

Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ALGEBRAIC CYCLES AND FANO THREEFOLDS OF GENUS 10

Robert Laterveer

Tome 151
Fascicule 4

2023

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 565-593

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 4, tome 151, décembre 2023

Comité de rédaction

Boris ADAMCZEWSKI
Christine BACHOC
François CHARLES
François DAHMANI
Clothilde FERMANIAN

Dorothee FREY
Wendy LOWEN
Béatrice de TILIÈRE
Eva VIEHMANN

Marc HERZLICH (Dir.)

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € (\$ 64)

Abonnement électronique : 135 € (\$ 202),

avec supplément papier : Europe 179 €, hors Europe 197 € (\$ 296)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France

Société Mathématique de France

Institut Henri Poincaré, 11, rue Pierre et Marie Curie

75231 Paris Cedex 05, France

Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96

bulletin@smf.emath.fr • smf.emath.fr

© Société Mathématique de France 2023

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

ALGEBRAIC CYCLES AND FANO THREEFOLDS OF GENUS 10

BY ROBERT LATERVEER

ABSTRACT. — We show that prime Fano threefolds Y of genus 10 have a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of Y injects into cohomology.

RÉSUMÉ (*Cycles algébriques et solides de Fano de genre 10*). — Soit Y un solide de Fano d'indice 1 et de genre 10. On montre que Y admet une décomposition de Chow–Künneth multiplicative, au sens de Shen–Vial. Il s'ensuit qu'un certain sous-anneau “tautologique” de l'anneau de Chow des puissances de Y s'injecte en cohomologie.

Texte reçu le 15 avril 2022, modifié le 25 mars 2023, accepté le 18 avril 2023.

ROBERT LATERVEER, Institut de Recherche Mathématique Avancée, CNRS – Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg CEDEX, FRANCE •
E-mail : robert.laterveer@math.unistra.fr

Mathematical subject classification (2020). — 14C15, 14C25, 14C30.

Key words and phrases. — Algebraic cycles, Chow group, motive, Beauville's “splitting property” conjecture, multiplicative Chow–Künneth decomposition, Fano threefolds, tautological ring, homological projective duality.

2020 Mathematics Subject Classification: 14C15, 14C25, 14C30. Supported by ANR grant ANR-20-CE40-0023.

1. Introduction

Given a smooth projective variety Y over \mathbb{C} , let

$$A^i(Y) := CH^i(Y)_{\mathbb{Q}}$$

denote the Chow groups of Y (i.e. the groups of codimension i algebraic cycles on Y with \mathbb{Q} -coefficients, modulo rational equivalence). The intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the Chow ring of Y [11]. In the case of K3 surfaces, this ring has a remarkable property:

THEOREM 1.1 (Beauville–Voisin [3]). — *Let S be a K3 surface. The \mathbb{Q} -subalgebra*

$$\langle A^1(S), c_j(S) \rangle \subset A^*(S)$$

injects into cohomology under the cycle class map.

The Chow ring of abelian varieties also exhibits particular behaviour: there is a multiplicative splitting [1]. Motivated by the cases of K3 surfaces and abelian varieties, Beauville [2] has conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting (and a certain subring should inject into cohomology). To make concrete sense of Beauville’s elusive “splitting property conjecture”, Shen–Vial [47] have introduced the concept of *multiplicative Chow–Künneth decomposition*; we will abbreviate this to “MCK decomposition” (for the precise definition, cf. Section 3 below).

It is something of a challenge to understand precisely which varieties admit an MCK decomposition. To give an idea of what is known: hyperelliptic curves have an MCK decomposition [47, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [9, Example 2.3]; K3 surfaces have an MCK decomposition, but certain high degree surfaces in \mathbb{P}^3 do not have an MCK decomposition (cf. the examples given in [41]). In this note, we will focus on Fano threefolds and ask the following question:

QUESTION 1.2. — *Let X be a Fano threefold with Picard number 1. Does X admit an MCK decomposition?*

The restriction on the Picard number is necessary to rule out a counterexample of Beauville [2, Examples 9.1.5]. The answer to Question 1.2 is affirmative for cubic threefolds [6], [9], for intersections of two quadrics [30], for intersections of a quadric and a cubic [31] and for prime Fano threefolds of genus 8 [29].

The main result of this note answers Question 1.2 for one more family:

THEOREM (= Theorem 5.1). — *Let Y be a prime Fano threefold of genus 10. Then Y has a multiplicative Chow–Künneth decomposition.*

The argument proving Theorem 5.1 is based on the connections between Y and a certain genus 2 curve, and between Y and an index 2 Fano threefold Z (cf. Theorem 2.2). The work of Kuznetsov [20], [21], [23], building these connections on a categorical level inside the set-up of *homological projective duality*, allows us to establish the instances of the *Franchetta property* that are needed to prove the theorem.

Reaping the fruits of Theorem 5.1, we obtain a result concerning the *tautological ring*, which is a certain subring of the Chow ring of powers of Y :

COROLLARY (= Corollary 7.1). — *Let Y be a prime Fano threefold of genus 10, and $m \in \mathbb{N}$. Let*

$$R^*(Y^m) := \langle (p_i)^*(h), (p_{ij})^*(\Delta_Y) \rangle \subset A^*(Y^m)$$

be the \mathbb{Q} -subalgebra generated by pullbacks of the polarization $h \in A^1(Y)$ and pullbacks of the diagonal $\Delta_Y \in A^3(Y \times Y)$. The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

This is the kind of injectivity result that motivated Beauville’s work on the “splitting property conjecture” [2]. To paraphrase Corollary 7.1, one could say that genus 10 Fano threefolds behave like hyperelliptic curves from the point of view of intersection theory (cf. Remark 7.2 below).

Conventions. — In this article, the word *variety* will refer to a reduced irreducible scheme of finite type over \mathbb{C} . A *subvariety* is a (possibly reducible) reduced subscheme that is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(Y)$ the Chow group of j -dimensional cycles on Y with \mathbb{Q} -coefficients; for Y smooth of dimension n the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notation $A_{hom}^j(Y)$ will be used to indicate the subgroup of homologically trivial cycles. For a morphism $f: X \rightarrow Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f .

The contravariant category of Chow motives (i.e. pure motives with respect to rational equivalence as in [46], [40]) will be denoted \mathcal{M}_{rat} .

2. Prime Fano threefolds of genus 10

The classification of Fano threefolds is one of the glories of twentieth century algebraic geometry [17]. Fano threefolds that are *prime* (i.e. with Picard group of rank 1 generated by the canonical divisor) come in 10 explicitly described families. In this paper, we will be concerned with one of these families:

THEOREM 2.1 (Mukai [38]). — *Let Y be a prime Fano threefold (i.e. a smooth projective Fano threefold with $\text{Pic}(Y) = \mathbb{Z}[K_Y]$), of genus 10. Then Y is a dimensionally transverse intersection*

$$Y = G_2 \text{Gr}(2, 7) \cap \mathbb{P}^{11} \subset \mathbb{P}^{13},$$

where $G_2 \text{Gr}(2, 7)$ is the minimal compact homogeneous space for the simple algebraic group of type G_2 (the variety $G_2 \text{Gr}(2, 7)$ can be realized as the zero locus of a section of a certain vector bundle on the Grassmannian $\text{Gr}(2, 7)$).

Conversely, any smooth dimensionally transverse intersection $G_2 \text{Gr}(2, 7) \cap \mathbb{P}^{11}$ is a prime Fano threefold of genus 10.

The Hodge diamond of Y is

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 0 & 0 & \\ & & & & 0 & 1 & 0 \\ & & & 0 & 2 & 2 & 0 \\ & & & 0 & 1 & 0 & \\ & & & 0 & 0 & & \\ & & & & 1 & & \end{array}$$

Proof. — The “conversely” statement is just because $G_2 \text{Gr}(2, 7)$ is a Fano variety of dimension 5 with Picard number 1, index 3 and degree 18; the codimension 2 complete intersection $Y := G_2 \text{Gr}(2, 6) \cap \mathbb{P}^{11}$ thus has index 1 and degree $d = 18$ (i.e. genus $g = d/2 + 1 = 10$). The first statement is proven in [38].

To see that $h^{2,1}(Y) = 2$, one can use Theorem 2.2 or Theorem 2.8 below. \square

2.1. Hilbert scheme of conics. —

THEOREM 2.2. — *Let Y be a prime Fano threefold of genus 10, and let $F := F(Y)$ be the Hilbert scheme parametrizing conics contained in Y .*

- (i) *F is an abelian surface, isomorphic to the intermediate Jacobian of Y .*
- (ii) *there is a genus 2 curve C , geometrically associated to Y , such that F is isomorphic to the Jacobian of C .*
- (iii) *There exists $P \in A^2(Y \times F)$ inducing an isomorphism*

$$P_* : H^3(Y, \mathbb{Q}) \xrightarrow{\cong} H^1(F, \mathbb{Q}).$$

- (iv) *There exists a Fano threefold Z of Picard number 1, index 2 and degree 4 (i.e. Z is a complete intersection of two quadrics in \mathbb{P}^5), such that the Fano surface $F_1(Z)$ of lines in Z is isomorphic to F :*

$$F_1(Z) \cong F(Y).$$

Proof. — Item (i) is proven for general Y in [16, Proposition 3]; the extension to arbitrary Y is done in [23, Theorem 1.1.1].

(ii) This follows from the proof of [16, Proposition 3] in the case Y is general; for arbitrary Y , it is part of the proof of [23, Proposition B.5.1]. Recall that Y is a codimension 2 linear section of the homogeneous variety $G_2 \text{Gr}(2, 7)$. The pencil of hyperplanes containing Y has six singular elements (because the projective dual to $G_2 \text{Gr}(2, 7)$ is a sextic hypersurface). The curve C is constructed as the double cover $C \rightarrow \mathbb{P}^1$ branched in these six points.

(iii) This follows from the Abel–Jacobi isomorphism (i); alternatively one could use Theorem 2.8.

(iv) This is part of a more general (and rather mysterious) phenomenon linking certain index 1 Fano threefolds and certain index 2 Fano threefolds, which was first discovered by Kuznetsov [21, Section 4.2], [23, Proposition B.5.1].

