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Simon Jacques

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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ORBIT CLOSURES IN FLAG VARIETIES FOR THE CENTRALIZER OF AN ORDER-TWO NILPOTENT ELEMENT: NORMALITY AND RESOLUTIONS FOR TYPES A, B, D

BY SIMON JACQUES

ABSTRACT. — Let G be a reductive algebraic group in classical types A, B, D. Let e be an element of the Lie algebra of G, with $Z \subset G$ its centralizer for the adjoint action. We assume that e identifies with a nilpotent matrix of order two, which guarantees that the number of Z-orbits in the flag variety of G is finite. For types B and D in characteristic two, we also assume that the image of e is totally isotropic. We show that the closure Y of such an orbit is normal. We also prove that Y is Cohen-Macaulay with rational singularities provided that the base field is of characteristic zero, and that Cohen–Macaulayness holds in any characteristic for type A. We exhibit a rational and birational morphism onto Y involving Schubert varieties. Our work generalizes a result by N. Perrin and E. Smirnov on the Springer fibers.

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SIMON JACQUES, Université Clermont Auvergne – LMBP UMR 6620, Université Clermont-Auvergne, Campus Universitaire des Cézeaux 3, place Vasarely TSA 60026 CS 60026, 63 178 Aubière Cedex, 63000 Clermont-Ferrand, France • *E-mail :* `simon.jacques.w@gmail.com`

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Résumé (*Adhérences de certaines orbites dans la variété de drapeaux, résolution et normalité dans les types classiques A, B, D*). — Soit G un groupe algébrique réductif en type A, B ou D. Soit e un élément de l'algèbre de Lie de G et $Z \subset G$ son centralisateur, agissant sur la variété de drapeaux G/B de G. Nous supposons que e s'identifie à une matrice nilpotente d'ordre deux, ce qui garantit un nombre fini de Z-orbites dans G/B. Pour les types B et D en caractéristique deux, nous supposons également que l'image de e est totalement isotrope. Nous montrons alors que toute adhérence Y de Z-orbite dans G/B est normale. Nous prouvons également que Y est de Cohen-Macaulay avec des singularités rationnelles sous l'hypothèse que la caractéristique du corps de base est zéro, et que cette propriété de Cohen-Macaulay est vraie en toute caractéristique pour le type A. Pour cela, nous construisons un morphisme rationnel et birationnel sur Y au moyen de variétés de Schubert. Notre travail généralise un résultat de N. Perrin et E. Smirnov sur les fibres de Springer.

Introduction

1. — Let $\Bbbk$ be an algebraically closed field and let G be a reductive connected algebraic group over $\Bbbk$, with B a Borel subgroup. Let e be a nilpotent element of the Lie algebra $\mathfrak{g}$ of G, and let Z be its centralizer in G for the adjoint action. When the number of Z-orbits in the flag variety G/B is finite, their closures are of particular interest. They include, in this case, the irreducible components of the so-called Springer fiber over e. It is the fiber of e under the proper birational morphism

$$\widetilde{\mathcal{N}} \to \mathcal{N},$$

called the Springer resolution, which is the projection onto the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ from the smooth variety $\widetilde{\mathcal{N}} := \{ (x, gB) \in \mathcal{N} \times G/B \mid Ad\ g^{-1} \cdot x \in \mathfrak{b} \}$, $\mathfrak{b}$ denoting the Lie algebra of B. The Springer fibers are of main interest in representation theory (see the seminal work of T.A. Springer [35], their link with the orbital varieties [34] and the Steinberg variety [38][1]). They are connected and equidimensional (see, for example, [34]), and their irreducible components have been the subject of numerous studies. For the classical cases and char($\Bbbk$) $\neq 2$, N. Spaltenstein (type A, [34]) and M. van Leeuwen (types B, C, D, [26]) showed that they are parameterized by standard and domino tableaux, whose shapes are given by Young diagrams relative to the nilpotent

1. Actually, the latter references deal with unipotent elements instead of nilpotent ones, regarding the Springer fibers as the variety of Borel subgroups containing a given unipotent element. However, recall that when G is the general linear group or is almost simple and simply connected, and the characteristic of $\Bbbk$ is good, the unipotent variety in the group G and the nilpotent cone in its Lie algebra $\mathfrak{g}$ can be identified with a G-equivariant isomorphism (see, for example, [35, Theorem 3.1] for an original but weaker statement and [20, Theorem 6.20] and [1, Corollary 9.3.3] for this more general one), so that the two notions of Springer fibers match exactly.

orbit in question. Subsequent studies of their singularities have often been based on these shapes and have mainly produced results for G the general linear group and $\Bbbk$ the field of complex numbers. For an example, F. Fung showed in [14] that they are all smooth in the so-called hook and two-line cases. A. Melnikov and L. Fresse gave a necessary and sufficient condition for this global smoothness in [12], while they gave a criterion for individual smoothness in [11, 12], under the additional assumption of being in the *two column case*. This is the first case where singularities appear. It also implies that the order of nilpotency of *ad e* is less than or equal to 3, which is a condition ensuring, after the work of Panyushev [30], that the number of Z-orbits is finite (in fact, this implication is established for $\mathrm{char}(\Bbbk) = 0$, but for our types A, B, D, it is still valid for the other characteristics; see Propositions 2.3 and 2.9).

2. — The *two-column case* is assumed in the article [31] by N. Perrin and E. Smirnov. For type A and $\mathrm{char}(\Bbbk) \neq 2$, they present rational resolutions of the components and show that they are normal and Cohen–Macaulay. They also give arguments for the same results in type D, but there is a gap in their proof of normality and the Cohen–Macaulay property, due to the nonalgebraicity of a certain map (see Appendix B for details and a counterexample). Nevertheless, their proof of the existence of a rational birational morphism onto the component is still valid for this type. Our work is mainly inspired by the latter, generalizing it in several directions. Retaining the assumption of the two-column case, we also prove normality and rationality, but for the much broader class of Z-orbit closures. For example, if $G = Gl_{n\Bbbk}$ is the general linear group, and r represents the rank of e considered as a nilpotent matrix of order two, then we can deduce from our Proposition 2.3, and the hook-length formula that the number of Z-orbits is $(n-r+1)(n-r)\dots(n-2r+2)$ times the number of irreducible components. In addition, we consider the three types A, B, D and we also deal with the case $\mathrm{char}(\Bbbk) = 2$ (with precautions regarding the nilpotent orbit considered; see below).

3. — Let us now state our main results. We assume that $\Bbbk$ is of arbitrary characteristic and we fix an integer n. Let $O_{n\Bbbk}$ be the group over $\Bbbk$ whose closed points are the invertible $n \times n$ matrices preserving the quadratic form

$$(1) \qquad \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} Y_k Y_{n-k+1}.$$

For even n, let us denote by $\triangle_n$ the Dickson invariant, as defined, for example, in [24, IV, §5]. This is the regular function on $O_{n\Bbbk}$ satisfying $det_n = 1 + 2\triangle_n$, where det_n is the restriction of the determinant to $O_{n\Bbbk}$ (see Section 1.3 for details). We then define the special orthogonal group $SO_{n\Bbbk}$ as the zero locus of $\triangle_n$ if n is even and as that of $det_n - 1$ if n is odd. Without these precautions, note that it fails to be connected and semi-simple of type B_n (odd n) or D_n (even n) in the case $\mathrm{char}(\Bbbk) = 2$ (see, for example, [18], [8, Appendix C] and Section 1.4).

Assume now that G is the general linear group $Gl_{n\Bbbk}$ or the special orthogonal group $SO_{n\Bbbk}$. We also assume that the nilpotent element e is identified with a nilpotent matrix of order two, which means that we are in the two-column case. If $\mathrm{char}(\Bbbk) = 2$ and $G = SO_{n\Bbbk}$, we make the additional assumption that the image of e is totally isotropic.

Recall that a proper morphism $f : X \to Y$ of locally noetherian schemes is called *rational* if $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$ and $R^i f_*\mathcal{O}_X = 0$ for $i > 0$. When the schemes are irreducible varieties with X smooth, such a rational morphism f is said to be a *rational resolution* if it is also birational with $R^i f_*\omega_X = 0$ for $i > 0$, where ω_X denotes the canonical bundle of X. If $\mathrm{char}(\Bbbk) = 0$, two rational resolutions can be dominated by a third, so being the target of a rational resolution leads to the intrinsic notion of *having rational singularities.* We prove the following.

Theorem 0.1. — *The Z-orbit closures in the flag variety of G are normal. In characteristic zero, they are Cohen–Macaulay with rational singularities. In any other characteristic, they remain Cohen–Macaulay in type A.*

This theorem is based on two results. The first is the construction of an explicit birational morphism using matrix models and involving Schubert varieties. It ensures the existence of a Borel subgroup B of G, containing a maximal torus T, and of a closed reductive subgroup H of G equipped with a retraction $\varpi : Z \to H$, having $B_H := B \cap H$ as a Borel subgroup and $T_H := T \cap H$ as the maximal torus, so that we have the following.

Theorem 0.2. — *For any Z-orbit closure Y in G/B, there exists w in the Weyl group of G such that $Y = \overline{HB \cdot wB} = \overline{Z \cdot wB}$ and*

$$H \times^{B_H} \overline{B \cdot wB} \to Y,\ [h, gB] \mapsto hgB \tag{2}$$

is rational, birational, Z-equivariant, with a Z-action on $H \times^{B_H} \overline{B \cdot wB}$ defined by $z \cdot [h, gB] = [\varpi(z)h, h^{-1}\varpi(z)^{-1}zhgB]$.

The second result is valid in a more general context, where we assume only that G is a connected reductive group over $\Bbbk$, and H a closed connected reductive subgroup of G. We make the same assumptions as before about T, B, T_H, B_H and fix any w in the Weyl group of G. We denote by ρ_G the half-sum of positive roots and, for any dominant character λ, by $V_G(\lambda)$ the dual Weyl G-module with lowest weight $-\lambda$. Let ρ_H and $V_H(\lambda)$ also be the corresponding objects for H. We refer to Section 3 for details of the notation and a stronger result that also deals with the vanishing of the canonical bundle.

Theorem 0.3. — *Let us assume the following.*

(i) *The morphism* $\pi\colon H \times^{B_H} \overline{B \cdot wB} \to \overline{HB \cdot wB}, [h, gB] \mapsto hgB$ *is birational.*
(ii) *The character* $2\rho_H - \rho_{G|T_H}$ *is dominant.*
(iii) $\mathrm{char}(\Bbbk) = 0$ *or*
(iii)′ $\mathrm{char}(\Bbbk) = p > 0$ *and the restriction* $V_G((p-1)\rho_G) \to V_H((p-1)\rho_{G|T_H})$ *is surjective.*

Then $\overline{HB \cdot wB}$ *is normal and* π *is rational.*

Remark 0.4. — It remains an open question whether the Cohen–Macaulay property is valid for types B and D and whether this property, rationality and normality are valid for type C and the exceptional types.

Remark 0.5. — If we take $H = T$ in Theorem 0.3, we find the well-known result on the normality of Schubert varieties. In fact, in this case, the sequence of arguments used in the proof coincides with that of M. Brion and S. Kumar in [4].

Remark 0.6. — In types B and D, if $\mathrm{char}(\Bbbk) \neq 2$, the assumption of being in the two-column case implies that the image of e is totally isotropic. However, this is not the case if $\mathrm{char}(\Bbbk) = 2$, as can be seen in type D by taking $e := \begin{pmatrix} 0&1&1&0 \\ 1&0&0&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{pmatrix}$ which is a matrix of nilpotency order two as $e' := \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$. Note that the dimensions of the centralizers of e and e' differ (they are 2 and 4, respectively, see [18, Theorem 4.5]), so we can see that the rank of nilpotent elements does not suffice to characterize nilpotent orbits in this case. Our assumption about the image of e is, therefore, necessary to work in our matrix model and then apply our reasoning (which crucially depends on the dimension of Z). It also turns out to be sufficient (see Section 2.2 and the result of [18, Theorem 3.8]).

Remark 0.7. — Let us assume that $\mathrm{char}(\Bbbk) \neq 2$, that the rank of e is odd and that the Z-orbit in question has a T-fixed point (which is not superfluous, since there are orbits without such points; see Proposition 2.9). Finally, let us replace Z by its neutral component Z^0. Theorem 0.2 (with the exception of rationality) is then again valid for $G = Sp_{n\Bbbk}$ the symplectic group over $\Bbbk$ (see the matrix models in Section 1 for a concrete description of this group). However, our proof of Theorem 0.1 cannot work because of the nondominance of the character involved in Theorem 0.3 (see Remark 4.2).

4. — To prove these results, we take up many of Perrin and Smirnov's arguments in [31] while developing new techniques. Our birational morphism (2) onto a Z-orbit closure is quite analogous to Perrin and Smirnov's morphism targeting an irreducible component of the Springer fiber (see (36) in Appendix B); in particular, each morphism involves a Schubert variety. However, our approach is different. Indeed, Perrin and Smirnov present an explicit description and a direct proof of birationality, depending on the type A or D and adapted to the particular framework of the irreducible components of the Springer fiber. Moreover, the presence of a Schubert variety is a simple consequence of their construction; it is detected in their source variety as the fiber of a certain projection. For us, in contrast, the Schubert variety is a starting point for producing more general resolutions, enabling us to deal simultaneously with types A, B, C, D (see Lemma 2.1 with the additional assumptions concerning type C, as indicated in the Remark 0.7), and recovering for type A, modulo some precautions, the previous construction (see (38) in Appendix B). To do this, we use results on parabolic induction (see [6] by P.-E. Chaput, L. Fresse, T. Gobet) and symmetric subgroups (see the work of R.W. Richardson and T.A. Springer [32]). Nevertheless, this general construction remains highly technical, since it is designed to be applied to a very specific matrix model.

Based on the previous birationality statement, our proof of normality, rationality and Cohen–Macaulay property closely follows that of Perrin and Smirnov, except for one point. We use the same reference [17] of X. He and J.-F. Thomsen to establish a Frobenius splitting[2] in order to obtain the surjectivity of a certain restriction of sections (see (26) in Section 3.2), and finally the desired results using the same inductive argument (Proposition 3.2). Moreover, our calculation for proving the Cohen–Macaulay property also follows their (with slight adaptations for types B, C, D; see Proposition 3.5).

The main difference lies in the arguments for extending the consequences of the Frobenius splitting from the positive characteristic to all characteristics. One way of doing this is to *realize* the desired varieties in mixed characteristics. For Perrin, Smirnov and us, this will mean producing flat schemes of finite presentation on sufficiently large bases, which are such that the collections of their geometric fibers account for different incarnations of the starting varieties, in positive and zero characteristics. But by virtue of the equations that define them, the Springer fibers and their irreducible components can be easily realized over $\mathbb{Z}$, and Perrin and Smirnov did not need to resort to more complicated arguments.

As for us, we must realize all varieties of the form $\overline{HB \cdot wB}$. This was done for $H = T$ (Schubert varieties) by V.-B. Mehta, A. Ramanathan in [29]. But the general case requires systematically dealing with a problem of scheme-theoretic image formation under nonflat base changes. By abandoning Spec $\mathbb{Z}$ for smaller bases Spec A, where A is an integral algebra of finite type over $\mathbb{Z}$, we present a solution for scheme-theoretic image realizations under some assumptions (such

2. Note that our approach is a little simpler since we use the main theorem of [17], whereas Perrin and Smirnov, in order to obtain a more precise splitting, combine several results from this reference.

that properness and having integral geometric fibers; see Theorem 5.6). This leads to fairly general realizations of closures, in a framework of actions of $\Bbbk$-groups on proper $\Bbbk$-schemes of finite type (Corollary 5.7) including the case of our $\overline{HB \cdot wB}$ varieties (see Lemma 5.5 and its proof in Appendix A).

5. — If the number of Z-orbits is finite, another interesting fact is that the G/Z-variety is spherical (i.e., it has a finite number of B-orbits). Transposed in terms of B-orbit closures in G/Z, our study then fits into the theory of spherical varieties, as originally developed by D. Luna in [27] and then by F. Knop in [23]. In [2], M. Brion presents for such B-orbit closures a powerful criterion for normality and the Cohen–Macaulay property, called the multiplicity free criterion. Applying it to our situation remains an open and interesting problem.

6. — Our article is organized as follows. We first introduce matrix models for type A on the one hand and types B, C, D on the other (Section 1). We then prove Theorem 0.2 with the exception of the rationality hypothesis, by applying two technical lemmas on matrix models (Section 2). With Theorem 3.1 in Section 3 we prove (a strong version of) Theorem 0.3, using our realization result presented in Appendix A. Some additional checks on matrix models allow us to combine theorems 0.2 and 3.1 and conclude in Section 4 with our main result (Theorem 0.1). In Appendix B, we present some comments on the comparison with Perrin and Smirnov's paper.

7. — Throughout this article, the varieties are reduced schemes of finite type over an algebraically closed field, and the (linear) algebraic groups are affine group schemes which are varieties. In the usual way, we often only consider closed points when working on varieties, and in this context, taking the $\cap$ intersection means that we are considering the reduced structure on the scheme-theoretic intersection (fiber product). We refer to [9, Exposés XIX to XXVI] for the definitions of algebraic group notions (Borel subgroup, maximal Torus, being reductive, ...) in the context of group schemes.

Notation. — Apart from Appendix A, $\Bbbk$ will denote an algebraically closed field.

1. Matrix models

1.1. General notation. — We first introduce the following notation and conventions.

- For any group G (group scheme or algebraic group) we will say that (T, B) is a Killing pair if B is a Borel subgroup of G containing a maximal torus T. If we fix such a pair and the group is assumed to be reductive, we denote by ρ_G the sum of all fundamental weights, that is the half sum of positive roots. If $P \supset B$ is a parabolic subgroup of G, we denote by W_P its Weyl group relative to T and by W^P the system of minimum-length representatives of the quotient W/W_P.

- If there is no ambiguity, we set $\bar{\imath} := m - i + 1$. We define $\check{\sigma}$ as the permutation $i \mapsto \overline{\sigma(\bar{\imath})}$ of $\{1, \dots, m\}$. We denote by $\ell_m(\sigma)$ the number of inversions of σ, that is its length in $\mathfrak{S}_m$: $\ell_m(\sigma) := \#\{1 \le i < j \le m \mid \sigma(i) > \sigma(j)\}$. For $\varsigma \in \mathfrak{S}_q$, $m \le q$, $0 \le k \le q - m$ we say that $\sigma \in \mathfrak{S}_m$ is *the induced permutation of* $\varsigma \in \mathfrak{S}_q$ *on* $\{k+1, \dots, k+m\} \subset \{1, \dots, q\}$ if the restriction of $\varsigma \left(\begin{smallmatrix} I_k & 0 & 0 \\ 0 & \sigma^{-1} & 0 \\ 0 & 0 & I_{q-(k+m)} \end{smallmatrix}\right)$ on $\{k+1, \dots, k+m\}$ is increasing.
- For a matrix M, we denote by ${}^\delta M$ the symmetric transform of M along the antidiagonal (namely, exchanging coefficients (i,j) and $(\bar{\jmath}, \bar{\imath})$). For any ring R, we often identify the permutation group $\mathfrak{S}_n$ with its image in $Gl_n(R)$, thanks to the action on R^n given by $\sigma \cdot (v_i)_i = (v_{\sigma(i)})_i$. From this point of view, we have $\check{\sigma} = {}^\delta\sigma^{-1}$ for any permutation σ. If $m \in \mathbb{N}$ and $\epsilon = \pm 1$, we denote $I_{\epsilon,m}$ the matrix $\left(\begin{smallmatrix} I_{\lfloor (m+1)/2 \rfloor} & 0 \\ 0 & \epsilon I_{\lfloor m/2 \rfloor} \end{smallmatrix}\right)$.
- For any ring R and $x \in \operatorname{Spec} R$, we denote by $\kappa(x)$ the residue field R_x/xR_x and by $\overline{\kappa(x)}$ an algebraic closure.

