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ÉDITORIAL

"We are all Hermite's heirs," claimed Cyparissos Stephanos (1857–1917), who studied with Hermite in Paris in the 1880s and became professor of mathematics at the University of Athens. [1]

And that would probably be even truer today. The databases MathSciNet and zbMATH contain more than 5,500 articles published between, say, 1950 and 2024, whose *title* includes the name "Hermite" and more than 6,000 articles whose title includes the derived adjective "Hermitian". They belong to an impressive variety of fields, from number theory to numerical analysis, from linear algebra to computer science, from functional analysis to probabiltity theory, from fluid mechanics to partial or ordinary differential equations and special functions. There are Hermite inequalities and a Hermite constant, Hermite polynomials, Hermite schemes, Hermitian forms, matrices or varieties.

Hermite's career, however, was not a smooth one. [2] Hermite was born in Dieuze (Lorraine) on December 24, 1822 as the son of a merchant family. He studied in Nancy, then in Paris and was admitted at the École polytechnique in 1842, without distinction (ranking 68th out of 134). He left the Polytechnique only a year later, both because of a disability that made him ineligible for the usual careers open to Polytechnique students, and because Joseph Liouville, who taught there, advised him to concentrate on his main passion, mathematics. Indeed, by this time, Hermite was already publishing on topics that would occupy him for the rest of his life: algebraic equations (with a proof of the impossibility of solving the general quintic equation by radicals) and elliptic functions and their generalization (which brought him

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1 See Phili (Christine), Sur le développement des mathématiques en Grèce durant la période 1850-1950, *Istorico-matematicheskie issledovania*, 3 (1997), pp. 80-102, quote p. 85.

2 On Hermite's life, aside the substantial introduction by his son-in-law Émile Picard to Hermite's complete works, one can mention among his numerous obituaries: Noether (Max), Charles Hermite, *Mathematische Annalen*, 55 (1902), pp. 337-385 and Darboux (Gaston), Notice historique sur Charles Hermite [1905], in *Éloges académiques et discours*, Paris: Hermann, 1912, pp. 117-172. In addition, two books have been devoted to Hermite's scientific biography: Ozhigova (Elena Petrovna), *Charles Hermite*, Leningrad: Nauka, 1982 (in Russian) and Brezinski (Claude), *Charles Hermite, père de l'analyse moderne*, Paris: Société française d'histoire des sciences et des techniques, 1990.

into epistolary contact with one of the most important mathematicians of the time, Carl Gustav Jacob Jacobi).[3]

During the 1850s, Hermite extended his research to number theory (in particular to the theory of quadratic forms), approximation, invariant theory and algebra.[4] He visited Berlin and made contact with several other mathematicians, from Gotthold Eisenstein, Carl Borchardt and Peter Gustav Lejeune-Dirichlet to James Sylvester, Arthur Cayley and Augustin Cauchy.[5]

Yet, he only secured a modest job as a *répétiteur* at the Polytechnique in 1848.[6] Nevertheless, it allowed him to marry Louise Bertrand, the sister of a co-disciple at Polytechnique, Joseph Bertrand, thus joining an influential academic family. They will have two daughters, one who will marry Georges Forestier, an engineer of the Ponts-et-Chaussées and an automobile expert

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3 On Hermite's work in these topics, see Houzel (Christian), Fonctions elliptiques et intégrales abéliennes, in Dieudonné (Jean) (dir.), *Abrégé d'histoire des mathématiques 1700-1900*, Paris: Hermann, 1978, vol. 2, pp. 1-113; Belhoste (Bruno), Autour d'un mémoire inédit: la contribution d'Hermite au développement de la théorie des fonctions elliptiques, *Revue d'histoire des mathématiques*, 2 (1996), pp. 1-66; Bottazzini (Umberto) and Gray (Jeremy), *Hidden Harmony—Geometric Fantasies: The Rise of Complex Function Theory*, New York: Springer, 2013.

4 On Hermite's works on these topics, see Brezinski (Claude), *History of Continued Fractions and Padé Approximants*, Berlin, New York: Springer, 1991 ; Serfati (Michel), *Quadrature du cercle, fractions continues, et autres contes: Sur l'histoire des nombres irrationnels et transcendants aux xviii^e et xix^e siècles*, Paris: Éditions de l'APMEP, 1992; Sinaceur (Hourya), *Corps et modèles*, Paris: Vrin, 1994 ; Houzel (Christian), L'équation générale du cinquième degré, in *La Géométrie algébrique: recherches historiques*, Paris: Blanchard, 2003, pp. 73-80; Brouzet (Robert), La double origine du groupe symplectique, *Expositiones mathematicae*, 22 (2004), pp. 55-82 ; Goldstein (Catherine), The Hermitian Form of Reading the *Disquisitiones*, in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim) (eds.), *The Shaping of Arithmetic after C. F. Gauss's* Disquisitiones Arithmeticae, Berlin: Springer, 2007, pp. 377-410 ; Brechenmacher (Frédéric), Autour de pratiques algébriques de Poincaré: héritages de la réduction de Jordan, 2011, hal-00630959v3 ; Goldstein (Catherine), Charles Hermite's Stroll through the Galois Fields, *Revue d'histoire des mathématiques*, 17 (2011), pp. 211-270.

5 On Hermite's role in the international community, see Archibald (Thomas), Charles Hermite and German mathematics in France, in Parshall (Karen Hunger) & Rice (Adrian), *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800–1945*, Providence, RI: American Mathematical Society, 2002, pp. 123-137; Šegan-Radonjić (Marija) & Todorčević (Vesna), Mihailo Petrović's Education in France and its Significance in Establishing the Petrović School of Mathematics, *Revue d'histoire des mathématiques*, 30 (2024), pp 31-69; and the editions of Hermite's correspondence with various mathematicians in the Bibliography of Charles Hermite, hereafter.

6 On this position, see Vincent (Yannick), "Les répétiteurs de mathématiques à l'École polytechnique de 1798 à 1900," thèse de doctorat, Université Paris-Saclay, 2019.

and the other, the mathematician Émile Picard.[7] This was also the period when Hermite moved from an apparently indifferent attitude to religion to a rigorous and committed Catholicism, partly under the influence of Cauchy.

In 1856, Hermite was elected to the French Academy of Sciences, but he had to wait until 1862 to obtain a position at the École normale supérieure. In the years 1869-1870 Hermite finally secured stable and important professorships, at the Polytechnique (until 1876) and at the Faculté des sciences de Paris, the Sorbonne, from which he retired only in 1892.[8] The war between France and Prussia and especially the advent of the Third Republic in 1871, against his conservative and Catholic views, left him deeply worried and pessimistic. During this time, however, he extended his work to various questions of analysis, in particular differential equations, and in 1873 he provided the first proof of the transcendence of a natural constant of analysis, the number e.[9] He was an influential figure in the reception of the works of Galois and Riemann in France and also maintained an intense international correspondence, both with his pairs and with younger colleagues, providing advice and often papers for their newly founded mathematical journals, for example *Mathesis, Jornal de Sciencias Mathematicas e Astronomicas, Acta mathematica* or *Archiv der Mathematik und Physik.* He died on January 14, 1901.

Hermite published his first papers in 1842, his last ones in 1901, a few days before his death, and many of them had a long and fruitful posterity. Although somewhat delayed from an institutional point of view, as mentioned above, Hermite's recognition during his lifetime was impressive: he was a member of several prestigious academies and the subscription for his jubilee gathered donations from 810 persons from 17 different countries. However, despite his central place in 19th century mathematics and the ubiquitous trace of his mathematics until now, Hermite has not been much studied by historians of mathematics. He is often portrayed as a bit out of

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7 On the Bertrand family, see Zerner (Martin), Le règne de Joseph Bertrand (1874-1900), in Gispert (Hélène), *La France mathématique,* Paris: SMF et SFHST, 1991, pp. 298-322.

8 On his courses, see Renaud (Hervé), "La fabrication d'un enseignement de l'analyse en France au xix^e^ siècle : acteurs, institutions, programmes et manuels," thèse de doctorat, Université de Nantes, 2017.

9 See Waldschmidt (Michel), Les débuts de la théorie des nombres transcendants, *Cahiers du séminaire d'histoire des mathématiques,* 4 (1983), pp. 93-115 ; Waldschmidt (Michel), La méthode de Charles Hermite en théorie des nombres transcendants, *Bibnum, Mathématiques,* (2009), `http://journals.openedition.org/bibnum/893`; Bottazzini (Umberto) and Gray (Jeremy), *Hidden Harmony—Geometric Fantasies,* op. cit.; Archibald (Tom), Hermite's "Concrete" Analysis: Research and Educational Themes in an Evolving Discipline, *Revue d'histoire des mathématiques* (2024), this issue.

step with the main innovations of his time, just missing out some important theorems, be it the structure of units in algebraic number fields or the transcendence of π, or underestimating the importance of some innovations, set theory, non-Euclidean geometry or teratological functions of analysis. A famous sentence in a letter to Thomas Stieltjes, in which Hermite "turns away with fright and horror from the lamentable plague of continuous functions without derivatives" is often taken out of context and used as evidence of his old-fashioned views on analysis, when in fact it was a simple joke after a miscalculation. [10]

In 2001, on the occasion of the centenary of Hermite's death, a special session of the seminar on the history of mathematics of the Institut Henri Poincaré in Paris was organized in the Hermite auditorium. [11] One of the most striking conclusions of this meeting was that little was actually known about Hermite. During the following decades, several investigations have revealed the complexity of his mathematics, both in its individual singularity and in its collective dimensions. Since 2001, Hermite's works have been discussed in the *Revue d'histoire des mathématiques* in connection with those of Évariste Galois, Henri Poincaré, Édouard Lucas, James Joseph Sylvester, Junius Masseau, Sylvestre François Lacroix, Élie Cartan, Leopold Kronecker, Charles-Ange Laisant, Giuseppe Peano, Hermann Minkowski, Camille Jordan, Louis Poinsot, Émile Picard, Gaston Darboux, Mihailo Petrović...

However, Charles Hermite was still the elephant in the room. The bicentenary of Hermite's birth in 2022 was thus an occasion to retrace his footsteps, to look back at some of his less studied contributions, at his institutional role in the Paris mathematical scene, [12] at his international relations

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10 On Hermite's general views on mathematics, see BREZINSKI (Claude), Charles Hermite: père de l'analyse mathématique moderne, op. cit. ; GOLDSTEIN (Catherine), Un arithméticien contre l'arithmétisation: les principes de Charles Hermite, in FLAMENT (Dominique) & NABONNAND (Philippe), *Justifier en mathématiques*, Paris, Éditions de la Maison des sciences de l'Homme, 2011, pp. 129-165.

11 Several aspects of Hermite's mathematical works had been discussed on this occasion, with contributions by Catherine Goldstein on number theory, Tom Archibald on theta functions and differential equations, Jean Mawhin on Hermite's patronage of Poincaré, Hélène Gispert and Philippe Nabonnand on his role in the French mathematical community. See the program on the website of the Institut Henri Poincaré, accessed December 18, 2024, `https://www.ihp.fr/fr/1995-2008`.

12 On the institutional roles played by Hermite in the 1880s at the Paris Academy of Science as well as on his patronage on the mathematical journals edited by Gaston Darboux and Camille Jordan, see CROIZAT (Barnabé), *Gaston Darboux: naissance d'un mathématicien, genèse d'un professeur, chronique d'un rédacteur*, Thèse de doctorat, Université Lille 1, 2016, and Frédéric Brechenmacher, Un journal de rang « élevé »: le *Journal de mathématiques pures et appliquées* sous la direction de Camille Jordan, 2021, hal-04252496v1.

with Italy, Russia, and Serbia, and at his resonances in today's mathematics. A conference, once again in the Hermite auditorium of the Institut Henri Poincaré, took place in December 2022 in Paris. [13] The speakers at this conference included, besides the two editors of this special issue, Karen Hunger Parshall, Yannick Vincent, Natalia Ingtem, Livia Giacardi, Rossana Tazzioli, Jean-Benoît Bost, Anne Quéguiner-Mathieu, Alin Bostan, Tom Archibald, Marija Šegan-Radonjić, Vesna Todorcevic, François Lê, Barnabé Croizat, and Hélène Gispert. [14]

The present special issue of the *Revue d'histoire des mathématiques* brings together specialists from different branches of the history of mathematics. It aims to highlight, in the light of recent and ongoing research, lesser-known aspects of Hermite's work, such as invariant theory and numerical equations, and further explores Hermite's conceptions of mathematics through an analysis of his writing style, the key role played by a concrete approach to analysis in both his academic works and his teaching, and the way he constructs a unity of mathematics. Hermite's preference for effective formulas and transfers by analogy from arithmetic to algebra to analysis contrasted with the contemporary development of foundational programs, abstract objects and general frameworks. This situation may have been one of the main factors troubling his overall image over the course of the 20th century. Taking seriously the way in which Hermite and many of his interlocutors conceived of mathematics, its disciplinary division, the introduction of new subjects of study, as well as his convictions on matters of proof and results, prompts us to rewrite the development of mathematics in the 19th century in a less linear way.

The scattering of Hermite's results among notes, lectures and letters, is another major obstacle to access his mathematical thought. His works, edited in four volumes by Picard in the early 20th century, do not contain a complete bibliography and there are sometimes substantial differences between the original version and the version given in the Œuvres. Moreover, several parts of Hermite's extensive correspondence with various mathematicians have been published in various monographs and journals from the 19th to the 21st century. We have therefore supplemented the papers in

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13 The conference was supported by the LinX Laboratory of the École polytechnique, the Institut de mathématiques de Jussieu-Paris Rive Gauche, the UFR of mathematics of Sorbonne Université, and the GDR 3398 of the History of Mathematics of the CNRS. We particularly thank Sylvie Benzoni, Émilie Faure, Sébastien Gauthier, Sitti Mchinda and Frédéric Zantonio for their help and support.

14 See the program on the website of CNRS, accessed 18 December 2024, `https://indico.math.cnrs.fr/event/8758/`.

this special issue with both a list of Hermite's original publications and of his published correspondence to date.

Frédéric Brechenmacher (LinX, École polytechnique)
Catherine Goldstein (IMJ-PRG, CNRS, Sorbonne Université, Université Paris Cité)

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HERMITE'S "CONCRETE" ANALYSIS: RESEARCH AND EDUCATIONAL THEMES IN AN EVOLVING DISCIPLINE

Tom Archibald

Abstract. — The emerging discipline of mathematical analysis exhibited various threads during the nineteenth century, with different values and priorities as to basic definitions and approaches being used in different national and local contexts. In this paper we examine the "concrete" analysis of Charles Hermite, looking at its roots in his own research and the developing pedagogical versions of it that appeared in his courses and the work of certain of his students.

Résumé (L'analyse « concrète » de Hermite : recherche et thématiques éducationnelles dans une discipline en évolution)

La discipline émergente d'analyse mathématique manifestait diverses versions au long du XIXᵉ siècle, avec différentes communautés locales adoptant des valeurs et priorités différents dans leurs choix de définitions et d'approches. Dans cet article on focalise sur l'analyse « concrète » de Charles Hermite, regardant les racines de son approche dans ses propres recherches, ainsi que les versions pédagogiques qui ont paru dans ses cours et dans les travaux de ses étudiants.

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Mots clefs. — Hermite, analyse, XIXᵉ siècle, pédagogie.

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1. INTRODUCTION

A great deal of the work of Charles Hermite falls under what we would now term analysis. As Catherine Goldstein and Norbert Schappacher argue in their landmark study of the legacy of Gauss, much of his work may reasonably be termed arithmetic algebraic analysis [Goldstein et al. 2007]. The term algebraic analysis was used already in the eighteenth century, and continues to be used by historians, to describe a certain body of work, vague in extent, that concerns itself with infinite and infinitesimal mathematics and its applications (for example in differential equations) while using techniques that even today we would naively term algebraic, focussing on issues having to do with symbol manipulation. Such work may indirectly address concerns that we would term geometric (that is, involving curves and surfaces) or even analytic (involving estimation and convergence questions) but this is not its primary focus [Fraser 1989; Jahnke 2003]. I will not dwell on the "arithmetic" label here, except to note that this refers to applications in number theory. In Hermite's case, this grew directly out of the work of Jacobi, shown in his early publications and in his correspondence with the latter [Hermite 1850b]. Hermite's work in number theory has been discussed in detail in [Goldstein 2007] and [Goldstein 2011a], where Hermite's views about the relation of this work to analysis are explored.

This focus was the starting point of a long career in which Hermite worked not only in this area, but in questions related to elliptic functions more generally, especially focussing on concrete representations of these, often in terms of theta functions. Other work of Hermite may be reasonably classed with the "analysis" label, notably involving differential equations and one famous result, the transcendence of e. Our purpose in this paper is not to survey all aspects of this work, but rather to sample it, looking at the features of what Hermite himself termed analysis, which is certainly not the analysis of today. Nor is it the analysis of Augustin-Louis Cauchy or Leonhard Euler, despite a certain backward-looking character. Our sample will be somewhat opportunistic, but will also include some discussion of the introductory teaching of the subject, pointing to various innovations Hermite attempted to introduce. We will also look very briefly at the work of some of the doctoral students he mentored, and comment on their role in shaping education in this area. Some of his research students taught in preparatory schools, a uniquely French form that emerged in the mid-nineteenth century with the particular aim of coaching students to succeed in entrance examinations to the most important of the state scientific or technical schools.

In what follows we will first give some examples of his practice. We then turn to some specific writings aiming at transmitting important parts of that practice: the appendix to the 1862 edition of Lacroix's *Calcul* on elliptic functions; the *Cours d'analyse* given at the École polytechnique in 1872, and the 1882 lectures at the Faculté des sciences of the Sorbonne, also referred to as a *Cours d'analyse* but very different from the earlier work. We will then look at three of the thesis student who had specific teaching roles. It is difficult, often, to determine which students had a thesis that might be thought of as directed by Hermite, but at least a dozen either acknowledge him directly or work on themes directly related to his own work.

In so doing, we want to point out also that the nineteenth century reforms in the foundations of analysis, for example those of Cauchy and Weierstrass, in no way led to the immediate extinction of all research in the older, more concrete and formula-based vein that had its origin in the eighteenth century. Consider a statement such as the one due to Giovanni Ferraro:

> [In the early 1820s] Cauchy published *Cours d'analyse* and *Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal*, which can be considered to mark the definitive abandonment of the eighteenth century formal approach to series theory. [Ferraro 2008, vii]

Not to blame Ferraro unfairly, since the next period was not his subject, but such casual remarks often make it seem that the innovations of Cauchy possessed the kind of rapid revolutionary effect that the statement seems to suggest. In what follows, we will see that in fact this older algebraic analysis lived on and thrived, in an influential and important Hermitian form that remained, and remains, part of the developing world of analysis.

In looking at aspects of the evolution of analysis, we may consider this label as describing a discipline, field, or research specialty, Distinguishing between these terms is not something we will attempt here in detail, but we note a few features. The idea of a discipline is one that has developed over time. In Aristotle, for example, the term is associated with something that must be learned, and in the Aristotelian context this meant that it was associated with a particular subject. By about 1900, the idea enters into sociology, so that Max Weber associates it with professional practice, and with the idea of how to reproduce a given set of practices. Newer sociological versions, for example by Pierre Bourdieu, retain this while analysing further the social character of what is being transmitted, refining ideas about the fundamental entity and attempting to cut through some confusions via the notion of a field. Our usage will concentrate on the basic feature of transmission of knowledge and approach.

In mathematics, the term discipline is used somewhat vaguely to distinguish what are often termed branches of the subject—things like algebraic topology or finite geometries. These remain associated with research specialties, and hence with practice both in research and teaching. One can thus look at a body of mathematical work as being associated with one or several fields of activity. From the viewpoint of the mathematical researcher (and trainer of researchers) this brings to the fore how to acquire the techniques of practice. The meaning of discipline in the context of the history of mathematics, and its relation to ideas of specialty or of mathematical practice, are examined in several works by Sébastien Gauthier, notably [Gauthier 2007] and [Gauthier 2009]. In the context of number theory, it is discussed in relation to the work of Gauss by Catherine Goldstein and Norbert Schappacher in the first two chapters of [Goldstein et al. 2007].

Analysis, rather than a discipline or specialty, was originally a method, the opposite of synthesis, and hence part of a practice of proof, or of description of proof. Algebra is an analytical method in that antique sense, and the addition of differential and integral techniques enriched the set of algebraic methods beginning (in Europe) in the seventeenth century. This set of novelties led to a body of work that supplemented older algebra and led to a new set of problems that could be attacked using these methods. Over the course of the nineteenth and twentieth centuries, this somewhat fluid picture hardened into a more or less codified set of practices that were to be mastered by the disciples.

2. HERMITE'S STARTING POINT: ANALYSIS IN FRANCE CIRCA 1840

As already observed, the older algebraic analysis emphasized formulas and symbol manipulation as a way to produce and control them. Already by the early nineteenth century, working with formulas had led to various paradoxical results, as is well known, and Hermite's mentor Cauchy was famously critical of what he had termed the "generalities of algebra," an approach from which he disassociated his own work in the Introduction to his *Analyse Algébrique* [Gilain 1989]. It is noteworthy, though, that Cauchy's own foundational efforts involving limits did not at once find general use. Nor, when used, did they provide a clear way to resolve many issues, notably in cases involving multiple limits at once. Similarly, despite the clarifications provided by Cauchy in the field of complex function theory, the uptake was slow [Bottazzini & Gray 2013, chs 2 and 3].

Hence the older, eighteenth-century forms held sway until at least 1850 in many contexts, both in France and elsewhere. Hermite, a "little Lagrange" in the words of his lycée mathematics professor Louis Paul Émile Richard, started research while at the École polytechnique [Hermite 1905–1917, vii, Préface by E. Picard]. The work of Lagrange and Abel were his main starting points, to judge from his early publications. His first good results built on work of C. G. J. Jacobi, and Hermite was encouraged by Joseph Liouville, an important mentor and supporter, to send these to Jacobi. These concerned the division of elliptic functions and the extension of some of Jacobi's results to hyperelliptic functions. Jacobi's positive response was read aloud at the Académie. To give some impression of context for at least some of Hermite's interests, we summarize some work of Jacobi.

The Context of Hermite's Early Work: Jacobi and Abel

Jacobi, in the 1829 *Fundamenta*, recalled the theorem of Euler, that if

$$\Pi(x) = \int_0^x \frac{dx}{\sqrt{X}},$$

where X is a polynomial of degree four, then we may write

$$\Pi(x) + \Pi(y) = \Pi(a),$$

where a is an algebraic function of x and y. [Jacobi 1829]

Niels Henrik Abel had extended this to polynomials of any degree. The more general result, called Abel's theorem by Jacobi, states:

Let X be a polynomial of degree $2m$ or $2m - 1$, and define

$$\Pi(x) = \int_0^x \frac{(A + A_1 x + \cdots + A_{m-2} x^{m-2})\,dx}{\sqrt{X}},$$

then given m values of the variable x it is possible to determine from them $m - 1$ quantities a_i such that

$$\Pi(x) + \Pi(x_1) + \cdots + \Pi(x_{m-1}) = \Pi(a) + \Pi(a_1) + \Pi(a_{m-2}).$$

An open question concerned what the inverses of the Abelian integrals are like, and what Abel's theorem can tell us about them.

Given our interest in Hermite's style of analysis, we note that his immersion in Jacobi's work must have influenced him considerably. Jacobi, emphasizing the production of formulas as a key part of investigating the properties of the functions whose analysis concerned him, frequently took what might be termed an algebraic point of view. He stressed for example that the values of functions are to be determined as the roots of algebraic equations.

We introduce some notation of Jacobi: If X is a polynomial of degree 5 or 6, and if

$$\int_0^x \frac{dx}{\sqrt{X}} = \Phi(x), \quad \int_0^x \frac{xdx}{\sqrt{X}} = \Phi_1(x)$$

and if we define

$$\Phi(x) + \Phi(y) = u, \quad \Phi_1(x) + \Phi_1(y) = v,$$

then on inversion we find that

$$x = \lambda(u, v)$$
$$y = \lambda_1(u, v).$$

These are the Abelian functions which are analogous in this higher-degree case to the trigonometric and elliptic functions.

Jacobi, in 1834, obtained concrete results from Abel's theorem, where I use the term concrete advisedly to indicate that he provided specific formulas (rather than, say, existence theorems or convergence results). If

$$X = x(1-x)(1-\varkappa^2 x)(1-\lambda^2 x)(1-\mu^2 x),$$

where $1 > \varkappa^2 > \lambda^2 > \mu^2$, then given the two equations

$$\int_0^x \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \int_{\frac{1}{\mu^2}}^y \frac{(\alpha+\beta x)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^z \frac{(\alpha+\beta x)dx}{\sqrt{X}}$$

$$\int_0^x \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} + \int_{\frac{1}{\mu^2}}^y \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^z \frac{(\alpha'+\beta' x)dx}{\sqrt{X}},$$

x, y and z are the roots of the quadratic

$$Ux^2 - U'x + U'' = 0,$$

where

$$U = \varkappa^2\mu^2(1-z)(1-\lambda^2 z),$$
$$U' = \varkappa^2 + \mu^2 + [\lambda^2 - (\varkappa^2+\mu^2)(1+\lambda^2) - \varkappa^2\mu^2]z + \varkappa^2\mu^2(1+\lambda^2)z^2,$$
$$U'' = (1-\varkappa^2 z)(1-\mu^2 z).$$

A final remark on this paper brings us to the immediate point of departure of Hermite. If we are given $x = \lambda(u, u')$ and $y = \lambda'(u, u')$, then by Abel's theorem the functions $x_n = \lambda(nu, nu')$ and $y_n = \lambda'(nu, nu')$ are given as the roots of a quadratic equation $U_n x^2 - U_n' x + U_n'' = 0$, where U_n, U_n', U_n'' and rational functions of $x, y, \sqrt{X}, \sqrt{Y}$ where X is a function of x and Y is a function of y. This multiplication theorem—given knowledge about u, draw conclusions about nu—is what Hermite inverted to get his division theorem.

We do not occupy ourselves with the details here, but rather simply remark on some aspects of Hermite's work in the 1840s. Hermite worked at first directly from Abel and Jacobi. The technical apparatus of Cauchy's complex analysis was not known to him until later, at least according to his (admittedly much later) preface to Riemann's collected works:

> La notion d'intégration le long d'une courbe avait été exposé ... dès 1825 dans un mémoire de Cauchy ... mais elle reste dans les mains de l'illustre Auteur; elle n'est connue ni de Jacobi , ni d'Eisenstein ... ; il faut attendre vingt-cinq ans, jusqu'aux travaux de Puiseux, de Briot et Bouquet, pour qu'elle prenne son essor et rayonne dans l'Analyse. [Hermite 1898]

However, in 1849 he wrote a paper, unpublished at the time, proving that analytic functions can't have more than two periods [Belhoste 1996]. This indicates that his knowledge of such matters was better than might be thought.

We have just mentioned Cauchy, Riemann, and Weierstrass, three important innovators whose impacts on analysis have been much discussed. While Hermite was certainly involved with, and to some degree informed about, the work of these three men, his own approaches generally contrast with at least aspects of their work. While Cauchy's limit-based foundations and careful definitions seem rooted in his pedagogical efforts of the 1820s, they were not always strictly adhered to in his own work, and took time to assume the foundational character that is now one of the chief associations with his name. This did not much interest Hermite, as we shall see later in our brief account of his lectures. Nor, despite Hermite's interest in those aspects of Weierstrass's work that bear on explicit representation of functions, did Hermite show any interest in a building up of the analysis on the basis of arithmetic-algebraic properties that was a *Leitmotif* of the career of Weierstrass.

Similarly, Riemann's foundational thinking about analysis involved a limit-based foundationalism, as seen in the *Habilitationsschrift* on Fourier series and the definition of the Riemann integral, where the influence of Cauchy and of Dirichlet are evident. In addition, Riemann's employment and reflection on the geometric or topological setting in both complex analysis and in the 1857 paper on abelian functions are absent from Hermite's work, even if he was to praise it as ground-breaking late in his career.

3. HERMITE'S MATURE ANALYSIS

3.1. *The solution of the quintic*

In the following decade, Hermite was to diversify his practice; during this time, he attained international recognition by his discussion of the solution of the quintic by means of elliptic functions, a topic that also interested Leopold Kronecker and Francesco Brioschi. To give an idea of the flavor of this work, with the end in mind of looking more closely at the content of his courses later on, we discuss the nature of the 1858 result on this subject [Hermite 1858a]. We note that there are several more detailed studies on aspects of the solution of the quintic via elliptic functions, including [Zappa 1995]. Further, Goldstein [2011a] discusses details of Hermite's use of Galois theory in his own effort.

We begin with some nomenclature: the inverse of

$$u(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}},$$

was called the *sinus amplitudinis* function, or (later) $x(u,k) = sn(u,k)$. The parameter k is called the *modulus* of the function. This affects the two periods of the function. The *period ratio*, frequently denoted ω, likewise is a function of k. Or, if you like, we can consider ω as an independent variable, and look at the modulus.

Now, we consider some features of the work, in which the analytic content will largely remain hidden. In 1827, Jacobi had provided a critical tool [Jacobi 1827]. He transformed the differential $\frac{dy}{\sqrt{P}}$ ($\deg P = 4$) by letting $y = \frac{A}{B}$, where A and B are polynomials of degree p. This was the basis of his multiplication theorems, that is, theorems that express $\mathrm{sn}(nu)$ in terms of $\mathrm{sn}(u)$. If one makes such a transformation and reduces the integral to standard form, the old and new moduli satisfy an equation, depending on the order of the transformation. If k is the original modulus and λ is the transformed modulus, and if we let $u = \sqrt[4]{k}$ and $v = \sqrt[4]{\lambda}$, then if $n = 5$, we obtain $u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$, a modular equation.

For a substitution of order p, the modular equation is of degree $p + 1$. In the 1827 paper, Jacobi gave the results for $p = 3$ and $p = 5$, while in the 1829 *Fundamenta*, Jacobi likewise obtained theta-function expansions for u and v, establishing relations between them.

The theta-function representation of an elliptic function was a basic tool for Jacobi, and a good deal of the *Fundamenta* is devoted to these. For Jacobi, the use was both theoretical and practical, since the rapid convergence of the theta-function expressions allowed calculations. Hermite built on his use of theta-functions for both theoretical and computational ends, though the published work frequently contains little or no detail of actual algebraic or numerical calculation.

Given $\phi(\omega) = \sqrt[4]{k}$, $\psi(\omega) = \sqrt[4]{k'}$, Hermite in 1858 used what he knew from Jacobi: that, for example, $\phi(n\omega)$ and $\phi(\omega)$ are linked by a modular equation of degree $n + 1$. Using a result of Sohncke, he then defined

$$\Phi(\omega) = \left[\phi(5\omega) + \phi\left(\frac{\omega}{5}\right)\right]\left[\phi\left(\frac{\omega + 16}{5}\right) - \phi\left(\frac{\omega + 4 \cdot 16}{5}\right)\right] \times \left[\phi\left(\frac{\omega + 2 \cdot 16}{5}\right) - \phi\left(\frac{\omega + 3 \cdot 16}{5}\right)\right].$$

He then *asserted* that the numbers $\Phi(\omega + k \cdot 16)$, $k = 0, \ldots, 4$ are the five roots of a quintic whose coefficients are rational functions of ϕ and ψ. This was the reduction of the sixth-order modular equation to one of order five, promised by Galois.

Where is the analysis? The analysis consists for the most part of θ-function identities. However, Hermite didn't tell the reader, in [Hermite 1858a], what he used. He produced a longer 1858 paper on this technique, [Hermite 1858b], where he left the final result as an exercise.This may be the reason that the publication was only in 1858, since he claimed to have had the Galois-theoretic reduction idea at least 10 years before. He also didn't tell the reader how the Galois theory was employed. What we know of how he conceived of this was explained to Jules Tannery in 1900 [Hermite 1902].

The theta-function identities, originating with Jacobi but refined and extended by Hermite and his students, were central to the theory of elliptic functions. They provide an excellent example of the kind of concrete formula that Hermite thought was the core of analysis, so we say a few words about them here. For convenience, consider the tidy formulation of McKean and Moll [McKean & Moll 1997, ch. 3], avoiding some complexities in writing them down just to give the flavor. If ω is the ratio of the periods of an elliptic function, and if $p = \exp(i\pi x), q = \exp(i\pi\omega)$ we can define

$$\theta_1 = i \sum (-1)^n p^{2n-1} q^{(n-1/2^2},$$

where the sum is over the integers. The other functions are defined similarly in such a way that shifts of half a period transform them into each other. The series converge rapidly, a key property, and another is that they have simple relations to one another and to the elliptic functions. For example,

$$sn(x,k) = \frac{\theta_3(0)\theta_1(x')}{\theta_2(0)\theta_4(x')},$$

where

$$x' = \frac{x}{\pi\theta_3^2(0)}.$$

This, and an enormous multitude of related results were used for various theoretical and computational purposes by all who worked in this line. Hence mastering the basic relations and formulas, and learning to work with them efficiently, was the core of an apprenticeship in this field. For this, an introductory work (or set of lectures) was the usual starting point.

3.2. *The appendix to Lacroix: A pedagogical work*

The first pedagogical work in which Hermite took on the problem of rendering his own ideas about analysis in a coherent form for others appeared in 1862. The occasion for this was a new edition of the *Traité du calcul différentiel et intégral* by Sylvestre-François Lacroix, edited by Joseph-Alfred Serret and Hermite, to which Hermite appended a long discussion of the theory of elliptic functions ([Hermite 1862a]). The point of the revision was to update this standard manual for learning analysis. The addition of a treatment of elliptic functions was novel in such a work, and the chapter was designed both for learners and as a reference. Hermite presented an outline of the basic properties of these functions, due not only to Jacobi and Abel but to Liouville, Cauchy, Riemann, and others. The expression of the elliptic functions via theta functions, modular equations, and the differential equations that they satisfy, were laid out in these pages. The starting points for his earlier researches in the area were likewise presented here in a connected way. Thus, for example, we find an account of the theta-function identities, of the different notations for elliptic functions, and of the basic properties. Abel's theorem, mentioned above, was also proved. In addition the "more convenient" Gudermann notation was described. There was also an account of Karl Weierstrass' methods of representation, in the form that they had appeared in the 1856 *Über die Theorie der analytischen Facultäten* [Weierstrass 1856].

4. HERMITE: A BROADENING INTERNATIONAL SCOPE

Hermite was elected to the Académie des sciences in 1856. His growing reputation in the late 1850s put him more and more in contact with the wider world of European mathematics, and already by the time of his election he had a considerable contact with mathematicians outside France [Archibald 2002]. As we already mentioned, Weierstrass' work was of particular interest to him. As Hermite repeatedly and convincingly denied his ability to read German, it is not immediately clear how he learned of this, but one of his fellow researchers in elliptic functions was Carl Wilhelm Borchardt, with whom Hermite had corresponded. The issue of translation of German work for French audiences is discussed in some detail by N. Verdier in [Chatzis et al. 2018].

Hermite noted the importance of theta functions already in the 1862 Appendix:

> ... on voit s'offrir un autre mode de représentation où les fonctions doublement périodiques sont exprimés par des quotients de séries rationalles en x et k^2 [k is the modulus] et convergentes quelles que soient les valeurs réelles ou imaginaires de ces deux quantités. Abel avait entrevu et rapidement indiqué la possibilité de ce nouveau mode d'expression des fonctions elliptiques, mais c'est à M. Weierstrass que revient l'honneur d'avoir mis dans la Science, au lieu d'un simple aperçu, une théorie profonde qui conduit directement à ces nouvelles fonctions ... [also for abelian transcendents, also for any number of variables]
> [Hermite 1862a, 208]

Thus Hermite, in 1862, appeared to have been unaware of the general efforts of Weierstrass to devise a better foundation for analysis, and indeed there is no reason that the publications of Weierstrass to that point would have drawn this to his attention, since he did not read German. Indeed, despite a general interest in Weierstrass' work as it related to elliptic functions and their generalizations, he showed relatively little interest in the foundational aspect of Weierstrass' work. By contrast, however, he was very interested in the general results on the representation of analytic functions that appeared in 1876 [Weierstrass 1876]. He was to learn of this in 1877 at the Gauss centennial, which he attended, via a verbal report from Hermann Amandus Schwarz. Hermite, despite his lack of German, quickly mastered this work. This and his earlier enthusiasm for the work of Weierstrass on elliptic functions was probably the basis of his recommendation, shortly afterward, that Gösta Mittag-Leffler study in Berlin. Mittag-Leffler's own extension of Weierstrass' representation theorem to meromorphic functions was likewise of great interest to Hermite.

5. RELATION TO TEACHING PRACTICE: TRANSMITTING NEW ELEMENTS OF DISCIPLINE

Hermite's teaching expressed many of his values and preferences. We shall not consider all of his courses here, but rather a small sample having to do with real and complex analysis, including differential and integral calculus. At the Sorbonne, this was not, strictly speaking, in his ambit, since his chair was in higher algebra. However, he taught these subjects regularly there and at the École polytechnique. His activity was of course embedded in a long tradition of French *cours d'analyse*, which has been examined extensively in the thesis of Hervé Renaud [Renaud 2017].

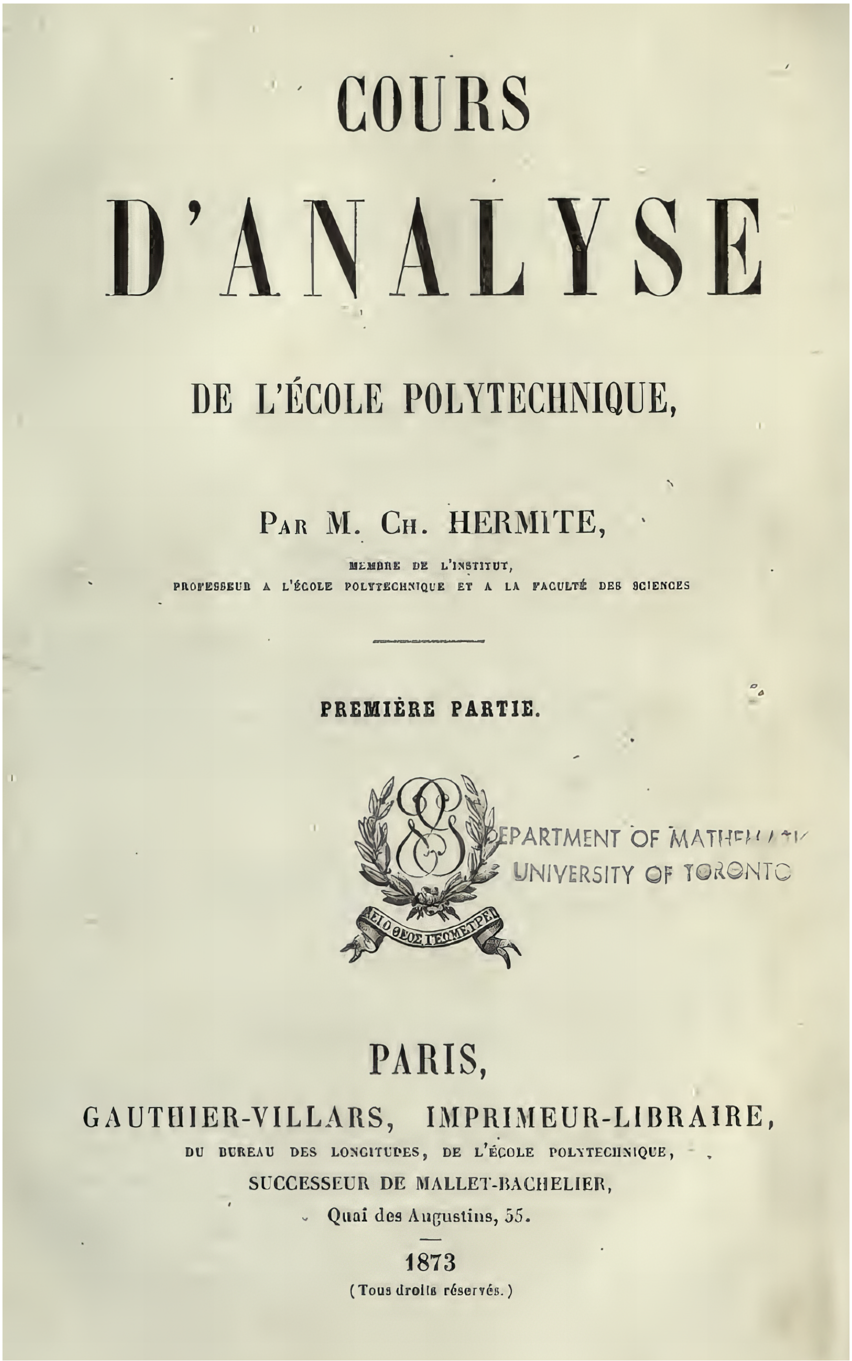
5.1. *The Cours at the École polytechnique, 1873*

The tightly controlled curriculum at the École polytechnique meant that Hermite was restricted there to a course in "Calcul différentiel et intégral" [Hermite 1873a]. Historical discussions of the curriculum at the École polytechnique in the preceding period may be found in [Gilain 1989] and [Belhoste et al. 1994], among others. Despite these curricular limits, his own tastes were very much in evidence. For example, he began with series expansions, an echo of the Lagrange he had studied himself. There is also at least passing reference to functions of a complex variable. Hermite also lectured at the Faculté des sciences in the 1870s on elliptic functions, following the approach of the Lacroix appendix. Notes for the latter from Mittag-Leffler have survived, and are discussed by Umberto Bottazzini and Jeremy Gray [Bottazzini & Gray 2013, 418-19].

The 1873 lecture notes at Polytechnique included a list of traditionally mandated topics, including the basics of differentiation and integration, the usual longish list of special integrals, and a number of geometric applications, including area and arc length. Elliptic functions were introduced too, as they arise in the arc length of conics and in the integrals of algebraic functions.

However, Hermite's ideas about the disciplinary structure are stated explicitly, in a pedagogical expression designed to inculcate his views in the young reader or teacher employing the volume.

> Les éléments des mathématiques présentent deux divisions bien tranchées: d'une part, l'Arithmétique et l'Algèbre; de l'autre, la Géométrie. Rien de plus différent, à leur début, que les considérations et les methodes propres à ces deux parties d'une même science, et, bien qu'associées dans la Géométrie analytique, elles restent essentiellement distinctes si loin qu'on les poursuive, et



COURS

D'ANALYSE

DE L'ÉCOLE POLYTECHNIQUE,

PAR M. CH. HERMITE,

MEMBRE DE L'INSTITUT,
PROFESSEUR A L'ÉCOLE POLYTECHNIQUE ET A LA FACULTÉ DES SCIENCES

PREMIÈRE PARTIE.

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE,
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
SUCCESSEUR DE MALLET-BACHELIER,
Quai des Augustins, 55.

1873

> paraissent se rapporter à des aptitudes et à des tendances intellectuelles spéciales. [Hermite 1873a, first page, unnumbered]

Despite this, differential and integral calculus, the subjects of the analysis course, brought the two threads together:

> Ce double point de vue de l'Algèbre et de la Géométrie se retrouve dans le Calcul différentiel et le Calcul intégral; on peut dire en effet des ces nouvelles branches de Mathématiques qu'elles sont comme une Algèbre plus vaste et plus féconde, appliquées à des questions de Géométrie inaccessibles au Calcul élémentaire, telles que la quadrature des courbes, la détermination des volumes limités par des surfaces quelconques, la rectification des courbes planes ou gauches, etc. [Ibid., idem]

In practice, however, the two fields were not on an equal footing in the calculus as presented by Hermite, as the preceding list suggests: geometry is not the source of the methods for these, nor is it the focus.

Continuing in this preface, Hermite insisted on an *Infinitesimal method* and left subtleties about infinities aside as unneeded, at least at this beginning level. After describing briefly some key results of the calculus, Hermite continued:

> C'est l'application répétée de ces mêmes propositions qui constitute ce qu'on nomme la *méthode infinitésimale* [his emphasis], méthode ... dont il sera dans ce Cours de nombreux exemples. Mais, dès à présent, nous devons dire qu'en se montrant de plus en plus féconde la notion de l'infini reste toujours simplement la notion d'une grandeur supérieure à toute grandeur donnée, et que les conditions de son emploi restent toujours celles des éléments de la Géométrie. [Hermite 1873a, second page, unnumbered]

What Hermite meant by the last point, that the basics of geometry are the constraint on the uses of infinity, remains a bit unclear. As Catherine Goldstein has pointed out, Hermite did not care for the invention of mathematical concepts, and thus shied away from discussing such things as the circles at infinity of Michel Chasles [Goldstein 2011b]. In any case, here he is addressing himself to beginners, and it might be possible to read too much into the statement.

However much he may have been constrained by the curriculum at the École, Hermite in many places showed his love of the formula, and via examples attempted to communicate his own flair and taste for combining algebra with a fairly naive notion of real and complex function theory in order to arrive at striking results in an elementary way. Consider the following example that applied the method of partial fractions to express the

Taylor coefficients of a rational function in trigonometric form [Hermite 1873b].

Hermite takes

$$f(x) = \frac{Ax + B}{x^2 - a^2}.$$

Rewriting using partial fractions, he obtained

$$\frac{f^{(n)}(x)}{n!} = \frac{(-1)^n}{2a}\left[\frac{Aa + B}{(x-a)^{(n+1)}} + \frac{Aa - B}{(x+a)^{(n+1)}}\right].$$

Hermite then set $a = i, A = 0, B = 1$ to get

$$\frac{f^{(n)}(x)}{n!} = \frac{(-1)^n}{2i}\left[\frac{1}{(x-i)^{(n+1)}} + \frac{1}{(x+i)^{(n+1)}}\right],$$

which, he noted, takes a particularly simple form if we make the substitution

$$x = \frac{\cos\phi}{\sin\phi}.$$

Three lines later, he obtained

$$\frac{f^{(n)}(x)}{n!} = (-1)^n \sin^{n+1}\phi \sin(n+1)\phi.$$

The 1873 *Cours* abounded with examples of this kind, Hermite taking the opportunity to display things that he found beautiful and that were consistent in this esthetic with the kind of results he obtained in his research.

5.2. *New directions in the lecture course after 1877*

As mentioned above, in 1877 Hermite attended a meeting in Göttingen in honor of the centennial of Gauss' birth; it was at this time that he made the acquaintance of the recent work of Weierstrass on the representation of analytic functions. Along with this scientific information, he returned deeply impressed with the status of the German professor in society—he could not imagine a French professor being honored as was Gauss—and envious of the German relationship between professors and students. This was much discussed in Hermite's correspondence, notable that with Paul Du Bois-Reymond [Lampe 1916].

He also seems to have absorbed at that time the German idea that the content of the lecture course should reflect the recent work of the professor, and that the professor could in fact choose the content freely. This was referred to as *Lehrfreiheit*. French courses, led by the curricula of the grandes écoles and rigidly controlled, had tended to focus on topics seen to be of use to the students in their career. This idea of *Lehrfreiheit* was embodied to at least some extent in Hermite's courses at the Faculté shortly afterwards, though the issue was complicated by the fact that others were dependent on his coverage of certain topics.

The version of the course recorded by Henri Andoyer in 1882 differed markedly from the one at the École polytechnique. [Hermite 1882] It was, of course, aimed at a different audience which, beyond the students at the Faculté des sciences, included students from the École normale supérieure. Here the lecture topics were not restricted by the school- and state-approved course outlines at polytechnique.

This lack of restriction, perhaps coupled with his enthusiasm for the German approach to teaching, led to a course that must have been challenging even for the best students. It would have required that the student be familiar with the techniques of finding derivatives and integrals for the usual elementary functions, or else that they very quickly acquaint themselves with those.

Some students apparently found this challenging, as we know from remarks published in an illustrated newspaper that appeared in connection with the new freedom of the press, *Le Passant.* Here we find, in a section called "Gazette de l'université," a remark from a student noting that

> Vous venez pour apprendre à intégrer, M. Hermite suppose que vous avez intégré toute votre vie. [Dunoiset 1882, 5]

This suggests that not all students were equipped with the prerequisite knowledge, though in fairness to Hermite we note that secondary school treatments would usually have included the requisite information about integrals. The same article also gave voice to other dissatisfactions:

> Son cours n'est pas un cours, c'est une conversation abracadabrante, faites à bâtons rompus, avec des digressions interminables, des chevauchées perpetuelles à travers l'Europe entière. [Dunoiset 1882, 5]

Hence the international character of the content was not so much appreciated, at least not by this witness.

As an illustration of the level, the course began with finding the area bounded by the ellipse. The integral of $\cos^2 x$ that results is handled by the comment that the method is an instance of a general formula that students would have needed to see in a preparatory course. One should use:

la méthode générale relative à la quantité $\int cos^m x \sin^n x dx$. On transforme les puissances ou produits des puissances du cosinus de x et sinus de x en expressions linéaires par rapport aux cosinus et sinus des multiples de l'arc.
[Hermite 1882, 6]

The resulting value is then stated. It is followed by a faster, less pedestrian, and more generally applicable trick to get the same result. This emphasis on "smart" calculation was likewise a feature of the entire course.

The course consisted of 25 lectures. Beginning with some area and arc length integrals, Hermite moved at once to elliptic integrals and their standard forms, followed by hyperelliptic integrals in the fourth lecture. The treatment of real variables concluded with double integrals, including the definitions and the application to the volume of the ellipsoid.

Lectures 6 through 13 included the basics of complex variable theory, using the ideas of Cauchy and expounding that theory much as it is sometimes done now, concluding with the residue calculus. Along the way, though, there were topics that could be described as advanced, such as the primary decomposition theorem of Weierstrass. The Mittag-Leffler canonical expression for meromorphic functions is also treated, as are important applications, for example the expression of the Legendre polynomials as integrals.

Lectures 14 through 20 covered a variety of results that in the mid-twentieth century would have been described as belonging mostly to the theory of special functions. Much of this work was quite up to date, including work that had been produced in the previous few years by such individuals as G. Mittag-Leffler, Edmond Laguerre, Bernhard Riemann, Pafnuti Chebyshev, Jules Tannery, and Paul Appell. One of the pathological examples given was due to Henri Poincaré. The final 5 lectures surveyed basic facts about elliptic functions and their expression using theta functions.

Thus, whether or not Hermite was directly inspired by the German example on which he remarked in 1877, in effect his choice of material was intended exactly to inculcate in his listeners not only a clear understanding of the basic theory of what he considered most important in analysis, but a connected synthesis of major current activity in the field internationally.

In general though, despite the negative individual reaction cited above (which also complained about Hermite's demonstrations of piety) the new curriculum was influential, and became the seed for later developments. It circulated widely, first in an affordable lithographed transcription. As a result of this publication, it had an influence beyond its auditors. For example, although it postdated the time of Picard as a student at the ENS, it also was influential on his later course. And some students, both from this version and from the earlier one, did in fact continue research work as thesis students.

6. THESES AND TEACHING AT THE LYCÉE

As the century developed, there was an expansion of interest in the doctorate in mathematics. This qualification offered a route to work in the expanding facultés as well as in the lycées and écoles préparatoires. Because of his position, Hermite was very much placed, after 1871, to play an influential role in the direction of research, and he was involved in a large number of theses, not always as director. Here we mention a few examples of students. These show aspects of Hermite's influence as a research leader, but they were also in a position to affect the direction of teaching at the lycée. Note that the role of the thesis director varied at that time, as it does today, and indeed the actual "supervisor" role that is now common did not exist as such. Hence the involvement of a given director is somewhat inferential, though in the cases discussed here the connection is well-attested via collaborations or specific acknowledgement.

The first example is Charles Joubert (1825-1906). Joubert was a Jesuit priest who, beginning in 1854, joined the faculty at the recently established École Ste. Géneviève, now often known as *Ginette*, then in Paris. Following the 1850 law allowing Catholic education to be re-instituted, the Jesuits established this school with the particular and specified aim of doing preparatory work for admission to the École polytechnique, to the military academy at St.-Cyr and to the École centrale des manufactures. In this the school achieved great success, with for example 27 admissions to Polytechnique in 1868. Joubert's activity at this school began very close to the time when Hermite committed anew to Catholicism, ostensibly at the urging of Cauchy during a severe illness of Hermite. Hermite was not to waiver in this, and he was often involved directly or indirectly with Catholic education, which had renewed problems after the revived secularism of the third republic. [Madelin 2004, Burnichon 1919, p. 448]

Joubert was an active mathematical researcher before he obtained the doctorate, in 1876. For example, he published a result in the *Comptes rendus* in 1867 that directly built on work of Hermite, and consists of what is now known as the Hermite-Joubert theorem. Without going into details, we note that this is a pendant to the work mentioned above on the equation of degree 5. [Hermite 1861a], [Joubert 1867]. Joubert's thesis subject, the transformation of elliptic functions, was directly linked to Hermite's work.

Joubert based his work on research of Hermite, and the basic treatise of Briot and Bouquet (1859) as well as Hermite's appendix to Lacroix. He also showed himself to have a good command of relevant German work, using Kronecker's ideas about complex multiplication, and citing Jacobi and Sohnke.

Two other thesis students, Charles Biehler (1845-1906) and Désiré André (1840-1917), had connections with the Collège Stanislas. This was likewise a Catholic school, associated with the Marianist order, with origins in the early nineteenth century, that likewise flourished anew in the 1850s. It focussed a portion of its efforts on becoming a preparatory school for the most influential institutions of higher education beginning in the 1860s.

Biehler was an early recruit, himself a Marianist postulant from Alsace, and he came to Stanislas to prepare *"spéciales"* from 1863 to 1866. This was at the same time that he was completing his novitiate. Biehler took his final vows in 1867, by which time he had already entered the teaching staff at Stanislas, becoming sous-directeur d'études in 1873. Along the way he had obtained a *licence* from the university by 1872. [Schelker 2018, 52] Biehler dedicated his 1879 thesis to Hermite: "mon maître". It concerns series expansions of elliptic functions. The text of Schelker (reproducing in part an older biography by Zinger) recounts Hermite praying actively in the front row of his thesis defense, which was attested to be brilliant. According to Schelket, Hermite and Biehler met weekly for the duration of Hermite's life.

One of his more famous students from Stanislas, the physicist Pierre Duhem, recalled him fondly:

> Sa thèse sur la théorie des fonctions elliptiques,véritable chef-d'œuvre d'élégance algébrique, avait soulevé en Sorbonne des applaudissements dont l'écho avait retenti jusqu'à nous ; nous savions que la vocation religieuse de l'enseignement, le désir de former des hommes, des Français et des chrétiens, l'avaient seuls décidé à renoncer à la science, à ses joies et à ses honneurs ; nous devinions combien ce sacrifice avait coûté, et nous sentions nos cœurs émus de reconnaissance. [Schelker 2018, 18]

A final example, Désiré André was a normalien who had obtained an *agrégation* in 1865. His 1877 thesis was likely undertaken to get a better position, since he took up a post as chargé de cours in applied mathematics at the faculté des sciences in Dijon in that year. His thesis also concerned series developments of elliptic functions. After 1879 he was "professeur libre" in Paris, presumably meaning that he taught in the Catholic schools and perhaps the Université libre, as Hermite also did. André joined Biehler at the Collège Stanislas in 1885 ([Schelker 2018, 14].

N° D'ORDRE
414.

THÈSES

PRÉSENTÉES

A LA FACULTÉ DES SCIENCES DE PARIS

POUR

LE DOCTORAT ÈS SCIENCES MATHÉMATIQUES,

PAR M. CH. BIEHLER,

Directeur des Études à l'École préparatoire du Collége Stanislas.

THÈSE D'ANALYSE. — SUR LES DÉVELOPPEMENTS EN SÉRIES DES FONCTIONS DOUBLEMENT PÉRIODIQUES DE TROISIÈME ESPÈCE.

THÈSE D'ALGÈBRE.

Soutenues le 8 Avril 1879, devant la Commission d'Examen.

MM. HERMITE, *Président.*

BOUQUET, } *Examinateurs.*
DARBOUX, }

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DE L'ÉCOLE POLYTECHNIQUE, DU BUREAU DES LONGITUDES,

SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1879

Figure 1. The thesis of Charles Biehler

Thus, students at highly competitive preparatory schools were being guided to success by masters who had studied Hermite's methods closely enough to be able to work in the same vein. While this stops short of documenting a direct effect of Hermite's vision of analysis on preparatory instruction, the vision articulated by Hermite in his courses would be well-known to these men. Further research is needed concerning the curriculum in these institutions to demonstrate conclusively whether any effects are direct or indirect. However, we can conclude even from these few examples that the Hermitian direction in the analysis of elliptic and theta-functions was being actively pursued by a community of interacting researchers who were likewise influential in the teaching world.

7. CONCLUDING REMARKS

The concrete analysis of Hermite's practice, and of his *Cours*, did not become the standard for basic instruction, though it has of course survived. In contrast, the abstract and axiomatic approach that he did not favor came very much into vogue in the first decade of the twentieth century, at least in the research field, with the work of Lebesgue informed by Hilbert's *Grundlagen der Geometrie*. Hilbert's paradise of set theory and his work on integral equations bore a great deal of fruit very rapidly in the realm of analysis. In France the work of Baire and Fréchet struck out in markedly different directions from that of Hermite.

Hermite's version of analysis has, however, enjoyed a healthy revival with the increased availability of powerful computational methods. This is not least because of the importance for applied mathematics of concrete representations. The British analysts, including Ramanujan, and the multinational group associated with the California-based Bateman project provide examples of this kind of work, and many others were presented during the anniversary conference of which the original of this paper was part. To focus on the reproductive function of disciplinary norms, though, we see that the Hermitian approach is not today the basic path, which follows instead the routes trodden by such as Paul Halmos, Jean Dieudonné, Lars Ahlfors, and Walter Rudin. There remains no unified approach to higher analysis. Instead, this "discipline" is a very large tent with many corners.