There is not a unique index 2 Fano threefold Z associated to Y (this is only true up to projective isomorphism), but there is a canonical way of giving a Z as in (iv), as explained in [23, Section B.5]: let $\lambda_0, \dots, \lambda_5$ denote the branch points of the double cover $C \rightarrow \mathbb{P}^1$. Choosing an embedding of \mathbb{A}^1 in \mathbb{P}^1 such that the six points λ_j are contained in \mathbb{A}^1 , let us write $\lambda_j \in \mathbb{A}^1$ for the affine coordinates of these six points. The intersection of quadrics

(1)

$$Z := \left\{ [x_0, x_1, \dots, x_5] \in \mathbb{P}^5 \mid x_0^2 + \dots + x_5^2 = \lambda_0 x_0^2 + \dots + \lambda_5 x_5^2 = 0 \right\} \subset \mathbb{P}^5$$

is smooth (and, hence, it is a Fano threefold as in (iv)), and the genus 2 curve C_Z (naturally associated to Z by looking at the six singular quadrics in the pencil defining Z) is isomorphic to C . Since it is known that $F_1(Z) \cong \text{Jac}(C_Z)$ [44], this gives the required isomorphism $F_1(Z) \cong F(Y)$. \square

REMARK 2.3. — Although they are related, there is an important difference between the index 2 Fano threefolds Z (as in Theorem 2.2(iv)) and the index 1 Fano threefolds Y . For the threefolds Z there is a Torelli theorem, i.e. Z is uniquely determined by $F_1(Z)$ [5, Section 3.6]. On the other hand, the (generic) Torelli theorem fails for prime Fano threefolds of genus 10: the threefold Y is *not* determined by the surface $F(Y)$. Indeed, the moduli space \mathcal{M}_{10} (of genus 10 prime Fano threefolds) has dimension 10, whereas the moduli space \mathcal{M}_2 of genus 2 curves has dimension 3, and so the surjective morphism

$$\mathcal{M}_{10} \rightarrow \mathcal{M}_2$$

(sending a Fano threefold Y to the curve C) has generic fibre of dimension 7.

In this context, Iliev and Manivel have proven a certain “modified Torelli statement”: the genus 10 prime Fano threefold Y containing a fixed K3 surface S as hyperplane section is uniquely determined by the image of $F(Y)$ in the Hilbert scheme $S^{[2]}$ [16, Theorem 6].

2.2. The Y - $F(Y)$ relation. —

PROPOSITION 2.4. — *Let Y be a prime Fano threefold of genus 10 and let $F := F(Y)$ be the Hilbert scheme of conics in Y . There is an isomorphism of Chow motives*

$$h(Y^{(2)}) \cong h(F)(-2) \oplus h(Y) \oplus h(Y)(-3) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Proof. — (NB: the symmetric product $Y^{(2)}$ is not smooth, but it is a projective Alexander scheme in the sense of [19], and so it makes sense to speak about the motive of $Y^{(2)}$ in the category \mathcal{M}_{rat} . If one prefers, one may think of the motive $h(Y^{(2)})$ as $(h(Y) \otimes h(Y))^{\mathfrak{S}_2}$.)

Letting $h \in A^1(Y)$ denote a hyperplane section (with respect to the embedding $Y \subset \mathbb{P}^{13}$ given by Theorem 2.1), let us write

$$(2) \quad \begin{aligned} \pi_Y^0 &:= \frac{1}{18} h^3 \times Y , \\ \pi_Y^2 &:= \frac{1}{18} h^2 \times h , \\ \pi_Y^4 &:= \frac{1}{18} h \times h^2 , \\ \pi_Y^6 &:= \frac{1}{18} Y \times h^3 , \\ \pi_Y^3 &:= \Delta_Y - \sum_{j \neq 3} \pi_Y^j \in A^3(Y \times Y) , \end{aligned}$$

and $h^j(Y) := (Y, \pi_Y^j, 0) \in \mathcal{M}_{\text{rat}}$. (This is the CK decomposition, which we will prove to be MCK in Theorem 5.1.)

As we have seen (Theorem 2.2), there is an isomorphism

$$P_* : H^3(Y, \mathbb{Q}) \xrightarrow{\cong} H^1(F, \mathbb{Q}) .$$

Since both Y and F verify the Lefschetz standard conjecture, the inverse isomorphism is also induced by a correspondence. (This is well known, cf., for instance, [51, Proof of Proposition 1.1], where I first learned this.) It follows that there is an isomorphism of homological motives

$$P : h^3(Y) \xrightarrow{\cong} h^1(F)(-1) \quad \text{in } \mathcal{M}_{\text{hom}} .$$

Since Y and F are Kimura finite-dimensional (in the sense of [45]), this can be upgraded to an isomorphism of Chow motives

$$(3) \quad P : h^3(Y) \xrightarrow{\cong} h^1(F)(-1) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Writing $h(Y) \cong h^1(F)(-1) \oplus \mathbf{1} \oplus \mathbf{1}(-1) \oplus \mathbf{1}(-2) \oplus \mathbf{1}(-3)$ and taking the symmetric power, one obtains

$$(4) \quad h(Y^{(2)}) \cong \text{Sym}^2 h^1(F)(-2) \oplus h^1(F)(-1) \oplus h^1(F)(-2) \\ \oplus h^1(F)(-3) \oplus h^1(F)(-4) \oplus \bigoplus \mathbf{1}(\ast) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

On the other hand, F being an abelian surface its motive decomposes

$$(5) \quad h(F) = \mathbf{1} \oplus h^1(F) \oplus \text{Sym}^2 h^1(F) \oplus h^1(F)(-1) \oplus \mathbf{1}(-2) \quad \text{in } \mathcal{M}_{\text{rat}}$$

(cf. for instance [46, Section 5]). Combining (4) and (5), we obtain the isomorphism of the proposition. □

REMARK 2.5. — Proposition 2.4 is formally similar to the relation between a cubic hypersurface $Y \subset \mathbb{P}^{n+1}$ and its Fano variety of lines $F := F(Y)$; in this case, it is known that

$$h(Y^{(2)}) \cong h(F)(-2) \oplus h(Y) \oplus h(Y)(-n) \quad \text{in } \mathcal{M}_{\text{rat}}$$

[8, Theorem 3]. This is the motivic version of the Y - $F(Y)$ relation in the Grothendieck ring of varieties that was discovered by Galkin–Shinder [13].

2.3. The Z - $F(Z)$ relation. —

PROPOSITION 2.6. — *Let $Z \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics and let $F := F_1(Z)$ be the Hilbert scheme of lines in Z . There is an isomorphism of Chow motives*

$$h(Z^{(2)}) \cong h(F)(-2) \oplus h(Z) \oplus h(Z)(-3) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Proof. — The argument is the same as that of Proposition 2.4, given that F is an abelian surface and that the universal line $P \subset Z \times F$ induces an Abel–Jacobi isomorphism

$$P_* : H^3(Z, \mathbb{Q}) \xrightarrow{\cong} H^1(F, \mathbb{Q})$$

[44, Theorem 4.14]. □

2.4. Conics on G . — Let us write $G := G_2 \text{Gr}(2, 7)$ for the G_2 -Grassmannian. We will need to understand conics on G :

PROPOSITION 2.7. — *Let $G := G_2 \text{Gr}(2, 7)$ and let $F(G)$ be the Hilbert scheme of conics (with respect to the Plücker embedding) contained in G . The scheme $F(G)$ is a smooth projective spherical variety of dimension 8. In particular, $A_{\text{hom}}^*(F(G)) = 0$.*

Proof. — The variety $F(G)$ is isomorphic to the so-called *Cayley Grassmannian* (cf. [37, Section 7.1], where this observation is attributed to Kuznetsov). The Cayley Grassmannian CG is known to be a smooth projective spherical variety [36, Theorem 1.1], hence it has trivial Chow groups. (Actually, it is

known that the Chow groups of CG are isomorphic to those of the Grassmanian $\text{Gr}(2, 6)$ [36, Theorem 1.2]. \square

2.5. Derived category. —

THEOREM 2.8. — *Let Y be a prime Fano threefold of genus 10. The derived category of Y admits a semi-orthogonal decomposition*

$$D^b(Y) = \langle D^b(C), \mathcal{O}_Y, \mathcal{U}^* \rangle,$$

where C is the genus 2 curve of Theorem 2.4, and \mathcal{U} is the restriction of the tautological rank 2 subbundle on $\text{Gr}(2, 7)$.

Proof. — This is proven in [20, Section 6.4], as an instance of *homological projective duality*. It will be important for us to understand how the curve C and the semi-orthogonal decomposition are constructed. As in loc. cit., let V be the 14-dimensional vector space such that $G := G_2 \text{Gr}(2, 7)$ is embedded in $\mathbb{P}(V)$ (writing $\text{Gr}(2, W) = \text{Gr}(2, 7)$, this V arises from the decomposition $\wedge^2 V = W^* \oplus V$ into G_2 -representations). The projective dual $G^\vee \subset \mathbb{P}(V^*)$ of G is a sextic hypersurface, with singular locus G_{sing}^\vee of dimension 10. Let

$$h: H \rightarrow \mathbb{P}(V^*) \setminus G_{\text{sing}}^\vee$$

be the double cover ramified along $G^\vee \setminus G_{\text{sing}}^\vee$. As explained in loc. cit., there is a certain sheaf of algebras \mathcal{A}_H on H such that the non-commutative variety

$$(H, \mathcal{A}_H)$$

is HPD dual to $G_2 \text{Gr}(2, 7)$. As proven in loc. cit., this entails in particular that given an r -dimensional linear subspace $L \subset V^*$, one can relate the derived categories of the linear sections $H_r := h^{-1}(\mathbb{P}(L))$ and $G_r := G \cap \mathbb{P}(L^*)$. Taking an L of dimension $r = 2$ and such that H_2 (and hence G_2) is smooth and dimensionally transverse, one has that H_2 is a genus 2 curve, and G_2 is a prime Fano threefold of genus 10 (and every prime Fano threefold of genus 10 arises in this way, cf. Theorem 2.1). This gives the semi-orthogonal decomposition as indicated.

