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REVISITING DERIVED CRYSTALLINE COHOMOLOGY

BY ZHOUHANG MAO

ABSTRACT. — We prove that the ∞-category of surjections of animated rings is projectively generated, introduce and study the notion of animated PD-pairs—surjections of animated rings with a "derived" PD-structure. This allows us to generalize classical results to nonflat and nonfinitely generated situations.

Using animated PD-pairs, we develop several approaches to derived crystalline cohomology and establish comparison theorems. As an application, we generalize the comparison between derived and classical crystalline cohomology from syntomic (affine) schemes (due to Bhatt) to quasisyntomic schemes.

We also develop a noncompleted animated analogue of prisms and prismatic envelopes. We prove a variant of the Hodge–Tate comparison for animated prismatic envelopes from which we deduce a result about flat cover of the final object for quasisyntomic schemes, which generalizes several known results under smoothness and finiteness conditions.

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Résumé (*Revisiter la cohomologie cristalline dérivée*). — Nous prouvons que la ∞-catégorie des surjections d'anneaux animés est projectivement générée, introduisons et étudions la notion de PD-paires animées – des surjections d'anneaux animés avec une PD-structure "dérivée". Cela nous permet de généraliser des résultats classiques à des situations non plates et non de type fini.

En utilisant les PD-paires animées, nous développons plusieurs approches de la cohomologie cristalline dérivée et établissons des théorèmes de comparaison. En tant qu'application, nous généralisons la comparaison entre la cohomologie cristalline dérivée et classique à partir de schémas syntomiques (affines) (due à Bhatt) à des schémas quasisyntomiques.

Nous développons également un analogue animé non complété des prismes et des enveloppes prismatiques. Nous prouvons une variante de la comparaison de Hodge-Tate pour les enveloppes prismatiques animées, à partir de laquelle nous déduisons un résultat sur le recouvrement plat de l'objet final pour les schémas quasisyntomiques, qui généralise plusieurs résultats connus sous des conditions de lissité et de finitude.

1. Introduction

In this introductory section, we start with a nontechnical discussion of the background, stating the main results in simplified forms. Then we explain the main techniques used in this article. After that, we present the main definitions and constructions used in this article.

1.1. Background and main results. — In this section, we discuss the background and the main results of the current work in a simplified form.

Regular sequences and local complete intersections play an important role in the study of Noetherian rings. However, in arithmetic geometry, Noetherianness is not preserved by operations related to perfectoids. Various generalizations to the non-Noetherian case are available. In [11], it has been shown that, the *quasiregularity* (à la Quillen) is a particularly good candidate to replace (Koszul) regularity in classical algebraic geometry: an ideal I of a ring A is called *quasiregular* (Definition 3.53) if the A/I-module I/I^2 is flat and the homotopy groups $\pi_i(L_{(A/I)/A})$ of the cotangent complex vanish for $i > 1$, or equivalently put, $L_{(A/I)/A} \simeq (I/I^2)[1]$. In particular, if an ideal is generated by a Koszul-regular sequence, then it is also quasiregular.

Let us briefly review some details in the simple case of characteristic p (instead of mixed characteristic). An $\mathbb{F}_p$-algebra R is called *perfect* if the Frobenius map $R \to R, x \mapsto x^p$ is bijective. An $\mathbb{F}_p$-algebra S is called *quasiregular semiperfect* if there exists a perfect $\mathbb{F}_p$-algebra R along with a surjective map $R \twoheadrightarrow S$ of rings of which the kernel $I \subseteq R$ is quasiregular. In this case, [11, Thm 8.12] shows that the derived de Rham cohomology of R with respect to the base $\mathbb{F}_p$ is concentrated in degree 0, and as a ring, it is equivalent to the

PD-envelope of (R, I). Since the cotangent complex $L_{R/\mathbb{F}_p}$ vanishes, the base $\mathbb{F}_p$ of the derived de Rham cohomology can be replaced by R.

This result was already known [9, Thm 3.27] when the kernel I of the map $R \twoheadrightarrow S$ in question is Koszul regular. In other words, [11] generalizes the classical results about Koszul-regular ideals to quasiregular ideals.

In this article, we develop a different approach, which works with greater generality: we do not need the base to be perfect, of characteristic p or even "p-local"Noetherian rings. such as $\mathbb{Z}_p$ or a perfectoid ring. We build a machinery to extend results about Koszul-regular ideals to quasiregular ideals in a systematic fashion. We say that a map $R \to S$ of animated rings [14, §5.1] is *surjective* if the induced map $\pi_0(R) \to \pi_0(S)$ is surjective (Definition 3.27).

THEOREM 1.1 (Theorem 3.29). — *The ∞-category of surjective maps of animated rings is projectively generated. The set $\{\mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_n] \twoheadrightarrow \mathbb{Z}[x_1, \dots, x_m] \mid m, n \in \mathbb{N}\}$ of objects forms a set of compact projective generators.*

For technical reasons, we will introduce the ∞-category of *animated pairs*, which is equivalent to the ∞-category of surjective maps of animated rings. By the formalism of left derived functors (Proposition A.14), given a functor defined for "standard" Koszul-regular pairs $(\mathbb{Z}[X, Y], (Y))$ where $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_m\}$[1], we get a functor defined on *all* animated pairs, and in particular, on classical ring-ideal pairs (A, I), and any comparison map between such functors is determined by the restriction to these Koszul-regular pairs. We learned the importance of such standard pairs from the proof of [8, Cor 4.14].

In order to formulate a reasonable generalization of [11, Thm 8.12], just as we need animated pairs, we also need *animated PD-pairs* (Definition 3.21), denoted by $(A \twoheadrightarrow A'', \gamma)$ (Notation 3.31). There is a canonical forgetful functor from the ∞-category of animated PD-pairs to the ∞-category of animated pairs, which preserves small colimits (Proposition 3.40). This is remarkable since the forgetful functor from the 1-category of PD-pairs to the 1-category of ring-ideal pairs does not preserve small colimits (Remark 3.41). Moreover, the forgetful functor admits a left adjoint, called the *animated PD-envelope functor*.

In general, the animated PD-envelope, considered as a kind of derived functor, is different from the PD-envelope. We will show that, there is a canonical filtration on the animated PD-envelope of $\mathbb{F}_p$-pairs[2] (i.e., pairs (A, I) where A is an $\mathbb{F}_p$-algebra), called the *conjugate filtration* (Definition 3.68), of which we can control the associated graded pieces:

1. In this article, the multivariable notations X and Y are used from time to time.
2. Or more generally, of animated $\mathbb{F}_p$-pairs.

THEOREM 1.2 (Corollaries 3.61 and 3.71). — *Let A be an $\mathbb{F}_p$-algebra and $I \subseteq A$ an ideal*[3]*. Then*

1. *The animated PD-envelope of (A, I) admits a natural animated $\varphi_A^*(A/I)$-algebra structure.*
2. *For every $i \in \mathbb{N}$, the $(-i)$-th associated graded piece of the animated PD-envelope of A is, as a $\varphi_A^*(A/I)$-module spectrum, naturally equivalent to $\varphi_A^*(\Gamma^i_{A/I}(L_{(A/I)/A}[-1]))$, where $\Gamma^i_{A/I}$ is the i-th derived divided power.*

As a corollary, the notion of quasiregularity provides an important *acyclicity condition*: along with a mild assumption, the animated PD-envelope coincides with the classical PD-envelope:

THEOREM 1.3 (Corollary 3.79). — *Let A be an $\mathbb{F}_p$-algebra, $I \subseteq A$ a quasiregular ideal. Suppose that the (derived) Frobenius twist $(A/I) \otimes^{\mathbb{L}}_{A,\varphi_A} A$ is concentrated in degree 0, i.e., $\mathrm{Tor}^i_A(A/I, A) \cong 0$ (where the last A is viewed as an A-module via the Frobenius $\varphi_A : A \to A$) for all $i \in \mathbb{N}_{>0}$. Then the animated PD-envelope of (A, I) coincides with the classical PD-envelope.*

We want to point out that $(A/I) \otimes^{\mathbb{L}}_{A,\varphi_A} A$ being concentrated in degree 0 is a very mild assumption. For example, when $I \subseteq A$ is generated by a Koszul-regular sequence, then this holds automatically [9, Lem 3.41]. This also happens when (A, I) comes from a "good" PD-envelope;see Remark 4.73. Using this, we show that

THEOREM 1.4 (Proposition 3.83). — *Let A be a ring and $I \subseteq A$ an ideal generated by a Koszul-regular sequence. Then the animated PD-envelope of (A, I) coincides with the classical PD-envelope.*

Moreover, this mild assumption is not needed if we are only interested in associated graded pieces of the PD-filtration, which answers a question of Illusie [24, VIII. Ques 2.2.4.2]:

THEOREM 1.5 (Propositions 3.90 and 3.98). — *Let A be an $\mathbb{F}_p$-algebra, $I \subseteq A$ a quasiregular ideal. Then there is a canonical comparison map from the animated PD-envelope to the classical PD-envelope (B, J, γ) of (A, I) compatible with PD-filtrations that induces equivalences on associated graded pieces. Furthermore, these associated graded pieces $\mathrm{gr}^*_{\mathrm{PD}} B$, as a graded commutative ring, are given by the free divided power A/I-algebra $\Gamma^*_{A/I}(I/I^2)$ generated by the A/I-module I/I^2.*

The key point is that animated PD-envelopes admit natural PD-filtrations of which we can control the associated graded pieces (Proposition 3.90).

3. In the introduction, for sake of simplicity, we usually replace the occurrences of animated pairs (resp. animated PD-pairs) by ring-ideal pairs (resp. PD-pairs) as input data.

Based on animated PD-pairs, we develop a theory of *derived crystalline cohomology* (Definition 4.21) based on a technical construction called *derived de Rham cohomology of a map of animated PD-pairs* (Definition 4.10) that generalizes the derived de Rham cohomology of a map of animated rings. In other words, our derived crystalline cohomology should be understood as a variant of derived de Rham cohomology, not site-theoretic cohomology. These functors preserve small colimits by Proposition 4.23 and Lemma 4.14; therefore formal properties such as base change compatibility and "Künneth" formula hold (Corollaries 4.24, 4.26 and 4.27).

In fact, the animated PD-envelope is, roughly speaking, a special case of derived crystalline cohomology:

THEOREM 1.6 (Proposition 4.75). — *Let (A, I, γ_A) be a PD-pair and $J \subseteq A$ be an ideal containing I. Let $(B \twoheadrightarrow A/J, \gamma_B)$ be the relative animated PD-envelope of (A, J) with respect to the PD-pair (A, I, γ_A). Then the underlying $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra of B is equivalent to the derived crystalline cohomology of A/J with respect to (A, I, γ_A).*

From this, we deduce a generalization of [11, Thm 8.12] under quasiregularity and the Tor-independent assumption mentioned above. To see this, similarly to the animated PD-envelope, we introduce the *conjugate filtration* on the derived crystalline cohomology (Definition 4.49) and on the relative animated PD-envelope (Definition 4.69) in characteristic p, and we have a similar control of associated graded pieces for the conjugate filtration on relative animated PD-envelopes (Corollary 4.70) and also on the derived crystalline cohomology, which is a crystalline variant of the *Cartier isomorphism* (cf. [9, Prop 3.5] over the base $(A, I, \gamma) = (\mathbb{F}_p, 0, 0)$):

THEOREM 1.7 (Proposition 4.54). — *Let (A, I, γ) be a PD-pair, where A is an $\mathbb{F}_p$-algebra. Note that the Frobenius map $\varphi_A : A \to A$ factors through $A \twoheadrightarrow A/I$, giving rise to a natural map $\varphi_{(A,I)} : A/I \to A$ (cf. Lemma 4.42). Then for every animated A/I-algebra R and $n \in \mathbb{N}$, the $(-i)$-th associated graded piece of the conjugate filtration on the derived crystalline cohomology of R relative to (A, I, γ) is, as a $\varphi^*_{(A,I)}(R)$-module spectrum, equivalent to $\varphi^*_{(A,I)} \left(\bigwedge^i_R L_{R/(A/I)} \right) [-i]$.*

On the other hand, similarly to [5], we develop an *affine crystalline site* (Definition 4.76) based on animated PD-pairs (Bhatt already indicated such a possibility;see the paragraph before [9, Ex 3.21]). Recall that a map $A \to R$ of rings is called *quasisyntomic* (Definition 4.96) if it is flat and the cotangent complex $L_{R/A}$, as an R-module spectrum, has Tor-amplitude in $[0, 1]$. We could also compare the derived crystalline cohomology to the site-theoretic cohomology:

THEOREM 1.8 (Propositions 4.77, 4.98 and 4.101). — *Let (A, I, γ_A) be a PD-pair and R an A/I-algebra.*

1. *There is a comparison map from the derived crystalline cohomology of R with respect to (A, I, γ_A) to the cohomology of the affine crystalline site, which is an equivalence when as an A/I-algebra, R is either of finite type, or quasisyntomic.*
2. *There is a comparison map from the cohomology of the affine crystalline site to the (classical) crystalline cohomology of R with respect to (A, I, γ_A). When R is a quasisyntomic A/I-algebra.*
   (a) *Supposing that p is nilpotent in A, then the comparison map is an equivalence.*
   (b) *Supposing that A is p-torsion free, then the comparison map becomes an equivalence after derived p-completion, or equivalently, after derived modulo p.*

The theorem above generalizes [9, Prop 3.25], which is established for syntomic algebras.

We do not know whether the derived crystalline cohomology and the cohomology of the affine crystalline site are equivalent without any assumption;we reduced this equivalence to a descent property of the derived crystalline cohomology "with respect to the base animated PD-pair" (Proposition 4.81).

In addition to PD-pairs and the crystalline cohomology, we also introduce *animated δ-rings* and *animated δ-pairs*, and a noncomplete but animated version of prisms, the static version of which was introduced in [12]. Similarly to animated PD-envelopes, the noncompleted animated prismatic envelope, which generalizes[4] the prismatic envelope for local complete intersections [12, Prop 3.13], admits the *conjugate filtration* of which the associated graded pieces are easily determined by a variant of the *Hodge–Tate comparison*:

THEOREM 1.9 (Theorem 5.49). — *Let (A, d) be a prism and $J \subseteq A/d$ an ideal. Then for every $i \in \mathbb{N}$, the $(-i)$-th associated graded piece of noncompleted prismatic envelope, as an $A/(d, J)$-module spectrum, is equivalent[5] to $\Gamma^i_{A/(d,J)}(L_{(A/(d,J))/(A,d)}[-1])$.*

As a corollary, similarly to animated PD-envelopes, when the ideal J is p-completely quasiregular, roughly speaking, the (p, d)-completed animated prismatic envelope satisfies the universal property of the prismatic envelope in [12, Prop 3.13] (Remark 5.54). Furthermore, the noncompleted prismatic envelope satisfies a faithful flatness (Proposition 5.52), which leads to a technical result that is essentially about the flat cover of the final object (Proposition 5.58), and a similar argument shows the (p, d)-completed variant:

4. More precisely, it is a noncompleted version.
5. Here we suppress the Breuil–Kisin twists.

THEOREM 1.10 (Proposition 5.59). — *Let (B, d) be a bounded oriented prism, R a derived p-complete and p-completely quasisyntomic B/d-algebra. Let P be a derived (p, d)-complete animated δ-B-algebra, which is (p, d)-completely quasismooth over B, equipped with a surjection $P \twoheadrightarrow R$ of B-algebras. Then the (completed) prismatic envelope of $P \twoheadrightarrow R$ exists and is a flat cover of the final object in the prismatic site.*

We stress that our theory is noncompleted. Technically, it is easier to deal with noncompleted version than with p-completed version because the ∞-category of p-completed objects is usually not projectively generated. For example, $\mathbb{Z}_p \in D_{\mathrm{comp}}(\mathbb{Z}_p)$ is not a compact object. We could overcome this issue by applying the techniques developed in Section 2.5, but it would make the theory inconvenient.

However, thanks to Clausen–Scholze's condensed mathematics, the noncompleted version could serve as a cornerstone of an analytic version that allows us to put "topologies" and "analytic structures" on our animated rings.

1.2. Main techniques. — We systematically adopt two techniques in this article: the *animation* and a kind of local-global principle for $\mathbb{Z}$. We briefly summarize them as follows.

There is a procedure to associate to 1-projectively generated 1-categories projectively generated ∞-categories, called the *animation*, a concept abstracted out in [14, §5.1], and defined by the *non-abelian derived category* [28, §5.5.8] of a *set of compact* 1*-projective generators*.

EXAMPLE. — The abelian category of R-modules admits a set of compact 1-projective generators given by free R-modules of finite rank. The animation of this category is the connective part $D_{\geq 0}(R)$ of the derived category $D(R)$.

EXAMPLE. — The 1-category of rings admits a set of compact 1-projective generators given by polynomial rings on finitely many variables.

REMARK. — It is not a coincidence that the sets of compact 1-projective generators above are given by "finite free objects". Indeed, it is a corollary of Proposition A.18, applied to the pairs $\mathrm{Set} \rightleftarrows \mathrm{Mod}_R$ and $\mathrm{Set} \rightleftarrows \mathrm{Ring}$ of adjoint functors.

We review the definition of animation and summarize its main properties in Appendix A.2. When applying this construction to the 1-category of rings, we get *the ∞-category of animated rings*. We apply this construction to the 1-category of δ-rings, obtaining the *∞-category of animated δ-rings* (Definition 5.4).

The technical advantage of this construction is that, to produce a sifted-colimit-preserving functor from a projectively generated ∞-category, it suffices

to produce a functor from the full subcategory spanned by a set of compact projective generators that, as we have seen, is given by "finite free objects".

Now we want to apply this procedure to the 1-category of ring-ideal pairs. Unfortunately, the 1-category of ring-ideal pairs might not be 1-projectively generated. However, it is reasonable to say that "standard" Koszul-regular pairs $(\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n], (y_1, \ldots, y_n))$ are "finite free objects". We pick the non-abelian derived category of the full category spanned by these pairs, and the 1-category of ring-ideal pairs embeds fully faithfully into it (Proposition 3.23). This ∞-category is equivalent to the ∞-category of surjections of animated rings (Theorem 3.29). Similarly, we apply this "modified animation" to the 1-category of PD-pairs, obtaining the *∞-category of animated PD-pairs*. The PD-envelope functor gives rise to the *animated PD-envelope* (Definition 3.21): a "good enough" pair of adjoint functors between 1-projectively generated 1-categories give rise to a pair of adjoint functors between animations (Corollary 2.3). However, here the story is slightly more complicated due to our "modification" of the animation.

In a similar fashion, we apply these animation techniques to δ-pairs, obtaining *animated δ-pairs* (Definition 5.9), and we use similar techniques to define and analyze noncompleted animated prismatic envelopes. We also use the animation techniques to define the "de Rham context" dRCon, the "crystalline context" CrysCon, the derived de Rham cohomology and the derived crystalline cohomology in Section 4.1.

Now we describe the second main technique that we use: the local-global principle for $\mathbb{Z}$. Some techniques are only valid in characteristic p. For example, we do not know how to define the conjugate filtration on the derived crystalline cohomology beyond characteristic p. However, these arithmetic objects, such as PD-structures, usually degenerate in characteristic 0. In view of these, we can usually then glue the results for each prime number $p \in \mathbb{N}$ and the result after rationalization. The simplest example of this technique is the following: let $X \in \mathrm{Sp}$ be a spectrum. Suppose that the spectrum $X/^{\mathbb{L}}p$ is equivalent to 0 for every prime number $p \in \mathbb{N}$, and that X is also contractible after rationalization. Then the spectrum X itself is contractible. We establish similar results (Lemmas 3.80 and 3.82) under connectivity assumptions. These results allow us to deduce integral results.

1.3. Main definitions and constructions. — In this section, we present the main definitions and constructions appearing in the current work, explaining the techniques mentioned above in more detail. As explained above, we introduce the following concept:

Definition 1.11 (Section 3.2). — The *∞-category of animated pairs* is the non-abelian derived category of the 1-category of ring-ideal pairs $(\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n], (y_1, \ldots, y_n))$. The *$\infty$-category of animated PD-pairs*

is the non-abelian derived category of the 1-category of PD-pairs $\Gamma_{\mathbb{Z}[x_1,\dots,x_m]}(y_1,\dots,y_n) \twoheadrightarrow \mathbb{Z}[x_1,\dots,x_m]$.

As mentioned before, later we will write these pairs as $(\mathbb{Z}[X,Y],(Y))$ for finite sets $X=\{x_1,\dots,x_m\}$ and $Y=\{y_1,\dots,y_n\}$. We will systematically identify the ring-ideal pairs (A,I) with the surjections $A \twoheadrightarrow A/I$ of rings. The notation $\Gamma_R(y_1,\dots,y_n)$ denotes the free PD-R-algebra[6] generated by $y_1,\dots,y_n$. One can show that the 1-category of ring-ideal pairs (resp. PD-pairs) embeds fully faithfully into the ∞-category of animated pairs (resp. animated PD-pairs);see Proposition 3.23. Then the first main theorem is the following

THEOREM 1.12 (Theorem 3.29). — *The left derived functor of sending the animated pair $(\mathbb{Z}[X,Y],(Y))$ to the surjection $\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X]$ of animated rings identifies the ∞-category of animated pairs with the ∞-category of surjections of animated rings.*

One can prove this directly. The proof presented in this article follows an indirect approach where one first shows an "linear analogue" for $D(\mathbb{Z})_{\geq 0}$ instead of $\mathrm{CAlg}^{\mathrm{an}}$ and then proves that this equivalence is "compatible with the multiplicative structure".

As explained before, this "modified animation" allows us to extend constructions for ring-ideal pairs $(\mathbb{Z}[X,Y],(Y))$ to animated pairs (or equivalently, surjections of animated rings). In particular, the (Y)-adic filtration gives rise to the *adic filtration* (Section 3.4), the equivalence $(Y^n)/(Y^{n+1}) \cong \mathrm{Sym}_{\mathbb{Z}[X]}((Y)/(Y^2))$ gives rise to its "derived version", and the computation of the cotangent complex $L_{\mathbb{Z}[X]/\mathbb{Z}[X,Y]} \simeq ((Y)/(Y^2))[1]$ gives rise to a similar property for arbitrary surjective maps of animated rings (Corollary 3.61). The PD-envelope functor, sending $(\mathbb{Z}[X,Y],(Y))$ to $\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]$, gives rise to the *animated PD-envelope functor* (Section 3.2). Similarly, the "modified animation" allows us to extend the classical PD-filtration on $\Gamma_{\mathbb{Z}[X]}(Y)$ to the *PD-filtration* on animated PD-pairs (Definition 3.85), and the *conjugate filtration* on animated PD-envelopes (Definition 3.68). The result about associated graded pieces of the PD-filtration (resp. the conjugate filtration) for the "standard case" $\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]$ (resp. $(\mathbb{Z}[X,Y],(Y))$) extends in a direct manner to animated PD-pairs (resp. animated pairs);see Lemma 3.87 (resp. Corollary 3.71). Furthermore, we know how to detect whether an animated pair (resp. animated PD-pair) is a classical ring-ideal pair (resp. PD-pair); see Proposition 3.35 (resp. Proposition 3.38). Consequently, when the input is a quasiregular ring-ideal pair with some mild conditions, we can deduce that the animated version coincides with the classical version (Corollary 4.71 and Propositions 3.83 and 3.98).

6. This is sometimes denoted by $R\langle y_1,\dots,y_n\rangle$ in the literature. We choose our notation to avoid confusion with that of topologically free algebras also usually denoted by $R\langle y_1,\dots,y_n\rangle$.

Based on animated pairs and animated PD-pairs, we define the *derived de Rham cohomology* associated to a map of animated PD-pairs. More precisely, we have

LEMMA 1.13 (Section 4.1). — *The ∞-category of maps of animated PD-pairs is projectively generated. A set of compact projective generators is given by* $(\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]) \to (\Gamma_{\mathbb{Z}[X,X']}(Y,Y') \twoheadrightarrow \mathbb{Z}[X,X'])$.

Therefore, we can extend the de Rham cohomology defined on these "standard" maps of "standard" PD-pairs to arbitrary maps of animated PD-pairs. It turns out that the derived de Rham cohomology of a map $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ of animated PD-pairs "does not depend on B" (Proposition 4.20), which is a generalization of the fact that the crystalline cohomology computed by the de Rham complex does not depend on the choice of the lift. This leads to the definition of the derived crystalline cohomology (Definition 4.21). The Hodge-filtration and the conjugate filtration also extends from "standard" maps of "standard" PD-pairs to maps of animated PD-pairs (Section 4.2). The associated graded pieces of the conjugate filtration is determined by a crystalline variant of the Cartier isomorphism (Proposition 4.54).

Similarly to the classical case, there is also a relative concept of animated PD-envelopes (Definition 4.57), and the relative animated PD-envelope could be reduced to the special case associated to the datum $((A \twoheadrightarrow A'', \gamma_A), A'' \twoheadrightarrow R)$, where $(A \twoheadrightarrow A'', \gamma_A)$ is an animated PD-pair and $A'' \twoheadrightarrow R$ is a surjection of animated rings (Lemma 4.64). Furthermore, we have the following lemma.

LEMMA 1.14 (Lemma 4.65). — *The ∞-category of data $((A \twoheadrightarrow A'', \gamma_A), A'' \twoheadrightarrow R)$ is projectively generated.*

Again, we can apply the techniques of non-abelian derived categories to this category, and along with the local-global principle for $\mathbb{Z}$ mentioned before, we show that the derived crystalline cohomology "coincides" with the relative animated PD-envelope (Proposition 4.75), which generalizes Bhatt's computation [9, Thm 3.27] of the derived de Rham cohomology $\mathrm{dR}_{\mathbb{F}_p[x]/\mathbb{F}_p}$.

In order to compare the derived crystalline cohomology with the classical crystalline cohomology, we introduce an animated variant of the crystalline site (Definition 4.76). Our main tools are the Čech–Alexander computation, the Katz–Oda filtration (Definition 4.86) introduced in [19] and the conjugate filtration. More precisely, the Katz–Oda filtration allows us to identify graded pieces associated to the Hodge filtration (Lemma 4.88), and the conjugate filtration allows us to establish a descent-type result in characteristic p (Lemma 4.90). Along with the local-global principle for $\mathbb{Z}$ mentioned before, we prove the comparison theorem for quasisyntomic schemes (Proposition 4.98).

Finally, we introduce the ∞-*category of animated* δ-*rings* (Definition 5.4), which is simply the animation of the 1-category of δ-rings, and the following definition.

DEFINITION 1.15 (Definition 5.9). — An *animated* δ-*pair* is an animated δ-ring A along with a surjection $A \twoheadrightarrow A''$ of animated rings.

Using similar techniques, we construct the noncomplete animated prismatic envelope (Corollary 5.27), the conjugate filtration (Lemma 5.38) and the Hodge–Tate comparison (Theorem 5.49). Similarly to "animated PD-envelope being classical under quasiregularity with some mild conditions", we deduce a flatness result of prismatic envelopes under quasiregularity (Proposition 5.52), which is sufficient for the flat cover results mentioned before (Propositions 5.58 and 5.59).

1.4. Structure of the article. — Here is a guideline to the article. Section 2 is devoted to technical preparations. We suggest the readers skip it in the first reading. Section 3 is devoted to the theory of animated pairs and animated PD-pairs and to the study of the animated PD-envelope. Section 4 is devoted to relative animated PD-envelopes, derived crystalline cohomology, cohomology of the affine crystalline site and their comparisons. Section 5 is devoted to animated δ-rings, animated δ-pairs, "noncomplete animated prisms", noncompleted animated prismatic envelope and a variant of the Hodge–Tate comparison. Appendix A is a collection of basic facts about animations and projectively generated categories (which we suggest the reader read first if they have not seen this concept before).

1.5. Notations and terminology. — In this article, since we often work in the ∞-category of certain "derived" categories, we try to distinguish the "ordinary" objects and "derived" objects by choosing different words.

Given an ∞-category $\mathcal{C}$ and a diagram $Y \leftarrow X \rightarrow Z$ in $\mathcal{C}$, the pushout of the diagram is denoted by $Y \amalg_X Z$. In particular, if $\mathcal{C}$ admits an initial object, the coproduct of two objects Y, Z is denoted by $Y \amalg Z$.

We will denote by An the ∞-category of (small) animæ, that is, the simplicial nerve of the simplicial category of (small) Kan complexes [28, Def 1.2.16.1].

We say that an anima $X \in \mathrm{An}$ or a spectrum $X \in \mathrm{Sp}$ is *static*[7] if $\pi_i(X) \cong 0$ for all $i \neq 0$. For two spectra $X, Y \in \mathrm{Sp}$, we will denote by $X \otimes^{\mathbb{L}} Y$ the smash product. *Rings* are always static and commutative, while $\mathbb{E}_n$-*rings* are $\mathbb{E}_n$-algebras in the symmetric monoidal ∞-category $(\mathrm{Sp}, \otimes^{\mathbb{L}})$.

7. This is usually called *discrete* in homotopy theory. We follow Clausen–Scholze's terminology in condensed mathematics to call them *static* to distinguish from the point-set topological discreteness. In particular, the *static* object $\mathbb{Z}_p$ might be equipped with the p-adic topology which is different from the *discrete* topology.

Given a ring A, we will refer to a "classical" A-module a *static A-module*. The category of static A-modules will be denoted by Mod_A. The category of *ring-module pairs* (A, M) where $M \in \mathrm{Mod}_A$ is denoted by Mod. An object in the derived ∞-category $D(A)$ an *A-module spectrum*.

Given an $\mathbb{E}_1$-ring A, the ∞-category of left (resp. right) A-module spectra will be denoted by LMod_A (resp. RMod_A). Given a right A-module spectrum M and a left A-module spectrum N, their relative tensor product is denoted by $M \otimes^{\mathbb{L}}_A N$, to avoid confusion with the ordinary tensor product of static modules.

Given an $\mathbb{E}_\infty$-ring A, the ∞-category of A-module spectra is denoted by $D(A)$. In particular, we have $\mathrm{Sp} = D(\mathbb{S})$. An *$\mathbb{E}_n$-$A$-algebra* is an $\mathbb{E}_n$-algebra in the symmetric monoidal ∞-category $(D(A), \otimes^{\mathbb{L}}_A)$.

Following [10], we will denote by $\mathrm{CAlg}^{\mathrm{an}}$ the ∞-category of animated rings, and $\mathrm{CAlg}^{\mathrm{an}}_R$ the ∞-category of animated R-algebras for an animated ring R.

2. Categorical preparations

In this section, we will do some technical preparations of ∞-categories, which will be used throughout this article. We try our best to refer to this section explicitly so that the reader could first skip this section and then read it when needed.

2.1. Animation of adjoint functors. — This section is devoted to proving that animation behaves well for certain "monadic" pairs of adjoint functors. Here is a general lemma.

LEMMA 2.1. — *Let $n \in \mathbb{N}_{>0} \cup \{\infty\}$. Let $\mathcal{C}$ be a small n-category that admits finite coproducts and $\mathcal{D}$ a locally small n-category, which admits small colimits. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite coproducts. Then, the following hold.*

1. *There is a pair of adjoint functors $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ (Notation A.10), where F is the left derived functor (Proposition A.14) of f and G is the functor given by $D \mapsto \mathrm{Map}_{\mathcal{D}}(f(\cdot), D) \in \mathcal{P}(\mathcal{C})$.*
2. *Suppose that for all objects $C \in \mathcal{C}$, the object $f(C) \in \mathcal{D}$ is compact and n-projective. Then the functor G preserves filtered colimits and geometric realizations. Under this assumption, if f is further assumed to be fully faithful, then so is F.*
3. *Suppose that the set $\{f(C) \,|\, C \in \mathcal{C}\} \subseteq \mathcal{D}$ generates $\mathcal{D}$ under small colimits. Then the functor G is conservative.*

Proof. — We exhibit the proof for $n = \infty$. First, the functor $f : \mathcal{C} \to \mathcal{D}$ extends uniquely to a functor $\tilde{F} : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$, which preserves small colimits by [28, Thm 5.1.5.6]. Since $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is stable under sifted colimits, it follows

that the functor F is equivalent to the composite $\mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C}) \xrightarrow{\tilde{F}} \mathcal{D}$. The functor $\tilde{F}$ admits a right adjoint by [28, Cor 5.2.6.5], which is equivalent to the composite $\mathcal{D} \xrightarrow{G} \mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C})$; therefore (F, G) is a pair of adjoint functors.

Part 2 follows from the fact that $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is stable under sifted colimits (Proposition A.11). The later statement follows from Proposition A.15.

Suppose that $\{f(C) \,|\, C \in \mathcal{C}\}$ generates $\mathcal{D}$ under small colimits, then for any map $X \to Y$ in $\mathcal{D}$, if the induced map $G(X) \to G(Y)$ is an equivalence in $\mathcal{P}_\Sigma(\mathcal{C})$, then for all objects $C \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{D}}(f(C), X) \to \mathrm{Map}_{\mathcal{D}}(f(C), Y)$ is an equivalence. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the full subcategory spanned by those $D \in \mathcal{D}$ such that the induced map $\mathrm{Map}_{\mathcal{D}}(D, X) \to \mathrm{Map}_{\mathcal{D}}(D, Y)$ is an equivalence. Then $\mathcal{D}'$ is stable under colimits, and $f(C) \in \mathcal{D}'$ for all $C \in \mathcal{C}$. The result follows. □

Then from Lemma A.13 and Corollary A.17 we have the following.

COROLLARY 2.2. — *Let $\mathcal{C}, \mathcal{D}$ be two small n-categories that admit finite coproducts and $f : \mathcal{C} \to \mathcal{D}$ a functor that preserves finite coproducts. Then, we have the following.*

1. *There is a pair of adjoint functors $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{P}_{\Sigma,n}(\mathcal{D})$, where G is given by $\mathcal{P}_{\Sigma,n}(\mathcal{D}) \ni H \mapsto H \circ F \in \mathcal{P}_{\Sigma,n}(\mathcal{C})$ and F is the left derived functor of the composite functor $\mathcal{C} \xrightarrow{f} \mathcal{D} \hookrightarrow \mathcal{P}_{\Sigma,n}(\mathcal{D})$.*
2. *The functor G preserves sifted colimits, and the canonical map $\tau_{\leq m} \circ G \to G \circ \tau_{\leq m}$ of functors is an equivalence for all $m \in \mathbb{N}$ (cf.* [28, Rem 5.5.8.26] *and the discussion before Lemma A.26).*
3. *If f is fully faithful, then so is the functor F.*
4. *If f is essentially surjective, then the functor G is conservative.*

Now we apply this to animations:

COROLLARY 2.3. — *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors between 1-categories such that the following hold.*

1. *The 1-category $\mathcal{D}$ admits filtered colimits and reflexive coequalizers (or equivalently, geometric realizations, by Remark A.7), and G preserves filtered colimits and reflexive coequalizers.*
2. *The 1-category $\mathcal{C}$ is projectively generated.*
3. *The functor G is conservative.*

Then $\mathcal{D}$ is 1-projectively generated, and we have a pair $\mathrm{Ani}(\mathcal{C}) \underset{\mathrm{Ani}(G)}{\overset{\mathrm{Ani}(F)}{\rightleftarrows}} \mathrm{Ani}(\mathcal{D})$ of adjoint functors between ∞-categories after animation. Furthermore, the functor $\mathrm{Ani}(G)$ is conservative, preserves sifted colimits, and the canonical map $\tau_{\leq 0} \circ \mathrm{Ani}(G) \to G \circ \tau_{\leq 0}$ of functors is an equivalence. If G preserves small colimits, then so does $\mathrm{Ani}(G)$.

Proof. — It follows from Proposition A.18 that the 1-category $\mathcal{D}$ is 1-projectively generated, and, therefore, $\mathcal{C}, \mathcal{D}$ admit small colimits that are preserved by F. Furthermore, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of a chosen set of compact 1-projective generators for $\mathcal{C}$, and $\mathcal{D}^0 \subseteq \mathcal{D}$ the full subcategory spanned by the images of objects of $\mathcal{C}$ under F, then there are equivalences $\mathcal{C} \simeq \mathcal{P}_{\Sigma,1}(\mathcal{C}^0)$ and $\mathcal{D} \simeq \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ of 1-categories by Proposition A.16 (note that F preserves finite coproducts).

Let $f : \mathcal{C}^0 \to \mathcal{D}^0$ be the functor induced by F, which preserves finite coproducts and is essentially surjective. It follows from Corollary 2.2 with $n = 1$ and the uniqueness of the right adjoint functor that the functor $G : \mathcal{D} \to \mathcal{C}$ is equivalent to $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \to \mathcal{P}_{\Sigma,1}(\mathcal{C}^0), H \mapsto H \circ f$.

We invoke again Corollary 2.2 with $n = \infty$ to obtain a pair of adjoint functors $\mathcal{P}_\Sigma(\mathcal{C}^0) \rightleftarrows \mathcal{P}_\Sigma(\mathcal{D}^0)$ induced by f. It follows from the definitions that $\mathrm{Ani}(\mathcal{C}) \simeq \mathcal{P}_\Sigma(\mathcal{C}^0)$, $\mathrm{Ani}(\mathcal{D}) \simeq \mathcal{P}_\Sigma(\mathcal{D}^0)$ and that the functor $\mathcal{P}_\Sigma(\mathcal{C}^0) \to \mathcal{P}_\Sigma(\mathcal{D}^0)$ obtained above is equivalent to $\mathrm{Ani}(F)$. Let $G' : \mathrm{Ani}(\mathcal{D}) \to \mathrm{Ani}(\mathcal{C})$ be the right adjoint to $\mathrm{Ani}(F)$. Since f is essentially surjective, G' is conservative. It remains to show that G' is equivalent to $\mathrm{Ani}(G)$.

Indeed, both G' and $\mathrm{Ani}(G)$ preserve sifted colimits. Since the functor $G : \mathcal{D} \to \mathcal{C}$ is equivalent to $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \to \mathcal{P}_{\Sigma,1}(\mathcal{C}^0), H \mapsto H \circ f$, the restrictions of G' and $\mathrm{Ani}(G)$ to the full subcategory $\mathcal{D}^0 \subseteq \mathrm{Ani}(\mathcal{D})$ are equivalent. It then follows from Proposition A.14 that G' and $\mathrm{Ani}(G)$ are equivalent. The colimit preserving properties follow from Corollary A.25. □

Now we look at two simple examples:

Example 2.4. — Let $R \to S$ be a map of rings. Then there is a pair $\mathrm{Mod}_R \overset{\cdot \otimes_R S}{\rightleftarrows} \mathrm{Mod}_S$ of adjoint functors between the categories of static modules. Since the forgetful functor $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is conservative, and preserves small colimits, we have the pair of adjoint functors $\mathrm{Ani}(\mathrm{Mod}_R) \overset{\mathrm{Ani}(\cdot \otimes_R S)}{\rightleftarrows} \mathrm{Ani}(\mathrm{Mod}_S)$. Under the equivalences $\mathrm{Ani}(\mathrm{Mod}_R) \simeq D_{\geq 0}(R)$ and $\mathrm{Ani}(\mathrm{Mod}_S) \simeq D_{\geq 0}(S)$, the functor $\mathrm{Ani}(\cdot \otimes_R S)$ is equivalent to the functor $\cdot \otimes^{\mathbb{L}}_R S$.

Example 2.5. — Let Ring be the 1-category of rings and Ab the 1-category of abelian groups. Then we have a pair $\mathrm{Ab} \overset{\mathrm{Sym}_{\mathbb{Z}}}{\rightleftarrows} \mathrm{Ring}$ of adjoint functors. Since the forgetful functor $\mathrm{Ring} \to \mathrm{Ab}$ is conservative, and preserves filtered colimits and reflexive coequalizers, we get a pair $D_{\geq 0}(\mathbb{Z}) \overset{\mathbb{L}\mathrm{Sym}_{\mathbb{Z}}}{\rightleftarrows} \mathrm{CAlg}^{\mathrm{an}}$ of adjoint functors.

In Corollary 2.3, the functor G (resp. $\mathrm{Ani}(G)$) exhibits $\mathcal{D}$ (resp. $\mathrm{Ani}(\mathcal{D})$) as monadic over $\mathcal{C}$ (resp. $\mathrm{Ani}(\mathcal{C})$). The associated endomorphism monad is given by $G \circ F$ (resp. $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \simeq \mathrm{Ani}(G \circ F)$ by Proposition A.27).

LEMMA 2.6. — *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors between ∞-categories. Let K be a small simplicial set. Then $G \circ F$ preserves K-indexed colimits if G preserves K-indexed colimits. The converse is true if G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.*

Proof. — If G preserves K-indexed colimits, since F is a left adjoint, it follows that so does $T := G \circ F$. Conversely, if G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$, then $\mathcal{D} \simeq \mathrm{LMod}_T(\mathcal{C})$, and the result follows from [30, Cor 4.2.3.5]. □

2.2. Diagram categories and undercategories. — In this section, we will show that diagram n-categories and undercategories of n-projectively generated categories are n-projectively generated, for which we give an explicit choice of n-projective generators. We first show the version for ∞-categories, then list the analogues for n-categories for which the proof is nearly verbatim. We start with diagram categories.

LEMMA 2.7. — *Let $(\mathcal{C}_\alpha)_{\alpha \in T}$ be a small collection of the projectively generated ∞-category. Then the ∞-category $\prod_{\alpha \in T} \mathcal{C}_\alpha$ is projectively generated. More precisely, let 1_α denote the initial objects of $\mathcal{C}_\alpha$. If the collections $S_\alpha \subseteq \mathcal{C}_\alpha$ of objects are sets of compact projective generators for $\mathcal{C}_\alpha$, then the collection $\{i_{s,\beta} \mid s \in S_\beta, \beta \in T\} \subseteq \prod_{\alpha \in T} \mathcal{C}_\alpha$ is a set of compact projective generators for $\prod_{\alpha \in T} \mathcal{C}_\alpha$, where $i_{s,\beta} \in \prod_{\alpha \in T} \mathcal{C}_\alpha$ is given by $\begin{pmatrix} \begin{cases} s & \beta' = \beta \\ 1_{\beta'} & \beta' \neq \beta \end{cases} \end{pmatrix}_{\beta' \in T}$.*

Proof. — Since the small colimits in $\prod \mathcal{C}_\alpha$ are computed pointwise, it follows that $\prod \mathcal{C}_\alpha$ is cocomplete. Now given S_α and $i_{s,\beta}$, let $\mathcal{D} \subseteq \prod \mathcal{C}_\alpha$ be the full subcategory generated by $\{i_{s,\beta}\}$ under colimits. For all $\beta \in T$, the fully faithful embedding $j_\beta : \mathcal{C}_\beta \to \prod C_\alpha$ given by $C \mapsto \begin{pmatrix} \begin{cases} C & \beta' = \beta \\ 1_{\beta'} & \beta' \neq \beta \end{cases} \end{pmatrix}_{\beta' \in T}$ preserves small colimits, and $j_\beta(s) = i_{s,t}$. Thus, the "skyscraper" functor $j_\beta(C)$ is an object of $\mathcal{D}$ for $C \in \mathcal{C}_\beta$.

Finally, we can write any object $F \in \prod \mathcal{C}_\alpha$ as a small colimit $\mathrm{colim}_{\beta \in T} j_\beta(F_\beta)$, and, therefore, $\mathcal{D} = \prod \mathcal{C}_\alpha$. □

Now let $\mathcal{C}$ be a cocomplete ∞-category, $K \in \mathrm{Set}_\Delta$ a small simplicial set and $K_0 \subseteq K$ the set of vertices. Then we have a pair of adjoint functors $\mathrm{Fun}(K_0, \mathcal{C}) \underset{(K_0 \to K)^*}{\overset{\mathrm{Lan}_{K_0 \to K}}{\rightleftarrows}} \mathrm{Fun}(K, \mathcal{C})$, where $\mathrm{Lan}_{K_0 \hookrightarrow K}$ is the functor of left Kan extension along the map $K_0 \to K$, and $(K_0 \to K)^*$ denotes the restriction along $K_0 \to K$.

WARNING 2.8. — *In an early draft, we called $K_0 \to K$ an "inclusion". However, any map of simplicial sets is equivalent to a cofibration up to a trivial fibration in Joyal model structure. That is to say, the concept of "nonfull*

subcategory" is not model-independent. We decided to suppress such model-dependent expressions.

Then, from Proposition A.18 and Lemma 2.7we have the following.

COROLLARY 2.9. — *Let $\mathcal{C}$ be a projectively generated ∞-category and $K \in \mathrm{Set}_\Delta$ a small simplicial set. Then the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ of functors is projectively generated.*

Next, we study undercategories.

LEMMA 2.10. — *Let $\mathcal{C}$ be a projectively generated ∞-category and $Z \in \mathcal{C}$ an object. Then the undercategory $\mathcal{C}_{Z/}$ is projectively generated. More precisely, letting $S \subseteq \mathcal{C}$ be a set of projective generators for $\mathcal{C}$, the set $\{Z \to X \amalg Z \mid X \in S\}$ is a set of compact projective generators for the undercategory $\mathcal{C}_{Z/}$.*

Proof. — Consider the pair $\mathcal{C} \underset{Y \leftarrow\shortmid (Z \to Y)}{\overset{X \mapsto (Z \to X \amalg Z)}{\rightleftarrows}} \mathcal{C}_{Z/}$ of adjoint functors. The forgetful functor $\mathcal{C}_{Z/} \to \mathcal{C}$:

- Is conservative, since an object in $\mathcal{C}_{Z/}$ could be identified with a map $\Delta^1 \to \mathcal{C}, 0 \mapsto Z$, and a map in $\mathcal{C}_{Z/}$ between two objects could be identified with a homotopy between two maps $\Delta^1 \rightrightarrows \mathcal{C}$; then we invoke [32, Tag 01DK] to conclude.
- Preserves sifted colimits, as it is a left fibration [32, Tag 018F], thus preserves weakly contractible colimits [32, Tag 02KT], and sifted diagrams are weakly contractible [32, Tag 02QL].

We then invoke Proposition A.18 to conclude[8]. □

Now we list the n-categorical analogues:

LEMMA 2.11. — *Let $\mathcal{C}$ be an n-projectively generated n-category and $K \in \mathrm{Set}_\Delta$ a small simplicial set. Then the n-category $\mathrm{Fun}(K, \mathcal{C})$ of functors is n-projectively generated.*

LEMMA 2.12. — *Let $\mathcal{C}$ be an n-projectively generated n-category and $Z \in \mathcal{C}$ an object. Then the undercategory $\mathcal{C}_{Z/}$ is n-projectively generated. More precisely, let $S \subseteq \mathcal{C}$ be a set of n-projective generators for $\mathcal{C}$, then the set $\{Z \to X \amalg Z \mid X \in S\}$ is a set of compact n-projective generators for the undercategory $\mathcal{C}_{Z/}$.*

Now we deduce the corollaries for animations.

8. We believe that our argument could be vastly simplified. However, we point out that the map $K \hookrightarrow \{*\} \star K$ is not necessarily cofinal if the simplicial set K is not sifted. For example, take K to be a discrete set with at least two elements.

COROLLARY 2.13. — *Let $\mathcal{C}$ be an n-projectively generated n-category. Then there is a canonical equivalence* $\mathrm{Ani}(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})) \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ani}(\mathcal{C}))$ *of ∞-categories, or equivalently, a canonical equivalence* $\mathrm{Ani}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathcal{C}))$ *of ∞-categories.*

Proof. — Let $S \subseteq \mathcal{C}$ be a set of compact n-projective generators for $\mathcal{C}$. Spelling out the proof of Corollary 2.9 (more precisely, its analogue Lemma 2.11), we extract an explicit set of compact n-projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, namely, $T := \{X \leftarrow 0 \,|\, X \in S\} \cup \{X \xleftarrow{\mathrm{id}_X} X \,|\, X \in S\}$. Note that $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ani}(\mathcal{C}))$ is a full subcategory, and again by the proof of Corollary 2.9, it follows that T is a set of compact projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ani}(\mathcal{C}))$. The result follows. □

The same proof leads to the following (compare with [38, Cons 4.3.4]).

COROLLARY 2.14. — *Let $\mathcal{C}$ be an n-projectively generated n-category. Then there are canonical equivalences*

$$\begin{aligned} \mathrm{Ani}(\mathrm{Fun}((\mathbb{Z}, \geq), \mathcal{C})) &\longrightarrow \mathrm{Fun}((\mathbb{Z}, \geq), \mathrm{Ani}(\mathcal{C})) \\ \mathrm{Ani}(\mathrm{Fun}(\mathbb{Z}, \mathcal{C})) &\longrightarrow \mathrm{Fun}(\mathbb{Z}, \mathrm{Ani}(\mathcal{C})) \\ \mathrm{Ani}(\mathrm{Fun}(\{0,1\}, \mathcal{C})) &\longrightarrow \mathrm{Fun}(\{0,1\}, \mathrm{Ani}(\mathcal{C})) \\ \mathrm{Ani}\left(\mathcal{C}_{Z/}\right) &\longrightarrow \mathrm{Ani}(\mathcal{C})_{Z/} \end{aligned}$$

of ∞-categories. The same goes for replacing $\mathbb{Z}$'s by $\mathbb{N}$'s.

2.3. Comma categories and base change. — In this section, we will discuss comma categories, which serves as our basic language to discuss various base changes.

DEFINITION 2.15. — Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. The *comma category*, sometimes denoted by $F \downarrow \mathcal{D}$, is given by the simplicial set $\mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$, where the map $\mathcal{C} \to \mathrm{Fun}(\{0\}, \mathcal{D})$ is given by F, and the map $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\{0\}, \mathcal{D})$ is induced by the vertex $\{0\} \to \Delta^1$.

EXAMPLE 2.16. — Consider the functor $\mathrm{id}_{\mathrm{CAlg}^{\mathrm{an}}} : \mathrm{CAlg}^{\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}}$. The comma category

$$\mathrm{CAlg}^{\mathrm{an}} \times_{\mathrm{Fun}(\{0\}, \mathrm{CAlg}^{\mathrm{an}})} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$$

is equivalent to $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$. An object is simply given by a base $A \in \mathrm{CAlg}^{\mathrm{an}}$ and an A-algebra $A \to R$.

EXAMPLE 2.17. — Consider the functor $\mathrm{Pair} \to \mathrm{Ring}, (A, I) \mapsto A/I$ and the composite functor $\mathrm{Pair}^{\gamma} \to \mathrm{Pair} \to \mathrm{Ring}$. Concretely, the objects in the comma category $\mathrm{Pair}^{\gamma} \times_{\mathrm{Fun}(\{0\}, \mathrm{Ring})} \mathrm{Fun}(\Delta^1, \mathrm{Ring})$ are given by a PD-pair (A, I, γ) along with an A/I-algebra $A/I \to R$. This is the nonanimated version of CrysCon that will be introduced in Section 4.1.

REMARK 2.18. — A similar comma category plays an role for prismatic cohomology. We will study a noncomplete version in Section 5.3.

LEMMA 2.19. — *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the simplicial set $\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ is an ∞-category.*

Proof. — It follows from [28, Corollary 2.3.2.5] applied to the inner fibration $\mathcal{D} \to \{*\}$ that $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\{0\}, \mathcal{D})$ is an inner fibration. Then it follows [28, Corollary 2.4.6.5] that $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\{0\}, \mathcal{D})$ is a categorical fibration. The result follows. □

REMARK 2.20. — The canonical projection $\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{C}$ admits a fully faithful section induced by $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D}), D \mapsto \mathrm{id}_D$, which is also a left adjoint of the projection in question.

LEMMA 2.21. — *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Suppose that $\mathcal{D}$ admits finite coproducts. Then the functor $\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{C} \times \mathcal{D}$ induced by $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\{1\}, \mathcal{D}) \simeq \mathcal{D}$ admits a left adjoint informally given by $(C, D) \mapsto (C, F(C) \to F(C) \amalg D)$.*

Proof. — We need the concept of relative adjunctions [30, §7.3.2]. In fact, the adjunction is relative to $\mathcal{C}$.

To see this, we start with the special case that $\mathcal{C} = \mathcal{D}$ and $F = \mathrm{id}_{\mathcal{D}}$. The point is that the pair $\mathcal{D} \times \mathcal{D} \rightleftarrows \mathrm{Fun}(\Delta^1, \mathcal{D})$ of adjoint functors satisfies [30, Prop 7.3.2.1], where the functor $\mathcal{D} \times \mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ is given by left Kan extension along the functor $\{0, 1\} \to \Delta^1$, and the functor $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ is simply given by the restriction along $\{0, 1\} \to \Delta^1$.

The general case follows from [30, Prop 7.3.2.5] by base change along $F : \mathcal{C} \to \mathcal{D}$. □

From Proposition A.18, we have the following corollary.

COROLLARY 2.22. — *Let $\mathcal{C}, \mathcal{D}$ be projectively generated ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the comma category $\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ is projectively generated. More precisely, let $S \subseteq \mathcal{C}$ and $T \subseteq \mathcal{D}$ be sets of compact projective generators. Then $\{(C, F(C) \to F(C) \amalg D) \,|\, C \in S, D \in T\}$ is a set of compact projective generators for $\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$.*

It follows from [28, Lem 5.4.5.5] that the colimits in comma categories exist and are easy to describe under the assumption that the functor in question preserves colimits.

LEMMA 2.23. — *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Let K be a simplicial set. Suppose that $\mathcal{C}, \mathcal{D}$ admits K-indexed colimits, which are preserved by F. Then the comma category $\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ admits K-indexed colimits, which are preserved by projection to either factor.*

REMARK 2.24 (Base change). — Let $\mathcal{C}, \mathcal{D}$ be ∞-categories ,which admit finite colimits, and $F : \mathcal{C} \to \mathcal{D}$ a functor, which preserves finite colimits. Given an object $(C, F(C) \to D) \in \mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$, there is a unique map $(C, \mathrm{id}_{F(C)}) \to (C, F(C) \to D)$ (which is, in fact, the unit map for the adjunction in Remark 2.20). For all maps $C \to C'$ in $\mathcal{C}$, we have the pushout of the diagram $(C', \mathrm{id}_{F(C')}) \leftarrow (C, \mathrm{id}_{F(C)}) \to (F(C) \to D)$ in $\mathcal{C}$, which is $(C', D \amalg_{F(C)} F(C'))$ by Lemma 2.23. At the beginning of this section, we said that the objects in $\mathcal{C}$ are considered as "bases". Thus we understand this pushout as 'a 'base change".

EXAMPLE 2.25. — In Example 2.16, given a map $A \to B$ of animated rings, the base change of $A \to R$ along $A \to B$ is $B \to R \otimes^{\mathbb{L}}_A B$. Since the cotangent complex functor $L_{./.} : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{Ani}(\mathrm{Mod})$ preserves small colimits (Lemma 2.35), we get the base change property: the natural map $L_{R/A} \otimes^{\mathbb{L}}_A B \to L_{R \otimes^{\mathbb{L}}_A B/B}$ is an equivalence (here we implicitly used Lemma 2.36). Similarly, we get the base change property $\mathrm{HH}(R/A) \otimes^{\mathbb{L}}_A B \simeq \mathrm{HH}(R \otimes^{\mathbb{L}}_A B/B)$ for Hochschild homology (the reader should feel free to ignore this since it will not be used in this article).

