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TRANSIENCE IN LAW FOR SYMMETRIC RANDOM WALKS IN INFINITE MEASURE

BY TIMOTHÉE BÉNARD

ABSTRACT. — We consider a random walk on a second countable locally compact topological space endowed with an invariant Radon measure. We show that if the walk is symmetric and if every subset that is invariant by the walk has zero or infinite measure, then one has escape of mass for almost every starting point. We then apply this result in the context of homogeneous random walks on infinite volume spaces and deduce a converse to the Eskin–Margulis recurrence theorem.

RÉSUMÉ (*Transience en loi des marches aléatoires symétriques en mesure infinie*). — On considère une marche aléatoire sur un espace topologique localement compact à base dénombrable muni d'une mesure de Radon invariante. On montre que si la marche est symétrique et si tout sous-ensemble invariant par la marche est de mesure nulle ou infinie, alors il y a fuite de masse pour presque tout point de départ. Nous appliquons ensuite ce résultat dans le contexte des marches aléatoires homogènes en volume infini, et déduisons une réciproque au théorème de récurrence d'Eskin-Margulis.

1. Introduction

The starting point of this text is an article published by Eskin and Margulis in 2004, which studies the recurrence properties of random walks on homogeneous spaces [11]. The space in question is a quotient G/Λ, where G is a real Lie group and $\Lambda \subseteq G$ a discrete subgroup. Given a probability measure

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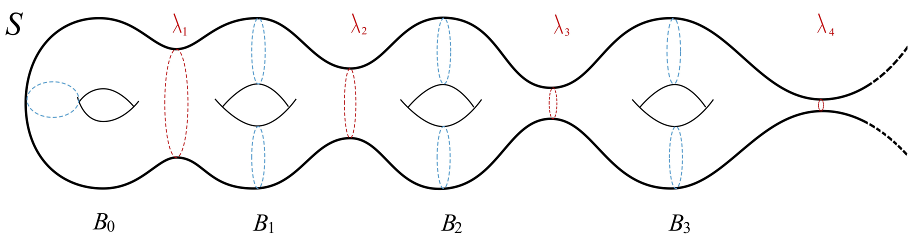
μ on G, we can define a random walk on G/Λ with transitional probability measures $(\mu * \delta_x)_{x\in G/\Lambda}$. In more concrete terms, a random step starting at a point $x \in G/\Lambda$ is performed by choosing an element $g \in G$ randomly with law μ and letting it act on x. The two authors ask about the position of the walk at time n for large values of n. They manage to show a surprising result: if G is a simple real algebraic group, if Λ has finite covolume in G, and if the support of μ is compact and generates a Zariski-dense subgroup of G, then for every starting point $x \in G/\Lambda$, the n-th step distribution of the walk $\mu^{*n} * \delta_x$ does not escape at infinity. More precisely, all the weak-$*$ limits of $(\mu^{*n} * \delta_x)_{n\geq 0}$ have mass 1. One says that there is no escape of mass. This reminds us of the behavior of the unipotent flow as highlighted by Dani and Margulis in [8, 13], proving that the trajectories of a unipotent flow on G/Λ spend most of their time inside compact sets. Eskin–Margulis' result is actually the starting point of a fruitful analogy with Ratner theorems, which led to the classification of stationary probability measures on X thanks to the work of Benoist and Quint [3, 4], followed by Eskin and Lindenstrauss [10].

This paper asks the question of a converse to Eskin–Margulis theorem.

> *Is the absence of mass escape characteristic of random walks on homogeneous spaces of finite volume, or could it also happen for walks in infinite volume?*

Let us illustrate the question with an *example*. Consider S a hyperbolic surface of the form



Such a surface is made of blocks $(B_i)_{i\geq 0}$ glued together along geodesic circles (in red) of respective length $(\lambda_i)_{i\geq 1} \in \mathbb{R}_{>0}^{\mathbb{N}^*}$. Each block comes with a pants decomposition, whose internal boundary components (in blue) are assumed to have length 1. We consider a (discretized) Brownian motion on (the unit tangent bundle of) S starting from B_0. If all the λ_i's are equal, the walk looks like the nearest neighbor random walk on $\mathbb{N}$, so we expect escape of mass. On the other hand, in the degenerate case where some λ_i is equal to 0, the walk evolves in a finite volume space so there is no mass escape by the Eskin–Margulis theorem, or more simply ergodic considerations in this case. Now, we may wonder what happens in intermediate situations where the sequence $(\lambda_i)_{i\geq 1}$ is positive but allowed to go to zero extremely fast. We will see that the n-th step distribution of the walk always escapes at infinity, regardless of the choice of $(\lambda_i)_{i\geq 1} \in \mathbb{R}_{>0}^{\mathbb{N}^*}$ (Theorem 1.2).

