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SKT HYPERBOLIC AND GAUDUCHON HYPERBOLIC COMPACT COMPLEX MANIFOLDS

BY SAMIR MAROUANI

ABSTRACT. — We introduce two notions of hyperbolicity for not necessarily Kähler even balanced n -dimensional compact complex manifolds X . The first, called *SKT hyperbolicity*, generalises Gromov's Kähler hyperbolicity by means of SKT metrics. The second, called *Gauduchon hyperbolicity* by means of Gauduchon metrics. Our first main result in this paper asserts that every SKT hyperbolic X is also Kobayashi/Brody hyperbolic and every Gauduchon hyperbolic X is divisorially hyperbolic. The second main result is to prove a vanishing theorem for the L^2 harmonic spaces on the universal cover of an SKT hyperbolic manifold.

RÉSUMÉ (*Hyperbolicité SKT et hyperbolicité de Gauduchon pour les variétés complexes compactes*). — Nous introduisons deux notions d'hyperbolicité pour les variétés complexes compactes X de dimension n , non nécessairement kählériennes ni équilibrées. La première, appelée hyperbolicité SKT, généralise l'hyperbolicité kählérienne au sens de Gromov à l'aide d'une métrique SKT. La seconde, appelée hyperbolicité de Gauduchon. Notre premier résultat principal dans cet article affirme que toute variété SKT hyperbolique est également hyperbolique au sens de Kobayashi/Brody, et toute variété Gauduchon hyperbolique est divisoriellement hyperbolique. Notre deuxième résultat principal démontre un théorème d'annulation pour les espaces harmoniques L^2 sur le revêtement universel d'une variété SKT hyperbolique.

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1. Introduction

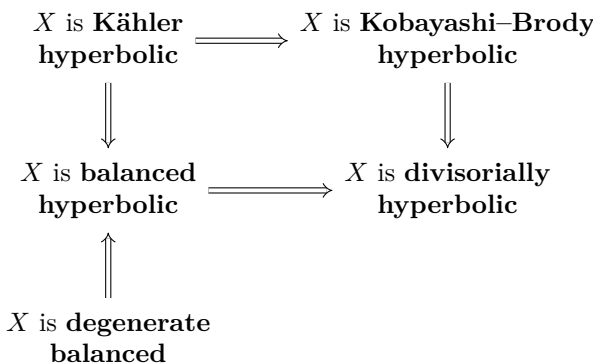
S. Kobayashi called a complex manifold X , which need not be either Kähler or compact, *hyperbolic* if the pseudo-distance he had introduced on X is actually a distance. Using the mapping decreasing property of this distance, one can show that every holomorphic map from the complex plane \mathbb{C} to a *Kobayashi hyperbolic* manifold is *constant*. Conversely, Brody observed that a compact complex manifold X is Kobayashi hyperbolic if every holomorphic map from \mathbb{C} to X is constant. The long-standing Kobayashi–Lang conjecture predicts that, for a compact Kähler manifold X , if X is Kobayashi hyperbolic then its canonical bundle K_X is *ample*.

M. Gromov introduced in one of his seminal papers [7], the notion of *Kähler hyperbolicity* for a compact Kähler manifold X . The manifold X is called *Kähler hyperbolic* if X admits a Kähler metric ω whose lift $\tilde{\omega}$ to the universal cover \tilde{X} of X can be expressed as

$$\tilde{\omega} = d\alpha$$

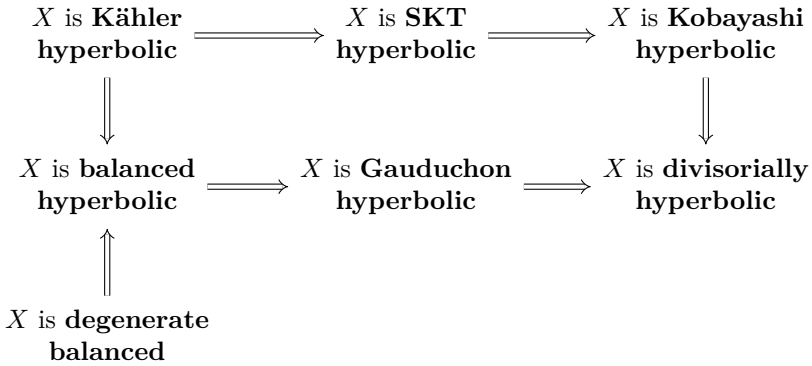
for a *bounded* 1-form α on \tilde{X} . As pointed out by Gromov, it is not hard to see that the Kähler hyperbolicity implies the Kobayashi hyperbolicity.

The Kähler hyperbolicity is generalized in [10] to what we call **balanced hyperbolicity**. This is done by replacing the Kähler metric in the Kähler hyperbolicity by a *balanced metric*. Meanwhile, a compact complex n -dimensional manifold X is said to be balanced hyperbolic if it carries a balanced metric ω such that ω^{n-1} is \tilde{d} -bounded. The Brody hyperbolicity is replaced by what we call **divisorial hyperbolicity**. A compact complex manifold X is called *divisorially hyperbolic* if there exists no non-trivial holomorphic map from \mathbb{C}^{n-1} to X satisfying certain *subexponential volume growth conditions*. Where the main result in [10] asserts that every balanced hyperbolic X is also divisorially hyperbolic (Thm 2.8), and therefore the following implication holds:



In this paper, we present a hyperbolicity theory where the Kähler metric is replaced by the SKT metric and the balanced metric is replaced by the Gauduchon metric on n -dimensional compact complex manifolds. The notions we introduce are weaker, thus more inclusive, than their classical counterparts. In particular, the setting does not need to be Kähler or even balanced. Our motivation stems from the existence of many interesting examples of non-Kähler compact complex manifolds that exhibit hyperbolicity features in a generalized sense, which we now set out to explain. Our first main result in this paper asserts that every SKT hyperbolic X is also Kobayashi hyperbolic, and every Gauduchon hyperbolic X is divisorially hyperbolic, and therefore the following implication holds:

THEOREM 1.1. — *Let X be a compact complex manifold. The following implications hold:*



Our second main result is to prove a vanishing theorem for the L^2 harmonic spaces on the universal cover of an **SKT hyperbolic** manifold. With conditions less stringent than Kähler hyperbolicity, we obtain the same result as in 1.4.A. Theorem in [7].

2. Aeppli cohomology and SKT metrics

Given a compact complex n -dimensional manifold X , recall that the Bott-Chern and Aeppli cohomology groups of any bidegree (p, q) of X are classically defined, using the spaces $C^{r,s}(X) = C^{r,s}(X, \mathbb{C})$ of smooth \mathbb{C} -valued (r, s) -forms on X , as

$$\begin{aligned}
 H_{BC}^{p,q}(X, \mathbb{C}) &= \frac{\ker(\partial : C^{p,q}(X) \rightarrow C^{p+1,q}(X)) \cap \ker(\bar{\partial} : C^{p,q}(X) \rightarrow C^{p,q+1}(X))}{\text{Im}(\partial\bar{\partial} : C^{p-1,q-1}(X) \rightarrow C^{p,q}(X))} \\
 H_A^{p,q}(X, \mathbb{C}) &= \frac{\ker(\partial\bar{\partial} : C^{p,q}(X) \rightarrow C^{p+1,q+1}(X))}{\text{Im}(\partial : C^{p-1,q}(X) \rightarrow C^{p,q}(X)) + \text{Im}(\bar{\partial} : C^{p,q-1}(X) \rightarrow C^{p,q}(X))}.
 \end{aligned}$$

The Bott–Chern Laplacian Δ_{BC} and the Aeppli Laplacian Δ_A are the 4^{th} order elliptic differential operators defined, respectively, as

$$\Delta_{BC} := \partial^* \partial + \bar{\partial}^* \bar{\partial} + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial^* \bar{\partial})^* (\partial^* \bar{\partial}) + (\partial^* \bar{\partial})(\partial^* \bar{\partial})^*,$$

and

$$\Delta_A := \partial \partial^* + \bar{\partial} \bar{\partial}^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial \bar{\partial}^*)^* (\partial \bar{\partial}^*) + (\partial \bar{\partial}^*)(\partial \bar{\partial}^*)^*,$$

where $\partial^* = -\star \bar{\partial} \star$, and $\star = \star_\omega : \Lambda^{p,q} T^* X \rightarrow \Lambda^{n-q,n-p} T^* X$ is the Hodge-star isomorphism defined by ω for arbitrary $p, q = 0, \dots, n$. Note that $\star \Delta_{BC} = \Delta_A \star$ and $\Delta_{BC} \star = \star \Delta_A$, then $u \in \ker \Delta_{BC}$ if and only if $\star u \in \ker \Delta_A$.

The Bott–Chern Laplacian is elliptic and essentially self-adjoint, so it induces a three-space decomposition

$$C_{p,q}^\infty = \ker \Delta_{BC} \oplus \text{Im} \partial \bar{\partial} \oplus (\text{Im} \partial^* + \text{Im} \bar{\partial}^*)$$

that is orthogonal w.r.t. the L^2 scalar product defined by ω . We have

$$\ker \partial \cap \ker \bar{\partial} = \ker \Delta_{BC} \oplus \text{Im} \partial \bar{\partial}.$$

We also have

$$\text{Im} \Delta_{BC} = \text{Im} \partial \bar{\partial} \oplus (\text{Im} \partial^* + \text{Im} \bar{\partial}^*).$$

Similarly, the 4^{th} order Aeppli Laplacian is elliptic and essentially self-adjoint, so it induces a three-space decomposition

$$C_{p,q}^\infty = \ker \Delta_A \oplus \text{Im} (\partial \bar{\partial})^* \oplus (\text{Im} \partial + \text{Im} \bar{\partial}),$$

that is orthogonal w.r.t. the L^2 scalar product defined by ω . We have

$$(1) \quad \ker (\partial \bar{\partial}) = \ker \Delta_A \oplus (\text{Im} \partial + \text{Im} \bar{\partial}).$$

We also have

$$\text{Im} \Delta_A = \text{Im} (\partial \bar{\partial})^* \oplus (\text{Im} \partial + \text{Im} \bar{\partial}).$$

2.1. Cones of classes of metrics. — Recall the classical notion introduced by Popovici [15, Definition 5.1]: the Gauduchon cone \mathcal{G}_X of a compact complex manifold X is the set of all Aeppli cohomology classes of Gauduchon metrics on X . As such, it is an open convex cone in $H_A^{n-1,n-1}(X, \mathbb{R})$, that generalises the Kähler cone \mathcal{K}_X . Moreover, it is never empty, thanks to the existence of the Gauduchon metric on all compact complex manifolds.

Recall that a Hermitian metric ω on X is said to be an *SKT metric* if $\partial \bar{\partial} \omega = 0$. Any such metric ω defines an Aeppli cohomology class associated with ω . We start by introducing the following analogue of the Gauduchon cone in bidegree $(1, 1)$.

DEFINITION 2.1. — Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The **SKT cone** of X is the set:

$$\mathcal{SKT}_X := \{[\omega]_A \in H_A^{1,1}(X, \mathbb{R}) \mid \omega \text{ is a SKT metric on } X\} \subset H_A^{1,1}(X, \mathbb{R}).$$

Any element $[\omega]_A$ of the SKT cone \mathcal{SKT}_X is called an **Aeppli-SKT class**.

