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Baptiste CHANTRAINE, Georgios DIMITROGLOU RIZELL, Paolo GHIGGINI & Roman GOLOVKO

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Annales Scientifiques de l'École Normale Supérieure,
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Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

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GEOMETRIC GENERATION OF THE WRAPPED FUKAYA CATEGORY OF WEINSTEIN MANIFOLDS AND SECTORS

BY Baptiste CHANTRAINE, Georgios DIMITROGLOU RIZELL, Paolo GHIGGINI AND Roman GOLOVKO

ABSTRACT. – We prove that the wrapped Fukaya category of any $2n$-dimensional Weinstein manifold (or, more generally, Weinstein sector) W is generated by the unstable manifolds of the index n critical points of its Liouville vector field. Our proof is geometric in nature, relying on a surgery formula for Floer cohomology and the fairly simple observation that Floer cohomology vanishes for Lagrangian submanifolds that can be disjoined from the isotropic skeleton of the Weinstein manifold. Note that we do not need any additional assumptions on this skeleton. By applying our generation result to the diagonal in the product $W \times W$, we obtain as a corollary that the open-closed map from the Hochschild homology of the wrapped Fukaya category of W to its symplectic cohomology is an isomorphism, proving a conjecture of Seidel. We work mainly in the "linear setup" for the wrapped Fukaya category, but we also extend the proofs to the "quadratic" and "localisation" setup. This is necessary for dealing with Weinstein sectors and for the applications.

RÉSUMÉ. – Nous démontrons que la catégorie de Fukaya enroulée d'une variété (ou plus généralement d'un secteur) de Weinstein W de dimension $2n$ est engendrée par les variétés instables des points critiques d'indice n de son champ de Liouville. Notre preuve, de nature géométrique, repose sur une formule pour la cohomologie de Floer d'une chirurgie et sur l'observation relativement simple que la cohomologie de Floer d'une lagrangienne disjointe du squelette isotrope de la variété de Weinstein s'annule (aucune condition supplémentaire n'est demandée au squelette). En appliquant le critère d'engendrement au produit $W \times W$ nous obtenons en corollaire que l'application ouverte-fermée de l'homologie de Hochschild de la catégorie de Fukaya enroulée de W vers sa cohomologie symplectique est un isomorphisme, prouvant une conjecture de Seidel. Nous travaillons principalement avec la définition « linéaire » de la catégorie de Fukaya enroulée mais nous étendons les preuves aux définitions « quadratique » et « par localisation ». Ces modifications sont nécessaires pour traiter les secteurs de Weinstein et pour certaines applications.

1. Introduction

The *wrapped Fukaya category* is an A_∞-category associated to any Liouville manifold. Its objects are exact Lagrangian submanifolds which are either compact or cylindrical at infinity, possibly equipped with extra structure, the morphism spaces are wrapped Floer

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chain complexes, and the A_∞ operations are defined by counting perturbed holomorphic polygons with Lagrangian boundary conditions. Wrapped Floer cohomology was defined by A. Abbondandolo and M. Schwarz [1], at least for cotangent fibers, but the general definition and the chain level construction needed to define an A_∞-category are due to M. Abouzaid and P. Seidel [4]. The definition of the wrapped Fukaya category was further extended to the relative case by Z. Sylvan, who introduced the notions of *stop* and *partially wrapped Fukaya category* in [39], and by S. Ganatra, J. Pardon and V. Shende, who later introduced the similar notion of *Liouville sector* in [23].

In this article we study the wrapped Fukaya category of Weinstein manifolds and sectors. In the absolute case our main result is the following.

THEOREM 1.1. – *If $(W, \theta, \mathfrak{f})$ is a $2n$-dimensional Weinstein manifold of finite type, then its wrapped Fukaya category $\mathcal{WF}(W, \theta)$ is generated by the Lagrangian cocore planes of the index n critical points of $\mathfrak{f}$.*

In the relative case (i.e., for sectors) our main result is the following. We refer to Section 2.3 for the definition of the terminology used in the statement.

THEOREM 1.2. – *The wrapped Fukaya category of the Weinstein sector $(S, \theta, \mathfrak{f})$ is generated by the Lagrangian cocore planes of its completion $(W, \theta_W, \mathfrak{f}_W)$ and by the spreading of the Lagrangian cocore planes of its belt $(F, \theta_F, \mathfrak{f}_F)$.*

REMARK 1.3. – Exact Lagrangian submanifolds are often enriched with some extra structure: Spin structures, grading or local systems. We ignore them for simplicity, but the same arguments carry over also when that extra structure is considered.

Generators of the wrapped Fukaya category are known in many particular cases. We will not try to give a comprehensive overview of the history of this recent but active subject because we would not be able to make justice to everybody who has contributed to it. However, it is important to mention that F. Bourgeois, T. Ekholm and Y. Eliashberg in [9] sketch a proof that the Lagrangian cocore disks split-generate the wrapped Fukaya category of a Weinstein manifold of finite type. Split-generation is a weaker notion than generation, which is sufficient for most applications, but not for all; see for example [28]. Moreover, Bourgeois, Ekholm and Eliashberg's proposed proof relies on their Legendrian surgery formula, whose analytic details are not complete (see [15] for recent development in that direction).

Most generation results so far, including that of Bourgeois, Ekholm and Eliashberg, rely on Abouzaid's split-generation criterion [2]. On the contrary, our proof is more direct and similar in spirit to Seidel's proof in [37] that the Lagrangian thimbles generate the Fukaya-Seidel category of a Lefschetz fibration or to Biran and Cornea's cone decomposition of Arnol'd type Lagrangian cobordisms [7]. Theorems 1.1 and 1.2 have been proved independently also by Ganatra, Pardon and Shende in [24, Theorem 1.10].

A product of Weinstein manifolds is a Weinstein manifold. Therefore, by applying Theorem 1.1 to the diagonal in a twisted product, and using results of S. Ganatra [22] and Y. Gao [25], we obtain the following result.

COROLLARY 1.4. – *Let* $(W, \theta, \mathfrak{f})$ *be a Weinstein manifold of finite type. Let* $\mathcal{D}$ *be the full* A_∞ *subcategory of* $\mathcal{WF}(W, \theta)$ *whose objects are the Lagrangian cocore planes. Then the open-closed map*

$$\mathcal{OC}\colon\ HH_*(\mathcal{D}, \mathcal{D}) \to SH^*(W) \tag{1}$$

is an isomorphism.

In Equation (1) HH_* denotes Hochschild homology, SH^* denotes symplectic cohomology and $\mathcal{OC}$ is the open-closed map defined in [2]. Corollary 1.4 in particular proves that

$$\mathcal{OC}\colon\ HH_*(\mathcal{WF}(W, \theta), \mathcal{WF}(W, \theta)) \to SH^*(W) \tag{2}$$

is an isomorphism. This proves a conjecture of Seidel in [38] for Weinstein manifolds of finite type. Note that a proof of this conjecture, assuming the Legendrian surgery formula of Bourgeois, Ekholm and Eliashberg was given by S. Ganatra and M. Maydanskiy in the appendix of [9].

The above result implies in particular that Abouzaid's generation criterion [2] is satisfied for the subcategory consisting of the cocore planes of a Weinstein manifold, from which one can conclude that the cocores split-generate the wrapped Fukaya category. In the exact setting under consideration this of course follows a fortiori from Theorem 1.1, but there are extensions of the Fukaya category in which this generation criterion has nontrivial implications. Notably, this is the case for the version of the wrapped Fukaya category for monotone Lagrangians, as we proceed to explain.

The wrapped Fukaya category as well as symplectic cohomology were defined in the monotone symplectic setting in [35] using coefficients in the Novikov field. When this construction is applied to exact Lagrangians in an exact symplectic manifold, a change of variables $x \mapsto t^{-\mathcal{A}(x)}x$, where $\mathcal{A}(x)$ is the action of the generator x and t is the formal Novikov parameter, allows for an identification of the Floer complexes and the open-closed map with the original complexes and map tensored with the Novikov field. The generalization of Abouzaid's generation criterion to the monotone setting established in [35] thus shows that

COROLLARY 1.5. – *The wrapped Fukaya category of monotone Lagrangian submanifolds of a Weinstein manifold which are unobstructed in the strong sense (i.e., with* $\mu^0 = 0$*, where* μ^0 *is the number of Maslov index two holomorphic disks passing through a generic point) is split-generated by the Lagrangian cocore planes of the Weinstein manifold.*

REMARK 1.6. – The strategy employed in the proof of Theorem 1.1 for showing generation fails for non-exact Lagrangian submanifolds in two crucial steps: in Section 7 and Section 8. First, there are well known examples of unobstructed monotone Lagrangian submanifolds in a Weinstein manifold which are Floer homologically nontrivial even if they are disjoint from the skeleton. Second, our treatment of Lagrangian surgeries requires that we lift the Lagrangian submanifolds in W to Legendrian submanifolds of $W \times \mathbb{R}$, and this is possible only for exact Lagrangian submanifolds. It is unclear to us whether it is true that the cocores generate (and not merely split-generate) the $\mu^0 = 0$ part of the monotone wrapped Fukaya category.

1.1. Comparison of setups

There are three "setups" in which the wrapped Fukaya category is defined: the "linear setup," where the Floer equations are perturbed by Hamiltonian functions with linear growth at infinity and the wrapped Floer chain complexes are defined as homotopy colimits over Hamitonians with higher and higher slope, the "quadratic setting," where the Floer equations are perturbed by Hamiltonian functions with quadratic growth at infinity, and the "localisation setting," where the Floer equations are unperturbed and the wrapped Fukaya category is defined by a categorical construction called *localisation*. The linear setup was introduced by Abouzaid and Seidel in [4] and the quadratic setup by Abouzaid in [2]. The latter is used also in Sylvan's definition of the partially wrapped Fukaya category and in the work of Ganatra [22] and Gao [25] which we use in the proof of Corollary 1.4. The localisation setup is used in [23] because the linear and quadratic setups are not available on sectors for technical reasons. All three setups are expected to produce equivalent A_∞ categories on Liouville manifolds. We chose to work in the linear setup for the proof of Theorem 1.1, but our proof can be adapted fairly easily to the other setups. Moreover these extensions are necessary to prove Theorem 1.2 and Corollary 1.4. We will describe the small modifications we need in the localisation setup in Section 10 and those we need in the quadratic setup in Section 11.

1.2. Strategy of the proof of Theorem 1.1

The strategy of the proof of Theorem 1.1 is the following. Given a cylindrical Lagrangian submanifold L, by a compactly supported Hamiltonian isotopy we make it transverse to the stable manifolds of the Liouville flow. Thus, for dimensional reasons, it will be disjoint from the stable manifolds of the critical points of index less than n and will intersect the stable manifolds of the critical points of index n in finitely many points $a_1, \dots, a_k$. For each a_i we consider a Lagrangian plane D_{a_i} passing through a_i, transverse both to L and to the stable manifold, and Hamiltonian isotopic to the unstable manifold of the same critical point. We assume that the Lagrangian planes are all pairwise disjoint. The unstable manifolds of the index n critical points are what we call the Lagrangian cocore planes.

At each a_i we perform a Lagrangian surgery between L and D_{a_i} so that the resulting Lagrangian $\overline{L}$ is disjoint from the skeleton of W. Since $\overline{L}$ will be in general immersed, we have to develop a version of wrapped Floer cohomology for immersed Lagrangian submanifolds. To do that we borrow heavily from the construction of Legendrian contact cohomology in [18]. In particular our wrapped Floer cohomology between immersed Lagrangian submanifolds uses augmentations of the Chekanov-Eliashberg algebras of the Legendrian lifts as bounding cochains. A priori there is no reason why such a bounding cochain should exist for $\overline{L}$, but it turns out that we can define it inductively provided that $D_{a_1}, \dots, D_{a_k}$ are isotoped in a suitable way. A large part of the technical work in this paper is devoted to the proof of this claim.

Then we prove a correspondence between twisted complexes in the wrapped Fukaya category and Lagrangian surgeries by realizing a Lagrangian surgery as a Lagrangian cobordism between the Legendrian lifts and applying the Floer theory for Lagrangian cobordisms we defined in [10]. This result can have an independent interest. Then we can conclude that $\overline{L}$ is isomorphic, in an appropriated triangulated completion of the wrapped Fukaya category, to a twisted complex $\mathbb{L}$ built from $L, D_{a_1}, \dots, D_{a_k}$.

Finally, we prove that the wrapped Floer cohomology of $\overline{L}$ with any other cylindrical Lagrangian is trivial. This is done by a fairly simple action argument based on the fact that the Liouville flow displaces $\overline{L}$ from any compact set because $\overline{L}$ is disjoint from the skeleton of W. Then the twisted complex $\mathbb{L}$ is a trivial object, and therefore some simple homological algebra shows that L is isomorphic to a twisted complex built from $D_{a_1}, \dots, D_{a_k}$.

This article is organized as follows. In Section 2 we recall some generalities about Weinstein manifolds and sectors. In Section 3 we recall the definition and the basic properties of Legendrian contact cohomology. In Sections 4 and 5 we define the version of Floer cohomology for Lagrangian immersions that we will use in the rest of the article. Despite their length, these sections contain mostly routine verifications and can be skipped by the readers who are willing to accept that such a theory exists. In Section 6 we define wrapped Floer cohomology for Lagrangian immersions using the constructions of the previous two sections. In Section 7 we prove that an immersed Lagrangian submanifold which is disjoint from the skeleton is Floer homologically trivial. In Section 8 we prove that Lagrangian surgeries correspond to twisted complexes in the wrapped Fukaya category. In Section 9 we finish the proof of Theorem 1.1 and in particular we construct the bounding cochain for $\overline{L}$. In Section 10 we prove Theorem 1.2. We briefly recall the construction of the wrapped Fukaya category for sectors in the localisation setup from [23] and show how all previous arguments adapt in this setting. Finally, in Section 11 we prove the isomorphism between Hochschild homology and symplectic cohomology. This requires that we adapt the proof of Theorem 1.1 to the quadratic setting.

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2. Geometric setup

In this section we revise some elementary symplectic geometry with the purpose of fixing notation and conventions.

2.1. Liouville manifolds

Let (W, θ) be a Liouville manifold of finite type, from now on called simply a Liouville manifold. This means that $d\theta$ is a symplectic form, the Liouville vector field $\mathcal{L}$ defined by the equation

$$\iota_{\mathcal{L}} d\theta = \theta$$

is complete and, for some $R_0 < 0$, there exists a proper smooth function $\mathfrak{t}\colon W \to [R_0, +\infty)$ such that, for $w \in W$,

(i) $d_w \mathfrak{t}(\mathcal{L}_w) > 0$ if $\mathfrak{t}(w) > R_0$, and

(ii) $d\mathfrak{t}_w(\mathcal{L}_w) = 1$ if $\mathfrak{t}(w) \geq R_0 + 1$.

In particular, R_0 is the unique critical value of $\mathfrak{t}$, which is of course highly nondegenerate, and every other level set is a contact type hypersurface.

We use the function $\mathfrak{t}$ to define some useful subsets of W.

Definition 2.1. – For every $R \in [R_0, +\infty)$ we denote $W_R = \mathfrak{t}^{-1}([R_0, R])$, $W_R^e = W \setminus \operatorname{int}(W_R)$ and $V_R = \mathfrak{t}^{-1}(R)$.

The subsets W_R^e will be called the *ends* of W. The Liouville flow of (W, θ) induces an identification

$$([R_0 + 1, +\infty) \times V, e^r \alpha) \cong (W^e_{R_0+1}, \theta), \tag{3}$$

where $V = V_0$ and α is the pull-back of θ to V_0. More precisely, if ϕ denotes the flow of the Liouville vector field, the identification (3) is given by $(r, v) \mapsto \phi_r(v)$. Let $\xi = \ker \alpha$ be the contact structure defined by α. Every V_R, for $R > R_0$, is contactomorphic to (V, ξ). Under the identification (3), the function $\mathfrak{t}$, restricted to $W^e_{R_0+1}$, corresponds to the projection to $[R_0 + 1, +\infty)$ in the sense that the following diagram commutes

$$\begin{array}{ccc} ([R_0 + 1, +\infty) \times V & \xrightarrow{\ \phi\ } & W^e_{R_0+1} \\ & \searrow \quad \swarrow_{\mathfrak{t}} & \\ & [R_0 + 1, +\infty). & \end{array}$$

Remark 2.2. – The choice of R_0 in the definition of $\mathfrak{t}$ is purely arbitrary because the Liouville flow is complete. In fact, for every map $\mathfrak{t}\colon W \to [R_0, +\infty)$ as above and for any $R_0' < R_0$ there is a map $\mathfrak{t}'\colon W \to [R_0', +\infty)$ satisfying (i) and (ii), which moreover coincides with $\mathfrak{t}$ on $\mathfrak{t}^{-1}([R_0 + 1, +\infty))$.

A diffeomorphism $\psi\colon W \to W$ is an *exact symplectomorphism* if $\psi^*\theta = \theta + dq$ for some function $q\colon W \to \mathbb{R}$. Flows of Hamiltonian vector fields are, of course, the main source of exact symplectomorphisms. Given a function $H\colon [-t_-, t_+] \times W \to \mathbb{R}$, where $t_\pm \geq 0$ and are allowed to be infinite, we define the Hamiltonian vector field X_H by

$$\iota_{X_H} d\theta = -dH.$$

Here dH denotes the differential in the directions tangent to W, and therefore X_H is a time-dependent vector field on W.

We spell out the change in the Liouville form induced by a Hamiltonian flow because it is a computation that will be needed repeatedly.

Lemma 2.3. – *Let $H\colon [-t_-, t_+] \times W \to \mathbb{R}$ be a Hamiltonian function and φ_t its Hamiltonian flow. Then, for all $t \in [t_-, t_+]$, we have $\varphi_t^*\theta = \theta + dq_t$, where*

$$q_t = \int_0^t (-H_\sigma + \theta(X_{H_\sigma})) \circ \varphi_\sigma d\sigma.$$

Proof. – We compute

$$\begin{aligned} \varphi_t^*\theta - \theta &= \int_0^t \frac{d}{d\sigma}(\varphi_\sigma^*\theta) d\sigma = \int_0^t \varphi_\sigma^*(L_{X_{H_\sigma}}\theta) d\sigma \\ &= \int_0^t \varphi_\sigma^*(\iota_{X_{H_\sigma}} d\theta + d\iota_{X_{H_\sigma}}\theta) d\sigma = \int_0^t \varphi_\sigma^*(-dH_\sigma + d(\theta(X_{H_\sigma}))) d\sigma. \end{aligned}$$

□

2.2. Weinstein manifolds

In this article we will be concerned mostly with Weinstein manifolds of finite type. We recall their definition, referring to [12] for further details.

Definition 2.4. – A Weinstein manifold $(W, \theta, \mathfrak{f})$ consists of:

(i) an even dimensional smooth manifold W without boundary,

(ii) a one-form θ on W such that $d\theta$ is a symplectic form and the Liouville vector field $\mathcal{L}$ associated to θ is complete, and

(iii) a proper Morse function $\mathfrak{f}\colon W \to \mathbb{R}$ bounded from below such that $\mathcal{L}$ is a pseudogradient of $\mathfrak{f}$ in the sense of [12, Equation (9.9)]: i.e.,

$$d\mathfrak{f}(\mathcal{L}) \geq \delta(\|\mathcal{L}\|^2 + \|d\mathfrak{f}\|^2),$$

where $\delta > 0$ and the norms are computed with respect to some Riemannian metric on W.

The function $\mathfrak{f}$ is called a *Lyapunov function* (for $\mathcal{L}$).

If $\mathfrak{f}$ has finitely many critical points, then $(W, \theta, \mathfrak{f})$ is a Weinstein manifold of finite type. From now on, Weinstein manifold will always mean Weinstein manifold of finite type.

Given a regular value M of $\mathfrak{f}$ the compact manifold $\{\mathfrak{f} \leq M\}$ is called a *Weinstein domain*. Any Weinstein domain can be completed to a Weinstein manifold in a canonical way by adding half a symplectisation of the contact boundary.

By Condition (iii) in Definition 2.4, the zeroes of $\mathcal{L}$ coincide with the critical points of $\mathfrak{f}$. If W has dimension $2n$, the critical points of $\mathfrak{f}$ have index at most n. For each critical point p of $\mathfrak{f}$ of index n, there is a stable manifold Δ_p and an unstable manifold D_p which are both exact Lagrangian submanifolds. We will call the unstable manifolds Δ_p of the critical points of index n the *Lagrangian cocore planes*.

Definition 2.5. – Let $W_0 \subset W$ be a Weinstein domain containing all critical points of $\mathfrak{f}$. The *Lagrangian skeleton* of $(W, \theta, \mathfrak{f})$ is the attractor of the negative flow of the Liouville vector field on the compact part of W, i.e.,

$$W^{\mathrm{sk}} := \bigcap_{t>0} \phi^{-t}(W_0),$$

where ϕ denotes the flow of the Liouville vector field $\mathcal{L}$. Alternatively, W^{sk} can be defined as the union of unstable manifolds of all critical points of $\mathfrak{f}$.

The stable manifolds of the index n critical points form the top dimensional stratum of the Lagrangian skeleton.

A Morse function gives rise to a handle decomposition. In the case of a Weinstein manifold $(W, \theta, \mathfrak{f})$, the handle decomposition induced by $\mathfrak{f}$ is compatible with the symplectic structure and is called the Weinstein handle decomposition of $(W, \theta, \mathfrak{f})$. By the combination of [12, Lemma 12.18] and [12, Corollary 12.21] we can assume that $\mathcal{L}$ is Morse-Smale. This implies that we can assume that handles of higher index are attached after handles of lower index. The deformation making $\mathcal{L}$ Morse-Smale can be performed without changing the symplectic form $d\theta$ and so that the unstable manifolds corresponding to the critical points of index n before and after such a deformation are Hamiltonian isotopic.

We will denote the union of the handles of index strictly less than n by W^{sc}. This will be called the *subcritical subdomain* of W. By construction, ∂W^{sc} is a contact type hypersurface in W.

We choose $\mathfrak{r}\colon W \to [R_0, +\infty)$, and we homotope the Weinstein structure so that

$$W_{R_0} = W^{\mathrm{sc}} \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l,$$

where $\mathcal{H}_1, \ldots, \mathcal{H}_l$ all are *standard* Weinstein handles corresponding to the critical points $p_1, \ldots, p_l$ of $\mathfrak{f}$ of Morse index n; see [40] for the description of the standard model, and [12, Section 12.5] for how to produce the Weinstein homotopy.

Remark 2.6. – We could easily modify $\mathfrak{f}$ so that it agrees with $\mathfrak{r}$ on W_0^e. However, this will not be necessary.

The *core* of the Weinstein handle $\mathcal{H}_i$ is the Lagrangian disk $C_i = \Delta_{p_i} \cap \mathcal{H}_i$. Let $D_\delta T^*C_i$ denote the disk cotangent bundle of C_i of radius $\delta > 0$. By the Weinstein neighborhood theorem, there is a symplectic identification $\mathcal{H}_i \cong D_\delta T^*C_i$ for some δ. However, the restriction of θ to $\mathcal{H}_i$ does not correspond to the restriction of the canonical Liouville form to $D_\delta T^*C_i$.

2.3. Weinstein sectors

In this section we introduce Weinstein sectors. These will be particular cases of Liouville sectors as defined in [23] characterized, roughly speaking, by retracting over a Lagrangian skeleton with boundary. In Section 2.4 below we will then show that any Weinstein pair as introduced in [19, Section 2] can be completed to a Weinstein sector.

Definition 2.7. – A *Weinstein sector* $(S, \theta, I, \mathfrak{f})$ consists of:

1. an even dimensional smooth manifold with boundary S;
2. a one-form θ on S such that $d\theta$ is a symplectic form and the associated Liouville vector field $\mathcal{L}$ is complete and everywhere tangent to ∂S;
3. a smooth function $I\colon \partial S \to \mathbb{R}$ which satisfies
   (a) $dI(\mathcal{L}) = \alpha I$ for some function $\alpha\colon \partial S \to \mathbb{R}_+$ which is constant outside a compact set and
   (b) $dI(C) > 0$, where C is a tangent vector field on ∂S such that $\iota_C d\theta|_{\partial S} = 0$ and $d\theta(C, N) > 0$ for an outward pointing normal vector field N;
4. a proper Morse function $\mathfrak{f}\colon S \to \mathbb{R}$ bounded from below having finitely many critical points, such that $\mathcal{L}$ is a pseudogradient of $\mathfrak{f}$ and satisfying moreover
   (a) $d\mathfrak{f}(C) > 0$ on $\{I > 0\}$ and $d\mathfrak{f}(C) < 0$ on $\{I < 0\}$,
   (b) the Hessian of a critical point of $\mathfrak{f}$ on ∂S evaluates negatively on the normal direction N, and
   (c) there is a constant $c \in \mathbb{R}$ whose sublevel set satisfies $\{\mathfrak{f} \leq c\} \subset S \setminus \partial S$ and contains all interior critical points of $\mathfrak{f}$.

For simplicity we will often drop part of the data from the notation. We will always assume that S is a Weinstein sector of *finite type*, i.e., that $\mathfrak{f}$ has only finitely many critical points. A Weinstein sector is a particular case of an exact Liouville sector in the sense of [23].

Example 2.8. – After perturbing the canonical Liouville form, the cotangent bundle of a smooth manifold Q with boundary admits the structure of a Weinstein sector.

To a Weinstein sector $(S, \theta, I, \mathfrak{f})$ we can associate two Weinstein manifolds in a canonical way up to deformation: the *completion* and the *belt*. The completion of S is the Weinstein manifold $(W, \theta_W, \mathfrak{f}_W)$ obtained by completing the Weinstein domain $W_0 = \{\mathfrak{f} \leq c\}$, which contains all interior critical points of $\mathfrak{f}$. The belt of S is the Weinstein manifold $(F, \theta_F, \mathfrak{f}_F)$ where $F = I^{-1}(0)$, $\theta_F = \theta|_F$ [(1)] and $\mathfrak{f}_F = \mathfrak{f}|_F$. To show that the belt is actually a Weinstein manifold it is enough to observe that $d\theta_F$ is a symplectic form because F is transverse to the vector field C, and that the Liouville vector field $\mathcal{L}$ is tangent to F because $dI(\mathcal{L}) = \alpha I$, and therefore the Liouville vector field of θ_F is $\mathcal{L}_F = \mathcal{L}|_F$.

Let $\kappa \in \mathbb{R}$ be a number such that all critical points of $\mathfrak{f}$ are contained in $\{\mathfrak{f} \leq \kappa\}$. We denote $S_0 = \{\mathfrak{f} \leq \kappa\}$ and $F_0 = F \cap S_0 = \{\mathfrak{f}_F \leq \kappa\}$. By Condition (4a) of Definition 2.7, the boundary ∂S_0 is a contact manifold with convex boundary with dividing set ∂F_0. Moreover $S \setminus S_0$ can be identified with a half symplectisation. Thus, given $R_0 \ll 0$, we can define a

(1) We abuse the notation by denoting the pull back by the inclusion as a restriction.

function $\mathfrak{r}\colon S \to [R_0, +\infty)$ satisfying the properties analogous to those in Section 2.1. We then write $S_R\colon eqq\mathfrak{r}^{-1}[R_0, R]$ and $S_R^e = S \setminus \operatorname{int}(S_R)$.

Definition 2.9. – Let ϕ be the flow of $\mathcal{L}$. The *skeleton* $S^{\mathrm{sk}} \subset S$ of a Weinstein sector $(S, \theta, \mathfrak{f})$ is given by

$$S^{\mathrm{sk}}\colon eqq \bigcap_{t>0} \phi^{-t}(S_0).$$

Remark 2.10. – Let W and F be the completion and the belt, respectively, of the Weinstein sector S. To understand the skeleton S^{sk} it is useful to note the following:

1. critical points of $\mathfrak{f}$ on ∂S are also critical points of $\mathfrak{f}|_{\partial S}$ and vice versa,
2. any critical point $p \in \partial S$ of $\mathfrak{f}$ lies inside $\{I = 0\} = F$ and is also a critical point of $\mathfrak{f}_F$,
3. the Morse indices of the two functions satisfy the relation
$$\operatorname{ind}_{\mathfrak{f}}(p) = \operatorname{ind}_{\mathfrak{f}_F}(p) + 1,$$
4. the skeleton satisfies $S^{\mathrm{sk}} \cap \partial S = F^{\mathrm{sk}}$.

The top stratum of the skeleton of $(S, \theta, \mathfrak{f})$ is given by the union of the stable manifolds of the critical points of $\mathfrak{f}$ of index n, where $2n$ is the dimension of S. Those are of two types: the stable manifolds Δ_p where p is an interior critical point of $\mathfrak{f}$, which are also stable manifolds for $\mathfrak{f}_W$ in the completion, and the stable manifolds Θ_p where p is a boundary critical point of $\mathfrak{f}$, for which $\Delta'_p = \Theta_p \cap \partial S$ is the stable manifold of p for $\mathfrak{f}_F$ in F.

Thus the Weinstein sector S can be obtained by attaching Weinstein handles, corresponding to the critical points of $\mathfrak{f}$ in the interior of ∂S, and Weinstein half-handles, corresponding to the critical points of $\mathfrak{f}$ in the boundary ∂S. We denote by S^{sc} the subcritical part of S, i.e., the union of the handles and half-handles of index less than n (where $2n$ is the dimension of S), by $\{\mathcal{H}_i\}$ the critical handle corresponding to Δ_i and by $\{\mathcal{H}'_j\}$ the half-handle corresponding to Θ_j. Finally we also choose the function $\mathfrak{r}\colon S \to [R_0, \infty)$ as in Section 2.1 which furthermore satisfies

$$S_{R_0} = S^{\mathrm{sc}} \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l \cup \mathcal{H}'_1 \cup \cdots \cup \mathcal{H}'_{l'}$$

and modify the Liouville form θ so that $\mathcal{H}_1, \ldots, \mathcal{H}_l, \mathcal{H}'_1, \ldots, \mathcal{H}'_{l'}$ are *standard* Weinstein handles.

It follows from the symplectic standard neighborhood theorem that a collar neighborhood of ∂S is symplectomorphic to

$$(F \times T^*(-2\epsilon, 0], d\theta_F + dp \wedge dq).$$

Definition 2.11. – Let S be a Weinstein sector and let L be a Lagrangian submanifold of its belt F. The *spreading* of L is

$$\operatorname{spr}(L) = L \times T^*_{-\epsilon}(-2\epsilon, 0] \subset F \times T^*(-2\epsilon, 0) \subset S.$$

Remark 2.12. – The spreading of L depends on the choice of symplectic standard neighborhood of the collar. However, given two different choices, the corresponding spreadings are Lagrangian isotopic. Furthermore, if L is exact in F, then $\operatorname{spr}(L)$ is exact in S, and thus two different spreadings are Hamiltonian isotopic.

EXAMPLE 2.13. – When the Weinstein sector is the cotangent bundle of a manifold with boundary, the spreading of a cotangent fiber of $T^*\partial Q$ is simply a cotangent fiber of T^*Q.

The proof of the following lemma is immediate.

LEMMA 2.14. – *The cocore planes of the index n half-handles of S are the spreading of the cocore planes of the corresponding index $n-1$ handles of F.*

2.4. Going from a Weinstein pair to a Weinstein sector

In this section we describe how to associate a Weinstein sector to a Weinstein pair. We recall the definition of Weinstein pair, originally introduced in [19].

DEFINITION 2.15. – A *Weinstein pair* (W_0, F_0) is a pair of Weinstein domains $(W_0, \theta_0, \mathfrak{f}_0)$ and $(F_0, \theta_F, \mathfrak{f}_F)$ together with a codimension one Liouville embedding of F_0 into ∂W_0.

We denote the completions of $(W_0, \theta_0, \mathfrak{f}_0)$ and $(F_0, \theta_F, \mathfrak{f}_F)$ by $(W, \theta_0, \mathfrak{f}_0)$ and $(F, \theta_F, \mathfrak{f}_F)$ respectively. Let $F_1 \subset F$ be a Weinstein domain retracting on F_0. If F_1 is close enough to F_0, the symplectic standard neighborhood theorem provides us with a Liouville embedding

$$((1-3\epsilon, 1] \times [-3\delta, 3\delta] \times F_1, s\,du + s\theta_F) \hookrightarrow (W_0, \theta_0). \tag{4}$$

Here s and u are coordinates on the first and second factors, respectively, and we require that the preimage of ∂W_0 is $\{s = 1\}$ and $F_1 \subset \partial W_0$ is identified with $\{(1, 0, x) : x \in F_1\}$. We denote by $\mathcal{U}$ the image of the embedding (4). After deforming $\mathfrak{f}_0$ we may assume that it is of the form $\mathfrak{f}_0(s, u, x) = s$ in the same coordinates.

Let $\mathcal{L}_F$ be the Liouville vector field of (F, θ_F). We choose a smooth function $\mathfrak{r}_F\colon F \to [R_0, +\infty)$, $R_0 \ll 0$, such that

— $F_0 = \mathfrak{r}_F^{-1}([R_0, 0])$,

— $d\mathfrak{r}_F(\mathcal{L}_F) > 0$ holds inside $\mathfrak{r}_F^{-1}(R_0, +\infty)$, and

— $d\mathfrak{r}_F(\mathcal{L}_F) = 1$ holds inside $\mathfrak{r}_F^{-1}(R_0 + 1, +\infty)$.

For simplicity of notation we also assume that

— $F_1 = \mathfrak{r}_F^{-1}([R_0, 1])$,

where F_1 denotes the manifold appearing in Formula (4). This condition is apparently a loss of generality because it cannot be satisfied for every Liouville form on F. However, the general case can be treated with minimal changes.

Consider the smooth function

$$r\colon [-3\delta, 3\delta] \times F_1 \to \mathbb{R},$$

$$r(u, x)\colon eqq 2 - \left(\frac{u}{3\delta}\right)^2 - \mathfrak{r}_F(x) - c$$

for some small number $c > 0$.

LEMMA 2.16. – *There exists a Weinstein domain $\widetilde{W}_0 \subset W$ containing all critical points of $\mathfrak{f}_0$ and which intersects some collar neighborhood of W_0 containing $\mathcal{U}$ precisely in the subset*

$$\mathcal{C}\colon eqq\{s \leq r(u, x)\} \subset \mathcal{U}.$$

The goal is now to deform the Liouville form θ_0 on

$$S_0: eqqW_0 \cap \widetilde{W}_0$$

to obtain a Liouville form θ so that the completion of (S_0, θ) is the sought Weinstein sector. The deformation will be performed inside $\mathcal{C}$. Given a smooth function $\rho\colon [1-3\epsilon, 1] \to \mathbb{R}$ such that:

— $\rho(s) = 0$ for $s \in [1-3\epsilon, 1-2\epsilon]$,

— $\rho(s) = 2s - 1$ for $s \in [1-\epsilon, 1]$, and

— $\rho'(s) \geq 0$ for $s \in [1-3\epsilon, 1]$,

we define a Liouville form $\theta_{\mathcal{U}}$ on $\mathcal{U}$ by

$$\theta_{\mathcal{U}} = s(du + \theta_F) - d(\rho(s)u).$$

The proof of the following lemma is a simple computation.

LEMMA 2.17. – *Let $\rho\colon [1-3\epsilon, 1] \to \mathbb{R}$ be a smooth function such that the Liouville vector field $\mathcal{L}_{\mathcal{U}}$ corresponding to the Liouville form $\theta_{\mathcal{U}}$ on $\mathcal{U}$ is equal to*

$$\mathcal{L}_{\mathcal{U}} = (s - \rho(s))\partial_s + \rho'(s)u\partial_u + \frac{\rho(s)}{s}\mathcal{L}_F.$$

We define the Liouville form θ on S_0 as $\theta|_{\mathcal{C}} = \theta_{\mathcal{U}}$ and $\theta|_{S_0 \setminus \mathcal{C}} = \theta_0$. By Lemma 2.17 the Liouville vector field $\mathcal{L}$ of θ is transverse to $\partial S_0 \setminus \partial W_0$ and is equal to

$$(1-s)\partial_s + 2u\partial_u + \frac{2s-1}{s}\mathcal{L}_F$$

in a neighborhood of $\partial S_0 \setminus \partial W_0$; in particular it is tangent to $\partial S_0 \setminus \partial W_0$. Thus we can complete (S_0, θ) to (S, θ) by adding a half-symplectisation of $\partial S_0 \setminus \mathrm{int}(\partial W_0)$. We define a function $I\colon \partial S \to \mathbb{R}$ by setting $I = u$ on $\partial S_0 \cap \partial W_0$ and extending it to ∂S so that $dI(\mathcal{L}) = 2I$ everywhere. It is easy to check that (S, θ, I) is an exact Liouville sector.

A Lyapunov function $\mathfrak{f}\colon S_0 \to \mathbb{R}_+$ for $\mathcal{L}$ can be obtained by interpolating between $\mathfrak{f}_0$ on $S_0 \setminus \mathcal{C}$ and $u^2 - (s-1)^2 + \mathfrak{f}_F + C$ on $\mathcal{C} \cap \{s \in [1-\epsilon, 1]\}$ for sufficiently large C. The Lyapunov function $\mathfrak{f}$ can then be extended to a Lyapunov function $\mathfrak{f}\colon S \to \mathbb{R}_+$ in a straightforward way. The easy verification that $(S, \theta, I, \mathfrak{f})$ is a Weinstein sector is left to the reader.

