Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

COXETER POLYTOPES AND BENJAMINI–SCHRAMM CONVERGENCE

Jean Raimbault

Tome 152
Fascicule 1

2024

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 33-44

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 1, tome 152, mars 2024

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 160 € ($ 240),
avec supplément papier : Europe 244 €, hors Europe 330 € ($ 421)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
152 (1), 2024, p. 33-44

COXETER POLYTOPES AND BENJAMINI–SCHRAMM CONVERGENCE

by Jean Raimbault

Abstract. — We observe that a large part of the volume of a hyperbolic polyhedron is taken by a tubular neighbourhood of its boundary and use this to give a new proof for the finiteness of arithmetic maximal reflection groups following a recent work with M. Frączyk and S. Hurtado. We also study in more depth the case of polygons in the hyperbolic plane.

Résumé (*Polytopes de Coxeter et convergence de Benjamini–Schramm*). — En partant de l'observation qu'au moins une proportion fixée du volume d'un polytope hyperbolique est concentrée dans un voisinage tubulaire de son bord, nous donnons une nouvelle démonstration de la finitude des groupes de réflexion arithmétiques, à la suite d'un travail en commun avec M. Frączyk and S. Hurtado. Nous effectuons aussi une étude plus poussée de ce phénomène dans le cas des polygones du plan hyperbolique.

Let X be a space of constant curvature, which is either a hyperbolic space $\mathbb{H}^d$, a Euclidean space $\mathbb{R}^d$ or a sphere $\mathbb{S}^d$. An hyperplane in X is a one-lower-dimensional complete totally geodesic subspace, and a polytope is a bounded (or in the case of $\mathbb{H}^d$, finite-volume) region delimited by a finite number of hyperplanes. A polytope in X is said to be Coxeter if the dihedral angles between its faces are each of the form π/m for some $m \geq 2$. For Coxeter polytopes in $\mathbb{R}^d$ or $\mathbb{S}^d$ there is a well-known, complete and very intelligible classification of Coxeter polytopes by Coxeter diagrams. On the other hand,

Texte reçu le 13 septembre 2022, modifié le 7 juin 2023, accepté le 26 juillet 2023.

Jean Raimbault, Institut de Mathématiques de Marseille, UMR 7373, CNRS, Aix-Marseille Université • *E-mail :* `jean.raimbault@univ-amu.fr`

Mathematical subject classification (2010). — 22E40, 51F15.

Key words and phrases. — Hyperbolic reflexion group, Benjamini–Schramm convergence.

This research was supported by ANR grant AGDE – ANR-20-CE40-0010-01.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2024/33/$ 5.00

doi:10.24033/bsmf.2884

Coxeter polytopes in $\mathbb{H}^d$ have a very different behaviour and are still quite mysterious. In the sequel, we will thus only be concerned with $X = \mathbb{H}^d$. We will study Coxeter polytopes from a metric viewpoint and establish some results about their shapes when the volume tends to infinity, especially when $d = 2$. A general survey on Coxeter groups in hyperbolic space is given in [13]; a more recent one is given in [3], which focuses on arithmetic aspects.

Let P be a Coxeter polytope in $\mathbb{H}^d$ and let Γ_P be the subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ generated by reflections in the faces of P. This is a discrete subgroup acting on X with fundamental domain P, by the Poincaré polyhedron theorem. Moreover, P is identified with the $\mathbb{H}^d$-orbifold given by $\Gamma_P\backslash\mathbb{H}^d$, by endowing each face of P with the local orbifold structure given by its pointwise stabiliser (the group generated by reflections in the maximal faces that contain it). For $R > 0$, the R-thin part of P is given by

$$P_{\leq R} = \{x \in P : \exists \gamma \in \Gamma_P,\ d(x, \gamma x) \leq R/2, \}$$

and it corresponds to the R-thin part of the orbifold. An easy exercise shows that $P_{\leq R}$ is equal to the set of points in P that are at distance at most $R/2$ from the boundary. The first result in this note is the following, which is essentially a consequence of the hyperbolic isoperimetric inequality as we prove in 1 below.

