|\eta_{\mathcal{F}}|^2$. This reminds us of the article [4] with the differences that in the semiholomorpic context, the metric arises naturally from the foliation but can be a singular metric.

In Section 3, we introduce some other geometric invariants of semiholomorphic foliations $\mathcal{F}$. Informally, we consider the smallest holomorphic family of holomorphic curves containing the leaves of $\mathcal{F}$, call it a *system of curves* defined by $\mathcal{F}$ and call *semirank* its number of parameters. These notions are invariant under biholomorphisms, and although a generic $\mathcal{C}^\infty$ foliation will be of infinite semirank, we show in Proposition 3.6 that real-analytic semiholomorphic foliations are of semirank 2. Note also that all the examples where the leaves are lines are of semirank 2, so this motivates us to restrict our attention to foliations of semirank 2 for the rest of the article.

This additional hypothesis gives us some new tools to study these objects; for example, we can use the duality between two-parameter families of curves. This topic has a long history, and we will refer the reader to [8] and the references therein for more details. In a local context, note that if we fix a two-parameter family of curves, the set of parameters is a complex surface $\check{U}$ and that a semiholomorphic foliation gives a real surface $S_{\mathcal{F}}$ in $\check{U}$. Studying $\mathcal{F}$ then reduces to studying $S_{\mathcal{F}}$. In particular, if $\mathcal{F}$ is the universal cover of a semiholomorphic foliation in the neighborhood of a curve C, then the fundamental group of C acts by holomorphic automorphisms of the system of curves, and its action on the dual $\check{U}$ stabilizes the real surface $S_{\mathcal{F}}$. These conditions are very strong, and we will use them extensively.

As we saw in Theorem 2.10, we can expect interesting examples to have complete leaves. At the end of this section, we try to describe what a semiholomorphic foliation in an open set $U \subset \mathbb{C}^2$ with complete leaves should look like in semirank 2, which should be the generic behavior for a semiholomorphic foliation of any semirank.

In the last section, we consider foliations whose system of curves is a projective structure in the sense of [8]. These projective structures are better understood than general families of curves, and, as we saw, they can arise naturally from global contexts. In particular, we classify all real-analytic foliations $\mathcal{F}$ on neighborhoods of compact curves C, which are locally diffeomorphic to foliations by lines, under the hypothesis that $i_C^*\eta_{\mathcal{F}}$ is not identically zero; see Theorems 4.1 and 4.2.

2. Bott's partial connection

2.1. Local expression. — Suppose in this section that $\mathcal{F}$ is transverse to the fibration $x = cte$. Consider the form $\omega_0 = dy - \lambda(x,y)dx$ defining the semi-holomorphic foliation $\mathcal{F}$. Note that

$$\begin{aligned} d\omega_0 &= \frac{\partial\lambda}{\partial y} dx \wedge dy + \frac{\partial\lambda}{\partial \bar{x}} dx \wedge d\bar{x} + \frac{\partial\lambda}{\partial \bar{y}} dx \wedge d\bar{y} \\ &= \left(\frac{\partial\lambda}{\partial y} dx\right) \wedge \omega_0 + \left(\frac{\partial\lambda}{\partial \bar{y}} dx\right) \wedge \overline{\omega_0}, \end{aligned}$$

and that for any function f, if $\omega = f\omega_0$, then

$$d\omega = \left(\frac{df}{f} + \frac{\partial\lambda}{\partial y} dx\right) \wedge \omega + \left(\frac{f^2}{|f|^2}\frac{\partial\lambda}{\partial \bar{y}} dx\right) \wedge \overline{\omega}.$$

If not for the factor $\frac{f^2}{|f|^2}$, the 1-form $\eta := \frac{\partial\lambda}{\partial\bar{y}} dx$ would be well defined modulo some multiples of ω and $\overline{\omega}$ and would define a $(1,0)$-form on the leaves of $\mathcal{F}$. Since this factor is of modulus 1, the metric on the leaves $|\eta|^2 = i\eta \wedge \overline{\eta}$ depends only on the foliation $\mathcal{F}$. We will write this 1-form $\eta_{\mathcal{F}}$ when the defining form ω is clear or irrelevant and call it the antiholomorphic part of Bott's connection.

The two following lemmas are immediate.

Lemma 2.1. — *The foliation $\mathcal{F}$ is holomorphic if and only if $\eta_{\mathcal{F}} = 0$; it is holomorphic at order 1 along a leaf L if and only if $i_L^*\eta_{\mathcal{F}} = 0$.*

Lemma 2.2. — *If Φ is a holomorphic germ of diffeomorphism around the origin, if η and $\widetilde{\eta}$ are the antiholomorphic parts of Bott's connection applied to ω and $\Phi^*\omega$, respectively, then $\widetilde{\eta} = \Phi^*\eta$.*

2.2. Tangential behavior. — Let us study more explicitly the coefficient $\frac{\partial\lambda}{\partial\bar{y}}$ in the form $\eta_{\mathcal{F}}$. First, let us introduce the differential operator

$$\partial_{\mathcal{F}} := \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y}.$$

We check that for every function $f : \mathbb{C}^2 \to \mathbb{C}$, the integrability of $\mathcal{F}$ guarantees $\partial_{\mathcal{F}}\overline{\partial_{\mathcal{F}}} f = \overline{\partial_{\mathcal{F}}}\partial_{\mathcal{F}} f$, so we can introduce the real operator

$$\Delta_{\mathcal{F}} = \partial_{\mathcal{F}}\overline{\partial_{\mathcal{F}}}.$$

Explicitly, we have $\Delta_{\mathcal{F}} f = \Delta_x f + |\lambda|^2\Delta_y f + \lambda\frac{\partial^2 f}{\partial\bar{x}\partial y} + \bar{\lambda}\frac{\partial^2 f}{\partial x\partial\bar{y}}$. This operator corresponds to the Laplacian in restriction to the leaves of $\mathcal{F}$; indeed, if $\{y = 0\}$ is a leaf of $\mathcal{F}$, then we have $\Delta_{\mathcal{F}} = \Delta_x$ on this leaf.

Put

$$a := \frac{\partial\lambda}{\partial y}, \quad b := \frac{\partial\lambda}{\partial\overline{y}}.$$

The operator $\overline{\partial_{\mathcal{F}}}$ does not commute with $\frac{\partial}{\partial y}$ (nor with $\frac{\partial}{\partial \bar{y}}$); more precisely, we have

$$\begin{cases} \overline{\partial_{\mathcal{F}}} a + \bar{b} b = 0 \\ \overline{\partial_{\mathcal{F}}} b + \bar{a} b = 0. \end{cases}$$

Introduce $\beta = \log(b) = \beta_1 + i\beta_2$, so that the second equation writes

$$\bar{a} = -\overline{\partial_{\mathcal{F}}}\beta,$$

and the first equation gives

$$\begin{cases} \Delta_{\mathcal{F}}\beta_1 = \exp(2\beta_1) \\ \Delta_{\mathcal{F}}\beta_2 = 0. \end{cases} \tag{3}$$

As we can see in [7, Thm 6.2 (a)], this equation gives:

LEMMA 2.3. — *The metric $|\eta_{\mathcal{F}}|^2$ on the leaves is of curvature -4 whenever it is not zero.*

Equation (3) will imply that b is very regular along the leaves; in fact, we can prove this directly using the point of view of holomorphic motions. Denote as before φ_x the holonomy transport of $\mathcal{F}$ between fibers of $\{x = cte\}$; the antiholomorphic part of Bott's connection is given by

$$\frac{\partial\lambda}{\partial\bar{y}}(x,y) = \frac{\partial^2\varphi}{\partial x\partial y}\left(x, \varphi_x^{-1}(y)\right)\frac{\partial\varphi_x^{-1}}{\partial\bar{y}}(y) + \frac{\partial^2\varphi}{\partial x\partial\bar{y}}\left(x, \varphi_x^{-1}(y)\right)\overline{\frac{\partial\varphi_x^{-1}}{\partial y}(y)}.$$

In restriction to a leaf $\{y = 0\}$, we can approximate φ_x by its linear part: $\varphi_x(y) = l_x(y) + O(|y|^2) = u(x)y + v(x)\bar{y} + O(|y|^2)$, where u and v are holomorphic functions of the parameter x. Note that

$$l_x^{-1}(y) = \frac{\overline{u}}{|u|^2 - |v|^2}y - \frac{v}{|u|^2 - |v|^2}\bar{y}.$$

Thus, we get in first-order approximation

$$\lambda(x,y) = u'(x)\frac{\overline{u}y - v\bar{y}}{|u|^2 - |v|^2} + v'(x)\frac{u\bar{y} - \bar{v}y}{|u|^2 - |v|^2} + O(|y|^2),$$

and

$$\frac{\partial\lambda}{\partial\bar{y}}(x,0) = \frac{uv' - u'v}{|u|^2 - |v|^2}(x).$$

In particular, we can see that the border of the leaf $\{y = 0\}$ is given by the real analytic curve $\{|u|^2 = |v|^2\}$; we can also deduce the following.