Now let us fix the integers r, n with $r \le \lfloor n/2 \rfloor$ and let $\varepsilon = \pm 1$. In this section, we introduce the matrix models $\mathcal{M}(n,r)$ and $\mathcal{M}(\varepsilon, n, r)$ for type A on the one hand, and types B, C, D on the other. They consist in giving $\mathbb{Z}$-group schemes $\mathcal{G}$, $\mathcal{B}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{L}$, $\mathcal{Z}$, $\mathcal{H}$, morphisms $\Theta : \mathcal{L} \to \mathcal{L}$ and $\Pi : \mathcal{Z} \to \mathcal{H}$, elements $\mathfrak{e}$ and σ_0, and sets $\mathcal{W}$, $\mathcal{W}_{\mathcal{P}}$, $\mathcal{W}^{\mathcal{P}}$ as follows.

1.2. Type A. — We define the matrix model $\mathcal{M}(n,r)$. For any ring R, we have on R-points:

$$\mathcal{G}(R) = \mathcal{G}_n(R) := Gl_n(R),$$

$$\mathcal{T}(R) = \mathcal{T}_n(R) := \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset \mathcal{G}(R),$$

$$\mathcal{B}(R) = \mathcal{B}_n(R) := \begin{pmatrix} * & * & * \\ & \ddots & * \\ 0 & & * \end{pmatrix} \subset \mathcal{G}(R),$$

$$\mathcal{P}(R) = \mathcal{P}_{n,r}(R) := \left\{ \begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & C \end{pmatrix} \in \mathcal{G}(R) \;\middle|\; \begin{smallmatrix} A, C \in \mathcal{G}_r(R) \\ B \in \mathcal{G}_{n-2r}(R) \end{smallmatrix} \right\},$$

$$\mathcal{L}(R) = \mathcal{L}_{n,r}(R) := \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \in \mathcal{P}(R) \right\} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \;\middle|\; \begin{smallmatrix} A, C \in \mathcal{G}_r(R) \\ B \in \mathcal{G}_{n-2r}(R) \end{smallmatrix} \right\},$$

$$\mathcal{Z}(R) = \mathcal{Z}_{n,r}(R) := \left\{ \begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & A \end{pmatrix} \in \mathcal{P}(R) \right\},$$

$$\mathcal{H}(R) = \mathcal{H}_{n,r}(R) := \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} \in \mathcal{P}(R) \right\},$$

and

$$\begin{array}{rccc} \Theta(R) = \Theta_{n,r}(R) : & \mathcal{L}(R) & \to & \mathcal{L}(R) \\ & \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} & \mapsto & \begin{pmatrix} C & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{pmatrix}, \\ \Pi(R) = \Pi_{n,r}(R) : & \mathcal{Z}(R) & \to & \mathcal{H}(R) \\ & \begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & A \end{pmatrix} & \mapsto & \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix}. \end{array}$$

We also define

$$\mathfrak{e} = \mathfrak{e}_{n,r} := \begin{pmatrix} 0 & 0 & I_r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is a square matrix of size n,

$$\sigma_0 = \sigma_{0,n} := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

which is a permutation in $\mathfrak{S}_n$, and, finally, we set

$$\mathcal{W} = \mathcal{W}_n := \mathfrak{S}_n,$$

$$\mathcal{W}_{\mathcal{P}} := \left\{ \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \;\middle|\; \sigma_1,\ \sigma_3 \in \mathfrak{S}_r,\ \sigma_2 \in \mathfrak{S}_{n-2r} \right\},$$

$$\mathcal{W}^{\mathcal{P}} := \left\{ u \in \mathcal{W} \;\middle|\; \begin{array}{c} u \text{ is increasing on} \\ \{1,\dots,r\},\ \{r+1,\dots,n-r\},\ \{n-r+1,\dots,n\} \end{array} \right\}.$$

1.3. Types B, C, D. — We define the matrix model $\mathcal{M}(\varepsilon, n, r)$ for integers $\varepsilon = \pm 1$ and $n \leq 2r$. Recall that it encompasses the types B, C, D for, respectively, $\varepsilon = 1$ and odd n, $\varepsilon = -1$ and even n, $\varepsilon = 1$ and even n. For $\mathcal{G}$ we adopt the picture presented in [7] in order to get smooth matrix groups over $\mathbb{Z}$. We explain some choices in the comments below. We need to introduce, for $m \in \mathbb{N}$, the orthogonal group O_{2m} over $\mathbb{Z}$. We describe it as the closed $\mathbb{Z}$-subscheme of the affine space of $2m \times 2m$ matrices with equations

$$ {}^{\delta}XX = I_{2m} \quad \text{and} \quad \sum_{k=1}^{m} X_{k,i}X_{2m-k+1,i} = 0,\ i \in \{ 1, \dots, 2m \} $$

meaning any fiber of O_{2m} is the group of linear transformations preserving the quadratic form

$$ \sum_{k=1}^{m} Y_k Y_{2m-k+1}. \tag{3} $$

Consider the determinant det_{2m} on O_{2m} as a regular function. There exists a unique regular function $\triangle_{2m}$, called the Dickson invariant, which satisfies $det_{2m} = 1 + 2\triangle_{2m}$; in other words, $\triangle_{2m}$ takes values 0 and -1 on the different components of O_{2m} (see [7, Lemma 4.1.4]). Let also $\dagger_{2m}$ be the difference between the mth and the $(m+1)$th vector of the usual basis of the $\mathbb{Z}$-free module $\mathbb{Z}^m$. We can now describe the data of our matrix model. For any ring R, we have on R-points:

$$ \mathcal{G}(R) = \mathcal{G}_{\varepsilon,n}(R) := \begin{cases} \{ A \in O_n(R) \mid \triangle_n(A) = 0 \} & \text{if } \varepsilon = 1 \text{ and } n \text{ is even} \\ \left\{ A \in O_{n+1}(R) \;\middle|\; \begin{matrix} \triangle_{n+1}(A) = 0 \\ A\dagger_{n+1} = \dagger_{n+1} \end{matrix} \right\} & \\ & \text{if } \varepsilon = 1 \text{ and } n \text{ is odd} \\ \left\{ A \in M_n(R) \;\middle|\; I_{-1,n}{}^{\delta}AI_{-1,n}A = I_n \right\} & \\ & \text{if } \varepsilon = -1 \text{ and } n \text{ is even,} \end{cases} $$

$$ \mathcal{T}(R) = \mathcal{T}_{\varepsilon,n}(R) := \begin{cases} \left\{ \begin{pmatrix} t_1 &&&&& \\ & \ddots &&&& \\ && t_{n/2} &&& \\ &&& t_{n/2}^{-1} && \\ &&&& \ddots & \\ &&&&& t_1^{-1} \end{pmatrix} \;\middle|\; \substack{t_i \in R^* \\ \forall i \in \{ 1,\dots,n/2 \}} \right\} & \\ & \text{if } n \text{ is even} \\ \left\{ \begin{pmatrix} t_1 &&&&&&& \\ & \ddots &&&&&& \\ && t_{(n-1)/2} &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& t_{(n-1)/2}^{-1} && \\ &&&&&& \ddots & \\ &&&&&&& t_1^{-1} \end{pmatrix} \;\middle|\; \substack{t_i \in R^* \\ \forall i \in \{ 1,\dots,(n-1)/2 \}} \right\} & \\ & \text{if } n \text{ is odd,} \end{cases} $$

$$\mathcal{B}(R)=\mathcal{B}_{\varepsilon,n}(R):=\left\{\begin{pmatrix} * & * & * \\ & \ddots & * \\ 0 & & * \end{pmatrix}\in\mathcal{G}(R)\right\},$$

$$\mathcal{P}(R)=\mathcal{P}_{\varepsilon,n,r}(R):=\left\{\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & C \end{pmatrix}\in\mathcal{G}(R)\;\middle|\;\substack{A,C\in\mathcal{G}_r(R)\\ B\in\mathcal{G}_{\varepsilon,n-2r}(R)}\right\}$$

$$=\left\{\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & {}^{\delta}A^{-1} \end{pmatrix}\in\mathcal{G}(R)\;\middle|\;\substack{A\in\mathcal{G}_r(R)\\ B\in\mathcal{G}_{\varepsilon,n-2r}(R)}\right\},$$

$$\mathcal{L}(R)=\mathcal{L}_{\varepsilon,n,r}(R):=\left\{\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\in\mathcal{P}(R)\right\}$$

$$=\left\{\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & {}^{\delta}A^{-1} \end{pmatrix}\;\middle|\;\substack{A\in\mathcal{G}_r(R)\\ B\in\mathcal{G}_{\varepsilon,n-2r}(R)}\right\},$$

$$\mathcal{Z}(R)=\mathcal{Z}_{\varepsilon,n,r}(R):=\left\{\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & I_{-\varepsilon,r}AI_{-\varepsilon,r} \end{pmatrix}\in\mathcal{P}(R)\right\}$$

$$=\left\{\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & I_{-\varepsilon,r}AI_{-\varepsilon,r} \end{pmatrix}\in\mathcal{P}(R)\;\middle|\;\substack{I_{-\varepsilon,r}{}^{\delta}AI_{-\varepsilon,r}A=I_r\\ B\in\mathcal{G}_{\varepsilon,n-2r}(R)}\right\},$$

$$\mathcal{H}(R)=\mathcal{H}_{\varepsilon,n,r}(R):=\left\{\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{-\varepsilon,r}AI_{-\varepsilon,r} \end{pmatrix}\in\mathcal{P}(R)\;\middle|\;I_{-\varepsilon,r}{}^{\delta}AI_{-\varepsilon,r}A=I_r\right\},$$

and

$$\begin{aligned}\Theta(R)=\Theta_{\varepsilon,n,r}(R):\quad &\mathcal{L}(R)\to\mathcal{L}(R)\\ &\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\mapsto\begin{pmatrix} I_{-\varepsilon,r}CI_{-\varepsilon,r} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_{-\varepsilon,r}AI_{-\varepsilon,r} \end{pmatrix},\\ \Pi(R)=\Pi_{\varepsilon,n,r}(R):\quad &\mathcal{Z}(R)\to\mathcal{H}(R)\\ &\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & I_{-\varepsilon,r}AI_{-\varepsilon,r} \end{pmatrix}\mapsto\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{-\varepsilon,r}AI_{-\varepsilon,r} \end{pmatrix}.\end{aligned}$$

We also define

$$\mathfrak{e}=\mathfrak{e}_{\varepsilon,n,r}:=\begin{pmatrix} 0 & 0 & I_{-\varepsilon,r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is a square matrix of size n for even n, of size $n+1$ for odd n,

$$\sigma_0 = \sigma_{0,\varepsilon,n} := \begin{cases} \begin{pmatrix} 0 & & & 1 \\ & & \begin{smallmatrix} & \cdot^{\cdot^{\cdot}} \end{smallmatrix} & \\ & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & & \\ \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & 0 \end{pmatrix} & \text{if } \varepsilon = 1,\ n \text{ is even and } n/2 \text{ is odd} \\ \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} & \text{else} \end{cases}$$

which is a permutation in $\mathfrak{S}_n$, and finally we set

$$\mathcal{W} = \mathcal{W}_{\varepsilon,n} := \left\{ \sigma \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \check{\sigma} = \sigma \text{ and, if } n \text{ is even and } \varepsilon = 1, \text{ then} \\ \#\{\, 1 \leq i \leq n/2 \mid \sigma(i) > n/2 \,\} \text{ is even} \end{array} \right\},$$

$$\mathcal{W}_{\mathcal{P}} := \left\{ \begin{pmatrix} \sigma & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & \check{\sigma} \end{pmatrix} \;\middle|\; \begin{array}{c} \sigma \in \mathfrak{S}_r,\ v \in \mathfrak{S}_{n-2r}, \\ \check{v} = v \text{ and, if } n \text{ is even and } \varepsilon = 1, \text{ then} \\ \#\{\, 1 \leq i \leq (n-2r)/2 \mid v(i) > (n-2r)/2 \,\} \text{ is even} \end{array} \right\},$$

$$\mathcal{W}^{\mathcal{P}} := \left\{ u \in \mathcal{W} \;\middle|\; \begin{array}{c} u \text{ is increasing on } \{\, 1, \dots, r \,\}, \{\, r+1, \dots, \lfloor n/2 \rfloor \,\} \\ \text{and } u(\lfloor n/2 \rfloor) < u(\lfloor n/2 \rfloor + 1 + \frac{\varepsilon + (-1)^n}{2}) \end{array} \right\}.$$

1.4. Comments. — Let us make some observations about the previous data. We prove the assumptions on flatness and smoothness below. The assumptions concerning the reductiveness, semi-simpleness, the types and the Killing pairs[3] then follow from the isomorphism theorem ([9, Théorème 1.1 Exposé XXV]) and the study of the geometric fibers of the involved groups.

General. —

- The couple $(\mathcal{T}, \mathcal{B})$ is a Killing pair for $\mathcal{G}$.
- The nilpotent matrix $\mathfrak{e}$ can be identified with a section of the Lie algebra of $\mathcal{G}$ over $\mathbb{Z}$ and with a closed element of its Lie algebra over any geometric fiber. It is of order 2 and rank r.
- We have $\mathcal{Z} = Z_{\mathcal{G}}(\mathfrak{e})$ the centralizer of $\mathfrak{e}$ in $\mathcal{G}$ for the adjoint action. Besides $\mathcal{L} \cap \mathcal{Z} = \mathcal{L}^{\Theta}$ and $\mathcal{Z} = \mathcal{L}^{\Theta} U_{\mathcal{P}}$ where $U_{\mathcal{P}}$ denotes the unipotent radical of $\mathcal{P}$.
- The couple $(\mathcal{T} \times_{\mathcal{G}} \mathcal{H}, \mathcal{B} \times_{\mathcal{G}} \mathcal{H})$ is a Killing pair for $\mathcal{H}$.
- The subgroup $\mathcal{P}$ of $\mathcal{G}$ is parabolic and $\mathcal{L}$ is its Levi subgroup containing $\mathcal{T}$.
- The Weyl group of $\mathcal{G}$ (respectively $\mathcal{P}$) related to $\mathcal{T}$ is naturally identified with $\mathcal{W}$ (respectively $\mathcal{W}_{\mathcal{P}}$). The set of minimum-length representatives of the quotient $\mathcal{W}/\mathcal{W}_{\mathcal{P}}$ then identifies with $\mathcal{W}^{\mathcal{P}}$. Besides, σ_0 is the longest element in $\mathcal{W}$.

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3. Note that at the level of schemes, all these notions demand smoothness, according to our setting, which follows [9].

- The isomorphism Θ is an involution of $\mathcal{L}$, which stabilizes $\mathcal{T}$ and $\mathcal{B}$ and Π is a retraction from $\mathcal{Z}$ to $\mathcal{H}$.

Type A. —

- The group $\mathcal{G}$ is reductive (not semi-simple) of type A_n.
- The group $\mathcal{H}$ is reductive (not semi-simple) of type A_r.
- On $W_{\mathcal{P}}$, Θ induces the bijection

$$\begin{pmatrix} \sigma & & \\ & v & \\ & & \sigma' \end{pmatrix} \mapsto \begin{pmatrix} \sigma' & & \\ & v & \\ & & \sigma \end{pmatrix}.$$

Types B, C, D. —

- The group $\mathcal{G}$ is semi-simple of type B_n, D_n in the cases $\varepsilon = 1$ and odd n, $\varepsilon = 1$ and even n respectively. The base change $\mathcal{G}_{\mathbb{Z}_{(2)}}$ is semi-simple of type C_n in the case $\varepsilon = -1$ and even n. In any case, the geometric fiber $\mathcal{G}_{\overline{x}}$ over any point $x \neq (2) \in \operatorname{Spec} \mathbb{Z}$ is isomorphic to the group whose closed points consist in $\left\{ A \in Sl_n(\overline{\kappa(x)}) \;\middle|\; I_{\varepsilon,n}{}^{\delta} A I_{\varepsilon,n} = A^{-1} \right\}$.
- We deduce that in the case $\varepsilon = 1$, for any algebraically closed field K, the base change $\mathcal{G}_K$ is also isomorphic to the special orthogonal group presented in the introduction. Then $\mathfrak{e}$ identifies with a $n \times n$ matrix whose image is totally isotropic.
- In the case of $\varepsilon = 1$, r is necessarily even. In the case of $\varepsilon = -1$, if r is odd then we have an isomorphism $Z^0 \rtimes \{ \pm 1 \} \simeq Z$ for any geometric fiber $Z := \mathcal{Z}_{\overline{x}}$ over x.
- In the case of $\varepsilon = 1$, $\mathcal{H}$ is semi-simple of type C_r. In the case of $\varepsilon = -1$, it is not even flat, and its geometric fibers are not connected, but the base change $\mathcal{H}_{\mathbb{Z}_{(2)}}$ is smooth and the neutral components of its geometric fibers are semi-simple of type B_r.
- On $W_{\mathcal{P}}$, Θ induces the bijection

$$\begin{pmatrix} \sigma & & \\ & v & \\ & & \check{\sigma} \end{pmatrix} \mapsto \begin{pmatrix} \check{\sigma} & & \\ & v & \\ & & \sigma \end{pmatrix}.$$

We outline the following propositions.

Proposition 1.1. — *In the models $\mathcal{M}(n,r)$ and $\mathcal{M}(1,n,r)$ (types A, B and D), $\mathcal{Z}$ and $\mathcal{L}^{\Theta}$ have geometric connected fibers. In $\mathcal{M}(-1,n,r)$ (type C) the fibers over $s \neq (2) \in \operatorname{Spec} \mathbb{Z}$ have two connected components. However, the unipotent part $(\mathcal{L}^{\Theta}_{\overline{s}})_u$ is contained in the neutral component $(\mathcal{L}^{\Theta}_{\overline{s}})^0$.*

Proof. — It is clear for types A, B, D. For type C, and any $s \neq (2) \in \operatorname{Spec} \mathbb{Z}$, we remark that

$$\mathcal{L}^{\Theta}_{\overline{s}} \simeq O_{r\overline{s}} \times Sp_{n-2r\overline{s}}$$

and

$$(\mathcal{L}_{\overline{s}}^{\Theta})^0 \simeq SO_{r\overline{s}} \times Sp_{n-2r\overline{s}}.$$

This implies

$$\mathcal{L}_{\overline{s}}^{\Theta} = (\mathcal{L}_{\overline{s}}^{\Theta})^0 \cup \gamma(\mathcal{L}_{\overline{s}}^{\Theta})^0,$$

for a suitable $\gamma \in \mathcal{L}_{\overline{s}}^{\Theta}$ with $det\ \gamma = -1$. Similar reasoning gives the same result for $\mathcal{Z}_{\overline{s}}$. We thus recognize two connected components. Since any unipotent matrix has its determinant equal to 1, the above isomorphisms also yield the desired inclusion. □

Proposition 1.2. — *In the models $\mathcal{M}(n,r)$ and $\mathcal{M}(1,n,r)$ (types A, B, D), $\mathcal{Z}$, $\mathcal{L}^{\Theta}$ and $\mathcal{Z} \times_{\mathcal{G}} {}^w\mathcal{B}$ for any $w \in \mathcal{W}$ are smooth. In $\mathcal{M}(-1,n,r)$ (type C), they are not even flat, but their base changes over $\mathbb{Z}_{(2)}$ are all smooth.*

Proof. — We establish the claimed smoothness and flatness in the following way. They follow from the same general argument, which also justifies the quite complicated picture of our groups in types B and D. It is based on the following lemma.

Lemma 1.3. — *Let S be a locally noetherian irreducible scheme and G be a S-group scheme of finite type. Let η denote the generic point of S. We assume that $G_{\overline{\eta}}$ is smooth. Then, for any $s \in S$ such that $\dim Lie(G_{\overline{s}}) \leq \dim Lie(G_{\overline{\eta}})$, $G_{\overline{s}}$ is smooth.*

As a consequence, if the dimensions of the Lie algebras of the geometric fibers are the same and if G is flat over S, then G is smooth over S.