(For later use, we further observe that taking a linear subspace L of dimension $r = 3$ and intersecting smoothly and transversely, the varieties G_3 and H_3 are K3 surfaces of degree 18 resp. 2 that are twisted derived equivalent: as shown in loc. cit., one has

$$(6) \quad D^b(G_3) \cong D^b(H_3, \alpha),$$

where α is a Brauer class. Moreover, the general K3 surfaces of degree 18 and of degree 2 arise in this way.) \square

3. MCK decomposition

DEFINITION 3.1 (Murre [39]). — Let X be a smooth projective variety of dimension n . We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \quad \text{in } A^n(X \times X),$$

such that the π_X^i are mutually orthogonal idempotents and $(\pi_X^i)_* H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

REMARK 3.2. — Murre has conjectured that any smooth projective variety should have a CK decomposition [39], [18].

DEFINITION 3.3 (Shen–Vial [47]). — Let X be a smooth projective variety of dimension n and let $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$ denote the class of the small diagonal

$$\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

An *MCK decomposition* is defined as a CK decomposition $\{\pi_X^i\}$ of X that is *multiplicative*, i.e. it satisfies

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) = 0 \quad \text{in } A^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k.$$

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

REMARK 3.4. — The small diagonal (when considered as a correspondence from $X \times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{sm} : h(X) \otimes h(X) \rightarrow h(X) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

Let us assume X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

By definition, this decomposition is multiplicative if for any i, j the composition

$$h^i(X) \otimes h^j(X) \rightarrow h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \quad \text{in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$.

If X has an MCK decomposition, then setting

$$A_{(j)}^i(X) := (\pi_X^{2i-j})_* A^i(X),$$

one obtains a bigraded ring structure on the Chow ring; that is, the intersection product sends $A_{(j)}^i(X) \otimes A_{(j')}^{i'}(X)$ to $A_{(j+j')}^{i+i'}(X)$.

It is conjectured that for any X with an MCK decomposition, one has

$$A_{(j)}^i(X) \stackrel{??}{=} 0 \quad \text{for } j < 0, \quad A_{(0)}^i(X) \cap A_{hom}^i(X) \stackrel{??}{=} 0;$$

this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition [39]. In particular, this would imply that the subring $A_{(0)}^*(X)$ injects into cohomology under the cycle class map.

REMARK 3.5. — The property of having an MCK decomposition is motivated by, and closely related to, Beauville's "splitting property" conjecture [2]. To give an idea of what is known: hyperelliptic curves have an MCK decomposition [47, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [9, Example 2.3]. It has been conjectured that all hyperkähler varieties have an MCK decomposition. For a more thorough discussion, and for more examples of varieties with an MCK decomposition, we refer to [47, Section 8], as well as [53], [48], [10], [25], [35], [26], [27], [28], [9], [33], [31], [34].

4. Franchetta property

DEFINITION 4.1. — Let $\mathcal{X} \rightarrow B$ be a smooth projective morphism, where \mathcal{X}, B are smooth quasi-projective varieties. We say that $\mathcal{X} \rightarrow B$ has the *Franchetta property in codimension j* if the following holds: for every $\Gamma \in A^j(\mathcal{X})$ such that the restriction $\Gamma|_{X_b}$ is homologically trivial for the very general $b \in B$, the restriction $\Gamma|_b$ is zero in $A^j(X_b)$ for all $b \in B$.

We say that $\mathcal{X} \rightarrow B$ has the *Franchetta property* if $\mathcal{X} \rightarrow B$ has the Franchetta property in codimension j for all j .

This property is studied in [42], [4], [7], [8].

DEFINITION 4.2. — Let $\mathcal{X} \rightarrow B$ be a family as above, with $X := X_b$ a fibre. We will write

$$GDA_B^j(X) := \text{Im}\left(A^j(\mathcal{X}) \rightarrow A^j(X)\right)$$

for the subgroup of *generically defined cycles*. (In a context where it is clear to which family we are referring, the index B will sometimes be dropped from the notation.)

With this definition, the Franchetta property amounts to saying that $GDA^*(X)$ injects into cohomology, under the cycle class map.

4.1. Franchetta property for Y . —

NOTATION 4.3. — Let G be the G_2 -Grassmannian $G := G_2 \text{Gr}(2, 7)$ and let $\mathcal{O}_G(1)$ be the polarization corresponding to the Plücker embedding $G \subset \mathbb{P}^{13}$. Let

$$B \subset \bar{B} := \mathbb{P}H^0(G, \mathcal{O}_G(1)^{\oplus 2})$$

denote the Zariski open subset parametrizing smooth dimensionally transverse complete intersections and let

$$\mathcal{Y} \rightarrow B$$

denote the universal family of smooth three-dimensional complete intersections (in view of Theorem 2.1, this is the universal family of prime Fano threefolds of genus 10).

PROPOSITION 4.4. — *Let $\mathcal{Y} \rightarrow B$ be the universal family of prime Fano threefolds of genus 10 (Notation 4.3). The family $\mathcal{Y} \rightarrow B$ has the Franchetta property.*

Proof. — We give two different proofs of this proposition. For the first proof, let $\bar{\mathcal{Y}} \subset \bar{B} \times G$ denote the projective closure. As the line bundle $\mathcal{O}_G(1)$ is base point free, the projection $\bar{\mathcal{Y}} \rightarrow G$ is a \mathbb{P}^r -fibration. Using the projective bundle formula, it is readily checked (cf., for instance, [42, Proof of Lemma 1.1]) that

$$GDA_B^*(Y) = \text{Im}(A^*(G) \rightarrow A^*(Y)) .$$

Since $A_{hom}^j(Y) = 0$ for $j \neq 2$, it only remains to ascertain that the cycle class map induces injections

$$(7) \quad \text{Im}(A^*(G) \rightarrow A^*(Y)) \rightarrow H^*(Y, \mathbb{Q}) .$$

But the Chow ring of the G_2 -Grassmannian G is as small as can be for a smooth projective variety; indeed, the fivefold G admits a full exceptional collection with six exceptional objects [20, Section 6.4], which means that G is a *minifold* in the sense of [12], [22, Section 1.4]. Taking Chow groups, we find that $A^*(G) = \mathbb{Q}^6$ and so $A^j(G) = \mathbb{Q}[h^j]$ for all j . This implies that (7) is injective and ends the first proof.

For the second proof of the proposition, we explore the relation with the genus 2 curve C given by the HPD framework of Theorem 2.8. This theorem gives an isomorphism

$$R_* : A^2(Y) \xrightarrow{\cong} A^1(C) .$$

Even better, this isomorphism exists *universally*: let us write $\mathcal{C} \rightarrow B$ for the universal family of smooth one-dimensional linear sections of H (where H is the double cover defined in the proof of Theorem 2.8). The correspondence R is generically defined (i.e. there exists $\mathcal{R} \in A^2(\mathcal{Y} \times_B \mathcal{C})$ such that the fibrewise

restriction $\mathcal{R}|_b$ is the correspondence $R \in A^2(Y \times C)$); this is because the semi-orthogonal decomposition of Theorem 2.8 exists universally (cf. [20, Proof of Theorem 1.2]). It follows that R_* sends generically defined cycles to generically defined cycles, and so there is a commutative diagram

$$\begin{CD} GDA_B^2(Y) @>R_*>> GDA_B^1(C) \\ @VVV @VVV \\ \mathbb{Q} \cong H^4(Y, \mathbb{Q}) @>R_*>> H^2(C, \mathbb{Q}) \cong \mathbb{Q} \end{CD}$$

(where vertical arrows are cycle class maps).

To prove the proposition, it thus suffices to prove the Franchetta property for the family of curves $C \rightarrow B$. Let $\bar{C} \subset \bar{B} \times \bar{H}$ denote the projective closure, then $\bar{C} \rightarrow \bar{H}$ is a \mathbb{P}^s -fibration, and so (as above) the projective bundle formula (plus the fact that C is contained in the smooth quasi-projective variety H , by construction) gives

$$GDA_B^1(C) = \text{Im}(A^1(H) \rightarrow A^1(C)) .$$

Let us check that the right-hand side is one-dimensional. In view of the usual spread lemma (cf. [54, Lemma 3.2]), it suffices to prove this for the very general C , and so we may suppose C is contained in a two-dimensional smooth linear section $C := H_2 \subset H_3 \subset H$. Such a surface H_3 is a degree 2 K3 surface. What is more, in view of Lemma 4.5 below, we may suppose H_3 has Picard number 1, and so

$$\text{Im}(A^1(H) \rightarrow A^1(C)) = \text{Im}(A^1(H_3) \rightarrow A^1(C)) = \mathbb{Q}[K_C] .$$

The proposition is now proven, modulo the following lemma:

LEMMA 4.5. — *The very general curve $C \subset H$ is contained in a linear section $H_3 \subset H$, where H_3 is a K3 surface with $\text{Pic}(H_3) = \mathbb{Z}$.*

To prove the lemma, we return to the HPD framework of Theorem 2.8. Thanks to Mukai, we know that the very general K3 surface of degree 18 arises as a linear section G_3 of the G_2 -Grassmannian G , and so the very general Fano threefold Y has an anticanonical section $G_3 \subset Y$ with $\text{Pic}(G_3) \cong \mathbb{Z}$. On the HPD dual side, the genus 2 curve $C \subset H$ associated to Y is contained in the K3 surface $H_3 \subset H$, which is twisted derived equivalent to G_3 , cf. equality (6). Twisted derived equivalent K3 surfaces are isogenous (and even have isomorphic Chow motives, cf. [15]), and so $\text{Pic}(H_3) \cong \mathbb{Z}$. □

4.2. Franchetta property for $Y \times Y$. —

PROPOSITION 4.6. — *Let $\mathcal{Y} \rightarrow B$ be as in Notation 4.3. The family $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$ has the Franchetta property.*