We will also introduce the *cone in cohomology of bidegree $(n - 1, n - 1)$* of X , which generalises the pseudo-effective cone \mathcal{E}_X of Bott–Chern cohomology classes of d -closed semi-positive $(1, 1)$ -currents introduced by Demailly [4].

DEFINITION 2.2. — Let X be a compact complex manifold. The **co-pseudo-effective cone** of X is the set

$$\mathcal{E}_X^{n-1, n-1} := \{[T]_{BC} \in H_{BC}^{n-1, n-1}(X, \mathbb{R}) \mid T \geq 0 \text{ } d\text{-closed } (n - 1, n - 1)\text{-current on } X\}.$$

Any element $[T]_{BC}$ of the co-pseudo-effective cone $\mathcal{E}_X^{n-1, n-1}$ is called a **co-pseudo-effective class**.

PROPOSITION 2.3. — *Let X be a compact complex manifold. Then*

- (i) *The SKT cone \mathcal{SKT}_X is an open convex subset of $H_A^{1,1}(X, \mathbb{R})$.*
- (ii) *The set $\mathcal{E}_X^{n-1, n-1}$ is a closed convex cone in $H_{BC}^{n-1, n-1}(X, \mathbb{R})$.*
- (iii) *The following two statements hold*
 - (a) *Given any class $\mathbf{c}_{BC}^{n-1, n-1} \in H_{BC}^{n-1, n-1}(X, \mathbb{R})$, the following equivalence holds:*

$$\begin{aligned} &\mathbf{c}_{BC}^{n-1, n-1} \in \mathcal{E}_X^{n-1, n-1} \\ \iff &\mathbf{c}_{BC}^{n-1, n-1} \cdot \mathbf{c}_A^{1,1} \geq 0 \text{ for every class } \mathbf{c}_A^{1,1} \in \mathcal{SKT}_X. \end{aligned}$$

- (b) *Given any class $\mathbf{c}_A^{1,1} \in H_A^{1,1}(X, \mathbb{R})$, the following equivalence holds:*

$$\begin{aligned} &\mathbf{c}_A^{1,1} \in \overline{\mathcal{SKT}}_X \\ \iff &\mathbf{c}_{BC}^{n-1, n-1} \cdot \mathbf{c}_A^{1,1} \geq 0 \text{ for every class } \mathbf{c}_{BC}^{n-1, n-1} \in \mathcal{E}_X^{n-1, n-1}. \end{aligned}$$

Proof. — (i) If ω_1 and ω_2 are **SKT** metrics on X , so is any linear combination $\lambda\omega_1 + \mu\omega_2$ with λ, μ non-negative reals. Therefore, \mathcal{SKT}_X is a convex cone. To show that \mathcal{SKT}_X is an open subset of $H_A^{1,1}(X, \mathbb{R})$, let us equip the finite-dimensional vector space $H_A^{1,1}(X, \mathbb{R})$ with an arbitrary norm $\| \cdot \|$ (e.g., the Euclidian norm after we have fixed a basis; at any rate, all the norms are equivalent). Let $[\omega]_A \in \mathcal{SKT}_X$ be an arbitrary element, where $\omega > 0$ is some SKT metric on X . Let $\alpha \in H_A^{1,1}(X, \mathbb{R})$ be a class such that $\|\alpha - [\omega]_A\| < \epsilon$ for some small $\epsilon > 0$. Fix any Hermitian metric $\tilde{\omega}$ on X and consider the Aeppli Laplacian Δ_A defined by $\tilde{\omega}$ inducing the Hodge isomorphism $H_A^{1,1}(X, \mathbb{R}) \simeq$

$H_{\Delta_A}^{1,1}(X, \mathbb{R})$. Let $\alpha_0 \in H_{\Delta_A}^{1,1}(X, \mathbb{R})$ be the Δ_A -harmonic representative of the class α . Since $\omega \in \ker(\partial\bar{\partial})$, (1) gives a unique decomposition

$$\omega = \omega_0 + (\partial u + \bar{\partial} v) \text{ with } \Delta_A \omega_0 = 0.$$

If we set $\gamma := \alpha_0 + (\partial u + \bar{\partial} v)$ (with the same forms u, v as for ω), then $\partial\bar{\partial}\gamma = 0$, γ represents the Aeppli-SKT class α , and we have

$$\|\gamma - \omega\|_{C^0} = \|\alpha_0 - \omega_0\|_{C^0} \leq C\|\alpha - [\omega]_A\| < C\epsilon,$$

for some constant $C > 0$ induced by the Hodge isomorphism. (We have chosen the C^0 norm on $H_{\Delta_A}^{1,1}(X, \mathbb{R})$ only for the sake of convenience). Thus, if $\epsilon > 0$ is chosen sufficiently small, the $(1, 1)$ -form γ must be positive definite since ω is. Thus γ is an **SKT** metric and represents the original Aeppli-SKT class α , so $\alpha \in \mathcal{SKT}_X$.

(ii) Let T_k be closed positive $(n - 1, n - 1)$ -currents such that the classes $[T_k]$ converge to a limit $[\Theta]$. Then $\int_X T_k \wedge \omega$ converges to $\int_X \Theta \wedge \omega$. In particular the sequence T_k is bounded in mass, and therefore weakly compact. If $T_{k_\nu} \rightarrow T$ is a weakly convergent sub-sequence, the limit T is a closed positive current, so $[T] = [\Theta]$ is co-pseudo-effective. Therefore, $\mathcal{E}_X^{n-1, n-1}$ is closed.

(iii) By the Serre-type duality (see, e.g., [15, Theorem 2.1]):

$$H_{BC}^{n-1, n-1}(X, \mathbb{C}) \times H_A^{1,1}(X, \mathbb{C}) \longrightarrow \mathbb{C}, \quad ([u]_{BC}, [v]_A) \mapsto [u]_{BC} \cdot [v]_A := \int_X u \wedge v,$$

we obtain (a) and (b) as a reformulation of a result in Lamari’s work (see [8, Lemme 3.3]). □

2.2. Some properties of SKT manifolds. — On a complex manifold X with $\dim_{\mathbb{C}} X = n$, we will often use the following standard formula (cf., e.g., [19, Proposition 6.29, p. 150]) for the Hodge star operator $\star = \star_{\omega}$ of any Hermitian metric ω applied to ω -primitive forms v of arbitrary bidegree (p, q) :

$$(2) \quad \star v = (-1)^{k(k+1)/2} i^{p-q} \omega_{n-p-q} \wedge v, \text{ where } k := p + q.$$

Recall that, for any $k = 0, 1, \dots, n$, a k -form v is said to be (ω) -primitive if $\omega_{n-k+1} \wedge v = 0$, and that this condition is equivalent to $\Lambda_{\omega} v = 0$, where Λ_{ω} is the adjoint of the operator $\omega \wedge \cdot$ (of multiplication by ω) w.r.t. the pointwise inner product $\langle \cdot, \cdot \rangle_{\omega}$ defined by ω .

LEMMA 2.4. — *Let ω be a Hermitian metric on a complex manifold X with $\dim_{\mathbb{C}} X = n$. Fix a primitive form $\phi \in C_{p,q}^{\infty}(X, \mathbb{C})$ with $p + q = n - 1$.*

*If ω is **SKT** and $\partial\phi = \bar{\partial}\phi = 0$, then*

$$(3) \quad \partial^*(\omega \wedge \phi) = 0, \quad \bar{\partial}^*(\omega \wedge \phi) = 0.$$

In particular if X is compact, then

$$\Delta_A(\omega \wedge \phi) = 0,$$

where Δ_A is the Aeppli Laplacian induced by ω .

Proof. — Let ϕ be the primitive form the standard formula (2); we get: $\star\phi = (-1)^{\frac{n(n-1)}{2}} i^{p-q} \omega \wedge \phi$, hence $\star(\omega \wedge \phi) = -(-1)^{\frac{n(n-1)}{2}} i^{q-p} \phi$. Meanwhile, $d^\star = -\star d \star$, so applying $-\star d$ to the previous identity, we get

$$d^\star(\omega \wedge \phi) = (-1)^{\frac{n(n+1)}{2}} i^{q-p} (\star\partial\phi + \star\bar{\partial}\phi).$$

By assumption, we have $\partial\phi = \bar{\partial}\phi = 0$, and then, on the one hand we have

$$d^\star(\omega \wedge \phi) = 0 \iff \partial^\star(\omega \wedge \phi) = 0 \text{ and } \bar{\partial}^\star(\omega \wedge \phi) = 0,$$

and on the other hand and since ω is **SKT**, we get

$$\partial\bar{\partial}(\omega \wedge \phi) = (\partial\bar{\partial}\omega) \wedge \phi = 0.$$

The proof of lemma is complete since on a compact complex manifold X ,

$$\alpha \in \text{Ker}(\Delta_A) \iff \partial^\star\alpha = 0, \quad \bar{\partial}^\star\alpha = 0, \quad \partial\bar{\partial}\alpha = 0. \quad \square$$

COROLLARY 2.5. — *Let (X, ω) be an **SKT** n -dimensional compact complex manifold and ϕ be a $(n - 1, 0)$ -form (respectively, $(0, n - 1)$ -form) on X such that $\partial\phi = 0$ and $\bar{\partial}\phi = 0$. Then,*

$$\Delta_A(\omega \wedge \phi) = 0,$$

where Δ_A is the Aeppli Laplacian induced by ω .

LEMMA AND DEFINITION 2.6. — *Let ω be an **SKT** metric on a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. The linear map:*

$$[\omega]_A \wedge \cdot : H_{BC}^{n-1, n-1}(X, \mathbb{C}) \longrightarrow H_A^{n, n}(X, \mathbb{C}) \simeq \mathbb{C}, \quad [\Gamma]_{BC} \longmapsto [\omega \wedge \Gamma]_A,$$

is **well defined** and depends only on the cohomology class $[\omega]_A \in H_A^{1, 1}(X, \mathbb{C})$. Moreover, any **SKT** metric gives rise to a non-zero class in Aeppli cohomology. We set:

$$H_{BC}^{n-1, n-1}(X, \mathbb{C})_{\omega\text{-prim}} := \ker([\omega]_A \wedge \cdot) \subset H_{BC}^{n-1, n-1}(X, \mathbb{C}),$$

and we call its elements **(ω) -primitive Bott-Chern $(n - 1, n - 1)$ -classes**.

Proof. — The well-definedness follows at once from the identities:

$$\begin{aligned} \partial\bar{\partial}(\omega \wedge \Gamma) &= \partial\bar{\partial}\omega \wedge \Gamma = 0, \quad \Gamma \in C_{n-1, n-1}^\infty(X, \mathbb{C}) \cap \ker d, \\ \omega \wedge \partial\bar{\partial}\Phi &= \partial(\omega \wedge \bar{\partial}\Phi) + \bar{\partial}(\Phi \wedge \partial\omega) \in \text{Im } \partial + \text{Im } \bar{\partial}, \quad \Phi \in C_{n-2, n-2}^\infty(X, \mathbb{C}), \end{aligned}$$

where the latter takes into account the fact that $\partial\bar{\partial}\omega = 0$.