Proof of Lemma 1.3. — Replacing G by its neutral component, we can assume that G is irreducible. Let $s \in S$. Applying [16, Lemma 13.1.1] to the dominant finite type morphism of irreducible schemes $G \to S$, we have the inequality of fiber dimensions

$$\dim\ G_{\overline{\eta}} \leq \dim\ G_{\overline{s}}.$$

If we assume the smoothness of the geometric generic fiber and the inequality between the dimensions of Lie algebras, we thus have

$$\dim\ Lie(G_{\overline{s}}) \leq \dim\ Lie(G_{\overline{\eta}}) = \dim\ G_{\overline{\eta}} \leq \dim\ G_{\overline{s}} \leq \dim\ Lie(G_{\overline{s}}),$$

which causes the smoothness of $G_{\overline{s}}$. The smoothness of G over S then follows from the smoothness fiber criterion for flat finite presentation morphisms. □

Let us now state our general argument. Let us consider the equations that define the desired group as a closed subscheme of the affine space of matrix $n \times n$. They are all linear or quadratic, with the exception of the one concerning the vanishing of the Dickson invariant. Some of the quadratic ones are given by

$$X^{\delta} X = I_{2m} \tag{4}$$

for some integers m and indeterminate $2m \times 2m$ matrices X. They lead to 2-torsion with the equations depending on $i \in \{1, \dots, 2m\}$

$$\sum_{k=1}^{2m} X_{ki} X_{2m-k+1,i} = \delta_{i,2m-i+1},$$

namely

$$2 \sum_{k=1}^{m} X_{ki} X_{2m-k+1,i} = \delta_{i,2m-i+1}.$$

This can be compensated by refining them with

$$\sum_{k=1}^{m} X_{ki} X_{2m-k+1,i} = 0. \tag{5}$$

This problem does not occur for the other quadratic equations arising from

$$I_{-1,m}{}^{\delta} X I_{-1,m} X = I_m. \tag{6}$$

Note that the Dickson invariant does not imply torsion, whereas its existence yields one for the equation $det_{2m} = 1$. Hence, the group at stake is flat or nonflat over $\mathbb{Z}$ depending on whether (4) or (6) is involved and potentially compensated. In any case, it is flat over $\mathbb{Z}_{(2)}$. On the other hand, it is finitely presented, and we can check that the Lie algebra of any of its geometric fibers is of constant dimension. Since the generic geometric fiber is smooth as an algebraic group in characteristic zero, we deduce the smoothness of all geometric fibers by the Lemma 1.3. Let us emphasize that the constancy of the Lie algebra dimension follows from the parity of $2m$. Indeed, it prevents the presence of squared terms in (4) and (5), so that the differentials do not bring 2-torsion and, therefore, a leap of dimension. This explains why we describe $\mathcal{G}_{1,n}$ into $\mathcal{G}_{1,n+1}$ for odd n. □

2. Birationality in types A, B, C and D

We start with proving Theorem 0.2 except for the claim on rationality. Our conclusions include a partial version of the result for type C; see Section 2.2.3.

2.1. Two lemmas. — We will use the two basic and technical lemmas below.

Lemma 2.1. — *Let G be a connected reductive algebraic group over $\Bbbk$ and (T, B) be a Killing pair. Let $H \subset Z \subset G$ be connected subgroups equipped with a retraction $\varpi : Z \to H$ and such that $B_H := B \cap H$ is a Borel subgroup of H. Let w be in W, the Weyl group of G. We assume the following.*

1. $\dim Z/\left(Z \cap {}^{w}B\right) = \ell(w) + \dim H/B_H$.
2. $z^{-1}\varpi(z) \in \overline{{}^{w}BB}$ *for each* $z \in Z$.
3. $Z \cap {}^{w}B \subset \varpi^{-1}(B_H)(Z \cap {}^{w}B)^0$.
4. *The scheme-theoretic intersection* $Z \times_G {}^{w}B$ *is reduced.*

Then Z acts on $H \times^{B_H} \overline{B \cdot wB}$ by $z \cdot [h, gB] = [\varpi(z)h, h^{-1}\varpi(z)^{-1}zhgB]$, and the map

$$\begin{aligned} \pi : \ H \times^{B_H} \overline{B \cdot wB} &\to \overline{Z \cdot wB} \\ [h, gB] &\mapsto hgB. \end{aligned} \tag{7}$$

is birational and Z-equivariant.

Lemma 2.2. — *Let G, B, T, W, H, Z, B_H, ϖ be as in the previous lemma. Let $P \supset B$ be a parabolic subgroup with U_P its unipotent radical and L its Levi subgroup containing T. Let θ be an involution of L, which stabilizes $B \cap L$ and T. We assume that Z is the subgroup $(L^\theta)^0 U_P \subset P$ and we fix $w \in W$.*

(i) *If the subvariety of unipotents L^θ_u of L^θ is contained in L_Z, then*
$$Z \cap {}^wB \subset \varpi^{-1}(B_H)(Z \cap {}^wB)^0.$$
(ii) *If $\mathrm{char}(\Bbbk) \neq 2$ and if $w = \tau v$ is the decomposition of w in $W_P(W^P)^{-1}$, then*
$$\dim Z/Z \cap {}^wB = \ell(w) + \dim Z/Z \cap B + \ell(\tau^{-1}\theta(\tau))/2 - \ell(\tau).$$

2.1.1. *Proof of Lemma 2.1.* — Let X be $\overline{Z \cdot wB}$ and $\hat{X}$ be the subvariety $(\iota \times id)^{-1}(\overline{G \cdot (eB, wB)})$ of $H/B_H \times G/B$, where ι is the immersion $H/B_H \hookrightarrow G/B$. We have the isomorphism

$$\hat{X} \simeq H \times^{B_H} \overline{B \cdot wB} \tag{8}$$

over H/B_H as H-equivariant bundles and over G/B thanks to π and the second projection $pr_2 : H/B_H \times G/B \to G/B$. Considering this isomorphism, it suffices to show that pr_2 induces a well-defined birational morphism $\hat{X} \to X$ and, transporting the action, to ensure that $\hat{X}$ is stable for the Z action on $H/B_H \times G/B$ given by $z \cdot (hB_H, gB) = (\varpi(z)hB_H, zgB)$. We will proceed in several steps.

1. With the hypothesis 1, we have $\dim \hat{X} = \ell(w) + \dim H/B_H = \dim Z \cdot wB = \dim X$.
2. If f denotes the morphism $g \mapsto (gB, wB)$, we have for all $b' \in {}^wB$ and $b \in B$, $f(b'b) = (b'bB, wB) = (b'B, b'wB) = b' \cdot (B, wB)$. Thus $f({}^wBB) \subset \overline{G \cdot (eB, wB)}$ and $f(\overline{{}^wBB}) \subset \overline{G \cdot (eB, wB)}$. Thanks to the hypothesis 2, we have then for all $z \in Z$
$$\begin{aligned} \iota \times id\ (z \cdot (eB_H, wB)) &= \iota \times id\ (\varpi(z)B_H, zwB) \\ &= z \cdot f(z^{-1}\varpi(z)) \in z \cdot f(\overline{{}^wBB}) \subset \overline{G \cdot (eB, wB)}. \end{aligned}$$
We deduce that $Z \cdot (eB_H, wB) \subset \hat{X}$.
3. The map pr_2 induces a surjective Z-equivariant morphism $Z \cdot (eB_H, wB) \to Z \cdot wB$. The previous points then give $\dim Z \cdot wB \leq \dim Z \cdot (eB_H, wB) \leq \dim \hat{X} = \dim Z \cdot wB$ so $\dim Z \cdot wB = \dim Z \cdot (eB_H, wB)$. This also

implies $\dim Z \cap {}^wB = \dim Z \cap {}^wB \cap \varpi^{-1}(B_H)$, so that $(Z \cap {}^wB)^0 = \big(Z \cap {}^wB \cap \varpi^{-1}(B_H)\big)^0$ and $(Z \cap {}^wB)^0 \subset \varpi^{-1}(B_H)$. With hypothesis 3, we deduce $\varpi(Z \cap {}^wB) \subset B_H$. Therefore,

$$\begin{aligned} pr_2^{-1}(wB) \cap Z \cdot (eB_H, wB) &= \{\, (\varpi(z)B_H, zwB) \mid \ z \in Z, zwB = wB \,\} \\ &= \{\, (\varpi(z)B_H, zwB) \mid \ z \in Z \cap {}^wB \,\} \\ &= \varpi(Z \cap {}^wB) \cdot B_H \times \{wB\} \\ &= \{(eB_H, wB)\}, \end{aligned}$$

and $Z \cdot (eB_H, wB) \to Z \cdot wB$ is bijective. But it is also separable thanks to hypothesis 4. It is finally an isomorphism by Zariski's main theorem.

4. The previous points imply, in particular, that $\dim \hat{X} = \dim Z \cdot (eB_H, wB)$, and then $\overline{Z \cdot (eB_H, wB)} = \hat{X}$. Hence, Z preserves $\hat{X}$ as desired.
5. To conclude, note that we have $Z \cdot wB \subset pr_2(\hat{X})$. Since H/B_H is complete, $pr_2(\hat{X})$ is closed in G/B, and we deduce $X \subset pr_2(\hat{X})$. Irreducibility and the dimension formula then give $\dim pr_2(\hat{X}) \leq \dim \hat{X} = \dim X$. Therefore, $\hat{X} \to X$ is well defined and surjective.

We have all the desired statements, with an isomorphism between the dense open orbits $Z \cdot (eB_H, wB)$ and $Z \cdot wB$.

2.1.2. *Proof of Lemma 2.2.* — Let Φ_L be the set of roots of L and put $L_Z := (L^\theta)^0$, $B_L := B \cap L$ and $T_Z := (T \cap Z)^0 = (T \cap L_Z)^0 = T^{\theta,0}$. The involution θ and the elements of W_P act linearly on the vector space $\mathbb{R} \otimes_{\mathbb{Z}} \Phi_L$ and let us denote these respective actions by $\star$ and $\diamond$. Note that B_L is a θ-stable Borel subgroup of L. Besides, since $T \subset B_L$ is a θ-stable maximal torus of L, T_Z is a regular subtorus of L (see, for example, [3, Lemma 4][4]), and it follows that it is a maximal torus of L_Z and of Z.

1. Let us first prove (i). Since $T \subset L$, there exists a cocharacter $\lambda : \mathbb{G}_m \to T$ such that $U_P = \big\{\, x \in G \;\big|\; \lim_{a\to 0} \lambda(a)x\lambda(a)^{-1} = 1 \,\big\}$ and $L = Z_G(\operatorname{Im}\lambda)$. Hence, if $z = lv \in Z \cap {}^wB$ with $l \in L_Z$ and $v \in U_P$ then $\lambda(a)z\lambda(a)^{-1} \in {}^wB$ for all $a \in \mathbb{G}_m$ and $\lambda(a)z\lambda(a)^{-1} = l\lambda(a)v\lambda(a)^{-1} \to l$ when $a \to 0$. We, therefore, have $l, v \in {}^wB$ and $z \in (L_Z \cap {}^wB)({}^wB \cap U_P)$ so that

$$Z \cap {}^wB \subset (L_Z \cap {}^wB)(Z \cap {}^wB)_u. \tag{9}$$

But we have

$$L_Z \cap {}^wB \subset (T \cap Z)(Z \cap {}^wB)_u. \tag{10}$$

To see this pick $x \in L_Z \cap {}^wB$ and let $x = tv$ be its decomposition with $t \in T$ and $v \in ({}^wB)_u$ in the connected solvable group ${}^wB$. We have $v = t^{-1}x \in TL_Z \subset L$, and we can apply θ on v. Since $x \in L_Z$

4. This is also valid for characteristic two.

we have $tv = \theta(t)\theta(v)$ and $\theta(v) = \theta(t)^{-1}tv \in {}^wB$ because θ stabilizes $T \subset {}^wB$. Hence the unipotent element $\theta(v)$ is in the group $({}^wB)_u$ and $\theta(t)^{-1}t = \theta(v)v^{-1}$ is also unipotent. It is thus the unit element that implies $t, v \in L^\theta$. By hypothesis $L_u^\theta \subset L_Z$ so that v, and then t, are in L_Z. Therefore, $x \in (T \cap L_Z)(({}^wB)_u \cap L_Z) \subset (T \cap Z)(Z \cap {}^wB)_u$. We also have

$$T \cap Z \subset \varpi^{-1}(B_H). \tag{11}$$

Indeed, since $\varpi : Z \to H$ is a retraction, we have $B_H = B \cap H \subset B \cap Z$ and $B_H = \varpi(B_H) \subset \varpi(B \cap Z)$, which is an equality because $\varpi(B \cap Z)$ is solvable. We also have

$$(Z \cap {}^wB)_u \subset (Z \cap {}^wB)^0. \tag{12}$$

Indeed, the unipotent subgroup $(Z \cap {}^wB)_u$ of the connected group Z is contained in a Borel subgroup $\widetilde{B}_Z$ of Z (see, for example, [19, Theorem 30.4]). Since it is stable under the action of the maximal torus T_Z, we can assume that $T_Z \subset \widetilde{B}_Z$ by the Borel fixed point theorem. As for P and U_P, the unipotent radical $(\widetilde{B}_Z)_u$ is described by $\{\, x \in Z \mid \lim_{a\to 0} \mu(a)x\mu(a)^{-1} = 1 \,\}$, where μ is a suitable cocharacter $\mathbb{G}_m \to T_Z$. Therefore, conjugating by $\mu(a)$ and taking the limit when $a \to 0$, any $x \in (Z \cap {}^wB)_u$ can be contracted, inside $Z \cap {}^wB$, to the unit element, and thus, belongs to $(Z \cap {}^wB)^0$.

Combining (9), (10), (11) and (12) we obtain the desired inclusion.

2. Let us now prove (ii) under the assumption $\mathrm{char}(\Bbbk) \neq 2$. On the one hand, we can use a result of Richardson and Springer ([33, Proposition 3.3.4] reformulating [32, Theorem 4.6]) on the involution fixed-points subgroup L^θ of L and we get (replacing harmlessly L^θ with L_Z):

$$\begin{aligned}\dim L_Z/L_Z \cap {}^\tau B = {} & \dim L_Z/L_Z \cap B \\ & + \left(\ell\left(\tau^{-1}\theta(\tau)\right) + \dim E_-\left(\tau^{-1}\theta(\tau)\right) - \dim E_-(1)\right)/2,\end{aligned}$$

where $E_-(\sigma) := \{\, v \in \mathbb{R} \otimes_{\mathbb{Z}} \Phi_L \mid \sigma \diamond (\theta \star v) = -v \,\}$ for any $\sigma \in W_P$. But for all $v \in \mathbb{R} \otimes_{\mathbb{Z}} \Phi_L$, we have $\theta(\tau) \diamond (\theta \star v) = \theta \star (\tau \diamond v)$ whence the equivalences

$$\begin{aligned} & (\tau^{-1}\theta(\tau)) \diamond (\theta \star v) = -v \\ \Leftrightarrow\ & \tau \diamond \tau^{-1} \diamond \theta(\tau) \diamond (\theta \star v) = -\tau \diamond v \\ \Leftrightarrow\ & \theta(\tau) \diamond (\theta \star v) = -\tau \diamond v \\ \Leftrightarrow\ & \theta \star (\tau \diamond v) = -\tau \diamond v \end{aligned}$$

which imply $E_-\left(\tau^{-1}\theta(\tau)\right) = \tau^{-1} \diamond E_-(1)$. Besides, since $U_P \subset B$, we have $L_Z/L_Z \cap B \simeq Z/Z \cap B$. We thus have

$$\dim L_Z/L_Z \cap {}^\tau B = \dim Z/Z \cap B + \ell\left(\tau^{-1}\theta(\tau)\right)/2. \tag{13}$$

In addition, since $\mathrm{char}(\Bbbk) \neq 2$, the fixed point subgroup L^θ has a finite number of orbits in the flag variety L/B_L (see [36, §4]). Thus L_Z is spherical in L. We can then apply a result[5] of Chaput, Fresse and Gobet ([6, Theorem 7.2 (c)]) for the subgroup Z constructed by parabolic induction $Z = L_Z U_P$. We get, for $\tau \in W_P$ and $\upsilon \in W^P$:

$$\dim Z/Z \cap {}^{\tau\upsilon}B = \ell(\upsilon) + \dim L_Z/L_Z \cap {}^{\tau}B.$$

But with the decomposition $w = \tau\upsilon$, we have $\ell(\upsilon) = \ell(w) - \ell(\tau)$ and, therefore,

$$\dim Z/Z \cap {}^{w}B = \ell(w) - \ell(\tau) + \dim L_Z/L_Z \cap {}^{\tau}B. \tag{14}$$

Combining (13) and (14) we obtain the desired formula.

2.2. Application to matrix models. — Let us now fix the framework presented in the Introduction for G, e and Z relative to types A, B, C, D. Recall that we assume that the image of e is totally isotropic for types B and D (which is automatically satisfied if *char* $k \neq 2$) and that we only consider $\mathrm{char}(\Bbbk) \neq 2$ for type C. Let r be the rank of e. From the comments in Section 1, we can, therefore, assume that G is the base change $\mathcal{G}_\Bbbk$ of $\mathcal{G}$ described in the matrix model $\mathcal{M}(n, r)$ or $\mathcal{M}(\varepsilon, n, r)$.

In addition, we know for $Gl_{n\Bbbk}$ that Young diagrams parametrize nilpotent orbits and that those related to elements of order two are characterized solely by the rank of these elements (two-column case). For $\mathrm{char}(\Bbbk) \neq 2$, this is also the case for the $O_{n\Bbbk}$ and $Sp_{n\Bbbk}$ groups (see, for example, [22, §1.6]). For $\mathrm{char}(\Bbbk) = 2$ and types B, D, if we want to characterize the orbit of a nilpotent element N, then we must add to the rank the datum of the sequence $(\chi_m^N)_{m \geq 1}$ whose m-th term is

$$\chi_m^N = \min\{\, l \geq 0 \mid N^l\,(\mathrm{Ker}\, N^m)\ \text{ is totally isotropic}\,\}$$

(see [18, Theorem 3.8]). But such a sequence is totally determined by the rank s of N, as soon as N is of order two with a totally isotropic image. Indeed, in this situation, we first obviously have $\chi_1^N \leq 1$ and $\chi_m^N = 1$ for all $m \geq 2$. Moreover, we have $2s \leq n$ since $\mathrm{Im}\, N \subset \mathrm{Ker}\, N$. If $2s = n$, the rank theorem implies that $\mathrm{Ker}\, N = \mathrm{Im}\, N$ so that $\mathrm{Ker}\, N$ is totally isotropic and $\chi_1^N = 0$. If $2s < n$, the dimension of $\mathrm{Ker}\, N$ exceeds $n/2$ so that it cannot be a totally isotropic subspace, and we deduce $\chi_1^N \geq 1$.

Thanks to an inner conjugation (for types A, C) or (potentially) an outer conjugation (for types B, D), we can, therefore, assume that e and then Z, respectively, coincide with $\mathfrak{e}$ and the base change $\mathcal{Z}_\Bbbk$ of the appropriate matrix model. We supplement this data with B, T, P, L, H, B_H, T_H, θ, ϖ and W, W_P, W^P, which stand for the various base changes $\mathcal{B}_\Bbbk$, $\mathcal{T}_\Bbbk$, $\mathcal{P}_\Bbbk$, $\mathcal{L}_\Bbbk$, $(\mathcal{H}_\Bbbk)^0$,

5. The article is written for characteristic zero but here we only need the second part, which is valid for any characteristic.

$(\mathcal{B}\times_{\mathcal{G}}\mathcal{H})_{\Bbbk}$, $(\mathcal{T}\times_{\mathcal{G}}\mathcal{H})_{\Bbbk}$, $\Theta_{\Bbbk}$, $\Pi_{\Bbbk}$ and the sets $\mathcal{W}$, $\mathcal{W}_{\mathcal{P}}$, $\mathcal{W}^{\mathcal{P}}$. We also set $L_Z := (L^{\theta})^0$, $B_L := B\cap L$. We keep the notation σ_0 for the longest element in W and denote by u_0 the permutation $\sigma_{0,n-2r}$ (type A), or $\sigma_{0,\varepsilon,n-2r}$ (types B, C, D).