2.5. Exact Lagrangian immersions with cylindrical end

DEFINITION 2.18. – Let (W, θ) be a $2n$-dimensional Liouville manifold. An *exact Lagrangian immersion with cylindrical end* (or, alternately, an *immersed exact Lagrangian submanifold with cylindrical end*) is an immersion $\iota\colon L \to W$ such that:

1. L is an n-dimensional manifold and ι is a proper immersion which is an embedding outside finitely many points,
2. $\iota^*\theta = df$ for some function $f\colon L \to \mathbb{R}$, called the *potential* of (L, ι), and
3. the image of ι is tangent to the Liouville vector field of (W, θ) outside a compact set of L.

In the rest of the article, *immersed exact Lagrangian submanifold* will always mean immersed exact Lagrangian submanifold with cylindrical end. Note that L is allowed to be compact, and in that case Condition (3) is empty: a closed immersed Lagrangian submanifold is a particular case of immersed Lagrangian submanifolds with cylindrical ends. With an abuse of notation, we will often write L either for the pair (L, ι) or for the image $\iota(L)$.

EXAMPLE 2.19. – Let $(W, \theta, \mathfrak{f})$ be a Weinstein manifold. The Lagrangian cocore planes D_p introduced in Section 2.2 are Lagrangian submanifolds with cylindrical ends.

Properness of ι and Condition (3) imply that for every immersed exact Lagrangian submanifold with cylindrical end $\iota\colon\ L \to W$ there is $R > 0$ sufficiently large such that $\iota(L) \cap W_R^e$ corresponds to $[R, +\infty) \times \Lambda$ under the identification

$$(W_R^e, \theta) \cong ([R, +\infty) \times V, e^r\alpha),$$

where Λ is a Legendrian submanifold of (V, ξ). Then we say that L is *cylindrical over* Λ. Here Λ can be empty (if L is compact) or disconnected.

There are different natural notions of equivalence between immersed exact Lagrangian submanifolds. The strongest one is Hamiltonian isotopy.

DEFINITION 2.20. – Two exact Legendrian immersions (L, ι_0) and (L, ι_1) with cylindrical ends are Hamiltonian isotopic if there exists a function $H\colon\ [0,1] \times W \to \mathbb{R}$ with Hamiltonian flow φ_t such that $\iota_1 = \varphi_1 \circ \iota_0$, and moreover $\iota_t = \varphi_t \circ \iota_0$ has cylindrical ends for all $t \in [0, 1]$.

REMARK 2.21. – If $f_0\colon\ L \to \mathbb{R}$ is the potential of (L, ι_0), by Lemma 2.3 we can choose

$$f_1 = f_0 + \int_0^1 (-H_\sigma + \theta(X_{H_\sigma})) \circ \varphi_\sigma d\sigma \tag{5}$$

as potential for (L, ι_1).

The weakest one is exact Lagrangian regular homotopy.

DEFINITION 2.22. – Two exact Legendrian immersions (L, ι_0) and (L, ι_1) with cylindrical ends are exact Lagrangian regular homotopic if there exists a smooth path of immersions $\iota_t\colon\ L \to W$ for $t \in [0, 1]$ such that (L, ι_t) is an exact Lagrangian immersion with cylindrical ends for every $t \in [0, 1]$.

We recall that any exact regular homotopy $\iota_t\colon\ L \to W$ can be generated by a local Hamiltonian defined on L in the following sense.

LEMMA 2.23. – *An exact regular Lagrangian homotopy ι_t induces a smooth family of functions $G_t\colon\ L \to \mathbb{R}$ determined uniquely, up to a constant depending on t, by the requirement that the equation*

$$\iota_t^*(d\theta(\cdot, X_t)) = dG_t$$

be satisfied, where $X_t\colon\ L \to TW$ is the vector field along the immersion that generates ι_t. When ι_t has compact support, then dG_t has compact support as well.

Conversely, any Hamiltonian $G_t\colon\ L \to \mathbb{R}$ generates an exact Lagrangian isotopy $\iota_t\colon\ L \to (W, \theta)$ for any initial choice of exact immersion $\iota = \iota_0$.

REMARK 2.24. – If ι_t is generated by an ambient Hamiltonian isotopy, then H extends to a single-valued Hamiltonian on W itself. However, this is not necessarily the case for an arbitrary exact Lagrangian regular homotopy.

The limitations of our approach to define Floer cohomology for exact Lagrangian immersions require that we work with a restricted class of exact immersed Lagrangian submanifolds.

DEFINITION 2.25. – We say that a Lagrangian immersion (L, ι) is *nice* if the singularities of $\iota(L)$ are all transverse double points, and for every double point p the values of the potential at the two points in the preimage of p are distinct. Then, given a double point p, we will denote $\iota^{-1}(p) = \{p^+, p^-\}$, where $f(p^+) > f(p^-)$.

REMARK 2.26. – If L is not connected, we can shift the potential on different connected components by independent constants. If $\iota^{-1}(p)$ is contained in a connected component of L, then $f(p^+) - f(p^-)$ is still well defined. However, if the points in $\iota^{-1}(p)$ belong to different connected components, the choice of p^+ and p^- in $\iota^{-1}(p)$, and $f(p^+) - f(p^-)$, depend of the choice of potential. For technical reasons related to our definition of Floer cohomology, it seems useful, although unnatural, to consider the potential (up to shift by an overall constant) as part of the data of an exact Lagrangian immersion.

For nice immersed exact Lagrangian submanifolds we define a stronger form of exact Lagrangian regular homotopy.

DEFINITION 2.27. – Let (L, ι_0) and (L, ι_1) be nice exact Lagrangian immersed submanifolds with cylindrical ends. An exact Lagrangian regular homotopy (L, ι_t) is a *safe isotopy* if (L, ι_t) is nice for every $t \in [0, 1]$.

Niceness can always be achieved after a C^1-small exact Lagrangian regular homotopy. In the rest of this article exact Lagrangian immersions will always be assumed nice.

2.6. Contactisation and Legendrian lifts

We define a contact manifold (M, β), where $M = W \times \mathbb{R}$, with a coordinate z on $\mathbb{R}$, and $\beta = dz + \theta$. We call (M, β) the *contactisation* of (W, θ). A Hamiltonian isotopy $\varphi_t \colon W \to W$ which is generated by a Hamiltonian function $H \colon [0, 1] \times W \to \mathbb{R}$ lifts to a contact isotopy $\psi_t^+ \colon M \to M$ such that

$$\psi_t^+(x, z) = (\psi_t(x), z - q_t(x)),$$

where $q_t \colon W \to \mathbb{R}$ is the function defined in Lemma 2.3.

An immersed exact Lagrangian (L, ι) with potential $f \colon L \to \mathbb{R}$ uniquely defines a Legendrian immersion

$$\iota^+ \colon L \to W \times \mathbb{R}, \quad \iota^+(x) = (\iota(x), -f(x)).$$

Moreover ι^+ is an embedding when (L, ι) is nice. We denote the image of ι^+ by L^+ and call it the *Legendrian lift* of L. Yet, any Legendrian submanifold of (M, β) projects to an immersed Lagrangian in W. This projection is called the *Lagrangian projection* of the Legendrian submanifold.

Double points of L correspond to Reeb chords of L^+, and the action (i.e., length) of the Reeb chord projecting to a double point p is $f(p^+) - f(p^-)$. If L is connected, different potentials induce Legendrian lifts which are contact isotopic by a translation in the z-direction. In particular, the action of Reeb chords is independent of the lift. But, if L is disconnected, different potentials can induce non-contactomorphic Legendrian lifts and the action of Reeb chords between different connected components depends on the potential.

3. Legendrian contact cohomology

In this section we provide an overview of Legendrian contact cohomology. We recall the notion of augmentation and explain how it is used to define bilinearized Legendrian contact cohomology.

3.1. The Chekanov-Eliashberg algebra

In view of the correspondence between Legendrian submanifolds of (M, β) and exact Lagrangian immersions in (W, θ), Floer cohomology for Lagrangian immersions will be a variation on the theme of Legendrian contact cohomology. The latter was proposed by Eliashberg and Hofer and later defined rigorously by Chekanov, combinatorially, in $\mathbb{R}^3$ with its standard contact structure in [11], and by Ekholm, Etnyre and Sullivan, analytically, in the contactisation of any Liouville manifold in [18]. In this subsection we summarize the analytical definition.

For $d > 0$, let $\widetilde{\mathcal{R}}^{d+1} = \mathrm{Conf}^{d+1}(\partial D^2)$ be the space of parametrized disks with $d+1$ punctures on the boundary. The automorphism group $Aut(D^2)$ acts on $\widetilde{\mathcal{R}}^{d+1}$ and its quotient is the Deligne-Mumford moduli space $\mathcal{R}^{d+1}$. Given $\boldsymbol{\zeta} = (\zeta_0, \ldots, \zeta_d) \in \widetilde{\mathcal{R}}^{d+1}$, we will denote

$$\Delta_{\boldsymbol{\zeta}} = D^2 \setminus \{\zeta_0, \ldots, \zeta_d\}.$$

Following [37], near every puncture ζ_i we will define positive and negative universal striplike ends with coordinates $(\sigma_i^+, \tau_i^+) \in (0, +\infty) \times [0,1]$ and $(\sigma_i^-, \tau_i^-) \in (-\infty, 0) \times [0,1]$ respectively. We will assume that $\sigma_i^- = -\sigma_i^+$ and $\tau_i^- = 1 - \tau_i^+$.

Remark 3.1. – Putting both positive and negative strip-like ends near each puncture will be useful for comparing wrapped Floer cohomology and contact cohomology, which use different conventions for positive and negative punctures.

Definition 3.2. – Let (V, α) be a contact manifold with contact structure ξ and Reeb vector field $\mathcal{R}$. An almost complex structure J on $\mathbb{R} \times V$ is *cylindrical* if

1. J is invariant under translations in $\mathbb{R}$,
2. $J(\partial_r) = \mathcal{R}$, where r is the coordinate in $\mathbb{R}$,
3. $J(\xi) \subset \xi$, and $J|_\xi$ is compatible with $d\alpha|_\xi$.

Definition 3.3. – An almost complex structure J on a Liouville manifold W is *compatible with* θ if it is compatible with $d\theta$ and, outside a compact set, corresponds to a cylindrical almost complex structure under the identification (3). We denote by $\mathcal{J}(\theta)$ the set of almost complex structures on W which are compatible with θ.

It is well known that $\mathcal{J}(\theta)$ is a contractible space.

Given an exact Lagrangian immersion (L,ι) in W, we will consider almost complex structures J on W which satisfy the following

(†) J is compatible with θ, integrable in a neighborhood of the double points of (L,ι), and for which L moreover is real-analytic near the double points.

We will denote the set of double points of (L,ι) by D.

Let $u\colon \Delta_{\boldsymbol{\zeta}} \to W$ be a J-holomorphic map with boundary in L. If u has finite area and no puncture at which the lift of $u|_{\partial\Delta_{\boldsymbol{\zeta}}}$ to L has a continuous extension, then $\lim_{z\to\zeta_i} u(z) = p_i$ for some $p_i \in D$. Since the boundary of u switches branch near p_i, the following dichotomy thus makes sense:

Definition 3.4. – We say that ζ_i is a *positive puncture* at p_i if

$$\lim_{\sigma_i^+\to+\infty} (\iota^{-1}\circ u)(\sigma_i^+,0) = p_i^+$$

and that ζ_i is a *negative puncture* at p_i if

$$\lim_{\sigma_i^-\to-\infty} (\iota^{-1}\circ u)(\sigma_i^-,0) = p_i^+.$$

Let L be an immersed exact Lagrangian. If $p_1,\dots,p_d$ are double points of L (possibly with repetitions), we denote by $\widetilde{\mathfrak{M}}_L(p_0;p_1,\dots,p_d;J)$ the set of pairs $(\boldsymbol{\zeta},u)$ where:

1. $\boldsymbol{\zeta}\in\widetilde{R}^{d+1}$ and $u\colon\Delta_{\boldsymbol{\zeta}}\to W$ is a J-holomorphic map,
2. $u(\partial\Delta_{\boldsymbol{\zeta}})\subset L$, and
3. ζ_0 is a positive puncture at p_0 and ζ_i, for $i=1,\dots,d$, is a negative puncture at p_i.

The group $\mathrm{Aut}(D^2)$ acts on $\widetilde{\mathfrak{M}}_L(p_0;p_1,\dots,p_d;J)$ by reparametrisations; the quotient is the moduli space $\mathfrak{M}_L(p_0;p_1,\dots,p_d;J)$. Note that the set $p_1,\dots,p_d$ can be empty. In this case, the elements of the moduli spaces $\mathfrak{M}(p_0;J)$ are called *teardrops*.

Given $u\in\widetilde{\mathfrak{M}}_L(p_0;p_1,\dots,p_d;J)$, let D_u be the linearisation of the Cauchy-Riemann operator at u. By standard Fredholm theory, D_u is a Fredholm operator with index $\mathrm{ind}(D_u)$. We define the *index* of u as

$$\mathrm{ind}(u)=\mathrm{ind}(D_u)+d-2.$$

It is locally constant, and we denote by $\mathfrak{M}^k_L(p_0;p_1,\dots,p_d;J)$ the subset of

$$\mathfrak{M}_L(p_0;p_1,\dots,p_d;J)$$

consisting of classes of maps u with $\mathrm{ind}(u)=k$.

The following proposition is a version of [18, Proposition 2.3]:

Proposition 3.5. – *For a generic J satisfying the condition (†), the moduli space*

$$\mathfrak{M}^k_L(p_0;p_1,\dots,p_d;J)$$

is a transversely cut out manifold of dimension k. In particular, if $k<0$ it is empty; if $k=0$ it is compact, and therefore consists of a finite number of points; and if $k=1$, it can be compactified in the sense of Gromov, see [18, Section 2.2].

The boundary of the compactification of the moduli space $\mathfrak{N}^1_L(p_0; p_1, \dots, p_d; J)$ *is*

$$\bigsqcup_{q\in D}\ \bigsqcup_{0\le i<j\le d} \mathfrak{N}^0_L(p_0; p_1, \dots, p_i, q, p_{j+1}, \dots, p_d; J) \times \mathfrak{N}^0_L(q; p_{i+1}, \dots, p_j; J). \tag{6}$$

If L is spin, the moduli spaces are orientable and a choice of spin structure induces a coherent orientation of the moduli spaces; see [17].

Definition 3.6. – We say that an almost complex structure J on W is L-regular if it satisfies (†) for L and all moduli spaces $\mathfrak{N}_L(p_0; p_1, \dots, p_d; J)$ are transversely cut out.

To a Legendrian submanifold L^+ of (M, β) we can associate a differential graded algebra $(\mathfrak{A}, \mathfrak{d})$ called the *Chekanov-Eliashberg algebra* (or *Legendrian contact cohomology algebra*) of L^+. As an algebra, $\mathfrak{A}$ is the free unital noncommutative algebra generated by the double points of the Lagrangian projection L or, equivalently, by the Reeb chords of L^+. The grading takes values in $\mathbb{Z}/2\mathbb{Z}$ and is simply given by the self-intersection of the double points. If $2c_1(W) = 0$ and the Maslov class of L^+ vanishes, it can be lifted to an integer valued grading by the Conley-Zehnder index. We will not make explicit use of the integer grading, and therefore we will not describe it further, referring the interested reader to [18, Section 2.2] instead.

The differential $\mathfrak{d}$ is defined on the generators as:

$$\mathfrak{d}(p_0) = \sum_{d\ge 0}\sum_{p_1,\dots,p_d} \#\mathfrak{N}^0_L(p_0; p_1, \dots, p_n; J)\, p_1 \cdots p_d.$$

According to [18, Proposition 2.6], the Chekanov-Eliashberg algebra is a Legendrian invariant:

Theorem 3.7 ([18]). – *If* L_0^+ *and* L_1^+ *are Legendrian-isotopic Legendrian submanifolds of* (M, β)*, then their Chekanov-Eliashberg algebras* $(\mathfrak{A}^0, \mathfrak{d}_0)$ *and* $(\mathfrak{A}^1, \mathfrak{d}_1)$ *are stable tame isomorphic.*

The definition of stable tame isomorphism of DGAs was introduced by Chekanov in [11], and then discussed by Ekholm, Etnyre and Sullivan in [18]. We will not use it in this article but note that on the homological level a stable tame isomorphism induces an isomorphism.

3.2. Bilinearized Legendrian contact cohomology

Differential graded algebras are difficult objects to manipulate, and therefore Chekanov introduced a linearisation procedure. The starting point of this procedure is the existence of an augmentation.

Definition 3.8. – Let $\mathfrak{A}$ be a differential graded algebra over a commutative ring $\mathbb{F}$. An *augmentation* of $\mathfrak{A}$ is a unital differential graded algebra morphism $\varepsilon\colon \mathfrak{A} \to \mathbb{F}$.

Let L_0^+ and L_1^+ be Legendrian submanifolds of (M, β) with Lagrangian projections L_0 and L_1 with potentials f_0 and f_1 respectively. We recall that the potential is the negative of the z coordinate. We will assume that L_0^+ and L_1^+ are *chord generic*, which means in this case that $L_0^+ \cap L_1^+ = \emptyset$ and all singularities of $L_0 \cup L_1$ are transverse double points.

Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be the Chekanov-Eliashberg algebras of L_0^+ and L_1^+ respectively. Let $\varepsilon_0\colon \mathfrak{A}_0 \to \mathbb{F}$ and $\varepsilon_1\colon \mathfrak{A}_1 \to \mathbb{F}$ be augmentations. Now we describe the construction of the *bilinearized Legendrian contact cohomology* complex $\mathrm{LCC}_{\varepsilon_0,\varepsilon_1}(L_0^+, L_1^+; J)$.

First, we introduce some notation. We denote by D_i the set of double points of L_i (for $i = 1, 2$) and by $\mathcal{C}$ the intersection points of L_0 and L_1 such that $f_0(q) < f_1(q)$. Double points in D_i correspond to Reeb chords of L_i^+, and double points in $\mathcal{C}$ correspond to Reeb chords from L_1^+ to L_0^+ (note the order!). We define the exact immersed Lagrangian $L = L_0 \cup L_1$ and, for $\mathbf{p}^0 = (p_1^0, \dots, p_{l_0}^0) \in D_0^{l_0}$, $\mathbf{p}^1 = (p_1^1, \dots, p_{l_1}^1) \in D_1^{l_1}$ and $q_\pm \in \mathcal{C}$, we denote

$$\mathfrak{M}_L^i(q_+; \mathbf{p}^0, q_-, \mathbf{p}^1; J) := \mathfrak{M}_L^i(q_+; p_1^0, \dots, p_{l_0}^0, q_-, p_1^1, \dots, p_{l_1}^1; J),$$

where J is an L-regular almost complex structure. If ε_i is an augmentation of $\mathfrak{A}_i$, we denote $\varepsilon_i(\mathbf{p}^i) := \varepsilon_i(p_1^i) \cdots \varepsilon_i(p_{l_i}^i)$.

As an $\mathbb{F}$-module, $\mathrm{LCC}_{\varepsilon_0,\varepsilon_1}(L_0^+, L_1^+; J)$ is freely generated by the set $\mathcal{C}$ and the differential of a generator $q_- \in \mathcal{C}$ is defined as

$$\partial_{\varepsilon_0,\varepsilon_1}(q_-) = \sum_{q_+ \in \mathcal{C}} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \#\mathfrak{M}_L^0(q_+; \mathbf{p}^0, q_-, \mathbf{p}^1; J)\varepsilon_0(\mathbf{p}^0)\varepsilon_1(\mathbf{p}^1) q_+. \tag{7}$$

The bilinearized Legendrian contact cohomology $\mathrm{LCH}_{\varepsilon_0,\varepsilon_1}(L_0^+, L_1^+)$ is the homology of this complex. The set of isomorphism classes of bilinearized Legendrian contact cohomology groups is independent of the choice of J and is a Legendrian isotopy invariant by the adaptation of Chekanov's argument from [11] due to the first author and Bourgeois [8].

4. Floer cohomology for exact Lagrangian immersions

In this section we define a version of Floer cohomology for exact Lagrangian immersions. Recall that our exact Lagrangian immersions are equipped with choices of potentials making their Legendrian lifts embedded. This is not new material; similar or even more general accounts can be found, for example, in [5], [6] and [31].

4.1. Cylindrical Hamiltonians

Definition 4.1. – Let W be a Liouville manifold. A Hamiltonian function $H\colon [0,1] \times W \to \mathbb{R}$ is *cylindrical* if there is a function $h\colon \mathbb{R}^+ \to \mathbb{R}$ such that $H(t, w) = h(e^{\mathfrak{r}(w)})$ outside a compact set of W.

The following example describes the behavior of the Hamiltonian vector field of a cylindrical Hamiltonian in an end of W, after taking into account the identification (3).

Example 4.2. – Let (V, α) be a contact manifold with Reeb vector field $\mathcal{R}$ and let $(\mathbb{R} \times V, d(e^r\alpha))$ be its symplectisation. Given a smooth function $h\colon \mathbb{R}^+ \to \mathbb{R}$, we define the autonomous Hamiltonian $H\colon \mathbb{R} \times V \to \mathbb{R}$ by $H(r, v) = h(e^r)$. Then the Hamiltonian vector field of H is

$$X_H(r, v) = h'(e^r)\mathcal{R}(r, v).$$

Let (L_0, ι_0) and (L_1, ι_1) be two immersed exact Lagrangian submanifolds of W with cylindrical ends over Legendrian submanifolds Λ_0 and Λ_1 of (V, α). Given a cylindrical Hamiltonian $H\colon [0,1] \times W \to \mathbb{R}$, we denote by $\mathcal{C}_H$—or simply $\mathcal{C}$ when there is no risk of confusion— the set of Hamiltonian chords $x\colon [0,1] \to W$ of H such that $x(0) \in L_0$ and $x(1) \in L_1$. If φ_t denotes the Hamiltonian flow of H, then $\mathcal{C}_H$ is in bijection with $\varphi_1(L_0) \cap L_1$.

Definition 4.3. – A cylindrical Hamiltonian $H\colon [0,1] \times W \to \mathbb{R}$, with Hamiltonian flow φ_t, is *compatible* with L_0 and L_1 if

(i) no starting point or endpoint of a chord $x \in \mathcal{C}_H$ is a double point of (L_0, ι_0) or (L_1, ι_1),

(ii) $\varphi_1(L_0)$ intersects L_1 transversely,

(iii) for ρ large enough $h'(\rho) = \lambda$ is constant, and

(iv) all time-one Hamiltonian chords from L_0 to L_1 are contained in a compact region.

Condition (iv) is equivalent to asking that λ should not be the length of a Reeb chord from Λ_0 to Λ_1.

Remark 4.4. – If cylindrical Hamiltonian H is *compatible* with L_0 and L_1, then $\mathcal{C}_H$ is a finite set.

4.2. Obstructions

If one tries to define Floer cohomology for immersed Lagrangian submanifolds by extending the usual definition naively, one runs into the problem that the "differential" might not square to zero because of the bubbling of teardrops in one-dimensional families of Floer strips. Thus, if (L, ι) is an immersed Lagrangian submanifold and J is L-regular, we define a map $\mathfrak{d}^0\colon D \to \mathbb{Z}$ by

$$\mathfrak{d}^0(p) = \#\mathfrak{M}^0_L(p; J)$$

and extend it by linearity to the free module generated by D. The map $\mathfrak{d}^0$ is called the *obstruction* of (L, ι). If $\mathfrak{d}^0 = 0$ we say that (L, ι) is *uncurved*.

Typically, asking that an immersed Lagrangian submanifold be uncurved is too much, and a weaker condition will ensure that Floer cohomology can be defined. We observe that $\mathfrak{d}^0$ is a component of the Chekanov-Eliashberg algebra of the Legendrian lift of L, and make the following definition.

Definition 4.5. – Let (L, ι) be an immersed exact Lagrangian submanifold. The *obstruction algebra* $(\mathfrak{D}, \mathfrak{d})$ of (L, ι)—or of (L, ι, J) when the almost complex structure is not clear from the context—is the Chekanov-Eliashberg algebra of the Legendrian lift L^+.

If L is connected, its obstruction algebra $(\mathfrak{D}, \mathfrak{d})$ does not depend on the potential. On the other hand, if L is disconnected, the potential differences at the double points of L involving different connected components, and therefore what holomorphic curves are counted in $(\mathfrak{D}, \mathfrak{d})$, depend on the choice of the potential.

Definition 4.6. – An exact immersed Lagrangian (L, ι) is *unobstructed* if $(\mathfrak{D}, \mathfrak{d})$ admits an augmentation.

Unobstructedness does not depend on the choice of L-regular almost complex structure and is invariant under general Legendrian isotopies as a consequence of Theorem 3.7 (this fact will not be needed). However, we will need the invariance statement from the following proposition.

PROPOSITION 4.7. – *If L_0 and L_1 are safe isotopic exact Lagrangian immersions, and J_0 and J_1 are L_0-regular and L_1-regular almost complex structures respectively, then the obstruction algebras $(\mathfrak{D}_0, \mathfrak{d}_0)$ of L_0 and $(\mathfrak{D}_1, \mathfrak{d}_1)$ of L_1 are isomorphic. In particular, there is a bijection between the augmentations of $(\mathfrak{D}_0, \mathfrak{d}_0)$ and the augmentations of $(\mathfrak{D}_1, \mathfrak{d}_1)$.*

Proof. – The safe isotopy between L_0 and L_1 induces a Legendrian isotopy between the Legendrian lifts L_0^+ and L_1^+ without birth or death of Reeb chords. By [18, Proposition 2.6] the Chekanov-Eliashberg algebras of L_0^+ and L_1^+ are stably tame isomorphic, and moreover stabilization occurs only at the birth or death of a Reeb chord. □

Proposition 4.7 is the main reason why we have made the choice of distinguishing between the obstruction algebra of a Lagrangian immersion and the Chekanov-Eliashberg algebra of its Legendrian lift. In fact Legendrian submanifolds are more naturally considered up to Legendrian isotopy. However, in this article we will consider immersed Lagrangian submanifolds only up to the weaker notion of safe isotopy.

4.3. The differential

We denote by Z the strip $\mathbb{R} \times [0,1]$ with coordinates (s,t).

Let $\widetilde{\mathcal{R}}^{l_0|l_1} \cong \mathrm{Conf}^{l_0}(\mathbb{R}) \times \mathrm{Conf}^{l_1}(\mathbb{R})$ be the set of pairs $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1)$ such that $\boldsymbol{\zeta}^0 = \{\zeta_1^0, \ldots, \zeta_{l_0}^0\} \subset \mathbb{R} \times \{0\}$ and $\boldsymbol{\zeta}^1 = \{\zeta_{l_1}^1, \ldots, \zeta_1^1\} \subset \mathbb{R} \times \{1\}$. We assume that the s-coordinates of ζ_j^i are increasing in j for ζ_j^0 and decreasing for ζ_j^1. We define

$$Z_{\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1} := Z \setminus \{\zeta_1^0, \ldots, \zeta_{l_0}^0, \zeta_1^1, \ldots, \zeta_{l_1}^1\}.$$

The group $\mathrm{Aut}(Z) = \mathbb{R}$ acts on $\widetilde{\mathcal{R}}^{l_0|l_1}$.

In the rest of this section we will assume that H is compatible with (L_0, ι_0) and (L_1, ι_1). We will consider a time-dependent almost complex structure $J_\bullet$ on W which satisfies the following

(††) (i) J_t is compatible with θ for all t,

(ii) $J_t = J_0$ for $t \in [0, 1/4]$ in a neighborhood of the double points of L_0 and $J_t = J_1$ for $t \in [3/4, 1]$ in a neighborhood of the double points of L_1,

(iii) J_0 satisfies (†) for L_0 and J_1 satisfies (†) for L_1.

Condition (ii) is necessary to ensure that $J_\bullet$ is independent of the coordinate $\sigma_{i,j}^-$ in some neighborhoods of the boundary punctures ζ_j^i, so that we can apply standard analytical results.

For the same reason we fix once and for all a nondecreasing function $\chi\colon \mathbb{R} \to [0,1]$ such that

$$\chi(t) = \begin{cases} 0 & \text{for } t \in [0, 1/4], \\ 1 & \text{for } t \in [3/4, 1] \end{cases}$$

and $\chi'(t) \leq 3$ for all t. It is easy to see that, if φ_t is the Hamiltonian flow of a Hamiltonian function H, then $\varphi_{\chi(t)}$ is the Hamiltonian flow of the Hamiltonian function H' such that $H'(t,w) = \chi'(t)H(\chi(t),w)$. We will use χ' to cut off the Hamiltonian vector field in the Floer equation to ensure that it has the right invariance properties in the strip-like ends corresponding to the boundary punctures ζ^i_j.

Given Hamiltonian chords $x_+, x_- \in \mathcal{C}$ and self-intersections $p^0_1, \ldots, p^0_{l_0}$ of L_0 and $p^1_1, \ldots, p^1_{l_1}$ of L_1 we define the moduli space

$$\widetilde{\mathfrak{M}}_{L_0,L_1}(p^1_1, \ldots, p^1_{l_1}, x_-, p^0_1, \ldots, p^0_{l_0}, x_+; H, J_\bullet)$$

of triples $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1, u)$ such that

— $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \widetilde{\mathcal{R}}^{l_0|l_1}$ and $u\colon Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1} \to W$ is a map satisfying the Floer equation

$$\frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - \chi'(t)X_H(\chi(t),u)\right) = 0, \tag{8}$$

— $\lim_{s\to\pm\infty} u(s,t) = x_\pm(\chi(t))$ uniformly in t,

— $u(s,0) \in L_0$ for all $(s,0) \in Z_{\boldsymbol{\zeta}^0, Z_{\boldsymbol{\zeta}^1}}$,

— $u(s,1) \in L_1$ for all $(s,1) \in Z_{\boldsymbol{\zeta}^0, Z_{\boldsymbol{\zeta}^1}}$,

— $\lim_{z\to\zeta^i_j} u(z) = p^i_j$, and

— ζ^i_j is a negative puncture at p^i_j for $i = 0, 1$ and $j = 1, \ldots, l_i$.

The group $\mathrm{Aut}(Z) = \mathbb{R}$ acts on $\widetilde{\mathfrak{M}}_{L_0,L_1}(p^1_1, \ldots, p^1_{l_1}, x_-, p^0_1, \ldots, p^0_{l_0}, x_+; H, J_\bullet)$ by reparametrisations. We will denote the quotient by

$$\mathfrak{M}_{L_0,L_1}(p^1_1, \ldots, p^1_{l_1}, x_-, p^0_1, \ldots, p^0_{l_0}, x_+; H, J_\bullet).$$

For $u \in \widetilde{\mathfrak{M}}_{L_0,L_1}(p^1_1, \ldots, p^1_{l_1}, x_-, p^0_1, \ldots, p^0_{l_0}, x_+; H, J_\bullet)$, we denote by F_u the linearisation of the Floer operator

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - \chi'(t)X_H(\chi(t),u)\right)$$

at u. By standard Fredholm theory, F_u is a Fredholm operator with index $\mathrm{ind}(F_u)$. We define

$$\mathrm{ind}(u) = \mathrm{ind}(F_u) + l_0 + l_1.$$

The index is locally constant, and we denote by

$$\mathfrak{M}^k_{L_0,L_1}(p^1_1, \ldots, p^1_{l_1}, x_-, p^0_1, \ldots, p^0_{l_0}, x_+; H, J_\bullet)$$

the subset of $\mathfrak{M}_{L_0,L_1}(p^1_1, \ldots, p^1_{l_1}, x_-, p^0_1, \ldots, p^0_{l_0}, x_+; H, J_\bullet)$ consisting of classes of maps u with index $\mathrm{ind}(u) = k$.

Observe that similar construction for closed, exact, graded, immersed Lagrangian submanifolds was considered by Alston and Bao in [6], where the regularity statement appears in [6, Proposition 5.2] and compactness is discussed in [6, Section 4]. In addition, the corresponding statement in the case of Legendrian contact cohomology in $P \times \mathbb{R}$ was proven by Ekholm, Etnyre and Sullivan, see [18, Proposition 2.3]. The following proposition translates those compactness and regularity statements to the settings under consideration.

PROPOSITION 4.8. – *For a generic time-dependent almost complex structure $J_\bullet$ satisfying (††), for which moreover J_0 is L_0-regular and J_1 is L_1-regular, the moduli space*

$$\mathfrak{M}^k_{L_0,L_1}(p^1_1,\dots,p^1_{l_1},x_-,p^0_1,\dots,p^0_{l_0},x_+;H,J_\bullet)$$

is a transversely cut-out manifold of dimension $k-1$. If $k=1$ it is compact, and therefore consists of a finite number of points. If $k=2$ it can be compactified in the sense of Gromov-Floer.

The boundary of the compactification of the one-dimensional moduli space

$$\mathfrak{M}^2_{L_0,L_1}(p^1_1,\dots,p^1_{l_1},x_-,p^0_1,\dots,p^0_{l_0},x_+;H,J_\bullet)$$

is

$$\begin{aligned}(9)\qquad &\bigsqcup_{y\in\mathcal{C}_H}\bigsqcup_{0\le h_i\le l_i}\mathfrak{M}^1_{L_0,L_1}(p^1_{h_1+1},\dots,p^1_{l_1},y,p^0_1,\dots,p^0_{h_0},x_+;H,J_\bullet)\\ &\qquad\qquad\times\mathfrak{M}^1_{L_0,L_1}(p^1_1,\dots,p^1_{h_1},x_-,p^0_{h_0+1},\dots,p^0_{l_0},y;H,J_\bullet)\\ &\bigsqcup_{q\in D_1}\bigsqcup_{0\le i\le j\le l_1}\mathfrak{M}^1_{L_0,L_1}(p^1_1,\dots,p^1_i,q,p^1_{j+1},\dots,p^1_{l_1},x_-,p^0_1,\dots,p^0_{l_0},x_+;H,J_\bullet)\\ &\qquad\qquad\times\mathfrak{N}^0_{L_1}(q;p^1_{i+1},\dots,p^1_j;J_1)\\ &\bigsqcup_{q\in D_0}\bigsqcup_{0\le i\le j\le l_0}\mathfrak{M}^1_{L_0,L_1}(p^1_1,\dots,p^1_{l_1},x_-,p^0_1,\dots,p^0_i,q,p^0_{j+1},\dots,p^0_{l_0},x_+;H,J_\bullet)\\ &\qquad\qquad\times\mathfrak{N}^0_{L_0}(q;p^0_{i+1},\dots,p^0_j;J_0).\end{aligned}$$

If both L_0 and L_1 are spin the moduli spaces are orientable, and a choice of spin structure on each Lagrangian submanifold induces a coherent orientation on the moduli spaces.

REMARK 4.9. – We use different conventions for the index of maps involved in the definition of the obstruction algebra and for maps involved in the definition of Floer cohomology. Unfortunately this can cause some confusion, but it is necessary to remain consistent with the standard conventions in the literature.

DEFINITION 4.10. – We say that a time-dependent almost complex structure $J_\bullet$ on W is (L_0,L_1)-regular if it satisfies (††) for L_0 and L_1 and all moduli spaces

$$\mathfrak{M}^k_{L_0,L_1}(p^1_1,\dots,p^1_{l_1},x_-,p^0_1,\dots,p^0_{l_0},x_+;H,J_\bullet)$$

are transversely cut out.

Note that, strictly speaking, the condition of being (L_0,L_1)-regular depends also on the Hamiltonian, even if we have decided to suppress it from the notation.

Suppose that the obstruction algebras $\mathfrak{D}_0$ and $\mathfrak{D}_1$ admit augmentations ε_0 and ε_1 respectively. To simplify the notation we write $\mathbf{p}^i=(p^i_1,\dots,p^i_{l_i})$,

$$\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H,J_\bullet)=\mathfrak{M}^k_{L_0,L_1}(p^1_1,\dots,p^1_{l_1},x_-,p^0_1,\dots,p^0_{l_0},x_+;H,J_\bullet)$$

and $\varepsilon_i(\mathbf{p}^i)=\varepsilon_i(p^i_1)\cdots\varepsilon_i(p^i_{l_i})$ for $i=0,1$. We also introduce the weighted count

$$\mathfrak{m}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+)=\#\mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H,J_\bullet)\varepsilon_0(\mathbf{p}^0)\varepsilon_1(\mathbf{p}^1).$$

Then we define the Floer complex over the commutative ring $\mathbb{F}$

$$\mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) = \bigoplus_{x \in \mathcal{C}} \mathbb{F}x,$$

with differential

$$\partial\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet),$$

defined on the generators by

$$\partial x_+ = \sum_{x_- \in \mathcal{C}_H} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+) x_-. \tag{10}$$

The algebraic interpretation of the Gromov-Floer compactification of the one-dimensional moduli spaces in Proposition 4.8 is that $\partial^2 = 0$.

We will denote the homology by $\mathrm{HF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H)$. The suppression of $J_\bullet$ from the notation is justified by Subsection 5.1.

4.4. Comparison with bilinearized Legendrian contact cohomology

In this subsection we compare the Lagrangian Floer cohomology of a pair of immersed exact Lagrangian submanifolds with the bilinearized Legendrian contact cohomology of a particular Legendrian lift of theirs.

Let L_0 and L_1 be exact Lagrangian immersions, $H\colon [0,1] \times W \to \mathbb{R}$ a Hamiltonian function compatible with L_0 and L_1 with Hamiltonian flow φ_t, and $J_\bullet$ an (L_0, L_1)-regular almost complex structure. We will introduce the "backward" isotopy $\overline{\varphi}_t = \varphi_1 \circ \varphi_t^{-1}$, where $\varphi_1 \circ \varphi_t^{-1} \circ \varphi_1^{-1}$ can be generated by the Hamiltonian $-H(t, \varphi_t^{-1} \circ \varphi_1^{-1})$.