Proof. — To prove this, we move to the HPD dual side, as in the second proof of Proposition 4.4. Kuznetsov’s work (Theorem 2.8) gives an isomorphism of motives

$$h^3(Y) \cong h^1(C)(-1) \quad \text{in } \mathcal{M}_{\text{rat}} ,$$

where C is a genus 2 curve. It follows that there is a split injection of motives

$$h(Y) \hookrightarrow h(C)(-1) \oplus \mathbb{1} \oplus \mathbb{1}(-1) \oplus \mathbb{1}(-2) \oplus \mathbb{1}(-3) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

As we have verified in the proof of Proposition 4.4, this isomorphism and split injection are generically defined (with respect to B), and so one obtains in particular split injections of Chow groups

$$GDA_B^j(Y \times Y) \hookrightarrow GDA_B^{j-2}(C \times C) \oplus \bigoplus GDA_B^*(C) \oplus \mathbb{Q}^t .$$

Since this injection is compatible with cycle class maps, it suffices to prove the Franchetta property for the family $C \times_B C \rightarrow B$ (where $C \rightarrow B$ is as in the proof of Proposition 4.4). To this end, we recall that $C \rightarrow B$ is the universal family of smooth one-dimensional linear sections of H (cf. the proof of Theorem 2.8). Since $\mathcal{O}_H(1)$ is very ample, this set-up verifies the property $(*_2)$ of [7], and so [7, Proposition 5.2] implies that there is equality

$$(8) \quad GDA_B^*(C \times C) = \langle \text{Im}(A^*(H \times H) \rightarrow A^*(C \times C)), \Delta_C \rangle .$$

It remains to check that the right-hand side of (8) injects into cohomology. We need a lemma:

LEMMA 4.7. — *Let $\mathcal{S} \rightarrow B'$ be the universal family of smooth two-dimensional linear sections of H (i.e. the fibres of $\mathcal{S} \rightarrow B'$ are the degree 2 K3 surfaces H_3 of (6)). The family $\mathcal{S} \times_{B'} \mathcal{S} \rightarrow B'$ has the Franchetta property.*

Proof. — The HPD set-up (cf. the proof of Theorem 2.8) shows that there is a “dual” family $\mathcal{T} \rightarrow B'$ over the same base, where the fibres of $\mathcal{T} \rightarrow B'$ are the degree 18 K3 surfaces $G_3 \subset G$ of (6). For any $b \in B'$, the fibres S_b and T_b are isogenous, and so they have isomorphic Chow motives [15]. This isomorphism of motives is universally defined (indeed, the Fourier–Mukai functor inducing the derived equivalence between S_b and T_b is universally defined, by construction), and so the Franchetta property for $\mathcal{S} \times_{B'} \mathcal{S} \rightarrow B'$ is equivalent to the Franchetta property for $\mathcal{T} \times_{B'} \mathcal{T} \rightarrow B'$. The latter property is proven in [7, Theorem 1.5]. □

Armed with Lemma 4.7, let us further analyze the right-hand side of (8). Up to shrinking the base B (and invoking the spread lemma [54, Lemma 3.2]), we may assume that the curve $C \subset H$ is contained in a smooth K3 surface $S \subset H$ such that S is a fibre of the family $\mathcal{S} \rightarrow B'$ of Lemma 4.7. Then we observe that (because of the inclusions $C \subset S \subset H$) the restriction map

$$A^*(H \times H) \rightarrow A^*(C \times C)$$

factors over $GDA_B^*(S \times S)$. Using (8) and Lemma 4.7, it follows that

$$GDA_B^1(C \times C) = \langle (p_i)^*(h), \Delta_C \rangle \cap A^1(C \times C) .$$

Since Δ_C is not decomposable in cohomology (otherwise $H^1(C, \mathbb{Q})$ would be zero), this shows the injectivity of $GDA_B^1(C \times C) \rightarrow H^2(C \times C, \mathbb{Q})$. Next, for the codimension 2 cycles we observe that

$$\begin{aligned} \text{Im}(A^2(H \times H) \rightarrow A^2(C \times C)) &\subset \text{Im}(GDA_B^2(S \times S) \rightarrow A^2(C \times C)) \\ &= \langle (p_i)^*(h), \Delta_S|_{C \times C} \rangle . \end{aligned}$$

(For the last equality, we used Lemma 4.7, which gives an equality $GDA_B^2(S \times S) = \langle (p_i)^*(h), \Delta_S \rangle$.) Since $C \subset S$ is a hyperplane section, the excess intersection formula gives the equality $\Delta_S|_{C \times C} = \Delta_C \cdot (p_1)^*(h)$. Now, h is proportional to the canonical divisor K_C in $A^1(C)$, and so

$$(9) \quad \Delta_C \cdot (p_1)^*(h) = \Delta_C \cdot (p_1)^*(K_C) \in \langle (p_i)^*(K_C) \rangle = \langle (p_i)^*(h) \rangle$$

(this inclusion is true more generally for hyperelliptic curves [50] but *not* for general curves of genus ≥ 4 ; cf. [14], [55]). It follows that

$$\text{Im}(A^2(H \times H) \rightarrow A^2(C \times C)) = \langle (p_i)^*(h) \rangle ,$$

and so (8) simplifies in codimension 2 to

$$\begin{aligned} GDA_B^2(C \times C) &= \langle \text{Im}(A^*(H \times H) \rightarrow A^*(C \times C)), \Delta_C \rangle \cap A^2(C \times C) \\ &= \langle (p_i)^*(h), \Delta_C \rangle \cap A^2(C \times C) \\ &= \langle (p_i)^*(h) \rangle \cap A^2(C \times C) \end{aligned}$$

(where in the last equality we used once more the inclusion (9)). In view of the Künneth decomposition of cohomology, it is now clear that $GDA_B^2(C \times C)$ injects into cohomology. This closes the proof. \square

COROLLARY 4.8. — *Let Y be a genus 10 prime Fano threefold. There exist $a_j \in \mathbb{Q}$ such that there is equality*

$$\Delta_Y \cdot (p_i)^*(K_Y) = \sum_{j=1}^3 a_j K_Y^j \times K_Y^{4-j} \quad \text{in } A^4(Y \times Y) \quad (i = 1, 2) .$$

Proof. — One can readily find $a_j \in \mathbb{Q}$ such that the equality of the corollary is true in cohomology (this is because $\Delta_Y \cdot (p_i)^*(K_Y)$ is the correspondence acting as cup product with K_Y ; this action is non-zero only on $H^{2j}(Y, \mathbb{Q})$, which is algebraic). The Franchetta property of Proposition 4.6 then allows to lift the equality to rational equivalence. \square

COROLLARY 4.9. — *Let $\mathcal{F} \rightarrow B$ denote the universal family of Hilbert schemes of conics contained in genus 10 prime Fano threefolds. The family $\mathcal{F} \rightarrow B$ has the Franchetta property.*

Proof. — The Y - $F(Y)$ relation of motives of Proposition 2.4 is universally defined (with respect to B); indeed, this relation is based on the isomorphism $R: h^3(Y) \cong h^1(F)(-1)$ of Theorem 2.8, which (as we saw in the proof of Proposition 4.4) is universally defined. This means that there is a commutative diagram

$$\begin{CD} GDA_B^j(F) @>>> GDA_B^{j+2}(Y^{(2)}) \oplus \bigoplus GDA_B^*(Y) \oplus \mathbb{Q}^t \\ @VVV @VVV \\ H^{2j}(F, \mathbb{Q}) @>>> H^{2j+4}(Y^{(2)}, \mathbb{Q}) \oplus \bigoplus H^*(Y, \mathbb{Q}) \oplus \mathbb{Q}^t, \end{CD}$$

where the horizontal arrows are injections, and the vertical arrows are the cycle class maps. The right vertical arrow is injective (this follows from the Franchetta property for Y and for Y^2 , cf. Propositions 4.4 and 4.6), and so the left vertical arrow is injective as well. □

4.3. Franchetta for $Z \times Z$. —

NOTATION 4.10. — Let $\mathcal{Y} \rightarrow B$ be the universal family as in Notation 4.3. Likewise, let

$$B_{(2,2)} \subset \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)^{\oplus 2})$$

be the Zariski open parametrizing smooth complete intersections of two quadrics and let

$$B_{(2,2)} \times \mathbb{P}^5 \supset \mathcal{Z} \rightarrow B_{(2,2)}$$

denote the universal family of smooth complete intersections of two quadrics in \mathbb{P}^5 .

The construction outlined in the proof of Theorem 2.2(iv) gives a non-empty Zariski open $B^\circ \subset B$ and a map

$$B^\circ \rightarrow B_{(2,2)}$$

associating to a prime genus 10 Fano threefold Y an intersection of two quadrics Z such that there is an isomorphism $F(Y) \cong F_1(Z)$. We will write

$$\mathcal{Z} \rightarrow B^\circ$$

for the family obtained by base change.

PROPOSITION 4.11. — *The family $\mathcal{Z} \times_{B^\circ} \mathcal{Z} \rightarrow B^\circ$ has the Franchetta property.*

Proof. — We will use the following motivic relation:

LEMMA 4.12. — *Let Y be a genus 10 Fano threefold parametrized by B° and let Z be the associated complete intersection of two quadrics (cf. Theorem 2.2). There is an isomorphism of motives*

$$\Gamma: h(Z) \xrightarrow{\cong} h(Y) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

Moreover, the correspondence Γ and the correspondence inducing the inverse isomorphism are generically defined with respect to B° .

The lemma obviously implies the proposition: the correspondence $\Gamma \times \Gamma$ induces an isomorphism

$$GDA_{B^\circ}^*(Z \times Z) \xrightarrow{\cong} GDA_{B^\circ}^*(Y \times Y)$$

compatible with cycle class maps, and so Proposition 4.11 follows from Proposition 4.6.