LEMMA 2.4. — *On a leaf of $\mathcal{F}$, the set of points x where $b(x) = 0$ is either discrete or the whole leaf. Moreover, if x is an isolated zero of b, then $\frac{\partial b}{\partial\bar{x}}(x) = 0$.*

2.3. Global setting. — In this section, S is a complex surface and $\mathcal{F}$ a smooth semiholomorphic foliation on S; we fix a covering $S = \cup U_i$, local coordinates (x_i, y_i) with $\mathcal{F}$ transverse to the fibration $x_i = cte$, and a $(1,0)$-form $\omega_i = f_i \cdot (dy_i - \lambda_i dx_i)$ defining $\mathcal{F}$ on each U_i.

The form $\eta_{\mathcal{F}}$ depends on the choice of the representant ω_i of $\mathcal{F}$, but as we have seen in Section 2.1, it is a section of a line bundle $T^{1,0}\mathcal{F} \otimes L$ where the transition functions of L are $\mathcal{C}^\infty$ functions of modulus 1. Putting together lemmas 2.1, 2.2, 2.3, and 2.4, we obtain the following:

Proposition 2.5. — *The metric $|\eta_{\mathcal{F}}|^2$ is intrinsically defined as a metric on the leaves of $\mathcal{F}$. On each leaf, this metric is either identically zero or only has isolated zeroes. If it is not identically zero, then it has curvature -4 away from its zeroes.*

The following corollary and Theorem 2.7 were already consequences of [7] and [10].

Corollary 2.6. — *Let S be a complex surface with a smooth holomorphic fibration π towards an elliptic curve. Then every smooth semiholomorphic foliation transverse to π is holomorphic.*

Theorem 2.7. — *Suppose S is a neighborhood of an elliptic curve C, and there exists a singular $\mathcal{C}^\infty$ foliation $\mathcal{H}$ by compact Levi-flat hypersurfaces such that C is invariant by $\mathcal{H}$, the foliation $\mathcal{H}$ is smooth outside C, and every leaf of $\mathcal{H}$ is the border of a neighborhood of C. Suppose, moreover, that for every sequence of points $p_n \in S$ with $p_n \to p_\infty \in C$ and such that the sequence $T_{p_n}\mathcal{H}$ has a limit H_∞, then $T_{p_\infty}C \subset H_\infty$.*

Then $\mathcal{H}$ is tangent to a holomorphic foliation.

Proof. — The field of complex directions $\mathcal{F} = T^{1,0}\mathcal{H}$ is integrable, so $\mathcal{F}$ is a semiholomorphic foliation. Under the hypotheses, for every local fibration $\{x_i = cte\}$ transverse to C on an open set $U_i \subset S$, this fibration is also transverse to $\mathcal{H}$ in a neighborhood of $C \cap U_i$. Thus, it is transverse to the foliation $\mathcal{F}$, which implies in particular that $\mathcal{F}$ is smooth in a neighborhood of C.

When $|\eta_{\mathcal{F}}|^2$ is not identically zero, it is a metric of curvature -4, so we only need to prove that every leaf of $\mathcal{F}$ is uniformized by $\mathbb{C}$. Choose a Kähler form ω on S so that we can compute the curvature of the leaves using ω. Consider a smooth $\mathcal{C}^\infty$ foliation $\mathcal{G}$ transverse to $\mathcal{F}$. For any base point $p_0 \in C$, if we denote by G_0 the leaf of $\mathcal{G}$ passing through p_0, we get an application $f : G_0 \times \widetilde{C} \to \widetilde{S}$ following the leaves of $\mathcal{F}$, where $\widetilde{C}$ is the universal cover of C and $\widetilde{S}$ that of S. Consider the map $f(y_0, \cdot) : C \cap U_i \to L_0 \cap U_i$ on a small open set U_i, where $y_0 \in G_0$ and L_0 is the leaf of $\mathcal{F}$ passing through y_0; by continuity of the metric induced by ω on the leaves of $\mathcal{F}$, this application is a quasi-isometry, and we can choose constants of quasi-isometry that do not depend on y_0. Since there are a finite number of these U_i, we see that $f(y_0,\cdot)$ is a quasi-isometry between $\widetilde{C}$ and the universal cover $\widetilde{L}$ of the leaf L of $\mathcal{F}$ passing through y_0, for any $y_0\in G_0$. We conclude that any leaf is uniformized by $\mathbb{C}$ as needed. □

Note that the proof is somewhat more general: we only need a smooth semiholomorphic foliation whose leaves stay inside a neighborhood of C.

We will see in the explicit examples of Section 4.1 that for a semiholomorphic foliation with a compact leaf, the normal and tangent bundles of this leaf are closely related; in general, we can prove the following.

Theorem 2.8. — *Suppose C is a compact leaf of genus g of a smooth semiholomorphic foliation $\mathcal{F}$ on a surface S, and $i_C^*\eta_{\mathcal{F}}\not\equiv 0$. Write n the number of zeroes of the form $i_C^*\eta_{\mathcal{F}}$, counted with multiplicity. Then $C\cdot C=1-g+\frac{n}{2}$.*

Note, in particular, that Camacho–Sad's theorem is false for smooth semiholomorphic foliations.

Proof. — The approximation at first order along C of $\mathcal{F}$ is a smooth semiholomorphic foliation on the normal bundle of C in S. The proposition only depends on this first-order approximation, so we can suppose that S is the normal bundle of C.

Consider local charts (x_i,y_i) with

$$(x_j,y_j)=\Phi_{ji}(x_i,y_i)=(\alpha_{ji}(x_i),\beta_{ji}(x_i)y_i).$$

Consider local $(1,0)$-forms $\omega_i=dy_i-\lambda_i dx_i$ defining $\mathcal{F}$, and the corresponding antiholomorphic parts of Bott's connection $\eta_i=\frac{\partial\lambda_i}{\partial\overline{y_i}}dx_i$. From Lemma 2.2 we know that

$$i_C^*(\Phi_{ji}^*\eta_j)=\frac{\beta_{ji}^2}{|\beta_{ji}|^2}i_C^*\eta_i.$$

Thus, if we write $b_i=\frac{\partial\lambda_i}{\partial\overline{y_i}}|_C$, we get

$$b_j\circ\alpha_{ji}=\frac{\beta_{ji}^2}{|\beta_{ji}|^2\alpha'_{ji}}b_i.$$

The cocycle $|\beta_{ji}|^{-2}$ is real positive, so its sections do not vanish; the cocycle β_{ji} defines the normal bundle and α'_{ji} the tangent bundle. On the other hand, we can see by Lemma 2.4 that b_i is a $\mathcal{C}^\infty$ function with holomorphic zeroes. It follows that $\deg(\Omega_C^1\otimes N^2)=n$, where n is the number of zeroes of the section b and, hence, the result. □

Example 2.9. — Consider a germ of surface (S,C) along a compact curve C of genus $g\geq 2$ and suppose there exists on S a smooth semiholomorphic foliation $\mathcal{F}$ leaving C invariant with $i_C^*\eta_{\mathcal{F}}$ nowhere vanishing. From the theorem we know that $C\cdot C=1-g$.

Consider then two curves c_1, c_2 in S cutting C transversally at two points $p_1 \neq p_2$ and the ramified covering $(\widetilde{S}, \widetilde{C})$ of order 2 over (S, C) ramifying along c_1 and c_2. The genus of $\widetilde{C}$ is $\widetilde{g} = 2g$; its self-intersection is $\widetilde{C}\cdot\widetilde{C} = 2-2g = 2-\widetilde{g}$, so that $\deg(\Omega^1_C \otimes N^2_C) = (2\widetilde{g}-2)+2(2-\widetilde{g}) = 2$. We can see from Lemma 2.2 that $i^*_{\widetilde{C}}\widetilde{\eta}$ does not vanish at points different from p_1, p_2, and a similar computation shows that it admits simple zeroes at p_1 and p_2.