Given an almost complex structure $J_\bullet$ and an arbitrary path ϕ_t of symplectomorphisms, we denote by $\phi_* J_\bullet$ the almost complex structure defined as $\phi_* J_t = d\phi_{\chi(t)} \circ J_t \circ d\phi_{\chi(t)}^{-1}$. The time rescaling by χ ensures that $\phi_* J_\bullet$ satisfies (††) for $\phi_1(L_0)$ and L_1 if and only if $J_\bullet$ does for L_0 and L_1.

LEMMA 4.11. – *Denote by* $\mathbf{0}$ *the constantly zero function on* W *and set*

$$q_\pm = x_\pm(1) \in \varphi_1(L_0) \cap L_1,$$

regarded as Hamiltonian chords of $\mathbf{0}$. *There is a bijection*

$$\overline{\varphi}_*\colon \mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, J_\bullet) \to \mathfrak{M}_{\varphi_1(L_0), L_1}(\mathbf{p}^1, q_-, \varphi_1(\mathbf{p}^0), q_+; \mathbf{0}, \overline{\varphi}_* J_\bullet)$$

defined by $(\overline{\varphi}_* u)(s,t) = \overline{\varphi}_t(u(s,t))$, *and moreover* $J_\bullet$ *is* (L_0, L_1)*-regular if and only if* $\overline{\varphi}_* J_\bullet$ *is* $(\varphi_1(L_0), L_1)$*-regular.*

Proof. – Since

$$\begin{aligned} \frac{\partial v}{\partial s}(s,t) &= d\overline{\varphi}_t \left(\frac{\partial u}{\partial s}(s,t) \right) \quad \text{and} \\ \frac{\partial v}{\partial t}(s,t) &= d\overline{\varphi}_t \left(\frac{\partial u}{\partial t}(s,t) - \chi'(t) X(\chi(t), u(s,t)) \right), \end{aligned}$$

u satisfies the Floer equation with Hamiltonian H if and only if $\overline{\varphi}_* u$ satisfies the Floer equation with Hamiltonian $\mathbf{0}$. The map $\overline{\varphi}_*$ is invertible because $\overline{\varphi}_t$ is for each t. Finally, we observe that $d\overline{\varphi}$ intertwines the linearized Floer operators at u and v. □

Given two Legendrian submanifolds Λ_0 and Λ_1 in $(W \times \mathbb{R}, dz + \theta) = (M, \beta)$, we say that Λ_0 is *above* Λ_1 if the z-coordinate of any point of Λ_0 is larger than the z-coordinate of any point of Λ_1.

LEMMA 4.12. – *Let L_0 and L_1 be immersed exact Lagrangian submanifolds of (W, θ) and H a cylindrical Hamiltonian compatible with L_0 and L_1. We denote by φ_t the Hamiltonian flow of H and $\widetilde{L}_0 = \varphi_1(L_0)$. We choose Legendrian lifts of $\widetilde{L}_0$ and L_1 to Legendrian submanifolds $\widetilde{L}_0^+$ and L_1^+ of (M, β) such that $\widetilde{L}_0^+$ is above L_1^+. If $\widetilde{J}$ is an L-regular almost complex structure on W for $L = \widetilde{L}_0 \cup L_1$, let $J_\bullet = (\overline{\varphi_t}^{-1})_* \widetilde{J}$. For every pair of augmentations ε_0 and ε_1 of the obstruction algebras of (L_0, J_0) and (L_1, J_1) respectively, there is an isomorphism of complexes*

$$\mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) \cong \mathrm{LCC}_{\widetilde{\varepsilon}_0, \varepsilon_1}(\widetilde{L}_0^+, L_1^+; \widetilde{J}),$$

where $\widetilde{\varepsilon}_0 = \varepsilon_0 \circ \varphi_1^{-1}$ is an augmentation of the obstruction algebra of $(\widetilde{L}_0, \widetilde{J})$.

Proof. – By Lemma 4.11 there is an isomorphism of complexes

$$\mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) \cong \mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (L_1, \varepsilon_1); \mathbf{0}, \widetilde{J}).$$

By definition the obstruction algebras of $\widetilde{L}_0$ and L_1 are isomorphic to the Chekanov-Eliashberg algebras of $\widetilde{L}_0^+$ and L_1^+. As the chain complexes $\mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (L_1, \varepsilon_1); \mathbf{0}, \widetilde{J})$ and $\mathrm{LCC}_{\widetilde{\varepsilon}_0, \varepsilon_1}(\widetilde{L}_0^+, L_1^+; \widetilde{J})$ are both generated by intersection points between $\widetilde{L}_0$ and L_1 it remains only to match the differentials.

For any $i = 1, \ldots, d$ and $\boldsymbol{\zeta} = \{\zeta_0, \ldots, \zeta_d\} \in \widetilde{\mathcal{R}}^d$ there is a biholomorphism $\Delta_{\boldsymbol{\zeta}} \cong Z_{\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1}$, where $\boldsymbol{\zeta}^0 = \{\zeta_1, \ldots, \zeta_{i-1}\}$, $\boldsymbol{\zeta}^1 = \{\zeta_{i+1}, \ldots, \zeta_d\}$, ζ_i is mapped to $s = +\infty$ and ζ_0 is mapped to $s = -\infty$. Such biholomorphisms induce bijections between the moduli spaces defining the boundary of $\mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (L_1, \varepsilon_1); \mathbf{0}, \widetilde{J})$ and the moduli spaces defining the boundary of $\mathrm{LCC}_{\widetilde{\varepsilon}_0, \varepsilon_1}(\widetilde{L}_0^+, L_1^+; \widetilde{J})$. □

4.5. Products

After the work done for the differential, the higher order products can be easily defined. For simplicity, in this section we will consider immersed exact Lagrangian submanifolds $L_0, \ldots, L_d$ which are pairwise transverse and cylindrical over chord generic Legendrian submanifolds. Thus the generators of the Floer complexes will be intersection points, which we will assume to be disjoint from the double points. The routine modifications needed to introduce Hamiltonian functions into the picture are left to the reader.

Given $d \leq 2$ and $l_i \geq 0$ for $i = 0, \ldots, d$ we define

$$\widetilde{\mathcal{R}}^{l_0|\cdots|l_d} = \mathrm{Conf}^{l_0+\cdots+l_d+d+1}(\partial D^2)$$

where $d+1$ points $\zeta_0^m, \ldots, \zeta_d^m$ (ordered counterclockwise) are labeled as *mixed* and the other ones ζ_j^i, with $i = 0, \ldots, d$ and $j = 1, \ldots, l_i$ (ordered counterclockwise and contained in the sector from ζ_i^m to ζ_{i+1}^m) are labeled as *pure*. Given $\boldsymbol{\zeta} \in \widetilde{\mathcal{R}}^{l_0|\cdots|l_d}$, we denote $\Delta_{\boldsymbol{\zeta}} = D^2 \setminus \boldsymbol{\zeta}$. For $i = 0, \ldots, d$ let $\partial_i \Delta_{\boldsymbol{\zeta}}$ be the subset of $\partial \Delta_{\boldsymbol{\zeta}}$ whose closure in ∂D^2 is the counterclockwise arc from ζ_i^m to ζ_{i+1}^m.

We will consider also a (generic) domain dependent almost complex structure $J_\bullet$ such that every J_z, $z \in \Delta_{\boldsymbol{\zeta}}$ satisfies (†), and moreover is of the form (††) at the strip-like ends of the mixed punctures and is constant in a neighborhood of the arcs $\partial_i \Delta_{\boldsymbol{\zeta}}$ outside those strip-like ends.

Finally we define the moduli spaces $\mathfrak{M}_{L_0,\dots,L_d}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d; J_\bullet)$ of pairs $(u, \boldsymbol{\zeta})$ (up to action of $Aut(D^2)$), where:

— $\boldsymbol{\zeta} \in \widetilde{\mathcal{R}}^{l_0|\cdots|l_d}$ and $u\colon\ \Delta_\zeta \to W$ satisfies $du + J_\bullet \circ du \circ i = 0$,

— $u(\partial_i \Delta_{\boldsymbol{\zeta}}) \subset L_i$,

— $\lim_{z \to \zeta_i^m} u(z) = x_i$,

— $\lim_{z \to \zeta_j^i} u(z) = p_j^i$, and

— p_j^i is a negative puncture at ζ_j^i for $i = 0, \dots, d$ and $j = 1, \dots, l_i$.

As usual, we denote by $\mathfrak{M}^0_{L_0,\dots,L_d}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d; J_\bullet)$ the zero-dimensional part of the moduli spaces.

If $\varepsilon_0, \dots, \varepsilon_d$ are augmentations for the corresponding Lagrangian immersions, we define the weighted count

$$\mathfrak{m}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d) = \#\mathfrak{M}^0_{L_0,\dots,L_d}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d; J_\bullet)\varepsilon_0(\mathbf{p}^0)\cdots\varepsilon_d(\mathbf{p}^d)$$

and define a product

$$\mu^d\colon\ \mathrm{CF}(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CF}(L_0, L_1) \to \mathrm{CF}(L_0, L_d), \tag{11}$$

where we wrote $\mathrm{CF}(L_i, L_j)$ instead of $\mathrm{CF}((L_i, \varepsilon_i), (L_j, \varepsilon_j))$ for brevity sake, via the formula

$$\mu^d(x_1, \dots, x_d) = \sum_{x_0 \in L_0 \cap L_d} \sum_{l_i \geq 0} \sum_{p_j^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d) x_0.$$

Remark 4.13. – The operations μ^d satisfy the A_∞ relations.

The following lemma will be useful in Section 8. It is a straightforward corollary of the existence of pseudoholomorphic triangles supplied by Corollary 4.18 below. The only point where we need to take some care is due to the fact that the Weinstein neighborhood considered is immersed.

Lemma 4.14. – *Let (L, ι) be an exact Lagrangian immersion. We extend ι to a symplectic immersion $\iota_*\colon\ (D_\delta T^*L, d\mathbf{q} \wedge d\mathbf{p}) \to (W, d\theta)$. Let (L, ι') be safe isotopic to (L, ι) and, moreover, assume that*

— *there exists a sufficiently C^1-small Morse function $g\colon\ L \to \mathbb{R}$ with local minima e_i, all whose critical points are disjoint from the double points of L, such that $\iota' = \iota_* \circ dg$, and*

— *outside some compact subset, L' is obtained by a small perturbation of L by the negative Reeb flow.*

We will denote $L = (L, \iota)$ *and* $L' = (L, \iota')$. *Then, if* L *admits an augmentation* ε *and* ε' *is the corresponding augmentation of* L', *for every cylindrical exact Lagrangian submanifold* T *such that* $\iota_*^{-1}(T)$ *is a union of cotangent disk fibers, the map*

$$\mu^2(e, \cdot)\colon \mathrm{CF}(T, (L, \varepsilon)) \to \mathrm{CF}(T, (L', \varepsilon')),$$
$$e := \sum_i e_i \in \mathrm{CF}((L, \varepsilon), (L', \varepsilon')),$$

is an isomorphism of complexes for a suitable almost complex structure on W *as in Corollary 4.18.*

In the case where L is closed and embedded, the element e is always a cycle which is nontrivial in homology as was shown by Floer (it is identified with the minimum class in the Morse cohomology of L). In general the following holds.

LEMMA 4.15. – *Under the hypotheses of Lemma 4.14, the element*

$$e \in \mathrm{CF}((L, \varepsilon), (L', \varepsilon'))$$

is a cycle. Furthermore, e is a boundary if and only if $\mathrm{CF}(T, (L, \varepsilon)) = 0$ *for every Lagrangian* T.

Proof. – The assumption that the augmentation ε' is identified canonically with the augmentation ε implies that e is a cycle by the count of pseudoholomorphic disks with a negative puncture at e supplied by Lemma 4.17.

Assume that $\partial E = e$. The last property is then an algebraic consequence of the Leibniz rule $\partial \mu^2(E, x) = \mu^2(\partial E, x)$ in the case where $\partial x = 0$, combined with the fact that $\mu^2(e, \cdot)$ is a quasi-isomorphism as established by the previous lemma. □

Later it will be useful to switch perspectives slightly, and instead of with the chain e, work with an augmentation induced by that chain. In general, given (L_0, ε_0), (L_1, ε_1) and a chain $c \in \mathrm{CF}((L_0, \varepsilon_0), (L_0, \varepsilon_1))$ we can consider Legendrian lifts L_0^+ and L_1^+ such that L_0^+ is above L_1^+ and the unital algebra morphism $\varepsilon_c\colon \mathfrak{A}(L_0^+ \cup L_1^+) \to \mathbb{F}$ uniquely determined

$$\varepsilon_c(x)\colon eqq \begin{cases} \varepsilon_i(x), & \text{if } x \in \mathfrak{A}(L_i), \\ \langle c, x \rangle, & \text{if } x \in L_0 \cap L_1, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the coefficient of x in c.

LEMMA 4.16. – *The element*

$$c \in \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); \mathbf{0}, J)$$

is a cycle if and only if

$$\varepsilon_c\colon \mathfrak{A}(L_0^+ \cup L_1^+) \to \mathbb{F}$$

is an augmentation, where the almost complex structure J *has been used to define the latter algebra as well, and the Legendrian lifts have been chosen so that no Reeb chord starts on* L_0^+ *and ends on* L_1^+.

Proof. – Note that the Floer complex under consideration has a differential which counts J-pseudoholomorphic strips, and that the obstruction algebra has a differential counting pseudoholomorphic disks with at least one boundary puncture. Identifying the appropriate counts of disks, the statement can be seen to follow by pure algebra, together with the fact that the differential of the DGA counts punctured pseudoholomorphic disks, and thus respects the filtration induced by the different components. The crucial property that is needed here is that, under the assumptions made on the Legendrian lifts, the differential of the Chekanov-Eliashberg algebra applied to a mixed chord is a sum of words, each of which contains precisely one mixed chord. □

4.6. Existence of triangles

In this section we prove an existence result for small pseudoholomorphic triangles with boundary on an exact Lagrangian cobordism and a small push-off. The existence of these triangles can be deduced as a consequence of the fact that the wrapped Fukaya category is homologically unital. Here we take a more direct approach based upon the adiabatic limit of pseudoholomorphic disks on a Lagrangian and its push-off from [16]; when the latter push-off becomes sufficiently small, these disks converge to pseudoholomorphic disks on the single Lagrangian with gradient-flow lines attached (called *generalized pseudoholomorphic disks* in the same paper).

Let $L \subset W$ be an exact immersed Lagrangian submanifold with cylindrical end. We recall that, as usual, we assume that every immersed Lagrangian submanifold is nice. Consider the Hamiltonian push-off $L_{\epsilon f}$, which we require to be again an exact immersed Lagrangian submanifold with cylindrical end, which is identified with the graph of $d(\epsilon f)$ for a Morse function $f\colon L \to \mathbb{R}$ inside a Weinstein neighborhood $(T^*_\delta L, -d(\mathbf{p}d\mathbf{q})) \looparrowright (W, d\theta)$ of L. We further assume that $df(\mathcal{L}) > 0$ outside a compact subset. (The assumption that the push-off is cylindrical at infinity does of course impose additional constraints on the precise behavior of the Morse function outside a compact subset.)

Now consider a Legendrian lift $L^+ \cup L^+_{\epsilon f}$ for which $(L_{\epsilon f})^+$ is above L^+. For $\epsilon > 0$ sufficiently small, it is the case that $L \cup L_{\epsilon f}$ again has only transverse double points. Moreover, the Reeb chords on the Legendrian lift can be classified as follows, using the notation from [16, Section 3.1]:

- Reeb chords $\mathcal{Q}(L) \cong \mathcal{Q}(L_{\epsilon f})$ on the lifts of L and $L_{\epsilon f}$ respectively, which stand in a canonical bijection;
- Reeb chords $\mathcal{C}$ being in a canonical bijection with the critical points of f; and
- two sets $\mathcal{Q}$ and $\mathcal{P}$ of Reeb chords from L to $L_{\epsilon f}$, each in canonical bijection with $\mathcal{Q}(L)$, and where the length of any Reeb chord in $\mathcal{Q}$ (resp. $\mathcal{P}$) is greater (resp. smaller) than the one of the corresponding chord in $\mathcal{Q}(L)$.

See the aforementioned reference for more details, as well as Figure 1.

Lemma 4.17 ([16]). – *For a suitable generic Riemannian metric g on L for which (f, g) constitutes a Morse-Smale pair and associated almost complex structure, which can be made to coincide with an arbitrary cylindrical almost complex structure outside a compact subset, there is a bijection between the set of pseudoholomorphic disks which have*

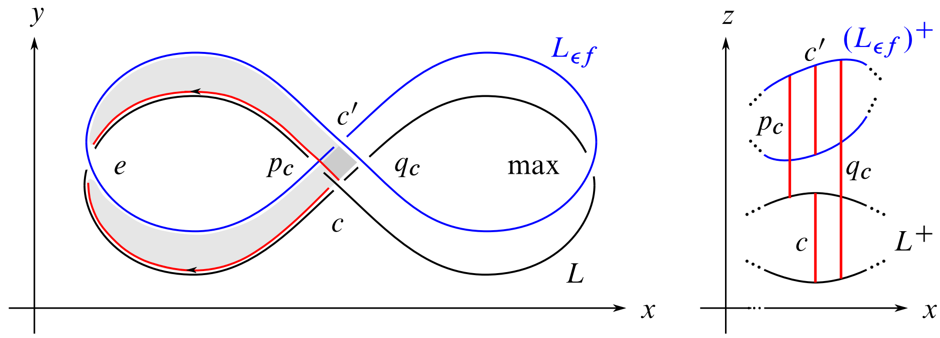

FIGURE 1. The small triangles on the two-copy living near gradient flow-lines of $-\nabla f$ shown in red. The upper copy with respect to the z-coordinate is shown in blue, while the lower copy is in black. The contact form used here is $dz - ydx$.

— *boundary on $L^+ \cup L^+_{\epsilon f}$ and precisely one positive puncture,*

— *at least one negative puncture at a local minimum $e \in \mathcal{C}$ of f, and*

— *form a moduli space of expected dimension zero,*

and the set of negative gradient flow-lines on (L, g) that connect either the starting point or the end point of a Reeb chord $c \in \mathcal{Q}(L)$ with the local minimum e, together with the set of negative gradient flow-lines that connect some critical point of index one with e.

More precisely, each such pseudoholomorphic disk lives in a small neighborhood of the aforementioned flow-line. In the first case, it is a triangle with a positive puncture at the Reeb chord $q_c \in \mathcal{Q}$ corresponding to c, and its additional negative punctures at e and either c (for the flow-line from the starting point *of c) or c' (for the flow-line from the* endpoint *of c); see Figure 1. In the second case, it is a Floer strip corresponding to the negative gradient flow-line connecting the saddle point and the local minimum.*

Proof. – This is an immediate application of Parts (3) and (4) of [16, Theorem 5.5]. A generalized pseudoholomorphic disk with a negative puncture at a local minimum can be rigid only if it is a flow-line connecting a saddle point to the minimum, or consists of a constant pseudoholomorphic disk located at one of the Reeb chords at $c \in \mathcal{Q}(L)$ together with a flow-line from that double point to the local minimum. The aforementioned result gives a bijection between such generalized pseudoholomorphic disks and pseudoholomorphic strips and triangles on the two-copy. □

Now consider an auxiliary exact immersed Lagrangian L' intersecting $L \cup L_{\epsilon f}$ transversely. For $\epsilon > 0$ sufficiently small there is a bijection between the intersection points $L \cap L'$ and $L_{\epsilon f} \cap L'$.

COROLLARY 4.18. – *For a suitable Morse-Smale pair (f, g) and almost complex structure as in Lemma 4.17 there is a unique rigid and transversely cut out pseudoholomorphic triangle with corners at $e \in L \cap L_{\epsilon f}$, $c \in L \cap L'$, and the corresponding double point $c' \in L_{\epsilon f} \cap L'$ for any connected gradient flow-line from $c \in L \cap L'$ to the local minimum $e \in \mathcal{C}$. The triangle is moreover contained inside a small neighborhood of the same flow-line.*

Proof. – We need to apply Lemma 4.17 in the case when L^+ is taken to be the Legendrian lift $(L\cup L')^+$, where $(L')^+$ is above L^+, and the push-off is taken to be $(L\cup L')^+_{\epsilon F}$ for a Morse function $F\colon L\cup L'\to\mathbb{R}$ that restricts to f on L. □

5. Continuation maps

In this section we analyze what happens to the Floer cohomology when we change J, H (in some suitable way) or move the Lagrangian submanifolds by a compactly supported safe exact isotopy.

5.1. Changing the almost complex structure

Following [18] (see also [20]) we will use the bifurcation method to prove invariance of Floer cohomology for Lagrangian intersection under change of almost complex structure. It seems, in fact, that the more usual continuation method is not well suited to describe how the obstruction algebras change when the almost complex structure changes.

Let us fix Lagrangian immersions (L_0,ι_0) and (L_1,ι_1) and a cylindrical Hamiltonian H compatible with L_0 and L_1. For a generic one-parameter family of time-dependent almost complex structure $J_\bullet^\bullet$ parametrized by an interval $[\delta_-,\delta_+]$ such that

— the extrema $J_\bullet^{\delta_-}$ and $J_\bullet^{\delta_+}$ are (L_0,L_1)-compatible, and

— $J_\bullet^{\delta}$ satisfies (††) for all $\delta\in[\delta_-,\delta_+]$

we define the *parametrized* moduli spaces

$$\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H,J_\bullet^\bullet)$$

consisting of pairs (δ,u) such that $\delta\in[\delta_-,\delta_+]$ and

$$u\in\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H,J_\bullet^\delta).$$

Using the zero-dimensional parametrized moduli spaces, we will define a continuation map

$$\Upsilon_{J_\bullet^\bullet}\colon \mathrm{LCC}((L_0,\varepsilon_0^+),(L_1,\varepsilon_1^+);H,J_\bullet^{\delta_+})\to \mathrm{LCC}((L_0,\varepsilon_0^-),(L_1,\varepsilon_1^-);H,J_\bullet^{\delta_-}).$$

PROPOSITION 5.1. – *For a generic one-parameter family $J_\bullet^\bullet$ of time-dependent almost complex structures as above, the parametrized moduli space*

$$\mathfrak{M}^k_{L_0,L_1}(\mathbf{q}^1,y_-,\mathbf{q}^0,y_+;H,J_\bullet^\bullet)$$

is a transversely cut-out manifold of dimension k. If $k=0$ it is compact, and therefore consists of a finite number of points. If $k=1$ it can be compactified in the sense of Gromov-Floer.

If both L_0 and L_1 are spin, the moduli spaces are orientable, and a choice of spin structure on each Lagrangian submanifold induces a coherent orientation on the parametrized moduli spaces.

In the following lemma we look more closely at the structure of the zero-dimensional parametrized moduli spaces. The analogous statement in the setting of Lagrangian Floer homology (for Lagrangian submanifolds) appears in [20, Section 3]. In the case of Legendrian contact cohomology, the corresponding construction appears in [18, Section 2.4].

LEMMA 5.2. – *For a generic $J_\bullet^\bullet$ there is a finite set $\Delta \subset (\delta_-, \delta_+)$ such that for $\delta \in \Delta$ exactly one of the following cases holds:*

(i) *there is a unique nonempty moduli space $\mathfrak{N}^{-1}_{L_0}(q_0^0; q_1^0, \dots, q_d^0; J_0^\delta)$ and all other moduli spaces are transversely cut out,*

(ii) *there is a unique nonempty moduli space $\mathfrak{N}^{-1}_{L_1}(q_0^1; q_1^1, \dots, q_d^1; J_1^\delta)$ and all other moduli spaces are transversely cut out, or*

(iii) *there is a unique nonempty moduli space $\mathfrak{M}^0_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\delta)$ and all other moduli spaces are transversely cut out,*

while for every $\delta \in [\delta_-, \delta_+] \setminus \Delta$ the moduli spaces of negative virtual dimension are empty.

(Of course, the self-intersection points q_j^i appearing in the three cases above have nothing to do with each other.) Note that the lemma does not claim that $J_\bullet^\delta$ is (L_0, L_1)-regular for $\delta \notin \Delta$: for example, if δ is a critical value of the projection

$$\mathfrak{M}^1_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\bullet) \to [\delta_-, \delta_+],$$

then $J_\bullet^\delta$ is not (L_0, L_1)-regular, but $\delta \notin \Delta$.

REMARK 5.3. – If $\Delta = \emptyset$, then the Floer chain complexes

$$\mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^{\delta_-}) \text{ and } \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^{\delta_+})$$

are isomorphic. In fact the one-dimensional parametrized moduli spaces have boundary points only at $J_\bullet^{\delta_-}$ and $J_\bullet^{\delta_+}$, and this implies that the algebraic count of elements of the zero-dimensional moduli spaces is the same for $J_\bullet^{\delta_-}$ and $J_\bullet^{\delta_+}$.

Next we describe what happens when we cross $\delta \in \Delta$ of type (i) or (ii). Since the cases are symmetric, we will describe only (i) and assume, without loss of generality, that $\delta = 0$.

LEMMA 5.4. – *Suppose that $\Delta = \{0\}$ and that the unique nontransversely cut out moduli space for $J_\bullet^0$ is $\mathfrak{N}^{-1}_{L_0}(q_0^0; q_1^0, \dots, q_d^0; J_0^0)$. Then for $\delta > 0$ the differentials of*

$$\mathrm{CF}((L_0, \varepsilon_0^-), (L_1, \varepsilon_1^-); H, J_\bullet^{-\delta}) \text{ and } \mathrm{CF}((L_0, \varepsilon_0^+), (L_1, \varepsilon_1^+); H, J_\bullet^{\delta})$$

are equal.

Proof. – We adapt the cobordism method of [17] (see also [18]). Given a positive Morse function $f\colon \mathbb{R} \to \mathbb{R}$ with local minima at ± 1 satisfying $f(1) = f(-1) = 1$, a local maximum at 0 and no other critical point, we define exact Lagrangian immersions $\widetilde{\iota}_i\colon \mathbb{R} \times L_i \to T^*\mathbb{R} \times W$ following [17, Subsection 4.3.2]. We denote $\widetilde{L}_i$ the image of $\widetilde{\iota}_i$.

Let (r, ρ) be the canonical coordinates on $T^*\mathbb{R}$. We consder the Hamiltonian function

$$\widetilde{H}\colon [0,1] \times T^*\mathbb{R} \times W \to \mathbb{R}$$

such that $\widetilde{H}(t, r, \rho, w) = H(t, w)$. Then its Hamiltonian vector field $X_{\widetilde{H}}$ is tangent to $\{(r, \rho)\} \times W$ for all $(r, \rho) \in T^*\mathbb{R}$. Each double point q of L_i gives rise to three double points $q[-1], q[0], q[+1]$ of $\widetilde{L}_i$ and each Hamiltonian chord x from L_0 to L_1 gives rise to three Hamiltonian chords $x[-1], x[-1], x[+1]$ from $\widetilde{L}_0$ to $\widetilde{L}_1$.

Let $\widetilde{J}_\bullet$ be the (time dependent) almost complex structure on $T^*\mathbb{R} \times W$ such that on $T_{(r,\rho,w)} \cong \mathbb{R}^2 \oplus T_w W$ it is given by $j_0 \oplus J_\bullet^{\alpha_\delta(r)}$ where j_0 is the standard almost complex structure on $\mathbb{R}^2$ and $\alpha_\delta\colon \mathbb{R} \to [-\delta, \delta]$ is as in Equation (4.9) of [17] for δ small. Proposition 4.7 provides isomorphisms

$$\mathfrak{Y}_i\colon (\mathfrak{D}_i^-, \mathfrak{d}_i^-) \to (\mathfrak{D}_i^+, \mathfrak{d}_i^+)$$

such that $\varepsilon_i^- = \varepsilon_i^+ \circ \mathfrak{Y}_i$. Those isomorphisms are the ones constructed in [17, Subsection 4.4.3] and denoted by ψ there. Let $\widetilde{\mathfrak{D}}_i$ be the obstruction algebra of $\widetilde{L}_i$ and define $\widetilde{\varepsilon}_i\colon \widetilde{\mathfrak{D}}_i \to \mathbb{F}$ as

$$\begin{cases} \widetilde{\varepsilon}_i(q[-1]) = \varepsilon_-(q), \\ \widetilde{\varepsilon}_i(q[0]) = 0, \\ \widetilde{\varepsilon}_i(q[+1]) = \varepsilon_+(q) \end{cases}$$

for every double point q of L_i. From the structure of the differential of the Chekanov-Eliashberg algebras of $\widetilde{L}_i$ described in [17, Subsection 4.4.3] it follows that $\widetilde{\varepsilon}_i$ is an augmentation of $\widetilde{\mathfrak{D}}_i$, and therefore the Floer chain complex $\mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (\widetilde{L}_1, \widetilde{\varepsilon}_1); \widetilde{H}, \widetilde{J}_\bullet)$ is well defined.

Lemmas 4.14, 4.15, 4.18 and 4.19 of [17] have direct counterparts for Floer solutions because the projection of a solution of the Floer equation with Hamiltonian $\widetilde{H}$ to $T^*\mathbb{R}$ is a holomorphic map. This implies that the differential $\widetilde{\partial}$ on $\mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (\widetilde{L}_1, \widetilde{\varepsilon}_1); \widetilde{H}, \widetilde{J}_\bullet)$ has the following form

$$\begin{cases} \widetilde{\partial}(x[-1]) = (\partial_- x)[-1], \\ \widetilde{\partial}(x[0]) = x[+1] - x[-1] + \sum_{y \in \mathcal{C}_H} a_y y[0], \\ \widetilde{\partial}(x[+1]) = (\partial_+ x)[+1], \end{cases} \tag{12}$$

where $\partial_\pm$ denotes the differentials of $\mathrm{CF}((L_0, \varepsilon_0^\pm), (L_1, \varepsilon_1^\pm); H, J_\bullet^{\pm\delta})$. The crucial point here is that no chord $y[\pm 1]$ contributes to $\widetilde{\partial}(x[0])$ if $y \neq x$. This is a consequence of the assumption that there is no Floer strip of index 0 for $J_\bullet^\bullet$ and of the Hamiltonian version of [17, Lemma 4.19]. From $\widetilde{\partial}^2 = 0$ it is easy to see that $\partial_+ = \partial_-$. □

Now we analyze how the complex changes when we cross $\delta \in \Delta$ of type (iii).

Lemma 5.5. – *Suppose that $\Delta = \{\delta_0\}$ and that the unique nontransversely cut out moduli space for $J_\bullet^{\delta_0}$ is $\mathfrak{M}^0_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\delta)$. Then the map*

$$\Upsilon_{J_\bullet^\bullet}\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^{\delta+}) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^{\delta-})$$

defined as

$$\Upsilon_{J_\bullet^\bullet}(x) = \begin{cases} x & \text{if } x \neq y_+, \\ y_+ + \#\mathfrak{M}^0_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\bullet)\varepsilon_0(\mathbf{q}^0)\varepsilon_1(\mathbf{q}^1) y_- & \text{if } x = y_+ \end{cases}$$

is an isomorphism of complexes.

Proof. – The proof is the same as in [20]. However, the proof in [20] holds only in the case of $\mathbb{Z}_2$-coefficients. For more general coefficients, we rely on the discussion in [17, 18]. □

Given a generic homotopy $J_\bullet^\bullet$, we split it into pieces containing only one point of Δ and compose the maps obtained in Lemma 5.4 and 5.5.

5.2. Changing the Hamiltonian

In this section we will keep the almost complex structure fixed. Let H_- and H_+ be time-dependent cylindrical Hamiltonian functions which are compatible with immersed Lagrangian submanifolds L_0 and L_1. From a one-parameter family of cylindrical Hamiltonians H_s such that

(i) $H_s = H_-$ for $s \ll 0$,

(ii) $H_s = H_+$ for all $s \gg 0$, and

(iii) $\partial_s h'_s(e^{\mathfrak{r}(w)}) \leq 0$ if $\mathfrak{r}(w)$ is sufficiently large,

we will define a continuation map

$$\Phi_{H_s}\colon \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_+) \to \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_-).$$

Given a time-dependent almost complex structure $J_\bullet$, an H_--Hamiltonian chord x_-, an H_+-Hamiltonian chord x_+ and double points $\mathbf{p}^0 = (p^0_1,\dots,p^0_{l_0})$ of L_0 and $\mathbf{p}^1 = (p^1_1,\dots,p^1_{l_1})$ of L_1, we define the moduli spaces

$$\mathfrak{M}_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H_s,J_\bullet)$$

as the set of triples $(\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1,u)$ such that:

— $(\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1) \in \widetilde{\mathcal{R}}^{l_0|l_1}$ and $u\colon Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1} \to W$ is a map satisfying the Floer equation

$$\frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - \chi'(t)X_{H_s}(\chi(t),u)\right) = 0, \tag{13}$$

— $\lim_{s\to\pm\infty} u(s,t) = x_\pm(\chi(t))$,

— $u(s,0) \in L_0$ for all $(s,0) \in Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1}$,

— $u(s,1) \in L_1$ for all $(s,1) \in Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1}$, and

— each ζ^i_j is a negative puncture at p^i_j for $i = 0,1$ and $j = 1,\dots,l_i$.

Note the only difference between Equation (13) and Equation (8) is that we made X_{H_s} depend on s in Equation (13). For this reason there is no action of $\mathrm{Aut}(Z)$ on the moduli spaces $\mathfrak{M}_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H_s,J_\bullet)$.

Let F_u be the linearisation at u of the Floer operator with s-dependent Hamiltonian. We define

$$\mathrm{ind}(u) = \mathrm{ind}(F_u) + l_0 + l_1,$$

and define $\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^0,x_-,\mathbf{p}^1,x_+;H_s,J_\bullet)$ as the subset of

$$\mathfrak{M}_{L_0,L_1}(\mathbf{p}^1,x_-,\mathbf{p}^0,x_+;H_s,J_\bullet)$$

consisting of the maps u with $\mathrm{ind}(u) = k$.

The following statement is analogous to the statement in Morse theory, [20, Section 3]. A similar boundary degeneration statement in the case of Legendrian contact cohomology appears in [18, Section 2.4].

PROPOSITION 5.6. – *Given H_s, for a generic time-dependent almost complex structure $J_\bullet$ satisfying (††) with respect to both H_+ and H_-, the moduli space $\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$ is a transversely cut-out manifold of dimension k. If $k = 0$ it is compact, and therefore consists of a finite set of points. If $k = 1$ it can be compactified in the sense of Gromov-Floer.*

If both L_0 and L_1 are spin, the choice of a spin structure on each induces a coherent orientation of the moduli space (see [17]*).*

We denote $\mathcal{C}_-$ the set of Hamiltonian chords of H_- and $\mathcal{C}_+$ the set of Hamiltonian chords of H_+. We also introduce the weighted count

$$\mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s) = \#\mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)\varepsilon_0(\mathbf{p}^0)\varepsilon_1(\mathbf{p}^1).$$

Given $x_+ \in \mathcal{C}_+$, we define the continuation map

$$\Phi_{H_s}(x_+) = \sum_{x_- \in \mathcal{C}_-} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s)x_-. \tag{14}$$

The Gromov-Floer compactification of the one-dimensional moduli spaces implies the following lemma.

LEMMA 5.7. – *The map Φ_{H_s} is a chain map.*

We denote by

$$\Phi^*_{H_-,H_+}\colon \mathrm{HF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_+) \to \mathrm{HF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_-)$$

the map induced in homology by Φ_{H_s}—soon it will be apparent that the notation is justified. As it happens in the more standard Floer cohomology for Lagrangian submanifolds, the continuation maps satisfy the following properties.

LEMMA 5.8. – 1. *Up to homotopy, Φ_{H_s} depends only on the endpoints H_+ and H_- of H_s,*

2. *$\Phi^*_{H,H}$ is the identity for every H, and*
3. *$\Phi^*_{H_-,H} \circ \Phi^*_{H,H_+} = \Phi^*_{H_-,H_+}$.*

Sketch of proof. – In order to prove (1.), we follow the standard procedure for defining chain homotopies in Floer theory; see [20] for more details. Given a homotopy H^δ_s, $\delta \in [0,1]$, between s-dependent Hamiltonian functions H^0_s and H^1_s with $H^\delta_s \equiv H_-$ for $s \ll 0$ and $H^\delta_s \equiv H_+$ for $s \gg 0$, we define the parametrized moduli spaces $\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^\bullet_s, J_\bullet)$ of pairs (δ, u) such that $\delta \in [0,1]$ and $u \in \mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^\delta_s, J_\bullet)$. We define the weighted count

$$\mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^\bullet_s) = \#\mathfrak{M}^{-1}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^\bullet_s, J_\bullet)\varepsilon_0(\mathbf{p}^0)\varepsilon_1(\mathbf{p}^1).$$

Then the chain homotopy

$$K\colon \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_+,J_\bullet) \to \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_-,J_\bullet)$$

between $\Phi_{H^0_s}$ and $\Phi_{H^1_s}$ is defined as

$$K(x_+) = \sum_{x_- \in \mathcal{C}_-} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^\bullet_s)x_-.$$

In order to prove (2.) we can choose $H_s \equiv H$: then the moduli space

$$\mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^0, x_-, \mathbf{p}^1, x_+; H_s, J_\bullet)$$

consists of constant strips.