In Section 2, we establish escape of mass in a very general framework, which does not rely on the algebraic setting mentioned previously. The measure μ is assumed to be symmetric, i.e., invariant under the inversion map $g \mapsto g^{-1}$.

Theorem 1.1. — *Let X be a locally compact second countable topological space equipped with a Radon measure λ, let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ, and let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ-invariant subset of X has zero or infinite λ-measure, then for λ-almost every starting point $x \in X$, one has the weak-$*$ convergence:

$$\mu^{*n} * \delta_x \xrightarrow[n\to+\infty]{} 0.$$

To put it in a nutshell, a symmetric random walk on a measured space without finite volume invariant subset is transient in law for almost every starting point. This result can be seen as an analogue in infinite measure of equidistribution results for random walks in finite measure obtained independently by Rota [16] and Oseledets [15].

In our statement, a measurable subset $A \subseteq X$ is considered as Γ-invariant if for every $g \in \Gamma$, $\lambda(gA\Delta A) = 0$. We will see later an equivalent characterization in terms of the Markov operator of the walk (Lemma 2.4).

Note also that the condition of symmetry on μ plays a role. Let (X, λ) be a locally compact space with an infinite Radon measure and endowed with a conservative ergodic measure-preserving $\mathbb{Z}$-action. If $\mu = \delta_1$ is the Dirac mass at $1 \in \mathbb{Z}$, then for λ-almost every $x \in X$, the sequence $(n.x)_{n\geq 0}$ comes back close to x infinitely often, so $\mu^{*n} * \delta_x = \delta_{n.x}$ cannot weakly converge to 0.

Without symmetry assumption on μ, the proof of Theorem 1.1 still yields convergence to 0 in Cesàro-averages.

In Section 3, we use Theorem 1.1 to address our original question concerning the escape of mass of homogeneous walks on infinite volume spaces. We obtain the following result.

Theorem 1.2. — *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G, and μ a probability measure on G whose support generates a group with unbounded projections in the noncompact factors of G.*

Then for almost every $x \in G/\Lambda$, one has the weak-$*$ convergence:

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x \xrightarrow[n\to+\infty]{} 0. \tag{1}$$

Moreover, if the probability measure μ is symmetric, then the convergence can be strengthened:

$$\mu^{*n} * \delta_x \xrightarrow[n\to+\infty]{} 0. \tag{2}$$

Note that convergence (1) is sufficient to ensure that Eskin–Margulis' observations cannot occur when the quotient G/Λ has infinite measure. Indeed, for almost every $x \in G/\Lambda$, we obtain the existence of an extraction $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\mu^{*\sigma(n)} * \delta_x \xrightarrow[n\to+\infty]{} 0.$$

Theorem 1.2 describes the asymptotic behavior of the probabilities of position for *almost every* starting point $x \in G/\Lambda$. One may not hope for transience in law for every starting point as it is possible that the orbit $\Gamma.x$ is finite.

To conclude this introduction, we emphasize that our paper focuses on the behavior *in law* of a random walk on G/Λ. A related natural theme of study is the behavior of the walk trajectories for which analogous notions of recurrence or transience exist. Although our conclusions support the idea that walks in infinite volume are always transient in law (the mass escapes), the picture becomes mixed when it comes to considering walk trajectories. Indeed, as observed in [7] or [2], pointwise recurrence or transience also depends on the nature of the ambient space.