EXAMPLE 2.26. — In Example 2.17, given a map $(A, I, \gamma) \to (B, J, \delta)$ of PD-pairs, the base change of $((A, I, \gamma), A/I \to R)$ along $(A, I, \gamma) \to (B, J, \delta)$ is $((B, J, \delta), B/J \to R \otimes_{A/I} (B/J))$.

REMARK 2.27. — We have a prismatic version of base change by Remark 2.18.

REMARK 2.28 (Colimits over a fixed base). — Let $\mathcal{C}, \mathcal{D}$ be cocomplete ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor that preserves small colimits. Given an object $C \in \mathcal{C}$, a small simplicial set K and a diagram $q : K \to \mathcal{D}_{F(C)/}$, we associate a diagram $K \to \mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ informally given by $k \mapsto (C, F(C) \to q(k))$ (the formal description necessitates a discussion of "fat" overcategories [28, §4.2.1]). By Lemma 2.23, the colimit of this diagram is given by $(C, \mathrm{colim}\, q)$. We understand this colimit as taking colimits over a fixed base.

EXAMPLE 2.29. — In Example 2.16, given an animated ring A and two A-algebras R, S, the map $(A \to R \otimes^{\mathbb{L}}_A S)$, seen as an object of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$, is the pushout of the diagram $(A \to R) \leftarrow (A, \mathrm{id}_A) \to (A \to S)$. Since the cotangent complex functor $L_{./.} : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{Ani}(\mathrm{Mod})$ (which we will review in Definition 2.33) preserves small colimits (Lemma 2.35), we get the "Künneth formula": the natural map $L_{R/A} \otimes^{\mathbb{L}}_R (R \otimes^{\mathbb{L}}_A S) \oplus L_{S/A} \otimes^{\mathbb{L}}_S (R \otimes^{\mathbb{L}}_A S) \to L_{(R \otimes^{\mathbb{L}}_A S)/A}$ is an equivalence (again, we used Lemma 2.36, and also the form of colimits in Ani(Mod)). Similarly, we have $\mathrm{HH}(R/A) \otimes^{\mathbb{L}}_A \mathrm{HH}(S/A) \simeq \mathrm{HH}(R \otimes^{\mathbb{L}}_A S/A)$ for Hochschild homology (again, Hochschild homology is not needed in this article).

Remark 2.30. — In view of Remark 2.18, prismatic cohomology has a similar "Künneth formula" [2, Prop 3.5.1].

Remark 2.31 (Transitivity). — Let $\mathcal{C}, \mathcal{D}$ be ∞-categories ,which admit finite colimits, and $F : \mathcal{C} \to \mathcal{D}$ a functor, which preserves finite colimits. Given a map $C \to C'$ in $\mathcal{C}$, any object $(C', F(C') \to D) \in \mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ could be written as the pushout of the diagram $(C, F(C) \to D) \leftarrow (C, F(C) \to F(C')) \to (C', \mathrm{id}_{F(C')})$. This is closely related to transitivity sequence in the cohomology theory, as shown in the examples below.

Example 2.32. — In Example 2.16, for any maps $A \to B \to R$ of animated rings, the "relative" map $B \to R$, viewed as an object of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$, is the pushout of the diagram $(A \to R) \leftarrow (A \to B) \to (\mathrm{id}_B : B \to B)$. Since the cotangent complex functor $L_{./.} : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{Ani}(\mathrm{Mod})$ preserves small colimits (Lemma 2.35), we get the transitivity sequence

$$L_{B/A} \otimes^{\mathbb{L}}_B R \longrightarrow L_{R/A} \longrightarrow L_{R/B}$$

(Lemma 2.36 was used) Similarly, we have $\mathrm{HH}(R/A) \otimes^{\mathbb{L}}_{\mathrm{HH}(B/A)} B \simeq \mathrm{HH}(R/B)$ for Hochschild homology.

Finally, we briefly review the theory of the cotangent complex of maps of animated rings and explain how this "coincides" with the theory of cotangent complex of maps of animated A-algebra for some ring A. By Corollary 2.13, the ∞-category $\mathrm{AniArr} := \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ is projectively generated, and the proof leads to a set $\{\mathbb{Z}[X] \to \mathbb{Z}[X, Y] \,|\, X, Y \in \mathrm{Fin}\}$ of compact projective generators. Let $\mathrm{AniArr}^0 \subseteq \mathrm{AniArr}$ denote the full subcategory spanned by those compact projective generators.

Definition 2.33. — The *cotangent complex functor* $\mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$ is defined to be the left derived functor (Proposition A.14) of the functor $\mathrm{AniArr}^0 \to \mathrm{Ani}(\mathrm{Mod}), (A \to B) \mapsto (B, \Omega^1_{B/A})$. The image of an object $(A \to B) \in \mathrm{AniArr}$ is denoted by $(B, L_{B/A})$.

Remark 2.34. — In fact, this functor is also the animation of the functor $\mathrm{Fun}(\Delta^1, \mathrm{Ring}) \to \mathrm{Mod}, (A \to B) \mapsto (B, \Omega^1_{B/A})$. We do not take this as the definition since later we will apply the same idea to functors that are not defined by the animation of a functor.

Since the functor $\mathrm{AniArr}^0 \to \mathrm{Ani}(\mathrm{Mod}), B \mapsto (B, \Omega^1_{B/A})$ preserves finite coproducts, by Proposition A.14, we get the following.

Lemma 2.35. — *The cotangent complex functor* $\mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$ *preserves small colimits.*

Now we consider the functor $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{AniArr}, A \mapsto (\mathrm{id}_A : A \to A)$. This functor preserves small colimits[9], thus so does the composite functor $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{AniArr} \xrightarrow{L_{\cdot/\cdot}} \mathrm{Ani}(\mathrm{Mod})$, concretely given by $A \mapsto (A, L_{A/A})$. The next simple[10] lemma is key to the "independence of the choice of the base", which was already used in examples before.

LEMMA 2.36. — *The composite functor* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{AniArr} \xrightarrow{L_{\cdot/\cdot}} \mathrm{Ani}(\mathrm{Mod})$ *above coincides with the functor* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Ani}(\mathrm{Mod}), A \mapsto (A, 0)$.

Proof. — By the colimit-preserving property above and Proposition A.14, it suffices to check this for polynomial rings $A = \mathbb{Z}[x_1, \ldots, x_n]$, but this follows directly from the definitions. □

We now consider the full subcategory $\mathcal{P}^0$ of $\mathrm{Fun}(\Delta^1, \mathrm{Ring})$ spanned by maps $A[X] \to A[X, Y]$ with $A \in \mathrm{Ring}$ and $X, Y \in \mathrm{Fin}$. The functor $\mathrm{Fun}(\Delta^1, \mathrm{Ring}) \to \mathrm{Ani}(\mathrm{Mod}), (A \to B) \mapsto (B, \Omega^1_{B/A})$ restricts to a functor $G : \mathcal{P}^0 \to \mathrm{Ani}(\mathrm{Mod})$. By Proposition A.14, the restriction F of the cotangent complex functor $\mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$ to the full subcategory $\mathcal{P}^0$ is left Kan extended from $\mathrm{AniArr}^0 \subseteq \mathcal{P}^0$, and, therefore, we have a comparison map $F \to G$, which becomes an equivalence after restriction to the full subcategory AniArr^0. By Example 2.25, this comparison map is an equivalence since G also has the "base change property".

Now we fix a ring A and let AniArr_A denote the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}_A)$. As before, by Corollary 2.13, it is projectively generated with a set $\{A[X] \to A[X, Y] \mid X, Y \in \mathrm{Fin}\}$ of compact projective generators, which spans a full subcategory $\mathrm{AniArr}^0_A \subseteq \mathrm{AniArr}_A$. Note that the functor $\mathrm{AniArr}^0_A \to \mathrm{Ani}(\mathrm{Mod}), (B \to C) \mapsto (C, \Omega^1_{C/B})$ coincides with the composite functor $\mathrm{AniArr}^0_A \to \mathcal{P}^0 \xrightarrow{G} \mathrm{Ani}(\mathrm{Mod})$, and since $F \simeq G$, this composite functor is just the cotangent complex functor applied to the underlying map of animated rings. From Proposition A.14 we have the following.

LEMMA 2.37. — *The composite functor* $\mathrm{AniArr}_A \to \mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$, $(B \to C) \mapsto (C, L_{C/B})$ *is equivalent to the left derived functor of* $\mathrm{AniArr}^0_A \to \mathrm{Ani}(\mathrm{Mod}), (A[X] \to A[X, Y]) \mapsto \Omega^1_{A[X,Y]/A[X]}$.

That is to say, the definition of the cotangent complex does not depend on the choice of the base. This argument applies to similar situations, such as animated PD-envelope, and such a phenomenon will appear frequently in this article.

9. In fact, this is fully faithful. However, in order to apply the same idea to later contexts, we only abstract out the colimit-preserving property.

10. We warn the reader that this lemma is not tautological.

2.4. ∞-category of graded and filtered objects. — In this section, we recollect basic properties of the ∞-category of graded and filtered objects. Our main reference is [38, §3].

The ∞-category of ($\mathbb{Z}$-)graded objects in an ∞-category $\mathcal{C}$ is the ∞-category $\mathrm{Gr}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ of functors, where $\mathbb{Z}$ is the set of integers as an ∞-category. Given a graded object $G \in \mathrm{Gr}(\mathcal{C})$, we will denote the value of G at $i \in \mathbb{Z}$ by X^i. This defines a functor $(\cdot)^i : \mathrm{Gr}(\mathcal{C}) \to \mathcal{C}$.

When the ∞-category $\mathcal{C}$ is presentable, for all $i \in \mathbb{Z}$, the functor $(\cdot)^i$ admits a fully faithful left adjoint $\mathrm{ins}^i : \mathcal{C} \to \mathrm{Gr}(\mathcal{C})$ simply given by $X \mapsto G$ where $G^j = \begin{cases} X & j = i \\ 0_{\mathcal{C}} & \text{otherwise} \end{cases}$ where $0_{\mathcal{C}} \in \mathcal{C}$ is the initial object.

We say that a graded object $G \in \mathrm{Gr}(\mathcal{C})$ is *nonnegatively graded* (resp. *nonpositively graded*) if the restriction $F|_{\mathbb{Z}_{<0}}$ (resp. $F|_{\mathbb{Z}_{>0}}$) is constantly $0_{\mathcal{C}}$. The full subcategory spanned by nonnegatively graded (resp. nonpositively graded) objects is denoted by $\mathrm{Gr}^{\geq 0}(\mathcal{C})$ (resp. $\mathrm{Gr}^{\leq 0}(\mathcal{C})$), which is canonically equivalent to $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$ (resp. $\mathrm{Fun}(\mathbb{Z}_{\leq 0}, \mathcal{C})$).

Similarly, the *∞-category of ($\mathbb{Z}$-)filtered objects* in an ∞-category $\mathcal{C}$ is the ∞-category $\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}((\mathbb{Z}, \geq), \mathcal{C})$ of functors. Given a filtered object $F \in \mathrm{Fil}(\mathcal{C})$, we will systematically denote the value of F at $i \in \mathbb{Z}$ by $\mathrm{Fil}^i F$ instead of $F(i)$ to indicate that we consider it as a filtered object. This defines a functor $\mathrm{Fil}^i : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$.

When the ∞-category $\mathcal{C}$ is presentable, for all $i \in \mathbb{Z}$, the functor Fil^i admits a fully faithful left adjoint $\mathrm{ins}^i : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ given by the left Kan extension along $\{i\} \to (\mathbb{Z}, \geq)$. Given $X \in \mathcal{C}$, $\mathrm{Fil}^j(\mathrm{ins}^i(X)) = \begin{cases} X & j \leq i \\ 0_{\mathcal{C}} & j > i \end{cases}$, where $0_{\mathcal{C}} \in \mathcal{C}$ is the initial object.

We say that a filtered object $F \in \mathrm{Fil}(\mathcal{C})$ is *nonnegatively filtered* if the restriction $F|_{\mathbb{Z}_{\leq 0}}$ is a constant functor $(\mathbb{Z}_{\leq 0}, \geq) \to \mathcal{C}$. We denote by $\mathrm{Fil}^{\geq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ the full subcategory spanned by nonnegatively filtered objects, which is canonically equivalent to $\mathrm{Fun}((\mathbb{Z}_{\geq 0}, \geq), \mathcal{C})$. Similarly, we say that a filtered object $F \in \mathrm{Fil}(\mathcal{C})$ is *nonpositively filtered* if the restriction $F|_{\mathbb{Z}_{>0}}$ is constantly $0_{\mathcal{C}}$. We denote by $\mathrm{Fil}^{\leq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ the full subcategory spanned by nonpositively filtered objects, which is canonically equivalent to $\mathrm{Fun}((\mathbb{Z}_{\leq 0}, \geq), \mathcal{C})$.

Given a filtered object $F \in \mathrm{Fil}(\mathcal{C})$, the *union* $\mathrm{Fil}^{-\infty}$ is defined to be the colimit $\mathrm{colim}_{(\mathbb{Z}, \geq)} F$ (when it exists). When $\mathcal{C}$ admits all sequential colimits, this defines a functor $\mathrm{Fil}^{-\infty} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$. Furthermore, when the ∞-category $\mathcal{C}$ is stable, a filtered object $F \in \mathrm{Fil}(\mathcal{C})$ is called *complete* [38, Def 3.2.12], or a *completely filtered object*, if $\lim F \simeq 0$ in $\mathcal{C}$. We will denote by $\mathrm{Fil}^{\wedge}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ the full subcategory spanned by completely filtered objects.

REMARK 2.38. — To avoid confusion our filtrations are always decreasing. When we need increasing filtrations, we invert the sign to get a decreasing filtration.

Now let $(\mathcal{C}, \otimes)$ be a presentable symmetric monoidal ∞-category. Note that $\mathbb{Z}$ (resp. $(\mathbb{Z}, \geq)$) has a symmetric monoidal structure given by the addition $+$, so the ∞-category $\mathrm{Gr}(\mathcal{C})$ (resp. $\mathrm{Fil}(\mathcal{C})$) admits a presentable symmetric monoidal structure given by the *Day convolution* $\otimes^{\mathrm{Day}}$ [36, §3]. Informally, given two graded (resp. filtered) objects F, G, we have $(F \otimes^{\mathrm{Day}} G)^i = \bigoplus_{j+k=i} F^j \otimes G^k$ (resp. $\mathrm{Fil}^i(F \otimes^{\mathrm{Day}} G) = \mathrm{colim}_{j+k \geq i} \mathrm{Fil}^j F \otimes \mathrm{Fil}^k G$). Under this symmetric monoidal structure, $(\cdot)^0 : \mathrm{Gr}(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathrm{Fil}^0 : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$) is lax symmetric monoidal, while the fully faithful left adjoint $\mathrm{ins}^0 : \mathcal{C} \to \mathrm{Gr}(\mathcal{C})$ (resp. $\mathcal{C} \to \mathrm{Fil}(\mathcal{C})$) is symmetric monoidal.

The stable subcategory $\mathrm{Gr}^{\geq 0}(\mathcal{C}) \subseteq \mathrm{Gr}(\mathcal{C})$ (resp. $\mathrm{Gr}^{\leq 0}(\mathcal{C}) \subseteq \mathrm{Gr}(\mathcal{C})$) inherits a presentable symmetric monoidal structure, and the 0-th piece $(\cdot)^0 : \mathrm{Gr}^{\geq 0}(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathrm{Gr}^{\leq 0}(\mathcal{C}) \to \mathcal{C}$) is symmetric monoidal.

Similarly, the stable subcategory $\mathrm{Fil}^{\geq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ (resp. $\mathrm{Fil}^{\leq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$) inherits a presentable symmetric monoidal structure, and the 0-th piece $\mathrm{Fil}^0 : \mathrm{Fil}^{\geq 0}(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathrm{Fil}^{\leq 0}(\mathcal{C}) \to \mathcal{C}$) is symmetric monoidal.

Now we study the relation between graded objects and filtered objects. First, the symmetric monoidal functor $\mathbb{Z} \to (\mathbb{Z}, \geq)$ induces a lax symmetric monoidal functor $\mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$, which admits a symmetric monoidal left adjoint $I : \mathrm{Gr}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$, the left Kan extension along $\mathbb{Z} \to (\mathbb{Z}, \geq)$. Concretely, it is given by $G \mapsto F$ where $\mathrm{Fil}^i F = \coprod_{j \geq i} G^j$.

All of the functors mentioned above preserve small colimits. From now on, let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category. Then these functors are exact. Now we consider the associated graded functor $\mathrm{gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}), F \mapsto G$, where $G^i = \mathrm{cofib}(F^{i+1} F \to \mathrm{Fil}^i F)$. It turns out that the functor gr behaves well.

PROPOSITION 2.39 ([29, Prop 3.2.1] [20, Prop 2.26]). — *Let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category. Then there exists a symmetric monoidal structure on the functor $\mathrm{gr}^* : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$. Moreover, this symmetric monoidal structure can be chosen so that the composite functor $\mathrm{Gr}(\mathcal{C}) \xrightarrow{I} \mathrm{Fil}(\mathcal{C}) \xrightarrow{\mathrm{gr}^*} \mathrm{Gr}(\mathcal{C})$ is homotopic to the identity as a symmetric monoidal functor.*

We also need the *Beilinson t-structure* on the ∞-category $\mathrm{Fil}(\mathcal{C})$ of filtered objects. As before, let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a presentable stable symmetric monoidal ∞-category. Moreover, we assume that $\mathcal{C}$ admits an accessible t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ compatible with filtered colimits such that $1_{\mathcal{C}} \in \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\geq 0}$ is closed under $\otimes$.

LEMMA 2.40. — *Under the assumptions above, the heart* $\mathcal{C}^\heartsuit := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ *admits a canonical symmetric monoidal structure* $\otimes^\heartsuit$ *given by* $X \otimes^\heartsuit Y := \tau_{\leq 0}(X \otimes Y)$*, and the embedding* $\mathcal{C}^\heartsuit \to \mathcal{C}$ *is then lax symmetric monoidal.*

The following is the ∞-categorical enrichment of [4, App].

PROPOSITION 2.41 ([38, Prop 3.3.11]). — *Let* $\mathrm{Fil}(\mathcal{C})^B_{\geq 0} \subseteq \mathrm{Fil}(\mathcal{C})$ *be the full subcategory spanned by* $X \in \mathrm{Fil}(\mathcal{C})$ *such that* $\mathrm{gr}^i(X) \in \mathcal{C}_{\geq -i}$ *for all* $i \in \mathbb{Z}$*. Then* $1_{\mathrm{ins}^0(1_\mathcal{C})} \in \mathrm{Fil}(\mathcal{C})^B_{\geq 0}$*,* $\mathrm{Fil}(\mathcal{C})^B_{\geq 0}$ *is closed under* $\otimes^{\mathrm{Day}}$ *and is the connective part of an accessible t-structure, called the Beilinson t-structure, whose heart is equivalent as symmetric monoidal* 1*-categories to the* 1*-category* $\mathrm{Ch}(\mathcal{C}^\heartsuit)$ *of chain complexes with "stupid" truncation* $\mathrm{Fil}^i K = K_{\leq -i}$ *for all* $i \in \mathbb{Z}$ *and* $K \in \mathrm{Ch}(\mathcal{C}^\heartsuit)$*.*

In particular, when $\mathcal{C}$ is the derived ∞-category of a ring R, the *filtered derived category* $\mathrm{DF}(R)$ is the ∞-category $\mathrm{Fil}(D(R))$ of filtered objects in the derived ∞-category $D(R)$ with the symmetric monoidal structure given by the derived tensor product $\cdot \otimes^{\mathbb{L}}_R \cdot$, and $\mathrm{DF}^{\geq 0}(R)$ is the ∞-category $\mathrm{Fil}^{\geq 0}(D(R))$ of nonnegatively filtered objects in $D(R)$. In this case, we will still denote by $\cdot \otimes^{\mathbb{L}}_R \cdot$ the Day convolution.

REMARK 2.42 ([38, Cons 4.3.4]). — Let R be a ring. The ∞-category $\mathrm{DF}(R)$ admits a structure of *derived algebraic context* [38, Def 4.2.1], of which the derived commutative algebras are called *filtered derived (commutative) R-algebras*. When $R = \mathbb{Z}$, they are also called *filtered derived rings*. Although we will not use this fact, we might comment when a filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra admits such a structure.

We need the following lemma, which follows from the fact that left Kan extensions are pointwise colimits that preserve cofibers and filtered colimits.

LEMMA 2.43. — *Let* $\mathcal{C}$ *be an* ∞*-category,* $\mathcal{C}^0 \subseteq \mathcal{C}$ *a full subcategory,* $\mathcal{E}$ *a stable* ∞*-category, which admits filtered colimits, and* $\tilde{F} : \mathcal{C} \to \mathrm{Fil}(\mathcal{E})$ *a functor left Kan extended along the fully faithful embedding* $\mathcal{C}^0 \hookrightarrow \mathcal{C}$*. Then we have the following.*

1. *The composite functor* $\mathrm{gr}^* \circ \tilde{F} : \mathcal{C} \to \mathrm{Fil}(\mathcal{E}) \xrightarrow{\mathrm{gr}^*} \mathrm{Gr}(\mathcal{E})$ *is left Kan extended along* $\mathcal{C}^0 \hookrightarrow \mathcal{C}$*.*
2. *The composite functor* $\mathrm{Fil}^{-\infty} \circ \tilde{F} : \mathcal{C} \to \mathrm{Fil}(\mathcal{E}) \xrightarrow{\mathrm{Fil}^{-\infty}} \mathcal{E}$ *is left Kan extended along* $\mathcal{C}^0 \hookrightarrow \mathcal{C}$*.*

2.5. Reflective subcategories. — In this section, we will develop the necessary machinery to deal with the (derived) p-complete or more generally I-complete situations. We start with the general formalism of reflective subcategories.

DEFINITION 2.44 ([28, Rem 5.2.7.9 & Def 5.2.7.2]). — Let $\mathcal{C}$ be an ∞-category. We say that a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *reflective* if the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{D}$. In such a case, we call the left adjoint $L : \mathcal{C} \to \mathcal{D}$ a *localization*.

PROPOSITION 2.45 ([28, Prop 5.2.7.8]). — *Let $\mathcal{C}$ be an ∞-category. A full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ is reflective if and only if for every object $C \in \mathcal{C}$, there exists an object $D \in \mathcal{C}^0$ along with a map $f : C \to D$, which induces an equivalence $\mathrm{Map}_{\mathcal{C}}(D, E) \to \mathrm{Map}_{\mathcal{C}}(C, E)$ for each object $E \in \mathcal{C}^0$ (in this case, $LC \simeq D$, where $L : \mathcal{C} \to \mathcal{C}^0$ is the localization).*

EXAMPLE 2.46. — Let $D_{\mathrm{comp}}(\mathbb{Z}_p) \subseteq D(\mathbb{Z})$ be the p-complete derived category of $\mathbb{Z}$, consisting of (derived) p-complete $\mathbb{Z}_p$-module spectra. Then $D_{\mathrm{comp}}(\mathbb{Z}_p) \subseteq D(\mathbb{Z})$ is reflective. The localization is the (derived) p-completion functor $D(\mathbb{Z}) \to D_{\mathrm{comp}}(\mathbb{Z}_p)$. Similarly, $D_{\mathrm{comp},\geq 0}(\mathbb{Z}_p) \subseteq D_{\geq 0}(\mathbb{Z})$ is the reflective subcategory of connective p-complete $\mathbb{Z}_p$-module spectra.

EXAMPLE 2.47. — More generally, let A be an animated ring and $I \subseteq \pi_0(A)$ a finitely generated ideal. Then the I-complete derived category $D_{\mathrm{comp}}(A)$ is a reflective subcategory of the derived category $D(A)$. The same is true for $D_{\mathrm{comp},\geq 0}(A) \subseteq D_{\geq 0}(A)$.

Now we study the left derived functors. Unfortunately, the localization does not, in general, map compact projective objects to compact projective objects. For example, $\mathbb{Z} \in D(\mathbb{Z})$ is compact and projective but $\mathbb{Z}_p \in D_{\mathrm{comp}}(\mathbb{Z}_p)$ is not. We suspect that $D_{\mathrm{comp}}(\mathbb{Z}_p)$ is not projectively generated, and, therefore, we are probably unable to left derive "arbitrary" functors as in the projectively generated case. However, most functors in practice are good enough to have a reasonable theory of left derived functors[11]. We start with a general discussion about the interaction between localization and left Kan extension [28, Def 4.3.2.2].

NOTATION 2.48. — Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a reflective (full) subcategory with the localization $L : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, $\mathcal{D}^0 \subseteq \mathcal{D}$ the full subcategory spanned by objects LC where C runs through all objects in $\mathcal{C}^0$. Let $\mathcal{C}^1 \subseteq \mathcal{C}$ be the full subcategory spanned by vertices of both $\mathcal{C}^0$ and $\mathcal{D}^0$.

From definitions we have the following.

LEMMA 2.49. — *In Setup 2.48, $\mathcal{D}^0 \subseteq \mathcal{C}^1$ is a reflective subcategory with localization $L|_{\mathcal{C}^1} : \mathcal{C}^1 \to \mathcal{D}^0$ being the restriction of $L : \mathcal{C} \to \mathcal{D}$.*

11. This approach is essentially depicted in the special case of p-completed rings in Bhatt's Eilenberg Lectures notes [7, Lecture VII]. We were informed of this approach by Yu MIN in private discussions.

Example 2.50. — One of the crucial examples for the setup above: $\mathcal{C}$ is the ∞-category of animated rings, $\mathcal{D}$ is the full subcategory of p-complete animated $\mathbb{Z}_p$-algebras, and $\mathcal{C}^0 \subseteq \mathcal{C}$ is the full subcategory spanned by polynomial rings $\mathbb{Z}[X_1, \ldots, X_n]$. More generally, let A be an animated ring and $I \subseteq \pi_0(A)$ a finitely generated ideal. Then we can consider the case that $\mathcal{C}$ is the ∞-category of animated A-algebras and $\mathcal{D} \subseteq \mathcal{C}$ is the full subcategory of I-complete animated A-algebras, and $\mathcal{C}^0 \subseteq \mathcal{C}$ is the full subcategory spanned by polynomial A-algebras $A[X_1, \ldots, X_n] := \mathbb{Z}[X_1, \ldots, X_n] \otimes^{\mathbb{L}}_{\mathbb{Z}} A$.

Lemma 2.51. — *In Setup 2.48, let $\mathcal{E}$ be an ∞-category and $\tilde{F} : \mathcal{C} \to \mathcal{E}$ a functor left Kan extended from the fully faithful embedding $\mathcal{C}^0 \hookrightarrow \mathcal{C}$. Then the restriction $\widetilde{F|_{\mathcal{D}}}$ is left Kan extended from the fully faithful embedding $\mathcal{D}^0 \hookrightarrow \mathcal{D}$.*

Proof. — It follows from [28, Lem 5.2.6.6] that the restriction $\tilde{F}|_{\mathcal{D}}$ is a left Kan extension of $\tilde{F}$ along $L : \mathcal{C} \to \mathcal{D}$, and, therefore, is left Kan extended from the composite functor $\mathcal{C}^0 \hookrightarrow \mathcal{C} \xrightarrow{L} \mathcal{D}$. The composite functor $\mathcal{C}^0 \hookrightarrow \mathcal{C} \to \mathcal{D}$ can be rewritten as the composite $\mathcal{C}^0 \xrightarrow{L} \mathcal{D}^0 \hookrightarrow \mathcal{D}$, and, therefore, $\tilde{F}|_{\mathcal{D}}$ is left Kan extended from $\mathcal{D}^0 \hookrightarrow \mathcal{D}$. □

Example 2.52. — In Example 2.50, the cotangent complex $L_{\cdot/\mathbb{Z}} : \mathcal{C} = \mathrm{CAlg}^{\mathrm{an}} \to D(\mathbb{Z})$ is left Kan extended from $\mathrm{Poly}_{\mathbb{Z}} \subseteq \mathrm{Ring}$. Consequently, the restriction $L_{\cdot/\mathbb{Z}}|_{\mathcal{D}} : \mathcal{D} \to D(\mathbb{Z})$ is left Kan extended from p-completed polynomial rings. Similarly, the p-completed cotangent complex $(L_{\cdot/\mathbb{Z}})^{\wedge}_p : \mathrm{CAlg}^{\mathrm{an}} \to D_{\mathrm{comp}}(\mathbb{Z}_p)$ is left extended from $\mathrm{Poly}_{\mathbb{Z}} \subseteq \mathrm{Ring}$, and, therefore, the restriction $(L_{\cdot/\mathbb{Z}})^{\wedge}_p|_{\mathcal{D}} : \mathcal{D} \to D_{\mathrm{comp}}(\mathbb{Z}_p)$ is left extended from p-completed polynomial rings.

Notation 2.53. — In Setup 2.48, let $F : \mathcal{D}^0 \to \mathcal{E}$ be a functor equipped with a left Kan extension $\tilde{F} : \mathcal{C} \to \mathcal{E}$ along the fully faithful inclusion $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ of the composite functor $\mathcal{C}^0 \xrightarrow{L} \mathcal{D}^0 \xrightarrow{F} \mathcal{E}$.

Remark 2.54. — In our applications, $\mathcal{C}$ will be a projectively generated ∞-category (Definition A.8) with a set S of compact projective generators. We will choose $\mathcal{C}^0 \subseteq \mathcal{C}$ to be the full subcategory spanned by finite coproducts of objects in S, and $\mathcal{E}$ will be a cocomplete ∞-category. In this case, the left Kan extension in question always exists (Propositions A.14 and A.16). More generally, if $\mathcal{C}^0$ is a small subcategory and that $\mathcal{C}$ is assumed to be locally small, then the left Kan extension exists.

In Setup 2.53, we first assume without loss of generality that $L|_{\mathcal{D}} = \mathrm{id}_{\mathcal{D}}$ by [28, Prop 5.2.7.4], and then $L^2 = L$. Now we let $F_1 : \mathcal{C}^1 \to \mathcal{E}$ denote the composite $\mathcal{C}^1 \to \mathcal{D}^0 \xrightarrow{F} \mathcal{E}$, which is an extension of the composite $\mathcal{C}^0 \to \mathcal{D}^0 \xrightarrow{F} \mathcal{E}$ along $\mathcal{C}^0 \to \mathcal{C}^1$. Since $\tilde{F}|_{\mathcal{C}^1} : \mathcal{C}^1 \to \mathcal{E}$ is, by definition, a left Kan extension of $\mathcal{C}^0 \to \mathcal{D}^0 \xrightarrow{F} \mathcal{E}$ along $\mathcal{C}^0 \to \mathcal{C}^1$, there exists an essentially unique comparison

map $\tilde{F}|_{\mathcal{C}^1} \to F_1$ of functors $\mathcal{C}^1 \rightrightarrows \mathcal{E}$. Restricting to the full subcategory $\mathcal{D}^0 \subseteq \mathcal{C}^1$, we get a comparison map $\tilde{F}|_{\mathcal{D}^0} \to F$. From Lemma 2.51 we have the following.

COROLLARY 2.55. — *In Setup 2.53, if we assume that the comparison map $\tilde{F}|_{\mathcal{D}^0} \to F$ is an equivalence, then $\tilde{F}|_{\mathcal{D}}$ is the left Kan extension of F along the fully faithful embedding $\mathcal{D}^0 \hookrightarrow \mathcal{D}$.*

We need the following concept:

PROPOSITION 2.56 ([28, Prop 5.2.7.12]). — *Let $\mathcal{C}$ be an ∞-category and let $L : \mathcal{C} \to L\mathcal{C} \subseteq \mathcal{C}$ be a localization functor. Let S denote the collection of all morphisms f in $\mathcal{C}$ such that Lf is an equivalence. Then for every ∞-category $\mathcal{D}$, composition with L induces a fully faithful functor $\psi : \mathrm{Fun}(L\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Moreover, the essential image of ψ consists of those functors $F : \mathcal{C} \to \mathcal{D}$ such that $F(f)$ is an equivalence in $\mathcal{D}$ for each $f \in S$.*

DEFINITION 2.57. — Let $\mathcal{C}$ be an ∞-category, $L : \mathcal{C} \to L\mathcal{C} \subseteq \mathcal{C}$ a localization functor and $\mathcal{D}$ an ∞-category. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *L-invariant* if for every morphism f in $\mathcal{C}$ such that Lf is an equivalence, then so is $F(f)$ in $\mathcal{D}$.

Now we come back to our previous discussion.

LEMMA 2.58. — *Under the above discussion, consider the following conditions.*

i. *The left Kan extension $\tilde{F} : \mathcal{C} \to \mathcal{E}$ is L-invariant.*
ii. *The comparison map $\tilde{F}|_{\mathcal{C}^1} \to F_1$ constructed above is an equivalence.*
iii. *The comparison map $\tilde{F}|_{\mathcal{D}^1} \to F$ constructed above is an equivalence.*

We have the following.

1. *Conditions 2.58 and 2.58 are equivalent.*
2. *Condition 2.58 implies condition 2.58.*
3. *Under the assumptions in Remark 2.54, condition 2.58 implies condition 2.58.*

Proof. — First, restricting the comparison map $\tilde{F}|_{\mathcal{C}^1} \to F_1$ to $\mathcal{C}^0$, we get the identity, so conditions 2.58 and 2.58 are equivalent.

If $\tilde{F}$ is L-invariant, then for all $X \in \mathcal{C}^1$, the unit map $X \to LX$ induces a commutative diagram

$$\begin{array}{ccc} \tilde{F}(X) & \longrightarrow & \tilde{F}(LX) \\ \downarrow & & \downarrow \\ F_1(X) & \longrightarrow & F_1(LX) \end{array}$$

with the horizontal maps being equivalences. In particular, for all $Y \in \mathcal{D}^0$, there exists $X \in \mathcal{C}^0$ such that $Y \simeq LX$. Then $\tilde{F}(X) \to F_1(X)$ is an equivalence,

and, therefore, so are $\tilde{F}(LX) \to F_1(LX)$ and $\tilde{F}(Y) \to F_1(Y)$, which proves condition 2.58.

We now assume that we are in the special case described in Remark 2.54. Suppose that condition 2.58 holds. Note that F_1 is, by definition, L-invariant, and, therefore, for all $X \in \mathcal{C}^0$, $\tilde{F}$ maps the unit map $X \to LX$ to an equivalence. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by those $X \in \mathcal{C}$ such that $\tilde{F}$ maps $X \to LX$ to an equivalence. Then $\mathcal{C}^0 \subseteq \mathcal{C}'$. It follows from Propositions A.14 and A.16 that $\tilde{F}$ preserves sifted colimits. Since L preserves small colimits, $\mathcal{C}'$ is closed under sifted colimits, and, therefore, $\mathcal{C}' = \mathcal{C}$ by Lemma A.13. □

Remark 2.59. — We conjecture that all conditions in Lemma 2.58 are equivalent under Setup 2.53 without the assumptions in Remark 2.54.

Now we describe how the setups above give rise to derived prismatic cohomology in [12]. Let (A, I) be a bounded prism [12, Def 3.2]. Let $\mathcal{C} = \mathrm{Ani}(\mathrm{Alg}_{A/I})$ be the ∞-category of A/I-algebras and $\mathcal{D} \subseteq \mathcal{C}$ the full subcategory of p-completed A/I-algebras. In this case, the localization functor $\mathcal{C} \to \mathcal{D}$ is simply given by the p-completion $(-)^\wedge_p$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory of polynomial A/I-algebras. Then $\mathcal{D}^0 \subseteq \mathcal{D}$ is the full subcategory of p-completed polynomial A/I-algebras. [12, §4.2] defines the functors $F := \mathbb{\Delta}_{\cdot/A} : \mathcal{D}^0 \to D_{\mathrm{comp}}(A)$ and $G := \overline{\mathbb{\Delta}}_{\cdot/A} : \mathcal{D}^0 \to D_{\mathrm{comp}}(A/I)$, where $D_{\mathrm{comp}}(A)$ is the ∞-category of (p, I)-complete A-module spectra, and $D_{\mathrm{comp}}(A/I)$ is the ∞-category of p-complete A/I-module spectra. In Setup 2.53 and Remark 2.54, we claim that the functor $\tilde{F}$ and $\tilde{G}$ are left Kan extended from $\mathcal{D}^0$ after restriction to $\mathcal{D}$. That is to say, $\tilde{F}$ and $\tilde{G}$ are left derived functors $\mathbb{L}\mathbb{\Delta}_{\cdot/A}$ and $\mathbb{L}\overline{\mathbb{\Delta}}_{\cdot/A}$ defined in [12, Cons 7.6]. Thanks to Lemma 2.58, it suffices to show that $\tilde{F}$ and $\tilde{G}$ are $(-)^\wedge_p$-invariant. We will first describe our proof and then we offer the lemmas used in the proof.

We start with $\tilde{G}$. Composing G with the Postnikov tower $D_{\mathrm{comp}}(A/I) \to \mathrm{DF}_{\mathrm{comp}}(A/I), X \mapsto (\tau_{\geq n}X)_{n\in(\mathbb{Z},\geq)}$,where $\mathrm{DF}_{\mathrm{comp}}(A/I) := \mathrm{Fil}(D_{\mathrm{comp}}(A/I))$ is the filtered derived ∞-category of p-completed A/I-module spectra, we get a functor $G^P : \mathcal{D}^0 \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ such that the *union* (see Corollary 2.61) $\mathrm{Fil}^{-\infty} G^P : \mathcal{D}^0 \to D_{\mathrm{comp}}(A/I)$ is equivalent to G. It follows from the Hodge–Tate comparison [12, Prop 6.2] that the functorial comparison map $\left(\bigwedge^i L_{\cdot/(A/I)}\{-i\}[-i]\right)^\wedge_p \to \mathrm{gr}^{-i} \circ G^P$ is an equivalence). Now Remark 2.54 shows that $G^P : \mathcal{D}^0 \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ gives rise to $\widetilde{G^P} : \mathcal{C} \to \mathrm{DF}_{\mathrm{comp}}(A/I)$, and the functor $\left(\bigwedge^i L_{\cdot/(A/I)}\{-i\}[-i]\right)^\wedge_p : \mathcal{D}^0 \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ gives rise to some $\mathcal{C} \to \mathrm{DF}_{\mathrm{comp}}(A/I)$, which is $\left(\bigwedge^i L_{\cdot/(A/I)}\{-i\}[-i]\right)^\wedge_p$ by Example 2.52, and in particular, $(-)^\wedge_p$-invariant. It follows from Lemma 2.43 that the associated graded pieces $\mathrm{gr}^{-i} \circ \widetilde{G^P}$ are $(-)^\wedge_p$-invariant, and, therefore, the $(-)^\wedge_p$-invariance of $\tilde{G}$ follows from Corollary 2.61.

Note that $\tilde{F}$ coincides with $\tilde{G}$ composed with the derived modulo I, that is, the composite functor $\mathrm{Ani}(\mathrm{Alg}_{A/I}) \xrightarrow{\tilde{F}} D_{\mathrm{comp}}(A) \xrightarrow{\cdot \hat{\otimes}^{\mathbb{L}}_A (A/I)} D_{\mathrm{comp}}(A/I)$. We deduce by derived Nakayama [39, Tag 0G1U] that $\tilde{F}$ is also $(-)^{\wedge}_p$-invariant.

Here are the lemmas that we used in the argument above.

LEMMA 2.60. — *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a reflective subcategory with localization $L : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{E}$ be a stable ∞-category. Let $F : \mathcal{C} \to \mathrm{Fil}^{\leq 0}(\mathcal{E})$ be a functor. If the associated graded pieces $\mathrm{gr}^i \circ \tilde{F}$ are L-invariant for all $i \in \mathbb{Z}$, then so is $\tilde{F}$.*

Proof. — For all $C \in \mathcal{C}$, we inductively show that the unit map $C \to LC$ induces an equivalence $\mathrm{Fil}^i(\tilde{F}(C)) \to \mathrm{Fil}^i(\tilde{F}(LC))$. By assumption, this is true for all $i > 0$. Now consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Fil}^{i+1}(\tilde{F}(C)) & \longrightarrow & \mathrm{Fil}^{i}(\tilde{F}(C)) & \longrightarrow & \mathrm{gr}^{i}(\tilde{F}(C)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fil}^{i+1}(\tilde{F}(LC)) & \longrightarrow & \mathrm{Fil}^{i}(\tilde{F}(LC)) & \longrightarrow & \mathrm{gr}^{i}(\tilde{F}(LC)) \end{array}$$

where the horizontal maps are fiber sequences. Suppose that the result is true for $i+1$. Then the leftmost and the rightmost vertical maps are equivalences, and, therefore, so is the middle vertical maps, which shows that the result is true for i. □

From definitions, the following hold.

COROLLARY 2.61. — *Under the assumptions in Lemma 2.60, if we further assume that $\mathcal{E}$ admits filtered colimits, then the union $\mathrm{Fil}^{-\infty} \circ \tilde{F} : \mathcal{C} \to \mathrm{Fil}(\mathcal{E}) \xrightarrow{\mathrm{Fil}^{-\infty}} \mathcal{E}$ is also L-invariant.*

3. Animated ideals and PD-pairs

In this section, we will first give an informal exposition of Smith ideals introduced in [22] in terms of ∞-categories. See also [41, 40] for various generalizations. Then we will show how to apply these ideas to define and study "ideals" of animated rings and animated PD-pairs, which are the cornerstones of the animated theory of crystalline cohomology.

3.1. Smith ideals. — We fix a presentable stable symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$. The reader should feel free to take the special case that $\mathcal{C} = \mathrm{Sp}$ is the ∞-category of spectra and $\otimes$ is the smash product of spectra.

Consider the 1-simplex Δ^1, which is simply the 1-category associated to the ordinal $[1] = \{0 < 1\}$. The opposite category $(\Delta^1)^{\mathrm{op}}$ has a symmetric monoidal structure given by $\max\{\cdot,\cdot\}$[12].

Thus, the presentable stable ∞-category $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$ admits a presentable symmetric monoidal structure given by the *Day convolution* $\otimes^{\mathrm{Day}}$ [36, §3].

Informally, the unit object $\mathbf{1}_{\mathrm{Fun}((\Delta^1)^{\mathrm{op}},\mathcal{C})}$ is given by $(\mathbf{1}_{\mathcal{C}} \leftarrow 0) \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, and given n functors $F_1, \dots, F_n \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, the Day convolution $F_1 \otimes^{\mathrm{Day}} \cdots \otimes^{\mathrm{Day}} F_n$ is given as follows: $F_1, \dots, F_n$ determines an n-cube $F : (\Delta^1)^{\mathrm{op}} \times \cdots \times (\Delta^1)^{\mathrm{op}} \to \mathcal{C}, (e_1, \dots, e_n) \mapsto F_1(e_1) \otimes \cdots \otimes F_n(e_n)$. This cube, except the final vertex, determines a "cubical pushout" mapping to the final vertex: $(F(0, \dots, 0) \leftarrow \mathrm{colim}_{(\Delta^1)^{\mathrm{op}} \times \cdots \times (\Delta^1)^{\mathrm{op}} \setminus (0,\dots,0)} F)$, which is $F_1 \otimes^{\mathrm{Day}} \dots \otimes^{\mathrm{Day}} F_n$.

In particular, when $n = 2$, the Day convolution of $(X_0 \leftarrow X_1)$ and $(Y_0 \leftarrow Y_1)$ is given by $(X_0 \otimes Y_0 \leftarrow (X_0 \otimes Y_1) \amalg_{X_1 \otimes Y_1} (X_1 \otimes Y_0))$. This is essentially equivalent to the *pushout product monoidal structure* in [22, Thm 1.2].

On the other hand, there exists a pointwise symmetric monoidal structure $\otimes$ on the stable ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$, where $F_1 \otimes \cdots \otimes F_n$ is given by the functor $e \mapsto F_1(e) \otimes \cdots \otimes F_n(e)$.

There is a comparison between these two stable symmetric monoidal ∞-categories.

Proposition 3.1 (M. Ramzi). — *Let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category. Then there is an equivalence* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})$ *in* $\mathrm{CAlg}_{\mathcal{C}}(\mathrm{Pr}^L)$. *On the level of underlying categories, the equivalence is given by* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \ni F \mapsto (F(0) \to \mathrm{cofib}(F(1) \to F(0))) \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ *of which the inverse is given by* $\mathrm{Fun}(\Delta^1, \mathcal{C}) \ni G \mapsto (G(0) \leftarrow \mathrm{fib}(G(0) \to G(1))) \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$.

Proof. — The pair of inverse functors are clearly well defined. It suffices to endow the functor

$$\begin{aligned} K : (\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}}) &\longrightarrow (\mathrm{Fun}(\Delta^1, \mathcal{C}), \otimes) \\ (F(1) \to F(0)) &\longmapsto (F(0) \to \mathrm{cofib}(F(1) \to F(0))) \end{aligned}$$

a symmetric monoidal structure. Such a structure exists for every pointed presentable symmetric monoidal ∞-categories. Indeed, since $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{An}_*) \otimes \mathcal{C}$, and the same for $\mathrm{Fun}(\Delta^1, \mathcal{C})$, without loss of generality, we may assume that $\mathcal{C} = \mathrm{An}_*$. By the universal property of Day convolution, it suffices to endow the composite functor

$$\Delta^1 \longrightarrow \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{An}_*) \xrightarrow{K} \mathrm{Fun}(\Delta^1, \mathrm{An}_*)$$

12. We were informed of this by Denis Nardin.

a symmetric monoidal structure, where the first functor is the Yoneda embedding combined with adjoining a point $\mathrm{An} \to \mathrm{An}_*$, but since Δ^1 is a 1-category, it is equivalent to endowing the composite functor

$$\Delta^1 \longrightarrow \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Set}_*) \xrightarrow{K} \mathrm{Fun}(\Delta^1, \mathrm{Set}_*)$$

a symmetric monoidal structure. Note that a symmetric monoidal functor out of $(\Delta^1, \max)$ is the same as a map of idempotent algebras, and this can be checked directly. □

Now we assume that $\mathcal{C}$ admits a symmetric monoidal t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, which is compatible with filtered colimits. This is the case when $\mathcal{C} = \mathrm{Sp}$, and $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is the canonical t-structure for spectra. Then so does $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, that is to say, $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})_{\geq 0} := \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}_{\geq 0})$ and $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})_{\leq 0} := \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}_{\leq 0})$. Transferring this t-structure along the equivalence in Proposition 3.1, we get a t-structure on $\mathrm{Fun}(\Delta^1, \mathcal{C})$ where $\mathrm{Fun}(\Delta^1, \mathcal{C})_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is spanned by edges $f : X \to Y$ in $\mathcal{C}$ such that $X \in \mathcal{C}_{\geq 0}$ and $\mathrm{fib}(Y \to X) \in \mathcal{C}_{\geq 0}$, or equivalently, $X, Y \in \mathcal{C}_{\geq 0}$, and f is 1-connective (that is to say, $\pi_0(f)$ is surjective). We summarize as follows.

COROLLARY 3.2. — *The equivalence in Proposition 3.1 induces an equivalence of presentable symmetric monoidal full subcategories* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}_{\geq 0}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})_{\geq 0}$*, where the full subcategory* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})_{\geq 0}$ *is spanned by maps* $Y \leftarrow X$ *in* $\mathcal{C}_{\geq 0}$*.*

Passing to $\mathbb{E}_n$-algebras for any $n \in \mathbb{N} \cup \{\infty\}$, we get the following.

COROLLARY 3.3. — *There is an equivalence between the* ∞*-category of* $\mathbb{E}_n$*-algebras in* $(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}})$ *and the* ∞*-category of* $\mathbb{E}_n$*-maps between* $\mathbb{E}_n$*-algebras in* $(\mathcal{C}, \otimes)$*. This equivalence induces an equivalence between the full subcategory spanned by connective* $\mathbb{E}_n$*-algebras in* $(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}})$ *and the full subcategory spanned by* 1*-connective* $\mathbb{E}_n$*-maps between connective* $\mathbb{E}_n$*-algebras in* $(\mathcal{C}, \otimes)$*.*

Explicitly, for any $\mathbb{E}_n$-algebra in $((\mathrm{Fun}(\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}})$ of which the underlying object is $(A \leftarrow I)$, the object $A \in \mathcal{C}$, the cofiber $\mathrm{cofib}(I \to A)$ and the map $A \to \mathrm{cofib}(I \to A)$ admit canonical $\mathbb{E}_n$-algebra structures. We can then understand I as an "ideal" of $\mathbb{E}_n$-algebra A. When A is connective, the previous identification also describes connective "ideals" of A. This is the Smith ideal in [22], which gives rise to a reasonable theory of ideals (resp. connective ideals) of $\mathbb{E}_n$-rings (resp. connective $\mathbb{E}_n$-rings) when $\mathcal{C}$ is the presentable stable symmetric monoidal ∞-category Sp of spectra.

In the rest of this section, we will study the animated analogue of the preceding equivalence, that is to say, "ideals" and "PD-ideals" of an animated ring. To do so, we need to exploit more structures of $D(\mathbb{Z})$.

3.2. Animated (PD-)pairs. — In this section, we introduce the central object of this section: animated pairs and (absolute) animated PD-pairs.

NOTATION 3.4. — Let Pair denote the 1-category of ring-ideal pairs (A, I), that is, a (commutative) ring A along with an ideal $I \subseteq A$. Let Pair^{γ} denote the 1-category of divided power rings (A, I, γ) [39, Tag 07GU]. The *(absolute) PD-envelope functor* [39, Tag 07H9] $\mathrm{Pair} \to \mathrm{Pair}^{\gamma}$, being the left adjoint to the forgetful functor $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}$, is denoted by $(A, I) \mapsto D_A(I)$.

NOTATION 3.5. — Let $\mathrm{Inj} \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ be the full subcategory spanned by injective maps $M \hookleftarrow M'$.

We note that there is a pair $\mathrm{Inj} \rightleftarrows \mathrm{Pair}$ of adjoint functors where $\mathrm{Pair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ is the forgetful functor $(A, I) \mapsto (A \leftarrow I)$, and $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Pair}$ is the "symmetric product" $(M \hookleftarrow M') \mapsto (\mathrm{Sym}_{\mathbb{Z}}(M), M'\,\mathrm{Sym}_{\mathbb{Z}}(M))$, where $\mathrm{Sym}_{\mathbb{Z}}(M) \hookleftarrow M'\,\mathrm{Sym}_{\mathbb{Z}}(M)$ is the ideal generated by elements in M'.

Unfortunately, the category Inj might not be 1-projectively generated. In particular, we cannot apply Corollary 2.3 to deduce that the category Pair is 1-projectively generated (we believe that it is not), and to construct "Ani(Pair)". In fact, we need to embed Pair as a full subcategory of a 1-projectively generated 1-category, and then the ∞-category of animated pairs coincides with the animation of that larger 1-category.

We begin by analyzing the full subcategory $\mathrm{Inj} \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$. Note that $\{\mathbb{Z} \leftarrow 0, \mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z}\} \subseteq \mathrm{Inj}$ is a set of compact 1-projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ by Lemma 2.11.

NOTATION 3.6. — Let $\mathrm{Inj}^{\mathrm{st}} \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ denote the full subcategory generated by $\{\mathbb{Z} \leftarrow 0, \mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z}\}$ under finite coproducts, which is, in fact, a full subcategory of Inj.

It follows from Proposition A.16 that there is an equivalence $\mathcal{P}_{\Sigma,1}(\mathrm{Inj}^{\mathrm{st}}) \xrightarrow{\simeq} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ of ∞-categories. It then follows from Lemma 2.1 that the fully faithful embedding $\mathrm{Inj} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathrm{Inj}^{\mathrm{st}})$ admits a left adjoint given by the left derived functor of the inclusion $\mathrm{Inj}^{\mathrm{st}} \hookrightarrow \mathrm{Inj}$. We claim the following.

LEMMA 3.7. — *The essential image of* $\mathrm{Inj} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathrm{Inj}^{\mathrm{st}})$ *is spanned by those finite-product-preserving functors* $F : (\mathrm{Inj}^{\mathrm{st}})^{\mathrm{op}} \to \mathrm{Set}$*, which maps the edge* $(\mathbb{Z} \leftarrow 0) \to (\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z})$ *in* $\mathrm{Inj}^{\mathrm{st}}$ *to an injective map of sets.*

Proof. — The functors $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab}) \rightrightarrows \mathrm{Set}$ corepresented by $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Inj}^{\mathrm{st}}$ and $(\mathbb{Z} \leftarrow 0) \in \mathrm{Inj}^{\mathrm{st}}$ are given by $(A \leftarrow A') \mapsto A'$ and $(A \leftarrow A') \mapsto A$, respectively, and the edge $(\mathbb{Z} \leftarrow 0) \to (\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z})$ gives rise to the natural map $A \leftarrow A'$ of the two functors. It follows that an object $F \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$

lies in Inj if and only if the value of the natural map on F is an injection. The result then follows from the equivalence $\mathcal{P}_{\Sigma,1}(\mathrm{Inj}^{\mathrm{st}}) \xrightarrow{\simeq} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$. □

NOTATION 3.8. — Let $\mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}$ denote the full subcategory spanned by *standard pairs*, being the images of $\mathrm{Inj}^{\mathrm{st}}$ under the functor $\mathrm{Inj} \to \mathrm{Pair}$. In other words, a standard pair is a pair of form $(\mathbb{Z}[X,Y],(Y))$ for finite sets X and Y.

Then by Corollary 2.2, we have the following.

LEMMA 3.9. — *The free-pair functor* $\mathrm{Inj}^{\mathrm{st}} \to \mathrm{Pair}^{\mathrm{st}}$*, being essentially surjective, gives rise to the forgetful functor* $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$*, which is conservative and preserves sifted colimits.*

CONSTRUCTION 3.10. — *Lemma 2.1 gives us a canonical pair of adjoint functors* $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \rightleftarrows \mathrm{Pair}$*, where* $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \to \mathrm{Pair}$ *is the left derived* 1*-functor (Proposition A.14) of the inclusion* $\mathrm{Pair}^{\mathrm{st}} \hookrightarrow \mathrm{Pair}$*, and* $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ *is given by the restricted Yoneda embedding* $(A,I) \mapsto \mathrm{Hom}_{\mathrm{Pair}}(\cdot,(A,I))$.

We first note that the forgetful functors are compatible.