One of these is certainly the corner associated with the characteristics of Hermite's approach, both to research and to teaching. As we saw above, some of these features are most easily described by what they are not. They don't appeal to some sort of foundational concept. Weierstrass, for example, sought to define functions along the lines of the ways in which numbers are defined, taking basic objects and combining them via finite and infinite operations. Further, while aspects of Cauchy's foundations are of course used, detailed arguments about multiple limits, for example, do not encumber the discussions of Hermite either in teaching or in his papers. Instead, taking functions as naturally occurring objects that are produced via the procedures of algebra and the differential and integral calculus, they are discussed and explored consistely with the ways in which other objects of creation are studied scientifically. Hermite's particularly strong interest in concrete representation, and the use of formulas displaying functions in ways that permit their comparison and the analysis of their individual characteristics, remains the central feature of his approaches. This is certainly consistent with his commitment to naturalism, and also is very much in line with his number-theoretic and algebraic work. This has been discussed in some detail by Goldstein, and we conclude with a citation from a work of Hermite in 1850 to which she has drawn attention:

> [T]he task for number theory and integral calculus [is] to penetrate into the nature of such a multitude of entities of reason, to classify them into mutually irreducible groups, to constitute them all individually through characteristic and elementary definitions.
>
> [Hermite, Letter to Jacobi, 1850, cited in [Goldstein 2018]]

REFERENCES

ARCHIBALD (Tom)
[2002] Charles Hermite and German Mathematics in France, in Rice (Adrian) & Parshall (K.), eds., *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800-1945*, Providence: AMS, 2002, pp. 123–137.

BELHOSTE (Bruno)
[1996] Autour d'un mémoire inédit : la contribution d'Hermite au développement de la théorie des fonctions elliptiques, *Revue d'histoire des mathématiques,* 2 (1996), pp. 1–66.

BELHOSTE (Bruno), DAHAN (A.) & PICON (A.)
[1994] *La Formation polytechnicienne 1794-1994*, Paris: Dunod, 1994.

BOTTAZZINI (U.) & GRAY (J.)
[2013] *Hidden Harmony–Geometric Fantasies: The Rise of Complex Function Theory,* Heidelberg: Springer, 2013.

Burnichon (Joseph)
[1919] *La compagnie de Jésus en France, t. III*, Paris: Gabriel Beauchesne, 1919.

Chatzis (K.), Morel (Thomas), Préveraud (Thomas) & Verdier (Norbert)
[2018] Traduire des mathématiques "pour et par des élèves" dans la première moitié du xix^e siècle: Acteurs et pratiques de traduction à travers trois cas d'étude en Europe et aux Etats-Unis , *Mémoires du Livre*, 24 (2018) URL `https://www.erudit.org/fr/revues/memoires/`.

Dunoiset (A.)
[1882] Gazette de l'université, *Le Passant*, 1(5) (1882), p. 5.

Ferraro (Giovanni D.), ed.
[2008] *The Rise and Development of the Theory of Series up to the Early 1820s*, Heidelberg: Springer, 2008.

Fraser (Craig)
[1989] The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century, *Archive for History of Exact Sciences*, 39 (1989), pp. 317–335.

Gauthier (Sébastien)
[2007] *La géométrie des nombres comme discipline (1890-1945)*, Paris: Thesis, Université Pierre et Marie Curie, 2007.
[2009] La géométrie dans la géométrie des nombres : histoire de discipline ou histoire de pratiques à partir des exemples de Minkowski, Mordell et Davenport, *Revue d'histoire des mathématiques*, 15 (2009), pp. 183–230.

Gilain (Christian)
[1989] Cauchy et le cours d'analyse de l'Ecole polytechnique, *Bulletin de la Sabix*, 5 (1989), pp. 3–31.

Goldstein (Catherine)
[2007] The Hermitian Form of Reading the *Disquisitiones*, in Goldstein (C.), Schappacher (N.) & Schwermer (J.), eds., *The Shaping of Arithmetic after C. F. Gauss's* Disquisitiones Arithmeticae, Heidelberg: Springer, 2007, pp. 376–411.
[2011a] Charles Hermite's Stroll through the Galois Fields, *Revue d'histoire des mathématiques*, 17 (2011), pp. 211–270.
[2011b] Un arithméticien contre l'arithmétisation: les principes de Charles Hermite, in Flamant (D.) & Nabonnand (P.), eds., *Justifier en mathématiques*, Paris: MSH, 2011, pp. 129–165.
[2018] Long-term History and Ephemeral Configurations, *Proceedings of the ICM*, 1 (2018), pp. 487–522.

Goldstein (C.), Schappacher (N.) & Schwermer (J.), eds.
[2007] *The Shaping of Arithmetic after C. F. Gauss's* Disquisitiones Arithmeticae, Heidelberg: Springer, 2007.

HERMITE (Charles)

[1850] Lettres de M. Hermite à M. Jacobi sur différents objets de la théorie des nombres, *Journal für Mathematik*, 40 (1850), pp. 261–315; Jacobi, C.G.J. *Mathem. Werke*, 2, Berlin, 1851, pp. 221–275; Hermite, Charles. *Œuvres*, 1, Paris, 1905, pp. 100–163.

[1858a] Sur quelques formules relatives à la transformation des fonctions elliptiques, *C. R.*, 46 (1858), pp. 171–175; *Œuvres*, 1, Paris, 1905, pp. 482–486.

[1858b] Sur quelques formules relatives à la transformation des fonctions elliptiques, *J. de mathém. (II)*, 3 (1858), pp. 26–36; *Œuvres*, 1, Paris, 1905, pp. 487–496.

[1861] Extrait de deux lettres à l'éditeur (M. Borchardt) sur l'invariant du dixhuitième ordre des formes du cinquième degré et sur le rôle qu'il joue dans la résolution de l'équation du cinquième degré, *Journal für Mathematik*, 59 (1861), pp. 304–305; *Œuvres*, 2, Paris, 1908, pp. 107–108.

[1862] *Note sur la théorie des fonctions elliptiques, in:* Lacroix (S.-F.), *Traité élémentaire du calcul différentiel et du calcul intégral*, 6 éd., Paris: Mallet-Bachelier, 1862; *Œuvres*, 2, Paris, 1908, pp. 125–238.

[1873a] *Cours d'analyse de l'École polytechnique*, Paris: Gauthier-Villars, 1873.

[1873b] *Sur l'élimination des fonctions arbitraires, in:* Hermite (Ch.), *Cours d'analyse de l'École polytechnique*, Paris: Gauthier-Villars, 1873; *Œuvres*, 2, Paris, 1908, pp. 501–515.

[1882] *Cours de M. Ch. Hermite, professé pendant le 2e semestre 1881–82. Rédigé par M. Andoyer*, Paris: Hermann, 1882 (Lithogr.).

[1898] Préface, *in: Œuvres mathématiques de Riemann*, tr. Léonce Laugel, 1898; *Œuvres*, 4, Paris, 1917, pp. 563–566.

[1902] Lettre de Ch. Hermite à M. Jules Tannery sur les fonctions modulaires, *in:* Tannery (J.) & Molk (J.), *Traité sur la théorie des fonctions elliptiques*, 4 (1902), pp. Paris, pp. 294–300; *Œuvres*, 2, Paris, 1908, pp. 13–21.

[1905–1917] *Œuvres*, vol. 1–4, Paris: Gauthier-Villars, 1905–1917.

JACOBI (C. G. J.)

[1827] Demonstratio Theorematis Ad Functionarum Ellipticarum Spectantis, in Borchardt (C. W.), ed., *Gesammelte Werke*, vol. 1, Berlin: Reimer, 1827, pp. 39–48.

[1829] *Fundamenta Nova Functionarum Ellipticarum*, Koenigsberg: Borntraeger, 1829.

JAHNKE (Niels)

[2003] Algebraic Analysis in the 18th Century, in Jahnke (Niels), ed., *A History of Analysis*, Providence, RI: Amer. Math. Soc., 2003, pp. 105–131.

JOUBERT (Charles)

[1867] Sur l'équation du sixième degré, *C. R.*, 64 (1867), pp. 1025–1029.

LAMPE (Emil)

[1916] Briefe von Ch. Hermite zu P. du Bois-Reymond aus den Jahren 1875-1888, *Archiv der Mathematik und Physik*, 1916, pp. 193–220.

MADELIN (Henri)
[2004] L'école Sainte-Geneviève: 150 ans d'histoire, *Études*, 400 (2004), pp. 477–487.

MCKEAN (H.) & MOLL (V.)
[1997] *Elliptic curves*, Cambridge: Cambridge Univ. Press, 1997.

RENAUD (H.)
[2017] *La fabrication d'un enseignement de l'analyse en France au XIX^{e} siècle: acteurs, institutions, programmes et manuels*, Nantes: Thesis, Universié de Nantes, 2017 URL `https://theses.hal.science/tel-01699107/`.

SCHELKER (N.)
[2018] Charles Biehler (1845-1906): Frère de Marie – Enseignant – Mathématicien, *L'Apôtre de Marie*, 2018, pp. 1–60.

WEIERSTRASS (Karl)
[1856] Über die Theorie der analytischen Facultäten, *Journ. reine und ang. Math.*, 51 (1856), pp. 1–60.
[1876] Zur Theorie der eindeutigen analytischen Functionen , *Mathematische Abhandlungen der Akademie der Wissenschaften zu Berlin*, 1876, pp. 11–60.

ZAPPA (Guido)
[1995] History of the solution of fifth- and sixth-degree equations, with an emphasis on the contributions of Francesco Brioschi, *Rend. Sem. Mat. Fis. Milano*, 65 (1995), pp. 89–107.

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CHARLES HERMITE'S PRACTICES AND THE PROBLEM OF THE UNITY OF MATHEMATICS

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Abstract. — The theme of the unity of mathematics developed during the nineteenth century as specialized articles proliferated and it has often been associated for this period with the definition of new types of mathematical objects in a structuralist setting. This article focusses on the almost opposite point of view of Charles Hermite. Although his work was praised by his contemporaries for beautifully contributing to and displaying the unity of mathematics, he himself strongly opposed the idea of free conceptual creation in mathematics and favored explicit, extensive computations with algebraic forms and classical functions. Hermite's way of testifying to the unity of mathematics must thus be reconstructed by a close reading of his papers, here based on a focus on a few keywords. The result appears proteiform; Hermite operates sometimes by constructing bridges within mathematics through formulas, sometimes by recycling and adapting well-known algebraic expressions, and even occasionally by providing alternative proofs of a theorem. The coherence of these practices with Hermite's general viewpoint on mathematics leads us to advocate for a richer history of the problem of the unity of mathematics.

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Mots clefs. — Charles Hermite, unité des mathématiques, formules mathématiques, analyse algébrique arithmétique.

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Résumé (Les pratiques de Charles Hermite et le problème de l'unité des mathématiques)

Le thème de l'unité des mathématiques s'est développé au cours du dix-neuvième siècle en même temps proliféraient les articles spécialisés et il a souvent été associé, pour cette période, à la définition de nouveaux types d'objets mathématiques dans un cadre structuraliste. Cet article se concentre sur le point de vue presque opposé de Charles Hermite. Bien que son travail ait été loué par ses contemporains pour avoir brillamment contribué à l'unité des mathématiques et même pour l'avoir mise en évidence, il s'est lui-même fermement opposé à l'idée d'une création conceptuelle libre en mathématiques et a privilégié les calculs explicites et étendus sur les formes algébriques et les fonctions classiques. La manière dont Hermite témoignait de l'unité des mathématiques doit donc être reconstituée par une lecture attentive de ses articles, ce que nous ferons ici en suivant les indications de quelques mots-clés. Le résultat apparaît protéiforme, Hermite opérant tantôt en construisant des ponts à l'intérieur des mathématiques par le biais de formules, tantôt en recyclant et en adaptant des expressions algébriques bien connues, et même occasionnellement en fournissant des preuves alternatives d'un théorème. La cohérence de ces pratiques avec le point de vue général d'Hermite sur les mathématiques nous conduit à plaider pour une histoire plus riche du problème de l'unité des mathématiques.

The theme of the unity of science became classic in the philosophy of science during the twentieth century, mathematics being from the start a key feature in this edifice. One of the most famous testimonies of this trend is, of course, the First International Congress for the Unity of Science, which took place in 1935 in Paris, in parallel with the long-term project of the *International Encyclopedia of Unified Science*. Besides Otto Neurath and Rudolf Carnap, the leading figures of the Vienna Circle at the origin of the project, the Congress gathered several representatives of the cream of the mathematical crop of the time: Elie Cartan, Jacques Hadamard, Federigo Enriques, Bertrand Russell, and Richard von Mises, among others. [1] "Recent years have witnessed a striking growth of interest in scientific enterprise and especially in the unity of science," says the front flap of the first volume of the *Encyclopedia*. Boosted in part by the successes and hopes of general relativity and its developments, the theme of the unity of science was often at the time associated with that of the unity of the world—both human and natural—on one side and, on the other, that of the unity of mathematics which was supposed to reflect and to warrant it.

In an address fittingly entitled "The Unity of Mathematics" at the American Association for the Advancement of Science in 1937, the mathematician James Byrnie Shaw, professor at the university of Illinois, claimed for

1 [Neurath et al. 1938]. For a brief history of this movement, see [Morris 1960]; see also [Kremer-Marietti 2003] and [Bourdeau et al. 2018].

instance that "there is unity in architecture, in sculpture, in painting, in poetry, in music, in drama, in dancing, in mathematics. This unity is due to the central ideas which permeate the whole work [...] Mathematics was so interwoven with life that its central ideas are also those of life" [Shaw 1937, p. 402]. Describing these ideas as those of form, identity, invariance, dependence and ideality, Shaw concluded on an epic tone:

> The primeval gods were born of chaos, but their immense power is hurrying the particles of chaos and the ripples of its ocean, its intense fields and its creative spirits, into the unity of a universe. Through the ages of human life mathematics has come to be the screen upon which we may glimpse this unity.
> [Shaw 1937, p. 411]

Besides the spiritual, and even sometimes theological, component illustrated in Shaw's quote, the theme of the unity of mathematics then operates in several ways. One is material: confronted with a potentially discouraging proliferation of results, some argued in favor of new classifications of knowledge embodied in appropriate textual tools, from reviewing journals, such as the *Jahrbuch über die Fortschritte der Mathematik* or the *Répertoire bibliographique des sciences mathématiques*, to all-encompassing encyclopedias, such as Felix Klein and Wilhelm Meyer's *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* or, of course, general books encapsulating new principles of unity, from Giuseppe Peano's *Formulario mathematico* to Nicolas Bourbaki's *Éléments de mathématique*, in whose title the singular "mathématique" emphasizes the unity of the domain.

But the two most well-known and well-studied components of this striving for unity in mathematics are methodological and conceptual. At the beginning of the twentieth century, the unity of mathematics was usually attached to a reduction process (often including a kind of axiomatization, as in Hilbert's program). At a conceptual level, this fostered the emergence of mathematical structures that were supposed to capture the bare bones of various objects or theories at the forefront of research. For the mathematicians promoting them, structures themselves possessed an intrinsic character of unity and at the same time helped to warrant the unity of mathematics, as they can be recognized and used in various mathematical subdomains and situations [Corry 2004].

Tenuous threads link together these different components, with an emphasis depending on the author, the genre of the texts or the time. In 1894, Richard Dedekind, introducing what will be soon considered a key structure, that of a field—the (unfortunate for our purpose here) translation of the word *Körper* in the original German, that is, “body”—already emphasized how it conceptually conveys unity:

> This name [“body”], similarly to that in the natural sciences, in geometry, and in the life of human society, is also intended here to designate a system that possesses a certain completeness, perfection, closure, whereby it appears as an organic whole, as a natural unity. [2]

That this “natural unity” at a conceptual level paves the road to a more global view of the unity of mathematics was elaborated by a number of mathematicians after Dedekind, from Hilbert to Emmy Noether to, of course, the Bourbaki group. [3] Charles Ehresmann (a co-disciple of Jean Dieudonné at the École normale supérieure and also a member of the Bourbaki group) would be very explicit a few decades later:

> This is a time of proliferation of mathematics; however, we can recognize also significant trends toward unity. [...] Considering the similarities of all theories, a unification is obtained by giving a general definition of the notion of a structure, or more precisely of a species of structures over sets. [Ehresmann 1966]

A decisive piece on the theme of the unity of mathematics and science—at least for the French scene, as witnessed for instance by a search of these terms on Gallica—is the double thesis of Albert Lautman, a philosopher close to Charles Ehresmann and other members of the Bourbaki group: the main thesis is devoted to structure and existence, the complementary one to the unity of mathematics itself, which contributes to strengthening the association between the various components of the theme. Lautman concludes:

> The unity of mathematics is essentially that of the logical patterns which govern the organization of its edifices. … The analogies of structure and adaptations of existences … have no other purpose than to help highlight the existence within mathematics of logical patterns, which are only knowable through mathematics itself, and ensure both its intellectual unity and its spiritual interest. [4]

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2 [Dedekind 1894, p. 452, footnote]: “Dieser Name soll, ähnlich wie in den Naturwissenschaften, in der Geometrie und im Leben der menschlichen Gesellschaft, auch hier ein System bezeichnen, das eine gewisse Vollständigkeit, Vollkommenheit, Abgeschlossenheit besitzt, wodurch es als ein organisches Ganzes, als eine natürliche Einheit erscheint.”

3 A huge literature has now been devoted to more tightly link Dedekind’s work and viewpoint to this trend that identifies Dedekind as one of its main precursors, see for instance [Ferreirós & Reck 2020; Sieg & Schlimm 2017].

4 [Lautman 1938, p. 198]: “L’unité des mathématiques est essentiellement celle des schémas logiques qui président à l’organisation de leurs édifices … Les analogies de

The same features, reorganized and illustrated in a variety of ways, were subsequently used in numerous texts, as for instance in this preface of a book aimed at a more general audience that the mathematician Georges Bouligand wrote with the engineer and teacher Jean Desbats just after World War II:

> We first notice the obstacles that mathematical activity constantly comes up against; the share of the plural and the diversified imposes a struggle at all times to reduce, simplify and encompass. [...] The unity of Mathematics requires an integral struggle. The diversity of objects subjected to reasoning, and also the plurality of hypothetico-deductive systems, come to an arrangement, and this is the essential point, with a *unitary structure* of Mathematics, whose fundamental terms are the two notions of *invariance* and *group*. This unification in the structure facilitates methodological unity. [5]

The theme of the unity of science and the key position of mathematics in it predate of course the 1930s (going back for some to Antiquity), the description of its relevant features and components varying greatly according to the period. [6] In the nineteenth century, one may think of Martin Ohm's *Versuch eines vollkommen consequenten Systems der Mathematik*, George Boole's *Laws of Thought* or of course Auguste Comte. Charles-Ange Laisant, whose 1898 book *La Mathématique: Philosophie, Enseignement* was reedited in 1907, envisioned the unification of all sciences, following Auguste Comte, through a universal method, leading from the concrete to the abstract and reciprocally, "which draws its inspiration from Mathematics only because Mathematics expresses the intrinsic traits of the

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structure et les adaptations d'existences ... n'ont pas d'autre but que de contribuer à mettre en lumière l'existence au sein des mathématiques de schémas logiques, qui ne sont connaissables qu'à travers les mathématiques elles-mêmes et en assurent à la fois l'unité intellectuelle et l'intérêt spirituel."

5 [Bouligand & Desbats 1947, p. 8]: "On constate d'abord les obstacles auquel se heurte constamment l'activité mathématique; la part du plural et du diversifié impose une lutte de tous les instants, en vue de réduire, de simplifier et d'englober. [...] L'unité de la Mathématique exige une lutte intégrale [...] La diversité des objets soumis au raisonnement, et aussi bien, la pluralité des systèmes hypothético-déductifs pactisent, c'est là le point essentiel, avec une *structure unitaire* de la Mathématique dont les termes fondamentaux sont les deux notions d'*invariance* et de *groupe*. Cette unification dans la structure facilite l'unité méthodologique."

6 See for a sample of this variety [Kremer-Marietti 2003; Krömer 2007; Maronne 2014; Stump 1997]. Still, it should be noted that there is no mention of this theme in, for instance, the eighteenth-century *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, edited by Denis Diderot, Jean Le Rond d'Alembert and Louis de Jaucourt. For an example of a different approach which can be retroactively linked to the theme of the unity of mathematics, see [Rabouin 2009].

positivist spirit, [and] nonetheless possesses a profound social and moral extension".[7]

The tensions between the effective work of the mathematicians and the explicit discourses on the components of the unity of mathematics, whether they come from philosophical studies or from general speeches made by mathematicians in the course of their mathematical life, are well-known. This situation poses a particular challenge for historians of mathematics: how to identify, and to account for, mathematical practices that, for their author, contribute to the unity of mathematics, without relying too hastily on a later (or simply a different) vision of what the unity of mathematics is or should be. And how not to consider the theme of unity as endowed with a synthetic explanatory capacity to more easily describe the work of a mathematician, but rather to perceive some modalities of this work which shape its specificity.

With these questions in mind, the case of Charles Hermite seems a particularly interesting one to tackle. His numerous mathematical heirs often emphasized his striving towards unity. At Hermite's 1892 Jubilee, for instance, Henri Poincaré commented:

> It can be said that the value of your discoveries is further enhanced by the care you have always taken to highlight the mutual support that all these apparently diverse sciences lend to each other.[8]

And Hermite's son-in-law and the editor of his complete works, the mathematician Émile Picard, introduced them along the same lines:

> These strange *rapprochements*, between questions of such different natures, exerted a sort of fascination on his mind. [...] Thus he wrote once about the work of Legendre and Gauss on the decomposition of numbers into squares: 'These illustrious mathematicians, by pursuing at the cost of so much effort their profound researches in this part of Higher Arithmetic, thus tended unwittingly towards another area of Science and gave a memorable example of this mysterious unity, which manifests itself sometimes in the most seemingly remote analytical works'.[9]

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7 I am here quoting Hamdi Mlika: "L'unité des sciences ne s'inspire de la Mathématique que parce que cette dernière exprime les traits intrinsèques de l'esprit positiviste, possède néanmoins une extension sociale et morale profonde" [Mlika 2009]. We again notice the use of the singular "mathématique".

8 [Jubilé 1893, p. 7]: "On peut dire en effet que le prix de vos découvertes est encore rehaussé par le soin que vous avez toujours eu de mettre en évidence l'appui mutuel que se prêtent les unes aux autres toutes ces sciences en apparence si diverses."

9 [Hermite 1905-1917, vol. 1, p. xxix]: "Ces rapprochements étranges, entre des questions de natures si différentes, exerçaient sur son esprit une sorte de fascination. [. . .] Aussi écrivait-il un jour à propos des travaux de Legendre et de Gauss sur

Although, to my knowledge, Hermite's work has never been used, nor even mentioned, by philosophers discussing the unity of mathematics, it would also be easy to spot aspects of it that would make Hermite a precursor of the versions of unity, particularly structuralist ones, mentioned above: the importance he attached to the observation of algebraic transformations and their effects [Goldstein 2019], his insistence on questions of invariance [Parshall 2024], his application of Galois's ideas to the monodromy of complex functions, prompting their later use in the study of differential equations [10], or his participation in a unified vision of what Norbert Schappacher and I have called with deliberate anachronism "arithmetic algebraic analysis" [Goldstein 2015; Goldstein & Schappacher 2007].

However, confronted in the second half of the nineteenth century with the first manifestations of this structuralist movement, Hermite repeatedly stated his skepticism towards it. He preferred, for instance, to compute with explicit, down-to-earth, representatives, the so-called reduced forms, rather than to deal with intrinsically defined classes of quadratic forms (one of the first steps in the advent of mathematical structures and of their operations). He would insist on the need to complete the study of the relations among roots of an algebraic equation, according to Galois's ideas—that is, for us, its Galois group—with an explicit, analytic, parametrization of these roots, in particular by elliptic functions in the quintic case. Moreover, he claimed to be indifferent, if not hostile, to philosophical issues. [11]

The main questions the present text addresses are thus: how can we get some access to Charles Hermite's viewpoint on the "mysterious unity of mathematics"? When and how was it expressed and constructed by a mathematician (of great influence in his time) who was hostile to systematic reflections on foundations, to axiomatization, to infinite sets considered as a whole, as well as to structural reductionism?

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la décomposition des nombres en carrés: 'Ces illustres géomètres, en poursuivant au prix de tant d'efforts leurs profondes recherches sur cette partie de l'Arithmétique supérieure, tendaient ainsi à leur insu vers une autre région de la Science et donnaient un mémorable exemple de cette mystérieuse unité, qui se manifeste parfois dans les travaux analytiques en apparence les plus éloignés'."

10 On this point, see [Archibald 2011] and, for Hermite himself, my own study of Hermite's underestimated importance in the reception of Galois [Goldstein 2011b].

11 Hermite's preferences and their effect on the creation of mathematical objects and his views on computations are discussed in more detail in [Goldstein 2011a].

1. THE WORDS OF UNITY

Tracking the word "unity" itself in the Hermitian corpus is quite misleading: "unity" ("unité" in French) also designates the number 1 and its surrogates, such as the various "roots of unity" which routinely appear in Hermite's algebraic works. In the published research works of Hermite, one single mention of "unity" in the sense we are interested in is to be found, that quoted above by Picard, which originally occurred in [Hermite 1864b]. [12] However, Picard's quote suggests to follow other words, such as "rapprochements".

Thanks to François Lê's seminal approach, [13] it is possible to locate and contextualize all words in Hermite's work. I have thus selected a number of them belonging to the semantic field of "rapprochements" and which are used several times in Hermite's articles: analogie (analogy), analogue (analogous), rapprochement, rapprocher (to bring closer), lien (link), liaison (connection), relation (relation). [14]

In the following table, the number of instances of several of these words is given: [15]

analogie	40
analogue(s)	157
liaison(s)	10
lien(s)	13
rapprochement(s)	10
rapprocher, rapproché(e,es)	35
relation(s)	990

These words also have a very local use, that is, as a substitute or a complement to one of the most frequent words in Hermite's articles, the

12 I have not found it either in the letters written by Hermite I have been able to consult, for instance to Thomas Stieltjes, Gösta Mittag-Leffler, Andreï Andreïevitch Markov, Ernesto Cesàro, Rudolf Lipschitz or Leopold Kronecker.

13 See in particular [Lê 2023; 2024].

14 Other words might play occasionally a similar role. For instance, Hermite once mentions "a sort of junction between the theory of elliptic sines and those of Göpel and Rosenhain's functions, [Hermite 1905-1917, vol. 3, p. 251]: "une sorte de jonction entre la théorie des sinus d'amplitude et celles des fonctions de Göpel and Rosenhain". For the sake of space, I have left aside here words with only one or very few occurrences and I will follow only those occurring more frequently and mentioned in the table.

15 I owe these data to François Lê whom I warmly thank. My table gathers singular and plural forms for the nouns and the conjugated forms for the verb. Let me also note that "unité," as in "roots of unity," appears c. 300 times in Hermite's published papers.

word "equation" [Lê 2024] ; it may be misleading for us. It is obvious with "relation," which Hermite routinely used to announce an algebraic or differential equation: "the quantities satisfy the relation so and so" (as for "unity" used as a surrogate of "1," this explains the high number of its instances). But it is also true of the word "liaison" (connection), again employed in exactly the same local context: "There exists between these periods, drawn from integral calculus, a *connection* expressed by the following equation ..." [16], or even of "rapprochement". [17]

What particularly interests us here is their occurrence, with a larger meaning, in comments about how parts of mathematics or concepts arising from different subdomains are interrelated. It is then necessary to consider these words as markers and to analyze the mathematics they are attached to through a direct and systematic reading of the texts where they appear. This presents the usual difficulties in reading closely Hermitian texts: sudden change of notations, allusive remarks, missing assumptions, etc. However, such an examination brings to light several interesting phenomena. For reasons of space, I will not report here on each occurrence, but only illustrate by one or two examples the main results of this enquiry.

This down-to-earth, but systematic, selection of texts based on explicit, verifiable criteria, has for me the advantage of somewhat stripping this selection of tacit presuppositions about what does or doesn't count as a contribution to the unity of mathematics. [18] A side benefit is to draw attention on little-studied texts which are nonetheless relevant and instructive for our purpose.

Hermite does not work in isolation. On the contrary, as this will appear here incidentally, he often reacts very quickly to writings of his predecessors or contemporaries. Here, I will not try and establish the genesis or mathematical environment of all Hermitian results included in the selected texts, and only a few necessary contextual elements will be provided: my limited purpose is not to study the collective dynamics involving a concept or a theorem, but only to assess what practices the chosen words designate within Hermitian mathematics. [19]

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16 [Hermite 1855, p. 251]: "Il existe entre ces périodes, telles qu'on les tire du calcul intégral, une *liaison* exprimée par l'équation suivante" ... My emphasis.

17 See, for instance, [Hermite 1871, p. 21].

18 For instance, Hermite's 1851 article interpreting Puiseux's previous work on monodromy in a Galois setting does not appear in the texts selected with our criteria.

19 For a global study of all the authors quoted by Hermite, see [Goldstein 2012]. Almost all sources used by Hermite, tacitly or explicitly, have now been included in

2. FROM WORDS TO MATHEMATICS

2.1. *Recognizing expressions*

A first noticeable feature that this enquiry brings to light is precisely the shifts from one operating level of relationships and rapprochements to another. Let us illustrate this point with a short and rather elementary article—an extract of a letter—Hermite sent to Ernst Leonard Lindelöf only a year before his death, in December 1899 [Hermite 1899-1900]. Lindelöf had answered a question on discrete probability coming from the *Intermédiaire des mathématiciens* [Lindelöf 1899-1900]: One has n sets of p balls, numbered from 1 to p in each set. One draws the balls one by one and randomly, while counting aloud from 1 to p, n times. What is the probability that the number called during the drawing coincides with the number written on the ball?

Lindelöf's solution is purely combinatorial: noticing first that the question is the same if one counts aloud n times 1, then n times 2, etc., and finally n times p, he sets up to establish the probability that there is no coincidence for the balls marked with the number 1 (that is, none of these balls is drawn in the first n draws); the repetition of the same procedure for the other marks $(2, \ldots, p)$ then provides him with the final solution of the initial question. Lindelöf computes the first step of his procedure, the probability that there is no coincidence for the balls marked with the number 1, in two different ways. One way is to deduce it from the probabilities to have one coincidence at least, then two coincidences at least, etc. An alternative path relies on a direct computation: dividing the number of drawings which *avoid* the balls marked with 1 in the first n places by the total number $(pn)!$ of possibilities. Because the two paths should arrive at the same result, Lindelöf deduces from these two computations the following identity, which he describes as "curious enough to be mentioned":

$$\frac{(r-n-1)(r-n-2)\ldots(r-2n)}{r(r-1)\ldots(r-n)} = \frac{1}{r} - \frac{n^2}{r(r-1)} + \frac{1}{1\cdot 2}\cdot\frac{n^2(n-1)^2}{r(r-1)(r-2)} - \cdots,$$

where $r = pn$ is the total number of balls. [20]

the database Thamous. References to historical studies of some results, their sources, and their developments will of course be given when they exist.

[20] I give this identity in the form provided in Hermite's article (equation (A)); it is in fact derived from Lindelöf's original one by changing r into $r-1$ and by dividing by r.

The right hand side of the equation can also be written as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \frac{k!}{r(r-1)\cdots(r-k)}.$$

It is this "curious" equation that triggers Hermite's interest. He immediately recognizes that the terms of the last sum are easily linked to special values of Eulerian integrals (here the beta functions):

$$\frac{k!}{r(r-1)\cdots(r-k)} = \int_0^1 x^{r-k-1}(1-x)^k dx = B(r-k, k+1)$$

and thus, with $x = \frac{y+1}{2}$,

$$\frac{k!}{r(r-1)\cdots(r-k)} = \frac{1}{2^r}\int_{-1}^1 (1+y)^{r-k-1}(1-y)^k dy.$$

Hermite then uses a well-known (at the time) expression of Legendre polynomials P_n:[21]

$$2^n P_n(y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 (1+y)^{n-k}(1-y)^k.$$

Lindelöf's identity thus becomes (with $s = r - n - 1$):

$$\frac{s(s-1)\cdots(s-n+1)}{(s+1)(s+2)\cdots(s+n+1)} 2^{s+1} = \int_{-1}^1 P_n(y)(1+y)^s dy.$$

The integral on the right is thus 0 for $s = 0, 1, \ldots, n-1$. This provides, in particular, a key property of orthogonality for the Legendre polynomials.

Hermite comments: "[Your identity] is tightly *linked* to the theory of polynomials $P_n(x)$ of Legendre and opens up a new path to their fundamental properties. … I thought you might be interested to see a *rapprochement*, a close *link* I might say, between the problem of probability that you solved and an important theory of analysis, the theory of spherical functions."[22]

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21 Legendre polynomials are for instance defined as the coefficients of the powers of t in the expansion $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$. This polynomial system is orthogonal for the inner product $\langle P, Q\rangle = \int_1^1 P(t)Q(t)dt$, among many other properties. Through another change of variables, Hermite gets the expression he needs from a 1837 paper of Peter Gustav Lejeune-Dirichlet, also mentioned in Hermite's 1878 presentation of the second edition of Eduard Heine's *Handbuch der Kugelfunctionen*, see [Dirichlet 1837; Hermite 1878].

22 [Hermite 1899-1900, p. 88, p. 90]: "[Votre identité] est *liée* étroitement à la théorie des polynômes $P_n(x)$ de Legendre, et ouvre une nouvelle voie pour parvenir à leurs propriétés fondamentales … J'ai pensé que vous ne verriez pas sans quelque intérêt un *rapprochement*, une étroite *liaison* je puis dire, entre le problème du calcul des

This rather marginal and late example nevertheless seems to me to highlight a way of working that we often find in in Hermite's work, including in his important achievements such as the creation of Hermitian forms [Goldstein 2019]: the starting point is a symbolic expression, a formula, which he observes and transforms, for instance by means of successive, often quite intricate, changes of variables. This allows him to locate them inside a stock of familiar expressions, sometimes in far-away theories, or to generalize them to other cases. The *rapprochement* on the domain level, as here between combinatorial probability theory and the theory of spherical functions, is constructed on the basis of *rapprochements* at the very local level of the expressions themselves, .

2.2. *Discovering the right analogy*

Legendre polynomials regularly appear in Hermite's work. As we will see, they also appear in association with the word "analogy," and in this context, they serve both as a mould and as a model for building bridges between fields or mathematical questions that may seem a priori far apart. They are not the only mathematical objects to play these roles, but I have chosen this example to illustrate how analogies are handled by Hermite, because it seems to me one of the simplest and shortest to present.

In 1864, struck by the importance of e^{-x^2} (or more generally of $e^{-\phi(x,y,z,\dots)}$ for a quadratic form ϕ) in the representation of elliptic and Abelian functions, Hermite notices that these exponentials "give rise, as the radical

$$(1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}} [\dots],$$

to a system of integral polynomials which may be used for the expansion of functions of any number of variables. [...] One will see, by the way, the most complete *analogy* between the properties of such expressions coming from so different origins".[23]

probabilités dont vous avez donné la solution, et une grande théorie de l'analyse, celle des fonctions sphériques." I have emphasized here our selected words, as well as "liée," the adjective derived from "lier" and "lien". Spherical functions (at the time) are functions on the sphere satisfying Laplace-type differential equations, among which are Legendre polynomials. They are the subject of a book by Eduard Heine, which Hermite praised and presented to the Academy of sciences [Hermite 1878].

23 [Hermite 1864b, p. 94]: "... les exponentielles e^{-x^2} et $e^{-\phi(x,y,z,\dots)}$ donnent naissance, comme le radical $(1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}}$ ... à un système de polynômes entiers, pouvant servir au développement des fonctions d'un nombre quelconque de variables. ... On verra, du reste, entre les propriétés d'expressions d'origine si différente, l'*analogie* la plus complète". My emphasis.

The coefficients of the radical $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$ developed into the powers of α indeed give rise to the Legendre polynomials. Hermite defines, for any non-zero real a, the polynomials U_n (in one variable x) such that

$$e^{-\frac{a}{2}(x-h)^2} = e^{-\frac{x^2 a}{2}}(U_0 + \frac{h}{1}U_1 + \frac{h^2}{1\cdot 2}U_2 + \cdots),$$

then states and proves for his U_n the same type of properties as for the classical Legendre polynomials, in particular:

— the U_n satisfy a recursion formula: $U_{n+1} - axU_n + anU_{n-1} = 0$,

— the U_n satisfy a second-order differential equation: $\frac{d^2U}{dx^2} - ax\frac{dU}{dx} + anU = 0$,

— the U_n form an orthogonal system, that is $\int_{-\infty}^{\infty} e^{\frac{ax^2}{2}} U_n U_{n'} = n!a^n\sqrt{\frac{2\pi}{a}}$ if $n = n'$ and 0 otherwise [Hermite 1864b, p. 267].

It is to be noticed that while the analogous elements in both situations are polynomials endowed with specific properties—and Hermite mentions several other families of such polynomials—, they come from quite distinct horizons: they arise from algebraic functions $(1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}}$ in Legendre's case, from transcendental exponential functions in Hermite's new case. The analytical similarities—development in series, differential equations, values of integrals—are again much more important for Hermite than the partition into algebra and analysis we have come to consider as decisive.

Hermite also extends his construction to several variables, beginning by the original Legendre polynomials themselves. For these polynomials, and two variables instead of one, a 1865 letter to Carl Borchardt (who published it in his journal [Hermite 1865a]) gives us a rare insight on the way rapprochements and analogies also guide Hermite's work to generalization. To begin with, and in order to mimic the construction of Legendre polynomials as the coefficients of the expansion with respect to a of $\frac{1}{\sqrt{1-2ax+a^2}}$, Hermite simply tries to replace $\frac{1}{\sqrt{1-2ax+a^2}}$ by $\frac{1}{\sqrt{1-2ax-2by+a^2+b^2}}$. However, the polynomials he thus obtains do not form an orthogonal family. "In order to reestablish the *analogy* with the functions of one variable which originates from the development of $\frac{1}{1-2ax+a^2}$ and seems to be lost here," Hermite explains,[24] he then replaces his original choice of the quadratic

[24] [Hermite 1865a, p. 295]: "Pour rétablir l'*analogie* avec les fonctions d'une variable ayant pour origine le développement de $\frac{1}{1-2ax+a^2}$ et qui me semble ici se perdre...". My emphasis.

form $1 - 2ax - 2by + a^2 + b^2$ in the variables a, b by its adjunct, that is, the quadratic form $(1 - ax - by)^2 - (a^2 + b^2)(x^2 + y^2 - 1)$.

This original idea of the role of the adjunct comes from Hermite's early work on Carl Friedrich Gauss's *Disquisitiones arithmeticae* and on the theory of quadratic forms, where the use of the adjunct form is a key ingredient.[25] The polynomials obtained with this adjunct form satisfy a formula close to the so-called Rodrigues' formula[26] for the Legendre polynomials X_n:

$$X_n = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n},$$

and Hermite can then use it to prove the other wanted properties and display "the *analogy* as complete as possible with Legendre functions" ([Hermite 1865a, p. 295], my emphasis). Again, the bridge is constructed by careful handling of symbolic expressions, the behavior of which is tested through various computations and the exact form corrected until it fits the desired scheme.

Transfers by analogy from arithmetic to algebra to analysis are frequent in Hermite's work. For instance, Hermite stresses in a letter to Paul Gordan the analogy between the arithmetical study of successive minima of $x + ay + bz$ (with a, b integers) and the study of $U \sin x + V \cos x + W$, where U, V, W are polynomials in x [Hermite 1873]: in this situation, the analogy is based on a continued fraction expansion (numerical or algebraic, according to which case is considered), a technique that Hermite will also use at the same date in his celebrated proof of the transcendence of e.[27]

Other examples involve the discovery of Hermitian forms in n variables, first introduced as a subclass of real quadratic forms in $2n$ variables [Goldstein 2019]: again, this is the concrete display of the coefficients of a specific family of quadratic forms which suggests to Hermite that a study using complex numbers may bring light to their characteristics, as it allows their arithmetic study to mimic that of quadratic forms with n variables and integral real coefficients. This, he concludes, is "adding new characters of similitude between real integers and complex numbers," at the time still a controversial issue [Hermite 1905-1917, vol. 1, p. 477]. The same kind of

25 [Brechenmacher 2011; Goldstein 2007]. In 1865, Hermite uses both Gauss's terminology of "adjunct form" and the terminology of the then blossoming invariant theory, "quadratic contravariant".

26 In his 1865 letter to Borchardt, dated from January 27, Hermite only attributes the formula to Carl Jacobi. He emphasizes Olinde Rodrigues's priority in a subsequent communication to the French Academy of Sciences on March 13, 1865 [Hermite 1865b, p. 517].

27 On this point, see [Goldstein 2015; Serfati 1992; Waldschmidt 1983].

construction is transferred by Hermite to forms associated to the theory of transformation of Abelian functions (now called symplectic forms): "This analytic theory of transformation is tightly *linked* to the arithmetical theory of quadratic forms I spoke about...," he writes. [28]

In some cases, links operate in several ways for the same question. Hermite's use of elliptic functions to solve general quintic equations is well-known, but it is perhaps less known that it perfectly illustrates Hermite's practice of analogy (and links) in mathematics. [29] His point of departure here is a quartic equation associated to the flex points of cubic curves: these nine flex points are aligned three by three on 12 lines, which in turn form 4 triangles (such that each of them contains the nine flex points), giving rise to a quartic equation. [30] Hermite comments:

> From the study of these equations, I noticed that they have the closest affinity with those found in the third-order transformation of elliptic functions, and so I thought it would not be useless to look into this *rapprochement*... This *analogy*, indeed, opened for me the way of representing by elliptic transcendental functions the roots of the general quartic equation. [31]

And so we are on familiar ground here: this is again by a direct comparison of two specific quartic equations (the quartic equation linked to the flex points and the multiplier equations for the transformation of elliptic functions of degree 3, whose roots are known in terms of special values of elliptic functions) that Hermite weaves links between two separate areas of mathematics. One consequence of this rapprochement is to express the roots of all these quartic equations as values of modular elliptic functions, that is, values of analytic, non-algebraic functions.

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28 [Hermite 1855, p. 784]: "cette théorie analytique de la transformation se trouve étroitement *liée* à la théorie arithmétique des formes quadratiques dont j'ai parlé". My emphasis. This case is studied in [Brouzet 2004].

29 On Hermite's place in the history of algebraic equations, see for instance [Gray 2018; Houzel 2002; Zappa 1995]. The context of Hermite's work on this topic, in particular its relation with Galois's viewpoint and the modular equations, is discussed in [Goldstein 2011b].

30 On the historical role of these equations arising from geometry, see [Lê 2015]. Here, as at other times, Hermite neglects the potential of a link to the advanced geometry of his time.

31 [Hermite 1905-1917, vol. 2, p. 22]: "L'étude de ces équations m'ayant fait remarquer qu'elles offrent la plus étroite affinité avec celles qu'on rencontre dans la transformation du troisième ordre des fonctions elliptiques, il ne m'a pas paru inutile de m'arrêter à ce *rapprochement* ... Cette *analogie*, d'ailleurs, m'a ouvert la voie pour représenter par les transcendantes elliptiques les racines de l'équation générale du quatrième degré." My emphasis. Let me remark that "affinité" (affinity) which belongs to the same semantic field as "rapprochement," is an hapax in Hermite's work.

This approach will be exported with success by Hermite to quintic equations, a case which, contrarily to the quartic one, is not in general solvable by radicals. This is the case that mobilized the attention of historians (and of contemporaries), and in particular the expression of the roots as explicit values of elliptic functions: depending on the authors, this analytic approach to an algebraic problem has been both praised, as establishing new bridges between far-away domains of mathematics and as allowing a concrete computation of the roots, and criticized, as introducing foreign elements to the solution. [32] Hermite's elliptic study of the quartic equation, for which algebraic expressions of the roots were already available, seems to have escaped to attention. However, for our issue, it is worth underlining how a famous Hermitian achievement again began with a close comparison between particular, well-known, equations. And that Hermite, here as before, forsakes the problem of the algebraic resolution of an algebraic equation for the constitution of a unique domain, with analysis both at the basis and at the lead.

2.3. *Rapprochement by the proofs*

There is still another instance of the word "rapprochement": it occurs when several proofs or methods of resolution of a problem are known. In the case of the quintic equation we just mentioned, "the theory of elliptic function leads to two methods for the resolution of the equation of the fifth degree [33]: the one proposed by Hermite in 1858 is based, as said earlier, on the possibility to link any quintic general equation, through adequate changes of variables, to the reduced form of a modular equation.

Jacobi had introduced his doubly-periodic functions (the so-called Jacobian elliptic functions) through integrals of the type $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, with $0 < k < 1$ a real number. Their periods are 2 or 4 times (depending on the function) the so-called complete integrals:

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}},$$

32 On this point, illustrated by Felix Klein's opinion, see [Goldstein 2011b, 46-51].

33 [Hermite 1905-1917, vol. 2, p. 347]: "La théorie des fonctions elliptiques conduit à deux méthodes pour la résolution de l'équation du cinquième degré." Hermite devoted to this issue 12 communications to the French Academy of Sciences in 1865 and 1866, later gathered in a booklet. To simplify, I will quote them from the second volume of his collected works where they are all reproduced.

with $k' = \sqrt{1-k^2}$. If one puts $\omega = \sqrt{-1}\frac{K'}{K}$, $\sqrt[4]{k}$ is a uniform function of ω, say $\phi(\omega)$, and there exists an algebraic equation of degree $n+1$ between $\phi(\omega)$ and $\phi(n\omega)$, for each integer $n > 1$. This is the modular equation of level n. [34]

Another method, due to Leopold Kronecker and completed by Francesco Brioschi, also in 1858, is based on a direct construction of cyclic functions of the roots of the quintic equation; this method finally links it to the so-called multiplier equation, which appears in the transformation theory of elliptic functions. [35]

In his work of 1865-1866, Hermite's aim is "to take M. Kronecker's method a step further and *bring it closer* to the previous one, using as a basis the remarkable and inventive work in which M. Brioschi set out its principles." [36] As in the rapprochements of the mathematical expressions, this work on the different proofs should provide alternative paths to a variety of properties. For instance, in the case of the quintic equation, Hermite comments: "I will add to this *rapprochement* between the two methods of solving the equation of the fifth degree, by deducing from the second the conditions for the reality of the roots," which he had previously studied by his own approach. [37]

More generally, and against the common idea of finding *the* right proof, Hermite advocates for multiple proofs of a result in order to fully understand it. [38] But reciprocally the very fact that there exist several available proofs is for him an incentive to clarify the links that this suggests.

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34 See [McKean & Moll 1999] for a clear presentation for a modern reader.

35 Definitions and details on this method are given in [Zappa 1995].

36 [Hermite 1905-1917, vol. 2, p. 348]: "approfondir la méthode de M. Kronecker et la *rapprocher* de la précédente, en prenant pour base le travail remarquable et plein d'invention dans lequel M. Brioschi en a exposé les principes".

37 [Hermite 1905-1917, vol. 2, p. 392]: "J'ajouterai encore à ce *rapprochement* entre les deux méthodes de résolution de l'équation du cinquième degré, en déduisant de la seconde les conditions de réalité des racines." My emphasis.

38 For instance, Hermite urges Thomas Stieltjes: "Il sera utile de donner pour parvenir aux mêmes conclusions deux procédés très différents," [Hermite & Stieltjes 1905, vol. 1, p. 41]. Or in a 1873 article,"On reconnaîtra volontiers que, dans le domaine mathématique, la possession d'une vérité importante ne devient complète et définitive qu'autant qu'on a réussi à l'établir par plus d'une méthode," [Hermite 1905-1917, vol. 3, p. 128]. Respectively: "It will be useful to consider two very different procedures to arrive at the same conclusions," and "It can be readily admitted that, in mathematics, the possession of an important truth only becomes complete and definitive when we have succeeded in establishing it by more than one method."

A good illustration is provided by Hermite's work on the number of classes of binary quadratic forms $ax^2 + 2bxy + cy^2$, with integer coefficients a, b, c. Two forms are said to be equivalent, or in the same class, when they can be deduced from each other by an invertible, linear change of variables with integer coefficients. The discriminant $b^2 - ac$ (called "determinant" by Gauss in the *Disquisitiones arithmeticae*, and by Hermite following him) is the same for equivalent forms; on the other hand, for a given determinant, there are a finite number of classes of equivalent forms. These class numbers had been computed by analytic means by Peter Gustav Lejeune-Dirichlet in a celebrated application of Fourier analysis to number theory. From the 1860s on, Kronecker established recurrence formulas for these class numbers by using elliptic modular functions.[39] In the wake of his involvement with the fifth-degree equation and the publication of Kronecker's article in French in 1860, Hermite writes to Joseph Liouville:

> M. Kronecker's beautiful theorems on the class numbers of quadratic forms [...] remained, however, isolated and belonging to a very distinct order of ideas to which only the theory of complex multiplication in elliptic functions seemed able to give access.[40]

Besides the doubly-periodic elliptic functions arising from the integrals

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

alluded to above, Jacobi had introduced theta-functions, that is, special quasi-periodic functions, whose quotients also provide these elliptic functions and which can be developed as series involving trigonometric functions and rational powers of $q = e^{-\pi \frac{K'}{K}}$. Following then Jacobi's model, Hermite sets out to derive some of Kronecker's results from the developments of quotients of theta functions in series of sinus and cosinus, such as:[41]

$$\frac{4\sqrt{q}\sin x}{1+q} - \frac{4\sqrt{q^3}\sin 3x}{1+q^3} + \frac{4\sqrt{q^5}\sin 5x}{1+q^5} + \cdots \tag{1}$$

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39 On these developments, see [Dickson 1919, ch. VI] and [Goldstein & Schappacher 2007].

40 [Hermite 1862, p. 25]: "[Les beaux théorèmes de M. Kronecker sur les nombres de classes de formes quadratiques ...] restaient cependant comme isolés et appartenant à un ordre d'idées très distinct où la théorie de la multiplication complexe dans les fonctions elliptiques paraissait seule pouvoir donner accès."

41 [Hermite 1864b, p. 146-147], formulas 5 and 15. Hermite provides forty-six formulas of this type, which he then combines.

or

$$\cot x + \frac{4q\sin 2x}{1-q} - \frac{4q^2\sin 4x}{1+q^2} + \frac{4q^3\sin 6x}{1-q^3} + \cdots . \tag{2}$$

In particular,[42] Hermite finds that $\sqrt{\frac{2K}{\pi}}$ can be developed as

$$\theta_1 =: \sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \cdots .$$

Multiplying two by two such developments adequately and integrating the result between 0 and $\frac{\pi}{2}$, Hermite obtains new series in powers of q such that the coefficient of the term in q^{Δ} (or $q^{\frac{\Delta}{4}}$) depends on the class numbers of binary quadratic forms $ax^2 + 2bxy + cy^2$, with integer coefficients a, b, c and discriminant $b^2 - ac = -\Delta$.

For instance, multiplying (1) and (2) above and integrating the result provides:

$$\theta_1^3 = 1 + 4\sum_{n>0} \frac{q^n}{1+(-q)^n} - 2\sum_{n>0}(-1)^n q^{n^2}\mathcal{B}_n + 4\sum_{n>0}\frac{q^{n^2+n}\mathcal{B}_n}{1+(-q)^n},$$

with $\mathcal{B}_n = 1 + 2q^{-1} + 2q^{-4} + \cdots + 2q^{-(n-1)^2}$.

Hermite then carefully studies the development into the powers of q of the three series composing θ_1^3. The third one, for instance, becomes after development $\sum_{n>0}(-1)^{a(n+1)}q^{n^2+n+an-b^2}$, displaying as the power of q the opposite Δ of the discriminant of the quadratic form $nx^2 + 2bxy + (n+1+a)y^2$, with $0 \leq |\, b \,| < n$. A careful study of the forms finally shows that

$$\theta_1^3 = 1 + 12\sum_{n>0}(\mathcal{H}_0 - \mathcal{H}_1)q^{\Delta},$$

with complementary terms if Δ is a square or the triple of a square.[43] Here $\mathcal{H}_0$ is the number of reduced binary quadratic forms with one of the extreme coefficients odd and $\mathcal{H}_1$ is the number of such forms with even extreme coefficients.[44]

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42 The notations in this domain change according to the authors and, in the case of Hermite, according to the date of the paper. A concordance table is given in [Dickson 1919, p. 93].

43 There is misprint or miscalculation in the original publication, the 1 is missing. It is corrected in the complete works, [Hermite 1905-1917, vol. 2, p 253].

44 A binary quadratic form $ax^2 + 2bxy + cy^2$ with integer coefficients and negative determinant is reduced when $2 \mid b \mid \leq a \leq c$, and $b \geq 0$ if $a = 2 \mid b \mid$ or $a = c$. There is in this case a unique reduced form in each class—the reduced form is a representative of all the forms of the class—and thus counting the reduced forms amounts to computing the class number. The extreme coefficients are a and c.

This corresponds (if one expresses it in the language of classes of forms) to one of Kronecker's recurrence formulas:

$$E(n) + 2E(n-1) + 2E(n-4) + 2E(n-9) + \cdots = \frac{2}{3}[2 + (-1)^n]X(n),$$

where $E(n) = 2F(n) - G(n)$, $G(n)$ is the number of classes of quadratic forms with determinant $-n$, $F(n)$ is the number of quadratic forms with determinant $-n$ and one odd extreme coefficient, $X(n)$ is the sum of the odd divisors of n.[45]

This number of classes, in special cases, had been linked by Legendre and Gauss to the number of decomposition of a number into the sum of three squares. Computing the cube of an expression made of exponentials of squares leads to exponentials of sums of three squares and thus Hermite's previous developments also provide, in particular, the number of representations of an integer as sums of three squares. This is indeed in such a context that Hermite used the word "unity" in our sense for the first and unique time in his published papers:

> We have two absolutely distinct methods which connect by a double *link* Legendre's and Gauss's propositions on the decomposition of numbers into three squares to the theory of elliptic functions. In so doing, these illustrious mathematicians were unknowingly reaching out to another region of science, and providing a memorable example of the mysterious *unity* that sometimes manifests itself in analytical work that is seemingly the furthest removed from the realm of science.[46]

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45 [Dickson 1919, p. 109]. Kronecker himself was particularly pleased to be able to "draw *only* from the theory of elliptic functions the beautiful propositions of higher arithmetic that until now were based on the deep considerations included in Gauss's *Disquisitiones arithmeticae*," [Kronecker 1860, p. 297]: "De cette manière, on peut tirer de la *seule* théorie des fonctions elliptiques les belles propositions d'arithmétique supérieure, qui jusqu'ici étaient fondées sur les profondes considérations que renferment les Disq. Arithm. de Gauss." Kronecker's paper does not include the detailed proofs, which were completed by Henry Smith in his Report to the British Association, [Smith 1865].

46 [Hermite 1905-1917, vol. 2, p. 254]: "On a deux méthodes absolument distinctes qui rattachent par un double *lien* à la théorie des fonctions elliptiques les propositions de Legendre et de Gauss sur la décomposition des nombres en trois carrés. Ces illustres géomètres ... tendaient ainsi à leur insu vers une autre région de la science et donnaient un mémorable exemple de cette mystérieuse *unité* qui se manifeste parfois dans les travaux analytiques en apparence les plus éloignés." My emphasis. This issue is not explored further here, but let us note the theological undertone of this passage, in line with Hermite's views that human beings are not the masters of a free mathematical creation: unity "manifests itself," the mathematicians "unknowingly" reach out etc. See [Goldstein 2011a].

In July 1862, Hermite again emphasizes that "the theory of elliptic functions presents two main points where it is *linked* to arithmetic and particularly to the theory of binary forms with negative determinant" [47], the first one being his own development in series, explained above, which he describes as elementary and the second one being Kronecker's use of complex multiplication.

Obtaining new proofs not only serves to solidify the links between the arithmetic of quadratic forms and elliptic functions, but also to extend their now common territory. In his letter to Liouville, Hermite explains:

> In arriving at these theorems of Mr. Kronecker by another route, I have attached them in the most direct way, i think, to the order of ideas which belongs to you, and, if I am not mistaken, in the very sense of your predictions, for the arithmetical notion of class is replaced by the much simpler and elementary idea of reduced forms. [... I would like to indicate], in conclusion, how I see the *connection* between the theory of elliptic functions, in its applications to arithmetic, and your general research on numerical functions. [48]

And a few months later, in November of the same year, he would again extend his viewpoint, this time to sew onto it Dirichlet's research on class numbers:

> M. Dirichlet's method for the determination of the class numbers of quadratic forms with the same determinant and those recently drawn from the consideration of elliptic functions in the case of negative determinants lead for the same question to such different solutions that it seems as difficult to find any *link* between their results as between the principles on which they are based. [49]

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47 This series of communications to the Academy is published again later under various titles and groupings and, in order to simplify, I quote them here from Hermite's complete works, [Hermite 1905-1917, vol. 2, p. 241]: "La théorie des fonctions elliptiques présente deux points principaux où elle vient *se lier* à l'Arithmétique et spécialement à la théorie des formes quadratiques à deux indéterminées de déterminant négatif." My emphasis. Again, let me note the grammatical choice of "se lier" which seems to exclude a human intervention.

48 [Hermite 1862, p. 25-26, p. 38]: "En parvenant par une autre voie à ces théorèmes de M. Kronecker, c'est à l'ordre d'idées qui vous appartient que je pense les avoir rattachés de la manière la plus directe, et, si je ne me trompe, dans le sens même de vos prévisions, car la notion arithmétique de classe se trouve remplacée par l'idée beaucoup plus simple et élémentaire des formes réduites. [... Je vous indique], en terminant, de quelle manière je conçois la *liaison* de la théorie des fonctions elliptiques, dans ses applications à l'arithmétique, avec vos recherches générales sur les fonctions numériques." My emphasis.

49 [Hermite 1905-1917, vol. 2, p. 255]: "La méthode de M. Dirichlet pour la détermination du nombre des classes de formes quadratiques de même déterminant, et celles qu'on a tirées récemment de la considération des fonctions elliptiques dans le cas des

To do this, Hermite proposes either to find a purely arithmetical proof of Kronecker's results or an elliptic-based one of Dirichlet's results, which he prepares by revisiting Dirichlet's original one.[50]

Hermite's involvement with Kronecker's formulas did not stop there. In 1884, for instance, he would tackle them again. His new procedure gives in particular an alternative for the quantity $(\sqrt{\frac{2kK}{\pi}})^3$, and a rapprochement between his formula and Kronecker's provides a new approach to some simple arithmetical relations involving class numbers and established by Gauss in the *Disquisitiones arithmeticae*. Once more, at this occasion, Hermite insists on the fact that they "reveal a tight *connection* between the arithmetical theory of quadratic forms and the analytic theory of elliptic transcendental functions"[51].