2. A general result of transience in law

This section is dedicated to the proof of Theorem 1.1. The proof results from a combination of the Dunford–Schwartz theorem [9] and Akcoglu–Sucheston's pointwise convergence of alternating sequences [1]. The latter guarantees that for λ-almost every $x \in X$, the sequence of probability measures $(\mu^{*n} * \check{\mu}^{*n} * \delta_x)_{n\geq 0}$ weak-$*$ converges toward a finite measure and is based on Rota and Oseledets' original idea to express this alternating sequence in terms of reversed martingales [16, 15]. We give a shorter proof than the one in [1]. Although our proof follows very closely the one of Rota [16] who considered walks on finite volume spaces, we use a different formalism that may be useful to illustrate the technique of "equidistribution of fibers" contained in the work of Benoist–Quint [4] (see also [5]).

2.1. Backwards martingales. — We first present a convergence theorem for backwards martingales on a σ-finite measured space. It will play a crucial role in the proof of the convergence of back-and-forths (2.2).

First, let us recall the definition of conditional expectation.

Definition (Conditional expectation). — Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathcal{Q}$ a sub-σ-algebra of $\mathcal{F}$, and m a positive measure on $(\Omega, \mathcal{F})$ whose restriction

$m_{|\mathcal{Q}}$ is σ-finite. Then, for every function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists a unique function $f' \in L^1(\Omega, \mathcal{Q}, m)$ such that for all $\mathcal{Q}$-measurable subset $A \in \mathcal{Q}$, one has $m(f\, 1_A) = m(f'\, 1_A)$. We denote this function by $\mathbb{E}_m(f|\mathcal{Q})$.

We have the following [12, page 533] (see also [6]).

Theorem 2.1 (Convergence of backwards martingales). — *Let $(\Omega, \mathcal{F}, m)$ be a measured space, $(\mathcal{Q}_n)_{n\geq 0}$ a decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that for all $n \geq 0$, the restriction $m_{|\mathcal{Q}_n}$ is σ-finite. Then, for any function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists $\psi \in L^1(\Omega, \mathcal{F}, m)$ such that we have the almost sure convergence:*

$$\mathbb{E}_m(f|\mathcal{Q}_n) \xrightarrow[n\to+\infty]{} \psi \qquad m\text{-}a.e.$$

Remark. If the measure m is σ-finite with respect to the tail-algebra $\mathcal{Q}_\infty := \bigcap_{n\geq 0} \mathcal{Q}_n$, then Theorem 2.1 can be deduced from the probabilistic case (by restriction to $\mathcal{Q}_\infty$-measurable domains of finite measure), and we can certify that $\psi = \mathbb{E}_m(f|\mathcal{Q}_\infty)$. On the opposite extreme, if every $\mathcal{Q}_\infty$-measurable subset of Ω has m-measure 0 or $+\infty$, then, the integrability of ψ implies that $\psi = 0$. The general picture is a direct sum of these two contrasting situations as $\Omega = \Omega_\sigma \amalg \Omega_\infty$ where Ω_σ is a countable union of $\mathcal{Q}_\infty$-measurable sets of finite measure, and the restricted measure $m_{|\Omega_\infty}$ takes only the values 0 or $+\infty$ on $\mathcal{Q}_\infty$ (see [12], footnote of page 533).

2.2. Convergence of back-and-forths. — We now state and show Theorem 2.2 about the convergence of back-and-forths of the μ-random walk on X. We denote by $\check{\mu} := i_*\mu$ the image of μ under the inversion map $i : \Gamma \to \Gamma$, $g \mapsto g^{-1}$.

Theorem 2.2 (Convergence of back-and-forths [1]). — *Let X be a locally compact second countable topological space equipped with a Radon measure λ, let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ, and let μ be a probability measure on Γ.*

There exists a family $(\nu_x)_{x\in X}$ of finite measures on X such that for λ-almost every $x \in X$, one has the weak-$*$ convergence:

$$(\mu^{*n} * \check{\mu}^{*n}) * \delta_x \xrightarrow[n\to+\infty]{} \nu_x.$$

Proof. — The following proof is inspired by [16] and [4]. Denote

$$B := \Gamma^{\mathbb{N}^*}, \quad \beta := \mu^{\mathbb{N}^*}, \quad T : B \to B, (b_i)_{i\geq 1} \mapsto (b_{i+1})_{i\geq 1}$$

the one-sided shift. One introduces a σ-finite fibered dynamical system (B^X, β^X, T^X) setting

- $B^X := B \times X$
- $\beta^X := \beta \otimes \lambda \in \mathcal{M}^{Rad}(B \times X)$
- $T^X : B^X \to B^X, (b, x) \mapsto (Tb, b_1^{-1}x)$.