LEMMA 3.11. — *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pair} & \longrightarrow & \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \\ \downarrow & & \downarrow \\ \mathrm{Inj} & \hookrightarrow & \mathrm{Fun}(\Delta^{1,\mathrm{op}}, \mathrm{Ab}) \end{array}$$

of 1*-categories, where the vertical arrows are forgetful functors, and the top horizontal arrow is described in Construction 3.10.*

Proof. — Given a pair $(A,I) \in \mathrm{Pair}$, the image in $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ is given by $\mathrm{Pair}^{\mathrm{st}} \ni (B,J) \mapsto \mathrm{Hom}_{\mathrm{Pair}}((B,J),(A,I))$, subsequently mapped to $\mathrm{Inj}^{\mathrm{st}} \ni (M \hookleftarrow M') \mapsto \mathrm{Hom}_{\mathrm{Pair}}((\mathrm{Sym}_{\mathbb{Z}}(M), M' \mathrm{Sym}_{\mathbb{Z}}(M)),(A,I)) \cong \mathrm{Hom}_{\mathrm{Fun}((\Delta^1)^{\mathrm{op}},\mathrm{Ab})}(M \hookleftarrow M', A \hookleftarrow I)$. The other composite is the same. This identification is functorial in (A,I). □

Now we show that $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ is an embedding to a 1-projectively generated 1-category that we want. The trick is to talk about maps $(\mathbb{Z}[X],0) \to (A,I)$ and $(\mathbb{Z}[X],(X)) \to (A,I)$ in place of elements in A and I, respectively, to do certain "element chasing". We remind the reader that $\mathrm{Poly}_{\mathbb{Z}}$ is a set of compact projective objects for Ring, which gives rise to an equivalence $\mathcal{P}_{\Sigma,1}(\mathrm{Poly}_{\mathbb{Z}}) \simeq$ Ring of 1-categories, where $\mathrm{Poly}_{\mathbb{Z}}$ is the 1-category of polynomial rings.

LEMMA 3.12. — *The functor* $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ *described in Construction 3.10 is fully faithful.*

Proof. — The faithfulness follows from Lemma 3.11 and the faithfulness of the forgetful functor Pair $\to$ Inj. Given two pairs $(A, I), (B, J)$ in Pair and a natural map

$$\mathrm{Hom}_{\mathrm{Pair}}(\cdot, (A, I))|_{(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}}} \to \mathrm{Hom}_{\mathrm{Pair}}(\cdot, (B, J))|_{(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}}}$$

of finite-product-preserving functors $(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$, we need to show that this is induced by some map $(A, I) \to (B, J)$ of pairs.

By Lemma 3.11, there exists a unique map $(A \leftharpoonup I) \to (B \leftharpoonup J)$ in Inj which corresponds to the natural transform after composition $(\mathrm{Inj}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$.

Similarly, since $\mathcal{P}_{\Sigma,1}(\mathrm{Poly}_{\mathbb{Z}}) \simeq \mathrm{Ring}$, there exists a unique map $A \to B$ of rings, which corresponds to the natural transform after composition $\mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}} \to (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$, where $\mathrm{Poly}_{\mathbb{Z}} \to \mathrm{Pair}^{\mathrm{st}}$ is given by $R \mapsto (R, 0)$.

It then follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Free}_{\mathbb{Z}}^{\mathrm{fin}} & \xrightarrow{\mathrm{Sym}} & \mathrm{Poly}_{\mathbb{Z}} \\ \downarrow & & \downarrow \\ \mathrm{Inj}^{\mathrm{st}} & \longrightarrow & \mathrm{Pair}^{\mathrm{st}} \end{array} \tag{1}$$

of 1-categories, where $\mathrm{Free}_{\mathbb{Z}}^{\mathrm{fin}}$ is the 1-category of finite free abelian groups, with finite-coproduct-preserving functors that the two maps $(A \leftharpoonup I) \to (B \leftharpoonup J)$ in Inj and $A \to B$ in Ring are compatible, which gives rise to a map $(A, I) \to (B, J)$ in Pair. □

Now we characterize the image of this embedding.

Lemma 3.13. — *The square in Lemma 3.11 is Cartesian. Equivalently by Lemma 3.7, the essential image of the fully faithful functor* $\mathrm{Pair} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ *is spanned by those finite-product-preserving functors* $F : (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \to \mathrm{Set}$*, which maps the edge* $(\mathbb{Z}[X], 0) \to (\mathbb{Z}[X], (X))$ *in* $\mathrm{Pair}^{\mathrm{st}}$ *to an injective map of sets.*

Proof. — Let $F : (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \to \mathrm{Set}$ be a functor that preserves finite products such that the composite $(\mathrm{Inj}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$ belongs to the full subcategory Inj under the identification $\mathcal{P}_{\Sigma}(\mathrm{Inj}^{\mathrm{st}}) \simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$. The goal is to show that there exists a pair $(A, I) \in \mathrm{Pair}$ that represents F.

Let $(A \leftharpoonup I) \in \mathrm{Inj}$ correspond to the composite functor $(\mathrm{Inj}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$, and the map $A \leftharpoonup I$ of underlying sets is precisely induced by the map $(\mathbb{Z}[X], 0) \to (\mathbb{Z}[X], (X))$ in $\mathrm{Pair}^{\mathrm{st}}$.

The ring structure is given as follows: the functor $\mathrm{Poly}_{\mathbb{Z}} \to \mathrm{Pair}^{\mathrm{st}}$ given by $R \mapsto (R, 0)$ preserves finite coproducts, and thus the composite functor $\mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}} \to (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$ preserves finite products, which corresponds to a ring structure on A. The compatibility follows from (1).

Now we show that $A \hookleftarrow I$ is an ideal, that is to say, the ring multiplication $A \times A \to A$ restricts to a map $A \times I \to I$. By the above construction, $A = F(\mathbb{Z}[Y], 0)$ and $I = F(\mathbb{Z}[X], (X))$, and since F preserves finite products, $A \times I = F(\mathbb{Z}[X, Y], (X))$. Consider $(\mathbb{Z}[T], (T)) \in \mathrm{Pair}^{\mathrm{st}}$. The map $(\mathbb{Z}[T], (T)) \to (\mathbb{Z}[X, Y], (X)), T \mapsto XY$ in $\mathrm{Pair}^{\mathrm{st}}$ induces a map $A \times I \to I$. The commutative diagram

$$\begin{array}{ccc} (\mathbb{Z}[X, Y], 0) & \longrightarrow & (\mathbb{Z}[T], 0) \\ \downarrow & & \downarrow \\ (\mathbb{Z}[X, Y], (X)) & \longrightarrow & (\mathbb{Z}[T], (T)) \end{array}$$

in $\mathrm{Pair}^{\mathrm{st}}$ shows that the preceding map $A \times I \to I$ is compatible with the ring structure and the inclusion $I \to A$.

It remains to construct an isomorphism $F \to \mathrm{Fun}_{\mathrm{Pair}}(\cdot, (A, I))|_{(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}}}$ of finite-product-preserving functors $(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$. Composing with the functor $(\mathrm{Inj}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}}$ denoted by j, we get a map $F \circ j \to \mathrm{Fun}_{\mathrm{Pair}}(\cdot, (A, I)) \circ j$ of functors $(\mathrm{Inj}^{\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$ which is an equivalence by construction (and the adjunction $\mathrm{Fun}(\Delta^1, \mathrm{Ab})_{\mathrm{inj}} \rightleftarrows \mathrm{Pair}$). We need to show that this equivalence descends along the essentially surjective functor j.

First, for any $(B, J) \in \mathrm{Pair}^{\mathrm{st}}$, by picking any lift under j, the map $F(B, J) \to \mathrm{Fun}_{\mathrm{Pair}}((B, J), (A, I))$ could be described as follows: for any $f \in F(B, J)$ and any $b \in B$, the element b corresponds uniquely to a map $\overline{b} : (\mathbb{Z}[t], 0) \to (B, J)$ of pairs. Note that $\overline{b}^*(f) \in F(\mathbb{Z}[t], 0) \cong A$. The image of f, as a map $(B, J) \to (A, I)$ of pairs, is concretely given by $b \mapsto \overline{b}^*(f)$, which is independent of the choice of the lift of (B, J).

Now it remains to show that, for any map $\varphi : (B, J) \to (C, K)$ of pairs, the diagram

$$\begin{array}{ccc} F(B, J) & \longrightarrow & \mathrm{Fun}_{\mathrm{Pair}}((B, J), (A, I)) \\ \uparrow & & \uparrow \\ F(C, K) & \longrightarrow & \mathrm{Fun}_{\mathrm{Pair}}((C, K), (A, I)) \end{array}$$

is commutative. Indeed, for any $f \in F(C, K)$, the image in $\mathrm{Fun}_{\mathrm{Pair}}((C, K), (A, I))$ is given by $c \mapsto \overline{c}^*(f)$, and the image in $\mathrm{Fun}_{\mathrm{Pair}}((B, J), (A, I))$ is given by $b \mapsto \overline{\varphi(b)}^*(f)$. On the other hand, the image of f in $F(B, J)$ is $\varphi^*(f)$, and the image in $\mathrm{Fun}_{\mathrm{Pair}}((B, J), (A, I))$ is given by $b \mapsto \overline{b}^*(\varphi^*(f))$. The result follows from the fact that $\varphi \circ \overline{b} = \overline{\varphi(b)}$ as maps $(\mathbb{Z}[t], 0) \rightrightarrows (C, K)$ of pairs. □

Remark 3.14. — The 1-category $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ contains more objects than Pair. They might be of independent interest. For example, let A be a ring and I an invertible A-module along with a map $j : I \to A$ of A-modules. If the

map j in question is not injective, then it does not "faithfully" correspond to a ring-ideal pair such as $(A, \mathrm{im}(j))$, that is to say, it represents an object in $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$,which is different from $(A, \mathrm{im}(j))$. In fact, the 1-category $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ could be identified with the 1-category of commutative algebra objects in $\mathrm{Fun}(\Delta^1, \mathrm{Ab})_{\mathrm{surj}}$ with pushout product monoidal structure, 1-categorical version of Section 3.1, or equivalently, the category of *quasi-ideals*[13] in [17, §3.3].

We now develop a PD analogue as follows.

NOTATION 3.15. — Let $\mathrm{Pair}^{\gamma,\mathrm{st}} \subseteq \mathrm{Pair}^{\gamma}$ denote the full subcategory spanned by the images of $(A, I) \in \mathrm{Pair}^{\mathrm{st}}$ under the functor of PD-envelope [39, Tag 07H9], denoted by $(D_A(I) \twoheadrightarrow A/I, \gamma)$ instead of the cumbersome notation $(D_A(I), \ker(D_A(I) \twoheadrightarrow A/I), \gamma)$.

CONSTRUCTION 3.16. — *By Lemma 2.1, we get a pair* $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}}) \rightleftarrows \mathrm{Pair}^{\gamma}$ *of adjoint functors. Explicitly, the objects in* $\mathrm{Pair}^{\gamma,\mathrm{st}}$ *are of the form* $D_{\mathbb{Z}[X,Y]}(Y) \cong \Gamma_{\mathbb{Z}[X]}(Y)$ *for finite sets* X *and* Y*.*

On the other hand, from Corollary 2.2 we have the following.

LEMMA 3.17. — *The PD-envelope functor* $\mathrm{Pair}^{\mathrm{st}} \to \mathrm{Pair}^{\gamma,\mathrm{st}}$*, being essentially surjective, gives rise to the forgetful functor* $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}}) \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$*, which is conservative and preserves sifted colimits.*

There is another *forgetful functor* $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}$. These functors are compatible.

LEMMA 3.18. — *The diagram*

$$\begin{array}{ccc} \mathrm{Pair}^{\gamma} & \longrightarrow & \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}}) \\ \downarrow & & \downarrow \\ \mathrm{Pair} & \hookrightarrow & \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \end{array}$$

is a commutative diagram of 1*-categories, where vertical arrows are forgetful functors, and the top horizontal arrow is described in Construction 3.16.*

Proof. — For any PD-pair $(A, I, \gamma) \in \mathrm{Pair}^{\gamma}$, the image in $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}})$ is given by $\mathrm{Pair}^{\gamma,\mathrm{st}} \ni (B, J, \delta) \mapsto \mathrm{Hom}_{\mathrm{Pair}^{\gamma}}((B, J, \delta), (A, I, \gamma))$, which is subsequently mapped to an object in $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ given by $\mathrm{Pair}^{\mathrm{st}} \ni (B, J) \mapsto \mathrm{Hom}_{\mathrm{Pair}^{\gamma}}(D_J(B), (A, I, \gamma))$. On the other hand, the image of (A, I, γ) in Pair is (A, I), which is subsequently mapped to an object in $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ given by $\mathrm{Pair}^{\mathrm{st}} \ni (B, J) \mapsto \mathrm{Hom}_{\mathrm{Pair}}((B, J), (A, I))$. It then follows from the functorial isomorphism $\mathrm{Hom}_{\mathrm{Pair}^{\gamma}}(D_J(B), (A, I, \gamma)) \cong \mathrm{Hom}_{\mathrm{Pair}}((B, J), (A, I))$ by adjunction. □

13. We thank Ofer GABBER for informing about this concept.

Similarly, we have the following embedding.

LEMMA 3.19. — *The functor* $\mathrm{Pair}^{\gamma} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}})$ *described in Construction 3.16 is fully faithful.*

Proof. — The proof is similar to that of Lemma 3.12. The faithfulness follows from Lemma 3.18 and the faithfulness of the forgetful functor $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}$. Given two PD-pairs (A, I, γ) and (B, J, δ) in Pair^{γ} and a map

$$F := \mathrm{Hom}_{\mathrm{Pair}^{\gamma}}(\cdot, (A, I, \gamma))|_{(\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}}} \to \mathrm{Hom}_{\mathrm{Pair}^{\gamma}}(\cdot, (B, J, \delta))|_{(\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}}} =: G$$

of finite-product-preserving functors $(\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$, we need to show that this is induced by some map $(A, I, \gamma) \to (B, J, \delta)$ of PD-pairs.

By Lemma 3.12, there exists a unique map $(A, I) \to (B, J)$ of pairs, which correspond to the natural transform after composition $(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}} \rightrightarrows \mathrm{Set}$. It remains to show that this map preserves the PD-structure.

Indeed, any $x \in I$ corresponds to a map $(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to (A, I, \gamma)$ of PD-pairs, i.e., an element $\overline{x} \in F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$, and the image y of $x \in I$ in J is given by the image $\overline{y} \in G(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$ under the map $F \to G$. For any $n \in \mathbb{N}_{>0}$, there is a canonical endomorphism $(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0), t \mapsto \gamma_n(t)$ of PD-pairs, which induces endomorphisms $F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$ and $G(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to G(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$ compatible with the map $F \to G$. In particular, the image, denoted by $\overline{x}_n$, of $\overline{x}$ under the first endomorphism maps to the image, denoted by $\overline{y}_n$, of $\overline{y}$ under the second endomorphism. We note that $\overline{x}_n$ corresponds to $\gamma_n(x)$, and $\overline{y}_n$ corresponds to $\gamma_n(y)$. Thus, the map $(A, I) \to (B, J)$ maps $\gamma_n(x)$ to $\gamma_n(y)$. □

We have the following description of the essential image (cf. Lemma 3.13).

LEMMA 3.20. — *The square in Lemma 3.18 is Cartesian. Equivalently by Lemma 3.13, the essential image of the fully faithful functor* $\mathrm{Pair}^{\gamma} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}})$ *(Lemma 3.19) is spanned by those finite-product-preserving functors* $F : (\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}} \to \mathrm{Set}$*, which maps the edge* $(\mathbb{Z}[X], 0, 0) \to (\Gamma_{\mathbb{Z}}(X) \twoheadrightarrow \mathbb{Z}, \gamma)$ *in* $\mathrm{Pair}^{\gamma,\mathrm{st}}$ *to an injective map of sets.*

Proof. — The proof is similar to that of Lemma 3.13. Let $F : (\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}} \to \mathrm{Set}$ be a functor such that the composite $(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$ lies in the essential image of the fully faithful functor $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$. We need to construct a PD-pair $(A, I, \gamma) \in \mathrm{Pair}^{\gamma}$ that represents the functor F.

Let (A, I) represent the composite functor $(\mathrm{Pair}^{\mathrm{st}})^{\mathrm{op}} \to (\mathrm{Pair}^{\gamma,\mathrm{st}})^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$. Unrolling the definitions, we see that $A = F(\mathbb{Z}[t], 0, 0)$, $I = F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma)$, and the map $I \to A$ is induced by the map $(\mathbb{Z}[t], 0, 0) \to (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma)$ of PD-pairs. We endow a PD-structure (A, I) as follows: there exists a canonical endomorphism $\gamma_n : (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma), t \mapsto \gamma_n(t)$ of PD-pair,

which induces a map $\gamma_n : I \to I$ for all $n \in \mathbb{N}_{>0}$. We need to check that $(\gamma_n)_{n\in\mathbb{N}_{>0}}$ satisfies the axioms of divided power structure [39, Tag 07GL], setting $\gamma_0 = \mathrm{id}$. We spell out the verification of two of them.

- $\gamma_n(x+y) = \sum_i \gamma_i(x)\gamma_{n-i}(y)$ for $(x,y) \in I^2$:
  First, in the PD-pair $(\Gamma_{\mathbb{Z}}(X,Y) \twoheadrightarrow \mathbb{Z}, \gamma)$, the identity $\gamma_n(X+Y) = \sum_i \gamma_i(X)\gamma_{n-i}(Y)$ holds. This implies that the composite

  $$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(T_0, \dots, T_n) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(X,Y) \twoheadrightarrow \mathbb{Z}, \gamma),$$

  where the first map is induced by $T \mapsto \sum_i T_i$, and the second map is induced by $T_i \mapsto \gamma_i(X)\gamma_{n-i}(Y)$, coincides with the composite

  $$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \xrightarrow{\gamma_n} (\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(X,Y) \twoheadrightarrow \mathbb{Z}, \gamma),$$

  where the second map is induced by $T \mapsto X+Y$. Applying F to the two compositions, using the fact that F preserves finite products, and that $(x,y) \in I^2$ corresponds to an element in $F(\Gamma_{\mathbb{Z}}(X,Y) \twoheadrightarrow \mathbb{Z}, \gamma)$, we get the result (for the part $T_i \mapsto \gamma_i(X)\gamma_{n-i}(Y)$,one need to separate $i = 0$ and $i > 0$).
- $\gamma_n(ax) = a^n\gamma_n(x)$ for $(a,x) \in A \times I$:
  In the PD-pair $(\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma)$, the identity $\gamma_n(YX) = Y^n\gamma_n(X)$ holds. This implies that the composite

$$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}[T_1,\dots,T_n]}(t) \twoheadrightarrow \mathbb{Z}[T_1, \dots, T_n], \gamma) \to (\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma),$$

  where the first map is induced by $T \mapsto T_1 \cdots T_n t$, and the second map is induced by $T_i \mapsto Y$ and $t \mapsto X$, coincides with the composite

  $$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \xrightarrow{\gamma_n} (\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma),$$

  where the second map is induced by $T \mapsto XY$. Applying F to the two compositions, using the fact that F preserves finite products, and that $(a,x) \in A \times I$ corresponds to an element in $F(\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma)$, we get the result.

Finally, the proof of the fact that (A, I, γ) represents F is parallel to the corresponding part of the proof of Lemma 3.13. □

Now we arrive at the definition of animated pairs and animated PD-pairs.

Definition 3.21. — The ∞-category $\mathrm{Pair}^{\mathrm{an}}$ of *animated pairs* is defined to be the ∞-category $\mathcal{P}_\Sigma(\mathrm{Pair}^{\mathrm{st}})$, and the ∞-category $\mathrm{Pair}^{\gamma,\mathrm{an}}$ of *animated PD-pairs* is defined to be the ∞-category $\mathcal{P}_\Sigma(\mathrm{Pair}^{\gamma,\mathrm{st}})$.

The *forgetful functor* $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}(\Delta^1, D(\mathbb{Z})_{\geq 0})$ is given by the pair $\mathcal{P}_\Sigma(\mathrm{Inj}^{\mathrm{st}}) \rightleftarrows \mathcal{P}_\Sigma(\mathrm{Pair}^{\mathrm{st}})$ obtained by applying Corollary 2.2 to the free-pair functor $\mathrm{Inj}^{\mathrm{st}} \to \mathrm{Pair}^{\mathrm{st}}$ (which is essentially surjective).

The *forgetful functor* $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ and the *animated PD-envelope functor* $\mathrm{Env}^{\gamma,\mathrm{an}} : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}}$ are given by the pair $\mathcal{P}_\Sigma(\mathrm{Pair}^{\mathrm{st}}) \rightleftarrows \mathcal{P}_\Sigma(\mathrm{Pair}^{\gamma,\mathrm{st}})$

obtained by applying Corollary 2.2 to the PD-envelope functor $\mathrm{Pair}^{\mathrm{st}} \to \mathrm{Pair}^{\gamma,\mathrm{st}}$ being essentially surjective.

From Corollary 2.2 we have the following.

COROLLARY 3.22. — *The forgetful functors* $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ *and* $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ *are conservative and preserve sifted colimits.*

These forgetful functors are compatible with canonical embeddings $\mathrm{Pair} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \hookrightarrow \mathrm{Pair}^{\mathrm{an}}$ and $\mathrm{Pair}^{\gamma} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}}) \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$.

PROPOSITION 3.23. — *The diagram*

$$\begin{array}{ccc} \mathrm{Pair}^{\gamma} & \hookrightarrow & \mathrm{Pair}^{\gamma,\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathrm{Pair} & \hookrightarrow & \mathrm{Pair}^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathrm{Inj} & \hookrightarrow & \mathrm{Fun}(\Delta^{1,\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) \end{array}$$

is a commutative diagram of ∞*-categories, where the vertical arrows are forgetful functors. Moreover, the squares are Cartesian.*

Proof. — The commutativity follows from Remark A.22 and Lemmas 3.11 and 3.18. The last claim follows from Lemmas 3.13 and 3.20. □

REMARK 3.24. — The embeddings $\mathrm{Pair} \hookrightarrow \mathrm{Pair}^{\mathrm{an}}$ and $\mathrm{Pair}^{\gamma} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$ admit left adjoints given by the composite functors $\mathrm{Pair}^{\mathrm{an}} \xrightarrow{\tau_{\leq 0}} \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \to \mathrm{Pair}$ and $\mathrm{Pair}^{\gamma,\mathrm{an}} \xrightarrow{\tau_{\leq 0}} \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}}) \to \mathrm{Pair}^{\gamma}$ (see Remark A.22 for $\tau_{\leq 0}$). We will give an explicit description of the functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$ in Proposition 3.35.

Taking the left adjoints to the upper square of the diagram in Proposition 3.23, we get the following.

COROLLARY 3.25. — *The diagram*

$$\begin{array}{ccc} \mathrm{Pair}^{\mathrm{an}} & \longrightarrow & \mathrm{Pair} \\ \downarrow & & \downarrow \\ \mathrm{Pair}^{\gamma,\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\gamma} \end{array}$$

is a commutative diagram of ∞*-categories, where the horizontal arrows are described in Remark 3.24.*

In particular, we rewrite the PD-envelope in terms of animated PD-envelope.

COROLLARY 3.26. — *The PD-envelope functor* $\mathrm{Pair} \to \mathrm{Pair}^{\gamma}, (A, I) \mapsto D_A(I)$ *coincides with the composite functor* $\mathrm{Pair} \hookrightarrow \mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\gamma}$, *where the last functor is described in Remark 3.24.*

In fact, there is a more concrete description of $\mathrm{Pair}^{\mathrm{an}}$, given by the following.

DEFINITION 3.27. — The ∞-*category of surjective maps of animated rings* is the full subcategory $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ of maps $A \to A''$ such that the induced map $\pi_0(A) \to \pi_0(A'')$ on the 0th homotopy groups is surjective.

We now show that the strategy to prove Corollary 3.3 adapts to our case. Indeed, by Corollary 3.2, we have the equivalence $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) \simeq \mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$ of ∞-categories; therefore a set of compact projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ gives rise to a set of compact projective generators for $\mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$: $\left\{\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \mathbb{Z} \to 0\right\}$. Now we study two adjunctions over these ∞-categories.

We have a pair $\mathrm{Fun}(\Delta^1, D(\mathbb{Z})_{\geq 0}) \rightleftarrows \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ of adjoint functors induced by the pair $D(\mathbb{Z})_{\geq 0} \overset{\mathbb{L}\mathrm{Sym}_{\mathbb{Z}}}{\rightleftarrows} \mathrm{CAlg}^{\mathrm{an}}$ of adjoint functors. Restricting to full subcategories, we get a pair $\mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0} \rightleftarrows \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$; the latter is defined before Corollary 3.2. We summarize the preceding discussion by the diagram

$$
\begin{array}{ccccc}
\mathcal{P}_{\Sigma}(\mathrm{Pair}^{\mathrm{st}}) & & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} & \subseteq & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0} & \subseteq & \mathrm{Fun}(\Delta^1, D(\mathbb{Z})_{\geq 0})
\end{array}
\tag{2}
$$

We note that both full subcategories are stable under small colimits, and, therefore, the forgetful functor $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \to \mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$ preserves sifted colimits. Since the forgetful functor is also conservative, it follows by Proposition A.18 that $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$ is projectively generated, for which $\left\{\mathbb{Z}[t] \xrightarrow{\mathrm{id}} \mathbb{Z}[t], \mathbb{Z}[t] \to \mathbb{Z}\right\}$ is a set of compact projective generators. Let $\mathcal{Z} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$ denote the full subcategory spanned by finite coproducts of these objects, which is effectively a full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Ring})$. The following lemma is then obvious.

LEMMA 3.28. — *There is an equivalence* $\mathrm{Pair}^{\mathrm{st}} \simeq \mathcal{Z}$ *of* 1*-categories given by* $\mathrm{Pair}^{\mathrm{st}} \to \mathcal{Z}, (A, I) \mapsto (A \twoheadrightarrow A/I)$ *and* $\mathcal{Z} \to \mathrm{Pair}^{\mathrm{st}}, (A \twoheadrightarrow A'') \mapsto (A, \ker(A \twoheadrightarrow A''))$.

From the previous discussion we have the following.

THEOREM 3.29. — *There is an equivalence* $\mathrm{Pair}^{\mathrm{an}} = \mathcal{P}_{\Sigma}(\mathrm{Pair}^{\mathrm{st}}) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$ *of* ∞*-categories which fits into* (2)*, making the left square a commutative square*[14].

In the proof of [3, Lem 3.16], Jorge ANTONÍO sketched a slightly different set of compact projective generators. It would be nice to compare the two choices.

REMARK 3.30. — Corollary 3.3 says that the ∞-category of $\mathbb{E}_\infty$-algebras in the symmetric monoidal ∞-category $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ is equivalent to that of $\mathbb{E}_\infty$-algebras in the symmetric monoidal ∞-category $\mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$ since two symmetric monoidal ∞-categories are equivalent. Our result essentially says that both ∞-categories admits endomorphism monads, which is also preserved under this equivalence, and, therefore, the module categories over these monads are equivalent.

NOTATION 3.31. — Given the equivalence in Theorem 3.29, we will symbolically denote an object in $\mathrm{Pair}^{\gamma,\mathrm{an}}$ by $(A \twoheadrightarrow A'', \gamma)$, where $A \twoheadrightarrow A''$ is the image under the forgetful functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}} \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$. When the PD-structure is the "obvious" one (like $\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]$), by abuse of notation, we will omit the γ in question. Under this notation, objects in $\mathrm{Pair}^{\mathrm{st}}$ could be identified with $\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X]$, and objects in $\mathrm{Pair}^{\gamma,\mathrm{st}}$ could be identified with $\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]$.

REMARK 3.32. — In Theorem 3.29, we can replace $D(\mathbb{Z})$ by any *derived algebraic context* $\mathcal{C}$ [38, Def 4.2.1], and then both $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Fun}(\Delta^1, \mathcal{C})$ admit canonical structures of derived algebraic contexts that are preserved under the equivalence $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, and Theorem 3.29 essentially generalizes to the equivalence between the ∞-categories of connective maps of *derived commutative algebras* [38, Rem 4.2.24] (note that $\mathrm{Pair}^{\mathrm{an}} \simeq \mathrm{DAlg}(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})))^{\mathrm{cn}}$ and $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \simeq \mathrm{DAlg}(\mathrm{Fun}(\Delta^1, D(\mathbb{Z})))^{\mathrm{cn}}$).

REMARK 3.33. — In fact, the machinery in [38, §4], due to Bhatt–Mathew and [13], allows us to define the ∞-category of *derived PD-pairs* of which the connective objects spans a full subcategory equivalent to the ∞-category of animated PD-pairs.

WARNING 3.34. — *We warn the reader that the heart* $\mathrm{DAlg}(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})))^{\heartsuit}$ *in* [38, Rem 4.2.24] *is equivalent to the* 1*-category* $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$*, not the* 1*-category* Pair.

We also identify the equivalence in Theorem 3.29 restricted to the full subcategory $\mathrm{Pair} \subseteq \mathrm{Pair}^{\mathrm{an}}$.

14. More precisely, there are two possible left squares in (2). However, by uniqueness of left/right adjoint, roughly speaking, one commutes if and only if the other commutes.

PROPOSITION 3.35. — *Let* $\mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ring})$ *be the full subcategory spanned by those surjective maps* $A \twoheadrightarrow A''$ *of rings. Then the equivalence* $\mathrm{Pair} \to \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}, (A, I) \mapsto (A \twoheadrightarrow A/I)$ *fits into a canonical commutative diagram*

$$\begin{array}{ccc} \mathrm{Pair} & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}} \\ \downarrow & & \downarrow \\ \mathrm{Pair}^{\mathrm{an}} & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \end{array}$$

of ∞*-categories. Furthermore, the localization* $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$ *(Remark 3.24) could be identified with* $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \to \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}, (A \twoheadrightarrow A'') \mapsto (\pi_0(A) \twoheadrightarrow \pi_0(A''))$.

Proof. — We note that $\mathrm{Fun}(\Delta^1, \mathrm{Ring}) \subseteq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ is the reflective subcategory (Definition 2.44) spanned by the 1-truncated objects, of which the localization is given by $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{Fun}(\Delta^1, \mathrm{Ring}), (A \to A'') \mapsto (\pi_0(A) \to \pi_0(A''))$ by Corollary 2.13 and Remark A.22. Restricting to the full subcategory $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$, we get a localization $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \to \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Pair}^{\mathrm{an}} & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \\ \downarrow & & \downarrow \\ \mathrm{Pair} & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}} \end{array}$$

of ∞-categories, where the vertical arrows are localizations (Remark 3.24). We claim that this is a commutative diagram. Indeed, both compositions commute with filtered colimits and geometric realizations (in fact, all small colimits, since both vertical arrows are localizations in Definition 2.44), and when restricting to $\mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}^{\mathrm{an}}$, both compositions are canonically equivalent. Then the claim follows from Proposition A.14.

Another way to show the commutativity is to show that the top right composition is $(\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair})$-invariant in Definition 2.57, then invoke Proposition 2.56.

Then the result follows by taking the right adjoints to the vertical arrows. □

COROLLARY 3.36. — *The lower square in Proposition 3.23 is left adjointable* [30, Def 4.7.4.13]*, which gives rise to a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pair}^{\mathrm{an}} & \longrightarrow & \mathrm{Pair} \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Delta^{1,\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) & \longrightarrow & \mathrm{Inj} \end{array}$$

of 1*-categories, where the vertical arrows are forgetful functors.*

WARNING 3.37. — *The upper square in Proposition 3.23 is not left adjointable. That is to say, the localizations* $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$ *and* $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\gamma}$ *are not compatible with forgetful functors; otherwise the forgetful functor* $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}$ *would commute with small colimits, which is false (see Remark 3.41).*

From Propositions 3.23 and 3.35 we have the following.

PROPOSITION 3.38. — *The essential image of the fully faithful embedding* $\mathrm{Pair}^{\gamma} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$ *is spanned by those animated PD-pairs* $(A \twoheadrightarrow A'', \gamma)$ *such that both* A *and* A'' *are static.*

To understand the difference between Pair and $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \simeq \tau_{\leq 0}(\mathrm{Pair}^{\mathrm{an}})$ better, we compute the following example.

EXAMPLE 3.39. — Consider $(\mathbb{Z}/4\mathbb{Z}, (2)) \in \mathrm{Pair}$ as an animated pair. By Proposition 3.35, this corresponds to the surjective map $\mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$ of rings. Let us study the coproduct $(\mathbb{Z}/4\mathbb{Z}, (2)) \amalg (\mathbb{Z}/4\mathbb{Z}, (2))$ taken in $\mathrm{Pair}^{\mathrm{an}}$. Thanks to Theorem 3.29, this corresponds to the surjective map $\mathbb{Z}/4\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{F}_2 \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_2$ of animated rings. The underlying map in $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$ is given by

$$(\mathbb{Z}/4\mathbb{Z})[1] \oplus \mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{F}_2[1] \oplus \mathbb{F}_2$$

induced by $0 : (\mathbb{Z}/4\mathbb{Z})[1] \to \mathbb{F}_2[1]$ and the canonical projection $\mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$. Under the forgetful functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$, the image of $(\mathbb{Z}/4\mathbb{Z}, (2)) \amalg (\mathbb{Z}/4\mathbb{Z}, (2))$ is thus given by

$$\begin{aligned}(\mathbb{Z}/4\mathbb{Z})[1] \oplus \mathbb{Z}/4\mathbb{Z} &\longleftarrow \mathrm{fib}((\mathbb{Z}/4\mathbb{Z})[1] \oplus \mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{F}_2[1] \oplus \mathbb{F}_2) \\ &\xleftarrow{\simeq} (\mathbb{Z}/4\mathbb{Z})[1] \oplus \mathbb{F}_2 \oplus 2\mathbb{Z}/4\mathbb{Z}\end{aligned}$$

induced by $\mathbb{F}_2 \to 0$, and other maps are canonical. Since the forgetful functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ commutes with $\tau_{\leq 0}$ (Remark A.22), we can identify the underlying object of $\tau_{\leq 0}((\mathbb{Z}/4\mathbb{Z}, (2)) \amalg (\mathbb{Z}/4\mathbb{Z}, (2)))$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ with $(\mathbb{Z}/4\mathbb{Z} \leftarrow \mathbb{F}_2 \oplus 2\mathbb{Z}/4\mathbb{Z})$, which is not injective. Roughly speaking, the localization $\mathcal{P}_{\Sigma}(\mathrm{Pair}^{\mathrm{st}}) \to \mathrm{Pair}$ will kill the kernel $\mathbb{F}_2$.

We now prove a stronger colimit-preserving property of the forgetful functor from animated PD-pairs to animated pairs, which does not seem to be obvious without this identification.

PROPOSITION 3.40. — *The forgetful functor* $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ *preserves small colimits.*

Proof. — By Proposition A.14, it suffices to show that the composite functor $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ preserves finite coproducts. We first "simplify" this composition, then we compute the finite coproducts by hand.

Since $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$ factors as $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$, it follows from Proposition 3.23 that the composite functor $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ is equivalent to the composite functor $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma} \to \mathrm{Pair} \hookrightarrow$

$\mathrm{Pair}^{\mathrm{an}}$. Under the equivalence in Theorem 3.29, this functor is concretely given by $\mathrm{Pair}^{\gamma,\mathrm{st}} \ni (A, I, \gamma) \mapsto (A \twoheadrightarrow A/I) \in \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$. Since $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ is stable under small colimits, we can take the finite coproducts in the larger ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$.

Every object in $\mathrm{Pair}^{\gamma,\mathrm{st}}$ is the PD-envelope of a pair of form $(\mathbb{Z}[X_1, \ldots, X_m, Y_1, \ldots, Y_n], (Y_1, \ldots, Y_n))$, which we will denote by $\Gamma_{\mathbb{Z}[X_1,\ldots,X_m]}(Y_1, \ldots, Y_n) \twoheadrightarrow \mathbb{Z}[X_1, \ldots, X_m]$. Now the result follows from the fact that

$$\Gamma_{\mathbb{Z}[X]}(Y) \otimes^{\mathbb{L}}_{\mathbb{Z}} \Gamma_{\mathbb{Z}[X']}(Y') \simeq \Gamma_{\mathbb{Z}[X,X']}(Y, Y')$$

and

$$\mathbb{Z}[X] \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}[X'] \simeq \mathbb{Z}[X, X']$$

where $X = (X_1, \ldots, X_m)$, $X' = (X'_1, \ldots, X'_{m'})$, $Y = (Y_1, \ldots, Y_n)$ and $Y' = (Y'_1, \ldots, Y'_{n'})$. □

Remark 3.41. — Proposition 3.40 implies that the forgetful functor $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\gamma,\mathrm{st}}) \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$ preserves small colimits, cf. Lemma 3.17. However, the forgetful functor $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}$ does not preserve small colimits, even pushouts [39, Tag 07GY]. The counterexample there is given by two PD-structures on the pair $(\mathbb{Z}/4\mathbb{Z}, (2))$. We explain the incompatibility of the localizations in Warning 3.37 by Example 3.39: the localization $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$ kills the kernel $\mathbb{F}_2$, while the localization $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\gamma}$ kills more, since the PD-structure does not necessarily pass to the quotient, so one needs to quotient out more relations.

Corollary 3.42. — *The composite functor* $\mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$, *where the second functor is the forgetful functor, preserves small colimits.*

3.3. Basic properties. — In this section, we will discuss basic properties of animated pairs (resp. animated PD-pairs).

First, we recall that, given a pair (A, I), let (B, J, γ) be its PD-envelope; then there is a canonical equivalence $A/I \cong B/J$ [39, Tag 07H7]. There is an analogue for animated PD-envelope:

Lemma 3.43. — *The composite functor* $F : \mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$, *where the second functor is the forgetful functor, is compatible with the evaluation* $\mathrm{ev}_{[1]} : \mathrm{Pair}^{\mathrm{an}} \simeq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \to \mathrm{CAlg}^{\mathrm{an}}$ *at* $[1] \in \Delta^1$. *That is to say, the composite functor* $\mathrm{Pair}^{\mathrm{an}} \xrightarrow{F} \mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{CAlg}^{\mathrm{an}}$ *is homotopy equivalent to the functor* $\mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{CAlg}^{\mathrm{an}}$.

Proof. — Both functors are left derived functors, and, therefore, it suffices to check on the full subcategory $\mathrm{Poly}_{\mathbb{Z}} \subseteq \mathrm{Pair}^{\mathrm{an}}$, which follows from a direct identification. □

We note that the functor $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}), A \mapsto (\mathrm{id}_A : A \to A)$ is fully faithful, admits a left adjoint $\mathrm{ev}_{[1]}$ and a right adjoint $\mathrm{ev}_{[0]}$. Restricting this to the fully faithful embedding $\mathrm{Pair}^{\mathrm{an}} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$, we get the following.

LEMMA 3.44. — *The functor* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}, A \mapsto (\mathrm{id}_A : A \to A)$ *is fully faithful and admits a left adjoint* $\mathrm{ev}_{[1]} : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}}, (A \twoheadrightarrow A'') \mapsto A''$ *and a right adjoint* $\mathrm{ev}_{[0]} : \mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}, (A \twoheadrightarrow A'') \mapsto A$.

This functor preserves small colimits, and, therefore, by Proposition A.14, it is the left derived functor of the composite functor $\mathrm{Poly}_{\mathbb{Z}} \to \mathrm{Pair}^{\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\mathrm{an}}, A \mapsto (A, 0)$. Applying Corollary 2.2 to the composite $\mathrm{Poly}_{\mathbb{Z}} \to \mathrm{Pair}^{\mathrm{st}} \to \mathrm{Pair}^{\gamma,\mathrm{st}}$, we get the following.

LEMMA 3.45. — *The composite functor* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} \mathrm{Pair}^{\gamma,\mathrm{an}}$ *is fully faithful, where the first functor is* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}, A \mapsto (\mathrm{id}_A : A \to A)$, *and a further composition* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$, *where the second functor is the forgetful functor, is equivalent to the fully faithful functor* $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}, A \mapsto (\mathrm{id}_A : A \to A)$.

Despite Warning 3.37, the image of an animated PD-pair $(A \twoheadrightarrow A'', \gamma)$ under the localization $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\gamma}$ is of the form $(-) \to \tau_{\leq 0}(A'')$:

LEMMA 3.46. — *There is a canonical commutative diagram*

$$\begin{array}{ccccc} \mathrm{Pair}^{\gamma,\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\mathrm{an}} & \xrightarrow{\mathrm{ev}_{[1]}} & \mathrm{CAlg}^{\mathrm{an}} \\ \downarrow & & & & \downarrow{\scriptstyle \tau_{\leq 0}} \\ \mathrm{Pair}^{\gamma} & \longrightarrow & \mathrm{Pair} & \xrightarrow{\mathrm{ev}_{[1]}} & \mathrm{Ring} \end{array}$$

of ∞*-categories, where the left horizontal arrows are forgetful functors, and the leftmost vertical arrow is the localization (Remark 3.24).*

Proof. — We first note that the composite functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Ring}$ preserves small colimits and, therefore, is a left derived functor (Proposition A.14), hence left Kan extended along $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$. The diagram is canonically commutative on the full subcategory $\mathrm{Pair}^{\gamma,\mathrm{st}} \subseteq \mathrm{Pair}^{\gamma,\mathrm{an}}$. It remains to show the existence of the extension of the equivalence in question.

Now consider the diagram

$$\begin{array}{ccccc} \mathrm{Ring} & \hookrightarrow & \mathrm{Pair} & \longrightarrow & \mathrm{Pair}^{\gamma} \\ \downarrow & & & & \downarrow \\ \mathrm{CAlg}^{\mathrm{an}} & \hookrightarrow & \mathrm{Pair}^{\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\gamma,\mathrm{an}} \end{array}$$

of ∞-categories, where the functors $\mathrm{Ring} \to \mathrm{Pair}$ and $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ are given by $A \mapsto (A, 0)$ and $A \mapsto (\mathrm{id}_A : A \to A)$, respectively, and the functor

$\mathrm{Pair} \to \mathrm{Pair}^{\gamma}$ and the functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}}$ are the PD-envelope (resp. animated PD-envelope) functors. This is a commutative diagram by Lemma 3.45. Taking the right adjoints, we get the commutativity by Lemma 3.44. □

Next, we show that animated PD-envelope "does nothing" after rationalization. More precisely, we have the following.

LEMMA 3.47. — *Consider the unit map η from the functor $\mathrm{id}_{\mathrm{Pair}^{\mathrm{an}}}$ to the composite functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$, where the first functor is the animated PD-envelope functor, and the second is the forgetful functor. Then the composite of η with the rationalization functor $\mathrm{Pair}^{\mathrm{an}} \simeq \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \xrightarrow{\cdot \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}} \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Alg}_{\mathbb{Q}}))_{\geq 0}$ is an equivalence of functors.*

Proof. — Since the rationalization functor preserves filtered colimits and geometric realizations, by Proposition A.14, it suffices to show the equivalence on $\mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}^{\mathrm{an}}$. Concretely, it is saying that the canonical map $\mathbb{Z}[X, Y] \to \Gamma_{\mathbb{Z}[X]}(Y)$ becomes an equivalence after rationalization, which follows from definitions. □

Now we consider the base change. Given a surjective map $(A \twoheadrightarrow A'') \in \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0}$ and a map $A \to B$ of animated rings, the base changed map $B \to A'' \otimes^{\mathbb{L}}_{A} B$ is also surjective. The key observation is that this base change is a pushout $(A \twoheadrightarrow A'') \amalg_{(\mathrm{id}_A : A \to A)} (\mathrm{id}_B : B \to B)$. Since the animated PD-envelope functor, being a left adjoint, and the forgetful functor preserve small colimits (Proposition 3.40), from Lemma 3.45 we have the following (to compare with Remark 2.24).

LEMMA 3.48. — *The composite functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ is compatible with base change, where the first functor is the animated PD-envelope functor, and the second is the forgetful functor. More precisely, there is an equivalence from $(C \otimes^{\mathbb{L}}_{A} B \twoheadrightarrow C'' \otimes^{\mathbb{L}}_{A} B)$ to the animated PD-e1.2: the technique of animation also appears in Lurie's higher topos theory envelope of $B \to A'' \otimes^{\mathbb{L}}_{A} B$ between animated pairs, where $(C \twoheadrightarrow C'', \gamma)$ is the animated PD-envelope of $(A \twoheadrightarrow A'')$, which is functorial with respect to the diagram*

$$\begin{array}{ccc} A & \twoheadrightarrow & A'' \\ \downarrow & & \\ B & & \end{array}$$

in $\mathrm{CAlg}^{\mathrm{an}}$.

REMARK 3.49 (General base). — Let R be a ring. We can then replace $\mathbb{Z}$ by R in the theory of animated pairs and PD-pairs. For example, the 1-category Ab is replaced by Mod_R, the ∞-category $D(\mathbb{Z})$ is replaced by $D(R)$, the ∞-category $\mathrm{CAlg}^{\mathrm{an}}$ is replaced by $\mathrm{CAlg}^{\mathrm{an}}_R$, the 1-category $\mathrm{Pair}^{\mathrm{st}}$ is replaced by

$\mathrm{Pair}_R^{\mathrm{st}}$ consisting the pairs of the form $(R[X,Y],(Y))$, and $\mathrm{Pair}^{\gamma,\mathrm{st}}$ is replaced by $\mathrm{Pair}_R^{\gamma,\mathrm{st}}$ consisting the PD-pairs of the form $\Gamma_{R[X]}(Y) \twoheadrightarrow R[X]$, etc. We get $\mathrm{Pair}_R^{\mathrm{an}}$ and $\mathrm{Pair}_R^{\gamma,\mathrm{an}}$. There are canonical base change functors $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{CAlg}_R^{\mathrm{an}}$, $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}_R^{\mathrm{an}}$ and $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}_R^{\gamma,\mathrm{an}}$ essentially induced by the base change $D(\mathbb{Z}) \xrightarrow{\cdot\otimes_{\mathbb{Z}}^{\mathbb{L}} R} D(R)$.

From Corollary 2.14 we have the following.

LEMMA 3.50. — *There are canonical equivalences of ∞-categories*

$$\begin{aligned}\mathrm{CAlg}_R^{\mathrm{an}} &\xrightarrow{\simeq} \mathrm{CAlg}_{R/}^{\mathrm{an}}\\ \mathrm{Pair}_R^{\mathrm{an}} &\xrightarrow{\simeq} \mathrm{Pair}^{\mathrm{an}}_{(\mathrm{id}_R:R\to R)/}\\ \mathrm{Pair}_R^{\gamma,\mathrm{an}} &\xrightarrow{\simeq} \mathrm{Pair}^{\gamma,\mathrm{an}}_{(\mathrm{id}_R:R\to R,0)/}\end{aligned}$$

By the proof of Lemma 2.37, from Lemma 3.48 we have the following.

LEMMA 3.51. — *Let R be a ring. Then there is a canonical commutative diagram*

$$\begin{array}{ccc}\mathrm{Pair}_R^{\mathrm{an}} & \longrightarrow & \mathrm{Pair}_R^{\gamma,\mathrm{an}}\\ \downarrow & & \downarrow\\ \mathrm{Pair}^{\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\gamma,\mathrm{an}}\end{array}$$

of ∞-categories, where the vertical arrows are forgetful functors, and the horizontal arrows are animated PD-envelope functors.

Moreover, again by Lemma 3.48, we have the following.

LEMMA 3.52. — *Let R be a ring. Then there is a canonical commutative diagram*

$$\begin{array}{ccc}\mathrm{Pair}^{\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\gamma,\mathrm{an}}\\ \downarrow & & \downarrow\\ \mathrm{Pair}_R^{\mathrm{an}} & \longrightarrow & \mathrm{Pair}_R^{\gamma,\mathrm{an}}\end{array}$$

of ∞-categories, where the horizontal arrows are animated PD-envelope functors and the vertical arrows are base change functors.

3.4. Quasiregular pairs. — This section is devoted to comparison of animated theory of pairs and PD-pairs with the classical version. Quasiregularity, introduced by Quillen, plays an important role.

Definition 3.53 ([16, Thm 6.13]). — We say that a pair $(A, I) \in \mathrm{Pair}$ is *quasiregular* if the shifted cotangent complex $L_{(A/I)/A}[-1] \in D(A/I)$ is a flat A/I-module. We will denote by $\mathrm{QReg}^{\mathrm{an}} \subseteq \mathrm{Pair}^{\mathrm{an}} \subseteq \mathrm{Pair}$ the full subcategory spanned by quasiregular pairs. The same for $\mathrm{QReg}^{\mathrm{an}}_{\mathbb{F}_p} \subseteq \mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p}$.

Example 3.54. — Let A be a ring and $I \subseteq A$ an ideal generated by a Koszul-regular sequence. Then $L_{(A/I)/A} \simeq (I/I^2)[1]$ [39, Tag 08SJ], and I/I^2 is a free A/I-module [39, Tag 062I]. We warn the reader that Quillen's quasiregular is different from "quasi-regular" in [39, Tag 061M], and the latter is not used in this article.

The first goal of this section is to show that there is a "derived" version of the adic filtration on animated pairs. Furthermore, for pairs, there is a natural comparison map from the "derived" version to the classical version (strictly speaking, our comparison is slightly more general), which becomes an equivalence when the pair in question is quasiregular. We refer to Section 2.4 for concepts and notations about filtrations. We need the following results, which relates the cotangent complex to powers of ideals.

Lemma 3.55 ([39, Tag 08RA]). — *There exists a map $(I/I^2)[1] \to L_{(A/I)/A}$ in $D(A/I)$ which is functorial in $(A, I) \in \mathrm{Pair}$, such that the composite map $(I/I^2)[1] \to L_{(A/I)/A} \to \tau_{\leq 1} L_{(A/I)/A}$ is an equivalence.*

Remark 3.56. — By abuse of terminology, by a map $M_{(A,I)} \to N_{(A,I)}$ in $D_{\geq 0}(A/I)$ being functorial in $(A, I) \in \mathrm{Pair}$, we mean that the map in question is a map between two functors $(A, I) \rightrightarrows \mathrm{Ani}(\mathrm{Mod})$ given by $(A, I) \mapsto (A/I, M_{(A,I)})$ and $(A, I) \mapsto (A/I, N_{(A,I)})$, respectively.

Lemma 3.57 ([39, Tag 08SI]). — *For any $(A, I) \in \mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}$, the cotangent complex $L_{(A/I)/A}$ is 1-truncated.*

Corollary 3.58. — *There exists an equivalence $(I/I^2)[1] \to L_{(A/I)/A}$ in $D_{\geq 0}(A/I)$ functorial in $(A, I) \in \mathrm{Pair}^{\mathrm{st}}$.*

Construction 3.59 (Adic filtration). — *Consider the classical adic filtration functor $\mathrm{Fil}^*_{\mathrm{ad,cl}} : \mathrm{Pair} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, $(A, I) \mapsto (I^n)_{n \in \mathbb{N}_{\geq 0}}$. Restricting this to the full subcategory $\mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}$ and applying Proposition A.14, we get a functor $\mathrm{Fil}^*_{\mathrm{ad}} : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, called the adic filtration functor. Such a construction appears in* [8, Cor 4.14] *in the language of model categories. In* [21, §6]*, Jeroen* Hekking *constructed the same filtration via different approaches.*

Remark 3.60. — By the same argument, there is a natural structure of filtered derived ring (Remark 2.42) on $\mathrm{Fil}^*_{\mathrm{ad}}(A \twoheadrightarrow A'')$, which we will not use in this article.

By Theorem 3.29, we can identify $\mathrm{Fil}^0_{\mathrm{ad}} : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}}$ with the composite functor

$$\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \xrightarrow{\mathrm{ev}_{[0]}} \mathrm{CAlg}^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}}, (A \twoheadrightarrow A'') \mapsto A$$

and $\mathrm{gr}^0_{\mathrm{ad}} : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}}$ with the composite functor

$$\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{CAlg}^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}}, (A \twoheadrightarrow A'') \mapsto A''$$

Combining Corollary 3.58, Proposition A.14, sifted-colimit-preserving properties of $\mathbb{L}\,\mathrm{Sym}^*$, and the concrete analysis of pairs in $\mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}^{\mathrm{an}}$, we get the following.

Corollary 3.61. — *For every $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}$, the shifted cotangent complex $L_{A''/A}[-1] \simeq \mathrm{gr}^1_{\mathrm{ad}}(A \twoheadrightarrow A'')$ is connective, and there exists an equivalence*

$$\mathbb{L}\,\mathrm{Sym}^*_{A''}(\mathrm{gr}^1_{\mathrm{ad}}(A \twoheadrightarrow A'')) \to \mathrm{gr}^*_{\mathrm{ad}}(A \twoheadrightarrow A'')$$

of graded $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras functorial in $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}$.

Now we construct a comparison map between the "derived" filtration $\mathrm{Fil}^*_{\mathrm{ad}}$ and the "nonderived" filtration $\mathrm{Fil}^*_{\mathrm{ad,cl}}$ on ring-ideal pairs. We apply a trick used in the proof of Proposition 3.35 and Lemma A.26.

Construction 3.62. — *Proposition A.14 and the universal property of left Kan extensions give us a comparison natural transform from the functor $\mathrm{Fil}^*_{\mathrm{ad}} : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ to the composite functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair} \xrightarrow{\mathrm{Fil}^*_{\mathrm{ad,cl}}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, where the first functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$ is the localization (Remark 3.24).*

Our next goal is to show that the comparison map is an equivalence after restriction to $\mathrm{QReg} \subseteq \mathrm{Pair}^{\mathrm{an}}$. Since the forgetful functor $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{DF}^{\geq 0}(\mathbb{Z})$ is conservative, we can show the equivalence after forgetting the $\mathbb{E}_\infty$-structure.

The previous discussion shows that the comparison map is an equivalence after composing with $\mathrm{Fil}^0 : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ and $\mathrm{gr}^0 : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ on the 1-category Pair (not only for quasiregular pairs). We define the functor $\mathrm{gr}^{[0,n)} : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z}), F \mapsto \mathrm{cofib}(\mathrm{Fil}^n(F) \to \mathrm{Fil}^0(F))$. Thus it suffices to prove that the comparison map is an equivalence after composing with $\mathrm{gr}^{[0,n)} : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ for all $n > 1$ for quasiregular pairs. Note that by definition, the essential image of $\mathrm{gr}^{[0,n)}_{\mathrm{ad}}$ already lies in $\mathrm{Ab} \subseteq D(\mathbb{Z})$. We show a more general statement (cf. the proof of Proposition 3.35 and Lemma A.26).

LEMMA 3.63. — *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pair}^{\mathrm{an}} & \xrightarrow{\mathrm{gr}_{\mathrm{ad}}^{[0,n)}} & D(\mathbb{Z})_{\geq 0} \\ \downarrow & & \downarrow{\scriptstyle \tau_{\leq 0}} \\ \mathrm{Pair} & \xrightarrow{\mathrm{gr}_{\mathrm{ad}}^{[0,n)}} & \mathrm{Ab} \end{array}$$

of ∞-categories, where the homotopy from the top-right composition to the bottom-left composition is induced by the comparison map in Construction 3.62.