Note that, by [7, Lemma 5.1], the self-intersection $C \cdot C$ should always be negative under the assumption $i^*_C \eta \not\equiv 0$. With this in mind, the theorem above seems incomplete, and for some reason, the form $\eta_{\mathcal{F}}$ cannot have an arbitrarily high number of zeroes for C fixed. It is not clear what values are admissible for $C \cdot C$ in $[1-g, -1]$, but to get an idea of this, it seems necessary to find examples that are not ramified coverings.

Theorem 2.10. — *Suppose that L is a leaf of $\mathcal{F}$ with compact adherence in S. Suppose, moreover, that $i^*_C \eta_{\mathcal{F}}$ is not identically zero for any leaf C in the adherence $\overline{L}$. Then L is complete for the metric $|\eta_{\mathcal{F}}|^2$.*

Proof. — Suppose $i^*_L \eta_{\mathcal{F}}$ is not identically zero. Then the set of points on L for which the metric degenerates is discrete, and we want to prove that for any geodesic $\gamma : \mathbb{R}^+ \to L$ that tends to the border of L, the length $\ell(\gamma) := \int_0^\infty |\eta(\gamma'(t))| dt$ is infinite.

Since the adherence of L is compact, we can cover it by a finite number of open sets U_i with $\eta_i = b_i dx_i$ on U_i. Remark first that for U_i small enough, we can suppose that γ passes through an infinite number of these U_i. On each leaf, the set of points where $b_i = 0$ is discrete, so we can find an $\varepsilon > 0$ such that $B_i := \{|b_i| < \varepsilon\}$ is relatively compact in U_i on each leaf. If γ passes an infinite number of times through these B_i, then γ must make an infinite number of times the path from within B_i to the border of U_i, and since $|b_i| \geq \varepsilon$ on the complement $U_i \setminus B_i$, we must have $\ell(\gamma) = \infty$.

Thus, we can suppose that γ only meets a finite number of these B_i. Then after a finite time $T > 0$, the geodesic γ only stays in the regions $|b_i| \geq \varepsilon$. The result follows easily. □

3. Semirank

3.1. Definitions. — We will define the semirank using the construction of jet spaces; let us recall this construction in our setting. Consider a holomorphic manifold X equipped with a holomorphic field of holomorphic 2-planes $\mathcal{P}$.

We can consider the field of 2-planes $\mathcal{P}$ as a rank 2 subbundle of TX; let $X' = \mathbb{P}\mathcal{P}$ be the $\mathbb{P}^1$-bundle obtained by projectivizing the rank 2 linear bundle $\mathcal{P}$. Then X' comes with a fibration $p : X' \to X$, a field of 3-planes $p^*\mathcal{P}$, and a field of 2-planes $\mathcal{P}'$ inside $p^*\mathcal{P}$ such that the value of $\mathcal{P}'$ at a point $(x, \lambda) \in X'$

is given by $\mathrm{Ker}(dv - \lambda du) \subset p^*\mathcal{P}$ if (u, v) are linear variables on $\mathcal{P}_x$ with λ corresponding to the direction in the kernel of $dv - \lambda du$.

The manifold X' satisfies the property that every holomorphic curve C in X tangent to $\mathcal{P}$ can be uniquely lifted to a curve C' in X' tangent to $\mathcal{P}'$ with $p(C') = C$.

Beginning by $(X_0, \mathcal{P}_0) = (U, TU)$, we can apply this construction inductively to obtain a sequence $(X_n, \mathcal{P}_n)$ of $(n+2)$-dimensional manifolds equipped with holomorphic fields of 2-planes and fibrations $p_n : X_n \to X_{n-1}$ with fibers $\mathbb{P}^1$. Consider $\pi_n : X_n \to U$ the composition of the p_n. By construction, each holomorphic curve in U can be uniquely lifted to a curve in X_n tangent to $\mathcal{P}_n$.

Now, suppose that $\mathcal{F}$ is a semiholomorphic foliation on U. It can be considered as a family of holomorphic curves with two real parameters; as such, it can be lifted to each X_n as a family of curves tangent to $\mathcal{P}_n$ and defines on each X_n a real four-dimensional submanifold Y_n of X_n. Note that any point $p \in U$ around which $\mathcal{F}$ is smooth can be uniquely lifted to the point in X_n corresponding to the lift of the leaf of $\mathcal{F}$ passing through p. By abuse of notation, we will still write p for this point whenever the foliation is clear from the context.

Definition 3.1. — We define the *semirank* of a semiholomorphic foliation $\mathcal{F}$ around a point $p \in U$ as the lowest integer n such that the germ of Y_n at p is not Zariski-dense in X_n.

By convention, the semirank is infinite if there exists no such integer. Note that holomorphic foliations correspond to semirank 1.

Proposition 3.2. — *Suppose that $\mathcal{F}$ is of semirank $n < \infty$ around a point p. Then the Zariski closure Z of Y_n in X_n is a hypersurface generically transverse to the fibers of $p_n : X_n \to X_{n-1}$.*

Proof. — Suppose on the contrary that Z is tangent to the fibration p_n or is not a hypersurface. Then $Z_{n-1} := p_n(Z)$ is a strict subvariety of X_{n-1}. By construction, Z_{n-1} contains Y_{n-1}; thus Y_{n-1} is not Zariski-dense in X_{n-1}, contradicting the minimality of n. □

From this proposition, we see that $TZ \cap \mathcal{P}_n$ defines a foliation by curves in Z tangent to $\mathcal{P}_n$, which can be projected to a holomorphic family of holomorphic curves in U with n parameters containing the leaves of $\mathcal{F}$. By construction, this n-parameter holomorphic family of curves is uniquely determined by $\mathcal{F}$.

Definition 3.3. — If $\mathcal{F}$ is of semirank n, the n-parameter holomorphic family of curves given by the Zariski closure of Y_n in X_n is called the *system of curves* $\mathscr{S}$ defined by $\mathcal{F}$. We will also say that $\mathcal{F}$ is tangent to $\mathscr{S}$.

Example 3.4. — For the foliation $\mathrm{Im}(x)dy - \mathrm{Im}(y)dx$, the induced system of curves is the set of complex affine lines.

To put this system into equations, take some coordinates $(x, y, \lambda_1, \dots, \lambda_n)$ of X_n centered around the point given by the leaf of $\mathcal{F}$ passing through the origin, so that λ_k is the coordinate of the $\mathbb{P}^1$-fiber of X_k corresponding to the direction $d\lambda_{k-1}/dx$. Since the Zariski closure Z of Y_n is generically transverse to the fibration p_n, we can express it around a generic point as the graph of a holomorphic application

$$\lambda_n = F(x, y, \lambda_1, \dots, \lambda_{n-1}).$$

In this case, the system of curves is given by solutions of the differential equation

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}),$$

where a curve is written as a function $y(x)$, and derivations are made with respect to the variable x. Note that some points can behave singularly for the system of curves, even if $\mathcal{F}$ is smooth. This happens exactly when Z is tangent to the fibration $p_n : X_n \to X_{n-1}$. We say that the family $\mathscr{S}$ is *smooth* when Z is smooth and transverse to the fibration p_n.

Example 3.5. — The system of curves associated to the foliation $\mathrm{Re}(1 + x^2)dy - 2x\mathrm{Re}(y)dx$ is the set of parabolas $\{y = ax^2 + b\}$. The differential equation satisfied by these curves is $xy'' = y'$ so Z has equation $x\lambda_2 - \lambda_1 = 0$ and is not transverse to the vector field $\frac{\partial}{\partial \lambda_2}$ above $x = 0$.

The proof of the following proposition is an adaptation of [5, Chapitre I, §I.4] in the context of semiholomorphic foliations.

Proposition 3.6. — *Let $\mathcal{F}$ be a real-analytic semiholomorphic foliation. Then the semirank of $\mathcal{F}$ is at most 2.*

Proof. — Suppose that (x, y) are coordinates around the origin such that $\mathcal{F}$ is nowhere vertical. Consider coordinates $(x, y, \lambda_1, \lambda_2)$ of X_2 where λ_i is a coordinate of the $\mathbb{P}^1$-fiber of X_i. Let $\lambda(x, y)$ be the slope of $\mathcal{F}$. The variety Y_2 satisfies the equations

$$\begin{cases} \lambda_1 = \lambda(x, y) \\ \lambda_2 = \dfrac{\partial \lambda}{\partial x}(x, y) + \lambda(x, y)\dfrac{\partial \lambda}{\partial y}(x, y). \end{cases}$$

Consider the four real parameters (x_1, x_2, v_1, v_2) with $l = l_1 + il_2$ for each $l = x, v$ and $y(x, v)$ the solution of the equations

$$\begin{cases} \dfrac{\partial y}{\partial x} = \lambda(x, y) \\ \dfrac{\partial y}{\partial \bar{x}} = 0 \\ y(0, v) = v \end{cases}$$

and put

$$\varphi(x,v) = (x, y, \lambda(x,y), \partial_x\lambda(x,y) + \lambda(x,y)\partial_y\lambda(x,y)),$$

where we wrote $y = y(x,v)$ for short. Now note that φ is a parametrization of Y_2, and since λ satisfies equation (2), the antiholomorphic derivative $\frac{\partial\varphi}{\partial\bar{x}}$ is zero.