Let $\mathcal{B}$ and $\mathcal{X}$ denote the Borel σ-algebras of B and X. The Borel σ-algebra of B^X is then the product algebra $\mathcal{B}\otimes\mathcal{X}$. For all $n \geq 0$, define the *sub-σ-algebra of the n-fibers* of T^X by setting

$$\mathcal{Q}_n := (T^X)^{-n}(\mathcal{B}\otimes\mathcal{X}).$$

It is a sub-σ-algebra of $\mathcal{B}\otimes\mathcal{X}$ such that for all $c \in B^X$, the smallest $\mathcal{Q}_n$-measurable subset of B^X containing c is the n-fiber $(T^X)^{-n}(T^X)^n(c)$. The restriction $\beta^X_{|\mathcal{Q}_n}$ is a σ-finite measure because β^X is σ-finite with respect to the σ-algebra $\mathcal{B}\otimes\mathcal{X}$ and is preserved by T^X.

As a first step, we will fix a continuous function with compact support $f \in C^0_c(X)$ and show that the sequence $((\mu^{*n} * \check{\mu}^{*n} * \delta_x)(f))_{n\geq 0}$ converges in $\mathbb{R}$ for λ-almost every x. To this end, we express $(\mu^{*n} * \check{\mu}^{*n} * \delta_x)(f)$ using a conditional expectation and we apply Theorem 2.1. Denote

$$\widetilde{f} : B^X \to \mathbb{R}, (b,x) \mapsto f(x), \quad \varphi_n := \mathbb{E}_{\beta^X}(\widetilde{f}|\mathcal{Q}_n) \in L^1(B^X, \mathcal{Q}_n).$$

We first give an explicit formula for the function φ_n. Intuitively, given a point $c = (b,x) \in B^X$, the value $\varphi_n(c)$ stands for the mean value of $\widetilde{f}$ on the smallest $\mathcal{Q}_n$-measurable subset of B^X containing c. By definition, this subset is the n-fiber going through c and is identified with the product Γ^n under the bijection

$$\begin{aligned} h_{n,c} : \Gamma^n &\to (T^X)^{-n}(T^X)^n(c), \\ a = (a_1,\dots,a_n) &\mapsto (aT^n b, a_1\dots a_n b_n^{-1}\dots b_1^{-1}.x). \end{aligned}$$

The following lemma asserts that $\varphi_n(c)$ is nothing else than the mean value of $\widetilde{f}$ on $(T^X)^{-n}(T^X)^n(c) \equiv \Gamma^n$ with respect to the measure $\mu^{\otimes n}$.

Lemma 2.3. — *Let $n \geq 0$. For β^X-almost every $(b,x) \in B^X$, one has*

$$\varphi_n(b,x) = \int_{\Gamma^n} f(a_1\dots a_n b_n^{-1}\dots b_1^{-1}x)\, d\mu^{\otimes n}(a).$$

Proof. — This result is extracted from [4] (Lemma 3.3). We recall the proof. Up to considering separately the positive and negative parts of f, one may assume $f \geq 0$. Denote by $\varphi'_n : B^X \to [0,+\infty]$ the map defined by the right-hand side of the above equation. We show that it coincides almost everywhere with φ_n by proving that it also satisfies the axioms for the conditional expectation characterizing φ_n.