For any integer m, and $\epsilon = \pm 1$, we will also denote more generally by G_m, B_m, T_m, $G_{\epsilon,m}$, $B_{\epsilon,m}$, $T_{\epsilon,m}$, W_m, $W_{\varepsilon,m}$ the base changes $\mathcal{G}_{m\Bbbk}$, $\mathcal{B}_{m\Bbbk}$, $\mathcal{T}_{m\Bbbk}$, $\mathcal{G}_{\epsilon,m_{\Bbbk}}$, $\mathcal{B}_{\epsilon,m_{\Bbbk}}$, $\mathcal{T}_{\epsilon,m_{\Bbbk}}$, and the sets $\mathcal{W}_m$, $\mathcal{W}_{\varepsilon,m}$. We also introduce the natural maps

$$\zeta_m : N_{G_m}(T_m) \to W_m,$$

which map a monomial matrix to the permutation (matrix) it induces.

In order to apply the previous lemmas to this situation, we begin by proving several properties for the type A and then for the three types B, C, D.

Restrictions for type C. — In type C, we assume that $\mathrm{char}(\Bbbk) \neq 2$, that r is odd, and we replace Z and H by their neutral components Z^0 and H^0 (see Remark 0.7).

2.2.1. *Type A.* — We begin with a proposition of independent interest.

Proposition 2.3. — *We have the following parametrization of the Z-orbits in G/B*

$$\begin{aligned} W_r \times W^P &\to Z\backslash(G/B)\\ (u,v) &\mapsto Z\cdot \begin{pmatrix} u & & \\ & 1 & \\ & & 1\end{pmatrix} v^{-1}B. \end{aligned}$$

As a consequence, the number of Z-orbits is finite. Besides, they all possess at least one T-fixed point.

Proof. — Let us recall that the Bruhat decomposition implies a bijection

$$\begin{aligned} W_r &\to G_r\backslash\,(G_r/B_r \times G_r/B_r)\\ u &\mapsto G_r\cdot(\sigma B_r, eB_r). \end{aligned}$$

But, considering the block shapes of L and L_Z, there is a bijection

$$\begin{aligned} G_r\backslash\,(G_r/B_r \times G_r/B_r) &\to L_Z\backslash\,(L/B_L)\\ G_r\cdot(xB_r, yB_r) &\mapsto L_Z\cdot \begin{pmatrix} x & & \\ & 1 & \\ & & y\end{pmatrix} B_L. \end{aligned}$$

We therefore have a bijection

$$\begin{aligned} W_r &\to L_Z\backslash\,(L/B_L)\\ u &\mapsto L_Z\cdot \begin{pmatrix} u & & \\ & 1 & \\ & & 1\end{pmatrix} B_L. \end{aligned}$$

We can then again apply [6, Theorem 7.2 (a)] to the spherical subgroup $Z = L_Z U_P$, which finally gives the desired parametrization

$$\begin{aligned} W_r \times W^P &\to Z\backslash(G/B)\\ (u,v) &\mapsto Z\cdot \begin{pmatrix} u & & \\ & 1 & \\ & & 1\end{pmatrix} v^{-1}B. \end{aligned}$$

The assertion on T-fixed points follows easily. □

Now, we consider the three essential results below.

Proposition 2.4. — *Let $v \in W$. Then, there exists $z_0 \in Z \cap N_G(T)$ such that $w := \zeta_n(z_0)^{-1}v$ satisfies the following.*

(a) *w^{-1} induces u_0 on $\{ r+1, \dots, n-r \}$; in other words, w^{-1} is decreasing on this set.*

(b) *w^{-1} is increasing on $\{ 1, \dots, r \}$.*

Proof. — If $v \in W$, there exists $\sigma_1 \in \mathfrak{S}_{n-2r}$ and $\sigma_2 \in \mathfrak{S}_r$ such that $v^{-1}\begin{pmatrix} 1 & & \\ & \sigma_1 & \\ & & 1 \end{pmatrix}$ decreases on $\{ r+1, \dots, n-r \}$, and $v^{-1}\sigma_2$ increases on $\{ 1, \dots, r \}$. We then fix

$$z_0 := \begin{pmatrix} \sigma_2 & & \\ & \sigma_1 & \\ & & \sigma_2 \end{pmatrix},$$

which is obviously an element of $Z \cap N_G(T)$ and which is also equal, as a matrix, to $\zeta_n(z_0)$ in W. Hence $v^{-1}\zeta_n(z_0)$ is $v^{-1}\sigma_1$ on $\{ r+1, \dots, n-r \}$ and is $v^{-1}\sigma_2$ on $\{ 1, \dots, r \}$, so that $\zeta_n(z_0)^{-1}v$ is the desired element. $\square$

Proposition 2.5. — *Suppose that $w \in W$ satisfies property (a) of Proposition 2.4. Then $z^{-1}\varpi(z) \in \overline{{}^wBB}$ for all $z \in Z$.*

Proof. — Let $w \in W$, satisfying the assumption. Since w induces u_0 we get

$$\begin{pmatrix} 1 & & \\ & {}^{u_0}B_{n-2r} & \\ & & 1 \end{pmatrix} \subset {}^wB. \tag{15}$$

But u_0 is the longest element in W_{n-2r}, so that

$${}^{u_0}B_{n-2r}B_{n-2r} \simeq B_{n-2r}u_0B_{n-2r}$$

is dense in G_{n-2r}, and (15) thus implies

$$\begin{pmatrix} 1 & & \\ & G_{n-2r} & \\ & & 1 \end{pmatrix} \subset \overline{{}^wBB}. \tag{16}$$

Now let $z \in Z$. The element $z^{-1}\varpi(z)$ has the shape

$$\begin{pmatrix} 1 & A_1 & A_2 \\ 0 & C & A_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & C & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & A_1 & A_2 \\ 0 & 1 & C^{-1}A_3 \\ 0 & 0 & 1 \end{pmatrix}$$

with $C \in G_{n-2r}$ and suitable matrices A_i. We conclude that $z^{-1}\varpi(z) \in \overline{{}^wBB}B = \overline{{}^wBB}$ by (16). $\square$

Proposition 2.6. — *Suppose $w \in W$ satisfies properties (a) and (b) of Proposition 2.4. Let $w = \tau\nu$ be the decomposition of w in $W_P(W^P)^{-1}$. Then*

$$\ell(\tau) - (\ell(\tau^{-1}\theta(\tau))/2 = \dim Z/Z \cap B - \dim H/B_H.$$

Proof. — Let $w \in W$, satisfying the assumptions. There exists $\sigma \in \mathfrak{S}_r$ such that $w^{-1}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \sigma \end{pmatrix}$ is increasing on $\{ n-r+1, \ldots, n \}$. We then fix

$$\tau := \begin{pmatrix} 1 & & \\ & u_0 & \\ & & \sigma \end{pmatrix},$$

which is in W_P. Besides, $w^{-1}\tau$ is in W^P because it restricts to w^{-1} on $\{ 1, \ldots, r \}$, to $w^{-1}\begin{pmatrix} 1 & & \\ & u_0^{-1} & \\ & & 1 \end{pmatrix}$ on $\{ r+1, \ldots, n \}$ and to $w^{-1}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \sigma \end{pmatrix}$ on $\{ n-r+1, \ldots, n \}$, which are all increasing permutations by assumptions on w. Besides, we have

$$\tau^{-1}\theta(\tau) = \begin{pmatrix} 1 & & \\ & u_0^{-1} & \\ & & \sigma^{-1} \end{pmatrix}\begin{pmatrix} \sigma & & \\ & u_0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \sigma & & \\ & 1 & \\ & & \sigma^{-1} \end{pmatrix}.$$

We thus have $\ell(\tau^{-1}\theta(\tau)) = 2\ell(\sigma)$. Besides, $\ell(\tau) = \ell(u_0) + \ell(\sigma)$. But an easy computation gives $\dim Z/Z \cap B - \dim H/B_H = \dim G_{n-2r}/B_{n-2r} = \ell(u_0)$. Combining all these equalities, we obtain the desired formula. □

2.2.2. *Types B, C and D.* — The previous properties have their analogs in types B, C and D. Before giving them, let us begin with some material adapted to this setting.

Some preliminaries. — For $v \in W$, we consider the quantity

$$d_v := \#\{ r+1 \leq i \leq \lfloor n/2 \rfloor \mid v^{-1}(i) > \lfloor n/2 \rfloor \}.$$

Then define $s_v \in \mathfrak{S}_{n-2r}$ as

$$s_v := \begin{cases} \left(\frac{n-2r}{2} \quad \frac{n-2r}{2}+1\right) & \text{if } \varepsilon = 1,\ n \text{ is even, } d_v \text{ is odd} \\ id & \text{else.} \end{cases}$$

We motivate our definition by the following fact.

Proposition 2.7. — *If $u \in \mathfrak{S}_{n-2r}$ is induced by v^{-1} on $\{ r+1, \ldots, n-r \}$, then $s_v u \in W_{\varepsilon,n-2r}$.*

Proof. — Indeed, let u be such an element. Since $\widetilde{v^{-1}} = v^{-1}$, we have $\check{u} = u$ and thus $\widetilde{s_v u} = s_v u$ in $\mathfrak{S}_{n-2r}$, so that it is enough to show the parity of $q := \#\{ 1 \leq i \leq (n-2r)/2 \mid s_v u(i) > (n-2r)/2 \}$ for $\varepsilon = 1$ and even n. In this case, $i \mapsto r+i$ identifies $\{ 1 \leq i \leq (n-2r)/2 \mid u(i) > (n-2r)/2 \}$ with $\{ r+1 \leq i \leq n/2 \mid v^{-1}(i) > n/2 \}$. Therefore, if d_v is even, then $s_v = id$ and $q = d_v$; if d_v is odd, then $s_v = \left(\frac{n-2r}{2} \quad \frac{n-2r}{2}+1\right)$ and $q = d_v \pm 1$. In each case, q is even, and we have proved the proposition. □

We will also use the following results concerning length and permutations.

Proposition 2.8. — *Let $d \in \mathbb{N}$, $\epsilon = \pm 1$, $v \in \mathfrak{S}_d$ and $u \in W_{\epsilon,d}$. We have the following.*

(i) $\ell_d(u) = 2\ell(u) + \epsilon \# \{ 1 \leq i \leq \lfloor d/2 \rfloor \mid u(i) > \lfloor d/2 \rfloor \}$.

(ii) *If $v(i) < v(j)$ or $v(\bar{\imath}) > v(\bar{\jmath})$ for any $1 \leq i < j \leq d$, then $\ell_d(\check{v}v^{-1}) = 2\ell_d(v)$.*

(iii) *There exists $\sigma \in \mathfrak{S}_d$ such that $\check{\sigma} = \sigma$ and $v\sigma(i) < v\sigma(j)$ or $v\sigma(\bar{\imath}) > v\sigma(\bar{\jmath})$ for any $1 \leq i < j \leq d$.*

Proof. — We prove the assertions separately.

(i) It is the result of taking proper account of the root systems associated with the different types.

(ii) We actually have

$$\ell_d(\check{v}v^{-1}) = 2\ell_d(v) - 2\# \{ 1 \leq i < j \leq d \mid v(i) > v(j) \text{ and } v(\bar{\imath}) < v(\bar{\jmath}) \}. \tag{17}$$

In fact, let N denote

$$N := \# \left\{ (s,t) \;\middle|\; s < t,\ v^{-1}(s) > v^{-1}(t),\ \check{v}v^{-1}(s) < \check{v}v^{-1}(t) \right\},$$

and let us compute the following numbers A and B

$$\begin{aligned}
A &:= \# \left\{ (i,j) \;\middle|\; i < j,\ v^{-1}(i) > v^{-1}(j),\ \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} \\
&= \# \left\{ (i,j) \;\middle|\; i < j,\ v^{-1}(i) > v^{-1}(j) \right\} \\
&\quad - \# \left\{ (i,j) \;\middle|\; i < j,\ v^{-1}(i) > v^{-1}(j),\ \check{v}v^{-1}(i) < \check{v}v^{-1}(j) \right\} \\
&= \ell_d(v) - N. \\
B &:= \# \left\{ (i,j) \;\middle|\; i < j,\ v^{-1}(i) < v^{-1}(j),\ \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} \\
&= \# \left\{ (i,j) \;\middle|\; v^{-1}(i) < v^{-1}(j),\ \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} \\
&\quad - \# \left\{ (i,j) \;\middle|\; i > j,\ v^{-1}(i) < v^{-1}(j),\ \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} \\
&= \# \{ l < k \mid \check{v}(l) > \check{v}(k) \} \\
&\quad - \# \left\{ (i,j) \;\middle|\; i > j,\ v^{-1}(i) < v^{-1}(j),\ \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} \\
&= \ell_d(\check{v}) - \# \left\{ (i,j) \;\middle|\; i > j,\ v^{-1}(i) < v^{-1}(j),\ \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} \\
&= \ell_d(v) - N.
\end{aligned}$$

We therefore have

$$\ell_d(\check{v}v^{-1}) = \# \left\{ 1 \leq i < j \leq d \;\middle|\; \check{v}v^{-1}(i) > \check{v}v^{-1}(j) \right\} = A + B = 2\ell_d(v) - 2N.$$

But

$$\begin{aligned}
N &= \# \left\{ (s,t) \;\middle|\; s < t,\ v^{-1}(s) > v^{-1}(t),\ \check{v}v^{-1}(s) < \check{v}v^{-1}(t) \right\} \\
&= \# \left\{ (s,t) \;\middle|\; s < t,\ \overline{v^{-1}(s)} < \overline{v^{-1}(t)},\ \overline{v\big(\overline{v^{-1}(s)}\big)} < \overline{v\big(\overline{v^{-1}(t)}\big)} \right\} \\
&= \# \left\{ (s,t) \;\middle|\; s < t,\ \overline{v^{-1}(s)} < \overline{v^{-1}(t)},\ v\big(\overline{v^{-1}(s)}\big) > v\big(\overline{v^{-1}(t)}\big) \right\} \\
&= \# \{ 1 \leq i < j \leq d \mid v(i) > v(j) \text{ and } v(\bar{\imath}) < v(\bar{\jmath}) \}.
\end{aligned}$$

Whence the equality (17).

(iii) We have the following facts:

- If $d=2k$, there exists $\sigma \in \mathfrak{S}_d$ such that $\check{\sigma}=\sigma$ and $v\sigma(i)=\min\{\, v\sigma(j) \mid j\in[i,\bar{\imath}]\,\}$ for all $i\in\{1,\dots,k\}$.
- If $d=2k+1$, there exists $\sigma \in \mathfrak{S}_d$ such that $\check{\sigma}=\sigma$ and $v\sigma(i)=\min\{\, v\sigma(j) \mid j\in[i,\bar{\imath}]\setminus\{\,k+1\,\}\,\}$ for all $i\in\{1,\dots,k\}$.

They imply the proposition. Indeed, let us assume that $\sigma\in\mathfrak{S}_d$ satisfies these properties and fix $1\le i<j\le d$. If $d=2k$, then if $i\le k$, we deduce from the minimality property of $v\sigma$ that $v\sigma(i)<v\sigma(j)$ when $j<\bar{\imath}$, and that $v\sigma(\bar{\jmath})<v\sigma(\bar{\imath})$ when $\bar{\imath}<j$. And if $k<i$ then $\bar{\jmath}<i$, $k\le\bar{\imath}$, and we again have $v\sigma(\bar{\jmath})<v\sigma(\bar{\imath})$ by minimality. In the case where $d=2k+1$ is odd, the same arguments give the alternative for $i\neq k+1$ and $j\neq k+1$. But if $i=k+1$, then $\bar{\jmath}<k+1<j$ and $v\sigma(\bar{\jmath})<v\sigma(j)$ by minimality. Since $\check{\sigma}=\sigma$, we have $\sigma(i)=\sigma(k+1)=k+1=\sigma(\bar{\imath})$, hence $v\sigma(i)=v(k+1)=v\sigma(\bar{\imath})$. Comparing $v\sigma(i)$ and $v\sigma(j)$ then gives the alternative, thanks to the previous inequality. If $j=k+1$, we apply the same argument with $\bar{\imath}$ in the place of j.

We now describe a suitable σ to prove the previous facts. First, let a be the composition of the transpositions $(i\ \bar{\imath})$ for $1\le i\le k$ such that $v(i)>v(\bar{\imath})$. Then, let b be the unique element of $\mathfrak{S}_d$ such that $\check{b}=b$ and that vab increases on $\{\,1,\dots,k\,\}$. The desired permutation is $\sigma:=ab$. $\square$

We are now able to state and prove the desired propositions, similar to Proposition 2.3, 2.4, 2.5 and 2.6. We begin with a parametrization. Let $w_0:=\sigma_{0,r}$ and let $\mathcal{I}_r^1$ (respectively $\mathcal{I}_r^{-1}$) denote the set of involutions in $\mathfrak{S}_r$ (respectively the set of involutions without fixed point). For $u\in W_r$, we also introduce the following subset of G_r/B_r:

$$\mathcal{O}(u):=\left\{\, xB_r \;\middle|\; {I_{-\varepsilon,r}}^{\delta}xI_{-\varepsilon,r}x\in B_rw_0uB_r \,\right\}.$$

Proposition 2.9. — *We have the following parametrization of the Z-orbits in G/B*

$$\begin{aligned}\mathcal{I}_r^{-\varepsilon}\times W^P &\to Z\backslash(G/B)\\ (u,v) &\mapsto Z\cdot\begin{pmatrix} x & & \\ & 1 & \\ & & {}^{\delta}x^{-1}\end{pmatrix}v^{-1}B \text{ with any } xB_r\in\mathcal{O}(u).\end{aligned}$$

As a consequence, the number of Z-orbits is finite. Besides, if $\varepsilon=1$ (types B, D), all Z-orbits contain a T-fixed point. If $\varepsilon=-1$ (type C), this is the case exactly for the Z-orbits parametrized by (u,v) with $u\in\mathcal{I}_r^{-1}$.

Proof. — In the literature, there exist several parametrizations of the $G_{-\varepsilon,r}$-orbits of the flag variety G_r/B_r; we will use the one presented in [13, Proposition 4][6] for $\varepsilon=1$ and the one presented in [32, Example 10.3] for $\varepsilon=-1$, $\mathrm{char}(\Bbbk)\neq 2$. Considering that $x\mapsto {I_{-\varepsilon,r}}^{\delta}x^{-1}I_{-\varepsilon,r}$ is an involution of G_r whose

6. Their article was written for the base field $\mathbb{C}$, but the result used here is valid for any base field of any characteristic; see the proof in [13, Section 3.5].

fixed-point set consists in the symplectic group (if $\varepsilon = 1$) or the (usual) special orthogonal group (if $\varepsilon = -1$, with $\mathrm{char}(\Bbbk) \neq 2$), these two works ensure the existence of a bijection

$$\begin{aligned} \mathcal{I}_r^{-\varepsilon} &\to G_{-\varepsilon,r}\backslash(G_r/B_r) \\ u &\mapsto \mathcal{O}(u). \end{aligned}.$$

But if we consider the block shapes of L and L_Z, we have a surjection

$$L/B_L \twoheadrightarrow G_r/B_r, \tag{18}$$

which is $L_Z \twoheadrightarrow G_{-\varepsilon,r}$ and $T \twoheadrightarrow T_r$ equivariant and which induces the bijection

$$\begin{aligned} G_{-\varepsilon,r}\backslash(G_r/B_r) &\to L_Z\backslash\,(L/B_L) \\ G_{-\varepsilon,r}\cdot xB_r \quad &\mapsto L_Z\cdot\begin{pmatrix} x & & \\ & 1 & \\ & & {}^{\delta}x^{-1}\end{pmatrix}B_L \end{aligned} \tag{19}$$

between the orbit sets. We deduce the bijection

$$\begin{aligned} \mathcal{I}_r^{-\varepsilon} &\to L_Z\backslash\,(L/B_L) \\ u &\mapsto L_Z(u) := L_Z\cdot\begin{pmatrix} x & & \\ & 1 & \\ & & {}^{\delta}x^{-1}\end{pmatrix}B_L \text{ with any } xB_r \in \mathcal{O}(u). \end{aligned}$$

As in type A, we can then apply [6, Theorem 7.2 (a)], and we finally have the bijection

$$\begin{aligned} \mathcal{I}_r^{-\varepsilon}\times W^P &\to Z\backslash(G/B) \\ (u,v) \quad &\mapsto Z(u,v) := Z\cdot\begin{pmatrix} x & & \\ & 1 & \\ & & {}^{\delta}x^{-1}\end{pmatrix}v^{-1}B \text{ with any } xB_r\in\mathcal{O}(u), \end{aligned}$$

which is the desired parametrization.