Thus, φ is holomorphic in x and real analytic in v_1, v_2; as such, it can be extended to a germ of holomorphic application $\varphi : (x, v_1, v_2) \to X_2$ defined in a neighborhood V of $(\mathbb{C}\times\mathbb{R}\times\mathbb{R}, 0)$ in $(\mathbb{C}^3, 0)$. Then $\varphi(V)$ is a complex three-dimensional subvariety of X_2 containing Y_2, which concludes the proof. □

Remark that when $\mathcal{F}$ is a semiholomorphic foliation and L is a leaf of $\mathcal{F}$, the first-order approximation of $\mathcal{F}$ along L is always real analytic and, thus, holomorphic or of semirank 2. If its system of curves is locally given by $\lambda' = F(x,y,\lambda)$, the first-order approximation is the limit of $h_t^*\mathcal{F}$ when t tends to 0, where h_t is the dilatation $h_t(x,y) = (x, ty)$. The system of curves defined by the first-order approximation is thus of the form $\lambda' = a_1(x)y + a_2(x)\lambda$; it is of order less than 3 in λ, so it is a projective structure. However, in general, this projective structure will have singular points along L.

After these generalities, we will restrict ourselves to foliations of semirank 2.

3.2. Duality. — Suppose given a holomorphic family of curves $\mathscr{S}$ with 2 parameters in an open set $U \subset \mathbb{C}^2$, represented by the hypersurface $Z \subset X_2$ and suppose that Z is smooth and transverse to the fibration $p_2 : X_2 \to X_1$. We will consider Z as a germ around the origin $0 \in X_2$. Write $\mathcal{G}$ the smooth holomorphic foliation of Z given by $\mathscr{S}$ and $\check{U}$ the contraction of Z in the direction $\mathcal{G}$. The space $\check{U}$ is a holomorphic surface parametrizing the curves of $\mathscr{S}$; we call it the *dual* of the system of curves $\mathscr{S}$.

This dual comes with a two-parameter family of curves; for each point $z \in U$, consider the $\mathbb{P}^1$-fiber F_z of X_1 over z. Since Z is transverse to the fibration p_2, the preimages of the F_z on Z form a smooth foliation by curves $\check{\mathcal{G}}$ on Z. This foliation is transverse to $\mathcal{G}$, so that it is projected to a two-parameter family of smooth curves on $\check{U}$.

This family is the dual family of $\mathscr{S}$ on $\check{U}$; since U is the contraction of Z in the direction $\check{\mathcal{G}}$, we can see that $\mathcal{G}$ and $\check{\mathcal{G}}$ play symmetric roles on Z, so that the bidual of $\mathscr{S}$ is $\mathscr{S}$ itself.

To any semiholomorphic foliation $\mathcal{F}$ tangent $\mathscr{S}$, we can associate a real surface $S_{\mathcal{F}} \subset \check{U}$ by looking at its leaves as points in $\check{U}$ (or equivalently, by projecting Y_2 to $\check{U}$). Of course, $\mathcal{F}$ is holomorphic if and only if $S_{\mathcal{F}}$ is a holomorphic curve.

Most of the time, this duality is incomplete, in the sense that both U and $\check{U}$ are small open sets. One can expect that when either U or $\check{U}$ are globally well defined, the situation is much more rigid. For example, when the family $\mathscr{S}$

is a projective structure (meaning that through each point and each direction passes a curve of $\mathscr{S}$), then $\check{U}$ is a neighborhood of a $\mathbb{P}^1$ of self-intersection 1, and the automorphism group of the family $\mathscr{S}$ is finite-dimensional; see [8] for generic automorphisms and the Zusatz of [6, Satz 4] to see that automorphisms are finitely determined.

Let us consider for one moment the most particular case: when $\mathscr{S}$ is the family of affine lines. We can thus suppose that $U = \mathbb{P}^2$ and $\check{U} = \check{\mathbb{P}}^2$, but the foliation is a priori only a foliation in the neighborhood of a point $0 \in \mathbb{P}^2$. Then $S_{\mathcal{F}}$ is a germ of a real surface in $\check{\mathbb{P}}^2$, and we can use [15] to form its dual $\check{S}_{\mathcal{F}} \subset \mathbb{P}^2$. In general, $\check{S}_{\mathcal{F}}$ is the hypersurface given by the envelope of the family of curves $\mathcal{F}$. The construction of [15] is, in fact, very general; we can do it for any system of curves $\mathscr{S}$. In this case, the definition of the dual of a real subvariety $M \subset U$ is

$$\check{M} = \{L \in \check{U} \mid L \text{ intersects } M \text{ at a point } p \text{ with } T_pL + T_pM \neq \mathbb{C}^2\}.$$

As stated above, the dual of a real surface is generically a real hypersurface. More precisely, we can say the following:

Lemma 3.7. — *Suppose that $\mathscr{S}$ is a smooth two-parameter system of curves in $U \subset \mathbb{C}^2$, and S is a real surface in U. We have the following possibilities for the dual $\check{S}$:*

1. *$\check{S}$ is a point, and S is a complex curve in the family $\mathscr{S}$.*
2. *$\check{S}$ is a complex curve, and S is also a complex curve.*
3. *$\check{S}$ is a real surface, in which case the intersection between curves of $\mathscr{S}$ and the surface S define a real two-parameter family of curves $\mathscr{S}_{\mathbb{R}}$ on S; when S is real analytic, $\mathscr{S}$ is the complexification of $\mathscr{S}_{\mathbb{R}}$.*
4. *$\check{S}$ is a real hypersurface.*

Proof. — The projection $p_2 : X_2 \to X_1$ induces a biholomorphism between the germs $(Z, 0)$ and $(X_1, 0)$ so we will work with X_1. In particular, the families $\mathscr{S}$ and $\check{\mathscr{S}}$ give two smooth transverse foliations $\mathcal{G}$ and $\check{\mathcal{G}}$ on X_1, and the contact structure $\mathcal{P}_1$ is given by $\mathcal{P}_1 = T\mathcal{G} \oplus T\check{\mathcal{G}}$. Consider also the two projections $p = p_1 : X_1 \to U$ and $p' : X_1 \to \check{U}$.

Now, as in [15], we define the lift $p^*S \subset X_1$ of S as the set of those $(z, \lambda) \in X_1$, where z is a coordinate in U and λ a coordinate of the fiber, such that $z \in S$, and λ is the complex direction of a real tangent vector in T_zS. Note that $\check{S} = p'(p^*S)$. As discussed in [15, §4], the lift p^*S can be non-transverse to the projection p' so that $\check{S}$ can be a priori very singular, in which case we could have difficulties defining the second lift $p'^*\check{S}$. However, as soon as it is well defined, we have the equality of germs $p^*S = p'^*\check{S}$. In particular, if $\check{S}$ is a point, then p^*S is a leaf of $\mathcal{G}$, and S is a complex curve in the family $\mathscr{S}$. We can also see that when $\check{S}$ is a holomorphic curve, then p^*S is the holomorphic

Legendrian lift of $\check{S}$, so that its projection $S = p(p^*S)$ is also a holomorphic curve.