Theorem 1. — *For every $d \geq 2$, there exists a constant $C(d)$ such that for every Coxeter polytope P of finite volume in $\mathbb{H}^d$, we have*

$$\mathrm{vol}(P_{\leq 2}) \geq C(d)\,\mathrm{vol}(P).$$

In a joint work with M. Frączyk and S. Hurtado [6] it was proven that $\mathrm{vol}(M_{\leq R}) = o(\mathrm{vol}\, M)$ uniformly for M a congruence arithmetic orbifold quotient of a given symmetric space. From this together with Theorem 1 we can quite easily deduce the following result, which was originally proved by Nikulin and Agol–Belolipetsky–Storm–Whyte ([11, 2], respectively). In fact, the inspiration for this note was provided by a recent work of Fisher–Hurtado [5], where they use a lower-level part of [6][1] to give a new proof of Nikulin and Agol–Belolipetsky–Storm–Whyte's result.

Corollary 2. — *For any d there are at most finitely many arithmetic maximal reflection groups in* $\mathrm{PO}(d, 1)$.

Proof. — Let Γ_{P_n} be a sequence of pairwise non-conjugated maximal Coxeter arithmetic lattices, that is P_n is a finite-volume Coxeter polyhedron in $\mathbb{H}^d$ and $\mathrm{vol}(P_n) \to +\infty$. Let Γ_n be the congruence closure of Γ_{P_n}. Theorem D in [6] states that if $\mathrm{vol}(\Gamma_n\backslash\mathbb{H}^d) \to +\infty$ then, putting $M_n = \Gamma_n\backslash\mathbb{H}^d$, we have that for any $R > 0$, $\lim_{n\to+\infty} \frac{\mathrm{vol}(M_n)_{\leq R}}{\mathrm{vol}\, M_n} = 0$. Since P_n is a finite (orbifold) cover of M_n

1. Namely the "arithmetic Margulis lemma", Theorem 3.1 in loc. cit., which is an essential ingredient in the proof of Theorem D in loc. cit.

and the ratio $\frac{\mathrm{vol}(\cdot)_{\leq R}}{\mathrm{vol}}$ is decreasing in finite covers, this would contradict the fact that $\mathrm{vol}(P_n)_{\leq 2} \geq C(d)\,\mathrm{vol}(P_n)$ if we knew that $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \to +\infty$.

So, to deduce the corollary we need only prove that $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \to +\infty$. This is the case if the adjoint trace fields of Γ_{P_n} are not of bounded degree. On the other hand, if these trace fields have bounded degree then by [2, Lemma 6.2] and standard arguments (see, e.g. Lemma 5.4 in loc. cit.) we have that $[\Gamma_n : \Gamma_{P_n}]$ is bounded, so that $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \gg \mathrm{vol}(P_n)$ goes to infinity. □

It is also well known that cofinite reflection groups cannot exist in large dimensions [12, 8], so we may as well say that there are only finitely many congruence (or maximal arithmetic) hyperbolic reflection groups.

In the case of polygons in $\mathbb{H}^2$, we can say more. Let $G = \mathrm{Isom}(\mathbb{H}^d) = \mathrm{PO}(d,1)$ and let μ_P be the G-invariant measure on the Chabauty space Sub_G of G supported on the conjugacy class of Γ_P. We will use the notion of Benjamini–Schramm convergence introduced in [1, Sections 2-3]; this is the notion of convergence induced by the topology of weak convergence of measures on Sub_G. In this language, Theorem 1 implies that the trivial subgroup is not a limit point of the measures μ_P. When $d = 2$ we can prove the following result, which is much more precise than Theorem 1.

Theorem 3. — *If μ belongs to the closure (in topology of weak convergence of measures on Sub_G) of the set of all μ_P for P a finite-volume Coxeter polygon in $\mathbb{H}^2$, then μ-almost every subgroup is non-trivial and generated by reflections.*

This seems likely to be true in higher dimensions as well, though our very elementary proof does not seem to immediately extend to this setting. Finally, note that any subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ generated by reflections is, in fact, generated by the reflections in the side of a Coxeter polytope (possibly with infinitely many faces), which is a well-known fact[2]

Organisation. — The proof of Theorem 1 is very short and given in Section 1. The rest of the article is dedicated to the proof of Theorem 3; first we collect a few useful facts on the geometry of hyperbolic Coxeter polygons in Section 2, and use them in Section 3 to prove that a Benjamini–Schramm limit of Coxeter polygons is almost surely non-trivial. Independently, we prove in Section 4 that the set of groups generated by reflections is closed in the Chabauty topology and deduce that a Benjamini–Schramm limit of Coxeter polygons is almost surely generated by reflections.