Let us now conclude about the claim on T-fixed points. As usual, a superscript will denote the set of fixed points for the involved action. Let (u,v) be a couple of parameters. Since w_0 is an involution without fixed point (recall r is even for $\varepsilon = 1$) and $I_{-\varepsilon,r}{}^{\delta}\sigma I_{-\varepsilon,r} = w_0\sigma^{-1}w_0$ for any $\sigma \in W_r$, it is not difficult to show that

$$\mathcal{O}(u)^{T_r} \neq \emptyset \Leftrightarrow u \text{ is conjugate to } w_0 \text{ in } W_r \Leftrightarrow u \in \mathcal{I}_r^{-1}.$$

Then it suffices to show that

$$Z(u,v)^T \neq \emptyset \Leftrightarrow \mathcal{O}(u)^{T_r} \neq \emptyset. \tag{20}$$

But, thanks to the previous reference, we have a (L and thus) T-equivariant isomorphism ([6, Proposition 6.4])

$$L/B_L \simeq \left(P\cdot v^{-1}B\right)^{\tau},$$

where τ is a suitable cocharacter $\mathbb{G}_m \to T$. Moreover, its restriction induces an isomorphism ([6, Theorem 7.2(b)])

$$L_Z(u) \simeq Z(u,v)^{\tau}.$$

Hence we get an isomorphism

$$L_Z(u)^T \simeq \left(Z(u,v)^\tau\right)^T = Z(u,v)^T.$$

Moreover, the equivariant surjection (18), which induces the bijection (19), gives an isomorphism

$$L_Z(u)^T \simeq \mathcal{O}(u)^{T_r}.$$

Composing the above isomorphisms, we finally get

$$Z(u,v)^T \simeq \mathcal{O}(u)^{T_r},$$

which leads to the desired equivalence (20). □

Proposition 2.10. — *Let $v \in W$. Then, there exists $z_0 \in Z \cap N_G(T)$ such that $w := \zeta_{\varepsilon,n}(z_0)^{-1}v$ satisfies the following.*

(a) *$s_v = s_w$ and w^{-1} induces $s_w u_0$ on $\{ r+1,\dots,n-r \}$.*
(b) *$w^{-1}(i) < w^{-1}(j)$ or $w^{-1}(r-i+1) > w^{-1}(r-j+1)$ for any $1 \le i < j \le r$.*

Proof. — Let $v \in W$. The inverse v^{-1} induces on $\{ r+1,\dots,n-r \}$ an element $u \in \mathfrak{S}_{n-2r}$ such that $s_v u$ is in $W_{\varepsilon,n-2r}$ (Proposition 2.7), so that $s_v u \sigma_1 = u_0$ for a suitable $\sigma_1 \in W_{\varepsilon,n-2r}$. Then $v^{-1}\begin{pmatrix} 1 & & \\ & \sigma_1 & \\ & & 1 \end{pmatrix}$ induces $s_v u_0$ on $\{ r+1,\dots,n-r \}$. Besides, there exists a monomial matrix $g_1 \in G_{\varepsilon,n-2r}$ such that $\zeta_{\varepsilon,n-2r}(g_1) = \sigma_1$. Applying Proposition 2.8 (iii) on v^{-1} also gives $\sigma_2 \in \mathfrak{S}_r$ such that $v^{-1}\begin{pmatrix} \sigma_2 & & \\ & I_{n-2r} & \\ & & \check{\sigma_2} \end{pmatrix}$ satisfies (b) with $\check{\sigma_2} = \sigma_2$ relatively to $\{ 1,\dots,r \}$. We can then find a monomial matrix $g_2 \in G_{-\varepsilon,r}$ such that $\zeta_{-\varepsilon,r}(g_2) = \sigma_2$. Indeed, in type C ($\varepsilon = -1$, odd r, $\mathrm{char}(\Bbbk) \neq 2$), we can define g_2 as the matrix product $\sigma_2\begin{pmatrix} 1 & & \\ & det(\sigma_2) & \\ & & 1 \end{pmatrix}$. In types B, D ($\varepsilon = 1$ and even r), since $\check{\sigma_2} = \sigma$ in $\mathfrak{S}_r$, there exists $\lambda \in T_{r/2}$, with coefficients ± 1, such that $I_{-1,r}\sigma_2 I_{-1,r} = \begin{pmatrix} \lambda & \\ & \delta_\lambda \end{pmatrix}\sigma_2$. We can then define g_2 as the matrix product $\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}\sigma_2$. Now we can introduce

$$z_0 := \begin{pmatrix} g_2 & & \\ & g_1 & \\ & & I_{-\varepsilon,r} g_2 I_{-\varepsilon,r} \end{pmatrix},$$

which is in $Z \cap N_G(T)$ with

$$\zeta_{\varepsilon,n}(z_0) = \begin{pmatrix} \zeta_{-\varepsilon,r}(g_2) & & \\ & \zeta_{\varepsilon,n-2r}(g_1) & \\ & & \zeta_{-\varepsilon,r}(I_{-\varepsilon,r} g_2 I_{-\varepsilon,r}) \end{pmatrix} = \begin{pmatrix} \sigma_2 & & \\ & \sigma_1 & \\ & & \check{\sigma_2} \end{pmatrix}.$$

We see that $v^{-1}\zeta_{\varepsilon,n}(z_0)$ and $v^{-1}\begin{pmatrix} 1 & & \\ & \sigma_1 & \\ & & 1 \end{pmatrix}$ induce the same element $s_v u_0$ on $\{ r+1,\dots,n-r \}$. Since $\sigma_1 \in W_{\varepsilon,n-2r}$, the quantities

$$\#\left\{ i \in \{ r+1,\dots,n-r \} \;\middle|\; v^{-1}(i) > \lfloor n/2 \rfloor \right\}$$

and

$$\#\left\{ i \in \{ r+1, \dots, n-r \} \;\middle|\; v^{-1}\begin{pmatrix} 1 & & \\ & \sigma_1 & \\ & & 1 \end{pmatrix}(i) > \lfloor n/2 \rfloor \right\}$$

have the same parity for $\varepsilon = 1$, even n. Hence $s_v = s_{\zeta_{\varepsilon,n}(z_0)^{-1}v}$. On the other hand, $v^{-1}\zeta_{\varepsilon,n}(z_0)$ and $v^{-1}\begin{pmatrix} \sigma_2 & & \\ & I_{n-2r} & \\ & & \check{\sigma_2} \end{pmatrix}$ induce the same element on $\{ 1, \dots, r \}$. We conclude that $w := \zeta_{\varepsilon,n}(z_0)^{-1}v$ satisfies the desired conditions. □

Proposition 2.11. — *Suppose $w \in W$ satisfies the property (a) of Proposition 2.10. Then $z^{-1}\varpi(z) \in \overline{{}^{w}BB}$ for all $z \in Z$.*

Proof. — Let w^{-1} be such an element and fix

$$w_0 := \begin{pmatrix} 1 & & \\ & u_0 s_w & \\ & & 1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 1 & & \\ & {}^{u_0}B_{\varepsilon,n-2r} & \\ & & 1 \end{pmatrix} \subset {}^{w}B. \tag{21}$$

Indeed, s_w stabilizes $B_{\varepsilon,n-2r}$ because, in the case $s_w \neq id$, even $n-2r$, we have for all $i, j \in \{ 1, \dots, n-2r \}$ with $1 \le i < j < \bar{\imath} = n-2r-i+1$ the inequality $i < (n-2r)/2$, so that $s_w(i) = i < s_w(j)$. The inclusion (21) is thus equivalent to:

$$w^{-1}w_0 \begin{pmatrix} 1 & & \\ & B_{\varepsilon,n-2r} & 0 \\ & & 1 \end{pmatrix} w_0^{-1}w \subset B,$$

that is

$$\forall r+1 \le i < j \le n-r,\ w^{-1}w_0(i) < w^{-1}w_0(j).$$

But, w^{-1} induces $s_w u_0$ on $\{ r+1, \dots, n-r \}$, which means that

$$w^{-1} \begin{pmatrix} 1 & & \\ & (s_w u_0)^{-1} & \\ & & 1 \end{pmatrix} = w^{-1}w_0$$

increases on $\{ r+1, \dots, n-r \}$. We get (21). But u_0 is the longest element in $W_{\varepsilon,n-2r}$ so that

$${}^{u_0}B_{\varepsilon,n-2r}B_{\varepsilon,n-2r} \simeq B_{\varepsilon,n-2r}u_0B_{\varepsilon,n-2r}$$

is dense in $G_{\varepsilon,n-2r}$, so that (21) implies

$$\begin{pmatrix} 1 & & \\ & G_{\varepsilon,n-2r} & \\ & & 1 \end{pmatrix} \subset \overline{{}^{w}BB}. \tag{22}$$

Now let $z \in Z$. The element $z^{-1}\varpi(z)$ has the shape

$$\begin{pmatrix} 1 & A_1 & A_2 \\ 0 & C & A_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & C & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & A_1 & A_2 \\ 0 & 1 & C^{-1}A_3 \\ 0 & 0 & 1 \end{pmatrix}$$

with $C \in G_{\varepsilon,n-2r}$ and suitable matrices A_i. We conclude $z^{-1}\varpi(z) \in \overline{{}^wBB}B = \overline{{}^wBB}$ by (22). □

Proposition 2.12. — *Suppose $w \in W$ satisfies properties (a) and (b) of Proposition 2.10. Let $w = \tau\nu$ be the decomposition of w in $W_P(W^P)^{-1}$. Then*

$$\ell(\tau) - \ell(\tau^{-1}\theta(\tau))/2 = \dim Z/Z \cap B - \dim H/B_H.$$

Proof. — Let w be such an element. There exists $\sigma \in \mathfrak{S}_r$ such that $w^{-1}\sigma$ is increasing on $\{1, \ldots, r\}$. We then fix

$$\tau := \left(\begin{smallmatrix} \sigma & 0 & 0 \\ 0 & u_0 & 0 \\ 0 & 0 & \check{\sigma} \end{smallmatrix}\right),$$

which is in W_P. With the decomposition

$$w^{-1}\tau = w^{-1}\begin{pmatrix} 1 & & \\ & (s_w u_0)^{-1} & \\ & & 1 \end{pmatrix}\begin{pmatrix} \sigma & & \\ & s_w & \\ & & \check{\sigma} \end{pmatrix}$$

we see that $w^{-1}\tau$ is increasing on $\{1, \ldots, r\}$, on $\{r+1, \ldots, n-r\}$ and that

$$w^{-1}\tau(\lfloor n/2 \rfloor) < w^{-1}\tau(\lfloor n/2 \rfloor + 1 + \frac{\varepsilon + (-1)^n}{2}).$$

Thus, $w^{-1}\tau$ is in W^P and $w^{-1}\tau = v^{-1}$. Applying θ, we find

$$\tau^{-1}\theta(\tau) = \begin{pmatrix} \sigma^{-1} & & \\ & u_0^{-1} & \\ & & \check{\sigma}^{-1} \end{pmatrix}\begin{pmatrix} \check{\sigma} & & \\ & u_0 & \\ & & \sigma \end{pmatrix} = \begin{pmatrix} \sigma^{-1}\check{\sigma} & & \\ & 1 & \\ & & \check{\sigma}^{-1}\sigma \end{pmatrix},$$

so that $\ell_n(\tau^{-1}\theta(\tau)) = 2\ell_r(\sigma^{-1}\check{\sigma})$. Besides $\ell_n(\tau) = \ell_{n-2r}(u_0) + 2\ell_r(\sigma)$. Using Proposition 2.8 (i) we see that $\ell_n(\tau^{-1}\theta(\tau)) = 2\ell(\tau^{-1}\theta(\tau))$ and $\ell_n(\tau) - 2\ell(\tau) = \ell_{n-2r}(u_0) - 2\ell(u_0)$. Since $\begin{pmatrix} \sigma & & \\ & I_{n-2r} & \\ & & \check{\sigma} \end{pmatrix} = \left(w^{-1}\begin{pmatrix} \sigma & & \\ & I_{n-2r} & \\ & & \check{\sigma} \end{pmatrix}\right)^{-1} w^{-1}$, where $\left(w^{-1}\begin{pmatrix} \sigma & & \\ & I_{n-2r} & \\ & & \check{\sigma} \end{pmatrix}\right)^{-1}$ increases on $w^{-1}(\{1, \ldots, r\})$, σ satisfies property (b) of Proposition 2.10. By Proposition 2.8 (ii) we have $\ell_r(\check{\sigma}^{-1}\sigma) = 2\ell_r(\sigma^{-1})$, that is $\ell_r(\sigma^{-1}\check{\sigma}) = 2\ell_r(\sigma)$. Therefore, we have $\ell(\tau^{-1}\theta(\tau))/2 = \ell_n(\tau^{-1}\theta(\tau))/4 = \ell_r(\sigma)$ and $\ell(\tau) = 1/2\big(\ell_n(\tau) - \ell_{n-2r}(u_0) + 2\ell(u_0)\big) = \ell_r(\sigma) + \ell(u_0)$. Hence $\ell(\tau) - \ell(\tau^{-1}\theta(\tau))/2 = \ell(u_0)$. An easy computation gives $\dim Z/Z \cap B - \dim H/B_H = \dim G_{\varepsilon,n-2r}/B_{\varepsilon,n-2r} = \ell(u_0)$, and we have the desired formula. □

2.2.3. *Conclusion.* — We can now prove the claims of Theorem 0.2 with the exception of rationality. We recall that G is the group $Gl_{n\Bbbk}$, $SO_{n\Bbbk}$, or $Sp_{n\Bbbk}$ under certain restrictions (see Section 2.2 for the full context). So let Y be a Z-orbit closure in G/B. In type C, we assume that the dense Z-orbit possesses a T-fixed point. By hypothesis (type C), Proposition 2.3 (type A) or Proposition 2.9 (types B, D), there exists $w \in W$ such that $Y = \overline{Z \cdot wB}$. By combining Propositions 2.4 and 2.5 (type A) or Propositions 2.10 and 2.11 (types B, C, D), we can assume without loss of generality that w satisfies the hypothesis 2 of Lemma 2.1. The hypothesis 4 is satisfied by Proposition 1.2, whereas the hypothesis 3 is satisfied by the point (i) of Lemma 2.2 with Proposition 1.1. For $\mathrm{char}(\Bbbk) \neq 2$, the point (ii) of the same Lemma together with Proposition 2.6 (type A) or 2.12 (types B, C, D) ensure that the hypothesis 1 is satisfied. But, thanks to the smoothness given by Proposition 1.2, the same dimension formula holds for $\mathrm{char}(\Bbbk) = 2$ and types A, B, D. We can thus apply Lemma 2.1 and get the desired result.

3. Normality, rationality, Cohen–Macaulayness, general case

We prove a stronger version of Theorem 0.3. We first clarify the notation used in this statement and the general context.

3.1. Context. — Let G be a connected semi-simple algebraic group over $\Bbbk$ and (T, B) a Killing pair. Let $H \subset G$ be a connected reductive subgroup such that $(T_H, B_H) := (T \cap H, B \cap H)$ is a Killing pair. For any character λ of T, let $\Bbbk_\lambda$ be the one-dimensional representation of B with weight λ and $\mathcal{L}_G(\lambda)$ be the G-equivariant line bundle on G/B corresponding to $G \times^B \Bbbk_{-\lambda} \to G/B$. Let

$$V_G(\lambda) := H^0(G/B, \mathcal{L}_G(\lambda))$$

denotes the dual Weyl G-module with lowest weight $-\lambda$ (if λ is dominant).

We fix w in the Weyl group W of G and consider the Schubert variety $\overline{B \cdot wB} \subset G/B$. We also consider the natural morphisms

$$q\colon H \times^{B_H} \overline{B \cdot wB} \to G/B, [h, gB] \mapsto hgB$$

and

$$k\colon H \times^{B_H} \overline{B \cdot wB} \to H/B_H, [h, gB] \mapsto hB_H.$$

Let $Z_{\mathfrak{w}}$ be the Bott–Samelson variety associated with the choice of a reduced word $\mathfrak{w}$ that decomposes w into simple reflections. The B_H-equivariant Bott–Samelson resolution $Z_{\mathfrak{w}} \to \overline{B \cdot wB}$ induces a birational morphism $H \times^{B_H} Z_{\mathfrak{w}} \to H \times^{B_H} \overline{B \cdot wB}$. By composing with q and k, we obtain morphisms

$$\widetilde{q}\colon H \times^{B_H} Z_{\mathfrak{w}} \to G/B,$$
$$\widetilde{k}\colon H \times^{B_H} Z_{\mathfrak{w}} \to H/B.$$

Let π and $\widetilde{\pi}$ be the restrictions of q and $\widetilde{q}$ onto their respective images

$$\pi\colon H \times^{B_H} \overline{B \cdot wB} \to \overline{HB \cdot wB}, \widetilde{\pi}\colon H \times^{B_H} Z_{\mathfrak{w}} \to \overline{HB \cdot wB}.$$

Theorem 3.1. — *We make the following assumptions.*

(i) *The morphism $\pi\colon H \times^{B_H} \overline{B \cdot wB} \to \overline{HB \cdot wB}$ is birational.*
(ii) *The character $2\rho_H - \rho_{G|T_H}$ is dominant.*
(iii) $\mathrm{char}(\Bbbk) = 0$ *or*
(iii)′ $\mathrm{char}(\Bbbk) = p > 0$, *and the restriction $V_G((p-1)\rho_G) \to V_H((p-1)\rho_{|T_H})$ is surjective.*

Then $\overline{HB \cdot wB}$ is normal and $\widetilde{\pi}$ is rational. Moreover, if

(iv) $\mathrm{char}(\Bbbk) = 0$ *or*
(iv)′ *there exists a line bundle $\mathcal{M}$ on G/B such that $k^*\mathcal{L}_H(\rho_{G|T_H} - 2\rho_H) \simeq q^*\mathcal{M}$,*

then $\overline{HB \cdot wB}$ is Cohen–Macaulay with dualizing sheaf $\widetilde{\pi}_*\omega_{H\times^{B_H}Z_{\mathfrak{w}}}$, and we have the vanishing $R^i\widetilde{\pi}_*\omega_{H\times^{B_H}Z_{\mathfrak{w}}} = 0$ for all $i > 0$.

3.2. Normality and rationality. — We start with the proof of the first part of Theorem 3.1 concerning normality and rationality.

3.2.1. *An inductive result by Perrin and Smirnov.* — We present here a general setup that contains as a particular case the Bott–Samelson resolution. Let Y be a scheme and n an integer. Let us consider, for $i \in [0, n]$, the following schemes and morphisms.

- A scheme T_i and a morphism $Y \xrightarrow{p_i} T_i$.
- Schemes $\widetilde{X}_i$ and X_i over Y.
- Morphism $\widetilde{X}_i \to X_i$ over Y.