That the map $[\omega]_A \wedge \cdot$ depends only on the Aeppli cohomology class $[\omega]_A$ follows from:

$$(\omega + \partial\bar{\alpha} + \bar{\partial}\alpha) \wedge \Gamma - \omega \wedge \Gamma = \partial(\bar{\alpha} \wedge \Gamma) + \bar{\partial}(\alpha \wedge \Gamma) \in \text{Im } \partial + \text{Im } \bar{\partial},$$

$$\Gamma \in C_{n-1, n-1}^\infty(X, \mathbb{C}) \cap \ker d.$$

Now we proof that any SKT metric gives rise to a non-zero class in Aeppli cohomology. Suppose there is a $(0, 1)$ -form α and a $(1, 0)$ -form β such that $\omega = \partial\alpha + \bar{\partial}\beta$. We get

$$\begin{aligned} (d(\alpha + \beta))^n &= (\omega + \bar{\partial}\alpha + \partial\beta)^n = \sum_{k=0}^n \sum_{j=0}^k C_n^k C_j^k (\bar{\partial}\alpha)^j \wedge (\partial\beta)^{k-j} \wedge \omega^{n-k} \\ &= \sum_{\{k=2j\}} C_j (\bar{\partial}\alpha)^j \wedge (\partial\beta)^j \wedge \omega^{n-2j}. \end{aligned}$$

To show that $\sum_{\{k=2j\}} C_j (\bar{\partial}\alpha)^j \wedge (\partial\beta)^j \wedge \omega^{n-2j} > 0$, it suffices to check that the real form $(\bar{\partial}\alpha)^j \wedge (\partial\beta)^j \wedge \omega^{n-2j}$ is weakly (semi)-positive at every point of X . (Recall that $\bar{\partial}\alpha$ is the conjugate of $\partial\beta$.) To this end, note that the $(2j, 2j)$ -form $(\bar{\partial}\alpha)^j \wedge (\partial\beta)^j$ is weakly semi-positive as the wedge product of a $(2, 0)$ -form and its conjugate (see [3, Chapter III, Example 1.2]). Therefore, the (n, n) -form is (semi)-positive since the product of a weakly (semi)-positive form and a strongly (semi)-positive form is weakly (semi)-positive and ω is strongly positive (see [3, Chapter III, Proposition 1.11]). (Recall that in bidegrees $(0, 1)$, $(1, 1)$, $(n - 1, n - 1)$ and (n, n) , the notions of weak and strong positivity coincide). By Stokes, we get

$$0 < \int_X \sum_{\{k=2j\}} C_j (\bar{\partial}\alpha)^j \wedge (\partial\beta)^j \wedge \omega^{n-2j} = \int_X d((\alpha + \beta) \wedge (d(\alpha + \beta))^{n-1}) = 0.$$

Consequently

$$(4) \quad [\omega]_A \neq 0. \quad \square$$

LEMMA 2.7. — *Let ω be an **SKT** metric on a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. Then, $H_{BC}^{n-1, n-1}(X, \mathbb{C})_{\omega\text{-prim}}$ is a complex hyperplane of $H_{BC}^{n-1, n-1}(X, \mathbb{C})$ depending only on the cohomology class $[\omega]_A$.*

Proof. — Suppose that $H_{BC}^{n-1, n-1}(X, \mathbb{C})_{\omega\text{-prim}} = H_{BC}^{n-1, n-1}(X, \mathbb{C})$. This translates to

$$(5) \quad \omega \wedge \Gamma \in \text{Im } \partial + \text{Im } \bar{\partial}, \quad \forall \Gamma \in C_{n-1, n-1}^\infty(X, \mathbb{C}) \cap \ker d.$$

Since ω is $(\partial\bar{\partial})$ -closed, it has a unique L_ω^2 -orthogonal decomposition:

$$(6) \quad \omega = (\omega)_h + (\partial\bar{\alpha}_\omega + \bar{\partial}\alpha_\omega),$$

with a $(1, 1)$ -form $(\omega)_h \in \ker \Delta_{A, \omega}$ and a $(1, 0)$ -form α_ω .

On the other hand, Γ is d -closed, so it has a unique L^2_ω -orthogonal decomposition:

$$\Gamma = \Gamma_h + \partial\bar{\partial}\Phi,$$

where Γ_h is $\Delta_{BC,\omega}$ -harmonic and Φ is a smooth $(n - 2, n - 2)$ -form on X .

Thus, for every $\Gamma \in C^\infty_{n-1,n-1}(X, \mathbb{C}) \cap \ker d$, we get:

$$\begin{aligned} \omega \wedge \Gamma &= (\omega)_h \wedge \Gamma + \partial(\bar{\alpha}_\omega \wedge \Gamma) + \bar{\partial}(\alpha_\omega \wedge \Gamma) \\ &= (\omega)_h \wedge \Gamma_h + \partial((\omega)_h \wedge \bar{\partial}\Phi) + \bar{\partial}(\Phi \wedge \partial(\omega)_h) + \partial(\bar{\alpha}_\omega \wedge \Gamma) + \bar{\partial}(\alpha_\omega \wedge \Gamma), \end{aligned}$$

where for the last identity we used the fact that $\bar{\partial}\partial(\omega)_h = 0$.

Thanks to assumption (5), the last identity implies that

$$(7) \quad (\omega)_h \wedge \Gamma_h \in \text{Im } \partial + \text{Im } \bar{\partial}, \quad \forall \Gamma_h \in C^\infty_{n-1,n-1}(X, \mathbb{C}) \cap \ker \Delta_{BC,\omega}.$$

Meanwhile, since $(\omega)_h$ is $\Delta_{A,\omega}$ -harmonic (and real), $\star_\omega(\omega)_h$ is $\Delta_{BC,\omega}$ -harmonic (and real). Hence,

$$\text{Im } \partial + \text{Im } \bar{\partial} \ni (\omega)_h \wedge \star_\omega(\omega)_h = |(\omega)_h|_\omega^2 dV_\omega \geq 0,$$

where the first relation follows from (7) by choosing $\Gamma_h = \star_\omega(\omega)_h$. Consequently, from Stokes's theorem we get:

$$\int_X |(\omega)_h|_\omega^2 dV_\omega = 0,$$

hence $(\omega)_h = 0$, which implies by the L^2 -orthogonal decomposition (6), that $[\omega]_A = 0$, which is absurd according to (4). \square

LEMMA 2.8. — *Suppose there exists an SKT metric ω on a compact complex manifold X . Then, for every $\Gamma \in C^\infty_{n-1,n-1}(X, \mathbb{C})$ such that $d\Gamma = 0$ and $[\Gamma]_{BC} \in H^{n-1,n-1}_{BC}(X, \mathbb{C})_{\omega\text{-prim}}$, we have:*

$$\langle\langle (\omega_{n-1})_h, \Gamma \rangle\rangle_\omega = 0,$$

where $\langle\langle \cdot, \cdot \rangle\rangle_\omega$ is the L^2 inner product induced by ω .

Proof. — Since $[\Gamma]_{BC} \in H^{n-1,n-1}_{BC}(X, \mathbb{C})_{\omega\text{-prim}}$ then, $\omega \wedge \Gamma$ is of the shape $\partial\Phi + \bar{\partial}\Psi$ for some $\Phi \in C^\infty_{n-1,n}$ and $\Psi \in C^\infty_{n,n-1}$. By Stokes, we get:

$$\begin{aligned} \langle\langle \Gamma, (\omega_{n-1})_h \rangle\rangle_\omega &= \int_X \Gamma \wedge \star_\omega(\omega_{n-1})_h = \int_X \Gamma \wedge (\omega)_h = \int_X \Gamma \wedge (\omega - \partial u - \bar{\partial} v) \\ &= \int_X \Gamma \wedge \omega + \int_X \partial(\Gamma \wedge u) + \int_X \bar{\partial}(\Gamma \wedge v) = \int_X \partial\Phi + \bar{\partial}\psi = 0, \end{aligned}$$

since $\partial\Gamma = 0$, $\bar{\partial}\Gamma = 0$, $\partial(\Gamma \wedge u) = d(\Gamma \wedge u)$ and $\bar{\partial}(\Gamma \wedge v) = d(\Gamma \wedge v)$. \square

We shall now get a *Lefschetz-type decomposition* of $H_{BC}^{n-1,n-1}(X, \mathbb{C})$, induced by an arbitrary SKT metric ω , with $H_{BC}^{n-1,n-1}(X, \mathbb{C})_{\omega\text{-prim}}$ as a direct factor.

PROPOSITION 2.9. — *Let X be an **SKT** compact complex manifold with $\dim_{\mathbb{C}} X = n$. Then, the Bott–Chern cohomology space of bidegree $(n - 1, n - 1)$ has a **Lefschetz-type L_{ω}^2 -orthogonal decomposition**:*

$$(8) \quad H_{BC}^{n-1,n-1}(X, \mathbb{C}) = H_{BC}^{n-1,n-1}(X, \mathbb{C})_{\omega\text{-prim}} \oplus \mathbb{C} \cdot [(\omega_{n-1})_h]_{BC},$$

where the (ω) -primitive subspace $H_{BC}^{n-1,n-1}(X, \mathbb{C})_{\omega\text{-prim}}$ is a complex hyperplane of $H_{BC}^{n-1,n-1}(X, \mathbb{C})$ depending only on the cohomology class $[\omega]_A \in H_A^{1,1}(X, \mathbb{C})$, while $(\omega_{n-1})_h$ is the $\Delta_{BC,\omega}$ -harmonic component of ω_{n-1} , and the complex line $\mathbb{C} \cdot [(\omega_{n-1})_h]_A$ depends on the choice of the SKT metric ω .