Proof. — The trick is to consider an auxiliary functor. Let $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}$ and let $I := \mathrm{fib}(A \twoheadrightarrow A'') \in D(\mathbb{Z})_{\geq 0}$. We recall that, by Theorem 3.29, and (2) in particular, the forgetful functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ is just $(A \twoheadrightarrow A'') \mapsto (A \leftarrow I)$.

Then the map $I \to A$ in $D(\mathbb{Z})_{\geq 0}$ induces a map $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to \mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} A$. Composing with the multiplication $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} A \to A$, we get the map $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to A$. We consider the functor $F : \mathrm{Pair}^{\mathrm{an}} \to D(\mathbb{Z})_{\geq 0}, (A \twoheadrightarrow A'') \mapsto \mathrm{cofib}(\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to A)$.

First, the functor F preserves filtered colimits and geometric realizations, since the functor $\mathbb{L}\,\mathrm{Sym}_{\mathbb{Z}}$ and the forgetful functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ do (Lemma 3.9). In fact, F is the left derived functor (Proposition A.14) of the functor $\mathrm{Pair}^{\mathrm{st}} \to D(\mathbb{Z})_{\geq 0}, (A, I) \mapsto \mathrm{cofib}(\mathrm{Sym}^n_{\mathbb{Z}} I \to A)$.

Next, note that for $(A, I) \in \mathrm{Pair}^{\mathrm{st}}$, the map $\mathrm{Sym}^n_{\mathbb{Z}} I \to A$ factors functorially as $\mathrm{Sym}^n_{\mathbb{Z}} I \to I^n \to A$, and the map $\mathrm{Sym}^n_{\mathbb{Z}} I \to I^n$ is surjective. It follows that there is a natural surjective map $\mathrm{cofib}(\mathrm{Sym}^n_{\mathbb{Z}} I \to A) \to A/I^n$, which gives rise to a map $F \to \mathrm{gr}_{\mathrm{ad}}^{[0,n)}$ of functors, which becomes an equivalence after composition with $\tau_{\leq 0} : D(\mathbb{Z})_{\geq 0} \to \mathrm{Ab}$.

We now show that the functor $\tau_{\leq 0} \circ F : \mathrm{Pair}^{\mathrm{an}} \to \mathrm{Ab}$ factors through the localization $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$. First, since Ab is a 1-category, it factors through $\mathrm{Pair}^{\mathrm{an}} \to \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$. Given $(A \twoheadrightarrow A'') \in \mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}})$, let $I = \mathrm{fib}(A \twoheadrightarrow A'')$ as before. Then $(A \leftarrow I) \in \mathcal{P}_{\Sigma,1}(\mathrm{Inj}^{\mathrm{st}}) \simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$, and, therefore, A, I are static. Let $I' = \mathrm{im}(I \to A)$. It follows that the localization $\mathcal{P}_{\Sigma,1}(\mathrm{Pair}^{\mathrm{st}}) \to \mathrm{Pair}$ maps $(A \twoheadrightarrow A'')$ to $(A, I') \in \mathrm{Pair}$. By Proposition 2.56, it suffices to show that F maps $(A \twoheadrightarrow A'') \to (A, I')$ to an equivalence. This simply follows from the fact that $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to \mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I'$ is a surjection on π_0, and the "multiplication" map $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to A$ factors as $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to \mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I' \to A$.

In conclusion, we have already shown that there exists an equivalence of two compositions in the diagram that we need to prove. To show that this equivalence is the equivalence that we want, we note that the top right composition preserves filtered colimits and geometric realizations, and then the first paragraph of the proof of Lemma 3.46 applies. □

In particular, when (A, I) is quasiregular, it follows from Corollary 3.61 that $\mathrm{gr}^n_{\mathrm{ad}}(A \twoheadrightarrow A/I) \in D(\mathbb{Z})_{\geq 0}$ is static for all $n \in \mathbb{N}$, which implies that $\mathrm{gr}^{[0,n)}_{\mathrm{ad}}(A \twoheadrightarrow A/I)$ is static for all $n \in \mathbb{N}$. Consequently, we have the following.

PROPOSITION 3.64. — *The comparison natural transformation in Construction 3.62 becomes an equivalence after restricting to the full subcategory* $\mathrm{QReg} \subseteq \mathrm{Pair}^{\mathrm{an}}$.

COROLLARY 3.65 ([16, 6.11]). — *For every quasiregular pair (A, I), the canonical map $\mathrm{Sym}^*_{A/I}(I/I^2) \to \bigoplus I^*/I^{*+1}$ of graded rings is an equivalence.*

Proof. — It suffices to show that the equivalence given by Corollary 3.61 coincides with the canonical map induced by the multiplicative structure on A. For any element $\overline{x_1} \cdots \overline{x_n} \in \mathrm{Sym}^n_{A/I}(I/I^2)$, we pick a lift $x_1, \ldots, x_n \in I$, which gives rise to a map $(B, J) := (\mathbb{Z}[X_1, \ldots, X_n], (X_1, \ldots, X_n)) \to (A, I)$ of pairs, which induces the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n_{B/J}(J/J^2) & \longrightarrow & J^n/J^{n+1} \\ \downarrow & & \downarrow \\ \mathbb{L}\,\mathrm{Sym}^n_{A/I}(\mathrm{gr}^1_{\mathrm{ad}}(A \twoheadrightarrow A/I)) & \longrightarrow & \mathrm{gr}^n_{\mathrm{ad}}(A \twoheadrightarrow A/I) \end{array}$$

in the ∞-category $D(\mathbb{Z})_{\geq 0}$. Taking $\tau_{\leq 0}$ and trace the element $\overline{X_1} \cdots \overline{X_n} \in \mathrm{Sym}^n_{B/J}(J/J^2)$, we get the result. □

We are unable to answer the following question in full generality.

QUESTION 1. — *Let (A, I) be a quasiregular pair. Let $(B \twoheadrightarrow B'', \gamma)$ denote its animated PD-envelope. Is it true that B, B'' are static, so by Proposition 3.38 and Corollary 3.26, it coincides with the classical PD-envelope?*

However, we are able to answer it under certain flatness. First, it follows from Lemma 3.47 that when A is a $\mathbb{Q}$-algebra, the animated PD-envelope of (A, I) is just $A \twoheadrightarrow A/I$, which is also the classical PD-envelope.

Now we consider the characteristic $p > 0$ case, switching the ground ring from $\mathbb{Z}$ to $\mathbb{F}_p$ (which is valid by Lemma 3.51). We will use the notations $\mathrm{Pair}^{\mathrm{st}}$ and $\mathrm{Pair}^{\gamma,\mathrm{st}}$ in Section 3.2, but the occurrences of $\mathbb{Z}$ are replaced by $\mathbb{F}_p$. We recall that the Frobenius map $A \to A, x \mapsto x^p$ of an $\mathbb{F}_p$-algebra A gives rise to an endomorphism $\varphi : \mathrm{id}_{\mathrm{Alg}_{\mathbb{F}_p}} \to \mathrm{id}_{\mathrm{Alg}_{\mathbb{F}_p}}$ of the identity functor $\mathrm{id}_{\mathrm{Alg}_{\mathbb{F}_p}} : \mathrm{Alg}_{\mathbb{F}_p} \to \mathrm{Alg}_{\mathbb{F}_p}$, which gives rise to an endomorphism $\mathrm{id}_{\mathrm{Ani}(\mathrm{Alg}_{\mathbb{F}_p})} \to \mathrm{id}_{\mathrm{Ani}(\mathrm{Alg}_{\mathbb{F}_p})}$ still denoted by φ. We now introduce the *conjugate filtration* on the animated PD-envelope of animated $\mathbb{F}_p$-pairs that we learned from [9].

CONSTRUCTION 3.66. — *Let (A, I) be an $\mathbb{F}_p$-pair such that the Frobenius $\varphi_A : A \to A$ is flat and let (B, J, γ) denote its PD-envelope. We first note that*

there is a $\varphi_A^*(A/I)$-algebra structure on B since $f^p = p\gamma_p(f) = 0$ for all $f \in J$. Then we have a filtration on B given by $\mathrm{Fil}^{-n}_{\mathrm{conj,cl}} B$ for $n \geq 0$ to be the $\varphi_A^*(A/I)$-submodule of B generated by $\{\gamma_{k_1 p}(f_1) \cdots \gamma_{k_m p}(f_m) \mid k_1 + \cdots + k_m \leq n$ and $f_1, \ldots, f_m \in I\}$, which gives rise to a structure of nonpositively filtered $\varphi_A^*(A/I)$-algebra. We note that the filtration is exhaustive, i.e.,$\mathrm{Fil}^{-\infty}_{\mathrm{conj,cl}} B = \mathrm{colim}_{n \in (\mathbb{Z}, \geq)} \mathrm{Fil}^{-n} B \to B$ is an isomorphism, and we can rephrase the nonpositively filtered $\varphi_A^*(A/I)$-algebra structure as a map $\varphi_A^*(A/I) \to \mathrm{Fil}\, B$[15] of a nonpositively filtered ring.

We need the following result.

LEMMA 3.67 ([9, Lem 3.42]). — *Let (A, I) be an $\mathbb{F}_p$-pair such that I/I^2 is a flat A/I-module and the Frobenius $\varphi_A : A \to A$ is flat and let (B, J, γ) denote the PD-envelope of (A, I).*

Then there is a comparison map $\varphi_A^*(\Gamma^i_{A/I}(I/I^2)) \to \mathrm{gr}^{-i}_{\mathrm{conj,cl}} B$ (as in Construction 3.66) of $\varphi_A^*(A/I)$-modules induced by the maps $(\gamma_{kp})_{k \in \mathbb{N}}$, which is functorial in (A, I). For example, when I/I^2 is free, an element in $\Gamma^i_{A/I}(I/I^2)$ represented by $\frac{f^{\otimes i}}{i!}$ will be mapped to $\gamma_{ip}(f)$ for $f \in I$.

Furthermore, if $I \subseteq A$ is generated by a Koszul-regular sequence[16]*, then the comparison map above is an isomorphism.*

Now we define the conjugate filtration on the animated PD-envelope.

DEFINITION 3.68. — The *conjugate filtration functor (on the animated PD-envelope)* $\mathrm{Fil}^*_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}} : \mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ together with the *structure map* of functors $\mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ from $(A \twoheadrightarrow A'', \gamma) \mapsto \varphi_A^*(A'') = A'' \otimes^{\mathbb{L}}_{A, \varphi_A} A$ to $\mathrm{Fil}^*_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}$, or equivalently, a functor

$$\begin{aligned} \mathrm{Pair}^{\mathrm{an}} &\longrightarrow \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))) \\ (A \twoheadrightarrow A'') &\longmapsto (\varphi_A^*(A'') \to \mathrm{Fil}^*_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'')) \end{aligned}$$

is defined to be the left derived functor (Proposition A.14) of $\mathrm{Pair}^{\mathrm{st}} \ni (A, I) \mapsto (\varphi_A^*(A/I) \to \mathrm{Fil}^*_{\mathrm{conj,cl}} D_A(I)) \in \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ in Construction 3.66, where $\varphi_A^*(A/I)$ is constantly filtered.

REMARK 3.69. — Informally speaking, the functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ in Definition 3.68 is capturing two pieces of data.

1. The conjugate filtration $\mathrm{Fil}^*_{\mathrm{conj}}$ on the animated PD-envelope $\mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'')$.
2. An $\mathbb{E}_\infty$-$\varphi_A^*(A'')$-algebra structure on the conjugate filtration.

That these data are *functorial* in $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}$.

15. We will from time to time suppress the asterisk in Fil^* to avoid confusion with φ^*.
16. We only need the special case that $(A, I) \in \mathrm{Pair}^{\mathrm{st}}$.

REMARK 3.70. — We note that the conjugate filtration is exhaustive, i.e., there is a canonical equivalence $\mathrm{Fil}^{-\infty}_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}} \to \mathrm{Env}^{\gamma,\mathrm{an}}$ of functors $\mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p))$, which follows either from Proposition A.14 and Lemma 2.43 or the fact that $\mathrm{Pair}^{\mathrm{an}} \simeq \mathcal{P}_\Sigma(\mathrm{Pair}^{\mathrm{st}}) \subseteq \mathcal{P}(\mathrm{Pair}^{\mathrm{st}})$ is stable under filtered colimits (Proposition A.11).

It follows from Lemma 3.67 that

COROLLARY 3.71. — *For every $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p}$, there exists an equivalence*

$$\varphi^*_A(\Gamma^i_{A''}(\mathrm{gr}^1_{\mathrm{ad}}(A \twoheadrightarrow A''))) \to \mathrm{gr}^{-i}_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'')$$

in $D(\varphi^*_A(A''))_{\geq 0}$ for all $i \in \mathbb{N}$ which is functorial in $(A \to A'') \in \mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p}$.

REMARK 3.72. — One might wonder what precisely the functor is, since the target category $D(\varphi^*_A(A''))_{\geq 0}$ depends on $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}_{\mathbb{F}_p}$. One can rigorously describe this $\varphi^*_A(A'')$-algebra structure in terms of structure maps (as in Definition 3.68). However, this is cumbersome, and we keep the current "imprecise" presentation.

We now apply this to a quasiregular pair $(A, I) \in \mathrm{QReg}_{\mathbb{F}_p}$. We first recall that

DEFINITION 3.73 ([30, Def 7.2.2.10]). — Let A be an $\mathbb{E}_1$-ring. We say that a right A-module spectrum M is *flat* if the following hold.

1. The homotopy group $\pi_0(M)$ is a flat right $\pi_0(A)$-module.
2. For each $n \in \mathbb{Z}$, the canonical map $\pi_0(M) \otimes_{\pi_0(A)} \pi_n(A) \to \pi_n(M)$ is an isomorphism of abelian groups.

The same concept applies to left A-module spectra.

REMARK 3.74 ([30, Rem 7.2.2.11 & 7.2.2.12]). — Let R be an $\mathbb{E}_1$-ring and M a flat right R-module spectrum. By definition, if R is connective (resp. static), then so is R. In particular, when R is static, a flat R-module spectrum is simply a flat R-module, and, therefore, we will sometimes refer to flat module spectra simply as flat modules since there is no ambiguity.

LEMMA 3.75. — *Let A be a connective $\mathbb{E}_1$-ring, and $M' \to M \to M''$ a fiber sequence of right A-module spectra. If M', M'' are flat right A-modules, then so is M.*

Proof. — First, M', M'' are connective by flatness, and, therefore, so is M. For every static left A-module N, we have a fiber sequence $N \otimes^{\mathbb{L}}_A M' \to N \otimes^{\mathbb{L}}_A M \to N \otimes^{\mathbb{L}}_A M''$. By flatness of M' and M'' and [30, Prop 7.2.2.13], the spectra $N \otimes^{\mathbb{L}}_A M'$ and $N \otimes^{\mathbb{L}}_A M''$ are static, and, therefore,, so is $N \otimes^{\mathbb{L}}_A M$. The result then follows from [30, Thm 7.2.2.15]. □

For future usage, we need to generalize the concept of quasiregular pairs slightly.

Definition 3.76. — We say that an animated pair $A \twoheadrightarrow A''$ is *quasiregular* if the shifted cotangent complex $L_{A''/A}[-1] \in D(A'')$ is a flat A''-module spectrum.

Corollary 3.77. — *Let $(A \twoheadrightarrow A'')$ be a quasiregular animated $\mathbb{F}_p$-pair, and let $(B \twoheadrightarrow B'', \gamma)$ denote its animated PD-envelope. Then B is a flat $\varphi_A^*(A'')$-module spectrum.*

Proof. — It follows from Corollary 3.71, Γ^* and base change preserving flatness ([31, Cor 25.2.3.3] & [30, Prop 7.2.2.16]) that $\mathrm{gr}^{-i}_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'')$ is a flat $\varphi_A^*(A'')$-module spectrum. The result follows from the fact that the full subcategory spanned by flat modules over a connective $\mathbb{E}_1$-ring is stable under extension (Lemma 3.75) and under filtered colimits by [30, Thm 7.2.2.14(1)]. □

Remark 3.78. — In fact, by Lemma 5.51, the map $\varphi_A^*(A'') \to B$ in Corollary 3.77 is faithfully flat.

From Proposition 3.38, Corollaries 3.26 and 3.77, and Remark 3.74 we have the following.

Corollary 3.79. — *Let $(A, I) \in \mathrm{QReg}_{\mathbb{F}_p}$ be a quasiregular pair. Suppose that $\varphi_A^*(A/I)$ is static. Then the animated PD-envelope $(B \twoheadrightarrow B'', \gamma)$ of $(A \twoheadrightarrow A/I)$ belongs to $\mathrm{Pair}^{\gamma}_{\mathbb{F}_p}$, and, therefore, coincides with the classical PD-envelope.*

We want to point out that such results for $\mathbb{F}_p$ will be used to deduce integral results, which are based on the following lemmas, cf. [31, Lem 6.1.2.4].

Lemma 3.80. — *Let $M \in \mathrm{Sp}_{\geq 0}$ be a connective spectrum. Suppose that the rationalization $M \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{Q}$ is static, and for every prime $p \in \mathbb{N}$, the homotopy groups of $M/^{\mathbb{L}}p := \mathrm{cofib}\left(M \xrightarrow{p} M\right)$ are concentrated in degree $0, 1$. Then M is static.*

Proof. — Since $\mathbb{Q}$ is $\mathbb{S}$-flat, $\pi_i(M) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_i(M \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{Q}) \cong 0$ when $i \neq 0$. On the other hand, $\pi_{i+1}(M/^{\mathbb{L}}p) \cong 0$ for $i > 0$ implies that the map $\pi_i(M) \xrightarrow{p} \pi_i(M)$ is injective for every prime $p \in \mathbb{N}$ and $i > 0$. It follows that $\pi_i(M) \cong 0$ for every $i > 0$. □

Warning 3.81. — *Lemma 3.80 is false if M is not assumed to be connective. A counterexample is given by $M = (\mathbb{Q}/\mathbb{Z})[-1]$, for which $M \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{Q} \simeq 0$ and $M/^{\mathbb{L}}p \simeq \mathbb{F}_p$ for every prime number $p \in \mathbb{N}$.*

LEMMA 3.82 (cf. [39, Tag 039C]). — *Let A be an animated ring and $M \in D_{\geq 0}(A)$ a connective A-module spectrum. Then the following conditions are equivalent.*

1. *M is a flat A-module.*
2. *$M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$ is a flat $A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$-module, and for every prime $p \in \mathbb{N}$, $M/^{\mathbb{L}}p$ is a flat $A/^{\mathbb{L}}p$-module.*

Proof. — The first implies the second by the stability of flatness under base change [30, Prop 7.2.2.16]. We now assume the second. By [30, Thm 7.2.2.15], it suffices to show that for each static A-module N, the tensor product $M \otimes^{\mathbb{L}}_A N$ is also static. Indeed,

$$(M \otimes^{\mathbb{L}}_A N) \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{Q} \simeq (M \otimes^{\mathbb{L}}_A N) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q} \simeq (M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}) \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}} (N \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q})$$

is static by the $\mathbb{Z}$-flatness of $\mathbb{Q}$ and the flatness of $M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$. On the other hand,

$$(M \otimes^{\mathbb{L}}_A N)/^{\mathbb{L}}p \simeq (M/^{\mathbb{L}}p) \otimes^{\mathbb{L}}_{A/^{\mathbb{L}}p} (N/^{\mathbb{L}}p)$$

for every prime $p \in \mathbb{N}$. Since $M/^{\mathbb{L}}p$ is $A/^{\mathbb{L}}p$-flat,

$$\pi_i((M/^{\mathbb{L}}p) \otimes^{\mathbb{L}}_{A/^{\mathbb{L}}p} (N/^{\mathbb{L}}p)) \simeq \pi_0(M/^{\mathbb{L}}p) \otimes_{\pi_0(A/^{\mathbb{L}}p)} \pi_i(N/^{\mathbb{L}}p) \cong 0$$

for all $i > 1$ by [30, Prop 7.2.2.13]. It then follows from Lemma 3.80 that $M \otimes^{\mathbb{L}}_A N$ is static. □

We record a simple consequence (compare with [12, Lem 2.42]):

PROPOSITION 3.83. — *Let A be a ring and $I \subseteq A$ an ideal generated by a Koszul-regular sequence. Then the animated PD-envelope $(B \twoheadrightarrow B'', \gamma)$ of $(A \twoheadrightarrow A/I)$ belongs to* Pair^{γ}*, and, therefore, coincides with the classical PD-envelope.*

Proof. — Note that $B'' \simeq A/I$ is static by Lemma 3.43. It follows from Lemma 3.47 that $B \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q} \simeq A$ is static. Let $(f_1, \ldots, f_r)$ be a Koszul-regular sequence which generates I. Fix a prime $p \in \mathbb{N}_{>0}$. Let A_0 denote $A/^{\mathbb{L}}p$. We follow the argument in [9, Lem 3.41]:

$$\begin{aligned}
\varphi^*_{A_0}((A/I)/^{\mathbb{L}}p) &\simeq \varphi^*_{A_0}(A_0/^{\mathbb{L}}(f_1)) \otimes^{\mathbb{L}}_{A_0} \cdots \otimes^{\mathbb{L}}_{A_0} \varphi^*_{A_0}(A_0/^{\mathbb{L}}f_r) \\
&\simeq (A_0/^{\mathbb{L}}f_1^p) \otimes^{\mathbb{L}}_{A_0} \cdots \otimes^{\mathbb{L}}_{A_0} (A_0/^{\mathbb{L}}f_r^p) \\
&\simeq A_0/^{\mathbb{L}}(f_1^p, \ldots, f_r^p) \\
&\simeq (A/^{\mathbb{L}}(f_1^p, \ldots, f_r^p))/^{\mathbb{L}}p
\end{aligned}$$

Note that since $(f_1, \ldots, f_r)$ is Koszul-regular, so is $(f_1^p, \ldots, f_r^p)$, which implies that $\pi_i(\varphi^*_{A_0}((A/I)/^{\mathbb{L}}p)) \cong 0$ for $i \neq 0, 1$. It follows from Corollary 3.77 and the base change property (Lemma 3.52) that $\pi_i(B/^{\mathbb{L}}p) \cong 0$ for $i \neq 0, 1$. The result then follows from Lemma 3.80. □

3.5. Illusie's question. — Given a ring A and an ideal $I \subseteq A$ generated by a Koszul-regular sequence, let (B, J, γ) denote the PD-envelope of (A, I). It is known that the canonical comparison map $\Gamma^*_{A/I}(I/I^2) \to J^{[*]}/J^{[*+1]}$ is an isomorphism, cf. [5, I. Prop 3.4.4], where $J^{[*]}$ are divided powers of J in B. In [24, VIII. Ques 2.2.4.2], Illusie asked whether this holds for quasiregular pairs (A, I). The answer is affirmative, and the goal of this section is to furnish a proof by our theory of animated PD-pairs.

Our strategy is similar to Section 3.4: both the animated PD-envelope and the PD-envelope of a pair (A, I) admit a canonical filtration, and there is a natural comparison between the two. Although for general quasiregular pairs (A, I) we do not know whether the animated PD-envelope coincides with the PD-envelope, the comparison map induces equivalences on graded pieces. The associated graded of the animated PD-envelope admits a natural structure of divided power algebra and an element tracing proves that the equivalence coincides with the comparison map in Illusie's question.

We start with the *PD-filtration* on animated PD-pairs. We refer to Section 2.4 for concepts and notations about filtrations.

Construction 3.84. — *Let $(A, I, \gamma) \in \mathrm{Pair}^{\gamma}$ be a PD-pair and $n \in \mathbb{N}$ a natural number. The classical divided power ideal $I^{[n]} \subseteq A$ is the ideal generated by elements $\gamma_{i_1}(x_1) \cdots \gamma_{i_k}(x_k)$, where $x_1, \ldots, x_k \in I$ and $(i_1, \ldots, i_k) \in \mathbb{N}^k$ with $i_1 + \cdots + i_k \geq n$. For example, for $(\Gamma_{\mathbb{Z}}(x) \twoheadrightarrow \mathbb{Z}) \in \mathrm{Pair}^{\gamma}$ with kernel I, the kernel $I^{[n]} \subseteq \Gamma_{\mathbb{Z}}(x)$ is generated by $\{\gamma_i(x) \,|\, i \geq n\}$ (which is different from the ideal $(\gamma_n(x))$). The classical PD-filtration on A is given by $A \supseteq I \supseteq I^{[2]} \supseteq \cdots$ endowing A with the structure of a filtered ring. A filtered ring is naturally a (nonnegatively) filtered $\mathbb{E}_\infty$-ring, and we get a functor $\mathrm{Fil}^*_{\mathrm{PD,cl}} : \mathrm{Pair}^{\gamma} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.*

Definition 3.85. — The *PD-filtration functor* $\mathrm{Fil}^*_{\mathrm{PD}} : \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ is defined to be the left-derived functor (Proposition A.14) of the composite functor $\mathrm{Pair}^{\gamma,\mathrm{st}} \hookrightarrow \mathrm{Pair}^{\gamma} \xrightarrow{\mathrm{Fil}^*_{\mathrm{PD,cl}}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$. For an animated PD-pair $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$, we will call the image $\mathrm{Fil}^*_{\mathrm{PD}}(A \twoheadrightarrow A'', \gamma)$ the $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra A with *PD-filtration*.

Remark 3.86. — By the same argument, the PD-filtration, in fact, gives rise to the structure of a filtered derived ring (Remark 2.42), which we will not use in this article.

Similarly to Corollary 3.61, by Proposition A.14, sifted-colimit-preserving property of the (derived) divided power functor $((A, M) \mapsto \Gamma_A M) : \mathrm{Ani}(\mathrm{Mod}) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(D(\mathbb{Z})))$ and the concrete analysis of $(A, I, \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{st}}$, we get the following.

LEMMA 3.87. — *For every* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$*, there exists an equivalence*

$$\Gamma^*_{A''}(\mathrm{gr}^1_{\mathrm{PD}}(A \twoheadrightarrow A'', \gamma))) \to \mathrm{gr}^*_{\mathrm{PD}}(A \twoheadrightarrow A'', \gamma)$$

of graded $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras, which is functorial in* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$.

Furthermore, we can compare the adic filtration on an animated pair and the PD-filtration on the animated PD-filtration. We first compare them on $\mathrm{Pair}^{\mathrm{st}}$ and then extend the comparison to $\mathrm{Pair}^{\mathrm{an}}$ by Proposition A.14, obtaining the following.

LEMMA 3.88. — *For every* $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}$*, let* $(B \twoheadrightarrow B'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ *denote its animated PD-envelope. Then there is a canonical comparison map*

$$\mathrm{gr}^*_{\mathrm{ad}}(A \twoheadrightarrow A'') \to \mathrm{gr}^*_{\mathrm{PD}}(B \twoheadrightarrow B'', \gamma)$$

of graded $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras, which is functorial in* $(A \twoheadrightarrow A'') \in \mathrm{Pair}^{\mathrm{an}}$*. Furthermore, this map induces equivalences in* $D(\mathbb{Z})$ *when* $* = 0, 1$.

CONSTRUCTION 3.89. — *Analogous to Section 3.4, by universal property of left Kan extensions, there exists an essentially unique comparison map* c_γ *from the composite functor* $\mathrm{Pair}^{\mathrm{an}} \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} \mathrm{Pair}^{\gamma,\mathrm{an}} \xrightarrow{\mathrm{Fil}^*_{\mathrm{PD}}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *to the composite functor* $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair} \xrightarrow{\mathrm{Env}^{\gamma}} \mathrm{Pair}^{\gamma} \xrightarrow{\mathrm{Fil}^*_{\mathrm{PD,cl}}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$*, where* $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}$ *is the localization in Remark 3.24.*

The main result of this section is the following.

PROPOSITION 3.90. — *The comparison map* c_γ *in Construction 3.89 becomes an equivalence after composition* $\mathrm{QReg} \hookrightarrow \mathrm{Pair}^{\mathrm{an}} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \xrightarrow{\mathrm{gr}^*} \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(\mathbb{Z}))$.

REMARK 3.91. — As seen in Question 1, we do not know whether the comparison is an equivalence when we replace $\mathrm{gr}^* : \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(\mathbb{Z}))$ by $\mathrm{Fil}^0 : \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{CAlg}_{\mathbb{Z}}$, although it is true under assumptions of Corollary 3.79, which is the only obstruction for the comparison map to be a filtered equivalence.

We start to prove this. Unfortunately, we are unable to establish a strong result like Lemma 3.63, essentially due to the complication discussed in Warning 3.37. Our trick is to show that after replacing gr^* by $\mathrm{gr}^{[0,n)}$, both functors satisfy a common universal property.

As in Section 3.4, we can forget the $\mathbb{E}_\infty$-algebra structure and then replace gr^* by $\mathrm{gr}^{[0,n)} : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z}), F \mapsto \mathrm{cofib}(\mathrm{Fil}^n F \to \mathrm{Fil}^0 F)$, i.e., it is equivalent to show that the natural comparison $c_\gamma^{[0,n)}$ from the composite functor

$$\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}} \xrightarrow{\mathrm{Fil}^*_{\mathrm{PD}}} \mathrm{DF}^{\geq 0}(D(\mathbb{Z})) \xrightarrow{\mathrm{gr}^{[0,n)}} D(\mathbb{Z})$$

to the composite functor

$$\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair} \hookrightarrow \mathrm{Pair}^{\gamma} \xrightarrow{\mathrm{Fil}^*_{\mathrm{PD,cl}}} \mathrm{DF}^{\geq 0}(D(\mathbb{Z})) \xrightarrow{\mathrm{gr}^{[0,n)}} D(\mathbb{Z}),$$

which is an equivalence after restricting to the full subcategory $\mathrm{QReg} \subseteq \mathrm{Pair}^{\mathrm{an}}$. Note that the composite functor $\mathrm{gr}^{[0,n)}_{\mathrm{PD,cl}} = \mathrm{gr}^{[0,n)} \circ \mathrm{Fil}^*_{\mathrm{PD,cl}}$ is concretely given by $(A, I, \gamma) \mapsto A/I^{[n]}$, which motivates the following definition.

Definition 3.92. — We say that a PD-pair (A, I, γ) is *PD-nilpotent of height* $n \in \mathbb{N}$ if $I^{[n]} = 0$. We will denote by $\mathrm{Pair}^{\gamma_n} \subseteq \mathrm{Pair}^{\gamma}$ the full subcategory spanned by PD-nilpotent PD-pairs of height n.

The following lemma could be checked directly or (as ∞-categories) by invoking [28, Prop 5.2.7.8].

Lemma 3.93. — *Let $n \in \mathbb{N}$ be a natural number. Then the full subcategory $\mathrm{Pair}^{\gamma_n} \hookrightarrow \mathrm{Pair}^{\gamma}$ is reflective of which the localization $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}^{\gamma_n}$, which will be denoted by $R^{[n]}_{\mathrm{cl}}$, is given by killing the higher divided powers: $(A, I, \gamma) \mapsto (A/I^{[n]}, IA/I^{[n]}, \overline{\gamma})$, where $\overline{\gamma}(\overline{x}) = \overline{\gamma(x)}$ for all $x \in I$ and $\overline{x}, \overline{\gamma(x)}$ are images of $x, \gamma(x)$ in $A/I^{[n]}$.*

Then the composite functor $\mathrm{gr}^{[0,n)}_{\mathrm{PD,cl}} : \mathrm{Pair}^{\gamma} \to D(\mathbb{Z}), (A, I, \gamma) \mapsto A/I^{[n]}$ can be rewritten as the composite $\mathrm{Pair}^{\gamma} \to \mathrm{Pair}^{\gamma_n} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z})$, where the last functor is the functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z}), (A \twoheadrightarrow A'', \gamma) \mapsto A$. We now show that the composite functor $\mathrm{gr}^{[0,n)}_{\mathrm{PD}} : \mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z})$ can also factor through $\mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z}), (A \twoheadrightarrow A'', \gamma) \mapsto A$. In fact, it is a "derived" version of the previous factorization.

Notation 3.94. — Let $n \in \mathbb{N}$ be a natural number. Then we will denote by $R^{[n]} : \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}}$ the left derived functor (Proposition A.14) of the composite functor $\mathrm{Pair}^{\gamma,\mathrm{st}} \to \mathrm{Pair}^{\gamma_n} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$, where the first functor $\mathrm{Pair}^{\gamma,\mathrm{st}} \to \mathrm{Pair}^{\gamma_n}$ is the restriction of the localization $R^{[n]}_{\mathrm{cl}} : \mathrm{Pair}^{\gamma} \to \mathrm{Pair}^{\gamma_n}$ to the full subcategory $\mathrm{Pair}^{\gamma,\mathrm{st}} \subseteq \mathrm{Pair}^{\gamma}$.

We compose $R^{[n]}$ with the functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z}), (A \twoheadrightarrow A'', \gamma) \mapsto A$. We get a functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z})$, which is equivalent to the composite functor $\mathrm{gr}^{[0,n)}_{\mathrm{PD}}$ by Proposition A.14 since both functors preserves filtered colimits and geometric realizations, and they are canonically identified on the full subcategory $\mathrm{Pair}^{\gamma,\mathrm{st}} \subseteq \mathrm{Pair}^{\gamma,\mathrm{an}}$.

Construction 3.95. — *Let $n \in \mathbb{N}$ be a natural number. Then there is an essentially unique comparison map $c^{[n]}_{\gamma}$ from the composite functor*

$$\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}} \xrightarrow{R^{[n]}} \mathrm{Pair}^{\gamma,\mathrm{an}},$$

which preserves filtered colimits and geometric realizations, to the composite functor

$$\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair} \hookrightarrow \mathrm{Pair}^{\gamma} \xrightarrow{R^{[n]}_{\mathrm{cl}}} \mathrm{Pair}^{\gamma_n} \hookrightarrow \mathrm{Pair}^{\gamma,\mathrm{an}}$$

which is equivalent to $c^{[0,n)}_{\gamma}$ *after composing the sifted-colimit-preserving functor* $\mathrm{Pair}^{\gamma,\mathrm{an}} \to D(\mathbb{Z})$ *by checking on the full subcategory* $\mathrm{Pair}^{\mathrm{st}} \subseteq \mathrm{Pair}^{\mathrm{an}}$ *and the universal property of the left Kan extension.*

It remains to show the following.

LEMMA 3.96. — *The comparison map* $c^{[n]}_{\gamma}$ *of functors* $\mathrm{Pair}^{\mathrm{an}} \rightrightarrows \mathrm{Pair}^{\gamma,\mathrm{an}}$ *becomes an equivalence after restricting to the full subcategory* $\mathrm{QReg} \subseteq \mathrm{Pair}^{\mathrm{an}}$.

Proof. — It follows from Lemmas 3.87 and 3.88, Corollary 3.61, and Proposition 3.38 and the fact that the derived divided powers Γ^* of a flat module are flat, and, therefore, static, that the essential image of the composite functor

$$\mathrm{QReg} \hookrightarrow \mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}} \xrightarrow{R^{[n]}} \mathrm{Pair}^{\gamma,\mathrm{an}} \tag{3}$$

lies in the full subcategory $\mathrm{Pair}^{\gamma} \subseteq \mathrm{Pair}^{\gamma,\mathrm{an}}$. We first show that the essential image further lies in the full subcategory $\mathrm{Pair}^{\gamma_n} \subseteq \mathrm{Pair}^{\gamma}$.

We fix a quasiregular pair $(A, I) \in \mathrm{QReg}$. Let $(C, K, \gamma) \in \mathrm{Pair}^{\gamma}$ denote the image of $(A, I) \in \mathrm{QReg}$ under the composite functor (3). Since (A, I) can be rewritten as a sifted colimit $\mathrm{colim}_{j\in\mathcal{I}}(B_j, J_j)$ taken in $\mathrm{Pair}^{\mathrm{an}}$, where $(B_j, J_j) \in \mathrm{Pair}^{\mathrm{st}}$. Let $(C_j, K_j, \gamma_j) \in \mathrm{Pair}^{\gamma,\mathrm{st}}$ be the PD-envelope of (B_j, J_j). Then $(C, K, \gamma) \simeq \mathrm{colim}_{j\in\mathcal{I}}(C_j/K_j^{[n]}, K_jC_j/K_j^{[n]}, \gamma_j)$ taken in $\mathrm{Pair}^{\gamma,\mathrm{an}}$. For every $x_1, \dots, x_m \in K$ and $i_1, \dots, i_m \in \mathbb{N}$ such that $i_1 + \cdots + i_m \geq n$, we need to show that $\gamma_{i_1}(x_1) \cdots \gamma_{i_m}(x_m) = 0$. The elements $x_1, \dots, x_m$ define a map $\varphi : (\Gamma_{\mathbb{Z}}(X_1, \dots, X_m) \twoheadrightarrow \mathbb{Z}, \delta) \to (C, K, \gamma)$ in $\mathrm{Pair}^{\gamma} \subseteq \mathrm{Pair}^{\gamma,\mathrm{an}}$. Since $(\Gamma_{\mathbb{Z}}(X_1, \dots, X_m) \twoheadrightarrow \mathbb{Z}, \delta) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ is compact and projective, and $\mathcal{I}$ is sifted, the map φ factors as $(\Gamma_{\mathbb{Z}}(X_1, \dots, X_m) \twoheadrightarrow \mathbb{Z}, \delta) \to (C_j/K_j^{[n]}, K_jC_j/K_j^{[n]}, \gamma_j) \to \mathrm{colim}_{k\in\mathcal{I}}(C_k/K_k^{[n]}, K_kC_k/K_k^{[n]}, \gamma_k)$ for some $j \in \mathcal{I}$. Then the element $\gamma_{i_1}(X_1) \cdots \gamma_{i_m}(X_m) \in \Gamma_{\mathbb{Z}}(X_1, \dots, X_m)$ is killed by the first map, and, hence, $\gamma_{i_1}(x_1) \cdots \gamma_{i_m}(x_m) = 0$.

Note that the composite of left adjoints $\mathrm{Pair} \to \mathrm{Pair}^{\gamma} \xrightarrow{R^{[n]}_{\mathrm{cl}}} \mathrm{Pair}^{\gamma_n}$ preserves small colimits, $(C, K, \gamma) \in \mathrm{Pair}^{\gamma_n}$ is isomorphic to the image of $(A, I) \in \mathrm{QReg} \subseteq \mathrm{Pair}$ under this composite functor, and the map $(A, I) \to (C, K)$ is the unit map under this isomorphism. The result then follows from the uniqueness of universal objects. □

REMARK 3.97. — In fact, there is an ∞-category $\mathrm{Pair}^{\gamma_n,\mathrm{an}}$ of *animated PD-pairs PD-nilpotent of height* n, defined to be the nonabelian derived category of the essential image of $\mathrm{Pair}^{\gamma,\mathrm{st}} \subseteq \mathrm{Pair}$ under the functor $R^{[n]}_{\mathrm{cl}} : \mathrm{Pair}^{\gamma} \to \mathrm{Pair}^{\gamma_n}$.

Then there is a pair of adjoint functors $\mathrm{Pair}^{\gamma_n,\mathrm{an}} \rightleftarrows \mathrm{Pair}^{\gamma_n,\mathrm{an}}$ by Corollary 2.2. Furthermore, by mimicking the proof of Lemma 3.19, the canonical functor $\mathrm{Pair}^{\gamma_n} \to \mathrm{Pair}^{\gamma_n,\mathrm{an}}$ is fully faithful. This leads to an alternative proof of Lemma 3.96. Although the functor $\mathrm{Pair}^{\gamma_n} \to \mathrm{Pair}^{\gamma}$ is fully faithful, we conjecture that the functor $\mathrm{Pair}^{\gamma_n,\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}}$ is not fully faithful, similarly to the fact that the forgetful functor $D(\mathbb{Z}/n\mathbb{Z}) \to D(\mathbb{Z})$ is not fully faithful although $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{Ab}$ is so.

Now we answer Illusie's question.

PROPOSITION 3.98. — *For every quasiregular pair $(A, I) \in \mathrm{QReg}$, let (B, J, γ) denote its PD-envelope. Then the canonical map $\Gamma^*_{A/I}(I/I^2) \to \bigoplus J^{[*]}/J^{[*+1]}$ of graded rings induced by $\gamma_* : I \to I$ is an equivalence.*

Proof. — It follows from Corollary 3.61, Lemmas 3.87 and 3.88, and Proposition 3.90 that there is a comparison map $\Gamma^*_{A/I}(I/I^2) \to \bigoplus J^{[*]}/J^{[*+1]}$ of graded rings. Then the result follows from element tracing, a modification of the proof of Corollary 3.65 by replacing $(\mathbb{Z}[X_1, \ldots, X_n], (X_1, \ldots, X_n))$ with $(\Gamma_{\mathbb{Z}}(X_1, \ldots, X_n) \twoheadrightarrow \mathbb{Z}, \gamma)$. □

4. Derived crystalline cohomology

In this section, we define and study the *Hodge-filtered derived crystalline cohomology*, a filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra functorially associated to an animated PD-pair $(A \twoheadrightarrow A'', \gamma)$ along with a map $A'' \to R$ of animated rings. To do so, we will introduce an auxiliary construction, the *Hodge-filtered derived de Rham cohomology*, functorially associated to a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ of animated PD-pairs, which will be proved independent of the choice of B, and then we define the Hodge-filtered derived crystalline cohomology for $(A \twoheadrightarrow A'', \gamma)$ along with $A'' \to R$ as the Hodge-filtered derived de Rham cohomology of the map $(A \twoheadrightarrow A'', \gamma) \to (\mathrm{id}_R : R \to R, 0)$. This is a generalization of the (usual) derived de Rham cohomology as introduced in Illusie's thesis [23].

Furthermore, we also define the *cohomology of the affine crystalline site*, which can be endowed with Hodge-filtration, whose definition is more similar to the classical crystalline cohomology in [5]. The Hodge-filtered derived de Rham cohomology is, roughly speaking, equivalent to the *relative animated PD-envelope* whenever $A'' \to R$ is surjective (Proposition 4.75), and the Hodge-filtered derived de Rham cohomology is equivalent to the cohomology of the affine crystalline site with Hodge filtration when $\pi_0(R)$ is a finitely generated $\pi_0(A'')$-algebra (Proposition 4.77) or when R is a quasisyntomic A''-algebra (Proposition 4.98). Furthermore, the cohomology of the affine crystalline site is equivalent to the classical crystalline cohomology when everything is classically given, at least up to p-completion, due to the fact that our theory is

noncompleted (Proposition 4.101). Our results can be understood as an extrapolation of techniques in [9].

REMARK 4.1. — Our theory is characteristic-independent. As a cost, the derived de Rham cohomology does not coincide with algebraic de Rham cohomology even under smoothness conditions, although this is true when the base is of characteristic p. In particular, for a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ of animated PD-pairs where A is an animated $\mathbb{Q}$-algebra, the underlying $\mathbb{E}_\infty$-ring of our Hodge-filtered derived de Rham cohomology is constantly A, cf. Lemma 4.13. However, in this case, the noncompleted crystalline cohomology (Definition 4.99) is also A, so the derived de Rham cohomology is as "bad" as the noncompleted derived crystalline cohomology. On the other hand, the Hodge-filtration allows us to recover the "correct" cohomology theory in characteristic 0 after taking Hodge completion by [9, Rem 2.6].

As a corollary, we deduce that the (usual) derived de Rham cohomology $\mathrm{dR}_{\mathbb{Z}/\mathbb{Z}[x]}$ is, as an $\mathbb{E}_\infty$-$\mathbb{Z}[x]$-algebra, equivalent to the PD-polynomial algebra $\Gamma_{\mathbb{Z}}(x)$. Bhatt showed an p-completed version of this [9, Thm 3.27].

REMARK 4.2. — In fact, our theory stems from the observation that the p-completed derived de Rham cohomology $(\mathrm{dR}_{\mathbb{Z}/\mathbb{Z}[x]})^\wedge_p$ coincides with the p-completed PD-polynomial ring $\Gamma_{\mathbb{Z}}(x)^\wedge_p$, and the rationalization becomes $\mathbb{Q}[x]$.

The virtue of our Hodge-filtered derived crystalline cohomology is that it preserves small colimits. We will show that this implies several properties of derived crystalline cohomology, such as "Künneth formula" and base change property (Corollaries 4.38 to 4.40).

REMARK 4.3. — In a forthcoming work [33], the Hodge-filtered derived de Rham cohomology admits a natural enrichment to derived PD-pairs, Remark 3.33, and the Hodge filtration is given by the PD-filtration of the derived PD-pair in question.

4.1. Derived de Rham cohomology. — In this section, we define the derived de Rham cohomology for maps of animated PD-pairs. We need the definition of *modules of PD-differentials*[17].

DEFINITION 4.4 ([39, Tag 07HQ]). — Let $(A, I, \gamma) \to (B, J, \delta)$ be a map of PD-pairs and M an B-module. A *PD A-derivation* into M is a map $\theta : B \to M$, which is additive, $\theta(a) = 0$ for $a \in A$, satisfies the Leibniz rule $\theta(bb') = b\theta(b') + b'\theta(b)$ and that

$$\theta(\delta_n(x)) = \delta_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and $x \in J$.

17. It is about differentials preserving PD-structure, rather than a module with a PD-structure.

In this situation, there exists a *universal PD A-derivation*

$$\mathrm{d}_{(B,J)/(A,I)} : B \to \Omega^1_{(B,J)/(A,I)}$$

such that for any PD A-derivation $\theta : B \to M$, and there exists a unique B-linear map $\xi : \Omega^1_{(B,J)/(A,I)} \to M$ such that $\theta = \xi \circ \mathrm{d}_{(B,J)/(A,I)}$. We also call $\Omega^1_{(B,J)/(A,I)}$ the *module of PD-differentials*.

Remark 4.5. — In Definition 4.4, the PD-structure on A is irrelevant. However, we will soon see that the derived version of module of PD-differentials does depend on the PD-structure on A.

Construction 4.6 ([39, Tag 07HZ]). — *Let $(A, I, \gamma) \to (B, J, \delta)$ be a map of PD-pairs such that $\Omega^1_{(B,J)/(A,I)}$ is a flat B-module*[18]*. The de Rham complex $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ is given by $\Omega^i_{(B,J)/(A,I)} = \bigwedge^i_B \Omega^1_{(B,J)/(A,I)}$ and $\mathrm{d} : \Omega^i_{(B,J)/(A,I)} \to \Omega^{i+1}_{(B,J)/(A,I)}$ is the unique A-linear map determined by*

$$\mathrm{d}(f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_i) = \mathrm{d}f_0 \wedge \cdots \wedge \mathrm{d}f_i.$$

We recall that a commutative differential graded A-algebra (abbrev. A-CDGA) is a commutative algebra object in the symmetric monoidal abelian 1-category $\mathrm{Ch}(\mathrm{Mod}_A)$ of chain complexes[19] *in static A-modules for a ring A. Then any nonpositively graded A-CDGA gives rise to an $\mathbb{E}_\infty$-A-algebra*[20]*, and in particular, the de Rham complex constructed above gives rise to the de Rham cohomology of $(A, I, \gamma) \to (B, J, \delta)$ as an $\mathbb{E}_\infty$-A-algebra. Furthermore, the truncation map $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d}) \to \Omega^0_{(B,J)/(A,I)} = B$ is a map of CDGAs, where B is concentrated in degree 0. Passing to the cohomology, we get a map of $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, called the augmentation map of the de Rham cohomology of $(A, I, \gamma) \to (B, J, \delta)$.*

Remark 4.7. — When we restrict ourselves to the full subcategory $\mathrm{Ch}_{\gg -\infty}(\mathrm{Mod}^\flat_A) \subseteq \mathrm{Ch}(\mathrm{Mod}_A)$ spanned by bounded below chain complexes of flat A-modules, the fully faithful embedding $\mathrm{Ch}_{\gg -\infty}(\mathrm{Mod}^\flat_A) \hookrightarrow \mathrm{DF}(A)$ is, in fact, symmetric monoidal. We will refer to this later.

18. We assume the flatness only to avoid the appearance of the ordinary tensor product $\otimes$ and the exterior power $\bigwedge$, since for flat modules, these coincide with the derived versions. In fact, we only need the very special case that $((A, I, \gamma) \to (B, J, \delta)) \in \mathrm{dRCon}$ defined before Definition 4.10.

19. We identify cochain complexes (K^*, d) with chain complexes (K_{-*}, d).

20. This is in [38, Notation 3.3.12], and we reproduce the argument as follows. We can identify the heart $\mathrm{DF}(A)^\heartsuit$ with respect to the Beilinson t-structure (Proposition 2.41) with the abelian 1-category $\mathrm{Ch}(\mathrm{Mod}_A)$. Furthermore, the fully faithful embedding $\mathrm{Ch}(\mathrm{Mod}_A) \hookrightarrow \mathrm{DF}(A)$ is lax symmetric monoidal (Lemma 2.40). Thus, an A-CDGA gives rise to an $\mathbb{E}_\infty$-algebra in $\mathrm{DF}(A)$. The embedding $\mathrm{Ch}(\mathrm{Mod}_A) \hookrightarrow \mathrm{DF}(A)$ restricts to a lax symmetric monoidal embedding $\mathrm{Ch}_{\leq 0}(\mathrm{Mod}_A) \to \mathrm{DF}^{\geq 0}(A)$. Thus, a nonpositively graded A-CDGA gives rise to an $\mathbb{E}_\infty$-algebra in $\mathrm{DF}^{\geq 0}(A)$, which is mapped to an $\mathbb{E}_\infty$-A-algebra by the symmetric monoidal functor $\mathrm{DF}^{\geq 0}(A) \to D(A)$.

REMARK 4.8. — The composite functor $\mathrm{Ch}_{\leq 0}(\mathrm{Mod}_A) \hookrightarrow \mathrm{DF}^{\geq 0}(A) \to D_{\leq 0}(A)$ maps any complex to its underlying module spectrum.

Now we define the derived de Rham cohomology for PD-pairs.

CONSTRUCTION 4.9. — *By Corollary 2.9, the* ∞*-category* $\mathrm{dRCon} := \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}})$ *(abbreviated for de Rham context) admits a set of compact projective generators given by maps of PD-pairs of the form* $(\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]) \to (\Gamma_{\mathbb{Z}[X,X']}(Y,Y') \twoheadrightarrow \mathbb{Z}[X,X'])$*, where each of* X, Y, X', Y' *consists of a finite set (including empty) of variables. These objects span a full subcategory* $\mathrm{dRCon}^0 \subseteq \mathrm{dRCon}$ *stable under finite coproducts. Then it follows from Proposition A.16 that there is an equivalence* $\mathcal{P}_\Sigma(\mathrm{dRCon}^0) \to \mathrm{dRCon}$ *of* ∞*-categories. The de Rham cohomology, equipped with the augmentation map in Construction 4.6, restricts us to a functor* $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}_{\mathbb{Z}})$*, where* $\mathrm{CAlg}_{\mathbb{Z}}$ *is the* ∞*-category of* $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras.*

DEFINITION 4.10. — The *derived de Rham cohomology functor* $\mathrm{dR}_{./.} : \mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ along with a canonical map $\mathrm{dR}_{(B \twoheadrightarrow B'', \delta)/(A \twoheadrightarrow A'', \gamma)} \to B$ of functors $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}_{\mathbb{Z}}$ is defined to be the left derived functor (Proposition A.14) of the functor $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}_{\mathbb{Z}})$ in Construction 4.9. Given a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ of animated PD-pairs, its *derived de Rham cohomology*, i.e., the image under the derived de Rham cohomology functor, is denoted by $\mathrm{dR}_{(B \to B'', \delta)/(A \to A'', \gamma)}$, or simply $\mathrm{dR}_{(B \to B'')/(A \twoheadrightarrow A'')}$ when there is no ambiguity.

We first explain that this is a generalization of classical derived de Rham cohomology.

REMARK 4.11. — We recall that the functor $\mathrm{CAlg}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}}, A \mapsto (\mathrm{id}_A : A \to A, 0)$ is fully faithful (Lemma 3.45), and, thus, so is the induced functor $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}}) = \mathrm{dRCon}$.

LEMMA 4.12. — *The composite functor* $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{dRCon} \xrightarrow{\mathrm{dR}_{./.}} \mathrm{CAlg}_{\mathbb{Z}}, (A \to B) \mapsto \mathrm{dR}_{(\mathrm{id}_B : B \to B, 0)/(\mathrm{id}_A : A \to A, 0)}$ *is equivalent to the classical derived de Rham cohomology functor* $(A \to B) \mapsto \mathrm{dR}_{B/A}$*.*

Proof. — The crucial point is that $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ is projectively generated, for which $\{(\mathbb{Z}[X] \to \mathbb{Z}[X,Y])\}$ forms a set of compact projective generators, which follows from Corollary 2.9 and Lemma 3.45. The result then follows from Proposition A.14 and the definition of the classical derived de Rham cohomology functor. □

We compute concretely the de Rham complex on dRCon^0. Fix an object $(\Gamma_{\mathbb{Z}[X']}(Y') \twoheadrightarrow \mathbb{Z}[X']) \to (\Gamma_{\mathbb{Z}[X,X']}(Y,Y') \twoheadrightarrow \mathbb{Z}[X,X']) \in \mathrm{dRCon}^0$,;to simplify notations, we will write $A := \Gamma_{\mathbb{Z}[X']}(Y'), A'' := \mathbb{Z}[X']$. Then this object can be rewritten as $(A \twoheadrightarrow A'', \gamma) \to (B := \Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \tilde{\gamma})$, where $X =$

$(x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_n)$ with the module of PD-differentials $\Omega^1 = B\mathrm{d}x_1 \oplus \cdots \oplus B\mathrm{d}x_n \oplus B\mathrm{d}y_1 \oplus \cdots \oplus B\mathrm{d}y_n$, and the universal PD-derivation $B \to \Omega^1$ is determined by $\mathrm{d}(X^\alpha \gamma_\beta(Y)) = \sum_{i=1}^m \alpha_i x_1^{\alpha_1} \cdots x_i^{\alpha_i - 1} \cdots x_m^{\alpha_m} \gamma_\beta(Y)\mathrm{d}x_i + \sum_{j=1}^n X^\alpha \gamma_{\beta_1}(y_1) \cdots \gamma_{\beta_j - 1}(y_j) \cdots \gamma_{\beta_n}(y_n)\mathrm{d}y_j$ (with multi-index product).

As we mentioned earlier, the derived de Rham cohomology is considered uninteresting in characteristic 0. Informally, the derived de Rham cohomology $\mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')}$ is functorially equivalent to A after rationalization. More precisely, we will show the following.