As the value φ'_n at a point $c \in B^X$ only depends on $(T^X)^n(c)$, the map φ'_n is $\mathcal{Q}_n$-measurable. It remains to show that for every $A \in \mathcal{Q}_n$, one has the equality $\beta^X(1_A\widetilde{f}) = \beta^X(1_A\varphi'_n)$. Writing A as $A = (T^X)^{-n}(E)$ where $E \in \mathcal{B}\otimes\mathcal{X}$ and remembering that the measure λ is preserved by Γ, one computes that:

$$
\begin{aligned}
\beta^X(1_A\varphi'_n) &= \int_{B\times X\times\Gamma^n} 1_A(b,x) f(a_1\dots a_n b_n^{-1}\dots b_1^{-1}x)\, d\mu^{\otimes n}(a) d\beta(b) d\lambda(x)\\
&= \int_{B\times X\times\Gamma^n} 1_E(T^n b, b_n^{-1}\dots b_1^{-1}x) f(a_1\dots a_n b_n^{-1}\dots b_1^{-1}x)\, d\mu^{\otimes n}(a) d\beta(b) d\lambda(x)\\
&= \int_{B\times X\times\Gamma^n} 1_E(T^n b, x) f(a_1\dots a_n x)\, d\mu^{\otimes n}(a) d\beta(b) d\lambda(x)\\
&= \int_{B\times X} 1_E(T^n b, x) f(b_1\dots b_n x)\, d\beta(b) d\lambda(x)\\
&= \int_{B\times X} 1_E(T^n b, b_n^{-1}\dots b_1^{-1}x) f(x)\, d\beta(b) d\lambda(x)\\
&= \beta^X(1_A\widetilde{f})
\end{aligned}
$$

which concludes the proof of Lemma 2.3. □

Theorem 2.3 implies that for λ-almost every $x\in X$,

$$
(**) \qquad \int_B \varphi_n(b,x)\, d\beta(b) = (\mu^{*n} * \check{\mu}^{*n} * \delta_x)(f).
$$

However, Theorem 2.1 on convergence of backwards martingales asserts that the sequence of conditional expectations $(\varphi_n)_{n\geq 0}$ converges β^X-almost-surely. Noticing that $\|\varphi_n\|_\infty \leq \|f\|_\infty$, the dominated convergence theorem and equation $(**)$ imply that for λ-almost every $x\in X$, the sequence

$$
((\mu^{*n} * \check{\mu}^{*n} * \delta_x)(f))_{n\geq 0}
$$

has a limit in $\mathbb{R}$.

We deduce from the previous paragraph that for λ-almost every $x\in X$, the sequence of probability measures $(\mu^{*n} * \check{\mu}^{*n} * \delta_x)_{n\geq 0}$ has a weak-$*$ limit (which is a measure on X whose mass is less or equal to 1, and possibly null). It is, indeed, a standard argument, which uses the separability of the space of continuous functions with compact support on X equipped with the supremum norm $(C^0_c(X), \|.\|_\infty)$ and the representation of non-negative linear forms on $C^0_c(X)$ by Radon measures (Riesz theorem). This concludes the proof of Theorem 2.2. □

2.3. Proof of Theorem 1.1. — We now prove Theorem 1.1, stating that a symmetric random walk on a measured space without finite volume invariant subset is almost everywhere transient in law. The proof will use the *Markov operator* P_μ attached to μ. It acts on the set of non-negative measurable functions on X via the formula

$$
P_\mu\varphi(x) := \int_G \varphi(gx)\, d\mu(g)
$$

and can be extended as a contraction on the spaces $L^p(X,\lambda)$ for $p\in[1,\infty]$.

Recall from the Introduction that a measurable subset $A \subseteq X$ is Γ-*invariant* if for all $g \in \Gamma$, one has $\lambda(A \Delta gA) = 0$. This condition can be rephrased in terms of the Markov operator:

Lemma 2.4. — *A measurable subset $A \subseteq X$ is* Γ-invariant *if and only if*

$$P_\mu 1_A = 1_A \qquad \lambda\text{-}a.e.$$

Proof. — The point is to show that P_μ-invariance implies Γ-invariance. Let A be a measurable subset such that $P_\mu 1_A = 1_A$ λ-a.e. The assumption on A means that for λ-almost every $x \in X$, μ-almost every $g \in G$, one has $1_A(gx) = 1_A(x)$. Fubini theorem then implies that for μ-almost every $g \in \Gamma$, one has $\lambda(A \Delta gA) = 0$. The subgroup $D \subseteq \Gamma$ generated by such elements g is dense in Γ and leaves the set A λ-a.e.-invariant. So we just need to check that the λ-a.e.-invariance is preserved by taking limits. Let $g \in \Gamma$, $(g_n) \in D^{\mathbb{N}}$ such that $g_n \to g$, let $\varphi \in C^0_c(X)$. By dominated convergence,

$$\int_{g_n A} \varphi \, d\lambda - \int_{gA} \varphi \, d\lambda = \int_A \varphi(g_n .) - \varphi(g.) \, d\lambda \xrightarrow[n \to +\infty]{} 0.$$