Such that, for all $i \in [0, n-1]$,

(23)
$$\begin{tikzcd} \widetilde{X}_{i+1} \arrow[r] \arrow[dd] & \widetilde{X}_i \arrow[d] \\ & Y \arrow[d, "p_{i+1}"] \\ Y \arrow[r, "p_{i+1}"'] & T_{i+1} \end{tikzcd}$$

is Cartesian, and for all $i \in [0, n]$, X_i is the scheme-theoretic image of $\widetilde{X}_i \to Y$:

(24)
$$\begin{tikzcd} \widetilde{X}_i \arrow[rr] \arrow[dr] & & Y. \\ & X_i \arrow[ur, hook] & \end{tikzcd}$$

Besides, we will say that a morphism $f : Z \to T$ satisfies (*) if the following hold:

$$(*) \qquad \begin{aligned} &f\colon Z \to T \text{ is faithfully flat and proper,}\\ &\text{its geometric fibers } Z_{\overline{t}} \text{ are connected, normal and reduced,}\\ &\text{with } \dim Z_t \leq 1,\ H^1(Z_t, \mathcal{O}_{Z_t}) = 0.\end{aligned}$$

Also let also **P** be a property of morphism of schemes that is preserved under any composition, any base change and that is satisfied by closed immersions.

The article [31] by Perrin and Smirnov then contains the following result, inspired by [4]. The proof consists of increasing and decreasing induction based on appropriate Cartesian diagrams and the stability under base change, descent, and cancellation for certain properties of morphisms of schemes. The authors state this result for varieties, but we present a statement in terms of schemes to emphasize the generality of their argument.

Proposition 3.2. — *We assume that for all i, p_i satisfy (*), T_i is locally noetherian, and there exists an ample line bundle $\mathcal{M}_i$ on T_i such that the restriction*

$$H^0\big(X_{i+1}, (p_{i+1}^*\mathcal{M}_{i+1})_{|X_{i+1}}\big) \to H^0\big(X_i, (p_{i+1}^*\mathcal{M}_{i+1})_{|X_i}\big)$$

is surjective. We also make the following assumptions.

- *The morphism $\widetilde{X}_0 \to X_0$ possesses **P** and is rational.*
- *The morphism $\widetilde{X}_0 \to X_0$ is birational.*
- *The scheme X_0 is normal.*

*Then for all i, the morphism $\widetilde{X}_i \to X_i$ possesses **P**, is rational, birational and the scheme X_i is normal.*

We will apply this result to suitable families. Recall the data G, H, B, T, w, ... that we have fixed and let $n := \ell(w)$. Let P_{α_k} be the minimal parabolic subgroup relative to the simple root α_k. Assume that we have $\mathfrak{w} = (s_{\alpha_{j_1}}, \dots, s_{\alpha_{j_n}})$. For all $i \in [0, n]$ let $\mathfrak{w}_i$ denote the subword $(s_{\alpha_{j_1}}, \dots, s_{\alpha_{j_i}})$ and $w_i := s_{\alpha_{j_1}} \dots s_{\alpha_{j_i}}$ the corresponding Weyl group element. Let $Z_{\mathfrak{w}_i}$ be the Bott–Samelson variety associated to $\mathfrak{w}_i$. As we did previously for $\mathfrak{w} = \mathfrak{w}_n$ with q and $\widetilde{q}$, we define q_i as the composition

$$q_i\colon H \times^{B_H} \mathcal{Z}_{\mathfrak{w}_i} \to G/B. \tag{25}$$

We then denote:

$$\begin{aligned} Y &:= G/B,\\ T_i &:= G/P_{\alpha_{j_i}},\\ p_i &:= G/B \to G/P_{\alpha_{j_i}},\end{aligned}$$

$$
\begin{aligned}
\widetilde{X}_i &:= H \times^{B_H} \mathcal{Z}_{\mathfrak{w}_i}, \\
\widetilde{X}_i \to Y &:= q_i, \\
X_i &:= \operatorname{Im} q_i = \overline{HB \cdot w_i B},
\end{aligned}
$$

and we choose an ample line bundle $\mathcal{M}_i$ on the projective variety T_i. By construction of the Bott–Samelson variety, we have:

$$
\begin{aligned}
\widetilde{X}_i \times_{T_{i+1}} Y &= \left(H \times^{B_H} \mathcal{Z}_{\mathfrak{w}_i}\right) \times_{G/P_{\alpha_{j_{i+1}}}} G/B \\
&\simeq H \times^{B_H} \left(\mathcal{Z}_{\mathfrak{w}_i} \times_{G/P_{\alpha_{j_{i+1}}}} G/B\right) \\
&\simeq H \times^{B_H} \mathcal{Z}_{\mathfrak{w}_{i+1}} \\
&= \widetilde{X}_{i+1}.
\end{aligned}
$$

Besides, we note that the $P_{\alpha_{j_i}}/B \simeq \mathbb{P}^1$-fibration p_i satisfies (*) and that

$$
\widetilde{X_0} \simeq X_0 \simeq H/B_H
$$

are normal varieties. On the other hand, the hypothesis (i) on the birationality of π ensures by composition that $\widetilde{\pi}$ is birational, that is:

$$
\widetilde{X}_n \to X_n \text{ is birational.}
$$

Because the X_i are embedded into Y in a way compatible with their mutual inclusions and the pullbacks $p_i^*\mathcal{M}_i$ are semi-ample on $Y = G/B$, it is now sufficient to prove the surjectivity

$$
H^0\left(G/B, \mathcal{L}\right) \twoheadrightarrow H^0\left(\overline{HB \cdot w_i B}, \mathcal{L}_{|\overline{HB \cdot w_i B}}\right) \tag{26}
$$

for all i and all semi-ample line bundles $\mathcal{L}$ on G/B. Once we show (26), we get the surjectivity

$$
H^0\big(X_{i+1}, (p_{i+1}^*\mathcal{M}_{i+1})_{|X_{i+1}}\big) \twoheadrightarrow H^0\big(X_i, (p_{i+1}^*\mathcal{M}_{i+1})_{|X_i}\big)
$$

for all i, which allows us to apply the previous proposition so that we obtain the first part of the theorem.

We fix now $i \in \{0, \dots, n\}$, a semi-ample line bundle $\mathcal{L}$ on G/B and we prove (26). We will need to distinguish between the zero and positive characteristic cases.

3.2.2. *Positive characteristic case.* — We first assume that $p := \operatorname{char}(\Bbbk) > 0$. The hypotheses (ii) and (iii)$'$ of the Theorem 3.1 enable us to apply the main result of He and Thomsen in [17, Theorem 20]. It gives a Frobenius splitting of G/B, relative to the line bundle $\mathcal{L}_G\left((p-1)\rho_G\right)$ and that compatibly splits our variety $\overline{HB \cdot w_i B}$. Because $\mathcal{L}_G\left((p-1)\rho_G\right)$ is ample and $\mathcal{L}$ semi-ample, [4, Theorem 1.4.8] gives:

$$H^1(G/B,\mathcal{L}) = 0, \tag{27}$$
$$H^0(G/B,\mathcal{L}) \twoheadrightarrow H^0\left(\overline{HB\cdot w_iB}, \mathcal{L}_{|\overline{HB\cdot w_iB}}\right),$$

and, in particular, we have the desired surjectivity (26).

3.2.3. *Characteristic zero.* — We now show how the previous surjectivity result can be extended from positive to zero characteristic. So let us assume in the sequel that $\mathrm{char}(\Bbbk) = 0$. We start by realizing our data G, B, H, X_i, w_i, $\mathcal{L}$, ..., as schemes, morphisms or sheaves over a suitable base ring. This can be done using the following lemma, which also preserves the dominance property of the character $2\rho_H - \rho_{G|T_H}$ on all geometric fibers.

Lemma 3.3. — *There exists a $\mathbb{Z}$-subalgebra A of $\Bbbk$ of finite type, group-schemes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{T}$, $\mathcal{B}$, $\mathcal{W}$ group schemes over A and a section $\mathcal{w}_i \in \mathcal{W}(A)$ such that the following hold.*

- *$\mathcal{G}$ is the semi-simple Chevalley group scheme over A with $\mathcal{G}_\Bbbk = G$, and with $(\mathcal{T},\mathcal{B})$ as a Killing pair satisfying $\mathcal{T}_\Bbbk = T$, $\mathcal{B}_\Bbbk = B$.*
- *$\mathcal{H}$ is a closed subgroup of $\mathcal{G}$ and is the reductive Chevalley group scheme over A with $\mathcal{H}_\Bbbk = H$ and with $(\mathcal{T}_\mathcal{H},\mathcal{B}_\mathcal{H}) := (\mathcal{T}\times_\mathcal{G}\mathcal{H}, \mathcal{B}\times_\mathcal{G}\mathcal{H})$ as a Killing pair satisfying $\mathcal{T}_{\mathcal{H}\Bbbk} = T_H$, $\mathcal{B}_{\mathcal{H}\Bbbk} = B_H$.*
- *The fppf quotient sheaf $\mathcal{G}/\mathcal{B}$ is representable by an A-scheme of finite presentation, smooth and projective, and such that its base change $(\mathcal{G}/\mathcal{B})_K$ over any algebraically closed field K is the flag variety $\mathcal{G}_K/\mathcal{B}_K$.*
- *$\mathcal{W}$ is the Weyl group of $\mathcal{G}$ related to $\mathcal{T}$ and the base change $\mathcal{w}_{i\Bbbk}$ recovers the element $w_i \in W$.*

Moreover, there also exists a line bundle $\mathbb{L}$ over $\mathcal{G}/\mathcal{B}$ and an A-scheme $\mathcal{X}_i$ flat and projective, equipped with a closed immersion $\mathcal{X}_i \to \mathcal{G}/\mathcal{B}$ over A, such that over G/B

$$\mathbb{L}_\Bbbk \simeq \mathcal{L}, \tag{28}$$

$$(\mathcal{X}_i)_\Bbbk \simeq \overline{HB\cdot w_iB} \tag{29}$$

and for all $s \in \mathrm{Spec}\ A$,

$$2\rho_{\mathcal{H}_{\overline{s}}} - \rho_{\mathcal{G}_{\overline{s}}|\mathcal{T}_{\mathcal{H}\overline{s}}} \text{ is dominant} \tag{30}$$

and we have over $(\mathcal{G}/\mathcal{B})_{\overline{s}}$

$$\mathcal{X}_{i\overline{s}} \simeq \overline{\mathcal{H}_{\overline{s}}\mathcal{B}_{\overline{s}}\cdot \mathcal{w}_{i\overline{s}}\mathcal{B}_{\overline{s}}}. \tag{31}$$

A proof is given in Appendix A. Taking into account classical results on general schemes ([16, §8]) and group schemes ([9, Exposés XIX to XXVI]) let us emphasize that our work focuses on the realization of the variety $X_i = \overline{HB\cdot w_iB}$ (see Corollary 5.8). It indeed demands caution as it raises a problem of scheme-theoretic image formation under nonflat base changes (see Theorem 5.6 and Corollary 5.7).

The base Spec$A$ is appropriate in the sense that the residue fields $\kappa(x)$ of its points x produce almost all characteristics.

Lemma 3.4. — *Let A be an integral $\mathbb{Z}$-algebra of finite type whose characteristic is zero. Then the set $\{ \operatorname{char}(\kappa(x)) \mid x \in \operatorname{Spec} A \}$ contains all but finitely many primes.*

Proof. — Since the characteristic is zero, we have $\mathbb{Z} \subset A$ and let $f\colon \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ be the finite type dominant morphism related to this inclusion. By Chevalley's Theorem $f(\operatorname{Spec} A)$ is constructible and then contains an open dense subset U of its closure $\operatorname{Spec} \mathbb{Z}$. Hence, the closed subset $\operatorname{Spec} \mathbb{Z} \setminus U$ is finite. On the other hand, we have $\operatorname{char}(\kappa(x)) = p$ for all prime $p \in \mathbb{Z}$ and $x \in f^{-1}(\{ p\mathbb{Z} \})$. We then have $\{ p \mid p \text{ prime}, p\mathbb{Z} \in U \} \subset \{ \operatorname{char}(\kappa(x)) \mid x \in \operatorname{Spec} A \}$ and the desired result. □

We can then choose $s \in \operatorname{Spec} A$ such that $p := \operatorname{char}(\kappa(s))$ is large enough to satisfy the assumption of [17, Lemma 14]. It gives the surjectivity:

$$V_{\mathcal{G}_{\overline{s}}}\big((p-1)\rho_{\mathcal{G}_{\overline{s}}}\big) \to V_{\mathcal{H}_{\overline{s}}}\big((p-1)\rho_{\mathcal{G}_{\overline{s}|\mathcal{T}_{\mathcal{H}_{\overline{s}}}}}\big).$$

We are now able to use the result (27) of the previous section on vanishing and surjectivity. Thanks to the isomorphism (31), we have

$$H^1\left((\mathcal{G}/\mathcal{B})_{\overline{s}}, \mathbb{L}_{\overline{s}}\right) = 0$$

and the surjectivity

$$H^0\left((\mathcal{G}/\mathcal{B})_{\overline{s}}, \mathbb{L}_{\overline{s}}\right) \twoheadrightarrow H^0\left(\mathcal{X}_{i\overline{s}}, \mathbb{L}_{\overline{s}|\mathcal{X}_{i\overline{s}}}\right).$$

By semi-continuity, the surjectivity can then pass from these geometric special fibers over s to the generic one ([4, Lemma 1.6.3]); we have

$$H^0\left((\mathcal{G}/\mathcal{B})_{\overline{\eta}}, \mathbb{L}_{\overline{\eta}}\right) \twoheadrightarrow H^0\left(\mathcal{X}_{i\overline{\eta}}, \mathbb{L}_{\overline{\eta}|\mathcal{X}_{i\overline{\eta}}}\right),$$

where $\eta \in \operatorname{Spec} A$ denotes the generic point. Tensorizing with the field extension $\overline{\kappa(\eta)} \subset \Bbbk$ then gives

$$H^0\left(\left((\mathcal{G}/\mathcal{B})_{\overline{\eta}}\right)_{\Bbbk}, (\mathbb{L}_{\overline{\eta}})_{\Bbbk}\right) \twoheadrightarrow H^0\left((\mathcal{X}_{i\overline{\eta}})_{\Bbbk}, (\mathbb{L}_{\overline{\eta}})_{\Bbbk|(\mathcal{X}_{i\overline{\eta}})_{\Bbbk}}\right),$$

but we have

$$\left((\mathcal{G}/\mathcal{B})_{\overline{\eta}}\right)_{\Bbbk} \simeq (\mathcal{G}/\mathcal{B})_{\Bbbk} \simeq G/B$$

and, with (28),

$$(\mathbb{L}_{\overline{\eta}})_{\Bbbk} \simeq (\mathbb{L})_{\Bbbk} \simeq \mathcal{L}$$

and, with (29),

$$(\mathcal{X}_{i\overline{\eta}})_{\Bbbk} \simeq \mathcal{X}_{i\Bbbk} \simeq \overline{HB \cdot w_i B}.$$

We therefore recognize the desired surjectivity (26).

3.3. Cohen–Macaulayness and rational singularities. — The second part of Theorem 3.1 will then follow if we can prove that

$$R^j\widetilde{\pi}_*\omega_{\widetilde{X}_n} = 0,\ j > 0. \tag{32}$$

In the characteristic zero case, (32) is automatically verified thanks to the Grauert–Riemenschneider theorem [15]. Now let us assume that the characteristic is positive and that $k^*\mathcal{L}_H(\rho_{G|T_H} - 2\rho_H) \simeq q^*\mathcal{M}$ for a suitable line bundle $\mathcal{M}$ (hypothesis (iv)$'$).

3.3.1. *Another Perrin and Smirnov argument.* — We again use the article [31] of Perrin and Smirnov. We consider the data involved in Proposition 3.2, to which we add the following. Denoting by ϕ_i the morphisms $\widetilde{X}_{i+1} \to \widetilde{X}_i$ we introduce their natural sections σ_i by the commutative diagrams

$$\begin{tikzcd} & & Y \arrow[dr, "p_{i+1}"] & \\ \widetilde{X}_i \arrow[urr, "q_i"] \arrow[r, "\sigma_i"] \arrow[drr, "id"'] & \widetilde{X}_{i+1} \arrow[ur, "q_{i+1}"] \arrow[dr, "\phi_i"] & & T_{i+1}. \\ & & \widetilde{X}_i \arrow[ur, "p_{i+1}q_i"'] & \end{tikzcd} \tag{33}$$

We assume that the morphisms $\widetilde{X}_i \to X_i$ are proper birational and that such p_i (and thus the ϕ_i) is a $\mathbb{P}^1$-fibration over a field. Hence the scheme-theoretic images $\operatorname{Im}\sigma_i$ are closed subschemes of codimension one of $\widetilde{X}_{i+1}$, and we can inductively define the divisors

$$\partial\widetilde{X}_{i+1} = \sigma_i(\widetilde{X}_i) \cup \phi_i^{-1}(\partial\widetilde{X}_i).$$

Finally, we put

$$\Phi_i := \phi_0\phi_1..\phi_{i-1}.$$

By an argument similar to that of Brion and Kumar in [4, Proposition 2.2.2][7], Perrin and Smirnov were able to give a general formula for the canonical sheaf of $\widetilde{X}_i$ involving a certain line bundle on Y; see [31, Lemma 4.7].

Proposition 3.5. — *Let $\mathcal{L}$ be a line bundle on Y such that $q_{i+1}^*\mathcal{L}^{-1}$ has degree one on the fibers of ϕ_i. Then*

$$\omega_{\widetilde{X}_i} = \mathcal{O}_{\widetilde{X}_i}(-\partial\widetilde{X}_i) \otimes q_i^*\mathcal{L} \otimes \Phi_i^*(f_0^*\mathcal{L}^{-1} \otimes \omega_{\widetilde{X}_0}). \tag{34}$$

7. That is, an induction combined with [25, Lemma A-18].

In our specific setting, we have $\omega_{\widetilde{X}_0} = \omega_{H/B_H} = \mathcal{L}_H(-2\rho_H)$. Moreover, we remark that $\Phi_n = \widetilde{p}$ and $f_n = \iota\widetilde{\pi}$, where ι denotes the immersion $\overline{HB \cdot wB} \hookrightarrow G/B$. Taking $\mathcal{L} = \mathcal{L}_G(-\rho_G)$ in the previous proposition we deduce with (iv)$'$

$$\omega_{\widetilde{X}_n} = \mathcal{O}_{\widetilde{X}_n}(-\partial\widetilde{X}_n) \otimes \widetilde{\pi}^*\iota^*(\mathcal{L}_G(-\rho_G) \otimes \mathcal{M}). \tag{35}$$

Thanks to the B-canonical splitting of the Bott–Samelson varieties (see [4, proposition 4.1.17]), and by hypotheses (ii) and (iii)$'$, we can still apply [17, Theorem 20] to obtain a splitting of $\widetilde{X_n}$ compatibly splitting $\partial\widetilde{X}_n$. We can then apply the last arguments of Perrin and Smirnov, inspired once again by Brion and Kumar (see [31, Lemma 5.6] with [4, Theorem 1.2.12]), to get the vanishings

$$R^j\widetilde{\pi}_*\mathcal{O}_{\widetilde{X}_n}(-\partial\widetilde{X}_n) = 0,\ j > 0,$$

and finally the desired (32) by the projection formula on (35).

4. Conclusion: normality, rationality, Cohen–Macaulayness, types A, B, D

In order to obtain Theorem 0.1 and the conclusion in Theorem 0.2 regarding rationality, it is now sufficient to apply Theorem 3.1 to the data of G, H, B, w defined by the matrix models and Section 2.2. Since the birationality assumption (i) is satisfied by the Section 2, it suffices to verify the hypothesis (ii) – or the hypothesis (iii)$'$ in the case of positive characteristic – and the hypothesis (iv)$'$ for the type A. This will be done thanks to the following two propositions. For $i \in \{1, \dots, n-1\}$ let ε_i denote the morphism

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i$$

from T_n to $\mathbb{G}_m$.