To prove the lemma, as both sides are Kimura finite-dimensional it suffices to construct an isomorphism of homological motives. In addition, since $h^j(Z) \cong h^j(Y) \cong \mathbf{1}(*)$ for $j \neq 3$, it suffices to construct an isomorphism between $h^3(Z)$ and $h^3(Y)$. To this end, we consider the composition

$$h^3(Z) \xrightarrow{t^P} h^1(F_1(Z))(-1) \xrightarrow{\cong} h^1(F(Y))(-1) \xrightarrow{Q} h^3(Y) \quad \text{in } \mathcal{M}_{\text{hom}},$$

where the middle isomorphism is induced by the isomorphism of varieties $F_1(Z) \cong F(Y)$ (Theorem 2.4), the first map is defined by the transpose of the universal line (which is an isomorphism by [44, Theorem 4.14]), and the last map is the inverse of the isomorphism of Theorem 2.4. A general Hilbert schemes argument (cf. [32, Proposition 2.11] or [24, Proposition 2.11]) implies that Q (and hence Γ) may be assumed to be generically defined; the same argument applies to the correspondence inducing the inverse isomorphism to Γ . \square

REMARK 4.13. — The Franchetta property for the family $\mathcal{Z} \times_{B(2,2)} \mathcal{Z} \rightarrow B(2,2)$ is established in [30, Proposition 3.6(ii)]. However, this does not imply anything for the base change to B° (there may be more algebraic cycles on the base-changed family).

The proof of Proposition 4.11 relies in an essential way on the relation between Z and the genus 10 Fano threefold Y .

4.4. Franchetta for $\mathcal{Z}^{(2)} \times \mathcal{Z}$. —

PROPOSITION 4.14. — *Let $\mathcal{Z} \rightarrow B^\circ$ be as above (Notation 4.10). The family*

$$(\mathcal{Z} \times_{B^\circ} \mathcal{Z} \times_{B^\circ} \mathcal{Z})/\mathfrak{S}_2 \rightarrow B^\circ$$

(where \mathfrak{S}_2 acts by exchanging the first 2 factors) has the Franchetta property.

Proof. — We are going to use the Fano surface of lines $F := F_1(Z)$. Let $\mathcal{Z} \rightarrow B^\circ$ be the universal family of intersections of two quadrics as in Notation 4.10 and let $\mathcal{F} \rightarrow B^\circ$ denote the universal family of Fano surfaces of lines. We now make the following claim:

CLAIM 4.15. — *The family*

$$\mathcal{F} \times_{B^\circ} \mathcal{Z} \rightarrow B^\circ$$

has the Franchetta property.

The claim suffices to prove Proposition 4.14; indeed, Proposition 2.6 gives a (generically defined) isomorphism of motives

$$h(Z^{(2)}) \cong h(F)(-2) \oplus h(Z) \oplus h(Z)(-3) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

In particular, this gives an isomorphism of Chow groups

$$GDA_{B^\circ}^j(Z^{(2)} \times Z) \cong GDA_{B^\circ}^{j-2}(F \times Z) \oplus GDA_{B^\circ}^j(Z \times Z) \oplus GDA_{B^\circ}^{j-3}(Z \times Z) ,$$

compatible with cycle class maps. Since we already know that $\mathcal{Z} \times_{B^\circ} \mathcal{Z} \rightarrow B^\circ$ has the Franchetta property (Proposition 4.11), the claim thus implies Proposition 4.14.

To prove Claim 4.15, we borrow the argument of the closely related [30, Proposition 3.10] (which, in its turn, is inspired by the work of Diaz [6]).

Let us write

$$P_{\mathbb{P}} \subset \text{Gr}(2, 6) \times \mathbb{P}^5$$

for the universal line and $P \subset F_1(Z) \times Z$ for the restriction of $P_{\mathbb{P}}$ to $F_1(Z) \times Z$.

Let $\bar{\mathcal{F}} \subset \bar{B} \times \text{Gr}(2, 6)$ and $\bar{\mathcal{Z}} \subset \bar{B} \times \mathbb{P}^5$ denote the projective closures. We now consider the projection

$$\pi: \bar{\mathcal{F}} \times_{\bar{B}} \bar{\mathcal{Z}} \rightarrow \text{Gr}(2, 6) \times \mathbb{P}^5 .$$

As a first step towards proving Claim 4.15, let us ascertain that π has the structure of a *stratified projective bundle* (in the sense of [7, Section 5.1]):

LEMMA 4.16. — *The morphism π has the structure of a \mathbb{P}^r -bundle over $(\text{Gr}(2, 6) \times \mathbb{P}^5) \setminus P_{\mathbb{P}}$, and a \mathbb{P}^s -bundle over $P_{\mathbb{P}}$.*

Proof. — Let $B_{(2,2)}$ be as in Notation 4.10. In [30, Proof of Proposition 3.10], it is proven that the family

$$\bar{\mathcal{F}} \times_{B_{(2,2)}} \bar{\mathcal{Z}} \rightarrow \text{Gr}(2, 6) \times \mathbb{P}^5$$

is a \mathbb{P}^r -bundle over $(\text{Gr}(2, 6) \times \mathbb{P}^5) \setminus P_{\mathbb{P}}$ and a \mathbb{P}^s -bundle over $P_{\mathbb{P}}$. As the morphism π is obtained by base changing this family, this establishes the lemma. □

As a second step towards proving Claim 4.15, let us identify the generically defined cycles on $F \times Z$.

LEMMA 4.17. — *There is equality*

$$\begin{aligned} GDA_{B^\circ}^*(F \times Z) &= \left\langle (p_F)^* GDA_{B^\circ}^*(F), (p_Z)^* GDA_{B^\circ}^*(Z) \right\rangle \\ &\quad + \left\langle (p_F)^* GDA_{B^\circ}^*(F), (p_Z)^* GDA_{B^\circ}^*(Z) \right\rangle \cdot P . \end{aligned}$$

Proof. — We have just seen (Lemma 4.16) that

$$\pi: \bar{\mathcal{F}} \times_{\bar{B}} \bar{\mathcal{Z}} \rightarrow \text{Gr}(2, 6) \times \mathbb{P}^5$$

is a stratified projective bundle, with strata $P_{\mathbb{P}}$ and $\text{Gr}(2, 6) \times \mathbb{P}^5$. Applying [7, Proposition 5.2] to this set-up, we find there is equality

$$(10) \quad \begin{aligned} GDA_{B^\circ}^*(F \times Z) &= \text{Im}\left(A^*(\text{Gr}(2, 6) \times \mathbb{P}^5) \rightarrow A^*(F \times Z)\right) \\ &+ \iota_* \text{Im}\left(A^*(P_{\mathbb{P}}) \rightarrow A^*(P)\right), \end{aligned}$$

where $\iota: P \hookrightarrow F \times Z$ is the inclusion morphism. The homogeneous varieties $\text{Gr}(2, 6)$ and \mathbb{P}^5 have trivial Chow groups, and so $A^*(\text{Gr}(2, 6) \times \mathbb{P}^5)$ is naturally isomorphic to $A^*(\text{Gr}(2, 6)) \otimes A^*(\mathbb{P}^5)$. This means that the first summand of (10) can be rewritten as

$$(11) \quad \begin{aligned} &\text{Im}\left(A^*(\text{Gr}(2, 6) \times \mathbb{P}^5) \rightarrow A^*(F \times Y)\right) \\ &= \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle. \end{aligned}$$

As for the second summand of (10), we make the following observation:

SUBLEMMA 4.18. — *The restriction map*

$$A^*(\text{Gr}(2, 6) \times \mathbb{P}^5) \rightarrow A^*(P_{\mathbb{P}})$$

is surjective.

Proof. — The universal line $P_{\mathbb{P}}$ is a \mathbb{P}^1 -bundle over $\text{Gr}(2, 6)$ with $p^*(h)$ relatively ample (where $h \in A^1(\mathbb{P}^5)$ is ample, and $p: P_{\mathbb{P}} \rightarrow \mathbb{P}^5$ is the morphism induced by projection). The sublemma thus follows from the projective bundle formula. □

Using the surjectivity of Sublemma 4.18, plus the equality (11), one reduces (10) to the equality of Lemma 4.17. This ends the proof. □

As a next step towards proving Claim 4.15, let us make a further simplification to the equality of Lemma 4.17:

LEMMA 4.19. — *There is equality*

$$\begin{aligned} GDA_{B^\circ}^*(F \times Z) &= \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle \\ &\oplus \mathbb{Q}[P] \oplus \mathbb{Q}[P \cdot (p_F)^*(h_F)]. \end{aligned}$$

Proof. — This is Lemma 4.17 in combination with the following two sublemmas:

SUBLEMMA 4.20. —

$$(P) \cdot (p_Z)^*GDA_{B^\circ}^*(Z) \subset \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle.$$

Proof. — We know (Proposition 4.4) that $GDA_{B^\circ}^*(Z) = \langle h \rangle$. Also, we know that there exist $a_j \in \mathbb{Q}$ such that

$$(12) \quad \Delta_Z \cdot (p_2)^*(h) = \sum_j a_j h^j \times h^{4-j} \quad \text{in } A^4(Z \times Z)$$

[30, Corollary 3.8].