1. Proof of Theorem 1

Fix $d \geq 2$. Let P be a Coxeter polytope in $\mathbb{H}^d$. We will denote by $F_i, i \in I$ the $(d-1)$-faces of P.

2. This is stated at the beginning of [13], and the proof is more or less obvious.

Now let

$$U = \{x \in P :\ d(x, \partial P) \le 1\}$$

and

$$P' = \{x \in P :\ d(x, \partial P) \ge 1\}.$$

We have $U \subset P_{\le 2}$ since every point of U is moved by at most 2 by a reflection in a face of P (as previously noted, it is clear that, in fact, $U = P_{\le 2}$).

For $x \in F_i$, let ν_x be the vector normal to F_i pointing inside P; note that $d(F_i, \exp_x(\nu_x)) = 1$, for all $x \in F_i$. Let $W_i \subset F_i$ defined by

$$W_i = \{x \in F_i : \exp_x(\nu_x) \in P',\ \forall\, j \neq i\, d(F_j, \exp_x(\nu_x)) > 1\}$$

and let

$$F_i' = \{\exp_x(\nu_x) :\ x \in W_i\}.$$

Then the F_i' are disjoint open subsets of $\partial P'$, and their complement $S = \partial P' \setminus \bigcup_{i \in I} F_i'$ is of measure 0 (with respect to the $(d-1)$-dimensional measure on $\partial P'$) as it is equal to the set of points in $\partial P'$ at distance 1 from at least two of the F_i.

For $y \in F_i'$, let ν_y' be the vector orthogonal to F_i' pointing outside P'; then the map

$$E : [0,1] \times \partial P' \setminus S \to \mathbb{H}^d,\ (t, y) \mapsto \exp_y(t\nu_y') \tag{1}$$

has its image inside U. Since the local geometry of the submanifolds F_i' of $\mathbb{H}^d$ depends only on d, we see that the Jacobian of E is uniformly bounded away from 0 (we prove this in detail in 1.1 at the end of the section); let $\varepsilon(d) > 0$ be a lower bound. Moreover, the sets $E([0,1] \times F_i')$ are pairwise disjoint (since a point in $E([0,1] \times F_i')$ is at distance ≤ 1 from exactly one face of ∂P, which is F_i, and at distance > 1 of all others). It follows that

$$\mathrm{vol}_d\left(E([0,1] \times \partial P' \setminus S)\right) \ge \varepsilon(d) \cdot 1 \cdot \mathrm{vol}_{d-1}(\partial P' \setminus S) = \varepsilon(d)\, \mathrm{vol}_{d-1}(\partial P'),$$

so that $\mathrm{vol}(U) \ge \varepsilon(d)\, \mathrm{vol}_{d-1}(\partial P')$. We finish the proof of the theorem with the following chain of inequalities:

$$\begin{aligned} \mathrm{vol}_d(P) &= \mathrm{vol}_d(U) + \mathrm{vol}_d(P') \\ &\le \mathrm{vol}_d(U) + \mathrm{vol}_{d-1}(\partial P') \\ &\le \left(1 + \varepsilon(d)^{-1}\right) \mathrm{vol}_d(U) \\ &\le C(d)\, \mathrm{vol}_d(P_{\le 2}). \end{aligned}$$

where the second inequality follows from the isoperimetric inequality for hyperbolic space [7, Proposition 6.6], which implies that $\mathrm{vol}_d(P') \le \mathrm{vol}_{d-1}(\partial P')$.

1.1. Exponential map on equidistant sets. — Let H be a geodesic hyperplane in $\mathbb{H}^d$ and H' a connected component of $\{x \in \mathbb{H}^d : d(x, H) = 1\}$. Let $E : [0, 1] \times H' \to \mathbb{H}^d$ be the map defined as in (1). Since all hyperplanes and their equidistant sets are related by isometries, and exponential maps are equivariant with respect to those, our claim will follow if we prove that the Jacobian $\det(DE(x, t))$ is uniformly bounded away from 0 for $x \in H', t \in [0, 1]$.