We deduce that $\int_A \varphi \, d\lambda = \int_{gA} \varphi \, d\lambda$. As this is true for every $\varphi \in C^0_c(X)$, one concludes that $\lambda(A \Delta gA) = 0$. □

We can now conclude.

Proof of Theorem 1.1. — It is enough to show that for λ-almost every $x \in X$, one has the convergence $\mu^{*2n} * \delta_x \to 0$. According to Theorem 2.2 and the symmetry of μ, the sequence $(\mu^{*2n} * \delta_x)_{n \geq 0}$ converges to a finite measure, so it is enough to check the following convergence in average: for λ-almost every $x \in X$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{*n} * \delta_x \longrightarrow 0.$$

As announced in the preceding remark, we show this last convergence *without using the assumption of symmetry on μ*. We need to check that for every non-negative continuous function with compact support $\varphi \in C^0_c(X)^+$,

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k_\mu \varphi \longrightarrow 0 \qquad \lambda\text{-a.e.},$$

where P_μ denotes the Markov operator of the walk.

The Dunford–Schwartz ergodic theorem [9, 14] implies that the sequence of functions $(\frac{1}{n} \sum_{k=0}^{n-1} P^k_\mu \varphi)_{n \geq 1}$ converges almost-surely to some function $\psi : X \to \mathbb{R}_+$. As the functions $P^k_\mu \varphi$ are uniformly bounded in $L^2(X, \lambda)$, Fatou lemma implies that $\psi \in L^2(X, \lambda)$. Furthermore, the function φ being bounded, the dominated convergence theorem applied to the probability space (Γ, μ) gives the P_μ-invariance

$$P_\mu \psi = \psi \qquad \lambda\text{-a.e.}$$

We now infer that ψ is Γ-invariant, meaning that for $g \in \Gamma$, one has the equality $\psi \circ g = \psi$ λ-a.e. on X. To this end, observe that the P_μ-invariance of ψ expresses ψ as a barycenter of translates $\psi \circ g$:

$$\int_\Gamma \psi \circ g \, d\mu(g) = \psi \qquad \lambda\text{-a.e.}$$

However, the functions $\psi \circ g$ all are in $L^2(X, \lambda)$ and have the same norm as ψ. The strict convexity of balls in a Hilbert space then gives for μ-almost every $g \in \Gamma$, the equality $\psi \circ g = \psi$ λ-almost everywhere. As the support of μ generates Γ as a closed subgroup, we infer as in Lemma 2.4 that for all $g \in \Gamma$, one has $\psi \circ g = \psi$ λ-a.e., which is the Γ-invariance announced above.

The Γ-invariance of ψ implies that for every constant $c > 0$, the set $\{\psi > c\}$ is Γ-invariant, so has zero or infinite λ-measure by hypothesis. As ψ^2 is integrable, we must have $\lambda\{\psi > c\} = 0$. Finally, we get that $\psi = 0$ λ-almost everywhere, which finishes the proof.

□

3. Application to homogeneous walks on infinite volume spaces

This section is dedicated to the proof of Theorem 1.2. We let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G, and $\Gamma \subseteq G$ a closed subgroup.

Let us recall the notion of factors of G used in this section.

Definition. — Denote by $\mathfrak{g}$ the Lie algebra of G. It can be uniquely decomposed as a direct sum of simple ideals: $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. The *factors* of G are the immersed connected subgroups $G_1, \dots, G_s$ of G whose Lie algebras are $\mathfrak{g}_1, \dots, \mathfrak{g}_s$. They are closed in G and commute mutually: for $i \neq j \in \{1, \dots, s\}$ and $g_i \in G_i, g_j \in G_j$ one has $g_i g_j = g_j g_i$. Lastly, the product map $\pi : G_1 \times \cdots \times G_s \to G, (g_1, \dots, g_s) \mapsto g_1 \dots g_s$ is a morphism of groups which is onto and has a finite kernel.