We fix s-dependent Hamiltonian functions H_s^+ and H_s^- such that $H_s^+ = H_+$ for $s \geq 1$ and $H_s^+ = H$ for $s \leq 0$, and $H_s^- = H$ for $s \geq 0$ and $H_s^- = H_-$ for $s \leq -1$. In order to prove (3.) we introduce the family of Hamiltonian functions

$$H_s^R = \begin{cases} H^+_{s-R} & \text{for } s \geq 0, \text{ and} \\ H_{s+R} & \text{for } s \leq 0 \end{cases}$$

with $R > 0$. By (1.), $\Phi_{H_s^R}$ induces $\Phi^*_{H_+,H_-}$ for all R. For $R \gg 0$ there is an identification

$$\begin{aligned} (15) \qquad & \mathfrak{M}^0_{L_0,L_1}(p^1_1, \dots, p^1_{l_1}, x_-, p^0_1, \dots, p^0_{l_0}, x_+; H^R_s, J_\bullet) \\ & \cong \coprod_{x \in \mathcal{C}_H} \coprod_{0 \leq h_i \leq l_i} \mathfrak{M}^0_{L_0,L_1}(p^1_{h_1+1}, \dots, p^1_{l_1}, x, p^0_1, \dots, p^0_{h_0}, x_+; H^+_s, J_\bullet) \\ & \times \mathfrak{M}^0_{L_0,L_1}(p^1_1, \dots, p^1_{h_1}, x_-, p^0_{1+h_0}, \dots, p^0_{l_0}, x; H^-_s, J_\bullet), \end{aligned}$$

which follows from standard compactness and gluing techniques, once we know that, for any $R' > 0$, there is R_0 such that, for all $R \geq R_0$, if

$$(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1, u) \in \mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^R_s, J_\bullet),$$

then $\zeta^i_j \notin [-R', R']$ for $i = 0, 1$ and $j = 1, \dots, l_i$.

This follows from a simple compactness argument: if there is R' and a sequence R_n with

$$(\boldsymbol{\zeta}^0_n, \boldsymbol{\zeta}^1_n, u_n) \in \mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H^{R_n}_s, J_\bullet)$$

and for every n there is some $\zeta^i_j \in [-R', R']$, then the limit for $n \to \infty$ has one level which is a solution of a Floer equation with s-invariant data and at least one boundary puncture. For index reasons this level must have index zero, but it cannot be constant because of the boundary puncture. This is a contradiction. □

With Lemma 5.8 at hand, we can prove the following invariance property in the usual formal way.

COROLLARY 5.9. – *If H_0 and H_1 are cylindrical Hamiltonian functions which are compatible with L_0 and L_1 and such that $h_0'(\mathfrak{r}(w)) = h_1'(\mathfrak{r}(w))$ for w outside of a compact set, then the continuation map*

$$\Phi^*_{H_0,H_1} \colon \mathrm{HF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_0) \to \mathrm{HF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_1)$$

is an isomorphism.

5.3. Compactly supported safe isotopies

Let $\psi_t\colon W \to W$ be a compactly supported smooth isotopy such that $\iota_t = \psi_t \circ \iota_1\colon L_1 \to W$ is a safe isotopy. By Lemma 2.23 there exists a local Hamiltonian G_t defined on L_1 which generates the ι_t and for which dG_t has compact support. (Recall that G_t may not extend to a single-valued function on W.)

In the following we will make the further assumption that the path

$$(\psi_t)_* J_1 = d\psi_t \circ J_1 \circ d\psi_t^{-1},\ \ t \in [0,1],$$

consists of compatible almost complex structures. This will cause no restriction, since we only need the case when ψ_t is equal to the Liouville flow, which is conformally symplectic.

REMARK 5.10. – In the following manner more general safe isotopies can be considered. Since it is possible to present any smooth isotopy as a concatenation of C^2-small isotopies, it then suffices to carry out the constructions here for each step separately. Namely, since tameness is an open condition, sufficiently C^2-small isotopies may be assumed to preserve any given tame almost complex structure. Further control near the double points can then be obtained by assuming that ψ_t actually is conformally symplectic there, which can be assumed without loss of generality.

Denote by $L'_1 = \psi_1 \circ \iota_1(L_1)$ the image. By the usual abuse of notation, we will write L'_1 or $\psi_1(L_1)$ instead of $(L_1, \psi_1 \circ \iota)$. From now on we will assume that the Hamiltonian H is compatible both with L_0 and L_1 and with L_0 and L'_1.

The obstruction algebras $\mathfrak{D}_1$ of (L_1, J_1) and $\mathfrak{D}'_1$ of $(L'_1, (\psi_1)_* J_1)$ are tautologically isomorphic because ψ_1 matches the generators and the holomorphic curves contributing to the differentials, and therefore any augmentation ε_1 of $\mathfrak{D}_1$ corresponds to an augmentation ε'_1 of $\mathfrak{D}'_1$.

We fix time-dependent almost complex structures $J^+_\bullet$ and $J^-_\bullet$ such that

— $J^\pm_t = J_0$ for $t \in [0, 1/4]$,

— $J^+_t = J_1$ and $J^-_t = (\psi_1)_* J_1$ for $t \in [3/4, 1]$,

— $J^+_\bullet$ is (L_0, L_1)-regular and $J^-_\bullet$ is (L_0, L'_1)-regular.

Given augmentations ε_0 for L_0 and ε_1 for L_1, we will define a chain map

$$\Psi_G\colon \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1); H, J^+_\bullet) \to \mathrm{CF}((L_0,\varepsilon_0),(L'_1,\varepsilon'_1); H, J^-_\bullet)$$

using a Floer equation with moving boundary conditions. The presence of self-intersection points of L_1 makes the construction of the moduli spaces more subtle than in the usual case because, in order to have strip-like ends, we need to make the moving boundary conditions constant near the boundary punctures, and therefore domain dependent.

Recall the sets

$$\mathrm{Conf}^n(\mathbb{R}) = \{(\zeta_1,\dots,\zeta_n) \in \mathbb{R}^n \,|\, \zeta_1 < \dots < \zeta_n\} \quad \text{and}$$
$$\overline{\mathrm{Conf}}^n(\mathbb{R}) = \{(\zeta_1,\dots,\zeta_n) \in \mathbb{R}^n \,|\, \zeta_1 \le \dots \le \zeta_n\}.$$

(Note that this is not how configuration spaces are usually compactified.) Given $n \in \mathbb{N}$, we denote $\mathbf{n} = \{1,\dots,n\}$ and for $m < n$ we denote $\hom(\mathbf{n},\mathbf{m})$ the set of nondecreasing and surjective function $\phi\colon \mathbf{n} \to \mathbf{m}$. Every $\phi \in \hom(\mathbf{n}, \mathbf{m})$ induces an embedding $\phi^*\colon \overline{\mathrm{Conf}}^m(\mathbb{R}) \to \overline{\mathrm{Conf}}^n(\mathbb{R})$ defined by

$$\phi^*(\zeta_1, \dots, \zeta_m) = (\zeta_{\phi(1)}, \dots, \zeta_{\phi(m)}).$$

The boundary of $\overline{\mathrm{Conf}}^n(\mathbb{R})$ is a stratified space with dimension m stratum

$$\bigsqcup_{\phi \in \hom(\mathbf{n},\mathbf{m})} \phi^*(\mathrm{Conf}^m(\mathbb{R})).$$

The embeddings ϕ defined above extend to diffeomorphisms

$$\overline{\phi}^*\colon \overline{\mathrm{Conf}}^m(\mathbb{R}) \times \mathbb{R}_+^{n-m} \to \overline{\mathrm{Conf}}^n(\mathbb{R})$$

such that

$$(\zeta_1', \dots, \zeta_n') = \overline{\phi}^*((\zeta_1, \dots, \zeta_m), (\epsilon_1, \dots, \epsilon_{n-m}))$$

if

$$\zeta_i' = \zeta_{\phi(i)} + \sum_{k=0}^{i-\phi(i)} \epsilon_k,$$

where $\epsilon_0 = 0$ for the sake of the formula.

LEMMA 5.11. – *Fix $\delta > 0$. There is a family of constants $\kappa_n > 0$ and smooth functions*

$$\nu_n\colon \overline{\mathrm{Conf}}^n(\mathbb{R}) \times \mathbb{R} \to [0,1]$$

such that, denoting by s the coordinate in the second factor,

1. $\nu_n(\zeta_1, \dots, \zeta_n, s) = 0$ *for $s > \kappa_n$,*
2. $\nu_n(\zeta_1, \dots, \zeta_n, s) = 1$ *for $s < -\kappa_n$,*
3. $\partial_s \nu_n(\zeta_1, \dots, \zeta_n, s) \in [-2, 0]$ *for all* $(\zeta_1, \dots, \zeta_n, s) \in \overline{\mathrm{Conf}}^n(\mathbb{R}) \times \mathbb{R}$,
4. $\partial_s \nu_n(\zeta_1, \dots, \zeta_n, s) = 0$ *if $|s - \zeta_i| \leq \frac{\delta}{2}$ for some $i = 1, \dots, n$, and*
5. $\nu_n \circ \phi^* = \nu_m$ *for all $m < n$ and all* $\phi \in \hom(\mathbf{n}, \mathbf{m})$.

Proof. – We can construct the sequences κ_n and ν_n inductively over n using the fact that the set of functions satisfying (1)–(4) is convex. ☐

Given $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n) \in \mathrm{Conf}^n(\mathbb{R})$, we define $\nu_{\boldsymbol{\zeta}}\colon \mathbb{R} \to [0,1]$ by $\nu_{\boldsymbol{\zeta}}(s) = \nu_n(\zeta_1, \dots, \zeta_n, s)$.

LEMMA 5.12. – *For every n, there is a contractible set of smooth maps*

$$\tilde{J}_n\colon \overline{\mathrm{Conf}}^n(\mathbb{R}) \times Z \to \mathcal{J}(\theta)$$

such that

1. $\tilde{J}_n(\boldsymbol{\zeta}, s, t) = J_t^+$ *if $s > \kappa_n + 1$,*
2. $\tilde{J}_n(\boldsymbol{\zeta}, s, t) = J_t^-$ *if $s < -\kappa_n - 1$,*
3. $\tilde{J}_n(\boldsymbol{\zeta}, s, t) = J_0$ *if* $t \in [0, 1/4]$,
4. $\tilde{J}_n(\boldsymbol{\zeta}, s, t) = d\psi_{\nu_{\boldsymbol{\zeta}}(s)} \circ J_1 \circ d\psi^{-1}_{\nu_{\boldsymbol{\zeta}}(s)}$ *if* $t \in [3/4, 1]$, *and*
5. *for all* $\phi \in \hom(\mathbf{n}, \mathbf{n-1})$, $\tilde{J}_n(\phi^*(\boldsymbol{\zeta}), s, t) = \tilde{J}_{n-1}(\boldsymbol{\zeta}, s, t)$.

Proof. – We build $\tilde{J}_n$ inductively on n. At each step, the map $\tilde{J}$ is determined in the complement of $\mathrm{Conf}^n(\mathbb{R}) \times [-\kappa_n - 1, \kappa_n + 1] \times [1/4, 3/4]$. We can extend it to $\overline{\mathrm{Conf}}^n(\mathbb{R}) \times Z$ because $\mathcal{J}(\theta)$ is contractible. □

Given $\boldsymbol{\zeta} \in \mathrm{Conf}^n(\mathbb{R})$, we will denote by $\tilde{J}_{\boldsymbol{\zeta}}$ the $s-$ and t-dependent almost complex structure obtained by restricting $\tilde{J}$ to $\{\boldsymbol{\zeta}\} \times Z$. Given $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \widetilde{\mathcal{R}}^{l_0|l_1}$, we will not distinguish between $\zeta^1_j \in \mathbb{R} \times \{1\}$ and its s-coordinate, and by this abuse of notation, to $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \widetilde{\mathcal{R}}^{l_0|l_1}$ we will associate $\nu_{\boldsymbol{\zeta}^1}$ and $\tilde{J}_{\boldsymbol{\zeta}^1}$. For simplicity, the s- and t-dependence of $\tilde{J}_{\boldsymbol{\zeta}^1}$ will be omitted in writing the Floer equation.

Consider the sets

$$\mathcal{C}_H = \{x\colon [0,1] \to W : x(0) \in L_0, x(1) \in L_1, \dot{x}(t) = X_H(x(t))\},$$
$$\mathcal{C}'_H = \{x\colon [0,1] \to W : x(0) \in L_0, x(1) \in L'_1, \dot{x}(t) = X_H(x(t))\}.$$

Definition 5.13. – Given $x_+ \in \mathcal{C}_H$, $x_- \in \mathcal{C}'_H$, and $\mathbf{p}^i \in D_i^{l_i}$ for $i = 0, 1$ and $l_i \geq 0$ we define the moduli space

$$\mathfrak{M}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$$

as the set of triples $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1, u)$ such that:

— $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \mathcal{R}^{l_0|l_1}$ and $u\colon Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1} \to W$ satisfies the Floer equation

$$\frac{\partial u}{\partial s} + \tilde{J}_{\boldsymbol{\zeta}^1}\left(\frac{\partial u}{\partial t} - \chi'(t) X_H(\chi(t), u)\right) = 0, \tag{16}$$

— $\lim_{s \to \pm\infty} u(s,t) = x_\pm(\chi(t))$,

— $u(s,0) \in L_0$ for all $(s,0) \in Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1}$,

— $u(s,1) \in \psi_{\nu_{\boldsymbol{\zeta}^1}(s)}(L_1)$ for all $(s,1) \in Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1}$, and

— each ζ^0_j is a negative puncture at p^0_j and each ζ^1_j is a negative puncture at $\psi_{\nu_{\boldsymbol{\zeta}^1}(\zeta^1_j)}(L_1)$.

(Recall that $\psi_t\colon W \to W$ here is a smooth isotopy satisfying the assumptions made in the beginning of this section, whose restriction to L_1 in particular is the compactly supported safe isotopy generated by $G\colon \mathbb{R} \times L \to \mathbb{R}$.) We denote by $\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$ the set of triples $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1, u)$ where $\mathrm{ind}(u) = k$.

Proposition 5.14. – *For a generic $\tilde{J}$ as in Lemma 5.12, the moduli space*

$$\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$$

is a transversely cut out manifold of dimension k. If $k = 0$, it is compact, and therefore consists of a finite set of points. If $k = 1$, it admits a compactification in the Gromov-Floer sense.

If L_0 and L_1 are spin, a choice of a spin structure on each induces a coherent orientation of the moduli space (see [17]*).*

DEFINITION 5.15. – We define the weighted count

$$\mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G) = \#\mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)\varepsilon_0(\mathbf{p}^0)\varepsilon_1(\mathbf{p}^1)$$

and then we define Ψ_G as

$$\Psi_G(x_+) = \sum_{x_- \in \mathcal{C}_H} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G) x_-.$$

REMARK 5.16. – The word $\mathbf{p}^1$ consists of double points living on the different Lagrangian immersions $\psi_{\nu_{\mathfrak{z}^1}(\zeta_j^1)}(L_1)$. However, when using the pushed forward almost complex structures $(\psi_{\nu_{\mathfrak{z}^1}(\zeta_j^1)})_* J_1$, their obstruction algebras all become canonically identified with $(\mathfrak{A}(L_1), \mathfrak{d})$ defined using J_1. This motivates our abuse of notation ε_1 for an augmentation induced by these canonical identifications.

A consideration of Proposition 5.14 together with a count of solutions of index -1 which arise in a one-parametric family of moduli spaces implies the following:

LEMMA 5.17. – *The map Ψ_G is a chain map. Moreover, up to chain homotopy, it does not depend on the choice of ν and on the homotopy class of ψ_t relative to the endpoints.*

LEMMA 5.18. – *Let $G^0, G^1 \colon L_1 \to \mathbb{R}$ be local Hamiltonian functions generating the safe isotopies $\psi_t^0 \circ \iota_1$ and $\psi_t^1 \circ \psi_1^0 \circ \iota_1$ respectively, and let $G^2 \colon L_1 \to \mathbb{R}$ be a local Hamiltonian function generating*

$$\psi_t^2 = \begin{cases} \psi_{2t}^1 & \text{for } t \in [0, 1/2], \\ \psi_{2t-1}^2 \circ \psi_1^1 & \text{for } t \in [1/2, 1]. \end{cases}$$

Then Ψ_{G^2} is chain homotopic to $\Psi_{G^0} \circ \Psi_{G^1}$.

The proof of Lemma 5.18 is analogous to the proof of Lemma 5.8.

COROLLARY 5.19. – *If $G \colon \mathbb{R} \times L \to \mathbb{R}$ satisfies $dG_t = 0$ outside a compact subset of $(0, 1) \times L$, then the map Ψ_G induces an isomorphism in homology.*

If $\bar{J}_\bullet^+$ is one (L_0, L_1)-regular almost complex structure and $\bar{J}_\bullet^-$ is another $(L_0, \psi_1(L_1))$-regular almost complex structure, instead of repeating the construction of $\tilde{J}$ with $\bar{J}_\bullet^\pm$ as starting point, we prefer to consider $\tilde{J}$ assigned once and for all to the triple (L_0, L_1, G) and define the continuation map

$$\mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, \bar{J}_\bullet^+) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1', \varepsilon_1); H, \bar{J}_\bullet^-)$$

as the composition $\Upsilon_- \circ \Psi_G \circ \Upsilon_+$, where

$$\Upsilon_+ \colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, \bar{J}_\bullet^+) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^+),$$
$$\Upsilon_- \colon \mathrm{CF}((L_0, \varepsilon_0), (L_1', \varepsilon_1'); H, J_\bullet^-) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1', \varepsilon_1'); H, \bar{J}_\bullet^-)$$

are the maps defined in Section 5.1.

5.4. Continuation maps for almost complex structures and Hamiltonian functions commute

Let H_s be a homotopy from H_- to H_+ as in Section 5.2 and $J_\bullet^\bullet$ a homotopy from $J_\bullet^{-1}$ to $J_\bullet^{+1}$ as in Section 5.1. For simplicity we will denote

$$\mathrm{CF}(H_\pm, J_\bullet^\pm) = \mathrm{CF}((L_0, \varepsilon_0^\pm), (L_1, \varepsilon_1^\pm); H_\pm, J_\bullet^{\pm 1}).$$

If $\varepsilon_i^+ = \varepsilon_i^- \circ \mathfrak{Y}_i$, we have defined continuation maps

$$\Upsilon_\pm \colon \mathrm{CF}(H_\pm, J_\bullet^+) \to \mathrm{CF}(H_\pm, J_\bullet^-),$$
$$\Phi_\pm \colon \mathrm{CF}(H_+, J_\bullet^\pm) \to \mathrm{CF}(H_-, J_\bullet^\pm)$$

and now we will to prove that they are compatible in the following sense.

Proposition 5.20. – *The diagram*

$$\begin{array}{ccc} \mathrm{CF}(H_+, J_\bullet^+) & \xrightarrow{\Phi_+} & \mathrm{CF}(H_-, J_\bullet^+) \\ {\scriptstyle \Upsilon_+}\big\downarrow & & \big\downarrow{\scriptstyle \Upsilon_-} \\ \mathrm{CF}(H_+, J_\bullet^-) & \xrightarrow[\Phi_-]{} & \mathrm{CF}(H_-, J_\bullet^-) \end{array} \tag{17}$$

commutes up to homotopy.

Proposition 5.20 will be proved by applying the bifurcation method to the definition of the continuation maps $\Phi_\pm$: i.e., we will study the parametrized moduli spaces

$$\mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet^\bullet)$$

consisting of pairs (δ, u) such that $\delta \in [0,1]$ and $u \in \mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet^\delta)$. For a generic homotopy $J_\bullet^\bullet$, these parametrized moduli spaces are transversely cut out manifolds of dimension one. As before, there is a finite set Δ of bifurcation points such that, for all $\delta \in \Delta$, there is a unique nonempty moduli space of one of the following types:

(i) $\mathfrak{N}^{-1}_{L_0}(q_0^0; q_1^0, \dots, q_d^0; J_0^\delta)$ or $\mathfrak{N}^{-1}_{L_1}(q_0^1; q_1^1, \dots, q_d^1; J_1^\delta)$,

(ii) $\mathfrak{M}^0_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_-, J_\bullet^\delta)$ or $\mathfrak{M}^0_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^\delta)$,

(iii) $\mathfrak{M}^{-1}_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J_\bullet^\delta)$.

To these moduli spaces correspond four types of boundary configuration for the compactification of the one-dimensional parametrized moduli spaces $\mathfrak{M}^0_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet^\bullet)$, which we write schematically as:

(i) $\mathfrak{N}^{-1}_{L_0}(J_0^\delta) \times \mathfrak{M}^0(H_s, J_\bullet^\delta)$ or $\mathfrak{N}^{-1}_{L_1}(J_1^\delta) \times \mathfrak{M}^0(H_s, J_\bullet^\delta)$,

(ii) $\mathfrak{M}^0(H_-, J_\bullet^\delta) \times \mathfrak{M}^0(H_s, J_\bullet^\delta)$ or $\mathfrak{M}^0(H_s, J_\bullet^\delta) \times \mathfrak{M}^0(H_+, J_\bullet^\delta)$,

(iii) $\mathfrak{M}^1(H_-, J_\bullet^\delta) \times \mathfrak{M}^{-1}(H_s, J_\bullet^\delta)$ or $\mathfrak{M}^{-1}(H_s, J_\bullet^\delta) \times \mathfrak{M}^1(H_+, J_\bullet^\delta)$,

(iii)′ $\mathfrak{N}^0_{L_0}(J_0^\delta) \times \mathfrak{M}^{-1}(H_s, J_\bullet^\delta)$ or $\mathfrak{N}^0_{L_1}(J_1^\delta) \times \mathfrak{M}^{-1}(H_s, J_\bullet^\delta)$.

There is also one fifth type of boundary configuration:

(iv) $\mathfrak{M}^0(H_s, J_\bullet^{-1})$ or $\mathfrak{M}^0(H_s, J_\bullet^1)$.

In order to prove Proposition 5.20 we split the homotopy $J_\bullet^\bullet$ into pieces, each of which contains only one element of Δ, and we prove that for each piece the corresponding diagram (17) commutes up to homotopy. Putting all pieces together, we will obtain the result. We rescale each piece of homotopy so that it is parametrized by $[-1,1]$ and the bifurcation point is $\delta = 0$.

LEMMA 5.21. – *Let* $\Delta = \{0\}$ *be of type* (i). *Then Diagram* (17) *commutes.*

Proof. – We have proved in Lemma 5.4 that $\Upsilon_\pm$ are the identity maps. Here we will prove that, under the hypothesis of the lemma, $\Phi_+ = \Phi_-$. We use the same construction, and the same notation, as in the proof of Lemma 5.4.

The homotopy of Hamiltonians H_s on W induces a homotopy of Hamiltonians $\widetilde{H}_s$ on $T^*\mathbb{R} \times W$. Let

$$\widetilde{\Phi}\colon \mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (\widetilde{L}_1, \widetilde{\varepsilon}_1); \widetilde{H}_+, \widetilde{J}_\bullet) \to \mathrm{CF}((\widetilde{L}_0, \widetilde{\varepsilon}_0), (\widetilde{L}_1, \widetilde{\varepsilon}_1); \widetilde{H}_-, \widetilde{J}_\bullet)$$

be the continuation map associated to the homotopy $\widetilde{H}_s$.

Solutions of the Floer equation perturbed by $\widetilde{H}_s$ are still holomorphic when projected to $T^*\mathbb{R}$, and therefore the counterpart of Lemmas 4.14 and 4.19 of [17] gives that, for $\kappa = -1, 0, +1$, $\widetilde{\Phi}(x[\kappa])$ is a linear combination of chords $y[\kappa]$. The statement for $\kappa = 0$ is a consequence of the assumption that there is no Floer strip for $\widetilde{H}_s$ of index -1. (Unlike in the proof of Lemma 5.4 we do not have the two strips between from $x[0]$ to $x[\pm 1]$ because in the definition of $\widetilde{\Phi}$ we consider solutions of index zero.) Then $\widetilde{\partial} \circ \widetilde{\Phi} = \widetilde{\Phi} \circ \widetilde{\partial}$ and Equation 12 give $\Phi_+ = \Phi_-$. □

LEMMA 5.22. – *Let* $\Delta = \{0\}$ *be of type (ii). Then Diagram* (17) *commutes.*

Proof. – We assume, without loss of generality, that the moduli space of negative formal dimension is

$$\mathfrak{M}^0_{L_1,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^0).$$

By Lemma 5.5 the continuation maps for the change of almost complex structure are $\Upsilon_- = \mathrm{Id}$ and

$$\Upsilon_+(x) = \begin{cases} x & \text{if } x \neq y_+, \\ y_+ + \#\mathfrak{M}^0_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^\delta)\varepsilon_0(\mathbf{q}^0)\varepsilon_1(\mathbf{q}^1)y_- & \text{if } x = y_+. \end{cases}$$

The structure of the compactification of one-dimensional parametrized moduli spaces implies that

$$\begin{aligned} \#\mathfrak{M}^0(\mathbf{q}^1\mathbf{p}^1, x_-, \mathbf{p}^0\mathbf{q}^0, y_+; H_s, J_\bullet^1) - \#\mathfrak{M}^0(\mathbf{q}^1\mathbf{p}^1, x_-, \mathbf{p}^0\mathbf{q}^0, y_+; H_s, J_\bullet^{-1}) \\ = \#\mathfrak{M}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^0)\#\mathfrak{M}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, y_-; H_s, J_\bullet^0), \end{aligned}$$

while the cardinality of all other moduli spaces remains unchanged. This implies that Diagram (17) commutes. □

We have dropped the Lagrangian labels from the notation in order to keep the formulas compact. We will do the same in the proofs of the following lemma.

LEMMA 5.23. – *Let* $\Delta = \{0\}$ *be of type (iii). Then Diagram* (17) *commutes up to homotopy.*

Proof. – Let $\mathfrak{M}^{-1}_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J^0_\bullet)$ be the nonempty moduli space of negative formal dimension. In this case $\Upsilon_\pm = \mathrm{Id}$ and we define a linear map

$$K\colon \mathrm{CF}(H_+, J^{+1}_\bullet) \to \mathrm{CF}(H_-, J^{-1}_\bullet)$$

by

$$K(x) = \begin{cases} 0 & \text{if } x \neq y_+, \\ \#\mathfrak{M}^{-1}_{L_0,L_1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J^0_\bullet)\varepsilon_0(\mathbf{q}^0)\varepsilon_1(\mathbf{q}^1)y_- & \text{if } x = y_+. \end{cases}$$

The structure of the boundary of the compactification of the one-dimensional parametrized moduli spaces implies that

$$\begin{aligned} &\#\mathfrak{M}^0(\mathbf{q}^1\mathbf{p}^1, x_-, \mathbf{p}^0\mathbf{q}^0, y_+; H_s, J^1_\bullet) - \#\mathfrak{M}^0(\mathbf{q}^1\mathbf{p}^1, x_-, \mathbf{p}^0\mathbf{q}^0, y_+; H_s, J^{-1}_\bullet) \\ &\qquad = \#\mathfrak{M}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J^0_\bullet)\#\mathfrak{M}^1(\mathbf{p}^1, x_-, \mathbf{p}^0, y_-; H_-, J^0_\bullet), \text{ and} \\ &\#\mathfrak{M}^0(\mathbf{p}^1\mathbf{q}^1, y_-, \mathbf{q}^0\mathbf{p}^0, x_+; H_s, J^1_\bullet) - \#\mathfrak{M}^0(\mathbf{p}^1\mathbf{q}^1, y_-, \mathbf{q}^0\mathbf{p}^0, x_+; H_s, J^{-1}_\bullet) \\ &\qquad = \#\mathfrak{M}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J^0_\bullet)\#\mathfrak{M}^1(\mathbf{p}^1, y_+, \mathbf{p}^0, x_+; H_+, J^0_\bullet). \end{aligned}$$

From this it follows that $\Phi_+ - \Phi_- = \partial K + K\partial$. □

The degenerations of type (iii)′ are canceled algebraically by the augmentations, and therefore we obtain the commutativity of the diagram (17) for a generic homotopy $J^\bullet_\bullet$.

Now we compare the continuation maps Φ for the change of Hamiltonian and the continuation maps Ψ for compactly supported safe isotopies of L_1. Let $G\colon \mathbb{R} \times L_1 \to \mathbb{R}$ be a local Hamiltonian function satisfying $dG_t = 0$ outside a compact subset of $(0,1) \times L_1$ which generates the safe isotopy $\psi_t \circ \iota_1$, and denote $L_1' = \psi_1(L_1)$. If H_+ and H_- are two Hamiltonian functions which are compatible both with L_0 and L_1 and with L_0 and L_1', then we have continuation maps

$$\Psi^\pm_G\colon\ \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1); H_\pm, J^+_\bullet) \to \mathrm{CF}((L_0,\varepsilon_0),(L_1',\varepsilon_1'); H_\pm, J^-_\bullet)$$

and continuation maps

$$\begin{aligned} \Phi&\colon\ \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1); H_+, J^+_\bullet) \to \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1); H_-, J^+_\bullet), \\ \Phi'&\colon\ \mathrm{CF}((L_0,\varepsilon_0),(L_1',\varepsilon_1'); H_+, J^-_\bullet) \to \mathrm{CF}((L_0,\varepsilon_0),(L_1',\varepsilon_1'); H_-, J^-_\bullet) \end{aligned}$$

induced by a homotopy of Hamiltonians H_s with $H_s = H_+$ for $s \geq 1$ and $H_s = H_-$ for $s \leq -1$.

LEMMA 5.24. – *The diagram*

$$\begin{array}{ccc} \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1); H_+, J^+_\bullet) & \xrightarrow{\Phi} & \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1); H_-, J^+_\bullet) \\ \Big\downarrow{\scriptstyle \Psi^+_G} & & \Big\downarrow{\scriptstyle \Psi^+_G} \\ \mathrm{CF}((L_0,\varepsilon_0),(L_1',\varepsilon_1'); H_+, J^-_\bullet) & \xrightarrow{\Phi'} & \mathrm{CF}((L_0,\varepsilon_0),(L_1',\varepsilon_1'); H_-, J^-_\bullet) \end{array}$$

commutes up to homotopy.

Sketch of proof. – For $R \in \mathbb{R}$ we define $H_s^R = H_{s-R}$. We define the moduli spaces $\mathfrak{M}^k_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^R, G, \tilde{J}, \nu)$ as in Definition 5.13 by replacing the Hamiltonian H by the s-dependent Hamiltonian H_s^R. Counting pairs (R, u) where $R \in \mathbb{R}$ and

$$u \in \mathfrak{M}^{-1}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^R, G, \tilde{J}, \nu),$$

weighted by the augmentations, we obtain a homotopy between $\Psi_G^- \circ \Phi$ and $\Phi' \circ \Psi_G^+$. □

6. Wrapped Floer cohomology for exact Lagrangian immersions

In this section we define wrapped Floer cohomology for unobstructed immersed exact Lagrangian submanifolds. With the preparation of the previous sections in place, the definition is not different from the usual one for Lagrangian submanifolds.

6.1. Wrapped Floer cohomology as direct limit

We start by defining wrapped Floer cohomology as a direct limit. This point of view will be useful in the vanishing theorem of the following section. A sketch of the chain level construction, following [4], will be given in the next subsection.

Let (L_0, ι_0) and (L_1, ι_1) be immersed exact Lagrangian submanifolds with augmentations ε_0 and ε_1 respectively. We assume that all intersection points between L_0 and L_1 are transverse, L_0 and L_1 are cylindrical over Legendrian submanifolds Λ_0 and Λ_1 respectively, and all Reeb chords between Λ_0 and Λ_1 are nondegenerate.

For every $\lambda \in \mathbb{R}$ we denote by $h_\lambda \colon \mathbb{R}^+ \to \mathbb{R}$ the function

$$h_\lambda(\rho) = \begin{cases} 0 & \text{if} \quad \rho \in (0, 1], \\ \lambda\rho - \lambda & \text{if} \quad \rho \geq 1. \end{cases} \tag{18}$$

We smooth h_λ inside the interval $[4/5, 6/5]$ (or any sufficiently small neighborhood of 1 independent of λ) and, by abuse of notation, we still denote the resulting function by h_λ. We assume also that the resulting smooth function satisfies $h''(\rho) \geq 0$ for all $\rho \in \mathbb{R}^+$. We define time-independent cylindrical Hamiltonians $H_\lambda \colon W \to \mathbb{R}$ by

$$H_\lambda(w) = h_\lambda(e^{\mathfrak{r}(w)}). \tag{19}$$

Hamiltonian functions of this form will be called *wrapping Hamiltonian functions*.

We fix a sequence of positive real number λ_n such that $\lim_{n\to+\infty} \lambda_n = +\infty$ and, for any n, λ_n is not the length of a Reeb chord from Λ_0 to Λ_1. The set of (L_0, L_1)-regular almost complex structures for every H_{λ_n} is dense, and we pick an element $J_\bullet$.

By Subsection 5.2, for every $m \geq n$ there are continuation maps

$$\Phi_{\lambda_n,\lambda_m} \colon \mathrm{HF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_n}, J_\bullet) \to \mathrm{HF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_m}, J_\bullet)$$

forming a direct system.

Definition 6.1. – The wrapped Floer cohomology of (L_0, ε_0) and (L_1, ε_1) is defined as

$$\mathrm{HW}((L_0, \varepsilon_0), (L_1, \varepsilon_1); J_\bullet) = \varinjlim \mathrm{HF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_n}, J_\bullet). \tag{20}$$

Wrapped Floer cohomology is well defined, in the sense that it is independent of the choice of the almost complex structure $J_\bullet$, and of the smoothing of the piecewise linear functions H_{λ_n} and of the sequence λ_n. Invariance of the almost complex structure follows from Proposition 5.20. Invariance of the smoothing of H_{λ_n} follows from Lemma 5.8 and Corollary 5.9. Finally, if $\lambda'_n \to +\infty$ is another sequence such that no λ'_n is not the length of a Reeb chord from Λ_0 to Λ_1, we can make both λ_n and λ'_n subsequences of a diverging sequence λ''_n and standard properties of the direct limit give canonical isomorphisms

$$\begin{aligned}\varinjlim \mathrm{HF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_{\lambda''_n},J_\bullet) &\cong \varinjlim \mathrm{HF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_{\lambda_n},J_\bullet)\\ &\cong \varinjlim \mathrm{HF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_{\lambda'_n},J_\bullet).\end{aligned}$$

Therefore $\mathrm{HW}((L_0,\varepsilon_0),(L_1,\varepsilon_1))$ does not depend on the sequence λ_n up to isomorphism. It can also be proved that it is invariant under safe isotopies, but we will need, and prove, only invariance under compactly supported ones.

LEMMA 6.2. – *Let $G\colon \mathbb{R}\times L_1 \to \mathbb{R}$ be a local Hamiltonian function which satisfies $dG_t = 0$ outside a compact subset of $(0,1)\times L_1$ and let $\psi_t \circ \iota_1$ be the exact regular homotopy it generates, which is assumed to be a safe isotopy. If $L'_1 = \psi_1(L_1)$, $J'_\bullet$ is an (L_0, L'_1)-regular almost complex structure and ε'_1 is the augmentation for L'_1 with respect to $J'_\bullet$ corresponding to ε_1, then there is an isomorphism*

$$\mathrm{HW}((L_0,\varepsilon_0),(L_1,\varepsilon_1);J_\bullet) \cong \mathrm{HW}((L_0,\varepsilon_0),(L'_1,\varepsilon'_1);J'_\bullet).$$

Proof. – It is enough to observe that, for every n, the isomorphisms

$$\mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);H_{\lambda_n},J_\bullet) \cong \mathrm{CF}((L_0,\varepsilon_0),(L'_1,\varepsilon'_1);H_{\lambda_n},J'_\bullet)$$

defined in Subsection 5.3 commute with the continuation maps $\Phi_{\lambda_n,\lambda_m}$ and therefore define isomorphisms of direct systems. This follows from Lemma 5.24 and Proposition 5.20. □

6.2. A sketch of the chain level construction

Here we recall very briefly the definition of the wrapped Floer complex and the A_∞-operations. Since Lagrangian immersions will appear only in an intermediate step of the proof of the main theorem, we will not try to make them objects of an enlarged wrapped Fukaya category. Presumably this can be done as in the embedded case, but we have not checked the details of the construction of the necessary coherent Hamiltonian perturbations.

Let L_0 and L_1 be exact Lagrangian immersions which intersect transversely and are cylindrical over chord generic Legendrian submanifolds. We fix a wrapping Hamiltonian $H \geq 0$ as in Equation (18) such that, for every $w \in \mathbb{N}$, the Hamiltonian wH is compatible with L_0 and L_1 (in the sense of Definition 4.3). We also fix an (L_0, L_1)-regular almost complex structure $J_\bullet$.

Let ε_0 and ε_1 be augmentations of the obstruction algebras of (L_0, J_0) and (L_1, J_1) respectively. Following [4] we define the *wrapped Floer chain complex* as the $\mathbb{F}[q]/(q^2)$-module

$$\mathrm{CW}((L_0,\varepsilon_0),(L_1,\varepsilon_1);J_\bullet) = \bigoplus_{w=0}^{\infty} \mathrm{CF}((L_0,\varepsilon_0),(L_1,\varepsilon_1);wH,J_\bullet)[q] \tag{21}$$

with a differential μ^1 such that, on $x + yq \in \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); wH, J_\bullet)[q]$, it is defined as

$$\mu^1(x + yq) = \partial x + y + \Phi_w(y) + (\partial y)q,$$

where

$$\Phi_w\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); wH, J_\bullet) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); (w + 1)H, J_\bullet)$$

is the continuation map for the change of Hamiltonian defined in Subsection 5.2.

REMARK 6.3. – The endomorphism ι_q (denoted ∂_q in [4]) defined as

$$\iota_q(x + yq) = y$$

is a chain map. However, its action in homology is trivial.