The group $\mathrm{Isom}(H)$ acts transitively on H', and the map E is equivariant with respect to this action. It follows that we need only to prove that $\det(DE(t, x))$ is uniformly bounded away from 0 for a fixed x and $t \in [0, 1]$. This is immediate by compacity, since $DE(t, x)$ is invertible for all $t \in \mathbb{R}$.

2. Lemmas on Coxeter polygons

We collect here some preliminary facts about Coxeter polygons in $\mathbb{H}^2$, and give complete proofs for all of them; though they are likely well known it seems more convenient to give their (short) proofs than locate sufficiently precise references for them. First we have a consequence of the collar/Margulis lemma.

Lemma 2.1. — *There exists $\eta > 0$ such that if P is a Coxeter polygon in $\mathbb{H}^2$ then:*

1. *if an edge of P has length $\leq \eta$, then its adjacent angles are right angles;*
2. *no two adjacent edges of P have both length $\leq \eta$;*
3. *any two non-consecutive vertices of P are at distance at least η.*

Proof. — Let $\Gamma = \Gamma_P$ be the discrete subgroup generated by the reflections σ_e in the sides e of P. Let δ be the constant given by the collar/Margulis lemma for $\mathbb{H}^2$, so that for any $x \in \mathbb{H}^2$, we have that

$$\Gamma_x := \langle \gamma \in \Gamma : d(x, \gamma x) \leq \delta \rangle$$

is virtually cyclic. So if e_2 is an edge of P with length $\leq \delta/2$ and e_1, e_3 the adjacent edges of P, then the subgroup generated by σ_{e_i} is virtually cyclic (as each of σ_{e_i} moves any vertex of e of less than δ). On the other hand, discrete virtually cyclic subgroups of $\mathrm{PGL}_2(\mathbb{R})$ cannot contain an element of finite order other than 2 (such a subgroup contains an hyperbolic isometry and any finite-order element in the group must preserve the two endpoints of its axis), and as this subgroup contains the rotations about the vertices of e_2, the angles between e_1, e_1 and e_2, e_3 must be right angles. This proves part 1 for any $\eta \leq \delta/2$.

We now prove part 2 for any $\eta \leq \delta/4$. Assume that there are two consecutive edges with lengths less than η, i.e. three consecutive vertices x_1, x_2, x_3 such that $d(x_1, x_2)$, $d(x_2, x_3) \leq \eta$. Then if $\sigma_i, 1 \leq i \leq 4$ are the reflections in the sides containing the x_j, each x_j is at distance less than 2η of the axis of each σ_i so $d(x_j, \sigma_i x_j) < 4\eta \leq \delta$. By the Collar Lemma it follows that the subgroup generated by the reflections σ_i must be virtually cyclic. On the other hand, it contains three rotations whose centres (namely x_1, x_2, x_3) are not colinear, which is impossible if it is virtually cyclic.

To prove the last point let v, w be two non-consecutive edges of P and let e' be the common perpendicular between v and w. Since P is a Coxeter polygon the sum of the angles of P at v and w is at most π, so there are two edges e_1, e_2 of P on the same side of e' such that the sum of the angles between e' and e_1, e_2 is at most $\pi/2$. Let β, γ be these angles. Since the hyperbolic triangle with angles $\pi/4, \beta, \gamma$ has an area at least $\pi/4$, at least one of its edges is of length $\geq d$ where d is the smallest diameter of a hyperbolic disc of area $\geq \pi/4$. Either it is the side length a opposite to the angle $\pi/4$, or we can assume that it is the side length b opposite to the angle β, and, in this case, we have $\sinh(a) = \frac{2\sqrt{2}\sinh(b)}{\sin(\beta)} \geq 2\sqrt{2}\sinh(r)$. In any case, we get that the side adjacent to the angles β, γ in this triangle has length bounded below by a constant a_0 independent of β, γ. It follows that if v, w are at distance less than a_0, then the half-lines supported by e_1, e_2 must intersect in $\mathbb{H}^2$. It follows that the subgroup generated by the reflections in the edges of P adjacent to v, w contain at least three rotations with non-aligned centre, so by Margulis lemma we must have $d(v, w) \geq \delta/2$. This proves that 3 holds for any $\eta \leq \min(\delta/2, a_0)$. □

Next, we will need the following lemma on the area of hyperbolic triangles.