Proposition 4.1. — *The character $2\rho_H - \rho_{G|T_H}$ is dominant. It is even zero in type A.*

Proof. — We begin with the type A case. We have $H \simeq G_r$ and easily

$$\begin{aligned} 2\rho_H &= \sum_{1\le i<j\le r} \varepsilon'_i - \varepsilon'_j, \\ 2\rho_{G|T_H} &= \sum_{1\le i<j\le n} \varepsilon'_i - \varepsilon'_j, \\ \varepsilon'_i &= \varepsilon'_{n-r+i}\ \forall i \in [1, r], \\ \varepsilon'_j &= 0\ \forall j \in [r+1, n-r]. \end{aligned}$$

We deduce

$$\begin{aligned}
2\rho_{G|T_H} &= \sum_{1\le i<j\le r} \varepsilon'_i - \varepsilon'_j + \sum_{n-r+1\le i<j\le n} \varepsilon'_i - \varepsilon'_j \\
&\quad + \sum_{1\le i\le r<j\le n-r} \varepsilon'_i - \varepsilon'_j + \sum_{r+1\le i\le n-r<j\le n} \varepsilon'_i - \varepsilon'_j \\
&\quad + \sum_{1\le i\le r<n-r+1\le j\le n} \varepsilon'_i - \varepsilon'_j \ + \sum_{r+1\le i<j\le n-r} \varepsilon'_i - \varepsilon'_j \\
&= \sum_{1\le i<j\le r} \varepsilon'_i - \varepsilon'_j + \sum_{1\le l<k\le r} \varepsilon'_{n-r+l} - \varepsilon'_{n-r+k} \\
&\quad + \sum_{1\le i\le r<j\le n-r} \varepsilon'_i - 0 + \sum_{1\le l\le r<i\le n-r} 0 - \varepsilon'_{n-r+l} \\
&\quad + \sum_{1\le i,l\le r} \varepsilon'_i - \varepsilon'_{n-r+l} + \sum_{r+1\le i<j\le n-r} 0 - 0 \\
&= 4\rho_H,
\end{aligned}$$

Hence $2\rho_H - \rho_{G|T_H} = 0$. For types B, D, the form of H leads to

$$\begin{aligned}
2\rho_H &= \sum_{1\le i\le r/2} (r-2i+2)\varepsilon'_i, \\
2\rho_{G|T_H} &= \sum_{1\le i\le \lfloor n/2\rfloor} (n-2i)\varepsilon'_i, \\
\varepsilon'_i &= -\varepsilon'_{r-i+1} \ \forall i \in [1,r], \\
\varepsilon'_j &= 0 \ \forall j \in [r+1, n-r].
\end{aligned}$$

We deduce

$$\begin{aligned}
2\rho_{G|T_H} &= \sum_{1\le i\le \lfloor n/2\rfloor} (n-2i)\varepsilon'_i = \sum_{1\le i\le r} (n-2i)\varepsilon'_i \\
&= \sum_{1\le i\le r/2} (n-2i)\varepsilon'_i + \sum_{r/2+1\le i\le r} (n-2i)\varepsilon'_i \\
&= \sum_{1\le i\le r/2} (n-2i)\varepsilon'_i + \sum_{1\le j\le r/2} (n-2(r-j+1))\varepsilon'_{r-j+1} \\
&= \sum_{1\le i\le r/2} \big((n-2i)-(n-2r+2i-2)\big)\varepsilon'_i \\
&= 2\sum_{1\le i\le r/2} (r-2i+1)\varepsilon'_i = 2\rho_H - \sum_{1\le i\le r/2} \varepsilon'_i,
\end{aligned}$$

Hence $2\rho_H - \rho_{G|T_H} = \sum_{1\le i\le r/2} \varepsilon'_i$. □

Remark 4.2. — Using similar calculations, we find for type C that $2\rho_H - \rho_{|T_H} = -\sum_{1\leq i\leq r/2}\varepsilon_i'$, a nondominant character.

Proposition 4.3. — *The restriction morphism*

$$V_G\left((p-1)\rho_G\right) \to V_H\left((p-1)\rho_{G|T_H}\right)$$

is surjective.

Proof. — We begin again with the type A case. Let us introduce the parabolic subgroup

$$P' = \left\{ \begin{pmatrix} A & * & * \\ 0 & C & * \\ 0 & 0 & D \end{pmatrix} \in G \;\middle|\; A, D \in G_r(R),\ C \in B_{n-2r} \right\}$$

containing B and

$$L' := \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix} \in G \;\middle|\; A, D \in G_r(R),\ C \in T_{n-2r} \right\}$$

its Levi subgroup related to T. Let us consider the subgroup

$$L'' := \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & 0 & D \end{pmatrix} \in G \;\middle|\; A, D \in G_r(R) \right\},$$

for which $B \cap L''$ is a Borel subgroup. Since P'/B is a Schubert variety, the restriction

$$V_G\left((p-1)\rho_G\right) \to V_{P'}\left((p-1)\rho_G\right)$$

is surjective (see [21, Proposition 14.15]). Besides, (P', L') is a Donkin pair in the sense of Donkin ([10]) because $L' \subset P'$ is a Levi subgroup ([28]). Thus, by [39, Remark 18], the restriction

$$V_{P'}\left((p-1)\rho_G\right) \to V_{L'}\left((p-1)\rho_G\right)$$

is surjective. But the inclusion $L'' \subset L'$ clearly induces an identification $L'/B \cap L' \simeq L''/B \cap L''$ so that

$$V_{L'}\left((p-1)\rho_G\right) \to V_{L''}\left((p-1)\rho_{G|T\cap L''}\right)$$

is an isomorphism. Since $H \hookrightarrow L''$ is a diagonal embedding, (L'', H) is also a Donkin pair ([28] again) and

$$V_{L''}\left((p-1)\rho_{G|T\cap L''}\right) \to V_H\left((p-1)\rho_{G|T_H}\right)$$

is surjective. By composition, the desired restriction is, therefore, surjective.

For types B and D, the argument is exactly the same as for type A except that we replace P', L' and L'' by

$$P' = \left\{ \begin{pmatrix} A & * & * \\ 0 & C & * \\ 0 & 0 & {}^{\delta}A^{-1} \end{pmatrix} \in G \;\middle|\; A \in G_r(R),\ C \in B_{\varepsilon,n-2r} \right\},$$

$$L' := \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & {}^{\delta}A^{-1} \end{pmatrix} \in G \;\middle|\; A \in G_r(R),\ C \in T_{\varepsilon,n-2r} \right\},$$

$$L'' := \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & 0 & {}^{\delta}A^{-1} \end{pmatrix} \in G \;\middle|\; A \in G_r(R) \right\},$$

and the pair (L'', H) is Donkin for a different reason. Here, it is no longer because we have a diagonal embedding, but because H is the fixed point set of a diagram automorphism of L'' (we get a pair of type (A_{n-1}, C_n)), see [5]. □

5. Appendix A: realizations

Notation. — Here $\Bbbk$ will denote a field that is not necessarily algebraically closed.

5.1. Definition of the notion. — For us, realizing a $\Bbbk$-variety X or finding a realization of X will simply mean finding finitely presented schemes on relatively large bases whose geometric fibres will provide different incarnations of X. The various schemes obtained will be gathered into coherent families. Our point of view is thus a special case of the one adopted in [16, §8], concerning projective families of schemes. As we are dealing with algebras, the term "large" will refer to the order for the inclusion property.

Definition 5.1. — Let X be a $\Bbbk$-scheme of finite type. A family $(X_A)_{A\in\mathcal{A}}$ is said to be a realization of X if:

1. There exists $A_0 \in \mathcal{C} := \{\,\mathbb{Z}\text{-subalgebras of } \Bbbk \text{ of finite type}\,\}$ such that $\mathcal{A} = \{\, A \in \mathcal{C} \mid A_0 \subset A \,\}$.
2. For any $A \subset A'$ in $\mathcal{A}$, X_A is a finitely presented A-scheme and

$$X_A \times_A \Bbbk \simeq X \quad \text{and} \quad X_A \times_A A' \simeq X_{A'}.$$

Example 5.2. — Let $X = \operatorname{Spec} \Bbbk[t_1, \dots, t_n]/(f_1, \dots, f_r)$. Let $\mathcal{A}$ be the set of all $\mathbb{Z}$-subalgebras of $\Bbbk$ of finite type containing $f_1, \dots, f_r$. For $A \in \mathcal{A}$, let $X_A := \operatorname{Spec} A[t_1, \dots, t_n]/(f_1, \dots, f_r)$. Then $(X_A)_{A\in\mathcal{A}}$ is a realization of X.

The definition 5.1 for $\Bbbk$-schemes of finite type extends naturally to the notion for morphisms between such schemes, for sheaves and modules, for $\Bbbk$-algebraic groups, etc, and for morphisms between these different kinds of objects. In addition, we can require that the objects of the family that *realizes* the data satisfy additional properties, such as being flat on the basis, being a closed immersion, being locally free, etc. Several results ensure the existence of pure realizations for schemes, morphisms, modules, morphisms of modules [16, Theorem 8.8.2, 8.5.2] and the fact that we can find several properties for the objects of the family as soon as we consider A sufficiently large [16, Theorem 8.10.5, Proposition 8.5.5]. Moreover, enlarging A preserves the commutativity of the diagrams and, thus, the various algebraic structures involved; in particular, the group structure [16, Scholia 8.8.3].

For reasons of convenience, we often omit to specify the index set $\mathcal{A}$.

5.2. Realization of scheme-theoretic images. — We refer to [37, Section 29.6] for the notion of scheme-theoretic image of a morphism of schemes and its fundamental properties. In particular, we will use the description of such an image when the source of the morphism is reduced [37, Lemma 29.6.7]. Here we aim to prove Theorem 5.6 below and its Corollary 5.7, which provide good realizations for suitable scheme-theoretic images. Our work is motivated by the fact that the construction of such images does not, in general, commute to (nonflat) base change, so their realization requires special attention. We begin with the following two preliminary propositions.

Proposition 5.3. — *Let $X \to S$ be a flat morphism and let $T \to S$ be a schematically dominant quasi-compact morphism. If X_T is integral, then X is also integral.*

Proof. — Consider a Cartesian diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "g"] & T \arrow[d, "f"] \\ X \arrow[r, "u"] & S, \end{tikzcd}$$

where u is flat and f is schematically dominant quasi-compact. By flatness, $X \simeq (\operatorname{Im} f)_X \simeq \operatorname{Im} g$ and g is also schematically dominant. But, if X_T is integral, then $\operatorname{Im} g$ is isomorphic to $\overline{g(X_T)}_{red}$ and is also integral. □

Proposition 5.4. — *Let S be a scheme and let $f : X \to Y$ be a finitely presented S-morphism with $\operatorname{Im} f \to S$ open. Then, there exists a nonempty open subset $U \subset S$ such that for any base change $\emptyset \neq T \to U$, we have*

If $(\operatorname{Im} f)_T$ is integral and X_T is reduced, then $\operatorname{Im} f_T \simeq (\operatorname{Im} f)_T$ over Y_T.

Proof. — Let $\widetilde{f} : X \to \operatorname{Im} f$ denote the restriction of f onto its scheme-theoretic image. By cancellation, $\widetilde{f}$ is finitely presented just as f. Since f is quasi-compact, $\operatorname{Im} f$ is topologically $\overline{f(X)} = \overline{\widetilde{f}(X)}$ and $\widetilde{f}$ is (topologically) dominant. Since $\operatorname{Im} f \to S$ is open, we can apply Lemma 5.5 below. It gives a nonempty open subset $U \subset S$ such that for any base change $\emptyset \neq T \to U$, $\widetilde{f}_T(X_T)$ contains a nonempty open subset of $(\operatorname{Im} f)_T$. If this last scheme is integral then $(\overline{\widetilde{f}_T(X_T)})_{red} \simeq (\operatorname{Im} f)_T$ and if X_T is reduced then $\operatorname{Im} \widetilde{f}_T \simeq (\overline{\widetilde{f}_T(X_T)})_{red}$. Under those assumptions $\widetilde{f}_T : X_T \to (\operatorname{Im} f)_T$ is thus schematically dominant. Since f_T factors as this morphism followed by the closed immersion $(\operatorname{Im} f)_T \hookrightarrow Y_T$, we deduce that $\operatorname{Im} f_T \simeq (\operatorname{Im} f)_T$. □

Lemma 5.5. — *Let S be a scheme and let $f : X \to Y$ be a finitely presented (topologically) dominant S-morphism with $Y \to S$ open. Then, there exists a nonempty open subset $U \subset S$ such that for any base change $\emptyset \neq T \to U$, $f_T(X_T)$ contains a nonempty open subset of Y_T.*

Proof. — By the Chevalley theorem $f(X)$ is a constructible subset of Y and thus contains a dense open subset V of its closure, which is Y since f is dominant. Since $Y \to S$ is open, V is sent onto a nonempty open subset U of S. By restricting over U, we get a diagram where the upper horizontal arrows are surjective and the right vertical one is an open immersion:

$$\begin{tikzcd} f_U^{-1}(V) \arrow[r, two heads] \arrow[d] & V \arrow[r, two heads] \arrow[d, hook] & U . \\ X_U \arrow[r, "f_U"] & Y_U \arrow[ur] & \end{tikzcd}$$

For any base change $\emptyset \neq T \to U$ the diagram

$$\begin{tikzcd} (f_U^{-1}(V))_T \arrow[r, two heads] \arrow[d] & V_T \arrow[r, two heads] \arrow[d, hook] & T \\ X_T \arrow[r, "f_T"] & Y_T \arrow[ur] & \end{tikzcd}$$

will then preserve those properties. We deduce $V_T \neq \emptyset$ and that $f_T(X_T)$ contains the open image of V_T in Y_T. □

Theorem 5.6. — *Let $f \colon X \to Y$ be a morphism of $\Bbbk$-schemes of finite type. Let $(f_A \colon X_A \to Y_A)_A$ be a realization of f. Assume that Y is proper over $\Bbbk$, and that there exists A such that X_A has integral geometric fibers (in particular $X \simeq ((X_A)_{\overline{\eta}})_{\Bbbk}$ is integral, where $\eta \in \operatorname{Spec} A$ denotes the generic point). Then, there exists A_0 such that the following hold.*

(i) $(\operatorname{Im} f_A)_{A_0\subset A}$ *realizes* $\operatorname{Im} f$.

(ii) *For any* A *containing* A_0*, we have the following.*

(a) X_A *is integral and flat over* A.

(b) $\operatorname{Im} f_A$ *is integral with integral geometric fibers and flat, proper, of finite presentation over* A.

(c) *The construction of the scheme-theoretic image of* f_A *commutes with any base change* $\emptyset \neq T \to \operatorname{Spec} A$ *such that* $(\operatorname{Im} f_A)_T$ *and* $(X_A)_T$ *are integral. In particular, for any* $s \in \operatorname{Spec} A$*, we have an isomorphism over* $(Y_A)_{\overline{s}}$*:*

$$\operatorname{Im}(f_A)_{\overline{s}} \simeq (\operatorname{Im} f_A)_{\overline{s}}.$$

Proof. — Let us first show that if A satisfies (ii); then this will also be the case for any B containing A and $(\operatorname{Im} f_B)_{A\subset B}$ will realize $\operatorname{Im} f$. By flatness, we have $(\operatorname{Im} f_A)_{\Bbbk} \simeq \operatorname{Im}(f_A)_{\Bbbk}$ and, therefore, $(\operatorname{Im} f_A)_{\Bbbk} \simeq \operatorname{Im} f$. Note also that the properties of flatness, properness, being of finite presentation and having integral geometric fibers are preserved for X_B and $(\operatorname{Im} f_A)_B$ on B. These two schemes are also integral by Proposition 5.3 applied on the two following Cartesian diagrams whose right vertical arrows are schematically dominant

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d] \\ X_B \arrow[r] & \operatorname{Spec} B \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Im} f \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d] \\ (\operatorname{Im} f_A)_B \arrow[r] & \operatorname{Spec} B. \end{tikzcd}$$

Besides, any base change $\emptyset \neq T \to \operatorname{Spec} B$ leads to isomorphisms

$$(X_B)_T \simeq ((X_A)_B)_T \simeq (X_A)_T,\ ((\operatorname{Im} f_A)_B)_T \simeq (\operatorname{Im} f_A)_T,\ \operatorname{Im}(f_B)_T \simeq \operatorname{Im}(f_A)_T.$$

By now applying (c) for f_A and the base change $\emptyset \neq \operatorname{Spec} B \to \operatorname{Spec} A$ we obtain the desired statements.

Let us now show the existence of an A satisfying (ii). Since $Y \to \operatorname{Spec} \Bbbk$ is proper, we can assume that $Y_A \to \operatorname{Spec} A$ and then $\operatorname{Im} f_A \to \operatorname{Spec} A$ are proper for A sufficiently large ([16, Theorem 8.10.5]). We will establish the other assertions by successive localizations.

(flatness) By generic flatness ([16, Proposition 8.9.4]) and localization of A at a suitable element, $\operatorname{Im} f_A$ and X_A can be supposed to be flat over A.

(fiber integrality) By flatness $(\operatorname{Im} f_A)_{\overline{\eta}} \simeq \operatorname{Im}(f_A)_{\overline{\eta}}$. But $\operatorname{Im}(f_A)_{\overline{\eta}}$ is integral since $(X_A)_{\overline{\eta}}$ is. Thus the set of $s \in \operatorname{Spec} A$ with $(\operatorname{Im} f_A)_{\overline{s}}$ integral is nonempty. It is an open subset ([16, Theorem 12.2.4]) and, localizing again, we can assume that it is the whole of $\operatorname{Spec} A$.

(base change) Since $\operatorname{Im} f_A \to \operatorname{Spec} A$ is open by flatness, we can apply Proposition 5.4 on the finitely presented morphism f_A :

$X_A \to Y_A$ and localizing, we can suppose that (c) is satisfied for f_A.

(integrality) We get that $\operatorname{Im} f_A$ and X_A are integral by applying Proposition 5.3 on the following Cartesian diagrams whose right vertical arrows are schematically dominant

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d] \\ X_A \arrow[r] & \operatorname{Spec} A \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Im} f \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d] \\ \operatorname{Im} f_A \arrow[r] & \operatorname{Spec} A. \end{tikzcd}$$

□

Corollary 5.7. — *Let G be a $\Bbbk$-group acting on a $\Bbbk$-scheme of finite type X. Let Z be a closed subscheme of X and H_1, H_2, …, H_n be closed subgroups of G. Let $(G_A)_A$, $(X_A)_A$, $(G_A \times_A X_A \to X_A)_A$, $(Z_A \hookrightarrow X_A)_A$, $(H_{iA} \hookrightarrow G_A)_A$, $i = 1, \ldots, n$ be families that realize these data. Assume that for large enough A, Z_A and all the H_{iA} have integral geometric fibers and that X is proper over $\Bbbk$. Then, there exists a family $(Y_A \hookrightarrow X_A)_A$, which realizes the closed subscheme $(\overline{H_1 H_2 \ldots H_n \cdot Z})_{red} \hookrightarrow X$ and such that, for any large enough A:*

(i) *Y_A is proper and flat over A, with integral geometric fibers.*

(ii) *For all $s \in \operatorname{Spec} A$,*

$$(Y_A)_{\overline{s}} \simeq (\overline{H_{1A\overline{s}} H_{2A\overline{s}} \ldots H_{nA\overline{s}} \cdot Z_{A\overline{s}}})_{red}$$

over $X_{A\overline{s}}$.

Proof. — By induction, it suffices to show the result for a single subgroup $H = H_1$ $(n = 1)$. For large enough A we consider the morphism $f_A \colon H_A \times_A Z_A \to X_A$, which denotes the restriction of the action $G_A \times_A X_A \to X_A$ over $H_A \times_A Z_A \hookrightarrow G_A \times_A X_A$. The family $(f_A)_A$ is clearly a realization of the $\Bbbk$-morphism of finite type $f : H \times_{\Bbbk} Z \to X$ defined in a similar way by action and restriction. Since the product of two integral schemes over a perfect field is again integral, $H_A \times_A Z_A$ has integral geometric fibers for A sufficiently large. We can, therefore, apply the previous Theorem 5.6 and we get a family of closed immersions $Y_A \hookrightarrow X_A$ realizing $\operatorname{Im} f \hookrightarrow X$, with Y_A flat and proper over A and isomorphims $(Y_A)_{\overline{s}} \simeq \operatorname{Im}(f_A)_{\overline{s}}$ over $(X_A)_{\overline{s}}$ for all $s \in \operatorname{Spec} A$. To conclude, all we need to do is to notice that we have $\operatorname{Im} f \simeq (\overline{H \cdot Z})_{red}$ and $\operatorname{Im}(f_A)_{\overline{s}} \simeq (\overline{H_{A\overline{s}} \cdot Z_{A\overline{s}}})_{red}$ for all $s \in \operatorname{Spec} A$. This comes from the reductiveness of $H_A \times_A Z_A$ and $H \times_{\Bbbk} Z$. □

5.3. Proof of Lemma 3.3. — We can now prove Lemma 3.3. Let us assume that $\Bbbk$ is algebraically closed and let G, H, T, B, T_H, B_H, W, w_i be as in the general setting of Section 3.1.