REMARK 6.4. – The direct sum (21) starts from $w = 1$ in [4]. It is equivalent to start from $w = 0$, when possible, by [4, Lemma 3.11]. The homology of $\mathrm{CW}((L_0, \varepsilon_0), (L_1, \varepsilon_1); J_\bullet)$ is isomorphic to $\mathrm{HW}((L_0, \varepsilon_0), (L_1, \varepsilon_1))$ defined as the direct limit in Equation (20) by [4, Lemma 3.12].

The A_∞-operations between wrapped Floer complexes are defined by counting pseudoholomorphic polygons with carefully constructed Hamiltonian perturbations. In the immersed case, those polygons will be allowed to have boundary punctures at double points and, as usual, must be counted with a weight coming from the augmentations. The only thing we need to know about the operations between wrapped Floer cohomology groups is that the component

$$\mu^d\colon \mathrm{CF}(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CF}(L_0, L_1) \to \mathrm{CF}(L_0, L_d)$$

of the operation

$$\mu^d_{\mathcal{WF}}\colon \mathrm{CW}(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CW}(L_0, L_1) \to \mathrm{CW}(L_0, L_d)$$

coincides with the operation μ^d defined in Equation (11). For simplicity of notation we have dropped the augmentations from the above formulas.

7. A trivial triviality result

An exact Lagrangian embedding with cylindrical ends which is disjoint from the skeleton is known to have vanishing wrapped Floer cohomology. This was proven in [13, Theorem 9.11(b)] but also follows from e.g., [4, Section 5.1]. Note that the statement is false in the more general case when the Lagrangian is only assumed to be monotone. In this section we extend this classical vanishing result to our setting of *exact* Lagrangian immersions.

7.1. Action and energy

In this subsection we define an action for double points of immersed exact Lagrangian submanifolds and for Hamiltonian chords and prove action estimates for various continuation maps. Let $p \in W$ be a double point of a Lagrangian immersion (L, ι) with potential f. We recall that there are points $p_\pm \in L$ characterized by $\iota^{-1}(p) = \{p_+, p_-\}$ and $f(p_+) > f(p_-)$. We define the *action* of p as

$$\mathfrak{a}(p) = f(p_+) - f(p_-).$$

If L is disconnected and p is in the intersection between the images of two connected components, then $\mathfrak{a}(p)$ depends on the choice of the potential function f, otherwise it is independent of it.

Given a holomorphic map $(\boldsymbol{\zeta}, u) \in \widetilde{\mathfrak{N}}_L(p_0; p_1, \dots, p_d; J)$, Stokes's theorem immediately yields

$$\int_{\Delta_{\boldsymbol{\zeta}}} u^* d\theta = \mathfrak{a}(p_0) - \sum_{i=1}^{d} \mathfrak{a}(p_i).$$

Since $\int_{\Delta_{\boldsymbol{\zeta}}} u^* d\theta > 0$ for a nonconstant J-holomorphic map, if

$$\widetilde{\mathfrak{N}}_L(p_0; p_1, \dots, p_d; J) \neq \emptyset,$$

we obtain

$$\mathfrak{a}(p_0) - \sum_{i=1}^{d} \mathfrak{a}(p_i) > 0. \tag{22}$$

Given two Lagrangian submanifolds L_0 and L_1 with potentials f_0 and f_1 and a Hamiltonian function H, we define the action of a Hamiltonian chord $x\colon [0,1] \to W$ from L_0 to L_1 as

$$\mathcal{A}(x) = \int_0^1 x^*\theta - \int_0^1 H(x(t))dt + f_0(x(0)) - f_1(x(1)). \tag{23}$$

Note that this is the negative of the action used in [34].

Example 7.1. – Let $H\colon W \to \mathbb{R}$ be a cylindrical Hamiltonian such that $H(w) = h(e^{\mathfrak{r}(w)})$, where $h\colon \mathbb{R}^+ \to \mathbb{R}$. Then a Hamiltonian chord $x\colon [0,1] \to W$ from L_0 to L_1 is contained in a level set $\mathfrak{r}^{-1}(r)$ and has action

$$\mathcal{A}(x) = h'(e^r)e^r - h(e^r) + f_0(x(0)) - f_1(x(1)). \tag{24}$$

The following lemma, which we prove in the more general case of the moduli spaces of Floer solutions with an s-dependent Hamiltonian, applies equally to the particular case of moduli spaces used in the definition of the Floer differential. We introduce the following notation. Given a set A and a function $f\colon A \to \mathbb{R}$, we denote $\|f\|_\infty^+ := \sup_{a \in A} \max\{f(a), 0\}$.

Lemma 7.2. – *Let $H_s\colon \mathbb{R} \times [0,1] \times W \to \mathbb{R}$ be an s-dependent cylindrical Hamiltonian function satisfying conditions (i), (ii), and (iii) of Subsection 5.2. We make the simplifying assumption that $\partial_s H_s \equiv 0$ if $s \notin [-1,1]$. If $\mathfrak{M}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet) \neq \emptyset$, then*

$$\mathcal{A}(x_-) \leq \mathcal{A}(x_+) + 6\|\partial_s H_s\|_\infty^+. \tag{25}$$

Note that Equation (25) is far from being sharp, but there will be no need for a sharper estimate.

Proof. – Let $(u, \boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \mathfrak{M}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$. Then, in a metric on u^*TW induced by $d\theta$ and $J_\bullet$,

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^1 |\partial_s u|^2 dt ds &= \int_{-\infty}^{+\infty}\int_0^1 d\theta(\partial_s u, \partial_t u - \chi'(t)X_{H_s}(\chi(t), u))dt ds \\ &= \int_{Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1}} u^* d\theta - \int_{-\infty}^{+\infty}\int_0^1 \chi'(t) dH_s(\chi(t)\partial_s u) dt ds.\end{aligned}$$

Using Stokes's theorem we obtain

$$\int_{Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1}} u^* d\theta = f_1(x_-(1)) - f_1(x_+(1)) - f_0(x_-(0)) + f_0(x_+(0)) - \sum_{i=0}^{1}\sum_{j=1}^{l_i} \mathfrak{a}(p_j^i).$$

Using the equality $\partial_s(H_s \circ u) = (\partial_s H_s) \circ u + dH_s(\partial_s u)$ we obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_0^1 \chi'(t) dH_s(\chi(t), (\partial_s u)) dt ds \\ &\quad = \int_{-\infty}^{+\infty}\int_0^1 \chi'(t)\partial_s(H_s(\chi(t), u(s,t))) dt ds - \int_{-\infty}^{+\infty}\int_0^1 \chi'(t)(\partial_s H_s)(\chi(t), u(s,t)) dt ds.\end{aligned}$$

We can compute

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_0^1 \chi'(t)\partial_s(H_s(\chi(t), u(s,t))) dt ds \\ &\qquad = \int_0^1 \chi'(t) H_+(\chi(t), x_+(\chi(t))) dt - \int_0^1 \chi'(t) H_-(\chi(t), x_-(\chi(t))) dt \\ &\qquad = \int_0^1 H_+(t, x_+(t)) dt - \int_0^1 H_-(t, x_-(t)) dt.\end{aligned}$$

Thus, rearranging the equalities, we have

$$\mathcal{A}(x_+) - \mathcal{A}(x_-) = \int_{-\infty}^{+\infty}\int_0^1 |\partial_s u|^2 dt ds + \sum_{i=0}^{1}\sum_{j=1}^{l_i} \mathfrak{a}(p_j^i) - \int_{-\infty}^{+\infty}\int_0^1 \chi'(t)\partial_s H_s(\chi(t) u(s,t)) ds dt.$$

Finally, we estimate

$$\int_{-\infty}^{+\infty}\int_0^1 \chi'(t)(\partial_s H_s)(\chi(t), u(s,t)) dt ds \leq 6\|\partial_s H_s\|_\infty^+$$

and obtain Equation (25). □

COROLLARY 7.3. – *The differential in* $\mathrm{CF}(L_0, L_1; H, J_\bullet)$ *decreases the action. If* $\partial_s H_s \leq 0$, *then the continuation map* Φ_{H_s} *also decreases the action.*

Now we turn our attention to the continuation map Ψ_G defined in Subsection 5.3. Let $G\colon \mathbb{R} \times L_1 \to \mathbb{R}$ be a local Hamiltonian function such that $dG_t = 0$ outside a compact subset of $(0,1) \times L_1$ and let $\psi_t \circ \iota_1$ be the compactly supported exact regular homotopy it generates. Now assume that ψ_t is a safe isotopy. Denote, as usual, $L_1' = \psi_1(L_1)$.

First we make the following remark about a special type of safe isotopy and the action of the image of the double points.

REMARK 7.4. – Let (L,ι) be an exact Lagrangian immersion and $\psi_t\colon W \to W$ a smooth isotopy. If $\psi_t^*\theta = e^{c(t)}\theta$, then

1. $\psi_t(L)$ is a safe isotopy, and
2. if p is a double point of (L,ι), then $\psi_t(p)$ is a double point of $(L, \psi_t \circ \iota)$ whose action satisfies $\mathfrak{a}(p) = e^{c(t)}\mathfrak{a}(\psi_t(p))$.

Given a Hamiltonian function $H\colon [0,1]\times W \to \mathbb{R}$ which is compatible both with L_0 and L_1, as well as with L_0 and L_1', let $\mathcal{C}_H$ be the set of Hamiltonian chords of H from L_0 to L_1 and let $\mathcal{C}_H'$ be the set of Hamiltonian chords of H from L_0 to L_1'.

Observe that any safe Lagrangian isotopy from L_1 to L_1' induces a continuous family of potentials f_1^s. Fixing a choice of local Hamiltonian $G\colon \mathbb{R}\times L_1 \to \mathbb{R}$ generating the safe isotopy makes the potential f_1' on L_1' determined by the choice of potential f_1 on L_1 via a computation as in the proof of Lemma 2.3.

LEMMA 7.5. – *For every chords $x_- \in \mathcal{C}_H'$ and $x_+ \in \mathcal{C}_H$ and for every sets of self-intersection points $\mathbf{p}^0 = (p_1^0,\dots,p_{l_0}^0)$ of L_0 and $\mathbf{p}^1 = (p_1^1,\dots,p_{l_1}^1)$ of L_1, if*

$$\mathfrak{M}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu) \neq \emptyset,$$

then

$$\mathcal{A}(x_-) \leq \mathcal{A}(x_+) + 2\mu\|G\|_\infty,$$

where $\|G\|_\infty$ is the supremum norm of G and $\mu \geq 0$ is the measure of the subset $\{s\in\mathbb{R}\}$ for which $G_s\colon L \to \mathbb{R}$ is not constantly zero.

Proof. – Consider $(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1, u) \in \mathfrak{M}_{L_0,L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$. Observe that the map $u\colon Z_{\boldsymbol{\zeta}^0,\boldsymbol{\zeta}^1} \to W$ extends to a continuous map $u\colon Z \to W$. We have:

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^1 |\partial_s u|^2 dt\,ds &= \int_Z du^*\theta - \int_0^1 \left(\int_{-\infty}^{+\infty}\left(\frac{\partial}{\partial s}(H\circ u)\right)ds\right)dt\\ &= \int_Z u^* d\theta - \int_0^1 H(x_+(t))dt + \int_0^1 H(x_-(t))dt.\end{aligned}$$

We denote by $u_i\colon \mathbb{R}\to W$, for $i=0,1$, the continuous and piecewise smooth maps $u_i(s) = u(s,i)$ and use Stokes theorem:

$$\int_Z u^* d\theta = \int_0^1 x_+^*\theta - \int_0^1 x_-^*\theta + \int_{\mathbb{R}} u_0^*\theta - \int_{\mathbb{R}} u_1^*\theta.$$

Let f_0 and f_1 be the potentials of L_0 and L_1 respectively, and $\tilde{f}_1$ the potential of $\psi_1(L_1)$. The map u_0 takes values in L_0, and therefore

$$\int_{\mathbb{R}} u_0^*\theta = f_0(x_+(0)) - \sum_{j=1}^{l_0}\mathfrak{a}(p_j^0) - f_0(x_-(0)).$$

We are left with the problem of estimating $\int_{\mathbb{R}} u_1^*\theta$, which is slightly more complicated here because $u_1(s) \in \psi_{\nu_{\boldsymbol{\zeta}^1}(s)}(L_1)$. We will denote $\psi_s^\nu\colon\ eqq\psi_{\nu_{\boldsymbol{\zeta}^1}(s)}$. Recall that ψ_s^ν is a smooth isotopy inducing a safe isotopy of L_1 generated by the local Hamiltonian function $G^\nu(s,w) = \nu_{\boldsymbol{\zeta}^1}'(s)G(\nu_{\boldsymbol{\zeta}^1}(s), w)$, $w\in L_1$.

We use the following trick. Consider $W \times \mathbb{R} \times \mathbb{R}$ with the Liouville form $\Theta\colon eqq\theta + \tau d\sigma$, where σ is the coordinate of the first copy of $\mathbb{R}$ and τ is the coordinate in the second copy. The notation here conflicts with the use of (σ, τ) as coordinates in the strip-like ends near the boundary punctures, but this will not cause confusion.

Consider the symplectic suspension

$$\Sigma\colon eqq\{(x,s,t) \in W \times \mathbb{R}^2;\ x \in \psi_s^\nu(y),\ y \in L_1,\ t = -G^\nu(s,y)\}$$

of the isotopy $\psi_s^\nu(L_1)$, which is an exact Lagrangian immersion. This should be seen as a corrected version of the trace of the isotopy, in order to make it Lagrangian.

Lift $u_1\colon \mathbb{R} \to W$ to $\tilde{u}_1\colon \mathbb{R} \to W \times \mathbb{R} \times \mathbb{R}$ by defining

$$\tilde{u}_1(s) = (u_1(s), s, -G^\nu(s, \overline{u}_1(s))) \in \Sigma,$$

and where $\overline{u}_1$ is the lift of u_1 to L which is smooth away from the punctures.

Using the computation in the proof of Lemma 2.3, together with the Lagrangian condition satisfied by Σ, we obtain

$$\int_{-\infty}^{+\infty} \widetilde{u}_1^* \Theta = f_1'(x_+(1)) - \sum_{j=1}^{l_1} \mathfrak{a}(p_j^1) - f_1(x_-(1)),$$

as well as

$$u_1^*\theta - \widetilde{u}_1^*\Theta = G^\nu(s, \overline{u}_1(s))d\sigma.$$

Observe that we here abuse notation, and use $\mathfrak{a}(p_j^1) > 0$ for the action computed with respect to the induced potential function on $\psi_s^\nu(L_1)$ for the corresponding value of $s \in \mathbb{R}$.

Since $\|\nu'_{\mathfrak{z}^1}\|_\infty \leq 2$, we finally obtain

$$\mathcal{A}(x_-) \leq \mathcal{A}(x_+) + 2\mu\|G\|_\infty,$$

where $\mu \geq 0$ is as required. □

7.2. Pushing up

In this subsection we prove the following proposition.

PROPOSITION 7.6. – *Let (L_0, ι_0) and (L_1, ι_1) be exact Lagrangian immersions in a Liouville manifold (W, θ) and let $J_\bullet$ be an (L_0, L_1)-regular almost complex structure. If the Liouville flow of (W, θ) displaces L_1 from any compact set, then, for all pair of augmentations ε_0 and ε_1 of the obstruction algebras of (L_0, J_0) and (L_1, J_1) respectively,*

$$\mathrm{HW}((L_0, \varepsilon_0), (L_1, \varepsilon_1), J_\bullet) = 0.$$

We postpone the proof to after a couple of lemmas. Given $\Lambda > \lambda > 0$ and $L > 0$, we define a function $h_{\lambda,\Lambda,R}\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$h_{\lambda,\Lambda,R}(\rho) = \begin{cases} 0 & \text{for } \rho \leq 1, \\ \lambda\rho - \lambda & \text{for } \rho \in [1, e^R], \\ \Lambda\rho - (\Lambda - \lambda)e^R - \lambda & \text{for } \rho \geq e^R. \end{cases} \tag{26}$$

See Figure 2 for the graph of $h_{\lambda,\Lambda,R}$.

The function $h_{\lambda,\Lambda,R}$ has two corners: one at $(1, 0)$ and one at $(e^R, \lambda e^R - \lambda)$. We smooth $h_{\lambda,\Lambda,R}$ in a small neighborhood of these corners so that the new function (which

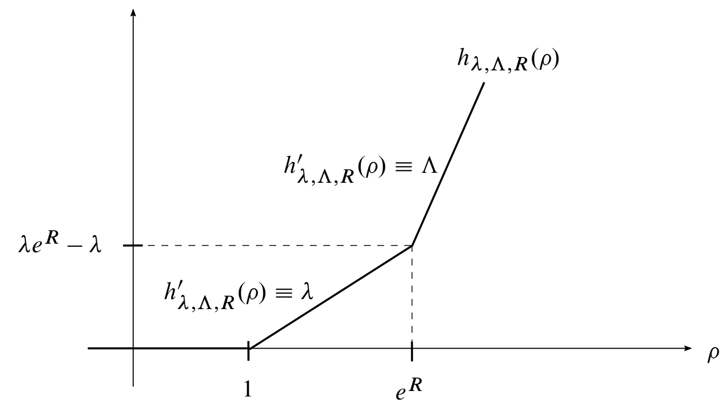

FIGURE 2. The graph of $h_{\lambda,\Lambda,R}$

we still denote by $h_{\lambda,\Lambda,R}$) satisfies $h''_{\lambda,\Lambda,R}(\rho) \geq 0$ for all ρ. We define the (time independent) cylindrical Hamiltonian $H_{\lambda,\Lambda,R}$: $W \to \mathbb{R}$ by $H_{\lambda,\Lambda,R}(w) = h_{\lambda,\Lambda,R}(e^{\mathfrak{r}(w)})$. We make the assumption that there is no Hamiltonian time-1 chord from L_0 to L_1 on ∂W_r when r satisfies either $h'_{\lambda,\Lambda,R}(e^r) = \lambda$ or $h_{\lambda,\Lambda,R}(e^r) = \Lambda$. This is equivalent to assuming that there is no Reeb chord from Λ_0 to Λ_1 of length either λ or Λ.

We assume, without loss of generality, that L_0 and L_1 intersect transversely, that $L_i \cap W_1^e$ is a cylinder over a Legendrian submanifold Λ_i and that all Reeb chords from Λ_0 to Λ_1 are nondegenerate. Then we have three types of chords:

— constant chords, i.e., intersection points between L_0 and L_1, which are contained in W_0,

— chords coming from smoothing the corner of $h_{\lambda,\Lambda,R}$ at $(1, 0)$, which are concentrated around ∂W_1, and

— chords coming from smoothing the corner of $h_{\lambda,\Lambda,R}$ at $(e^R, \lambda e^R - \lambda)$, which are concentrated around ∂W_R.

Constant chords and chords coming from smoothing the first corner will be called *type I chords*, while chords coming from smoothing the second corner will be called *type II chords*. We say that a chord of $H_{\Lambda,\lambda,L}$ *appears at slope* s if it is contained in ∂W_r for r such that $h'_{\Lambda,\lambda,L}(e^r) = s$. By abuse of terminology, we will consider the intersection points between L_0 and L_1 as chords appearing at slope zero.

LEMMA 7.7. – *Given $\lambda > 0$, there exists $C > 0$ such that, for every $\Lambda > \lambda$ and every $R \geq C$, every chord of type II of $H_{\lambda,\Lambda,R}$ has larger action than any chord of type I.*

Proof. – If x is a Hamiltonian chord contained in ∂W_r, then the action of x is

$$\mathcal{A}(x) = h'_{\lambda,\Lambda,R}(e^r)e^r - h_{\lambda,\Lambda,R}(e^r) + f_0(x(0)) - f_1(x(1)). \tag{27}$$

Observe that $|f_0(x(0)) - f_1(x(1))|$ is uniformly bounded because f_0 and f_1 are locally constant outside a compact set. The Hamiltonian chords of type I appear at slope $\lambda_- < \lambda$ and near ∂W_0, and therefore r in Equation (27) is close to zero. Then, there is a constant C_-, depending on f_0, f_1 and the smoothing procedure at the first corner such that, if x is a chord of type I, then $\mathcal{A}(x) \leq \lambda_- + C_-$.

On the other hand, if x is a chord of type II, then it appears at slope $\lambda_+ > \lambda$ and around $r = R$. Then there is a constant C_+, depending on f_0, f_1 and the smoothing procedure at the second corner such that $\mathcal{A}(x) \geq e^R\lambda_+ - \lambda e^R + \lambda - C_+ = e^R(\lambda_+ - \lambda) + \lambda - C_+$. The lemma follows from $\lambda_+ - \lambda > 0$ and the fact that chords arise at a discrete set of slopes. □

From now on we will always take $R \geq C$. The consequence of Lemma 7.7 is that the chords of type I generate a subcomplex of

$$\mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet),$$

which we will denote by $\mathrm{CF}^I((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet)$. The main ingredient in the proof of Proposition 7.6 is the following lemma.

Lemma 7.8. – *If the Liouville flow of (W, θ) displaces L_1 from any compact set, then the inclusion map*

$$\mathrm{CF}^I((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet) \hookrightarrow \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet)$$

is trivial in homology whenever $\Lambda \gg 0$ is sufficiently large.

Proof. – The Liouville flow applied to L_1 gives rise to a compactly supported safe isotopy from L_1 to L_1', and is generated by the time-dependent local Hamiltonian $G\colon \mathbb{R} \times L \to \mathbb{R}$ for which $dG_t = 0$ outside a compact subset of $(0, 1) \times L \to \mathbb{R}$; see Lemma 2.23. Since the Liouville form is *conformally* symplectic, it actually preserves the space of compatible almost complex structures which are cylindrical at infinity.

We will choose to apply the Liouville flow so that $L_1' \subset \{\rho \geq e^R\}$; recall that this is possible by our assumptions.

The continuation map

$$\Psi_G\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1', \varepsilon_1'); H_{\lambda,\Lambda,R}, J_\bullet')$$

defined in Subsection 5.3 induces an isomorphism in homology if $J_\bullet'$ is an (L_0, L_1')-regular almost complex structure such that $J_0' = J_0$ and $J_1' = (\psi_1)_* J_1$, and ε_1' is the augmentation of the obstruction algebra of (L_1', J_1') defined by $\varepsilon_1' = \varepsilon_1 \circ \psi_1^{-1}$.

By Lemma 7.5, there is a constant C, independent of Λ, such that $\Psi_G(x)$ is a linear combination of chords of action at most C whenever x is a chord from L_0 to L_1 of type I.

On the other hand, $\mathrm{CF}((L_0, \varepsilon_0), (L_1', \varepsilon_1); H_{\lambda,\Lambda,R})$ is generated by Hamiltonian chords x of action

$$\mathcal{A}(x) \geq (\Lambda - \lambda)e^R + \lambda + f_0(x(0)) - f_1'(x(1)).$$

Here we have used $L_1' \subset \{\rho \geq e^R\}$, together with the particular form of $H_{\lambda,\Lambda,R}$ in the same subset. This implies that, for Λ large enough, $\Psi_G(x) = 0$ for all chords x of type I. □

Proof of Proposition 7.6. – For every λ and Λ there are continuation maps

$$\Phi^{(1)}_{\lambda,\Lambda}\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\lambda, J_\bullet) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet),$$

$$\Phi^{(2)}_{\lambda,\Lambda}\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\Lambda, J_\bullet) \text{ and}$$

$$\Phi_{\lambda,\Lambda}\colon \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\lambda, J_\bullet) \to \mathrm{CF}((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\Lambda, J_\bullet)$$

such that there is a chain homotopy between $\Phi_{\lambda,\Lambda}$ and $\Phi^{(2)}_{\lambda,\Lambda} \circ \Phi^{(1)}_{\lambda,\Lambda}$.

We can assume that $\Phi_{\lambda,\Lambda}$, $\Phi^{(1)}_{\lambda,\Lambda}$ and $\Phi^{(2)}_{\lambda,\Lambda}$ are defined using s-dependent Hamiltonians H^*_s ($* = \emptyset, (1), (2)$) such that $\partial_s H^*_s \leq 0$, and therefore they decrease the action. Hence, the image of $\Phi^{(1)}_{\lambda,\Lambda}$ is contained in

$$\mathrm{CF}^I((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet)$$

(here we use Lemma 7.7) and therefore it follows from Lemma 7.8 that $\Phi_{\lambda,\Lambda} = 0$ in homology. By the definition of wrapped Floer cohomology as a direct limit, this implies that

$$\mathrm{HW}((L_0, \varepsilon_0), (L_1, \varepsilon_1), J_\bullet) = 0. \qquad \square$$

8. Floer cohomology and Lagrangian surgery

Lalonde and Sikorav in [27] and then Polterovich in [33] defined a surgery operation on Lagrangian submanifolds. It is expected that Lagrangian surgery should correspond to a twisted complex (i.e., an iterated mapping cone) in the Fukaya category. Results in this direction have been proved by Seidel in [36], Fukaya, Oh, Ohta and Ono in [21] and by Biran and Cornea in [7]. After a first version of this article had appeared, Palmer and Woodward gave a more comprehensive treatment of Lagrangian surgery in [32]. Our goal in this section is to establish Proposition 8.16, which provides us with a result along these lines in the generality that we need.

The difficult point in handling the Lagrangian surgery from the Floer theoretic perspective is that, except in very favorable situations, the Lagrangian submanifolds produced are not well behaved from the point-of-view of pseudoholomorphic disks. In our situation, we turn out to be lucky, since only surgeries that preserve exactness are needed. Nevertheless, there still is a complication stemming from the fact that the resulting Lagrangian is only immersed (as opposed to embedded). This is the main reason for the extra work needed, and here we rely on the theory developed in the previous sections.

The bounding cochains that we will consider in this exact immersed setting are those corresponding to augmentations of the corresponding obstruction algebras introduced in Section 4.2. This turns out to be a very useful perspective, since it enables us to apply techniques from Legendrian contact cohomology in order to study them.

8.1. The Cthulhu complex

In this subsection we recall, and slightly generalize, the definition of Floer cohomology for Lagrangian cobordisms we defined in [10].

Definition 8.1. – Given cylindrical exact Lagrangian immersions L_+ and L_- in (W, θ) which coincide outside a compact set, an exact Lagrangian cobordism Σ from L^+_- to L^+_+ is a properly embedded submanifold

$$\Sigma \subset (\mathbb{R} \times M, d(e^t \beta)) = (\mathbb{R} \times W \times \mathbb{R}, d(e^t(dz + \theta)))$$

such that, for C and R sufficiently large,

1. $\Sigma \cap (-\infty, -C] \times W \times \mathbb{R} = (-\infty, -C] \times L^+_-$,
2. $\Sigma \cap [C, +\infty) \times W \times \mathbb{R} = [C, +\infty) \times L^+_+$,

3. $\Sigma \cap (\mathbb{R} \times W_R^e \times \mathbb{R})$ is tangent to both ∂_s and the lift of the Liouville vector field $\mathcal{L}$ of (W, θ), and
4. $e^s\alpha|_\Sigma = dh$ for a function $h\colon \Sigma \to \mathbb{R}$ which is constant on $\Sigma \cap (-\infty, -C] \times W \times \mathbb{R}$.

The intersection

$$\Sigma \cap (\mathbb{R} \times W_R^e \times \mathbb{R}) = \mathbb{R} \times (L_\pm^+ \cap (W_R^e \times \mathbb{R}))$$

defined in (3) is called the *lateral end* of Σ.

The surgery cobordism $\Sigma(a_1, \dots, a_k)$ that we will define in the next subsection clearly satisfies all these properties when $\mathbb{L}(a_1, \dots, a_k)$ is connected.

Given two exact Lagrangian cobordisms Σ^0 and Σ^1 from $(L_-^0)^+$ to $(L_+^0)^+$ and from $(L_-^1)^+$ to $(L_+^1)^+$ with augmentations ε_-^0 and ε_-^1 of $(L_-^0)^+$ and $(L_-^1)^+$ respectively, we define the *Cthulhu complex* $\mathrm{Cth}_{\varepsilon_-^0, \varepsilon_-^1}(\Sigma^0, \Sigma^1)$ which, as an $\mathbb{F}$-module, splits as a direct sum

$$\mathrm{Cth}_{\varepsilon_-^0,\varepsilon_-^1}(\Sigma^0, \Sigma^1) = \mathrm{LCC}_{\varepsilon_+^0,\varepsilon_+^1}((L_+^0)^+, (L_+^1)^+) \oplus \mathrm{CF}_{\varepsilon_-^0,\varepsilon_-^1}(\Sigma^0, \Sigma^1) \oplus \mathrm{LCC}_{\varepsilon_-^0,\varepsilon_-^1}((L_-^0)^+, (L_-^1)^+),$$

where ε_+^i is the augmentation of $\mathfrak{A}((L_+^i)^+)$ induced by ε_-^i and Σ^i, and $\mathrm{CF}_{\varepsilon_-^0,\varepsilon_-^1}(\Sigma^0, \Sigma^1)$ is the $\mathbb{F}$-module freely generated by the intersection points $\Sigma^0 \cap \Sigma^1$, which we assume to be transverse. Furthermore, we assume that $L_\pm^0 \cap L_\pm^1 \cap W_R^e = \emptyset$, which is not a restriction since the ends are cylinders over Legendrian submanifolds.

The differential on the Cthulhu complex can be written as a matrix

$$\mathfrak{d}_{\varepsilon_-^0,\varepsilon_-^1} = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & d_{-0} & d_{--} \end{pmatrix},$$

where d_{++} and d_{--} are the differentials of $\mathrm{LCC}_{\varepsilon_+^0,\varepsilon_+^1}((L_+^0)^+, (L_+^1)^+)$ and $\mathrm{LCC}_{\varepsilon_-^0,\varepsilon_-^1}((L_-^0)^+, (L_-^1)^+)$ respectively, and the other maps are defined by counting J-holomorphic disks in $\mathbb{R} \times M$ with boundary on $\Sigma^0 \cup \Sigma^1$ and boundary punctures asymptotic to Reeb chords from $(L_\pm^1)^+$ to $(L_\pm^0)^+$ and intersection points between Σ^0 and Σ^1. See [10, Section 6] for the detailed definition. The cobordisms considered in [10] have the property that $\Sigma^i \cap [-C, C] \times M$ is compact for every $C > 0$, while here we consider cobordisms with a lateral end. The theory developed in [10] can be extended to the present situation thanks to the following maximum principle.

Lemma 8.2. – *Let J and $\widetilde{J}$ be almost complex structures on $\mathbb{R} \times W_R^e \times \mathbb{R}$ and W_R^e, respectively, each cylindrical inside the respective symplectisation $\mathbb{R} \times W_R^e$ and half-symplectisation W_R^e for some $R > 0$. We moreover require that the canonical projection*

$$(\mathbb{R} \times W_R^e \times \mathbb{R}, J) \to (W_R^e, \widetilde{J})$$

is holomorphic. Then every J-holomorphic map $u\colon \Delta \to \mathbb{R} \times W \times \mathbb{R}$ with

- $\Delta = D^2 \setminus \{\zeta_0, \dots, \zeta_d\}$ *where* $(\zeta_0, \dots, \zeta_d) \in \mathrm{Conf}^{d+1}(\partial D^2)$,
- $u(\partial\Delta) \subset \Sigma^0 \cup \Sigma^1$, *and*
- *u maps some neighborhood of the punctures $\{\zeta_0, \dots, \zeta_d\}$ into $\mathbb{R} \times W_R \times \mathbb{R}$,*

has its entire image contained inside $\mathbb{R} \times W_R \times \mathbb{R}$.

Proof. – By the assumptions the image of the curve $u|_{u^{-1}(\mathbb{R}\times W_R^e\times\mathbb{R})}$ under the canonical projection

$$(\mathbb{R}\times W_R^e\times\mathbb{R}, J)\to (W_R^e, \widetilde{J})$$

is compact with boundary on $\mathbb{R}\times\partial W_R^e\times\mathbb{R}$. The statement is now a consequence of the maximum principle for pseudoholomorphic curves inside $W_R^e\cong[R,+\infty)\times V$ which

— satisfy a cylindrical boundary condition, and

— are pseudoholomorphic for a cylindrical almost complex structure.

Namely, by e.g., [26, Lemma 5.5], the symplectisation coordinate $\mathfrak{r}\colon\ W_R^e\to[R,+\infty)$ restricted to such a curve cannot have a local maximum. □

With Lemma 8.2 at hand, the arguments of [10] go through, and therefore we have the following result.

THEOREM 8.3 ([10]). – *The map $\mathfrak{d}_{\varepsilon^0_-,\varepsilon^1_-}$ is a differential and the Cthulhu complex*

$$(\mathrm{Cth}_{\varepsilon^0_-,\varepsilon^1_-}(\Sigma^0,\Sigma^1),\mathfrak{d}_{\varepsilon^0_-,\varepsilon^1_-})$$

is acyclic.

The consequence of interest for us is the following.

COROLLARY 8.4. – *If $\Sigma^0\cap\Sigma^1=\emptyset$, then the map*

$$d_{+-}\colon\ \mathrm{LCC}_{\varepsilon^0_-,\varepsilon^1_-}((L^0_-)^+,(L^1_-)^+)\to\mathrm{LCC}_{\varepsilon^0_+,\varepsilon^1_+}((L^0_+)^+,(L^1_+)^+)$$

is a quasi-isomorphism.

Proof. – If $\Sigma^0\cap\Sigma^1=\emptyset$, the Cthulhu differential simplifies as follows:

$$\mathfrak{d}_{\varepsilon^0_-,\varepsilon^1_-}=\begin{pmatrix} d_{++} & 0 & d_{+-}\\ 0 & 0 & 0\\ 0 & 0 & d_{--}\end{pmatrix}$$

and thus the Cthulhu complex becomes the cone of d_{+-}. Since it is acyclic, it follows that d_{+-} is a quasi-isomorphism. □

8.2. The surgery cobordism

In this subsection we describe the Lagrangian surgery of [27] and [33] from the Legendrian viewpoint. In particular, we interpret it as a Lagrangian cobordism between the Legendrian lifts of the Lagrangian submanifolds before and after the surgery. We refer to [14] for more details.

We first describe the local model for Lagrangian surgery. Given $\eta,\delta>0$, we consider the open subset

$$\mathcal{V}_{\eta,\delta}:=\{|q|<\eta,|p|<2\delta,z\in\mathbb{R}\}$$

of $\mathcal{J}^1(\mathbb{R}^n)$. Given $\zeta>0$, we denote by $\Lambda^+_{\eta,\delta,\zeta}$ the (disconnected) Legendrian submanifold of $\mathcal{V}_{\eta,\delta}$ given by the two sheets

$$\{(q,\pm df_{\eta,\delta,\zeta}(|q|),\pm f_{\eta,\delta,\zeta}(|q|)) : |q|<\eta\},$$

where

$$f_{\eta,\delta,\zeta}(s) = \frac{\delta}{2\eta}s^2 + \frac{\zeta}{2}.$$

This is a Legendrian submanifold with a single Reeb chord of length ζ. Note that $\Lambda^+_{\eta,\delta,\zeta}$ is described by the generating family $F^+_{\eta,\delta,\zeta}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$F^+_{\eta,\delta,\zeta}(q,\xi) = \frac{\xi^3}{3} - g^+(|q|)\xi,$$

where

$$g^+(s) = \left(\frac{3}{2}f_{\eta,\delta,\zeta}(s)\right)^{\frac{2}{3}}.$$

Note that g^+ is smooth because $g^+(s) > 0$ holds for every s. Let $g^- : \mathbb{R}_+ \to \mathbb{R}$ be a function such that

(i) $g^-(s) = \left(\frac{3}{2}f_{\eta,\delta,\zeta}(s)\right)^{\frac{2}{3}}$ for $s > 3\eta/4$,
(ii) $g^-(s) < 0$ for $s < \eta/2$, and
(iii) $0 < (g^-)'(s) < 2\frac{\delta\eta}{\delta\eta+\zeta}$.

Note that Condition (iii) can be achieved if $\zeta < \frac{7\delta\eta}{16}$. The Legendrian submanifold $\Lambda^-_{\eta,\delta,\zeta}$ of $\mathcal{V}_{\delta,\eta}$ generated by

$$F^-_{\eta,\delta,\zeta}(q,\xi) = \frac{\xi^3}{3} - g^-(|q|)\xi$$

coincides with $\Lambda^+_{\eta,\delta,\zeta}$ near $|q| = \epsilon$ and has no Reeb chords (see Figure 3). Note that indeed $\Lambda^-_{\eta,\delta,\zeta} \subset \mathcal{V}_{\delta,\eta}$ because Condition (iii) ensures that the p-coordinates of $\Lambda^-_{\eta,\delta,\zeta}$, given by $\frac{\partial F^-_{\eta,\delta,\zeta}}{\partial p_i}$ along critical values of $F(q,\cdot)$, are smaller than 2δ.

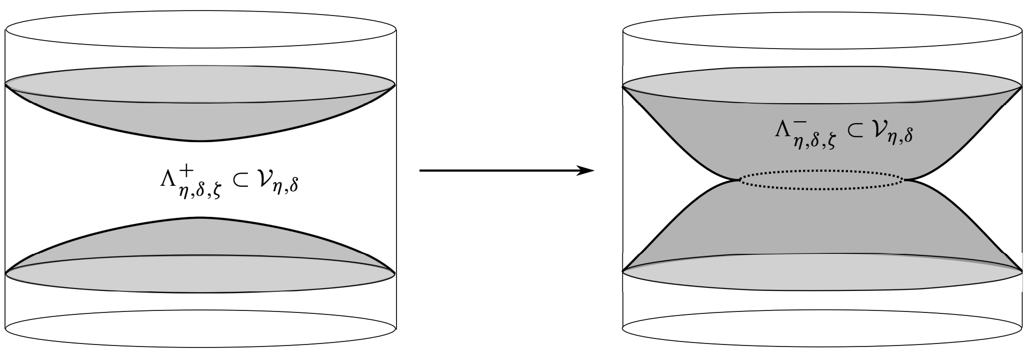

FIGURE 3. The front projections of Λ^+ and Λ^-

On Figure 4 we see the front and Lagrangian projections of the one-dimensional version of Λ^+ and Λ^-.