Lemma 2.2. — *For any ℓ_0, there exists a constant $A > 0$ such that any hyperbolic triangle with edges of length at least ℓ_0 and angles at most $\pi/2$ has an area at least A.*

Proof. — Let a, b, c be the edge lengths of such a triangle T and define

$$s = \frac{a+b+c}{2},\ s_a = s - a,\ s_b = s - b,\ s_c = s - c.$$

The hyperbolic Heron formula [10, Theorem 1.1(i)] states that

$$\sin(\mathrm{Area}(T)) = \frac{\sqrt{\sinh(s)\sinh(s_a)\sinh(s_b)\sinh(s_c)}}{4\cosh(a/2)\cosh(b/2)\cosh(c/2)}.$$

Using the hyperbolic cosine law it follows from our hypotheses on T that there exists a constant $\ell_1 > 0$ (independent of a, b, c) such that $s, s_a, s_b, a_c \geq \ell_1$. It follows that we have $\sinh(s_a) \gg e^{s_a}$ and similarly for the other terms. As $s_a + s_b + s_c = 2s$ we get that

$$\sinh(s)\sinh(s_a)\sinh(s_b)\sinh(s_c) \gg e^{3s}.$$

Similarly,

$$\cosh(a/2)\cosh(b/2)\cosh(c/2) \ll e^s,$$

so in the end

$$\operatorname{Area}(T) \geq \sin(\operatorname{Area}(T)) \gg e^{s/2},$$

which finishes the proof. □

Finally, we will use the following lemma.

LEMMA 2.3. — *Let $\varepsilon > 0$. There exists a constant $\eta' > 0$ such that for any Coxeter polygon P in $\mathbb{H}^2$ and any R there exists a polygon P' such that*

1. *P' has no edge length smaller than η';*
2. $\frac{\operatorname{Area}(P'_{\leq R})}{\operatorname{Area} P'} \leq 2\frac{\operatorname{Area}(P_{\leq R})}{\operatorname{Area} P}$.

Proof. — We construct P' as follows: let $v_1, \dots, v_m$ be a cyclic ordering of the vertices of P. Let η be the constant from Lemma 2.1, and $0 < \alpha < 1/2$ (to be determined later). For each pair (v_i, v_{i+1}) of adjacent vertices such that $d(v_i, v_{i+1}) \leq \alpha \cdot \eta$, we remove the vertex v_{i+1} from P, that is, if $i_1, \dots, i_k$ are those indices such that $d(v_{i_j}, v_{i_j+1}) \leq \alpha \cdot \eta$, we take P' to be the polygon spanned by the $v_i, i \notin \{i_1, \dots, i_k\}$. Since $d(v_{i_j+1}, v_{i_j+2}) > \eta$ by Lemma 2.1, it follows from the triangle inequality that $d(v_{i_j}, v_{i_j+2}) > (1-\alpha)\eta > \alpha\eta$, and so P' satisfies condition 1 for any $\eta' \leq \alpha\eta$.

We now prove that P' satisfies condition 2 for sufficiently small α. First we estimate the area of each removed triangle. To do this let i such that $d(v_i, v_{i+1}) < \alpha\eta$; we want to estimate the area of the triangle T_i spanned by v_i, v_{i+1}, v_{i+2}. Let γ be its angle at v_i, $c = d(v_{i+1}, v_{i+2})$, $a = d(v_i, v_{i+2})$; by the hyperbolic sine law we have that $\sinh(a) = \frac{\sinh(c)}{\sin(\gamma)}$. We compute that

$$\begin{aligned} \sin(\gamma) &\geq \frac{\sinh(a-\eta)}{\sinh(a)} \\ &\geq \frac{\sinh(a) - \eta\cosh(a)}{\sinh(a)} \geq 1 - u \cdot \alpha, \end{aligned}$$

where $u > 0$ on the last line depends only on η. It follows that $\gamma \geq \frac{\pi}{2} - u' \cdot \alpha$, and finally that

$$\operatorname{Area}(T_i) \leq u' \cdot \alpha, \tag{2}$$

for some u' independent of P, α.