LEMMA 4.13. — *There is a comparison map $A \to \mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')}$, functorial in $((A \twoheadrightarrow A'') \to (B \twoheadrightarrow B'')) \in \mathrm{dRCon}$, as a natural transformation of two functors $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}_{\mathbb{Z}}$. This natural transformation becomes an equivalence after composing with the rationalization $(-) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q} : \mathrm{CAlg}_{\mathbb{Z}} \to \mathrm{CAlg}_{\mathbb{Q}}$.*

Proof. — We first construct the comparison map in question. We have the composite of forgetful functors $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}}$, $(A \twoheadrightarrow A'', \gamma) \mapsto A$. Further composing with the evaluation map $\mathrm{dRCon} \to \mathrm{Pair}^{\gamma,\mathrm{an}}$ at $[0] \in \Delta^1$, we get a functor $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$, $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \mapsto A$. We restrict this functor to dRCon^0, getting a functor $\mathrm{dRCon}^0 \to \mathrm{CAlg}_{\mathbb{Z}}$, which coincides with the composite functor $\mathrm{dRCon}^0 \to \mathrm{Ring} = \mathrm{CAlg}(\mathrm{Ab}) \hookrightarrow \mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab})) \to \mathrm{CAlg}_{\mathbb{Z}}$ given by the "same" formula $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \mapsto A$. Note that there is a canonical map of functors from $\mathrm{dRCon}^0 \to \mathrm{Ring} \to \mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$ to the de Rham complex functor $\mathrm{dRCon}^0 \to \mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$, which is given by the A-CDGA structure on the de Rham complex. Now Proposition A.14 gives us a comparison map of the left derived functors $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}_{\mathbb{Z}}$.

It remains to see that this comparison map is an equivalence after rationalization. First, we note that the rationalization $\mathrm{CAlg}_{\mathbb{Z}} \to \mathrm{CAlg}_{\mathbb{Q}}$ preserves small colimits, and in particular, filtered colimits and geometric realizations; it follows from Proposition A.14 that both functors are still left derived functors after rationalization, and, therefore, it suffices to check the equivalence on dRCon^0. The Poincaré lemma implies that the comparison map of functors $\mathrm{dRCon}^0 \rightrightarrows \mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$ becomes a homotopy equivalence after composing with $\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab})) \to \mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Mod}_{\mathbb{Q}}))$, which implies that it becomes an equivalence after composing with $\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab})) \to \mathrm{CAlg}_{\mathbb{Q}}$ by Remark 4.8. □

Another consequence of this computation is that the de Rham cohomology functor $\mathrm{dRCon}^0 \to \mathrm{CAlg}_{\mathbb{Z}}$ preserves finite coproducts, which follows from the fact that the de Rham cohomology functor $\mathrm{dRCon}^0 \to \mathrm{CAlg}(\mathrm{Ch}_{\gg -\infty}(\mathrm{FreeAb}))$ preserves finite coproducts, and that the composite functor $\mathrm{Ch}_{\gg -\infty}(\mathrm{FreeAb}) \hookrightarrow \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ is symmetric monoidal; cf. Remark 4.7. By Proposition A.14, we have the following.

LEMMA 4.14. — *The derived de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ *preserves small colimits.*

Now we show that the derived de Rham cohomology associated to the map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ does not depend on B and define the derived crystalline cohomology. To formally define the ∞-category of animated PD-pairs $(A \twoheadrightarrow A'', \gamma)$ along with a map $A'' \to R$ of animated rings, we need the concept of comma categories in Section 2.3. Consider the comma category $\mathrm{CrysCon} := \mathrm{Pair}^{\gamma,\mathrm{an}} \times_{\mathrm{CAlg}^{\mathrm{an}}} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$ (abbreviation for crystalline context), where the functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}}$ is the composite functor $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}}, (A \twoheadrightarrow A'', \gamma) \mapsto A''$ and the functor $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{CAlg}^{\mathrm{an}}$ is the evaluation $(A'' \to R) \mapsto A''$ at $0 \in \Delta^1$. It follows from Corollary 2.22 that CrysCon admits a set of compact projective generators of the form $((\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X], \gamma), \mathbb{Z}[X] \to \mathbb{Z}[X, Z])$, where each of X, Y, Z consists of a finite set of variables, which spans a full subcategory $\mathrm{CrysCon}^0 \subseteq \mathrm{CrysCon}$ stable under finite coproducts.

CONSTRUCTION 4.15. — *We note that there is a canonical functor* $\mathrm{dRCon} \to \mathrm{CrysCon}$ *induced by the evaluation* $\mathrm{dRCon} = \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}}) \xrightarrow{\mathrm{ev}_{[0]}} \mathrm{Pair}^{\gamma,\mathrm{an}}$ *and the functor* $\mathrm{dRCon} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})$*, which is itself induced by the composite of the forgetful functors* $\mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}^{\mathrm{an}}, (A \twoheadrightarrow A'', \gamma) \mapsto A''$*. Concretely, the functor* $\mathrm{dRCon} \to \mathrm{CrysCon}$ *is given by* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \mapsto ((A \twoheadrightarrow A'', \gamma), A'' \to B'')$*. Since both functors preserve small colimits (we have used Proposition 3.40), we deduce the following.*

LEMMA 4.16. — *The functor* $\mathrm{dRCon} \to \mathrm{CrysCon}$ *in Construction 4.15 preserves small colimits.*

It follows from Proposition A.14 that $\mathrm{dRCon} \to \mathrm{CrysCon}$ is the left derived functor of the composite functor $\mathrm{dRCon}^0 \to \mathrm{CrysCon}^0 \to \mathrm{CrysCon}$, $((\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X'])) \mapsto ((\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]), \mathbb{Z}[X] \to \mathbb{Z}[X, X'])$. From Corollary 2.2 we have the following.

LEMMA 4.17. — *The functor* $\mathrm{dRCon} \to \mathrm{CrysCon}$ *in Construction 4.15 admits a right adjoint* $\mathrm{CrysCon} \to \mathrm{dRCon}$*, which preserves sifted colimits.*

One can verify (see also Lemma 3.45) the following.

LEMMA 4.18. — *The right adjoint* $\mathrm{CrysCon} \to \mathrm{dRCon}$ *is concretely given by* $((A \twoheadrightarrow A'', \gamma), A'' \to R) \mapsto ((A \twoheadrightarrow A'', \gamma) \to (\mathrm{id}_R : R \to R, 0))$.

The independence of $\mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')}$ with respect to B is formally formulated as follows.

CONSTRUCTION 4.19. — *The counit map of the adjunction* $\mathrm{dRCon} \rightleftarrows \mathrm{CrysCon}$ *in Lemma 4.17 is an equivalence by Lemma 4.18, and, therefore, the functor* $\mathrm{CrysCon} \to \mathrm{dRCon}$ *is fully faithful. The unit map between functors* $\mathrm{dRCon} \rightrightarrows$

dRCon *is concretely given by* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \to ((A \twoheadrightarrow A'', \gamma) \to (\mathrm{id}_{B''} : B'' \to B'', 0))$. *Applying the derived de Rham functor* $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$, *we get the comparison map* $\mathrm{dR}_{(B \twoheadrightarrow B'', \delta)/(A \twoheadrightarrow A'', \gamma)} \to \mathrm{dR}_{(\mathrm{id}_{B''}, 0)/(A \twoheadrightarrow A'', \gamma)}$ *which is functorial in* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \in \mathrm{dRCon}$, *and the comparison map is viewed as a natural transformation between two functors* $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}_{\mathbb{Z}}$.

Proposition 4.20. — *The natural transformation in Construction 4.19. In other words, the derived de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ *is* $(\mathrm{dRCon} \to \mathrm{CrysCon})$*-invariant (Definition 2.57).*

Proof. — Both functors preserve sifted colimits, so by Proposition A.14, it suffices to establish the equivalence for the full subcategory $\mathrm{dRCon}^0 \subseteq \mathrm{dRCon}$. For every $(\Gamma_{\mathbb{Z}[X']}(Y') \twoheadrightarrow \mathbb{Z}[X']) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X']) \in \mathrm{dRCon}^0$ simply denoted by $((A \twoheadrightarrow A'', \gamma) \to (\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \gamma))$, we need to show that the map

$$\mathrm{dR}_{(\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X])/(A \twoheadrightarrow A'')} \to \mathrm{dR}_{(\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)/(A \twoheadrightarrow A'')}$$

is an equivalence. Note that the constructed map

$$(\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \gamma) \to (\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)$$

in $\mathrm{Pair}^{\gamma,\mathrm{an}}_{/(A \twoheadrightarrow A'', \gamma)}$ factors as

$$(\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \gamma) \xrightarrow{\alpha} (A[X] \twoheadrightarrow A''[X], \gamma) \xrightarrow{\beta} (\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)$$

Thus it suffices to show that both maps α and β induce equivalences after passing to the functor $\mathrm{dR}_{./(A \twoheadrightarrow A'', \gamma)} : \mathrm{Pair}^{\gamma,\mathrm{an}}_{/(A \twoheadrightarrow A'', \gamma)} \to \mathrm{CAlg}_{\mathbb{Z}}$. Note that $(A[X] \twoheadrightarrow A''[X], \gamma) \in \mathrm{dRCon}^0$, $\mathrm{dR}_{\alpha/(A \twoheadrightarrow A'', \gamma)}$ could be computed by de Rham complexes, which corresponds a homotopy equivalence of de Rham complexes by the divided power Poincaré lemma [39, Tag 07LC].

It remains to show that $\mathrm{dR}_{\beta/(A \twoheadrightarrow A'', \gamma)}$ is also an equivalence. For this, we need to resolve $(\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)$ simplicially under $(A[X] \twoheadrightarrow A''[X], \gamma)$. Recall that $A = \Gamma_{\mathbb{Z}[X']}(Y')$ and $A'' = \mathbb{Z}[X']$. The key point is that we can resolve A'' simplicially by divided power polynomial A-algebras, in the same way as resolving $\mathbb{Z}$ simplicially by polynomial $\mathbb{Z}[t]$-algebras, which essentially follows from a bar construction of $\mathbb{N}$, see [9, Rem 3.31]. For every divided power polynomial A-algebra $\Gamma_A(Z)$, $(\Gamma_{A[X]}(Z) \twoheadrightarrow A''[X], \gamma)$ belongs to dRCon^0, and the map $\mathrm{dR}_{(A[X] \twoheadrightarrow A''[X])/(A \twoheadrightarrow A'')} \to \mathrm{dR}_{(\Gamma_{A[X]}(Z) \twoheadrightarrow A''[X])/(A \twoheadrightarrow A'')}$ (functorial in $\Gamma_A(Z)$) is an equivalence again by the divided power Poincaré lemma [39, Tag 07LC]. It follows that $\mathrm{dR}_{\beta/(A \twoheadrightarrow A'', \gamma)}$ is, indeed, an equivalence. □

In view of Proposition 2.56, we define the derived crystalline cohomology functor, which corresponds to the $(\mathrm{dRCon} \to \mathrm{CrysCon})$-invariant functor $\mathrm{dR}_{./.}$.

DEFINITION 4.21. — The *derived crystalline cohomology functor* $\mathrm{CrysCoh} : \mathrm{CrysCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ is defined to be the composite $\mathrm{CrysCon} \to \mathrm{dRCon} \xrightarrow{\mathrm{dR}_{\cdot/\cdot}} \mathrm{CAlg}_{\mathbb{Z}}$.

NOTATION 4.22. — We will denote the derived crystalline cohomology of $((A \twoheadrightarrow A'', \gamma), A'' \to R) \in \mathrm{CrysCon}$ by $\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma)}$ (or $\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}$ even $\mathrm{CrysCon}_{R/A}$ when there is no ambiguity).

Now we show that

PROPOSITION 4.23. — *The derived crystalline cohomology functor* $\mathrm{CrysCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ *preserves small colimits.*

Proof. — The functor $\mathrm{CrysCon} \to \mathrm{dRCon}$ preserves sifted colimits and $\mathrm{dR}_{\cdot/\cdot} : \mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ preserves small colimits. It follows that the derived crystalline cohomology functor CrysCoh preserves sifted colimits. By Proposition A.14, and it remains to show that $\mathrm{CrysCoh}|_{\mathrm{CrysCon}^0}$ preserves finite coproducts. The point is that every $(\Gamma_{\mathbb{Z}[X']}(Y') \twoheadrightarrow \mathbb{Z}[X'], \mathbb{Z}[X'] \to \mathbb{Z}[X, X']) \in \mathrm{CrysCon}^0$ lifts to $(\Gamma_{\mathbb{Z}[X']}(Y') \twoheadrightarrow \mathbb{Z}[X']) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X']) \in \mathrm{dRCon}^0$, the functor $\mathrm{dRCon}^0 \to \mathrm{CrysCon}^0$ preserves finite coproducts, and the functor $\mathrm{dR}_{\cdot/\cdot}$ preserves finite coproducts. □

Now we apply the discussions in Section 2.3 to deduce some formal properties. First, by Remark 2.24, we have the following.

COROLLARY 4.24. — *The derived crystalline cohomology is compatible with base change. More precisely, let* $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$ *and let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs. Then the canonical map*

$$\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \otimes^{\mathbb{L}}_{A} B \longrightarrow \mathrm{CrysCoh}_{(R \otimes^{\mathbb{L}}_{A''} B'')/(B \twoheadrightarrow B'', \gamma_B)}$$

is an equivalence.

REMARK 4.25. — Let R be a (finitely generated) polynomial $\mathbb{F}_p$-algebra. Then by the p-completed derived crystalline cohomology of R with respect to the PD-pair $(\mathbb{Z}_p, (p))$ is equivalent to the usual (p-completed) crystalline cohomology of R with respect to $\mathbb{Z}_p$. In fact, later (Propositions 4.98 and 4.101), we will show that the same p-completed comparison holds for quasisyntomic $\mathbb{F}_p$-algebras R (and in particular, for smooth $\mathbb{F}_p$-algebras). However, our noncompleted derived crystalline cohomology is not necessarily p-complete.

Next, by Remark 2.28, we have the following.

Corollary 4.26. — *The derived crystalline cohomology is symmetric monoidal. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ *and let* $A \to R$, $A \to S$ *be two maps of animated rings. Then the canonical map*

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{A} \mathrm{CrysCoh}_{S/(A\twoheadrightarrow A'')} \longrightarrow \mathrm{CrysCoh}_{(R\otimes^{\mathbb{L}}_{A''}S)/(A\twoheadrightarrow A'')}$$

is an equivalence.

Finally, by Remark 2.31, we have the following.

Corollary 4.27. — *The derived crystalline cohomology is transitive. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs, and let* $B'' \to R$ *be a map of animated rings. Then the canonical map*

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{CrysCoh}_{B''/(A\twoheadrightarrow A'')}} B \longrightarrow \mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')}$$

is an equivalence, where the map $\mathrm{CrysCoh}_{B''/(A\twoheadrightarrow A'')} \to B$ *is*

$$\mathrm{CrysCoh}_{B''/(A\twoheadrightarrow A'')} \to \mathrm{CrysCoh}_{B''/(B\twoheadrightarrow B'')} \simeq B.$$

Remark 4.28. — In particular, if we take $(A \twoheadrightarrow A'', \gamma_A) = (\mathbb{Z}, 0, 0)$ in Corollary 4.27, we see that, fixing an animated PD-pair $(B \twoheadrightarrow B'', \gamma_B)$, any derived crystalline cohomology $\mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')}$ is completely determined by the derived de Rham cohomology $\mathrm{dR}_{R/\mathbb{Z}}$. However, without the theory of derived crystalline cohomology, we do not know how to construct the map $\mathrm{dR}_{B''/\mathbb{Z}} \to B$ in terms of the PD-structure on $B \twoheadrightarrow B''$.

4.2. Filtrations. — In this section, we will define the *Hodge filtration* on the derived de Rham cohomology and show that most of our previous discussions are compatible with the Hodge filtration. Furthermore, in characteristic p, we will define the *conjugate filtration*, which is of technical importance to control the cohomology. We start with the definition of the Hodge filtration.

Definition 4.29 (cf. [6, §6.13]). — Let $(A, I, \gamma) \to (B, J, \delta)$ be a map of PD-pairs such that $\Omega^1_{(B,J)/(A,I)}$ is a flat B-module. The *Hodge filtration* Fil^*_H on the de Rham complex $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ is given by the differential graded ideals $\mathrm{Fil}^m_H \Omega^*_{(B,J)/(A,I)} := J^{[m-*]}\Omega^*_{(B,J)/(A,I)} \subseteq \Omega^*_{(B,J)/(A,I)}$.

Construction 4.30. — *As CDGAs give rise to* $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras, (nonnegatively) filtered CDGAs give rise to (nonnegatively) filtered* $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras. Moreover, the truncation map* $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d}) \to B$ *is a map of filtered CDGAs, which gives rise to a map of filtered* $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras. Thus we get a functor* $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})))$.

Definition 4.31. — The *Hodge-filtered derived de Rham cohomology functor* $\mathrm{Fil}^*_H \mathrm{dR}_{./.} : \mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ together with a canonical map $\mathrm{Fil}^*_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')} \to \mathrm{Fil}^*_{\mathrm{PD}} B$ is defined to be the left derived functor

(Proposition A.14) of the functor $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})))$ in Construction 4.30, where $\mathrm{Fil}^*_{\mathrm{PD}} B$ is an abbreviation of $\mathrm{Fil}^*_{\mathrm{PD}}(B \twoheadrightarrow B'', \gamma)$ (Definition 3.85).

Most of properties in Section 4.1 hold with a similar proof:

LEMMA 4.32. — *The composite functor*

$$\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})),$$
$$(A \to B) \mapsto \mathrm{Fil}^*_H \mathrm{dR}_{(\mathrm{id}_B : B \to B, 0)/(\mathrm{id}_A : A \to A, 0)}$$

is equivalent to the classical Hodge-filtered derived de Rham cohomology functor $(A \to B) \mapsto \mathrm{Fil}^*_H \mathrm{dR}_{B/A}$.

LEMMA 4.33. — *The map in Lemma 4.13 admits a natural enrichment, that is to say, a map* $\mathrm{Fil}^*_{\mathrm{PD}} A \to \mathrm{Fil}^*_H \mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')}$ *of functors* $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.[21]

LEMMA 4.34. — *The Hodge-filtered derived de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *preserves small colimits.*

This allows us to define the Hodge-filtration on the derived crystalline cohomology, due to the following proposition, which follows from the proof of Proposition 4.20 by replacing the Poincaré lemma by the filtered Poincaré lemma, cf. [6, Thm 6.13]:

PROPOSITION 4.35. — *The map*

$$\mathrm{Fil}^*_H \mathrm{dR}_{(B \twoheadrightarrow B'', \delta)/(A \twoheadrightarrow A'', \gamma)} \to \mathrm{Fil}^*_H \mathrm{dR}_{(\mathrm{id}_{B''}, 0)/(A \to A'', \gamma)}$$

of functors $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *induced by the counit map associated to* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \in \mathrm{dRCon}$ *is an equivalence. In other words, the Hodge-filtered de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *is* $(\mathrm{dRCon} \to \mathrm{CrysCon})$*-invariant (Definition 2.57).*

DEFINITION 4.36. — The *Hodge-filtered derived crystalline cohomology functor* $\mathrm{Fil}^*_H \mathrm{CrysCoh} : \mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ is defined to be the composite $\mathrm{CrysCon} \to \mathrm{dRCon} \xrightarrow{\mathrm{Fil}^*_H \mathrm{dR}_{./.}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.

PROPOSITION 4.37. — *The Hodge-filtered derived crystalline cohomology functor* $\mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *preserves small colimits.*

Similarly to Corollaries 4.24, 4.26 and 4.27, we have

21. Corrected thanks to a message from Lenny TAELMAN.

Corollary 4.38. — *The Hodge-filtered derived crystalline cohomology is compatible with base change. More precisely, let* $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$ *and let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs. Then the canonical map*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \otimes^{\mathbb{L}}_{\mathrm{Fil}_{\mathrm{PD}} A} \mathrm{Fil}_{\mathrm{PD}} B \longrightarrow \mathrm{Fil}_H \mathrm{CrysCoh}_{(R \otimes^{\mathbb{L}}_{A''} B'')/(B \twoheadrightarrow B'', \gamma_B)}$$

is an equivalence.

Corollary 4.39. — *The derived crystalline cohomology is symmetric monoidal. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ *and let* $A \to R$, $A \to S$ *be two maps of animated rings. Then the canonical map*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')} \otimes^{\mathbb{L}} \mathrm{Fil}_H \mathrm{CrysCoh}_{S/(A \twoheadrightarrow A'')} \to \mathrm{Fil}_H \mathrm{CrysCoh}_{(R \otimes^{\mathbb{L}}_{A''} S)/(A \twoheadrightarrow A'')}$$

is an equivalence, where the tensor product on the left is relative to $\mathrm{Fil}_{\mathrm{PD}} A$.

Corollary 4.40. — *The derived crystalline cohomology is transitive. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs and let* $B'' \to R$ *be a map of animated rings. Then the canonical map*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_H \mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'')}} \mathrm{Fil}_{\mathrm{PD}} B \longrightarrow \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(B \twoheadrightarrow B'')}$$

is an equivalence, where the map $\mathrm{Fil}_H \mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'')} \to B$ *is equivalent to the map*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'')} \to \mathrm{Fil}_H \mathrm{CrysCoh}_{B''/(B \twoheadrightarrow B'')} \simeq \mathrm{Fil}_{\mathrm{PD}} B.$$

Now we come to the characteristic $p > 0$ case. We start with an analysis of the Frobenius map on an animated PD-$\mathbb{F}_p$-pair. Let $(A, I, \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{st}}_{\mathbb{F}_p}$ be an animated PD-$\mathbb{F}_p$-pair of the form $\Gamma_{\mathbb{F}_p[X]}(Y) \twoheadrightarrow \mathbb{F}_p[X]$. We also have similar definitions for $\mathrm{dRCon}_{\mathbb{F}_p}, \mathrm{dRCon}^0_{\mathbb{F}_p}$ and $\mathrm{CrysCon}_{\mathbb{F}_p}, \mathrm{CrysCon}^0_{\mathbb{F}_p}$, and a parallel theory for $\mathbb{F}_p$-stuff. We first point out that, by Corollary 4.24 along with the proof of Lemma 2.37 (to compare with Lemmas 3.51 and 3.52), we have the following.

Lemma 4.41. — *The derived crystalline cohomology* $\mathrm{CrysCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p))$ *fits into the commutative diagram*

$$\begin{array}{ccc} \mathrm{CrysCon}_{\mathbb{F}_p} & \longrightarrow & \mathrm{CrysCon} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}(D(\mathbb{F}_p)) & \longrightarrow & \mathrm{CAlg}(D(\mathbb{Z})) \end{array}$$

of ∞-categories, where the horizontal arrows are forgetful functors. The same goes for the derived de Rham cohomology. Furthermore, this diagram is left adjointable (roughly speaking, if we replace the horizontal arrows by their left adjoints, it is still a commutative diagram of ∞-categories).

Then the Frobenius map $\varphi_A : A \to A$ factors uniquely through the quotient map $A \twoheadrightarrow A/I$, which gives rise to a map $A/I \to A$. From Proposition A.14 we then have the following.

LEMMA 4.42. — *For any animated PD-$\mathbb{F}_p$-pair $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}_{\mathbb{F}_p}$, the Frobenius map $\varphi_A : A \to A$ factors functorially through the map $A \twoheadrightarrow A''$, which gives rise to the a map $A'' \to A$, denoted by $\varphi_{(A \twoheadrightarrow A'',\gamma)}$ or $\varphi_{A \twoheadrightarrow A''}$ when there is no ambiguity (when $(A \twoheadrightarrow A'', \gamma)$ comes from a PD-$\mathbb{F}_p$-pair (A, I, γ); it will also be denoted by $\varphi_{(A,I,\gamma)}$ or $\varphi_{(A,I)}$).*

Now we point out that in the char p-case, the de Rham complex is "Frobenius-linear" (compare with Definition 3.68).

CONSTRUCTION 4.43. — *Let $(A, I, \gamma) \to (B, J, \delta)$ be an object in $\mathrm{dRCon}^0_{\mathbb{F}_p}$. Each graded piece $\Omega^i_{(B,J,\delta)/(A,I,\gamma)}$ admits a natural B-module structure and, therefore, also a $\varphi^*_{(A,I)}(B/J)$-module structure induced by the map $\varphi^*_{(A,I)}(B/J) := (B/J) \otimes^{\mathbb{L}}_{A/I,\varphi_{(A,I)}} A \to B$, the linearization of $\varphi_{(B,J)} : B/J \to B$. Furthermore, the differential d is $\varphi^*_{(A,I)}(B/J)$-linear, which makes the de Rham complex $(\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$ a $\varphi^*_{(A,I)}(B/J)$-CDGA. In other words, there is a map $\varphi^*_{(A,I)}(B/J) \to (\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$ of $\mathbb{F}_p$-CDGAs, where $\varphi^*_{(A,I)}(B/J)$ is concentrated in degree 0.*

CONSTRUCTION 4.44. — *Let $(A, I, \gamma) \to (B, J, \delta)$ be an object in $\mathrm{dRCon}^0_{\mathbb{F}_p}$. The derived de Rham cohomology $\mathrm{dR}_{(B,J,\delta)/(A,I,\gamma)}$ is computed by the de Rham complex $(\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$. The Whitehead tower $(\tau_{\geq n} \mathrm{dR}_{(B,J,\delta)/(A,I,\gamma)})_{n\in(\mathbb{Z},\geq)}$ defines a nonpositive[22] exhaustive filtration, and, thus, the map $\varphi^*_{(A,I)}(B/J) \to (\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$ is a map of filtered $\mathbb{F}_p$-CDGAs (where $\varphi^*_{(A,I)}(B/J)$ is trivially filtered), which gives rise to a map of filtered $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, and, thus, a functor $\mathrm{dRCon}^0_{\mathbb{F}_p} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$.*

DEFINITION 4.45. — The *conjugate-filtered derived de Rham cohomology functor* $\mathrm{Fil}^*_{\mathrm{conj}} \mathrm{dR}_{./.} : \mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ along with the *structure map* $\varphi^*_{(A \twoheadrightarrow A'')}(B'') \to \mathrm{Fil}^*_{\mathrm{conj}} \mathrm{dR}_{(B \twoheadrightarrow B'',\delta)/(A \twoheadrightarrow A'',\gamma)}$ is defined to be the left derived functor (Proposition A.14) of the functor $\mathrm{dRCon}^0_{\mathbb{F}_p} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ in Construction 4.44.

22. In the literature, the conjugate filtration is increasing. We make it decreasing by negating the sign.

From either Proposition A.14 and Lemma 2.43 or the fact that $\mathrm{Pair}^{\mathrm{an}} \simeq \mathcal{P}_\Sigma(\mathrm{Pair}^{\mathrm{st}}) \subseteq \mathcal{P}(\mathrm{Pair}^{\mathrm{st}})$ is stable under filtered colimits (Proposition A.11) , we have the following.

LEMMA 4.46. — *The conjugate filtration on the derived de Rham cohomology is exhaustive.*

We now prove the corresponding results of Section 4.1 for the conjugate filtration.

LEMMA 4.47. — *The conjugate-filtered derived de Rham cohomology functor* $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *preserves small colimits (note that so does the functor* $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}_{\mathbb{F}_p}, ((A \twoheadrightarrow A', \gamma) \to (B \twoheadrightarrow B', \delta)) \mapsto \varphi^*_{(A \twoheadrightarrow A')}(B')$*).*

Proof. — First, we note that, for any connective $\mathbb{E}_\infty$-ring A, the Whitehead-tower functor $D(A) \to \mathrm{DF}(A), M \mapsto (\tau_{\geq n} M)_{n \in (\mathbb{Z}, \geq)}$ is canonically lax symmetric monoidal (recall that $\mathrm{DF}(A)$ is endowed with the Day convolution). We give an informal description: given $M, N \in D(A)$, for all $m, n \in \mathbb{Z}$, the canonical map $\tau_{\geq m} M \to M$ and $\tau_{\geq n} N \to N$ gives rise to a map $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N) \to M \otimes^{\mathbb{L}}_A N$. Since $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N)$ is $(m+n)$-connective, this gives rise to a map $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N) \to \tau_{\geq m+n}(M \otimes^{\mathbb{L}}_A N)$. Assembling these maps, we get the lax symmetric monoidal structure. Next, when A is given by a field, in particular, $A = \mathbb{F}_p$, the structure above is, in fact, symmetric monoidal, since $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N) \to \tau_{\geq m+n}(M \otimes^{\mathbb{L}}_A N)$ is an equivalence for all $m, n \in \mathbb{Z}$.

Now recall that in a symmetric monoidal ∞-category, finite coproducts of commutative algebra objects are given by tensor products. It follows from Lemma 4.14 that the conjugate-filtered derived de Rham cohomology functor $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ is the left derived functor of a finite-coproduct-preserving functor, and then the result follows from Proposition A.14. □

Note that by the divided power Poincaré lemma [39, Tag 07LC], the conjugate filtration on the divided power polynomial algebra is trivial. The proof of Proposition 4.20 leads to the following.

PROPOSITION 4.48. — *The natural transformation*

$$\mathrm{Fil}^*_{\mathrm{conj}}\, \mathrm{dR}_{(B \twoheadrightarrow B', \delta)/(A \twoheadrightarrow A', \gamma)} \to \mathrm{Fil}^*_{\mathrm{conj}}\, \mathrm{dR}_{(\mathrm{id}_{B'}, 0)/(A \twoheadrightarrow A', \gamma)}$$

of functors $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{Z}))$ *induced by the counit map associated to* $((A \twoheadrightarrow A', \gamma) \to (B \twoheadrightarrow B', \delta)) \in \mathrm{dRCon}$ *is an equivalence. In other words, the conjugate-filtered de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{Z}))$ *is* $(\mathrm{dRCon} \to \mathrm{CrysCon})$*-invariant (Definition 2.57) (note that so is the functor* $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}_{\mathbb{F}_p}, ((A \twoheadrightarrow A', \gamma) \to (B \twoheadrightarrow B', \delta)) \mapsto \varphi^*_{(A \twoheadrightarrow A')}(B')$*).*

DEFINITION 4.49. — The *conjugate-filtered derived crystalline cohomology functor* $\mathrm{Fil}^*_{\mathrm{conj}} \mathrm{CrysCoh} : \mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ along with the *structure map* $\varphi^*_{(A \twoheadrightarrow A', \gamma)}(R) \to \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A', \gamma)}$ is defined to be the composite

$\mathrm{CrysCon} \to \mathrm{dRCon} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$, where the later functor is the conjugate-filtered derived de Rham cohomology functor combined with the structure map.

By Lemma 4.46, we have the following.

LEMMA 4.50. — *The conjugate filtration on the derived crystalline cohomology is exhaustive.*

Similarly to Proposition 4.23, we have the following.

PROPOSITION 4.51. — *The conjugate-filtered derived crystalline cohomology functor* $\mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *preserves small colimits.*

Now we analyze the associated graded pieces of the conjugate filtration. Let $(A, I, \gamma) \to (B, J, \delta)$ be an element in dRCon^0. We recall the inverse[23] Cartier map $C^{-1} : \varphi^*_{(A,I)}(\Omega^\star_{(B/J)/(A/I)}) \to H^\star(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ of graded $\varphi^*_{(A,I)}(B/J)$-algebras (where $\star$ is the grading), then we deduce that this is, in fact, an isomorphism. Our presentation has been adapted from the proof of [25, Thm 7.2].

- $\star = 0$: This is the composite map $\varphi^*_{(A,I)}(B/J) \to B \to H^0(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$, i.e., the $\varphi^*_{(A,I)}(B/J)$-algebra structure on $H^0(\Omega_{(B,J)/(A,I)}, \mathrm{d})$.
- $\star = 1$: Consider the map $B \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ of sets given by $f \mapsto [f^{p-1}\mathrm{d}f]$. We first check that this map is additive. In $\Omega^1_{\mathbb{Z}[u,v]/\mathbb{Z}}$, we have

$$\begin{aligned}(u+v)^{p-1}\mathrm{d}(u+v) - u^{p-1}\mathrm{d}u - v^{p-1}\mathrm{d}v &= \frac{1}{p}(\mathrm{d}((u+v)^p) - \mathrm{d}(u^p) - \mathrm{d}(v^p)) \\ &= \frac{1}{p}\mathrm{d}\left(\sum_{j=1}^{p-1}\binom{p}{j}u^j v^{p-1-j}\right) \\ &= \mathrm{d}\left(\sum_{j=1}^{p-1}\frac{1}{p}\binom{p}{j}u^j v^{p-1-j}\right)\end{aligned}$$

  We deduce the additivity by the map $\mathbb{Z}[u, v] \to B, u \mapsto f, v \mapsto g$.

  Now we note that the map $f \mapsto [f^{p-1}\mathrm{d}f]$ satisfies Leibniz' rule (recall that $H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ is a $\varphi^*_{(A,I)}(B/J)$-module; therefore a B/J-module). Indeed, $[(fg)^{p-1}\mathrm{d}(fg)] = f^p[g^{p-1}\mathrm{d}g] + g^p[f^{p-1}\mathrm{d}f]$.

  Thus we get a derivation $B/J \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$, which gives rise to a B/J-linear map $\Omega^1_{(B/J)/(A/I)} \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ and after linearization, we get $\varphi^*_{(A,I)}\Omega^1_{(B/J)/(A,I)} \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$.
- $\star > 1$: Taking the exterior power of the map for $\star = 1$.

23. A priori, the "inverse" Cartier map C^{-1} is not defined to be the inverse of a map, but just defined to be a map.

Now we show the Cartier isomorphism.

LEMMA 4.52. — *Let* $(A, I, \gamma) \to (B, J, \delta)$ *be an element in* $\mathrm{CrysCon}^0_{\mathbb{F}_p}$. *Then the inverse Cartier map* $C^{-1} : \varphi^*_{(A,I)}(\Omega^\star_{(B/J)/(A/I)}) \to H^\star(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ *is an isomorphism of graded* $\varphi^*_{(A,I)}(B/J)$*-algebras.*

Proof. — Recall that (B, J, δ) is of the form $(\Gamma_{A[X]}(Y) \twoheadrightarrow (A/I)[X], \delta)$. It is then direct to check that the inverse Cartier map C^{-1} factors as $\varphi^*_{(A,I)}(\Omega^\star_{(B/J)/(A/I)}) \to H^\star(\Omega^*_{(A[X],IA[X])/(A,I)}, \mathrm{d}) \to H^\star(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$, where the first map is the inverse Cartier map associated to $(A \twoheadrightarrow A/I, \gamma) \to (A[X], IA[X], \gamma)$, and the second map is an isomorphism by the divided power Poincaré lemma [39, Tag 07LC].

Thus we can assume that $(B, J, \delta) = (A[X], IA[X], \gamma)$. In this case, the inverse Cartier map is base-changed from that for $(A, 0, 0) \to (A[X], 0, 0)$ along $(A, 0, 0) \to (A, I, \gamma)$, thus we can assume that $I = 0$, which is [25, Thm 7.2]. □

From Proposition A.14 we then have the following.

PROPOSITION 4.53. — *There exists a natural isomorphism*[24]

$$C^{-1} : \varphi^*_{(A \twoheadrightarrow A'')} \left(\bigwedge_{B''}^{\star} L_{B''/A''} \right) [-\star] \to \mathrm{gr}^{-\star}_{\mathrm{conj}} \, \mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')}$$

in $\mathrm{CAlg}(\mathrm{Gr}^{\leq 0}(D(\varphi^*_{A \twoheadrightarrow A''}(B''))))$, *called the derived Cartier isomorphism (cf.* [9, Prop 3.5]*), which is functorial*[25] *in* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \in \mathrm{dRCon}_{\mathbb{F}_p}$.

Note that both functors are $(\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CrysCon}_{\mathbb{F}_p})$-invariant (Definition 2.57); from Proposition 2.56 we have the following.

PROPOSITION 4.54. — *There exists a natural isomorphism*

$$C^{-1} : \varphi^*_{(A \twoheadrightarrow A'')} \left(\bigwedge_{R}^{\star} L_{R/A''} \right) [-\star] \to \mathrm{gr}^{-\star}_{\mathrm{conj}} \, \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}$$

in $\mathrm{CAlg}(\mathrm{Gr}^{\leq 0}(D(\varphi^*_{A \twoheadrightarrow A''}(R))))$, *called the derived Cartier isomorphism, which is functorial in* $((A \twoheadrightarrow A'', \gamma), A'' \to R) \in \mathrm{CrysCon}_{\mathbb{F}_p}$.

24. To avoid the ambiguity of symbols, we suppress the asterisk on Fil^* to avoid confusion with the pullback symbol φ^*.

25. Here we use the same convention as in Remark 3.72.

4.3. Relative animated PD-envelope. — Recall that, in the classical case [39, Tag 07H9], given a PD-pair $(A, I, \gamma) \in \mathrm{Pair}^{\gamma}$ and an ideal J of A containing I, there is a relative PD-envelope $D_{(A,I,\gamma)}(J)$, which is essentially the same as $D_A(J) \otimes_{D_A(I)} A$, where the map $D_A(I) \to A$ is induced by the PD-structure γ on A. We seek an animated version of this construction, which is needed to study the derived crystalline cohomology. Roughly speaking, we first construct a relative PD-envelope for any animated PD-pair $(A \twoheadrightarrow A'', \gamma_A)$ along with a map $(A \twoheadrightarrow A'') \to (B \twoheadrightarrow B'')$ of animated pairs, and then we restrict to the special case that the map $A \to B$ is an equivalence. It turns out that the general case can be recovered from the special case.

Similarly to the classical case, the relative animated PD-envelope of $(A \twoheadrightarrow A'', \gamma_A)$ along with $(A \twoheadrightarrow A'') \to (B \twoheadrightarrow B'')$ is obtained by the base change $\mathrm{Env}^{\gamma,\mathrm{an}}(B \twoheadrightarrow B'') \amalg_{\mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'')} (A \twoheadrightarrow A'', \gamma_A)$ of the (absolute) animated PD-envelope, where the map $\mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'') \to (A \twoheadrightarrow A'', \gamma_A)$ is induced by the PD-structure γ_A. More formally, we start with reviewing the formal category-theoretic fact about relative adjunctions.

LEMMA 4.55 (dual to [28, Prop 5.2.5.1]). — *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjoint pair of ∞-categories. Assume that the ∞-category $\mathcal{D}$ admits pushouts and let $D \in \mathcal{D}$ be an object. Then*

1. *The induced functor $g : \mathcal{D}_{D/} \to \mathcal{C}_{GD/}$ admits a left adjoint f.*
2. *The functor f is equivalent to the composition*

$$\mathcal{C}_{GD/} \xrightarrow{f'} \mathcal{D}_{FGD/} \xrightarrow{f''} \mathcal{D}_{D/},$$

where f' is induced by F and f'' is induced by the pushout along the counit map $FGD \to D$.

We note that this construction is functorial in $D \in \mathcal{D}$.

NOTATION 4.56. — We denote the comma category $\mathrm{Pair}^{\gamma,\mathrm{an}} \times_{\mathrm{Pair}^{\mathrm{an}}, \mathrm{ev}_{[0]}} \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\mathrm{an}})$ by PDEnvCon, an object of which is denoted by $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'')$, instead of the cumbersome notation $((A \twoheadrightarrow A'', \gamma), (A \twoheadrightarrow A'') \to (B \twoheadrightarrow B''))$.

DEFINITION 4.57. — Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ be an animated PD-pair. The *(relative) animated PD-envelope* of an animated pair in $\mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/}$ is the image under the functor $\mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/} \to \mathrm{Pair}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma)/}$ induced by the animated PD-envelope functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}^{\gamma,\mathrm{an}}$ by Lemma 4.55.

Concretely, let $B \twoheadrightarrow B''$ be an object in $\mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/}$. Then the relative animated PD-envelope of $B \twoheadrightarrow B''$ is given by

$$\mathrm{Env}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma_A)}(B \twoheadrightarrow B'') := \mathrm{Env}^{\gamma,\mathrm{an}}(B \twoheadrightarrow B'') \amalg_{\mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'')} (A \twoheadrightarrow A'', \gamma_A)$$

where the map $\mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'') \to (A \twoheadrightarrow A'', \gamma)$ is the counit map associated

to $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$, and the map $\mathrm{Env}^{\gamma,\mathrm{an}}(A \twoheadrightarrow A'') \to \mathrm{Env}^{\gamma,\mathrm{an}}(B \twoheadrightarrow B'')$ is the image of $(A \twoheadrightarrow A'') \to (B \twoheadrightarrow B'')$ under the animated PD-envelope functor.

This defines the *(relative) animated PD-envelope functor* $\mathrm{PDEnvCon} \to \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}}), ((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'')) \mapsto \mathrm{Env}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma)}(B \twoheadrightarrow B'')$.

Example 4.58. — Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ be an animated PD-pair. Then the animated PD-envelope of $A \twoheadrightarrow A''$ relative to $(A \twoheadrightarrow A'', \gamma)$ is given by $(A \twoheadrightarrow A'', \gamma)$. This follows from the fact that $A \twoheadrightarrow A''$ relative to $(A \twoheadrightarrow A'', \gamma)$ is the base change of $\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ relative to $(\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}, 0)$ along the map $(\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}, 0) \to (A \twoheadrightarrow A'', \gamma)$ of animated PD-pairs. Compare with this Lemma 4.66.

Example 4.59. — Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ be an animated PD-pair. Then the animated PD-envelope of $\mathrm{id}_{A''} : A'' \twoheadrightarrow A''$ relative to $(A \twoheadrightarrow A'', \gamma)$ is given by $(\mathrm{id}_{A''} : A'' \to A'', 0)$. This follows from checking the universal property of the unit map at $\mathrm{id}_{A''} : A'' \twoheadrightarrow A''$ of the adjunction $\mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/} \rightleftarrows \mathrm{Pair}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma)/}$.

Immediately from Lemma 3.47 we have the following.

Lemma 4.60. — *Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{an}}$ be an animated PD-pair, $(B \twoheadrightarrow B'') \in \mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/}$ an animated pair under $A \twoheadrightarrow A''$. Let $(C \twoheadrightarrow B'', \delta)$ denote its relative animated PD-envelope. Then the unit map $(B \twoheadrightarrow B'') \to (C \twoheadrightarrow B'')$ becomes an equivalence after rationalization.*

Recall that given a PD-pair (A, I, γ) and a map $(A, I) \to (B, J)$ of pairs with $A \to B$ being flat, the PD-structure γ extends to B, i.e, there exists a unique PD-structure $\overline{\gamma}$ on (B, IB) such that the map $(A, I) \to (B, J)$ of pairs gives rise to a map $(A, I, \gamma) \to (B, IB, \overline{\gamma})$ of PD-pairs. Then the PD-envelope of (B, J) with respect to (A, I, γ) is the same as that with respect to the PD-pair $(B, IB, \overline{\gamma})$, which corresponds to the crystalline cohomology of B/J with respect to $(B, IB, \overline{\gamma})$. We now show an animated analogue (without flatness).

Notation 4.61. — Let $\mathrm{CrysCon}_{\mathrm{surj}}$ denote the full subcategory $\mathrm{Pair}^{\gamma,\mathrm{an}} \times_{\mathrm{CAlg}^{\mathrm{an}}} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}})_{\geq 0} \subseteq \mathrm{CrysCon}$ spanned by objects $((A \twoheadrightarrow A'', \gamma), A'' \to R)$ such that $A'' \to R$ is also surjective.

Construction 4.62. — *There is a canonical functor* $\mathrm{PDEnvCon} \to \mathrm{CrysCon}_{\mathrm{surj}}$ *given as follows: for every object* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'')) \in \mathrm{PDEnvCon}$,

we get the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow\!\!\!\!\rightarrow & A'' \\ \downarrow & & \downarrow \\ B & \longrightarrow\!\!\!\!\rightarrow & B'' \end{array}$$

in $\mathrm{CAlg}^{\mathrm{an}}$*, which gives rise to two surjective maps* $B \otimes^{\mathbb{L}}_A A'' \twoheadrightarrow B''$ *and* $B \twoheadrightarrow B \otimes^{\mathbb{L}}_A A''$*. Furthermore, the latter admits a PD-structure: it is the underlying animated pair of the pushout* $(\mathrm{id}_B : B \to B, 0) \amalg_{(\mathrm{id}_A : A \to A, 0)} (A \twoheadrightarrow A'', \gamma)$ *in* $\mathrm{Pair}^{\gamma,\mathrm{an}}$*. We denote by* $(B \twoheadrightarrow B \otimes^{\mathbb{L}}_A A'', \delta)$ *this pushout. Then we get an object* $((B \twoheadrightarrow B \otimes^{\mathbb{L}}_A A'', \delta), B \otimes^{\mathbb{L}}_A A'' \twoheadrightarrow B'')$ *in* $\mathrm{CrysCon}_{\mathrm{surj}}$*.*

One verifies the following.

Lemma 4.63. — *The functor* $\mathrm{PDEnvCon} \to \mathrm{CrysCon}_{\mathrm{surj}}$ *in Construction 4.62 admits a fully faithful right adjoint* $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{PDEnvCon}$ *given by* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto ((A \twoheadrightarrow A'', \gamma) \to (A \twoheadrightarrow R))$*.*

Thus $\mathrm{CrysCon}_{\mathrm{surj}}$ can be seen as a reflective subcategory (Definition 2.44) of PDEnvCon. Now we claim the following.

Lemma 4.64. — *The relative animated PD-envelope functor* $\mathrm{PDEnvCon} \to \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}})$ *is* $(\mathrm{PDEnvCon} \to \mathrm{CrysCon}_{\mathrm{surj}})$*-invariant (Definition 2.57).*

Proof. — For every object $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'')$ in PDEnvCon, we have a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow A'' \otimes^{\mathbb{L}}_A B, \delta)$ of animated PD-pairs. By the concrete description of the relative animated PD-envelope functor, it suffices to show that this map along with the counit maps forms a pushout diagram of animated PD-pairs. As discussed in Construction 4.62, $(B \twoheadrightarrow A'' \otimes^{\mathbb{L}}_A B, \delta)$ is the pushout $(\mathrm{id}_B : B \to B, 0) \amalg_{(\mathrm{id}_A : A \to A, 0)} (A \twoheadrightarrow A'', \gamma)$. The counit maps for $(\mathrm{id}_A, 0)$ and $(\mathrm{id}_B, 0)$ are identities (Lemma 3.45). The result then follows from Proposition 3.40, which implies that counit maps are compatible with small colimits. □

Consequently, in order to study the relative animated PD-envelope functor, it suffices to study the composite $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{PDEnvCon} \to \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}})$. By abuse of terminology, we will simply denote this functor as RelPDEnv as well and call the image (or after evaluation at $1 \in \Delta^1$) *the animated PD-envelope* of an object $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathrm{surj}}$. We remark that the functor $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{PDEnvCon}$ preserves small colimits by Proposition 3.40, and, therefore, so does the composite functor.

We note that $\mathrm{CrysCon}_{\mathrm{surj}}$ is projectively generated: let $\mathrm{CrysCon}^0_{\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathrm{surj}}$ be the full subcategory spanned by objects $((\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y,Z], \gamma), \mathbb{Z}[Y,Z] \twoheadrightarrow \mathbb{Z}[Z])$ for all finite sets X, Y, Z.

LEMMA 4.65. — *The full subcategory* $\mathrm{CrysCon}^0_{\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathrm{surj}}$ *constitutes a set of compact projective generators for* $\mathrm{CrysCon}_{\mathrm{surj}}$.

Proof. — We only sketch the proof, which is similar to that of Theorem 3.29. The key observation is that the composite of forgetful functors $\mathrm{CrysCon}^0_{\mathrm{surj}} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}^{\mathrm{an}})_{\mathrm{surj}} \to \mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}, ((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto (A \twoheadrightarrow A'' \twoheadrightarrow R)$, which preserves filtered colimits and geometric realizations by Proposition 3.40, admits a left adjoint, where $\mathrm{Fun}(\Delta^2, \mathcal{C})_{\mathrm{surj}} \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ is the full subcategory spanned by $(X \to Y \to Z) \in \mathrm{Fun}(\Delta^2, \mathcal{C})$ such that $X \to Y$ and $Y \to Z$ are surjective, for $\mathcal{C} = D(\mathbb{Z})_{\geq 0}$ and $\mathcal{C} = \mathrm{CAlg}^{\mathrm{an}}$.

The ∞-category $\mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}$ admits a set $\{\mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \twoheadrightarrow \mathbb{Z}Y \oplus \mathbb{Z}Z \twoheadrightarrow \mathbb{Z}Z \mid X, Y, Z \in \mathrm{Fin}\}$ of compact projective generators that spans the full subcategory $\mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})^0_{\mathrm{surj}}$, which follows from the fact that the left adjoint to the left derived functor $\mathcal{P}_\Sigma(\mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})^0_{\mathrm{surj}}) \to \mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})^0_{\mathrm{surj}}$ is conservative (cf.,the proof of [30, Prop 25.2.1.2]).

The result then follows from Proposition A.18. □

By Proposition A.14, the functor $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}})$ is the left derived functor of the restricted functor $\mathrm{CrysCon}^0_{\mathrm{surj}} \to \mathrm{Fun}(\Delta^1, \mathrm{Pair}^{\gamma,\mathrm{an}})$, which is concretely given as follows.

LEMMA 4.66. — *The relative animated PD-envelope of an object* $((\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y,Z], \gamma), \mathbb{Z}[Y,Z] \twoheadrightarrow \mathbb{Z}[Z]) \in \mathrm{CrysCon}^0_{\mathrm{surj}}$ *is functorially given by* $(\Gamma_{\mathbb{Z}[Z]}(X,Y) \twoheadrightarrow \mathbb{Z}[Z], \tilde{\gamma}) \in \mathrm{Pair}^{\gamma,\mathrm{an}}_{(\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y,Z], \gamma)/}$, *i.e., coincides with the classical relative PD-envelope.*

Proof. — First, by the adjointness, there exists a functorial comparison map from the relative animated PD-envelope to $(\Gamma_{\mathbb{Z}[Z]}(X,Y) \twoheadrightarrow \mathbb{Z}[Z], \tilde{\gamma})$. It suffices to show that this is an equivalence.

In this case, $((\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y,Z], \gamma), \Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Z]) \in \mathrm{PDEnvCon}$ is the base change of $((\mathrm{id}_{\mathbb{Z}[Y,Z]}, 0), \mathbb{Z}[Y,Z] \twoheadrightarrow \mathbb{Z}[Z]) \in \mathrm{PDEnvCon}$ along $(\mathrm{id}_{\mathbb{Z}[Y,Z]}, 0) \to (\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y,Z], \gamma)$. The result then follows from the base-change property of the relative adjunction, along with the simple fact that the (absolute) animated PD-envelope of $\mathbb{Z}[Y,Z] \twoheadrightarrow \mathbb{Z}[Z]$ is $(\Gamma_{\mathbb{Z}[Z]}(Y) \twoheadrightarrow \mathbb{Z}[Z], \gamma)$. □

As a generalization of Definition 3.68, we now introduce the *conjugate filtration* on the relative animated PD-envelope in $\mathrm{char}\, p$.

CONSTRUCTION 4.67. — *Let* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma}_{\mathbb{F}_p}$ *be a PD-pair and* $I \subseteq A''$ *an ideal. We recall that there is a canonical "Frobenius" map* $\varphi_{A \twoheadrightarrow A''} : A'' \to A$ *by Lemma 4.42. We suppose that* $\varphi_{A \twoheadrightarrow A''}$ *is flat*[26]. *Let* (B, J, δ)

26. This is satisfied when $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma,\mathrm{st}}$, which is the only case that we need to develop the theory. For more examples, see Remark 4.73.

denote the classical PD-envelope of $(A \twoheadrightarrow A''/I)$ *relative to* $(A \twoheadrightarrow A'', \gamma)$. *We note that* $B/J \cong A''/I$. *As in the absolute case, due to the PD-structure* (B, J, δ), *there is a canonical* $\varphi^*_{A \twoheadrightarrow A''}(A''/I)$*-algebra structure on* B, *and we consider the nonpositive filtration on* B *given by* $\mathrm{Fil}^{-n}_{\mathrm{conj,cl}}(B)$ *for* $n \in \mathbb{N}$ *being the* $\varphi^*_{A \twoheadrightarrow A''}(A''/I)$*-submodule of* B *generated by* $\{\gamma_{i_1 p}(f_1) \cdots \gamma_{i_m p}(f_m) \mid i_1 + \cdots + i_m \leq n \text{ and } f_1, \ldots, f_m \in I\}$.

We have the following relative version of Lemma 3.67, for which the proof of [9, Lem 3.42] adapts.

LEMMA 4.68. — *Let* $(A \twoheadrightarrow A'', \gamma)$ *be a PD-*$\mathbb{F}_p$*-pair such that* $\varphi_{A \twoheadrightarrow A''}$ *is flat and let* $I \subseteq A''$ *be an ideal such that* I/I^2 *is a flat* A''/I*-module. The relative PD-envelope* (B, J, δ) *and the filtration* $\mathrm{Fil}^*_{\mathrm{conj,cl}} B$ *are as in Construction 4.67.*

Then there is a comparison map $\varphi^*_{A \twoheadrightarrow A''}(\Gamma^i_{A''/I}(I/I^2)) \to \mathrm{gr}^{-i} B$ *of* $\varphi^*_{A \twoheadrightarrow A''}(A''/I)$*-modules induced by the maps* $(\gamma_{ip})_{i \in \mathbb{N}}$ *(as in Lemma 3.67), which is functorial in* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow A''/I)$ *in a subcategory of* $\mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$. *Furthermore, if* $I \subseteq A''$ *is generated by a Koszul-regular sequence*[27]*, and then this comparison map is an isomorphism.*

DEFINITION 4.69. — The *conjugate filtration functor (on the animated PD-envelope)* $\mathrm{Fil}_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}_{(-)}(-) : \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ together with the *structure map* of functors $\mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ from $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto \varphi^*_{A \twoheadrightarrow A''}(R) = R \otimes^{\mathbb{L}}_{A'', \varphi_{A \twoheadrightarrow A''}} A$ to $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto \mathrm{Fil}_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma)}(R)$ is defined to be the left derived functor (Proposition A.14) of $\mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}} \ni ((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow A''/I) \mapsto (\varphi^*_{A \twoheadrightarrow A''}(A''/I) \to \mathrm{Fil}^*_{\mathrm{conj,cl}} B) \in \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ as in Construction 4.67.

As in the absolute case (including Remark 3.72), from Lemma 4.68, we have the following.

COROLLARY 4.70. — *For every* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$, *there exists an equivalence*

$$\varphi^*_{A \twoheadrightarrow A''}(\Gamma^i_R(\mathrm{gr}^1_{\mathrm{ad}}(A'' \twoheadrightarrow R))) \to \mathrm{gr}^{-i}_{\mathrm{conj}} \mathrm{Env}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma)}(R)$$

in $D(\varphi^*_{A \twoheadrightarrow A''}(R))_{\geq 0}$ *for all* $i \in \mathbb{N}$, *which is functorial in* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$.

As in the absolute case, we have the following.

COROLLARY 4.71. — *For every* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$ *such that* $A'' \twoheadrightarrow R$ *is a quasiregular animated pair, let* $(B \twoheadrightarrow R, \delta)$ *denote the relative animated PD-envelope. Then* B *is a flat* $\varphi^*_{A \twoheadrightarrow A''}(R)$*-module.*

27. We only need the simple case that $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow A''/I) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{conj}}$, which "simplifies" the proof in the sense that a "brute-force" computation suffices.

Similarly to Proposition 3.83, we have the following.