Let $\mathbb{L}$ be an exact Lagrangian immersion in (W,θ) with double points $a_1,\dots,a_k$, and let $\mathbb{L}^+$ be a Legendrian lift of $\mathbb{L}$. The double points of $\mathbb{L}$ lift to Reeb chords of $\mathbb{L}^+$ which we will denote with the same name by an abuse of notation.

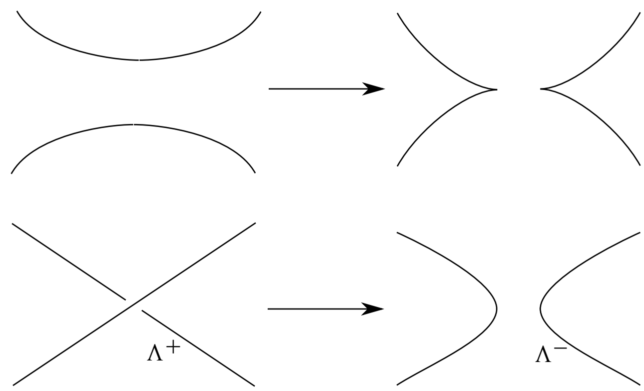

FIGURE 4. Front (top) and Lagrangian (bottom) projections of the Lagrangian surgery

DEFINITION 8.5. – A set of Reeb chords $\{a_1, \dots, a_k\}$ on $\mathbb{L}^+$ is called *contractible* if, for all $i = 1, \dots, k$, there is a neighborhood $\mathcal{U}_i$ of the Reeb chord a_i in the contactisation (M, β) of (W, θ) and a strict contactomorphism $(\mathcal{U}_i, \mathcal{U}_i \cap \mathbb{L}^+) \cong (\mathcal{V}_{\eta_i,\delta_i}, \Lambda^1_{\eta_i,\delta_i,\zeta_i})$ for numbers $\eta_i, \delta_i, \zeta_i$ satisfying $\zeta_i < \frac{7\delta_i\eta_i}{16}$.

REMARK 8.6. – This is a restrictive assumption because, in general, the lengths of the chords $a_1, \dots, a_k$ cannot be modified independently. An example when this is possible, and which will be the case in our main theorem, is when $\mathbb{L}^+$ is a link with $k+1$ components, all a_i are mixed chords, and each component contains either the starting point or the end point of at least one of the a_i. In this situation we can indeed modify the Legendrian link by Legendrian isotopies of each of his components so that its Lagrangian projection is unchanged and all the previous conditions on the neighborhoods are satisfied. (Note that this might not be an isotopy of the Legendrian *link*.)

In the following we assume that $\{a_1, \dots, a_k\}$ is a set of contractible Reeb chords on $\mathbb{L}^+$. We denote by $\mathbb{L}^+(a_1, \dots, a_k)$ the Legendrian submanifold of (M, β) obtained by replacing each of the $\Lambda^+_{\eta_i,\delta_i,\zeta_i}$ by the corresponding $\Lambda^-_{\eta_i,\delta_i,\zeta_i}$ and by $\mathbb{L}(a_1, \dots, a_k)$ the Lagrangian projection of $\mathbb{L}^+(a_1, \dots, a_k)$. Observe that here we need to make use of the identifications with the standard model, which exists by the contractibility condition.

Then $\mathbb{L}(a_1, \dots, a_k)$ is an exact Lagrangian immersion in (W, θ) which is the result of Lagrangian surgery on $\mathbb{L}$ along the self-intersection points $a_1, \dots, a_k$. It is evident from the construction that $\mathbb{L}(a_1, \dots, a_k)$ coincides with $\mathbb{L}$ outside a neighborhood of the a_i's and has k self-intersection points removed. The latter fact follows from the fact that since ζ_i can be chosen arbitrarily small, no Reeb chords are created when going from $\Lambda^+_{\eta_i,\delta_i,\zeta_i}$ to $\Lambda^-_{\eta_i,\delta_i,\zeta_i}$.

Next we construct an exact Lagrangian cobordism $\Sigma(a_1, \dots, a_k)$ in the symplectisation of (M, β) with $\mathbb{L}$ at the positive end and $\mathbb{L}(a_1, \dots, a_k)$ at the negative end. Fix $T > 0$ and choose a function $G\colon (0, \epsilon) \times \mathbb{R}_+ \to \mathbb{R}$ such that:

— $G(t, s) = g^-(s)$ for $t < 1/T$,

— $G(t, s) = g^+(s)$ for $t > T$,

— $\frac{\partial G}{\partial t}(t, s) \geq 0$ with strict inequality at $s = 0$, and

— $G(t,s) = g^+(s) = g^-(s)$ for $s > 3\eta/4$.

We consider the Lagrangian submanifold of $T^*(\mathbb{R}^+ \times B^n(\eta))$ described by the generating family

$$F(t,q,\xi) = t \cdot \left(\frac{\xi^3}{3} + G(t,|q|)\xi\right),$$

which is mapped by the symplectomorphism $T^*(\mathbb{R}^+ \times B^n(\eta)) \cong \mathbb{R} \times \mathcal{J}^1(B^n(\eta))$ to a Lagrangian cobordism $\Sigma_{\eta,\delta,\zeta}$ in the symplectisation of (M,β) from $\Lambda^-_{\eta_k,\delta_k,\zeta_k}$ at the negative end to $\Lambda^+_{\eta_k,\delta_k,\zeta_k}$ at the positive end. Self-intersections of $\Sigma_{\eta,\delta,\zeta}$ are given by the critical points of the function

$$\Delta_F(t,q,\xi_1,\xi_2) = F(t,q,\xi_1) - F(t,q,\xi_2)$$

with non-zero critical value, and such points do not exist because of the third condition on G. Thus this cobordism is embedded.

In the trivial cobordism $\mathbb{R}\times\mathbb{L}^+$ we replace $\mathbb{R}\times(\mathcal{U}_i \cap \mathbb{L}^+)$ with $\Sigma_{\eta_i,\delta_i,\zeta_i}$, for all $i = 1,\dots,k$, to get a cobordism $\Sigma(a_1,\dots,a_k)$ from $\mathbb{L}^+(a_1,\dots,a_k)$ at the negative end to $\mathbb{L}^+$ at the positive end.

8.3. Effect of surgery on Floer cohomology

In this subsection we use $\Sigma(a_1,\dots,a_k)$ and our Floer theory for Lagrangian cobordisms to relate the Floer cohomology of $\mathbb{L}$ with the Floer homology of $\mathbb{L}(a_1,\dots,a_k)$. The Lagrangian cobordism $\Sigma(a_1,\dots,a_k)$ induces a dga morphism

$$\Phi_\Sigma\colon \mathfrak{A}(\mathbb{L}^+) \to \mathfrak{A}(\mathbb{L}^+(a_1,\dots,a_k)).$$

If follows from [14, Theorem 1.1] that, for a suitable almost complex structure on the cobordism that has been obtained by perturbing an arbitrary cylindrical almost complex structure, we have

$$\begin{aligned}\Phi_\Sigma(a_i) &= 1 \text{ for } i = 1,\dots,k,\\ \Phi_\Sigma(c) &= c + \mathbf{w} \text{ if } c \neq a_i,\end{aligned} \tag{28}$$

where $\mathbf{w}$ is a linear combination of products $c_1 \cdots c_m$ with

$$\mathfrak{a}(c_1) + \cdots + \mathfrak{a}(c_m) < \mathfrak{a}(c).$$

LEMMA 8.7. – *If $\varepsilon\colon \mathfrak{A}(\mathbb{L}^+) \to \mathbb{F}$ is an augmentation such that $\varepsilon(a_i) = 1$ for $i = 1,\dots,k$, then there is an augmentation $\overline{\varepsilon}\colon \mathfrak{A}(\mathbb{L}^+(a_1,\dots,a_k)) \to \mathbb{F}$ such that $\varepsilon = \overline{\varepsilon} \circ \Phi_\Sigma$.*

Proof. – Let $\mathfrak{I}$ be the bilateral ideal generated by $a_i - 1,\dots,a_k - 1$: then ε induces an augmentation

$$\overline{\varepsilon}\colon \mathfrak{A}(\mathbb{L}^+)/\mathfrak{I} \to \mathbb{F}.$$

By Equation (28) Φ_Σ is surjective and its kernel is $\mathfrak{I}$. Surjectivity is proved by a sort of Gauss elimination using the action filtration. Then there is an isomorphism between $\mathfrak{A}(\mathbb{L}^+)/\mathfrak{I}$ and $\mathfrak{A}(\mathbb{L}^+(a_1,\dots,a_k))$, and therefore the augmentation $\overline{\varepsilon}\colon \mathfrak{A}(\mathbb{L}^+)/\mathfrak{I} \to \mathbb{F}$ induces an augmentation on $\mathfrak{A}(\mathbb{L}^+(a_1,\dots,a_k))$, which we still denote by $\overline{\varepsilon}$. □

The construction of $\overline{\varepsilon}$ is not explicit because the isomorphism $\mathfrak{A}(\mathbb{L}^+)/\mathfrak{I} \cong \mathfrak{A}(\mathbb{L}^+(a_1,\dots,a_k))$ is not explicit.

PROPOSITION 8.8. – *For any immersed cylindrical exact Lagrangian submanifold $T \subset W$ with augmentation ε' there is a quasi-isomorphism*

$$\mathrm{LCC}_{\varepsilon',\overline{\varepsilon}}(T^+, \mathbb{L}^+(a_1, \ldots, a_k)) \xrightarrow{\simeq} \mathrm{LCC}_{\varepsilon',\varepsilon}(T^+, \mathbb{L}^+),$$

under the assumption that the augmentations ε and $\overline{\varepsilon}$ are as in Lemma 8.7.

Proof. – We denote by Σ_T the trivial cobordism $\Sigma_T = \mathbb{R} \times T^+ \subset \mathbb{R} \times M$. Recall that the surgery cobordism goes from $\mathbb{L}^+(a_1, \ldots, a_k)$ to $\mathbb{L}^+$. Since the surgery is localized to a neighborhood of the intersection points $a_1, \ldots, a_k$, by a Hamiltonian isotopy we can assume that

$$\Sigma_T \cap \Sigma(a_1, \ldots, a_k) = \emptyset.$$

Then Corollary 8.4 implies that the map d_{+-} in the Cthulhu differential for the cobordisms Σ_T and $\Sigma(a_1, \ldots, a_k)$ is a quasi-isomorphism. □

LEMMA 8.9. – *Let $\mathbb{L}'$ be an immersed exact Lagrangian submanifold with an augmentation ε' and let $\mathbb{L}^+, (\mathbb{L}')^+$ be Legendrian lifts such that $\mathbb{L}^+$ is above $(\mathbb{L}')^+$. When Lemma 8.7 is applied to an augmentation*

$$\varepsilon_c \colon \mathfrak{A}(\mathbb{L}^+ \cup (\mathbb{L}')^+) \to \mathbb{F}$$

induced by the cycle

$$c \in \mathrm{CF}((\mathbb{L}, \varepsilon), (\mathbb{L}', \varepsilon'))$$

as in Lemma 4.16, then the push-forward of the augmentation under the DGA morphism denoted by

$$\overline{\varepsilon_c} = \overline{\varepsilon}_{\overline{c}} \colon \ \mathfrak{A}(\mathbb{L}^+(a_1, \ldots, a_k) \cup (\mathbb{L}')^+) \to \mathbb{F}$$

is induced by a cycle

$$\overline{c} \in \mathrm{CF}((\mathbb{L}(a_1, \ldots, a_k), \overline{\varepsilon}), (\mathbb{L}', \overline{\varepsilon'})),$$

which moreover is mapped to c under the quasi-isomorphism from Proposition 8.8 .

Proof. – There is no Reeb chord starting on either $\mathbb{L}^+$ or $\mathbb{L}^+(a_1, \ldots, a_k)$ and ending on $(\mathbb{L}')^+$, so the pushed-forward augmentation is automatically of the form $\overline{\varepsilon}_{\overline{c}}$. Lemma 4.16 then implies that $\overline{c}$ is a cycle.

The last statement is an algebraic consequence of the fact that the disks counted by the DGA morphism Φ_Σ induced by the surgery cobordism can be identified with the disks counted by the quasi-isomorphism from Proposition 8.8. □

Now assume that $\mathbb{L}'$ is a push off of $\mathbb{L}$ as constructed in Lemma 4.14, and let $e \in \mathrm{CF}(\mathbb{L}, \mathbb{L}')$ be the "unit" defined by the sum of the local minima e_i of the Morse function on the connected components of $\mathbb{L}$; i.e., $e = \sum e_i$.

COROLLARY 8.10. – *The cycle*

$$\overline{e} \in \mathrm{LCC}_{\overline{\varepsilon},\varepsilon'}(\mathbb{L}^+(a_1, \ldots, a_k), (\mathbb{L}')^+)$$

provided by Lemma 8.9 (which is mapped to e under the quasi-isomorphism by Proposition 8.8) satisfies the property that

$$\mu^2(\overline{e}, \cdot) \colon \ \mathrm{CF}(T, (\mathbb{L}(a_1, \ldots, a_k), \overline{\varepsilon})) \to \mathrm{CF}(T, (\mathbb{L}', \varepsilon'))$$

is a quasi-isomorphism for any exact Lagrangian submanifold T with cylindrical end.

Proof. – Consider the Legendrian lift $\mathbf{L}^+ = \mathbb{L}^+ \cup (\mathbb{L}')^+$ such that $\mathbb{L}^+$ is above $(\mathbb{L}')^+$. Then the lift $\mathbf{L}^+(a_1, \ldots, a_k) = \mathbb{L}^+(a_1, \ldots, a_k) \cup (\mathbb{L}')^+$ is specified uniquely by the requirement that it coincides with the first lift outside a compact subset.

Recall that e is closed by Lemma 4.15 and by Lemma 4.16 there is thus an induced augmentation ε_e of $\mathfrak{A}(\mathbf{L}^+)$. Recall that this augmentation coincides with ε and ε' when restricted to the generators on the components $\mathbb{L}^+$ and $(\mathbb{L}')^+$, respectively, while $\varepsilon_e(e_i) = 1$ holds for any chord corresponding to a local minimum while $\varepsilon_e(c) = 0$ for every other chord c between $\mathbb{L}^+$ and $(\mathbb{L}')^+$.

Applying Proposition 8.8 to the Legendrian $\mathbf{L}^+(a_1, \ldots, a_k)$ obtained by surgery on $\mathbf{L}$, yields a quasi-isomorphism

$$\mathrm{LCC}_{\overline{\varepsilon}_{\overline{e}}}(T^+, \mathbf{L}^+(a_1, \ldots, a_k)) \xrightarrow{\cong} \mathrm{LCC}_{\varepsilon_e}(T^+, \mathbf{L}^+).$$

(Here we use that $\overline{\varepsilon}_{\overline{e}} = \overline{\varepsilon_e}$ by Lemma 8.9.) The complex on the right-hand side is acyclic by Lemma 4.14, and hence so is the complex on the left-hand side. The sought statement is now a consequence of the straight-forward algebraic fact that the complex

$$\mathrm{LCC}_{\overline{\varepsilon}_{\overline{e}}}(T^+, \mathbf{L}^+(a_1, \ldots, a_k))$$

is equal to the mapping cone of

$$\mu^2(\overline{e}, \cdot)\colon\ \mathrm{CF}(T, (\mathbb{L}(a_1, \ldots, a_k), \overline{\varepsilon})) \to \mathrm{CF}(T, (\mathbb{L}', \varepsilon')). \qquad \square$$

8.4. Twisted complexes

The aim of this section is to relate the geometric notion of Lagrangian surgery to the algebraic notion of twisted complex in the wrapped Fukaya category. We first recall the definition of a twisted complex in an A_∞-category.

Given a unital A_∞-category $\mathcal{A}$, we describe the category $\mathrm{Tw}\,\mathcal{A}$ of twisted complexes over $\mathcal{A}$ and recall its basic properties. We introduce the following notation: given a number d of matrices A_i with coefficients in the morphism spaces of an A_∞-algebra, we denote by $\mu^d_{\mathcal{A}}(A_d, \ldots, A_1)$ the matrix whose entries are obtained by applying $\mu^d_{\mathcal{A}}$ to the entries of the formal product of the A_i's.

Definition 8.11. – A *twisted complex* over $\mathcal{A}$ is given by the following data:

- a finite collection of objects $L_0, \ldots, L_k$ of $\mathcal{A}$ for some k,
- integers κ_i for $i = 0, \ldots, k$, and
- a matrix $X = (x_{ij})_{0 \le i,j \le k}$ such that $x_{ij} \in \hom_{\mathcal{A}}(L_i, L_j)$ and $x_{ij} = 0$ if $i \ge j$, which satisfies the *Maurer-Cartan equation*

$$\sum_{d=1}^{k} \mu^d_{\mathcal{A}}(\underbrace{X, \ldots, X}_{d \text{ times}}) = 0.$$

The integers κ_i are degree shifts and are part of the definition only if the morphism spaces $\hom_{\mathcal{A}}(L_i, L_j)$ are graded, and otherwise are suppressed.

Given two twisted complexes $\mathfrak{L} = (\{L_i\}, \{\kappa_i\}, X)$ and $\mathfrak{L}' = (\{L'_i\}, \{\kappa'_i\}, X')$ we define

$$\hom_{\mathrm{Tw}\,\mathcal{A}}(\mathfrak{L}, \mathfrak{L}') := \bigoplus_{i,j} \hom_{\mathcal{A}}(L_i, L_j)[\kappa_i - \kappa'_j]$$

and, given $d+1$ twisted complexes $\mathcal{L}_0, \dots, \mathcal{L}_d$, we define A_∞ operations

$$\mu^d_{\mathrm{Tw}\,\mathcal{A}} \colon \hom_{\mathrm{Tw}\,\mathcal{A}}(\mathfrak{L}_{d-1}, \mathfrak{L}_d) \otimes \cdots \otimes \hom_{\mathrm{Tw}\,\mathcal{A}}(\mathfrak{L}_0, \mathfrak{L}_1) \to \hom_{\mathrm{Tw}\,\mathcal{A}}(\mathfrak{L}_0, \mathfrak{L}_d)$$

by

(29)
$$\mu^d_{\mathrm{Tw}\,\mathcal{A}}(q_d, \dots, q_1) = \sum_{k_1,\dots,k_d \geq 0} \mu_{\mathcal{A}}^{k_1+\cdots+k_d+d}(\underbrace{X_d, \dots, X_d}_{k_d}, q_d, X_{d-1}, \dots, X_1, q_1, \underbrace{X_0 \dots, X_0}_{k_0}).$$

It is shown in [37, Section (3l)] that the set of twisted complexes with operations $\mu^d_{\mathrm{Tw}\,\mathcal{A}}$ constitutes an A_∞-category $\mathrm{Tw}\,\mathcal{A}$ which contains $\mathcal{A}$ as a full subcategory. Furthermore it is shown in [37, Lemma 3.32 and Lemma 3.33] that $\mathrm{Tw}\,\mathcal{A}$ is the triangulated envelope of $\mathcal{A}$ and thus $H^0\,\mathrm{Tw}(\mathcal{A})$ is the derived category of $\mathcal{A}$.

Definition 8.12. – We say that a collection of objects $L_1, \dots, L_k$ of $\mathcal{A}$ generates $\mathcal{A}$ if and only if any object L of $\mathcal{A}$ is quasi-isomorphic in $\mathrm{Tw}\,\mathcal{A}$ to a twisted complex built from the object L_i's.

Lemma 8.13. – *If there is a twisted complex $\mathfrak{L}$ built from $L_0, \dots, L_k$ such that, for every object T of $\mathcal{A}$ we have $H \hom_{\mathrm{Tw}\,\mathcal{A}}(T, \mathfrak{L}) = 0$, then L_0 is quasi-isomorphic in* $\mathrm{Tw}\,\mathcal{A}$ *to a twisted complex built from $L_1, \dots, L_k$.*

Proof. – This follows from the iterated cone description of twisted complexes from [37, Lemma 3.32]. More precisely, from the definition of twisted complexes, for any object T we have that $\hom_{\mathcal{A}}(T, L_0)$ is a quotient complex of $\hom_{\mathrm{Tw}\,\mathcal{A}}(T, \mathfrak{L})$ by the twisted complex $\mathfrak{L}'$ built from $\mathfrak{L}$ starting at L_1 (i.e., "chopping" out L_0 from the twisted complex $\mathfrak{L}$), and thus those three objects fit in an exact triangle. The vanishing of $H \hom_{\mathrm{Tw}\,\mathcal{A}}(T, \mathfrak{L})$ implies then that

$$H \hom_{\mathcal{A}}(T, L_0) \cong H \hom_{\mathrm{Tw}\,\mathcal{A}}(T, \mathfrak{L}').$$

The result follows now because the map from L_0 to $\mathfrak{L}'$, which is given by the maps (x_{0j}), is a map of twisted complexes. □

We now relate twisted complexes in the wrapped Fukaya category with certain augmentations of the Chekanov-Eliashberg algebra of the Legendrian lift of the involved Lagrangian submanifolds.

Remark 8.14. – In the following lemma we will make a slight abuse of notation by building twisted complexes from immersed exact Lagrangian submanifolds: to our knowledge, the wrapped Fukaya category has not yet been extended to include also exact *immersed* Lagrangian submanifolds. However, since the statements and proofs only concern transversely intersecting Lagrangian submanifolds, there are no additional subtleties arising when considering the A_∞ operations. In other words, we only consider morphisms between *different* objects in the category. We can thus think of twisted complexes in the "Fukaya pre-category". Of course if all Lagrangian submanifolds L_i involved are embedded, the statements make sense also in the ordinary wrapped Fukaya category.

LEMMA 8.15. – *Let (L_i, ϵ_i), for $i = 0, \dots, k$, be unobstructed exact immersed Lagrangian submanifolds which are assumed to be equipped with fixed potentials f_i.*

We denote $\mathbb{L} = L_1 \cup \dots \cup L_k$ and $\mathbb{L}^+$ its Legendrian lift determined by the given potentials. We assume that $\mathbb{L}^+$ is embedded.

If $\boldsymbol{\varepsilon}\colon \mathfrak{A}(\mathbb{L}^+) \to \mathbb{F}$ is an augmentation such that:

1. *$\boldsymbol{\varepsilon}(p) = \varepsilon_i(p)$ for every pure chord p of L_i^+, and*
2. *$\boldsymbol{\varepsilon}(a) = 0$ for every mixed chord a from L_i^+ to L_j^+ such that $i > j$,*

we define

$$x_{ij} := \begin{cases} \sum\limits_{a \in L_i \cap L_j} \boldsymbol{\varepsilon}(a) a & \text{if } i < j, \\ 0 & \text{if } i \geq j, \end{cases}$$

and $X = (x_{ij})_{0 \leq i,j \leq k}$, where the double point a is considered as an element in the summand with wrapping parameter $w = 0$ (see Section 6.2). Then (ignoring the degrees for simplicity) the pair $\mathfrak{L} = (\{(L_i, \varepsilon_i)\}, X)$ is a twisted complex in the wrapped Fukaya category. Moreover, for any test Lagrangian submanifold T,

$$H \hom_{\mathrm{Tw}\,\mathcal{WF}}(T, \mathfrak{L}) = \mathrm{HW}(T, (\mathbb{L}, \boldsymbol{\varepsilon})).$$

Proof. – Denote by $\boldsymbol{\varepsilon}_0$ the augmentation of $\mathfrak{A}(\mathbb{L}^i)$ which vanishes on the mixed chords, while taking the value ε_i on the generators living on the component L_i. Recall the chain model for wrapped Floer complex described in Subsection 6.2, where the homotopy direct limit $\mathrm{CW}((\mathbb{L}, \boldsymbol{\varepsilon}_0), (\mathbb{L}, \boldsymbol{\varepsilon}_0); J_\bullet)$ is an infinite direct sum starting with the term having a wrapping parameter $w = 0$, i.e., the complex

$$\mathrm{CF}((\mathbb{L}, \boldsymbol{\varepsilon}_0), (\mathbb{L}, \boldsymbol{\varepsilon}_0); \mathbf{0}, J_\bullet) \oplus \mathrm{CF}((\mathbb{L}, \boldsymbol{\varepsilon}_0), (\mathbb{L}, \boldsymbol{\varepsilon}_0); \mathbf{0}, J_\bullet) q.$$

The bounding cochain X can be identified to a sum of elements in the leftmost summand by definition.

Note that $\mathbb{L}$, of course, is only immersed. However, in the case where it consists of a union of embeddings, it still represents an object in the twisted complexes of the ordinary wrapped Fukaya category; namely, it is the "direct sum" of the Lagrangian submanifolds L_i, $i = 1, \dots, m$.

First we prove that X satisfies the Maurer-Cartan equation. The Maurer-Cartan equation involves a count of holomorphic polygons in moduli spaces

$$\mathfrak{M}^0_{L_0,\dots,L_d}(\mathbf{p}^d, a_0, \mathbf{p}^1, a_1, \dots, \mathbf{p}^{d-1}, a_d; J)$$

as in Section 4.5. On the other hand, the equation $\boldsymbol{\varepsilon} \circ \mathfrak{d} = 0$ counts holomorphic polygons in the moduli spaces $\mathfrak{N}^0_{\mathbb{L}}(a_0; \mathbf{p}^1, a_1, \dots, a_d, \mathbf{p}^d)$, which are the subset of the previous moduli spaces consisting of those holomorphic polygons which satisfy the extra requirement that the intersection points $a_1, \dots, a_d$ should be negative punctures (in the sense of Definition 3.4). Condition (2) in the definition of $\boldsymbol{\varepsilon}$ however implies that $\#\mathfrak{M}^0_{L_0,\dots,L_d}(\mathbf{p}^d, a_0, \mathbf{p}^1, a_1, \dots, \mathbf{p}^{d-1}, a_d; J)$ is multiplied by a nonzero coefficient only if $a_1, \dots, a_d$ are negative punctures. This proves that X satisfies the Maurer-Cartan equation.

For the second part, note that the differential in $\hom_{\mathrm{Tw}\,\mathcal{WF}}(T, \mathfrak{L})$ counts the same holomorphic polygons (with Hamiltonian perturbations) as the differential in $\mathrm{CW}(T, (\mathbb{L}, \boldsymbol{\varepsilon}))$

because the Maurer-Cartan element X involves only elements in the Floer complexes defined with wrapping parameter $w = 0$, and hence vanishing Hamiltonian term. □

The previous lemma together with Proposition 8.8 implies the following result, which is the main result of this section:

PROPOSITION 8.16. – *Let* $(L_1, \varepsilon_1), \ldots, (L_m, \varepsilon_m)$ *be unobstructed immersed exact Lagrangian submanifolds with preferred choices of potentials* f_i, *and let* $a_1, \ldots, a_k$ *be a set of intersection points lifting to contractible Reeb chords on the induced Legendrian lift* $\mathbb{L}^+$, *where* $\mathbb{L} := L_1 \cup \cdots \cup L_m$. *Assume that there is an augmentation* $\boldsymbol{\varepsilon}$ *of the Chekanov-Eliashberg algebra of* $\mathbb{L}^+$ *such that:*

(1) $\boldsymbol{\varepsilon}(c) = \varepsilon_i(c)$ *if* c *is a double point of* L_i,

(2) $\boldsymbol{\varepsilon}(a_i) = 1$ *for* $i = 1, \ldots, k$, *and*

(3) $\boldsymbol{\varepsilon}(q) = 0$ *if* $q \in L_i \cap L_j$ *is an intersection point, with* $i > j$, *at which* $f_i(q) > f_j(q)$ *(i.e., q corresponds to a Reeb chord from* L_i^+ *to* L_j^+*).*

Then for any other exact Lagrangian submanifold T *there is a quasi-isomorphism*

$$\mathrm{CW}(T, (\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}})) \cong \hom(T, \mathfrak{L}),$$

with $\overline{\boldsymbol{\varepsilon}}$ *induced by* $\boldsymbol{\varepsilon}$ *as in Lemma 8.7, and where* $\mathfrak{L}$ *is a twisted complex built from the* L_i *with* $i = 1, \ldots, m$.

REMARK 8.17. – Conditions (2) and (3) of Proposition 8.16 imply that if a_k is an intersection point between different Lagrangians L_i and L_j for $i < j$, then $f_i(a_k) < f_j(a_k)$. Conditions (1) and (2) of Proposition 8.16 imply that if a_k is a self-intersection point of L_i, then augmentation ε_i evaluates to 1 on a_k.

Proof of Proposition 8.16. – We consider the twisted complex $\mathfrak{L}$ built from L_i, $i = 1, \ldots, m$, that is constructed by an application of Lemma 8.15 with the augmentation $\boldsymbol{\varepsilon}$. In other words, the twisted complex is defined using the Maurer-Cartan element

$$X := a_1 + \cdots + a_k \in \mathrm{CF}((\mathbb{L}, \boldsymbol{\varepsilon}_0), (\mathbb{L}, \boldsymbol{\varepsilon}_0); \mathbf{0}, J_\bullet) \subset \mathrm{CW}((\mathbb{L}, \boldsymbol{\varepsilon}_0), (\mathbb{L}, \boldsymbol{\varepsilon}_0); J_\bullet)$$

living in the summand with wrapping parameter $w = 0$. The quasi-isomorphism

$$\hom(T, \mathfrak{L}) \cong \mathrm{CW}(T, (\mathbb{L}, \boldsymbol{\varepsilon}))$$

is then a consequence of the same lemma.

What remains is constructing a quasi-isomorphism

$$\mathrm{CW}(T, (\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}})) \cong \hom(T, \mathfrak{L})$$

for all test Lagrangian submanifolds T. This is done by considering the twisted complex corresponding to the cone of the "unit" $\overline{e}$ from Corollary 8.10. We proceed to give the details.

Let $\mathbb{L}'$ be the push-off of $L_1 \cup \cdots \cup L_m$ as considered in Lemma 4.14.

Consider the cycle $\overline{e} \in \mathrm{CW}((\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}}), (\mathbb{L}', \boldsymbol{\varepsilon}'))$ supplied by Corollary 8.10. As above, $\overline{e}$ is an element in the summand

$$\mathrm{CF}((\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}}), (\mathbb{L}', \boldsymbol{\varepsilon}'); \mathbf{0}, J_\bullet) \subset \mathrm{CW}((\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}}), (\mathbb{L}', \boldsymbol{\varepsilon}')); J_\bullet)$$

with wrapping parameter $w = 0$. Then

$$\big(\{(\mathbb{L}(a_1, \dots, a_k), \overline{\boldsymbol{\varepsilon}}), (\mathbb{L}', \boldsymbol{\varepsilon}')\}, \overline{e}\big)$$

is a twisted complex $\mathfrak{L}'$ corresponding to the cone of $\mu^2(\overline{e}, \cdot)$. The last part of Corollary 8.10 combined with Lemma 8.13 then establishes the sought quasi-isomorphism. Indeed, every summand

$$\mathrm{CF}(T, (\mathbf{L}, \boldsymbol{\varepsilon}_{\overline{e}}); w \cdot H, J_\bullet) \subset \mathrm{CW}(T, (\mathbf{L}, \boldsymbol{\varepsilon}_{\overline{e}}); J_\bullet)$$

in the homotopy direct limit which computes the homology of the cone is acyclic by Corollary 8.10. (Here $\mathbf{L} = \mathbb{L}(a_1, \dots, a_k) \cup \mathbb{L}'$ as in the proof of the latter corollary.)

Here we remind the reader that the components of $\mathbb{L}(a_1, \dots, a_k)$ typically are only immersed, as opposed to embedded. The statement that

$$\mathrm{CW}(T, (\mathbb{L}(a_1, \dots, a_k), \overline{\boldsymbol{\varepsilon}})) \cong \hom(T, \mathfrak{L})$$

is hence established on the level of twisted complexes on the pre-category level; cf. Remark 8.14. □

9. Generating the Wrapped Fukaya category

In this section we prove Theorem 1.1.

9.1. Geometric preparation

Before proving the main theorems we need some geometric preparation which will be used in the technical work of Section 9.3. Recall that the Liouville form θ has been modified in order to make $(\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l, \theta, \mathfrak{f})$ into a union of standard critical Weinstein handles. After adding the differential of a function supported in a small neighborhood of $\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l$, we change the Liouville form once again so that the symplectomorphism between $(\mathcal{H}_i, d\theta)$ and $(D_\delta T^* C_i, d\,\mathbf{p} \wedge d\,\mathbf{q})$ maps the new Liouville form θ_c to $\mathbf{p} d\,\mathbf{q}$. We make the modification so that the new Liouville vector field $\mathcal{L}_c$ is still positively transverse to ∂W_0, has no zeros outside W_0, and so that the new and old Lagrangian skeleta coincide. (On the other hand $\mathcal{L}_c$ is no longer a pseudo-gradient vector field for $\mathfrak{f}$, but this will not impair the proof of Theorem 1.1.) Note that the above identification maps the core of a handle to the zero section and the cocore into a cotangent fiber. Further, we perform the construction of the new Liouville form so that the corresponding Liouville vector field is still everywhere tangent to D_i. The reason for changing θ to θ_c is to simplify the arguments of Subsection 9.3.

The set of cylindrical exact Lagrangian submanifolds of (W, θ) coincides with that of (W, θ_c) and the wrapped Floer cohomology between any two such Lagrangian submanifolds is unaffected by the modification of θ by the invariance properties of wrapped Floer homology; see [4, Section 5]. This means that $\mathcal{WF}(W, \theta)$ is quasi-equivalent to $\mathcal{WF}(W, \theta_c)$.

With a new Liouville vector field we will choose a new function $\mathfrak{r}\colon W \to [R_0, +\infty)$ satisfying Conditions (i) and (ii) of Section 2.1 for $R_0 \ll 0$ such that, on ∂W_0, the old and new $\mathfrak{r}$ coincide. From now on, $\mathfrak{r}$ will always be defined using the new Liouville vector field $\mathcal{L}_c$. Later in the proof of Proposition 9.3, we will modify $\mathfrak{r}$ so that the new $R_0 \ll 0$ becomes sufficiently small, while keeping $\mathfrak{r}$ fixed outside a compact subset.

Let ψ_t be the Liouville flow of (W, θ_c) and let

$$\widehat{\mathcal{H}}_i := \bigcup_{t \geq 0} \psi_t(\mathcal{H}_i).$$

It follows that $\widehat{\mathcal{H}}_i \subset W$ are pairwise disjoint, embedded codimension zero manifolds. Moreover, there are exact symplectomorphisms

$$(\widehat{\mathcal{H}}_i, \theta_c) \cong (T^*C_i, \mathbf{p}d\mathbf{q})$$

with the standard symplectic cotangent bundles.

Recall Conditions (i) and (ii) from Subsection 2.1.

In particular, $\mathfrak{r}^{-1}(R_0) = W^{\mathrm{sc}} \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l$, while $\mathfrak{r}|_{\mathfrak{r}^{-1}[R_0+1,+\infty)}$ is a symplectisation coordinate induced by the hypersurface $\mathfrak{r}^{-1}(R_0+1)$ of contact type. In the following we make the further assumption that

$$\mathfrak{r}^{-1}(R_0+1) \cap \widehat{\mathcal{H}}_i = S^*_{r_0} T^*C_i \tag{30}$$

for some $r_0 > 0$, where the latter radius-r_0 spherical cotangent bundle is induced by the flat metric on C_i. This means that

$$\mathfrak{r}(\mathbf{p}, \mathbf{q}) = \log \|\mathbf{p}\| - \log r_0 + R_0 + 1, \quad \|\mathbf{p}\| \geq r_0, \tag{31}$$

holds in the above canonical coordinates.

Given a point $a \in C_i$ (for some $i = 1, \ldots, l$), we denote by D_a the Lagrangian plane which satisfies $D_a \cap C_i = \{a\}$ while being everywhere tangent to the Liouville vector field. In particular, $D_a \cap \mathcal{H}_i$ corresponds to the cotangent fiber $D_\delta T^*_a C_i \subset D_\delta T^* C_i$ under the identification $\mathcal{H}_i \cong D_\delta T^* C_i$.

Lemma 9.1. – *For every $i = 1, \ldots, l$ and $a \in C_i$, the Lagrangian plane D_a is isotopic to D_i by a cylindrical Hamiltonian isotopy.*

Proof. – Recall that $(\widehat{\mathcal{H}}_i, \theta_c)$ is isomorphic to $(T^*C_i, \mathbf{p}d\mathbf{q})$ as a Liouville manifold and D_a and D_i correspond to two cotangent fibers. Therefore they are clearly isotopic by a cylindrical Hamiltonian isotopy. □

In particular, D_a and D_i are isomorphic objects in the wrapped Fukaya category when $a \in C_i$.

The next lemma is immediate.