Now the triangle S_i spanned by v_{i-1}, v_i, v_{i+2} has all its edge lengths at least η, by Lemma 2.1. Its angles are at most $\pi/2$ since it is inscribed in the Coxeter polygon P. So by Lemma 2.2 we have that $\operatorname{Area}(S_i) \geq A$. By (2) this implies that $\operatorname{Area}(T_i) = O(\operatorname{Area}(S_i)\alpha)$ uniformly in i, P, and as $\operatorname{Area}(P') = \operatorname{Area}(P) - \sum_{l=1}^{k} \operatorname{Area}(T_{i_l})$ it follows that

$$\operatorname{Area}(P') = (1 - O(\alpha)) \operatorname{Area}(P).$$

On the other hand, we have that $P'_{\leq R}$ is contained in the $\alpha\eta$-neighbourhood of $P_{\leq R}$ so that

$$\mathrm{Area}((P')_{\leq R} \leq (1+O(\alpha))\,\mathrm{Area}(P_{\leq R}).$$

From these two inequalities it follows that

$$\frac{\mathrm{Area}((P')_{\leq R})}{\mathrm{Area}(P')} \leq (1+O(\alpha))\frac{\mathrm{Area}(P_{\leq R})}{\mathrm{Area}(P)},$$

which finishes the proof by taking $\eta' = \alpha\eta$ for α small enough (independently of P, R). ☐

3. Non-triviality of BS-limits of polygons

In this section, we give the proof of the first part of Theorem 3 that a Benjamini–Schramm limit of a sequence of Coxeter polygons is almost surely non-trivial. We will prove that for any sequence P_n of Coxeter polygons in $\mathbb{H}^2$, we have

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{\mathrm{Area}\left((P_n)_{\geq R}\right)}{\mathrm{vol}\, P_n} = 0, \tag{3}$$

from which the first statement follows immediately.

Let P'_n be the polygons obtained from the P_n by applying Lemma 2.3; it follows from the condition (1) that it suffices to prove (3) for the P'_n. Note that in any triangulation[3] of P'_n all angles are at most $\pi/2$ (they are smaller than those of the P_n, and the latter are of the form π/k, $k \geq 2$), and by using point (3) of Lemma 2.1 in addition we see that all edges in the triangulation have length at least η, so by Lemma 2.2 we have a uniform lower bound for all areas of triangles occurring in any triangulation of any P'_n.

We triangulate P'_n as follows: we choose a vertex, and as long as possible add an edge between the current vertex and the second-to-next one (clockwise). See Figure 3.1 for an illustration.

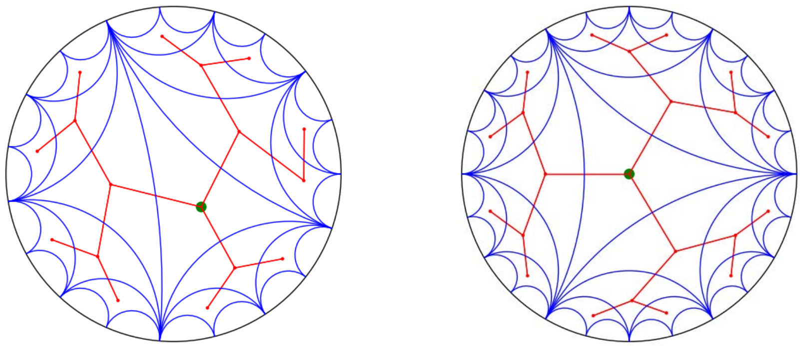
The tree T_n dual to this triangulation, rooted at the last triangle, has radius at most $\log_2(n)+1$ by [9, Proposition 2.1]. Moreover, the distance of the closest leaf to this triangle is at least $\log_2(n)-1$ as well[4].

Let $S > 0$. Let $(T_n)_{\leq S}$ be the set of vertices in T_n, which are at distance at most S from ∂T_n[5]. Then $T_n \setminus (T_n)_{\leq S}$ is contained in the ball of radius $\log_2(n)+1-S$ since every leaf is at a distance of at most $\log_2(n)+1$ from the

3. By this we mean a decomposition of a polygon into triangles whose vertices are vertices of the polygon.

4. This can easily be seen by observing that these functions are monotonic, and for a 2^k-gon they are equal to $\log_2(n) \pm 1$ (it would be more natural in this case to centre at the edge between the last two triangles).