When $\check{S}$ is a real surface that is not holomorphic, the lift p^*S is a real three-dimensional subvariety of X_1, which is everywhere non-transverse to both p and p'. In particular, the fibers of p' define a foliation by real curves $\mathcal{G}_\mathbb{R}$ on p^*S. This foliation is generically transverse to the projection p, so $p(\mathcal{G}_\mathbb{R})$ is a foliation by real curves on S. By definition, the leaves of $\mathcal{G}_\mathbb{R}$ are included in leaves of $\mathcal{G}$, so that the leaves of the induced foliation on S are included in curves of the system $\mathscr{S}$. When S is real analytic, we can suppose modulo biholomorphism that $S = \mathbb{R}^2$ around a generic point, and in this case, it is clear that $\mathscr{S}$ is the complexification of $\mathscr{S}_\mathbb{R}$. □

Note, however, that if $\mathcal{F}$ is a smooth semiholomorphic foliation in an open set U, and $S_\mathcal{F}$ is its dual surface, then the bidual $\check{S}_\mathcal{F}$ will not intersect U. We will explain this in more detail in the next section, but for now, let us study some cases when $\check{S}_\mathcal{F}$ does intersect U, which is very exceptional, as we can see in the following result:

Proposition 3.8. — *Let $\mathscr{S}$ be a smooth system of curves in an open set $U \subset \mathbb{C}^2$ and $\mathcal{F}$ a semiholomorphic foliation tangent to $\mathscr{S}$ with a singular point $p \in U$. If p is an isolated singular point, then $\mathcal{F}$ is the pencil of curves of $\mathscr{S}$ passing through p; it is holomorphic and $\check{S}_\mathcal{F} = \{p\}$. Suppose $p \in U$ is a non-isolated singular point of $\mathcal{F}$. Then the bidual $\check{S}_\mathcal{F}$ is a real surface passing through p, $\operatorname{Sing}(\mathcal{F}) = \check{S}_\mathcal{F}$, the system of curves $\mathscr{S}$ induces a real system of curves $\mathscr{S}_\mathbb{R}$ on $\check{S}_\mathcal{F}$ and when $\mathcal{F}$ is real analytic, $\mathscr{S}$ is the complexification of $\mathscr{S}_\mathbb{R}$.*

Proof. — Since $\mathcal{F}$ is a foliation, the real codimension of its singular set Σ is at least 2. Consider a leaf L_0 of $\mathcal{F}$ intersecting Σ at a point p_0. By definition, $\mathcal{F}$ has another leaf L_1 intersecting L_0 at p_0. Since L_0 and L_1 are curves in $\mathscr{S}$, and $\mathscr{S}$ is smooth, their intersection at p_0 is transversal, and any leaf L of $\mathcal{F}$ close to L_1 intersects L_0 at a point $p \in \Sigma$ close to p_0. If this point p is always equal to p_0, then all of the leaves pass through $\{p_0\}$ and $\Sigma = \{p_0\}$. In this case, we can see that $\check{\Sigma} = S_\mathcal{F}$ is the holomorphic curve of the family $\check{\mathscr{S}}$ corresponding to p_0, so that $\mathcal{F}$ is holomorphic.

Otherwise the set of these points p is a curve on Σ, and, in particular, $L \cap \Sigma$ is a curve. We can say the same thing for all leaves, so that each leaf L intersects Σ along a curve. Since the intersection points between leaves are points, Σ cannot be a real curve, and it must be a real surface. Necessarily, these intersections define a real two-parameter family of real curves on Σ, that is, a real system of curves $\mathscr{S}_\mathbb{R}$. In particular, there is an open set V in the real tangent bundle $T\Sigma$ such that for $(p, v) \in V$, there is a leaf L of $\mathcal{F}$ intersecting Σ at p in the real direction v. By definition of the dual of Σ, we have $\check{\Sigma} = S_\mathcal{F}$, and thus $\Sigma = \check{S}_\mathcal{F}$.

Finally, note that the curves of the system $\mathscr{S}_{\mathbb{R}}$ are all contained in leaves of $\mathcal{F}$, so that the complexified of $\mathscr{S}_{\mathbb{R}}$ is a complex system of curves containing all the leaves of $\mathcal{F}$. Since $\mathcal{F}$ is not holomorphic, it must be equal to $\mathscr{S}$. □

Proposition 3.9. — *Let C be an elliptic curve and S a two-dimensional neighborhood of C. Suppose there exists a smooth semiholomorphic foliation $\mathcal{F}$ of semirank 2 in S admitting C as a leaf. Suppose also that the system of curves $\mathscr{S}$ defined by $\mathcal{F}$ is smooth in the neighborhood of the curve C, and the dual system $\check{\mathscr{S}}$ is a projective structure. Then $\mathcal{F}$ is holomorphic.*

Proof. — Consider the universal cover $(U, \tilde{C})$ of (S, C), equipped with the pull-back semiholomorphic foliation, and the induced system of curves, which we will still denote by $\mathcal{F}$ and $\mathscr{S}$. If the dual system $\check{\mathscr{S}}$ is a projective structure, then the dual $V \supset U$ of $\check{U}$ contains a compact curve $L \simeq \mathbb{P}^1$; the curve L is the compactification $L = \tilde{C} \cup \{\infty\}$, the surface V is a neighborhood of L, we have $L \cdot L = 1$, and the system $\mathscr{S}$ extends naturally as a two-parameter family of deformations of L.

It follows that any curve $L' \in \mathscr{S}$ close to L intersects L at one point, so that if the foliation $\mathcal{F}$ is smooth, then $L' \cap L \in L \setminus \tilde{C} = \{\infty\}$. We can see that $\mathcal{F}$ is exactly the pencil of curves $L' \in \mathscr{S}$, which intersect L at infinity, so that it is, indeed, holomorphic. □

3.3. Complete local models. — As we saw in Theorem 2.10, in many interesting cases, the leaves will be complete for the metric $|\eta_{\mathcal{F}}|^2$. It would then be interesting to study local models of foliations $\mathcal{F}$ in open sets $U \subset \mathbb{C}^2$ such that the leaves of $\mathcal{F}$ are complete.

Example 3.10. — Consider the foliation given by the 1-form $\omega = \mathrm{Im}(x)dy - \mathrm{Im}(y)dx$. This is, in fact, a singular foliation on the whole of $\mathbb{P}^2(\mathbb{C})$. Its singular set is equal to $\mathrm{Sing}(\mathcal{F}) = \mathbb{P}^2(\mathbb{R})$. Each complex line L tangent to the foliation is cut into two pieces by $\mathrm{Sing}(\mathcal{F})$, and each piece equipped with the metric $|\eta|^2$ is equal to Poincaré's half plane.

This example is, in fact, very special; as we saw in Proposition 3.8, it corresponds to the case when the dual of $\check{S}_{\mathcal{F}}$ is a real surface, which is a degenerate case. In the rest of this section, we suppose that we are in the generic case when $\check{S}_{\mathcal{F}}$ is a real hypersurface.

By duality, the hypersurface $\check{S}_{\mathcal{F}}$ is the envelope of the family of curves parametrized by $S_{\mathcal{F}}$; each leaf L of $\mathcal{F}$ intersects $\check{S}_{\mathcal{F}}$ along a curve $\gamma \subset L$ and is tangent to $\check{S}_{\mathcal{F}}$ along γ (see [15] for more details); it follows that the leaves of $\mathcal{F}$ are tangent to $\check{S}_{\mathcal{F}}$, but we cannot extend them outside the curve γ or different leaves will intersect L. This behavior should be the generic behavior for complete models, even when the foliation is not of semirank 2.

More precisely, consider an open set $V \subset \mathbb{C}^2$, a smooth two-parameter family of curves $\mathscr{S}$, and a germ of semiholomorphic foliation $\mathcal{F}$ along a leaf d_0, tangent to $\mathscr{S}$ and giving a germ of real surface $(S_{\mathcal{F}}, d_0)$ in the dual surface $\check{V}$. Now suppose that the family $\mathscr{S}$ can be extended to a smooth system still denoted $\mathscr{S}$ on some open set $U \supset V$, and that the dual $\check{S}_{\mathcal{F}}$ is a real hypersurface in U that intersects d_0 along a real curve. Inside the germ of surface (U, d_0), the hypersurface $\check{S}_{\mathcal{F}}$ has an interior that can be considered the biggest domain of definition of $\mathcal{F}$, and each leaf of $\mathcal{F}$ cuts tangentially $\check{S}_{\mathcal{F}}$ along a real curve. Note that, using notations of Section 1.2, this real curve has equation $\{|u|^2 = |v|^2\}$, so that a generic geodesic along a leaf that tends to $\check{S}_{\mathcal{F}}$ has infinite length. Thus, when the real curve $\check{S}_{\mathcal{F}} \cap d_0$ is compact, $\mathcal{F}$ can be extended to a semiholomorphic foliation in the interior of $\check{S}_{\mathcal{F}}$ whose leaves are complete. Conversely, note that if leaves of $\mathcal{F}$ do not adhere to the hypersurface $\check{S}_{\mathcal{F}}$, then they are not complete.