3. CIRCULATING ALGEBRAIC EXPRESSIONS: A DETAILED EXAMPLE

The preceding exemples clearly suggest that Hermite works on analogies or constructs links and rapprochements at the level of the symbolic expressions themselves, even when he states them in terms of connections between fields or mathematical sub-disciplines. For example, he doesn't try to derive from Legendre polynomials a list of properties and criteria that would define a general class of objects to be studied; he adapts, through transformations and calculations, explicit expressions to new situations, thus weaving, thread by thread, the new, associated, connection between algebraic and analytic functions.

This feature has some serious historiographical consequences, if only in terms of the writing adapted to discuss this type of practice. To gain a deeper understanding of how expressions circulate, creating the unity Hermite's successors would enthusiastically praise, we need to take a closer

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déterminants négatifs, conduisent pour la même question à des solutions tellement différentes, qu'il semble aussi difficile de trouver un *lien* quelconque entre leurs résultats qu'entre les principes sur lesquels elles se fondent." My emphasis.

50 It is here interesting to note that Dirichlet's approach is described as arithmetical (despite his use of real series). In his own version, Hermite emphasizes only his restructuration of the (arithmetical) formulas, neglecting to prove a (necessary) convergence result which allows the identification of his formulas with Dirichlet class number formula. For details on this point and references to the authors who corrected it, see [Dickson 1919, p. 114-115].

51 [Hermite 1905-1917, vol. 4, p. 138-139]: "Les théorèmes dont je vais m'occuper consistent dans les relations suivantes [...] qui révèlent une *liaison* étroite entre la théorie arithmétique des formes quadratiques et la théorie analytique des transcendantes elliptiques." My emphasis.

look at some technical nooks and crannies of Hermite's work. In this section, I would like to analyze an example of such a circulation, based on avatars of Lagrange's resolvents.

This example concerns expressions of the kind

$$\phi(\alpha)(x_0, x_1, \ldots, x_n) = x_0 + x_1\alpha + x_2\alpha^2 + \cdots x_n\alpha^n,$$

that is, linear forms in the x_i, where $x_0, \ldots, x_n$ are indeterminates or variables (usually set at integral values) and α a root of an algebraic equation of degree $n+1$. Hermite usually takes into account, at the same time, all the $\phi(\xi)$, ξ running over all the roots of the algebraic equation.

To consider these expressions is of course not original with Hermite. For instance, they appear, for $n = 1$ or $n = 2$, in Joseph-Louis Lagrange's last Addition to Leonhard Euler's *Algebra*[52] or, for any degree, in a letter addressed to Joseph Liouville by Dirichlet, and published twice in French.[53] Dirichlet, in particular, uses the notation $\phi(\alpha)$ and considers the value at integers of the form obtained as the product of the $\phi(\xi)$, for all the roots ξ.

3.1. *Decomposition of prime numbers*

As for Hermite, these forms appear quite early in his work, in 1847, in a (successful) attempt to prove a statement of Jacobi. In a 1839 communication to the Berlin Academy, translated and published in French in 1843, Jacobi had commented on how Gauss used complex numbers $a + b\sqrt{-1}$, with a and b ordinary integers, and developed their arithmetic in order to establish quartic reciprocity laws; it led Jacobi to advocate for a study of the decomposition of ordinary prime numbers into (different types of) complex numbers.[54] If p is a prime of the form $4n+1$, such as 5, 13 or

52 [Lagrange 1774, t. 2, p. 651, 655 in the French 1774 edition]. To solve a cubic equation, for instance, Lagrange uses the fact that $x_1 + x_2 j + x_3 j^2$, with j a primitive cubic root of 1, takes 6 values when the x_i are permuted, but that the 6th-degree equation of which these 6 values are the roots is in fact quadratic in x^3, and thus easily solvable.

53 See for instance [Dirichlet 1840]. Liouville's closeness to Hermite and the publication of the letter both in the *Comptes rendus* of the French Academy and in Liouville's *Journal de mathématiques pures et appliquées* suggests that Hermite was probably aware of these texts; an explicit reference to them was added by Jacobi, when he published some letters from Hermite to him in 1850. On the other hand, Hermite certainly knew Lagrange's algebraic texts very well.

54 [Jacobi 1839; 1843]. Despite its shortness and elementary character, this paper proved to give a key impulse to the development of algebraic number theory. It is already discussed within this context in [Goldstein 2007; Goldstein & Schappacher 2007]. For the convenience of the readers, I repeat here briefly this presentation, emphasizing the work on specific algebraic expressions.

17, it can be written as a sum of two squares of integers, $p = a^2 + b^2$; thus $p = a^2+b^2 = (a+b\sqrt{-1})(a-b\sqrt{-1})$. It is true in particular if p is a prime of the form $8n+1$, such as 17. However, in this case, p can also be represented by other quadratic forms besides $x^2 + y^2$: $p = c^2 + 2d^2$ and $p = e^2 - 2f^2$. For example, $17 = 16 + 1 = 9 + 2 \times 4 = 25 - 2 \times 4$.

Jacobi proved that a further decomposition can provide the needed coherence in these representations. More precisely, in his paper, he factorizes $a + b\sqrt{-1} = \phi(\alpha)\phi(-\alpha)$, where α a primitive 8th-root of unity (that is, as written at the time, "$\alpha = \sqrt[4]{-1}$") and $\phi(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3$, with integers a_0, a_1, a_2, a_3.

As $\alpha^4 + 1 = 0$, $\phi(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3$ is exactly an expression of the type we are interested in. One has also: $a + b\sqrt{-1} = \phi(\alpha)\phi(\alpha^5)$, and, similarly, $a - b\sqrt{-1} = \phi(\alpha^3)\phi(\alpha^7)$. Jacobi then obtains $p = 8n + 1 = a^2 + b^2 = \phi(\alpha)\phi(\alpha^5)\phi(\alpha^3)\phi(\alpha^7)$. And other groupings of the $\phi(\alpha^k)$ provide the other decompositions:

$$p = [\phi(\alpha)\phi(\alpha^3)][\phi(\alpha^5)\phi(\alpha^7)] = (c + d\sqrt{-2})(c - d\sqrt{-2}) = c^2 + 2d^2$$
$$p = [\phi(\alpha)\phi(\alpha^7)][\phi(\alpha^3)\phi(\alpha^5)] = (e + f\sqrt{2})(e - f\sqrt{2}) = e^2 - 2f^2.$$

To summarize, in order to get the various decompositions of a prime of the form $8n + 1$, Jacobi had to use not only what was considered at the time as "the" (only known, and even only thought of) complex integers, that is, Gaussian complex integers $a+b\sqrt{-1}$ with a, b ordinary integers, but also complex numbers built with 8th-roots of unity. A natural question then arises: what are, for each kind of primes, the relevant complex integers to consider and how? At the end of his paper, Jacobi asks the question for another case, that of the primes p of the form $5n + 1$ and states that in this case

$$p = N_1N_2N_3N_4,$$

where each N_i is of the form $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3$, with $\alpha^5 = 1, \alpha \neq 1$. However, he does not give any proof of this result.

A well-known development arising from Jacobi's paper is the arithmetical study of cyclotomic integers (that is, linear combinations of roots of unity, as above), which led to the introduction of ideal complex factors by Ernst Eduard Kummer, also c. 1847, and then to Dedekind's theory of ideals. Hermite, however, attacked the problem from a quite different angle. In a letter to Jacobi, he explains:

> It is in some very elementary properties of quadratic forms, with any number of variables, that I encountered the principles of analysis which I ask your permission to discuss. I have drawn from these principles a proof of your beautiful

theorem on the decomposition of prime numbers $5m + 1$, into four complex factors, built with the fifth roots of unity[55].

Hermite clearly had hopes to be able to handle a much more general case than the particular ones mentioned by Jacobi. In his letter, he considers any irreducible algebraic equation of degree n with integer coefficients

$$F(x) = x^n + Ax^{n-1} + \cdots + Kx + L = 0$$

the roots of which are designated by $\alpha, \beta, \ldots, \lambda$.

In particular, $F(x) = (x - \alpha)(x - \beta) \cdots (x - \lambda)$.

He then fixes an integer N and assumes that there exists an integer a such that $F(a) \equiv 0 \pmod{N}$ (in other terms, N divides $F(a)$). Let us notice that if N is a prime number of the form $pm + 1$, for a prime number p and an integer m, Fermat's Little Theorem shows that $x^{N-1} \equiv 1 \pmod{N}$, for all integers x prime to N; thus the condition that there exists an integer a such that $F(a) \equiv 0 \pmod{N}$ is satisfied with $F(x) = \frac{x^p - 1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x + 1$.[56]

A decisive move is now to introduce the quantities which will replace the $\phi(\alpha)$ of Jacobi's paper. Hermite *adapts* the linear forms we are interested in and defines:

$$\phi(\alpha) = Nx_0 + (\alpha - a)x_1 + (\alpha^2 - a^2)x_2 + \cdots + (\alpha^{n-1} - a^{n-1})x_{n-1}.$$

He also considers their product over all roots of F: $\mathcal{F} = \phi(\alpha)\phi(\beta) \cdots \phi(\lambda)$. A key point is that the coefficients of $\mathcal{F}$, which are already by construction symmetric functions of the roots $\alpha, \beta, \ldots, \lambda$ and thus integers, are now also all multiples of N. Indeed, these coefficients are obtained by products of terms which contain either N or a product $(\alpha - a)(\beta - a) \cdots (\lambda - a)$. Up to its sign, this last quantity is $F(a)$, which is, as explained, divisible by N.

However, the main object, as Hermite mentioned in the quote above, is not this product, but a (positive definite) quadratic form Hermite associated to it: If the roots $\alpha, \beta, \ldots, \lambda$ are real, he puts

$$f(x_0, x_1, \ldots, x_{n-1}) = \phi^2(\alpha) + \phi^2(\beta) + \cdots + \phi^2(\lambda).$$

[55] [Hermite 1850, p. 262]: "C'est dans quelques propriétés très élémentaires des formes quadratiques, à un nombre quelconque de variables, que j'ai rencontré les principes d'Analyse dont je vous demande la permission de vous entretenir. J'ai tiré de ces principes une démonstration de votre beau théorème sur la décomposition des nombres premiers $5m + 1$, en quatre facteurs complexes, formés des racines cinquièmes de l'unité."

[56] In more modern terms, there exists an integer a whose image in $(\mathbb{Z}/N\mathbb{Z})^*$ is exactly of order p; such an element exists because this group is of order $N - 1$, which is divisible by p.

If β and γ, say, are complex conjugate roots, he uses the term $\phi(\beta)\phi(\gamma)$ instead of the squares.[57]

Generalizing Gauss's theory of reduction of binary quadratic forms to n-ary quadratic forms, Hermite had proved earlier that, for each quadratic form, there exist integers such that the value of the quadratic form at these integers is bounded, the bound depending only on the value of the determinant of the form and the number of variables.[58] More precisely, there exist integers $x_0, x_1, \ldots, x_{n-1}$ such that

$$0 < f(x_0, x_1, \ldots, x_{n-1}) < \left(\frac{4}{3}\right)^{\frac{n(n-1)}{2}} \sqrt[n]{|D|},$$

with D the determinant of the form f.

It is then easy to deduce bounds for the various terms $\phi^2(\alpha)$, $\phi^2(\beta)$, ... (or $\phi(\beta)\phi(\gamma)$) which compose the form, and thus for their product. Finally, taking into account that here the coefficients of $\mathcal{F}$ are multiples of N, as explained above, Hermite concludes that there exist integers $x_0, x_1, \ldots, x_{n-1}$ such that

$$\mathcal{F}(x_0, x_1, \ldots, x_{n-1}) = MN, \qquad M < \left(\frac{4}{3}\right)^{\frac{n(n-1)}{4}} (\Delta n^{-n})^{\frac{1}{2}},$$

where M is a non zero integer and Δ is the discriminant of the equation $F = 0$ (that is, the product of the difference of the roots, taken two by two).

To address Jacobi's statement, Hermite chooses

$$F(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1.$$

Let $n = 4$ and N a prime number of the form $5k + 1$. Then the construction of the $\phi, \mathcal{F}, \ldots$ above applies. One has $\Delta = 5^3$ and $M < 1.65$, thus $M = 1$. This provides:

$$N = \mathcal{F}(x_0, x_1, x_2, x_3) = \phi(\alpha)\phi(\beta)\phi(\gamma)\phi(\delta),$$

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57 Hermite introduces in fact a family of quadratic forms, built with terms $D_\alpha\phi^2(\alpha)$, or $D_{\beta,\gamma}\phi(\beta)\phi(\gamma)$, with $D_\alpha, D_{\beta,\gamma}$ positive real numbers. It provides infinitely many solutions, by varying the coefficients $D_\alpha, D_{\beta,\gamma}$, but as it is not relevant to our example, we will ignore this variant, see [Goldstein 2007] for details.

58 This is what is now called the Hermite-Minkowski theorem, as Hermann Minkowski established later a better bound than Hermite's original one. It is a generalization of the bound given above for the first coefficient of a reduced form and thus a key result for the classification of quadratic forms at the time. To state this theorem, Hermite had first to define the determinant of a n-ary quadratic form (our modern discriminant).

where $\alpha, \beta, \gamma, \delta$ are the primitive 5th-roots of unity, and

$$\phi(\alpha) = Nx_0 + (\alpha - a)x_1 + (\alpha^2 - a^2)x_2 + (\alpha^3 - a^3)x_3,$$
$$\phi(\beta) = Nx_0 + (\beta - a)x_1 + (\beta^2 - a^2)x_2 + \cdots, \qquad \phi(\gamma) = \cdots,$$

that is, the decomposition of N into complex factors of the required type.

Hermite also handles in the same way new cases, that of prime numbers N of the form $7k \pm 1$. If $N = 7m + 1$, he puts:

$$\phi(\zeta^k)(x_0, \ldots, x_5) = Nx_0 + (\zeta^k - a)x_1 + \cdots + (\zeta^{5k} - a^5)x_5,$$

where ζ^k now runs over the roots of the cyclotomic equation $F(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$, that is, the primitive 7-th roots of unity and a is an integer such that $a^7 \equiv 1 \pmod N$ but $a \not\equiv 1 \pmod N$, and thus $F(a) \equiv 0 \pmod N$.

With the same construction as before, *mutatis mutandis*, the determinant D of the quadratic form $f = \phi(\zeta)\phi(\zeta^6) + \phi(\zeta^2)\phi(\zeta^5) + \phi(\zeta^3)\phi(\zeta^4)$ is, in absolute value, $\frac{7^5}{2^6}N^2$, and there exists integers such that the value of $\mathcal{F}$ at these integers is MN, with $M < (\frac{4}{3})^{\frac{15}{2}}(\frac{7^5}{6^6})^{\frac{1}{2}}$, that is, finally $M < 5{,}192$. Moreover, Hermite proves that the value at integers of $\mathcal{F}$ is also either divisible by 7 or $\equiv 1 \pmod 7$, thus again $M = 1$ and N is a product of six linear combinations of primitive 7th-roots of unity.

In the case $N = 7m - 1$, Hermite finds that N can be expressed as a product of three linear combinations of the roots of the cubic equation

$$x^3 + x^3 - 2x - 1 = 0.$$ [59]

At that time already, Hermite placed great hopes in the use of "forms whose coefficients depend on roots of algebraic equations with integral coefficients. Maybe we will succeed to deduce from them a complete system of characters for each species of this kind of quantities".[60] In the same series of letters to Jacobi, he uses the same construction, this time with the roots α, β, γ of a cubic equation, to study what we now call the units of the associated cubic field: for Hermite, the problem is to find the solutions of the equation

$$(x + \alpha y + \alpha^2 z)(x + \beta y + \beta^2 z)(x + \gamma y + \gamma^2 z) = 1.$$

59 These roots are the three periods of two roots of the 7-th cyclotomic equation, as explained in the last section of Gauss's *Disquisitiones arithmeticae* on cyclotomic equations.

60 [Hermite 1850, p. 286]: "... des formes dont les coefficients dépendent de racines d'équations algébriques à coefficients entiers. Peut-être parviendra-t-on à déduire de là un système complet de caractères pour chaque espèce de ce genre de quantités".

He shows in particular that, up to roots of unity, the solutions are the powers of one of them only (in the case of one real root) or the products of the powers of at most two of them, in the case of three real roots.[61] He also uses the relation between the product of the $x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots + \alpha^{n-1} x_{n-1}$, when α runs over the solutions of an algebraic equation with integral coefficients, and the associated quadratic forms (according to the procedure explained above between f and $\mathcal{F}$) to deduce from the theory of forms that the infinitely many algebraic equations with integral coefficients and a given discriminant define only a finite number of irrational numbers (their roots), up to a rational change of variables [Hermite 1905-1917, vol. 1, p. 225].

The results I just outlined announced (among other contemporary results, in particular by Dirichlet) what will be known later as "algebraic number theory". We are thus not surprised now that Hermite combined algebraic tools and notions to be able to use number-theoretical techniques (the theory of integral quadratic forms, in particular), in order to answers questions arising both in number theory and algebra (according to the classification of domains in his time). However, in the middle of the nineteenth century, this mixture, which Gauss used with spectacular effects in the last section of his *Disquisitiones arithmeticae*, was still worthy of note.[62] Hermite was very aware of this fusion and of its promises and challenges:

> In the immense range of research opened up by M. Gauss, algebra and number theory seem to me to merge into a same order of analytical notions, which our present knowledge does not yet allow us to form a correct idea of.[63]

3.2. *Sturm's theorem*

This merger will further extend its reach when Hermite will use his favorite quadratic forms to tackle a key result of algebra at the time, Sturm's

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61 Hermite will tackle later the general case, for algebraic equations of all degrees. However, he did not prove the *existence* of the independent basic solutions, a result known as Dirichlet's unit theorem.

62 On the place of number theory c. 1800 and the change brought about by the *Disquisitiones*, see [Goldstein et al. 2007].

63 [Hermite 1850, p. 291]: "Dans cette immense étendue de recherches qui nous a été ouverte par M. Gauss, l'Algèbre et la Théorie des nombres me paraissent devoir se confondre dans un même ordre de notions analytiques, dont nos connaissances actuelles ne nous permettent pas encore de nous faire une juste idée."

theorem, which provides the exact number of real solutions, located in a given interval, of an algebraic equation with real coefficients. [64]

Hermite has been interested in this result during all his professional life, both in research and teaching ; he called it still in 1890 "one of the most important propositions of the theory of algebraic equations ... which had the good fortune to become immediately classic." [65]

According to Sturm's theorem, the number of distinct real roots located between, say, x_1 and x_2 of a polynomial V in one variable with real coefficients is equal to the difference in the number of sign variations when one evaluates, at x_2 and at x_1 respectively, a finite sequence of polynomial functions V_i. In Charles Sturm's original memoir, the sequence V_i was obtained through successive Euclidean divisions, which required quite laborious computations, and the first proof of the theorem used the intermediate value theorem.

Around 1840, James Sylvester expressed Sturm's auxiliary polynomials V_i directly in function of the squares of differences of the roots of V ; if V is of degree m and its roots are $a, b, \ldots, k, l$, one has: [66]

$$
\begin{aligned}
\frac{V_1}{V} &= \sum \frac{1}{x-a} \\
\frac{V_2}{V} &= \sum \frac{(a-b)^2}{(x-a)(x-b)} \\
\frac{V_3}{V} &= \sum \frac{(a-b)^2(a-c)^2(b-c)^2}{(x-a)(x-b)(x-c)} \\
&\vdots \\
\frac{V_m}{V} &= \frac{(a-b)^2(a-c)^2\cdots(k-l)^2}{(x-a)(x-b)\cdots(x-l)}.
\end{aligned}
$$

Hermite brought several innovations to the theorem: he extended it to a system of several equations, he displayed infinitely many Sturm's series of auxiliary functions, he proved all the theorems without any recourse

64 This theorem, its complex genesis and its far-reaching reformulations have been studied in detail and in depth in [Sinaceur 1994].

65 [Hermite 1905-1917, vol. 4, p. 291]: ... un théorème qui est l'une des plus importantes propositions de la théorie des équations algébriques [... et qui] a eu le bonheur de devenir classique". On this point, see [Vincent 2020].

66 [Hermite 1853]. To simplify, I follow as much as possible here Hermite's notations, despite their ambiguity and the lack of precision on the summation range. We can assume here that the roots are distinct and ordered.

to continuity arguments (in particular, without the intermediate value theorem). Moreover, through the use of his favorite quadratic forms, he unified Sturm's theorem on the number of real roots of an equation in a given interval and Cauchy theorem on the number of imaginary roots in a bounded given domain. He also connected Sturm's theorem with the classification of algebraic forms, a topic at the forefront of research at the time. [67]

Here as elsewhere, working as closely as possible with formulas (and their multiple interpretations) is a decisive factor in Hermite's mathematical practice, a factor he himself often emphasizes. About his extension of Sturm's theorem to several variables, where he begins by replacing the factors of the type $\frac{1}{x-a}$ in the original Sturm's theorem by $\frac{1}{(x-a)(y-a')}$, he says for instance:

> I must first mention the beautiful expressions discovered by Mr. Sylvester for the auxiliary functions that appear in Mr. Sturm's theorem, and those deduced by Mr. Cayley, as having opened up a new path for me. These are, indeed, formulas *analogous* to those of these two learned geometers, which will be posited a priori for simultaneous equations, and from which properties all similar to those of Mr. Sturm's functions will be easily concluded. [68]

Let us just reconstruct briefly how the same quadratic forms we met above are used by Hermite in this other context. [69] Let us consider a polynomial V with real coefficients and m distinct roots a_i. Hermite associated to it (a family of) quadratic m-ary forms:

$$\sum \frac{1}{x-a_i}(x_0 + a_i x_1 + a_i^2 x_2 + \cdots + a_i^{m-1} x_{m-1})^2.$$

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67 See in particular [Hermite 1905-1917, vol. 1, pp. 281-287, 397-414, 415-428, 479-481]. As for other topics, Hermite's work is tightly connected with the works of other mathematicians exactly at the same time, in this case Sturm himself, Carl Borchardt, Arthur Cayley, James Sylvester, and others [Sinaceur 1994]. They exchange letters, which are partially published, and complement each other's work, sometimes day by day. Given our focus on specific Hermite's tools here, this environment is left aside.

68 [Hermite 1905-1917, vol. 3, p. 2]: "Je dois indiquer d'abord les belles expressions découvertes par M. Sylvester pour les fonctions auxiliaires qui figurent dans le théorème de M. Sturm, et celles que M. Cayley en a déduites, comme m'ayant ouvert une voie nouvelle. Ce sont, en effet, des formules *analogues* à celles de ces deux savants géomètres qui seront posées a priori pour des équations simultanées et dont on conclut avec facilité des propriétés toutes semblables à celles des fonctions de M. Sturm." My emphasis.

69 Hermite's published papers on this topic are extracts of letters, with different notations and only allusive outlines of his arguments. For simplicity, I have standardized slightly the notation and detailed some calculations.

Here x is a real number, different from the roots a_i, and it is treated as a parameter; I will designate these expressions by $f_x(x_0, x_1, \ldots, x_{m-1})$. We recognize the type of forms, built with the roots of an algebraic equation, introduced before, except for the coefficients of the squares (here $\frac{1}{x-a_i}$) which may be here negative as well as positive. Negative coefficients occur when the root a_i is greater than x, positive ones when a_i is smaller than x, and thus the number of positive and negative coefficients keep track of the number of roots smaller and bigger than x (and by combination of f_x and $f_{x'}$ of the number of roots in the interval $\left]x, x'\right[$). Let us note that the forms f_x are rational symmetrical functions of the roots of V, thus are real quadratic forms when x is real.

The determinant $\Delta_{m-1,x}$ of f_x is

$$\Delta_{m-1,x} = \frac{1}{\prod(x-a_i)} \begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{m-1} \\ 1 & a_2 & a_2^2 & \ldots & a_1^{m-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & a_m & a_m^2 & \ldots & a_m^{m-1} \end{vmatrix}^2 .$$

The Vandermonde determinant is equal as usual to the product

$$\prod_{1 \leq i < j \leq m} (a_i - a_j)$$

and thus $\Delta_{m-1,x}$ is V_m/V. Let us designate by $\Delta_{0,x}$, $\Delta_{1,x}$, $\Delta_{2,x}$, etc., the determinants corresponding to the partial forms $\sum \frac{x_0^2}{x-a_i}$, $\sum \frac{(x_0+a_i x_1)^2}{x-a_i}$, $\sum \frac{(x_0+a_i x_1+a_i^2 x_2)^2}{x-a_i}$, etc. ; they similarly correspond to Sylvester's auxiliary functions for Sturm's theorem.

Hermite explains to Borchardt: "the reduction of a quadratic form to a sum of squares, which has been the topic of your memoir [...] plays the main role in my research". [70]

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70 [Hermite 1856, p. 39]: "La réduction d'une forme quadratique à une somme de carrés, qui a été le sujet de votre Mémoire [...] joue le principal rôle dans mes recherches". The classification of real quadratic forms, more precisely the transformation by a linear, invertible change of variables, of any real quadratic form into a linear combination of squares with coefficients, say, +1 or -1, was a hot topic at the time. As well-known now, the number of coefficients of each sign is an invariant of the form: this is Sylvester law of inertia. On these developments, see [Brechenmacher 2007].

Indeed, the form f_x can be reduced to a "sum of squares" (up to real constants) via a triangular change of variables of determinant 1. One has:

$$\begin{aligned} f_x = {} & \epsilon_0 (x_0 + \alpha_{1,0} x_1 + \alpha_{2,0} x_2 + \ldots + \alpha_{m-1,0} x_{m-1})^2 \\ & + \epsilon_1 (x_1 + \alpha_{2,1} x_2 + \alpha_{3,1} x_3 + \ldots + \alpha_{m-1,1} x_{m-1})^2 \\ & + \cdots \\ & + \epsilon_{m-2} (x_{m-2} + \alpha_{m-1,m-2} x_{m-1})^2 \\ & + \epsilon_{m-1} x_{m-1}^2 . \end{aligned}$$

The ϵ_i are real coefficients. Thus $\Delta_{m-1,x} = \epsilon_0 \epsilon_1 \ldots \epsilon_{m-1}$.

Moreover, putting $x_{m-1} = 0$ gives $\Delta_{m-2,x} = \epsilon_0 \ldots \epsilon_{m-2}$, putting $x_{m-2} = x_{m-1} = 0$ gives $\Delta_{m-3,x} = \epsilon_0 \ldots \epsilon_{m-3}$, etc. Finally, the real quadratic form f_x can be reduced by a linear transformation with real coefficients to a form of the type:

$$\Delta_{0,x} X_0^2 + \frac{\Delta_{1,x}}{\Delta_{0,x}} X_1^2 + \frac{\Delta_{2,x}}{\Delta_{1,x}} X_2^2 + \ldots + \frac{\Delta_{m-1,x}}{\Delta_{m-2,x}} X_{m-1}^2 .$$

The signs of the coefficients provide the sign of the values of the Sturm's functions. In other terms, Sturm's theorem for a polynomial is now seen as an explicit version of the law of inertia for our special quadratic forms, built with the roots of this polynomial.

3.3. *Toward complex analysis*

Hermite then handles the complex case by a variant of the same expressions. Let F be any algebraic equation with complex coefficients, this time of degree n, $F(z) = Az^n + Bz^{n-1} + \cdots + Kz + L = 0$, with roots a, b, ..., k.[71] Hermite introduces the quadratic form:

$$\begin{aligned} \phi(x, y, \ldots, u) = {} & \frac{i}{F_0(a) F'(a)} (x + ay + \cdots + a^{n-1} u)^2 \\ & + \frac{i}{F_0(b) F'(b)} (x + by + \cdots + b^{n-1} u)^2 \\ & + \cdots \\ & + \frac{i}{F_0(k) F'(k)} (x + ky + \cdots + k^{n-1} u)^2 , \end{aligned}$$

where F_0 is the polynomial the coefficients of which are the complex conjugate of those of F. The quadratic form ϕ is real and Hermite proves that its signature provides the number of roots in the upper, resp. lower,

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71 In Hermite's fickle notations ; see [Hermite 1856, p. 40].

half plane. He deduces the number of roots of an algebraic equation contained in various domains of the complex plane, such as a rectangle or a hyperbole and, again by means of astute transformations, obtains particular cases of Cauchy's integral formula.

Hermite emphasizes the role of his special quadratic forms in his procedure:

> I came to this study largely by a search on arithmetic questions, which since the year 1847 have called my attention on the quadratic forms composed of a sum of squares of similar functions of the roots of the same equation.
>
> I have thus felt a real satisfaction to relate to these forms these magnificent theorems of M. Sturm and M. Cauchy, which open the new era of modern Algebra. [72]

3.4. *Toward invariant theory*

This account of the number of roots of an equation through the study of quadratic forms is used by Hermite in another work which leads this time to invariant theory. [73] The context is the resolution (by elliptic modular functions) of the fifth-degree equation which Hermite and Kronecker, and then Brioschi, approached as we have seen above, in different ways—triggering Hermite's attempt to link and re-interpret the various methods of resolution. His point of departure here is a general binary form of the fifth degree $f(x,y) = \alpha x^5 + 5\beta x^4 y + 10\gamma x^3 y^2 + 10\gamma' x^2 y^3 + 5\beta' x y^4 + \alpha' y^5$, and the associated fifth-degree equation $f(x,1) = 0$, with its roots $a, b, \ldots, k$. [74]

Hermite first shows that if, in this equation, one replaces x by $z = \frac{\phi(x,1)}{f'_x(x,1)}$, where $\phi(x,y)$ is a covariant of degree 3 in x, y of f, the coefficients of the transformed equation in z are invariants of $f(x,y)$. [75] Then, he adapts once

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72 [Hermite 1856, p. 50]: "J'ai été amené à cette étude en grande partie par des recherches sur des questions arithmétiques, qui, depuis l'année 1847, ont appelé mon attention sur les formes quadratiques composées d'une somme de carrés de fonctions semblables des racines d'une même équation. Aussi ai-je éprouvé une véritable satisfaction à rattacher à la considération de ces formes ces magnifiques théorèmes de M. Sturm et M. Cauchy, qui ouvrent l'ère nouvelle de l'Algèbre moderne."

73 On the history of this topic, see [Dieudonné 1971; Parshall 1989; Wolfson 2008]. On Hermite's contributions to it, see [Parshall 2024].

74 [Hermite 1866]. This booklet gathers 12 communications to the French Academy in 1865 and 1866. They are also reproduced in the second volume of Hermite's works.

75 Covariants (resp. invariants) are at the time polynomials in the coefficients of f and x, y (resp. in the coefficients of f), which are invariant under a linear change of variables x, y of determinant 1. A first example of an invariant is Gauss's determinant of a binary quadratic form, $b^2 - ac$.

more the expression of his favorite quadratic forms: he replaces the expression $(t_0 + at_1 + \cdots + a^4 t_4)^2$ in his special quadratic forms by

$$\left(t_0 + \frac{t_1\phi_1(a,1) + t_2\phi_2(a,1) + t_3\phi_3(a,1) + t_4\phi_4(a,1)}{f'_x(a,1)}\right)^2,$$

where the ϕ_i are four cubic covariants of f that Hermite provides explicitly.

Adding these new expressions for all roots $a, b, \ldots, k$, Hermite obtains a form whose coefficients are invariants of f. The signature of this form (that is, the number of positive and negative coefficients in its transformation as a linear combination of squares, according to Sylvester's law of inertia) provides, as previously, the number of real and imaginary roots of $f(x,1) = 0$.

To go further, Hermite also adapts the expression of the functions V which intervene in Sturm's theorem.

First of all, he replaces simply

$$V_1 = (x-a)(x-b)\cdots(x-k)\sum \frac{1}{x-a}$$

by

$$\mathcal{V}_1 = (x-a)(x-b)\cdots(x-k)\sum \frac{x'-a}{x-a},$$

$$V_2 = (x-a)(x-b)\cdots(x-k)\sum \frac{(a-b)^2}{(x-a)(x-b)}$$

by

$$\mathcal{V}_2 = (x-a)(x-b)\cdots(x-k)\sum \frac{(x'-a)(x'-b)}{(x-a)(x-b)}(a-b)^2,$$

and so on. As in the original Sturm's theorem, where they were expressed as we have seen as determinants, the various terms

$$\frac{\mathcal{V}_{i+1}}{(x-a)(x-b)\cdots(x-k)}$$

are here invariants of the quadratic forms (which are again variants of the forms built with the roots of an equation, this time with parameters x and x'):

$$\sum \frac{x'-a}{x-a}(t_0 + at_1 + a^2t_2 + \cdots + a^it_i)^2.$$

More generally they share the basic properties of the Sturm's functions.

> It's easy to see how these new functions are *closely related* to those of Sturm's theorem, whose analytical properties they reproduce. They then serve as a

natural and easy transition to those [...] which are double covariants of the form $f(x,y)$, the proposed equation being $f(x,1)=0$.[76]

To obtain them, Hermite adapts once more his favorite quadratic forms, considering this time

$$\sum \frac{x'-ay'}{x-ay}\Big(t_0+\frac{t_1\phi_1(x,1)+t_2\phi_2(x,1)+t_3\phi_3(x,1)+t_4\phi_4(x,1)}{f'_x(x,1)}\Big)^2.$$

These new functions, being composed of covariants in (x,y) and in (x',y') (the expressions $\frac{x'-ay'}{x-ay}$) and of invariants, are covariants which can replace Sturm's functions, as desired.

3.5. *Interpolation*

Perhaps more surprising is the use of the same kind of quadratic forms to approach problems of interpolation. More precisely, the objective is to approximate a (sufficiently regular) function F by a polynomial P of degree $m \leq n$, such that the distance between this polynomial P and F at $n+1$ given points x_i—that is, here, the sum of the squares of the differences between the value of P and F at the x_i—is minimum. Hermite was inspired to deal with this question through a communication of Pafnuti Tchebichef at the Academy of sciences of Saint-Petersburg on January 12, 1855, which was translated into French in 1858.[77]

Let us assume that the function F takes the values u_i at the given points x_i, $i = 0,1,\ldots,n$. Hermite's first step is to use Lagrange's interpolation formula to construct a polynomial Π of degree n with the same values u_i at the points x_i.

Let $f(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$ a polynomial of degree $n+1$ with the x_i as roots. Hermite puts:

$$\Pi(x) = \frac{f(x)}{(x-x_0)f'(x_0)}u_0 + \frac{f(x)}{(x-x_1)f'(x_1)}u_1 + \cdots + \frac{f(x)}{(x-x_N)f'(x_N)}u_n.$$

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76 [Hermite 1905-1917, vol .2, p. 382]: "On voit assez [...] l'étroite *liaison* de ces nouvelles fonctions avec celles du théorème de Sturm dont elles reproduisent les propriétés analytiques. Elles servent ensuite de transition naturelle et facile pour arriver à celles [...] qui sont des covariants doubles de la forme $f(x,y)$, l'équation proposée étant $f(x,1)=0$." My emphasis.

77 [Tchebichef & Bienaymé 1858]. Hermite's communication to the French Academy follows the French publication closely, taking place on January 10, 1859 [Hermite 1859]. Both Tchebichef and Hermite work in a slightly more general context than reported here as they multiply the squares by positive real numbers that may represent errors in the measures of the values $F(x_i)$. For the sake of simplicity, I shall ignore these factors here.

By a linear transformation of the u_i which preserves the Euclidean norm $\sum u_i^2$ (that is, by an orthogonal linear transformation), one can also write, with the new variables v_i:

$$\Pi(x) = \Phi_0(x)v_0 + \Phi_1(x)v_1 + \cdots + \Phi_n(x)v_n,$$

where the Φ_k are polynomials of degree n.

Hermite remarks that $\sum \Pi(x_i)^2 = \sum u_i^2 = \sum v_i^2$. And that this, in turn, implies that $\sum_i \Phi_k(x_i)^2 = 1$ and $\sum_i \Phi_k(x_i)\Phi_{k'}(x_i) = 0$, for all $0 \leq k \leq n$ and $0 \leq k' \leq n$, $k \neq k'$.

His second step implements the condition of minimality, that is, that the distance between F and Π should be the smallest possible. More precisely, Hermite explains how to choose coefficients $A, B, \ldots, H$ such that:

$$\sum_{i=0}^{n} \left[F(x_i) - A\Phi_0(x_i) - B\Phi_1(x_i) - \cdots - H\Phi_m(x_i)\right]^2$$

is minimum.

Finally, Hermite shows that the Φ_k can be constructed with $m = n$. It could be deduced from a counting argument on the number of variables and equations, but Hermite, as usual, looks for an explicit solution. Hermite says:

> It would be difficult in this way to explicitly express the new functions $[\Phi_i]$ by the quantities $x_0, x_1, \ldots, x_n$. It is [...] by making use of the properties of quadratic forms that we can achieve this." [78]

Once again, a variant of our usual quadratic forms, built this time with the roots x_i of f, will provide the expression of the Φ_i. Hermite puts:

$$\sum_{i=0}^{n} (x - x_i)(v_0 + x_i v_1 + x_i^2 v_2 + \cdots + x_i^{k-1} v_{k-1})^2.$$

For $1 < k \leq m$, let Δ_k be the determinant of the quadratic form ; it is a polynomial of degree k in x, and if δ_k is the coefficient of x^k in this polynomial, Hermite proves that $\Phi_k(x) = \frac{\Delta_k}{\sqrt{\delta_k \delta_{k+1}}}$ (with specific values for Φ_0 and Φ_1). He concludes:

> Finally, I would like to point out that the sequence of quantities $1, \Delta_1, \Delta_2, \ldots, \Delta_m$ has the properties of Sturm's functions with respect to the equation $f(x) = 0$. [79]

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78 [Hermite 1859, pp. 65-66]: "Cependant il serait difficile par cette voie de parvenir à exprimer explicitement les nouvelles fonctions $[\Phi_i]$ par les quantités $x_0, x_1, \cdots x_n$. C'est [...] en faisant usage des propriétés des formes quadratiques qu'on y arrive."

79 [Hermite 1859, pp. 66-67]: "Je remarquerai enfin, en terminant, que la suite des quantités $1, \Delta_1, \Delta_2, \ldots, \Delta_m$ possède à l'égard de l'équation $f(x) = 0$ les propriétés des fonctions de Sturm."

This work, again, also illustrates how playing with explicit algebraic formulas is intertwined with relations at another level. In his article on interpolation, Tchebichef used expansions into continued fractions. Bienaymé had already interpreted this feature as a step to a unification, as he emphasized how this work "spreads a new light on the hidden links that unite the various parts of the analysis of series or of interpolation". [80]

In Hermite's youth, the expansion into continued fractions was the basic technique for approximation. [81] In his letters to Jacobi on the theory of numbers [Hermite 1850], Hermite proudly explains how to replace this tool with his own, that is, again, quadratic forms with appropriate coefficients. This context is relevant to understanding why and how Tchebichef's 1858 article (entitled "On continued fractions") would immediately attract Hermite's attention and direct him towards his favorite quadratic forms: the thematic level, here the theme of approximation, suggests him to use a specific process, and in turn this process which operates at the level of the algebraic expressions themselves reinforces a thematic rapprochement between mathematical subdomains, from arithmetic to algebra to analysis. [82]

4. REVISITING THE UNITY OF MATHEMATICS THROUGH HERMITE'S PRACTICES

The evolution of mathematics during the 19th century has sometimes been described as the replacement of an older approach, based on formulas, with a modern conceptual one. Such a description is often supported by some of the statements made by the mathematicians of the time, for instance that of Dirichlet noting: "the constantly increasing tendency of

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80 [Tchebichef & Bienaymé 1858, p. 290, note]: "Le travail de M. Tchebichef, en reliant aux fractions continues une classe au moins des fonctions ou des coefficients mis en évidence par Gauss dans la méthode des moindres carrés, répand une clarté nouvelle sur les liens cachés qui réunissent les diverses parties de l'analyse des séries ou de l'interpolation."

81 For Hermite, it will also be an important tool to adapt in a variety of contexts, either for numbers or for functions, launching a whole field of research for his students, in particular Henri Padé, see [Brezinski 1991].

82 Many years later—but again recycling some constructions and results of his letters to Jacobi on the theory of numbers—, Hermite tackles in the same way another result of Tchebichef, this time on the minima of $x - ay - b$, with a and b real constants and x and y integers to be found, and its algebraic counterpart, the best approximations of V by $X - UY$, where U and V are sufficiently regular given functions, and X and Y polynomials to be found, [Hermite 1880].

the new analysis is to put thoughts in the place of calculations."[83] Hermite offers of course a good counter-example to the relevance of such a dichotomy, if we wish to account for the actual work of mathematicians. In many of his articles, a stock of algebraic expressions and formulas, adapted to each specific mathematical situation, weaves like a garland through mathematical themes. They also serve as intermediary steps, enabling Hermite to arrive, by successive replacement, at the expressions needed to conclude, but also at new concepts, as though "naturally".[84] This point of view is consistent with Hermite's creative vision of computations and, more generally, his view of mathematics as a natural science, a nature guaranteed by an underlying divine order.[85] It is of course worth noting that other mathematicians, in the second half of the nineteenth century, extoll the role of formulas and computations too, as witnessed by a famous passage of a letter from Kronecker to Cantor in August 1884:

> I recognize true scientific value—in the field of mathematics—only in concrete mathematical truths, or to put it more sharply, "only in mathematical formulas". [...]The various theories for the foundations of mathematics have been blown away by time, but Lagrange's resolvent has remained.[86]

It is notable that Hermite's focus on explicit expressions extends beyond his articles, in particular to his teaching. If his son-in-law Émile Picard emphasizes how in his lectures, "on the most elementary of questions, [Hermite] suddenly opened up immense horizons, and alongside the Science

[83] "Die immer mehr hervortretende Tendenz der neueren Analysis ist Gedanken an die Stelle der Rechnung zu setzen." See a synthetic reminder of this viewpoint, as well as relevant references, in the introduction of [Sørensen 2005]. Let us note the debatable substitution of "thought" by "concepts" in many interpretations of Dirichlet's quote. The opposition has already been criticized by historians for several decades, see [Gilain & Guilbaud 2015; Goldstein et al. 2007].

[84] The importance of a construction presented as natural is essential for Hermite. For the example of Hermitian forms, see [Goldstein 2019].

[85] See his criticism of Louis Poinsot's statement that "computation is an instrument that does not produce anything from itself and that somehow gives back only the ideas entrusted to it" in a letter to Leo Königsberger, [Goldstein 2011a, p. 147].

[86] See [Jahnke 1987]: "Einen wahren wissenschaftlichen Werth erkenne ich auf dem Felde der Mathematik nur in concreten mathematischen Wahrheiten, oder schärfer ausgedrückt, 'nur in mathematischen Formeln'. [...] Die verschiedenen Theorien für die Grundlagen der Mathematik sind von der Zeit weggeweht, aber die Lagrange'sche Resolvente ist geblieben." For a deeper analysis of the role of formulas and computations in Kronecker's work, see [Edwards 2009; 1995; Vergnerie 2019]. Another interesting case, that of Dedekind, is studied in [Haffner 2021].

of today, we saw the Science of tomorrow,"[87] a more ironic and critical account of Hermite's courses testifies to the continuity between his teaching and his mathematical practice. Charles Rabut, a Ponts-et-Chaussées engineer, who had followed these courses at Polytechnique around 1871, complains:

> As for mathematical dreams, several of my schoolmates and I had them following Hermite's abominable lessons on Eulerian, elliptic, ultra-elliptic and other functions; they consisted of assimilations of algebraic symbols to things from real life. I remember a certain function $C(x)$, where C was a caravan. I believe that this phenomenon, which was quite widespread among our fellow students at the time, was a reflex protest by the cerebral organism against the inoculation of a veritable intellectual poison.[88]

With its caravan of embodied symbols, Rabut's nightmare seems remarkably resonant with what has been discussed here. As we have seen, Hermite's work on specific algebraic expressions constructs a unity from below, supported, in a less technical but decisive way, by a configuration of links and transfers operating at other levels. Whether in copying and adapting a tool developed in one branch or in reshaping a theorem in a new frame, the threads created by this very work on expressions is consubstantial with the identification of analogies and rapprochements between large swathes of mathematics, more specifically among arithmetic, algebra, and analysis (as defined in his time). While Hermite's proof of Sturm's theorem is often praised today for avoiding the use of analysis (in a perspective that privileges the purity of methods with respect to the statement or the disciplinary situation), we have seen that more important for Hermite was first the link between his number-theoretical research and a theorem of algebra, then the possibility of extending it to Cauchy's setting. In other cases, as in the resolution of the fifth-degree equation, Hermite would rejoice

87 [Picard 1901, p. 29]: "à propos de la question la plus élémentaire il faisait surgir tout d'un coup d'immenses horizons, et où à côté de la Science d'aujourd'hui on apercevait la Science de demain".

88 [Maillet 1905, p. 26]: "Quant aux rêves mathématiques, plusieurs de mes camarades de salle à l'École et moi en avons fait à la suite des abominables leçons d'Hermite sur les fonctions eulériennes, elliptiques, ultra-elliptiques et autres; ils consistaient en des assimilations de symboles algébriques à des choses de la vie réelle. Ainsi je me souviens d'une certaine fonction $C(x)$, où C était une caravane. J'estime que ce phénomène, assez répandu parmi nos camarades à cette époque, est une protestation réflexe de l'organisme cérébral contre l'inoculation d'un véritable poison intellectuel".

about bringing analytic means into a number-theoretical question.[89] On this point, Paul Painlevé comments:

> The memoir [on the transformation of Abelian transcendents], in which the theory of functions is interwoven with Arithmetic and Algebra, is in some ways *representative* of Hermite's work. No one has shown more strikingly, through his methods and discoveries, the intimate relations that unite these three branches of Science, and the mutual support they can and must lend each other.[90]

While Hermite was particularly effective in building bridges between these domains, he was far from alone. Echoes of this can also be found for instance in Emile Borel's commentary on Edmond Laguerre's work:

> [O]ne is at first astonished to see Laguerre's research on whole transcendents, and others besides, appear among the memoirs of Algebra. This astonishment disappears when, instead of reading only the Table of Contents, we study the text itself; we then see that one of Laguerre's characteristic traits is the ease with which he solves many questions of Analysis by the methods of the most elementary Algebra, and we hardly notice the transition between the Algebra of polynomials and the Algebra of transcendental functions, if I may express myself in this fashion.[91]

However, simply highlighting the connection between integers in arithmetic, polynomials in algebra, and functions in analysis does not give access to the practices involved in establishing this connection, which vary widely depending on the moment and the author. Although Kronecker, Hermite, Laguerre, but also Dedekind and Heinrich Weber all inherited the mid-century research field of arithmetic algebraic analysis, they deployed it, interpreted it and worked within or beyond it in very different ways, prioritizing or not the various aspects. André Weil's trilingual dictionary, or

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89 On this role of analysis, taken in a very broad sense and not limited to continuous processes, see [Archibald 2024; Vincent 2024].

90 [Painlevé 1905, p. 50-51]: "Le mémoire [sur la transformation des transcendantes abéliennes], où la théorie des fonctions se mêle à l'Arithmétique et à l'Algèbre, est en quelques sorte *représentatif* de l'œuvre d'Hermite. Nul n'a montré, d'une façon plus éclatante, par ses méthodes et ses découvertes, les relations intimes qui unissent ces trois branches de la Science, l'appui mutuel qu'elles peuvent et doivent se prêter."

91 The quote comes from [Vincent 2024] to which I also refer for other relations between Hermite's and Laguerre's works: "[O]n est tout d'abord étonné de voir les recherches de Laguerre sur les transcendantes entières et d'autres encore, figurer parmi les mémoires d'Algèbre. Cet étonnement disparaît lorsque, au lieu de lire seulement la Table des matières, on étudie le texte même; on voit alors que l'un des traits caractéristiques de Laguerre est l'aisance avec laquelle il résout bien des questions d'Analyse par les méthodes de l'Algèbre la plus élémentaire, et l'on s'aperçoit à peine de la transition entre l'Algèbre des polynômes et l'Algèbre des fonctions transcendantes, si l'on peut ainsi s'exprimer."

Rosetta stone, half a century later, is yet another formulation of these possible analogies—even it would be tempting to see Hermite or Kronecker as its precursors.[92] Such a comparison, which would be necessary to understand the dynamics of unification between parts of these fields, goes far beyond the scope of this article, where I have only tried to better understand Hermite's related practice by following a few key words.

As the quotations in the introduction to this article testify, the question of the unity of mathematics has mainly been discussed by mathematicians or philosophers as a philosophical question, based on their epistemological priorities or on a global vision of mathematics, whether to describe its encompassing modalities or to define its present or future rules. More recently, the variety of mathematical activities in time and place has led others to criticize or to throw into doubt the very idea of a unity of mathematics [Booß-Bavnbek & Davis 2013]. Hermite's (counter-)example, however, suggests another conclusion than considering the unity of mathematics as a given to be studied, as an horizon the norms of which are to be established, or as an illusion to be dismissed. Examining the configuration of Hermite's writings closely, on a micro-scale,[93] reveals how the sense of unity in mathematics that drove him and was emphasized by his epigones was concretely put into practice and reinforced in his day-to-day mathematical work. It also suggests that the topic of the unity of mathematics can be associated with a richer set both of work practices and of representations than is usually taken into account, and which, interweaving together, also calls for a *historical* investigation.

REFERENCES

ARCHIBALD (Tom)

[2011] Differential Equations and Algebraic Transcendents: French Efforts at the Creation of a Galois Theory of Differential Equations 1880–1910, *Revue d'histoire des mathématiques*, 17 (2) (2011), pp. 373–401.

[2024] Hermite's "Concrete" Analysis: Research and Educational Themes in an Evolving Discipline, *Revue d'histoire des mathématiques*, 2024; this issue.

BOOSS-BAVNBEK (Bernhelm) & DAVIS (Philip)

[2013] Unity and Disunity of Mathematics, *EMS Newsletter*, 87 (2013), pp. 28–31.

92 See [Dedekind & Weber 1882; Kronecker 1882; Weil 1979].

93 In the sense discussed in [Elias 1971; Grendi 1977; Lepetit 1995].

Bottazzini (Umberto) & Gray (Jeremy)
[2013] *Hidden Harmony - Geometric Fantasies: The Rise of Complex Function Theory*, New York: Springer, 2013.

Bouligand (Georges) & Desbats (Jean)
[1947] *La Mathématique et son unité*, Paris: Payot, 1947.

Bourdeau (Michel), Heinzmann (Gerhard) & Wagner (Pierre)
[2018] *Sur la philosophie scientifique et l'unité de la science : Le congrès de Paris 1935 et son héritage*, Paris: Kimé, 2018; Numéro spécial de *Philosophia Scientiae*, https://doi.org/10.4000/philosophiascientiae.1517.

Brechenmacher (Frédéric)
[2007] L'identité algébrique d'une pratique portée par la discussion sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes (1766–1874), *Sciences et Techniques en Perspective*, 1 (2007), pp. 5–85.
[2011] Autour de pratiques algébriques de Poincaré: héritages de la réduction de Jordan, 2011; hal-00630959v3.
[2014] Lagrange e l'equazione secolare, *Lettera Pristem* (2014), pp. 80–93; English transl., Lagrange and the secular equation, *Lett. Mat. Inst.*, 2 (2014), p. 79–91.
[2016] Algebraic generality vs arithmetic generality in the 1874 controversy between C. Jordan and L. Kronecker, in Chemla (Karine), Chorlay (Renaud) & Rabouin (David), eds., *The Oxford Handbook of Generality in Mathematics and the Science*, Oxford: Oxford Univ. Press, 2016, pp. 433–467.

Brezinski (Claude)
[1991] *History of Continued Fractions and Padé Approximants*, Berlin: Springer, 1991.

Brouzet (Robert)
[2004] La double origine du groupe symplectique, *Expositiones mathematicae*, 22-1 (2004), pp. 55–82.

Corry (Leo)
[2004] *Modern Algebra and the Rise of Mathematical Structures*, Basel: Birkhäuser, 2nd edition, 2004.

Dedekind (Richard)
[1894] Über die Theorie der ganzen algebraischen Zahlen (Supplement XI), in *Dirichlets Vorlesungen über Zahlentheorie,* 4th ed., Braunschweig: Friedrich Vieweg und Sohn, 1894, pp. 434–657.

Dedekind (Richard) & Weber (Heinrich)
[1882] Theorie der algebraischen Functionen einer Veränderlichen, *Journal für die reine und angewandte Mathematik*, 92 (1882), pp. 181–290.

Dickson (Eugene), Leonard
[1919] *History of the Theory of Numbers,* vol. III, *Quadratic and Higher Forms,* Washington: The Carnegie Institute of Washington, 1919; repr. New York, Chelsea Publishing Company, 1952.

Dieudonné (Jean)
[1971] La théorie des invariants au xix[e] siècle, *Séminaire Nicolas Bourbaki,* 23 (1971), pp. 257–274.
[1985] Fractions continuées et polynômes orthogonaux dans l'œuvre de E. N. Laguerre, in Brezinski (Claude), Draux (André), Magnus (Alphonse), Maroni (Pascal) & Ronveaux (André), eds., *Polynômes orthogonaux et applications: Proceedings of the Laguerre Symposium,* Berlin, Heidelberg: Springer, 1985, pp. 1–15.

Dirichlet (Peter Gustav Lejeune)
[1837] Sur les séries dont le terme général dépend de deux angles, et qui servent à exprimer des fonctions arbitraires entre des limites données, *Journal für die reine und angewandte Mathematik,* 12 (1837), pp. 35–56.
[1840] Sur la théorie des nombres, *Comptes rendus hebdomadaires des séances de l'Académie des sciences,* 9 (1840), pp. 285–288; repr. in *Journal de mathématiques pures et appliquées* 5 (1840), pp. 72-74.

Dunlop (Katherine)
[2017] Poincaré on the foundations of Arithmetic and Geometry, part II: Intuition and Unity in Mathematics, *HOPOS,* 7(1) (2017), pp. 88–107.

Edwards (Harold M.)
[2009] The Construction of Solvable Polynomials, *Bulletin of the American Mathematical Society,* 46(3) (2009), pp. 397–411.
[1995] Kronecker on the Foundations of Mathematics, in Hintikka (Jaakko), ed., *From Dedekind to Gödel,* Dordrecht: Springer, 1995, pp. 45–52.

Ehresmann (Charles)
[1966] Trends toward unity in mathematics, *Cahiers de topologie et géométrie différentielle,* 8 (1966), pp. 1–7.

Elias (Norbert)
[1971] *Was ist Soziologie,* München: Juventa, 1971.

Ferreirós (José) & Reck (Erich H.)
[2020] Dedekind's Mathematical Structuralism: From Galois Theory to Numbers, Sets, and Functions, in Reck (Erich H.) & Schiemer (Georg), eds., *The Prehistory of Mathematical Structuralism,* New York: Oxford Academic, 2020, pp. 59–87.

Gilain (Christian) & Guilbaud (Alexandre), eds.
[2015] *Sciences mathématiques 1750-1850: continuités et ruptures,* Paris: CNRS Éditions, 2015.

Goldstein (Catherine)
[2007] The Hermitian Form of Reading the D. A., in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. G. Gauss's* Disquisitiones Arithmeticae, Berlin: Springer, 2007, pp. 375–410.
[2011a] Un arithméticien contre l'arithmétisation: les principes de Charles Hermite, in Flament (Dominique) & Nabonnand (Philippe), eds., *Justifier en mathématiques,* Paris: Editions de la Maison des sciences de l'Homme, 2011, pp. 129–165.

[2011b] Charles Hermite's Stroll through the Galois fields, *Revue d'histoire des mathématiques*, 17 (2011), pp. 211–270.

[2012] Les autres de l'un : deux enquêtes prosopographiques sur Charles Hermite, in Nabonnand (Philippe) & Rollet (Laurent), eds., *Les uns et les autres*, Nancy: Presses universitaires de Nancy, 2012, pp. 509–540.

[2015] Axel Thue in context, *Journal de théorie des nombres de Bordeaux*, 27 (2) (2015), pp. 309–337.

[2019] Long term history and ephemeral configurations, in Sirakov (Boyan), Ney de Souza (Paulo) & Viana (Marcelo), eds., *Proceedings of the International Congress of Mathematicians- Rio de Janeiro, 2018* World Scientific, 2019, pp. 487–522.

GOLDSTEIN (Catherine) & SCHAPPACHER (Norbert)

[2007] A Book in Search of a Discipline, in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. G. Gauss's* Disquisitiones Arithmeticae, Berlin: Springer, 2007, pp. 3–65.

GOLDSTEIN (Catherine), SCHAPPACHER (Norbert) & SCHWERMER (Joachim), eds.

[2007] *The Shaping of Arithmetic after C. G. Gauss's* Disquisitiones Arithmeticae, Berlin: Springer, 2007.

GRAY (Jeremy)

[2018] *A History of Abstract Algebra*, Cham: Springer Nature, 2018.

GRENDI (Edoardo)

[1977] Micro-analisi e storia sociale, *Quaderni storici*, 12(35) (1977), pp. 506–520.

HAFFNER (Emmylou)

[2021] The Shaping of Dedekind's Rigorous Mathematics : What Do Dedekind's Drafts Tell Us About His Ideal of Rigor?, *Notre Dame Journal of Formal Logic*, 62(1) (2021), pp. 5–31.

HERMITE (Charles)

[1850] Extraits de lettres de M. Ch. Hermite à M. Jacobi sur différents objets de la théorie des nombres, *Journal für die reine und angewandte Mathematik*, 40 (1850), pp. 261–315.

[1853] Remarques sur le théorème de Sturm, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 36 (1853), pp. 294–297.

[1855] Sur la transformation des fonctions abéliennes, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 40 (1855), pp. 249–254.

[1856] Sur le nombre des racines d'une équation algébrique comprises entre des limites données, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 39–51.

[1859] Sur l'interpolation, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 48 (1859), pp. 62–67.

[1862] Sur la théorie des fonctions elliptiques et ses applications à l'Arithmétique, *Journal de mathématiques pures et appliquées*, s. 2, 7 (1862), pp. 25–44.

[1864a] Sur un nouveau développement en série des fonctions, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 58 (1864), pp. 93–100.

[1864b] Sur les théorèmes de M. Kronecker relatifs aux formes quadratiques, *Journal de mathématiques pures et appliquées*, s. 2, 9 (1864), pp. 145–159.

[1865a] Extrait d'une lettre de M. Hermite à M. Borchardt, *Journal für die reine und angewandte Mathematik*, 64 (1865), pp. 294–296.