First let $\underline{G}$, $\underline{H}$, $\underline{T}$, $\underline{B}$, $\underline{T_H}$, $\underline{B_H}$, $\underline{W}$ be the group schemes over $\mathbb{Z}$ and let $\underline{w}_i \in \underline{W}(\mathbb{Z})$ be the section such that (see [9, Exposés XIX to XXVI]) the following hold:

- $\underline{G}$ is the semi-simple Chevalley group scheme with $\underline{G}_{\Bbbk} = G$.
- $\underline{H}$ is the reductive Chevalley group scheme with $\underline{H}_{\Bbbk} = H$.
- $(\underline{T}, \underline{B})$ is the Killing pair of $\underline{G}$ with $\underline{T}_{\Bbbk} = T$, $\underline{B}_{\Bbbk} = B$.
- $(\underline{T_H}, \underline{B_H})$ is the Killing pair of $\underline{H}$ with $\underline{T_H}_{\Bbbk} = T_H$, $\underline{B_H}_{\Bbbk} = B_H$.
- $\underline{W}$ is the Weyl group of $\underline{G}$ related to $\underline{T}$.
- The base change $\underline{w}_{i\Bbbk}$ is the element $w_i \in W$.

We know that the fppf quotient sheaf $\underline{G}/\underline{B}$ is representable by a $\mathbb{Z}$-scheme of finite presentation, smooth and projective and such that its base change $(\underline{G}/\underline{B})_K$ over any algebraically closed field K is the flag variety $\underline{G}_K/\underline{B}_K$ (see [9, Exposé XXIV, Théorème 1.3]).

We now make use of [16, §8]. By enlarging A at each step, we successively prove the following assertions. Since there is a closed immersion of groups $H \hookrightarrow G$, we can suppose that $\underline{H}_A$ is a closed subgroup of $\underline{G}_A$. Besides, since the base changes over $\Bbbk$ of $\underline{T_H}_A$ and $\underline{T}_A \times_{\underline{G}_A} \underline{H}_A$ give T_H, we can assume that these two groups identify over $\underline{H}_A$. Similarly $\underline{B_H}_A$ and $\underline{B}_A \times_{\underline{G}_A} \underline{H}_A$ can be identified over $\underline{H}_A$. Finally, the root data are preserved and, for any $s \in \operatorname{Spec} A$,

$$2\rho_{\underline{H}_{A\overline{s}}} - \rho_{\underline{G}_{A\overline{s}}|\underline{T_H}_{A\overline{s}}} \text{ is dominant.}$$

On the other hand, there exist semi-ample line bundles $\mathcal{L}_A$ on $(\underline{G}/\underline{B})_A$, for A sufficiently large so that

$$(\mathcal{L}_A)_A \text{ realizes } \mathcal{L}.$$

To finish the proof of the lemma, we now just need to find a good realization of our variety $X_i = \overline{HB \cdot w_i B}$. This will be done thanks to the following consequence of Corollary 5.7.

Corollary 5.8. — *There exists $(Y_{iA} \hookrightarrow (\underline{G}/\underline{B})_A)_A$, which realizes the closed subvariety $X_i \hookrightarrow G/B$ and such that, for large enough A, we have the following.*

(i) *Y_{iA} is projective and flat over A.*

(ii) *For all $s \in \operatorname{Spec} A$,*

$$(Y_{iA})_{\overline{s}} \simeq \overline{\underline{H}_{A\overline{s}}\underline{B}_{A\overline{s}} \cdot \underline{w}_{A\overline{s}}\underline{B}_{A\overline{s}}}$$

over $\underline{G}_{A\overline{s}}/\underline{B}_{A\overline{s}}$.

Proof. — Suppose A is large enough to guarantee all the above statements and existences. Let $Z := \operatorname{Spec} \Bbbk$ and $Z \hookrightarrow G/B$ be the section that corresponds to the closed point $w_i B \in G/B$. The choice of $\underline{B}_A$ gives a morphism $\underline{W}_A \to (\underline{G}/\underline{B})_A$ defined by the natural transformation $(n\underline{T}_A(S) \mapsto n\underline{B}_A(S))_S$ on the corresponding sheaves. Composing with $\underline{w}_{iA} \colon \operatorname{Spec} A \to \underline{W}_A$ produces a section $\operatorname{Spec} A \to (\underline{G}/\underline{B})_A$. Its geometric fiber over $s \in \operatorname{Spec} A$ is integral and corresponds to the closed point $\underline{w}_{iA\overline{s}}\underline{B}_{A\overline{s}} \in (\underline{G}/\underline{B})_{A\overline{s}}$. Besides, the family of these sections realizes $Z \hookrightarrow G/B$. Moreover $(\underline{G}/\underline{B})_A$ is proper over A and $\underline{B}_A$ and $\underline{H}_A$ have integral geometric fibers as Borel and reductive group schemes.

Finally, the natural action of G on G/B is realized by the family of the natural actions of $\underline{G}_A$ on $(\underline{G}/\underline{B})_A$. By applying Corollary 5.7 with $H_1 := H$, $H_2 := B$ we obtain all the desired assertions, except for the projectivity of the Y_{iA}, which is automatically satisfied as they are closed subschemes of $(\underline{G}/\underline{B})_A$. □

6. Appendix B: on Perrin and Smirnov's arguments

Notation. — Here $\Bbbk$ denotes an algebraically closed field with $\mathrm{char}(\Bbbk) \neq 2$.

6.1. An identical birational morphism. — We present Perrin and Smirnov's construction of a birational morphism onto an irreducible component of a Springer fiber, in type A. We also show that these morphisms are special cases of those we have constructed.

Perrin and Smirnov's version. — Let V be an n-dimensional $\Bbbk$-vector space. Let N be a nilpotent endomorphism of V of nilpotency order two, with rank r and let Z_N be its centralizer in the general linear group $\mathrm{Gl}(V)$. In this setting, let

$$\mathcal{F} := \mathcal{F}(V) = \{\, V_1 \subset \cdots \subset V_{n-1} \mid V_i \subset V,\ \dim V_i = i,\ \forall i \,\}$$

be the flag variety and let

$$\mathcal{F}_N := \{\, V_\bullet \in \mathcal{F} \mid N(V_i) \subset V_i,\ \forall i \,\}$$

be the Springer fiber over N. Let

$$\tau = \begin{array}{cc} \boxed{n} & \boxed{p_r} \\ \vdots & \vdots \\ \boxed{\cdot} & \boxed{p_1} \\ \vdots & \\ \boxed{1} & \end{array}$$

be a standard tableau (with decreasing numbers from left to right and from top to bottom) and let

$$X := X_\tau = \big\{\, F_\bullet \in \mathcal{F}(V) \;\big|\; \dim F_{p_i} \cap \mathrm{Im}\, N \geq i,\ \dim F_{p_i} \cap \mathrm{Ker}\, N \geq p_i - i + 1, \\ F_{p_i} \subset N^{-1}(F_{p_{i-1}}), \forall i \in [1, r] \big\}$$

be the related irreducible component of the Springer fiber over N. We then define the variety

$$\hat{X} := \hat{X}_\tau = \big\{ (F'_\bullet, F_\bullet) \in \mathcal{F}(\mathrm{Im}\, N) \times \mathcal{F}(V) \big|\ F'_i \subset F_{p_i} \subset N^{-1}(F'_{i-1})\ \forall i \in [1, r] \big\}.$$

In [31], Perrin and Smirnov show that $\hat{X}$ is smooth, irreducible and that the projection to $\mathcal{F}$ induces a proper birational Z_N-equivariant morphism

$$\hat{X} \to X \tag{36}$$

as soon as $p_{i+1} > p_i + 1$ for all i (if this is not the case, the result is valid for an irreducible component of $\hat{X}$).

Our version. — Let G, H, B_H, Z, e be as in the matrix setting of Section 2 (type A). Let be integers $q_1, \dots, q_r$ and $s_1, \dots, s_{n-2r}$ such that the following hold.

- $q_r := p_r + 1 = \min\big(p_r + 1, \dots, n\big)$ and by decreasing induction on $i = r-1, \dots, 1$, $q_i := \min\big(\{p_i + 1, \dots, n\} \setminus \{p_{i+1}, q_{i+1}, \dots, p_r, q_r\}\big)$.
- $\{s_1, \dots, s_{n-2r}\} := \{1, \dots, n\} \setminus \{p_1, q_1, \dots, p_r, q_r\}$ with $s_i < s_{i+1}$.

We then define $w := w_\tau$ the unique element of $\mathfrak{S}_n$ such that the following hold.

- $w(p_i) := i$, $\forall i \in \{1, \dots, r\}$.
- $w(q_i) := n - r + i$, $\forall i \in \{1, \dots, r\}$.
- $w(s_j) := n - r + 1 - j$, $\forall j \in \{1, \dots, n - 2r\}$.

Let us remark that w induces the increasing bijection from $\{p_1, \dots, p_r\}$ to $\{1, \dots, r\}$ and the decreasing bijection from $\{s_1, \dots, s_{n-2r}\}$ to $\{r+1, \dots, n-r\}$. Here are some examples:

a) $$\tau = \begin{array}{|c|c|} \hline 5 & 4 \\ \hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad w_\tau = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,1\,4\,2\,5 \end{pmatrix}.$$

b) $$\tau = \begin{array}{|c|c|} \hline 6 & 4 \\ \hline 5 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad w_\tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,1\,5\,2\,6\,3 \end{pmatrix}.$$

c) $$\tau = \begin{array}{|c|c|} \hline 7 & 5 \\ \hline 6 & 4 \\ \hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad w_\tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 4\,1\,5\,2\,3\,7\,6 \end{pmatrix}.$$

We can check that w satisfies the conditions of Proposition 2.6 so that we can apply the proof of Section 2 and get a birational Z-equivariant morphism as (2):

$$H \times^{B_H} \overline{B \cdot wB} \to \overline{Z \cdot wB}. \tag{37}$$

Let us show that the morphisms (36) and (37) identify.

Identification. — There is a basis (f_i) of V such that:

$$\operatorname{Im} N = \langle f_1, \dots, f_r \rangle$$
$$\operatorname{Ker} N = \langle f_1, \dots, f_{n-r} \rangle$$
$$N(f_{n-r+i}) = f_i, \ \forall i \in \{\, 1, \dots, r \,\}.$$

By choosing it so that G acts on V and then on $\mathcal{F}$, we identify G with $\mathrm{Gl}(V)$, e with N, Z with Z_N, H with a diagonal embedding of $\mathrm{Gl}(\mathrm{Im}\, N)$ into $\mathrm{Gl}(V)$ and B with the stabilizer of $F := \langle f_1 \rangle \subset \cdots \subset \langle f_1, \dots, f_{n-1} \rangle \in \mathcal{F}$. Hence, in particular,

$$G/B \simeq \mathcal{F}.$$

Besides, the element w act on $\mathcal{F}$. Note that

$$wF \in X$$

so that $Z \cdot wF \subset X$. Since

$$\dim Z \cdot wB = \dim H/B_H + \ell(w)$$

thanks to (37), the computations of $\dim H/B_H = \binom{r}{2}$ and $\ell(w) = \binom{n-r}{2}$ lead to the equality

$$\overline{Z \cdot wF} = X.$$

On the other hand, we can check (when $p_{i+1} > p_i + 1$ for all i) that the fiber of (36) over wF is isomorphic to the Schubert variety defined by w, i.e.

$$\overline{B \cdot wF}.$$

This gives the following diagram

$$\begin{tikzcd} H \times^{B_H} \overline{B \cdot wB} \arrow[r] \arrow[d, "\wr"'] & \overline{Z \cdot wB} \arrow[r, hook] \arrow[d, "\wr"'] & G/B \arrow[d, "\wr"'] \\ \hat{X} \arrow[r] & X \arrow[r, hook] & \mathcal{F}, \end{tikzcd} \tag{38}$$

identifying (37) and (36) through the base (f_i) and through a natural isomorphism between H-equivariant bundles with isomorphic fibers.

6.2. The problem of normality, a counterexample. — While the general arguments presented in Proposition 3.2 and 3.5 are taken from the article of Perrin and Smirnov, the authors actually apply them in a different way for the type D. The reason is that they embed the irreducible component of the Springer fiber into a larger variety, which does not live in the flag variety of the ambient group but in its product with a Lagrangian space (see their Proposition 3.18). They thus deal with a X_n different from ours, which allows them to obtain a formula analogous to (35) without any additional assumption on the pullback sheaf like hypothesis (iv)$'$ of Theorem 3.1 (see their Lemma 4.7). The problem is that the embedding they present does not exist in general as an algebraic morphism, so they cannot carry out their argument and even ensure normality. Let us present a counterexample in the context of their article.

Context. — Let V be a $2n$-dimensional $\Bbbk$-vector space, and $\mathrm{SO}(\omega)$ be the group of unimodular linear operators preserving a symmetric nondegenerate bilinear form ω. Let $N \in \mathfrak{g}$ be a nilpotent antiadjoint endomorphism. Let Z_N be the stabilizer of N in $\mathrm{SO}(\omega)$:

$$Z_N := \left\{ g \in \mathrm{SO}(\omega) \;\middle|\; gNg^{-1} = N \right\}.$$

If $2r = \dim \operatorname{Im} N$, there exists a basis (f_i) of V such that:

$$\begin{aligned} \operatorname{Im} N &= \langle f_1, \ldots, f_{2r} \rangle \\ \operatorname{Ker} N &= \langle f_1, \ldots, f_{2n-2r} \rangle \\ N(f_{2n-2r+i}) &= f_i, \ \forall i \in \{ 1, \ldots, r \} \\ N(f_{2n-2r+i}) &= -f_i, \ \forall i \in \{ r+1, \ldots, 2r \} \\ \omega(f_i, f_j) &= \delta_{i,2n-j+1}, \ \forall i, j \in \{ 1, \ldots, 2n \} \end{aligned}$$

Now let $\mathcal{OF}$ be the variety of orthogonal flags defined by:

$$\mathcal{OF} := \left\{ V_1 \subset \ldots \subset V_{n-1} \subset V_{n+1} \subset \ldots \subset V_{2n-1} \;\middle|\; V_{2n-i} = V_i^{\perp}, \ \dim V_i = i \ \forall i \right\},$$

and $\mathcal{OF}_N$ be the closed subvariety of N-stable flags, i.e., the Springer fiber over N:

$$\mathcal{OF}_N := \{ V_\bullet \in \mathcal{OF} \mid N(V_i) \subset V_i \ \forall i \}.$$

We can endow the vector space $\operatorname{Im} N$ with a skew-symmetric nondegenerate bilinear form α satisfying:

$$\alpha(u, N(v)) = \omega(u, N(v)) \ \forall u \in \operatorname{Im} N, \ v \in V.$$

Let then $\mathcal{L}$ be the variety of Lagrangian subspaces of $\operatorname{Im} N$ for α:

$$\mathcal{L} := \{ W \subset \operatorname{Im} N \mid W \text{ is totally } \alpha\text{-isotropic} \}$$

If (f'_i) is any basis of V such that $\omega(f'_i, f'_j) = 0$ for $i + j \neq 2n + 1$, we will denote by $F(f'_1, \ldots, f'_{2n})$ the flag in $\mathcal{OF}$ such that, for all i,

$$F(f'_1, \ldots, f'_{2n})_i := \langle f'_1, \ldots, f'_i \rangle,$$

and merely by $F_\bullet$ the flag $F(f_1, \ldots, f_{2n})$. We consider the application ϕ defined as follows:

$$\phi : \ \mathcal{OF}_N \to \mathcal{L}, \ V_\bullet \mapsto \sum_{i=1}^{n-1} V_i \cap N(V_i^{\perp})$$

According to [31, Remark 3.13], ϕ is well defined. It is clear that Z_N acts on $\mathcal{OF}$ and $\mathcal{L}$ and that ϕ is Z_N-equivariant.

Non-continuity. — Now, if the embedding of the authors exists as an algebraic morphism, then ϕ must exist as well. But we will show that ϕ is not continuous in the case $n = 2r = 4$.

Let w and s be the operators in $\mathrm{SO}(\omega)$ that correspond respectively to the permutation 15263748 and 13245768[8] of (f_i), and let $\{\, U(t) \,\}$ be the one-parameter subgroup of Z_N acting on (f_i) with the matrix

$$U(t) := \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & 0 & t & 0 & & \\ & & & 1 & 0 & -t & & \\ & & & & 1 & 0 & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}.$$

We can check that for all $t \neq 0$,

$$\begin{aligned} U(t)wF_\bullet &= F(f_1, tf_3 + f_5, f_2, -tf_4 + f_6, f_3, f_7, f_4, f_8) \\ &= F(f_1, f_3 + 1/tf_5, f_2, f_4 - 1/tf_6, f_3, f_7, f_4, f_8) \end{aligned}$$

and

$$sF_\bullet = F(f_1, f_3, f_2, f_4, f_5, f_7, f_6, f_8),$$

and we see that

$$\lim_{t\to\infty} U(t)wF_\bullet = sF_\bullet.$$

But $wF_\bullet$ is N-stable and

$$\phi(wF_\bullet) = \langle f_1, f_2\rangle.$$

We have also

$$\phi(U(t)wF_\bullet) = U(t)\langle f_1, f_2\rangle = \langle f_1, f_2\rangle$$

and

$$\phi(sF_\bullet) = s\langle f_1, f_2\rangle = \langle f_1, f_3\rangle.$$

Hence,

$$\lim_{t\to\infty} \phi(U(t)wF_\bullet) = \langle f_1, f_2\rangle \neq \langle f_1, f_3\rangle = \phi(\lim_{t\to\infty} U(t)wF_\bullet).$$

8. Let T be the maximal torus of $\mathrm{SO}(\omega)$ related to (f_i), ε_i be the characters on T defined by $t \mapsto f_i^*(t(f_i))$ and W be the Weyl group attached to T. Let s_2 and s_4 in W be, respectively, the reflections associated to the roots $\epsilon_2 - \epsilon_3$ and $\epsilon_3 + \epsilon_4$. Then, w and s can be, respectively, seen as representatives of the Weyl group elements $s_4 s_2$ and s_2.

6.3. Some additional comments. — In our context, the Lagrangian space of Perrin and Smirnov corresponds exactly to H/P_H, where P_H is the parabolic subgroup of H containing B_H, and characterized by the $(r/2)^{th}$ simple root $(\varepsilon'_{r/2} - \varepsilon'_{r/2+1})$ according to Section 4). The problem of embedding then amounts to the existence of an algebraic morphism ψ from G/B to H/P_H making the following diagram commutative

$$\begin{tikzcd} H \times^{B_H} \overline{B \cdot wB} \arrow[r, "k"] \arrow[d, "q"'] & H/B_H \arrow[d, hook] \\ G/B \arrow[r, "\psi"] & H/P_H . \end{tikzcd}$$

Let us remark that the line bundle $\mathcal{L}_H(\rho_{G|T_H} - 2\rho_H)$ on H/B_H is the pullback of an equivariant line bundle on H/P_H so that such a diagram would guarantee that we satisfy the hypothesis (iv)$'$ of Theorem 3.1. Therefore, the embedding problem seems to be crucial for obtaining rational resolutions and the Cohen–Macaulay property, in Perrin and Smirnov's argument, as well as in ours. While the previous counterexample does not rigorously forbid the existence of ψ, it does indicate that another approach might be necessary if we want to prove these two additional results.

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