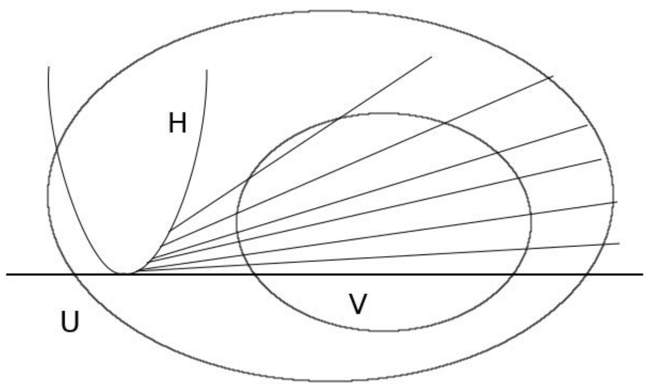

FIGURE 3.1. The hypersurface $H = \check{S}_{\mathcal{F}}$ is tangent to leaves of $\mathcal{F}$

DEFINITION 3.11. — We call a germ of a *smooth non-degenerated local complete model* along a leaf L the triples $(U, V, \mathcal{F})$, where $V \subset U \subset \mathbb{C}^2$ are germs of open sets along L, $\mathcal{F}$ is a germ of smooth semiholomorphic foliation on V tangent to a system of curves $\mathscr{S}$, $L \cap V$ is a leaf of $\mathcal{F}$, the system $\mathscr{S}$ can be extended to a smooth system of curves on U, and the border $H = \partial V$ is a germ of compact hypersurface that satisfies $H = \check{S}_{\mathcal{F}}$ as germs along $L \cap H$.

If a smooth semiholomorphic foliation tangent to a regular system of curves on a global surface has complete leaves, then its universal covering is either of the form described in Proposition 3.8 or is a smooth nondegenerate local complete model along each leaf.

4. Foliations tangent to projective structures

4.1. Examples of foliations by lines. — We try to construct some examples of foliations $\mathcal{F}$ on germs of surfaces (S, C) around a hyperbolic compact complex curve C, such that C is a leaf of $\mathcal{F}$. To do so, we will take $\mathcal{F}$ locally modeled on the foliation $\mathcal{F}_0$ given by the $(1,0)$-form $\omega_0 = \mathrm{Im}(x)dy - \mathrm{Im}(y)dx$ on $\mathbb{P}^2(\mathbb{C})$ around the leaf $L_0 = \{y = 0\}$. Equivalently, we want to find a group G of germs of diffeomorphisms of the surface $\mathbb{P}^2(\mathbb{C})$ around an open set $V_0 \subset L_0$, such that the quotient of V_0 by $G|_{V_0}$ is diffeomorphic to C. Of course, if we want the foliation $\mathcal{F}_0$ to give a foliation $\mathcal{F}$ on the quotient, the group G must send leaves of $\mathcal{F}_0$ to leaves of $\mathcal{F}_0$. This means, in particular, that G must preserve the set of affine lines in $\mathbb{P}^2(\mathbb{C})$, so G is a subgroup of $\mathrm{PSL}_3(\mathbb{C})$.

The group G must also preserve the singular set $\mathbb{P}^2(\mathbb{R})$ of $\mathcal{F}_0$, so $G \subset \mathrm{PSL}_3(\mathbb{R})$. Note that by the duality explained in Section 3.2, the fact for G to preserve $\mathbb{P}^2(\mathbb{R})$ is equivalent for its action on the dual $\check{\mathbb{P}}^2(\mathbb{C})$ to preserve the dual surface $S_{\mathcal{F}}$. This means exactly that G sends leaves of $\mathcal{F}_0$ to leaves of $\mathcal{F}_0$, i.e., the group of automorphisms of $\mathcal{F}_0$ is $\mathrm{PSL}_3(\mathbb{R})$. Write an element $M \in \mathrm{SL}_3(\mathbb{R})$ as

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

If we want M to stabilize L_0, we must have $a_{21} = a_{23} = 0$, and $M|_{L_0}$ is given in an affine coordinate x by

$$M|_{L_0}(x) = \frac{a_{11}x + a_{13}}{a_{31}x + a_{33}}.$$

Hence the construction: take any Fuchsian subgroup $G_0 \subset \mathrm{PSL}_2(\mathbb{R})$ such that the quotient of the half-plane $\mathbb{H}_0 \subset L_0$ by G_0 is a hyperbolic compact curve C. Write $a_{11}(g), a_{13}(g), a_{31}(g), a_{33}(g)$ as the coefficients of the elements $g \in G_0$. Choose any extension G of G_0 to $\mathrm{PSL}_3(\mathbb{R})$ (i.e., $a_{22}(g) = 1$, and $(a_{12}(g), a_{32}(g))$ is a cocycle for the group G_0); the most simple extension being, of course, $a_{12}(g) = a_{32}(g) = 0$ and $a_{22}(g) = 1$. Then the quotient of a neighborhood U of $\mathbb{H}_0$ in $\mathbb{P}^2(\mathbb{C})$ by G is a surface S containing a curve C quotient of $\mathbb{H}_0$, and the foliation $\mathcal{F}_0$ descends to a smooth semiholomorphic foliation $\mathcal{F}$ on S having C as a leaf.

THEOREM 4.1. — *Let $C = \mathbb{H}_0/G_0$ be a compact curve of genus $g \geq 2$, where $G_0 \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup. Let $\mathscr{M}$ denote the moduli space of neighborhoods $(S, C, \mathcal{F})$ of C in complex surfaces equipped with a smooth semi-holomorphic foliation $\mathcal{F}$ locally diffeomorphic to $\mathrm{Im}(x)dy - \mathrm{Im}(y)dx$, modulo biholomorphism. The construction explained above induces a bijection*

$$\mathscr{M} \simeq H^1(G_0, \mathbb{R}^2).$$

Proof. — As we explained above, every example comes from a cocycle $Z^1(G_0, \mathbb{R}^2)$, so that we only need to prove that equivalence modulo biholomorphism for $(S, C, \mathcal{F})$ corresponds to equivalence modulo coboundaries for cocycles. We keep the notations above in this proof.

The holonomy of $\mathcal{F}$ in S defines a permutation of the leaves, which is exactly the action of G on the dual surface $S_{\mathcal{F}}$; in particular, from this point of view, the holonomy is given by elements of $\mathrm{PSL}_3(\mathbb{R})$. It follows that this holonomy is well defined modulo conjugacy in $\mathrm{PSL}_3(\mathbb{R})$; we can suppose that this conjugacy fixes L_0 and the group $G_0 = G|_{L_0}$. The action by conjugacy of the subgroup of $\mathrm{PSL}_3(\mathbb{R})$ fixing L_0 and G_0 is exactly the action of coboundaries on cocycles, so that the theorem is proved. □

Note that, if we have coordinates $[x : y : z]$ on $\mathbb{P}^2$, in the affine chart $z = 1$ the action of the group given by the zero cocycle writes as

$$M(x, y) = \left(\frac{a_{11}x + a_{13}}{a_{31}x + a_{33}}, \frac{y}{a_{31}x + a_{33}} \right) =: (\alpha(x), \beta(x)y).$$

The tangent bundle of the leaf $L_0 = \{y = 0\}$ will be given by $\alpha'(x) = \frac{a_{11}a_{33} - a_{13}a_{31}}{(a_{31}x + a_{33})^2}$, and its normal bundle by $\beta(x)$. We see here explicitly that the tangent bundle and the normal bundle are closely related, as stated in Theorem 2.8.

Note also that the cocycles only intervene at higher order.

4.2. Generic foliations by lines. —

THEOREM 4.2. — *Consider a real-analytic semiholomorphic foliation by lines $\mathcal{F}$ defined in an open set $U \subset \mathbb{P}^2(\mathbb{C})$ neighborhood of the leaf L_0. Suppose that there is a subgroup $G < \mathrm{PSL}_3(\mathbb{C})$ stabilizing $\mathcal{F}$ and L_0, such that the restriction to L_0 is injective, $G|_{L_0}$ is a Fuchsian group, and the quotient of L_0 by G is a compact Riemann surface of genus $g \geq 2$.*

Suppose, moreover, that $\mathcal{F}$ is not holomorphic at first order along L_0, i.e., $i_{L_0}^* \eta_{\mathcal{F}} \not\equiv 0$. Then $\mathcal{F}$ is biholomorphic to the foliation given by the $(1,0)$-form $\mathrm{Im}(x)dy - \mathrm{Im}(y)dx$.