5. For us, the boundary of a finite tree is its set of leaves.



(A) $n = 19$ (B) $n = 24$

FIGURE 3.1. Triangulations of ideal n-gons and their dual tree

root. It follows that $|T_n \setminus (T_n)_{\leq S}| \leq 2^{\log_2(n)+1-S} = O(2^{-S}|T_n|$ since $|T_n| = n$. So we have

$$|(T_n)_{\leq S}| \geq (1 - O(2^{-S}))|T_n|. \tag{4}$$

On the other hand, if $x \in P'_n$ lies in a triangle corresponding to a triangle $t \in (T_n)_{\leq S}$, then we may construct a path c as follows: let $t = t_0$; choose a side of t_0 closest to x such that the corresponding edge of T_n points away from the root, and let c_0 be the geodesic from x to this side. If the side lies on the boundary let $c = c_0$; otherwise let t_1 be the triangle on the other side, let x_1 be the foot of c_0 on the side $t_0 \cap t_1$ and iterate the construction until the boundary is reached, say in l steps, and let c be the concatenation of $c_0, \dots, c_l$, where c_i is the path obtained at the $(i+1)$th step. As $t \in (T_n)_{\leq S}$ and the edges (t_i, t_{i+1}) point away from the root we must have that $l \leq S + 1$. Moreover, the length of each c_i is at most $\log(3)$ since $\mathbb{H}^2$ is $\log(3)/2$-hyperbolic (which implies that in any geodesic triangle any point lies at distance at most $\leq \log(3)$ from any union of two edges; see, for instance, [4, Section 6.1]) so at each step at least one of the two edges pointing away from the root is at distance $\leq \log(3)/2$ from the foot of c_i. It follows that c has a length of at most $(S + 1)\log(3)$, and we conclude that x is at a distance of at most $\log(3) \cdot (S + 1)$ from the boundary. From this and (4), we deduce that at least $(1 - O(2^{-S}))|T_n|$ triangles of T_n lie entirely in $(P'_n)_{\leq R}$, for $S = R/\log(3)$. If A is a lower bound for the area of triangles in T_n (which is independent of n by the remarks above), we thus have that

$$\mathrm{Area}(P'_n) \geq (1 - O(2^{-R})A.$$

On the other hand, we have that

$$\operatorname{Area}(P_n' \setminus (P_n')_{\le R}) \le \pi \cdot 2^{\log_2(n)+1-S},$$

so that

$$\operatorname{Area}((P_n')_{\le R}) \ge (1 - O(2^{-R}))\operatorname{Area}(P_n'),$$

from which (3) follows immediately.

4. Chabauty limits of Coxeter polygons

In this section, we prove the following result, which immediately implies the second part of Theorem 3 since limits of discrete invariant random subgroups are themselves supported on discrete subgroups (as follows from [1, Proposition 2.2, Theorem 2.9]).

Proposition 4.1. — *The set of discrete groups generated by reflections is closed in the Chabauty space of discrete subgroups of* $\mathrm{PO}(2,1)$.

This will follow from the next lemma.

Lemma 4.2. — *Let P be a Coxeter polygon in $\mathbb{H}^2$, S the set of reflections in its faces and $\Gamma = \Gamma_P = \langle S \rangle$. If $w \in \Gamma$ and T is the subset of S containing all elements occurring in a minimal expression for w as a word in the elements of S, then $d(x, wx) \ge d(x, sx)$, for all $s \in T$.*

Proof. — Assume that $s \in S$ occurs in a minimal expression for w; then x and wx are separated by a hyperplane that is Γ-equivalent to the hyperplane W_s supporting the side of P corresponding to s. Thus, we need only show the following statement: the smallest distance between two points in the orbit $\Gamma \cdot x$ separated by W_s is realised by (x, sx).