[1865b] Sur quelques développements en série de fonctions de plusieurs variables, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 60 (1865), pp. 370–377.

[1866] *Sur l'équation du cinquième degré*, Paris: Gauthier-Villars, 1866.

[1871] Sur la construction géométrique de l'équation relative à l'addition des intégrales elliptiques de première espèce, *Bulletin des sciences mathématiques et astronomiques*, 2 (1871), pp. 21–23.

[1873] Extrait d'une lettre à M. Paul Gordan (sur l'expression $U \sin x + V \cos x + W$), *Journal für die reine und angewandte Mathematik*, 76 (1873), pp. 303–311.

[1878] Sur la théorie des fonctions sphériques, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, 86 (1878), pp. 1515–1519.

[1880] Sur une extension donnée à la théorie des fractions continues par M. Tchebychef, *Journal für die reine und angewandte Mathematik*, 88 (1880), pp. 10–15.

[1899-1900] Extrait d'une lettre adressée à L. Lindelöf, *Öfversigt af Finska vetenskaps-societetens förhandlingar*, 42 (1899-1900), pp. 88–90.

[1905-1917] *Œuvres*, Paris: Gauthier-Villars, 1905-1917; 4 volumes.

HERMITE (Charles) & STIELTJES (Thomas)

[1905] *Correspondance*, Paris: Gauthier-Villars, 1905; 2 volumes.

HOUZEL (Christian)

[2002] *La Géométrie algébrique. Recherches historiques*, Paris: Blanchard, 2002.

JACOBI (Carl Gustav Jacob)

[1839] Über die complexen Primzahlen, welche in der Theorie der Reste der 5ten, 8ten und 12ten Potenzen zu betrachten sind, *Bericht über die zur Bekanntmachung geeigneten Verhandlungen der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 1839, pp. 86–91; Repr. in *Journal für die reine und angewandte Mathematik*, 19 (1839), pp. 314–318.

[1843] Sur les nombres premiers complexes que l'on doit considérer dans la théorie des résidus de cinquième, huitième et douzième puissance, *Journal de mathématiques pures et appliquées*, 8 (1843), pp. 268–272; trad. by Hervé Faye of [Jacobi 1839].

JAHNKE (Hans Niels)

[1987] Motive und Probleme der Arithmetisierung der Mathematik in der ersten Hälfte des 19. Jahrhunderts, *Archive for History of Exact Sciences*, 37-2 (1987), pp. 101–182.

JUBILÉ

[1893] *Jubilé de M. Hermite (24 décembre): 1822-1892*, Paris: Gauthier-Villars, 1893.

Kremer-Marietti (Angèle)
[2003] De l'unité de la science à la science unifiée : De Comte à Neurath, in Petit (Annie), ed., *Auguste Comte : Trajectoires positivistes 1798-1998*, Paris: L'Harmattan, 2003, pp. 189–203.

Krömer (Ralf)
[2007] *Tool and Object: A History and Philosophy of Category Theory*, Basel, Boston, Berlin: Birkhäuser, 2007.

Kronecker (Leopold)
[1860] Sur le nombre des classes différentes de formes quadratiques à déterminants négatifs, *Journal de mathématiques pures et appliquées*, s. 2, 5 (1860), pp. 289–299.
[1882] Grundzüge einer arithmetischen Theorie der algebraischen Grössen, *Journal für die reine und angewandte Mathematik*, 92 (1882), pp. 1–122.

Lagrange (Joseph-Louis)
[1774] Paragraphe IX : De la manière de trouver des fonctions algébriques de tous les degrés, qui étant multipliées ensemble produisent toujours des fonctions semblables, in *Élémens d'algèbre, par M. Léonard Euler, traduits de l'allemand avec des notes et des additions, tome second : de l'analyse indéterminée*, Lyon: Jean-Marie Bruyset, 1774, pp. 636–658.

Lautman (Albert)
[1938] *Essai sur l'unité des sciences mathématiques dans leur développement actuel*, Paris: Hermann, 1938; Thèse complémentaire pour le doctorat es lettres (philosophie), Faculté des lettres de l'Université de Paris ; quoted from the reedition, *Essai sur l'unité des mathématiques et divers écrits*, Paris, Christian Bourgois 1018/Union générale d'éditions, 1977, pp. 155-202.

Lê (François)
[2015] "Geometrical Equations": Forgotten Premises of Felix Klein's *Erlanger Programm*, *Historia Mathematica*, 42(3) (2015), pp. 315–342.
[2023] Words, history and mathematics, *European Mathematical Society Magazine*, 130 (2023), pp. 26–35.
[2024] On Charles Hermite's style, *Revue d'histoire des mathématiques*, 2024; this issue.

Lepetit (Bernard), ed.
[1995] *Les formes de l'expérience*, Paris: Albin Michel, 1995.

Lindelöf (Ernst Leonard)
[1899-1900] Un problème du calcul des probabilités, *Öfversigt af Finska vetenskaps-societetens förhandlingar*, 42 (1899-1900), pp. 79–87.

Maillet (Edmond)
[1905] Les rêves et l'inspiration mathématique, *Bulletin de la société philomathique*, s. 9, 7 (1905), pp. 19–62.

Maronne (Sébastien)
[2014] Pierre Samuel et Jules Vuillemin : mathématiques et philosophie, *Revue d'Auvergne*, 2014, pp. 151–173.

McKean (Henry) & Moll (Victor)
[1999] *Elliptic Curves*, Cambridge: Cambridge Univ. Press, 1999.

Mlika (Hamdi)
[2009] Charles Laisant, Auguste Comte et l'unité des mathématiques, in Kremer-Marietti (Angèle), ed., *Auguste Comte: La science, la société,* Paris: L'Harmattan, 2009, pp. 65–84.

Morris (Charles William)
[1960] On the history of the International Encyclopedia of Unified Science, *Synthese*, 12 (1960), pp. 517–521.

Neurath (Otto), Carnap (Rudolf) & Morris (Charles William), eds.
[1938] *International Encyclopedia of Unified Science vol. 1, part 1*, Chicago: The University of Chicago Press, 1938; Combined edition, 1955.

Painlevé (Paul)
[1905] Charles Hermite, *Nouvelles annales de mathématiques,* s. 4, 5 (1905), pp. 49–53.

Parshall (Karen)
[2024] A Convergence of Paths: Arthur Cayley, Charles Hermite, James Joseph Sylvester, and the Early Development of a Theory of Invariants, *Revue d'histoire des mathématiques*, 2024; this issue.
[1989] Toward a History of Nineteenth-Century Invariant Theory, in Rowe (David) & Mc Cleary (John), eds., *The History of Modern Mathematics*, Boston: Academic Press, 1989, pp. 157–206.

Picard (Émile)
[1901] L'œuvre scientifique de Charles Hermite, *Annales scientifiques de l'École normale supérieure*, 8 (1901), pp. 9–34.

Rabouin (David)
[2009] *Mathesis universalis : L'idée de "mathématique universelle" d'Aristote à Descartes*, Paris: Presses Universitaires de France, 2009.

Serfati (Michel)
[1992] *Quadrature du cercle, fractions continues, et autres contes. Sur l'histoire des nombres irrationnels et transcendants aux XVIII^e et XIX^e siècles*, Paris: Éditions de l'Association des Professeurs de Mathématiques, 1992.

Shaw (James Byrnie)
[1937] The Unity of Mathematics, *The Scientific Monthly*, 45-5 (1937), pp. 402–411.

Sieg (Wilfried) & Schlimm (Dirk)
[2017] Dedekind's Abstract Concepts: Models and Mappings, *Philosophia Mathematica*, 25-3 (2017), pp. 292–317 .

Sinaceur (Hourya)
[1994] *Corps et modèles*, Paris: Vrin, 1994.

Smith (Henry)
[1865] Report on the Theory of Numbers, Part VI, *Report of the British Association*, 1865, pp. 322–375.

Sørensen (Henrik Kragh)
[2005] Exceptions and counterexamples: Understanding Abel's comment on Cauchy's Theorem, *Historia Mathematica*, 32(4) (2005), pp. 453–480.

Stump (David)
[1997] Reconstructing the Unity of Mathematics circa 1900, *Perspectives on Science*, 5-3 (1997), pp. 383–417.

Tchebichef (Pafnuti) & Bienaymé (Irénée-Jules) (trad.)
[1858] Sur les fractions continues, *Journal de mathématiques pures et appliquées*, s. 2, 3 (1858), pp. 289–323.

Vergnerie (Cedric)
[2019] L'algèbre sans les fictions des racines : Kronecker et la théorie des caractéristiques dans les *Vorlesungen über die algebraischen Gleichungen*, *Revue d'histoire des mathématiques*, 25 (1) (2019), pp. 1–107.

Vincent (Yannick)
[2020] Ce que nous apprend une étude autour des équations numériques à propos de l'École polytechnique au xix^e siècle, *Philosophia Scientiae*, 24-1 (2020), pp. 59–74; en ligne, `https://www.cairn.info/revue-philosophia-scientiae-2020-1-page-59.htm`.
[2024] Les travaux de Laguerre sur les équations polynomiales et l'influence hermitienne, *Revue d'histoire des mathématiques*, 2024; this issue.

Waldschmidt (Michel)
[1983] Les débuts de la théorie des nombres transcendants (à l'occasion du centenaire de la transcendance de π), *Cahiers du séminaire d'histoire des mathématiques*, 4 (1983), pp. 93–115.

Weil (André)
[1979] Une lettre et un extrait de lettre à Simone Weil, in *Œuvres scientifiques – Collected Papers,* vol. 1, Berlin, Heidelberg: Springer, 1979, pp. 244–255.

Wolfson (Paul)
[2008] George Boole and the origins of invariant theory, *Historia Mathematica*, 35-1 (2008), pp. 37–46.

Zappa (Guido)
[1995] Storia della risoluzione delle equazioni di quinto e sesto grado, *Rendiconti del Seminario Matematico e Fisico di Milano*, 65 (1995), pp. 89–107.

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ON CHARLES HERMITE'S STYLE

François Lê

Abstract. — This paper tackles the issue of Charles Hermite's style of writing. The approach, supported by the techniques of statistical textual data analysis, is quantitative and comparative, the chosen point of comparison being Camille Jordan. Two main facets of the notion of style are scrutinized. On the one hand, the investigation focuses on lexical richness, with a particular stress on lexical diversity and the use of hapaxes, that is, words appearing only once in each corpus. On the other hand, specificities of words and of grammatical categories are examined. It is shown that Hermite's prose is characterized by a higher lexical diversity than Jordan, and reflects a lively mathematical narration where the first person and other words which describe the mathematical processes are of great importance.

Résumé (Sur le style de Charles Hermite). — Cet article aborde la question du style d'écriture de Charles Hermite à l'aide de techniques quantitatives issues de l'analyse statistique des données textuelles. L'approche est comparative, et confronte Hermite à Camille Jordan. Deux aspects de la notion de style sont abordés. D'une part, la question de la richesse lexicale est discutée, notamment vis-à-vis de la diversité lexicale et l'utilisation d'hapax, c'est-à-dire de mots apparaissant une seule fois dans chaque corpus. D'autre part, il s'agit de considérer les mots et catégories grammaticales spécifiques aux deux auteurs considérés. Nous montrons que le discours hermitien possède un plus grande diversité lexicale que celui de Jordan, et reflète une narration

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mathématique dynamique, où la première personne et les mots qui décrivent les processus mathématiques à l'œuvre jouent un rôle de grande importance.

In 1899, historian of science Paul Tannery published a paper entitled "Stylometry: its origins and its present" in *Revue philosophique de la France et de l'étranger*, [Tannery 1899]. This paper was a critical review of a book by Wincenty Lutosławski which had appeared shortly before, and which aimed at establishing the writing chronology of Plato's works by studying (the evolution of) his style, [Lutosławski 1897]. Following a statistical approach, Lutosławski investigated textual data which were supposed to characterize this style, such as rare words, sentence length, or the mutual ratios of the numbers of nouns, verbs, adjectives, and adverbs employed by Plato. Before him, other scholars had already taken into account such textual statistics to determine the chronology of the platonic dialogues, but Lutosławski differentiated himself by proposing basic rules meant to ensure the soundness of this method, which he christened "stylometry." [1] Although Tannery expressed a guarded opinion on the quality of these rules, he still admitted that "stylometry would be invaluably helpful if it was grounded scientifically." Here he referred not so much to chronological matters as to authenticity questions and authorship attribution, for stylometry would allow "bringing to light the particular and multiple causes which create the overall impression left by the style of an author." [2]

More than a century later, thanks to the development of computers and of the statistical analysis of textual data (sometimes called, with nuances, lexicometry or textometry), stylometry has extended its field of action, and researchers have been reflecting on both its theoretical basis and its technical implementation. Among other kinds of results, semantic and syntactic specificities of literary writers, which were hard to detect with the naked eye, have been revealed; literary genres have been correlated to the over- or underuse of some parts of speech; and quantified approaches of the phenomena of rhythm and rhyme have renewed the analyses of poetic corpuses. [3] Such

1 For a global presentation of Lutosławski and an analysis of his research in linguistics, see [Pawłowski 2008].

2 « La stylométrie ne peut en effet que prétendre à mettre en évidence les causes particulières et multiples qui produisent l'impression générale que laisse le style d'un auteur. » Further: « Que la stylométrie puisse rendre d'inappréciables services, si elle est scientifiquement justifiée et appliquée suivant des lois reconnues valables, cela va de soi. » [Tannery 1899, 161-162].

3 See for instance [Brunet 1978; Muller 1967] or, more recently, [Beaudouin 2002; Labbé & Labbé 2018]. I am indebted to Catherine Goldstein for having brought Valérie Baudouin's works to my attention.

statistical techniques thus offer a particular way to address the thorny question of style, of which the difficulties and the resistance both to be theorized and to be turned into an univocal, operative category are notorious—an observation which does not mean that other highly interesting stylistic issues cannot be dealt with successfully. [4]

In the history and the philosophy of mathematics, the notion of style has also been tackled several times during the last decades. Without entering into details, let me just recall that most of these contributions proposed to characterize mathematical styles with the help of criteria linked to the manner of how mathematicians of the past thought of mathematical objects and correspondingly integrated them in their works, how their demonstrations were made with respect to certain values, methods, or disciplinary preferences, or how they included more or less examples in their publications. [5]

These proposals, with all their nuances, have obviously their own merits, and there is no question of diminishing or discussing them here. My intention in this article is to approach the issue of style (and indeed, that of Charles Hermite) by following an alternative path, that of stylometry.

Hence, both my focus and methodology are different from those of the cited historical and philosophical research. Quite paradoxically, they appear to be more literary and more mathematical, respectively—or, if one prefers, less mathematical and less literary. Let me explain what I mean.

I say more literary because my aim is not to dissect some of Hermite's proofs, nor to understand how a given theorem is stated with regard to some mathematical values, nor to account for particular disciplinary articulations in his work. [6] Rather, I would like to tackle the notion of style, understood as the "set of the expressive traits which denote the author in a writing," [7] by scrutinizing the words of Hermite that are *not* directly linked

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4 The literature on this topic is huge. Here I only refer to the introduction and the different contributions of [Himy-Piéri et al. 2014], to [Herschberg Pierrot 2005], or to the interesting investigations on the figure of speech paradox presented in [Gallard 2019].

5 See the synthesis [Mancosu 2009/2021] and its references, as well as [Rowe et al. 2010] and [Rabouin 2017]. Paolo Mancosu's article includes a discussion of works from the history and the philosophy of science, such as [Gayon 1996], where the notion of style has also been challenged many times.

6 Moreover, it is not about establishing a Hermitian lexicon with the help of statistical tools, as it has been made for Francis Bacon for instance, [Fattori 1980]. Among other recent research devoted to the vocabulary of scientists, see [Giacomotto-Charra & Marrache-Gouraud 2021].

7 This definition comes from the section related to language and linguistics in the entry "Style" of *Trésor de la langue française*: "Ensemble des traits expressifs qui dénotent l'auteur dans un écrit." See `https://www.cnrtl.fr/definition/style`. I summon

to mathematical objects. To characterize the way Hermite expresses himself, the attention will thus be placed on the words from the natural language which have a more functional role in the mathematical discourse: personal and demonstrative pronouns, conjunctions, non-technical nouns and verbs, and so forth. Lexical richness will also be taken into account to describe Hermite's style. This richness will be evaluated from three different viewpoints: vocabulary extent, number and nature of the hapaxes, i.e., of words which are used only once by Hermite, and, to a lesser extent, vocabulary sophistication. [8]

I also say more mathematical, because the analysis will be constantly supported by statistical calculations and indicators, which will help quantify the description of Hermite's mathematical prose and, to a certain degree, objectify "the overall impression left by the style of [this] author," to borrow Tannery's words again. Of course, this does not mean that the statistical, computer-aided tool is able to provide purely objective results: the human researcher remains present throughout the whole process, from the initial technical conventions to the very selection of the questions to be tackled, and to the interpretation of the given numbers. In particular, any blind reliance on such numbers will be excluded: to understand and put these numbers into perspective, the corresponding words will always be studied within their textual environment.

Hermite's style will be appraised from the French, technical texts gathered in his *Œuvres complètes*: this excludes two papers written in English and Italian, as well as texts such as addresses, obituary notices, and prefaces of books by other mathematicians. [9] This operation leads to a corpus of 186 texts, published between 1842 and 1901. [10]

To determine whether the different textual data that can be measured in this corpus are actually characteristic of Hermite, a comparative corpus has to be considered. In the absence of a bigger set of mathematical texts

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up this definition simply to fix ideas, my aim not being to discuss in detail what an optimal definition of style might be in a historical-mathematical framework.

8 As will be seen, indeed, vocabulary sophistication has been harder to handle. I will explain why, and comment on what I tried to do.

9 The *Œuvres* also contain a report written by Augustin-Louis Cauchy on a memoir by Hermite, which has obviously been dismissed. Therefore the resulting corpus is almost the same as that studied in a prosopographic perspective in [Goldstein 2012].

10 Here and in the rest of the paper, the word "text" will refer to the different items that are distinguished in the *Œuvres*, where several series of notes published in *Comptes rendus hebdomadaires des séances de l'Académie des sciences* have been fused. Moreover, the content of these texts is not strictly identical with that of the original publications: a number of misprints have been corrected, and Émile Picard, the editor of the *Œuvres*, occasionally deleted parts which he indicated to be inaccurate.

which would be available and ready for the textometric treatment, the point of comparison that has been chosen is Camille Jordan: this French mathematician was more or less a contemporary of Hermite, shared a number of research topics with him, and produced a work whose dimensions are roughly similar to Hermite's. Selecting Jordan's papers from his *Œuvres complètes* in the same way as has been done for Hermite yields a corpus of 122 texts published between 1861 and 1920. Thus the temporal width of Jordan's corpus is exactly the same as Hermite's, although it begins two decades later. Incidentally, these two decades correspond approximately to the age difference of the two mathematicians, since Hermite is born in 1822 and Jordan in 1838.

In accordance with what has been announced above, the investigation is divided in two main parts, which deal with the notion of lexical richness and with that of grammatical and lexical specificities, respectively—the technical definitions of these terms will be given below. Among other results, we will see that, compared to Jordan, Hermite possesses a higher lexical richness, both in consideration with the vocabulary extent and the hapaxes. Moreover, we will see that the two corpuses differ largely with respect to the use of some grammatical categories. In particular, such unbalanced grammatical distributions will serve to characterize Hermite's mathematical writing as a narrative involving to a great degree the person of Hermite himself through the use of many personal pronouns, and of many specific verbs which describe the mathematical processes in a lively way. On the contrary, it will be shown that Jordan is more effaced in his writings, where the third person is privileged and the verbs tend to describe the mathematical progression in a more distant way.

Such a difference between Hermite and Jordan will be briefly discussed in the conclusion in the light of some basic concepts coming from narratology. I will then come back to the soundness of the approach proposed in this paper. In particular, a number of issues which arise naturally will be made explicit: that of the non-synchronicity of Hermite and Jordan (which can be linked to the collective dimension of style), but also that of the possible influence of mathematical domains on the question of style. To close the paper, some reflections on the very interest of turning away from the mathematical core and focusing instead on what could appear as insignificant details will eventually be proposed.

1. AN OVERVIEW

For the needs of TXM, the textometry software that has been used, [11] each text of the corpus must correspond to an appropriate computer file. Among the possibilities, I chose to work with `txt` files, formatted with the conventions described in [Lê 2022]. Thus, apart from correcting misprints and standardizing a few words (such as the noun "Tchebichef," originally present in different transliterated forms), the main preliminary operation consisted in deleting the content of each mathematical formula and replacing it by a mere symbol $*$ or #, if the formula was inline or displayed, respectively. The software then spotted the words of every text, counted them, and associated them with their lemma (i.e., the entry which would correspond to the given word in a dictionary) and the grammatical category to which they belong. [12]

These data form the ground for all the functionality of TXM: exhaustive searches and counts of (sequences of) words, lemmas, or grammatical categories related to diverse queries (lemmas that belong to a given grammatical category, words of which the lemma begins with, or contains, a given chain of characters, etc.); lists of concordances, which situate the results of such queries in their close textual neighborhood and allow to sort them in a variety of ways; inspection of texts in their whole; and other, more advanced tools, of which some will be used and presented later in this paper.

For Hermite, the preliminary treatment inventories 364412 words counted with repetition, which correspond to 6334 distinct forms (called the tokens) and 2740 lemmas. Among these words, 19542 symbols $*$ and 10869 symbols # are to be found.

In spite of the lower number of his texts, Jordan's corpus counts more words, since they are 591732 in number, distributed into 6983 tokens and 2852 lemmas. The mathematical substitutes are divided into 55252 symbols $*$ and 6714 symbols #.

In both cases, apart from mathematical symbols, the most frequent words are functional words of the French language: articles, conjugated forms of the auxiliary verbs *être* and *avoir*, pronouns, prepositions, usual adverbs, and punctuation marks. The first-ranked substantives in the lexicons of Hermite and Jordan are nouns of mathematical objects, which reveal at once some of the thematic predilections of our two authors:

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11 Its technical presentation is given in [Heiden 2010].

12 In this paper, "word" is taken in its textometric meaning. In particular, punctuation marks and symbols (including those standing for mathematical formulas) are counted as words.

substitutions and groups for Jordan, equations, functions, and forms [13] for Hermite (see table 1).

Hermite		Jordan	
Word	Frequency	Word	Frequency
équation	1395	substitutions	4816
forme	1368	groupe	3295
fonctions	1304	substitution	2682
fonction	1146	forme	2299
expression	942	nombre	2052
nombre	905	ordre	1951
degré	894	lettres	1915
formes	871	variables	1616
racines	870	cas	1425
coefficients	834	système	1372

Table 1. The ten first common nouns in Hermite's and Jordan's lexicons

As stated above, this angle of the textual analysis, related to the technical, mathematical words, will not be investigated further. Instead, and before delving into the issue of the lexical richness, let me consider table 2. It shows how the frequency of words and the contribution of the latter to the vocabulary are correlated—to make things clear, the term "frequency" designates the absolute number of occurrences of a word (or a lemma...) in a given corpus, and the term "vocabulary" refers to the set of all the tokens in such a corpus.

Frequency class	Words		Tokens	
$f \geqslant 1500$	182590	50,1%	29	0,5%
$1000 \leqslant f \leqslant 1499$	20064	5,5%	16	0,3%
$500 \leqslant f \leqslant 999$	30795	8,5%	42	0,7%
$100 \leqslant f \leqslant 499$	71586	19,6%	323	5,1%
$2 \leqslant f \leqslant 99$	57332	15,7%	3879	61,2%
$f = 1$	2045	0,6%	2045	32,3%

Table 2. Distribution of Hermite's lexicon into frequency classes. The given percentages are relative to the total numbers of words (364412) and of tokens (6334), respectively.

13 Just as in English, the French word *forme* can designate either the mathematical object (e.g., a quadratic form) or the aspect of something. Without entering into details, let me just remark that a great number of *forme* do refer to the mathematical objects which bear this name; consistently, the plural *formes*, which is more likely to refer to these objects, is also present in the list of the most frequent words used by Hermite.

The very high frequencies [14] ($f \geqslant 1500$) take up half of Hermite's corpus although they represent only 0.5% of the vocabulary; conversely, the hapaxes embody 0.6% of the word mass but almost one third of the tokens. Said differently, a handful of words are repeated extremely often, but the vocabulary is concentrated in the very low frequencies: almost all the entries of the vocabulary occur in the corpus with a frequency lower than 100. This phenomenon is quite general, and can also be seen in literary texts, with some nuances: for instance, hapaxes tend to represent a tiny part of such texts, but they contribute up to 50% of the vocabulary. [15]

The very high frequencies correspond to the two symbols that signal the existence of mathematical formulas, and to diverse functional words of the language such as punctuation marks, articles, conjunctions, and prepositions (*de*, *la*, *et*, etc.: "of," "the," "and"). The high frequencies ($1000 \leqslant f \leqslant 1499$), for their part, are mostly composed of pronouns, together with the four substantives *équation*, *forme*, *fonction*, *fonctions*, and the word *deux* ("two"). Verbs that are not auxiliary verbs begin to appear in the medium frequencies ($500 \leqslant f \leqslant 999$), with conjugated forms of *pouvoir* et *donner* ("can / to be able to," "to give"). Other such verbs occupy more and more room as frequencies get lower; eventually, they represent about half of the hapaxes. [16]

Since I do not wish to make a detailed comparison with Jordan on this point, the analogous numbers which correspond to his corpus will not be provided, and I will just indicate that the ratios are very similar to those of Hermite, either for the distribution of words or for that of the tokens.

That said, two components of the notion of lexical richness are related to some of the numbers which have been given above for our two mathematicians. The first one concerns the number of the tokens in the corpuses, while the second one investigates more closely the category of the hapaxes.

2. LEXICAL RICHNESS

Two main aspects form the notion of lexical richness in its traditional meaning. [17] On the one hand, one is interested in the numerical side of

14 The number and the extent of the frequency classes of table 2 are arbitrary. They roughly follow the model given in [Kastberg Sjöblom 2002, § 2.3].

15 See the numbers given in [Kastberg Sjöblom 2002, § 2.3] [Lebart et al. 2019, 51], as well as the examples of *Les Misérables* and *Germinal* which we present below.

16 The description being based on words, and not lemmas, this observation is to be linked to the fact that in French, there exist many more inflected forms of verbs than inflected forms of nouns and adjectives, for example.

17 See [Muller 1977, 115].

the matter exclusively, in the sense that the vocabulary and some of its subsets (typically, the hapaxes) are considered with respect to their size only: this aspect is usually referred to as vocabulary diversity. On the other hand, the semantic content of the vocabulary is at the core of the issue of sophistication, where one tries to evaluate the degree of refinement, or eccentricity, of the terms used by an author. Both of these aspects are most relevant when they are put into a comparative framework: the extent of a vocabulary and the rarity of words are notions which need external references to be gauged. Moreover, even if vocabulary diversity and sophistication are often related to one another in practice, they must be clearly differentiated: in a given text, a writer might use a lot of different, yet completely banal words or, conversely, repeat some advanced terms over and over.

Before beginning the analysis, three preliminary remarks on our situation should be made explicit. The first one concerns the problem posed by mathematical symbols. Because of the text formatting that has been mentioned earlier, it is impossible to take into account the diversity of these symbols as they appear in the original publications of Hermite and Jordan. Therefore, even if it would be interesting to include them in our reflection, the lexical richness will be evaluated only on the basis of the words expressed in the natural language. Accordingly, in this whole section, the numbers of words and tokens will always exclude the symbols $*$ and #. [18]

The second point to bear in mind stems from the very structure of the Hermitian corpus. Indeed, this corpus contains 70 letters or extracts of letters that have been published in journals at the time: they represent 38% of the number of texts and 26% of the number of words. [19] On the contrary, only one such letter is to be found in Jordan's corpus—a letter which happens to be very short. [20] A question, then, is to ascertain if and how the epistolary format may influence the lexical richness. Consequently, the case of the letters will be treated separately when needed.

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18 Actually, every calculation has been made twice, to see the possible influence of the mathematical symbols in the results. It turns out that, on the whole, these results are very similar.

19 All these (extracts of) letters but one have been published in mathematical journals at Hermite's time. The exception is a 1900 letter to Jules Tannery, which was possibly not intended to be published by Hermite, but has been reproduced posthumously in the 1902 fourth volume of *Éléments de la théorie des fonctions elliptiques* by Tannery and Jules Molk, [Hermite 1902]. As will be seen, this letter contains many hapaxes, but taking it into account or not in the corpus does not change anything regarding the global numbers and the comparison with Jordan.

20 As Frédéric Brechenmacher pointed out to me, this asymmetry echoes the fact that Hermite and Jordan had a different conception of what is, or what should be, a mathematical publication. See [Brechenmacher 202?].

The last remark concerns the difference between the sizes of Hermite's and Jordan's corpuses (which count 334001 and 529766 non-mathematical words, respectively). In fact, this well-known issue goes beyond our case study. It is rooted in the fact that the vocabulary extent of a corpus is not a linear function of the number of words: as new words are added to a text, its vocabulary grows too, but this growth is slower since a part of the adjoined words have already been used before. Hence, a crucial issue is to be able to confront, in a relevant way, the extents of the vocabularies of two corpuses whose sizes are markedly different.

2.1. *Lexical diversity*

In particular, comparing the ratios between the numbers of words and of tokens can be seen as a first indicator of lexical diversity, but it has to be refined in most cases. To put things into perspective, let me (naively) compare Hermite's corpus with two French novels of the second half of the nineteenth century, namely Victor Hugo's *Les Misérables* (1862) and Émile Zola's *Germinal* (1885).[21] A direct examination of the numbers of table 3 shows that even if *Germinal* possesses fewer words than Hermite's texts, it has more than twice as many tokens and more than three times as many hapaxes: this configuration leaves no doubt of the fact that *Germinal* has a higher lexical diversity. The comparison with *Les Misérables* is of the same vein, although this case is a bit different: even if the numbers of words and of tokens are ordered in the same way as in Hermite's corpus, the orders of magnitude of the data do seem to indicate clearly a higher richness for Hugo's book.

	Words	Tokens	Hapaxes
Hermite	334001	6332	2045
Jordan	529766	6981	2014
Germinal	211379	14458	6305
Les Misérables	645243	31685	14394

Table 3. Numbers of words, tokens, and hapaxes which are not the symbols $*$ and #

More delicate is the comparison between Hermite and Jordan: the latter's texts contain more words and more tokens than Hermite's, but the

[21] The data presented in the following lines are those which I obtained with the help of TXM, by using the versions of these novels which are available on `https://fr.wikisource.org`.

numbers of tokens are relatively close to one another, and there is no inversion of their order compared with the sizes of the corpuses. This is a typical case where caution has to be taken: to what extent is the superiority of Jordan's vocabulary extent due to the bigger size of the corpus itself?

To answer such a question, several solutions have been proposed by researchers working in lexical statistics.[22] Among these solutions, the techniques of text shortening consist in estimating what would be the vocabulary extent of the longest of two (corpuses of) texts if, considering its frequency structure, its size was the same as the shortest one—for the reasons that have been evoked above, acting by mere linearity is not seen as adequate. Here I decided to use the method founded on what has been called the coefficient of vocabulary partition, [Labbé & Hubert 1997].[23]

By reducing Jordan's corpus to the size of Hermite's, the corresponding calculations[24] give the numbers listed in table 4. They mean that, knowing the actual and complete structure of Jordan's corpus, one estimates that it would count 5825 tokens if it was made of 334001 words. This represents a relative difference of about 8% in comparison with Hermite.[25]

	Hermite		Jordan (reduced)
	Words	Tokens	Tokens
Letters	85540	3824	3434
Non-letters	248461	5539	5196
Whole corpus	334001	6332	5825

Table 4. Expected vocabulary extents of Jordan's corpus, if it was reduced to the sizes of Hermite's whole corpus and of its two sub-corpuses made of the letters and the non-letters

Moreover, the same observation holds when Jordan's corpus is reduced to the sizes of Hermite's sub-corpuses made of the letters and the non-letters, respectively. The relative differences change a little bit, however, since the one associated with the letters equals approximately 10%, while

22 Apart from those which will be cited and used below, see the references given in [Lebart et al. 2019, 50].

23 This method refines, in the case of corpuses with a certain vocabulary "specialization," the classical technique of Charles Muller based on a probabilistic, binomial model, [Muller 1977, ch. 20].

24 Since the software TXM does not include such a shortening process, I proceeded to the calculations on my own.

25 A mere linear process would have been way more violent, since the vocabulary of Jordan would have been estimated to 4401 tokens, i.e., 30% less than Hermite's 6332 tokens.

the other one is close to 6%. In particular, it is interesting that even though the epistolary format does seem to favor a higher lexical diversity, Jordan is still characterized with a lower diversity when compared to the non-epistolary part of Hermite's corpus.

To complete these observations, let me eventually remark that Hermite's letters do have a greater lexical diversity than his other publications. This can already be seen in the previous indicators (*via* an intermediate comparison with Jordan). But it is also possible to apply the shortening technique to these sub-corpuses: the one made of the non-letters would count 3522 tokens, that is, 8% less than the letters. [26]

2.2. *Hapaxes: numerical comparisons*

The presence of a great number of hapaxes in a corpus is often considered as a mark of a high lexical diversity. In the case of the comparison between Hermite and Jordan, a remarkable phenomena is to be observed: even though the corpus of the former is shorter than that of the latter, it contains more hapaxes (see the data already given in table 3). However, to have a clearer picture of the situation, a possibility is, again, to shorten Jordan's corpus. In fact, when it comes to the hapaxes only, the method which has been used above coincides with a simple linear operation. Thus if Jordan's corpus was reduced to the size of Hermite's, it would count 1270 hapaxes, which represent nearly 62% of Hermite's 2045 hapaxes: from this viewpoint, Hermite's vocabulary appears as being much more diverse than Jordan's, which echoes the previous results. [27]

That said, an examination of the lists of the hapaxes of our mathematical authors reveals that the grammatical categories to which they belong are distributed differently. On both sides, a significant part of the hapaxes are just numbers (typically, these numbers stand for pages and years, and appear when Hermite and Jordan cite other publications). They are 175 in Hermite and 110 in Jordan. Furthermore, the quantity of hapaxes in Jordan is inflated by about a hundred of ordinal numeral adjectives written in the form *157ème*. All these adjectives come from one paper where Jordan establishes a long enumeration of groups and synthesizes it with the help of phrases such as "*157ème groupe*," "*161ème à 163ème groupes*," etc. [Jordan

26 By shortening Hermite's whole corpus to the size of *Germinal*, its theoretical vocabulary extent would be of 5181 tokens, a number which is radically smaller than the 14458 tokens of Zola's novel.

27 Of course, this result derived from the hapaxes and that on the vocabulary diversity as presented in the previous subsection are not alien to one another, since the hapaxes contribute up to 30% of the vocabulary, in our cases.

1868/1869]. Reciprocally, and contrary to Jordan's corpus, that of Hermite possesses a large number of foreign words (603), of which 218 are hapaxes. Most of these foreign words are constituents of titles and extracts that are cited by Hermite; a handful of them correspond to Latin phrases that Hermite uses here and there. Such non-French terms thus also contribute to extend the hapax number of Hermite.

These hapaxes being neglected,[28] the remaining ones are almost exclusively content words, i.e., nouns, verbs, (non-numeral) adjectives, and some adverbs. Hermite has 1640 of them, a smaller number than the 1808 of Jordan. The relative order of these numbers is thus the opposite of that of the total numbers of hapaxes; however, reducing Jordan's corpus to the size of Hermite's leads to an estimated number of 1140 hapaxes, which is markedly less than the 1640 of Hermite.

Let me finally note that nearly 30% of Hermite's hapaxes come from his letters, whereas these particular texts represent 26% of the total mass of the words of the corpus. The letters, thus, are slightly richer in hapaxes that the rest of the corpus. To have a finer interpretation of these numerical observations, I now consider the meanings of the hapaxes more closely.

2.3. *Semantic content of the hapaxes*

An important preliminary remark is that a big part of the hapaxes are words which seem to be completely ordinary, and whose very low frequency is surprising at first sight. For example, the words *formée, parlant, rencontrés, essai,* and *Comparaison* ("formed," "talking," "met," "attempt," "Comparison"[29]) are hapaxes for Hermite. One should keep in mind, indeed, that the notion of hapax relates to the very graphical form of words, and not to the associated lemmas: *Comparaison* is not the same as *comparaison*, the verb *parler* ("to talk") actually appears 41 times in Hermite's texts in different conjugated forms, etc.

I will now focus on more particular hapaxes, which seem to characterize to a greater extent Hermite's personal writing.[30]

Here again, it is useful to distinguish the letters from the other publications of Hermite. Indeed, a certain number of hapaxes which come from

28 The case of the Latin phrases is obviously interesting with respect to the vocabulary sophistication. Their number, however, is too small to have any effect on the global counts and comparisons of hapaxes.

29 The French "formée" and "rencontrés" are past principles. The former is feminine singular, the latter is masculine plural.

30 The following lines thus can be seen as a first view of Hermite's lexical sophistication, although the impression of such a sophistication is totally subjective.

the letters seem to be in direct connection with a special way of writing, where the marks of personal involvement and anecdotes are multiplied. In this respect, the most emblematic and most extreme text is a 1900 letter to Jules Tannery, which begins as follows—the hapaxes are in bold characters:

> Saint-Jean-de-Luz, **villa Bel-air**, 24 septembre 1900.
>
> Mon cher ami,
>
> Je viens **dégager** ma **parole** et m'**acquitter** bien **tardivement**, il me faut l'**avouer**, de ma **promesse** de vous démontrer les formules concernant les quantités $\varphi\left(\frac{c+d\omega}{a+b\omega}\right)$ données dans mon **ancien** article *Sur l'équation du cinquième degré.*
>
> Le bon **air** de la **mer** m'a aidé à surmonter la **torpeur** qui **faisait obstacle** à mon travail ; j'en profite pour **échapper** aux **remords** de ma **conscience**, et, en pensant que vous avez sous les yeux cet article, j'aborde comme il suit la question.[31] [Hermite 1902, 13]

The following pages of the letter are more technical and contain almost no hapax. Such words appear again massively in the conclusion of the letter:

> Et nous **causerons** aussi d'autre chose que d'Analyse, nous **argumenterons**, nous nous **disputerons**. De ma **proximité** de l'**Espagne**, je rapporte des **cigarettes** d'**Espagnoles** ; si vous ne **venez** pas en **fumer** avec votre **collaborateur** d'aujourd'hui, votre professeur d'autrefois, c'est que vous avez le cœur d'un **tigre**. ***Totus tuus*** *et* ***toto*** *corde.*[32] [Hermite 1902, 21]

The two themes that are revealed by these hapaxes—that of the delay and the excuses associated with the epistolary answers, and that of the sociable chat on occasional topics—can be observed in other papers. For example, at the other tip of the Hermitian chronology, in one of his letters to Carl Gustav Jacob Jacobi on number theory:

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31 "Saint-Jean-de Luz, villa Bel-air, September 24 1900. My dear friend, I am coming to free my word and to fulfill, with, I must confess, a great delay, my promise of demonstrating the formulas on the quantities $\varphi\left(\frac{c+d\omega}{a+b\omega}\right)$ that I gave in my old article *On the equation of the fifth degree.* The good air of the sea helped me overcome the torpor which hindered my work; I take advantage of the situation to escape the remorse of my consciousness, and, imagining that you have this article before your eyes, I tackle the question as follows." The pages given in my citations refer to the pages in Hermite's and Jordan's *Œuvres complètes.*

32 "And we will chat about other things than Analysis, we will argue, we will quarrel with each other. From my proximity with Spain, I bring back cigarettes of Spanish women; if you do not come smoking them with your colleague of today, your professor of the past, then you have the heart of a tiger. *Totus tuus et toto corde.*"

> **Près** de deux années se sont **écoulées**, sans que j'aie encore **répondu** à la lettre pleine de bonté que vous m'avez fait l'honneur de m'écrire. **Aujourd'hui** je viens vous **supplier** de me pardonner ma longue **négligence** et vous exprimer toute la joie que j'ai **ressentie** en me **voyant** une place dans le recueil de vos Œuvres.[33]
> [Hermite 1850, 100]

Or, as he wrote to Eugenio Beltrami in 1881 about Domenico Chelini:

> Je n'ai point connu, **personnellement**, l'homme excellent et le géomètre si distingué dont vous **voulez honorer** la mémoire, mais j'ai **recueilli** l'**éloge** de son talent et de ses **vertus** de la **bouche** de votre éminent **compatriote** M. Brioschi.[34]
> [Hermite 1881b, 87]

However, the hapaxes of the letters contain many other terms which have nothing to do with such themes, and which deal with mathematical questions more directly. For instance, as he was commenting a result of Leopold Kronecker (about a certain function) in a letter to Joseph Liouville in 1862, Hermite declared:

> M. Kronecker, en la donnant comme l'expression analytique d'un de ses théorèmes, avait bien évidemment **pressenti** la signification qu'elle **recevrait** dans la théorie des fonctions elliptiques, et, à cet égard, je ne puis trop **admirer** la **pénétration** dont il a fait **preuve**.[35] [Hermite 1862, 120]

This kind of meliorative comments seems to be more present in the letters, which thus appear to encourage Hermite's personal expression.

Hapaxes which reflect such comments can also be found in the other publications, yet to a lesser extent: without citing the texts in which they are contained, let me note that terms such as *mystère, paradoxe, prestige, lumière, guide, inattendu, magnifiques,* and *stérile* ("mystery," "paradox," "prestige," "light," "guide," "unexpected," "magnificent," and "sterile") are examples of hapaxes that are associated with sentences where Hermite develops his viewpoints on his own works, on some of his colleagues, or on the mathematical objects, theorems, and theories themselves.

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33 "Almost two years have passed, and I have not yet responded to the letter, filled with kindness, that you made me the honor to write to myself. I am coming today to beg you to forgive my long negligence, and to express all the joy that I have felt by seeing for myself some room in the collection of your works."

34 "I have not known personally the excellent man and the so distinguished geometer whose memory you want to honor, but I gathered the praise of his talent and his virtues from the mouth of your eminent countryman Mr. Brioschi."

35 "Mr. Kronecker, by giving it as the analytic expression of one of his theorems, had obviously foreseen the sense that it would receive in the theory of elliptic functions, and, in this respect, I cannot but admire the insight he demonstrated."

Finally, a non-negligible number of hapaxes are terms of a purely technical nature, which reveal some thematic specializations: *Émanants* is the name, proposed by James Joseph Sylvester, of objects of the theory of forms and invariants; the word *couronnes* ("annulus") appears (together with two occurrences of the singular *couronne*) in a paper on Laurent series; and the *[points] stationnaires [d'une] quadrique* ("stationary [points of a] quadric") are just mentioned in the post-scriptum of a letter to Lazarus Fuchs on elliptic functions and, to a lesser extent, cubic curves.

On Jordan's side (which will be treated without providing the same amount of detail), a particularity is that the hapaxes comprise many more words which correspond to technical terms, and which are associated with topics to which Jordan devoted one or two papers in the corpus. It is the case of a memoir on the stability of floating bodies [Jordan 1867/1868], where the semantic field of navigation manifests itself through the intermediary of hapaxes such as *navires*, *émersion*, *submergé*, etc. ("ships," "emersion," "submerged"). Similarly, the theme of mountainous geography is developed in a paper called "On the lines of crest and thalweg", [Jordan 1872], which contains the (transparent) hapaxes *cirques*, *Grenoble*, *Isère*, *torrents*, etc.

The hapaxes that refer to the expression of Jordan's personal viewpoints are much scarcer than in Hermite's case. In fact, most of them are linked to the lexical field of polemics, and come from the comments that Jordan writes during the 1874 controversy with Kronecker on bilinear forms [36]: the words *contradicteur*, *excusable*, *objective*, *incontestable*, *jugé*, and *complaisance* ("detractor," "excusable," "objective," "unquestionable," "judged," "complacency") are but a few examples of them.

Hermite's corpus, hence, is characterized with a higher lexical diversity, and this diversity comes in part from a greater number of hapaxes which, contrary to Jordan, concern as much the mathematical objects as the expression of the author's viewpoints, anecdotes, and ways of opening his letters.

2.4. *On lexical sophistication*

It is more difficult to me to draw solid conclusions about the lexical sophistication of Hermite and Jordan. One of the principal reasons is that the affectation of a word is a characteristic trait whose evaluation is linked, *a priori*, to a high degree of subjectivity, which is something I wish to avoid as much as I can. A possible way to bypass this problem is to connect it with

36 On this controversy, see [Brechenmacher 2007].

the notion of rarity of use in corpuses which could be taken as representatives of the writing norms of a given time period. Unfortunately, as already stated, I do not possess enough mathematical texts of the nineteenth century for such a quantitative treatment yet.

Nevertheless, I would like to explain what I tried to do to tackle the problem, and what obstacle stood in the way.

The starting point was to consider the two components of the symmetrical differences of the lexicons of Jordan and Hermite, that is, the sets of the words that are used by one of them and not the other. Assuming that technical words do not contribute to the sophistication issue, I removed them from these sets. Then, for each of the remaining terms, I noted how many times it was used in corpuses of reference which I constructed with the help of the online database Frantext, made of literary works mostly: [37] the idea was to assess how often the terms which are proper to Hermite or Jordan are used in the literary production of their time, in order to get rid of my own (anachronistic) impression of sophistication.

The experience turned out to be difficult to interpret. For instance, the verb *vaincre* ("to vanquish / to defeat"), which appears only in Hermite's texts, is common in the literary production of the period 1842–1901, but its occurrences in mathematical texts seem to be much more singular, and give a particular taste to the Hermitian writing, as in: "*Les formes de degrés* pairs *m'ont présenté de plus grandes difficultés, que dès longtemps je ne puis espérer vaincre.*" [38] On the contrary, a verb like *ensuivre* ("to ensue") is relatively rare in the literature of the time but quite usual in Hermite's corpus. The same phenomena can be observed in Jordan's case. To take but one example, the use of *condamner* ("to condemn") in the phrase: "*L'hypothèse dont nous venons de partir se condamne d'elle-même*" [39] sounds precious in a mathematical text even though the verb is really usual in novels, poems, and plays from the period 1861–1920.

As these examples show, the question of the lexical sophistication must wait until larger corpuses of mathematical texts are ready to be investigated

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37 In February 2022, Frantext counted 5555 French references, for a total of 264 millions of words. I considered two comparative corpuses adapted to Hermite's and Jordan's periods of publication, in order to erase the possible effects of the time shift between these periods. The reference corpus for Hermite is composed of 696 texts and about 42 millions of words; the one for Jordan comprises 750 texts and 38 millions of words.

38 "The forms of *even* degree presented greater difficulty, which I cannot hope to defeat since a long time." [Hermite 1856b, 351].

39 "The hypothesis from which we just started condemns itself." [Jordan 1861, 151].

and taken as points of comparison, if one wants to treat it in a quantitative way.

Finally, no clear conclusion came out of a direct, non-quantified examination of the symmetric difference of Hermite's and Jordan's vocabularies, in particular because both of them seem to possess advanced words which could be substituted with one another. For example, to the adverbs *éminemment, hardiment* and *obscurément* ("eminently," "boldly," "obscurely") which appear only on Hermite's side, respond Jordan's *subsidiairement, prodigieusement* and *promptement* ("additionally," "prodigiously," "promptly"): it is delicate to tip the scales in favor of one or the other side.

Thus I turn away, with terror and horror, from this lamentable plague of lexical sophistication, and come to the grammatical and lexical specificities of our two authors.

3. SPECIFICITIES

To begin with, it may be helpful to explain on an example the meaning of the technical term "specificities." Let us suppose that we want to assess if Hermite uses markedly more adverbs than Jordan, taking into account the dissimilarity between the sizes of their corpuses. We know that there are 14158 adverbs in Hermite and 25032 adverbs in Jordan, but to compare these numbers by a simple linear reduction is not seen as a satisfactory solution.

One way to deal with this issue[40] is to consider the union of Hermite's and Jordan's corpuses: it contains many subsets whose cardinality is equal to that of Hermite's corpus, and, obviously, the latter is one of them. Then, supposing that the 39190 adverbs are equidistributed in the union, one evaluates the expected number of adverbs in any such subset in the framework of a hypergeometric distribution. If the number of Hermite's adverbs is larger (resp. smaller) than this expected number, there is an over-representation (resp. under-representation) of adverbs in Hermite's corpus. In the process, a coefficient called the specificity score is calculated. It helps quantify the over- or under-representation, which correspond to a positive or negative score, respectively.

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[40] The model has first been proposed by Pierre Lafon [Lafon 1980]. Among others, see the caveat explained on p. 137. The software TXM integrates the calculation of specificities.

Naturally, the adverbs that have been taken for the example can be replaced by the results of any textual query: words, lemmas, grammatical categories, sequences of words, etc. In any case, it must be emphasized that the over-representation of a word in Hermite does not mean that it is not used by Jordan: it means that it is abnormally more used by Hermite than by Jordan, from the statistical point of view explained above.

In this section, specificity calculations are used at several scales. Firstly, they serve to make a general comparison of the over- and under-uses of grammatical categories in our two corpuses: the results are presented in table 5. For instance, one sees in this table that proper nouns are over-represented in Hermite (and thus under-represented in Jordan) whereas common nouns are banal, or that Hermite favors displayed mathematical formulas, while inline formulas proliferate in Jordan's texts. [41]

At this stage, comparisons pertain only to the numbers of proper nouns, common nouns, etc., and not to the words that compose these grammatical categories. To help interpret such results, a possibility is to enter into details by inspecting both the (absolute) frequencies of the constituents of a given category and the specificities of these constituents within the category.

In general, stylometry suggests taking particular care of grammatical categories which correspond to function words, that is, words that are not nouns, verbs, adjectives, and adverbs: words that correspond to those categories of content words, indeed, would convey the actual content of a text and would thus take the researcher away from the stylistic inquiry. [42] In our case, however, to include content words in the analysis allows highlighting some writing features which are not of purely functional nature, and do not relate to the actual mathematical content either. Moreover, content words will sometimes be necessary to interpret correctly some over- or under-representations of function words; conversely, such imbalances of function words can reflect differences of mathematical content.

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41 The imbalance between the two types of mathematical symbols is in part due to the fact that Hermite's corpus include many papers of analysis, where formulas are larger and thus need to be displayed. Without going further into detail here, I refer to [Lê 2022], where this point is discussed in another case.

42 As is well known, however, content words can have a purely functional role in a text. This is obviously the case of some adverbs (such as "then," in English) or some nouns which are parts of fixed syntagmas, such as "as a result."

Category	Frequency	Frequency H.	Spec. score
#	17583	10869	1000,0
Proper nouns	2427	1842	313,9
Pers. pronouns	44630	20153	211,6
Weak punct.	80462	34540	186,9
Verbs, present	44722	19587	139,1
Foreign words	764	603	116,8
Verbs, infinitive	15774	6815	39,2
Verbs, pres. part.	12941	5576	30,9
Prepositions	108375	43034	30,0
Verbs, past part.	17560	7297	20,7
Articles	78598	31159	19,7
Abbreviations	2543	1183	17,5
Symbols	414	234	13,6
Conjunctions	47555	18699	7,8
Quot. marks	54	36	4,7
Demonstratives	18214	7097	2,1
Verbs, sple past	20	10	0,7
Rel. pronouns	18553	7092	0,4
Poss. pronouns	18	7	0,3
Common nouns	144114	54849	−0,5
Prep. + det.	24673	9356	−0,6
Pronouns (other)	39	6	−2,7
Verbs, imperative	3260	1116	−5,6
Adjectives	61534	22917	−5,7
Verbs, impf.	1149	331	−10,7
Verbs, subj. impf.	178	19	−15,8
Adverbs	39190	14158	−16,2
Poss. adj.	4619	1220	−63,1
Verbs, subj. pres.	5932	1488	−100,6
Verbs, cond.	2388	395	−117,4
Strong punct.	38165	11882	−184,9
Verbs, sple fut.	18474	5032	−216,6
Numerals	16592	4318	−241,4
Indef. prns & qtfs.	9836	1950	−1000,0
*	74794	19542	−1000,0

Table 5. Distribution and specificity scores of grammatical categories. The second column lists the frequencies in the reunion of Hermite's and Jordan's corpuses. The third one is related to Hermite only.

3.1. *Proper nouns and marks of citations*

Let me first very briefly comment on the case of proper nouns, which are clearly over-represented in Hermite. As has been explained elsewhere, [43]

[43] See the second section of [Goldstein 2012], of which I take the results. The conventions used in the present paper leads to 1842 proper nouns, which are represented

proper nouns of persons have different status in Hermite's *Œuvres*: they can be noun complements in the designation of mathematical objects, they can designate journals through the name of their editor, and they can refer to the people whom Hermite is writing to, to translators, or to authors of works upon which Hermite expands.

Citations, at least in the way in which they are formulated by Hermite and Jordan, seem to play a role in the imbalance of proper nouns, as indicate other grammatical specificities too. Thus the overabundance of foreign words in Hermite, which we already mentioned, is mostly supplied by titles of cited works in German, Latin, Italian, or English, and by quotes of words and sentences written in these languages—this aspect is also revealed by the (lighter) over-use of quotation marks in Hermite. Moreover, the positive specificity of the category of abbreviations echoes such observations, and is due to a massive use of *M.*, *p.*, and *t.*, which stand for *Monsieur*, *page*, and *tome*.

In any case, the multitude of proper nouns has the effect of marbling Hermite's texts with the presence of individuals and collectives of all sorts, and thus brings to these texts a certain human color, which appears to be less bright in Jordan's corpus.

3.2. *Common nouns, adjectives, and adverbs*

Common nouns, adjectives, and adverbs are categories whose specificity scores are not high, compared to the others in table 5: common nouns, as stated above, are actually completely banal, while adverbs and, in a more minor way, adjectives are a bit under-represented in Hermite. However, the examination of the words composing these three categories, and especially the words that are specific to one author or the other, reveals interesting phenomena.

A great number of these specific terms are of technical, mathematical nature: it is the case of *polynôme*, *intégrale*, *elliptique*, and *doublement* ("polynomial," "integral," "elliptic," "doubly") for Hermite, and of *groupe*, *lettres*, *échangeable*, or *transitivement* ("group," "letters," "exchangeable," "transitively") for Jordan.[44] Since these words reflect mathematical topics themselves, I will disregard them in order to focus on features that are closer to the issue of style.

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by 232 tokens: this is more than in the given reference, which concentrates on the nouns of persons. Our 1842 occurrences, indeed, include places, institutions, and (abbreviations of) first names.

44 For the sake of brevity, I will not systematically present tables with frequencies and specificity scores when examining particular categories.

Some nouns, adjectives, and adverbs, although being specific to Hermite or Jordan, activate in fact the same meanings: for instance, Hermite is fond of the adverbs *immédiatement* and *facilement* ("immediately," "easily"), which certainly energize his writing, while Jordan prefers to use *évidemment* ("obviously") over and over: these adverbs do not seem to have really different senses,[45] and thus appear as mere personal preferences of our two authors.

Other specific terms create lexical fields that are only present on one side. Concerning Hermite, a whole set of nouns and adjectives reflects the frequent expression of his personal views on the sequence of mathematical events, on objects, theorems, and works of the past: *méthode*, *recherche*, *facile*, *important*, *beau*, *essentiel* ("method," "research," "easy," "important," "beautiful," "essential") are but a few such specific terms through which the person of Hermite is made visible in the text, and of which there is no equivalent in Jordan. The corresponding semantic field is thus the same as that which has been already detected with the hapaxes.

The terms that are specific to Jordan and that are not directly linked to mathematical objects relate to the *reductio ad absurdum*, with the nouns and adjectives *hypothèse*, *absurde*, *inadmissible*, *contraire* ("hypothesis," "absurd," "inadmissible," "contrary").[46] As for the adverbs, the over-representation of the twin negative markers *ne* and *pas* seems to correspond to the same characteristic.[47] The proof by contradiction thus appears to be carefully avoided by Hermite. Examining the absolute frequencies shows it even more clearly, since *inadmissible* is never employed by Hermite, while *absurde* is used four times only—*contraire* is a bit more frequent, with 81 occurrences of words having this lemma, but most these occurrences concern quantities of opposite signs or refer to opposite types of monotony of functions.

3.3. *Verbs, personal pronouns, conjunctions*

The quasi absence of proof by contradiction in Hermite has also an impact on the specificities of verb tenses and moods. Indeed, as table 5

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[45] Such assertions have systematically been supported by a direct inspection of the occurrences of the said words in their textual context.

[46] If *contraire* appears both as a specific adjective and a specific noun, *absurde* is present only in its adjectival form.

[47] In terms of absolute frequencies, these two words inundate Jordan's adverbs, which explains why the category itself is over-represented in the latter. Moreover, the word *si* ("if"), which is the introducer of hypotheses *par excellence*, is among the conjunctions that are largely over-represented in Jordan's texts.

shows, the conditional, the subjunctive, and, to a lesser degree, the imperfect indicative are under-represented in Hermite's texts. But conditional and subjunctive are two ways to formulate hypotheses and their possible consequences,[48] which is in accordance with Jordan's predilection (compared with Hermite) of the proof by contradiction.[49]

To carry on with the discussion on verbs, let me enumerate the first ones that are specific to Hermite, in their word form, and by decreasing order of specificity: *ai, savoir,*[50] *conduit, tire, faisant, donne, supposant, vais, conclut, trouve, obtient, obtenir, écrire, observe, a, été, remarque, employant, parvenir, trouvera...* Conversely, for Jordan : *sera, contient, formé, contiendra, pourra, contenu, aura, Soient, contenant, déplace, Supposons, forme, être, existe, serait, déplacent, seront, transforme, succéder...*

Several lessons can be learned from the comparison of these lists. First, Jordan's specific verbs include more technical terms (in particular with the diverse forms of *contenir* and *déplacer*: "to contain," "to move"), while those on Hermite's side rather evoke the description of processes: *conduire, tirer, observer, écrire* ("to lead," "to draw," "to observe," "to write") as well as *[je] vais,* ("[I] am going to"). Hermite's mathematical narration, therefore, is marked by such verbs which contribute to vitalize the action and to recall the involvement of a human person in charge of this action.[51]

Furthermore, the specificities of the personal pronouns used by Hermite and Jordan agree with this conclusion. Indeed, Hermite over-uses

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48 Moreover, in French, the imperfect indicative usually accompanies the conditional, when it comes to sentences that are introduced by *si*: "*Si le groupe était abélien, il serait résoluble.*" Jordan's over-use of *Supposons* ("Let us suppose"), which will be seen in a few lines, is to be linked to that of the subjunctive, which is the mood of many verbs that follow the phrase *Supposons que.*

49 Of course, one may wonder whether Hermite and Jordan use other ways of expressing this kind of reasoning. A systematic survey of the two corpora shows that this is the case only in minimal proportions. In particular, this survey led me to count 35 proofs by contradiction in Hermite's corpus, and 599 ones in Jordan's, which confirms the imbalance revealed by our statistical clues.