We make the assumption that Γ *has unbounded projections in the noncompact factors of* G, which means that the projection of $\pi^{-1}(\Gamma) \subseteq G_1 \times \cdots \times G_s$ in any G_i is unbounded if G_i is noncompact.

Theorem 1.2 expresses escape of mass for a walk on G/Λ induced by a probability measure μ whose supports generate a dense subgroup of Γ. We will obtain it as a consequence of Theorem 1.1 together with its comment about the nonsymmetric case. To apply them, we need to check the assumption that every subset of G/Λ that is invariant by the walk has zero or infinite Haar measure.

This would be obvious if the action of Γ on G/Λ were ergodic. However, this is not always the case, even when Γ and Λ are Zariski-dense in G.

Example. — Denote by $\mathbb{D}$ the Poincaré disk, set $G = PSL_2(\mathbb{R}) \equiv \mathrm{Isom}^+(\mathbb{D}) \equiv T^1\mathbb{D}$ and consider a Schottky subgroup $S_0 \subseteq G$ whose limit set $\mathscr{L}_0$ on the boundary of $\mathbb{D}$ is contained under four geodesic arcs, which are disjoint and small enough. Set $\Gamma = \Lambda = S_0$. For some nonzero measure subset of unit vectors $x \in T^1\mathbb{D}$, the set $\overline{x\Lambda} \cap \partial\mathbb{D} = x\mathscr{L}_0$ does not intersect the limit set $\mathscr{L}_0$ of Γ. Given such an x and looking in the quotient space, the orbital map $\Lambda \to \Gamma\backslash G, g \mapsto \Gamma xg$ is proper, so its image cannot be dense. Thus, the right action of Λ on $\Gamma\backslash G$ is not ergodic or, equivalently, the left action of Γ on G/Λ is not ergodic.

The absence of finite volume invariant subspaces will be a consequence of the Howe–Moore theorem [17, Theorem 2.2.20], which we now recall.

Theorem (Howe–Moore). — *Let G be a semisimple connected real Lie group with finite center, and π a continuous morphism from G to the unitary group of a separable Hilbert space $(\mathcal{H}, \langle .,. \rangle)$. Assume that every noncompact factor G_i of G has a trivial set of fixed points, i.e., $\mathcal{H}^{G_i} := \{x \in \mathcal{H},\ G_i.x = x\}$ is $\{0\}$.*

Then for every $v, w \in \mathcal{H}$, one has

$$\langle \pi(g).v, w \rangle \xrightarrow[g\to\infty]{} 0.$$

In the statement, the unitary group $U(\mathcal{H})$ is endowed with the strong operator topology, and the notation $g \to \infty$ means that g leaves every compact subset of G.

The Howe–Moore theorem implies a lemma of rigidity.

Lemma 3.1. — *Assume that G has no compact factor. Let $(\mathcal{H}, \rho)$ be a unitary representation of G on a separable Hilbert space.*

$$\textit{If}\ \ \mathcal{H}^G = \{0\}\ \ \textit{then}\ \ \mathcal{H}^\Gamma = \{0\}.$$

Proof. — Denote by $G_1, \dots, G_s$ the factors of G. Up to pulling back the representation of G by the product map $\pi : G_1 \times \cdots \times G_s \to G, (g_1, \dots, g_s) \mapsto g_1 \dots g_s$, one may suppose that $G = G_1 \times \cdots \times G_s$.