PROPOSITION 4.72. — *Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{Pair}^{\gamma}$ be a PD-pair and $I \subseteq A''$ an ideal generated by a Koszul-regular sequence. Let $(B \twoheadrightarrow B'', \delta)$ denote the relative animated PD-envelope of $((A \twoheadrightarrow A'', \gamma), A'' \to A''/I) \in \mathrm{CrysCon}$. Then $(B \twoheadrightarrow B'', \delta)$ is a PD-pair and, therefore, coincides with the classical relative PD-envelope.*

REMARK 4.73. — More precisely, in Corollary 3.77, the map $\varphi^*_{A \twoheadrightarrow A''}(R) \to B$ is induced by the Frobenius map $\varphi_{B \twoheadrightarrow R} : R \to B$ (which can be seen by left deriving the special case that $((A \twoheadrightarrow A'', \gamma), A'' \to R) \in \mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}}$). In particular, if the Frobenius map $\varphi_{A \twoheadrightarrow A''} : A'' \to A$ is flat, then so is the Frobenius map $\varphi_{B \twoheadrightarrow R} : R \to B$.

For example, when R is a quasiregular, semiperfect $\mathbb{F}_p$-algebra [11, Def 8.8], we set $(A \twoheadrightarrow A'', \gamma) = (\mathrm{id}_{R^\flat} : R^\flat \to R^\flat, 0)$ and the map $A'' \to R$ to be the canonical map; by definition, $R^\flat$ is a perfect $\mathbb{F}_p$-algebra, and, therefore, $\varphi_{R^\flat}$ is flat. Then the animated PD-envelope $B \twoheadrightarrow R$ of $A'' \to R$ satisfies the condition that the Frobenius map $\varphi_{B \to R} : R \to B$ is flat, and hence B is static. It follows that $(R \twoheadrightarrow B, \delta)$ is a PD-pair (Proposition 3.38).

Note that the associated graded pieces of derived crystalline cohomology and relative animated PD-envelope of a "surjective" crystalline context $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$, with respect to conjugate filtrations, are equivalent by Corollaries 3.61 and 4.70 and Proposition 4.54. In fact, we have the following.

LEMMA 4.74. — *There is a canonical equivalence*

$$\mathrm{Fil}_{\mathrm{conj}}\, \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma)} \to \mathrm{Fil}_{\mathrm{conj}}\, \mathrm{Env}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma)}(R)$$

in $\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\varphi^*_{A \twoheadrightarrow A''}(R)))$*, which is functorial*[28] *in* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$.

Proof. — We first point out how to produce the comparison map of underlying $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, i.e., ignoring the $\varphi^*_{A \twoheadrightarrow A''}(R)$-algebra structures and conjugate filtrations. This is logically not necessary, but it benefits our understanding. Given $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R)$, let $(B \twoheadrightarrow R, \delta)$ denote its relative animated PD-envelope. It follows from Proposition 4.20 that the crystalline cohomology $\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma)}$ is naturally equivalent to the derived de Rham cohomology $\mathrm{dR}_{(B \twoheadrightarrow R, \delta)/(A \twoheadrightarrow A'', \gamma)}$, and by definition, it is equipped with a map $\mathrm{dR}_{(B \twoheadrightarrow R, \delta)/(A \twoheadrightarrow A'', \gamma)} \to B$ of $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, which gives rise to the underlying comparison map that we want.

By Lemma 4.65 and Proposition A.14, it suffices to construct the equivalence restricted to the full subcategory $\mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$, i.e.,

28. Here, we apply the same convention as in Remark 3.72.

to establish the equivalence for all $((\Gamma_{\mathbb{F}_p[Y,Z]}(X) \twoheadrightarrow \mathbb{F}_p[Y,Z], \gamma_0), \mathbb{F}_p[Y,Z] \twoheadrightarrow \mathbb{F}_p[Z]) \in \mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}}$. This is essentially [9, Lem 3.29 & Thm 3.27]. We will briefly sketch the argument. The preceding paragraph already established a comparison map of underlying $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras. The key point is that both sides are static: the relative animated PD-envelope is static by definition, and the derived crystalline cohomology is static by Cartier isomorphism (Proposition 4.54) and the fact that static modules are closed under extension and filtered colimits; see Corollary 3.79 for a similar argument. Then the result follows from explicit simplicial resolution. □

We now deduce the integral version of Lemma 4.74. We recall that $\mathrm{Fil}_{\mathrm{PD}} : \mathrm{Pair}^{\gamma,\mathrm{an}} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ is the PD-filtration functor (Definition 3.85), and Fil_H is the Hodge-filtration.

Proposition 4.75. — *There is a canonical equivalence*

$$\mathrm{Fil}_H \mathrm{CrysCoh} \to \mathrm{Fil}_{\mathrm{PD}} \circ \mathrm{Env}^{\gamma,\mathrm{an}}_{(-)}(-)$$

of functors $\mathrm{CrysCon}_{\mathrm{surj}} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.

Proof. — The comparison map is established in the same way as in the proof of Lemma 4.74. It suffices to show that this is an equivalence.

We first show that this becomes an equivalence after passing to underlying $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, i.e., ignoring the Hodge filtration. By conservativity of the forgetful functor $\mathrm{CAlg}_{\mathbb{Z}} \to D(\mathbb{Z})$, it suffices to show the equivalence for underlying $\mathbb{Z}$-module spectra. Note that it is an equivalence after $(-) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p$ and rationalization by Lemmas 4.13, 4.60 and 4.74, and thus it is itself an equivalence.

To establish the equivalence of filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, it remains to show that the comparison map induces equivalences after passing to associated graded pieces, and by Lemma 2.1, it suffices to prove the result restricted to the full subcategory $\mathrm{CrysCon}^0_{\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathrm{surj}}$, which is essentially due to [24, Cor VIII.2.2.8], see [9, Rem 3.33]. □

4.4. Affine crystalline site. — We now turn to the site-theoretic aspects of the derived crystalline cohomology by showing that the derived crystalline cohomology is equivalent to the cohomology of the *affine crystalline site* under a mild smoothness condition. We warn the reader again that our theory is noncompleted. Fix a crystalline context $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$.

Definition 4.76. — The *affine crystalline site* $\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))$ is defined to be the opposite ∞-category of animated PD-pairs $(B \twoheadrightarrow B'', \gamma_B)$ under $(A \twoheadrightarrow A'', \gamma_A)$ along with an equivalence $R \xrightarrow{\simeq} B''$ of A-algebras, depicted by

the diagram

$$\begin{array}{ccccc} A & & \longrightarrow & & B \\ \downarrow & & & & \downarrow \\ A'' & \longrightarrow & R & \xrightarrow{\simeq} & B'' \end{array}$$

which we will simply denoted by $\left(R \xrightarrow{\simeq} B'' \twoheadleftarrow B\right) \in \mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))$. More formally, it is the homotopy fiber of the functor $\mathrm{Pair}^{\gamma,\mathrm{an}}_{(A\twoheadrightarrow A'')/} \to \mathrm{CAlg}^{\mathrm{an}}_{A''}, (B \twoheadrightarrow B'', \gamma_B) \mapsto B''$ at the object $R \in \mathrm{CAlg}^{\mathrm{an}}_{A''}$. The endowed Grothendieck topology is indiscrete.

The *structure presheaf* $\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))^{\mathrm{op}} \to \mathrm{CAlg}_A$, denoted by $\mathcal{O}_{\mathrm{CRIS}(R/(A\twoheadrightarrow A'', \gamma_A))}$ (or simply $\mathcal{O}$ when there is no ambiguity), is induced by the evaluation $\mathrm{Pair}^{\gamma,\mathrm{an}}_{(A\twoheadrightarrow A'', \gamma_A)/} \to \mathrm{CAlg}^{\mathrm{an}}_A \to \mathrm{CAlg}(D(A))$. Concretely, it is given by $\left(R \xrightarrow{\simeq} B'' \twoheadleftarrow B\right) \mapsto B$.

Although the affine crystalline site is not small, the cohomology of the structure presheaf exists in CAlg_A by Čech–Alexander calculation (which we will reproduce in Proposition 4.81). We will simply call it the *cohomology of the crystalline site* and denote by $R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A)), \mathcal{O})$. Furthermore, the structure sheaf admits the PD-filtration (Definition 3.85), which gives rise to a filtration on the cohomology of the crystalline site, called the *Hodge-filtration* and denoted by Fil_H. We now have a comparison between the derived crystalline cohomology and the cohomology of the crystalline site, which becomes an equivalence after Hodge-completion.

Proposition 4.77. — *There is a natural comparison map*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A)), \mathcal{O})$$

in the ∞*-category* $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$. *After passing to the associated graded pieces, i.e., composition with the functor* $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A)) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(A))$, *this comparison map becomes an equivalence. Moreover, when* $\pi_0(R)$ *is a finitely generated* $\pi_0(A'')$*-algebra, then the comparison map is an equivalence.*

We need some preparation about cosimplicial objects in ∞-categories.

Definition 4.78 ([28, Def 6.1.2.2]). — Let $\mathcal{C}$ be an ∞-category. A *cosimplicial object* of $\mathcal{C}$ is a functor $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$. The value of this functor at $[\nu] \in \mathbf{\Delta}$ is denoted by X^ν. A *map of cosimplicial objects* $X^\bullet \to Y^\bullet$ is simply a map of functors.

We note that there are two inclusions $\{0\} \hookrightarrow [1] \hookleftarrow \{1\}$ viewed as two maps $[0] \rightrightarrows [1]$, and a constant map $[1] \to [0]$, which induce three functors i_0, i_1: $\mathbf{\Delta} \simeq \mathbf{\Delta}_{/[0]} \rightrightarrows \mathbf{\Delta}_{/[1]}$ and $\delta : \mathbf{\Delta}_{/[1]} \to \mathbf{\Delta}_{/[0]} \simeq \mathbf{\Delta}$. For any ∞-category $\mathcal{C}$, let

i_0^* (resp. i_1^*) denote the induced functor $\mathrm{Fun}(\mathbf{\Delta}_{/[1]}, \mathcal{C}) \to \mathrm{Fun}(\mathbf{\Delta}, \mathcal{C})$ and let δ^* denote the induced functor $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{C}) \to \mathrm{Fun}(\mathbf{\Delta}_{/[1]}, \mathcal{C})$.

DEFINITION 4.79 ([30, Def 7.2.1.6]). — Let $\mathcal{C}$ be an ∞-category and let f and g be two maps $X^\bullet \rightrightarrows Y^\bullet$ of cosimplicial objects. A *simplicial homotopy from f to g* is a map $h : \delta^*(X^\bullet) \to \delta^*(Y^\bullet)$ of functors $\mathbf{\Delta}_{/[1]} \rightrightarrows \mathcal{C}$ such that the map $i_0^*(h) : X^\bullet \to Y^\bullet$ (resp. $i_1^*(h) : X^\bullet \to Y^\bullet$), being a map of cosimplicial objects, is equivalent to f (resp. g). When $X^\bullet = Y^\bullet$, we say that the simplicial homotopy $h : \delta^*(X^\bullet) \to \delta^*(X^\bullet)$ is *constant* if it is equivalent to $\mathrm{id}_{\delta^*(X^\bullet)}$.

LEMMA 4.80. — *Let $\mathcal{C}$ be an ∞-category and $X^\bullet, Y^\bullet$ two cosimplicial objects of which the totalization exists in $\mathcal{C}$. Let f and g be two maps $X^\bullet \rightrightarrows Y^\bullet$ of cosimplicial objects such that there exists a simplicial homotopy from f to g. Then the maps f, g induce equivalent*[29] *maps $\lim_{\mathbf{\Delta}} X^\bullet \rightrightarrows \lim_{\mathbf{\Delta}} Y^\bullet$ of totalizations.*

Denis NARDIN. — For every cosimplicial object $X^\bullet$ in $\mathcal{C}$, there are two observations.

1. The canonical map $\lim_{\mathbf{\Delta}_{/[1]}} \delta^*(X^\bullet) \to \lim_{\mathbf{\Delta}} X^\bullet$ is an equivalence (this involves the existence of the limit as the source). Indeed, it suffices to show that the map $\delta : \mathbf{\Delta}_{/[1]} \to \mathbf{\Delta}$ is coinitial. By Joyal's version of Quillen's theorem A [28, Thm 4.1.3.1], it suffices to show that, for every $[n] \in \mathbf{\Delta}$, the category $\mathbf{\Delta}_{/[1]} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]}$ is weakly contractible. Its geometric realization is $\Delta^1 \times \Delta^n$, which is known to be weakly contractible.
2. The two maps $\lim_{\mathbf{\Delta}} X = \lim_{\mathbf{\Delta}} i_\nu^* \delta^*(X^\bullet) \to \lim_{\mathbf{\Delta}_{/[1]}} \delta^*(X^\bullet)$ for $\nu = 0, 1$ are equivalences, and these two maps are equivalent. Indeed, both are inverses of the equivalence $\lim_{\mathbf{\Delta}_{/[1]}} \delta^*(X^\bullet) \to \lim_{\mathbf{\Delta}} X^\bullet$ above.

Note that the map $\lim_{\mathbf{\Delta}} f$ (resp. $\lim_{\mathbf{\Delta}} g$) can be identified with the composite

$$\lim_{\mathbf{\Delta}} X = \lim_{\mathbf{\Delta}} i_\nu^* \delta^*(X^\bullet) \longrightarrow \lim_{\mathbf{\Delta}_{/[1]}} \delta^*(X^\bullet) \xrightarrow{\lim_{\mathbf{\Delta}_{/[1]}}(h)} \lim_{\mathbf{\Delta}_{/[1]}} \delta^*(Y^\bullet) \longrightarrow \lim_{\mathbf{\Delta}} Y^\bullet$$

for $\nu = 0$ (resp. $\nu = 1$). The result then follows. □

Proof of Proposition 4.77. — There is a map from the constant presheaf $\underline{\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)}}$ on the affine crystalline site to the structure presheaf $\overline{\mathcal{O}}$ given by the canonical map in Definition 4.31, which induces the comparison map in question.

Now we show that this map becomes an equivalence after passing to the associated graded pieces. We first note that, when the map $A'' \to R$ is surjective, i.e.,$((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}_{\mathrm{surj}}$, the result follows directly from Proposition 4.75. Our strategy is to reduce the general case to this special case via Čech–Alexander computation.

29. Or called "homotopic". We avoid the terminology "homotopic" to avoid confusion with the simplicial homotopy.

We pick a polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras. Let $P^\bullet \to R$ denote the Čech conerve of the object $P \to R$ in the ∞-category $\mathrm{CAlg}^{\mathrm{an}}_{A/}$. Concretely, it is given by $P^\nu := P^{\otimes^{\mathbb{L}}_A(\nu+1)}$, and the map $P^\nu \to R$ is simply given by the composite map $P^\nu \to P \to R$, which is surjective. In other words, we get a cosimplicial object $(P^\bullet \twoheadrightarrow R) \in \mathrm{Fun}\left(\Delta, \mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/}\right)$. Let $(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet}) \in \mathrm{Fun}\left(\Delta, \mathrm{Pair}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma_A)/}\right)$ denote the cosimplicial relative animated PD-envelope, i.e., applying the functor $\mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A'')/} \to \mathrm{Pair}^{\gamma,\mathrm{an}}_{(A \twoheadrightarrow A'', \gamma_A)/}$ (Definition 4.57) pointwise. This effectively gives rise to a cosimplicial object $\Delta \to \mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))^{\mathrm{op}}$. Composing with the Hodge-filtered presheaf $\mathrm{Fil}_H \mathcal{O} : \mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$, we get a cosimplicial filtered $\mathbb{E}_\infty$-A-algebra $\Delta \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$, the limit of which computes the cohomology $\mathrm{Fil}_H R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A)), \mathcal{O})$. In plain terms, this cosimplicial filtered $\mathbb{E}_\infty$-A-algebra is just the PD-filtration of the cosimplicial animated PD-pair $(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})$.

For this cosimplicial object, the comparison map constructed above is concretely given by

$$\mathrm{Fil}_H \mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(A \twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_{\mathrm{PD}} D^\bullet \tag{4}$$

Now Proposition 4.75 and Lemma 4.63 gives us an equivalence

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \to \mathrm{Fil}_{\mathrm{PD}} D^\bullet$$

which is effectively given by

$$\mathrm{Fil}_H \mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \to \mathrm{Fil}_{\mathrm{PD}} D^\bullet$$

by chasing the proof. In other words, (4) can be rewritten as the natural map

$$\mathrm{Fil}_H \mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(A \twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H \mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})}$$

or equivalently, the natural map

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})}$$

It remains to show that this cosimplicial map gives rise to an equivalence after taking the limit, i.e., the totalization, and passing to associated graded pieces. We isolate the remaining part into Lemma 4.88. □

Before proving Lemma 4.88, we isolate an important observation in the previous proof into a proposition.

PROPOSITION 4.81. — *For every crystalline context* $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$*, the following are equivalent:*

1. *The comparison map in Proposition 4.77 is an equivalence.*
2. *There exists a polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras, and letting $P^\bullet \to R$ denote the Čech conerve of $P \to R$ in the ∞-category $\mathrm{CAlg}_A^{\mathrm{an}}$ as in the proof of Proposition 4.77, then the natural maps*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \tag{5}$$

   form a limit diagram in $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$*.*
3. *For all polynomial A-algebras P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras, and letting $P^\bullet \to R$ denote the Čech conerve of $P \to R$ in the ∞-category $\mathrm{CAlg}_A^{\mathrm{an}}$, then the natural maps* (5) *form a limit diagram in* $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$*.*
4. *(After proving Lemma 4.88) There exists a (or equivalently, for every) polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras, and letting $P^\bullet \to R$ denote the Čech conerve of $P \to R$ in the ∞-category $\mathrm{CAlg}_A^{\mathrm{an}}$ as in the proof of Proposition 4.77, then the natural maps*

$$\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \to \mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})}$$

   form a limit diagram in $\mathrm{CAlg}(D(A))$*.*

In order to deal with associated graded pieces of the Hodge filtration, we need a variant of the *Katz–Oda filtration* in [19, Cons 3.12]. We need an auxiliary construction:

DEFINITION 4.82. — The *cotangent complex functor* $L_{./.} : \mathrm{dRCon} \to \mathrm{Ani}(\mathrm{Mod})$ is defined to be the left derived functor (Proposition A.14) of the functor $\mathrm{dRCon}^0 \to \mathrm{Ani}(\mathrm{Mod}), ((A, I, \gamma_A) \to (B, J, \gamma_B)) \mapsto (B, \Omega^1_{(B,J)/(A,I)})$.

The proof of Lemma 4.12 leads to the following,

LEMMA 4.83. — *The composite functor* $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\mathrm{an}}) \to \mathrm{dRCon} \to \mathrm{Ani}(\mathrm{Mod})$ *is equivalent to the classical cotangent complex functor.*

We now introduce the *"stupid" filtration* Fil_B on the Hodge-filtered derived de Rham cohomology $\mathrm{Fil}_H \mathrm{dR}_{./.}$. For each $((A, I, \gamma_A) \to (B, J, \gamma_B)) \in \mathrm{dRCon}^0$, consider the filtration $(\Omega^{\geq n}_{(B,J,\gamma_B)/(A,I,\gamma_A)}, \mathrm{d})_{n \in (\mathbb{N}, \geq)}$ of the Hodge-filtered CDGA, which gives rise to a bifiltered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra. By Proposition A.14, we get a functor $\mathrm{CAlg}(\mathrm{Fun}((\mathbb{N}, \geq) \times (\mathbb{N}, \geq), D(\mathbb{Z}))), ((A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)) \mapsto \mathrm{Fil}_B \mathrm{Fil}_H \mathrm{dR}_{(B \to B'')/(A \to A'')}$.

WARNING 4.84. — *Unlike the Hodge filtration, the "stupid" filtration does not descend to* CrysCon*, that is to say, it depends on the choice of B in question.*

We now analyze the associated graded pieces with respect to the "stupid" filtration.

LEMMA 4.85. — *Let* $((A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)) \in \mathrm{dRCon}$ *be a de Rham context. Then associated graded pieces* $\mathrm{gr}^i_B \mathrm{Fil}_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$ *can be functorially identified with* $\mathrm{ins}^i\left(\bigwedge^i_B L_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}[-i]\right) \otimes^{\mathbb{L}}_B \mathrm{Fil}_{\mathrm{PD}} B$ *as a* $\mathrm{Fil}_{\mathrm{PD}} B$*-module in* $\mathrm{DF}^{\geq 0}(B)$ *(where* $\cdot \otimes^{\mathbb{L}}_B \mathrm{Fil}_{\mathrm{PD}} B$ *is the base change from* $\mathrm{DF}^{\geq 0}(B)$ *to the* ∞*-category of* $\mathrm{Fil}_{\mathrm{PD}} B$*-modules). Furthermore,* $\mathrm{Fil}^i_B \mathrm{gr}^j_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')} \simeq 0$ *when* $i > j$.

Proof. — By Proposition A.14, it suffices to check the equivalences on dRCon^0, which follow from definitions. □

We are now ready to introduce the Katz–Oda filtration.

DEFINITION 4.86 (cf. [19, Cons 3.12]). — Let $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ be a map of animated PD-pairs and $B'' \to R$ a map of animated rings. The *Katz–Oda filtration* on the Hodge-filtered derived crystalline cohomology $\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ rewritten as

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}} \mathrm{Fil}_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$$

is the nonnegative filtration induced by the "stupid" filtration on $\mathrm{Fil}_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$.

We now have

LEMMA 4.87 (cf. [19, Lem 3.13]). — *Let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs and* $B'' \to R$ *a map of animated rings. Then we have the following.*

1. *The associated graded pieces* $\mathrm{gr}^i_{\mathrm{KO}} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ *are functorially equivalent to*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_{\mathrm{PD}} B} \left(\mathrm{ins}^i\left(\bigwedge^i_B L_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}[-i]\right) \otimes^{\mathbb{L}}_B \mathrm{Fil}_{\mathrm{PD}} B\right)$$

   as $\mathrm{Fil}_{\mathrm{PD}} B$*-modules in* $\mathrm{DF}^{\geq 0}(\mathbb{Z})$ *for all* $i \in \mathbb{N}$*, where the functor* ins^i *is defined in Section 2.4.*
2. *The induced Katz–Oda filtration on* $\mathrm{gr}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ *is complete. In fact, for* $i > j$*, we have*

$$\mathrm{Fil}^i_{\mathrm{KO}} \mathrm{gr}^j_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \simeq 0.$$

Proof. — We have seen (Corollary 4.40) that the canonical map

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_H \mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}} \mathrm{Fil}_{\mathrm{PD}} B \to \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')}$$

is an equivalence. Then both follow from Lemma 4.85. □

The convergence of Katz–Oda filtration on associated graded pieces is the key to Lemma 4.88.

LEMMA 4.88. — *In Proposition 4.81, the maps* (5) *form a limit diagram after passing to the associated graded pieces, i.e., after passing along the functor* $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A)) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(A))$. *Furthermore, if the* $\pi_0(A'')$*-algebra* $\pi_0(R)$ *is of finite type, then the maps* (5) *form a limit diagram.*

Proof. — For every $[\nu] \in \mathbf{\Delta}$, let $\mathrm{Fil}_{\mathrm{KO},\nu} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}$ denote the Katz–Oda filtration with respect to the setup $((A \twoheadrightarrow A'', \gamma_A) \to (P^\nu \to P^\nu \otimes^{\mathbb{L}}_A A'', \gamma_{P^\nu}), P^\nu \to R)$. This construction is canonically functorial in $[\nu] \in \mathbf{\Delta}$. Note that the map (5) is the canonical map

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')} = \mathrm{Fil}^0_{\mathrm{KO},\nu} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')} \longrightarrow \mathrm{gr}^0_{\mathrm{KO},\nu} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}. \tag{6}$$

Now we show that (6) becomes an equivalence after replacing Fil_H by gr_H and taking limit over $[\nu] \in \mathbf{\Delta}$. That is, by the completeness in Lemma 4.87, for every $i \in \mathbb{N}_{>0}$, the totalization $\lim_{\nu \in \mathbf{\Delta}} \mathrm{gr}^i_{\mathrm{KO},\nu} \mathrm{gr}_H \mathrm{CrysCoh}_{R/(A \to A'')}$ is contractible. We show the slightly stronger statement that, for every $i \in \mathbb{N}_{>0}$, the totalization

$$\lim_{\nu \in \mathbf{\Delta}} \mathrm{gr}^i_{\mathrm{KO},\nu} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')} \tag{7}$$

is contractible.

The key observation is that $\mathrm{ins}^i \left(\bigwedge^i L_{(P^\bullet \to P^\bullet \otimes^{\mathbb{L}}_A A'')/(A \twoheadrightarrow A'')}[-i] \right)$ is homotopy equivalent to 0 as a cosimplicial $B^\bullet$-module spectrum by [8, Lem 2.6] when $i > 0$ (this is, of course, false when $i = 0$). It follows that the cosimplicial object $\mathrm{gr}^{i,(\bullet)}_{\mathrm{KO}} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}$ is homotopy equivalent to 0 as $\mathrm{Fil}_{\mathrm{PD}} B$-modules by [39, Tag 07KQ] and Lemma 4.87.

Finally, when $\pi_0(R)$ is a finitely generated $\pi_0(A'')$-algebra, we pick a polynomial A-algebra P of finitely many variables along with a surjection $P \twoheadrightarrow R$. For every $[\nu] \in \mathbf{\Delta}$, since the animated A-algebra P is polynomial of finite type, so is the animated A-algebra P^ν, and thus the Katz–Oda filtration is finite, i.e., $\mathrm{Fil}_{\mathrm{KO},\nu} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}$ is finite in the $\mathrm{Fil}_{\mathrm{KO},\nu}$-direction, and in particular, it is a complete filtration. Since completely filtered objects are stable under small limits, it follows that the object

$$\lim_{[\nu] \in \mathbf{\Delta}} \mathrm{Fil}_{\mathrm{KO},\nu} \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')}$$

is completely filtered in the "$\lim_{[\nu] \in \mathbf{\Delta}} \mathrm{Fil}_{\mathrm{KO},\nu}$"-direction. Now for every $i \in \mathbb{N}_{>0}$, since the totalization (7) is contractible, the functorial map (6) becomes an equivalence after taking $\lim_{[\nu] \in \mathbf{\Delta}}$, and the result follows. □

WARNING 4.89. — *One should be careful about homotopy equivalences. In an earlier draft of this article, we came up with the following "proof": the Hodge-filtered derived de Rham cohomology* $\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ *can be rewritten as*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_H \mathrm{dR}_{(P^\bullet\to P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}} \mathrm{Fil}_H \mathrm{dR}_{(P^\bullet\to P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}$$

and since the map $A \to P^\bullet$ *is a homotopy equivalence as* A*-algebras, the map* $\mathrm{Fil}_H \mathrm{dR}_{(P^\bullet\to P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')} \to \mathrm{Fil}_{\mathrm{PD}} P^\bullet$ *is also a homotopy equivalence, and, "therefore", the constant cosimplicial algebra* $\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ *is homotopy equivalent to*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_H \mathrm{dR}_{(P^\bullet\to P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}} \mathrm{Fil}_{\mathrm{PD}} P^\bullet \simeq \mathrm{Fil}_H \mathrm{CrysCoh}_{R/(P^\bullet\to P^\bullet\otimes^{\mathbb{L}}_A A'')}$$

and, therefore, the conditions in Proposition 4.81.

This argument is incorrect: when playing with homotopy equivalences, one cannot replace the base cosimplicial algebra by a homotopy equivalent algebra without justification. In fact, the last homotopy equivalence obtained above is also incorrect: if it were *the case, we consider the special case that* $(A \twoheadrightarrow A'', \gamma_A)$ *is given by* $(\mathrm{id}_A : A \to A, 0)$*, and* $\mathrm{CrysCoh}_{R/P^\bullet}$ *is just the animated PD-envelope of* $P^\bullet \twoheadrightarrow R$ *(see the proof of Proposition 4.77). We inspect the homotopy equivalence of cosimplicial objects that we assumed:*

$$\mathrm{dR}_{R/A} \simeq^{\mathrm{HoEq}} \mathrm{dR}_{R/P^\bullet}$$

when A *is a static* $\mathbb{F}_p$*-algebra and* R *is a smooth* A*-algebra such that* $\mathrm{dR}_{R/A}$ *is not static, the map* $P^\bullet \twoheadrightarrow R$ *is Koszul regular and the derived de Rham cohomology* $\mathrm{dR}_{R/P^\bullet}$ *is simply the PD-envelope and, therefore, static. Applying* π_i *to the homotopy equivalence, where* $i \neq 0$ *is so chosen that* $\pi_i(\mathrm{dR}_{R/A}) \neq 0$*, we get a contradiction.*

In view of this warning, our proof of Lemma 4.88 tells us that the associated graded pieces with respect to the Katz–Oda filtration are homotopy equivalent, but the homotopy equivalences cannot be glued, even after forgetting all the richer structures to the underlying ∞*-category* $D(\mathbb{Z})$*.*

When the $\pi_0(A)$-algebra $\pi_0(R)$ is not of finite type, we can still prove that the comparison map is an equivalence with mild smoothness of $A'' \to R$ (Proposition 4.98). We start with another sufficient condition in characteristic p, which is essentially a variant of [26, Prop 2.17] by Proposition 4.81.

LEMMA 4.90. — *Let* $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}_{\mathbb{F}_p}$. *Suppose the following.*

1. *The cotangent complex* $L_{R/A''} \in D_{\geq 0}(R)$ *has* Tor*-amplitude in* $[0,1]$.
2. *The derived Frobenius twist* $\varphi^*_{A \twoheadrightarrow A''}(R)$ *(see Lemma 4.42) is bounded above, i.e.,* $\pi_i(\varphi^*_{A \twoheadrightarrow A''}(R)) \cong 0$ *for* $i \gg 0$.

Then the comparison map in Proposition 4.77 is an equivalence.

Proof. — Our proof is also adapted from [26, Prop 2.17]. By Proposition 4.81 and Lemma 4.88, it suffices to say that the natural maps

$$\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \longrightarrow \mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \tag{8}$$

form a limit diagram in $\mathrm{CAlg}(D(A))$. We endow both sides with conjugate filtration (Definition 4.49) and show that this, in fact, forms a limit diagram in $\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(A))$.

We show that, after passing to associated graded pieces with respect to the conjugate filtration, the maps (8) form a limit diagram, which implies that the natural maps (8) form limit diagrams after passing to finite level of quotients, and then we control the convergence to deduce the result. To show the result for associated graded pieces, by Proposition 4.54, it suffices to show that the maps

$$\varphi^*_{A \twoheadrightarrow A''} \left(\bigwedge_R^\star L_{R/A''} \right) [-\star] \longrightarrow \varphi^*_{P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A''} \left(\bigwedge_R^\star L_{R/(P^\bullet \otimes^{\mathbb{L}}_A A'')} \right) [-\star] \tag{9}$$

form a limit diagram in $\mathrm{Gr}^{\geq 0}(D(A))$.

Let $R_1 := \varphi^*_{A \twoheadrightarrow A''}(R)$. Note that the Frobenius map $\varphi_{P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A''}$ factors as $P^\bullet \otimes^{\mathbb{L}}_A A'' \to \varphi^*_A(P^\bullet) \to P^\bullet$ where the second map is the Frobenius map of $P^\bullet$ relative to A. Then the maps (9) can be rewritten as the maps

$$\bigwedge_{R_1}^\star L_{R_1/A} [-\star] \longrightarrow \left(\bigwedge_{R_1}^\star L_{R_1/\varphi^*_A(P^\bullet)} \right) [-\star] \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet$$

or equivalently, the maps

$$\mathrm{gr}^\star_H \, \mathrm{dR}_{R_1/A} \longrightarrow \mathrm{gr}^\star_H \, \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet$$

by an inverse application of Lemma 4.85 (recall that for derived de Rham cohomology of animated rings, the "stupid" filtration coincides with the Hodge filtration). We again consider the Katz–Oda filtration associated to the cosimplicial system $A \to \varphi^*_A(P^\bullet) \to R_1$ (Lemma 4.87) and by completeness, we can pass to associated graded pieces for $i = 0$:

$$\mathrm{gr}^\star_H \, \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \longrightarrow \mathrm{gr}^\star_H \, \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet \tag{10}$$

and $i \in \mathbb{N}_{>0}$:

$$\mathrm{gr}^\star_H \, \mathrm{dR}_{R_1/\varphi_A^*(P^\bullet)} \otimes^{\mathbb{L}}_{\varphi_A^*(P^\bullet)} \left(\bigwedge^{i}_{\varphi_A^*(P^\bullet)} L_{\varphi_A^*(P^\bullet)/A}[-i] \right) \longrightarrow 0$$

As in Lemma 4.88, the latter maps constitute a homotopy equivalence by [8, Lem 2.6] and [39, Tag 07KQ] and, therefore, constitute a limit diagram by Lemma 4.80. On the other hand, by Lemma 4.93, the maps (10) constitute a limit diagram.

Now we control the convergence. Again by Lemma 4.85, we rewrite the maps (10) as the maps

$$\bigwedge^{\star}_{R_1} L_{R_1/\varphi_A^*(P^\bullet)}\,[-\star] \longrightarrow \left(\bigwedge^{\star}_{R_1} L_{R_1/\varphi_A^*(P^\bullet)}\,[-\star] \right) \otimes^{\mathbb{L}}_{\varphi_A^*(P^\bullet)} P^\bullet$$

Now consider the transitivity sequence

$$L_{\varphi_A^*(P^\bullet)/A} \otimes^{\mathbb{L}}_{\varphi_A^*(P^\bullet)} R_1 \longrightarrow L_{R_1/A} \longrightarrow L_{R_1/\varphi_A^*(P^\bullet)}$$

For every static R_1-module M, we get the fiber sequence

$$L_{\varphi_A^*(P^\bullet)/A} \otimes^{\mathbb{L}}_{\varphi_A^*(P^\bullet)} M \longrightarrow L_{R_1/A} \otimes^{\mathbb{L}}_{R_1} M \longrightarrow L_{R_1/\varphi_A^*(P^\bullet)} \otimes^{\mathbb{L}}_{R_1} M$$

Since $L_{R/A''} \in D_{\geq 0}(R)$ has Tor-amplitude in $[0,1]$, so does $L_{R_1/A} \in D_{\geq 0}(R_1)$, and, therefore, $\pi_j(L_{R_1/A} \otimes^{\mathbb{L}}_{R_1} M) \cong 0$ for $j \neq 0, 1$. Note that $L_{\varphi_A^*(P^\bullet)/A}$ is a flat $\varphi_A^*(P^\bullet)$-module. It follows that $\pi_j(L_{R_1/\varphi_A^*(P^\bullet)} \otimes^{\mathbb{L}}_{R_1} M) \cong 0$ for $j \neq 0, 1$. Furthermore, since $\varphi_A^*(P^\bullet) \to R_1$ is surjective, $\pi_0(L_{R_1/\varphi_A^*(P^\bullet)} \otimes^{\mathbb{L}}_{R_1} M) \cong 0$. It follows that $L_{R_1/\varphi_A^*(P^\bullet)}[-1]$ is a flat R_1-module, and so is $\bigwedge^\star_{R_1} L_{R_1/\varphi_A^*(P^\bullet)}\,[-\star] \simeq \Gamma^\star_{R_1}(L_{R_1/\varphi_A^*(P^\bullet)}[-1])$. By assumption, R_1 is bounded above, and, therefore, so is $\bigwedge^\star_{R_1} L_{R_1/\varphi_A^*(P^\bullet)}\,[-\star]$.

It remains to show that the associated graded pieces are uniformly bounded above, which implies that (8) form a limit diagram, by Lemma 4.95 and that the conjugate filtration is exhaustive (Lemma 4.50). Suppose that the homotopy groups of R_1 are concentrated in the range $[a, b]$; then the associated graded pieces of the target can be rewritten as $\Gamma^\star_{R_1}(L_{R_1/\varphi_A^*(P^\bullet)}[-1]) \otimes^{\mathbb{L}}_{\varphi_A^*(P^\bullet)} P^\bullet$, where $\Gamma^\star_{R_1}(L_{R_1/\varphi_A^*(P^\bullet)}[-1])$ is a flat R_1-module , and, therefore, the homotopy groups of it are also concentrated in the range $[a, b]$. Since the relative Frobenius $\varphi_A^*(P^\bullet) \to P^\bullet$ is flat, we get $\pi_j(\Gamma^\star_{R_1}(L_{R_1/\varphi_A^*(P^\bullet)}[-1]) \otimes^{\mathbb{L}}_{\varphi_A^*(P^\bullet)} P^\bullet) \cong \pi_j(\Gamma^\star_{R_1}(L_{R_1/\varphi_A^*(P^\bullet)}[-1])) \otimes_{\pi_0(\varphi_A^*(P^\bullet))} \pi_0(P^\bullet) \cong 0$ for $j \notin [a, b]$. □

We need the following lemmas.

LEMMA 4.91. — *Let $\mathcal{C}$ be an ∞-category that admits finite coproducts. Let $\varnothing$ denote the initial object of $\mathcal{C}$, and let X, Y be two objects of $\mathcal{C}$. Then for any two maps $g_0, g_1 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the induced maps $X^\bullet \rightrightarrows Y^\bullet$ of Čech conerves*

$X^\bullet$ of X (i.e. of $\varnothing \to X$) and $Y^\bullet$ of Y (i.e., of $\varnothing \to Y$) are homotopic. More precisely, there exists a simplicial homotopy from $g_0^\bullet$ to $g_1^\bullet$ that is functorial in g_0 and g_1. In particular, if $X = Y$ and $g_0 = g_1$, and then the simplicial homotopy is constant.

Proof. — We start with the special case that $\mathcal{C}$ is a 1-category. We define the simplicial homotopy $h : \delta^*(X^\bullet) \to \delta^*(Y^\bullet)$ as follows: for every $(\alpha_n : [n] \to [1]) \in \mathbf{\Delta}_{/[1]}$, we note that $(X^\bullet)(\alpha_n) = X^n = X \amalg \cdots \amalg X$ and $(Y^\bullet)(\alpha_n) = Y^n = Y \amalg \cdots \amalg Y$, and we set $h_{\alpha_n} = \coprod_{i=0}^n g_{\alpha_n(i)} : X^n \to Y^n$. By construction, $i_0^*(h)^\nu = h_{[\nu] \xrightarrow{0} [1]} = \coprod_{i=0}^n g_0 = g_0^\nu$ and $i_1^*(h)^\nu = g_1^\nu$.

We need to check that this is a map of functors. For every map $\psi : (\alpha_n : [n] \to [1]) \to (\alpha_m : [m] \to [1])$ in $\mathbf{\Delta}_{/[1]}$, we need to check that the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{h_{\alpha_n}} & Y^n \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} \\ X^m & \xrightarrow{h_{\alpha_m}} & Y^m \end{array}$$

commutes, where the vertical maps $\psi_* : X^n \to X^m$ and $\psi_* : Y^n \to Y^m$ are induced by ψ. To this end, let $j_i : X \to X^n$ be the i-th canonical map for $0 \leq i \leq n$.

Then the composite $h_{\alpha_n} \circ j_i : X \to X^n \to Y^n$ can be rewritten as the composite $j_i \circ g_{a_n(i)} : X \to Y \to Y^n$, and the composite $\psi_* \circ h_{\alpha_n} \circ j_i : X \to Y^m$ is equivalent to the composite $j_{\psi(i)} \circ g_{\alpha_n(i)} : X \to Y \to Y^m$. Similarly, the composite $\psi_* \circ j_i : X \to X^n \to X^m$ is equivalent to the $\psi(i)$-th canonical map $j_{\psi(i)} : X \to X^m$, and the composite $h_{\alpha_m} \circ \psi_* \circ j_i$ can be identified with the composite $j_{\psi(i)} \circ g_{\alpha_m(\psi(i))} : X \to Y \to Y^m$.

Since $\alpha_m(\psi(i)) = \alpha_n(i)$, it follows that $\psi_* \circ h_{\alpha_n} \circ j_i = h_{\alpha_m} \circ \psi_* \circ j_i$ for every $0 \leq i \leq n$. It follows that $\psi_* \circ h_{\alpha_n} = h_{\alpha_m} \circ \psi_*$. The other claims for the 1-category $\mathcal{C}$ follow directly from the construction.

Now we claim that the result for ∞-categories follows from that for 1-categories. The point is that there exists a universal 1-category[30] $\mathcal{C}_0$ along with two objects $X_0, Y_0 \in \mathcal{C}$ and two maps $X_0 \rightrightarrows Y_0$, which admits all finite products, such that for every ∞-category $\mathcal{C}$ as in the assumption of this lemma, there exists an essentially unique functor $\mathcal{C}_0 \to \mathcal{C}$ that preserves finite coproducts: let K be the diagram $\bullet \rightrightarrows \bullet$ and then take the presheaf ∞-category $\mathcal{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathrm{An})$. Then we can take $\mathcal{C}_0$ to be the full subcategory of $\mathcal{P}(K)$ spanned by finite coproducts of the two vertices of K. □

Corollary 4.92. — *Let $\mathcal{C}$ be an ∞-category with finite coproducts and two objects X, Y in $\mathcal{C}$. Let $i : X \to Y$ be a map that admits a left inverse $r : Y \to X$. Then there is a "strong deformation retract", i.e., a simplicial homotopy from*

30. We were informed about this by Denis Nardin.

$\mathrm{id}_{Y^\bullet}$ *to* $i^\bullet \circ r^\bullet$*, which restricts to a constant simplicial homotopy of* $X^\bullet$ *along* $i^\bullet : X^\bullet \to Y^\bullet$*, where* $X^\bullet$ *(resp.* $Y^\bullet$*) is the Čech conerve of* X *(resp.* Y*), and* $i^\bullet : X^\bullet \to Y^\bullet$ *and* $r^\bullet : Y^\bullet \to X^\bullet$ *are induced simplicial maps.*

Proof. — We apply Lemma 4.91 to $\mathrm{id}_Y, i \circ r \in \mathrm{Hom}_{\mathcal{C}}(Y, Y)$, getting the desired simplicial homotopy. To see the latter statement, it suffices to inspect the commutative diagram

$$\begin{array}{ccc} Y & \overset{\mathrm{id}_Y}{\underset{i\circ r}{\rightrightarrows}} & Y \\ \uparrow & & \uparrow \\ X & \overset{\mathrm{id}_X}{\underset{\mathrm{id}_X}{\rightrightarrows}} & X \end{array}$$

and invoke the functoriality. □

Lemma 4.93. — *Let* $A \in \mathrm{CAlg}^{\mathrm{cn}}$ *be a connective* $\mathbb{E}_\infty$*-ring and let* $B \to C$ *be a faithfully flat map of connective* $\mathbb{E}_\infty$*-*A*-algebras. Let* $B^\bullet$ *(resp.* $C^\bullet$*) denote the Čech conerve of the map* $A \to B$ *(resp.* $A \to C$*). Then for any cosimplicial* $B^\bullet$*-module* $N^\bullet$*, the natural cosimplicial map*

$$N^\bullet \longrightarrow N^\bullet \otimes^{\mathbb{L}}_{B^\bullet} C^\bullet$$

induces an equivalence after totalization $\lim_{\bullet \in \mathbf{\Delta}}$ *in* $D(A)$*, where the cosimplicial map* $B^\bullet \to C^\bullet$ *is induced by* $B \to C$*.*

Proof. — Let $D^{\bullet,\bullet}$ denote the cosimplicial Čech conerve of $B^\bullet \to C^\bullet$ (each $D^{\nu,\bullet}$ is the Čech conerve of $B^\nu \to C^\nu$), which is a bicosimplicial object in CAlg_A. We note that there is a unique cosimplicial map $B^\bullet \to D^{\bullet,\mu}$ for all $[\mu] \in \mathbf{\Delta}$. Consider the bicosimplicial object $M^{\bullet,\bullet}$:

$$\begin{aligned} \mathbf{\Delta}^2 &\longrightarrow D(A) \\ ([\nu], [\mu]) &\longmapsto N^\nu \otimes^{\mathbb{L}}_{B^\nu} D^{\nu,\mu} \end{aligned}$$

and its limit $I := \lim_{([\nu],[\mu])} M^{\nu,\mu}$. The map that we need to show to be an equivalence factors as $\lim_{[\nu]} N^\nu \to \lim_{([\nu],[\mu])} M^{\nu,\mu} \to \lim_{[\nu]} M^{\nu,0} \simeq \lim_{[\nu]} N^\nu \otimes^{\mathbb{L}}_{B^\nu} C^\nu$. It suffices to show that both maps are equivalences.

For the first map, in fact, for every $[\nu] \in \mathbf{\Delta}$, the map $N^\nu \to \lim_{[\mu]\in\mathbf{\Delta}} M^{\nu,\mu}$ is an equivalence by faithfully flat descent.

For the second map, since $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathbf{\Delta}^{\mathrm{op}}$ is cofinal [28, Lem 6.5.3.7], where $\mathbf{\Delta}_{\mathrm{inj}} \subseteq \mathbf{\Delta}$ is the (nonfull) subcategory with strictly increasing maps $[m] \to [n]$, we can replace $\lim_{\mathbf{\Delta}}(\cdot)$ by $\lim_{\mathbf{\Delta}_{\mathrm{inj}}}(\cdot)$. By [28, Cor 4.4.4.10], it suffices to show that, for every injective map $[\mu_1] \to [\mu_2]$ in $\mathbf{\Delta}$, the induced map $\lim_{[\nu]} M^{\nu,\mu_1} \to \lim_{[\nu]} M^{\nu,\mu_2}$ is an equivalence. Every injective map $[\mu_1] \to [\mu_2]$ admits a retract in $\mathbf{\Delta}$, and, therefore, by Corollary 4.92, the induced map $D^{\bullet,\mu_1} \to D^{\bullet,\mu_2}$ is a homotopy equivalence of cosimplicial $\mathbb{E}_\infty$-$B^\bullet$-algebras, and,

therefore, $M^{\bullet,\mu_1} \to M^{\bullet,\mu_2}$ is a homotopy equivalence of cosimplicial A-modules by [39, Tag 07KQ]. The result then follows from Lemma 4.80. □

REMARK 4.94. — When A is a static $\mathbb{F}_p$-algebra, B is a polynomial A-algebra and $C = B \otimes^{\mathbb{L}}_{A,\varphi_A} A$ is the Frobenius twist of B, we recover [12, Lem 5.4].

LEMMA 4.95. — *Let $(M_i^\bullet)_{i\in(\mathbb{Z}_{\leq 0},\geq)} \in \mathrm{Fun}(\mathbf{\Delta} \times (\mathbb{Z}_{\leq 0},\geq), \mathrm{Sp})$ be a cosimplicial filtered spectra. Suppose that it is uniformly bounded above, i.e., there exists $N \in \mathbb{N}$ such that for every $i \in \mathbb{Z}_{\leq 0}$ and $\nu \in \mathbf{\Delta}$, we have $\pi_j(M_i^\nu) \cong 0$ for all $j > N$. Let $M^\bullet := \mathrm{colim}_{i\to-\infty} M_i^\bullet$. Then the canonical map $\mathrm{colim}_{i\to-\infty} \lim_{\nu\in\mathbf{\Delta}} M_i^\nu \to \lim_{\nu\in\mathbf{\Delta}} M^\nu$ is an equivalence.*

Proof. — We can rewrite $\lim_{\nu\in\mathbf{\Delta}}$ as $\lim_{n\to\infty} \lim_{\nu\in\mathbf{\Delta}_{\leq[n]}}$. Furthermore, the functor $\mathbf{\Delta}^{\mathrm{inj}}_{\leq[n]} \to \mathbf{\Delta}_{\leq[n]}$ is right cofinal, and, therefore, we can replace $\lim_{\nu\in\mathbf{\Delta}_{\leq[n]}}$ by $\lim_{\nu\in\mathbf{\Delta}^{\mathrm{inj}}_{\leq[n]}}$, which is a finite limit, and, therefore, commutes with $\mathrm{colim}_{i\to-\infty}$. For any cosimplicial spectrum $X^\bullet$, there is a canonical map $\lim_{\nu\in\mathbf{\Delta}} X^\nu \to \lim_{\nu\in\mathbf{\Delta}_{\leq[n]}} X^\nu$. If $X^\bullet$ is assumed to be uniformly bounded above, then the coconnectivity of $\mathrm{fib}(\lim_{\nu\in\mathbf{\Delta}} X^\nu \to \lim_{\nu\in\mathbf{\Delta}_{\leq[n]}} X^\nu)$ tends to $-\infty$ as $n \to \infty$ by [30, Cor 1.2.4.18]. The result then follows. □

For the integral version, we need to extend quasisyntomicity to animated rings.

DEFINITION 4.96 (cf. [11, Def 4.9]). — We say that a map $R \to S$ of animated rings is *quasisyntomic* if it is flat, and the cotangent complex $L_{S/R}$ has Tor-amplitude in $[0, 1]$.

EXAMPLE 4.97. — Any smooth map, or more generally, any syntomic map of static rings is quasisyntomic. Quasisyntomic maps are stable under base change.

We now phrase the integral comparison.

PROPOSITION 4.98. — *Let $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$ such that A is bounded above (that is, $\pi_n(A) \cong 0$ for $n \gg 0$), and the map $A'' \to R$ is quasisyntomic. Then the comparison map in Proposition 4.77 is an equivalence.*

Proof. — We again appeal to Proposition 4.81. It suffices to show that the map

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma_A)} \longrightarrow \lim_{\nu\in\mathbf{\Delta}} \mathrm{CrysCoh}_{R/(P^\nu\twoheadrightarrow P^\nu\otimes^{\mathbb{L}}_A A'',\gamma_{P^\nu})}$$

is an equivalence of $\mathbb{Z}$-module spectra (since the forgetful functor is conservative), which can be checked by base change along $\mathbb{Z} \to \mathbb{Z}/p$ for all prime numbers $p \in \mathbb{N}_{>0}$ and along $\mathbb{Z} \to \mathbb{Q}$. The latter follows from Lemma 4.13 and that the map $A \to P$ is faithfully flat, and, therefore the canonical map $A \to \lim_{\nu\in\mathbf{\Delta}} P^\nu$ is an equivalence (in fact, this is induced by a homotopy

equivalence of cosimplicial objects, but we do not need this). For every prime number p, by base change property (Lemma 4.41) and Lemma 4.90, where the flatness of $A'' \to R$ implies the flatness of $A/^{\mathbb{L}}p \to \varphi^*_{A/^{\mathbb{L}}p \to A''/^{\mathbb{L}}p}(R/^{\mathbb{L}}p)$, and, therefore, the Frobenius twist in question is bounded above. □

Finally, we want to compare the cohomology of the affine crystalline site and the classical crystalline cohomology. We first describe a noncomplete variant of the classical affine crystalline site, which we will name after *static affine crystalline site*.

Definition 4.99. — Let $(A, I, \gamma_A) \in \mathrm{Pair}^\gamma$ be a PD-pair and let $A/I \to R$ be a map of rings. Note that $((A \twoheadrightarrow A/I, \gamma_A), A/I \to R) \in \mathrm{CrysCon}$ is a crystalline context. The *static affine crystalline site* $\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A))$ is the full subcategory of $\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A))$ spanned by those $(B \twoheadrightarrow B/J, \gamma_B) \in \mathrm{Pair}^\gamma$ along with a map $R \to B/J$, i.e., the animated PD-pair in question is given by a PD-pair, equipped with the indiscrete topology.

We note that the structure presheaf $\mathcal{O}$ on $\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A))$ restricts to a presheaf $\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A))$, still called the *structure presheaf*, which is canonically equipped with PD-filtration, of which the cohomology is called the *cohomology of the static crystalline site* (resp. *Hodge-filtered cohomology of the static crystalline site*), denoted by $R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$ (resp. $\mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$.

Warning 4.100. — *Here, the PD-filtration is that for animated PD-envelope, although we are considering PD-pairs. However, when $I = 0$, thanks to Proposition 3.90, we can consider the classical PD-envelope instead.*

Now the cohomology of the affine crystalline site coincides with the classical version.

Proposition 4.101. — *Let $(A, I, \gamma_A) \in \mathrm{Pair}^\gamma$ be a PD-pair and $A/I \to R$ a quasisyntomic map of rings (R is static by flatness). Then the comparison map*

$$\mathrm{Fil}_H R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A)), \mathcal{O}) \to \mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$$

of filtered $\mathbb{E}_\infty$-A-algebras induced by the inclusion $\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)) \hookrightarrow \mathrm{Cris}(R/(A, I, \gamma_A))$ is an equivalence.

Proof. — We adapt the Čech–Alexander computation in Proposition 4.77. We pick a polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$. Let $P^\bullet \to R$ denote the Čech conerve of the object $P \to R$ in $(\mathrm{Alg}_A)_{/R}$. Concretely, $P^\nu = P^{\otimes_A(\nu+1)}$. Note that since $A \to P$ is flat, the classical tensor product coincides with the derived tensor product, and, therefore, the cosimplicial pair $P^\bullet \to R$ coincides with the cosimplicial animated pair in the proof of Proposition 4.77, and then $\mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$ is

computed by the classical PD-envelope of $P^\bullet \twoheadrightarrow R$ with respect to (A, I, γ_A), equipped with the PD-filtration.

Let $(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})$ denote the cosimplicial animated PD-envelope of $(P^\bullet \twoheadrightarrow R)$ relative to (A, I, γ_A). It suffices to show that $(D^\nu \twoheadrightarrow R, \gamma_{D^\bullet})$ is given by a PD-pair for all $\nu \in \mathbf{\Delta}$, or equivalently, the underlying animated ring D^ν is static, by virtue of Proposition 3.38 and Lemma 3.43, which follows from Lemma 4.102 below. □

LEMMA 4.102. — *Let $(A \twoheadrightarrow A'', \gamma_A)$ be an animated PD-pair, $A'' \to R$ a quasisyntomic map of animated rings, P a polynomial A-algebra (of possibly infinitely many variables) and $P \twoheadrightarrow R$ a surjection of A-algebras. Let $(D \twoheadrightarrow R, \gamma_D)$ denote the animated PD-envelope of $P \twoheadrightarrow R$ relative to $(A \twoheadrightarrow A'', \gamma_A)$. Then D is a flat A-module.*

Proof. — This is a "quasi" variant of "flatness of PD-envelope" [12, Lem 2.42]. By Lemma 3.82, it suffices to show that $D \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$ is a flat $A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$-module, and for every prime $p \in \mathbb{N}$, $D/^{\mathbb{L}}p$ is a flat $A/^{\mathbb{L}}p$-module.

By Lemma 4.60, the map $P \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q} \to D \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$ is an equivalence. Since $A \to P$ is flat, so is the map $A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q} \to D \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$.