Lemma 9.2. – *Let $L \subset W$ be a cylindrical exact Lagrangian submanifold. Then, up to a (compactly supported) Hamiltonian isotopy, we can assume that $L \cap (C_1 \cup \cdots \cup C_l) = \{a_1, \ldots, a_k\}$, the intersections are transverse and $L \cap W^{\mathrm{sc}} = \emptyset$.*

Now we are going to normalize the intersections between L and the planes D_{a_i}. For every a_i we choose the natural symplectomorphism between a neighborhood

$$\mathcal{D}_{a_i} \subset (\widehat{\mathcal{H}}_i, \theta_c) \cong (T^*C_i, \mathbf{p}d\mathbf{q})$$

of $D_{a_i} \cong T^*_{a_i} C_i$ and $(D_\eta T^* D_{a_i}, -d\tilde{\mathbf{p}} \wedge d\tilde{\mathbf{q}})$ for some $\eta > 0$ small, where $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ are the canonical coordinates on $T^*D_{a_i}$. It is clearly possible to make this identification so that

$$\mathfrak{r}(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \log \|\tilde{\mathbf{q}}\| - \log r_0 + R_0 + 1, \quad \|\tilde{\mathbf{q}}\| \geq r_0, \tag{32}$$

is satisfied.

We redefine $\mathfrak{r}$ as in Remark 2.2, without deforming it outside a compact subset. After making $R_0 \ll 0$ sufficiently small in this manner, we may assume that:

- $R_0 + k + 3 \leq 0$,
- $L \cap W_{R_0+k+3}$ is the union of k disjoint disks with centers at $a_1, \dots, a_k$, and
- the connected component of $L \cap W_{R_0+k+3}$ containing a_i is identified inside $\mathcal{D}_{a_i} \cong D_\eta T^* D_{a_i}$ with the graph of the differential of a function $g_{a_i}\colon D_{a_i} \to \mathbb{R}$ for $i = 1, \dots, k$.

Then we modify L by a compactly supported Hamiltonian isotopy so that it satisfies the following properties:

(L1) The connected component of $L \cap W_{R_0+k+3}$ containing a_i is contained inside the Weinstein neighborhood

$$\mathcal{D}_{a_i} \cap W_{R_0+k+3} \cong D_\eta T^*(D_{a_i} \cap \{\|\tilde{\mathbf{q}}\| \leq e^{k+2} r_0\}),$$

where it is described by the graph of the differential of a function

$$g_{a_i}\colon D_{a_i} \cap \{\|\tilde{\mathbf{q}}\| \leq e^{k+2} r_0\} \to \mathbb{R}$$

with a nondegenerate minimum at a_i and no other critical points,

(L2) the connected components of $L \cap W_{R_0+k+3} \setminus W_{R_0+k+2}$ are cylinders which are disjoint from all the cocores D_{a_i}; moreover, these cylinders are tangent to the Liouville vector field $\mathcal{L}_c$ in the same subset; and

(L3) $\|g_{a_i}\|_{C^2} \leq \epsilon'$ for $i = 1, \dots, k$ and $\epsilon' > 0$ small which will be specified in Lemma 9.4.

Conditions (L1)–(L3) provide sufficient control of the intersections of L and the Lagrangian skeleton. Later in Lemma 9.4 we will use this in order to perform a deformation of the immersed Lagrangian submanifold

$$L \cup D_{a_1} \cup \dots \cup D_{a_k}$$

by Hamiltonian isotopies applied to the different components D_{a_i}. The goal is to obtain an exact Lagrangian immersion admitting a suitable augmentation; the corresponding bounding cochain (see Lemma 8.15) will then give us the twisted complex which exhibits L as an object built out of the different D_{a_i}.

9.2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 assuming the results of Section 9.3. The result is a corollary of the following proposition.

PROPOSITION 9.3. – *Let $L \subset W$ be an exact Lagrangian submanifold with cylindrical end. If $L \cap W^{\mathrm{sk}} = L \cap (C_1 \cup \dots \cup C_l) = \{a_1, \dots, a_k\}$ and the intersections are transverse, then L is isomorphic in* $\mathrm{Tw}\,\mathcal{WF}(W, \theta)$ *to a twisted complex built from the objects $D_{a_1}, \dots, D_{a_k}$.*

Proof. – We assume that L satisfies Conditions (L1), (L2) and (L3) from the previous section. Then by Lemma 9.4 combined with Lemma 9.5 there exist Lagrangian planes $D^w_{a_1}, \dots, D^w_{a_k}$ satisfying the following properties. First $D^w_{a_i}$ is Hamiltonian isotopic to D_{a_i} (possibly after re-indexing the $a_1, \dots, a_k$) by a cylindrical Hamiltonian isotopy supported in $W \setminus W_{R_0+i}$. Second, for an appropriate Legendrian lift $\mathbb{L}^+$ of $\mathbb{L} = L \cup D^w_{a_1} \cup \dots \cup D^w_{a_k}$ to $(W \times \mathbb{R}, \theta + dz)$ such that the intersection point a_i lifts to a Reeb chord from $(D^w_{a_i})^+$ to L^+ of length $\epsilon > 0$ for $i = 1, \dots, k$—see Lemma 9.4 for more details—there exists an augmentation $\boldsymbol{\varepsilon}\colon \mathfrak{A}(\mathbb{L}^+) \to \mathbb{F}$ for which

1. $\boldsymbol{\varepsilon}(a_i) = 1$ for $i = 1, \dots, k$, and
2. $\boldsymbol{\varepsilon}(d) = 0$ if d is a chord from L^+ to $(D^w_{a_i})^+$ for $i = 1, \dots, k$, or a chord from $(D^w_{a_i})^+$ to $(D^w_{a_j})^+$ with $i > j$.

Moreover, using Property (L3) above for $\epsilon' > 0$ sufficiently small, it follows that the Reeb chords a_i all are contractible (cf. Definition 8.5).

By Proposition 8.16 the augmentation $\boldsymbol{\varepsilon}$ induces a twisted complex $\mathfrak{L}$ built from $L_1 = D^w_{a_1}, \dots, L_k = D^w_{a_k}, L_{k+1} = L$, for which

$$\hom(T, \mathfrak{L}) \cong \mathrm{CW}(T, (\mathbb{L}(a_1, \dots, a_k), \overline{\boldsymbol{\varepsilon}})).$$

The right-hand side is an acyclic complex by Proposition 7.6. Using this acyclicity, Proposition 8.16 implies that L is quasi-isomorphic to a twisted complex built from the different $D^w_{a_i} \cong D_{a_i}$ (this last isomorphism follows from the invariance properties for wrapped Floer cohomology under cylindrical Hamiltonian isotopy; see e.g., [4, Section 5]). □

We can therefore complete the proof of Theorem 1.1:

Proof of Theorem 1.1. – Lemma 9.2 and Proposition 9.3 imply that L is isomorphic to a twisted complex built out of the Lagrangian planes D_{a_i}. Lemma 9.1 and the fact that Hamiltonian isotopies generated by cylindrical Hamiltonians induce isomorphisms in the wrapped Fukaya category (see e.g., [4, Section 5]) imply that each D_{a_i} is isomorphic to one of the cocores of W. □

9.3. Constructing the augmentation

We start by assuming that the modifications from Section 9.1 have been performed, so that in particular (L1)–(L3) are satisfied. When considering potentials in this subsection, recall that we have modified the Liouville form from θ to θ_c.

Let $f\colon L \to \mathbb{R}$ be a potential function for L. We order the intersection points $a_1, \dots, a_k$ such that

$$f(a_k) \leq \dots \leq f(a_1).$$

The Morse function $g_{a_i}\colon\ D_{a_i} \cap \{\tilde{\mathbf{q}} \leq e^{k+2} r_0\} \to \mathbb{R}$ from (L1) can be assumed to be sufficiently small by (L3), so that df is almost zero inside $L \cap W_{R_0+k+1}$.

We fix functions $\mathfrak{h}_i\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$\mathfrak{h}_i(\rho) = \begin{cases} 0, & \text{if } \rho \leq R_0 + i, \\ -\rho + R_0 + i + \frac{1}{2}, & \text{if } \rho \geq R_0 + i + 1, \end{cases}$$

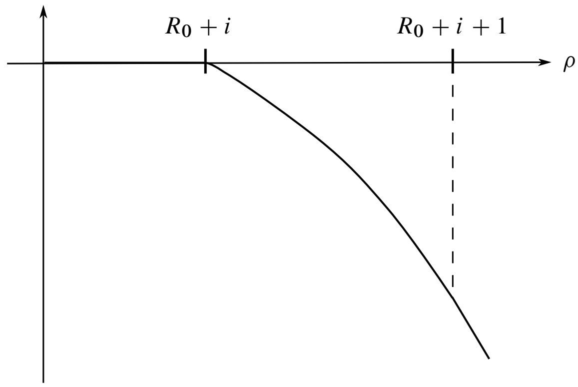

FIGURE 5. The graph of $\mathfrak{h}_i$

and $\mathfrak{h}_i''(\rho) \leq 0$ for all $\rho \in \mathbb{R}^+$. Then we define the cylindrical Hamiltonians $H^i \colon W \to \mathbb{R}$, $i = 1, \ldots, k$, by

$$H^i(w) = \mathfrak{h}_i(e^{\mathfrak{r}(w)}).$$

The graph of $\mathfrak{h}_i$ appears in Figure 5.

We will denote by ϕ_t^i the flow of the Hamiltonian vector field of H^i. Given $T_i \in \mathbb{R}$, we denote $D_{a_i}^w = \phi_{T_i}^i(D_{a_i})$.

We fix $\epsilon > 0$, and on each Lagrangian plane D_{a_i} we choose the potential function $f_i \colon D_{a_i} \to \mathbb{R}$ such that

$$f_i = f(a_i) + \epsilon.$$

Note that the functions f_i indeed are constant, since the Liouville vector field is tangent to the planes D_{a_i}. Let $f_i^w \colon D_{a_i}^w \to \mathbb{R}$ be the potential function on $D_{a_i}^w$ induced by f_i using Equation (5).

We denote by $\mathbb{L} = L \cup D_{a_1}^w \cup \cdots \cup D_{a_k}^w$, which we regard as an exact Lagrangian immersion, and by $\mathbb{L}^+$ the Legendrian lift of $\mathbb{L}$ to $(W \times \mathbb{R}, \theta + dz)$ defined using the potential functions $f, f_1^w, \ldots, f_k^w$. Note that an intersection point $d \in D_{a_i}^w \cap D_{a_j}^w$ lifts to a chord *starting* on $D_{a_i}^w$ and *ending* on $D_{a_j}^w$ if and only if $f_i^w(d) > f_j^w(d)$, and similarly if one of the two disks is replaced by L and its potential is replaced by f.

LEMMA 9.4. – *There exist real numbers $0 < T_k < \cdots < T_1$ and $\epsilon, \epsilon' > 0$ such that, if L satisfies (L1)–(L3), then each chord of $\mathbb{L}^+$ is of one of the following types:*

1. *type a: the chords a_i, going from $(D_{a_i}^w)^+$ to L^+ for $i = 1, \ldots, k$, of length ϵ,*
2. *type b: chords b_{ij}^m consisting of all other chords from $(D_{a_i}^w)^+$ to L^+ for $1 \leq i < j \leq k$ and $1 \leq m \leq m_0(i, j)$ for some $m_0(i, j)$,*
3. *type c: chords c_{ij}^m from $(D_{a_i}^w)^+$ to $(D_{a_j}^w)^+$ for $1 \leq i < j \leq k$ and $1 \leq r \leq m_0(i, j)$, and*
4. *"order-reversing" type: chords from L^+ to $(D_{a_i}^w)^+$ for $i = 1, \ldots, k$ or chords from $(D_{a_i}^w)^+$ to $(D_{a_j}^w)^+$ for $i = 1, \ldots, k$ and $i > j$.*

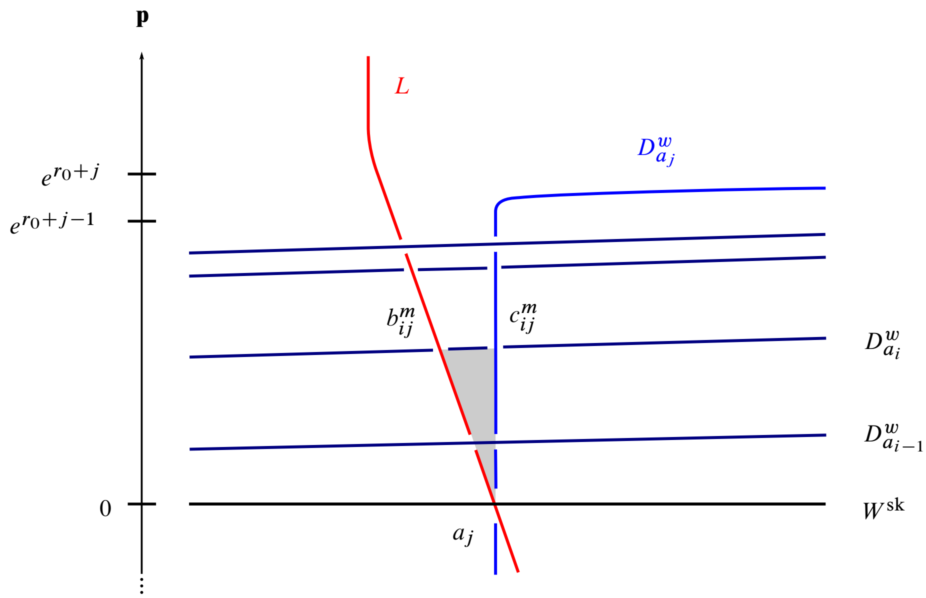

FIGURE 6. A schematic picture of the wrapping and of the small triangle with $i < j$. (Also cf. Equation (31) combined with Figure 5.)

(see Figure 6). Moreover, for every $i < j$ and m, there exists a unique rigid and transversely cut out pseudoholomorphic triangle in W having boundary on $L \cup D^w_{a_i} \cup D^w_{a_j}$, a positive puncture at b^m_{ij}, and negative punctures at a_j and c^m_{ij}, in the order following the boundary orientation. (Positivity and negativity is determined by our choice of Legendrian lift.)

Note that the set $\{c^m_{ij}\}$ could be empty for some i, j. In that case, we say that $m_0(i, j) = 0$.

Proof. – Recall that Properties (L1)–(L3) from Subsection 9.1 have been made to hold; in particular $L \cap W_{R_0+k+2}$ consists of a k number of disks which may be assumed to be close to the disks D_{a_i}, $i = 1, \ldots, k$.

The proof of the lemma at hand is easier to see if one starts by Hamiltonian isotoping L to make it *coincide* with $D_{a_1} \cup \cdots \cup D_{a_k}$ inside W_{R_0+k+2}. (Thus, we can argue about the intersection points of the deformations $D^w_{a_i}$ and D_{a_j}, as opposed to the intersection points of $D^w_{a_i}$ and the different parts of L.) By Property (L2) it suffices to deform L in such a way that it becomes the graph dg_{a_i} for a function satisfying $g_{a_i} \equiv 0$ inside the subsets $D_{a_i} \cap W_{R_0+k+2}$. Note that, in the case where L and $D_{a_j} \cap W_{R_0+k+2}$ *coincide*, the intersection points b^m_{ij} and c^m_{ij} coincide as well.

First, we observe that $L, D^w_{a_1}, \ldots, D^w_{a_k}$ are embedded exact Lagrangian submanifolds, and therefore there is no Reeb chord either from L^+ to L^+ or from $(D^w_{a_i})^+$ to $(D^w_{a_i})^+$ for any $i = 1, \ldots, k$.

From Equation (5), the potential of $D^w_{a_i}$ is

$$f^w_i = f(a_i) + \epsilon + T_i(\mathfrak{h}'_i(e^{\mathfrak{r}})e^{\mathfrak{r}} - \mathfrak{h}_i(e^{\mathfrak{r}})).$$

Note that the quantity $T_i(\mathfrak{h}_i'(e^r)e^r - \mathfrak{h}_i(e^r))$ is nonincreasing in r because $\mathfrak{h}_i'' \leq 0$. Therefore f_i^w satisfies

$$\begin{aligned} &f_i^w(w) = f(a_i) + \epsilon && \text{if } w \in D_{a_i}^w \cap W_{R_0+i},\\ &f_i^w(w) \in [f(a_i) + \epsilon - T_i(R_0 + i + \tfrac{1}{2}), f(a_i) + \epsilon] && \text{if } w \in D_{a_i}^w \cap (W_{R_0+i+1} \setminus W_{R_0+i}),\\ &f_i^w(w) = f(a_i) + \epsilon - T_i(R_0 + i + \tfrac{1}{2}) && \text{if } w \in D_{a_i}^w \cap W_{R_0+i+1}^e. \end{aligned}$$

Note that $D_{a_i}^w \cap W_{R_0+i} = D_{a_i} \cap W_{R_0+i}$ and that $D_{a_i}^w \cap W_{R_0+i+1}^e$ is a cylinder over a Legendrian submanifold.

We choose positive numbers $0 < T_k < \cdots < T_1$ such that

1. $f(a_1) + \epsilon - T_1(R_0 + 1 + \frac{1}{2}) < \cdots < f(a_k) + \epsilon - T_k(R_0 + k + \frac{1}{2})$,
2. $f(a_i) + \epsilon - T_i(R_0 + i + \frac{1}{2}) < \min_L f$ for all $i = 1, \ldots, k$,
3. there are no intersection points between $L, D_{a_1}, \ldots, D_{a_k}, D_{a_1}^w, \ldots D_{a_k}^w$ in their cylindrical parts, and
4. at every intersection point between $L, D_{a_1}, \ldots, D_{a_k}, D_{a_1}^w, \ldots D_{a_k}^w$ the respective potential functions are different, except for intersection points $p \in L \cap D_{a_i}^w$ where $H^i(p) = \mathfrak{h}_i(e^{\mathfrak{r}}) = 0$.

The last two conditions are achieved by choosing $T_1, \ldots, T_k$ generically.

We observe that, for any point $c \in D_{a_i}^w \cap D_{a_j}$ and any $i, j = 1, \ldots, k$, the quantity $\mathfrak{a}(c) = |f_i^w(c) - f_j(c)|$ is independent of ϵ. Then we choose $\epsilon > 0$ sufficiently small so that

$$\epsilon < \min\{\mathfrak{a}(c) : c \in D_{a_i}^w \cap D_{a_j} \text{ and } \mathfrak{a}(c) \neq 0 \text{ for } i, j = 1, \ldots, k\}.$$

This implies that, for all $c \in D_{a_i}^w \cap D_{a_j}$ such that $\mathfrak{a}(c) \neq 0$, the signs of $f_i^w(c) - f_j(c)$ and of $f_i^w(c) - f(c)$ are equal. (Recall that $f = f_j - \epsilon$ holds there by construction.)

Consider the set of points $c_{ij}^m \in D_{a_i}^w \cap D_{a_j}$ with positive action difference

$$0 < f_i^w(c_{ij}^m) - f_j(c_{ij}^m) = f_i^w(c_{ij}^m) - (f(a_j) + \epsilon).$$

(Here m is an index distinguishing the various points with the required property.) Then $i < j$ and $c_{ij}^m \in W_{R_0+i+1} \setminus W_{R_0+i}$; in particular $c_{ij}^m \in D_{a_i}^w \cap D_{a_j}^w$. See Figure 7.

The intersection points b_{ij}^m now coincide with c_{ij}^m, but seen as intersections of $L = D_{a_j}$ and $D_{a_i}^w$. We now perturb L back to make it coincide with the graph of dg_{a_i} of a sufficiently small Morse function g_{a_i} near each D_{a_i} having a unique critical point consisting of a global minimum. Recall that this global minimum corresponds to the intersection point $a_i \in L \cap D_{a_i}$.

We can make the Morse function satisfy $\|g_{a_i}\|_{C^2} \leq \epsilon'$ for $\epsilon' > 0$ sufficiently small. In particular, this means that each intersection point c_{ij}^m still corresponds to a unique intersection point $b_{ij}^m \in D_{a_i}^w \cap L$, such that moreover $f_i^w(b_{ij}^m) - f(b_{ij}^m) > 0$ is satisfied. Conversely, any intersection point $d \in D_{a_i}^w \cap L$ with $f_i^w(d) - f(d) > 0$ is either a_i or one of the b_{ij}^m. This is the case because the only intersection point in $D_{a_i}^w \cap L \cap W_{R_0+i}$ is a_i and for any intersection point $d \in D_{a_i}^w \cap L \cap W_{R_0+i+1}^e$ we must have $f_i^w(d) - f(d) < 0$.

The existence of the triangle follows now by applying Corollary 4.18 to $L \cup D_{a_i}^w \cup D_{a_j}^w$ intersected with the subset $W_{R_0+j} \subset W$. Note that, inside this Liouville subdomain, our deformed Lagrangian L is given as the graph of the differential of a small Morse function g_{a_j} on D_{a_j} (using a Weinstein neighborhood of the latter); hence the lemma indeed applies. Here

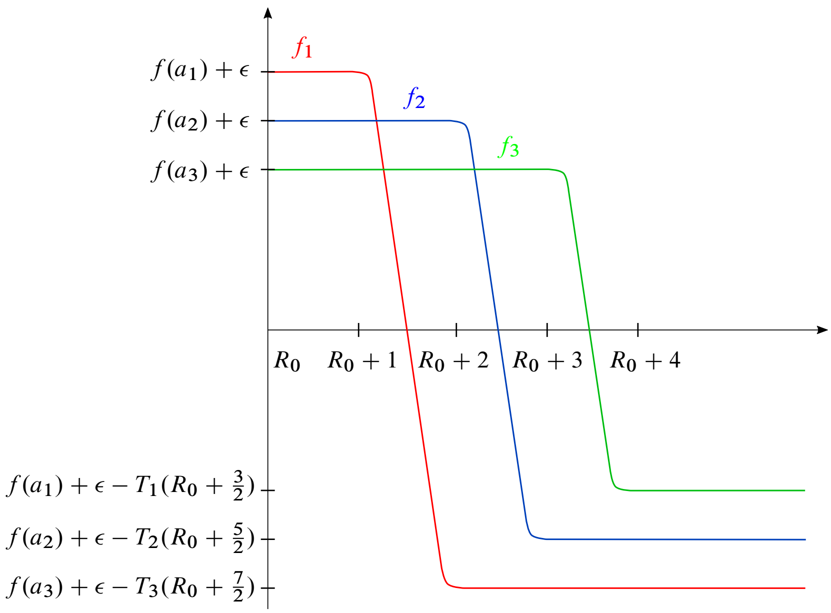

FIGURE 7. The profiles of f_i^w

the monotonicity property for the symplectic area of a pseudoholomorphic disk can be used in order to deduce that the triangles of interest can be a priori confined to the same Liouville subdomain. □

The triangle provided by the previous lemma is the stepping stone in the inductive construction of an augmentation for $\mathfrak{A}(\mathbb{L}^+)$.

LEMMA 9.5. – *The Chekanov-Eliashberg algebra* $(\mathfrak{A}(\mathbb{L}^+), \mathfrak{d})$ *of* $\mathbb{L}^+$ *admits an augmentation*

$$\varepsilon \colon \mathfrak{A}(\mathbb{L}^+) \to \mathbb{F}$$

such that $\varepsilon(a_i) = 1$ *for all* $i = 1, \dots, k$*. Moreover, this augmentation vanishes on the order reversing chords.*

Proof. – Set $L_i := D^w_{a_i}$ and $L_{k+1} := L$. recall that each of the L_i is embedded, and therefore there is no Reeb chord from L_i^+ to itself for any i. Thus all Reeb chords go between different connected components of $\mathbb{L}^+$ and are as described in Lemma 9.4.

The bilateral ideal of $\mathfrak{A}(\mathbb{L})$ generated by the order reversing chords is preserved by the differential, and therefore the quotient algebra, which we will denote by $\mathfrak{A}^{\rightarrow}$, inherits a differential $\mathfrak{d}^{\rightarrow}$. We can identify $\mathfrak{A}^{\rightarrow}$ with the subalgebra of $\mathfrak{A}$ generated by the chords of type a, b and c, and $\mathfrak{d}^{\rightarrow}$ to the portion of the differential of $\mathfrak{A}(\mathbb{L})$ involving only generators of $\mathfrak{A}^{\rightarrow}$.

On $\mathfrak{A}^{\rightarrow}$ we define a filtration of algebras

$$\mathbb{Z} = \mathfrak{A}^{\rightarrow}_{k+1} \subset \mathfrak{A}^{\rightarrow}_{k} \subset \cdots \subset \mathfrak{A}^{\rightarrow}_{0} = \mathfrak{A}^{\rightarrow}, \tag{33}$$

where $\mathfrak{A}^{\rightarrow}_i$ is generated by all chords a_s, b^m_{sj}, c^m_{sj} with $s \geq i$.

Given a chord c of $\mathbb{L}^+$, we denote its action by $\mathfrak{a}(c)$. The differential $\mathfrak{d}^{\to}$ preserves the action filtration on $\mathfrak{A}^{\to}$ (and on all its subalgebras). We assume that

(i) the actions of all chords b_{ij}^m and c_{ij}^m are pairwise distinct and,

(ii) for all i, j, m, the actions $\mathfrak{a}(b_{ij}^m)$ and $\mathfrak{a}(c_{ij}^m)$ are close enough that, whenever $\mathfrak{a}(c_{i_- j_-}^{m-}) < \mathfrak{a}(c_{ij}^m) < \mathfrak{a}(c_{i_+ j_+}^{m+})$, we also have $\mathfrak{a}(c_{i_- j_-}^{m-}) < \mathfrak{a}(b_{ij}^m) < \mathfrak{a}(c_{i_+ j_+}^{m+})$.

The first is a generic assumption, and the second is achieved by choosing $\epsilon' > 0$ sufficiently small in Lemma 9.4.

For each fixed i we define a total order on the pairs (j, m) by declaring that $(h, l) \prec_i (j, m)$ if $\mathfrak{a}(c_{ih}^l) < \mathfrak{a}(c_{ij}^m)$. When the index i is clear from the context, we will simply write $\prec$.

We know that $\mathfrak{d}^{\to} a_i = 0$ for action reasons and $\langle \mathfrak{d}^{\to} b_{ij}^m, a_j c_{ij}^m \rangle = 1$ by the last part of Lemma 9.4. Combining this partial information on the differential $\mathfrak{d}^{\to}$ and the assumptions (i) and (ii) above with the action filtration, we obtain the following structure for the differential:

$$\begin{aligned}
\mathfrak{d}^{\to} a_i &= 0,\\
\mathfrak{d}^{\to} b_{ij}^m &= \alpha_j^m a_i + \sum_{(h,l)\prec_i (j,m)} \beta_{jl}^{mh} b_{ih}^l + a_j c_{ij}^m + \sum_{(l,h)\prec_i (j,m)} w_{lj}^{hm} c_{ih}^l,\\
\mathfrak{d}^{\to} c_{ij}^m &= \sum_{(l,h)\prec_i (j,m)} \widetilde{w}_{lj}^{hm} c_{ih}^l
\end{aligned}$$

with $\alpha_j^m, \beta_{jl}^{mh} \in \mathbb{Z}$ and $w_{lj}^{hm}, \widetilde{w}_{lj}^{hm} \in \mathfrak{A}_{i+1}^{\to}$.

Then the filtration (33) is preserved by $\mathfrak{d}^{\to}$. We want to define an augmentation $\varepsilon\colon \mathfrak{A}^{\to} \to \mathbb{Z}$ such that $\varepsilon(a_i) = 1$ for all $i = 1, \dots, k$ working by induction on i.

For $i = k + 1$, there is nothing to prove since $\mathfrak{A}_{k+1}^{\to} = \mathbb{Z}$.

Suppose now we have defined an augmentation $\varepsilon\colon \mathfrak{A}_{i+1}^{\to} \to \mathbb{Z}$. We will extend it to an augmentation $\varepsilon\colon \mathfrak{A}_i^{\to} \to \mathbb{Z}$ by an inductive argument over the action of the chords c_{ij}^m. For this reason in the following discussion i will be fixed.

We define $\varepsilon(a_i) = 1$ and $\varepsilon(b_{ij}^m) = 0$ for all j and m. To define ε on c_{ij}^m we work inductively with respect to the order $\prec$ induced by the action. Suppose that we have defined $\varepsilon(c_{ih}^l)$ for all c_{il}^h such that $(h, l) \prec (j, m)$. Then we can achieve $\varepsilon(\mathfrak{d}^{\to} b_{ij}^m) = 0$ by prescribing an appropriate value to

$$\varepsilon(c_{ij}^m) = \varepsilon(a_j c_{ij}^m),$$

since the values of ε on all other chords appearing in the expression of $\mathfrak{d}^{\to} b_{ij}^m$ already have been determined.

Now we have defined ε on all generators of $\mathfrak{A}_i^{\to}$ and, by construction, $\varepsilon(\mathfrak{d}^{\to} d) = 0$ for every chord d in $\mathfrak{A}_i^{\to}$ except possibly for the chords c_{ij}^m. We will prove that in fact $\varepsilon(\mathfrak{d}^{\to} c_{ij}^m) = 0$ holds as well, and thus show that ε is an augmentation on $\mathfrak{A}_i^{\to}$. Once again we will argue by induction on the action of the chords c_{ij}^m.

If (j, m) is the minimal element for the order $\prec$, then $\mathfrak{d}^{\to} c_{ij}^m = 0$ and therefore $\varepsilon(\mathfrak{d}^{\to} c_{ij}^k) = 0$. Suppose now that we have verified that $\varepsilon(\mathfrak{d}^{\to} c_{ih}^l) = 0$ for all $(h, l) \prec (j, m)$. From $\mathfrak{d}^{\to}(\mathfrak{d}^{\to} b_{ij}^m) = 0$ and $\varepsilon(\mathfrak{d}^{\to}(a_j c_{ij}^m)) = \varepsilon(\mathfrak{d}^{\to} c_{ij}^m)$ we obtain

$$\varepsilon(\mathfrak{d}^{\to} c_{ij}^m) + \sum_{(h,l)\prec(j,m)} \beta_{jl}^{mh} \varepsilon(\mathfrak{d}^{\to} b_{ih}^l) + \sum_{(l,h)\prec(j,m)} \varepsilon(\mathfrak{d}^{\to}(w_{lj}^{hm} c_{ih}^l)) = 0.$$

We have $\varepsilon(\mathfrak{d}^{\rightarrow} b^l_{ih}) = 0$ by construction and $\varepsilon(\mathfrak{d}^{\rightarrow}(w^{hm}_{lj} c^l_{ih})) = 0$ by the induction hypothesis. From this we conclude that $\varepsilon(\mathfrak{d}^{\rightarrow} c^m_{ij}) = 0$.

Finally we simply precompose ε with the projection $\mathfrak{A}(\mathbb{L}) \to \mathfrak{A}^{\rightarrow}$ and obtain an augmentation of $\mathfrak{A}(\mathbb{L})$ satisfying the required conditions. □

10. Generation of the Wrapped Fukaya category of Weinstein sectors.

In this section we prove Theorem 1.2. We recall that the "linear setup," introduced by Abouzaid and Seidel in [4] and used in the proof of Theorem 1.1, is not available for sectors; instead, Ganatra, Pardon and Shende in [23] define the wrapped Fukaya category of a Liouville sector by a localisation procedure. However, the strategy of the proof of Theorem 1.1 applies to the "localisation setup" as well, with only minor modifications of some technical details. The goal of this section is to explain those modifications, which in most cases will be simplifications. Before proceeding, we recall that the proof of Theorem 1.1 had four main steps:

1. an extension of the construction of wrapped Floer cohomology to certain exact immersed Lagrangian submanifolds,
2. triviality of wrapped Floer cohomology for immersed exact Lagrangian submanifolds which are disjoint from the skeleton ("trivial triviality"),
3. identification of certain twisted complexes in the wrapped Fukaya category with Lagrangian surgeries, and
4. construction of the bounding cochain after a suitable modification of the Lagrangian cocores.

10.1. The wrapped Fukaya category for sectors

In this subsection we recall briefly the definition of the wrapped Floer cohomology and the wrapped Fukaya category for sectors following [23] and show that our construction of the wrapped Floer cohomology of an exact immersed Lagrangian submanifold can be carried over to this setting.

Given $\epsilon > 0$ we, denote $\mathbb{C}_{0 \leq \Re < \epsilon} = \{x + iy \in \mathbb{C} : 0 \leq x < \epsilon\}$. If (S, θ, I) is a Liouville sector, by [23, Proposition 2.24] there is an identification

$$(\mathrm{Nbd}(\partial S), \theta) \cong \left(F \times \mathbb{C}_{0 \leq \Re < \epsilon}, \theta_F + \frac{1}{2}(x dy - y dx) + df \right), \tag{34}$$

where $f\colon F \times \mathbb{C}_{0 \leq \Re < \epsilon} \to \mathbb{R}$ satisfies the following properties:

— the support of f is contained in $F_0 \times \mathbb{C}_{0 \leq \Re < \epsilon}$ for some Liouville domain $F_0 \subset F$, and

— f coincides with $f_{\pm\infty}\colon F \to \mathbb{R}$ for $|y| \gg 0$.

We denote $\pi\colon \mathrm{Nbd}(\partial S) \to \mathbb{C}_{0 \leq \Re < \epsilon}$ the projection induced by the identification (34).

We will consider almost complex structures J on S which are cylindrical with respect to the Liouville vector field $\mathcal{L}$ of θ and make the projection π holomorphic (where, of course, we endow $\mathbb{C}$ with its standard complex structure). It is easy to see that this choice of almost complex structures constrains the holomorphic curves with boundary in $\mathrm{int}(S)$ so that they stay away from the boundary ∂S; see [23, Lemma 2.41]. Thus, if L_0, L_1 are two transversely intersecting exact Lagrangian submanifolds with cylindrical ends, the Floer chain complex *with zero Hamiltonian* $\mathrm{CF}(L_0, L_1)$ is defined.

Let $L_\bullet = \{L_t\}_{t\in I}$ be an isotopy of exact Lagrangian submanifolds which are cylindrical at infinity over Legendrian submanifolds Λ_t in the contact manifold (V, α) which is the boundary at infinity of (S, θ, I). Let X_t be a vector field along L_t directing the isotopy. We can choose this vector field so that, where the isotopy is cylindrical, it is the lift of a vector field along Λ_t which we denote by X_t^∞. We say that the Lagrangian isotopy is *positive* if $\alpha(X_t^\infty) \geq 0$ everywhere. We say that the isotopy is *small* if its trace $\bigcup_{t\in I} \Lambda_t$ is embedded. This implies that Legendrian links $\Lambda_0 \cup \Lambda_t$ are embedded and thus Legendrian isotopic to each other for all $t \in I \setminus \{0\}$.

Following [23, Subsection 3.3], to any positive isotopy $L_\bullet$ of exact Lagrangian submanifolds with cylindrical ends, we associate a *continuation element* $c(L_\bullet) \in \mathrm{HF}(L_1, L_0)$ as follows. If the isotopy is small, there is a map $H^*(L_0) \to \mathrm{HF}(L_1, L_0)$, and we define $c(L_\bullet)$ as the image of the unit in $H^*(L_0)$ under this map. If $L_\bullet$ is not small, then we decompose it into a concatenation of small isotopies and define $c(L_\bullet)$ as the composition (by the triangle product) of the continuation elements of the small isotopies. Then, for any Lagrangian submanifold K which is transverse with both L_0 and L_1, we define the continuation map

$$\mathrm{HF}(L_0, K) \xrightarrow{[\mu_2(\cdot, c(L_\bullet))]} \mathrm{HF}(L_1, K). \tag{35}$$

See [23, Lemma 3.26] for the properties of the continuation element.

Given a Lagrangian submanifold L with cylindrical end, following [23, Subsection 3.4] we consider its *wrapping category* $(L \to -)^+$, which is the category whose objects are isotopies of Lagrangian submanifolds $\phi\colon L \to L^w$ and morphisms from $(\phi\colon L \to L^w)$ to $(\phi'\colon L \to L^{w'})$ are *homotopy classes* of *positive* isotopies $\psi\colon L^w \to L^{w'}$ such that $\phi\#\psi = \phi'$.

With all this in place, wrapped Floer cohomology is defined as

$$\mathrm{HW}(L, K) = \varinjlim_{(L\to L^w)^+} \mathrm{HF}(L^w, K), \tag{36}$$

where the maps in the direct system are the continuation maps defined above.

Now suppose that K is immersed and ε is an augmentation of its obstruction algebra. Then $\mathrm{CF}(L, (K, \epsilon))$ is defined as in Section 4, as long as we use the trivial Hamiltonian $H = \mathbf{0}$ in the definition—being in a Liouville sector makes no difference in any other aspect of the construction. The definition of $\mathrm{HW}(L, (W, \varepsilon))$ is then the same as in Equation (36) using the product μ_2 defined in Section 4.5.

Remark 10.1. – This definition is sufficient for our needs because in the proof of Theorem 1.2 we only need wrapped Floer cohomology with immersed Lagrangian submanifolds in the right entry. However, it is possible to extend the definition to the case of immersed Lagrangian submanifolds on the left by identifying augmentations of L with augmentations of L^w and defining the continuation element for small isotopies using Lemmas 4.14 and 4.15. Note that these lemmas are stated for Floer cohomology with trivial Hamiltonian, and therefore they extend immediately to Liouville sectors.

Now we sketch the construction of the wrapped Fukaya category following [23, Subsection 3.5]. We recall that we do not need to extend the definition so that it includes immersed Lagrangian submanifolds, even if it would probably not be too difficult. We fix a countable set I of exact Lagrangian submanifolds with cylindrical ends so that any cylindrical Hamiltonian isotopy class has at least one representative and, for every $L \in I$, we fix a cofinal sequence $L = L^{(0)} \to L^{(1)} \to \cdots$ in $(L \to -)^+$. We denote by $\mathcal{O}$ the set of all these Lagrangian submanifolds. We assume that we have chosen the elements in $\mathcal{O}$ so that all finite subsets $\{L_1^{i_1}, \dots, L_k^{i_k}\}$ with $i_1 < \cdots < i_k$ consist of mutually transverse Lagrangian submanifolds.