Assume $s = 2$. The hypothesis $\mathcal{H}^G = \{0\}$ implies that $\mathcal{H}^{G_1} \cap \mathcal{H}^{G_2} = \{0\}$. Thus, we can decompose

$$\mathcal{H} = \mathcal{H}^{G_1} \oplus \mathcal{H}^{G_2} \oplus \mathcal{H}',$$

where $\mathcal{H}'$ is the orthogonal of $\mathcal{H}^{G_1} \oplus \mathcal{H}^{G_2}$ in $\mathcal{H}$. Moreover, each subspace is invariant by G. Let $v \in \mathcal{H}$ be a Γ-invariant vector. Decompose v as $v = v_1 + v_2 + v'$ with $v_i \in \mathcal{H}^{G_i}$, $v' \in \mathcal{H}'$. The representation of G on $\mathcal{H}$ leads to a unitary representation of G_2 on $\mathcal{H}^{G_1}$, and the Γ invariance of v implies that v_1 is invariant under $p_2(\Gamma)$, the projection of Γ on the factor G_2. As $p_2(\Gamma)$ is unbounded in G_2, one can apply the Howe–Moore theorem to obtain $v_1 = 0$. In the same way, $v_2 = 0$. Thus, $v = v' \in \mathcal{H}'$. The representations of G_1 and G_2 induced by G on $\mathcal{H}'$ have no nontrivial fixed point. Hence, we can apply the Howe–Moore theorem one more time to infer that $v' = 0$. Finally, $\mathcal{H}^\Gamma = \{0\}$.

For the general case where $s \geq 1$, argue by induction on s using the previous method and the decomposition of $\mathcal{H}$ as $\mathcal{H}^{G_1\times\cdots\times G_{s-1}} \oplus \mathcal{H}^{G_s} \oplus \mathcal{H}'$. □

We deduce that for a group G with no compact factor, the action of Γ on G/Λ does not have finite volume invariant subspaces.

Lemma 3.2. — *Assume that G has no compact factor. Then every Γ-invariant subset of G/Λ has a zero or infinite Haar-measure.*

Proof. — Argue by contradiction assuming that there exists a Γ-invariant subset $A \subseteq G/\Lambda$ such that $\lambda(A) \in (0, +\infty)$ for some G-invariant Radon measure λ on G/Λ. Consider the regular unitary representation of G on $L^2(G/\Lambda)$, given by the formula $g.f = f(g^{-1}.)$. The characteristic function $1_A \in L^2(G/\Lambda)$ is a nonzero fixed point for the action of Γ. As G has no compact factor, Theorem 3.1 and the assumption on Γ imply that there exists a nonzero fixed point $\varphi \in L^2(G/\Lambda)$ for the action of G. Such a function is λ-a.e. constant, implying that λ has finite mass. Absurd. □

We can now conclude with the

Proof of Theorem 1.2. — Assume first that the group G has no compact factor. If the probability measure μ is symmetric, then convergence (2) comes from Lemma 3.2 and Theorem 1.1. If there is no assumption of symmetry, we still get the convergence in Cesàro average (1) via the remark following Theorem 1.1.

We now explain how to reduce Theorem 1.2 to the case where G has no compact factor. Denote by $G_1, \ldots, G_s$ the factors of G, and π the induced finite cover of G, i.e., $\pi : G_1 \times \cdots \times G_s \to G, (g_1, \ldots, g_s) \mapsto g_1 \ldots g_s$. There exists a probability measure $\widetilde{\mu}$ on $\Pi_{i=1}^s G_i$ whose support is $\pi^{-1}(\operatorname{supp}\mu)$ and such that the $\widetilde{\mu}$-walk on $\Pi_{i=1}^s G_i/\pi^{-1}(\Lambda)$ lifts the μ-walk on G/Λ. It is enough to show escape of mass for this $\widetilde{\mu}$-walk. Denote by $G_1, \ldots, G_k$ the noncompact factors of G and $p : \Pi_{i=1}^s G_i \to \Pi_{i=1}^k G_i, (g_i)_{i\leq s} \mapsto (g_i)_{i\leq k}$ the projection on their product (notice that $k \geq 1$, otherwise G would not have a discrete subgroup of infinite covolume). Then the projection $p(\pi^{-1}(\Lambda))$ is a discrete subgroup of infinite covolume in $\Pi_{i=1}^k G_i$. It is enough to prove escape of mass for the projection $p_*\widetilde{\mu}$ on $\Pi_{i=1}^k G_i$. Note that this probability measure generates a group with unbounded projections in the G_i's for $i = 1, \ldots, k$. Hence, we have reduced Theorem 1.2 to the case of a group with no compact factor, which finishes the proof. □

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