In turn, this is implied by the statement that the closest point to x on a Γ-translate of W_s is its projection on W_s. If this were not the case, there would be a billiard trajectory in P starting at x and ending on W_s shorter than the segment from x orthogonal to W_s. This is impossible: indeed, there is no trajectory from x to W_s at all that is shorter than this segment. This finishes the proof. □

Proof of Proposition 4.1. — Let H be a discrete Chabauty limit of a sequence Γ_n of Coxeter groups in $\mathbb{H}^2$. Since H is discrete, it follows from the Kazhdan–Margulis theorem that there exists $\varepsilon > 0$ such that, by conjugating the Γ_n, we may assume there is a point $o \in \mathbb{H}^2$ that maps to the ε-thick part of $\Gamma_n \backslash \mathbb{H}^2$ for every n. Let P_n be the Coxeter polygon of Γ_n containing o.

Fix a $g \in H$: we want to prove that g is a product of reflections in H. We know that g is a limit of a sequence $g_n \in \Gamma_n$. The distances $d(o, g_n o)$ are uniformly bounded, say $d(o, g_n o) \le R$ for all n, and it follows that the word length of g_n in the reflections in the sides of P_n must be bounded, say by some $l \in \mathbb{N}$ (since there are uniformly finitely many points in the orbit $\Gamma_n \cdot o$ at a distance of at most R from o by a packing argument). By Lemma 4.2 it follows that we have $g_n = s_{i_1,n} \cdots s_{i_{l_n},n}$, where $s_{i,n}$ are reflections in the sides of P_n such that $d(o, s_{i,n}o) \leq R$ and $l_n \leq l$ for all n. We can thus pass to a subsequence and assume that the sequences $s_{i,n}$, $1 \leq i \leq l$ are convergent. The limits are reflections belonging to H, and g can be written as a product of them. □

Acknowledgments. — I am grateful to David Fisher for comments on earlier versions of the paper and to a referee for pointing out several imprecisions in the submitted version.

BIBLIOGRAPHY

[1] M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault & I. Samet – "On the growth of L^2-invariants for sequences of lattices in Lie groups", *Ann. Math. (2)* **185** (2017), no. 3, p. 711–790 (English).

[2] I. Agol, M. Belolipetsky, P. Storm & K. Whyte – "Finiteness of arithmetic hyperbolic reflection groups", *Groups Geom. Dyn.* **2** (2008), no. 4, p. 481–498 (English).

[3] M. Belolipetsky – "Arithmetic hyperbolic reflection groups", *Bull. Am. Math. Soc., New Ser.* **53** (2016), no. 3, p. 437–475 (English).

[4] B. H. Bowditch – *A course on geometric group theory.*, MSJ Mem., vol. 16, Tokyo: Mathematical Society of Japan (MSJ), 2006 (English).

[5] D. Fisher & S. Hurtado – "A new proof of finiteness of maximal arithmetic reflection groups", 2022.

[6] M. Fraczyk, S. Hurtado & J. Raimbault – "Benjamini–Schramm convergence of arithmetic locally symmetric spaces", In preparation, see `https://arxiv.org/abs/2202.13619`, 2023.

[7] M. Gromov – *Metric structures for Riemannian and non-Riemannian spaces. Transl. from the French by Sean Michael Bates. With appendices by M. Katz, P. Pansu, and S. Semmes. Edited by J. LaFontaine and P. Pansu*, 3rd printing ed., Mod. Birkhäuser Classics, Basel: Birkhäuser, 2007 (English).

[8] A. G. Khovanskij – "Hyperplane sections of polyhedra, toroidal manifolds, and discrete groups in Lobachevskij space", *Funct. Anal. Appl.* **20** (1986), p. 41–50 (English).

[9] M. Korman, S. Langerman, W. Mulzer, A. Pilz, M. Saumell & B. Vogtenhuber – "The dual diameter of triangulations", *Comput. Geom.* **68** (2018), p. 243–252 (English).

[10] A. D. Mednykh – "Brahmagupta formula for cyclic quadrilaterials in the hyperbolic plane", *Sib. Èlektron. Mat. Izv.* **9** (2012), p. 247–255 (English).

[11] V. V. Nikulin – "Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces", *Izv. Math.* **71** (2007), no. 1, p. 53–56 (English).

[12] È. B. Vinberg – "Absence of crystallographic reflection groups in Lobachevskiĭ spaces of large dimension", *Funkts. Anal. Prilozh.* **15** (1981), no. 2, p. 67–68 (Russian).

[13] ———, "Hyperbolic reflection groups", *Russ. Math. Surv.* **40** (1985), no. 1, p. 31–75 (English).