LEMMA 2.10. — *For every $\Gamma \in C_{n-1,n-1}^{\infty}(X, \mathbb{C}) \cap \ker d$, the coefficient of $[(\omega_{n-1})_h]_{BC}$ in the Lefschetz-type decomposition of $[\Gamma]_{BC} \in H_{BC}^{n-1,n-1}(X, \mathbb{C})$ according to (8), namely in*

$$[\Gamma]_{BC} = [\Gamma]_{BC,prim} + \lambda [(\omega_{n-1})_h]_{BC},$$

is given by

$$(9) \quad \lambda = \lambda_{\omega}([\Gamma]_{BC}) = \frac{[\omega]_A \cdot [\Gamma]_{BC}}{\|\omega_h\|_{\omega}^2} = \frac{1}{\|\omega_h\|_{\omega}^2} \int_X \Gamma \wedge \omega.$$

Proof. — Since $[\Gamma]_{BC,prim} \in H_{BC}^{n-1,n-1}(X, \mathbb{C})_{\omega\text{-prim}}$, we have $[\omega]_A \cdot [\Gamma]_{BC,prim} = 0$, so

$$\begin{aligned} [\omega]_A \cdot [\Gamma]_{BC} &= \lambda \int_X \omega \wedge (\omega_{n-1})_h = \lambda \int_X \omega_h \wedge (\omega_{n-1})_h = \lambda \int_X \omega_h \wedge \star_{\omega}(\omega)_h \\ &= \lambda \|(\omega)_h\|_{\omega}^2. \end{aligned}$$

This gives (2.10). □

Formula (9) implies that $\lambda_{\omega}([\Gamma]_{BC})$ is *real* if the class $[\Gamma]_{BC} \in H_{BC}^{n-1,n-1}(X, \mathbb{C})$ is real. Thus, we can define a *positive side* and a *negative side* of the hyperplane

$$H_{BC}^{n-1,n-1}(X, \mathbb{R})_{\omega\text{-prim}} := H_{BC}^{n-1,n-1}(X, \mathbb{C})_{\omega\text{-prim}} \cap H_{BC}^{n-1,n-1}(X, \mathbb{R})$$

in $H_{BC}^{1,1}(X, \mathbb{R})$ by

$$\begin{aligned} H_{BC}^{n-1,n-1}(X, \mathbb{R})_{\omega}^{+} &:= \{[\Gamma]_{BC} \in H_{BC}^{n-1,n-1}(X, \mathbb{R}) \mid \lambda_{\omega}([\Gamma]_{BC}) > 0\}, \\ H_{BC}^{n-1,n-1}(X, \mathbb{R})_{\omega}^{-} &:= \{[\Gamma]_{BC} \in H_{BC}^{n-1,n-1}(X, \mathbb{R}) \mid \lambda_{\omega}([\Gamma]_{BC}) < 0\}. \end{aligned}$$

These are open subsets of $H_{BC}^{n-1,n-1}(X, \mathbb{R})$ that depend only on the cohomology class $[\omega]_A \in H_A^{1,1}(X, \mathbb{R})$.

Since $[\Gamma]_{BC}$ is ω -primitive if and only if $\lambda_\omega([\Gamma]_{BC}) = 0$, we get a *partition* of $H_{BC}^{n-1,n-1}(X, \mathbb{R})$:

$$H_{BC}^{n-1,n-1}(X, \mathbb{R}) = H_{BC}^{n-1,n-1}(X, \mathbb{R})_\omega^+ \cup H_{BC}^{n-1,n-1}(X, \mathbb{R})_{\omega\text{-prim}} \cup H_{BC}^{n-1,n-1}(X, \mathbb{R})_\omega^-$$

depending only on the cohomology class $[\omega]_A \in H_A^{1,1}(X, \mathbb{R})$.

As a consequence of these considerations, we get:

PROPOSITION 2.11. — *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

The cone $\mathcal{E}_X^{n-1,n-1} \subset H_{BC}^{n-1,n-1}(X, \mathbb{R})$ of X is the intersection of all the non-negative sides

$$H_{BC}^{n-1,n-1}(X, \mathbb{R})_\omega^{\geq 0} := H_{BC}^{n-1,n-1}(X, \mathbb{R})_\omega^+ \cup H_{BC}^{n-1,n-1}(X, \mathbb{R})_{\omega\text{-prim}}$$

of hyperplanes $H_{BC}^{n-1,n-1}(X, \mathbb{R})_{\omega\text{-prim}}$ determined by cohomology classes $[\omega]_A \in H_A^{1,1}(X, \mathbb{R})$:

$$\mathcal{E}_X^{n-1,n-1} = \bigcap_{[\omega]_A \in H_A^{1,1}(X, \mathbb{R})} H_{BC}^{n-1,n-1}(X, \mathbb{R})_\omega^{\geq 0}.$$

Proof. — By the duality between the co-pseudo-effective cone $\mathcal{E}_X^{n-1,n-1}$ and the closure $\overline{\mathcal{SKT}}_X$ of the SKT cone, we know that a given class $[T]_{BC} \in H_{BC}^{n-1,n-1}(X, \mathbb{R})$ lies in $\mathcal{E}_X^{n-1,n-1}$ (i.e., $[T]_{BC}$ can be represented by a closed semi-positive $(n-1, n-1)$ -current) if and only if

$$\int_X T \wedge \omega \geq 0 \quad \text{for all } [\omega]_A \in \mathcal{SKT}_X.$$

The last condition is equivalent to $\lambda_\omega([T]_{BC}) \geq 0$, hence to $[T]_{BC} \in H_{BC}^{n-1,n-1}(X, \mathbb{R})_\omega^{\geq 0}$, for all $[\omega]_A \in \mathcal{SKT}_X$, so the contention follows. \square

3. SKT and Gauduchon hyperbolicity

In this section, we introduce and discuss two notions of hyperbolicity for not necessarily Kähler even balanced n -dimensional compact complex manifolds that generalise Gromov’s Kähler hyperbolicity and balanced hyperbolicity.

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary Hermitian metric (i.e., a C^∞ positive definite $(1, 1)$ -form) ω on X . The metric ω is said to be **Kähler** $d\omega = 0$. The manifold X is said to be *Kähler* if it carries a *Kähler metric*. There are several well-studied generalisations of the Kählerness condition:

- i) ω is *balanced*, if $d\omega^{n-1} = 0$.
- ii) ω is *Gauduchon*, if $\bar{\partial}\partial\omega^{n-1} = 0$, such a metric always exists on a compact complex manifold.

- iii) ω is *strongly Gauduchon* if there is a $(n, n-2)$ -form Γ such that $\partial\omega^{n-1} = \bar{\partial}\Gamma$.
- iv) ω is Hermitian symplectic, if there is a $(0, 2)$ -form α such that $\partial\alpha = 0$ and $\bar{\partial}\omega = -\bar{\partial}\alpha$.
- v) ω is **SKT** (or pluriclosed), if $\partial\bar{\partial}\omega = 0$.

Throughout the text, $\pi_X : \tilde{X} \rightarrow X$ will stand for the universal cover of X and $\tilde{\omega} = \pi_X^*\omega$ will be the Hermitian metric on \tilde{X} , which is the lift of ω . Recall that a C^∞ k -form α on X is said to be \tilde{d} (bounded) with respect to ω if $\pi_X^*\alpha = d\beta$ on \tilde{X} for some C^∞ $(k-1)$ -form β on \tilde{X} , which is bounded w.r.t. $\tilde{\omega}$; (see [7] and [10]). In general, we propose the following definition, which generalizes that of \tilde{d} -bounded of a differential form.

DEFINITION 3.1. — A C^∞ k -form ϕ on X is said to be $(\tilde{\partial} + \bar{\tilde{\partial}})$ -bounded with respect to ω if $\pi_X^*\phi = \partial\alpha + \bar{\partial}\beta$ on \tilde{X} for some C^∞ $(k-1)$ -forms α and β on \tilde{X} that are bounded w.r.t. $\tilde{\omega}$.

3.1. SKT hyperbolic manifolds. — The first notion that we introduce in this work is the following.

DEFINITION 3.2. — Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. A Hermitian metric ω on X is said to be **SKT hyperbolic** if ω is SKT and $(\tilde{\partial} + \bar{\tilde{\partial}})$ -bounded with respect to ω .

The manifold X is said to be SKT hyperbolic if it carries a *SKT hyperbolic metric*.

Let us first notice the following implication:

$$X \text{ is Kähler hyperbolic} \implies X \text{ is SKT hyperbolic.}$$

To see this, note the obvious fact that every Kähler metric is SKT and every \tilde{d} -bounded form is, in particular, $(\tilde{\partial} + \bar{\tilde{\partial}})$ -bounded.

LEMMA 3.3. — *Let (X, ω) be a compact complex Hermitian manifold with $\dim_{\mathbb{C}}X = n$. Let $k \in \{1, \dots, 2n\}$ and $\alpha \in C_k^\infty(X, \mathbb{C}) \cap \ker \partial \cap \ker \bar{\partial}$. If α is $(\tilde{\partial} + \bar{\tilde{\partial}})$ -bounded (with respect to ω), then α^p is $(\tilde{\partial} + \bar{\tilde{\partial}})$ -bounded (with respect to ω) for every non-negative integer p .*

Proof. — By the $(\tilde{\partial} + \bar{\tilde{\partial}})$ -boundedness assumption on α , $\pi_X^*\alpha = \partial\beta + \bar{\partial}\gamma$ on \tilde{X} for some smooth $\tilde{\omega}$ -bounded $(k-1)$ -forms β and γ on \tilde{X} . Note that $\partial\beta + \bar{\partial}\gamma$ is trivially $\tilde{\omega}$ -bounded on \tilde{X} since it equals $\pi_X^*\alpha$ and α is ω -bounded on X thanks to X being compact. We get:

$$\pi_X^*\alpha^p = (\partial\beta + \bar{\partial}\gamma) \wedge (\pi_X^*\alpha)^{p-1} = \partial(\beta \wedge (\pi_X^*\alpha)^{p-1}) + \bar{\partial}(\gamma \wedge (\pi_X^*\alpha)^{p-1}),$$

where both β, γ and $\pi_X^* \alpha$ are $\tilde{\omega}$ -bounded, and hence so is $(\partial\beta + \bar{\partial}\gamma) \wedge (\pi_X^* \alpha)^{p-1}$. □

COROLLARY 3.4. — *Let (X, ω) be a compact complex Kähler manifold with $\dim_{\mathbb{C}} X = n$. The following implication holds*

$$\omega \text{ is SKT hyperbolic} \implies \omega^k \text{ is } (\partial + \bar{\partial})\text{-bounded for all } k \in \{1, \dots, n\}.$$

THEOREM 3.5. — *Every SKT hyperbolic compact complex manifold is Kobayashi hyperbolic.*

Proof. — Let X be a compact complex manifold, with $\dim_{\mathbb{C}} X = n$, equipped with a SKT hyperbolic metric ω . This means that, if $\pi_X : \tilde{X} \rightarrow X$ is the universal cover of X , we have

$$\pi_X^* \omega = \partial\alpha + \bar{\partial}\beta \quad \text{on } \tilde{X},$$

where α and β are a smooth, $\tilde{\omega}$ -bounded $(0, 1)$ -form and $(1, 0)$ -form on \tilde{X} and $\tilde{\omega} = \pi_X^* \omega$ is the lift of the metric ω to \tilde{X} . We now adapt the proof of Theorem 4.1 in [2] to our context.

Suppose there exists a non-constant holomorphic map $f : \mathbb{C} \rightarrow X$. We will prove that $f^* \omega = 0$ on \mathbb{C} , which implies f is a constant, a contradiction.