The foliation $\mathcal{F}$ defines a real surface $S = S_{\mathcal{F}}$ in $\check{\mathbb{P}}^2$, and the action of G on the dual space $\check{\mathbb{P}}^2$, denoted by $\check{G} < \mathrm{PSL}_3(\mathbb{C})$ stabilizes S. Since G stabilizes a line L_0, the group $\check{G}$ stabilizes a point p_0. If $g \in G$ is hyperbolic in restriction to L_0, then its action $\varphi \in \mathrm{PSL}_3(\mathbb{C})$ on $\check{\mathbb{P}}^2$ is such that $d_{p_0}\varphi$ is hyperbolic. Most real surfaces do not have a lot of symmetries in $\mathrm{PSL}_3(\mathbb{C})$; we begin by examining those having one hyperbolic symmetry.

LEMMA 4.3. — *Let $(S, p_0) \subset \check{\mathbb{P}}^2$ be a smooth germ of real-analytic surface such that $T_{p_0}S$ is not complex. Suppose that $\varphi \in \mathrm{PSL}_3(\mathbb{C})$ stabilizes S and $d_{p_0}\varphi$ is hyperbolic. Then either S is a real affine plane or S is equipped with a real codimension 1 foliation whose leaves are invariant by φ. Moreover, these* *leaves are intersections between S and real affine planes passing through p_0; the foliation has exactly two separatrices at p_0, and they are tangent to eigenvectors of $d_{p_0}\varphi$.*

Proof. — Since $\varphi(p_0) = p_0$, we can write φ in some coordinates

$$\varphi = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{C}).$$

Now, the differential $d_{p_0}\varphi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is hyperbolic and, thus, diagonalizable: $d_{p_0}\varphi \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, for example with $|\lambda| > |\mu|$. By hypothesis, $T_{p_0}S$ is not a complex direction, so there is a conjugacy ψ with $\psi \in \mathrm{PSL}_3(\mathbb{C})$ fixing p_0, such that $T_{p_0}\psi(S) = p_0 + \mathbb{R}^2$. This implies $d_{p_0}(\psi\varphi\psi^{-1})(\mathbb{R}^2) = \mathbb{R}^2$, and since λ and μ are not complex conjugates, we see that λ and μ are real.

There are two cases to examine: either φ is diagonalizable and in some coordinates,

$$\varphi = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

or

$$\varphi = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

In both cases, if we consider coordinates (x, y) centered at p_0 with $d_0\varphi(x, y) = (\lambda x, \mu y)$, if we write $x = x_1 + ix_2$, $y = y_1 + iy_2$, then for any equations of the tangent plane $T_{p_0}S$

$$\begin{cases} l_{11}x_1 + l_{12}x_2 + l_{13}y_1 + l_{14}y_2 = 0 \\ l_{21}x_1 + l_{22}x_2 + l_{23}y_1 + l_{24}y_2 = 0, \end{cases}$$

by the equation $d\varphi(T_{p_0}S) = T_{p_0}S$, we get

$$\begin{cases} (\lambda/\mu - 1)(l_{11}x_1 + l_{12}x_2) = 0 \\ (\mu/\lambda - 1)(l_{23}y_1 + l_{24}y_2) = 0. \end{cases}$$

We deduce that when $T_{p_0}S$ is not complex, it has equations $\{x_2/x_1 = \tan(\alpha_1), y_2/y_1 = \tan(\alpha_2)\}$ for some constants $\alpha_1, \alpha_2 \in S^1$. The application $(x, y, z) \mapsto (e^{-i\alpha_1}x, y, z)$ stabilizes φ, so we can suppose that $\alpha_1 = 0$ in both cases. In the diagonalizable case, we can also suppose that $\alpha_2 = 0$.

Suppose now that we are in the diagonalizable case; in some coordinates (x, y) centered at p_0, we have $\varphi(x, y) = (\lambda x, \mu y)$. Note first that the real functions x_2/x_1 and y_2/y_1 are stable by φ and induce real foliations by invariant curves on S whenever they are not constant on S. We can write S as a graph

$$\begin{cases} x_2 = x_1 f_1(x_1, y_1) \\ y_2 = y_1 f_2(x_1, y_1). \end{cases}$$

The functions f_1, f_2 are unique, and since S is stable by φ, we get for every x_1, y_1,

$$\begin{cases} f_1(x_1, y_1) = f_1(\lambda x_1, \mu y_1) \\ f_2(x_1, y_1) = f_2(\lambda x_1, \mu y_1). \end{cases}$$

There are two cases: either the diffeomorphism $\varphi|_{\mathbb{R}^2}$ has a nonconstant real-analytic first integral or not. In the first case, every first integral is of the form $f(x_1^p y_1^q)$ for some integers p, q, so that the two foliations induced by x_2/x_1 and y_2/y_1 on S are, in fact, equal. This means that every leaf of this common foliation is a component of the intersection between S and a real affine plane $\{x_2 = \tan(\theta)x_1, y_2 = \tan(\alpha)y_1\}$. Note that, since this foliation has a first integral $x_1^p y_1^q$, it has two separatrices at p_0, and the complexification of the tangents at p_0 of these separatrices are the eigenvectors of φ corresponding to λ and μ. In the second case, every first integral is constant, so that S is included in a real affine plane $\{x_2 = \tan(\theta)x_1, y_2 = \tan(\alpha)y_1\}$.

In the nondiagonalizable case, φ can be expressed in some coordinates (x, y) centered at p_0 as

$$\varphi(x, y) = \left(\frac{\lambda x}{1 + y}, \frac{y}{1 + y} \right).$$

In the coordinates $(z, w) = (x/y, 1/y)$, this expression becomes $\varphi(z, w) = (\lambda z, w + 1)$. As before, we notice that the functions z_2/z_1 and w_2 are invariant by φ.

The tangent plane to S can be parametrized by $(s, t) \in \mathbb{R}^2 \mapsto (x, y) = (s, te^{i\alpha_2})$, so that in a neighborhood of p_0, S is close to the surface parametrized by $(z, w) = (se^{-i\alpha_2}/t, e^{-i\alpha_2}/t)$. In particular, if $\alpha_2 \neq \pm\pi/2$, in a neighborhood of p_0, we can write $S \setminus \{p_0\}$ as a graph

$$\begin{cases} z_2 = z_1 f_1(z_1, w_1) \\ w_2 = f_2(z_1, w_1). \end{cases}$$

As in the diagonalizable case, the functions f_i are unique, and S is stabilized by φ, so we get

$$\begin{cases} f_1(z_1, w_1) = f_1(\lambda z_1, w_1 + 1) \\ f_2(z_1, w_1) = f_2(\lambda z_1, w_1 + 1). \end{cases}$$

The local first integrals of $\varphi|_{\mathbb{R}^2}$ are all of the form $f(z_1 \exp(-w_1 \log(\lambda)))$. Note that $z_1 \exp(-w_1 \log(\lambda)) = (x/y)\exp(-\log(\lambda)/y)$, so that these first integrals are not real analytic at $(x,y)=(0,0)$. It follows that z_2/z_1 and w_2 are constant on S. Since $[x:y:1]=[z:1:w]$, these equations, indeed, define real affine subspaces.

If α_2 is equal to $\pm\pi/2$, we can write S as a graph

$$\begin{cases} z_1 = z_2 f_1(z_2,w_2) \\ w_1 = f_2(z_2,w_2), \end{cases}$$

and we get

$$\begin{cases} f_1(z_2,w_2) = f_1(\lambda z_2,w_2) \\ f_2(z_2,w_2) = f_2(\lambda z_2,w_2) - 1. \end{cases}$$

From the second equation we get that $f_2|_{\{z_2=0\}} = \infty$ so that in fact $\alpha_2 \neq \pm\pi/2$. □

Proof of Theorem 4.2. — Aiming at a contradiction, we suppose in this proof that the dual surface S is not contained in any affine real plane. The group $G|_{L_0}$ is generated by $2g$ hyperbolic elements h_i, and two different h_i have different pairs of fixed points on the border ∂L_0. These fixed points are the eigenvectors of the differentials $d_{p_0}\varphi_i$ of the actions of h_i on $\check{\mathbb{P}}^2$.