50 This infinitive is exceptional in this list, for it is the only one to be employed almost exclusively within the fixed phrase *à savoir,* used as a synonym of *c'est-à-dire* ("that is / that is to say").

51 On this point, see [Goldstein 2007, 398]: "More difficult to pinpoint, but quite characteristic, the flavour of Hermite's mathematical prose itself reminds the reader strongly of these French authors [Lagrange, Legendre, Cauchy, Fourier]. The style is discursive and oriented towards the description of processes." Moreover, the over-representation of *observer* may be linked to Hermite's predilection of observing formulas, although phrases such as *j'observe que* do not necessarily introduce reflections on what is observed, but rather serve as a way to state intermediary results in the course of a proof, for instance. On Hermite and observation, see [Goldstein 2011].

those which are linked to the first person singular, the (semi-)impersonal *on* ("one"), and the second person plural, which is associated with the French *vouvoiement*. Interestingly, if *vous* is a definite trace of the epistolary genre, it is not the case of *je*: the letters being excluded from the corpus, the first person singular is still overabundant in Hermite, whereas the second person plural disappears almost completely from the counts. On the contrary, Jordan makes considerable use of *il*, *elle* ("he / it," "she") and their diverse plural and reflexive declensions. About the *il*, it is helpful to delineate its different uses in Jordan's texts: almost none of them stand for a person, about a third represent mathematical objects, and the rest is assigned with an impersonal value in phrases such as *il y a*, *il faut*, and *il existe* ("there is / there are," "one has to," "there exists / there exist").

These are the specificities of personal pronouns within their own category, but table 5 shows that, when it comes to global numbers, Hermite over-employs these pronouns. This can be explained by the fact that Hermite's specific verbs, which have been listed above, cannot have mathematical objects as subjects, and are almost systematically associated with the pronouns *je* and *on*. No similar phenomenon seems to exist on Jordan's side: as mathematical objects are often the subjects of the verbs, some of them are represented by personal pronouns, but others are written as a noun or as a mathematical symbol, as in: *Ainsi, G contient...* ("Thus, G contains..."), and this makes the total number of personal pronouns decrease.

Finally, the lists of Hermite's and Jordan's specific verbs also reflect the global imbalance of the verb tenses that can be seen in table 5. The present indicative, as well as the infinitives and the past and present participles are overabundant in Hermite, while the simple future is Jordan's feature. At this point, it is perhaps useful to recall that in French, the simple future is one way among others to express the future; another one is to combine a conjugated form of *aller* with an infinitive (e.g., *je vais observer* and *j'observerai*: "I am going to observe" and "I will observe"). Hermite and Jordan both use these two ways of expressing the future, but the specificities of the tenses show that they use them in different proportions.

The over-representation of the simple future in Jordan is supplied in part by verbs that express mathematical facts: such simple futures have a gnomic value, as in "*Ce système ne contiendra donc en général qu'une fraction des substitutions du système primitif*" [Jordan 1861, 132], or "*Il est clair qu'une partie quelconque d'une ligne géodésique sera elle-même géodésique*" [52] [Jordan 1866].

52 "Thus, this system will generally contain only a fraction of the substitutions of the primitive system"; "It is clear that any part of a geodesic line will be geodesic itself".

On the other hand, the futures that are expressed with the verb *aller* and an infinitive are over-used in Hermite's texts, and are often associated with the first person (singular or plural). The verbs are in great part those of the description of processes; other examples than those which we already saw include: "*Cette remarque faite, je vais étudier de plus près les quotients...*" [Hermite 1856a, 381] and "*La notion de coupure se présente de la manière la plus simple dans un cas particulier que je vais maintenant considérer.*" [Hermite 1881a, 63].[53]

Such an expression of the future colors Hermite's texts with a certain vitality, with an immediacy of the described mathematical action, which is also fueled by the use of the present indicative, and of the different participles.[54] In this respect, it is particularly telling that the present participle *supposant* is characteristic of Hermite, while the imperative *Supposons* is preferred by Jordan: both have the same meaning, of course, but the former is a trace of a prose which is more energetic than that conveyed by the latter. As for the present indicative, the same effects were already noticeable in the verbs that describe processes, and which we mentioned above.

The vigor of Hermite's writing can also be detected, although to a lesser extent, in the conjunctions that are specific to him. Among those that he tends to over-use, one can list *comme, et, lorsque, or, afin,* and *quand* ("as," "and," "when," "but / now," "so (that)," "when"); on Jordan's side, *donc, car, ou,* and *ni* ("so," "for," "or," "nor") prevail. In particular, Hermite favors subordinating conjunctions; those such as *quand* and *lorsque,* which are used relatively rarely by Jordan, are often parts of phrases such as *quand on ajoute, quand on remplace,* or *lorsqu'on suppose* ("when one adds," "when one replaces," "when one supposes"). In general, such conjunctions could be replaced by *si* ("if"), but *quand* et *lorsque* have a temporal connotation which, again, evoke the dynamics of Hermite's prose.

3.4. *Sentences, demonstrative categories, favorite phrases*

But the overabundance, in Hermite's corpus, of the category of conjunctions itself is linked to another disequilibrium displayed in table 5: that between the weak punctuation marks, which are over-represented in Hermite, and the strong punctuation marks, which proliferate in Jordan's corpus. These clues point to a difference between the average length of the

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53 "This remark being made, I am going to study more closely the quotients..."; "The notion of cut presents itself most simply in a particular case that I am now going to consider."

54 These tenses and moods are still over-represented in Hermite if the different forms of *aller, être,* and *avoir,* which are associated with the future and the past participles, are not taken into account for the calculation of the specificities.

sentences written by Hermite and Jordan. Incidentally, this is easily confirmed and refined by considering the absolute numbers in question: about 10247 "actual" marks of strong punctuation (among which 10227 periods, 18 question marks and 2 exclamation marks) can be counted on Hermite's side, which represent the same number of sentences. [55] In relating this number to that of the words in the corpus, one finds that the Hermitian sentence counts about 36 words on average. The analogous estimations for Jordan yield a number of 23473 sentences (almost the double of Hermite's), each of them having 25 words on average. Hence the Hermitian sentence is wider, and this width is supported with the over-representation of weak punctuation marks (mostly commas) and of conjunctions.

For its part, the category of demonstratives, which encompasses demonstrative pronouns and determinants, is quite banal in terms of specificity. However, inside this category, the words *c'*, *cette*, *C'*, *cet*, and *Cela*[56] are overused in Hermite. Inspecting the occurrences of these words within their context brings to light several phrases which Hermite seems to favor: *Cela étant*, *pour cela*, *c'est-à-dire*, and *à cet effet* ("That being said," "for this," "that is (to say)," "to this end") are examples of phrases which are commonly used by Hermite, and almost never by Jordan. [57]

Furthermore, the phrases *pour cela* and *à cet effet* are often preceded or followed by the characteristic verbs that we listed previously, which mark the liveliness of the process depiction: "*J'observe, à cet effet*," "*À cet effet, nous remarquerons que...*," "*Je vais établir pour cela que...*" etc. [58] The *C'* is also typical of Hermite, who writes it four times more frequently than Jordan; it expresses the introduction and the presentation of the information in a

55 This number is the result of the subtraction, from the total number of strong punctuation marks, of the numbers of abbreviating dots and of dots that are appended to numbers which numerate paragraphs and sections. This is an approximate manner of counting the number of sentences, which does not include a reflection on what is a sentence. In particular, it is blind to the extreme cases of word-sentences such as "*Théorème*." These cases are characteristic of Jordan but, since they are marginal from a numeric point of view, they do not change the proposed interpretation on sentence lengths.

56 All these words can be translated by "this," "that," or "it." The words *c'* and *C'* are the elided forms of *ce* and *Ce*; on the contrary, the final *t* in *cet* appears when the following word begins with a vowel or a silent *h*.

57 Another phrase whose Hermite has the quasi-exclusivity—and which does not involve a demonstrative pronoun—is *par conséquent* ("consequently"). It is used 318 times by Hermite and only 5 times by Jordan. More generally, it is surprising to see that some words or phrases are completely absent from one author or the other, although they seem to be absolutely commonplace. For instance, Hermite never employs the adverb *pourtant* ("yet")!

58 "I observe, to this end,..."; "To this end, we remark that..."; "For this, I am going to establish that.."

very active way: "*C'est à ce même résultat que je dois parvenir en me plaçant dans la seconde hypothèse*" and "*C'est ce qui résulte immédiatement des expressions…*" [59] are examples of beginnings of sentences which energize the mathematical speech.

Speaking of beginnings of sentences, it is also possible to investigate those, made of two words, that are specific to Hermite and Jordan (see table 6). Somehow, they sum up a number of the results that have been described until now, as they clearly show some of the most striking differences between our two mathematicians. Indeed, while the beginnings of Jordan's sentences involve (or make us guess the involvement of) mathematical objects as subjects, the first person singular is apparent already in the attacks of Hermite's sentences, and is associated with some of the characteristic verbs and syntagmas that we already brought to light. The other ones that appear in this list, such as *De là*, *Effectivement*, *Maintenant*, and *Voici maintenant* ("From this," "Indeed," "Now," "Here [is] now"), are yet other testimonies of the liveliness of the Hermitian prose.

3.5. *The case of indefinite pronouns and quantifiers*

Among the few grammatical categories which are deeply unbalanced in table 5, the case of indefinite pronouns and quantifiers has not been examined yet. [60] Actually, this case seems to be quite particular with regard to the issue of style: even if indefinite pronouns and quantifiers are function words, their over-representation in Jordan could be explained by the very content of the mathematics he develops.

Indeed, the inspection of the absolute numbers of these words reveals that the category is over-used by Jordan because of the massive presence of *toutes*, *une*, *tous*, *un*, *chacun*, etc. ("every," "one," "all," "each"), that is, because of pronouns and quantifiers which relate to wholes by referring to their constituents (whether these constituents are individuated or not). Now, the nouns that are most frequently associated with these words are letters, systems, substitutions, and groups, and we saw that some of Jordan's specific verbs are the diverse conjugated forms of *contenir*. All these clues seem to indicate that the over-representation of indefinite pronouns and quantifiers could be rooted in that fact that, in comparison with Hermite,

59 "It is this same result that I must reach by placing me in the second hypothesis"; "It is what immediately results from the expressions…"

60 Another such category is that of the determinants, whose overabundance in Hermite seems to be more difficult to interpret. In particular, it is not clear to me whether its over-representation is to be linked to the deficit of indefinite pronouns.

Sentence beginning	Freq.	Freq H.	Spec. score
Cela étant	198	195	93,5
C'est	289	234	68,0
Or,	319	283	61,0
De là	88	80	31,0
Je me	56	52	21,4
J'observe	39	39	19,7
Effectivement,	45	43	19,1
Je remarque	40	39	18,3
J'ai	65	53	16,1
Ainsi,	82	62	15,8
On trouve	45	40	14,9
Maintenant,	33	32	14,8
Voici maintenant	29	29	14,7
⋮	⋮	⋮	⋮
D'autre	183	3	-24,9
Soit *	580	68	-28,1
Soient *	451	42	-29,4
En effet	621	74	-29,4
D'ailleurs	414	29	-33,0
Les substitutions	290	5	-38,9
On aura	566	33	-50,4
Le groupe	334	2	-50,6
Donc *	421	4	-61,2
Si *	824	11	-115,2

Table 6. Some beginnings of sentences made of two words, ordered by specificity score

Jordan deals more often with questions having to do with different kinds of sets, their subsets and their constituents.

4. CONCLUDING REMARKS

The characteristics of Hermite's writing that have been brought to light and quantified throughout this paper may be summed up as follows. Hermite favors the use of the first person singular, which, to a large extent, is associated with many verbs in the present indicative that relate, step by step, the mathematical process as Hermite thinks of it: observations, remarks, deductions, and conclusions are closely tied to the person of Hermite, whose presence remains visible in many other places of the mathematical writing. The individual person of Hermite, further, is not the only human trace in his corpus, since many mathematicians are, in one way or the other, summoned through the different proper nouns. Hermite's writing is also characterized by a remarkable lexical diversity. This diversity is particularly due to the many hapaxes that are employed, and which appear to be in part linked to the expression of Hermite's viewpoints on mathematics, in part to the personal anecdotes and the sociable talk that are mostly visible in letters.

An interesting perspective on all this can be gained by considering, with Lucien Vinciguerra [Vinciguerra 2019], mathematical texts as special cases of narratives: from this point of view, a mathematician appears as an author who is also a narrator, that is, an authority who takes charge of the production and the enunciation of a mathematical discourse. The above analysis, then, allows us to interpret Hermite as a homodiegetic narrator in the sense of Gérard Genette's classical typology of narrators [Genette 1972]. Indeed, all the clues that have been gathered indicate that Hermite takes part of the universe, of the action that he relates in his texts: the words that he uses to construct his writings invite us to hear his own voice, to follow him making progress in his reasoning, organizing the construction of his argumentation, and commenting on objects and theorems. In contrast, Jordan appears as a heterodiegetic narrator, with a more distant, overhanging position regarding the described action. Obviously, this does not mean that Jordan is less involved or less enthusiastic than Hermite with the mathematics he deals with: the point is that each of them has a specific way of narrating mathematics which generate different impressions for the reader.

As has been emphasized several times, and as the preceding paragraph recalls, our conclusions on Hermite's style have been drawn mostly from the confrontation with Jordan. Inevitably, this raises questions related to the decision of having chosen the latter as a point of comparison.

Jordan, indeed, is a mathematician whose period of activity began about two decades after Hermite's. Thus one naturally wonders whether features such as the higher personalization of Hermite's prose actually reflect a difference between two authors, or between two representatives of two generations. The latter hypothesis would accredit (or would be accredited by) the fact that mathematicians from newer generations would globally favor impersonal phrasings in their technical publications. This question deals with the collective dimensions of style, with how social constraints of writing evolve, and how a given mathematician such as Hermite differentiates himself within his contemporaries: to answer such a question, several studies, involving a sufficiently great number of mathematical texts by different authors of different generations would be necessary.

Another issue would be to take into account the possible impact of the existence of different research themes in the works of the considered authors. Indeed, as has been shown in [Lê 2022], grammatical specificities of two corpuses associated with two disciplines can be interpreted in the light of different, collective writing practices: for instance, texts from the theory of algebraic surfaces are markedly richer in common nouns than texts from invariant theory because of the different ways of writing geometry and invariant theory at the time.

In this perspective, it must be admitted that the results that have been presented in this paper are completely blind to specificities which would be internal to Hermite's corpus, especially in regards with (sub-)disciplines. Since I have no room to develop this point, I will confine myself with a teaser on this issue. Hermite's papers can be grouped into several subsets, according to their classification in the *Catalogue of scientific papers*. Among these subsets, the largest ones (in terms of word number) deal with number theory, linear substitutions, equation theory, algebraic functions, other special functions, complex functions, and the foundation of analysis. [61] Then, for instance, when confronted to the other subsets, number theory is characterized by a certain over-representation of indefinite pronouns and verbs in conditional and simple future forms, and an under-representation in verbs to the present indicative. Said differently, within Hermite's corpus, number theory has some of the specificities that Jordan has in comparison with Hermite. [62] The question of interpreting such results correctly and finely remains open for the moment.

Mathematical style has often been described as the manner of expressing or presenting mathematical truths, or mathematical facts: a theorem, a proof, a more or less coherent set of results, could thus be expressed algebraically or geometrically, rigorously or intuitively, in a set-theoretic way, by starting from axioms, by following a given method.

Such a position differs from the one that has been adopted here, not so much on the fact that studying the style of a mathematician comes down to studying how the latter expresses himself or herself, but rather on the issue of what is the "expression" and, simultaneously, what is the object of this expression. Or, considering that traces of styles appear as soon as the

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61 These keywords are parts of the corresponding sections of the *Catalogue*.

62 The scores of the grammatical specificities observed in the case of the subsets of Hermite's corpus, however, are way lower than those relative to the Hermite–Jordan comparison.

same content is written in different ways, the question is to decide what is "the same," in a mathematical context. [63]

Following the literary path, I tried to turn away from what is *a priori* linked to mathematical objects, results, and other disciplinary features. Instead, I chose to focus on the diverse facets of lexical richness, as well as on the specificities of certain categories of words which do not belong to the technical, mathematical lexicon, [64] assuming that it would be an appropriate way to detect writing peculiarities of another nature than those concerning the mathematical content itself, and thus to analyze the impression that one feels when reading Hermite's mathematical works.

But, having excluded from the stylistic investigation everything tied to the objects, the theorems, and the values themselves, the question arises as to whether there is anything significant and interesting left: are the mentioned reading impressions even relevant for the history of mathematics?

My personal conviction is that they are. I do believe that the favorite expressions of a mathematician, his or her writing peculiarities, and, more generally, the way he or she elaborates the mathematical narration, are part and parcel of his or her works, even if they are not directly connected with the mathematical core. Then, for the very reason that such elements concern impressions of reading, taking them as the object of inquiry requires to construct an adequate methodological framework that allows accounting for them, by making as explicit as possible the decisions that must be made, the different steps of the investigation, and its limitations. The stylometric approach that I followed here is one proposal among others which, by its statistical nature and its systematic inclination, appears to me as a good candidate to make progress in this direction.

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63 For this view on style, see [McCleary 2010]. About the issue of sameness in the history of mathematics, see in particular [Goldstein 1995].

64 However, the examples of the indefinite pronouns and of the proof by contradiction showed that there is no question of associating, without due process, content words with mathematical content, and function words with the expression of this content.

REFERENCES

BEAUDOUIN (Valérie)
[2002] *Mètre et rythmes du vers classique*, Paris: Honoré Champion, 2002.

BRECHENMACHER (Frédéric)
[202?] Un journal de « rang élevé »: le *Journal de mathématiques pures et appliquées* sous la direction de Camille Jordan, in Gispert (Hélène), Nabonnand (Philippe) & Peiffer (Jeanne), eds., *Circulation des mathématiques dans et par les journaux*, 202?; to appear.
[2007] La controverse de 1874 entre Camille Jordan et Leopold Kronecker, *Revue d'histoire des mathématiques*, 13(2) (2007), pp. 187–257.

BRUNET (Étienne)
[1978] *Le vocabulaire de Jean Giraudoux, structure et évolution*, Genève: Slatkine, 1978.

FATTORI (Marta)
[1980] *Lessico del* Novum organum *di Francesco Bacone*, Rome: Edizioni dell'Ateneo e Bizzarri Roma, 1980; Two volumes.

GALLARD (Pierre-Yves)
[2019] *Paradoxes et style paradoxal : l'âge des moralistes*, Paris: Classiques Garnier, 2019.

GAYON (Jean)
[1996] De la catégorie de style en histoire des sciences, *Alliage*, 26 (1996), pp. 13–25.

GENETTE (Gérard)
[1972] *Figures III*, Paris: Éditions du Seuil, 1972.

GIACOMOTTO-CHARRA (Violaine) & MARRACHE-GOURAUD (Myriam), eds.
[2021] *La Science prise aux mots : Enquête sur le lexique scientifique de la Renaissance*, Paris: Classiques Garnier, 2021.

GOLDSTEIN (Catherine)
[2007] The Hermitian Form of Reading the *Disquisitiones*, in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C.F. Gauss's Disquisitiones Arithmeticae*, Berlin: Springer, 2007, pp. 377–410.
[2011] Un arithméticien contre l'arithmétisation : les principes de Charles Hermite, in Flament (Dominique) & Nabonnand (Philippe), eds., *Justifier en mathématiques*, Paris: Maison des Sciences de l'Homme, 2011, pp. 129–165.
[2012] Les autres de l'un : deux enquêtes prosopographiques sur Charles Hermite, in Nabonnand (Philippe) & Rollet (Laurent), eds., *Les uns et les autres... Biographies et prosopographies en histoire des sciences*, Nancy: Presses Universitaires de Nancy, 2012, pp. 509–540.
[1995] *Un théorème de Fermat et ses lecteurs*, Saint-Denis: Presses Universitaires de Vincennes, 1995.

HEIDEN (Serge)
[2010] The TXM Platform: Building Open-Source Textual Analysis Software Compatible with the TEI Encoding Scheme, in *24th Pacific Asia Conference on Language, Information and Computation*, Sendai, Japan, 2010, pp. 389–398.

HERMITE (Charles)
[1850] Extraits de lettres de M. Ch. Hermite à M. Jacobi sur différents objets de la théorie des nombres, *Journal für die reine und angewandte Mathematik*, 40 (1850), pp. 261–278; Œuvres I, 100–163.
[1856a] Sur la théorie des fonctions homogènes à deux indéterminées. Second mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 18–38; Œuvres I, 372–396.
[1856b] Sur la théorie des fonctions homogènes à deux indéterminées. Premier mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 1–17; Œuvres I, 350–371.
[1862] Sur la théorie des fonctions elliptiques et ses applications à l'Arithmétique. Lettre adressée à M. Liouville., *Journal de Mathématiques pures et appliquées*, 7 (1862), pp. 25–48; Œuvres II, 109–124.
[1881a] Sur quelques points de la théorie des fonctions (extrait d'une lettre à M. Mittag-Leffler), *Journal für die reine und angewandte Mathematik*, 91 (1881), pp. 54–78; Œuvres IV, 48–75.
[1881b] Sur les fonctions $\Theta(x)$ et $H(x)$ de Jacobi (estratto di lettera al Prof. E. Beltrami), in Cremona (Luigi) & Beltrami (Eugenio), eds., *In memoriam Dominici Chelini. Collectanea mathematica*, Milan, Naples, Pise: Hoepli, 1881, pp. 1–5; Œuvres IV, 87–91.
[1902] Lettre de M. Charles Hermite à M. Jules Tannery sur les fonctions modulaires, in *Éléments de la théorie des fonctions elliptiques*, vol. 4, Paris: Gauthier-Villars, 1902, pp. 294–302; Œuvres II, 13–21.

HERSCHBERG PIERROT (Anne)
[2005] *Le style en mouvement. Littérature et art*, Belin, 2005.

HIMY-PIÉRI (Laure), CASTILLE (Jean-François) & BOUGAULT (Laurence), eds.
[2014] *Le style, découpeur de réel : Faits de langue, effets de style*, Rennes: Presses universitaires de Rennes, 2014.

JORDAN (Camille)
[1861] Mémoire sur le nombre de valeurs des fonctions, *J. Éc. polytech.*, 22 (cahier 38) (1861), pp. 113–194; Œuvres I, 1–82.
[1866] Recherches sur les polyèdres, *Journal für die reine und angewandte Mathematik*, 66 (1866), pp. 22–85; Œuvres IV, 15–78.
[1867/1868] Mémoire sur la stabilité de l'équilibre des corps flottants, *Annali di Matematica Pura ed Applicata*, 1 (1867/1868), pp. 170–221; Œuvres IV, 179–230.
[1868/1869] Mémoire sur les groupes de mouvements, *Annali di Matematica Pura ed Applicata*, 2 (1868/1869), pp. 167–215; Œuvres IV, 231–302.
[1872] Sur les lignes de faîte et de thalweg, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 74, 75 (1872), pp. 1457–1459; Œuvres IV, 323–327.

Kastberg Sjöblom (Margareta)
[2002] *L'écriture de J. M. G. Le Clézio, une approche lexicométrique*, Ph.D. Thesis, Université de Nice, 2002 URL `http://www.revue-texto.net/Corpus/Publications/Kastberg/Kastberg_LeClezio.html`.

Labbé (Cyril) & Labbé (Dominique)
[2018] Les phrases de Marcel Proust, *Proceedings of the 14th International Conference on Statistical Analysis of Textual Data*, 2018, pp. 400–410.

Labbé (Dominique) & Hubert (Pierre)
[1997] Vocabulary richness, *Lexicometrica*, 0 (1997) URL `http://lexicometrica.univ-paris3.fr/article/numero0/VocabRichness.pdf`.

Lafon (Pierre)
[1980] Sur la variabilité de la fréquence des formes dans un corpus, *Mots*, 1 (1980), pp. 127–165.

Lê (François)
[2022] La théorie des surfaces algébriques dans les *Mathematische Annalen* à l'épreuve de la textométrie (1869-1898), *Revue d'histoire des mathématiques*, 28(1) (2022), pp. 1–45.

Lebart (Ludovic), Pincemin (Bénédicte) & Poudat (Céline)
[2019] *Analyse des données textuelles*, Presses de l'Université du Québec, 2019.

Lutosławski (Wincenty)
[1897] *The Origin and Growth of Plato's Logic: With an Account of Plato's Style and of the Chronology of his Writings*, London, New York, Bombay: Longmans, Green, and Co., 1897.

Mancosu (Paolo)
[2009/2021] Mathematical Style, in Zalta (Edward N.), ed., *The Stanford Encyclopedia of Philosophy*, 2009/2021 URL `https://plato.stanford.edu/entries/mathematical-style/`.

McCleary (John)
[2010] Milnor, Serre, and the Cartan Seminar, *Oberwolfach Reports*, 7 (2010), pp. 629–631.

Muller (Charles)
[1967] *Étude de statistique lexicale. Le vocabulaire du théâtre de Pierre Corneille*, Paris: Larousse, 1967.
[1977] *Principes et méthodes de statistique lexicale*, Paris: Hachette, 1977.

Pawłowski (Adam)
[2008] Les aspects linguistiques dans l'œuvre scientifique de Wincenty Lutosławski, *Organon*, 37(40) (2008), pp. 149–176.

Rabouin (David)
[2017] Styles in Mathematical Practice, in Chemla (Karine) & Keller (Evelyn Fox), eds., *Cultures without Culturalism: The Making of Scientific Knowledge*, Durham, London: Duke University Press, 2017, pp. 262–306.

Rowe (David E.), Volkert (Klaus), Nabonnand (Philippe) & Remmert (Volker), eds.
[2010] Disciplines and Styles in Pure Mathematics, 1800–2000, *Oberwolfach Reports*, 7 (2010), pp. 581–665.

Tannery (Paul)
[1899] La stylométrie : ses origines et son présent, *Revue philosophique de la France et de l'étranger*, 47 (1899), pp. 159–169.

Vinciguerra (Lucien)
[2019] *Celui qui parle. Science et roman*, Paris: Hermann, 2019.

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A CONVERGENCE OF PATHS: CAYLEY, HERMITE, SYLVESTER, AND EARLY INVARIANT THEORY

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In loving memory of Brian J. Parshall
(28 October, 1945 – 17 January, 2022)

Abstract. — This paper considers the beginnings of a *theory* of invariants in the early 1850s in the broader contexts of individual pathways toward the establishment of reputation and of the professionalization of mathematics in the nineteenth century. In particular, it treats the different, but intersecting, mathematical paths by which two Englishmen, Arthur Cayley and James Joseph Sylvester, and one Frenchman, Charles Hermite, came to focus on an analysis *per se* of the transformation of homogeneous forms by linear substitutions. It then looks at the intense mathematical exchanges in the first half of the 1850s that resulted in their early invariant-theoretic results. Although by the close of the 1850s, Cayley, Hermite, and Sylvester had largely gone their own separate mathematical ways, the three remained united in their sense of having created what they called the "New Algebra."

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Mots clefs. — Arthur Cayley, Charles Hermite, James Joseph Sylvester, history of invariant theory, nineteenth century, professionalization and internationalization of mathematics.

Résumé. — Cet article considère les origines d'une *théorie* d'invariants au début des années 1850 dans les contextes plus larges des chemins suivis pour établir une réputation mathématique et de la professionnalisation des mathématiques au XIX^e^ siècle. En particulier, il s'occupe des différentes voies mathématiques, mais néanmoins voies croisées, par lesquelles deux Anglais, Arthur Cayley et James Joseph Sylvester, et un Français, Charles Hermite, en vinrent à se concentrer sur une analyse de la transformation de formes homogènes par les substitutions linéaires. Il se penche ensuite sur les échanges mathématiques intenses dans la première moitié des années 1850, échanges qui ont abouti aux premiers résultats proprement dits invariant-théoriques. Bien qu'à la fin des années 1850, les chemins mathématiques de Cayley, Hermite, et Sylvester avaient largement divergés, les trois mathématiciens sont restés unis dans le sentiment d'avoir créé ce qu'ils ont appellé la « nouvelle algèbre. »

Arthur Cayley was a twenty-two-year-old assistant tutor at and fellow of Trinity College, Cambridge in June 1844 when he wrote to George Boole, a Lincolnshire schoolteacher more than six year's his senior. The 1842 Cambridge Senior Wrangler had been reading a paper published by the largely self-taught Boole in the *Cambridge Mathematical Journal* and had produced "a few formulae relative to it" that he hoped would spark Boole's interest. [1] Following on his reading primarily of Joseph-Louis Lagrange's *Méchanique analitique* and the *Mécanique céleste* of Pierre Simon de Laplace, Boole had been intrigued by what he styled an "important and oft recurring problem of analysis," namely, "[t]he transformation of homogeneous functions by linear substitutions" [Boole 1841-1843a, p. 1]. Cayley, spurred to a large extent by "analytical geometry, his growing passion in mathematics" [Crilly 2006, p. 86], carried Boole's ideas further with the "formulae" that he published in 1845 and that marked, in some sense, his entrée into what would later become a theory of invariants [Cayley 1845].

Instances of the transformation of homogeneous functions by linear substitutions also cropped up in settings other than the mathematization of mechanics. Carl Friedrich Gauss had explored, in the number-theoretic context of his *Disquisitiones arithmeticae* of 1801, the question of how a binary quadratic form with integer coefficients was affected by a linear transformation [Gauss 1966, pp. 111-112]. It was Charles Hermite's independent reading of that source, as well as of Lagrange's *Traité de la résolution des équations numériques de tous les degrés* (first published in 1789 and revised by the author in 1808), that had exposed the *collégien* to both higher algebra and number theory while at Paris's Collège Louis-le-Grand

1 The letter is quoted, among other places, in [Crilly 2006, p. 86] and [Wolfson 2008, p. 43].

in the early 1840s [Picard 1905, p. viii]. By 1848, Hermite, who was one year Cayley's junior, had passed his *baccalauréat* and *licence*, had become a *répétiteur* and admissions examiner at the École polytechnique, and had published a note in which he, too, focused on the problem of transformation, but from a number-theoretic point of view [Hermite 1848].

A year earlier, yet another mathematician, James Joseph Sylvester, had also come to the matter of transformation from number theory [Sylvester 1847a], although he had cut his mathematical eyeteeth in the 1830s on a purely algebraic approach to the theory of elimination, that is, the theory involved, for example, in finding when two polynomial equations of degrees m and n in one variable have a common root or in determining the real roots of an algebraic equation $f(x) = 0$ of degree n.[2] Almost seven years Cayley's senior, Sylvester, a Jew and the Second Wrangler in 1837, had had a checkered career as he had tried, ultimately unsuccessfully, to establish himself as a mathematician in academe both in England and the United States over the course of the late 1830s and early 1840s. Back in England by the close of 1843, he took a position in 1844 as an actuary at the Equity and Law Life Assurance Society in London. This soon put him in close proximity to Cayley, who had left Cambridge in 1846 to prepare for a career at the Bar. By 1847, the two had met and started up what would become the mathematical correspondence they would maintain for essentially the rest of their lives.[3]

It is not clear exactly when Sylvester and Cayley met Hermite. Both Englishmen were, however, intent on making mathematical reputations for themselves in England and beyond. They had both participated in the French mathematical scene in the 1840s, and both would publish regularly in European journals. It is clear that Sylvester united the three of them in print in 1851 under the common rubric of "transformation" in his paper, "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms" [Sylvester 1851e]. There, he gave an early statement of "invariance" as it had emerged in the work of Boole and Cayley, at the same time that he referred to his "admirable friend M. Hermite" [Sylvester 1851e, pp. 185 and 190, resp.].[4] Over the course of the first half of the 1850s, these three young mathematicians—separated by the English Channel—made common cause in the development of a new

2 See the discussion in [Parshall 2006, pp. 59-62].

3 This early period in Sylvester's life is treated in [Parshall 2006, pp. 49-94]. For a glimpse of Cayley and Sylvester's correspondence, see [Parshall 1998].

4 Page references given for papers by Sylvester, Cayley, and Hermite in what follows refer to the pagination in their respective collected works.

theory—the theory of invariants—or what they would unabashedly term "the New Algebra" [Sylvester 1851a, p. 252]. At the same time, they worked to establish their respective careers in mathematics.

Although much has been written on Cayley and Sylvester's roles in the early development of invariant theory, [5] neither Hermite's part in that development nor his relationship with Cayley and Sylvester and with what became a British school of invariant theory has received particular historical scrutiny. What mathematical paths led them to focus on an analysis *per se* of the transformation of homogeneous forms by linear substitutions? What was the dynamic of the mathematical interchange between them—two in England and one in France—in the first half of the 1850s that resulted in their early invariant-theoretic results? In addressing these questions, this paper not only highlights Hermite's participation in the early development of what was later recognized as the British strain of invariant theory but also provides an interesting case study of how new mathematical ideas could develop in the mid-nineteenth century.

CAYLEY'S ANALYTICAL-GEOMETRICAL PATH

Boole's 1841 paper, "Exposition of a General Theory of Linear Transformations," focused on the determination of "the relations by which" the coefficients of a homogeneous polynomial of degree n in m unknowns "are held in mutual dependence" before and after a linear transformation of its variables is applied. [6] The general technique that he developed to address this problem involved the elimination of the variables from the given polynomial via partial differentiation with respect to each of its unknowns, and he illustrated it in a number of specific examples before stating a general result.

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5 See, for example, [Crilly 1986], [Parshall 1989], [Parshall 1998], [Crilly 2006], and [Parshall 2006]. The latter three works also provide details on the subsequent evolution of invariant theory at the hands of Cayley and Sylvester as does Crilly [1988], while Parshall [1989] analyzes and compares the contemporaneous British and German approaches to the field. And, although David Hilbert purportedly struck a near fatal blow to invariant theory in 1890 with the publication of his paper "Über die Theorie der algebraischen Formen" [Hilbert 1890] and actually "killed" it in 1893 [Hilbert 1893], sociologist of science Charles Fischer pronounced the field "dead" only by the 1920s in [Fischer 1966] and [Fischer 1967]. See [Parshall 1990], however, for more on the "death" argument.

6 See [Boole 1841-1843a, p. 3], as quoted in [Parshall 1989, p. 161]. In what follows, I have massaged the notation in the original papers in order to generate a notation for this paper more consistent across the various works discussed.

Consider, as did Boole, the simplest case of the binary quadratic form

$$Q = ax^2 + 2bxy + cy^2, \tag{1}$$

that is, the homogeneous polynomial of degree two in two unknowns, where a, b, and c are implicitly real numbers [Boole 1841-1843a, p. 6]. Calculating the partial derivatives of Q with respect to x and y and setting the results equal to zero generated two equations,

$$\frac{\partial Q}{\partial x} = 2ax + 2by = 0$$
$$\frac{\partial Q}{\partial y} = 2bx + 2cy = 0,$$

from which Boole eliminated the variables to get the expression

$$\theta(Q) = b^2 - ac,$$

that is, one of the desired relations by which the coefficients of Q "are held in mutual dependence" or what was later termed by Sylvester the discriminant of Q.[7] He next applied the linear transformation

$$\begin{aligned} x &= mx' + ny' \\ y &= m'x' + n'y', \end{aligned} \tag{2}$$

for $m, n, m', n' \in \mathbb{R}$ (and $mn' - m'n \neq 0$, although he assumed, but did not explicitly note, this restriction) to Q to get a new binary quadratic form

$$R = A(x')^2 + 2Bx'y' + C(y')^2.$$

Clearly, calculating the partial derivatives of R yielded

$$\theta(R) = B^2 - AC.$$

Later in his paper [Boole 1841-1843a, p. 19], Boole proved, by explicit calculation, that $\theta(R)$ and $\theta(Q)$ were equal up to a power of the determinant of the linear transformation (2).

Repeating the same analysis *mutatis mutandis* for the binary cubic $Q = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ shows that its discriminant

$$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$$

also remains unchanged up to a power of the determinant of the linear transformation. In language that would emerge only later (see below), but

[7] Sylvester coined the term "discriminant" in a letter to Cayley dated 25 August, 1851. As he put it, the purpose of that letter was to "submit" for Cayley's "approval & ratification" a number of terms in addition to "discriminant," among them, "invariant" and "resultant." See [Parshall 1998, pp. 35-37]. Sylvester used the term "discriminant" in print for the first time in [Sylvester 1851d, p. 280].

that will now be adopted here, Boole concluded in general that the discriminant is an invariant of the homogeneous binary form of degree n. His paper's sequel closed with the suggestive remark that "[a]n equally important subject of inquiry presents itself in the connection between linear transformations and an extensive class of theorems [i.e., expressions] depending on partial differentials, particularly such as are met with in Analytical Geometry. ... To those who may be disposed to engage in the investigation, it will, I believe, present an ample field of research and discovery." [8]

Enter Cayley. On reading Boole's paper, he had seen how to generalize the ideas presented there from homogeneous polynomials of degree n in m unknowns to multilinear forms, that is, to forms made up of "n sets of m variables," in which "the variables of each set [enter] linearly." [9] In so doing, he had developed a computationally nightmarish construct—he called it the hyperdeterminant—that allowed him to produce the "few formulae" about which he had written to Boole. His method, presented in "On the Theory of Linear Transformations" in the *Cambridge Mathematical Journal* in 1845, thus generated invariants for multilinear forms. By suitably identifying the coefficients and the variables in a multilinear form, however, a homogeneous polynomial in m variables results. Similar identification of the coefficients of an invariant of a multilinear form produces an invariant of the underlying polynomial. [10] It was general, but it was complicated, and even Cayley had to admit that he had "not yet succeeded in obtaining the general expression of a hyperdeterminant" and "[could] do so" only in three cases [Cayley 1845, p. 85].

Still, he had found a particularly interesting expression, $ae - 4bd + c^2$, in the coefficients of the binary quartic form

$$Q = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

that was an invariant different from its discriminant $\theta(Q)$ [Cayley 1845, p. 89]. He apparently communicated his new finding to Boole directly, for in an addendum to his paper, he noted that Boole rather quickly not only isolated a third such expression, namely, $ace - b^2e - ad^2 - c^3 + 2bdc$, but also showed by a brute force calculation that

$$(3) \qquad \theta(Q) = (ae - 4bd + c^2)^3 - 27(ace - b^2e - ad^2 - c^3 + 2bdc)^2.$$

8 See [Boole 1841-1843b, p. 119], as quoted in [Wolfson 2008, p. 43].

9 See [Cayley 1845, p. 80] for the quotation. In more familiar terms, Cayley considered monic polynomials in n sets of m variables each: $f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, \ldots, z_1, z_2, \ldots, z_m)$. If $x_i = y_i = \cdots = z_i$, for $1 = 1, 2, \ldots, m$, then f becomes an m-ary form of degree mn.

10 Tony Crilly discussed this method in [Crilly 1986, 243-244].

As Cayley remarked, "[h]ence the two functions on which the linear transformation of functions of the fourth order ultimately depend are the very simple ones $ae - 4bd + 3ca$, $ace - ad^2 - eb^2 - c^3 + 2bdc$, the [discriminant] being merely a derivative from these" [Cayley 1845, p. 94].[11] He viewed it as an "absolute necessity" to his calculational goals to understand this phenomenon completely [Cayley 1845, p. 94].

Cayley quickly followed his 1845 paper with a sequel "On Linear Transformations" early in 1846. There, he not only made explicit what he deemed the central problem of the emerging theory, namely, (a) "[t]o find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables" but also foresaw (b) the "very great difficulties" in "determining the *independent* derivatives, and the relation" as in (3) "between these and the remaining ones" [Cayley 1846b, p. 95 (his emphasis)]. Relative to (a), he introduced yet another calculationally unwieldy technique for generating invariants, this one called hyperdeterminant derivation and applicable only to homogeneous polynomials of degree n in 2 variables. "[M]ainly of theoretical significance and not particularly suited to calculation," however, not even the inveterate calculator Cayley was "overly fond of it as a calculating mechanism" [Crilly 2006, p. 110]. Real progress on (b) would only come in the 1850s in concert especially with Sylvester (see below).

This flurry of invariant-theoretic activity in 1845 and 1846, so exuberantly communicated by Cayley in his private correspondence to Boole,[12] came to naught in the short term. The papers generated no apparent research interest among mathematicians in the British Isles, and the French translation of them that Cayley published in Crelle's *Journal für die reine und angewandte Mathematik* in 1846 failed to elicit a Continental response [Cayley 1846a]. The young mathematician, intent on establishing his reputation at home and abroad, would have to bide his time. In 1846, Cayley not only moved from Cambridge to London to pursue a legal career but also put invariant theory temporarily aside.

11 In equation (13) in [Parshall 1989, p. 164], I inadvertently gave not the discriminant, which is the invariant that Boole's method generated, but rather the new cubic invariant $ace - b^2e - ad^2 - c^3 + 2bdc$ that Boole later found and communicated to Cayley. Note that Cayley is asserting here that, associated to the *binary quartic*, is a set of two independent invariants. His argument *in this case* for what would later be termed a "finite basis" preceded the remark. Note, too, that for Cayley in this context and at this point in time, a "derivative" was what would later be called an "invariant."

12 See [Crilly 1986, pp. 243-245] and [Crilly 2006, pp. 86-92]. On Cayley's correspondence with Boole, see [MacHale 1985, p. 55-58].

HERMITE'S NUMBER-THEORETIC PATH

Forty years before Boole published his work, Gauss, in the exhaustive exploration of quadratic forms that made up the fifth section of his *Disquisitiones arithmeticae*, had considered how a linear transformation (2) affected the variables of a binary quadratic form (1), where, for him, $a, b, c, m, n, m', n' \in \mathbb{Z}$ not $\mathbb{R}$. Substituting (2) into (1), he produced a new binary quadratic form

$$a'(x')^2 + 2b'x'y' + c'(y')^2, \tag{4}$$

explicitly exhibiting the new coefficients:

$$\begin{aligned} a' &= am^2 + 2bmm' + c(m')^2 \\ b' &= amn + b(mn' + nm') + cm'n' \\ c' &= an^2 + 2bnn' + c(n')^2. \end{aligned}$$

He then noted, first, that "[m]ultiplying the second equation by itself, the first by the third, and subtracting we get

$$b'b' - a'c' = (b^2 - ac)(mn' - nm')^2,\text{"}$$

second, that the discriminant of (4) is divisible by the discriminant of (1), and, third, that their quotient is square [Gauss 1966, pp. 111-112].[13] As early as 1801, then, Gauss had, without the terminology and within a different theoretical framework, already discovered that the discriminant of the binary quadratic form is an invariant.

By 1843, Hermite had become an assiduous student of Gauss's (as well as Lagrange's) work and, as he would later put it, it was from these two masters that he "learned Algebra."[14] In the 1840s, however, this and other reading, especially of the work on elliptic functions of Carl Gustav Jacob Jacobi, had led Hermite "to study algebraic complex numbers in general and to classify them in the spirit of Lagrange's and Gauss's classification of binary quadratic forms," while "keep[ing] close to elliptic functions" [Goldstein & Schappacher 2007a, p. 42]. By the end of the decade, moreover, Hermite

13 Gauss's observations are also discussed in [Parshall 1989, p. 159]. As Catherine Goldstein and Norbert Schappacher noted, Gauss treated quadratic forms in and of themselves and not just, as Lagrange and Legendre had done, as tools for studying the representation of integers [Goldstein & Schappacher 2007a, p. 8].

14 "C'est surtout dans ces deux livres ... que j'ai appris l'Algèbre" [Picard 1905, p. viii]. Compare also [Goldstein 2007, pp. 377-378].

had come more specifically to consider number theory *per se* both in correspondence with Jacobi and in a number of short papers.[15] It was in one of the latter, "Note sur la réduction des fonctions homogènes à coefficients entiers et à deux indéterminées," that the transformation of a binary form of degree n arose.

Hermite considered [Hermite 1848, pp. 84-86] a homogeneous form of degree n in two variables with integer coefficients

$$f(x,y) = Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 + \cdots + Kxy^{n-1} + Ly^n \tag{5}$$

and the transformation (2) (where, however, we have that m, n, m', $n' \in \mathbb{Z}$ and $mn' - m'n = \pm 1$) that takes (5) to

$$f(x',y') = A'x'^n + B'x'^{n-1}y' + C'x'^{n-2}y'^2 + \cdots + K'x'y'^{n-1} + L'y'^n.$$

Letting $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n (complex) roots (he then assumed they were all real) of the equation

$$Az^n + Bz^{n-1} + Cz^{n-2} + \cdots + Kz + L = 0,$$

he defined what he called the "determinant" of (5) to be a particular expression D in the differences of the roots. Writing $f(x,y)$ in terms of the roots as $A(x-\alpha_1 y)(x-\alpha_2 y)\cdots(x-\alpha_n y)$ and applying (2), he showed that "the value of D stays absolutely the same" under the action of (2).[16] Hermite's "determinant" was what Sylvester would later term the discriminant, but in a special, number-theoretically-oriented setting as opposed to that in which Boole had isolated it in 1841. Hermite's binary form had *integer* coefficients, and his linear transformation had determinant ± 1.

At roughly the same time that Hermite published this paper, he was writing to Jacobi about his emerging ideas. Excerpts from four of these (undated) letters, likely written at some point in 1847,[17] were published in Crelle's *Journal* in 1850. In the third, he confessed to Jacobi that although "the goal of [his] first researches had been to examine the new method of approximation that you have given in establishing the impossibility of a function with three imaginary periods," he had come to recognize that that project "paled in comparison with major issues in the theory of

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15 For a beautiful, finely grained account of Hermite's early number-theoretic work and its links to Gauss's *Disquisitiones arithmeticae*, see [Goldstein 2007].

16 "la valeur de D reste absolument la même" [Hermite 1848, p. 86].

17 Catherine Goldstein argued convincingly for this dating in [Goldstein 2007, p. 383 (note 22)].

forms, considered from a general point of view."[18] "In this vast expanse of research opened up for us by Mr. Gauss," he continued, "Algebra and the Theory of numbers seem to me to become merged into one [and the] same order of analytical ideas, of which our actual knowledge does not yet allow us to form an adequate idea."[19] Although he had touched on it in his first letter to Jacobi, in his third, he began to formulate just such an idea through the development of what, after Gauss, he termed "the notion of *formes adjointes*" [Hermite 1850, pp. 137ff] (see (7) below), that is, his construct for isolating what would later be termed contravariants. Given these developments, it is little wonder that the first half of the 1850s found Hermite both at work on the new field of research that Catherine Goldstein and Norbert Schappacher termed "arithmetic algebraic analysis" [Goldstein & Schappacher 2007a, p. 52] and, at the same time, engaged in the invariant-theoretic ideas coming out of England.

SYLVESTER'S ALGEBRAIC PATH

If Cayley was largely motivated in what would become his invariant-theoretic research by analytical-geometrical concerns in the 1840s and Hermite came to his via more number-theoretical interests, Sylvester tended to approach his mathematical research from a more purely algebraic point of view. This was made manifest in the series of papers he wrote on elimination as early as 1839 and 1840 from the post of professor of natural philosophy that he had secured in 1839 at University College, London. In particular, he had closely examined the auxiliary functions $f_i(x)$ that arise in the application of Sturm's theorem for finding the real roots α_i between any two real numbers a and b of an algebraic equation $f(x) = 0$. Inspired in this work by, as he put it, "happening to be present at a sitting of the French Institute" [Sylvester 1839, p. 44 (note ‡)], Sylvester succeeded not only in explicitly exhibiting the $f_i(x)$ as algebraic expressions in terms of the squares of the differences of the roots

18 "le but de mes premières recherches avait été d'examiner le nouveau mode d'approximation que vous avez donné en établissant l'impossibilité d'une fonction à trois périodes imaginaires" and "les problèmes si vastes que j'avais cru me proposer m'ont semblé peu de chose à côté des grandes questions de la théorie des formes, considérée d'une manière générale" [Hermite 1850, p. 136].

19 "Dans cette immense étendu de recherches qui nous a été ouverte par M. Gauss, l'Algèbre et la Théorie de nombres me paraissent devoir se confondre dans un même ordre de notions analytiques, dont nos connaissances actuelles ne nous permettent pas encore de nous faire une juste idée" [Hermite 1850, p. 136].

but also in developing some of the basic properties of determinants.[20] Hourya Sinaceur was quite right when she remarked that “Sylvester was convinced, in effect, that all of analysis could be presented, finally, in the language of the theory of determinants, this ‘algebra of algebra.’ … At a time when the Paris school was dominated, a few exceptions aside, by the analytic spirit, Sylvester wanted to promote the algebraic spirit to the point of subjecting all of analysis to it!”[21]

Sylvester’s early research activity, as well as his ability to participate directly in the Parisian mathematical scene, came to an end in the fall of 1841 when he made a bold transatlantic move by accepting what he hoped would be a more congenial professorship of *mathematics* at the University of Virginia. Those hopes, however, were quickly dashed. Sylvester quit the job after less than five months and spent the next year and a half trying, ultimately in vain, to find a new academic position in the United States.[22] Having returned to England by the beginning of 1844, he finally landed the actuarial job in London that had allowed him to recover, by 1846, what he described as his “footing in the world’s slippery path” and to “secure a landing place whereon to breathe and calmly survey and determine upon [his] future course” [Parshall 1998, p. 15]. A year later, he had reengaged with both mathematics and mathematical Paris.

During “the course of a rapid tour of the continent,” Sylvester was in the City of Lights and in conversation with his “illustrious friend M. Sturm” about new ideas he had had on cubic equations of the form

$$Ax^3 + By^3 + Cz^3 = Dxyz \tag{6}$$

(for integers A, B, C and D) and that he, in an effort to build his reputation in France, had asked Sturm to convey for him to the Institute [Sylvester 1847a, p. 109]. Sketches of his results subsequently appeared in print in three short papers published in the *Philosophical Magazine* in 1847.

Sylvester opened the first with the statement of what he styled “a general theorem of transformation.” Assuming in (6) that “A and B are equal, or in the ratio of two cube numbers to one another” and that the expression $27ABC - D^3$, which he called the “determinant,” satisfies certain specific conditions, he claimed (he gave no proofs in this paper)

20 For more details and all of the pertinent references, see [Parshall 1998, pp. 5-9].

21 “Sylvester était convaincu, en effet, que toute l’Analyse pouvait être, en dernier ressort, présentée dans le langage de la théorie des déterminants, cette ‘algèbre de l’algèbre.’ … À une époque où l’école de Paris était, en dépit de quelques exceptions, dominée par l’esprit analytique, Sylvester veut promouvoir l’esprit algébrique jusqu’à y soumettre l’analyse toute entière!” [Sinaceur 1991, p. 126].

22 On this period in Sylvester’s life, see [Parshall 2006, pp. 64-80].

that (6) can "be made to depend upon another [expression] of the form $A'u^3 + B'v^3 + C'w^3 = D'uvw$, where $A'B'C' = ABC$, $D' = D$, [and] $uvw =$ some factor of z" [Sylvester 1847a, p. 107]. From this, he concluded (in [Sylvester 1847a] and [Sylvester 1847b]) that certain particular cases of (6) had no integer solutions, while in [Sylvester 1847c], he laid out, given a particular integral solution of (6), a process for generating the rest.[23]

Unfortunately, the "[p]ressing avocations" that had prevented him from providing actual proofs of his results on the cubic overwhelmed him thereafter [Sylvester 1847a, p. 109]. Sylvester was employed as an actuary; he had embarked on legal studies at the Inner Temple in London in 1846; he was instrumental in setting up the Institute of Actuaries, also in London, beginning in 1848. He did meet Cayley during this period of intense professional activity and, as their mathematical and personal friendship evolved, tried to think at least sporadically about mathematics. In a letter written with a formal air to "My dear Sir" and dated 24 November, 1847—the earliest known letter in the Sylvester-Cayley correspondence—Sylvester shared some number-theoretic thoughts with this other mathematician-turning-lawyer that had been sparked by his reading of Legendre's *Théorie des nombres* [Parshall 1998, pp. 18-19]. Still, three years would elapse before Sylvester, the mathematician, would be heard from again in print.

PATHS CONVERGE

The close proximity that Sylvester and Cayley enjoyed—the former at 26 Lincoln's Inn Fields, the latter at 2 Stone Buildings, both within the walls of the Inns of Court—as well as their shared mathematical interests made natural the convergence of their two paths. By the spring of 1850, they were in daily communication about ideas primarily of an algebraic-geometric cast that hearkened back to Cayley's work on linear transformations as well as to Sylvester's research on the theory of elimination. On Boxing Day in 1850, for example, Sylvester wrote to Cayley with both ideas and questions.

In his letter, he revealed his awareness of connections between Boole's 1841 "Exposition of a General Theory of Linear Transformations" and of work that the German mathematician, Otto Hesse, had published in 1844 [Parshall 1998, pp. 30-32]. As just noted, in 1847, Sylvester had considered the cubic Diophantine equation $Ax^3 + By^3 + Cz^3 = Dxyz$. In 1850, he was thinking about how to effect a transformation of an *arbitrary*

[23] Compare [Dickson 1921–1923, 2: p. 589].

cubic form in three variables x, y and z into what, based on his reading of Hermite [Sylvester 1851e, p. 190], he termed the canonical form $a^3\xi^3 + b^3\eta^3 + c^3\zeta^3 + 6\epsilon abc\xi\eta\zeta$ in the variables ξ, η, and ζ. As Sylvester had apparently learned from Cayley [Parshall 1998, pp. 30 and 32], Hesse had undertaken an analysis of third-order plane curves in 1844 and, along the way, had considered exactly the same question about transformation [Hesse 1844, pp. 90-95]. For Sylvester, an application of Boole's technique gave him the desired result, but Sylvester asked Cayley "whether it is identical with Hesse's solution" [Parshall 1998, p. 30]. It was not, but by 1851, Sylvester had published a "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms" in which he had taken the first tentative steps toward uniting the work of Boole, Cayley, Hermite, himself, and others within a common framework. [24]

Sylvester was coming to appreciate that a number of very different techniques had led to similar results. As he vaguely put it, "from dependencies of equations, transition may be made to the relations of functional forms, and *vice versâ*" [Sylvester 1851e, p. 184]. One of those "dependencies" was the so-called resultant of which the discriminant is a special case. As noted, Boole and Cayley each had a method for producing the discriminant, and both had discovered that the discriminant so generated had the invariantive property. Moreover, Hermite, but "in a more restricted sense" and via his *formes adjointes*, had made a similar discovery [Sylvester 1851e, pp. 185-186]. In explicitly highlighting this confluence, Sylvester linked his own developing ideas with those of Cayley and Hermite for the first time in print. This marked at least the symbolic convergence of three distinct paths toward a theory of invariants.

At this juncture, Sylvester, ebullient about getting back into mathematical research after a long dry spell, served as the catalyst. In the two years from 1851 to 1853, he produced an impressive body of work that culminated in massive papers: "On the Principles of the Calculus of Forms" in 1852 [Sylvester 1852] and "On the Theory of Syzygetic Relations of Two

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24 Sylvester also included Hesse's technique in his overview of results, although he actually accused his German counterpart of having appropriated, without acknowledgment, the "method of finding the resultant of any set of three equations of degrees equal or differing only by a unit, one from those of the other two" that he had presented in work published four years earlier in 1851 [Sylvester 1851e, p. 189]. At this point, however, Sylvester seemed unaware of Gotthold Eisenstein's work on the binary cubic that also foreshadowed invariant-theoretic notions. See [Eisenstein 1844] and compare the discussion in [Parshall 1989, pp. 170-171]. Ultimately and contemporaneously, a parallel school of invariant theory developed in Germany around, in particular, Hesse, Siegfied Aronhold, Alfred Clebsch, and Paul Gordan (see below). Italians like Francesco Brioschi were also soon drawn into invariant-theoretic research.

Rational Integral Functions" in 1853 [Sylvester 1853b]. As the correspondence makes clear, Cayley was Sylvester's primary sounding board, but Sylvester's published works clearly show that he was also in regular contact with Hermite and was increasingly absorbing the work of his new French friend.

Sylvester followed his "Sketch" in the May 1851 issue of the *Cambridge and Dublin Mathematical Journal* with a short paper "On the General Theory of Associated Algebraical Forms" in the journal's November number. There, he began to roll out the evolving terminology of what was becoming a *theory* of invariants [Sylvester 1851b]. He also explicitly connected the work of Cayley and Hermite with those new concepts. In particular, he introduced the notions of "covariant" and "contravariant." Given a homogeneous form in n variables and a linear transformation of those variables, a covariant is an expression in the coefficients *and* variables of the form that remains unaltered by the linear transformation (up to a power of the linear transformation's determinant), while a contravariant is an expression in the coefficients and variables that remains so unaltered by that transformation's transpose.[25] As Sylvester noted, "[c]ovariants are Mr. Cayley's hyperdeterminants; contravariants, include, but are not coincident with, M. Hermite's formes-adjointes, if we understand by the last-named term such forms as may be derived by the process described by M. Hermite in the third of his letters to M. Jacobi" [Sylvester 1851b, p. 200]. Still, Hermite had also used his idea of a *forme adjointe* in that letter "in a sense as wide as that" of Sylvester's notion of contravariant in the following "most remarkable theorem" [Sylvester 1851b, p. 201]: "If we have a function of any number of letters, say of x, y, z as

$$ax^m + mbx^{m-1}y + mcx^{m-1}z + \frac{m(m-1)}{2}dx^{m-1}y^2 + \&\text{c.},$$

and if I be any invariant of this function, then will

$$\left(x^m \frac{d}{da} + x^{m-1}y\frac{d}{db} + x^{m-1}z\frac{d}{dc} + x^{m-2}y^2\frac{d}{dd}\ \&c\right)^r I \tag{7}$$

be a '*forme adjointe*' of the given function." As Sylvester showed in the first (February 1852) installment of his paper "On the Principles of the Calculus of Forms," (7) is a contravariant [Sylvester 1852, p. 289].