Let p and q denote the projections from P to F resp. Z . Using equality (12), we find that

$$\begin{aligned} P \cdot (p_Z)^*(h) &= (p \times \Delta_Z)_*(q \times \Delta_Z)^*\left(\Delta_Z \cdot (p_2)^*(h)\right) \\ &= (p \times \Delta_Z)_*(q \times \Delta_Z)^*\left(\sum_j a_j h^j \times h^{4-j}\right) \\ &= \sum_j a_j (p_F)^*P^*(h^j) \cdot (p_Z)^*(h^{4-j}) \quad \text{in } A^3(F \times Z). \end{aligned}$$

Since P and h are generically defined, we have that $P^*(h^j) \in GDA_{B^\circ}^*(F)$, and so we get

$$P \cdot (p_Z)^*(h) \in \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle.$$

It follows that likewise

$$(13) \quad P \cdot (p_Z)^*(h^i) \in \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle \quad \forall i,$$

which proves the sublemma. □

SUBLEMMA 4.21. —

$$(P) \cdot (p_F)^*GDA_{B^\circ}^2(F) \subset \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle.$$

Proof. — In this proof, let us drop the B° subscript, since all generically defined cycles are with respect to B° . Since $\mathcal{F} \rightarrow B$ has the Franchetta property, $GDA^1(F)$ and $GDA^2(F)$ are one-dimensional, generated by h_F resp. h_F^2 . It is readily checked (cf. [30, Sublemma 3.14]) that $GDA^2(F)$ is also generated by $c_2(Q)|_F$, where Q is the universal quotient bundle on $\text{Gr}(2, 6)$. To prove the lemma, we thus need to check that

$$(14) \quad P \cdot (p_F)^*(c_2(Q)|_F) \in \left\langle (p_F)^*GDA^*(F), (p_Z)^*GDA^*(Z) \right\rangle.$$

The morphism $p: P \rightarrow F$ being a \mathbb{P}^1 -bundle (with $q^*(h)$ being relatively ample), we find that

$$p^*(c_2(Q)|_F) = -q^*(h^2) - q^*(h)p^*(c_1(Q)|_F) \quad \text{in } A^2(P).$$

Pushing forward under the closed immersion $P \hookrightarrow F \times Z$, this implies that

$$P \cdot (p_F)^*(c_2(Q)|_F) = -P \cdot (p_Z)^*(h^2) - P \cdot (p_Z)^*(h) \cdot (p_F)^*(h_F) \quad \text{in } A^4(F \times Z).$$

Using equation (13), we see that the right-hand side is decomposable, and so (14) is proven.

(An alternative argument is as follows: up to a finite base change, we may assume that $\mathcal{F} \rightarrow B$ is an abelian scheme, i.e. there is a zero section. Let $o \in F$ denote the origin and let $C_o \subset Z$ denote the line corresponding to $o \in F$. There is equality

$$P \cdot (p_F)^*(o) = (p_F)^*(o) \cdot (p_Z)^*(C_o) \quad \text{in } A^4(F \times Z).$$

Moreover, the class $o \in A^2(F)$ is generically defined, and hence is a generator for $GDA_{B^\circ}^2(F) = A_{(2)}^2(F) \cong \mathbb{Q}$. Also, the class $C_o \in A^2(Z)$ is generically defined (it is the fibrewise restriction of the relative cycle

$$(p_Z)_*((p_{\mathcal{F}})^*(o_{\mathcal{F}}) \cdot \mathcal{P}) \in A^2(Z),$$

where $o_{\mathcal{F}} \in A^2(\mathcal{F})$ is the zero-section, and $\mathcal{P} \subset \mathcal{F} \times_B Z$ is the relative universal line). The sublemma is proven. □

Combining Sublemmas 4.20 and 4.21, one obtains a proof of Lemma 4.19. □

We are now in position to wrap up the proof of Claim 4.15: thanks to Lemma 4.19, combined with the Künneth decomposition in cohomology, the Franchetta property in codimension ≥ 4 for $\mathcal{F} \times_{B^\circ} Z \rightarrow B^\circ$ reduces to the Franchetta property for $\mathcal{F} \rightarrow B^\circ$ and that for $Z \rightarrow B^\circ$. The second follows from Proposition 4.11; the first follows from Proposition 4.11 combined with the generically defined isomorphism of the Z - $F(Z)$ relation (Proposition 2.6).

For the Franchetta property in codimension 3, one needs to check that

$$(15) \quad GDA_{B^\circ}^3(F \times Z) = \text{Dec}^3(F \times Z) \oplus \mathbb{Q}[P \cdot (p_F)^*(h_F)] \rightarrow H^6(F \times Z, \mathbb{Q})$$

is injective, where we have used the shorthand

$$\text{Dec}^j(F \times Z) := \left\langle (p_F)^*GDA_{B^\circ}^*(F), (p_Z)^*GDA_{B^\circ}^*(Z) \right\rangle \cap A^j(F \times Z)$$

for the *decomposable* cycles. However, the class $P \cdot (p_F)^*(h_F)$ is linearly independent from the decomposable cycles $\text{Dec}^3(F \times Z)$ in cohomology, because

$$(P \cdot (p_F)^*(h_F))_* : H^1(F, \mathbb{Q}) \rightarrow H^3(F, \mathbb{Q}) \rightarrow H^3(Z, \mathbb{Q})$$

is an isomorphism, while D_* is zero on $H^1(F, \mathbb{Q})$ for any decomposable cycle D . The injectivity of (15) thus reduces to the Franchetta property for F and for Z .

The argument in codimension 2 is similar: one needs to check that

$$(16) \quad GDA^2(F \times Z) = \text{Dec}^2(F \times Z) \oplus \mathbb{Q}[P] \rightarrow H^4(F \times Z, \mathbb{Q})$$

is injective. However, the class P is linearly independent from $\text{Dec}^2(F \times Z)$ in cohomology, because of the isomorphism

$$P_* : H^3(F, \mathbb{Q}) \xrightarrow{\cong} H^3(Z, \mathbb{Q}).$$

The injectivity of (16) thus reduces to the Franchetta property for F and for Z . □

4.5. Franchetta for $Y^{(2)} \times Y$. —

PROPOSITION 4.22. — *Let $\mathcal{Y} \rightarrow B$ be as above. The family*

$$(\mathcal{Y} \times_B \mathcal{Y} \times_B \mathcal{Y})/\mathfrak{S}_2 \rightarrow B$$

(where \mathfrak{S}_2 acts by exchanging the first two factors) has the Franchetta property.

Proof. — In view of the spread lemma [54, Lemma 3.2], it suffices to prove the Franchetta property over B° . As we have seen, there is an isomorphism of motives

$$h(Y) \cong h(Z) \quad \text{in } \mathcal{M}_{\text{rat}} ,$$

which is generically defined with respect to B° (cf. Lemma 4.12 above). This isomorphism of motives induces isomorphisms of Chow groups

$$GDA_{B^\circ}^*(Y^{(2)} \times Y) \xrightarrow{\cong} GDA_{B^\circ}^*(Z^{(2)} \times Z) ,$$

compatible with cycle class maps. The result thus follows from Proposition 4.14. □

COROLLARY 4.23. — *Let $\mathcal{Y} \rightarrow B$ be as above and let $\mathcal{F} \rightarrow B$ denote the universal family of Fano varieties of conics. The family*

$$\mathcal{F} \times_B \mathcal{Y} \rightarrow B$$

has the Franchetta property.

Proof. — This follows from Proposition 4.22 in view of the generically defined Y - $F(Y)$ relation (Proposition 2.4). □

REMARK 4.24. — The Franchetta type result central to this paper (with the purpose of establishing MCK in new cases) is Proposition 4.22, which concerns the genus 10 Fano threefold Y . However, in order to prove Proposition 4.22 we were compelled to first prove the analogous result for the index 2 Fano threefold Z (Proposition 4.14). The reason for this detour via Z can be explained as follows: the argument of Proposition 4.14 (roughly speaking: once one has the Franchetta property for F and for Z one also has it for $F \times Z$, as the only “extra cycles” come from the universal line $P \subset F \times Z$) does *not* apply directly to Y . Indeed, working with the Fano threefold Y one runs into the double trouble that (1) we do not know whether the correspondence P of Theorem 2.2 is the universal conic, (2) even if we knew this, the analogue of Sublemma 4.18 is not clear for the universal conic on Y .

5. Main result

THEOREM 5.1. — *Let Y be a prime Fano threefold of genus 10. Then Y has a multiplicative Chow–Künneth decomposition. The induced bigrading on the Chow ring is such that*

$$A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)] .$$

Proof. — Let $\{\pi_Y^j\}$ be the CK decomposition for Y defined in (2). We observe that this CK decomposition is *generically defined* with respect to the family $\mathcal{Y} \rightarrow B$ (Notation 4.3), i.e. it is obtained by restriction from “universal projectors” $\pi_{\mathcal{Y}}^j \in A^3(\mathcal{Y} \times_B \mathcal{Y})$. (This is just because h and Δ_Y are generically defined.)

Writing $h^j(Y) := (Y, \pi_Y^j) \in \mathcal{M}_{\text{rat}}$, we have

$$(17) \quad h^{2j}(Y) \cong \mathbb{1}(-j) \quad \text{in } \mathcal{M}_{\text{rat}} \quad (j = 0, \dots, 3) ,$$

i.e. the interesting part of the motive is concentrated in $h^3(Y)$.

What we need to prove is that this CK decomposition is MCK, i.e.

$$(18) \quad \pi_Y^k \circ \Delta_Y^{sm} \circ (\pi_Y^i \times \pi_Y^j) = 0 \quad \text{in } A^6(Y \times Y \times Y) \quad \text{for all } i + j \neq k ,$$

or equivalently that

$$h^i(Y) \otimes h^j(Y) \xrightarrow{\Delta_Y^{sm}} h(Y)$$

coincides with

$$h^i(Y) \otimes h^j(Y) \xrightarrow{\Delta_Y^{sm}} h(Y) \rightarrow h^{i+j}(Y) \rightarrow h(Y) ,$$

for all i, j .

As a first step, let us assume that we have three integers (i, j, k) and at most one of them is equal to 3. The cycle in (18) is generically defined and homologically trivial. The isomorphisms (17) induce an injection

$$(\pi_Y^i \times \pi_Y^j \times \pi_Y^k)_* A^6(Y \times Y \times Y) \hookrightarrow A^*(Y)$$

and send generically defined cycles to generically defined cycles (this is because the isomorphisms (17) are generically defined). As a consequence, the required vanishing (18) follows from the Franchetta property for Y (Proposition 4.4).

In the second step, let us assume that among the three integers (i, j, k) , exactly two are equal to 3. In this case, using the isomorphisms (17) we find an injection

$$(\pi_Y^i \times \pi_Y^j \times \pi_Y^k)_* A^6(Y \times Y \times Y) \hookrightarrow A^*(Y \times Y) ,$$

respecting the generically defined cycles. As such, the required vanishing (18) follows from the Franchetta property for $Y \times Y$ (Proposition 4.6).