For every prime $p \in \mathbb{N}$, by the base change property (a relative version of Lemma 3.52, with a similar proof), $D_0 := D/^{\mathbb{L}}p$ is the animated PD-envelope of $P/^{\mathbb{L}}p \twoheadrightarrow R/^{\mathbb{L}}p$ relative to the animated PD-pair $(A \twoheadrightarrow A'', \gamma_A)$. To simplify notations, we let $P_0 := P/^{\mathbb{L}}p, R_0 := R/^{\mathbb{L}}p, A_0 := A/^{\mathbb{L}}p, A''_0 := A''/^{\mathbb{L}}p$. Since $A \to R$ is quasisyntomic, so is $A_0 \to R_0$. Consider the transitivity sequence

$$L_{P_0/A_0} \otimes^{\mathbb{L}}_{P_0} R_0 \longrightarrow L_{R_0/A_0} \longrightarrow L_{R_0/P_0}$$

For every static R_0-module M, we get a fiber sequence

$$L_{P_0/A_0} \otimes^{\mathbb{L}}_{P_0} M \longrightarrow L_{R_0/A_0} \otimes^{\mathbb{L}}_{R_0} M \longrightarrow L_{R_0/P_0} \otimes^{\mathbb{L}}_{R_0} M$$

Since P_0 is a polynomial A_0-algebra, $\mathbb{L}_{P_0/A_0}$ is a flat P_0-module. The map $A_0 \to R_0$ is quasisyntomic, and, therefore, $\pi_*(L_{R_0/A_0} \otimes^{\mathbb{L}}_{R_0} M) \cong 0$ for $* \neq 0, 1$. It follows that $\pi_*(L_{R_0/P_0} \otimes^{\mathbb{L}}_{R_0} M) \cong 0$ for $* \neq 0, 1$. Furthermore, since $P_0 \to R_0$ is surjective, $\pi_0(L_{R_0/P_0} \otimes^{\mathbb{L}}_{R_0} M) \cong 0$. It follows that $P_0 \twoheadrightarrow R_0$ is a quasiregular animated pair. By Corollary 4.71, D_0 is a flat $\varphi^*_{P_0 \twoheadrightarrow P_0 \otimes^{\mathbb{L}}_{A_0} A''_0}(R_0)$-module where $\varphi_{P_0 \twoheadrightarrow P_0 \otimes^{\mathbb{L}}_{A_0} A''_0} : P_0 \otimes^{\mathbb{L}}_{A_0} A''_0 \to P_0$ is the Frobenius map (Lemma 4.42). It remains to see that the composite map $A_0 \to P_0 \to \varphi^*_{P_0 \twoheadrightarrow P_0 \otimes^{\mathbb{L}}_{A_0} A''_0}(R_0) = R_0 \otimes^{\mathbb{L}}_{P_0 \otimes^{\mathbb{L}}_{A_0} A''_0} P_0$ is flat, where the second map is the "map into the second factor".

We note that the Frobenius $\varphi_{P_0 \twoheadrightarrow P_0 \otimes^{\mathbb{L}}_{A_0} A''_0}$ factors as $P_0 \otimes^{\mathbb{L}}_{A_0} A''_0 \to \varphi^*_{A_0}(P_0) \to P_0$, where the second map is the Frobenius of P_0 relative to A_0. Let R_1 denote

$R_0 \otimes^{\mathbb{L}}_{A_0'', \varphi_{A_0} \to A_0''} A_0$. Since $A_0'' \to R_0$ is flat, so is $A_0 \to R_1$, and we have

$$\varphi^*_{P_0 \to P_0 \otimes^{\mathbb{L}}_{A_0} A_0''}(R_0) \simeq R_1 \otimes^{\mathbb{L}}_{\varphi^*_{A_0}(P_0)} P_0$$

as a pushout of A_0-algebras. The relative Frobenius $\varphi^*_{A_0}(P_0) \to P_0$ is flat, and, therefore, the map $R_1 \to R_1 \otimes^{\mathbb{L}}_{\varphi^*_{A_0}(P_0)} P_0$. The result then follows since flatness is stable under composition. □

Remark 4.103. — If we examine the proof of Lemma 4.102 closely, we can see that, instead of being a polynomial, what we really need to impose on the map $A \to P$ is that the map is *quasismooth* (i.e., it is flat and $L_{P/A}$ is a flat P-module), and for every prime $p \in \mathbb{N}$, the Frobenius of $P/^{\mathbb{L}}p$ relative to $A/^{\mathbb{L}}p$ is flat.

5. Animated prismatic structures

We fix a prime $p \in \mathbb{N}$. In this section, we will develop the theory of animated δ-rings, that of animated δ-pairs, a noncomplete theory of prisms and prove a variant of the Hodge–Tate comparison, from which we deduce a result about "flat covers of the final object". Almost every ring that we will discuss is a $\mathbb{Z}_{(p)}$-algebra; we will simply denote $\mathrm{Pair}^{\mathrm{an}}_{\mathbb{Z}_{(p)}}$ by $\mathrm{Pair}^{\mathrm{an}}$ and $\mathrm{Pair}^{\gamma,\mathrm{an}}_{\mathbb{Z}_{(p)}}$ by $\mathrm{Pair}^{\gamma,\mathrm{an}}$.

5.1. Animated δ-rings and δ-pairs. — In this section, we will define *animated δ-rings* and *animated δ-pairs* and discuss the interaction between the δ-structure and the PD-structure. Recall the following.

Definition 5.1 ([12, Def 2.1]). — A *δ-ring* is a pair (R, δ), where R is a $\mathbb{Z}_{(p)}$-algebra and $\delta : R \to R$ is an endomorphism of the underlying set R such that

1. $\delta(x+y) = \delta(x)+\delta(y)-P(x,y)$ for all $x, y \in R$ where $P(X,Y) \in \mathbb{Z}[X,Y]$ is the polynomial
$$\frac{(X+Y)^p - X^p - Y^p}{p} := \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} X^{p-j}Y^j$$
2. $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.
3. $\delta(1) = 0$.

A *map* $f : (R, \delta) \to (S, \delta)$ *of δ-rings* is a map $f : R \to S$ of rings such that $f \circ \delta = \delta \circ g$ as maps of sets. These form the 1-*category of δ-rings*, denoted by Ring_δ.

Remark 5.2 ([12, Rem 2.2]). — Given a δ-ring (R, δ), we write $\varphi : R \to R$ for the map $x \mapsto x^p + p\delta(x)$. Then φ is a ring endomorphism of R, which lifts the Frobenius map $R/p \to R/p$, i.e. $\varphi(x) \equiv x^p \pmod p$ for every $x \in R$.

The 1-category Ring_δ admits an initial object $\mathbb{Z}_{(p)}$ [12, Ex 2.6], and more generally, all small colimits and small limits, and the forgetful functor $\mathrm{Ring}_\delta \to \mathrm{Alg}_{\mathbb{Z}_{(p)}}$ preserves them [12, Rem 2.7]. The forgetful functor $\mathrm{Ring}_\delta \to \mathrm{Set}$ admits a left adjoint $\mathrm{Set} \to \mathrm{Ring}_\delta$, which sends a set S to the free δ-ring generated by S, denoted by $\mathbb{Z}_{(p)}\{S\}$. Indeed, when $S = \{x\}$ is a singleton, it is given by the free δ-ring $\mathbb{Z}_{(p)}\{x\}$ of which the underlying $\mathbb{Z}_{(p)}$-algebra is isomorphic to the polynomial $\mathbb{Z}_{(p)}$-algebra $\mathbb{Z}_{(p)}[x, \delta(x), \delta^2(x), \ldots]$ [12, Lem 2.11], and the general case follows by taking the coproduct of S-copies of $\mathbb{Z}_{(p)}\{x\}$. From Corollary 2.3 we have the following.

LEMMA 5.3. — *The* 1*-category* Ring_δ *is* 1*-projectively generated and, therefore, presentable.*

DEFINITION 5.4. — The ∞*-category of animated* δ*-rings*, denoted by $\mathrm{CAlg}_\delta^{\mathrm{an}}$, is defined to be the animation $\mathrm{Ani}(\mathrm{Ring}_\delta)$. An *animated* δ*-ring* is formally denoted by (R, δ), where R is the image of (R, δ) under the forgetful functor $\mathrm{CAlg}_\delta^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$, or simply by R when the δ-structure is unambiguously obvious.

REMARK 5.5. — In [10, App A], an alternative description of animated δ-rings in terms of derived Frobenius lift, similar to Remark 5.2, is given.

By the adjoint functor theorem, the forgetful functor $\mathrm{Ring}_\delta \to \mathrm{Alg}_{\mathbb{Z}_{(p)}}$ admits a left adjoint. A further application of Corollary 2.3 leads to the following.

LEMMA 5.6. — *There is a pair* $\mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}} \rightleftarrows \mathrm{CAlg}_\delta^{\mathrm{an}}$ *of adjoint functors, being the animation of the pair* $\mathrm{Alg}_{\mathbb{Z}_{(p)}} \rightleftarrows \mathrm{Ring}_\delta$ *of adjoint functors. We will call the functor* $\mathrm{CAlg}_\delta^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$ *the free animated* δ*-ring functor*[31]*. The functor* $\mathrm{CAlg}_\delta^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$*, called the forgetful functor, is conservative and preserves small colimits (and as a right adjoint, it preserves small limits as well).*

Concretely, a set of compact projective generators for $\mathrm{CAlg}_\delta^{\mathrm{an}}$ is given by free δ-rings generated by a finite set, which spans a full subcategory $\mathrm{Ring}_\delta^0 \subseteq \mathrm{Ring}_\delta$. Recall that $\mathrm{Ring}_\delta \hookrightarrow \mathrm{CAlg}_\delta^{\mathrm{an}}$ is a full subcategory (Remark A.22). Now we characterize this full subcategory in terms of the underlying animated ring.

LEMMA 5.7. — *Let* $(R, \delta) \in \mathrm{CAlg}_\delta^{\mathrm{an}}$ *be an animated* δ*-ring. Then the following are equivalent.*

1. *The animated* δ*-ring* $(R, \delta) \in \mathrm{CAlg}_\delta^{\mathrm{an}}$ *is* n*-truncated.*
2. *The underlying animated ring* $R \in \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$ *is* n*-truncated.*
3. *For every* $m \in \mathbb{N}_{>n}$*, the homotopy group* $\pi_m(R)$ *vanishes.*

31. The nonanimated version was called the "δ-envelope" in [18, Def 1.1].

Proof. — The equivalence of parts 2 and 3 is [31, Prop 25.1.3.3]. On the other hand, part 1 is equivalent to saying that, for every free δ-ring F generated by a finite set, the mapping anima $\mathrm{Map}_{\mathrm{CAlg}_\delta^{\mathrm{an}}}(F, R)$ is n-truncated by [28, Rem 5.5.8.26]. Since any such F is a finite coproduct of $\mathbb{Z}_{(p)}\{x\}$, it is equivalent to $\mathrm{Map}_{\mathrm{CAlg}_\delta^{\mathrm{an}}}(\mathbb{Z}_{(p)}\{x\}, R)$ being n-truncated, which is equivalent to part 3 since

$$\mathrm{Map}_{\mathrm{CAlg}_\delta^{\mathrm{an}}}(\mathbb{Z}_{(p)}\{x\}, R) \simeq \mathrm{Map}_{\mathrm{An}}(\{x\}, R) \simeq R \qquad \square$$

Now we define the Frobenius map on animated δ-rings. We note that the identity functor $\mathrm{id} : \mathrm{CAlg}_\delta^{\mathrm{an}} \to \mathrm{CAlg}_\delta^{\mathrm{an}}$ is the animation of the identity functor $\mathrm{id} : \mathrm{Ring}_\delta \to \mathrm{Ring}_\delta$.

Definition 5.8. — The *Frobenius endomorphism* is the endomorphism of the identity functor $\mathrm{id} : \mathrm{CAlg}_\delta^{\mathrm{an}} \to \mathrm{CAlg}_\delta^{\mathrm{an}}$ defined to be the animation of the Frobenius endomorphism (described in Remark 5.2) of the identity functor $\mathrm{id} : \mathrm{Ring}_\delta \to \mathrm{Ring}_\delta$.

Recall that a *δ-pair* is the datum (A, I) of a δ-ring A along with an ideal $I \subseteq A$ [12, Def 3.2]. Similarly to animated pairs, we have an "animated version" of δ-pairs:

Definition 5.9. — The ∞-category $\mathrm{Pair}_\delta^{\mathrm{an}}$ of *animated δ-pairs* is defined to be the fiber product $\mathrm{CAlg}_\delta^{\mathrm{an}} \times_{\mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}} \mathrm{Pair}^{\mathrm{an}}$ where the functor $\mathrm{CAlg}_\delta^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$ is the forgetful functor, and the functor $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$ is the evaluation $(A \twoheadrightarrow A'') \mapsto A$. An *animated δ-pair* is an object in $\mathrm{Pair}_\delta^{\mathrm{an}}$, which we will denote by $((A, \delta), A \twoheadrightarrow A'')$ or simply by $A \twoheadrightarrow A''$ when there is no ambiguity.

From Lemma 5.6 and [28, Lem 5.4.5.5], which characterizes colimits in the fiber products, we have the following.

Lemma 5.10. — *The ∞-category $\mathrm{Pair}_\delta^{\mathrm{an}}$ is cocomplete, and the forgetful functor $\mathrm{Pair}_\delta^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ is conservative and preserves small colimits.*

Explicitly, an *animated δ-pair* is given by an animated δ-ring (A, δ) along with a surjection $A \twoheadrightarrow A''$ of animated $\mathbb{Z}_{(p)}$-algebras. Since $\mathrm{Pair} \subseteq \mathrm{Pair}^{\mathrm{an}}$ is a full subcategory (Proposition 3.23), and so is $\mathrm{Ring}_\delta \subseteq \mathrm{CAlg}_\delta^{\mathrm{an}}$ (Remark A.22), the 1-category of δ-pairs is a full subcategory of the ∞-category of animated δ-pairs. Similarly to the ∞-category of animated pairs, we have the following.

Lemma 5.11. — *The forgetful functor $\mathrm{Pair}_\delta^{\mathrm{an}} \to \mathrm{Pair}^{\mathrm{an}}$ admits a left adjoint, and the ∞-category $\mathrm{Pair}_\delta^{\mathrm{an}}$ is projectively generated.*

Proof. — The left adjoint $\mathrm{Pair}^{\mathrm{an}} \to \mathrm{Pair}_\delta^{\mathrm{an}}$ concretely given by $(A \twoheadrightarrow A'') \mapsto ((A^\delta, \delta), (A^\delta \twoheadrightarrow A'' \otimes_A^{\mathbb{L}} A^\delta))$, where A^δ is the image of $A \in \mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}}$ under the free animated δ-ring functor $\mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{an}} \to \mathrm{CAlg}_\delta^{\mathrm{an}}$. Now the result follows from Corollary 2.3 and Lemmas 5.3 and 5.6. $\square$

Concretely, a set of compact projective generators for $\mathrm{Pair}_\delta^{\mathrm{an}}$ is given by the set $\{(\mathbb{Z}_{(p)}\{X,Y\},(Y)) \mid X,Y \in \mathrm{Fin}\}$ of *standard δ-pairs*, which spans a full subcategory $\mathrm{Pair}_\delta^{\mathrm{st}} \subseteq \mathrm{Pair}_\delta^{\mathrm{an}}$. Now we turn to the PD-structure. Recall that

LEMMA 5.12 ([12, Lem 2.11]). — *The Frobenius endomorphism $\varphi_{\mathbb{Z}_{(p)}\{x\}} : \mathbb{Z}_{(p)}\{x\} \to \mathbb{Z}_{(p)}\{x\}$ on the free δ-ring $\mathbb{Z}_{(p)}\{x\}$, which is, in fact, induced by $x \mapsto \varphi(x) = x^p + p\delta(x)$, is faithfully flat. The same holds for free δ-rings generated by arbitrary sets (not-necessarily finite).*

We remark that, thanks to Lemma 3.82, it is not necessary to pass to the polynomial ring of finitely many variables to invoke the fiberwise criterion of flatness.

We now relate δ-structure to divided powers. Note that, for any p-torsion free $\mathbb{Z}_{(p)}$-algebra A, any element $y \in A$ and any $n \in \mathbb{N}$, we have

$$\frac{y^n}{n!} \in \frac{y^n}{p^{v_p(n!)}} \mathrm{GL}_1(\mathbb{Z}_{(p)})$$

In particular, $y^p/p!$ (resp. $y^{p^2}/(p^2)!$) differs multiplicatively from y^p/p (resp. y^{p^2}/p^{p+1}) by a unit. When A is a p-torsion free δ-ring, we have $\varphi(y) = y^p + p\delta(y)$, and $y^p/p! \in A[p^{-1}]$ belongs to A if and only if $\varphi(y)$ is divisible by p.

Now we define the animated δ-ring $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$ to be the pushout of the diagram

$$\begin{array}{ccc} \mathbb{Z}_{(p)}\{y\} & \xrightarrow{y \mapsto pz} & \mathbb{Z}_{(p)}\{z\} \\ \downarrow{\scriptstyle y \mapsto \varphi(x)} & & \\ \mathbb{Z}_{(p)}\{x\} & & \end{array}$$

in the ∞-category $\mathrm{CAlg}_\delta^{\mathrm{an}}$. Since the Frobenius map $\varphi : \mathbb{Z}_{(p)}\{y\} \to \mathbb{Z}_{(p)}\{x\}$ is faithfully flat, so is the map $\mathbb{Z}_{(p)}\{z\} \to \mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$. It follows that $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$ is static and p-torsion-free by Remark 3.74, and, therefore, it is a δ-ring by Lemma 5.7 (this is essentially [12, Lem 2.36]). We need another characterization of the underlying ring of $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$.

LEMMA 5.13 ([12, Lem 2.36]). — *There is a natural isomorphism*

$$D_{\mathbb{Z}_{(p)}\{x\}}(x) \longrightarrow \mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$$

of p-torsion-free $\mathbb{Z}_{(p)}$-algebras.

This map transfers the surjective map $D_{\mathbb{Z}_{(p)}\{x\}}(x) \twoheadrightarrow \mathbb{Z}_{(p)}\{x\}/(x)$ to a surjective map $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\} \twoheadrightarrow \mathbb{Z}_{(p)}\{x\}/(x)$, the existence of which does not seem to be *a priori* clear (which is implicitly involved in [12, Lem 2.35]).

Note that since $x \in D_{\mathbb{Z}_{(p)}\{x\}}(x)$ is a nonzero-divisor, the map from the animated PD-envelope of $(\mathbb{Z}_{(p)}\{x\},(x))$ to the classical PD-envelope is an equivalence, by base change of $(\mathbb{Z}_{(p)}[x],(x))$ along the flat map $\mathbb{Z}_{(p)}[x] \to \mathbb{Z}_{(p)}\{x\} \simeq$

$\mathbb{Z}_{(p)}[x, \delta(x), \delta^2(x), \ldots]$, or, alternatively, by Proposition 3.83. We can replace x by a finite number of variables, which leads to the following.

COROLLARY 5.14. — *There exists a canonical δ-pair structure on the animated PD-envelope of every δ-pair $(\mathbb{Z}_{(p)}\{X, Y\}, (Y)) \in \mathrm{Pair}_\delta^{\mathrm{st}}$. More formally, there exists a canonical functor $\mathrm{Pair}_\delta^{\mathrm{st}} \to \mathrm{Pair}_\delta^{\mathrm{an}}$, which fits into a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Pair}_\delta^{\mathrm{st}} & \dashrightarrow & & & \mathrm{Pair}_\delta^{\mathrm{an}} \\ \downarrow & & & & \downarrow \\ \mathrm{Pair}^{\mathrm{an}} & \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} & \mathrm{Pair}^{\gamma,\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\mathrm{an}} \end{array}$$

of ∞-categories.

Proof. — The functoriality of $\mathrm{Pair}_\delta^{\mathrm{st}} \to \mathrm{Pair}_\delta^{\mathrm{an}}$ needs explanation: a map $(\mathbb{Z}_{(p)}\{X, Y\}, (Y)) \to (\mathbb{Z}_{(p)}\{X', Y'\}, (Y'))$ of δ-pairs induces a map $(\mathbb{Q}\{X, Y\}, (Y)) \to (\mathbb{Q}\{X', Y'\}, (Y'))$ of pairs after inverting p, which is "Frobenius"-equivariant, where $\mathbb{Q}\{X, Y\} := \mathbb{Z}_{(p)}\{X, Y\}[p^{-1}]$. A careful v_p-analysis implies that this map restricts to a map $\mathbb{Z}_{(p)}\{X, Y, \varphi(Y)/p\} \to \mathbb{Z}_{(p)}\{X', Y', \varphi(Y')/p\}$ of $\mathbb{Z}_{(p)}$-subalgebras, which gives rise to the functoriality. □

From Propositions 3.40 and A.14, Lemma 5.10, and Corollary 5.14 we have the following.

COROLLARY 5.15. — *There exists a canonical animated δ-pair structure on the animated PD-envelope of every animated δ-pair. More formally, there exists a canonical functor $\mathrm{Pair}_\delta^{\mathrm{an}} \to \mathrm{Pair}_\delta^{\mathrm{an}}$, which fits into a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Pair}_\delta^{\mathrm{an}} & \dashrightarrow & & & \mathrm{Pair}_\delta^{\mathrm{an}} \\ \downarrow & & & & \downarrow \\ \mathrm{Pair}^{\mathrm{an}} & \xrightarrow{\mathrm{Env}^{\gamma,\mathrm{an}}} & \mathrm{Pair}^{\gamma,\mathrm{an}} & \longrightarrow & \mathrm{Pair}^{\mathrm{an}} \end{array}$$

of ∞-categories. Moreover, the functor $\mathrm{Pair}_\delta^{\mathrm{an}} \to \mathrm{Pair}_\delta^{\mathrm{an}}$ preserves small colimits.

We give an analysis of the conjugate filtration on the PD-envelope of $(\mathbb{F}_p\{x\}, (x))$, where $\mathbb{F}_p\{x\} := \mathbb{Z}_{(p)}\{x\}/^{\mathbb{L}}p$, which is the base change of the PD-envelope of $(\mathbb{Z}_{(p)}\{x\}, (x))$ along $\mathbb{Z}_{(p)} \to \mathbb{F}_p$. Recall the following.

1. The (animate) PD-envelope $D_{\mathbb{F}_p[x]}(x)$ is a free $\mathbb{F}_p[x]/(x^p)$-module generated by the set $\{\gamma_{kp}(x) \mid k \in \mathbb{N}\}$ of divided powers of x.
2. For $i \in \mathbb{N}_{\geq 0}$, the $(-i)$-th piece of the conjugate filtration of $D_{\mathbb{F}_p[x]}(x)$ is generated by $\{\gamma_{kp}(x) \mid k \leq i\}$ as an $\mathbb{F}_p[x]/(x^p)$-submodule.

By the base change property (Lemma 3.52), we have the following.

1. The (animate) PD-envelope $D_{\mathbb{F}_p\{x\}}(x)$ is a free $\mathbb{F}_p\{x\}/(x^p)$-module generated by the set $\{\gamma_{kp}(x) \mid k \in \mathbb{N}\}$.
2. For $i \in \mathbb{N}_{\geq 0}$, the $(-i)$-th piece of the conjugate filtration of $D_{\mathbb{F}_p\{x\}}(x)$ is generated by $\{\gamma_{kp}(x) \mid k \leq i\}$ as an $\mathbb{F}_p\{x\}/(x^p)$-submodule.

We follow the argument of [12, Lem 2.35]: for every $y \in \mathbb{Z}_{(p)}\{x\}$ with $y^p/p \in \mathbb{Z}_{(p)}\{x\}$, we have

$$\begin{aligned}
\delta\left(\frac{y^p}{p}\right) &= \frac{1}{p}\left(\frac{\varphi(y)^p}{p} - \left(\frac{y^p}{p}\right)^p\right) \\
&= \frac{(y^p + p\delta(y))^p}{p^2} - \frac{y^{p^2}}{p^{p+1}} \\
&= \frac{1}{p^2}\left(y^{p^2} + p^2 y^{p(p-1)}\delta(y) + \sum_{k=0}^{p-2}\binom{p}{k} y^{kp}(p\delta(y))^{p-k}\right) - \frac{y^{p^2}}{p^{p+1}} \\
&= \frac{p^{p-1}-1}{p^{p+1}} y^{p^2} + y^{p(p-1)}\delta(y) + \sum_{k=0}^{p-2} p^{p-2-k}\binom{p}{k} y^{kp}\delta(y)^{p-k}
\end{aligned} \tag{11}$$

Letting $z = x^p/p$, from $p^{p-1} - 1 \in \mathrm{GL}_1(\mathbb{Z}_{(p)})$ we have the following.

1. The set $\{z^{a_0}\delta(z)^{a_1}(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r} \mid r \in \mathbb{N}, 0 \leq a_0, a_1, \ldots, a_r < p\}$ forms a basis of the free $\mathbb{F}_p\{x\}/(x^p)$-module $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}/^{\mathbb{L}}p \simeq D_{\mathbb{F}_p\{x\}}(x)$.
2. For every $i \in \mathbb{N}$, the $(-i)$-th piece of the conjugate filtration of $D_{\mathbb{F}_p\{x\}}(x)$ is generated by $\{z^{a_0}\delta(z)^{a_1}(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r} \mid 0 \leq a_0, a_1, \ldots, a_r < p, a_0 + a_1 p + a_2 p^2 + \cdots + a_r p^r \leq i\}$.

Remark 5.16. — In a bit more imprecise terms, $\delta^k(z)$ differs from $\gamma_{p^k}(x)$ up to a unit, modulo "lower terms".

This generalizes to the multivariable case with the same argument.

Lemma 5.17. — *Let $(A, I) := (\mathbb{Z}_{(p)}\{X, Y\}, (Y)) \in \mathrm{Pair}^{\mathrm{st}}_\delta$ be a standard δ-pair and let (B, J, γ) be the (animated) PD-envelope of $(\mathbb{F}_p\{X, Y\}, (Y))$. Let $Y = \{y_1, y_2, \ldots\}$ and $z_j := \varphi(y_j)/p$. Then we have the following.*

1. *The $\varphi^*_A(A/I)$-module B is freely generated by the subset*

$$\left\{\prod_{j,k}(\delta^k(z_j))^{a_{j,k}} \mid 0 \leq a_{j,k} < p\right\} \subseteq B.$$

2. *For every $i \in \mathbb{N}$, the $(-i)$-th piece of the conjugate filtration of B is generated by*

$$\left\{ \prod_{j,k} (\delta^k(z_j))^{a_{j,k}} \mid 0 \leq a_{j,k} < p, \sum_{j,k} a_{j,k} p^k \leq i \right\}$$

as a $\varphi_A^*(A/I)$-submodule.

5.2. Oriented prisms. — In this section, we will study animated δ-rings viewed as "noncomplete oriented animated prisms"[32]. Recall that a *orientable prism* is a δ-pair (A, I) such that the ideal $I \subseteq A$ is principal, the δ-ring A is I-torsion free, derived (p, I)-complete, and $p \in I + \varphi(I)A$ [12, Def 3.2]. For technical reasons, we will study the "noncomplete" analogues where the completeness and the torsion-freeness are dropped.

We fix a δ-ring A along with a chosen nonzero-divisor $d \in A$. In practice, we are only interested in the special case that $A = \mathbb{Z}_{(p)}\{d\}$ and some variants like $A = \mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}$. We denote by $\mathrm{Ring}_{\delta,A}$ the 1-category $(\mathrm{Ring}_\delta)_{A/}$ of δ-A-algebras. From Lemma 2.10 we have the following.

Lemma 5.18. — *The* 1*-category* $\mathrm{Ring}_{\delta,A}$ *is* 1*-projectively generated and, therefore, presentable. A set of compact* 1*-projective generators is given by* $\{A\{X\} := A \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}\{X\} \mid X \in \mathrm{Fin}\}$*, which spans a full subcategory of* $\mathrm{Ring}_{\delta,A}$ *denoted by* $\mathrm{Ring}^0_{\delta,A}$.

Definition 5.19. — Let B be an animated δ-ring. The ∞*-category of animated δ-B-algebras*, denoted by $\mathrm{CAlg}^{\mathrm{an}}_{\delta,B}$, is defined to be the undercategory $\mathrm{Ani}(\mathrm{Ring}_\delta)_{B/}$. When B is static, it is equivalent to the animation $\mathrm{Ani}(\mathrm{Ring}_{\delta,B})$ by Corollary 2.14.

By Lemma 4.55, we get an adjunction $\mathrm{CAlg}^{\heartsuit}_A \rightleftarrows \mathrm{Ring}_{\delta,A}$, where the forgetful functor $\mathrm{Ring}_{\delta,A} \to \mathrm{CAlg}^{\heartsuit}_A$ preserves all small colimits (and as a right adjoint, it preserves small limits as well). From Corollary 2.3 we have the following.

Lemma 5.20. — *There is a pair* $\mathrm{CAlg}^{\mathrm{an}}_A \rightleftarrows \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$ *of adjoint functors, being the animation of the pair* $\mathrm{CAlg}^{\heartsuit}_A \rightleftarrows \mathrm{Ring}_{\delta,A}$ *of adjoint functors. We will call the functor* $\mathrm{CAlg}^{\mathrm{an}}_A \to \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$ *the free animated δ-A-algebra functor. The functor* $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A} \to \mathrm{CAlg}^{\mathrm{an}}_A$*, called the forgetful functor, is conservative and preserves small colimits (and as a right adjoint, it preserves small limits as well).*

Definition 5.21. — The ∞*-category of animated δ-A-pairs* $\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$ is defined to be the undercategory $(\mathrm{Pair}^{\mathrm{an}}_\delta)_{(\mathrm{id}_A : A \to A)/}$, which is equivalent to the fiber product $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A} \times_{\mathrm{CAlg}^{\mathrm{an}}_A} \mathrm{Pair}^{\mathrm{an}}_A$ by [28, Lem 5.4.5.4].

32. They are noncomplete oriented analogues of [10, Def 2.4].

REMARK 5.22. — By [10, Cor 2.10], our ∞-category $\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$ is a noncomplete version of the ∞-category of animated prisms over (A,d). Their setup is more general in the sense that the base prism can be both animated and nonorientable.

The set $\{(A\{X,Y\},(Y))\,|\,X,Y\in\mathrm{Fin}\}$ forms a set of compact projective generators for $\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$ by Lemma 2.10, which spans a full subcategory $\mathrm{Pair}^{\mathrm{st}}_{\delta,A}\subseteq\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$. From Lemmas 4.55 and 5.11 we have the following.

LEMMA 5.23. — *The forgetful functor* $\mathrm{Pair}^{\mathrm{an}}_{\delta,A}\to\mathrm{Pair}^{\mathrm{an}}_{A}$ *admits a left adjoint.*

There is a canonical functor $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}\to\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$ given by $B\mapsto(B\twoheadrightarrow B/^{\mathbb{L}}d)$. We observe the following.

LEMMA 5.24. — *The functor* $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}\to\mathrm{Pair}^{\mathrm{an}}_{\delta,A}, B\mapsto(B\twoheadrightarrow B/^{\mathbb{L}}d)$ *admits a left adjoint* $\mathrm{Pair}^{\mathrm{an}}_{\delta,A}\to\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$*, given by the left derived functor (Proposition A.14) of* $\mathrm{Pair}^{\mathrm{st}}_{\delta,A}\to\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}, (A\{X,Y\},(Y))\mapsto A\{X,Y/d\}$*, where* $A\{X,Y/d\}$ *is an abbreviation for the free* δ*-*A*-algebra* $A\{x_1,x_2,\dots,y_1/d,y_2/d,\dots\}$[33]*.*

Proof. — Let G denote the functor $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}\to\mathrm{Pair}^{\mathrm{an}}_{\delta,A}, B\mapsto(B\twoheadrightarrow B/^{\mathbb{L}}d)$. Then we have a functor $F:\mathrm{Pair}^{\mathrm{an}}_{\delta}\to\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{an}}_{\delta},\mathrm{An})^{\mathrm{op}}$, which preserves small colimits and sends $(B\twoheadrightarrow B'')\in\mathrm{Pair}^{\mathrm{st}}_{\delta,A}$ to the functor $\mathrm{Map}_{\mathrm{Pair}^{\mathrm{an}}_{\delta,A}}(B\twoheadrightarrow B'',G(\cdot))$. By Proposition A.14, it is the left derived functor of its restriction to the full subcategory $\mathrm{Pair}^{\mathrm{st}}_{\delta,A}\subseteq\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$.

We now show that, for every $(A\{X,Y\},(Y))\in\mathrm{Pair}^{\mathrm{st}}_{\delta,A}$, the functor $F((A\{X,Y\},(Y)))$ is equivalent to the functor $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}}(A\{X,Y/d\},\cdot)$. In other words, the essential image of $F|_{\mathrm{Pair}^{\mathrm{st}}_{\delta,A}}$ lies in the full subcategory $\mathrm{Ring}^{0}_{\delta,A}\hookrightarrow\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}\hookrightarrow\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{an}}_{\delta,A},\mathrm{An})^{\mathrm{op}}$. By adjunctions $\mathrm{Fun}((\Delta^1)^{\mathrm{op}},D(\mathbb{Z})_{\geq 0})\rightleftarrows\mathrm{Pair}^{\mathrm{an}}_{A}\rightleftarrows\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$ (Definition 3.21 and Lemma 5.23), we have

$$\begin{aligned}F(A\{X,Y\},(Y))(B)&\simeq\mathrm{Map}_{\mathrm{Pair}^{\mathrm{an}}_{A}}(A[X,Y]\twoheadrightarrow A[X],B\twoheadrightarrow B/^{\mathbb{L}}d)\\&\simeq\mathrm{Map}_{\mathrm{Fun}((\Delta^1)^{\mathrm{op}},D(\mathbb{Z})_{\geq 0})}\left(X\mathbb{Z}\oplus Y\mathbb{Z}\leftarrow Y\mathbb{Z},B\xleftarrow{d}B\right)\\&\simeq B^{\mathrm{Card}(Y)}\times B^{\mathrm{Card}(X)}\\&\simeq\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}}(A\{X,Y/d\},B)\end{aligned}$$

which are functorial in $B\in\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$ (note that naively speaking, the "values" of Y/d correspond to the "preimages" of Y under the map $B\xleftarrow{d}B$ and, therefore, the formal notation Y/d).

33. These generators y_i/d are, in fact, formal variables z_i. This notation indicates that the counit map $(A\{X,Y\},(Y))\to(A\{X,Y/d\}\twoheadrightarrow A\{X,Y/d\}/^{\mathbb{L}}d)$ is induced by $x_i\mapsto x_i$ and $y_i\mapsto z_id$.

Since the Yoneda embedding $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}, \mathrm{An})^{\mathrm{op}}$ is stable under small colimits, it follows that the essential image of F lies in $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$, which proves that G admits a left adjoint $L : \mathrm{Pair}^{\mathrm{an}}_{\delta,A} \to \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$, and that $L(A\{X, Y/d\}) \simeq A\{X, Y/d\}$.

We still need to show that $L|_{\mathrm{Pair}^{\mathrm{st}}_{\delta,A}}$ coincides with the functor defined in the obvious way. We have shown this objectwise, and since $L(\mathrm{Pair}^{\mathrm{st}}_{\delta,A})$ lies in the full subcategory $\mathrm{Ring}^{0}_{\delta,A} \hookrightarrow \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$, which is a 1-category, we only need to show that the image of morphisms coincide with the "obvious" choice, i.e., without higher categorical complication. This can be checked putting different d-torsion-free δ-A-algebras $B \in \mathrm{Ring}_{\delta,A}$ into the functorial isomorphism

$$\mathrm{Hom}_{\mathrm{Pair}_{\delta,A}}((A\{X, Y\}, (Y)), (B, (d))) \cong \mathrm{Hom}_{\mathrm{Ring}_{\delta,A}}(A\{X, Y/d\}, B)$$

given by the adjunction. □

Now we introduce a variant of Definition 5.21.

Definition 5.25. — The *∞-category of animated δ-(A, d)-pairs* $\mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ is defined to be the undercategory $(\mathrm{Pair}^{\mathrm{an}}_{\delta,A})_{(A \twoheadrightarrow A/^{\mathbb{L}}d)/}$, which is equivalent to the fiber product $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A} \times_{\mathrm{CAlg}^{\mathrm{an}}_{A}} \mathrm{Pair}^{\mathrm{an}}_{(A \twoheadrightarrow A/^{\mathbb{L}}d)/}$ by [28, Lem 5.4.5.4].

By Lemma 2.10, we have the following.

Lemma 5.26. — *The ∞-category* $\mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ *is projectively generated. A set of compact projective generators is given by* $\{(A\{X, Y\}, (d, Y)) \mid X, Y \in \mathrm{Fin}\}$*, which spans a full subcategory of* $\mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ *denoted by* $\mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$*.*

Note that A is initial in $\mathrm{Pair}^{\mathrm{an}}_{\delta,A}$. From Lemmas 4.55 and 5.24 we have the following.

Corollary 5.27. — *The functor* $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A} \to \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}, B \mapsto (B \twoheadrightarrow B/^{\mathbb{L}}d)$ *admits a left adjoint* $\mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)} \to \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$*, which will be denoted by* $\mathrm{Env}^{\mathbb{\Delta}}$[34]*, given by the left derived functor (Proposition A.14) of* $\mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)} \to \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$*,* $(A\{X, Y\}, (d, Y)) \mapsto A\{X, Y/d\}$*.*

Furthermore, for every $B \in \mathrm{CAlg}^{\mathrm{an}}_{\delta,A}$, by unrolling the definitions, the counit map $\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B/^{\mathbb{L}}d) \to B$ is an equivalence, and, therefore, we have the following.

Lemma 5.28. — *The functor* $\mathrm{CAlg}^{\mathrm{an}}_{\delta,A} \to \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ *is, in fact, fully faithful, the image of which is a reflective subcategory (Definition 2.44).*

The following concept is not strictly necessary, but it would help us to understand when we need to "divide by d".

34. This is understood as a "noncomplete" prismatic envelope when p lies in the Jacobson radical $\mathrm{Rad}(A)$, and $d \in A$ is weakly distinguished (Definition 5.30).

DEFINITION 5.29. — Let A be a δ-ring and $d \in A$ a nonzero-divisor. Let $M \in D(A/^{\mathbb{L}}d)$ be a $A/^{\mathbb{L}}d$-module spectrum. For every $n \in \mathbb{Z}$, the *n-th Breuil–Kisin twist* of M with respect to (A, d), denoted by $M\{n\}$, is defined to be $M \otimes^{\mathbb{L}}_{A/^{\mathbb{L}}d} (dA/d^2A)^{\otimes^{\mathbb{L}}_{A/^{\mathbb{L}}d} n}$.

Note that when $d \in A$ is a nonzero-divisor, the $A/^{\mathbb{L}}d$-module $d^nA/d^{n+1}A$ is free of rank 1 and therefore, equivalent to $A/^{\mathbb{L}}d$. The Breuil–Kisin twists are strictly necessary when we want to generalize to nonorientable prisms. In our case, we understand $M\{1\}$ "formally multiplied by d" and $M\{-1\}$ "formally divided by d", just as the formal notations y_i/d in Lemma 5.24.

Finally, we introduce a variant of the concept of *distinguished elements* [12, Def 2.19].

DEFINITION 5.30. — Let A be a δ-ring. We say that an element $d \in A$ is *weakly distinguished* if the ideal $(d, \delta(d))$ is the unital ideal A, or equivalently, $\delta(d)$ is invertible in A/d.

REMARK 5.31. — Let A be a δ-ring and $d \in \mathrm{Rad}(A)$ an element in the Jacobson radical. Then d is weakly distinguished if and only if it is distinguished.

The following lemma is a motivation for the introduction of weakly distinguished elements.

LEMMA 5.32 (cf. [12, Lem 2.23]). — *Let A be a δ-ring and, $I = (d) \subseteq A$ a principal ideal. Then for any invertible element $u \in \mathrm{GL}_1(A)$, the principal ideals $\delta(d)(A/I)$ and $\delta(ud)(A/I)$ are the same. In particular, when I is generated by a nonzero-divisor, the principal ideal $\delta(d)(A/I)$ does not depend on the choice of the generator $d \in I$.*

Proof. — We have $\delta(ud) = \varphi(u)\delta(d) + \delta(u)d^p \equiv \varphi(u)\delta(d) \pmod{ud}$. Since u is invertible, so is $\varphi(u)$, and the result follows. □

COROLLARY 5.33. — *Let A be a δ-ring, $I \subseteq A$ a principal ideal generated by a nonzero-divisor. Then the following are equivalent.*

1. *There exists a weakly distinguished generator d of I.*
2. *Every generator d of I is weakly distinguished.*

REMARK 5.34 (Bhatt). — We have a variant of Corollary 5.33 that does not involve nonzero-divisors, by replacing a principal ideal I by the equivalence classes of maps $A \to A$ of A-modules, and the proof of Lemma 5.32 implies that the concept of "weakly distinguished" is invariant under this equivalence. More generally, we can consider the equivalence classes of an invertible A-module I along with a map $I \to A$, and define the concept of such a map $I \to A$ being weakly distinguished when $p \in \mathrm{Rad}(A)$. This generalizes to animated δ-rings.

Recall that an animated ring A is *p-local* if the element $p \in \pi_0(A)$ lies in the Jacobson radical[35] $\mathrm{Rad}(\pi_0(A))$.

LEMMA 5.35. — *Let A be a p-local δ-ring and $d \in A$ a weakly distinguished element. Then for every $n \in \mathbb{N}$, $\varphi^n(\delta(d))$ is invertible in A/d.*

Proof. — By induction, it suffices to show that, for every $u \in A$ of which the image in A/d is invertible, then so is the image of $\varphi(u)$ in A/d. It follows from the identity $\varphi(u) = u^p + p\delta(u)$, since the image of u^p in A/d invertible, and $p \in \mathrm{Rad}(A/d)$. □

5.3. Conjugate filtration. — In this section, we will introduce the *conjugate filtration* on "noncomplete prismatic envelopes", which plays a role similar to that of conjugate filtrations on animated PD-envelopes and derived crystalline cohomology. Let A be a p-local δ-ring and $d \in A$ a weakly distinguished nonzero-divisor. To simplify the presentation, we mostly concentrate on the "single variable" case: $\mathrm{Env}^{\mathbb{\Delta}}(A\{y\}, (d, y))/^{\mathbb{L}}d \simeq A\{y/d\}/^{\mathbb{L}}d$ as an $A\{y\}/^{\mathbb{L}}(d, y)$-algebra (or module).

First, note that the identity

$$\begin{aligned}\delta(u^p) &= (\varphi(u^p) - u^{p^2})/p \\ &= ((u^p + p\delta(u))^p - u^{p^2})/p \\ &= \sum_{k=1}^{p} \binom{p}{k} u^{p(p-k)} p^{k-1} \delta(u)^k\end{aligned} \tag{12}$$

holds in the free δ-ring $\mathbb{Z}_{(p)}\{u\}$, and, therefore, it is an identity in any δ-ring.

We now compute $\delta^n(y)$ in terms of $\delta^n(z)$, where $y = zd$ in the free δ-A-algebra $A\{y\}$:

$$\begin{aligned}\delta(y) &= \delta(zd) \\ &= \delta(z)\varphi(d) + z^p\delta(d) \\ \delta^2(y) &= \delta(\delta(z)\varphi(d) + z^p\delta(d)) \\ &= \delta(\delta(z)\varphi(d)) + \delta(z^p\delta(d)) - \underbrace{\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} (\delta(z)\varphi(d))^{p-k} (z^p\delta(d))^k}_{=:R_2} \\ &= \delta^2(z)\varphi^2(d) + \delta(z)^p\delta(\varphi(d)) + \delta(z^p)\delta(\varphi(d)) + z^{p^2}\delta^2(d) - R_2 \\ &= \delta^2(z)\varphi^2(d) + \varphi(\delta(d))(1 + p^{p-1})\delta(z)^p + \sum_{k=1}^{p-1} \cdots + z^{p^2}\delta^2(d) - R_2\end{aligned}$$

35. The *Jacobson radical* $\mathrm{Rad}(A)$ of a ring A is defined to be the subset (and *a fortiori*, the ideal) of elements $x \in A$ such that for every $a \in A$, the element $1 + ax$ is invertible in A.

where we used the fact that $\varphi\circ\delta=\delta\circ\varphi$ and (12) (which leads to the summand $\sum_{k=1}^{p-1}\cdots$), and in general, we have the following.

LEMMA 5.36. — *Let $A\{z\}$ be the free δ-A-algebra and $y:=zd$. For every $n\in\mathbb{N}$, there exists a unique polynomial $P_n\in A[X_0,\dots,X_{n-1}]$ with $\deg_{X_{n-1}}P_n\leq p$ such that*

$$\delta^n(y)=\delta^n(z)\varphi^n(d)+P_n(z,\delta(z),\dots,\delta^{n-1}(z))$$

Moreover, there exists a unique $Q_n\in A[X_0,\dots,X_{n-1}]$ with $\deg_{X_{n-1}}Q_n<p$ such that $P_n=a_n\varphi^{n-1}(\delta(d))X_{n-1}^p+Q_n$, where a_n are partial sums $\sum_{k=0}^{n-1}p^{k(p-1)}$ of the geometric progression $(p^{k(p-1)})_{k\in\mathbb{N}}$. Note that $a_n\in\mathrm{GL}_1(\mathbb{Z}_{(p)})$ for $n>0$. On the other hand, if we endow X_i with degree p^i, then P_n is homogeneous of degree p^n.

Proof. — The uniqueness follows from the freeness. We prove the existence inductively on $n\in\mathbb{N}$. When $n=0$, this is obvious. Now let $n\in\mathbb{N}_{>0}$ and assume that this is true for every $m<n$, Now we have

$$\begin{aligned}\delta^n(y)&=\delta(\delta^{n-1}(y))\\&=\delta(\delta^{n-1}(z)\varphi^{n-1}(d)+P_{n-1}(z,\delta(z),\dots,\delta^{n-2}(z)))\\&=\delta(\delta^{n-1}(z)\varphi^{n-1}(d))+\delta(P_{n-1}(z,\delta(z),\dots,\delta^{n-2}(z)))-R_n\end{aligned}$$

where $\delta(\delta^{n-1}(z)\varphi^{n-1}(d))=\delta^n(z)\varphi^n(d)+(\delta^{n-1}(z))^p\varphi^{n-1}(\delta(d))$ and

$$R_n:=\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}(\delta^{n-1}(z)\varphi^{n-1}(d))^{p-k}(P_{n-1}(z,\delta(z),\dots,\delta^{n-2}(z)))^k$$

Note that the "degree" of $\delta^{n-1}(z)$ in R_n is strictly less than p. Let $b_{n-1}=a_{n-1}\varphi^{n-2}(\delta(d))$; we have

$$\begin{aligned}\delta(P_{n-1}(z,\delta(z),\dots))&=\delta(b_{n-1}(\delta^{n-2}(z))^p+Q_{n-1}(z,\delta(z),\dots))\\&=\delta(b_{n-1}(\delta^{n-2}(z))^p)+\delta(Q_{n-1}(z,\delta(z),\dots))-\underbrace{\sum_{k=1}^{p-1}\cdots}_{=:R'_n}\\&=\varphi(b_{n-1})\delta((\delta^{n-2}(z))^p)+\delta(Q_{n-1}(\cdots))\\&\quad+\delta(b_{n-1})(\delta^{n-2}(z))^{p^2}-R'_n\end{aligned}$$

and only $\varphi(b_{n-1})\delta((\delta^{n-2}(z))^p)$ has a contribution on $\delta^{n-1}(z)^p$, and

$$\delta((\delta^{n-2}(z))^p)=\sum_{k=1}^{p}\binom{p}{k}(\delta^{n-2}(z))^{p(p-k)}p^{k-1}(\delta^{n-1}(z))^k$$

has a contribution on $\delta^{n-1}(z)^p$ only at $k=p$, i.e., $p^{p-1}\delta^{n-1}(z)^p$. Note that $\varphi(b_{n-1})=a_{n-1}\varphi^{n-1}(\delta(d))$, the result then follows. □

Construction 5.37. — *We now rewrite $A\{y\} \to A\{z\}, y \mapsto zd$ as the sequential composite (i.e., the $A\{y\}$-algebra $A\{z\}$ is equivalent to the sequential colimit of)*

$$A\{y\} \longrightarrow A\{y\} \otimes^{\mathbb{L}}_{B_0} C_0 \longrightarrow A\{y\} \otimes^{\mathbb{L}}_{B_1} C_1 \longrightarrow \cdots \tag{13}$$

where $A_n := A[z, \dots, \delta^{n-1}(z)], B_n := A_n[\delta^n(y)]$ and $C_n := A_n[\delta^n(z)]$ are polynomial algebras, and the map $B_n \to C_n$ is given by the evaluation $\delta^n(y) \mapsto \delta^n(z)\varphi^n(d) + P_n(z, \delta(z), \dots, \delta^{n-1}(z))$ by Lemma 5.36. Thus, $B_n \to C_n$ could be written as the composite (where we replace $\delta^n(y)$ by u and $\delta^n(z)$ by v)

$$\begin{aligned} B_n &= A_n[u] \\ &\to A_n[u, v]/(u - \varphi^n(d)v - P_n(z, \delta(z), \dots, \delta^{n-1}(z))) \cong A_n[v] = C_n \end{aligned} \tag{14}$$

In other words, $B_n \to C_n$ is essentially formally adjoining[36] *$(\delta^n(y) - P_n(z, \delta(z), \dots, \delta^{n-1}(z)))/\varphi^n(d)$ to B_n as an (animated) A-algebra, and the $A\{y\}$-algebra $A\{z\}$ is obtained by formally adjoining $(\delta^n(y) - P_n(z, \delta(z), \dots, \delta^{n-1}(z)))/\varphi^n(d)$ iteratively from $A\{y\}$. The conjugate filtration on $A\{y/d\}/^{\mathbb{L}}d$ is given by $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{y/d\}/^{\mathbb{L}}d)$ being the $A\{y\}/^{\mathbb{L}}(y, d)$-submodule of $A\{y/d\}/^{\mathbb{L}}d$ spanned by $\{(y/d)^{a_0}\delta(y/d)^{a_1}(\delta^2(y/d))^{a_2} \cdots (\delta^r(y/d))^{a_r} \mid r \in \mathbb{N}, 0 \le a_0, a_1, \dots, a_r < p\}$.*

Passing to the multivariable version, we get the following.

Lemma 5.38. — *Let A be a p-local δ-ring and $d \in A$ a weakly distinguished nonzero-divisor. Then there exists a canonical functor $\mathrm{Fil}^*_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(\cdot)/^{\mathbb{L}}d\right) : \mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)} \to \mathrm{CAlg}(\mathrm{DF}^{\le 0}(A/^{\mathbb{L}}d))$, which preserves finite coproducts, along with a functorial map $\mathrm{Fil}^*_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d\right) \to \mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d$, understood as the conjugate filtration on $\mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d$, such that*

1. *The conjugate filtration is exhaustive, that is to say, the induced map $\mathrm{Fil}^{-\infty}_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d\right) \to \mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d$ is an equivalence in $D(A)$.*
2. *The filtration $\mathrm{Fil}^*_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(A\{y\} \twoheadrightarrow A\{y\}/^{\mathbb{L}}(y, d))/^{\mathbb{L}}d\right)$ coincides with the filtration $\mathrm{Fil}^*_{\mathrm{conj}}(A\{y/d\}/^{\mathbb{L}}d)$ in Construction 5.37.*
3. *The maps $\mathrm{Fil}^{-i}_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(A\{x\} \twoheadrightarrow A\{x\}/^{\mathbb{L}}d)/^{\mathbb{L}}d\right) \to \mathrm{Env}^{\mathbb{\Delta}}(A\{x\} \twoheadrightarrow A\{x\}/^{\mathbb{L}}d)/^{\mathbb{L}}d \simeq A\{x\}/^{\mathbb{L}}d$ are equivalences for all $i \in \mathbb{N}$; that is to say, the conjugate filtration on $(A/^{\mathbb{L}}d)\{x\}$ is "constant*[37]*".*

Proof. — The conjugate filtration on each object $\mathrm{Env}^{\mathbb{\Delta}}(B, J)$ for $(B, J) \in \mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$ is completely determined by these properties and that the functor preserves finite coproducts, since every (B, J) can be written as a coproduct of

36. Note that this is true although $\varphi^n(d)$ is not necessarily a nonzero-divisor.

37. More precisely, it is constant after restriction to $\mathbb{Z}_{\le 0}$, but this restriction is expected as the conjugate filtration is nonpositive.

$A\{x\} \twoheadrightarrow A\{x\}/^{\mathbb{L}}d$ and $A\{y\} \twoheadrightarrow A\{y\}/^{\mathbb{L}}(y,d)$. Concretely, $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X,Y/d\}/^{\mathbb{L}}d)$ are generated, as an $A\{X,Y\}/(d,Y)$-submodule, by "standard monomials" $\prod_{(r,y)\in E}\delta^r(y/d)$ "of total degree$\leq i$", where $E \subseteq \mathbb{N}\times Y$ is a finite subset, and the element $\delta^r(y/d)$ is of degree p^r for $r\in\mathbb{N}$ and $y\in Y$. One verifies that this, indeed, gives rise to a functor.

Alternatively, if we further assume that d is weakly transversal (Definition 5.42), then we can invoke Lemma 5.45 to reduce the computations significantly. □

Definition 5.39. — Let A be a p-local δ-ring and $d\in A$ a weakly distinguished nonzero-divisor. Then the *conjugate filtration* on $\mathrm{Env}^{\mathbb{\Delta}}(B\twoheadrightarrow B'')/^{\mathbb{L}}d$ for $(B\twoheadrightarrow B'')\in \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ is given by the left derived functor (Proposition A.14) $\mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}\to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(A/^{\mathbb{L}}d))$ of the functor $\mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}\to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(A/^{\mathbb{L}}d))$ in Lemma 5.38.

From Lemma 2.43 we have the following.

Lemma 5.40. — *Let A be a p-local δ-ring and $d\in A$ a weakly distinguished nonzero-divisor. Then the conjugate filtration on $\mathrm{Env}^{\mathbb{\Delta}}(B\twoheadrightarrow B'')/^{\mathbb{L}}d$ for every $(B\twoheadrightarrow B'')\in \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ is exhaustive, i.e., $\mathrm{Fil}^{-\infty}\mathrm{Env}^{\mathbb{\Delta}}(B\twoheadrightarrow B'')/^{\mathbb{L}}d \to \mathrm{Env}^{\mathbb{\Delta}}(B\twoheadrightarrow B'')/^{\mathbb{L}}d$ is an equivalence.*

We now analyze the "denominators" $\varphi^n(d)$ when A is p-local, and d is weakly distinguished.