There exists a lift \tilde{f} of f to \tilde{X} , namely a holomorphic map $\tilde{f} : \mathbb{C} \rightarrow \tilde{X}$ such that $f = \pi_X \circ \tilde{f}$. The $(1, 1)$ -form $f^* \omega$ is ≥ 0 on \mathbb{C} and may not be > 0 on countably many points $\Sigma \subset \mathbb{C}$. Thus, it can be written in the form (a) below for some C^∞ function $\mu : \mathbb{C} \rightarrow [0, +\infty)$:

$$f^* \omega \stackrel{(a)}{=} \mu(z) \frac{i}{2} dz \wedge d\bar{z} \stackrel{(b)}{=} \tilde{f}^*(\pi_X^* \omega) = \tilde{f}^*(\partial\alpha + \bar{\partial}\beta) \stackrel{(c)}{=} d(\tilde{f}^*(\alpha + \beta)) \quad \text{on } \mathbb{C},$$

where (b) follows from $f = \pi_X \circ \tilde{f}$, and (c) follows since f^* commutes with d and by the fact that a $(1, 1)$ -form is of maximal degree on \mathbb{C} . We have the following

CLAIM 3.6. — *The 1-form $\tilde{f}^*(\alpha + \beta)$ is $(f^* \omega)$ -bounded on \mathbb{C} .*

Proof of claim. — For any tangent vector v , we have:

$$\begin{aligned} |(\tilde{f}^*(\alpha + \beta))(v)|^2 &= |(\alpha + \beta)(\tilde{f}_* v)|^2 \stackrel{(a)}{\leq} (C_1 + C_2) |\tilde{f}_* v|_{\tilde{\omega}}^2 \\ &= C |v|_{\tilde{f}^* \tilde{\omega}}^2 \stackrel{(b)}{=} C |v|_{f^* \omega}^2, \end{aligned}$$

where $C > 0$ is a constant independent of v that exists such that inequality (a) holds thanks to the $\tilde{\omega}$ -boundedness of α and β on \tilde{X} , while (b) follows from $\tilde{f}^* \tilde{\omega} = f^* \omega$. □

End of proof of Theorem 3.5. — For any bounded open subset $\Omega \subset \mathbb{C}$, we let $A_\mu(\Omega)$ and $L_\mu(\partial\Omega)$ stand for the area of Ω , respectively the length of $\partial\Omega$ w.r.t. the measure $f^*\omega = \mu(z)(i/2) dz \wedge d\bar{z}$ on \mathbb{C} . For any $r > 0$, letting $D_r := \{z \in \mathbb{C} \mid |z| < r\}$ and $S_r := \{z \in \mathbb{C} \mid |z| = r\}$, we get:

$$A_\mu(D_r) = \frac{i}{2} \int_{D_r} \mu(z) dz \wedge d\bar{z} = \int_0^r \left(\int_{S_t} \mu d\sigma \right) dt \stackrel{(a)}{\geq} \int_0^r \left(\int_{S_t} \mu^{\frac{1}{2}} d\sigma \right)^2 \frac{1}{\left(\int_{S_t} d\sigma \right)} dt,$$

where $d\sigma$ is the length element of the circle, and inequality (a) follows from Hölder’s inequality applied to the functions $\mu^{1/2}$ and 1.

Therefore, we get:

$$(10) \quad A_\mu(D_r) \geq \int_0^r L_\mu^2(S_t) \frac{1}{2\pi t} dt.$$

Meanwhile, for every $r > 0$, we also have:

$$(11) \quad \begin{aligned} A_\mu(D_r) &= \int_{D_r} f^*\omega = \int_{D_r} d(\tilde{f}^*(\alpha + \beta)) \\ &= \int_{S_r} \tilde{f}^*(\alpha + \beta) \stackrel{(a)}{\leq} C \int_{S_r} \mu^{\frac{1}{2}} d\sigma \stackrel{(b)}{=} C L_\mu(S_r), \end{aligned}$$

where $C > 0$ is a constant that exists such that inequality (a) holds thanks to Claim 3.6, and equality (b) holds since the (1,1)-form $f^*\omega$ already has maximal degree on \mathbb{C} and is, therefore, proportional to $\frac{i}{2} dz \wedge d\bar{z}$.

Putting (10) and (11) together, we get:

$$(12) \quad A_\mu(D_r) \geq \frac{1}{2\pi C^2} \int_0^r \frac{A_\mu(D_t)^2}{t} dt := \frac{1}{2\pi C^2} F(r),$$

where the last equality is the definition of a function $F : (0, +\infty) \rightarrow (0, +\infty)$.

Deriving F , we get for every $r > 0$:

$$t \frac{d}{dt} F(t) = A_\mu(D_t)^2 \geq \frac{1}{4\pi^2 C^4} (F(t))^2,$$

where the last inequality follows from (12). This amounts to

$$\frac{d}{dt} \left(\frac{-1}{F(t)} \right) \geq \frac{1}{4\pi^2 C^4} \frac{1}{t}, \quad t > 0.$$

Integrating the last identity over $t \in [a, b]$, with $0 < a < b$ arbitrary, we get:

$$\frac{1}{F(a)} \geq \frac{1}{F(b)} + \frac{1}{4\pi^2 C^4} \log \frac{b}{a} \geq \frac{1}{4\pi^2 C^4} (\log b - \log a), \quad 0 < a < b.$$

For every fixed $a > 0$, letting $b \rightarrow +\infty$ in the last identity, we get $F(a) = 0$. This can only happen if $f^*\omega = 0$ on \mathbb{C} , which contradicts the fact that f is non-constant on \mathbb{C} . \square

PROPOSITION 3.7. — *Let G be a complex Lie group and Γ a co-compact subgroup of G ; then there is no **SKT hyperbolic metric** on G/Γ . Moreover, G/Γ can not be Kobayashi hyperbolic.*

Proof.^f — A Lie group G has one-dimensional Lie subgroups H passing to the neutral element e in any tangent direction. The latter subgroup has for its universal covering a connected, simply connected group of dimension 1. Now, such a group is the complex line \mathbb{C} . So, the universal covering is $\mathbb{C} \rightarrow H$, and the H 's cover a neighbourhood U of e . Kobayashi's metric vanishes along an H . Hence, it is identically zero in U because the distance from any point to e is zero. In particular, the infinitesimal Kobayashi–Royden pseudometric on the tangent bundle TU vanishes identically. We can cover G by translates of U . Since the left multiplication is an isometry with respect to the Kobayashi and the Kobayashi–Royden pseudometrics, the Kobayashi–Royden pseudometric of G is identically zero. Then also the Kobayashi pseudometric is. It follows that it is identically zero on any quotient G/Γ where Γ is a subgroup of G , that is, on any homogeneous space of G . \square

We now adapt in a straightforward way to our context the first part of the proof of [2, Proposition 2.11], where the non-existence of rational curves in compact Kähler hyperbolic manifolds was proved, and get the same result with a weaker condition on ω .

PROPOSITION 3.8. — *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and $\pi_X : \tilde{X} \rightarrow X$ be the universal covering map of X . Suppose that $\pi_X^*\omega = \partial\alpha + \bar{\partial}\beta$ for some smooth $(0, 1)$ -form α and $(1, 0)$ -form β on \tilde{X} . Then, there is no non-constant holomorphic map $f : \mathbb{P}^1 \rightarrow X$.*

Proof. — Suppose there exists a non-constant holomorphic map $f : \mathbb{P}^1 \rightarrow X$. We will show that $f^*\omega = 0$ on \mathbb{P}^1 , contradicting the non-constancy of f .

Let $\tilde{f} : \mathbb{P}^1 \rightarrow \tilde{X}$ be a lift of f to \tilde{X} , namely a holomorphic map such that $f = \pi_X \circ \tilde{f}$. From

$$f^*\omega = \tilde{f}^*(\pi_X^*\omega),$$

we get by integration:

$$\begin{aligned} \int_{\mathbb{P}^1} f^*\omega &= \int_{\mathbb{P}^1} \tilde{f}^*(\pi_X^*\omega) = \int_{\mathbb{P}^1} \tilde{f}^*(\partial\alpha + \bar{\partial}\beta) \\ &= \int_{\mathbb{P}^1} \tilde{f}^*(d(\alpha + \beta)) = \int_{\mathbb{P}^1} d(\tilde{f}^*(\alpha + \beta)) = 0, \end{aligned}$$

where the last identity follows from Stokes's theorem.

Meanwhile, $f^*\omega \geq 0$ at every point of \mathbb{P}^1 . Therefore, $f^*\omega = 0$ on \mathbb{P}^1 , a contradiction. \square

An immediate observation is that, since a *SKT hyperbolic* manifold X contains no rational curves, then by Mori's cone theorem we get K_X is **nef**.

3.2. Gauduchon hyperbolic manifolds. — The second notion that we introduce in this work is the following.

DEFINITION 3.9. — Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is said to be **Gauduchon hyperbolic** if ω^{n-1} is $(\partial + \bar{\partial})$ -bounded with respect to ω .

The manifold X is said to be Gauduchon hyperbolic if it carries a *Gauduchon hyperbolic metric*.

Let us first notice the following implication:

$$X \text{ is balanced hyperbolic} \implies X \text{ is Gauduchon hyperbolic.}$$

To see this, note the obvious fact that every balanced metric is Gauduchon and every \tilde{d} -bounded form is, in particular, $(\partial + \bar{\partial})$ -bounded.

If X is a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$ and ω is a Hermitian metric on X , for any holomorphic map $f : \mathbb{C}^{n-1} \rightarrow X$ that is *non-degenerate* at some point $x \in \mathbb{C}^{n-1}$ (in the sense that its differential map $d_x f : \mathbb{C}^{n-1} \rightarrow T_{f(x)} X$ at x is of maximal rank), we consider the smooth $(1, 1)$ -form $f^*\omega$ on \mathbb{C}^{n-1} . The assumptions made on f imply that the differential map $d_z f$ is of maximal rank for every point $z \in \mathbb{C}^{n-1} \setminus \Sigma$, where $\Sigma \subset \mathbb{C}^{n-1}$ is an analytic subset. Thus, $f^*\omega$ is ≥ 0 on \mathbb{C}^{n-1} and is > 0 on $\mathbb{C}^{n-1} \setminus \Sigma$. Consequently, $f^*\omega$ can be regarded as a *degenerate metric* on \mathbb{C}^{n-1} . Its degeneration locus, Σ , is empty if f is non-degenerate at every point of \mathbb{C}^{n-1} , in which case, $f^*\omega$ is a genuine Hermitian metric on \mathbb{C}^{n-1} . However, in our case, Σ will be non-empty in general, so $f^*\omega$ will only be a genuine Hermitian metric on $\mathbb{C}^{n-1} \setminus \Sigma$. Fix an arbitrary integer $n \geq 3$. For any $r > 0$, let $B_r := \{z \in \mathbb{C}^{n-1} \mid |z| < r\}$ and $S_r := \{z \in \mathbb{C}^{n-1} \mid |z| = r\}$ stand for the open ball, resp. the sphere, of radius r centred at $0 \in \mathbb{C}^{n-1}$. Moreover, for any $(1, 1)$ -form $\gamma \geq 0$ on a complex manifold and any positive integer p , we will use the notation:

$$\gamma_p := \frac{\gamma^p}{p!}.$$

For a holomorphic map $f : \mathbb{C}^{n-1} \rightarrow (X, \omega)$ in the above setting and for $r > 0$, we consider the (ω, f) -volume of the ball $B_r \subset \mathbb{C}^{n-1}$:

$$\text{Vol}_{\omega, f}(B_r) := \int_{B_r} f^*\omega_{n-1} > 0.$$

Meanwhile, for $z \in \mathbb{C}^{n-1}$, let $\tau(z) := |z|^2$ be its squared Euclidean norm. At every point $z \in \mathbb{C}^{n-1} \setminus \Sigma$, we have:

$$(13) \quad \frac{d\tau}{|d\tau|_{f^*\omega}} \wedge \star_{f^*\omega} \left(\frac{d\tau}{|d\tau|_{f^*\omega}} \right) = f^*\omega_{n-1},$$