Things were definitely heating up at the end of 1851. Sylvester and Hermite were in active conversation [Sylvester 1851c, p. 246]; Sylvester

25 See [Sylvester 1851b, p. 200]. As noted, the terminology was in flux. Instead of the term "transpose," the early invariant-theorists used "inverse." See, for example, Salmon's definition of the contravariant in [Salmon 1859, pp. 117-118].

was writing almost daily to Cayley to fill him in on his latest ideas as he worked to write them up (see [Parshall 1998, pp. 30-37] for a sample of their correspondence); Cayley, actively engaged in the give-and-take, was separately preparing his own research for publication [Crilly 2006, pp. 168-179]. As Tony Crilly explained, however, "[a] modern arrangement such as a joint paper ... was not the custom in Britain in the middle of the nineteenth century. Both young men [and Hermite, too] nursed mathematical ambition, but they were individualists" who "kept close guard on their own ideas" [Crilly 2006, p. 173].

In 1851, as Sylvester pushed his research agenda, Cayley was flooding Crelle's *Journal* with a series of notes mostly on topics in analytic geometry. By the end of the year, however, Sylvester had apparently succeeded in drawing Cayley back into what was becoming an invariant-theoretic fray. On 21 November, Cayley returned to his earlier work on hyperdeterminants in a "Note sur la théorie des hyperdéterminants" that he sent to Crelle. There, he considered "the case of a homogeneous polynomial in two variables, using Mr. Sylvester's new terms" of "Covariant" and "Invariant" and essentially translated his earlier hyperdeterminant-theoretic work into this new language. [26] Two weeks later, he was writing to Sylvester with what may be seen as the birth certificate of nineteenth-century, British invariant theory. "Every invariant [U]," he wrote, "satisfies the partial diff[erentia]l equations

$$\Big(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\cdots+nj\frac{d}{dk}\Big)U=0$$
$$\Big(b\frac{d}{db}+2c\frac{d}{dc}+3d\frac{d}{dd}+\cdots+nk\frac{d}{dk}\Big)U=\frac{1}{2}nsU$$

(s the degree of the Invariant) & of course the two equations formed by taking the coeff[icien]ts in a reverse order. This will constitute the foundation of a new theory of Invariants." [27] Why? It gave a general calculational

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[26] "le cas d'une fonction homogène à deux variables, et en me servant des nouveaux termes de M. Sylvester, je nomme 'Covariant' d'une fonction donnée, toute fonction qui ne change pas de forme en faisant subir aux variables des transformations linéaires quelconques, et 'Invariant' toute fonction des seuls coefficients qui a la propriété mentionnée" [Cayley 1851, p. 368].

[27] For the quotation, see [Parshall 1998, p. 37]. The first of these operators is the functionally important one. To get a feel for the computation, let $U=(ad-bc)^2-4(b^2-ac)(c^2-bd)$, the discriminant of the binary quartic equation considered above. It is not hard to see that applying the operators $a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}$ and $d\frac{d}{dc}+2c\frac{d}{db}+3b\frac{d}{da}$ ("taking the coefficients in reverse order") to U yields 0 in both instances.

means for identifying invariants. The race was on, and the two friends were neck in neck.

The February 1852 number of the *Cambridge and Dublin*, for example, found Cayley's "On the Theory of Permutants" [Cayley 1852] immediately preceding the first installment of Sylvester's "On the Principles of the Calculus of Forms." In his paper, Cayley "was chiefly concerned with the organization of the algebra of forms," "assembl[ing] the various types of determinants he had dealt with previously and group[ing] them together under one umbrella concept, the algebraic form he called a permutant" [Crilly 2006, p. 173]. In his, Sylvester was, in addition to establishing much of the new language of a theory of invariants, presenting the technique for generating invariants that he called compound permutation. The two were actually publishing versions of the same technique, although a priority dispute, evidence of which may be found in both papers, was quickly diffused.[28] Sylvester's paper, moreover, also reflected his mathematical interactions—through publications, letters, and likely in person during trips across the Channel to Paris—with his "valued friend M. Hermite" in the early 1850s [Sylvester 1852, p. 296]. Sylvester was actively drawing Hermite not only into the fray but also into the British mathematical scene; by 1852 at least, Hermite had also been in correspondence with Cayley [Crilly 2006, p. 186].

Sylvester continued work on the next installment of "On the Principles of the Calculus of Forms" through the spring of 1852, made a concerted effort to build his mathematical reputation in France, became more fully aware of the parallel development of a theory of invariants in Germany at the hands initially of Hesse and Siegfried Aronhold and later of Alfred Clebsch and Paul Gordan, and nurtured a mathematical relationship with the Irish mathematician, George Salmon.[29] The summer of 1852, however, found him derailed by what he called the problem of syzygies, that is, the existence, that Boole and Cayley had recognized as early as 1845, of dependence relations like (3) among the invariants associated with a homogeneous polynomial of given degree in two variables. Sylvester mulled over that problem into 1853, ultimately presenting his enormous paper "On the Theory of Syzygetic Relations of Two Rational Integral Functions" to the Royal Society in June [Sylvester 1853b]. There, he explored questions like

[28] See [Sylvester 1852, p. 318] and [Cayley 1852, p. 26], and compare [Parshall 1998, pp. 33-35] and [Crilly 2006, p. 173].

[29] On this period as a whole, see [Parshall 2006, pp. 112-122]. On Sylvester's reputation-building efforts, see [Parshall & Seneta 1997]. The non-British development of a theory of invariants is discussed in [Parshall 1989, pp. 170-180].

when do such relations exist among in- or covariants, and, if they exist, how can they be determined explicitly? He also used his new findings to reinterpret, in an invariant-theoretic context, some of his earlier findings on Sturm functions, incorporating then-recent work of Hermite.[30] It was a formidable paper and its completion, combined with the press of duties associated with his position at the Equity Law and Life, had left Sylvester exhausted by the close of 1853.

Cayley and Hermite had followed Sylvester's progress, while largely continuing to pursue their own respective lines of research. Cayley generated a number of short papers on a variety of mathematical topics from group theory to analytical geometry to the theory of probability, but by 1854, he had embarked on what would ultimately be a series of ten memoirs on quantics (Cayley's new term for homogeneous polynomials in two or more variables). The first of his papers appeared in the *Philosophical Transactions of the Royal Society* in 1854 [Cayley 1854]. For his part, Hermite continued to develop the new "arithmetic algebraic analysis" until 1854, when he, too, came out with a major invariant-theoretic study. Indicative of the fact that he had been drawn into Cayley and Sylvester's mathematical world just as much as they had been drawn into his, Hermite published "Sur la théorie des fonctions homogènes à deux indéterminées" [Hermite 1854] in the *Cambridge and Dublin.*[31]

Cayley's 1854 paper, "An Introductory Memoir upon Quantics," like Sylvester's "Sketch" three years earlier, laid out the vocabulary and basic definitions of the theory of invariants, introduced the general notation for quantics that he would continue to use in his invariant-theoretic work, and recast some of his earlier results in this new guise. It also reflected Cayley's knowledge of results that Hermite was only making known to the mathematical public at essentially the same time.[32]

In "On the Principles of the Calculus of Forms," Sylvester had alluded to a time- and calculation-saving theorem that he had discovered in 1852 "in the course of a most instructive and suggestive correspondence with Mr Salmon" and that he promised to publish "along with other very important matter, in the next number of the" *Cambridge and Dublin* [Sylvester

30 See [Sinaceur 1991, pp. 129-140] as well as, especially, [Hermite 1853].

31 Hermite also wrote up his findings for the broader European mathematical audience in 1854, although his two installments of that version only appeared in the *Journal für die reine und angewandte Mathematik* in 1856. See [Hermite 1856a] and [Hermite 1856b].

32 In a postscript to his paper dated 7 October, 1854. Cayley remarked that he had "deduced as a corollary, the law of reciprocity of MM. Sylvester and Hermite" [Cayley 1854, p. 139]. Hermite's result also appeared in print in 1854 [Hermite 1854].

1852, p. 339]. When that number appeared in February 1853, however, Sylvester had to confess that "[a]ccidental causes have prevented me from composing the additional sections on the Calculus of Forms, which I had destined for the present Number of this *Journal*" [Sylvester 1853a, p. 403]. At issue was the so-called law of reciprocity: "To every covariant of a form of degree m, which is, relative to the coefficients of this form, of degree p, corresponds a covariant of degree p relative to the coefficients of a form of degree m."[33] When the dust finally settled, it had been Hermite—as Sylvester acknowledged in 1853 followed by Cayley in 1854—who had actually given the theorem's proof.[34] With characteristic hyperbole, Sylvester declared that "[t]o M. Hermite, therefore, belongs the honor of reviving and establishing—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms" [Sylvester 1853a, p. 403]. "Our new calculus." Sylvester viewed the creation of a theory of invariants in the first half of the 1850s as a joint venture among the three friends, and Hermite echoed that sentiment in acknowledging how "encouraged" he was "by the very kind manner" in which his "friend Mr. Sylvester" had welcomed his new results.[35]

Although Hermite opened his 1854 paper "Sur la théorie des fonctions homogènes à deux indéterminées" with his proof of the law of reciprocity, that paper contained much more. In particular, he tackled the problem of the binary quintic form, that is, a homogeneous polynomial

$$g(x,y) = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5. \tag{8}$$

As was well-known at mid-century thanks to the work of Niels Henrik Abel, a general fifth-degree polynomial

$$h(x) = ax^5 + bx^4 + cx^3y + dx^2 + ex + f = 0 \tag{9}$$

33 "A tout covariant d'une forme de degré m, et qui par rapport aux coefficients de cette forme est du degré p, correspond un covariant de degré $[p]$ par rapport aux coefficients d'une forme du degré $[m]$" [Hermite 1854, p. 297]. (In Hermite's paper, the p and the m are incorrectly switched in the second clause; I have corrected them here.) As is evident in this quotation, the terminology was still in flux at this point, with "degree" referring to both the degree of a polynomial and the order of an invariant in the coefficients of its underlying homogeneous polynomial.

34 See [Sylvester 1853a, p. 403] and [Cayley 1854, p. 232], respectively. For Hermite's proof, see [Hermite 1854, pp. 297-299]. Both Sylvester and Cayley also noted in these papers that Cayley had come up with a proof different from Hermite's.

35 "Depuis que la première Partie de ces recherches a été terminée, encouragé par la manière si bienveillante dont elles ont été accueilles par mon ami M. Sylvester ..." [Hermite 1854, p. 315].

is not solvable by radicals, that is, it is impossible to give a formula for finding the roots of (9) in terms of the binary operations of addition, subtraction, multiplication, and division together with the extraction of roots. But, was it possible to approach the quintic some other way and, in so doing, gain further insights into it? Cayley had thought there might be as early as the 1840s in light of his proto-invariant-theoretic work [Crilly 2006, p. 89]. By the 1850s, as a theory of invariants actually evolved and as Cayley and his partners worked toward, among other things, the early goal of finding "all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables" [Cayley 1846b, p. 95], the quintic had come increasingly under scrutiny.

In 1846, Cayley had already isolated a quintic invariant of order four in the coefficients of (8), namely, [36]

$$\begin{aligned} G = a^2f^2 &- 10abef + 4acdf + 16ace^2 + 16ace^2 \\ &- 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f \\ &- 76bcde + 48bd^3 + 48c^3e - 32c^2d^2. \end{aligned}$$

By 1854, fifteen different invariants and covariants of the quintic had been discovered and explicitly calculated by Cayley, Sylvester, Hermite, and others as they had proceeded with their agenda of collecting and classifying invariants and covariants—and of detecting the syzygies between them—for binary quantics of progressively higher degrees. [37] There had been three known invariants—of orders 4, 8, and 12—of the binary quintic until Hermite had stunned Cayley and Sylvester with his discovery of a new one of order 18 [Hermite 1854, pp. 305-306 and 310-314]. As Hermite put it in his published exposition of 1854, "Mr. Cayley, Mr. Sylvester and I had long thought that, in general, the invariants of forms of the m^{th} degree should be expressible as whole functions of $m-2$ of them, and it is this that prevented Mr. Sylvester from seeking to prove the law of reciprocity, of which he, too, had presumed the existence, a necessary

36 See [Crilly 2006, p. 98] and compare [Cayley 1846b, pp. 107-108], where the invariant in question is denoted by D_{410}.

37 Tony Crilly noted the similarity between the collecting impulse of nineteenth-century invariant theorists and that of the nineteenth-century chemists and naturalists in his doctoral dissertation of 1981 [Crilly 1981] as well as in his biography of Cayley [Crilly 2006, pp. 193-196]. Catherine Goldstein focused on this aspect of Hermite's mathematical agenda, in particular, in [Goldstein 2016].

contradiction having emerged between this law and that of the number of fundamental invariants."[38] In his "Second Memoir on Quantics" submitted in the spring of 1855 but published in 1856, Cayley not only presented the combinatorial formula that he had discovered sometime prior to October 1854 for determining, for a homogeneous polynomial of given degree in two variables, the number of its linearly independent in- and covariants [Cayley 1854, p. 234] but also explicitly laid out those covariants for quantics up to and including the quintic [Cayley 1856].[39]

PATHS DIVERGE

The first half of the 1850s had found Cayley, Hermite, and Sylvester in intense conversation as they developed their new theory of invariants.[40] It seems clear that, in the process, they had come to share the sense that being a mathematician in the mid-nineteenth century not only meant generating new mathematical ideas but also working to make those ideas known both at home and abroad. It was important to establish a reputation.

To the latter end, all three young men took care to give their work wide exposure. Cayley and Sylvester published their research in English-language journals like the *Cambridge and Dublin* and the *Philosophical Transactions* at the same time that they publicized it abroad, in Cayley's case through publication in Crelle's *Journal* and in Sylvester's through direct participation in the French mathematical scene and, after 1852, in foreign journals.[41]

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38 "M. Cayley, M. Sylvester et moi avions longtemps pensé qu'en général les invariants des formes de $m^{\text{ième}}$ degré devaient s'exprimer par les fonctions entières de $m-2$ d'entre eux, et c'est même ce qui a empêché M. Sylvester de chercher à démontrer la loi de réciprocité dont il avait aussi présumé l'existence, une contradiction nécessaire s'étant manifestée entre cette loi et celle du nombre des invariants fondamentaux" [Hermite 1854, p. 312].

39 As is well-known, though, Cayley mistakenly believed, on the basis of this work, that there were an infinite number of irreducible covariants associated with the binary quintic equation. For more on this and on Cayley's error, see [Hawkins 1987], [Parshall 1989, pp. 169-170], and [Parshall 1998, pp. 184-188]. Tony Crilly also discusses this in [Crilly 1986, pp. 248-249] and [Crilly 2006, pp. 204-207].

40 According to Hermite's biographer, Émile Picard, Sylvester said much later that "We thus formed, Cayley, Hermite, and I, an invariant trinity [Nous formions alors, Cayley, Hermite, et moi, une trinité invariantive]" [Picard 1905, p. xx]. Picard would not have found such a statement from Sylvester surprising, given the intense period of interaction the three men had had in the 1850s. The phrase "invariant trinity," however, has generally been used in reference to Cayley, Salmon, and Sylvester, and although its origins are unclear, that usage has often been attributed to Hermite.

41 Tony Crilly touched on this issue throughout his biography of Cayley [Crilly 2006], while I focused on it in Sylvester's case in [Parshall & Seneta 1997] and [Parshall 2006].

Similarly, Hermite published in the *Comptes rendus* of Paris's Académie des Sciences as well as in Crelle's *Journal* and in the *Cambridge and Dublin*. [42] When the latter journal ceased publication for financial reasons in 1854 and the new *Quarterly Journal of Pure and Applied Mathematics* took its place in 1855 under the co-editorship of Sylvester and Norman Ferrers, Hermite and Cayley joined the editorial team. [43] Despite this professional linkage, however, the mathematical paths of Sylvester, Hermite, and Cayley had already begun to diverge.

Sylvester's research push in 1852 and 1853, combined with his increasing sense that in order to pursue his mathematics most effectively he needed to be in an academic setting, left him at sea in 1854 and much of 1855 as he first tried and failed to leave the Equity Law and Life but then secured the professorship of mathematics at the Royal Military Academy in Woolwich. His mathematical malaise was only heightened when he realized that he had been a mere hair's breadth away of being the first to find the invariant of order 18 of the binary quintic. Looking back on things in 1870, he recounted that he had "discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. … The canonizant of the quartic (its cubic covariant) was the first thing to offer itself in the inquiry. I had but to think the words 'Resultant of Quintic and its Canonizant,' and the octodecadic skew invariant would have fallen spontaneously into my lap" [Sylvester 1904-1912, 2: 714]. [44] With his assumption of the Woolwich professorship in the fall of 1855, moreover, Sylvester found himself distracted by his new duties and unfocused in his research, although as the second volume of his collected works attests, he did continue to produce new results. He would really only come back to invariant theory in the mid-1860s, and then, in particular, to the problem that he, Cayley, and Hermite had considered in the 1850s, namely, "a complete invariantive determination of the character of the roots of the general equation of the fifth degree." [45]

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42 As Tom Archibald argued [Archibald 2002], Hermite was particularly interested in establishing a reputation among the mathematicians in Germany.

43 On the history of this journal, see [Despeaux 2002, pp. 162-184] and [Crilly 2004]. Irish physicist and mathematician, George Gabriel Stokes, rounded out the editorial team.

44 Tony Crilly noted, and I missed, this cause (of several) of Sylvester's depression of the mid-1850s. See [Crilly 2006, pp. 205-206] and compare [Parshall 2006, pp. 131-138].

45 See [Sylvester 1864, p. 376] for the quotation and [Sylvester 1864, pp. 418-475] for the mathematics.

For his part, Hermite continued to work out various invariant-theoretic ramifications in a number of short papers into the 1860s, despite the setback brought on by the case of smallpox he contracted in 1856. While he produced a number of isolated results on forms—for example in three instead of two variables—his flashiest result reflected *his* continued interest in things quintic. In 1858, he succeeded in showing how to solve a general fifth-degree polynomial equation (9) using an interesting amalgam of invariant-theoretic notions and elliptic function theory [Hermite 1858].[46] The end of the 1850s and into the 1860s also saw him return to more properly number-theoretic concerns as he took up the position of *maître de conférences* (in 1862) at the École normale supérieure. Hermite finally assumed an actual professorship in analysis at the École polytechnique and in the *Faculté des Sciences de Paris* in 1869 [Freudenthal 1970–1990].

Of the three, it was Cayley who remained the most steadfast to the theory of invariants after the mid-1850s. In quick succession, and in addition to his voluminous work particularly in geometry, he had followed his "Second Memoir on Quantics" with five more by 1861. The eighth, ninth, and tenth memoirs would appear in 1867, 1870, and 1878, respectively. He also continued to think seriously about the quintic as a cache of letters he wrote to Congregationalist minister, Robert Harley, between 1859 and 1863 makes evident [Crilly et al. 2017]. In addition to Harley, several other mathematicians took an interest in invariant theory in the aftermath of the intense early 1850s, but it was perhaps George Salmon who proved the most faithful, producing numerous editions of his *Lessons Introductory to the Modern Higher Algebra* following the first in 1859 [Salmon 1859]. And, like Sylvester and Hermite, Cayley, finally secured an academic post, his the new Sadleirian professorship of pure mathematics at Cambridge in 1863.

* * *

The research paths of Cayley, Hermite, and Sylvester may have diverged by the end of the 1850s, but the three mathematicians remained friends for the rest of their lives. The bonds they had forged in the early 1850s had been strong. They had shared the desire for—and had succeeded in making—national and international reputations for themselves. Their examples, as well as the examples of others, can be seen as defining

46 By 1865, Hermite had seen how to unify his elliptic-function-theoretic approach to the solution of the quintic, which hinged on the modular equation, and that of Francesco Brioschi and Leopold Kronecker, which depended on the multiplier equation, via invariant theory. See [Hermite 1865–1866] and compare the discussion in [Goldstein 2011b, pp. 248-249]. See also [Goldstein 2011a].

what, over the course of the last half of the nineteenth century, became one of the new professions, the mathematician, namely, a person employed in academe—ideally as a professor of mathematics—who actively produced and published new mathematical results and who fostered and participated in accoutrements like journals—as authors, editors, and referees—and societies like the Royal Society of London and Paris's Académie des Sciences. [47] Yet, first and foremost at least in their minds, Cayley, Hermite, and Sylvester had been instrumental in the creation of a new area of mathematics, the invariant theory that they deemed the nineteenth century's "New Algebra."

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REFERENCES

Archibald (Thomas)

[2002] Charles Hermite and German Mathematics in France, in *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800–1945*, New York: American Mathematical Society and London: London Mathematical Society, 2002, pp. 122–137.

Boole (George)

[1841-1843a] Exposition of a General Theory of Linear Transformations: Part I, *Cambridge Mathematical Journal*, 3 (1841-1843), pp. 1–20.

[1841-1843b] Exposition of a General Theory of Linear Transformations: Part II, *Cambridge Mathematical Journal*, 3 (1841-1843), pp. 106–119.

47 Much has been written on the rise of professions, but see, for example, [Reader 1966] for the British context and [Geison 1984] for the French.

Cayley (Arthur)

[1845] On the Theory of Linear Transformations, *Cambridge and Dublin Mathematical Journal*, 4 (1845), pp. 193–209; or [Cayley 1889–1898, 1: 80-94].

[1846a] Mémoire sur les hyperdéterminants (traduction d'un mémoire anglais inséré dans le *Cambridge Mathematical Journal* avec quelques additions de l'auteur), *Journal für die reine und angewandte Mathematik*, 30 (1846), pp. 1–37; or [Cayley 1889–1898, 1: 117] (which does not give the translation).

[1846b] On Linear Transformations, *Cambridge and Dublin Mathematical Journal*, 1 (1846), pp. 104–122; or [Cayley 1889–1898, 1: 95-112].

[1851] Note sur la théorie des hyperdéterminants, *Journal für die reine und angewandte Mathematik*, 42 (1851), pp. 368–371; or [Cayley 1889–1898, 1: 577-580].

[1852] On the Theory of Permutants, *Cambridge and Dublin Mathematical Journal*, 7 (1852), pp. 40–51; or [Cayley 1889–1898, 2: 16-26].

[1854] An Introductory Memoir Upon Quantics, *Philosophical Transactions of the Royal Society of London*, 144 (1854), pp. 245–258; or [Cayley 1889–1898, 2: 221-234].

[1856] A Second Memoir on Quantics, *Philosophical Transactions of the Royal Society of London*, 146 (1856), pp. 101–126; or [Cayley 1889–1898, 2: 250-275].

[1889–1898] *The Collected Mathematical Papers of Arthur Cayley*, 14 vols., Cambridge: University Press, 1889–1898.

Crilly (Tony)

[1981] *The Mathematics of Arthur Cayley with Particular Reference to Linear Algebra*, Ph.D. Thesis, Middlesex Polytechnic, 1981.

[1986] The Rise of Cayley's Invariant Theory (1841–1862), *Historia Mathematica*, 13 (1986), pp. 241–254.

[1988] The Decline of Cayley's Invariant Theory (1863–1895), *Historia Mathematica*, 15 (1988), pp. 332–347.

[2004] 'The *Cambridge Mathematical Journal* and Its Descendants: The Linchpin of a Research Community in the Early and Mid-Victorian Age, *Historia Mathematica*, 31 (2004), pp. 455–497.

[2006] *Arthur Cayley: Mathematician Laureate of the Victorian Age*, Baltimore: Johns Hopkins University Press, 2006.

Crilly (Tony), Weintraub (Steven) & Wolfson (Paul)

[2017] Arthur Cayley, Robert Harley and the Quintic Equation: Newly Discovered Letters 1859–1863, *Historia Mathematica*, 44 (2017), pp. 150–169.

Despeaux (Sloan)

[2002] *The Development of a British Publication Community: Nineteenth-Century Mathematics in British Scientific Journals*, Ph.D. Thesis, University of Virginia, 2002.

Dickson (Leonard E.)

[1921–1923] *History of the Theory of Numbers*, Washington, D. C.: 3 vols. : Carnegie Institution of Washington, 1921-1923. Providence: AMS Chelsea Publishing, Reprint edition, 1921–1923.

EISENSTEIN (Gotthold)
[1844] Théorèmes sur les formes cubiques et solution d'une équation du quatrième degré à quatre indéterminées, *Journal für die reine und angewandte Mathematik*, 27 (1844), pp. 75–79.

FISCHER (Charles S.)
[1966] The Death of a Mathematical Theory: A Study in the Sociology of Knowledge, *Archive for History of Exact Sciences*, 3 (1966), pp. 137–159.
[1967] 'The Last Invariant Theorists: A Sociological Study of the Collective Biographies of Mathematical Specialists, *Archives européennes de sociologie*, 8 (1967), pp. 216–244.

FREUDENTHAL (Hans)
[1970–1990] Entry on "Hermite, Charles", in Gillispie (Charles C.), ed., *Dictionary of Scientific Biography*, New York: Charles Scribner's Sons, 1970–1990.

GAUSS (Carl Friedrich)
[1966] *Disquisitiones arithmeticae*, Trans. Arthur A. Clarke, New Haven: Yale University Press, 1966.

GEISON (Gerald L.), ed.
[1984] *Professions and the French State, 1700–1900*, Philadelphia: University of Pennsylvania Press, 1984.

GOLDSTEIN (Catherine)
[2007] The Hermitian Form of Reading the *Disquisitiones*, in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007, pp. 377–410.
[2011a] Un arithméticien contre l'arithmétisation: les principes de Charles Hermite, in Flament (Dominque) & Nabonnand (Philippe), eds., *Justifier en mathématiques*, Paris: Éditions de la Maison des sciences de l'homme, 2011, pp. 129–165.
[2011b] Charles Hermite's Stroll through the Galois fields, *Revue d'histoire des mathématiques*, 17 (2011), pp. 211–270.
[2016] 'Découvrir des principes en classant': La classification des formes quadratiques selon Charles Hermite., in Lê (François) & Paumier (Anne-Sandrine), eds., *La classification comme practique scientifique*, 2016, pp. 103–135; Cahiers François Viète, Ser. III. No. 1.

GOLDSTEIN (Catherine) & SCHAPPACHER (Norbert)
[2007a] A Book in Search of a Discipline (1801–1860), in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007, pp. 3–65.
[2007b] Several Disciplines and a Book (1860–1901), in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007, pp. 67–103.

GOLDSTEIN (Catherine), SCHAPPACHER (Norbert) & SCHWERMER (Joachim), eds.
[2007] *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007.

HAWKINS (Thomas)
[1987] Cayley's Counting Problem and the Representation of Lie Algebras, in *Proceedings of the International Congress of Mathematicians—Berkeley* 2 vols., N.p. Amer. Math. Soc., 1987, pp. 1642–1656.

HERMITE (Charles)
[1848] Note sur la réduction des fonctions homogènes à coefficients entiers et à deux indéterminées, *Journal für die reine und angewandte Mathematik*, 36 (1848), pp. 357–364; or [Hermite 1905–1917, 1: 84-93].
[1850] Extraits de lettres de M. Ch. Hermite à M. Jacobi sur différents objets de la théorie des nombres, *Journal für die reine und angewandte Mathematik*, 40 (1850), pp. 261–315; [Hermite 1905–1917, 1: 100-163].
[1853] Remarques sur le théorème de M. Sturm, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*, 36 (1853), pp. 294–297; or [Hermite 1905–1917, 1: 284-287].
[1854] Sur la théorie des fonctions homogènes à deux indéterminées, *Cambridge and Dublin Mathematical Journal*, 9 (1854), pp. 172–217; or [Hermite 1905–1917, 1: 296-349].
[1856a] Sur la théorie des fonctions homogènes à deux indéterminées: Premier mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 1–17; or [Hermite 1905–1917, 1: 350-371].
[1856b] Sur la théorie des fonctions homogènes à deux indéterminées: Second mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 18–38; or [Hermite 1905–1917, 1: 372-396].
[1858] Sur la résolution de l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*, 46 (1858), pp. 508–515; or [Hermite 1905–1917, 2: 5-12].
[1865–1866] Sur l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*, 61 (1865), 877–882, 965–972, 1073–1081 and 62 (1866), 65–72, 157–162, 245–252, 715–722, 919–924, 959–966, 1054–1059, 1161–1167, 1213–1215 (1865–1866), pp. 877–882; or [Hermite 1905–1917, 2: 347–424].
[1905–1917] *Œuvres de Charles Hermite*, Paris: Gauthier-Villars, 1905–1917; 4 vols.

HESSE (Otto)
[1844] Über die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln, *Journal für die reine und angewandte Mathematik*, 28 (1844), pp. 68–96.

HILBERT (David)
[1890] Über die Theorie der algebraischen Formen, *Mathematische Annalen*, 36 (1890), pp. 473–534.
[1893] Über die vollen Invariantensysteme, *Mathematische Annalen*, 42 (1893), pp. 313–373.

MacHale (Desmond)
[1985] *George Boole: His Life and Work*, Dublin: Boole Press Limited, 1985.

Parshall (Karen Hunger)
[1989] Toward a History of Nineteenth-Century Invariant Theory, in *The History of Modern Mathematics*, vol. 1, Boston: Academic Press, 1989, pp. 157–206.
[1990] The One-Hundredth Anniversary of the Death of Invariant Theory?, *The Mathematical Intelligencer*, 12(4) (1990), pp. 10–16.
[1998] *James Joseph Sylvester: Life and Work in Letters*, Oxford: Clarendon Press, 1998.
[2006] *James Joseph Sylvester: Jewish Mathematician in a Victorian World*, Baltimore: Johns Hopkins University Press, 2006.

Parshall (Karen Hunger) & Rice (Adrian), eds.
[2002] *Mathematics Unbound: The Evolution of an International Mathematical Research Community 1800–1945*, HMATH, vol. 23, Providence: Amer. Math. Soc. and London: London Mathematical Society, 2002.

Parshall (Karen Hunger) & Seneta (Eugene)
[1997] Building an International Mathematical Reputation: The Case of J. J. Sylvester (1814–1897), *The American Mathematical Monthly*, 104 (1997), pp. 210–222.

Picard (Émile)
[1905] Préface in [Hermite 1905–1917, 1: vii-xl], 1905.

Reader (William J.)
[1966] *Professional Men: The Rise of the Professional Classes in Nineteenth-Century England*, London and New York: Weidenfield and Nicolson and Basic Books, 1966.

Salmon (George)
[1859] *Lessons Introductory to the Modern Higher Algebra*, 1859; 5th edition, Dublin: Hodges, Figgis, and Co., 1885; Reprint Ed. Bronx: Chelsea Publishing Co., 1964.

Sinaceur (Hourya)
[1991] *Corps et modèles: Essai sur l'histoire de l'algèbre réelle*, Paris: Librairie philosophique J. Vrin, 1991.

Sylvester (James Joseph)
[1839] On Rational Derivation from Equations of Coexistence, That Is to Say, a New and Extended Theory of Elimination. Part I, *Philosophical Magazine*, 15 (1839), pp. 428–435; or [Sylvester 1904-1912, 1: 40-46].
[1847a] An Account of a Discovery in the Theory of Numbers Relative to the Equation $Ax^3 + By^3 + Cz^3 = Dxyz$, *Philosophical Magazine*, 31 (1847), pp. 189–191; [Sylvester 1904-1912, 1: 107-109].
[1847b] On the equation in numbers $Ax^3 + By^3 + Cz^3 + Dxyz$, and Its Associate System of Equations, *Philosophical Magazine*, 31 (1847), pp. 293–296; or [Sylvester 1904-1912, 1: 110-113].

[1847c] On the General Solution (in Certain Cases) of the Equation $x^3 + y^3 + Az^3 = Mxyz$, &c., *Philosophical Magazine*, 31 (1847), pp. 467–471; or [Sylvester 1904-1912, 1: 114-118].
[1851a] On a Certain Fundamental Theorem of Determinants, *Philosophical Magazine*, 2 (1851), pp. 142–145; or [Sylvester 1904-1912, 1: 252-255].
[1851b] On the General Theory of Associated Algebraical Forms, *Cambridge and Dublin Mathematical Journal*, 6 (1851), pp. 289–293; or [Sylvester 1904-1912, 1: pp. 198-202].
[1851c] On the Relation between the Minor Determinants of Linear Equivalent Quadratic Functions, *Philosophical Magazine*, 1 (1851), pp. 295–305; or [Sylvester 1904-1912, 1: 241-250].
[1851d] On a Remarkable Discovery in the Theory of Canonical Forms and of Hyperdeterminants, *Philosophical Magazine*, 2 (1851), pp. 391–410; or [Sylvester 1904-1912, 1: 265-283].
[1851e] Sketch of a Memoir on Elimination, Transformation, and Canonical Forms, *Cambridge and Dublin Mathematical Journal*, 6 (1851), pp. 186–200; or [Sylvester 1904-1912, 1: 184-197].
[1852] On the Principles of the Calculus of Forms, *Cambridge and Dublin Mathematical Journal*, 7 (1852), pp. 52–97, 179-217, and 8 (1852), pp. 256–269; or [Sylvester 1904-1912, 1: pp. 284-327, 328-363, and 411-422].
[1853a] Note on the Calculus of Forms, *Cambridge and Dublin Mathematical Journal*, 8 (1853), pp. 62–64; or [Sylvester 1904-1912, 1: pp. 402-403].
[1853b] On a Theory of the Syzygetic Relations of Two Rational Integral Functions, Comprising an Application to the Theory of Sturm's Functions, and That of the Greatest Algebraical Common Measure, *Philosophical Transactions of the Royal Society of London*, 143 (1853), pp. 407–548; or [Sylvester 1904-1912, 1: pp. 429-586].
[1864] Algebraical Researches, Containing a Disquisition on Newton's Rule for the Discovery of Imaginary Roots, and an Allied Rule Applicable to a Particular Class of Equations, Together with a Complete Invariantive Determination of the Character of the Roots of the General Equation of the Fifth Degree, &c., *Philosophical Transactions of the Royal Society of London*, 154 (1864), pp. 579–666; or [Sylvester 1904-1912, 2: pp. 376-479].
[1904-1912] *The Collected Mathematical Papers of James Joseph Sylvester*, Cambridge: Cambridge Univ. Press, 1904-1912; 4 vols., Reprint Ed. New York: Chelsea Publishing Co, 1973.

WOLFSON (Paul)
[2008] George Boole and the Origins of Invariant Theory, *Historia Mathematica*, 35 (2008), pp. 37–46.

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LES TRAVAUX DE LAGUERRE SUR LES ÉQUATIONS POLYNOMIALES ET L'INFLUENCE HERMITIENNE

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Résumé. — Il existe un certain nombre de points communs entre Edmond Laguerre et Charles Hermite, deux mathématiciens français du XIX^e siècle, tant du point de vue de leurs parcours au sein des institutions mathématiques du XIX^e siècle que de leurs travaux. Ils se sont tous les deux intéressés au sujet des équations polynomiales et nous décrivons dans cet article le rôle qu'a pu jouer Hermite dans les travaux de Laguerre. Nous nous attachons ainsi à décrire l'ensemble des résultats publiés par Laguerre sur le sujet tout en pointant l'importance tant quantitative que qualitative de Hermite. Une dernière partie permettra de saisir les points communs et les similarités entre les approches de Hermite et de Laguerre. Plus généralement, il s'agira de montrer qu'ils ont des conceptions similaires tant sur la question de la généralité en mathématiques que sur les interactions entre algèbre et analyse par exemple.

Abstract. — Edmond Laguerre and Charles Hermite are both French mathematicians of the 19th century. They shared a common professional background in various institutions and the way they did mathematics. They were both interested in the subject of polynomial equations and we shall see in this article what role Hermite played in Laguerre's works. We describe how the results published by Laguerre emphasized the importance of Hermite. A last part deals with common points and similarities between the approaches of Laguerre and

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Hermite. More globally, we show that they had comparable conceptions of various issues: about the question of generality in mathematics and about interactions between calculus and algebra.

INTRODUCTION

Pour bon nombre de mathématiciennes et de mathématiciens aujourd'hui, un des éléments rapprochant Charles Hermite d'Edmond Laguerre est sans doute qu'ils ont tous les deux donné leur nom à une famille de polynômes. Ceux de Hermite vérifient l'équation différentielle $y'' - xy' + ny = 0$ tandis que ceux de Laguerre vérifient l'équation différentielle $xy'' + (1 - x)y' + ny = 0$. Ils ont en fait bien d'autres points communs. Tous deux ont intégré l'École polytechnique, en tant qu'élèves, à seulement un peu plus d'une dizaine d'années d'intervalle[1]. Hermite y a été répétiteur de 1848 à 1853 puis professeur de 1869 à 1876 et Laguerre y a été répétiteur de 1864 à 1886. Nous savons qu'il pouvait arriver à Laguerre de remplacer Hermite à l'amphithéatre en cas d'absence[2] et qu'Hermite reprenait des démonstrations de Laguerre dans son propre cours[3]. Ils se rencontraient aussi dans le cadre de la Société mathématique de France dont ils étaient tous les deux membres [Hermite & Mittag-Leffler 1984, p. 141].

Tous les deux étaient également reconnus par le monde académique de leur temps en devenant, par exemple, membres de l'Académie des Sciences dans la section Géométrie. Hermite est ainsi entré à l'Académie des Sciences en 1856 à l'âge de 34 ans, rejoint plus tard par Laguerre en 1885 à l'âge de 51 ans. Entre 1856 et 1885, un grand nombre de notes publiées par Laguerre dans les *Comptes rendus de l'Académie des sciences* ont été présentées en séance par Hermite[4]. Hermite a d'ailleurs soutenu la

1 Hermite fait partie de la promotion 1842 et Laguerre de la promotion 1853 de l'École polytechnique.

2 Poincaré explique par exemple dans une lettre envoyée à sa mère : « Une fois Hermite était malade et Laguerre nous faisant l'amphi nous fit une certaine question. Mais comme il écrit très mal à la planche, je n'avais pu prendre de notes ». [Rollet 2017, p. 84].

3 Plus précisément, Stieltjes écrit à Hermite dans une lettre datée du 2 juillet 1889 : « il semble qu'il doit être plus facile d'obtenir les théorèmes de Fuchs que les développements en séries, c'est aussi ce qui résulte des démonstrations de Laguerre et de Goursat que vous donnez dans votre Cours » [Hermite & Stieltjes 1898, p. 292].

4 Chaque note publiée dans les *Comptes rendus de l'Académie des sciences* devait être présentée par un membre de l'Académie des Sciences. S'il n'était pas nécessaire de connaître un académicien pour publier, la proximité entre Hermite et Laguerre peut

candidature de Laguerre à l'Académie. Il indique par exemple dans une lettre à Gosta Mittag-Leffler datée de 1881 avoir « l'intention de proposer à la section de Géométrie de mettre en seconde ligne et seul [derrière Darboux], Mr Laguerre qui a beaucoup grandi depuis quelques temps par les recherches algébriques » [Hermite & Mittag-Leffler 1984, p. 110].

Cette même correspondance témoigne de la proximité personnelle entre les deux mathématiciens, lorsqu'Hermite écrit qu'il est « en [rapports d'amitié] avec Mr Laguerre qui est un excellent homme, et aussi un géomètre du plus rare mérite » [Hermite & Mittag-Leffler 1985, p. 105] puis que « la mort de Mr Laguerre, avec qui [il] était lié affectueusement depuis longtemps [le] prive malheureusement d'un appui important » [Hermite & Mittag-Leffler 1985, p. 128]. Quelques années plus tard, Hermite sera d'ailleurs l'un des éditeurs des *Œuvres* de Laguerre. Réciproquement, Eugène Rouché, camarade de Laguerre à l'École polytechnique, note que Laguerre « professait une si légitime admiration » pour Hermite [Rouché 1887, p. 148].

A la lecture des *Œuvres* de Laguerre, outre le fait qu'Hermite est l'un des éditeurs, on constate que le nom de Hermite apparaît très souvent. Nous reviendrons sur le fait que c'est même le nom le plus cité par Laguerre dans ses travaux. Cela traduit là aussi une certaine proximité, Laguerre faisant par exemple mention de discussions avec Hermite. Dans [Laguerre 1882a], il écrit par exemple :

> M. Hermite m'a dit tenir de M. Genocchi que la méthode généralement attribuée à Plana appartient en réalité à F. Chio. [Laguerre 1882a, p. 161]

De même, dans [Laguerre 1882b], Laguerre écrit :

> Je tiens de M. Hermite, à qui j'avais communiqué ces résultats, qu'il les avait obtenu de son côté et par la même voie. [Laguerre 1882b, p. 636]

Ces quelques éléments illustrent non seulement la présence de Hermite dans les travaux de Laguerre par le biais de citations mais également la collaboration et les échanges directs entre les deux mathématiciens.

On peut toutefois se demander pourquoi Laguerre cite autant Hermite alors que leurs travaux portent globalement sur des domaines assez différents. Laguerre a en effet publié majoritairement des articles en Géométrie tandis que ce sujet n'est que peu traité par Hermite. Inversement, Hermite s'est davantage intéressé à la théorie des nombres, sujet fort peu exploré

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toutefois expliquer que le nom de Hermite revienne souvent parmi ceux qui présentent les notes de Laguerre avant sa propre élection [Crosland 1992, p. 288-293].

par Laguerre. En fait, dans les travaux de Laguerre, les références à Hermite sont surtout importantes à propos d'un sujet spécifique, celui des équations polynomiales[5]. Cela n'est d'ailleurs pas étranger au fait qu'ils ont fréquenté le même cadre scolaire. La résolution pratique des équations et l'approximation des racines avaient en effet une place importante dans la formation mathématique de l'École polytechnique et des classes préparatoires. On peut notamment citer les théorèmes de Hermite et de Laguerre qui portent sur les équations numériques, c'est-à-dire sur les équations résolues de manière approchée. Laguerre s'est beaucoup intéressé à ce sujet à partir des années 1870 lorsqu'il est devenu répétiteur à l'École polytechnique et examinateur du concours d'admission. De son côté, Hermite a su trouver et démontrer un résultat concernant les racines des équations dont les coefficients forment une progression arithmétique. Il l'a fait alors qu'il était encore élève en classes préparatoires. Nous aurons l'occasion d'y revenir lorsque nous aborderons la question des points communs entre les deux mathématiciens mais il est clair que la thématique des équations polynomiales est un sujet qui a intéressé tant Hermite que Laguerre. Notre objectif sera alors de comprendre pourquoi Laguerre fait autant de références à Hermite sur ce sujet. Il s'agira, d'une part, de présenter les différents articles de Laguerre au sujet des équations et, d'autre part, de capter l'influence et l'importance de Hermite sur ces travaux.

Les deux premières parties de notre article sont essentiellement factuelles. La première présente un panorama général des travaux de Laguerre et revient sur l'importance quantitative de Hermite dans ces travaux. La seconde partie apporte des détails sur les résultats dûs à Laguerre au sujet des équations numériques et explicite l'objet des références à Hermite. La dernière partie vise enfin à présenter les points de convergence entre les deux mathématiciens, tant dans leurs objets d'étude que dans leur conception des mathématiques. L'exemple des équations numériques permettra notamment d'illustrer cette convergence de point de vue.

1. PANORAMA GÉNÉRAL DES TRAVAUX DE LAGUERRE ET DE L'INFLUENCE HERMITIENNE

1.1. *Les mathématiques de Laguerre*

Publiées en 1898 par Charles Hermite, Henri Poincaré et Eugène Rouché, les *Œuvres* de Laguerre contiennent 148 articles, classés en trois

[5] Les deux mathématiciens ont aussi tous les deux travaillé sur les fonctions elliptiques et Laguerre a régulièrement cité Hermite à ce sujet (les articles de Laguerre sur la questions sont classés dans la rubrique Calcul intégral de ses *Œuvres*). Nous n'aborderons néanmoins pas cette thématique ici.

parties : 45 dans la rubrique « Algèbre », 21 dans celle de « Calcul intégral » et 82 en « Géométrie ». À cela, nous pouvons aussi ajouter un article de Charles Hermite intitulé « Sur un mémoire de Laguerre concernant les équations algébriques » et inséré à la fin du premier volume des *Œuvres*. Cette présence de Hermite dans les *Œuvres* de Laguerre renforce encore un peu plus l'impression de proximité évoquée en introduction. Cela signifie en outre que si Laguerre a été influencé par les mathématiques de Hermite, ce dernier s'est, réciproquement, intéressé aux travaux du premier. Deux articles de Laguerre sont d'ailleurs cités à plusieurs reprises par Hermite, à savoir [Laguerre 1877] et [Laguerre 1880b], en particulier au moment de l'édition des *Œuvres*.

Dans le cadre de cet article, et sauf mention explicite du contraire, les termes Algèbre, Calcul intégral et Géométrie désigneront les trois parties des *Œuvres*. Les travaux de Laguerre en Géométrie ont globalement précédé ceux d'Algèbre et de Calcul intégral[6]. Après avoir rédigé quelques articles alors qu'il était encore élève (ces articles portent sur la théorie des foyers), Laguerrre a recommencé à publier en 1865. Entre temps, de 1853 à 1864, il s'est écoulé une décennie où, engagé dans l'armée, il ne publie aucun texte mathématique. En 1864, il obtint alors un poste de répétiteur de Géométrie descriptive à l'École polytechnique, moment qui correspond à une reprise de ses publications. Dans un premier temps, jusqu'en 1874, ses articles sont en fait très majoritairement classés en Géométrie et ne seront donc que peu utiles dans le cadre de notre étude sur les équations polynomiales[7]. À partir de 1874, Laguerre s'est en revanche d'avantage intéressé à des sujets d'Algèbre et de Calcul intégral tout en continuant à publier des articles de Géométrie pendant quelques années. C'est ainsi qu'en 1874, il a publié son premier article sur les équations numériques dans les *Comptes rendus de l'Académie des sciences* : « Sur une formule nouvelle permettant d'obtenir, par approximations successives, les racines d'une équation dont toutes les racines sont réelles ».

À partir de là, Laguerre a publié régulièrement des articles dont le titre comporte les mots « équations numériques », ce sujet constituant alors une sorte de fil rouge dans ses travaux. Autour de ce fil rouge, il est de plus possible de distinguer des périodes de trois ou quatre années durant lesquelles Laguerre s'est consacré à un sujet plus particulier. Par exemple, à la fin des années 1870, entre 1877 et 1880, il a publié plusieurs articles

6 Pour plus d'informations concernant les travaux de Laguerre, voir [Vincent 2019].

7 De 1852 à 1875, 51 articles sur 62 ont été classés dans la rubrique Géométrie des *Œuvres*.

sur les développements des fonctions en séries ou en fractions continues. D'ailleurs, l'idée que ce groupe d'articles constitue un sujet de recherche à part est confirmé par le fait que les éditeurs des *Œuvres* ont choisi de tous les rassembler à la fin de la rubrique Algèbre. Ensuite, de 1880 à 1884, là où Laguerre a rédigé essentiellement des articles d'Algèbre, on retrouve de nombreux articles faisant référence aux équations transcendantes. À la fin de sa vie, de 1880 jusqu'à sa mort en 1886, il s'est en tout cas principalement consacré à des sujets d'Algèbre et n'a plus guère publié en Géométrie[8].

1.2. *L'importance quantitative de Hermite*

Le nom de Hermite est celui qui est le plus fréquemment cité par Laguerre. On dénombre en effet, dans l'ensemble des *Œuvres* de Laguerre pas moins de 32 occurences du nom « Hermite ». Parfois, il s'agit de références assez vagues à un résultat ou à une méthode attribué à Hermite mais bien souvent, Laguerre donne aussi la référence précise d'un article ou d'un ouvrage. Il est d'ailleurs fait mention de Hermite dans les titres de deux articles de Laguerre : « Sur le calcul des systèmes linéaires, extrait d'une lettre adressée à Charles Hermite » [Laguerre 1867] et « Sur quelques théorèmes de M. Hermite, lettres à Borchardt » [Laguerre 1880c].

Jusqu'au début des années 1870, le nom de Hermite apparaît en fait très peu dans les articles de Laguerre. On retrouve par exemple bien plus souvent des citations de travaux de Chasles, de Steiner, de Clebsch ou de Mannheim. Ceci s'explique avant tout par les sujets auxquels s'intéresse Laguerre, portant davantage sur la Géométrie alors qu'Hermite a relativement peu publié sur ce sujet. Ainsi, avant 1870, seules trois articles de Laguerre citent explicitement Hermite. La première référence apparaît dans [Laguerre 1867]. Plus précisément, dans cet article, Laguerre cite un article de Hermite de 1843 « Sur la division des fonctions abeliennes ou ultra elliptiques » [Hermite 1843] ainsi qu'un article de 1854 « Sur la théorie des formes quadratiques » [Hermite 1854]. Par la suite, toutes les références à des articles de Hermite seront faites entre 1878 et 1882. La majorité d'entre elles réfèrent d'ailleurs aux quatre articles suivants, qui sont donc cités plusieurs fois :

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8 Sur les 37 articles publiés par Laguerre à cette période, 26 sont effectivement classés dans la rubrique Algèbre par les éditeurs des *Œuvres*. Seuls quelques articles de Géométrie, portant en particulier sur la géométrie de direction, font exception.

Table 1. Nombre de références à des mathématiciens dans les volumes 1 et 2 des *Œuvres* de Laguerre

Références du Volume 1	Nombre d'occurences
Hermite	32
Jacobi	28
Descartes	18
Sturm	9
Cauchy	7
Rolle	7
Liouville	5
Weierstrass	3
Biehler	2
Eisenstein	2
Liouville	1
Jonquières	1
Sylvester	1
Références du Volume 2	**Nombre d'occurences**
Chasles	19
Steiner	15
Joachimsthal	11
Cayley	9
Darboux	8
Mannheim	5
Liouville	3
Sturm	1

– « Sur un nouveau développement en série des fonctions », 1864, *Comptes rendus hebdomadaires des séances de l'Académie des sciences* [Hermite 1864].

– « Extrait d'une lettre à Mr Borchardt (sur quelques approximations algébriques) », 1873, *Journal für die reine und angewandte Mathematik* [Hermite 1873a].

– « Sur la fonction exponentielle », 1873, *Comptes rendus hebdomadaires des séances de l'Académie des sciences* [Hermite 1873b].

– « Extrait d'une lettre à M. Fuchs de Gottingue sur quelques équations différentielles linéaires », 1875, *Journal für die reine und angewandte Mathematik* [Hermite 1875].

Quelques autres articles de Hermite sont cités en revanche une seule fois par Laguerre. Là aussi, il s'agit en tout cas de citations datant du début des années 1880.

Finalement, la grande majorité des interactions avec Hermite s'est faite au sujet de l'Algèbre et du Calcul intégral. Il semblait ainsi raisonnable de s'intéresser à ces sujets et plus particulièrement aux équations polynomiales où le nombre de références à Hermite est effectivement important.

Réciproquement, et sans rentrer dans les détails, on peut ajouter que Laguerre est régulièrement cité par Hermite. Il s'agit d'ailleurs d'un des rares auteurs nouveaux cités sur plusieurs décennies par Hermite après 1860, ce qui s'explique principalement par le fait qu'Hermite ait relu les articles de Laguerre au moment de la publication des *Œuvres* [Goldstein 2012, p. 533].

La seconde partie de cet article visera à décrire plus en détails les travaux de Laguerre. Il existe toutefois une difficulté :

> Si la variété des sujets rend les Mémoires difficiles à classer, la concision avec laquelle il les rédigeait les rend encore plus impropre à l'analyse ; et, pour mettre son oeuvre dans tout son jour, il faudrait en reproduire la majeure partie. [Rouché 1887, p. 106]

Pour cette raison, et plûtot que de décrire les *Œuvres* de Laguerre linéairement, article après article, nous avons préféré utiliser des synthèses rédigés par deux des éditeurs des *Œuvres* : Henri Poincaré et Eugène Rouché[9]. Ces documents nous permettront d'avoir une première approche des travaux de Laguerre sur les équations et d'en saisir les points importants.

2. LES TRAVAUX DE LAGUERRE SUR LES ÉQUATIONS ET L'OBJET DES RÉFÉRENCES À HERMITE

2.1. *Le point de vue de deux des éditeurs des Œuvres : Henri Poincaré et Eugène Rouché*

2.1.1. *La structure des exposés de Poincaré et de Rouché*

Deux documents rédigés par les éditeurs des *Œuvres* synthétisent et présentent les travaux de Laguerre dans leur globalité. Le premier est signé Henri Poincaré : il s'agit de la préface des *Œuvres* [Laguerre 1898]. Le second, d'Eugène Rouché, est paru dans les *Nouvelles Annales* en 1887, un an après la mort de Laguerre [Rouché 1887].

La préface de Poincaré comporte onze pages et traite essentiellement de mathématiques. Elle balaie l'ensemble des sujets abordés par Laguerre au cours de sa carrière. Il commence par parler de la Géométrie des substitutions linéaires, qu'il rattache à l'étude des formes quadratiques et des fonctions abéliennes. Ensuite, il cite les idées générales de Laguerre sur

9 Malheuresement il n'existe pas de telle synthèse rédigée par Charles Hermite luimême.

la théorie des équations différentielles avec la notion d'invariants avant de s'attarder un peu plus longuement sur la théorie des équations numériques. Enfin, il termine sa présentation par un paragraphe sur l'étude des fractions continues et leur utilisation pour approcher des fonctions.

L'article de Rouché est quant à lui plus long, composé de 69 pages. Après avoir présenté succinctement la vie de Laguerre, Rouché passe à la description détaillée de ses travaux. Rouché divise son exposé en sept parties :

– Emploi des imaginaires en Géométrie.
– Application du Calcul intégral à la théorie des formes en Géométrie.
– Géométrie infinitésimale.
– Géométrie de direction.
– Méthodes d'approximation pour certaines fonctions analytiques.
– Résolution numérique des équations.
– Équations différentielles et fonctions elliptiques.

On observe une similitude dans la façon dont Poincaré et Rouché ont choisi de diviser leur exposé. En effet, ils distinguent quatre grandes parties liées à la Géométrie auxquelles ils ajoutent une concernant les équations numériques, une sur les approximations de fonctions avec les fractions continues notamment et une sur les équations différentielles. Ils proposent ainsi un classement des travaux de Laguerre plus détaillé que la simple division des *Œuvres* en trois parties (Algèbre, Calcul Intégral et Géométrie) qui ne permettait que peu de nuances.

Si cette différence pouvait s'expliquer par la nature fondamentalement différente des deux classements (celui permettant d'éditer des œuvres complètes et celui permettant de présenter des idées générales sur les travaux d'une vie), elle n'en reste pas moins intéressante à souligner. Cela permet par exemple à Rouché de regrouper l'ensemble des articles sur les fractions continues dans une même section, distincte des travaux sur les équations numériques et distincte également de ceux sur les équations différentielle. Les éditeurs des *Œuvres* ont au contraire regroupé certaines de ces références dans la rubrique Algèbre alors que d'autres étaient placés dans la rubrique Calcul intégral. Cela vient en tout cas conforter l'idée que le sujet des fractions continues pourrait constituer une thématique relativement autonome et faisant un lien entre des sujets d'Algèbre et de Calcul intégral dans les travaux de Laguerre.

2.1.2. *Les équations numériques*

Poincaré et Rouché semblent considérer que la résolution des équations numériques constitue une part importante des travaux de Laguerre.

Rouché introduit la partie sur les équations numériques en disant qu'il s'agit de « la plus considérable de son œuvre, et peut-être celle à laquelle il attachait le plus de prix » [Rouché 1887, p. 153], quand Poincaré la présente comme « la partie la plus remarquable de [son] œuvre » [Laguerre 1898, preface, p. xii]. Ajoutons que le sujet des équations numériques n'est pas seulement important pour Laguerre lui-même mais est au cœur du système d'enseignement à l'École polytechnique au XIX^{e} siècle. Et s'il est enseigné de manière assez routinière en général, il donne aussi l'occasion de recherches nouvelles qui peuvent être intégrées dans les exercices ou les cours. On verra notamment qu'un théorème attribué à Laguerre au sujet des équations numériques pouvait être utilisé par les candidats lors des examens du concours d'admission et a fait l'objet de publications de professeur et d'élèves dans des journaux comme les *Nouvelles annales de mathématiques.* Chose intéressante : Hermite a lui aussi démontré un résultat sur les équations numériques dans un cadre proprement scolaire, alors qu'il était encore élève de classe préparatoire. Là aussi, la découverte du résultat a été suivie d'un certain nombre de publications dans les *Nouvelles annales de mathématiques.* Si les deux résultats ne sont pas directement liés, c'est en tout cas un point commun remarquable entre les deux hommes sur lequel nous reviendrons.

Les deux éditeurs, Rouché et Poincaré, semblent en outre s'entendre sur le fait que Laguerre n'a pas complètement achevé ses travaux sur le sujet. Poincaré parle « d'une méthode ingénieuse pour séparer et calculer les racines imaginaires, mais dont Laguerre n'a pas eu le temps de tirer toutes les conséquences » quand Rouché explique que le mémoire [Laguerre 1883] paru dans le *Journal de mathématiques pures et appliquées* était censé former les premiers chapitres d'un ouvrage plus général sur la question des équations numériques mais dont la rédaction est restée inachevée[10] : « Il se proposait avant que la mort vînt le surprendre, de coordonner ces recherches et de les réunir en un Volume qui en eut renfermé l'exposition complète. »

Rouché explique que Laguerre avait prévu d'organiser son ouvrage en trois parties. La première concernait la généralisation du théorème de Descartes et ses applications. C'est justement l'objet du mémoire [Laguerre 1883] qui synthétise l'ensemble des travaux de Laguerre sur le sujet et qui reprend le contenu de publications antérieures parues dans les *Nouvelles annales de mathématiques* et dans les *Comptes rendus de l'Académie*

10 On pourra consulter [Vincent 2019, p. 263-279] pour une étude détaillée de ce mémoire et [Vincent 2022] pour un exposé « moderne » des résultats de Laguerre sur les équations numériques.

des sciences, en particulier [Laguerre 1879a ; 1880a ; 1881a] et [Laguerre 1881b]. La seconde partie présentait des méthodes d'approximations des racines d'une équation reprenant une partie du mémoire [Laguerre 1883] mais présentant également des résultats supplémentaires développés dans [Laguerre 1880b] et [Laguerre 1880h]. La troisième partie enfin était consacrée à la recherche des racines imaginaires. Moins développée que les deux premières parties, Laguerre avait tout de même publié quelques articles sur le sujet, notamment [Laguerre 1874c], [Laguerre 1878a] et [Laguerre 1880g].