In the third and final step, let us treat the case $i = j = k = 3$. For this case, we observe that

$$\begin{aligned} \pi_Y^3 \circ \Delta_Y^{sm} \circ (\pi_Y^3 \times \pi_Y^3) &= (\pi_Y^3 \times \pi_Y^3 \times \pi_Y^3)_*(\Delta_Y^{sm}) \\ &\in GDA_B^6(Y^{(2)} \times Y) \cap A_{hom}^6(Y^{(2)} \times Y). \end{aligned}$$

The required vanishing (18) for $i = j = k = 3$ thus follows from the Franchetta property for $Y^{(2)} \times Y$ (Proposition 4.22).

Finally, let us prove that $A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)]$. We remark that $c_2(Y) \in A_{(0)}^2(Y)$ because

$$(\pi_Y^i)_*c_2(Y) = (\pi_Y^i)_*c_2(T_{\mathcal{Y}/B})|_Y = 0 \quad \text{in } A^2(Y) \quad \text{for all } i \neq 4,$$

as follows from the Franchetta property for $\mathcal{Y} \rightarrow B$ (Proposition 4.4). One readily checks that $c_2(Y)$ is non-zero (e.g. one can take a smooth anticanonical section $S \subset Y$; if $c_2(Y)$ were zero, then by adjunction also $c_2(S) = 0$, which is absurd since S is a K3 surface). Since $A_{(0)}^2(Y)$ injects into $H^4(Y, \mathbb{Q}) \cong \mathbb{Q}$, it follows that $A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)]$. □

6. Compatibilities

In this short section, we show that the MCK decomposition that we have constructed for Y is compatible with the one on the Fano surface $F(Y)$ (Proposition 6.1) and with the one on the associated index 2 Fano threefold Z (Proposition 6.2).

PROPOSITION 6.1. — *Let Y be a prime Fano threefold of genus 10 and let $F := F(Y)$ be the Fano surface of conics and let $P \subset F \times Y$ be the universal conic. Then*

$$P_*A_{(j)}^i(F) \subset A_{(j)}^{i-1}(Y), \quad P^*A_{(j)}^i(Y) \subset A_{(j)}^i(F),$$

where $A_{(*)}^*(Y)$ is the bigrading induced by the MCK decomposition of Theorem 5.1, and $A_{(*)}^*(F)$ is the Beauville bigrading for the abelian surface F .

Proof. — Let π_Y^* and π_F^* denote the MCK decomposition of Theorem 5.1 resp. the MCK decomposition of the abelian variety F . We will prove that the correspondence P is of pure degree 0, in the sense of Shen–Vial [48, Definition 1.2], i.e.

$$(19) \quad (\pi_F^i \times \pi_Y^j)_*P = 0 \quad \forall i + j \neq 6.$$

This implies the statement in view of [48, Proposition 1.6].

To prove the vanishing (19), we observe that the cycle in (19) is generically defined (with respect to the base B) and homologically trivial. The vanishing thus follows from the Franchetta property for $F \times Y$ (Corollary 4.23). □

PROPOSITION 6.2. — *Let Y be a prime Fano threefold of genus 10 and let Z be the index 2 Fano threefold associated to Y (in the sense that $F_1(Z) \cong F(Y)$, cf. Theorem 2.2). Then there are isomorphisms*

$$A^i_{(j)}(Y) \cong A^i_{(j)}(Z),$$

where $A^*_{(*)}(Y)$ is the bigrading induced by the MCK decomposition of Theorem 5.1, and $A^*_{(*)}(Z)$ is the bigrading constructed in [30].

Proof. — Let π_Y^* and π_Z^* denote the MCK decomposition of Theorem 5.1 resp. the MCK decomposition of [30]; both are generically defined with respect to B° . As in Proposition 6.1, it suffices to prove that the correspondence Γ of Lemma 4.12 is of pure degree 0. This follows from the Franchetta property for $Y \times Z$, which is equivalent to the Franchetta property for $Y \times Y$ in view of Lemma 4.12. □

7. The tautological ring

COROLLARY 7.1. — *Let Y be a prime Fano threefold of genus 10 and let $m \in \mathbb{N}$. Let*

$$R^*(Y^m) := \langle (p_i)^*(h), (p_{ij})^*(\Delta_Y) \rangle \subset A^*(Y^m)$$

be the \mathbb{Q} -subalgebra generated by pullbacks of the polarization $h \in A^1(Y)$ and pullbacks of the diagonal $\Delta_Y \in A^3(Y \times Y)$. (Here, p_i and p_{ij} denote the various projections from Y^m to Y resp. to $Y \times Y$). The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

Proof. — This is inspired by the analogous result for cubic hypersurfaces [8, Section 2.3], which in turn is inspired by analogous results for hyperelliptic curves [49], [50] (cf. Remark 7.2 below) and for K3 surfaces [43].

As in [8, Section 2.3], let us write $o := \frac{1}{18}h^3 \in A^3(Y)$, and

$$\tau := \Delta_Y - \frac{1}{18} \sum_{j=0}^3 h^j \times h^{3-j} \in A^3(Y \times Y)$$

(this cycle τ is nothing but the projector on the motive $h^3(Y)$ considered above). Moreover, let us write

$$\begin{aligned} o_i &:= (p_i)^*(o) \in A^3(Y^m), \\ h_i &:= (p_i)^*(h) \in A^1(Y^m), \\ \tau_{i,j} &:= (p_{ij})^*(\tau) \in A^3(Y^m). \end{aligned}$$

We define the \mathbb{Q} -subalgebra

$$\bar{R}^*(Y^m) := \langle o_i, h_i, \tau_{ij} \rangle \subset H^*(Y^m, \mathbb{Q})$$

(where i ranges over $1 \leq i \leq m$, and $1 \leq i < j \leq m$). One can prove (just as [8, Lemma 2.11] and [43, Lemma 2.3]) that the \mathbb{Q} -algebra $\bar{R}^*(Y^m)$ is isomorphic to the free graded \mathbb{Q} -algebra generated by $o_i, h_i, \tau_{i,j}$, modulo the following relations:

$$(20) \quad o_i \cdot o_i = 0, \quad h_i \cdot o_i = 0, \quad h_i^3 = 18 o_i;$$

$$(21) \quad \tau_{i,j} \cdot o_i = 0, \quad \tau_{i,j} \cdot h_i = 0, \quad \tau_{i,j} \cdot \tau_{i,j} = 4 o_i \cdot o_j;$$

$$(22) \quad \tau_{i,j} \cdot \tau_{i,k} = \tau_{j,k} \cdot o_i;$$

$$(23) \quad \sum_{\sigma \in \mathfrak{S}_6} \prod_{i=1}^6 \tau_{\sigma(2i-1), \sigma(2i)} = 0.$$

To prove Corollary 7.1, we need to check that these relations are also verified modulo rational equivalence. The relations (20) take place in $R^*(Y)$, and so they follow from the Franchetta property for Y (Proposition 4.4). The relations (21) take place in $R^*(Y^2)$. The first and the last relations are trivially verified, because Y being Fano one has $A^6(Y^2) = \mathbb{Q}$. As for the second relation of (21), this follows from the Franchetta property for $Y \times Y$ (Proposition 4.4(ii)). (Alternatively, one can deduce the second relation from the MCK decomposition: the product $\tau \cdot h_i$ lies in $A^4_{(0)}(Y^2)$, and it is readily checked that $A^4_{(0)}(Y^2)$ injects into $H^8(Y^2, \mathbb{Q})$.)

Relation (22) takes place in $R^*(Y^3)$ and follows from the MCK relation. Indeed, we have

$$\Delta_Y^{sm} \circ (\pi_Y^3 \times \pi_Y^3) = \pi_Y^6 \circ \Delta_Y^{sm} \circ (\pi_Y^3 \times \pi_Y^3) \quad \text{in } A^6(Y^3),$$

which (using Lieberman’s lemma) translates into

$$(\pi_Y^3 \times \pi_Y^3 \times \Delta_Y)_* \Delta_Y^{sm} = (\pi_Y^3 \times \pi_Y^3 \times \pi_Y^6)_* \Delta_Y^{sm} \quad \text{in } A^6(Y^3),$$

which means that

$$\tau_{1,3} \cdot \tau_{2,3} = \tau_{1,2} \cdot o_3 \quad \text{in } A^6(Y^3).$$

Finally, relation (23), which takes place in $R^*(Y^{12})$, expresses the fact that

$$\text{Sym}^6 H^3(Y, \mathbb{Q}) = 0,$$

where $H^3(Y, \mathbb{Q})$ is viewed as a super vector space. To check that this relation is also verified modulo rational equivalence, we observe that relation (23) involves a cycle contained in

$$A^*(\text{Sym}^6(h^3(Y))).$$

But we have vanishing of the Chow motive

$$\text{Sym}^6 h^3(Y) = 0 \quad \text{in } \mathcal{M}_{\text{rat}},$$

because $\dim H^3(Y, \mathbb{Q}) = 4$, and $h^3(Y)$ is oddly finite-dimensional (all Fano threefolds have finite-dimensional motive [52, Theorem 4]). This establishes relation (23), modulo rational equivalence, and ends the proof. \square

REMARK 7.2. — Given a curve C and an integer $m \in \mathbb{N}$, one can define the *tautological ring*

$$R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \subset A^*(C^m)$$

(where p_i, p_{ij} denote the various projections from C^m to C resp. $C \times C$). Tavakol has proven [50, Corollary 6.4] that if C is a hyperelliptic curve, the cycle class map induces injections

$$R^*(C^m) \hookrightarrow H^*(C^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

On the other hand, there are many (non hyperelliptic) curves for which the tautological ring $R^*(C^3)$ does *not* inject into cohomology (this is related to the non-vanishing of the Ceresa cycle, cf. [50, Remark 4.2] and also [9, Example 2.3 and Remark 2.4]).

Corollary 7.1 shows that genus 10 Fano threefolds behave similarly to hyperelliptic curves. It would be interesting to understand what happens for other Fano threefolds: is Corollary 7.1 true for all of them or not? (This is related to Question 1.2.)