where $\star_{f^*\omega}$ is the Hodge star operator induced by $f^*\omega$. Thus, the $(2n-3)$ -form

$$d\sigma_{\omega,f} := \star_{f^*\omega} \left(\frac{d\tau}{|d\tau|_{f^*\omega}} \right)$$

on $\mathbb{C}^{n-1} \setminus \Sigma$ is the area measure induced by $f^*\omega$ on the spheres of \mathbb{C}^{n-1} . This means that its restriction

$$(14) \quad d\sigma_{\omega,f,t} := \left(\star_{f^*\omega} \left(\frac{d\tau}{|d\tau|_{f^*\omega}} \right) \right)_{|_{S_t}}$$

is the area measure induced by the degenerate metric $f^*\omega$ on the sphere $S_t = \{\tau(z) = t^2\} \subset \mathbb{C}^{n-1}$ for every $t > 0$. In particular, the area of the sphere $S_r \subset \mathbb{C}^{n-1}$ w.r.t. $d\sigma_{\omega,f,r}$ is

$$A_{\omega,f}(S_r) = \int_{S_r} d\sigma_{\omega,f,r} > 0, \quad r > 0.$$

In this case, there is no reason for the measure on the sphere that "disintegrates" the volume form $f^*(\omega^{n-1})$, and the area measure induced by the partially degenerate metric $f^*(\omega)$ to coincide (contrary to the case of the equality (b) obtained in (11), which is valid except when working on \mathbb{C}). In fact, if $\tau(z) = |z|^2$, the area measure induced by the degenerate metric $f^*(\omega)$ on S_t is almost by definition

$$d\sigma_{\omega,f,t} := \left(\star_{f^*\omega} \left(\frac{d\tau}{|d\tau|_{f^*\omega}} \right) \right)_{|_{S_t}}.$$

On the other hand, if $d\mu_{\omega,f,t}$ is the positive measure that disintegrates $f^*(\omega^{n-1})$ on the sphere $S_t = \{\tau(z) = t^2\}$, we have $t = \tau^{\frac{1}{2}}$, $d\tau = 2tdt$, and then we get

$$\text{Vol}_{\omega,f}(B_r) = \int_{B_r} f^*\omega_{n-1} = \int_0^r \left(\int_{S_t} d\mu_{\omega,f,t} \right) dt = \int_{B_r} d\mu_{\omega,f,t} \wedge \frac{d\tau}{2t}.$$

As a consequence, we get

$$(f^*\omega^{n-1})_{|_{S_t}} = \frac{1}{2t} (d\mu_{\omega,f,t} \wedge d\tau)_{|_{S_t}}.$$

Let us recall the two following definitions introduced in [10].

DEFINITION 3.10. — Let (X, ω) be a **compact** complex Hermitian manifold with $\dim_{\mathbb{C}} X = n \geq 2$ and let $f : \mathbb{C}^{n-1} \rightarrow X$ be a holomorphic map that is **non-degenerate** at some point $x \in \mathbb{C}^{n-1}$.

1) We say that f has **subexponential growth** if the following two conditions are satisfied:

(i) There exist constants $C_1 > 0$ and $r_0 > 0$ such that

$$(15) \quad \int_{S_t} |d\tau|_{f^*\omega} d\sigma_{\omega,f,t} \leq C_1 t \text{Vol}_{\omega,f}(B_t), \quad t > r_0.$$

(ii) For every constant $C > 0$, we have:

$$(16) \quad \limsup_{b \rightarrow +\infty} \left(\frac{b}{C} - \log F(b) \right) = +\infty,$$

where

$$F(b) := \int_0^b \text{Vol}_{\omega,f}(B_t) dt = \int_0^b \left(\int_{B_t} f^* \omega_{n-1} \right) dt, \quad b > 0.$$

2) We say that X is **divisorially hyperbolic** if there is no holomorphic map f that has **subexponential growth**.

Building upon Theorem 2.8 presented in [10], we establish a profound generalisation, yielding the following remarkable result.

THEOREM 3.11. — *Every Gauduchon hyperbolic compact complex manifold is also divisorially hyperbolic.*

Proof. — Let X be a compact complex manifold, with $\dim_{\mathbb{C}} X = n$, equipped with a Gauduchon hyperbolic metric ω . This means that, if $\pi_X : \tilde{X} \rightarrow X$ is the universal cover of X , we have

$$\pi_X^* \omega^{n-1} = \partial \Gamma_{n-2,n-1} + \bar{\partial} \Gamma_{n-1,n-2} \quad \text{on } \tilde{X},$$

where $\Gamma_{n-2,n-1}$ is an $\tilde{\omega}$ -bounded C^∞ $(n-2, n-1)$ -form on \tilde{X} , $\Gamma_{n-1,n-2}$ is an $\tilde{\omega}$ -bounded C^∞ $(n-1, n-2)$ -form on \tilde{X} , and $\tilde{\omega} = \pi_X^* \omega$ is the lift of the metric ω to \tilde{X} .

Suppose there exists a holomorphic map $f : \mathbb{C}^{n-1} \rightarrow X$ that is non-degenerate at some point $x \in \mathbb{C}^{n-1}$ and has subexponential growth in the sense of Definition 3.10. We will prove that $f^* \omega^{n-1} = 0$ on \mathbb{C}^{n-1} , in contradiction to the non-degeneracy assumption made on f at x .

Since \mathbb{C}^{n-1} is simply connected, there exists a lift \tilde{f} of f to \tilde{X} , namely a holomorphic map $\tilde{f} : \mathbb{C}^{n-1} \rightarrow \tilde{X}$ such that $f = \pi_X \circ \tilde{f}$. In particular, $d_x \tilde{f}$ is injective since $d_x f$ is.

The smooth $(n-1, n-1)$ -form $f^* \omega^{n-1}$ is ≥ 0 on \mathbb{C}^{n-1} and > 0 on $\mathbb{C}^{n-1} \setminus \Sigma$, where $\Sigma \subset \mathbb{C}^{n-1}$ is the proper analytic subset of all points $z \in \mathbb{C}^{n-1}$ such that $d_z f$ is not of maximal rank. We have:

$$f^* \omega_{n-1} = \tilde{f}^* (\pi_X^* \omega^{n-1}) = d(\tilde{f}^* \Gamma_{n-2,n-1} + \Gamma_{n-1,n-2}) \quad \text{on } \mathbb{C}^{n-1}.$$

With respect to the degenerate metric $f^*\omega$ on \mathbb{C}^{n-1} , we have the following.

CLAIM 3.12. — *The $(2n - 3)$ -form $\tilde{f}^*(\Gamma_{n-2,n-1} + \Gamma_{n-1,n-2})$ is $(f^*\omega)$ -bounded on \mathbb{C}^{n-1} .*

Proof of claim. — For any tangent vectors v_1, \dots, v_{2n-3} in \mathbb{C}^{n-1} , we have:

$$\begin{aligned} & |(\tilde{f}^*(\Gamma_{n-2,n-1} + \Gamma_{n-1,n-2}))(v_1, \dots, v_{2n-3})|^2 \\ &= |(\Gamma_{n-2,n-1} + \Gamma_{n-1,n-2})(\tilde{f}_*v_1, \dots, \tilde{f}_*v_{2n-3})|^2 \\ &\stackrel{(a)}{\leq} C |\tilde{f}_*v_1|_{\tilde{\omega}}^2 \dots |\tilde{f}_*v_{2n-3}|_{\tilde{\omega}}^2 \\ &= C |v_1|_{f^*\omega}^2 \dots |v_{2n-3}|_{f^*\omega}^2 \stackrel{(b)}{=} C |v_1|_{f^*\omega}^2 \dots |v_{2n-3}|_{f^*\omega}^2, \end{aligned}$$

where $C > 0$ is a constant independent of the v_j 's that exists such that inequality (a) holds thanks to the $\tilde{\omega}$ -boundedness of $(\Gamma_{n-2,n-1}$ and $+\Gamma_{n-1,n-2}$ on \tilde{X} , while (b) follows from $\tilde{f}^*\tilde{\omega} = f^*\omega$. □

End of proof of Theorem 3.5. — On the one hand, we have $d\tau = 2t dt$ and

$$(17) \quad \text{Vol}_{\omega,f}(B_r) = \int_{B_r} f^*\omega_{n-1} = \int_0^r \left(\int_{S_t} d\mu_{\omega,f,t} \right) dt = \int_{B_r} d\mu_{\omega,f,t} \wedge \frac{d\tau}{2t},$$

where $d\mu_{\omega,f,t}$ is the positive measure on S_t defined by

$$\frac{1}{2t} d\mu_{\omega,f,t} \wedge (d\tau)|_{S_t} = (f^*\omega_{n-1})|_{S_t}, \quad t > 0.$$

Comparing this with (13) and (14), this means that the measures $d\mu_{\omega,f,t}$ and $d\sigma_{\omega,f,t}$ on S_t are related by

$$(18) \quad \frac{1}{2t} d\mu_{\omega,f,t} = \frac{1}{|d\tau|_{f^*\omega}} d\sigma_{\omega,f,t}, \quad t > 0.$$

Now, the Hölder inequality yields:

$$\int_{S_t} \frac{1}{|d\tau|_{f^*\omega}} d\sigma_{\omega,f,t} \geq \frac{A_{\omega,f}^2(S_t)}{\int_{S_t} |d\tau|_{f^*\omega} d\sigma_{\omega,f,t}},$$

so together with (17) and (18) this leads to:

$$\begin{aligned} (19) \quad \text{Vol}_{\omega,f}(B_r) &= \int_0^r \left(\int_{S_t} \frac{1}{2t} d\mu_{\omega,f,t} \right) dt = \int_0^r \left(\int_{S_t} \frac{1}{|d\tau|_{f^*\omega}} d\sigma_{\omega,f,t} \right) d\tau \\ &\geq 2 \int_0^r \frac{A_{\omega,f}^2(S_t)}{\int_{S_t} |d\tau|_{f^*\omega} d\sigma_{\omega,f,t}} t dt, \quad r > 0. \end{aligned}$$

On the other hand, for every $r > 0$, we have:

$$\begin{aligned}
 (20) \quad \text{Vol}_{\omega,f}(B_r) &= \int_{B_r} f^* \omega_{n-1} = \int_{B_r} d(\tilde{f}^*(\Gamma_{n-2,n-1} + \Gamma_{n-1,n-2})) \\
 &= \int_{S_r} \tilde{f}^*(\Gamma_{n-2,n-1} + \Gamma_{n-1,n-2}) \stackrel{(a)}{\leq} C \int_{S_r} d\sigma_{\omega,f} = C A_{\omega,f}(S_r),
 \end{aligned}$$

where $C > 0$ is a constant that exists such that inequality (a) holds thanks to Claim 3.12.