Rouché décide ainsi de structurer son exposé en reprenant successivement les travaux de Laguerre dans ces trois parties. Il est intéressant de noter que Poincaré semble avoir retenu une structuration similaire pour présenter sa préface. Nuance tout de même avec la présentation de Rouché, il termine sa présentation sur les équations en présentant la résolution des équations transcendantes alors que Rouché incluait cette question dans la première partie.

Le théorème de Descartes[11] *et ses applications*

Concernant la généralisation du théorème de Descartes, Rouché indique qu'il « s'agit de la plus complète et Laguerre semble avoir dit son dernier mot sur le sujet ». Il commence par citer le théorème de Descartes sous la forme suivante : « $F(x)$ désignant un polynôme ordonné suivant les puissances décroissantes de x, le nombre des racines positives de l'équation $F(x) = 0$ est au plus égal au nombre des variations du polynôme $F(x)$ » (le nombre des variations d'un polynôme désigne ici le nombre de changements de signes dans la suite des coefficients de ce polynôme). Par exemple, pour le polynôme $x^5 + 4x^3 - x^2 + 3$ admet au plus deux racines positives étant donné que la suite des coefficients $(1, 0, 4, -1, 0, 3)$ présente deux changements de signes.

Rouché détaille ensuite en quelques lignes la nouvelle démonstration proposée par Laguerre « non seulement à cause de sa simplicité, mais surtout parce qu'on y trouve l'origine de l'extrême généralisation que Laguerre est parvenu à donner au théorème de Descartes ». Cette démonstration se fait par récurrence sur le nombre de variations du polynôme F et utilise principalement le théorème de Rolle. Plus précisément, Laguerre considère un polynôme F dont le nombre de variations est n et

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11 Le théorème de Descartes est aussi parfois appelé « règle des signes de Descartes ».

utilise, pour un réel α quelconque, l'équivalence suivante :

$$F(x) = 0 \Longleftrightarrow \frac{F(x)}{x-\alpha} = 0.$$

Pour un réel α bien choisi, il montre que la dérivée de $\frac{F(x)}{x-\alpha}$ admet au plus $n-1$ variations donc, par hypothèse de récurrence, que l'équation $\left(\frac{F(x)}{x-\alpha}\right)' = 0$ admet au plus $n-1$ solutions. D'après le théorème de Rolle, c'est que l'équation $\frac{F(x)}{x-\alpha} = 0$ admet au plus n solutions, ce qui est par conséquent également le cas de l'équation $F(x) = 0$ et permet à Laguerre de conclure la récurrence.

Dans cette démonstration, Laguerre ne suppose pas que l'équation soit polynomiale, utilisant principalement la continuité de F à travers le théorème de Rolle. Ainsi, cela signifie que l'énoncé proposé par Laguerre est donc identique à celui de la règle de Descartes si ce n'est que $F(x)$ peut désigner une série entière. Rouché énumère alors différents cas pour lesquels Laguerre peut appliquer son résultat généralisé : pour l'évaluation du nombre de solutions des équations de la forme

$$\frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{L}{x-l} = 0$$

(où les nombres $A, B, \ldots, L$, $a, b, \ldots, l$ sont des nombres réels), celles de la forme

$$\int_a^b e^{-zx}\Phi(z)dz = 0$$

(où a, b et x sont des nombres réels et d'après Laguerre, $\Phi(z)$ désigne une fonction entièrement arbitraire, continue ou discontinue d'une façon quelconque, pouvant par exemple être nulle dans autant d'intervalles qu'on veut » [Laguerre 1883][12]), celles de la forme

$$\int_a^b \frac{\Phi(z)}{(z-x)^n} dz = 0,$$

ou encore celles de la forme

$$A_1 F(\alpha_1 x) + \cdots + A_n F(\alpha_n x) = 0$$

(où pour tout $1 \leqslant i \leqslant n$, A_i et α_i sont des nombres réels et F est une série entière).

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12 En termes modernes, il faut bien sûr considérer en pratique que cette fonction Φ est intégrable.

Rouché cite ensuite le « cas où le premier membre de l'équation est exprimé linéairement au moyen des polynômes de Legendre, et plus généralement au moyen de polynômes entiers satisfaisant à certaines équations différentielles linéaires du second ordre ». Pour conclure cette partie sur le théorème de Descartes, Rouché parle du cas général où le premier membre de l'équation est une fonction entière de x.

Avec ces différentes applications, il apparaît donc qu'un des liens établi par Laguerre entre Algèbre et Calcul intégral provient de la généralisation du théorème de Descartes aux séries entières. Poincaré s'exprime d'ailleurs en des termes similaires en considérant que « le théorème de Descartes devient un instrument d'une flexibilité merveilleuse ; manié par Laguerre, il le conduit à des règles élégantes, bien plus simples que celles de Sturm et s'appliquant à des classes très étendues d'équations ».

Approximation des racines

Après cette première partie sur la règle de Descartes et ses applications, Rouché passe donc à la seconde : l'approximation des racines d'une équation algébrique ou transcendante. Il commence par rappeler les résultats classiques fournissant une limite supérieure des racines, notamment la « méthode de Newton[13] » qui consiste « à trouver une quantité qui rende positive la fonction $f(x)$ et ses dérivées successives ». Pointant les difficultés de calculer ces dérivées, il énonce alors le résultat de majoration des racines suivant, connu sous le nom de « théorème de Laguerre » au XIXe siècle et présentant justement l'intérêt de se calculer facilement :

> Considérant une équation
>
> $$f(x) = A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m = 0$$
>
> et forme la suite de polynômes suivants :
>
> $$\begin{aligned} f_m(x) &= A_0, \\ f_{m-1}(x) &= A_0x + A_1 \\ &\vdots \\ f_1(x) &= A_0x^{m-1} + \cdots + A_{m-1} \\ f(x) &= A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m. \end{aligned}$$
>
> Tout nombre positif a qui rend positifs les polynômes de la suite [ci-dessus], est une limite supérieure des racines de l'équation $\left[f(x) = 0\right]$.
>
> [Rouché 1887, p. 157]

13 Nous attirons l'attention sur le fait qu'il ne s'agit pas ici de la méthode algorithmique portant le même nom et permettant d'approximer le zéro d'une fonction.

La démonstration de ce résultat découle d'ailleurs de la règle des signes de Descartes généralisée, appliquée au développement en série de la fraction $\frac{f(x)}{x-a}$. Comme application de ce résultat, on peut par exemple citer le cas d'une équation présentée par Laguerre lui-même dans [Laguerre 1883] :

> En considérant l'équation
>
> $$f(x) = x^5 - 3x^3 + x^2 - 8x - 10 = 0,$$
>
> avec $a = 3$, on obtient $f_5(a) = 1$, $f_4(a) = 3$, $f_3(a) = 6$, $f_2(a) = 19$, $f_1(a) = 49$ et $f(a) = 137$. Cela signifie par conséquent qu'il n'existe pas de solution supérieure à 3.

Comme on le voit, cette règle fournissant une majoration de la plus grande des racines ne constitue pas une approximation des racines à proprement parler mais une sorte de préalable afin de localiser les racines et de pouvoir ensuite en proposer une approximation. C'est sans doute en le voyant comme une étape préliminaire que Rouché a décidé d'inclure ce résultat dans la partie « approximation des racines ». Après avoir présenté quelques variantes à cette première règle, il passe donc à la méthode d'approximation des racines en tant que telle. Il rappelle la méthode d'approximation de Newton qui donne, à condition d'avoir une valeur suffisamment approchée d'une racine, « un moyen commode et rapide d'approcher indéfiniement de cette racine ». Laguerre cherche un moyen d'établir une méthode qui marche à tous les coups, quel que soit le point de départ considéré. Rouché explicite le résultat établi par Laguerre, insistant sur le fait que cette méthode offre « l'avantage de n'être jamais en défaut, quelle que soit la valeur de départ α :

> En désignant par $f(x) = 0$ une équation de degré n dont toutes les racines sont réelles et par α une quantité arbitraire, les deux valeurs de x déterminées par l'équation
>
> $$(*) \qquad \frac{1}{x-\alpha} = -\frac{f'(\alpha)}{f(\alpha)} \pm \frac{1}{nf(\alpha)}\sqrt{(n-1)H(\alpha)}$$
>
> sont respectivement comprises entre α et les deux racines de l'équation proposée qui avoisinent x.
>
> Dans la formule [ci-dessus], $H(x)$ représente le Hessien de $f(x)$:
>
> $$H(x) = f'^2(x) - nf(x)f''(x). \text{ »}$$
>
> [Rouché 1887, p. 157]

Partant d'une valeur réelle quelconque, cela permet donc à Laguerre d'obtenir une approximation des racines de manière itérative. Dans sa préface, Poincaré n'est pas aussi précis que Rouché et se contente de présenter l'idée générale que Laguerre a mise en place. Il compare notamment la méthode de Laguerre avec celle de Newton :

> La méthode de Newton consiste à remplacer l'équation à résoudre par une équation du premier degré qui en diffère très peu ; Laguerre la remplace par une équation du deuxième degré qui en diffère moins encore. L'approximation est plus rapide ; de plus, la méthode n'est jamais en défaut au moins quand toutes les racines sont réelles. Le procédé nouveau est surtout avantageux quand le premier membre de l'équation est un des polynômes qui satisfont à une équation différentielle linéaire et dont le rôle analytique est si important.
> [Laguerre 1898, préface, p. xiii]

Cette citation nous permet de mieux appréhender la formule d'approximation précédente $(*)$. En effet, la première partie de l'égalité $(\frac{1}{x-\alpha} = -\frac{f'(\alpha)}{f(\alpha)} + \cdots)$ rappelle la méthode d'approximation de Newton alors que la seconde partie $(\frac{1}{nf(\alpha)}\sqrt{(n-1)H(\alpha)})$ fait apparaître une dérivée seconde, correpondant à l'ordre deux dont parle Poincaré.

Avant de passer à la question des racines imaginaires, Rouché termine cette deuxième partie en écrivant quelques lignes sur les équations dont toutes les racines sont réelles, expliquant qu'elles « s'offrent de manière fréquente en Analyse ». Il parle notamment d'une méthode élaborée par Laguerre pour déterminer si, oui ou non, une équation donnée n'admet que des racines réelles.

Recherche des racines imaginaires

Rappelons que cette partie est considérée comme étant inachevée par Rouché et par Poincaré. Rouché explique toutefois que pour déterminer les racines imaginaires d'une équation, Laguerre a utilisé « une notion absolument nouvelle, celle des points dérivés ». En fait, partant d'une équation polynomiale, il forme « l'équation $f(x,y) = 0$ où y a été introduit pour rendre le polynôme f homogène ». Il nomme alors « point dérivée » d'un point M d'affixe x le point d'affixe ζ définie par la relation

$$\zeta\frac{df}{dx} + \frac{df}{dy} = 0.$$

Cette notion lui permet alors d'établir le théorème suivant :

> Tout cercle passant par un point quelconque du plan et par le point dérivé renferme au moins une racine de l'équation ; et il y a aussi au moins une racine en dehors du cercle. [Rouché 1887, p. 161]

C'est ensuite ce résultat « géométrique » qui permet à Laguerre de présenter une méthode d'approximation successives des racines imaginaires.

Les travaux sur les développements de fonctions en lien avec la résolution des équations

Dans la partie sur les développements de fonctions exposée par Rouché, deux paragraphes abordent explicitement la résolution des équations polynomiales en lien avec des développements en séries ou en fraction continues. Nous verrons d'ailleurs qu'il s'agit d'une approche que l'on retrouve dans les travaux de Hermite[14].

Le premier paragraphe de Rouché fait état d'une démonstration nouvelle proposée par Laguerre au sujet des fonctions symétriques des racines. Plus exactement, Rouché explique que :

> Une démonstration, faite par la théorie des fractions continues algébriques, du théorème fondamental de la théorie des fonctions symétriques des racine d'une équation, théorème qui donné d'abord par Cauchy, avait été démontré par Borchardt au moyen de la théorie des fonctions ultra-elliptiques.
>
> [Rouché 1887, p. 152]

Il s'agit donc d'une utilisation de développements en fractions continues dans l'objectif de résoudre un problème sur les équations, et plus particulièrement d'exprimer les coefficients d'un polynôme en fonction des sommes de Newton formées à partir des racines[15]. Cette démarche est sans doute à replacer dans un contexte plus général des XVIIIe et XIXe siècles où la question de la résolution des équations polynomiales par des techniques de développements de fonctions, que ce soit un développement en séries ou en fraction continues, était souvent posée. Pour ne donner que quelques exemples, dès le XVIIIe siècle, Lagrange proposait une manière de résoudre les équations par des développements en série dans [Lagrange 1770]. Cette « formule » est ensuite reprise, développée et discutée, entre autres, par Laplace, Fourier, Chio et Cauchy[16]. Sur la question de la résolution d'équations à l'aide des fractions continues, Lagrange est aussi considéré comme un précurseur avec son article [Lagrange 1769][17]. Il y démontre qu'un irrationnel est racine d'une équation de degré deux si et

14 Voir [Goldstein 2011] et [Sinaceur 1991].

15 On rappelle que si l'on note $(x_i)_{1 \leqslant i \leqslant n}$ les racines d'un polynôme de degré n, les sommes de Newton sont les $S_k = 1 \leq i \leq n x_i^k$.

16 Pour plus de détails sur les travaux de Lagrange et des autres mathématiciens sur cette question, la lectrice pourra consulter l'article [Chabert 2015].

17 Chabert [2015] donne également des informations à ce sujet.

seulement si son développement en fraction continue est périodique à partir d'un certain rang. Plusieurs décennies après, Galois complétera ce théorème en s'intéressant au cas particulier des irrationels dont le développement en fractions continues est « purement périodique »[18]; Le travail de Laguerre dont fait mention Rouché semble s'inscrire dans ce cadre plus général où l'on utilise les développements de fonctions comme des outils pour la résolution d'équations.

Le deuxième passage dans lequel Rouché fait allusion à un problème lié à celui de la résolution des équations numériques est assez différent dans la démarche. Le résultat en question, que Rouché juge « très important », est le suivant :

> $F(x)$ désignant un polynôme de degré $n\mu$, tellement choisi que les fractions
>
> $$\frac{\Phi_1(x)}{F(x)}, \frac{\Phi_2(x)}{F(x)}, \dots, \frac{\Phi_n(x)}{F(x)}$$
>
> approchent le plus les transcendantes
>
> $$e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x},$$
>
> le polynôme $F(x)$ est entièrement caractérisé par cette propriété que, dans le développement de $F(x)e^{zx}$ suivant les puissances croissantes de x, le coefficient de $x^{\mu(n-1)}$ est
>
> $$z^\mu(z-a_1)^\mu \cdots (z-a_n)^\mu.$$
>
> L'expression de $F(x)$ résulte de là fort aisément. [Rouché 1887, p. 151]

En fait, Rouché explique que Laguerre a utilisé un autre résultat pour le démontrer. Il s'agit d'un théorème de Hermite dont Laguerre propose une nouvelle démonstration, qualifiée par Rouché de « géométrique ». Ce théorème donne une condition suffisante à la réalité des racines d'une équation. Plus précisément :

> Si pour toutes les racines de l'équation
>
> $$F(x) + i\Phi(x) = 0,$$
>
> le coefficient de i a le même signe, l'équation
>
> $$pF(x) + q\Phi(x) = 0,$$
>
> où p et q désignent deux nombres réels arbitraires, a toutes ses racines réelles. [Rouché 1887, p. 151]

---

18 À ce sujet, on pourra consulter l'article de Catherine Goldstein sur Galois et Hermite [Goldstein 2011].

Pour démontrer ce résultat, Laguerre note $\alpha+ai$, $\beta+bi, \ldots$ les racines de $F(x)+i\Phi(x)$ et il remarque alors qu'en posant $H(x-\alpha-ai) = F(x)+i\Phi(x)$, l'équation $pF(x) + q\Phi(x) = 0$ implique que

$$(p - iq)H(x - \alpha - ai) + (p + iq)H(x - \alpha + ai) = 0$$

et par suite que $H(x - \alpha + ai)$ et $H(x - \alpha - ai)$ ont le même module. Il peut alors conclure que toute racine de $pF(x) + q\Phi(x)$ est à équidistance de $\alpha+ai$ et $\alpha-ai$ dans le plan complexe et, se situant sur l'axe des abscisses, est donc réelle [Laguerre 1880c].

Ainsi, dans ce cas, il apparaît que Laguerre ne se propose pas de résoudre une équation à l'aide de développements de fonctions mais cherche, au contraire, à utiliser un résultat relatif à la résolution des équations pour établir un développement en série. Cette perspective nous paraît assez remarquable et mérite une attention particulière.

Nous avons jusque là présenté les travaux de Laguerre et vu que le nom de Hermite apparaissait fréquemment dans les articles sur les équations numériques. Il s'agit maintenant de comprendre, d'un point de vue plus qualitatif, comment et pourquoi Laguerre cite Hermite dans ses articles.

2.2. *L'objet des références à Hermite*

Les articles de Laguerre citent Hermite pour différentes raisons. Un premier ensemble de citations concerne notamment le résultat évoqué par Rouché sous le nom de « théorème de M. Hermite » et publié dans [Hermite 1879]. Ainsi, dans [Laguerre 1880c], Laguerre commence par rappeler le résultat puis par remarquer que « M. Biehler a obtenu en même temps ce théorème [...] et [que] sa démonstration, comme celle de Hermite, repose sur la considération de l'indice de la fraction $\frac{\Phi(x)}{F(x)}$ » [Laguerre 1880c, p. 239]. Il propose alors une autre démonstration du résultat, utilisant la représentation géométrique dans le plan des racines complexes et des considérations sur les modules. C'est d'ailleurs la même démonstration qu'il reproduit dans [Laguerre 1880d] où il présente différents types d'équations dont toutes les racines sont réelles. Aussi, à d'autres moments, Laguerre propose une application de ce théorème de Hermite. Par exemple, dans [Laguerre 1884b], Laguerre utilise ce résultat afin de prouver que « Si l'équation $f(x) = a_0 + a_1 x + \ldots a_n x^n = 0$ a toutes ses racines réelles et de même signe, l'équation

$$a_0 \cos(\lambda) + a_1 \cos(\lambda + \theta)x + a_2 \cos(\lambda + 2\theta)x^2 + \ldots + a_n \cos(\lambda + n\theta)x^n = 0,$$

où λ et θ désignent deux arcs arbitraires, a toutes ses racines réelles » [Laguerre 1884b, p. 121].

Il existe quelques autres exemples, assez rares, où Laguerre cite un résultat de Hermite pour l'appliquer ou pour le démontrer d'une autre manière. Ainsi, dans [Laguerre 1880e], Laguerre fait référence à un autre résultat de Hermite paru dans [Hermite 1873a] :

> Soient $a, b, c, \ldots, l, m$ quantités arbitraires et $F, F_1, F_2, \ldots, F_{m-1}$, m polynômes entiers en x. Si l'on désigne respectivement par $\alpha, \beta, \gamma, \ldots, \lambda$ les degrés de ces polynômes et si l'on pose, pour abréger,
>
> $$\mu = \alpha + \beta + \gamma + \cdots + \lambda,$$
>
> on voit que, dans l'expression
>
> $$V = Fe^{ax} + F_1 e^{bx} + F_2 e^{cx} + \cdots + F_{m-1} e^{lx},$$
>
> figurent les $\mu + m$ coefficients des polynômes $F, F_1, F_2, \ldots F_{m-1}$. [...] M. Hermite a donné une méthode très simple et très élégante pour déterminer les valeurs de ces polynômes. [Laguerre 1880e, p. 11]

Dans la suite de l'article, Laguerre fait remarquer qu'il a traité le cas où tous les coefficients a, b, c, λ sont égaux dans [Laguerre 1880c] puis il propose, dans le cas général, une méthode différente de celle de Hermite pour retrouver les valeurs des polynômes.

Un autre exemple intéressant est donné par l'article [Laguerre 1880c] où Laguerre indique que la formule qu'il avait présentée dans un précédent article « appartient en réalité à M. Hermite ». Il développe et explique également les différences d'approches qu'il a avec Hermite :

> Seulement, tandis que M. Hermite prend pour point de départ les propriétés des réduites de la fonction $\log\left(\frac{x-1}{x}\right)$, je m'appuie sur les propriétés des réduites de e^x ; on apperçoit, dans cette circonstance, le premier indice d'une liaison singulière entre les réduites de fonctions si différentes.
>
> [Laguerre 1880c, p. 241]

Les réduites de la fonction exponentielle sont en fait des fractions rationnelles telles que leur développement en série coïncide avec les premiers termes de la série définissant la fonction exponentielle, à savoir $\sum_{k \geqslant 0} \frac{x^k}{k!}$. La référence aux résultats de Hermite à ce sujet est ici doublement intéressante. Tout d'abord, il s'agit des résultats qui ont mené Hermite à la célèbre démonstration de la transcendance du nombre e. Pour autant, il est tout aussi important de souligner que Laguerre, tout comme Hermite lui-même, s'intéresse avant tout à ces résultats en ce qu'ils donnent un moyen d'approximer des fonctions [Serfati 1992]. Il ne mentionne pas en revanche la transcendance de e.

Ainsi, il existe un certain nombre d'exemples qui montrent que Laguerre et Hermite travaillent sur des sujets similaires et que Laguerre utilise parfois des résultats de Hermite, tout en proposant également d'autres preuves de ces résultats. Cela ne doit toutefois pas masquer le fait qu'une grande partie des références à Hermite sont d'un autre type. En effet, bien souvent, Laguerre cite Hermite non pas pour utiliser un de ses résultats mais plutôt pour faire référence à un objet introduit par Hermite et que Laguerre utilise dans un tout autre contexte. Ainsi, il fait référence aux polynômes de Hermite à plusieurs reprises afin, pour ainsi dire, de tester ses propres résultats sur les équations. Par exemple, il illustre les propriétés qu'il vient de démontrer dans [Laguerre 1880f, p. 810] en les appliquant aux polynômes de Hermite et en faisant référence à [Hermite 1864] dans lequel Hermite avait justement introduit les polynômes qui portent son nom. C'est également cette même démarche qu'adopte Laguerre dans [Laguerre 1880g, p. 251], dans [Laguerre 1878c] ou encore dans [Laguerre 1880h, p. 306] où Laguerre détermine une limite inférieure de la différence entre deux racines consécutives des polynômes de Hermite en application d'un résultat présenté notamment dans [Laguerre 1880b]. Un autre exemple de citations du même type concerne les références à l'article « Sur la fonction exponentielle » [Hermite 1873b]. Laguerre cite souvent cet article pour prendre l'exemple des réduites de la fonction exponentielle c'est-à-dire « une fraction $\frac{\Phi(x)}{F(x)}$ dont les deux termes sont des polynômes de degré n tels que le développement de cette fraction suivant les puissances croissantes de x coïncide, jusqu'au terme du degré $2n$ inclusivement, avec le développement de e^x » [Laguerre 1880a, p. 102]. Dans [Laguerre 1880a], Laguerre applique justement la règle des signes de Descartes et sa généralisation au cas des séries aux réduites de l'exponentielle. Dans [Laguerre 1880d, p. 230], il établit un résultat portant sur le nombre de racines réelles de ces réduites. Ici, Laguerre prend donc des polynômes définis par Hermite comme des exemples sur lesquels il applique des théorèmes divers et variés. La référence à Hermite ne correspond donc pas exactement à une référence directe à ses travaux mais illustre malgré tout l'influence que pouvait avoir Hermite sur les mathématiques de l'époque.

Un autre type de citations consiste enfin, pour Laguerre, à faire référence à Hermite et à l'un de ses articles tout en expliquant qu'il ne souhaite pas développer davantage. Dans [Laguerre 1880i, p. 44], avant de faire référence à un article de Hermite, Laguerre écrit par exemple : « il serait facile d'exprimer Δ au moyen d'une intégrale multiple, mais je ne m'étendrai pas sur le sujet ». De la même manière, dans [Laguerre 1880c, p. 242], il indique qu'« il ne [s']étendra pas davantage sur ce sujet » avant, là aussi, de citer Hermite.

Enfin, à plusieurs reprises Laguerre cite le cours de Hermite professé à l'École polytechnique. Dans ce cas, il s'agit plutôt de renvoyer le lecteur à une référence que l'on peut qualifier de classique. Il y fait par exemple référence dans [Laguerre 1880d, p. 229] lorsqu'il utilise le théorème de Rolle.

Finalement, il est clair que Hermite est très souvent cité par Laguerre et ce pour des raisons assez diverses. La description précédente donne néanmoins l'impression que les résultats pour lesquels Hermite est cité ne concernent pas toujours les aspects les plus importants développés par les deux mathématiciens autour des équations. Pour le dire autrement, bien que Laguerre cite parfois des articles célèbres d'Hermite, il ne le fait en général pas pour en reprendre les résultats principaux. Ainsi, lorsqu'il cite le fameux article « Sur la fonction exponentielle » de 1873, ce n'est jamais pour faire référence à la transcendance du nombre e mais plutôt pour parler des réduites de l'exponentielle. Aussi, il cite le *Mémoire sur l'équation du cinquième degré* [Hermite 1866] dans [Laguerre 1878b], non pas pour faire référence à la résolutibilité des équations (aux fonctions elliptiques ou aux équations modulaires par exemple[19]) mais simplement à propos d'une remarque faite par Hermite permettant de remplacer, dans l'application du théorème de Sturm, le polynôme dérivé par une combinaison linéaire du polynôme dérivé et du polynôme lui-même. Le fait que Laguerre cite ainsi Hermite sur des questions assez spécifiques voires techniques donnent en tout cas l'impression que Laguerre avait l'habitude de lire dans les détails les travaux de Hermite et était familier de ses publications.

3. SIMILARITÉS ENTRE LAGUERRE ET HERMITE

Les éléments présentés jusqu'ici étaient principalement factuels. Il s'agit maintenant d'ouvrir le propos en soulignant, de manière plus générale, les points communs qui peuvent exister entre les deux mathématiciens. Nous disposons pour cela d'un autre document : un compte rendu d'Émile Borel où il commente les travaux de Laguerre d'un point de vue, pour ainsi dire, plus philosophique.

3.1. *Le point de vue d'Émile Borel*

Émile Borel peut être qualifié de contemporain d'Edmond Laguerre, étant né en 1871 alors que Laguerre est mort en 1886. Toutefois, Borel

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19 À propos des travaux de Hermite sur les équations, on pourra consulter [Goldstein 2011].

n'a pas cotoyé Laguerre en tant que mathématicien étant donné qu'il n'avait que 15 ans à la mort de Laguerre. En 1898, Borel a donc 27 ans lorsque les *Œuvres* de Laguerre sont publiées et que lui revient de faire un compte rendu dans le *Bulletin des Sciences mathématiques*. Le compte rendu qu'il propose [Borel 1898, p. 305] en un peu moins de sept pages est relativement différent des textes de Rouché et même de Poincaré. Il procède en trois partie, la première sur les équations algébriques, la seconde sur les équations transcendantes, et la dernière sur les fractions continues. Autre différence notable avec les présentations de Rouché et de Poincaré, Borel ne présente que peu de résultats et ne les détaille pour ainsi dire quasiment pas mais insiste sur les motivations de Laguerre et pour ainsi dire, les aspects philosophiques liés à ses travaux. C'est d'ailleurs la raison pour laquelle nous avons préféré présenter séparément ce document : après avoir focalisé notre présentation autour des résultats mathématiques de Laguerre et de leur structuration, nous allons maintenant élargir notre propos à partir du point de vue personnel de Borel.

3.1.1. *La division des Œuvres entre Algèbre et Calcul Intégral*

Rappelons que les éditeurs des *Œuvres* n'ont pas retenu le même plan pour structurer les articles de Laguerre et pour les présenter dans [Rouché 1887] et [Laguerre 1898, preface]. Borel donne un certain nombre d'éléments permettant de comprendre ces choix :

> Les Mémoires de Laguerre sont classés sous deux rubriques distinctes : Algèbre, Calcul Intégral; et, dans cette classification, on est tout d'abord étonné de voir les recherches de Laguerre sur les transcendantes entières, et d'autres encore, figurer parmi les mémoires d'Algèbre. Cet étonnement disparaît lorsque, au lieu de lire seulement la Table des matières, on étudie le texte même; on voit alors que l'un des traits caractéristiques de Laguerre est l'aisance avec laquelle il résout bien des questions d'Analyse par les méthodes de l'Algèbre la plus élémentaire, et l'on s'aperçoit à peine de la transition entre l'Algèbre des polynômes et l'Algèbre des fonctions transcendantes, si l'on peut ainsi s'exprimer. [Borel 1898, p. 305]

Ainsi, lorsqu'il s'agit d'organiser la table des matières des *Œuvres*, les éditeurs ont retenus deux grandes parties, Algèbre et Calcul Intégral dans lesquelles le critère de classement semble être lié aux méthodes utilisées. D'un côté, les résultats faisant intervenir des méthodes « élémentaires ». De l'autre, des travaux mobilisant des méthodes et des technique liées au calcul d'intégrales ou à la résolution d'équations différentielles. En revanche, lorsque ces mêmes éditeurs, Rouché et Poincaré, ont voulu présenter une synthèse des travaux de Laguerre, ils n'ont pas retenus les mêmes distinctions. Dans ce cas, ils ont plutôt classés les résultats mathématiques de Laguerre en fonction des objets mathématiques abordés dans les articles en séparant par exemple les équations polynomiales, les équations différentielles et les développements de fonctions en séries ou en fractions continues. Cela permet donc de comprendre un peu mieux la structuration des *Œuvres* et la place que peut occuper un sujet tel que les fractions continues, à l'intersection entre Algèbre et Calcul intégral dans les *Œuvres* de Laguerre.

La remarque de Borel est d'ailleurs à rapprocher de conceptions similaires chez Hermite. Ce dernier considère en effet qu'il existe une certaine continuité entre l'étude des nombres entiers, des polynômes et des séries entières. Il considère les séries comme des généralisations des polynômes, eux même généralisations des entiers, et fait ainsi un lien entre arithmétique, algèbre et analyse [Goldstein et al. 2007, pp. 399-401]

3.1.2. *Aspects théoriques et pratiques chez Laguerre*

Émile Borel revient à plusieurs reprises sur la tension qui peut exister entre théorie et pratique chez Laguerre, notamment ceux sur les équations algébriques. Afin d'introduire son propos sur les équations, Borel rappelle d'ailleurs les difficultés calculatoires souvent rencontrées lorsqu'il est question d'équations numériques :

> La détermination exacte du nombre des racines réelles d'une équation algébrique et la séparation de ces racines ont longtemps été au nombre des plus importantes préoccupations des géomètres. La beauté de la solution de Sturm, perfectionnée par M. Hermite, et aussi la multitude d'autres sujets nouveaux qui sollicitaient l'attention ont, dès le milieu de ce siècle, diminué beaucoup l'importance relative de ces questions. La complication rapidement inextricable qu'atteignent les calculs dans les applications n'était d'ailleurs pas faits pour encourager l'étude. [Borel 1898, pp. 305-306]

Ainsi, le théorème de Sturm apparaît à la fois comme utile pour répondre à une question mathématique « importante » mais en même temps peu propice aux calculs pratiques. À l'opposé de ce résultat, Borel cite la règle des signes de Descartes qui est très facile d'application mais qui ne donne pas toujours une majoration très intéressante du nombre de racines. Borel explique alors que Laguerre souhaitait trouver une sorte de compromis entre ces deux règles :

> Laguerre pensa cependant qu'entre les méthodes imparfaites, mais simples, dont la règle des signes de Descartes fut le premier exemple, et les méthodes parfaites, mais pratiquement compliquées, dont Sturm a donné le type, il y avait

> place pour d'autres méthodes moins imparfaites que les premières et plus aisément utilisables que les secondes.
>
> Ces nouvelles méthodes devaient permettre, au prix de calculs assez simples, de résoudre *dans la plupart des cas,* le problème pratique de la séparation des racines. [Borel 1898, p. 306]

Pour Borel, les travaux de Laguerre sur les équations sont donc, en quelque sorte, un juste milieu entre l'impératif théorique d'exprimer le nombre exact des racines en fonction des coefficients de l'équation et l'impératif pratique de mener à bien le calcul de ce nombre de racines.

3.1.3. *Conception de la généralité chez Laguerre et place accordée aux cas particuliers*[20]

Borel insiste sur le positionnement de Laguerre concernant la question de la généralité en mathématiques en donnant l'exemple de ses travaux sur les équations :

> L'étude des travaux de Laguerre sur la théorie des équations met déjà en évidence une qualité qu'il possédait au plus haut degré et que l'on retrouve dans le reste de son œuvre : je veux parler de sa faculté de généralisation, à la fois prudente et hardie. [Borel 1898, p. 307]

Borel explicite ensuite ce point de vue en précisant ce qu'il entend par le terme « généralisation » :

> Généraliser une proposition est souvent très facile, si l'on entend par là donner un énoncé assujetti à la seule condition de contenir le premier comme cas particulier. Mais la généralisation ainsi comprise sera rarement intéressante, rarement féconde.
>
> Presque toujours généraliser une propriété d'une certaine classe d'être algébriques (ou géométriques, etc.), c'est étendre cette propriété à une classe plus large. Cette extension ne sera le plus souvent possible que si l'on modifie la forme de la proposition, si on la réduit à ce qu'elle a d'*essentiel,* et voilà un premier point où la sagacité de l'analyste devra s'exercer. [Borel 1898, p. 307]

Ainsi, Borel pense que la capacité de Laguerre à généraliser une proposition de manière utile est liée à sa compréhension des « raisons essentielles » de ce résultat. Il explique qu'une généralisation pertinente consiste à « étendre une proposition autant que possible en surface, tout en la réduisant le moins possible en profondeur ». Là encore, il s'agit, d'une certaine manière, de « juste mesure ». Il poursuit :

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[20] À propos de la question de la généralité en mathématiques, on pourra consulter [Chemla et al. 2016].

> Mais ce n'est pas tout; il est clair que, plus on élargit la classe d'êtres auxquels on veut étendre la proposition, plus cette proposition devra elle même être restreinte dans son énoncé. Il y a donc à garder une mesure et c'est là que la lecture de Laguerre peut nous fournir des modèles incomparables.
>
> [Borel 1898, p. 307]

Une généralisation adaptée consiste donc à élargir la classe d'objets considérés sans que l'énoncé ne devienne inintéressant. Pour mieux faire comprendre ce double impératif, il donne alors quelques exemples, notamment celui des équations transcendantes. Laguerre a en fait généralisé certains résultats concernant les polynômes aux fonctions entières. Pour cela, il a repris les travaux de Weierstrass et proposé une classification des fonctions entières par une notion nouvelle : leur genre. Borel explique que cela lui permet d'établir un résultat, légèrement modifié par rapport à la proposition initiale, et valable pour les fonctions de genre fini. En cela, Laguerre a dû définir la classe d'objet adaptée au problème qu'il considère en introduisant un nouveau concept. C'est en cela que la recherche de généralité n'est pas prévisible mais au contraire, résulte de l'exercice de « la sagacité de l'analyste ».

3.2. *Deux résultats de Laguerre et de Hermite au sujet des équations numériques*

Nous avons déjà mentionné le théorème de Laguerre permettant de calculer une approximation des racines d'un polynôme. Ce résultat et certains de ses corollaires ont fait l'objet de nombreux articles dans les *Nouvelles Annales*, revue de mathématiques à destination de professeurs et d'élèves de classes préparatoires au concours de l'École polytechnique notamment. Laguerre a publié ses résultats dans les *Nouvelles Annales* mais ils ont également été repris, discutés ou améliorés par d'autres auteurs de ce journal. Un des élèves de Laguerre à l'École polytechnique, Gabriel Candèze a notamment publié une contribution, comparant l'efficacité du théorème de Laguerre avec les autres règles déjà connues à l'époque comme celles de Budan et de Fourier [Candèze 1880].

De son côté, Hermite a découvert et démontré un résultat ensuite associé à son nom au sujet des équations numériques alors qu'il était élève en classes préparatoires. Plus précisément, il était élève du collège Louis-le-Grand et suivait les cours du professeur Richard lorsqu'il a participé au Grand concours de 1842. Chaque année, ce concours proposait une question pour les élèves en classe de mathématiques élémentaires et une pour ceux de mathématiques spéciales. Cette année, le sujet de mathématiques spéciales portait sur « la règle des signes de Descartes » et Hermite se distingua en remarquant un résultat nouveau concernant certaines équations particulières :

> Lorsque les coefficients de quatre termes consécutifs d'une équation forment une progression arithmétique, l'équation a nécessairement des racines imaginaires. [Terquem 1843]

S'il ne s'agit sans doute pas du résultat de Hermite ayant le plus marqué l'histoire de l'algèbre, il est toutefois intéressant de noter un certain nombres de points commun avec Laguerre. Tout comme le résultat de Laguerre, celui de Hermite est repris et discuté dans les *Nouvelles Annales.* Dans [Guilmin 1846], Adrien Guilmin, normalien de la promotion 1836 en propose par exemple une démonstration avant d'énoncer un corollaire : « si une équation a trois coefficients consécutifs en progression géométrique, elle a des racines imaginaires ». De même, le professeur lyonnais de Virieu généralise le théorème de Hermite en proposant une nouvelle démonstration [de Virieu 1846]. En outre, nous avons retrouvé dans les papiers d'Abel Transon, professeur de classes préparatoires puis examinateur d'admission à l'École polytechnique, les traces du résultat de Hermite. Le brouillon en question contient l'énoncé du théorème ainsi qu'une brève allusion à la démonstration de De Virieu[21].

Finalement, et bien qu'il n'y ait pas de lien directs entre les deux résultats de Laguerre et de Hermite, cela témoigne d'une présence commune dans les milieux d'enseignement des classes préparatoires et de l'École polytechnique. D'après l'*Enseignement mathématique* [1914, p. 352], les deux résultats ont d'ailleurs fait l'objet de questions au concours d'admission de l'École polytechnique dans les années 1880. Cela illustre également le fait que le lien entre les mathématiques de Hermite et de Laguerre ne sauraient se comprendre uniquement du point de vue des idées et des références intertextuelles présentes dans les articles. Ce lien traduit aussi un ancrage commun dans un même milieu professionnel (celui des enseignants) et institutionnel (celui de l'École polytechnique).

3.3. *Une similarité d'approche et de conception des mathématiques*

Dans les articles de Hermite, ce ne sont pas les résultats vus généralement comme les plus importants qui sont repris par Laguerre. Cela s'explique sans doute par le fait qu'Hermite et Laguerre ne s'intéressent pas

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21 Pour plus de détails concernant le résultat de Hermite et sa réception, consulter [Vincent 2020].

toujours au même domaines des mathématiques. Laguerre ne publie par exemple quasiment pas en théorie des nombres et n'est pas amené à considérer des questions de transcendances de nombres.

Pourtant, il y a une approche et des conceptions similaires chez les deux mathématiciens. Pour Laguerre comme pour Hermite, la résolution des équations polynomiales n'est pas un sujet d'algèbre isolé du reste de leurs travaux mathématiques. Chez Laguerre, nous avons vu qu'il aborde cette question d'une manière très « analytique » en basant par exemple la démonstration de son théorème sur les équations numériques sur le théorème de Rolle. Cela le mène à généraliser certains résultats aux séries entières et à s'intéresser aux zéros de fonctions analytiques. De la même manière, les travaux de Hermite sur les équations ne se limitent pas à des considérations uniquement « algébriques » [Goldstein et al. 2007, p. 403-406].

De ce point de vue, la démarche de Laguerre, consistant à utiliser les développements en séries et les fractions continues pour résoudre des problèmes relatifs aux équations, est à rapprocher de la démarche d'Hermite lorsqu'il démontre la transcendance du nombre e en utilisant et généralisant les fractions continues [Goldstein et al. 2007, p. 381]. Il n'existe malheuresement pas de commentaire de Laguerre lui-même, dans ses *Œuvres* ou ses correspondances, faisant référence à une influence qu'aurait pu avoir Hermite sur sa conception des mathématiques. Cependant, il est clair qu'il existe une forme d'interdépendance et de similarité entre les approches des deux mathématiciens. Cette proximité n'était d'ailleurs pas nécessairement visible au début de leurs carrières, lorsqu'ils s'intéressaient à des sujets différents, mais devient de plus en en plus claire par la suite, dans les années 1870-1880. S'il existe une proximité en termes d'approches, l'étude des citations dans les *Œuvres* de Laguerre a également permis de montrer que Laguerre faisait aussi bien souvent référence à des objets spécifique définis par Hermite (les polynômes de Hermite, les réduites de la fonction exponentielle, etc.). Pour ainsi dire, les objets qu'Hermite a introduit ou sur lesquels il a travaillé faisaient partie du paysage mathématique, incontournable à qui travaille sur les équations à son époque. Pour Laguerre en tout cas, il est une source d'inspiration du quotidien, avec qui il a par ailleurs l'occasion d'échanger. Cela est donc à relier à une autre proximité qu'a permis de mettre en évidence notre étude : Laguerre et Hermite évoluent dans les mêmes institutions, tant à l'École polytechnique qu'à l'Académie des Sciences et certains de leurs résultats comme ceux sur les équations numériques en sont fortement marqués.

RÉFÉRENCES

BOREL (Émile)
[1898] « Compte rendus et analyses, Œuvres de Laguerre », *Bulletin des Sciences mathématiques*, xxii (1898), p. 304–310.

CANDÈZE (Pierre Antoine Gabriel)
[1880] « Sur une règle de M. Laguerre », *Nouvelles annales de mathématiques*, s. 2, 19 (1880), p. 307–311.

CHABERT (Jean-Luc)
[2015] « Sur la résolution numérique des équations », dans Gilain (Christain) & Guilbaud (Alexandre), éds., *Sciences mathématiques, 1750 - 1850, continuités et ruptures*, Paris : CNRS éditions, 2015, p. 475–507.

CHEMLA (Karine), CHORLAY (Renaud) & RABOUIN (David)
[2016] *The Oxford Handbook of Generality in Mathematics and the Sciences*, Oxford : Oxford Univ. Press, 2016.

CROSLAND (Maurice)
[1992] *Science under control, the French Academy of sciences, 1795–1914*, Cambridge : Cambridge Univ. Press, 1992.

GILAIN (Christian) & GUILBAUD (Alexandre), éds.
[2015] *Sciences mathématiques : 1750-1850 : continuités et ruptures*, Paris : CNRS éditions, 2015.

GOLDSTEIN (Catherine)
[2011] « Charles Hermite's stroll through the Galois fields », *Revue d'histoire des mathématiques*, 17 (2011), p. 211–270.
[2012] Les autres de l'un : deux enquêtes prosopographiques sur Charles Hermite, dans Rollet (Laurent) & Nabonnand (Philippe), éds., *Les uns et les autres…Biographies et prosopographies en histoire des sciences*, Nancy : PUN - Editions Universitaire de Lorraine, 2012, p. 509–540.

GOLDSTEIN (Catherine), SCHAPPACHER (Norbert) & SCHWERMER (Joachim), éds.
[2007] *The Shaping of Arithmetic*, Berlin, Heidelberg, New York : Springer Science, 2007.

GUILMIN (Adrien)
[1846] « Conséquences de la règle des signes de Descartes », *Nouvelles annales de mathématiques*, s. 1, 5 (1846), p. 239–244.

HERMITE (Charles)
[1843] « Sur la division des fonctions abeliennes ou ultra elliptiques », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 17 (1843), p. 82.
[1854] « Sur la théorie des formes quadratiques », *Journal für die reine und angewandte Mathematik*, 47 (1854), p. 313–368.
[1864] « Sur un nouveau développement en série des fonctions », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 58 (1864), p. 93–100.

[1866] *Sur l'équation du cinquième degré*, Paris : Gauthier-Villars, 1866.

[1873a] « Extrait d'une lettre à Mr Borchardt (sur quelques approximations algébriques) », *Journal für die reine und angewandte Mathematik*, 76 (1873), p. 342–344.

[1873b] « Sur la fonction exponentielle », *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences*, 77 (1873), p. 18–24.

[1875] « Extrait d'une lettre à M. Fuchs de Gottingue sur quelques équations différentielles linéaires », *Journal für die reine und angewandte Mathematik*, 79 (1875), p. 324–338.

[1879] « Sur l'indice des fractions rationnelles », *Bulletin de la Société Mathématique de France*, 7 (1879), p. 128–131.

HERMITE (Charles) & MITTAG-LEFFLER (Gösta)

[1984] « Lettres de Charles Hermite à Gösta Mittag-Leffler (1874-1883) », *Cahiers du séminaire d'histoire des mathématiques*, s. 1, 5 (1984), p. 49–285.

[1985] « Lettres de Charles Hermite à Gösta Mittag-Leffler (1884-1891) », *Cahiers du séminaire d'histoire des mathématiques*, s. 1, 6 (1985), p. 79–217.

HERMITE (Charles) & STIELTJES (Thomas)

[1898] *Correspondance d'Hermite et de Stieltjes*, vol. 2, Gauthier-Villars, 1898.

LAGRANGE (Joseph-Louis)

[1769] « Sur la résolution des équations numériques », *Histoire de l'Académie royale des sciences et des belles-lettres de Berlin, année 1767*, 23 (1769), p. 311–352.

[1770] « Nouvelle méthode pour résoudre les équations littérales par le moyen des séries », *Histoire de l'Académie royale des sciences et des belles-lettres de Berlin, année 1768*, 24 (1770), p. 204–233.

LAGUERRE (Edmond Nicolas)

[1867] « Sur le calcul des systèmes linéaires, extrait d'une lettre à M. Hermite », *J. Éc. polytech.*, 62 (1867), p. 215–264.

[1874a] « Sur la théorie des équations numériques », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 78 (1874), p. 278–280.

[1874b] « Sur la résolution des équations numériques dont toutes les racines sont réelles », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 79 (1874), p. 996–998.

[1874c] « Sur le rôle des émanants dans la théorie des équations numériques », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 78 (1874), p. 278–280.

[1877] « Sur la résolution des équations numériques », *Bulletin de la Société Mathématique de France*, 5 (1877), p. 78–92.

[1878a] « Sur la résolution des équations numériques », *Nouvelles annales de mathématiques*, s. 2, 17 (1878), p. 20–25.

[1878b] « Remarques sur quelques points de la théorie des équations numériques », *Nouvelles annales de mathématiques*, s. 2, 17 (1878), p. 104–106 ; signé « Un abonné ».

[1878c] « Sur le développement suivant les puissances d'un polynôme », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 86 (1878), p. 295–297.

[1879a] « Sur la règle des signes de Descartes », *Nouvelles annales de mathématiques*, s. 2, 18 (1879), p. 5–13.

[1879b] « Sur la séparation des racines d'une équation algébrique à coefficients numériques », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 89 (1879), p. 635–637.

[1880a] « Sur la détermination d'une limite supérieure des racines d'une équation et sur la séparation des racines », *Nouvelles annales de mathématiques*, s. 2, 19 (1880), p. 49–57.

[1880b] « Sur une méthode pour obtenir par approximation les racines d'une équation algébrique qui a toutes ses racines réelles », *Nouvelles annales de mathématiques*, s. 2, 19 (1880), p. 161–171.

[1880c] « Sur quelques théorèmes de M. Hermite, extrait d'une lettre adressée à M. Borchardt », *Journal für die reine und angewandte Mathematik*, 89 (1880), p. 239–242.

[1880d] « Sur quelques propriétés des équations algébriques qui ont toutes leurs racines réelles », *Nouvelles annales de mathématiques*, s. 2, 19 (1880), p. 224–236.

[1880e] « Sur la fonction exponentielle », *Bulletin de la Société Mathématique de France*, 8 (1880), p. 11–18.

[1880f] « Sur les équations algébriques dont le premier membre satisfait à une équation linéaire du second ordre », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 90 (1880), p. 809–812.

[1880g] « Théorèmes généraux sur les équations algébriques », *Nouvelles annales de mathématiques*, s. 2, 19 (1880), p. 241–253.

[1880h] « Sur l'approximation des fonctions circulaires au moyen des fonctions algébriques », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 90 (1880), p. 304–307.

[1880i] « Sur le développement d'une fonction suivant les puissances croissantes d'un polynôme », *Journal für die reine und angewandte Mathematik*, 88 (1880), p. 35–48.

[1881a] « Sur la séparation des racines des équations numériques », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 92 (1881), p. 1146–1149.

[1881b] « Sur une extension de la règle des signes de Descartes », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 92 (1881), p. 230–233.

[1882a] « Sur quelques équations transcendantes », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 94 (1882), p. 160–163.

[1882b] « Sur la détermination du genre d'une fonction transcendante entière », *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 94 (1882), p. 635–638.

[1883] « Sur la théorie des équations numériques », *Journal de mathématiques pures et appliquées*, s. 3, 4 (1883), p. 99–146.

[1884a] « Sur l'approximation des racines des équations algébriques », *Nouvelles annales de mathématiques*, s. 3, 3 (1884), p. 113–118.

[1884b] « Sur quelques points de la théorie des équations numériques », *Acta Mathematica*, 4 (1884), p. 97–121.

[1898] *Œuvres*, vol. 1, Paris : Gauthier-Villars et fils, 1898.

L'ENSEIGNEMENT MATHÉMATIQUE
[1914] « Discussion sur la préparation mathématique des ingénieurs », 16 (1914), p. 328–355.

ROLLET (Laurent)
[2017] *La correspondance de jeunesse d'Henri Poincaré. Les années de formation. De l'École polytechnique à l'École des Mines (1873-1878),* Cham : Birkhäuser, 2017.

ROLLET (Laurent) & NABONNAND (Philippe), éds.
[2012] *Les uns et les autres – biographies et prosopographies en histoire des sciences,* Collection Histoire des institutions scientifiques, Nancy : Presses universitaires de Nancy : Editions universitaires de Lorraine, 2012 ; OCLC : ocn847520832.

ROUCHÉ (Eugène)
[1887] « Edmond Laguerre, sa vie et ses travaux », *Nouvelles annales de mathématiques,* s. 3, 6 (1887), p. 105–173.

SERFATI (Michel)
[1992] *Fragments d'histoire des mathématiques, Quadrature du cercle, fractions continues et autres contes sur l'histoire des nombres irrationnels et transcendants aux XVIII^e et XVIII^e siècles,* vol. tome 4, APMEP, 1992.

SINACEUR (Hourya)
[1991] *Corps et modèles : essai sur l'histoire de l'algèbre réelle,* Mathesis, Paris : J. Vrin, 1991.

TERQUEM (Olry)
[1843] « Grand concours de 1843 », *Nouvelles annales de mathématiques,* s. 1, 2 (1843), p. 374–376.

VINCENT (Yannick)
[2019] *Les répétiteurs de mathématiques à l'École polytechnique de 1798 à 1900,* Thèse, École polytechnique, 2019 ; soutenue le 12 décembre 2019.
[2020] « Ce que nous apprend une étude autour des équations numériques à propos de l'École polytechnique au XIX^e siècle », *Philosophia Scientiae,* 1 (2020), p. 59–74.
[2022] « Perfectionnement de la règle des signes de Descartes », *Quadrature,* 125 (2022), p. 29–35.

DE VIRIEU
[1846] « Solution de la question 711 IV », *Nouvelles annales de mathématiques,* s. 2, 4 (1846), p. 79–80.

Revue d'histoire des mathématiques
30 (2024), p. 293–312

BIBLIOGRAPHY OF CHARLES HERMITE

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Charles Hermite's *Œuvres*, edited by his son-in-law and pupil Emile Picard, do not contain a complete bibliography, but only (sometimes partial) indications of the original places and dates of publication of the articles reproduced. Moreover, there are sometimes substantial differences between the original version and the version given in the *Œuvres*, paragraphs containing Hermite's errors having been either corrected or omitted. Below is a list of Hermite's original publications and their references in the *Œuvres*, where available (in the form : "Rep. in [Hermite, OC]", followed by volume and page numbers). We have supplemented this with a list of Hermite's most important published correspondence to date with various mathematicians. Several other sets of letters to or from Hermite are known, but have not yet been published.

1. CHARLES HERMITE'S ARTICLES AND BOOKS

Cayley (Arthur) & Hermite (Charles), On Cubic forms, *Quarterly Journal of Pure and Applied Mathematics*, 1 (1857), pp. 85–91. Rep. in [Hermite, OC], I, pp. 437–439 (only Hermite's letter is reproduced).

Fuchs (Lazarus) & Hermite (Charles), Sur un développement en fraction continue, *Acta Mathematica*, 4 (1884), pp. 89–92. Rep. in [Hermite, OC], IV, pp. 172–174.

Hermite (Charles), Lieu géométrique des poles d'une section conique par rapport à une autre, *Nouvelles annales de mathématiques*, 1 (1842), pp. 263–264. Rep. in [Hermite, OC], I, pp. 1–2.

HERMITE (Charles), Considérations sur la résolution algébrique de l'équation du cinquième degré, *Nouvelles annales de mathématiques,* 1 (1842), pp. 329–336. Rep. in [Hermite, OC], I, pp. 3–9.

HERMITE (Charles), Sur la division des fonctions abéliennes ou ultra-elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences,* 17 (1843), p. 82.

HERMITE (Charles), Sur la théorie des transcendantes à différentielles algébriques, *Journal de mathématiques pures et appliquées,* 9 (1844), pp. 353–368. Rep. in [Hermite, OC], I, pp. 49–63.

HERMITE (Charles), Principaux théorèmes de l'analyse des fonctions elliptiques, *Mémoires de la Société royale des sciences, lettres et arts de Nancy* (1845), pp. 201–211. Rep. in [Hermite, OC], I, pp. 64–70.

HERMITE (Charles), Extraits de deux lettres de M. Charles Hermite à M. C.G.J. Jacobi, *Journal für die reine und angewandte Mathematik,* 32 (1846), pp. 277–299. Rep. in [Hermite, OC], I, pp. 10–37.

HERMITE (Charles), Note sur un théorème relatif aux nombres entiers, *Journal de mathématiques pures et appliquées,* 13 (1848), p. 15. Rep. Un théorème de Fermat, *Nouvelles annales de mathématiques,* 12 (1853), pp. 45–46. Rep. in [Hermite, OC], I, p. 264.

HERMITE (Charles), Note sur la théorie des fonctions elliptiques, *The Cambridge and Dublin Mathematical Journal* 3 (1848), pp. 54–56. Rep. in [Hermite, OC], I, pp. 71–73.

HERMITE (Charles), Sur la division des fonctions abéliennes ou ultra-elliptiques, *Mémoires présentés par divers savants à l'Académie royale des sciences de l'Institut de France : Sciences mathématiques et physiques,* 10 (1848), pp. 563–574. Rep. in [Hermite, OC], I, pp. 38–48.

HERMITE (Charles), Note sur la réduction des fonctions homogènes à coefficients entiers et à deux indéterminées, *Journal für die reine und angewandte Mathematik,* 36 (1848), pp. 357–364. Rep. in [Hermite, OC], I, pp. 84–93.

HERMITE (Charles), Démonstration élémentaire d'une proposition relative aux diviseurs de $x^2 + Ay^2$. *Journal de mathématiques pures et appliquées,* 14 (1849), pp. 451–452. Rep. in [Hermite, OC], I, pp. 274–275.

HERMITE (Charles), Sur une question relative à la théorie des nombres, *Journal de mathématiques pures et appliquées,* 14 (1849), pp. 21–30. Rep. in [Hermite, OC], I, p. 265–273.

HERMITE (Charles), Sur la théorie des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences,* 29 (1849), pp. 594–595. Rep. in [Hermite, OC], I, p. 74 (Augustin Cauchy's report on the memoir is reproduced p. 75–83).

HERMITE (Charles), Extraits de lettres de M. Ch. Hermite à M. Jacobi sur différents objets de la théorie des nombres, *Journal für die reine und angewandte Mathematik,* 40 (1850), pp. 261–278, 279–315. Rep. in [Hermite, OC], I, pp. 100–163.

HERMITE (Charles), Sur la théorie des formes quadratiques ternaires, *Journal für die reine und angewandte Mathematik,* 40 (1850), pp. 173–177. Rep. in [Hermite, OC], I, pp. 94–99.

HERMITE (Charles), Sur l'introduction des variables continues dans la théorie des nombres, *Journal für die reine und angewandte Mathematik*, 41 (1851), pp. 191–216. Rep. in [Hermite, OC], I, pp. 164–192.

HERMITE (Charles), Sur les fonctions à double période, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 32 (1851), pp. 442–450. [Only Cauchy's report on this memoir submitted to the Academy is available here].

HERMITE (Charles), Sur les fonctions algébriques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 32 (1851), pp. 458–461. Rep. in [Hermite, OC], I, pp. 276–280.

HERMITE (Charles), Sur l'introduction des variables continues dans la théorie des nombres, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 34 (1852), p. 133.

HERMITE (Charles), Sur l'extension du théorème de M. Sturm à un système d'équations simultanées, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 35 (1852), pp. 52–54. Rep. in [Hermite, OC], I, pp. 281–283.

HERMITE (Charles), Sur la décomposition d'un nombre en quatre carrés. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 37 (1853), pp. 133–134. Rep. in [Hermite, OC], I, pp. 288–289.

HERMITE (Charles), Remarques sur le théorème de Sturm. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 36 (1853), pp. 294–297. Rep. in [Hermite, OC], I, pp. 284–287.

HERMITE (Charles), Remarques sur un théorème de M. Cayley relatif aux déterminants gauches, *The Cambridge and Dublin Mathematical Journal*, 9 (1854), pp. 63–66. Rep. in [Hermite, OC], I, pp. 290–295.

HERMITE (Charles), Sur la théorie des fonctions homogènes à deux indéterminées, *The Cambridge and Dublin Mathematical Journal*, 9 (1854), pp. 172–217. Rep. in [Hermite, OC], I, pp. 296–349.

HERMITE (Charles), Sur la théorie des formes quadratiques ternaires indéfinies, *Journal für die reine und angewandte Mathematik*, 47 (1854), pp. 307–312. Rep. in [Hermite, OC], I, pp. 193–199.