Consider three among them: $\varphi_1,\varphi_2,\varphi_3$. The foliations defined by them on S have different separatrices, so they are different. Thus, by a generic point $p\in S$ pass three curves contained in real affine planes also passing through p_0. Consider the projection $\pi:(\mathbb{P}^2(\mathbb{C}),p_0)\setminus\{p_0\}\to\mathbb{P}^3(\mathbb{R})$. Since S is not contained in any real plane, the space $\pi(S)$ is a surface; and we just saw that through a generic point $p\in\pi(S)$ pass three different affine lines contained in $\pi(S)$. By [9, §16.5], this implies that $\pi(S)$ is a real affine plane, so that S is contained in a real affine hyperplane H. This hyperplane H is obviously unique, so that it is stable by $d_{p_0}\varphi$ for each $\varphi\in G$. It follows that the unique complex direction tangent to H is stable by each $d_{p_0}\varphi$, and that all of the elements of G share an eigenvector. This is obviously not possible for the fundamental group of a smooth compact curve. □

In the proof of this result, the hypothesis of real analyticity is only used in Lemma 4.3 to show that when two real functions on S are invariant by an automorphism φ, they must define the same foliation. This result is false in the $\mathcal{C}^\infty$ context. Indeed, consider a diffeomorphism $\varphi(x_1,y_1)=(\lambda x_1,\mu y_1)$ with for example $\lambda>1>\mu>0$ and $\lambda^p\mu^q=1$ for some integers $p,q\in\mathbb{N}^*$.

Write $l=\log(\lambda), m=\log(\mu)$ and consider the functions $r=x^py^q$ and $\theta=\frac{1}{m-l}\log\left(\frac{y}{x}\right)$. Then $r\circ\varphi=r$ and $\theta\circ\varphi=\theta+1$; the foliation $\{r=cte\}$ is stable by φ, has two separatrices $L_x=\{y=0\}, L_y=\{x=0\}$, and these separatrices cut the plane $\mathbb{R}^2$ into four invariant quadrants $\mathbb{R}^2\setminus(L_x\cup L_y)=\bigcup Q_i$.

If f is a function defined in a quadrant Q_i that is invariant by φ, its restriction to a curve $\{r=c\}$ is periodic in θ, and we can develop it in Fourier series:

$$f|_{Q_i}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n^{(i)}(r)e^{in\theta}.$$

We know by Fourier theory that $f|_{Q_i}$ is $\mathcal{C}^\infty$ if and only if $|c_n^{(i)}(r)| = o(n^k)$ for each $k \in \mathbb{N}$. The question is then to find those tuples (f_1, f_2, f_3, f_4) that can be glued to a $\mathcal{C}^\infty$ function f on $\mathbb{R}^2$. Note here that if $|c_n^{(i)}(r)| = o(r^k)$ for each i, n, k, we can, indeed, glue the functions f_i to a function f: when (x, y) tends to one of the separatrices L_x or L_y, a quick computation shows that all the derivatives $\frac{\partial^{k_1+k_2} f_i}{\partial x^{k_1}\partial y^{k_2}}$ tend to zero.

This shows that there are, indeed, germs of real surfaces invariant by an automorphism φ, which contradict Lemma 4.3 in the $\mathcal{C}^\infty$ context.

4.3. Another example. — We can try to find other examples of semiholomorphic foliations whose leaves are curves of a projective structure. In general, projective structures could have isolated symmetries, so we will only consider symmetries that are exponentials of infinitesimal automorphisms, as studied in [8, §6.2]. The only groups of infinitesimal automorphisms that can give rise to a compact curve of genus $g \geq 2$ are $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. The only projective structure (modulo biholomorphisms) having a symmetry group $\mathfrak{sl}_3(\mathbb{C})$ is the family of lines in $\mathbb{P}^2$, which we already considered. As for $\mathfrak{sl}_2(\mathbb{C})$, it only occurs as infinitesimal symmetries for the structure $\mathscr{S}$ whose curves have equations $y = y(x)$ in a neighborhood U of the origin with

$$y'' = (xy' - y)^3.$$

However, the action of the group is the linear action of $G = \mathrm{SL}_2(\mathbb{C})$, so it fixes a point. Thus, we cannot build an example of neighborhood of a compact curve of genus $g \geq 2$ from this structure.

Consider then the dual family $\check{\mathscr{S}}$ of curves of this projective structure; it is defined in the surface S_0 described in [8, §5.4]. Recall that S_0 is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ along the diagonal Δ, and the dual family $\check{\mathscr{S}}$ corresponds to graphs of Möbius functions tangent to Δ. The curves in $\check{\mathscr{S}}$ are rational curves of self-intersection 1 in S_0. In the following, we will consider S_0 as a neighborhood of the curve L_0 corresponding to Δ. Since this family is the dual of $\mathscr{S}$, it also has symmetry group $\check{G} = \mathrm{SL}_2(\mathbb{C})$; in U the automorphism group fixed the origin $0 \in U$, so that the action of $\check{G}$ on S_0 stabilizes the curve L_0 and acts as Möbius transforms on it.

In some coordinates (x, y), a curve L in $\check{\mathscr{S}}$ has the equation

$$y = \frac{a(x - x_0)}{\sqrt{1 + a^2(x - x_0)}}, \tag{4}$$

where the two parameters x_0 and a correspond respectively to the point of intersection between L and L_0 and to the slope of L at this point.

Now consider the real plane $\{(x_0, a) \in \mathbb{R}^2\} \subset U$, which corresponds to a semiholomorphic foliation $\mathcal{F}$ in S_0 whose leaves are precisely those curves for which a and x_0 are real. We can check that when a and x_0 are real, we can recover x_0 from the system of two real equations (4) by eliminating a. Indeed, we find the quadratic equation

$$0 = \mathrm{Im}(y^2)x_0^2 - 2\mathrm{Im}(\bar{x}y^2)x_0 + \left[\mathrm{Im}(x)|y^2|^2 + \mathrm{Im}(\bar{x}^2y^2)\right],$$

which we can solve to obtain

$$x_0 = \frac{\mathrm{Im}(\bar{x}y^2) - |y|^2\mathrm{Im}(x)\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}}}{\mathrm{Im}(y^2)} =: f(x, y).$$

In particular, the real function f is constant along the leaves of $\mathcal{F}$, so that its level hypersurfaces are Levi-flat hypersurfaces foliated by leaves of $\mathcal{F}$. We deduce that the field of complex directions $T^{(1,0)}\mathcal{F}$ is the unique complex direction contained in $\mathrm{Ker}(df)$, so that it is the kernel of the $(1,0)$-form ∂f. We can compute

$$\frac{\partial f}{\partial x} = \frac{-\bar{y}}{2i\mathrm{Im}(y^2)\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}}}\left[\bar{y}\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}} + y\left(1 - \frac{\mathrm{Im}(y^2)}{2\mathrm{Im}(x)}\right)\right],$$

$$\frac{\partial f}{\partial y} = \frac{\bar{y}}{2i\mathrm{Im}(y^2)^2\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}}} \times \left[2\mathrm{Im}(x)|y|^2\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}} + \left(2\mathrm{Im}(x)\mathrm{Re}(y^2) - \bar{y}^2\mathrm{Im}(y^2)\right)\right].$$

Alternatively, we can express the slope $\lambda = dy/dx$ as

$$\lambda = \frac{\mathrm{Im}(y^2)}{2\mathrm{Im}(x)} \frac{\bar{y}\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}} + y\left(1 - \frac{\mathrm{Im}(y^2)}{2\mathrm{Im}(x)}\right)}{|y|^2\sqrt{1 - \frac{\mathrm{Im}(y^2)}{\mathrm{Im}(x)}} + \left(\mathrm{Re}(y^2) - \bar{y}^2\frac{\mathrm{Im}(y^2)}{2\mathrm{Im}(x)}\right)}.$$

The surface S_0 is defined over $\mathbb{R}$ and is the complexified of the real-analytic surface $S_\mathbb{R}$ obtained as the closure of $\{(x, y) \in \mathbb{R}^2\}$ in S_0. As before, $L_0 \setminus S_\mathbb{R}$ is the union of two hyperbolic planes; consider one of them $\mathbb{H}_0 \subset L_0$ and a small neighborhood $V \subset S_0$ of $\mathbb{H}_0$. One can check that if V is small enough, $\mathcal{F}$ is smooth in V, and that its automorphism group is $\mathrm{SL}_2(\mathbb{R})$ acting as Möbius transformations on $\mathbb{H}_0$.

We conclude that for any compact curve C of genus $g \geq 2$ there is exactly one example (up to biholomorphism) of a surface (S, C) equipped with a smooth semiholomorphic foliation $\mathcal{F}$ coming from the construction above. Remember that the construction above only gives those examples that come in a family; there might be exceptional examples coming from automorphisms that are not exponentials of infinitesimal automorphisms.

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