Lemma 5.41 (cf. [1, Lem 3.5]). — *Let A be a p-local δ-ring and $d\in A$ a weakly distinguished element. Then for every $n\in\mathbb{N}_{>0}$, there exists a unit $u\in \mathrm{GL}_1(A/d)$ such that $\varphi^n(d)\equiv pu \pmod d$.*

Proof. — We will construct inductively on $n\in\mathbb{N}_{>0}$ a sequence $(u_n)_n\in A^{\mathbb{N}_{>0}}$ such that for every $n\in\mathbb{N}_{>0}$, the image of u_n in A/d is invertible, and $\varphi^n(d)-d^{p^n}=pu_n$. We take $u_1=\delta(d)$, and suppose that u_m are already constructed for $1\leq m<n$, then

$$\begin{aligned}
\varphi^n(d) &= \varphi^{n-1}(\varphi(d))\\
&= \varphi^{n-1}(d^p+p\delta(d))\\
&= (\varphi^{n-1}(d))^p + p\varphi(\delta(d))\\
&= (d^{p^{n-1}}+pu_{n-1})^p + p\varphi(\delta(d))\\
&= d^{p^n} + p\left(\varphi(\delta(d)) + \sum_{k=1}^{p}\binom{p}{k} d^{p^{n-1}(p-k)}p^{k-1}u_{n-1}^k\right)
\end{aligned}$$

We pick $u_n=\delta(d)+\sum_{k=1}^{p}\binom{p}{k}d^{p^{n-1}(p-k)}p^{k-1}u_{n-1}^k$. Note that the second summand $\sum_{k=1}^{p}\cdots$ is canonically divisible by p (separating the cases $k=0$ and

$k > 1$), thus $u_n \equiv \delta(d) \pmod{p}$ of which the image in $A/(p,d)$ is invertible. The result then follows from the fact that $p \in \mathrm{Rad}(A/d)$. □

We introduce the following temporary terminology.

Definition 5.42. — Let A be a δ-ring. We say that an element $d \in A$ is *weakly transversal* if it is weakly distinguished, and the sequence (d,p) is regular in A, that is to say, d is a nonzero-divisor, and A/d is p-torsion-free.

Recall that for a ring A, the *Zariski localization* of A along an ideal $I \subseteq A$ is defined to be the localization of A at the multiplicative set $1+I$. The image of I in $(1+I)^{-1}A$ lies in the Jacobson radical.

Example 5.43. — The element d in the p-local δ-ring $\mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}_{(p)}$ is weakly transversal. In fact, this special case suffices for our applications.

Now we assume that $d \in A$ is weakly transversal. In the "single variable" case $A\{y\}/^{\mathbb{L}}(d,y) \to A\{y/d\}/^{\mathbb{L}}d$, by Lemmas 5.36 and 5.41, the sequence $(z, \delta(z), \delta^2(z), \ldots)$ forms a system similar to that of divided p^r-powers $(\gamma_{p^r})_{r\in\mathbb{N}}$ up to a multiplication of a unit after modulo d:

$$\begin{aligned} p\delta(z) &\equiv -a_1\delta(d)z^p \pmod{B} \\ p\delta^2(z) &\equiv -a_2\varphi(\delta(d))\delta(z)^p \pmod{B[\delta(z)]} \\ p\delta^3(z) &\equiv -a_3\varphi^2(\delta(d))\delta^2(z)^p \pmod{B[\delta(z), \delta^2(z)]} \end{aligned}$$

where $B := A\{y\}/^{\mathbb{L}}(d,y)$ and $a_n\varphi^{n-1}(\delta(d)) \in \mathrm{GL}_1(A/d)$ (cf. Remark 5.16). We now translate this observation to an analysis of the conjugate filtration, which seems hard to attack directly. We look at the maps $B_0/^{\mathbb{L}}(d,y) \to C_0/^{\mathbb{L}}d$ and $B_n/^{\mathbb{L}}d \to C_n/^{\mathbb{L}}d$ for $n \in \mathbb{N}_{>0}$ induced by the map (14). We first note that the map $B_0/^{\mathbb{L}}(d,y) \to C_0/^{\mathbb{L}}d$ is the polynomial algebra in single variable z.

If we further (derived) modulo p, we see that $B_n/^{\mathbb{L}}(d,p) \to C_n/^{\mathbb{L}}(d,p)$ for $n \in \mathbb{N}_{>0}$ is killing a polynomial $\delta^n(y) - P_n(z, \delta(z), \ldots, \delta^{n-1}(z))$ monic in $\delta^{n-1}(z)$ of degree p, and then adjoining a formal variable $\delta^n(z)$. In view of (13), we see that the map $A\{y\}/^{\mathbb{L}}(d,p,y) \xrightarrow{y\mapsto zd} A\{z\}/^{\mathbb{L}}(d,p)$ is the composition of consecutively adjoining a root of a monic polynomial of degree p, and consequently, as a $A\{y\}/^{\mathbb{L}}(d,p,y)$-module, $A\{z\}/^{\mathbb{L}}(d,p)$ is freely generated by

$$\{z^{a_0}\delta(z)^{a_1}(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r} \mid r \in \mathbb{N}, 0 \le a_0, a_1, \ldots, a_r < p\}.$$

On the other hand, if we invert p, we see that, for every $n \in \mathbb{N}_{>0}$, the maps $(B_n/^{\mathbb{L}}d)[p^{-1}] \to (C_n/^{\mathbb{L}}d)[p^{-1}]$ are equivalences, and, therefore, $(A\{y\}/^{\mathbb{L}}(d,y))[p^{-1}] \to (A\{z\}/^{\mathbb{L}}d)[p^{-1}]$ is the polynomial algebra in one variable z.

The mod p conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{z\}/^{\mathbb{L}}(d,p))/^{\mathbb{L}}p$ is then freely generated by

$$\big\{z^{a_0}\delta(z)^{a_1}(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r} \mid r\in\mathbb{N}, 0\leq a_0,a_1,\ldots,a_r<p, \\ a_0+pa_1+\cdots+p^ra_r\leq i\big\}.$$

On the other hand, the rationalized conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{z\}/^{\mathbb{L}}(d,p))[p^{-1}]$ is given by the $(A\{y\}/^{\mathbb{L}}(d,y))[p^{-1}]$-polynomials in z of degree$\leq i$. This follows from the following lemma, which can be established by induction on n.

Lemma 5.44. — *In the rationalized free δ-ring $\mathbb{Z}_{(p)}\{x\}[p^{-1}]\cong\mathbb{Q}[x,\varphi(x),\varphi^2(x),\ldots]$, for every $n\in\mathbb{N}$, the image of $\delta^n(x)\in\mathbb{Z}_{(p)}\{x\}$ in $\mathbb{Q}[x,\varphi(x),\varphi^2(x),\ldots]$ is given by a polynomial $D_n(x,\varphi(x),\ldots,\varphi^n(x))$ such that $\deg_x D_n=p^n$ with leading term $(-p^{-1})^{1+p+\cdots+p^n}x^{p^n}$ for all $n\in\mathbb{N}$.*

We summarize the "multi-variable" version as follows.

Lemma 5.45. — *Let A be a p-local δ-ring and $d\in A$ a weakly transversal element. Let $(A\{X,Y\},(d,Y))\in\mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$. Then we have the following.*

1. *The generator $\big\{\prod_{(r,y)\in E}\delta^r(y/d)\big\}_E$ for $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X,Y/d\}/^{\mathbb{L}}d)$ as an $A\{X,Y\}/^{\mathbb{L}}(d,Y)$-submodule, "of total degree$\leq i$", where $E\subseteq\mathbb{N}\times Y$ is a finite subset, and the element $\delta^r(y/d)$ is of degree p^r, becomes a basis after (derived) modulo p. This also holds for $i=+\infty$.*
2. *The $(-i)$-th piece of the rationalized conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X,Y/d\}/^{\mathbb{L}}d)[p^{-1}]\subseteq(A\{X,Y/d\}/^{\mathbb{L}}d)[p^{-1}]$ is given by the $A\{X,Y\}/^{\mathbb{L}}(d,Y)$-polynomials in variables Y/d of total degree$\leq i$. This also holds for $i=+\infty$.*

Furthermore, an element $x\in A\{X,Y/d\}/^{\mathbb{L}}d$ belongs to the $(-i)$-th piece of the conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X,Y/d\}/^{\mathbb{L}}d)$ if and only if so does it after (derived) modulo p and after rationalization.

Remark 5.46. — In some vague terms, in Lemma 5.45, the derived modulo p is about "controlling the denominators", and the rationalization is about "controlling the degree".

Recall that for every $(B,J)\in\mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$, there exists a canonical map $B/^{\mathbb{L}}d\simeq B\otimes^{\mathbb{L}}_A(A/^{\mathbb{L}}d)\to B/J$, which is, in fact, surjective. Then we have the following "multivariable" version.

Lemma 5.47. — *Let A be a p-local δ-ring and $d\in A$ a weakly transversal element. For every $(A\{X,Y\},(d,Y))=:(B,J)\in\mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$, let $K:=\ker(B/^{\mathbb{L}}d\to B/J)$. Note that K/K^2 is naturally a B/J-module. Then there exists a comparison map*

$$\Gamma^*_{B/J}((K/K^2)\{-1\})\longrightarrow\mathrm{gr}^{-*}_{\mathrm{conj}}\big(\mathrm{Env}^{\mathbb{\Delta}}(B,J)/^{\mathbb{L}}d\big)$$

of graded B/J-algebras induced by $[\gamma_n(z_i)] \mapsto \prod_{j=0}^r \left(\frac{\delta^j(z_i)}{-a_j\varphi^j(\delta(d))}\right)^{n_j}$, where $Y = \{y_1, \ldots\}$, $z_i = y_i/p$ and $n = \sum_{j=0}^r n_j p^j$ is the p-adic expansion of n. The comparison map is functorial in $(B, J) \in \mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$.

Proof. — The comparison map is induced by $\left[\gamma_n\left(\frac{y}{d}\right)\right] \mapsto \prod_{j=0}^r \left(\frac{\delta^j(y/d)}{-a_j\varphi^j(\delta(d))}\right)^{n_j}$ for every y in the ideal (d, Y). To see that this is well defined, the most non-trivial part is to show that this vanishes when $y \in (d, Y^2)$. By the multiplicity of the conjugate filtration, we can assume that $n_r = 1$ and $n_j = 0$ for $j \neq r$, and it suffices to analyze $\delta^r(y/d)$ when $y \in (d, Y^2)$, which can be reduced to the special case that $y = y_1y_2$ where $y_1, y_2 \in Y$.

By Lemma 5.44, the element $\delta^r(y_1y_2/d) \in A[p^{-1}][X, Y/d, \varphi(Y), \varphi^2(Y/d), \ldots]$ is a polynomial in $y_1y_2/d = y_1z_2, \varphi(y_1y_2), \ldots, \varphi^r(y_1y_2)$. The crucial point is that $y_1y_2/d = (y_1/d)(y_2/d)d = 0$ in $A\{X, Y/d\}/^{\mathbb{L}}d$, and, therefore, after rationalization, $\delta^r(y_1y_2/d)$ lies in $\mathrm{Fil}^0_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d\right)[p^{-1}]$.

By Lemma 5.36, $\delta^r(y_1z_2) = \delta^r(z_2)\varphi^r(y_1) + P_r(z_2, \ldots, \delta^{r-1}(z_2))$, where P_r is an $A\{y_1\}$-polynomial. Note that $\varphi^r(y_1) = \varphi^r(z_1d) = \varphi^r(z_1)\varphi^r(d) \equiv 0 \pmod{(d, p)}$ by Lemma 5.41. Since P_r is homogeneous of degree p^r when $\deg(\delta^j(z_2)) = p^j$, it follows that for every monomial $\prod_j T_j^{n_j}$ of P_r, there exists a j such that $n_j \geq p$, but then $\delta^j(z_2)^{n_j}$ is a linear combination of basis elements in Lemma 5.45, which shows that $\prod_j(\delta^j(z_2))^{n_j} \in \mathrm{Fil}^{-(p^r-1)}_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B, J)/^{\mathbb{L}}d\right)/^{\mathbb{L}}p$. The result then follows from the last part of Lemma 5.45. □

Again, from Lemma 5.45, via derived modulo p and rationalization, we have the following.

Lemma 5.48. — *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. For every $(B, J) \in \mathrm{Pair}^{\mathrm{st}}_{\delta,(A,d)}$, the comparison map in Lemma 5.47 is an equivalence.*

After such a long march, let us harvest the Hodge–Tate comparison, which is a prismatic analogue of Corollary 3.71. Note that for every $(B \twoheadrightarrow B'') \in \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$, and note that the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/^{\mathbb{L}}d & \longrightarrow & B'' \end{array}$$

induces a natural map $B/^{\mathbb{L}}d \simeq B \otimes^{\mathbb{L}}_A (A/^{\mathbb{L}}d) \to B''$ which is surjective; that is to say, $B/^{\mathbb{L}}d \twoheadrightarrow B''$ is an animated pair. Then from Lemma 5.48 and Proposition A.14 we have the following.

THEOREM 5.49 (Hodge–Tate). — *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. Then for every animated δ-(A,d)-pair $(B \twoheadrightarrow B'') \in \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$, there exists a canonical equivalence*

$$\Gamma^i_{B''}(\mathrm{gr}^1_{\mathrm{ad}}(B/^{\mathbb{L}}d \twoheadrightarrow B''))\{-1\}) \longrightarrow \mathrm{gr}^{-i}_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d\right)$$

which is functorial in $(B \twoheadrightarrow B'') \in \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$, where $\mathrm{Fil}_{\mathrm{ad}}$ is the adic filtration functor defined in Construction 3.59.

Let R be an $\mathbb{E}_1$-ring. Recall that a right R-module M is *faithfully flat* if it is flat (Definition 3.73), and $\pi_0(M)$ is a faithfully flat right $\pi_0(R)$-module. A map $R \to S$ of $\mathbb{E}_\infty$-rings is *faithfully flat* if S is faithfully flat as an R-module. There is a useful characterization of faithfully flat algebras.

LEMMA 5.50 ([27, Lemma 5.5]). — *Let $f : R \to S$ be a map of static (commutative) rings. Then f is faithfully flat if and only if f is flat, injective, and that* $\mathrm{coker}(f)$ *taken in the category of R-modules is flat.*

LEMMA 5.51. — *Let $f : R \to S$ a map of $\mathbb{E}_\infty$-rings. If f is faithfully flat, then* $\mathrm{cofib}(f)$ *taken in the ∞-category of R-module spectra is flat. The converse is true if R is supposed to be connective.*

Proof. — Assume first that f is faithfully flat. Let $M := \mathrm{coker}(\pi_0(R) \to \pi_0(S))$. By Lemma 5.50, the map $\pi_0(R) \to \pi_0(S)$ is injective, and the $\pi_0(R)$-module M is flat. Then for every $n \in \mathbb{Z}$, we have the exact sequence

$$\mathrm{Tor}_1^{\pi_0(R)}(\pi_n(R), M) \to \pi_n(R) \to \pi_n(R) \otimes_{\pi_0(R)} \pi_0(S) \to \pi_n(R) \otimes_{\pi_0(R)} M \to 0$$

which implies that the map $\pi_n(R) \to \pi_n(R) \otimes_{\pi_0(R)} \pi_0(S)$ is injective. Since f is flat, the canonical map $\pi_n(R) \otimes_{\pi_0(R)} \pi_0(S) \to \pi_n(S)$ is an isomorphism, and, therefore, the map $\pi_n(R) \to \pi_n(S)$ is injective. Then the long exact sequence associated to the fiber sequence $R \to S \to \mathrm{cofib}(f)$ splits into short exact sequences

$$0 \longrightarrow \pi_n(R) \longrightarrow \pi_n(S) \longrightarrow \pi_n(\mathrm{cofib}(f)) \longrightarrow 0$$

which implies that the canonical map $M \to \pi_0(\mathrm{cofib}(f))$ is an isomorphism. Furthermore, we have a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_n(R) & \longrightarrow & \pi_n(R) \otimes_{\pi_0(R)} \pi_0(S) & \longrightarrow & \pi_n(R) \otimes_{\pi_0(R)} M & \longrightarrow & 0 \\
 & & \downarrow \sim & & \downarrow \sim & & \downarrow & & \\
0 & \longrightarrow & \pi_n(R) & \longrightarrow & \pi_n(S) & \longrightarrow & \pi_n(\mathrm{cofib}(f)) & \longrightarrow & 0
\end{array}$$

By the short five lemma, the map $\pi_n(R) \otimes_{\pi_0(R)} M \to \pi_n(\mathrm{cofib}(f))$ is an isomorphism, and, therefore, $\mathrm{cofib}(f)$ is flat.

Now we assume that R is connective and that $\mathrm{cofib}(f)$ is flat. By definition, $\mathrm{cofib}(f)$ is connective and so is S by the fiber sequence $R \to S \to \mathrm{cofib}(f)$. For every static R-module M, we have the fiber sequence

$$M \longrightarrow M \otimes^{\mathbb{L}}_R S \longrightarrow M \otimes^{\mathbb{L}}_R \mathrm{cofib}(f)$$

By flatness of $\mathrm{cofib}(f)$ and [30, Prop 7.2.2.13], $M \otimes^{\mathbb{L}}_R \mathrm{cofib}(f)$ is static, and, therefore, so is $M \otimes^{\mathbb{L}}_R S$. It then follows from [30, Thm 7.2.2.15] that S is a flat R-module. It remains to show that the map $\pi_0(R) \to \pi_0(S)$ is faithfully flat. By Lemma 5.50, it suffices to show that $\pi_0(R) \to \pi_0(S)$ is injective and $\mathrm{coker}(\pi_0(R) \to \pi_0(S))$ is flat. The first follows from the connectivity of $\mathrm{cofib}(f)$, and the latter follows from the isomorphism $\mathrm{coker}(\pi_0(R) \to \pi_0(S)) \cong \pi_0(\mathrm{cofib}(f))$ and the flatness of $\mathrm{cofib}(f)$. □

Now we have a prismatic analogue of Corollary 3.77, with a similar argument.

Proposition 5.52. — *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. Let $(B \twoheadrightarrow B'') \in \mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)}$ be an animated δ-(A,d)-pair such that the canonical animated pair $B/^{\mathbb{L}}d \twoheadrightarrow B''$ is quasiregular. Then the unit map $B'' \to \mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ is faithfully flat.*

Proof. — By Theorem 5.49 and the quasiregularity of $B/^{\mathbb{L}}d \twoheadrightarrow B''$, for every $i \in \mathbb{N}$, the B''-module $\mathrm{gr}^{-i}_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d\right)$ is flat. By Lemma 3.75, for every $i \in \mathbb{N}_{>0}$, $\mathrm{cofib}\left(B'' \to \mathrm{Fil}^{-i}_{\mathrm{conj}}\left(\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d\right)\right)$ is flat. Since the conjugate filtration is exhaustive (Lemma 5.38), and the collection of flat modules is stable under filtered colimits [30, Lem 7.2.2.14(1)], we get $\mathrm{cofib}\left(B'' \to \mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d\right)$ is a flat B''-module. Then the result follows from Lemma 5.51. □

Remark 5.53. — In Proposition 5.52, if we further assume that B'' is static, then so is $\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$. This does not imply that $\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')$ is static. However, it implies that, after taking d-completion, $\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')$ becomes static, which should be understood as a "static d-completed envelope".

Remark 5.54. — There is a p-completed analogue of Proposition 5.52: suppose that the animated pair $B/^{\mathbb{L}}d \twoheadrightarrow B''$ is p-completely[38] quasiregular; that is to say, the shifted cotangent complex $L_{B''/(B/^{\mathbb{L}}d)}[-1]$ is a p-completely flat B''-module, then the same proof shows that the unit map $B'' \to \mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ is p-completely faithfully flat (i.e., it becomes faithfully flat after derived modulo p).

In particular, if (B,d) is a bounded oriented prism [12, Def 3.2] and that B'' is static and has bounded p-power torsion, then the p-completion of $\mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow$

38. "p-complete" concepts are usually applied to p-complete objects. However, this is not necessary because we can always derive p-complete a noncomplete object.

$B'')/^{\mathbb{L}}d$ is static. Moreover, by [12, Lem 3.7(2,3)], the (p,d)-completion of $C := \mathrm{Env}^{\mathbb{\Delta}}(B \twoheadrightarrow B'')$ is static, and thus it follows from a formal argument that $(C^{\wedge}_{(p,d)}, d)$ is the prismatic envelope of the δ-pair $B \twoheadrightarrow B''$ as long as it is d-torsion free. In other words, we generalize [12, Prop 3.13] by weakening regularity to quasiregularity.

We record a simple corollary that furnishes a quite general class of "flat covers of the final object" in the *affine prismatic site* (similarly to Definition 4.76). For this, we need the following definition.

Definition 5.55. — Let A be a δ-ring, $d \in A$ an element and B an animated δ-A-algebra. The *∞-category of δ-(B,d)-pairs*, denoted by $\mathrm{Pair}^{\mathrm{an}}_{\delta,(B,d)}$, is defined to be the undercategory $(\mathrm{Pair}^{\mathrm{an}}_{\delta,(A,d)})_{(B \twoheadrightarrow B/^{\mathbb{L}}d)/}$.

Let A be a p-local δ-ring and $d \in A$ a weakly distinguished nonzero-divisor. Let B be an animated δ-A-algebra and R an animated $B/^{\mathbb{L}}d$-algebra. Similarly to Definition 4.76, we can consider the category of animated δ-B-algebras C along with a map[39] $R \to C/^{\mathbb{L}}d$ of animated $B/^{\mathbb{L}}d$-algebras, which we will denote by $R \to C/^{\mathbb{L}}d \twoheadleftarrow C$, depicted by the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B/^{\mathbb{L}}d & \longrightarrow & R \longrightarrow C/^{\mathbb{L}}d \end{array}$$

More formally, this is the fiber product $\mathrm{Ani}(\mathrm{Ring}_{\delta})_{B/} \times_{\mathrm{Ani}(\mathrm{Ring})_{(B/^{\mathbb{L}}d)/}} \mathrm{Ani}(\mathrm{Ring})_{R/}$ of ∞-categories, the opposite category of which will be denoted by $\mathbb{\Delta}(R/(B,d))$[40].

Now let P be an animated δ-B-algebra along with a surjection $P \twoheadrightarrow R$ of animated B-algebras such that the cotangent complex $L_{P/B}/^{\mathbb{L}}d$ is a flat $P/^{\mathbb{L}}d$-module.

Remark 5.56. — We note that such P exists in abundance. For example, this happens when R is a smooth $B/^{\mathbb{L}}d$-algebra which admits a smooth B-lift P with a δ-structure compatible with that on B, or P is a polynomial B-algebra $B[x_i]$ (of possibly infinitely many variables) with $\delta(x_i) = 0$ along with a surjection $P \twoheadrightarrow R$ of animated B-algebras.

Then the animated pair $P \twoheadrightarrow R$ admits a canonical animated δ-(B,d)-pair structure, and thus the animated δ-ring $\mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ gives rise to an object of $\mathbb{\Delta}(R/(B,d))$ (by abuse of notation, we will still denote by $\mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ the object of $\mathbb{\Delta}(R/(B,d))$).

39. Unlike the crystalline case, here we do not assume that the map $R \to C/^{\mathbb{L}}d$ is an equivalence.

40. In [12], they used the notation $(R/A)_{\mathbb{\Delta}}$. However, this notation is usually devoted to topoi (such as X_{et} and X_{cris}). We therefore adopt the traditional notation for sites.

REMARK 5.57. — By Lemma 5.28, when $P \twoheadrightarrow R$ is "already" a noncompleted prism in the sense that the induced map $P/^{\mathbb{L}}d \to R$ is an equivalence, the noncompleted prismatic envelope $\mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ is equivalent to P itself.

For any object $(R \to C/^{\mathbb{L}}d \twoheadleftarrow C) \in \mathbb{\Delta}(R/(B,d))$, by unrolling the definitions, the product of $(R \to C/^{\mathbb{L}}d \twoheadleftarrow C)$ and $\mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ in $\mathbb{\Delta}(R/(B,d))$ is given by $\mathrm{Env}^{\mathbb{\Delta}}(P \otimes^{\mathbb{L}}_B C \twoheadrightarrow R)$. We therefore have a map $C/^{\mathbb{L}}d \to \mathrm{Env}^{\mathbb{\Delta}}(P \otimes^{\mathbb{L}}_B C \twoheadrightarrow C/^{\mathbb{L}}d)/^{\mathbb{L}}d$ of animated R-algebras. The following proposition is essentially equivalent to the "flat cover of the final object", cf. [15, Prop 1.1.2][41].

PROPOSITION 5.58. — *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. Let B be an animated δ-A-algebra and $P \twoheadrightarrow R$ an animated δ-(B,d)-pair such that the cotangent complex $L_{P/B}/^{\mathbb{L}}d$ is a flat $P/^{\mathbb{L}}d$-module. Then, for every $(R \to C/^{\mathbb{L}}d \twoheadleftarrow C) \in \mathbb{\Delta}(R/(B,d))$, the map $C/^{\mathbb{L}}d \to \mathrm{Env}^{\mathbb{\Delta}}(P \otimes^{\mathbb{L}}_B C \twoheadrightarrow C/^{\mathbb{L}}d)/^{\mathbb{L}}d$ is faithfully flat.*

Proof. — By Proposition 5.52, it suffices to show that the map $(P/^{\mathbb{L}}d) \otimes^{\mathbb{L}}_{B/^{\mathbb{L}}d} (C/^{\mathbb{L}}d) \to C/^{\mathbb{L}}d$ is quasiregular. To simplify the notations, let $P'' := P/^{\mathbb{L}}d$, $B'' := B/^{\mathbb{L}}d$ and $C'' := C/^{\mathbb{L}}d$. We have the transitivity sequence

$$L_{(P''\otimes^{\mathbb{L}}_{B''}C'')/C''} \otimes^{\mathbb{L}}_{P''\otimes^{\mathbb{L}}_{B''}C''} C'' \to L_{C''/C''} \simeq 0 \to L_{C''/(P''\otimes^{\mathbb{L}}_{B''}C'')}$$

associated to the maps $C'' \to P'' \otimes^{\mathbb{L}}_{B''} C'' \to C''$, whose composite is $\mathrm{id}_{C''}$. Note that $L_{(P''\otimes^{\mathbb{L}}_{B''}C'')/C''} \simeq L_{P''/B''} \otimes^{\mathbb{L}}_{B''} C''$ is a flat $P'' \otimes^{\mathbb{L}}_{B''} C''$-module. It follows that $L_{C''/(P''\otimes^{\mathbb{L}}_{B''}C'')}[-1]$ is a flat C''-module. □

We first learned the possibility of such kind of result from [34, Prop 3.4] (which is closely related to [15, Prop 1.1.2]). Later we came up with an argument which is essentially equivalent to the proof of Proposition 5.58, but the foundation was lacking then, therefore the current article could be understood as paving the way to this proof. Now we want to point out that, with minor modifications, this proof would imply [34, Prop 3.4] and the relevant technical lemmas in the recent works by Y. Tian and by A. Ogus [37] announced in Illusie conference. Furthermore, when the proper foundation is laid, the same proof would lead to a flat cover of the final object in the *absolute prismatic site*, and in particular, it would recover [2, Lem 5.2.8]. We now show this implication.

As in Remark 5.54, we assume that (B,d) is a bounded oriented prism, R is derived p-complete and the map $B/^{\mathbb{L}}d \to R$ is a p-completely quasisyntomic (i.e. the map $B/^{\mathbb{L}}d \to R$ is p-completely flat and the cotangent complex $L_{R/(B/^{\mathbb{L}}d)}$ has p-complete Tor-amplitude in $[0,1]$ as an R-module spectrum). Then by [11, Lem 4.7], R is static and has bounded p-power torsion. Let P be

41. This characterization was already implicit in Faltings's proof of "independence of the choice of the framing".

a derived (p,d)-complete animated δ-B-algebra which is (p,d)-completely quasismooth (i.e. the map $B \to P$ is (p,d)-completely flat and the cotangent complex $L_{P/B}$ is a (p,d)-completely flat B-module). Then by [12, Lem 3.7(2,3)], P is static and for every $n \in \mathbb{N}$, the multiplication map $d^n : P \to P$ is injective and P/d^n has bounded p-power torsion.

Now suppose that we are given a surjection $P \twoheadrightarrow R$ of B-algebras. Then by Remark 5.54, the derived (p,d)-completion of $\mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ is static and the prism defined by this (p,d)-completed algebra is the prismatic envelope in the sense of [12, Prop 3.13], where the d-torsion-freeness follows from the complete flatness of $B \to P$ and [12, Lem 3.7(2)]. Moreover, since both $B/^{\mathbb{L}}d \to R$ and $R \to \mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)/^{\mathbb{L}}d$ are p-completely flat, the map $B \to \mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ is (p,d)-completely flat (this in fact generalizes the flatness in [12, Prop 3.13]). The proof of Proposition 5.58 shows that

PROPOSITION 5.59. — *Let (B,d) be a bounded oriented prism, R a derived p-complete and p-completely quasisyntomic B/d-algebra. Let P be a derived (p,d)-complete animated δ-B-algebra which is (p,d)-completely quasismooth over B, equipped with a surjection $P \twoheadrightarrow R$ of B-algebras. Then the (p,d)-completion of $\mathrm{Env}^{\mathbb{\Delta}}(P \twoheadrightarrow R)$ is static which gives rise to a bounded prism (C,d) in the prismatic site*[42] *defined in* [12, Def 4.1] *of R relative to the base prism (B,d). Furthermore, (C,d) is a flat cover of the final object in this site.*

This implies virtually all the similar technical cover results for relative prismatic site mentioned above, cf. Remark 5.57.

REMARK 5.60. — For the absolute prismatic site, the proof also works in the special case of [2, Lem 5.2.8], but we are not aware of a statement as general as Proposition 5.59.

Appendix A. Animations and projectively generated categories

In this appendix, we recollect basic category-theoretic facts about animations [14] and projectively generated categories [28, §5.5.8] needed in the text.

A.1. Projectively generated n-categories. — In this subsection, we will briefly recollect basic facts about projectively generated n-categories. We will denote by An the ∞-category of animæ (see Section 1), and by $\widehat{\mathrm{An}}$ the ∞-category of large animæ. We say that an anima X is *n-truncated* for $n \in \mathbb{N}_{\geq 0}$ if the homotopy groups $\pi_i(X,x) = 0$ for every point $x \in X$ and every $i \in \mathbb{N}_{>n}$, and (-1)-*truncated* if X is either empty or contractible, and (-2)-*truncated* if $X = \varnothing$. An ∞-category $\mathcal{C}$ is an *n-category* [28, Prop 2.3.4.18] if for every pair

42. It is the nonanimated but (p,d)-completed version of our $\mathbb{\Delta}(R/(B,d))$.

$(X, Y) \in \mathcal{C} \times \mathcal{C}$ of objects, the mapping anima $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is $(n-1)$-truncated. We will denote by $\mathrm{An}_{\leq n}$ the ∞-category of n-truncated animae, and by $\widehat{\mathrm{An}}_{\leq n}$ the ∞-category of large n-truncated animae.

Remark A.1. — 1-categories are just categories in the classical category theory. If we define ∞-categories as quasicategories as in [28], this identification is given by the nerve construction. Since in our texts, categories often mean ∞-categories, we usually add "1-" to avoid possible ambiguities.

In fact, for the text, we only need results for $n = 1$ and $n = \infty$, but the generalization to general $n \in \mathbb{N}_{>0} \cup \{\infty\}$ is quite cost-free.

Proposition A.2 ([28, Cor 2.3.4.8]). — *Let $\mathcal{C}$ be an n-category and K a simplicial set. Then* $\mathrm{Fun}(K, \mathcal{C})$ *is an n-category.*

Definition A.3 ([28, Rem 5.5.8.20]). — Let $\mathcal{C}$ be a cocomplete n-category and $X \in \mathcal{C}$ an object. We say that X is *compact and n-projective*, or that X is a *compact n-projective object*, if the functor $\mathcal{C} \to \mathrm{An}_{\leq n-1}, Y \mapsto \mathrm{Map}_{\mathcal{C}}(X, Y)$ corepresented by X commutes with filtered colimits and geometric realizations.

Remark A.4. — Here we need $\widehat{\mathrm{An}}$ in lieu of An because the ∞-category $\mathcal{C}$ is not necessarily locally small. In practice, the ∞-categories that we encounter, e.g. projectively generated ∞-categories, are *a fortiori* locally small, but not necessarily a priori locally small.

Remark A.5. — In fact, an object $X \in \mathcal{C}$ is called n-projective if and only if the functor $\mathcal{C} \to \mathrm{An}_{\leq n-1}, Y \mapsto \mathrm{Map}_{\mathcal{C}}(X, Y)$ corepresented by X commutes with geometric realizations. In particular, when $\mathcal{C}$ is an abelian 1-category, an object $X \in \mathcal{C}$ is 1-projective if and only if it is a "projective object" of the abelian 1-category $\mathcal{C}$.

Remark A.6. — Let $\mathcal{C}$ be a cocomplete n-category and $X \in \mathcal{C}$ a compact n-projective object. In general, X is not a compact projective object of $\mathcal{C}$ as an ∞-category. In fact, the inclusion $\widehat{\mathrm{An}}_{\leq n-1} \to \mathrm{An}$ does *not* commute with geometric realizations. That is to say, for general simplicial objects $Y_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, the geometric realization $|\mathrm{Map}_{\mathcal{C}}(X, Y_\bullet)|_{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}$ is not in general $(n-1)$-truncated.

Remark A.7. — There is another way to characterize geometric realizations in an n-category $\mathcal{C}$. In fact, the fully faithful embedding $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is "n-cofinal", and, therefore, the geometric realization of a simplicial object $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ exists if and only if colimit of the composite functor $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ exists, and the two colimits are equivalent. Furthermore, for any diagram $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \to \mathcal{C}$, the left Kan extension along $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ always exists. Thus for a cocomplete n-category $\mathcal{C}$, an object $X \in \mathcal{C}$ is n-projective if and only if

the functor $\mathrm{Map}_{\mathcal{C}}(X,\cdot)$ corepresented by X preserves $\mathbf{\Delta}^{\mathrm{op}}_{\leq[n]}$-indexed colimits. See [35] and the proof of [30, Lem 1.3.3.10].

Definition A.8 ([28, Def 5.5.8.23]). — Let $\mathcal{C}$ be a cocomplete n-category and $S \subseteq \mathcal{C}$ a (small) collection of objects of $\mathcal{C}$. We say that S is a *set of compact n-projective generators* for $\mathcal{C}$ if the following conditions are satisfied:

1. Each element of S is a compact n-projective object of $\mathcal{C}$.
2. The full subcategory of $\mathcal{C}$ spanned by finite coproducts of elements of S is essentially small.
3. The set S generates $\mathcal{C}$ under small colimits.

We say that an n-category $\mathcal{C}$ is *n-projectively generated* if it is cocomplete and there exists a set S of compact n-projective generators for $\mathcal{C}$.

Remark A.9. — Let $\mathcal{C}$ be a cocomplete n-category and $\mathcal{C}_0 \subseteq \mathcal{C}$ an essentially small full subcategory. Then we will abuse the terminology by saying that $\mathcal{C}_0$ is a set of compact n-projective generators for $\mathcal{C}$ if a skeleton of $\mathcal{C}_0$ is a set of compact n-projective generators for $\mathcal{C}$.

Notation A.10. — Let $\mathcal{C}$ be a small n-category which admits finite coproducts. We let $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ denote the full subcategory of $\mathcal{P}_n(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}_{\leq n-1})$ spanned by those functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{An}_{\leq n-1}$ which preserves finite products. When $n = \infty$, we will omit the subscript n.

Proposition A.11. — *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. Then*

1. *The ∞-category $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ is an accessible localization of $\mathcal{P}_n(\mathcal{C})$ and, therefore, presentable.*
2. *The Yoneda embedding $j : \mathcal{C} \to \mathcal{P}_n(\mathcal{C})$ factors through $\mathcal{P}_{\Sigma,n}(\mathcal{C})$. Moreover, the induced functor $\mathcal{C} \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ preserves finite coproducts.*
3. *Let $\mathcal{D}$ be a presentable n-category and let $\mathcal{P}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors. Then G factors through $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ if and only if $f = F \circ j : \mathcal{C} \to \mathcal{D}$ preserves finite coproducts.*
4. *The full subcategory $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \subseteq \mathcal{P}_n(\mathcal{C})$ is stable under sifted colimits*[43].

We recall that, for a small ∞-category $\mathcal{C}$, $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is the full subcategory generated under filtered colimits by the essential image of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$, [28, Prop 5.3.5.3 & Cor 5.3.5.4]. It follows from [28, Prop 5.3.5.11] that

43. We do not introduce n-sifted diagrams, so *a priori* it is a sifted diagram defined in [28, Def 5.5.8.1]. However, here one can replace sifted diagrams by n-sifted diagram. See Remark A.7.

LEMMA A.12. — *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. Then the fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{P}_{\Sigma,n}(\mathcal{C})$ extends uniquely to a functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ which preserves filtered colimit. This functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ is fully faithful.*

LEMMA A.13. — *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. Then the n-category $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ is n-projectively generated for which $\mathcal{C} \subseteq \mathcal{P}_{\Sigma,n}(\mathcal{C})$ is a set of n-projective generators. In fact, for any $X \in \mathcal{P}_{\Sigma,n}(\mathcal{C})$, there exists a simplicial object $U_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{C})$ (or equivalently, a diagram $\mathbf{\Delta}^{\mathrm{op}}_{\leq n} \to \mathrm{Ind}(\mathcal{C})$ by Remark A.7) whose colimit is X.*

Proof. — First, since $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \subseteq \mathcal{P}_n(\mathcal{C})$ is an accessible localization, $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ is presentable [28, Rem 5.5.1.6] and, therefore, cocomplete. Since $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \subseteq \mathcal{P}_n(\mathcal{C})$ is stable under sifted colimits (Proposition A.11), the objects of $\mathcal{C}$ are compact and n-projective. The last statement then follows from [28, Lem 5.5.8.14]. □

PROPOSITION A.14. — *Let $\mathcal{C}$ be a small n-category which admits finite coproducts and let $\mathcal{D}$ be an n-category which admits filtered colimits and geometric realizations. Let $\mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}), \mathcal{D})$ denote the full subcategory spanned by those functors $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \to \mathcal{D}$ which preserve filtered colimits and geometric realizations. Then we have the following.*

1. *Composition with the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ induces an equivalence $\theta : \mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of categories. The inverse θ^{-1} is given by the left Kan extension along j. In this case, we will call $\theta^{-1}(f)$ the left derived functor of $f \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*
2. *Any functor $g \in \mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}), \mathcal{D})$ preserves sifted colimits.*
3. *Assume that $\mathcal{D}$ admits finite coproducts. A functor $g \in \mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}), \mathcal{D})$ preserves small colimits if and only if $g \circ j$ preserves finite coproducts.*

PROPOSITION A.15. — *Let $\mathcal{C}$ be a small n-category which admits finite coproducts, $\mathcal{D}$ an n-category which admits filtered colimits and geometric realizations, and $F : \mathcal{P}_{\Sigma,n}(\mathcal{C}) \to \mathcal{D}$ a left derived functor of $f = F \circ j : \mathcal{C} \to \mathcal{D}$, where $j : \mathcal{C} \to \mathcal{P}_{\Sigma,n}(\mathcal{D})$ denotes the Yoneda embedding. Consider the following conditions.*

1. *The functor f is fully faithful.*
2. *The essential image of f consists of compact n-projective objects of $\mathcal{D}$.*
3. *The n-category $\mathcal{D}$ is generated by the essential image of f under filtered colimits and geometric realizations.*

If 1 and 2 are satisfied, then F is fully faithful. Moreover, F is an equivalence if and only if 1, 2 and 3 are satisfied.

PROPOSITION A.16. — *Let $\mathcal{C}$ be a n-projectively generated n-category with a set S of compact n-projective generators for $\mathcal{C}$. Then we have the following.*

1. *Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then $\mathcal{C}^0$ is essentially small, and the left derived functor $F : \mathcal{P}_{\Sigma,n}(\mathcal{C}^0) \to \mathcal{C}$ is an equivalence of n-categories. In particular, $\mathcal{C}$ is a compactly generated presentable n-category.*
2. *Let $C \in \mathcal{C}$ be an object. The following conditions are equivalent.*
   (a) *The object $C \in \mathcal{C}$ is compact and n-projective.*
   (b) *The functor $\mathcal{C} \to \mathrm{An}_{\leq n-1}$ corepresented by C preserves sifted colimits.*
   (c) *There exists an object $C' \in \mathcal{C}^0$ such that C is a retract of C'.*

Proof. — We explain more details of the first point than [28, Prop 5.5.8.25]. It follows from definitions that $\mathcal{C}^0$ is essentially small. Then it follows from Proposition A.15 that the left derived functor $F : \mathcal{P}_{\Sigma,n}(\mathcal{C}^0) \to \mathcal{C}$ is fully faithful. Since $\mathcal{C}^0 \subseteq \mathcal{C}$ is stable under finite coproducts taken in $\mathcal{C}$, the embedding $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ preserves finite coproducts. It follows from Proposition A.14 that F preserves small colimits, thus the essential image of F is stable under small colimits. By assumption, S generates $\mathcal{C}$ under small colimits, and, therefore, F is essentially surjective. □

COROLLARY A.17. — *Let $\mathcal{C}$ be a projectively generated n-category and let $\mathcal{D}$ be an n-category which admits filtered colimits and geometric realizations. If a functor $\mathcal{C} \to \mathcal{D}$ preserves filtered colimits and geometric realizations, then it also preserve sifted colimits.*

The following proposition is extremely useful to detect projectively generated n-categories.

PROPOSITION A.18 ([30, Cor 4.7.3.18]). — *Given a pair $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ of adjoint functors between n-categories. Assume that*

1. *The n-category $\mathcal{D}$ admits filtered colimits and geometric realizations, and the functor G preserves filtered colimits and geometric realizations.*
2. *The n-category $\mathcal{C}$ is n-projectively generated.*
3. *The functor G is conservative.*

Then

1. *The n-category $\mathcal{D}$ is n-projectively generated.*
2. *An object $D \in \mathcal{D}$ is compact and n-projective if and only if there exists a compact n-projective object $C \in \mathcal{C}$ such that D is a retract of $F(C)$.*
3. *The functor G preserves all sifted colimits.*

A.2. Animation of n-projectively generated n-categories. — In this subsection, we describe a procedure, called *animation*, introduced in [14, §5.1], to produce a projectively generated ∞-category from an n-projectively generated n-category. Roughly speaking, this projectively generated ∞-category is determined by a set of compact n-projective generators for the n-category in question.

Definition A.19. — Let $\mathcal{C}$ be an n-projectively generated n-category. We choose a set $S \subseteq \mathcal{C}$ of compact n-projective generators for $\mathcal{C}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then the *animation* of $\mathcal{C}$, denoted by $\mathrm{Ani}(\mathcal{C})$, is defined to be the projectively generated ∞-category $\mathcal{P}_\Sigma(\mathcal{C}^0)$.

Remark A.20. — The definition of the animation does not depend on the choice of the set of compact n-projective generators. The key is that if S' is another compact n-projective generators, then it follows from Proposition A.16 that every object $X' \in S'$ is a retract of an object $X \in \mathcal{C}^0$ in Definition A.19. The same applies to the discussions below.

Example A.21. — Let Ab be the abelian category of abelian groups. Then $\mathrm{Ani}(\mathrm{Ab})$ coincides with the (connective) derived category $D_{\geq 0}(\mathrm{Ab})$.

Remark A.22. — In the context of Definition A.19, we have $\mathcal{C} \simeq \mathcal{P}_{\Sigma,n}(\mathcal{C}^0)$ by Proposition A.16 and $\mathrm{Ani}(\mathcal{C}) \simeq \mathcal{P}_\Sigma(\mathcal{C}^0)$. It follows that the n-category $\mathcal{C}$ could be identified with n-truncated objects in $\mathrm{Ani}(\mathcal{C})$. In particular, there exists a left adjoint $\tau_{\leq n-1} : \mathrm{Ani}(\mathcal{C}) \to \mathcal{C}$ to the fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Ani}(\mathcal{C})$, cf. [28, Rem 5.5.8.26].

We now discuss the animation of functors.

Definition A.23 ([14, §5.1.4]). — Let $\mathcal{C}, \mathcal{D}$ be two n-projectively generated n-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the *animation* of the functor F, denoted by $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$, is defined as follows:

We choose a set $S \subseteq \mathcal{C}$ of compact n-projective generators for $\mathcal{C}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then the functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to the composite $\mathcal{C}^0 \to \mathcal{C} \to \mathcal{D} \to \mathrm{Ani}(\mathcal{D})$. We define $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$ to be the left derived functor (in Proposition A.14) of $\mathcal{C}^0 \to \mathrm{Ani}(\mathcal{D})$.

Example A.24. — Let $F : \mathrm{Ab} \to \mathrm{Ab}$ be an additive functor. Then the animation $\mathrm{Ani}(F) : \mathrm{Ani}(\mathrm{Ab}) \to \mathrm{Ani}(\mathrm{Ab})$ coincides with the left derived functor $\mathbb{L}F : D_{\geq 0}(\mathrm{Ab}) \to D_{\geq 0}(\mathrm{Ab})$ in homological algebra.

It follows from Propositions A.14 and A.16 that

Corollary A.25. — *In Definition A.23, if F preserves sifted colimits (cf. Corollary A.17), then so does* $\mathrm{Ani}(F)$. *Furthermore, if F preserves small colimits, then so does* $\mathrm{Ani}(F)$.

In homological algebra, there is a natural comparison map $H_0 \circ \mathbb{L}F \to F \circ H_0$, which becomes an equivalence when F is assumed to be right exact. Now we study the animated analogue. In the context of Definition A.23, the composite functor $\mathrm{Ani}(\mathcal{C}) \xrightarrow{\tau_{\leq n-1}} \mathcal{C} \xrightarrow{F} \mathcal{D} \overset{j_{\mathcal{D}}}{\hookrightarrow} \mathrm{Ani}(\mathcal{D})$ is an extension of the composite functor $\mathcal{C} \xrightarrow{F} \mathcal{D} \hookrightarrow \mathrm{Ani}(\mathcal{D})$. Since $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$ is the left Kan extension, there exists an essentially unique map $\mathrm{Ani}(F) \to j_{\mathcal{D}} \circ F \circ \tau_{\leq n-1}$ of functors $\mathrm{Ani}(\mathcal{C}) \rightrightarrows \mathrm{Ani}(\mathcal{D})$. By adjunction, we get a canonical map $\tau_{\leq n-1} \circ \mathrm{Ani}(F) \to F \circ \tau_{\leq n-1}$ of functors $\mathrm{Ani}(\mathcal{C}) \rightrightarrows \mathcal{D}$.

Lemma A.26 ([14, §5.1.4]). — *In Definition A.23, suppose that the functor $F : \mathcal{C} \to \mathcal{D}$ (between n-categories) preserves sifted colimits. Then the map $\tau_{\leq n-1} \circ \mathrm{Ani}(F) \to F \circ \tau_{\leq n-1}$ of functors constructed above is an equivalence of functors.*

Proof. — First, note that the map $\tau_{\leq n-1} \circ \mathrm{Ani}(F) \to F \circ \tau_{\leq n-1}$ of functors $\mathrm{Ani}(\mathcal{C}) \rightrightarrows \mathcal{D}$ is an equivalence of functors after composing with the inclusion $\mathcal{C}^0 \hookrightarrow \mathrm{Ani}(\mathcal{C})$. We claim that both functors $\tau_{\leq n-1} \circ \mathrm{Ani}(F)$ and $F \circ \tau_{\leq n-1}$ preserve sifted colimits, thus belonging to $\mathrm{Fun}_{\Sigma}(\mathrm{Ani}(\mathcal{C}), \mathcal{D})$ which becomes an equivalence after mapped along $\mathrm{Fun}_{\Sigma}(\mathrm{Ani}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and hence by Proposition A.14, the constructed map of functors is an equivalence.

In fact, since $\tau_{\leq n-1}$ is a left adjoint and, therefore, commutes with small colimits, which implies that $\tau_{\leq n-1} \circ \mathrm{Ani}(F)$ commutes with sifted colimits. On the other hand, $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves sifted colimits and, therefore, also preserves sifted colimits since $\mathcal{C}, \mathcal{D}$ are n-categories. Thus $F \circ \tau_{\leq n-1}$ also preserves sifted colimits. □

In homological algebra, left deriving functors is not compatible with compositions, and, therefore, neither is animation of functors in general. However, recall that with some acyclicity conditions [39, Tag 015M], there is a compatibility of left deriving functors and compositions. Here is such a condition in the world of animations.

Proposition A.27 ([14, Prop 5.1.5]). — *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three n-projectively generated n-categories and $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{E}$ two functors preserving sifted colimits (cf. Corollary A.17). Then*

1. *There is a natural transformation from the composite $\mathrm{Ani}(G) \circ \mathrm{Ani}(F)$ to $\mathrm{Ani}(G \circ F)$ (In fact, for this, we only need that G preserves sifted colimits).*
2. *Let $\mathcal{C}^0 \subseteq \mathcal{C}$ and $\mathcal{D}^0 \subseteq \mathcal{D}$ be full subcategories determined by a choice of set of compact n-projective generators as in Definition A.19. If either $F(\mathcal{C}^0) \subseteq \mathrm{Ind}(\mathcal{D}^0)$ in $\mathcal{D}$ or $(\mathrm{Ani}(G))(F(\mathcal{C}^0)) \subseteq \mathcal{E}$ in $\mathrm{Ani}(\mathcal{E})$, then the natural transformation $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \to \mathrm{Ani}(G \circ F)$ is an equivalence.*

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BIBLIOGRAPHY

[1] J. Anschütz & A.-C. L. Bras – "The p-completed cyclotomic trace in degree 2", *Ann. K-Theory* **5** (2020), no. 3, p. 539–580.
[2] ———, "Prismatic Dieudonné theory", *Forum Math. Pi* **11** (2023), p. 0.
[3] J. António – "Spreading out the Hodge filtration in non-archimedean geometry", 2020, ArXiv e-prints.
[4] A. Beilinson – "On the derived category of perverse sheaves", Lecture Notes in Math., no. 1289, p. 27–41, Springer, Berlin, 1987.
[5] P. Berthelot – *Cohomologie cristalline des schémas de caractéristique $p > 0$*, Lecture Notes in Mathematics, no. 407, Springer-Verlag, Berlin-New York, 1974.
[6] P. Berthelot & A. Ogus – *Notes on crystalline cohomology*, Princeton University Press; University of Tokyo Press, Princeton, N.J.; Tokyo, 1978.
[7] Bhatt – "Prismatic cohomology", 2018, `http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/`.
[8] B. Bhatt – "Completions and derived de Rham cohomology", 2012, ArXiv e-prints.
[9] ———, "p-adic derived de Rham cohomology", 2012, ArXiv e-prints.
[10] B. Bhatt & J. Lurie – "The prismatization of p-adic formal schemes", 2022, ArXiv e-prints.
[11] B. Bhatt, M. Morrow & P. Scholze – "Topological Hochschild homology and integral p-adic Hodge theory", *Publications Mathématiques. Institut de Hautes Études Scientifiques* **129** (2019), p. 199–310.
[12] B. Bhatt & P. Scholze – "Prisms and prismatic cohomology", *Ann. of Math. (2)* **196** (2022), no. 3, p. 1135–1275.
[13] L. Brantner & A. Mathew – "Deformation theory and partition Lie algebras", 2019, ArXiv e-prints.
[14] K. Česnavičius & P. Scholze – "Purity for flat cohomology", *Ann. of Math. (2)* **199** (2024), no. 1, p. 51–180.
[15] A. Chatzistamatiou – "Q-crystals and q-connections", 2020, ArXiv e-prints.

[16] Daniel G. Quillen – *Homotopical algebra*, Lecture Notes in Mathematics, no. 43, Springer-Verlag, Berlin-New York, 1967.
[17] V. Drinfeld – "On a notion of ring groupoid", 2021, ArXiv e-prints.
[18] M. Gros, B. Le Stum & A. Quirós – "Twisted differential operators and q-crystals", in *p-Adic Hodge theory, singular varieties, and non-abelian aspects*, Simons Symp., Springer, Cham, 2023, p. 183–238.
[19] H. Guo & S. Li – "Period sheaves via derived de Rham cohomology", *Compos. Math.* **157** (2021), no. 11, p. 2377–2406.
[20] O. Gwilliam & D. Pavlov – "Enhancing the filtered derived category", *Journal of Pure and Applied Algebra* **222** (2018), no. 11, p. 3621–3674.
[21] J. Hekking – "Graded algebras, projective spectra and blow-ups in derived algebraic geometry", 2021, ArXiv e-prints.
[22] M. Hovey – "Smith ideals of structured ring spectra", 2014, ArXiv e-prints.
[23] L. Illusie – *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, no. 239, Springer-Verlag, Berlin-New York, 1971.
[24] ———, *Complexe cotangent et déformations. II*, Lecture Notes in Mathematics, no. 283, Springer-Verlag, Berlin-New York, 1972.
[25] N. M. Katz – "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin", *Institut des Hautes Études Scientifiques. Publications Mathématiques* **39** (1970), p. 175–232.
[26] S. Li & T. Liu – "Comparison of prismatic cohomology and derived de Rham cohomology", *J. Eur. Math. Soc.* (2023), https://doi.org/10.4171/JEMS/1377.
[27] J. Lurie – "Tannaka duality for geometric stacks", 2004, ArXiv Mathematics e-prints.
[28] ———, *Higher topos theory*, Annals of Mathematics Studies, no. 170, Princeton University Press, Princeton, NJ, 2009.
[29] ———, "Rotation invariance in algebraic K-theory", 2015, https://www.math.ias.edu/~lurie/papers/Waldhaus.pdf.
[30] ———, "Higher algebra", 2017, https://www.math.ias.edu/~lurie/papers/HA.pdf.
[31] ———, "Spectral algebraic geometry", 2018, https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.
[32] ———, "Kerodon", 2023, https://kerodon.net.
[33] K. Magidson – "Divided powers and derived de Rham cohomology", 2024, ArXiv e-prints.
[34] M. Morrow & T. Tsuji – "Generalised representations as q-connections in integral p-adic Hodge theory", ArXiv e-prints.
[35] D. Nardin – "Difficulties with descent data as homotopy limit of image of Čech nerve", 2016, MathOverflow. https://mathoverflow.net/a/236600/.

[36] T. Nikolaus – "Stable ∞-operads and the multiplicative Yoneda lemma", 2016, ArXiv e-prints.
[37] A. Ogus – "Prismatic and p-de Rham cohomology: comparison theorems and antecedents", 2021, Illusie conference. https://server.mcm.ac.cn/~zheng/LI/Ogus.pdf.
[38] A. Raksit – "Hochschild homology and the derived de Rham complex revisited", 2020, ArXiv e-prints.
[39] StacksProjectAuthors – "The Stacks project", 2021, https://stacks.math.columbia.edu.
[40] D. White & D. Yau – "Arrow categories of monoidal model categories", *Math. Scand.* **125** (2019), no. 2, p. 185–198.
[41] D. White & D. Yau – "Smith ideals of operadic algebras in monoidal model categories", *Algebr. Geom. Topol.* **24** (2024), no. 1, p. 341–392.