We make $\mathcal{O}$ into a strictly unital A_∞-category by defining

$$\hom_{\mathcal{O}}(L^{(i)}, K^{(j)}) = \begin{cases} \mathrm{CF}(L^{(i)}, K^{(j)}) & \text{if } i > j, \\ \mathbb{Z} & \text{if } L^{(i)} = K^{(j)}, \\ 0 & \text{otherwise.} \end{cases}$$

If $i > j$, the continuation element of the positive isotopy $L^{(j)} \to L^{(i)}$ belongs to $H \hom_{\mathcal{O}}(L^{(i)}, L^{(j)}) = \mathrm{HF}(L^{(i)}, L^{(j)})$. We will write $L^{(i)} > K^{(j)}$ if $i > j$.

We denote by C the set of all closed morphisms of $\mathcal{O}$ which represent a continuation element. Thus we define the wrapped Fukaya category of (S, θ, I) as $\mathcal{WF}(S, \theta) = \mathcal{O}_{C^{-1}}$, where $\mathcal{O}_{C^{-1}}$ is the A_∞-category obtained by dividing $\mathcal{O}$ by all cones of morphisms in C: i.e., $\mathcal{O}_{C^{-1}}$ has the same objects as $\mathcal{O}$ and its morphisms are defined as the morphisms of the image of $\mathcal{O}$ in the quotient of the triangulated closure of $\mathcal{O}$ by its full subcategory of cones of elements of C. This construction has the effect of turning all elements of C into quasi-isomorphisms. See [23, Subsection 3.1], and in particular Definition 3.1 therein, for a precise definition of the localisation of an A_∞-category. In the following lemma we summarize the properties of the localisation that we will need.

LEMMA 10.2. – *The categories $\mathcal{O}$ and $\mathcal{WF}(S, \theta)$ are related as follows:*

1. *$\mathcal{WF}(S, \theta)$ and $\mathcal{O}$ have the same objects,*
2. *$H(\hom_{\mathcal{WF}}(L, K)) \cong \mathrm{HW}(L, K)$,*
3. *the category $\mathcal{WF}(S, \theta)$ is independent of all choices up to quasi-equivalence, and*
4. *The localisation functor $\mathcal{O} \to \mathcal{WF}(S, \theta)$ is the identity on objects and has trivial higher order terms (i.e., it matches A_∞ operations on the nose). Moreover, when* $\hom_{\mathcal{O}}(L, K) = \mathrm{CF}(L, K)$, *it induces the natural map* $\mathrm{HF}(L, K) \to \mathrm{HW}(L, K)$.

Proof. – (1) follows from the definition of localisation. (2) is the statement of [23, Lemma 3.37]. (3) is the statement of [23, Proposition 3.39]. (4) follows from the definition of $\mathcal{A}_\infty$-products in $\mathcal{A}_\infty$-quotients; see [30, Corollary 2.4]. □

10.2. Trivial triviality

In this section we prove Proposition 7.6 for sectors. The proof is the same as in Section 7.2 except for few details which must be adjusted because we need to use geometric wrapping of the Lagrangian submanifolds instead of Hamiltonian perturbations of the Floer equation and the continuation maps from Equation (35) instead of those from Subsection 5.2.

For Liouville sectors we need to modify the definition of a cylindrical Hamiltonian in order to have a complete flow. To that aim, the crucial notion is that of a *coconvex set*.

DEFINITION 10.3. – Let X be a vector field on a manifold V. We say that a subset $\mathcal{N} \subset V$ is *coconvex* (for X) if every finite time trajectory of the flow of X with initial and final point in $V \setminus \mathcal{N}$ is contained in $V \setminus \mathcal{N}$.

The following lemma is a rewording of [23, Proposition 2.34].

LEMMA 10.4. – *Given a contact manifold (V, α) with convex boundary, it is possible to find a function $g\colon V \to \mathbb{R}_{\geq 0}$ such that*

1. $g > 0$ *outside the boundary ∂V and $g \equiv 1$ outside a collar neighborhood $\partial V \times [0, \delta)$ on which $\alpha = dt + \beta$ where β is a one-form on ∂V,*
2. $g = t^2 G$ *on $\partial V \times [0, \delta)$, where $G > 0$ and t is the coordinate of $[0, \delta)$, and*
3. *there is a collar neighborhood $\mathcal{N}$ of ∂V, contained in $\partial V \times [0, \delta)$, which is coconvex for the contact Hamiltonian X_g of g.*

Note that X_g vanishes along ∂V. It is called a *cut off Reeb vector field* in [23] because it is the Reeb vector field of the contact form $g^{-1}\alpha$ on $\operatorname{int}(V)$. From now on we will assume that g and $\mathcal{N}$ are fixed once and for all for the contact manifold (V, α) arising as boundary at infinity of (S, θ, I). We will also extend g to the complement of a compact set of S so that it is invariant under the Liouville flow.

DEFINITION 10.5. – Let S be a Liouville sector. A Hamiltonian function $H\colon [0,1] \times S \to \mathbb{R}$ is *cylindrical* if there is a function $h\colon \mathbb{R}^+ \to \mathbb{R}$ such that $H(t, w) = g(w)h(e^{\mathfrak{r}(w)})$ outside a compact set of S.

The definition of cylindrical Hamiltonian compatible with two immersed exact Lagrangian submanifolds in the case of sectors is the same as Definition 4.3. Condition (iv) in the latter definition becomes equivalent to asking that λ should not be the length of a chord of the cut off Reeb vector field. In this section, cylindrical Hamiltonian will be used to define positive Hamiltonian isotopies of Lagrangian submanifolds and not to deform the Floer equation.

We say that an exact Lagrangian submanifold of S (possibly immersed) is *safe* if it is cylindrical over a Legendrian submanifold contained in $V \setminus \mathcal{N}$. Since $\mathcal{N}$ is strictly contained in an invariant neighborhood of ∂V, every cylindrical exact Lagrangian submanifold of S is Hamiltonian isotopic to one which is safe by a cylindrical Hamiltonian isotopy. We will assume that all Lagrangian submanifolds with cylindrical end are safe unless we explicitly state the contrary.

Fix an exact Lagrangian submanifold L in S with cylindrical end. Given $\lambda, \Lambda, R \in \mathbb{R}$ such that $0 < \lambda \leq \Lambda$ and $0 < R$, let $h_{\lambda,\Lambda,R}$ be the function defined in Equation (26) and consider the cylindrical Hamiltonian $H_{\lambda,\Lambda,R}$ induced by $h_{\lambda,\Lambda,R}$ as in Definition 10.5. Note that, when $\Lambda = \lambda$, we obtain the Hamiltonian function H_λ induced by the function h_λ of Equation (18) independently of R. We denote by $L_\bullet^{\lambda,\Lambda,R} = \{L_t^{\lambda,\Lambda,R}\}_{t\in\mathbb{R}}$ the positive Hamiltonian isotopy generated by $H_{\lambda,\Lambda,R}$ such that $L_0^{\lambda,\Lambda,R} = L$. When $\Lambda = \lambda$ we write $L_\bullet^\lambda$ instead.

LEMMA 10.6. – *Let $\lambda_n \to +\infty$ be an increasing sequence. Then the Lagrangian submanifolds $L_1^{\lambda_n}$ form a cofinal collection in $(L \to -)^+$.*

Proof. – Since the Hamiltonian H_λ is autonomous, for every $\kappa \in \mathbb{R}$ we have $L_1^{\kappa\lambda} = L_\kappa^\lambda$. The family $\{L_t^\lambda\}_{t\geq 0}$ is cofinal by [23, Lemma 3.30] because the Hamiltonian vector field of H_λ in the cylindrical end of S is the lift of a (strictly) positive multiple of the Reeb vector field of the contact form $g^{-1}\alpha$. □

Let K be an immersed exact Lagrangian submanifold with cylindrical end, and let ε be an augmentation of the obstruction algebra of K. Often we will drop ε from the notation: even if the Floer complex depends on it, the arguments in this subsection do not. Given $\lambda < \Lambda$, we call the intersection points in $L_1^{\lambda,\Lambda,R} \cap K \cap \mathfrak{r}^{-1}((-\infty, R/2))$ *intersection points of type I* and the intersection points in $L_1^{\lambda,\Lambda,R} \cap K \cap \mathfrak{r}^{-1}((R/2, +\infty)$ *intersection points of type II*. They correspond to the Hamiltonian chords of type I or II in Subsection 7.2. We denote by $\mathrm{CF}^I(L_1^{\lambda,\Lambda,R}, K)$ the subcomplex of $\mathrm{CF}(L_1^{\lambda,\Lambda,R}, K)$ generated by the intersection points of type I. The following lemma is the equivalent of Lemma 7.8 in this context.

LEMMA 10.7. – *If the Liouville flow of (S,θ) displaces K from every compact set, then the inclusion $\mathrm{CF}^I(L_1^{\lambda,\Lambda,R}, K) \hookrightarrow \mathrm{CF}(L_1^{\lambda,\Lambda,R}, K)$ is trivial in homology when Λ and R are sufficiently large.*

Sketch of proof. – The proof is the same as that of Lemma 7.8, whose main ingredients are Equation (27), which computes the action of the generators of the Floer complex, and Lemma 7.5 which estimates the action shift of the continuation maps for compactly supported safe isotopies from Subsection 5.3. Both ingredients are still available for Liouville sectors: in fact intersection points between $L_1^{\lambda,\Lambda,R}$ and K are in bijection with Hamiltonian chords of $H_{\lambda,\Lambda,R}$ and the action of an intersection point is the same as the action of the corresponding chord by Equation (5) and Equation (23). Thus (27) still gives bounds on the action of the generators of $\mathrm{CF}(L_1^{\lambda,\Lambda,R}, K)$, after taking into account the fact that the extra factor involving g coming from Definition 10.5 is uniformly bounded because $L_1^{\lambda,\Lambda,R}$ and K are safe.

Moreover, the definition of the continuation maps for compactly supported safe isotopies from Subsection 5.3 and the proof of Lemma 7.5 do not depend on the Hamiltonian deformation in the Floer equation and therefore, setting $H \equiv 0$ in the Floer equations, they hold also for sectors. □

Now we can finish the proof of the equivalent of Proposition 7.6 for sectors.

PROPOSITION 10.8. – *Let (S,θ,I) be a Liouville sector and let K and L be exact Lagrangian submanifolds of S with cylindrical ends. We allow K to be immersed, and in that case we assume its obstruction algebra admits an augmentation ε. If the Liouville flow of (S,θ) displaces K from every compact set of S, then* $\mathrm{HW}(L,(K,\varepsilon)) = 0$.

Proof. – For any fixed $\lambda < \Lambda$ there is a natural homotopy class of positive isotopies $L_1^\bullet$ from L_1^λ to L_1^Λ. We need to show that, for Λ large enough with respect to λ, the continuation map associated to this class is trivial.

We represent this class by a concatenation of positive isotopies $L_1^{\lambda,\bullet,R}$ from L_1^{λ} to $L_1^{\lambda,\Lambda,R}$ and $L_1^{\bullet,\Lambda,R}$ from $L_1^{\lambda,\Lambda,R}$ to L_1^{Λ} which lead to continuation maps

$$[\mu^2(\cdot, c^{\lambda,\bullet,R})]\colon \mathrm{HF}(L_1^{\lambda}, K) \to \mathrm{HF}(L_1^{\lambda,\Lambda,R}, K)$$
$$[\mu^2(\cdot, c^{\bullet,\Lambda,R})]\colon \mathrm{HF}(L_1^{\lambda,\Lambda,R}, K) \to \mathrm{HF}(L_1^{\Lambda}, K).$$

To prove the proposition, it is sufficient to prove that, for any fixed λ. $[\mu^2(\cdot, c^{\lambda,\bullet,R})]$ is trivial if Λ and R are large enough.

It follows from [23, Lemma 3.27] and the definition of $H_{\lambda,\Lambda,R}$ that the map $\mu^2(\cdot, c(L_1^{\lambda,\bullet,R}))$ is the natural inclusion $\mathrm{CF}(L_1^{\lambda}, K) \subset \mathrm{CF}(L_1^{\lambda,\Lambda,R}, K)$ whose image is the subcomplex $\mathrm{CF}^I(L_1^{\lambda,\Lambda,R}, K)$. Thus, the triviality of $[\mu^2(\cdot, c(L_1^{\lambda,\bullet,R})]$ follows from Lemma 10.7. □

Remark 10.9. – In view of Remark 10.1, we expect Proposition 10.8 to hold also when L is immersed, as long as its obstruction algebra admits an augmentation. However, we haven't checked the details.

10.3. Twisted complexes

In this subsection we extend to sectors the results of Section 8 identifying certain Lagrangian surgeries with twisted complexes.

The first step is to observe that the constructions in Subsecton 8.1 can be extended to Lagrangian cobordisms in the symplectisation of the contactisation of a Liouville sector. In fact, we can work with almost complex structures on $\mathbb{R} \times S \times \mathbb{R}$ satisfying the hypothesis of Lemma 8.2 and such that the projection $\mathbb{R}\times S\times\mathbb{R} \to S$ is holomorphic near $\mathbb{R}\times\partial S\times\mathbb{R}$. Then, by [23, Lemma 2.14], the holomorphic curves appearing in the definition of the Cthulhu complex do not approach $\mathbb{R} \times \partial S \times \mathbb{R}$.

In Subsection 8.3 we work only with Floer complexes with trivial Hamiltonian, and therefore the results of that subsection extend to Liouville sectors without effort. Moreover, some contortions which were needed to apply those results in the linear setup are no longer necessary in the localisation setup. Let $L_1, \dots, L_m$ be exact Lagrangian submanifolds with cylindrical ends and denote $\mathbb{L} = L_1 \cup \cdots \cup L_m$. Unlike in Section 8, here we do not need to consider the case of immersed L_i.

We recall some notation from Section 8. Given a set of intersection points $\{a_1, \dots, a_k\}$ which corresponds to a set of contractible chords (see Definition 8.5) for a Legendrian lift $\mathbb{L}^+$ of $\mathbb{L}$ to the contactisation of S, we denote by $\mathbb{L}(a_1, \dots, a_k)$ the result of Lagrangian surgery performed on $a_1, \dots, a_k$ as explained in 8.2. If $\mathbb{L}^+$ admits an augmentation $\boldsymbol{\varepsilon}$ such that $\boldsymbol{\varepsilon}(a_i) = 1$ for $i = 1, \dots, k$, then by Lemmas 8.7 and 8.8 there exists an augmentation $\overline{\boldsymbol{\varepsilon}}$ of $\mathbb{L}(a_1, \dots, a_k)^+$ such that, for any exact Lagrangian submanifold T with cylindrical ends, there is an isomorphism

$$\Phi_*\colon \mathrm{HF}(T, (\mathbb{L}(a_1, \dots, a_k), \overline{\boldsymbol{\varepsilon}})) \xrightarrow{\cong} \mathrm{HF}(T, (\mathbb{L}, \boldsymbol{\varepsilon})). \tag{37}$$

Moreover, this isomorphism preserves the triangle products in the sense that, given two exact Lagrangian submanifolds T_0 and T_1, the diagram

$$\begin{array}{ccc} \mathrm{HF}(T_0, (\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}})) \otimes \mathrm{HF}(T_1, T_0) & \xrightarrow{[\mu_2]} & \mathrm{HF}(T_1, (\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}})) \\ \downarrow {\scriptstyle \Phi\otimes\mathrm{Id}} & & \downarrow {\scriptstyle \Phi} \\ \mathrm{HF}(T_0, (\mathbb{L}, \boldsymbol{\varepsilon})) \otimes \mathrm{HF}(T_1, T_0) & \xrightarrow{[\mu_2]} & \mathrm{HF}(T_1, (\mathbb{L}, \boldsymbol{\varepsilon})) \end{array}$$

commutes. This is a particular case of [29, Theorem 2]. Then the isomorphisms (37) induce isomomorphisms

$$\mathrm{HW}(T, (\mathbb{L}(a_1, \ldots, a_k), \overline{\boldsymbol{\varepsilon}})) \cong \mathrm{HW}(T, (\mathbb{L}, \boldsymbol{\varepsilon})) \tag{38}$$

for every exact Lagrangian submanifold T with cylindrical ends.

Assume that $\boldsymbol{\varepsilon}(q) = 0$ for all Reeb chord from L_i^+ to L_j^+ with $i > j$. We add enough objects to $\mathcal{O}$ so that $L_1, \ldots, L_m$ are objects of $\mathcal{O}$ and $L_m > \cdots > L_1$. By Lemma 10.2(3) this operation does not change $\mathcal{WF}(S, \theta)$ up to quasi-equivalence.

Then $X = (x_{ij})_{0 \le i,j \le m} \in \bigoplus_{0 \le i,j \le m} \hom_{\mathcal{O}}(L_j, L_i)$ defined as

$$x_{ij} = \begin{cases} \sum_{a \in L_i \cap L_j} \boldsymbol{\varepsilon}(a) a & \text{if } i < j \\ 0 & \text{if } i \ge j \end{cases}$$

satisfies the Maurer-Cartan equation in $\mathcal{O}$ by the same argument as in the proof of Lemma 8.15 and our choice of ordering of the objects $L_1, \ldots, L_m$. Thus X satisfies the Maurer-Cartan equation also in $\mathcal{WF}(S, \theta)$ by Lemma 10.2(4). We denote by $\mathfrak{L} = (\{L_i\}, X)$ the corresponding twisted complex both in $\mathcal{O}$ and in $\mathcal{WF}(S, \theta)$. For any object T of $\mathcal{O}$ with $T > L_m$ there is a tautological identification of chain complexes $\hom_{\mathrm{Tw}\,\mathcal{O}}(T, \mathfrak{L}) = \mathrm{CF}(T, (\mathbb{L}, \boldsymbol{\varepsilon}))$ which, moreover, respects the triangle products; this follows from a direct comparison of the holomorphic polygons counted by the differentials on the left and on the right as in the proof of Lemma 8.15. Thus the isomorphisms are compatible with the multiplication by continuation elements and in the limit we obtain a map

$$\mathrm{HW}(T, (\mathbb{L}, \boldsymbol{\varepsilon})) \to H \hom_{\mathrm{Tw}\,\mathcal{WF}}(T, \mathfrak{L}),$$

which is an isomorphism by [23, Lemma 3.37] and a simple spectral sequence argument. We can summarize these results in the following lemma, which is the analogue of Proposition 8.16 in the context of sectors.

LEMMA 10.10. – *Let $L_1, \ldots, L_m$, be embedded exact Lagrangian submanifolds with cylindrical ends. If there exist a Legendrian lift $\mathbb{L}^+$ of $\mathbb{L} = L_1 \cup \cdots \cup L_m$, an augmentation $\boldsymbol{\varepsilon}$ of the Chekanov-Eliashberg algebra of $\mathbb{L}^+$ and a set of contractible Reeb chord $\{a_1, \ldots, a_k\}$ such that:*

(1) $\boldsymbol{\varepsilon}(a_i) = 1$ *for* $i = 1, \ldots, k$, *and*

(2) $\boldsymbol{\varepsilon}(q) = 0$ *if q is a Reeb chord from L_i^+ to L_j^+,*

then there exist a twisted complex $\mathfrak{L}$ *built from* $L_1, \dots, L_m$ *and an augmentation* $\overline{\boldsymbol{\varepsilon}}$ *of the Chekanov-Eliashberg algebra of* $\mathbb{L}(a_1, \dots, a_k)^+$ *such that, for any other exact Lagrangian submanifold with cylindrical end* T *there is an isomorphism*

$$\mathrm{HW}(T, (\mathbb{L}(a_1, \dots, a_k), \overline{\boldsymbol{\varepsilon}})) \cong H \hom_{\mathcal{WF}}(T, \mathfrak{L}).$$

10.4. Construction of the augmentation

In this section we finish the proof of Theorem 1.2. Let $(S, \theta, I, \mathfrak{f})$ be a Weinstein sector. We modify θ so that it coincides with the standard Liouville form on the handles $\mathcal{H}_i$ and half-handles $\mathcal{H}_i'$.

We denote $\partial_+\mathcal{H}_i' \cong S_\delta T^* H_i$. The Reeb vector field on $\partial_+\mathcal{H}_i$ induced by the canonical Liouville form is the cogeodesic flow of H_i for the flat metric, and therefore $\partial_+\mathcal{H}_i' \cap \partial W$ has a coconvex collar. This is an important observation, because it allows us to choose, once and for all, g and a corresponding coconvex collar $\mathcal{N}$ for ∂V as in Definition 10.3 on the contact manifold (V, α) which is the boundary at infinity of $(S, \theta, I, \mathfrak{f})$ such that $g \equiv 1$ on $\mathcal{H}_i' \setminus \mathcal{N}$ for all Weinstein half-handles $\mathcal{H}_i'$.

As in Section 9, we isotope L so that it is disjoint from the subcritical part of S^{sk} and intersects the cores of the critical Weinstein handles and half-handles transversely in a finite number of points $a_i, \dots, a_k$, and for each point a_i we consider the cocore plane D_{a_i} passing through it. The wrapping of the cocore planes taking place in the proof of Lemma 9.4 is the point where the proof requires a little more work than the case of a Weinstein manifold.

We define the Hamiltonian functions H^i of Lemma 9.4 to be $H^i(w) = g(w)\mathfrak{h}_i(e^{\mathfrak{r}(w)})$ and denote by $D_{a_i}^w$ the image of D_{a_i} under the Hamiltonian flow of H^i for a sufficiently large time. The flow can push $D_{a_i}^w$ close to ∂W, and therefore we can no longer assume that the wrapped planes $D_{a_i}^w$ are safe. However, L and the planes D_{a_i} were safe, and therefore all intersection points of type a, b, and c (defined in Lemma 9.4) between L and the planes $D_{a_1}^w, \dots, D_{a_k}^w$ correspond to Hamiltonian chords contained in the complement of $\mathcal{N}$ because the Hamiltonian vector field of H^i is a negative multiple of the cut off Reeb vector field outside a compact set. Since $g = 1$ outside $\mathcal{N}$, the energy estimates of Subsection 9.3 still hold in the case of sectors, and that allows us to construct an augmentation $\boldsymbol{\varepsilon}$ for a suitable Legendrian lift $\mathbb{L}^+$ of $\mathbb{L} = L \cup D_{a_i}^w \cup \dots \cup D_{a_k}^w$ as in Lemma 9.5. At this point the proof of Theorem 1.2 proceeds in the same way as the proof of Theorem 1.1.

11. Hochschild homology and symplectic cohomology

In this section we use the work of Ganatra [22] and Gao [25] to derive Corollary 1.4 from Theorem 1.1. Since Ganatra and Gao work in the quadratic setup of the wrapped Fukaya category, we must extend the proof of Theorem 1.1 to that setup first.

11.1. Wrapped Floer cohomology in the quadratic setup

On the level of complexes, wrapped Floer cohomology in the quadratic setup is in some sense the simplest one to define; in this case the wrapped Floer complex $\mathrm{CW}(L_0, L_1)$ is the Floer complex $\mathrm{CF}(L_0, L_1; H)$ for a *quadratic* Hamiltonian $H\colon W \to \mathbb{R}$, by which we mean that $H = C \cdot e^{2\mathfrak{r}}$ is satisfied outside some compact subset of W for some constant $C > 0$. This construction of the wrapped Floer complexes can be generalized to the case where L_1 has transverse double points in the same manner as in the linear case, by using the obstruction algebra.

In the following we assume that all Lagrangians are cylindrical inside the noncompact cylindrical end $W^e_{R-1} \subset W$ for some $R \gg R_0$. Denote by $\psi_t\colon (W, \theta) \to (W, e^{-t}\theta)$ the Liouville flow of (W, θ) and recall that $\mathfrak{r} \circ \psi_t = \mathfrak{r} + t$ in W^e_{R-1} and hence $\psi_t(W^e_r) = W^e_{r+t}$ for all $r \geq R - 1$.

11.2. Trivial triviality

In this subsection we prove Proposition 7.6, i.e., “trivial triviality,” in the quadratic setting. In fact, since the quadratic wrapped Floer cohomology complex does not involve continuation maps, the proof becomes even simpler here.

Assume that $L_1 \subset W$ is disjoint from the skeleton, and thus that $\psi_t(L_1)$ is a safe exact isotopy that displaces L_1 from any given compact subset. The Liouville flow ψ_t is conformally symplectic with the conformal factor e^t, i.e., $(\psi_t)^* d\theta = e^t \cdot d\theta$. Since L_1 is cylindrical outside a compact subset, the aforementioned safe exact isotopy is generated by a locally defined Hamiltonian function G_t which satisfies the following action estimate.

Lemma 11.1. – *The locally defined generating Hamiltonian $G_t\colon \psi_t(L_1) \to \mathbb{R}$ from Section 7 is of the form $G_t = e^t \cdot G_0 \circ \psi^{-t}$ for a function $G_0\colon W \to \mathbb{R}$ which vanishes outside a compact set, and thus satisifes the bound $\|G_t\|_\infty \leq C_{L_1} e^t$ for some constant C_{L_1} that only depends on L_1. In particular, the Hamiltonian isotopy whose image of L_1 at time $t = 1$ is equal to $\psi_t(L_1)$ can be generated by a compactly supported $\widetilde{G}_t$ that satisfies $\|\widetilde{G}_t\|_\infty \leq t C_{L_1} e^t$.*

The conformal symplectic property implies that the primitive of θ satisfies $\theta_{|T\psi_t(L_1)} = e^t d(f_1 \circ \psi_t)$ for the primitive $f_1\colon L_1 \to \mathbb{R}$ of $\theta_{|TL_1}$ that vanishes in the cylindrical end. The formula for the action of the Reeb chords (23), in particular cf. (24) in the example thereafter applied with $h(x) = x^2$, now readily implies that:

Lemma 11.2. – *The generators of* $\mathrm{CF}(L_0, \psi_t(L_1); H)$ *have action bounded from below by $C'_{L_1} e^{2t}$ for some constant $C'_{L_1} > 0$ whenever $t \gg 0$ is sufficiently large.*

A construction using moving boundary conditions (as in Section 5.3) yields quasi-isomorphisms

$$\Psi_{G_t}\colon \mathrm{CF}(L_0, L_1; H) \to \mathrm{CF}(L_0, \psi_t(L_1); H).$$

By an action estimate (cf. Lemma 7.5) in conjunction with the above two lemmas, any subcomplex of : $\mathrm{CF}(L_0, L_1; H)$ that is spanned by the generators below some given action level is contained in the kernel of Ψ_{G_t} whenever $t \gg 0$ is taken sufficiently large. In conclusion, we obtain our sought result:

PROPOSITION 11.3. – *If $\psi_t(L_1)$ displaces L_1 from any given compact subset, then* $\mathrm{HW}(L_0, L_1) = 0$.

11.3. Twisted complexes and surgery formula

In order to obtain the surgery formula in Proposition 8.16 in the quadratic setting one needs to take additional care. The reason is that, since the products and higher A_∞-operations in this case are defined using a trick that involves rescaling by the Liouville flow, it is a priori not so clear how to relate these operations to operations defined by counts of ordinary pseudoholomorphic polygons, as in the differential of the Chekanov-Eliashberg algebra. (Recall that our surgery formula is proven by identifying bounding cochains with augmentations for the Chekanov-Eliashberg algebra of an exact Lagrangian immersion.) Our solution to this problem is to amend the construction of the A_∞-structure to yield a quasi-isomorphic version, for which the compact part of the Weinstein manifold is left invariant by the rescaling (while in the cylindrical end we still apply the Liouville flow, as is necessary for compactness issues). The upshot is that the new A_∞-structure is given by counts of ordinary pseudoholomorphic disks (together with small Hamiltonian perturbations) inside the compact part.

Here we show how to modify the definition of the A_∞-operations of the wrapped Fukaya category in a way that allows us to apply our strategy for proving the surgery formula. It will follow that the cones constructed can be quasi-isomorphically identified with cones in the original formulation of the quadratic wrapped Fukaya category, which is sufficient for establishing the generation result.

In order to modify the definition of the A_∞-operations we begin by constructing a family $\psi_{s,t}\colon W \to W$ of diffeomorphisms parametrized by $t \geq 0$ and $s \in [0,1]$ that satisfies the following:

— $\psi_{s,t}|_{W_{R-1}} = \psi_{st}$ inside W_{R-1};
— $\psi_{s,t}|_{\partial W_r} = \psi_{(s+(1-s)\beta(r))t}$ for any $r \in [R-1, R]$; and
— $\psi_{s,t}|_{W_R^e} = \psi_t$ for any $r \geq R$,

where $\beta(r)$ is a smooth cut off function that satisfies

— $\beta(r) = 0$ near $r = R-1$;
— $\beta(r) = 1$ near $r = R$; and
— $\partial_r \beta(r) \geq 0$.

The above flow $\psi_{s,t}$ is a conformal symplectomorphism only when $s = 1$, in which case $\psi_{1,t} = \psi_t$ is the Liouville flow. For general values of s it is the case that $\psi_{s,t}$ is a conformal symplectomorphism only outside a compact subset. Notwithstanding, it is the case that:

LEMMA 11.4. – 1. *If L is an exact Lagrangian immersion that is cylindrical inside W^e_{R-1}, it follows that $L_\tau := \psi^{-1}_{s(\tau),t(\tau)}(L)$ is a safe exact isotopy, which moreover is fixed setwise inside W^e_{R-1};*

2. *Any compatible almost complex structure J_t which is cylindrical inside W^e_{R-1+st} satisfies the property that $\psi^*_{s,t}J_t$ is a compatible almost complex structure which is cylindrical inside W^e_R and, moreover, equal to J_t inside W^e_{R-1+st}; and*
3. *Conjugation $\psi^{-1}_{s_0,t_0} \circ \varphi^H_t \circ \psi_{s_0,t_0}$ with the diffeomorphism ψ_{s_0,t_0} induces a bijective correspondance between Hamiltonian isotopies of W generated by Hamiltonians which depend only on $\mathfrak{r}$ inside $W^e_{R-1+s_0t_0}$ and Hamiltonian isotopies of W generated by Hamiltonians which only depend on $\mathfrak{r}$ inside W^e_{R-1}. More precisely, if the former Hamiltonian is given by $H\colon W \to \mathbb{R}$ then the latter is given by $(f_{s_0,t_0} \cdot H) \circ \psi_{s_0,t_0}$ for a smooth function $f_{s_0,t_0}\colon W \to \mathbb{R}_{>0}$ which only depends on $\mathfrak{r}$ inside $W^e_{R-1+s_0t_0}$, while $f_{s_0,t_0}|_{W_{R-1+s_0t_0}} \equiv e^{-s_0t_0}$.*

REMARK 11.5. – The function $f_{s_0,t_0}\colon W \to \mathbb{R}_{>0}$ in Part (3) of the previous lemma can be determined as follows. First note that $f_{1,t} \equiv e^{-t}$, while for general $s \in [0,1]$ the equality $f_{s,t} \equiv e^{-t}$ still holds outside a compact subset. Inside $W^e_{R-1+s_0t_0}$ the simple ordinary differential equation

$$e^{-\mathfrak{r}}\partial_{\mathfrak{r}}((f_{s_0,t_0}(\mathfrak{r}) \cdot H) \circ \psi_{s_0,t_0}) = (e^{-\mathfrak{r}}\partial_{\mathfrak{r}}H(\mathfrak{r})) \circ \psi_{s_0,t_0}$$

then determines $f_{s,t}$.

By the properties described in Lemma 11.4, it follows that we can use $\psi_{s_0,t}$ for any fixed s_0 instead of the Liouville flow ψ_t in Abouzaid's construction of the wrapped Fukaya category [3]. We illustrate this in the case of the product μ^2. One first defines a morphism

(39)

$$\begin{aligned}\mathrm{CF}(L_1, L_2; H, J_t) \otimes \mathrm{CF}(L_0, L_1; H, J_t)\\ \to \mathrm{CF}\left((\psi_{s_0,\log 2})^{-1}(L_0), (\psi_{s_0,\log 2})^{-1}(L_2); (f_{s_0,\log 2} \cdot H) \circ \psi_{s_0,\log 2}, \psi^*_{s_0,\log 2}J_t\right)\end{aligned}$$

for any fixed $s_0 \in [0,1]$ that is defined by a count of three-punctured disks with a suitable moving boundary condition, and where

$$f_{s_0,\log 2}\colon W \to \mathbb{R}_{>0}$$

is the function from Part (2) of Lemma 11.4. (In particular,

$$f_{s_0,\log 2} \equiv e^{-\log 2}$$

holds outside a compact subset.)

REMARK 11.6. – The fact that $(f_{s_0,\log w}(\mathfrak{r}) \cdot H) \circ \psi_{s_0,\log w}$ coincides with $\frac{1}{w}e^{2(t+\log w)} = we^{2t}$ outside a compact subset is crucial for the maximum principle (and thus compactness properties) of the Floer curves involved in the definition of the morphism of Equation (39).

It is immediate that an analogous version of [3, Lemma 3.4] now also holds in the present setting, giving rise to a canonical identification between the Floer complexes

$$\mathrm{CF}\left((\psi_{s_0,\log 2})^{-1}(L_0), (\psi_{s_0,\log 2})^{-1}(L_2); (f_{s_0,\log 2} \cdot H) \circ \psi_{s_0,\log 2}, \psi^*_{s_0,\log 2}J_t\right)$$

and

$$\mathrm{CF}(L_1, L_2; H, J_t).$$

In this manner we obtain the operation μ^2. The general case follows similarly, by an adaptation of the construction [3] based upon $\psi_{s_0,t}$.

Finally, to compare the A_∞ structures defined by different values of the parameter $s_0 \in [0,1]$, one can use [37]. Here Part (1) of Lemma 11.4 is crucial. Also, note that the family $(f_{s_0,\log w} \cdot H) \circ \psi_{s_0,\log w}$ of Hamiltonians is independent of the parameter s_0 outside a compact subset for any fixed value of w.

Finally, since the intersection points of type a, b, c of Lemma 9.4 belong to the region where we have turned off the rescaling by the Liouville flow, the construction of the augmentation in Lemma 9.5 remains unchanged. From this point, the proof of Theorem 1.1 in the quadratic setup procedes as in the linear setup.

11.4. Proof of Corollary 1.4

The proof of Corollary 1.4 is based on Theorem 1.1 and the following trivial observation. If $(W, \theta, \mathfrak{f})$ is a Weinstein manifold and $\pi_i \colon W \times W \to W$, for $i = 1, 2$, are the projections to the factors, we consider the Weinstein manifold $(W \times W, \pi_1^*\theta - \pi_2^*\theta, \mathfrak{f} \circ \pi_1 + \mathfrak{f} \circ \pi_2)$. Note that the sign in the Liouville form has been chosen so that the diagonal $\Delta \subset W \times W$ is an exact Lagrangian submanifold with cylindrical end. The following lemma is a direct consequence of the definition.

LEMMA 11.7. – *The Weinstein manifold $(W \times W, \pi_1^*\theta - \pi_2^*\theta, \mathfrak{f} \circ \pi_1 + \mathfrak{f} \circ \pi_2)$ has a Weinstein handle decomposition for which the Lagrangian cocores are precisely the products of the cocores $D_i \times D_j$, where D_i denotes a Lagrangian cocore in the Weinstein decomposition of* $(W, \theta, \mathfrak{f})$.

Let $\mathcal{W}^2$ be the version of the wrapped Fukaya category for the product Liouville manifold $(W \times W, \pi_1^*\theta - \pi_2^*\theta)$ defined in [22], where the wrapping is performed by a split Hamiltonian. If $\mathcal{B}$ is a full subcategory of $\mathcal{WF}(W, \theta)$, we denote by $\mathcal{B}^2$ the full subcategory of $\mathcal{W}^2$ whose objects are products of objects in $\mathcal{B}$. Then [22, Proposition 14.1] implies the following.

PROPOSITION 11.8. – *If Δ is generated by $\mathcal{B}^2$ in $\mathcal{W}^2$, then the map*

$$[\mathcal{OC}] \colon HH_{n-*}(\mathcal{B}, \mathcal{B}) \to SH^*(W) \tag{40}$$

has the unit in the symplectic cohomology in its image.

Proof of Corollary 1.4. – Let $(W, \theta, \mathfrak{f})$ be a Weinstein manifold and let $\mathcal{D}$ be the collection of the Lagrangian cocore disks of W. Then, by Lemma 11.7 and Theorem 1.1 the collection $\mathcal{D}^2$ of products of cocore disks of W generates $\mathcal{WF}(W \times W, \pi_1^*\theta - \pi_2^*\theta)$ and so, in particular, generates the diagonal Δ. By [25, Theorem 1.1] the category $\mathcal{WF}(W \times W, \pi_1^*\theta - \pi_2^*\theta)$ is equivalent to the category $\mathcal{W}^2$ defined above, and therefore the collection $\mathcal{D}^2$ generates the diagonal also in $\mathcal{W}^2$. Thus Proposition11.8 implies that the image of the open-closed map (40) contains the unit and therefore Corollary 1.4 follows from [22, Theorem 1.1]. □

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Baptiste Chantraine
Nantes Université
CNRS, Laboratoire de Mathématiques Jean Leray
UMR 6629
44000 Nantes, France
E-mail: baptiste.chantraine@univ-nantes.fr

Georgios Dimitroglou Rizell
Uppsala University
Sweden
E-mail: georgios.dimitroglou@math.uu.se

Paolo Ghiggini
Univ. Grenoble Alpes
CNRS, IF
38000 Grenoble, France
E-mail: paolo.ghiggini@univ-grenoble-alpes.fr

Roman Golovko
Charles University
Czech Republic.
E-mail: golovko@karlin.mff.cuni.cz