Putting (19) and (20) together, we get for every $r > r_0$:

$$\begin{aligned}
 (21) \quad \text{Vol}_{\omega,f}(B_r) &\geq \frac{2}{C^2} \int_0^r \text{Vol}_{\omega,f}(B_t) \frac{t \text{Vol}_{\omega,f}(B_t)}{\int_{S_t} |d\tau|_{f^*\omega} d\sigma_{\omega,f,t}} dt \\
 &\stackrel{(a)}{\geq} \frac{2}{C_1 C^2} \int_{r_0}^r \text{Vol}_{\omega,f}(B_t) dt \stackrel{(b)}{:=} C_2 F(r),
 \end{aligned}$$

where (a) follows from the growth assumption (15), and (b) is the definition of a function $F : (r_0, +\infty) \rightarrow (0, +\infty)$ with $C_2 := 2/(C_1 C^2)$.

By taking the derivative of F , we get for every $r > r_0$:

$$F'(r) = \text{Vol}_{\omega,f}(B_r) \geq C_2 F(r),$$

where the last inequality is (21). This amounts to

$$\frac{d}{dt} \left(\log F(t) \right) \geq C_2, \quad t > r_0.$$

Integrating this over $t \in [a, b]$, with $r_0 < a < b$ arbitrary, we get:

$$(22) \quad -\log F(a) \geq -\log F(b) + C_2(b-a), \quad r_0 < a < b.$$

Now, fix an arbitrary $a > r_0$ and let $b \rightarrow +\infty$. Thanks to the *subexponential growth* assumption (16) made on f , there exists a sequence of reals $b_j \rightarrow +\infty$ such that the right-hand side of inequality (22) for $b = b_j$ tends to $+\infty$ as $j \rightarrow +\infty$. This forces $F(a) = 0$ for every $a > r_0$, and hence $\text{Vol}_{\omega,f}(B_r) = 0$ for every $r > r_0$. This amounts to $f^* \omega^{n-1} = 0$ on \mathbb{C}^{n-1} , in contradiction to the non-degeneracy assumption made on f at a point $x \in \mathbb{C}^{n-1}$. \square

3.3. Some properties of SKT hyperbolic manifolds. — It is a classical fact due to Gaffney [5] that certain basic facts in the Hodge theory of compact Riemannian manifolds remain valid on *complete* such manifolds. The main ingredient in the proof of this fact is the following *cut-off trick* of Gaffney's, which played a key role in [7, §.1]. It also appears in [3, VIII, Lemma 2.4].

LEMMA 3.13. — ([5]) *Let (X, g) be a Riemannian manifold. Then, (X, g) is **complete** if and only if there exists an exhaustive sequence $(K_\nu)_{\nu \in \mathbb{N}}$ of **compact** subsets of X :*

$$K_\nu \subset \overset{\circ}{K}_{\nu+1} \quad \text{for all } \nu \in \mathbb{N} \quad \text{and} \quad X = \bigcup_{\nu \in \mathbb{N}} K_\nu,$$

and a sequence $(\psi_\nu)_{\nu \in \mathbb{N}}$ of C^∞ functions $\psi_\nu : X \rightarrow [0, 1]$ satisfying, for every $\nu \in \mathbb{N}$, the conditions:

$$\begin{aligned} \psi_\nu &= 1 \text{ in a neighbourhood of } K_\nu, \quad \text{Supp } \psi_\nu \subset \overset{\circ}{K}_{\nu+1}, \quad \text{and} \\ \|d\psi_\nu\|_{L_g^\infty} &:= \sup_{x \in X} |(d\psi_\nu)(x)|_g \leq \varepsilon_\nu \end{aligned}$$

for some constants $\varepsilon_\nu > 0$ such that $\varepsilon_\nu \downarrow 0$ as ν tends to $+\infty$.

In particular, the cut-off functions ψ_ν are compactly supported. One can choose $\varepsilon_\nu = 2^{-\nu}$ for each ν (see, e.g., [3, VIII, Lemma 2.4]), but this will play no role here.

An immediate consequence of Gaffney’s cut-off trick is the following classical generalisation of Stokes’s theorem to possibly non-compact, but complete, Riemannian manifolds when the forms involved are L^1 .

LEMMA 3.14. — ([7, Lemma 1.1.A.]) *Let (X, g) be a **complete** Riemannian manifold of real dimension m . Let η be an L_g^1 -form on X of degree $m - 1$ such that $d\eta$ is again L_g^1 . Then*

$$\int_X d\eta = 0.$$

We will split this section into two parts: one for the Kähler case and one for the general case.

3.4. Case of non-Kählerian manifolds. —

PROPOSITION 3.15. — *Let (X, ω) be an **SKT hyperbolic manifold** and let $\pi : \tilde{X} \rightarrow X$ be the universal cover of X . There exists no non-zero d -closed positive $(n - 1, n - 1)$ -current $\tilde{T} \geq 0$ on \tilde{X} such that \tilde{T} is $L_{\tilde{\omega}}^1$, where $\tilde{\omega} := \pi^*\omega$ is the lift of ω to \tilde{X} .*

Proof. — Let $n = \dim_{\mathbb{C}} X$. The SKT hyperbolic assumption on X means that $\pi^*\omega = \partial\alpha + \bar{\partial}\beta$ on \tilde{X} for some smooth $L_{\tilde{\omega}}^\infty$ -forms α and β of degree $(0, 1)$ and $(1, 0)$, respectively, on \tilde{X} . By the closedness assumption on \tilde{T} as in the

statement, we get

$$\int_{\tilde{X}} \tilde{T} \wedge \pi^* \omega = \int_{\tilde{X}} \tilde{T} \wedge (\partial\alpha + \bar{\partial}\beta) = \int_{\tilde{X}} d(\tilde{T} \wedge (\alpha + \beta)).$$

We would have

$$(23) \quad 0 < \int_{\tilde{X}} \tilde{T} \wedge \pi^* \omega = \int_{\tilde{X}} d(\tilde{T} \wedge (\alpha + \beta)) = 0,$$

which is contradictory.

The last identity in (23) follows from Lemma 3.14 applied on the complete manifold $(\tilde{X}, \tilde{\omega})$ to the $L^1_{\tilde{\omega}}$ -current $\eta := \tilde{T} \wedge (\alpha + \beta)$ of degree 1 whose differential $d\eta = \tilde{T} \wedge \pi^* \omega$ is again $L^1_{\tilde{\omega}}$. That η is $L^1_{\tilde{\omega}}$ follows from $\tilde{T} \in L^1_{\tilde{\omega}}$ (by hypothesis) and from α and β being both $L^\infty_{\tilde{\omega}}$, while $d\eta$ being $L^1_{\tilde{\omega}}$ follows from \tilde{T} being $L^1_{\tilde{\omega}}$ and from $\pi^* \omega$ being $L^\infty_{\tilde{\omega}}$ (as a lift of the smooth, hence bounded, form ω on the compact manifold X). \square

Similarly, we can replace the SKT hyperbolic manifold with a Gauduchon hyperbolic manifold, and using the same approach, we can demonstrate that there does not exist a non-zero d -closed positive $(1, 1)$ -current $\tilde{T} \geq 0$ on \tilde{X} such that \tilde{T} is $L^1_{\tilde{\omega}}$.

THEOREM 3.16. — *Let X be a compact complex **SKT hyperbolic** manifold with $\dim_{\mathbb{C}} X = n$. Let $\pi : \tilde{X} \rightarrow X$ be the universal cover of X and $\tilde{\omega} := \pi^* \omega$ the lift to \tilde{X} of a SKT hyperbolic metric ω on X . Fix a primitive $L^2_{\tilde{\omega}}$ -form ϕ on \tilde{X} of bidegree (p, q) with $p + q = n - 1$ such that*

$$\partial\phi = 0, \quad \bar{\partial}\phi = 0.$$

Then $\phi = 0$.

Proof. — Let $n = \dim_{\mathbb{C}} X$. The SKT hyperbolic assumption on X means that $\pi^* \omega = \partial\alpha + \bar{\partial}\beta$ on \tilde{X} for some smooth $L^\infty_{\tilde{\omega}}$ -forms α and β of degree $(0, 1)$ and $(1, 0)$, respectively, on \tilde{X} . Let ϕ be as in the statement of the theorem. Then, by (3) of Lemma 2.4 we get

$$(24) \quad \begin{aligned} 0 &= \langle\langle \partial^* (\tilde{\omega} \wedge \phi), \psi_\nu \alpha \wedge \phi \rangle\rangle = \langle\langle \tilde{\omega} \wedge \phi, \partial(\psi_\nu \alpha \wedge \phi) \rangle\rangle \\ &= \langle\langle \tilde{\omega} \wedge \phi, \partial\psi_\nu \wedge \alpha \wedge \phi \rangle\rangle + \langle\langle \tilde{\omega} \wedge \phi, \psi_\nu \wedge \partial\alpha \wedge \phi \rangle\rangle, \end{aligned}$$

and

$$(25) \quad \begin{aligned} 0 &= \langle\langle \bar{\partial}^* (\tilde{\omega} \wedge \phi), \psi_\nu \beta \wedge \phi \rangle\rangle = \langle\langle \tilde{\omega} \wedge \phi, \bar{\partial}(\psi_\nu \beta \wedge \phi) \rangle\rangle \\ &= \langle\langle \tilde{\omega} \wedge \phi, \bar{\partial}\psi_\nu \wedge \beta \wedge \phi \rangle\rangle + \langle\langle \tilde{\omega} \wedge \phi, \psi_\nu \wedge \bar{\partial}\beta \wedge \phi \rangle\rangle. \end{aligned}$$

Putting (24) and (25) together, we get

$$(26) \quad \langle\langle \tilde{\omega} \wedge \phi, d(\psi_\nu) \wedge (\alpha + \beta) \wedge \phi \rangle\rangle + \langle\langle \tilde{\omega} \wedge \phi, \psi_\nu \wedge (\partial\alpha + \bar{\partial}\beta) \wedge \phi \rangle\rangle = 0.$$

Therefore,

$$(27) \quad \begin{aligned} & \left| \int_{\tilde{X}} \langle \tilde{\omega} \wedge \phi, d(\psi_\nu) \wedge (\alpha + \beta) \wedge \phi \rangle dV \right| \\ & \leq \int_{\tilde{X}} \|\tilde{\omega}\|_{L^\infty} \|d\psi_\nu\|_{L^\infty} \|\alpha + \beta\|_{L^\infty} \|\phi\|_{L^2} dV \\ & \leq \varepsilon_\nu \|\tilde{\omega}\|_{L^\infty} \|\alpha + \beta\|_{L^\infty} \|\phi\|_{L^2} \downarrow 0 \text{ as } \nu \rightarrow +\infty. \end{aligned}$$

Hence, combining (27) and (26) we get,

$$\lim_{\nu \rightarrow +\infty} \langle\langle \tilde{\omega} \wedge \phi, \psi_\nu \wedge (\partial\alpha + \bar{\partial}\beta) \wedge \phi \rangle\rangle = \langle\langle \tilde{\omega} \wedge \phi, \tilde{\omega} \wedge \phi \rangle\rangle = 0.$$

On the other hand, the pointwise map $\tilde{\omega} \wedge \cdot : \Lambda^{n-1}T^*\tilde{X} \rightarrow \Lambda^{n+1}T^*\tilde{X}$ is bijective, so we get $\phi = 0$ from $\tilde{\omega} \wedge \phi = 0$.