8. Question

QUESTION 8.1. — *Let Y be a prime Fano threefold of genus 10 and let $S \subset Y$ be a smooth anticanonical divisor. Then S is a K3 surface (and the general genus 10 K3 surface arises in this way). Is it true that*

$$\begin{aligned} \operatorname{Im}(A^1(S) \rightarrow A^2(Y)) &= \mathbb{Q}[h^2], \\ \operatorname{Im}(A^2(Y) \rightarrow A^2(S)) &= \mathbb{Q}[o_S]? \end{aligned}$$

(To prove this, it would suffice to prove that the graph of the inclusion morphism $S \hookrightarrow Y$ is in $A^3_{(0)}(S \times Y)$, with respect to the product MCK decomposition. I have not been able to settle this.)

This question also makes sense for other Fano threefolds and their anticanonical divisors (e.g. the question is interesting for cubic threefolds).

Acknowledgements. — Thanks to Laurent Manivel for kindly answering my questions and alerting me to [37]. Thanks to Kai and Len for being great experts on Fort Boyard.

BIBLIOGRAPHY

- [1] A. BEAUVILLE – “Sur l’anneau de Chow d’une variété abélienne”, *Math. Ann.* **273** (1986), no. 647, p. 651.
- [2] ———, “On the splitting of the Bloch–Beilinson filtration”, in *Algebraic cycles and motives* (J. Nagel & C. Peters, eds.), London Math. Soc. Lecture Notes, no. 344, Cambridge University Press, 2007.
- [3] A. BEAUVILLE & C. VOISIN – “On the Chow ring of a K3 surface”, *J. Alg. Geom.* **13** (2004), no. 417, p. 426.
- [4] N. BERGERON & Z. LI – “Tautological classes on moduli space of hyperkähler manifolds”, *Duke Math. J.*, arXiv:1703.04733.
- [5] O. DEBARRE – “Periods and moduli”, in *Current Developments in Algebraic Geometry* (L. Caporaso et al., eds.), MSRI Publications, no. 59, Cambridge University Press, 2012, p. 65–84.
- [6] H. DIAZ – “The Chow ring of a cubic hypersurface”, *International Math. Research Notices* **2021** (2021), no. 22, p. 17071–17090.
- [7] L. FU, R. LATERVEER & C. VIAL – “The generalized Franchetta conjecture for some hyper-Kähler varieties (with an appendix joint with M. Shen)”, *Journal Math. Pures et Appliquées* **130** (2019), p. 1–35.
- [8] ———, “The generalized Franchetta conjecture for some hyper-Kähler varieties, II”, *Journal de l’Ecole Polytechnique–Mathématiques* **8** (2021), p. 1065–1097.
- [9] ———, “Multiplicative Chow–Künneth decompositions and varieties of cohomological K3 type”, *Annali Mat. Pura ed Applicata* **200** (2021), no. 5, p. 2085–2126.
- [10] L. FU, Z. TIAN & C. VIAL – “Motivic hyperkähler resolution conjecture: I. generalized Kummer varieties”, *Geometry & Topology* **23** (2019), no. 1, p. 427–492.
- [11] W. FULTON – *Intersection theory*, Ergebnisse der Mathematik, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.
- [12] S. GALKIN, L. KATZARKOV, A. MELLIT & E. SHINDER – “Minifolds and phantoms”, arXiv:1305.4549.
- [13] S. GALKIN & E. SHINDER – “The Fano variety of lines and rationality problem for a cubic hypersurface”, arXiv:1405.5154.
- [14] M. GREEN & P. GRIFFITHS – “An interesting 0-cycle”, *Duke Math. J.* **119** (2003), no. 2, p. 261–313.
- [15] D. HUYBRECHTS – “Motives of isogenous K3 surfaces”, *Comment. Math. Helvetici* **94** (2019), no. 3, p. 445–458.
- [16] A. ILIEV & L. MANIVEL – “Prime Fano threefolds and integrable systems”, *Math. Ann.* **339** (2007), no. 4, p. 937–955.
- [17] V. ISKOVSKIĖ & Y. PROKHOROV – *Algebraic geometry V: Fano varieties*, Encyclopaedia of Math. Sciences, no. 47, Springer-Verlag, Berlin, 1999.

- [18] U. JANNSEN – “On finite-dimensional motives and Murre’s conjecture”, in *Algebraic cycles and motives* (J. Nagel & C. Peters, eds.), Cambridge University Press, Cambridge, 2007.
- [19] S.-I. KIMURA – “On the characterization of Alexander schemes”, *Comp. Math.* **92** (1994), no. 3, p. 273–284.
- [20] A. KUZNETSOV – “Hyperplane sections and derived categories”, *Izv. Math.* **70** (2006), no. 447, p. 547.
- [21] ———, “Derived categories of Fano threefolds”, *Proc. V. A. Steklov Inst. Math* **264** (2009), no. 110, p. 122.
- [22] ———, “On linear sections of the spinor tenfold, I”, *Izvestiya: Mathematics* **82** (2018), no. 4, p. 694–751.
- [23] A. KUZNETSOV, Y. PROKHOROV & C. SHRAMOV – “Hilbert schemes of lines and conics and automorphism groups of Fano threefolds”, *Japanese Journal of Mathematics* **13** (2018), no. 1, p. 109–185.
- [24] R. LATERVEER – “A family of cubic fourfolds with finite-dimensional motive”, *Journal of the Mathematical Society of Japan* **70** (2018), no. 4, p. 1453–1473.
- [25] ———, “A remark on the Chow ring of Küchle fourfolds of type $d3$ ”, *Bulletin Australian Math. Soc.* **100** (2019), no. 3, p. 410–418.
- [26] ———, “Algebraic cycles and Verra fourfolds”, *Tohoku Math. J.* **72** (2020), no. 3, p. 451–485.
- [27] ———, “On the Chow ring of certain Fano fourfolds”, *Ann. Univ. Paedagog. Crac. Stud. Math.* **19** (2020), p. 39–52.
- [28] ———, “On the Chow ring of Fano varieties of type S^2 ”, *Abh. Math. Semin. Univ. Hambg.* **90** (2020), p. 17–28.
- [29] ———, “Algebraic cycles and Fano threefolds of genus 8”, *Portugal. Math. (N.S.)* **78** (2021), no. 3-4, p. 255–280.
- [30] ———, “Algebraic cycles and intersections of 2 quadrics”, *Mediterranean Journal of Mathematics* **18** (2021), p. 4.
- [31] ———, “Algebraic cycles and intersections of a quadric and a cubic”, *Forum Mathematicum* **33** (2021), no. 3, p. 845–855.
- [32] ———, “Motives and the Pfaffian–Grassmannian equivalence”, *Journal of the London Math. Soc.* **104** (2021), no. 4, p. 1738–1764.
- [33] ———, “Algebraic cycles and intersections of three quadrics”, *Mathematical Proceedings of the Cambridge Philosophical Society* **173** (2022), no. 2, p. 349–367.
- [34] ———, “On the Chow ring of Fano varieties on the Fatighenti-Mongardi list”, *Communications in Algebra* **50** (2022), no. 1, p. 131–145.
- [35] R. LATERVEER & C. VIAL – “On the Chow ring of Cynk–Hulek Calabi–Yau varieties and Schreieder varieties”, *Canadian Journal of Math.* **72** (2020), no. 2, p. 505–536.

- [36] L. MANIVEL – “The Cayley Grassmannian”, *J. of Algebra* **503** (2018), p. 277–298.
- [37] ———, “The double Cayley Grassmannian”, *Math. Research Letters* **28** (2021), p. 1765–1792.
- [38] S. MUKAI – “Curves, K3 surfaces and Fano 3-folds of genus ≤ 10 ”, p. 357–377, Kinokuniya, Tokyo, 1988.
- [39] J. MURRE – “On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II”, *Indag. Math.* **4** (1993), p. 177–201.
- [40] J. MURRE, J. NAGEL & C. PETERS – *Lectures on the theory of pure motives*, Amer. Math. Soc. University Lecture Series, no. 61, Providence, 2013.
- [41] K. O’GRADY – “Decomposable cycles and Noether-Lefschetz loci”, *Doc. Math.* **21** (2016), p. 661–687.
- [42] N. PAVIC, J. SHEN & Q. YIN – “On O’Grady’s generalized Franchetta conjecture”, *Int. Math. Res. Notices* (2016), p. 1–13.
- [43] Q. YIN – “Finite-dimensionality and cycles on powers of K3 surfaces”, *Comment. Math. Helv.* **90** (2015), p. 503–511.
- [44] M. REID – “The complete intersection of two or more quadrics”, Ph.D. thesis, Univ. of Cambridge, 1972.
- [45] S.-I. KIMURA – “Chow groups are finite dimensional, in some sense”, *Math. Ann.* **331** (2005), no. 1, p. 173–201.
- [46] T. SCHOLL – “Classical motives”, in *Motives* (U. Jannsen et al., eds.), no. 55, 1994, Part 1.
- [47] M. SHEN & C. VIAL – “The Fourier transform for certain hyperKähler fourfolds”, *Memoirs of the AMS* **240** (2016), no. 1139.
- [48] ———, “The motive of the Hilbert cube $X^{[3]}$ ”, *Forum Math. Sigma* **4** (2016).
- [49] M. TAVAKOL – “The tautological ring of the moduli space M_2^{gt} ”, *International Math. Research Notices* **2014** (2014), 178.
- [50] ———, “Tautological classes on the moduli space of hyperelliptic curves with rational tails”, *J. Pure Applied Algebra* **222** (2018), no. 8, p. 2040–2062.
- [51] C. VIAL – “Niveau and coniveau filtrations on cohomology groups and Chow groups”, *Proceedings of the LMS* **106** (2013), no. 2, p. 410–444.
- [52] ———, “Projectors on the intermediate algebraic Jacobians”, *New York J. Math.* **19** (2013), p. 793–822.
- [53] ———, “On the motive of some hyperkähler varieties”, *J. Reine Angew. Math.* **725** (2017), p. 235–247.
- [54] C. VOISIN – *Chow rings, decomposition of the diagonal, and the topology of families*, Princeton University Press, Princeton and Oxford, 2014.
- [55] Q. YIN – “The generic nontriviality of the Faber–Pandharipande cycle”, *International Math. Res. Notices* **2015** (2015), no. 5, p. 1277–1263.