HERMITE (Charles), Sur la théorie des formes quadratiques. Premier mémoire, *Journal für die reine und angewandte Mathematik*, 47 (1854), pp. 313–342. Rep. in [Hermite, OC], I, pp. 200–233.

HERMITE (Charles), Sur la théorie des formes quadratiques. Second mémoire, *Journal für die reine und angewandte Mathematik*, 47 (1854), pp. 343–368. Rep. in [Hermite, OC], I, pp. 234–263.

HERMITE (Charles), Sur la théorie de la transformation des fonctions abéliennes, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 40 (1855), pp. 249–254, 304–309, 365–369, 427–431, 485–489, 536–541, 704–707, 784–787. Rep. in [Hermite, OC], I, pp. 444–478.

HERMITE (Charles), Remarque sur un théorème de M. Cauchy, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 41 (1855), pp. 181–183. Rep. in [Hermite, OC], I, pp. 479–481.

Hermite (Charles), Sur la théorie des fonctions homogènes à deux indéterminées. Premier mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 1–17. Rep. in [Hermite, OC], I, pp. 350–371.

Hermite (Charles), Sur la théorie des fonctions homogènes à deux indéterminées. Second mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 18–38. Rep. in [Hermite, OC], I, pp. 372–396.

Hermite (Charles), Extrait d'une lettre de Mr. Ch. Hermite de Paris à Mr. Borchardt de Berlin sur le nombre des racines d'une équation algébrique comprises entre des limites données, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 39–51. Rep. in [Hermite, OC], I, pp. 397–414.

Hermite (Charles), Sur la théorie des polynômes homogènes du second degré, in Gerono, Camille-Christophe & Roguet, Charles, *Programme détaillé d'un cours d'arithmétique, d'algèbre et de géométrie analytique, comprenant les connaissances exigées pour l'admission aux écoles du gouvernement, et suivi de notes et des énoncés d'un grand nombre de problèmes*, Paris : Mallet-Bachelier, 1856, 4[e] éd., note VI. Rep. in [Hermite, OC], II, pp. 435–478.

Hermite (Charles), Extract of a letter from M. Hermite to Prof. Sylvester : Sur les solutions de l'équation $ax + by = n$, *Quarterly Journal of Pure and Applied Mathematics*, 1 (1857), pp. 370–373. Rep. in [Hermite, OC], I, pp. 440–443.

Hermite (Charles), Sur les formes cubiques à deux indéterminées, *Quarterly Journal of Pure and Applied Mathematics*, 1 (1857), pp. 20–22. Rep. in [Hermite, OC], I, pp. 434–436.

Hermite (Charles), Extrait d'une lettre à M. Borchardt sur le nombre limité d'irrationalités auxquelles se réduisent les racines des équations à coefficients entiers complexes d'un degré et d'un discriminant donnés, *Journal für die reine und angewandte Mathematik*, 53 (1857), pp. 182–192. Rep. in [Hermite, OC], I, pp. 415–428.

Hermite (Charles), Nouvelles Recherches de M. Kummer. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 45 (1857), pp. 1010–1011.

Hermite (Charles), Extrait d'une lettre à M. Borchardt sur l'invariabilité du nombre des carrés positifs et des carrés négatifs dans la transformation des polynômes homogènes du second degré, *Journal für die reine und angewandte Mathematik*, 53 (1857), pp. 271–274. Rep. in [Hermite, OC], I, pp. 429–433.

Hermite (Charles), Sur quelques formules relatives à la transformation des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 46 (1858), pp. 171–175 and *Journal de mathématiques pures et appliquées*, s. 2, 3 (1858), pp. 26–36. Rep. in [Hermite, OC], I, pp. 487–497.

Hermite (Charles), Sur la théorie des formes cubiques à trois indéterminées, *Journal de mathématiques pures et appliquées*, s. 2, 3 (1858), pp. 37–40. Rep. in [Hermite, OC], II, pp. 1–4.

Hermite (Charles), Sur la résolution de l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 46 (1858), pp. 508–515. Rep. in [Hermite, OC], II, pp. 5–12.

HERMITE (Charles), Sur la résolution de l'équation du quatrième degré, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 46 (1858), pp. 715–722. Rep. in [Hermite, OC], II, pp. 22–29.

HERMITE (Charles), Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 46 (1858), pp. 961–967. Rep. in [Hermite, OC], II, pp. 30–37.

HERMITE (Charles), Sur l'abaissement de l'équation modulaire du huitième degré, *Annali di Matematica pura ed applicata*, 2 (1859), pp. 59–61. Rep. in [Hermite, OC], II, pp. 83–86.

HERMITE (Charles), Sur la réduction des formes cubiques à deux indéterminées, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 48 (1859), pp. 351–357. Rep. in [Hermite, OC], II, pp. 93–99.

HERMITE (Charles), Sur l'interpolation, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 48 (1859), pp. 62–67. Rep. in [Hermite, OC], II, pp. 87–92.

HERMITE (Charles), Sur la théorie des équations modulaires, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 48 (1859), pp. 940–947, 1079–1084, 1095–1102. Rep. in [Hermite, OC], II, pp. 38–61.

HERMITE (Charles), Sur la théorie des équations modulaires (suite et fin), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 49 (1859), pp. 16–24, 110–118, 141–144. Rep. in [Hermite, OC], II, pp. 61–82.

HERMITE (Charles), Sur la théorie des équations modulaires et la résolution de l'équation du cinquième degré, Paris : Mallet-Bachelier, 1859.

HERMITE (Charles), Sur le résultant de trois formes quadratiques ternaires (extrait de lettre à M. Borchardt), *Journal für die reine und angewandte Mathematik*, 57 (1860), pp. 371–375. Rep. in [Hermite, OC], II, pp. 100–106.

HERMITE (Charles), Sur la transformation du troisième ordre des fonctions elliptiques, extrait d'une lettre de M. Hermite à l'éditeur, *Journal für die reine und angewandte Mathematik*, 60 (1861), p. 304. Rep. in *Annali di Matematica pura ed applicata*, 4 (1861), p. 176. Rep. in [Hermite, OC], II, pp. 239–240.

HERMITE (Charles), Sur l'invariant du dix-huitième ordre des formes du cinquième degré et sur le rôle qu'il joue dans la résolution de l'équation du cinquième degré, *Journal für die reine und angewandte Mathematik*, 59 (1861), pp. 304–305. Rep. in [Hermite, OC], II, pp. 107–108.

HERMITE (Charles), Lettre à M. Liouville, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 53 (1861), pp. 214–228. Rep. in [Hermite, OC], II, pp. 109–124.

HERMITE (Charles), Note sur la théorie des fonctions elliptiques, in LACROIX (Sylvestre François), *Traité élémentaire de calcul différentiel et de calcul intégral* (6e édition), Paris : Mallet-Bachelier, 1862, pp. 365–491. Rep. *Note sur le calcul différentiel et le calcul intégral*, Paris : Mallet-Bachelier, 1862.

HERMITE (Charles), Sur la théorie des formes quadratiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 55 (1862), pp. 684–692. Rep. in [Hermite, OC], II, pp. 255–263.

HERMITE (Charles), Sur la théorie des fonctions elliptiques et ses applications à l'arithmétique, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 55 (1862), pp. 11–18, 85–91. Rep. in [Hermite, OC], II, pp. 241–254.

HERMITE (Charles), Sur la théorie des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 57 (1863), pp. 613–618. Rep. in [Hermite, OC], II, pp. 264–270 (with another title).

HERMITE (Charles), Sur la théorie des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 57 (1863), pp. 993–1000. Rep. in [Hermite, OC], II, pp. 271–279 (with another title),.

HERMITE (Charles), Sur les fonctions de sept lettres, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 57 (1863), pp. 750–757. Rep. in [Hermite, OC], II, pp. 280–288.

HERMITE (Charles), Sur quelques formules relatives au module dans la théorie des fonctions elliptiques, *Journal de mathématiques pures et appliquées*, s. 2, 9 (1864), pp. 313–320. Rep. in [Hermite, OC], II, pp. 271–279.

HERMITE (Charles), Remarque sur le développement de $\cos amx$, *Journal de mathématiques pures et appliquées*, s. 2, 9 (1864), pp. 289–295. Rep. in [Hermite, OC], II, pp. 264–270.

HERMITE (Charles), Sur les théorèmes de M. Kronecker relatifs aux formes quadratiques, *Journal de mathématiques pures et appliquées*, s. 2, 9 (1864), pp. 145–159. Rep. in [Hermite, OC], II, pp. 241–254.

HERMITE (Charles), Sur un nouveau développement en série des fonctions, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 58 (1864), pp. 93–100, 266–273. Rep. in [Hermite, OC], II, pp. 293–308.

HERMITE (Charles), Extrait d'une lettre de M. Hermite à M. Brioschi, *Journal für die reine und angewandte Mathematik*, 63 (1864), pp. 30–32. Rep. in [Hermite, OC], II, pp. 289–292.

HERMITE (Charles), Sur l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 61 (1865), pp. 877–882, 965–972, 1073–1081. Rep. in [Hermite, OC], II, pp. 347–371.

HERMITE (Charles), Extrait d'une lettre de M. Hermite à M. Borchardt, *Journal für die reine und angewandte Mathematik*, 64 (1865), pp. 294–296. Rep. in [Hermite, OC], II, pp. 309–312.

HERMITE (Charles), Sur deux intégrales doubles, *Annales scientifiques de l'École normale supérieure*, 2 (1865), pp. 49–53. Rep. in [Hermite, OC], II, pp. 313–318.

HERMITE (Charles), Sur quelques développements en série de fonctions de plusieurs variables, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 60 (1865), pp. 370–377, 432–440, 461–466, 512–518. Rep. in [Hermite, OC], II, pp. 319–346.

HERMITE (Charles), Questions 778–779, *Nouvelles annales de mathématiques*, s. 2, 5 (1866), pp. 479–480.

Hermite (Charles), Sur le rayon de courbure des courbes gauches, *Nouvelles annales de mathématiques*, s. 2, 5 (1866), pp. 297–298. Rep. in [Hermite, OC], II, pp. 479–480.

Hermite (Charles), Sur l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 62 (1866), pp. 65–72, 157–162, 245–252, 715–722, 919–924, 959–966, 1054–1059, 1161–1167, 1213–1215. Rep. in [Hermite, OC], II, pp. 371–424.

Hermite (Charles), *Sur l'équation du cinquième degré*, Paris : Gauthier-Villars, 1866.

Hermite (Charles), Note sur l'observation en mathématiques, in Chevreul (Michel-Eugène), Distribution des connaissances humaines du ressort de la philosophie naturelle, *Mémoires de l'Académie des sciences*, s. 2, 35 (1866), pp. 519–584, note pp. 528–529. Rep. in [Hermite, OC], IV, pp. 586–590.

Hermite (Charles), Sur l'intégrale $\int \frac{x^m dx}{\sqrt{1-x^2}}$, *Annali di Matematica pura ed applicata*, s. 2, 1 (1867), pp. 155–158. Rep. in [Hermite, OC], II, pp. 481–485.

Hermite (Charles), Sur les invariants des formes du cinquième degré, in Salmon (George) & Bazin (Henry Emile) (trad.), *Leçons d'algèbre supérieure*, Paris : Gauthier-Villars, 1868, pp. 237–246. Rep. in [Hermite, OC], II, pp. 425–433.

Hermite (Charles), Sur l'invariant gauche des formes du sixième degré, in Salmon (George) & Bazin (Henry Emile) (trad.), *Leçons d'algèbre supérieure*, Paris : Gauthier-Villars, 1868, p. 247. Rep. in [Hermite, OC], II, p. 434.

Hermite (Charles), Préface, in Valson (Claude-Alphonse), *La vie et les travaux du Baron Cauchy, membre de l'Académie des sciences*, Paris : Gauthier-Villars, 1868, vol. 1, p. v. Rep. in [Hermite, OC], IV, p. 556.

Hermite (Charles), Sur le développement en série des intégrales elliptiques de première et de seconde espèce (extrait d'une lettre à M. Brioschi), *Annali di Matematica pura ed applicata*, s. 2, 2 (1868–1869), pp. 97–98. Rep. in [Hermite, OC], II, pp. 486–488.

Hermite (Charles), Sur l'intégrale $\int_{-1}^{+1} \frac{dx}{(a-x)\sqrt{1-x^2}}$, *Annali di Matematica pura ed applicata*, s. 2, 3 (1869), p. 83. Rep. in [Hermite, OC], II, p. 492.

Hermite (Charles), Sur l'expression du module des transcendantes elliptiques, en fonction du quotient des deux périodes, *Annali di Matematica pura ed applicata*, s. 2, 3 (1869), pp. 81–82. Rep. in [Hermite, OC], II, pp. 489–491.

Hermite (Charles), Sur la transcendante E_n, *Annali di Matematica pura ed applicata*, s. 2, 3 (1869), p. 83. Rep. in [Hermite, OC], II, p. 493.

Hermite (Charles), Sur l'intégrale $\int_{-1}^{+1} \frac{\sin \alpha dx}{1-2x\cos\alpha+x^2}$, *Bulletin des sciences mathématiques et astronomiques*, 1 (1870), pp. 320–323. Rep. in [Hermite, OC], II, pp. 494–497.

Hermite (Charles), Sur la construction géométrique de l'équation relative à l'addition des intégrales elliptiques de première espèce, *Bulletin des sciences mathématiques et astronomiques*, 2 (1871), pp. 21–23. Rep. in [Hermite, OC], II, pp. 498–500.

HERMITE (Charles), Sur l'intégration des fonctions circulaires, *Proceedings of the London Mathematical Society*, 4 (1871), pp. 164–175. Rep. in [Hermite, OC], III, pp. 66–114.

HERMITE (Charles), Sur l'équation $x^3 + y^3 = z^3 + u^3$, *Nouvelles annales de mathématiques*, s. 2, 11 (1872), pp. 5–8. Rep. in [Hermite, OC], III, pp. 115–117.

HERMITE (Charles), Sur l'intégration des fractions rationnelles, *Nouvelles annales de mathématiques*, s .2, 11 (1872), pp. 145–148. Rep. *Annales scientifiques de l'École normale supérieure*, s. 2, 1 (1872), pp. 215–218. See [Hermite, OC], III, p. 46.

HERMITE (Charles), On an application of the theory of unicursal curves. *Proceedings of the London Mathematical Society*, 4 (1873), pp. 343–345. Rep. in [Hermite, OC], III, pp. 123–126.

HERMITE (Charles), Sur l'élimination des fonctions arbitraires, *Report of the forty-second meeting of the British association for the advancement of science*, 42 (1873), Report on the state of science, pp. 233–238.

HERMITE (Charles), On the elimination of arbitrary functions, *Messenger of Mathematics*, 2 (1873), pp. 69–70.

HERMITE (Charles), Sur la fonction exponentielle, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 77 (1873), pp. 18–24, 74–79, 226–233, 285–293. Rep. in [Hermite, OC], III, pp. 150–181.

HERMITE (Charles), Sur une équation transcendante, *Bulletin des sciences mathématiques et astronomiques*, 4 (1873), pp. 61–64. Rep. in [Hermite, OC], III, pp. 131–134.

HERMITE (Charles), Extrait d'une lettre à Mr Borchardt (sur quelques approximations algébriques), *Journal für die reine und angewandte Mathematik*, 76 (1873), pp. 342–344. Rep. in [Hermite, OC], III, pp. 146–149.

HERMITE (Charles), *Cours d'analyse de l'École polytechnique*, Paris : Gauthier-Villars, 1873.

HERMITE (Charles), Sur le contact des surfaces, in HERMITE (Charles), *Cours d'analyse de l'École polytechnique*, 1873, pp. 139–149. Rep. in [Hermite, OC], III, pp. 485–495.

HERMITE (Charles), Extrait d'une lettre à M. Paul Gordan (sur l'expression $U \sin x + V \cos x + W$), *Journal für die reine und angewandte Mathematik*, 76 (1873), pp. 303–311. Rep. in [Hermite, OC], III, pp. 135–145.

HERMITE (Charles), Sur l'irrationalité de la base des logarithmes hyperboliques, *Report of the forty-third meeting of the British association for the advancement of science*, 43 (1874), pp. 22–23. Rep. in [Hermite, OC], III, pp. 127–130.

HERMITE (Charles), Proof of the irrationality of e^x when x is an integer, *Messenger of Mathematics*, 3 (1874), pp. 98–100.

HERMITE (Charles), Sur l'intégrale $\int_0^\pi (\frac{\sin^2 x}{1 - -2a\cos x + a^2})^m dx$ (extrait d'une lettre), *Nouvelle correspondance mathématique de M. E. Catalan*, 1 (1874), pp. 33–35. Rep. in [Hermite, OC], III, pp. 182–184.

HERMITE (Charles), Sur quelques intégrales définies, *Journal de l'École polytechnique*, 27 (Cahier 44) (1874), pp. 181–196.

HERMITE (Charles), Extrait d'une lettre à M. Borchardt sur la transformation des formes quadratiques ternaires en elles-mêmes, *Journal für die reine und angewandte Mathematik*, 78 (1874), pp. 325–328. Rep. in [Hermite, OC], III, pp. 185–189.

HERMITE (Charles), Extrait d'une lettre à M. Borchardt sur la réduction des formes quadratiques ternaires, *Journal für die reine und angewandte Mathematik*, 79 (1874), pp. 17–20. Rep. in [Hermite, OC], III, pp. 190–193.

HERMITE (Charles), *Sur la fonction exponentielle*, Paris : Gauthier-Villars, 1874.

HERMITE (Charles), Sur les développements de $F(x) = sn^a x cn^b x dn^c x$ où les exposants sont des entiers, *Bihang till Kongl. svenska vetenskaps-akademiens Handlingar (Académie royale des sciences de Stockholm)*, 3 (10) (1875), pp. 3–10. Rep. in [Hermite, OC], III, pp. 222–231.

HERMITE (Charles), Extrait d'une lettre à M. Fuchs de Gottingue sur quelques équations différentielles linéaires, *Journal für die reine und angewandte Mathematik*, 79 (1875), pp. 324–338. Rep. in [Hermite, OC], III, pp. 194–210.

HERMITE (Charles), Lettre à M. Borchardt sur la fonction de Jacob Bernoulli, *Journal für die reine und angewandte Mathematik*, 79 (1875), pp. 339–344. Rep. in [Hermite, OC], III, pp. 215–221.

HERMITE (Charles), Sur le développement de l'inverse du sinus d'amplitude et de son carré, suivant les puissances croissantes de la variable, in *Association française pour l'avancement des sciences : compte-rendu de la 4e session, Nantes 1875*, Paris : AFAS, 1876, pp. 131–136.

HERMITE (Charles), Sur un exemple de réduction d'intégrales abéliennes aux fonctions elliptiques, *Annales de la Société scientifique de Bruxelles*, 1 (1876), pp. 1–16. Rep. in [Hermite, OC], III, pp. 249–261.

HERMITE (Charles), Sur l'aire d'un segment de courbe convexe. Question 95, *Nouvelle correspondance mathématique de M. E. Catalan*, 2 (1876), p. 217. Rep. in [Hermite, OC], III, p. 248.

HERMITE (Charles), Sur une formule de M. Delaunay (extrait d'une lettre à M. Paul Mansion), *Nouvelle correspondance mathématique*, 2 (1876), pp. 54–55. Rep. in [Hermite, OC], III, pp. 246–247.

HERMITE (Charles), Extrait d'une lettre à M. Königsberger sur le développement des fonctions elliptiques suivant les puissances croissantes de la variable, *Journal für die reine und angewandte Mathematik*, 81 (1876), pp. 220–228. Rep. in [Hermite, OC], III, pp. 236–245.

HERMITE (Charles), Sur un théorème d'Eisenstein, *Proceedings of the London Mathematical Society*, 7 (1876), pp. 173–175. Rep. in [Hermite, OC], III, pp. 232–235.

HERMITE (Charles), Extrait d'une lettre à M. Borchardt (sur les nombres de Bernoulli), *Journal für die reine und angewandte Mathematik*, 81 (1876), pp. 93–95. Rep. in [Hermite, OC], III, pp. 211–214.

Hermite (Charles), Présentation d'un Mémoire de M. P. du Bois-Reymond, intitulé : "Recherches sur la convergence et la divergence des formules de représentation de Fourier", *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 82 (1876), pp. 756–757.

Hermite (Charles), Lettre à M. P. Gordan, *Mathematische Annalen*, 10 (1876), pp. 287–288.

Hermite (Charles), Extrait d'une lettre à M. L. Fuchs, *Journal für die reine und angewandte Mathematik*, 82 (1877), pp. 343–347. Rep. in [Hermite, OC], III, pp. 421–424.

Hermite (Charles), Note sur une formule de Jacobi, *Mémoires de la Société royale des sciences de Liège*, 6 (1877), pp. 1–7. Rep. in [Hermite, OC], III, pp. 262–265.

Hermite (Charles), Sur quelques applications des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 85 (1877), pp. 689–695, 728–732, 821–826, 870–875, 984–990, 1085–1091, 1185–1191. Rep. in [Hermite, OC], III, pp. 267–310.

Hermite (Charles), Études de M. Sylvester sur la théorie algébrique des formes, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 84 (1877), pp. 974–975. Rep. in [Hermite, OC], III, p. 419.

Hermite (Charles), Sur les formules de M. Frenet, *Jornal de Sciencias Mathematicas e Astronomicas de M. Gomes Teixeira*, 1 (1877), pp. 65–70. Rep. in [Hermite, OC], IV, pp. 508–511.

Hermite (Charles), Sur l'intégrale $\int_0^{2\pi} f(\sin x, \cos x)\,dx$, *Jornal de Sciencias Mathematicas e Astronomicas de M. Gomes Teixeira*, 2 (1878), pp. 65–67. Rep. in [Hermite, OC], IV, pp. 505–507.

Hermite (Charles), Extrait d'une lettre à M. Gylden, de Stockholm : Sur le pendule, *Journal für die reine und angewandte Mathematik*, 85 (1878), pp. 246–249. Rep. in [Hermite, OC], III, pp. 447–450.

Hermite (Charles), Extrait d'une lettre à M. Lindemann : Observations algébriques sur les courbes planes, *Journal für die reine und angewandte Mathematik*, 84 (1878), pp. 298–299. Rep. in [Hermite, OC], III, pp. 444–446.

Hermite (Charles), Sur la décomposition d'une fraction rationnelle en éléments simples, *Annales de la Société scientifique de Bruxelles*, 2 (1878), Seconde partie : Mémoires, pp. 157–160.

Hermite (Charles), Sur quelques applications des fonctions elliptiques (suite), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 86 (1878), pp. 271–277, 422–427, 622–628, 777–780, 850–854. Rep. in [Hermite, OC], III, pp. 309–335.

Hermite (Charles), Sur l'équation de Lamé (extrait d'une lettre à M. Brioschi), *Annali di Matematica pura ed applicata*, s. 2, 9 (1878), pp. 21–24. Rep. in [Hermite, OC], III, pp. 475–478.

Hermite (Charles), Sur la formule de Maclaurin (extrait d'une lettre à M. Borchardt), *Journal für die reine und angewandte Mathematik*, 84 (1878), 6pp. 4–69. Rep. in [Hermite, OC], III, pp. 425–431.

HERMITE (Charles), Sur l'intégrale $\int_0^1 \frac{z^{a-1}-z^{-a}}{1-z}dz$, *Atti della Reale Accademia di Scienze di Torino*, 14 (1878), pp. 91–115. Rep. in [Hermite, OC], III, pp. 455–474.

HERMITE (Charles), Sur la théorie des fonctions sphériques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 86 (1878), pp. 1515–1519. Rep. in [Hermite, OC], III, p. 451–454.

HERMITE (Charles), Sur la formule d'interpolation de Lagrange (extrait d'une lettre à M. Borchardt), *Journal für die reine und angewandte Mathematik*, 84 (1878), pp. 70–79. Rep. in [Hermite, OC], III, pp. 432–443.

HERMITE (Charles), Sur la différentiation des fonctions elliptiques par rapport au module, *Astronomische Nachrichten*, 96 (21) (1879), pp. 321–326. Rep. in [Hermite, OC], IV, pp. 1–7.

HERMITE (Charles), Sur quelques applications des fonctions elliptiques (suite), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 89 (1879), pp. 1001–1005, 1092–1097. Rep. in [Hermite, OC], III, pp. 335–344.

HERMITE (Charles), Sur quelques applications des fonctions elliptiques (suite), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 90 (1880), pp. 106–112, 201–208, 478–483, 643–649, 761–766. Rep. in [Hermite, OC], III, pp. 351–377.

HERMITE (Charles), Équations différentielles linéaires, *Bulletin des sciences mathématiques et astronomiques*, s. 2, 3 (1879), pp. 311–325. Rep. in [Hermite, OC], III, pp. 496–508.

HERMITE (Charles), Sur l'indice des fractions rationnelles, *Bulletin de la Société mathématique de France*, 7 (1879), pp. 128–131. Rep. in [Hermite, OC], III, pp. 509–512.

HERMITE (Charles), Sur une extension donnée à la théorie des fractions continues par M. Tchebychef (Extrait d'une lettre à M. Borchardt), *Journal für die reine und angewandte Mathematik*, 88 (1880), pp. 10–15. Rep. in [Hermite, OC], III, pp. 513–519.

HERMITE (Charles), Sur l'intégration de l'équation différentielle de Lamé (Extrait d'une lettre à M. E. Heine), *Journal für die reine und angewandte Mathematik*, 89 (1880), pp. 9–18. Rep. in [Hermite, OC], IV, pp. 8–15.

HERMITE (Charles), Sur une proposition de la théorie des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 90 (1880), pp.1096–1098. Rep. in [Hermite, OC], IV, p. 16–21.

HERMITE (Charles), Sur la série de Fourier et autres représentations analytiques des fonctions d'une variable réelle, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 91 (1880), pp. 1018–1019. Rep. in [Hermite, OC], IV, pp. 22–24.

HERMITE (Charles), Sur une formule d'Euler, *Journal de mathématiques pures et appliquées*, s. 3, 6 (1880), pp. 5–18. Rep. in [Hermite, OC], IV, pp. 25–38.

HERMITE (Charles), Sur une représentation analytique des fonctions au moyen des transcendantes elliptiques (extrait d'une lettre à M. U. Dini), *Annali di*

Matematica pura ed applicata, s. 2, 10 (1880), pp. 137–144. Rep. in [Hermite, OC], IV, pp. 39–47.

HERMITE (Charles), Sur les équations différentielles linéaires du second ordre (extrait d'une lettre à M. Brioschi), *Annali di Matematica pura ed applicata*, s. 2, 10 (1880), pp. 101–103. Rep. in [Hermite, OC], IV, pp. 516–518.

HERMITE (Charles), Extrait d'une lettre sur la série $\frac{1}{(\log 2)^n} + \frac{1}{(\log 3)^n} + \cdots + \frac{1}{(\log x)^n} + \cdots$, *Mathesis*, 1 (1881), p. 37. Rep. in [Hermite, OC], IV, pp. 503–504.

HERMITE (Charles), Sur les fonctions $\Theta(x)$ et $H(x)$ de Jacobi (estratto di lettera al Prof. E. Beltrami), in CREMONA (Luigi) & BELTRAMI (Eugenio) (éd.), *In memoriam Dominici Chelini : Collectanea mathematica*, Milan, Naples, Pise : Hoepli, 1881, pp. 1–5. Rep. in [Hermite, OC], IV, pp. 87–91.

HERMITE (Charles), Extrait d'une lettre à M. Gyldén sur la détermination de l'intégrale $\int \frac{dn^4 u}{(sn^2 a - sn^2 u)} du$, *Astronomische Nachrichten*, 101 (2) (1881), pp. 17–18. Rep. in [Hermite, OC], IV, pp. 85–86.

HERMITE (Charles), Sur l'intégrale eulérienne de seconde espèce (extrait d'une lettre adressée à M. Schwarz de Göttingen), *Journal für die reine und angewandte Mathematik*, 90 (1881), pp. 332–338. Rep. in [Hermite, OC], IV, pp. 76–84.

HERMITE (Charles), Sur quelques applications de la théorie des fonctions elliptiques (suite), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 93 (1881), pp. 920–925, 1098–1103. Rep. in [Hermite, OC], III, pp. 377–388.

HERMITE (Charles), Sur quelques points de la théorie des fonctions (extrait d'une lettre à M. Mittag-Leffler), *Journal für die reine und angewandte Mathematik*, 91 (1881), pp. 54–78. Rep. in [Hermite, OC], IV, pp. 48–75.

HERMITE (Charles), Sur une relation donnée par M. Cayley, dans la théorie des fonctions elliptiques (extrait d'une lettre adressée à M. Mittag-Leffler), *Acta Mathematica*, 1 (1882), pp. 368–370. Rep. in [Hermite, OC], IV, pp. 108–110.

HERMITE (Charles), Sur quelques applications de la théorie des fonctions elliptiques (suite), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 94 (1882), pp. 186–192, 372–377, 477–482, 594–600, 753–759. Rep. in [Hermite, OC], III, pp. 388–418.

HERMITE (Charles), Sur l'intégrale elliptique de troisième espèce, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 94 (1882), pp. 901–904. Rep. in [Hermite, OC], IV, pp. 104–107.

HERMITE (Charles), Sur une application du théorème de M. Mittag-Leffler dans la théorie des fonctions, *Journal für die reine und angewandte Mathematik*, 92 (1882), pp. 145–155. Rep. in *Acta Societatis Scientiarum Fennicæ*, 12 (1883), pp. 425–436. Rep. in [Hermite, OC], IV, pp. 92–103.

HERMITE (Charles), Sur la réduction des intégrales hyperelliptiques aux fonctions de première, de seconde et de troisième espèce, *Bulletin des sciences mathématiques et astronomiques*, s. 2, 7 (1883), pp. 36–42. Rep. in [Hermite, OC], IV, pp. 120–126.

Hermite (Charles), Sur quelques points de la théorie des nombres (extrait d'une lettre à M. Lipschitz), *Acta Mathematica*, 2 (1883), pp. 299–300. Rep. in [Hermite, OC], IV, pp. 127–128.

Hermite (Charles), Sur la fonction $sn^a x$ (extrait d'une lettre à M. Mittag-Leffler), *Acta Societatis Scientiarum Fennicæ*, 12 (1883), pp. 437–443. Rep. in [Hermite, OC], IV, pp. 114–119.

Hermite (Charles), Remarques au sujet de la communication de M. Lipschitz sur un point de la théorie des fonctions elliptiques, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 97 (1883), pp. 1414–1415. Rep. in [Hermite, OC], IV, pp. 111–113.

Hermite (Charles), Sur deux limites d'une intégrale définie, *Mathesis*, 3 (1883), p. 82.

Hermite (Charles), Sur une formule relative à la théorie des fonctions d'une variable, *American Journal of Mathematics*, 6 (1884), pp. 60–62. Rep. in [Hermite, OC], IV, pp. 133–135.

Hermite (Charles), Extract from a Letter to Mr. Sylvester from M. Hermite, *American Journal of Mathematics*, 6 (1883–1884), pp. 173–174. Rep. in [Hermite, OC], IV, pp. 136–137.

Hermite (Charles), Sur la réduction des intégrales hyperelliptiques (extrait d'une lettre à M. le prof. Chrystal), *Proceedings of the Royal Society of Edinburgh*, 12 (1884), pp. 642–646. Rep. in [Hermite, OC], IV, pp. 201–205.

Hermite (Charles), Sur l'usage des produits infinis dans la théorie des fonctions elliptiques (extrait d'une lettre à M. Lipschitz), *Acta Mathematica*, 4 (1884), p. 193. Rep. in [Hermite, OC], IV, p. 175.

Hermite (Charles), Sur quelques conséquences arithmétiques des formules de la théorie des fonctions elliptiques, *Bulletin de l'Académie des sciences de St Pétersbourg*, 29 (1884), pp. 325–352. Rep. in *Acta Mathematica* 5 (1884), pp. 297–330, cor. *Acta Mathematica*, 8 (1886), p. II. Rep. in [Hermite, OC], IV, pp. 138–168.

Hermite (Charles), Sur une identité trigonométrique, *Nouvelles annales de mathématiques*, s. 3, 4 (1885), pp. 57–59. Rep. in [Hermite, OC], IV, pp. 206–208.

Hermite (Charles), Sur une fonction uniforme (note au sujet de la communication de M. Stieltjes), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 101 (1885), pp. 112–115. Rep. in [Hermite, OC], IV, pp. 184–187.

Hermite (Charles), Sur la théorie des fractions continues, *Bulletin des sciences mathématiques*, 9 (= 20) (1885), pp. 11–13. Rep. in [Hermite, OC], IV, pp. 178–180.

Hermite (Charles), Sur une application de la théorie des fonctions doublement périodiques de seconde espèce, *Annales scientifiques de l'École normale supérieure*, 2 (1885), pp. 303–314. Rep. in [Hermite, OC], IV, pp. 190–200.

Hermite (Charles), Sur les fonctions holomorphes, *Journal de mathématiques pures et appliquées*, s. 4, 1 (1885), pp. 9–10. Rep. in [Hermite, OC], IV, pp. 188–189.

Hermite (Charles), Sur les polynômes de Legendre (extrait d'une lettre à M. F. Gomes Teixeira), *Jornal de Sciencias Mathematicas e Astronomicas de M. Gomes Teixeira*, 6 (1885), pp. 81–84. Rep. in [Hermite, OC], IV, pp. 169–171.

Hermite (Charles), Note (sur une lettre de M. Obrastzoff, de Saint-Pétersbourg, à propos de la fraction continue de Lambert, qui donne tan x), *Bulletin des sciences mathématiques*, 9 (= 20) (1885), pp. 135–137. Rep. in [Hermite, OC], IV, pp. 181–183.

Hermite (Charles), Discours prononcé aux obsèques de M. Bouquet, au nom de la Faculté des Sciences, *Bulletin des sciences mathématiques*, 9 (= 20) (1885), pp. 303–304. Rep. in [Hermite, OC], IV, pp. 176–177.

Hermite (Charles), Sur un théorème de Galois relatif aux équations solubles par radicaux, in Serret (Joseph-Alfred), *Cours d'algèbre supérieure* (5e éd), Paris : Gauthier-Villars, 1885, t. 2, p. 677–680.

Hermite (Charles), Sur les valeurs asymptotiques de quelques fonctions numériques (Extrait d'une lettre adressée à M. Fuchs), *Journal für die reine und angewandte Mathematik*, 99 (1886), pp. 324–328. Rep. in [Hermite, OC], IV, pp. 209–214.

Hermite (Charles), Remarques sur les formes quadratiques de déterminant négatif, *Bulletin des sciences mathématiques*, 10 (= 21) (1886), pp. 23–30. Rep. in [Hermite, OC], IV, p. 215–222.

Hermite (Charles), *Sur quelques applications des fonctions elliptiques*, Paris : Gauthier-Villars, 1886.

Hermite (Charles), Extrait de deux lettres adressées à M. Craig, *American Journal of Mathematics*, 9 (1887), ppm. 381–388. Rep. in [Hermite, OC], IV, pp. 241–250.

Hermite (Charles), Rosenhain, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 104 (1887), p. 891. Rep. in [Hermite, OC], IV, p. 240.

Hermite (Charles), Remarques arithmétiques sur quelques formules de la théorie des fonctions elliptiques, *Journal für die reine und angewandte Mathematik*, 100 (1887), pp. 51–65. Rep. in [Hermite, OC], IV, pp. 223–239.

Hermite (Charles), Remarques sur la décomposition en éléments simples des fonctions doublement périodiques, *Annales de la Faculté des Sciences de Toulouse*, 2 (1888), C1-C12. Rep. in [Hermite, OC], IV, pp. 262–273.

Hermite (Charles), Sur un mémoire de Laguerre concernant les équations algébriques, *Memorie della pontificia accademia dei nuovi Lincei*, 3 (1888), pp. 155–164. Rep. in [Hermite, OC], IV, pp. 251–259.

Hermite (Charles), Démonstration nouvelle d'une formule relative aux intégrales eulériennes de seconde espèce (extrait d'une lettre à M. Lerch), *Sitzungsberichte der Königlichen Böhmischen Gesellschaft der Wissenschaften* 1 (1888), pp. 364–365. Rep. in [Hermite, OC], IV, pp. 260–261.

Hermite (Charles), Sur la transformation de l'intégrale elliptique de seconde espèce (Extrait d'une lettre adressée à M. Matyas Lerch), *Abhandlungen der Königlichen Böhmischen Gesellschaft der Wissenschaften*, s. 7, 2 (1888), pp. 1–7. Rep. in [Hermite, OC], IV, p. 274–279.

Hermite (Charles), Discours prononcé aux funérailles de M. Halphen le 23 mai 1889, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 108 (1889), pp. 1079–1081. Rep. in [Hermite, OC], IV, pp. 280–282.

HERMITE (Charles), Discours prononcé lors de la séance publique annuelle du lundi 30 décembre 1889, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 109 (1889), pp. 991–999. Rep. in [Hermite, OC], IV, pp. 567–575.

HERMITE (Charles), Sur les racines de la fonction sphérique de seconde espèce (extrait d'une lettre adressée à M. Lerch), *Annales de la Faculté des sciences de Toulouse*, 4 (2) (1890), pp. 1–10. Rep. in [Hermite, OC], IV, pp. 327–336.

HERMITE (Charles), Sur les polynômes de Legendre, *Rendiconti del Circolo Matematico di Palermo*, 4 (1890), pp. 146–152. Rep. in [Hermite, OC], IV, pp. 314–320.

HERMITE (Charles), Discours prononcé devant le Président de la République le 5 août 1889, à l'inauguration de la nouvelle Sorbonne, *Bulletin des sciences mathématiques*, 14 (24) (1890), pp. 6–36. Rep. in [Hermite, OC], IV, pp. 283–313.

HERMITE (Charles), Discours prononcé lors de la séance publique annuelle du lundi 29 décembre 1890, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 111 (1890), pp. 1015–1019. Rep. in [Hermite, OC], IV, pp. 576–581.

HERMITE (Charles), Sur la transformation des fonctions elliptiques (extrait d'une lettre adressée à M. S. Pincherle), *Rendiconti del Circolo Matematico di Palermo*, 5 (1891), pp. 155–157. Rep. in [Hermite, OC], IV, pp. 337–339.

HERMITE (Charles), Sur les polynômes de Legendre (extrait d'une lettre à M. F. Caspary), *Journal für die reine und angewandte Mathematik*, 107 (1891), pp. 80–83. Rep. in [Hermite, OC], IV, pp. 321–326.

HERMITE (Charles), Sur une formule de Jacobi concernant l'intégrale elliptique à module imaginaire, *Mitteilungen der Mathematischen Gesellschaft in Hamburg*, 3 (1891–1900) [1890], pp. 22–24. Rep. in [Hermite, OC], IV, pp. 521–522.

HERMITE (Charles), Sur l'addition des arguments dans les fonctions elliptiques, *Jornal de Sciencias Mathematicas e Astronomicas de M. Gomes Teixeira*, 11 (1892), pp. 65–66. Rep. in [Hermite, OC], IV, pp. 512–513.

HERMITE (Charles), Sur la transformation des fonctions elliptiques, *Annales de la Faculté des Sciences de Toulouse*, 6 (4) (1892), L1-L13. Rep. *Rozpravy České Akademie cisaře Františka Josefa*, 1 (1892), pp. 619–638 (with a Czech transl.). Rep. in [Hermite, OC], IV, pp. 343–354.

HERMITE (Charles), Note sur M. Kronecker, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 114 (1892), pp. 19–21. Rep. in [Hermite, OC], IV, pp. 340–342.

HERMITE (Charles), Discours prononcé à l'occasion de son Jubilé, in *1822–1892. Jubilé de M. Hermite (24 décembre)*, Paris : Gauthier-Villars, 1893, pp. 13–17. Rep. in [Hermite, OC], IV, pp. 582–585.

HERMITE (Charles), Sur quelques propositions fondamentales de la théorie des fonctions elliptiques, in MOORE (Eliakim Hastings), BOLZA (Oskar), MASCHKE (Heinrich) & WHITE (Henry S.) (eds), *Mathematical Papers Read at the International Mathematical Congress held in connection with the World's Columbian Exposition (Chicago 1893)*, New York : MacMillan, American Mathematical Society, 1893, pp. 105–115. Repr. in [Hermite, OC], IV, pp. 451–462.

Hermite (Charles), Sur une intégrale définie (extrait d'une lettre à M. Ed. Weyr), *Časopis pro Pěstování Matematiky a Fysiky*, 23 (1893), pp. 273–274. Rep. in [Hermite, OC], IV, pp. 399–400.

Hermite (Charles), Sur une extension de la formule de Stirling, *Mathematische Annalen*, 41 (1893), pp. 581–590. Rep. in [Hermite, OC], IV, pp. 378–388.

Hermite (Charles), Sur la généralisation des fractions continues algébriques (extrait d'une lettre à M. Pincherle), *Annali di Matematica pura ed applicata*, s. 2, 21 (1893), pp. 289–308. Rep. in [Hermite, OC], IV, pp. 357–377.

Hermite (Charles), Notice sur les travaux de M. Kummer, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 16 (1893), pp. 1163–1164. Rep. in [Hermite, OC], IV, pp. 355–356.

Hermite (Charles), Remarque sur la définition du logarithme des quantités imaginaires (extrait de deux lettres à M. Ed. Weyr), *Časopis pro Pěstování Matematiky a Fysiky*, 22 (1894), pp. 225–232. Rep. in [Hermite, OC], IV, pp. 496–499.

Hermite (Charles), Sur la fonction eulérienne (extrait d'une lettre à M. Ed. Weyr), *Časopis pro Pěstování Matematiky a Fysiky*, 23 (1894), pp. 65–66. Rep. in [Hermite, OC], IV, pp. 397–398.

Hermite (Charles), Remarque sur les nombres de Bernoulli et les nombres d'Euler (extrait de deux lettres à M. Ed. Weyr), *Sitzungsberichte der Königlichen Böhmischen Gesellschaft der Wissenschaften* (1894), pièce XXXVII, pp. 1–4. Rep. in [Hermite, OC], IV, pp. 393–396.

Hermite (Charles), Sur les polynômes entiers à une variable, *Nyt Tidsskrift for Matematik* 5 (B) (1894), pp. 1–4. Rep. in [Hermite, OC], IV, pp. 389–392.

Hermite (Charles), Sur la fonction $\Gamma(x)$ (extrait d'une lettre à M. Lerch), *Bulletin international de l'Académie de l'empereur François Joseph, Prague* (1895), pp. 214–219. Rep. in [Hermite, OC], IV, pp. 554–555.

Hermite (Charles), Extrait d'une lettre à M. Craig, *American Journal of Mathematics*, 17 (1895), pp. 6–10. Rep. in [Hermite, OC], IV, pp. 488–494.

Hermite (Charles), Sur le logarithme de la fonction Gamma, *American Journal of Mathematics*, 17 (1895), pp. 111–116. Rep. in [Hermite, OC], IV, pp. 423–430.

Hermite (Charles), Sur la fonction $\log\Gamma(a)$ (extrait d'une lettre à M. K. Hensel), *Journal für die reine und angewandte Mathematik*, 115 (1895), pp. 201–208. Rep. in [Hermite, OC], IV, pp. 412–422.

Hermite (Charles), Sur l'équation bicarrée, *Mathesis* 5 (1895), pp. 11–12. Rep. in [Hermite, OC], IV, pp. 403–404.

Hermite (Charles), Sur les nombres de Bernoulli, *Mathesis*, 5 (Supplément II) (1895), pp. 1–7. Rep. in [Hermite, OC], IV, pp. 405–411.

Hermite (Charles), Notice sur M. Cayley, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 120 (1895), pp. 233–234. Rep. in [Hermite, OC], IV, pp. 401–402.

Hermite (Charles), Préface, in Appell (Paul) & Goursat (Édouard), *Théorie des fonctions algébriques et de leurs intégrales : étude des fonctions analytiques sur une surface* *de Riemann*, Paris : Gauthier-Villars, 1895, pp. a-g. Rep. in [Hermite, OC], IV, pp. 557–562.

Hermite (Charles), Sur une formule de M. G. Fontené, *Bulletin des sciences mathématiques*, 20 (1896), pp. 218–220. Rep. in [Hermite, OC], IV, pp. 448–450.

Hermite (Charles), Sur une extension de théorème de Laurent (extrait d'une lettre adressée à M. L. Fuchs), *Journal für die reine und angewandte Mathematik*, 116 (1896), pp. 85–89. Rep. in [Hermite, OC], IV, pp. 431–436.

Hermite (Charles), Question 951, *Intermédiaire des mathématiciens*, 4 (1897), p. 1. Rep. in [Hermite, OC], IV, p. 495.

Hermite (Charles), Sur quelques développements en série de la théorie des fonctions elliptiques, *Bulletin de la société physico-mathématique de Kasan*, s. 2, 6 (1897), p. 1–21. Rep. in [Hermite, OC], IV, pp. 470–487.

Hermite (Charles), Notice sur M. Weierstrass, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 124 (1897), pp. 430–433. Rep. in [Hermite, OC], I, pp. 463–466.
textscHermite (Charles), Notice sur M. Brioschi, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 125 (1897), pp. 1139–1141. Italian trans. in *Bolletino di bibliografia e storia delle science matematiche*, 1, (1898), pp. 62–73. Rep. in [Hermite, OC], IV, pp. 467–469.

Hermite (Charles), De la sommation d'une série considérée par Abel [in Russian], *Bulletin de la société physico-mathématique de Kasan*, s. 2, 8 (1898), p. 107–108. Rep. in French in [Hermite, OC], IV, pp. 514–515.

Hermite (Charles), Expression approchée de l'aire d'un segment de courbe [in Russian], *Bulletin de la société physico-mathématique de Kasan*, s. 2, 8 (1898), p. 109.

Hermite (Charles), Préface, in *Œuvres mathématiques de Riemann*, tr. Léonce Laugel, Paris : Gauthier-Villars, 1898, pp. vii-xi.

Hermite (Charles), Extrait d'une lettre adressée à L. Lindelöf, *Öfversigt af Finska vetenskaps-societetens förhandlingar*, 92 (1899–1900), pp. 88–90. Rep. in [Hermite, OC], IV, pp. 500–502.

Hermite (Charles), Lettre adressée à M. B. Anissimov, *Matematicheskii Sbornik (Bulletin de la société mathématique de Moscou)*, 21 (1) (1900), pp. 66–67. Rep. in [Hermite, OC], IV, pp. 519–520.

Hermite (Charles), [Lettre à Jules Tannery sur les fonctions modulaires], 24 septembre 1900, in Tannery (Jules) & Molk (Jules) Éléments de la théorie des fonctions elliptiques (4 vols), Paris : Gauthier-Villars, (1893–1902), vol. 4, pp. 294–302. Rep. in [Hermite, OC], II, pp. 13–21.

Hermite (Charles), Sulle Frazioni continue, *Le Matematiche pure ed applicate*, 1 (1901), p. 1. Rep. in [Hermite, OC], IV, pp. 552–553.

Hermite (Charles), Extrait d'une lettre à M. Jahnke, *Archiv der Mathematik und Physik*, 1 (1901), p. 20. Rep. in [Hermite, OC], I, pp. 544–545.

Hermite (Charles), Sur une équation transcendante, *Archiv der Mathematik und Physik*, 1 (1901), p. 22. Rep. in [Hermite, OC], IV, pp. 546–551.

HERMITE (Charles), Extraits de quelques lettres à M. Pincherle, *Annali di Matematica pura ed applicata*, s. 3, 5 (1901), pp. 57–72. Rep. in [Hermite, OC], IV, pp. 529–543.

HERMITE (Charles), Mémoire sur l'extension du théorème de M. Sturm à un système d'équations simultanées, in [Hermite, OC], III, pp. 1–34. (posthum. pub.)

HERMITE (Charles), & ANDOYER (Henri) (réd.), *Cours à la Faculté des sciences de Paris*, Paris : Hermann, 1882.

HERMITE (Charles), & ANDOYER (Henri) (réd.), *Cours à la Faculté des sciences de Paris* (2e éd.), Paris : Hermann, 1883.

HERMITE (Charles), & ANDOYER (Henri) (réd.), *Cours à la Faculté des sciences de Paris* (3e éd.), Paris : Hermann, 1887.

HERMITE (Charles), & ANDOYER (Henri) (réd.), *Cours à la Faculté des sciences de Paris* (4e éd.), Paris : Hermann, 1891.

HERMITE (Charles), & LIOUVILLE (Joseph), Lettre de M. Hermite à M. Liouville et Réponse de M. Liouville, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 53 (1861), pp. 214–231. Rep. in [Hermite, OC], II, pp. 109–124 (only Hermite's letter).

HERMITE (Charles), & LIOUVILLE (Joseph), Sur la théorie des fonctions elliptiques et ses applications à l'Arithmétique et Réponse et Note de M. Liouville. *Journal de mathématiques pures et appliquées*, s. 2, 7 (1862), pp. 25–48. Rep. in [Hermite, OC], II, pp. 109–124 (only Hermite's letter).

HERMITE (Charles), & NATANI (Leopold) (trad.), *Uebersicht der Theorie der elliptischen Funktionen*, Berlin : Wiegandt & Hempel, 1863.

HERMITE (Charles), POINCARÉ (Henri) & ROUCHÉ (Eugène) (éd.), *Œuvres de Laguerre*, 2 vols., Paris : Gauthier-Villars, 1898 et 1905.

HERMITE (Charles), & SONIN (Nicolai Yakovlevich), Sur les polynômes de Bernoulli (extrait de lettres), *Journal für die reine und angewandte Mathematik*, 116 (1896), pp. 133–156. Rep. in [Hermite, OC], IV, pp. 437–447 (only Hermite's letter).

HERMITE (Charles) & TANNERY (Jules), Sur quelques points de la théorie des fonctions, *Bulletin des sciences mathématiques*, 5 (1881), pp. 312–320.

2. ŒUVRES

PICARD (Émile) (éd.), *Œuvres de Charles Hermite publiées sous les auspices de l'Académie des sciences*, 4 vols., Paris : Gauthier-Villars, 1905–1917.

3. CORRESPONDENCE

Let us remark that letters from or to Hermite were also published as articles in journals during the lifetime of their authors. Those from Hermite are listed above among his regular articles.

3.1. *Complete editions*

HERMITE (Charles), & STIELTJES (Thomas), *Correspondance*, éd. BAILLAUD (Benjamin) & BOURGET (Henri), 2 vols., Paris : Gauthier-Villars, 1905. A supplement has been published by DUGAC (Pierre), Partie inédite de la correspondance d'Hermite avec Stieltjes, *Cahiers du séminaire d'histoire des mathématiques*, 4 (1983), pp. 75–87.

HERMITE (Charles), Briefe zu P. Du Bois-Reymond aus den Jahren 1875–1888, ed. LAMPE (Emil), *Archiv der Mathematik und Physik*, 24–3 (1916), pp. 193–220, 289–310.

TCHEBICHEFF (Pafnuti Lvovitch), *Polnoe sobranie cotchinenii* [*Complete Collected Works*], Moscow, Leningrad : Izdatelʹstvo Akademii Nauk SSSR, vol. V, 1951, pp. 424–436. It contains 11 letters from Hermite to Tchebichef, translated into Russian. On this, see INGTEM (Natalia), La correspondance entre Ch. Hermite et P. L. Tchébysheff, contribution to Hermite's Days for the bicentenary of his birth, December 2022.

HERMITE (Charles), Lettres à A. Markov, 1885–1899, ed. OGIGOVA (Helen), *Revue d'histoire des sciences et de leurs applications*, 20–1 (1967), pp. 1–32.

HERMITE (Charles), Lettres à Gösta Mittag-Leffler (1874–1883) [ed. DUGAC (Pierre)], *Cahiers du Séminaire d'histoire des mathématiques*, 5 (1984), pp. 49–285.

HERMITE (Charles), Lettres à Gösta Mittag-Leffler (1884–1891) [ed. DUGAC (Pierre)], *Cahiers du Séminaire d'histoire des mathématiques*, 6 (1985), pp. 79–217.

HERMITE (Charles), Lettres à Gösta Mittag-Leffler (1892–1900) [ed. DUGAC (Pierre)], *Cahiers du Séminaire d'histoire des mathématiques*, 10 (1989), pp. 1–82.

BUTZER (Paul Leo), CARBONE (Luciano), JONGMANS (François) & PALLADINO (Franco), Les relations épistolaires entre Charles Hermite et Ernesto Cesàro, *Bulletin de l'Académie royale de Belgique – Classe des sciences*, 11, nos 7–12 (2000), pp. 377–417.

HERMITE (Charles), *Le Lettere di Charles Hermite a Angelo Genocchi (1868–1887)*, ed. MICHELACCI (Giacomo), *Bollettino di storia delle scienze matematiche*, XXV, 1/2 (2005), pp. 1–196.

FREITAS (Pedro), Les lettres de Charles Hermite à Francisco Gomes Teixeira, *Revue d'histoire des mathématiques*, 26 (2020), pp. 73–95.

3.2. *Selected letters*

BOTTAZZINI (Umberto), Le funzioni a periodi multipli nella corrispondenza tra Hermite e Casorati, *Archive for History of Exact Sciences* 18 (1977), pp. 3988. It contains a correspondence between Hermite and Felice Casorati, as well as letters from Hermite to Eugenio Beltrami. On the letters from Beltrami to Hermite in general, see GIACARDI (Livia) & TAZZIOLI (Rossana), "Pardonnez-moi ce bavardage…" : les lettres de Beltrami à Hermite (1878–1899), contribution to Hermite's Days for the bicentenary of his birth, December 2022, online `https://www.researchgate.net/publication/377663256`.

SCHARLAU (Winfried), *Rudolf Lipschitz : Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass und anderen*, Braunschweig : Vieweg & Sohn, 1986. It contains a selection of the letters exchanged with Rudolf Lipschitz. On their correspondence in general, see also GOLDSTEIN (Catherine), Hermite and Lipschitz : A Correspondence and Its Echoes, in BORGATO (Maria-Teresa), NEUENSCHWANDER (Erwin) & PASSERON (Irène), eds., *Mathematical Correspondences and Critical Editions*, Cham : Springer, 2018, pp. 167–193.

PARSHALL (Karen Hunger), *James Joseph Sylvester : Life and Work in Letters*, Oxford : Oxford University Press, 1998. It contains letters exchanged with James Joseph Sylvester.

DECAILLOT (Anne-Marie), *Cantor et la France*, Paris : Kimé, 2008. It contains letters exchanged with Georg Cantor.

GAUTHIER (Sébastien) & GOLDSTEIN (Catherine), in preparation. It contains a few letters exchanged with Léonce Laugel.

Sommaire

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Revue d'histoire des mathématiques / *Journal for the History of Mathematics*

Éditée par la Société Mathématique de France, la *Revue d'histoire des mathématiques* publie des articles originaux (en français ou en anglais) consacrés à l'histoire des mathématiques, de l'Antiquité à nos jours. Dans ces textes, les sciences mathématiques peuvent être considérées aussi bien dans leur développement propre que dans leurs rapports à d'autres disciplines ou dans leurs contextes (culturel, institutionnel, social). La *Revue d'histoire des mathématiques* a l'ambition de servir la communauté internationale des historiens des mathématiques en offrant un espace de débat critique ouvert à des bilans historiographiques et des notes prospectives ou programmatiques. Elle s'adresse, au-delà de cette communauté, aux mathématiciens, aux historiens et philosophes des sciences, aux sociologues, aux anthropologues, et à tous ceux qu'intéresse une réflexion sur les mathématiques et leur développement.

Edited under the auspices of the French Mathematical Society (Société Mathématique de France), the Journal for the History of Mathematics *publishes original papers (in French or in English) devoted to the history of mathematics, from Antiquity to the present. The* Journal *welcomes manuscripts dealing with the development of the mathematical sciences proper as well as papers bearing on relationships to other disciplines or on the institutional, cultural, and social contexts. The ambition of the* Journal for the History of Mathematics *is to serve the historians of mathematics' international community by offering a forum for critical debate, open to historiographic essays and programmatic contributions. Beyond the professional community, the* Journal *is addressed to mathematicians, historians and philosophers of science, sociologists, anthropologists, and to all those interested in understanding mathematics and its development.*

À propos des indicateurs bibliographiques / *Bibliographic indicators*

Le Comité de rédaction de la *Revue d'histoire des mathématiques* souhaite exprimer sa position sur les facteurs d'impact et autres indicateurs présents sur le marché des revues scientifiques, comme le taux d'acceptation des articles. En 2009, la *Revue d'histoire des mathématiques* avait déjà, comme la très grande majorité des revues d'histoire des sciences, signé l'appel « Journals under Threat : A Joint Response from HSTM Editors », contre le classement en A, B, C de ces revues. Tant les mathématiciens que les spécialistes de sciences humaines et sociales ont établi que les indicateurs bibliométriques usuels n'ont pas de pertinence individuelle — en particulier parce qu'ils varient d'une discipline à une autre, et même d'une sous-discipline à une autre —, qu'ils ne permettent pas d'évaluer la qualité scientifique d'articles ou d'auteurs, et que la plupart d'entre eux sont faciles à manipuler. Dans un domaine en plein développement théorique comme l'histoire des mathématiques, le Comité de rédaction estime aussi que la qualité d'une revue n'est pas mesurée par son refus d'une grande quantité d'articles (ce qu'il est toutefois amené à faire), mais par sa capacité à améliorer les articles par des rapports détaillés et par l'encadrement des auteurs jusqu'à la publication. Il ne rendra donc public aucun indicateur de ce type.

The Editorial Board of the Revue d'histoire des mathématiques *wishes to express its point of view concerning impact factors—and other indicators such as acceptance rates—that are currently being used in the scientific journal market. In 2009, the* Revue d'histoire des mathématiques, *like most history of science journals, signed the call "Journals under Threat: A Joint Response from HSTM Editors" against the ranking of these journals on an A, B, and C scale. Mathematicians as well as scholars in the social sciences and in the humanities have established that standard bibliometric indicators are meaningless for ranking individual papers; they vary from one discipline to another, and even from one sub-discipline to another, they also do not assess the scientific quality of articles and authors, and most are easy to tamper with. In a field in full conceptual development such as the history of mathematics, the Editorial Board also believes that the quality of a journal is not measured by its rejection of a large number of articles (which it is always obliged to do), but by its ability to improve articles through detailed referee reports and through working with authors at each step of the publication process. The Editorial Board of the* Revue d'histoire des mathématiques *will thus not make public any indicators of this kind.*

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