The proof is complete. □

COROLLARY 3.17. — *Let ϕ be a $(n-1, 0)$ -form (respectively, a $(0, n-1)$ -form) on a connected complete manifold $(\tilde{X}, \tilde{\omega})$ such that*

$$\phi \in L^2(\tilde{X}), \quad \partial\phi = 0, \quad \bar{\partial}\phi = 0.$$

If $\tilde{\omega} = \partial\alpha + \bar{\partial}\beta$ where α and β are bounded 1-forms on \tilde{X} , then

$$\phi = 0.$$

Similarly, since all 1-forms are primitive, we get the following.

PROPOSITION 3.18. — *Let ϕ be a $(1, 0)$ -form (respectively, a $(0, 1)$ -form) on a connected complete manifold $(\tilde{X}, \tilde{\omega})$ such that*

$$\phi \in L^2(\tilde{X}), \quad \partial\phi = 0, \quad \bar{\partial}\phi = 0.$$

If $\tilde{\omega}_{n-1} = \partial\alpha + \bar{\partial}\beta$, where α and β are bounded $(2n-3)$ -forms on \tilde{X} , then

$$\phi = 0.$$

Proof. — Consider a $(1, 0)$ -form ϕ . Since all 1-forms are primitive, the standard formula (2) we get: $\star\phi = -i\tilde{\omega}_{n-1} \wedge \phi$, hence $\star(\tilde{\omega}_{n-1} \wedge \phi) = -i\phi$. Meanwhile, $\partial^* = -\star\bar{\partial}\star$ (resp. $\bar{\partial}^* = -\star\partial\star$), so applying $-\star\bar{\partial}$ (resp. $-\star\partial$) to the previous identity, we get:

$$\partial^*(\tilde{\omega}_{n-1} \wedge \phi) = 0 \quad (\text{resp. } \bar{\partial}^*(\tilde{\omega}_{n-1} \wedge \phi) = 0).$$

Meanwhile, the pointwise map $\tilde{\omega}_{n-1} \wedge \cdot : \Lambda^1T^*\tilde{X} \rightarrow \Lambda^{2n-1}T^*\tilde{X}$ is bijective.

By following a technique similar to the one used in the previous theorem, we can now conclude that the desired result holds. □

3.5. Case of Kähler manifolds. — Let (X, ω) be a Kähler manifold of dimension $2n$. The operator $L^k : \Lambda^p T^*X \rightarrow \Lambda^{2k+p} T^*X$ defined by $L^k(\phi) = \omega^k \wedge \phi$ for all p -forms $\phi \in \Lambda^p T^*X$ commutes with d and Δ . We first recall the following result

THEOREM 3.19. — (Lefschetz.) *The map L^k is injective on harmonic forms for $2p + 2k \leq 2n = \dim_{\mathbb{R}}(X)$ and surjective for $2p + 2k \geq 2n$.*

This result shows that L^k is bijective for $2p + 2k = 2n$. Moreover, the injectivity (in particular the bijectivity) shows that L is a **quasi-isometry**, i.e.,

$$C^{-1} \|\phi\|_{L^2} \leq \|L^k \phi\|_{L^2} \leq C \|\phi\|_{L^2} \text{ for all } \phi \in \Lambda^p T^*X \text{ and some constant } C > 0.$$

For more details, see Theorem 1.2.A. in [7].

THEOREM 3.20. — *Let (X, ω) be a complete Kähler manifold of dimension $2n$ and $\omega = \partial\alpha + \bar{\partial}\beta$, where α and β are, respectively, a bounded $(0, 1)$ and $(1, 0)$ forms on X . Then every L^2 -form $\psi \in \text{Dom } \Delta$ on X of degree $p \neq n$ satisfies the inequality*

$$\langle \psi, \Delta\psi \rangle_{L^2} \geq \lambda_0^2 \langle \psi, \psi \rangle_{L^2},$$

where λ_0 is a strictly positive constant that depends only on $n = \dim_{\mathbb{C}} X$, α and β .

Proof. — We can assume that ψ has compact support because Δ is essentially self-adjoint.

For a given $p < n$, let $k = n - p$, we get $2p + 2k = 2n$, so by the Lefschetz theorem L^k is a bijective quasi-isometry, and so every L^2 -form ψ of degree $2k + p$ is the product $\psi = L^k \phi = \omega^k \wedge \phi$, where $\phi = L^{-k} \psi$ and $C^{-1} \|\phi\|_{L^2} \leq \|L^k \phi\|_{L^2} \leq C \|\phi\|_{L^2}$. Since L^k commutes with Δ , we also have

$$\langle \langle \Delta\phi, \phi \rangle \rangle \leq C_1 \langle \langle \Delta\psi, \psi \rangle \rangle.$$

Meanwhile, we have: $\psi = \omega^k \wedge \phi = (\partial\alpha + \bar{\partial}\beta) \wedge \omega^{k-1} \wedge \phi$. In other words,

$$(28) \quad \psi = d\theta - \psi',$$

where

$$\begin{aligned} \theta &:= (\alpha + \beta) \wedge \omega^{k-1} \wedge \phi \quad \text{and} \\ \psi' &:= (\alpha + \beta) \wedge \omega^{k-1} \wedge d\phi + (\bar{\partial}\alpha + \partial\beta) \wedge \omega^{k-1} \wedge \phi. \end{aligned}$$

To estimate θ , we write:

$$(29) \quad \|\theta\| \leq \|\alpha + \beta\|_{L^\infty} \|\phi\|_{L^2} \leq C \|\alpha + \beta\|_{L^\infty} \|\psi\|_{L^2}.$$

To estimate the left part of ψ' , since $\|d\phi\|_{L^2}^2 \leq \langle \langle \Delta\phi, \phi \rangle \rangle \leq C_1 \langle \langle \Delta\psi, \psi \rangle \rangle$, we have:

$$(30) \quad \|(\alpha + \beta) \wedge \omega^{k-1} \wedge d\phi\|_{L^2} \leq \|\alpha + \beta\|_{L^\infty} \|d\phi\|_{L^2} \leq \|\alpha + \beta\|_{L^\infty} \langle \langle \Delta\psi, \psi \rangle \rangle^{\frac{1}{2}}.$$

To find an upper bound for $\|\psi\|_{L^2}$, we write:

$$(31) \quad \|\psi\|^2 = \langle\langle \psi, d\theta - \psi' \rangle\rangle \leq |\langle\langle \psi, d\theta \rangle\rangle| + |\langle\langle \psi, \psi' \rangle\rangle|,$$

where (28) was used to get the first equality.

For the first term on the r.h.s. of (31), we get:

$$(32) \quad |\langle\langle \psi, d\theta \rangle\rangle| = |\langle\langle d^* \psi, \theta \rangle\rangle| \leq \|d^* \psi\|_{L^2} \|\theta\|_{L^2} \\ \leq C \langle\langle \Delta \psi, \psi \rangle\rangle^{\frac{1}{2}} \|\alpha + \beta\|_{L^\infty} \|\psi\|_{L^2},$$

where (29) was used to get the last inequality.

For the second term on the r.h.s. of (31), we get:

$$(33) \quad |\langle\langle \psi, \psi' \rangle\rangle| \leq |\langle\langle \psi, (\alpha + \beta) \wedge \omega^{k-1} \wedge d\phi \rangle\rangle| + |\langle\langle \psi, \bar{\partial}\alpha \wedge \omega^{k-1} \wedge \phi \rangle\rangle| \\ + |\langle\langle \psi, \partial\beta \wedge \omega^{k-1} \wedge \phi \rangle\rangle| \\ \leq \|\psi\|_{L^2} \|\alpha + \beta\|_{L^\infty} \langle\langle \Delta \psi, \psi \rangle\rangle^{\frac{1}{2}} + |\langle\langle \psi, \bar{\partial}\alpha \wedge \omega^{k-1} \wedge \phi \rangle\rangle| \\ + |\langle\langle \psi, \partial\beta \wedge \omega^{k-1} \wedge \phi \rangle\rangle|,$$

where (30) was used to get the last inequality.

To estimate the last two terms on the right in (33), we write:

$$(34) \quad |\langle\langle \psi, \bar{\partial}\alpha \wedge \omega^{k-1} \wedge \phi \rangle\rangle| \leq |\langle\langle \psi, \bar{\partial}(\alpha \wedge \omega^{k-1} \wedge \phi) \rangle\rangle| + |\langle\langle \psi, \alpha \wedge \omega^{k-1} \wedge \bar{\partial}\phi \rangle\rangle| \\ = |\langle\langle \bar{\partial}^* \psi, \alpha \wedge \omega^{k-1} \wedge \phi \rangle\rangle| + |\langle\langle \psi, \alpha \wedge \omega^{k-1} \wedge \bar{\partial}\phi \rangle\rangle| \\ \leq C \|\bar{\partial}^* \psi\|_{L^2} \|\alpha\|_{L^\infty} \|\psi\|_{L^2} + C \|\bar{\partial}\phi\|_{L^2} \|\alpha\|_{L^\infty} \|\psi\|_{L^2} \\ \leq 2C.C_2 \|\alpha\|_{L^\infty} \|\psi\|_{L^2} \langle\langle \Delta \psi, \psi \rangle\rangle^{\frac{1}{2}},$$

since

$$\|\bar{\partial}^* \psi\|_{L^2}^2 \leq \langle\langle \Delta \phi, \phi \rangle\rangle \leq C_2 \langle\langle \Delta \psi, \psi \rangle\rangle \text{ and } \|\bar{\partial}\phi\|_{L^2}^2 \leq \langle\langle \Delta \phi, \phi \rangle\rangle \leq C_2 \langle\langle \Delta \psi, \psi \rangle\rangle.$$

Using the same technique, we obtain the following estimate

$$(35) \quad |\langle\langle \psi, \partial\beta \wedge \omega^{k-1} \wedge \phi \rangle\rangle| \leq 2C.C_2 \|\beta\|_{L^\infty} \|\psi\|_{L^2} \langle\langle \Delta \psi, \psi \rangle\rangle^{\frac{1}{2}}.$$

Adding up (32), (33), (34) and (35) and using (31), we get

$$\|\psi\|^2 \leq (\lambda_0^2)^{-1} \langle\langle \Delta \psi, \psi \rangle\rangle,$$

for the forms ψ of degree $2k + p > n$.

The case $2k + p < n$ follows by the Poincare duality as the operator $\star : \Lambda^p T^* X \rightarrow \Lambda^{n-p} T^* X$ commutes with Δ and is isometric for the L_2 -norms.

The proof is complete. □

As a simple consequence we get the following **Lefschetz vanishing theorem**.

COROLLARY 3.21. — *Let $(\tilde{X}, \tilde{\omega})$ be a connected complete Kähler manifold. If $\tilde{\omega} = \partial\alpha + \bar{\partial}\beta$, where α and β are bounded 1-forms on \tilde{X} , then $\mathcal{H}_{\tilde{\omega}}^p(\tilde{X}, \mathbb{C}) = \{0\}$, unless $p = n$, where $\mathcal{H}_{\tilde{\omega}}^p(\tilde{X}, \mathbb{C}) := \{u \in L_p^2(\tilde{X}, \mathbb{C}) \mid \Delta_{\tilde{\omega}} u = 0\}$ is the space of $\Delta_{\tilde{\omega}}$ -harmonic L^2 -forms of degree p .*

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