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CHARLES HERMITE'S PRACTICES AND THE PROBLEM OF THE UNITY OF MATHEMATICS

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Abstract. — The theme of the unity of mathematics developed during the nineteenth century as specialized articles proliferated and it has often been associated for this period with the definition of new types of mathematical objects in a structuralist setting. This article focusses on the almost opposite point of view of Charles Hermite. Although his work was praised by his contemporaries for beautifully contributing to and displaying the unity of mathematics, he himself strongly opposed the idea of free conceptual creation in mathematics and favored explicit, extensive computations with algebraic forms and classical functions. Hermite's way of testifying to the unity of mathematics must thus be reconstructed by a close reading of his papers, here based on a focus on a few keywords. The result appears proteiform; Hermite operates sometimes by constructing bridges within mathematics through formulas, sometimes by recycling and adapting well-known algebraic expressions, and even occasionally by providing alternative proofs of a theorem. The coherence of these practices with Hermite's general viewpoint on mathematics leads us to advocate for a richer history of the problem of the unity of mathematics.

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Résumé (Les pratiques de Charles Hermite et le problème de l'unité des mathématiques)

Le thème de l'unité des mathématiques s'est développé au cours du dix-neuvième siècle en même temps proliféraient les articles spécialisés et il a souvent été associé, pour cette période, à la définition de nouveaux types d'objets mathématiques dans un cadre structuraliste. Cet article se concentre sur le point de vue presque opposé de Charles Hermite. Bien que son travail ait été loué par ses contemporains pour avoir brillamment contribué à l'unité des mathématiques et même pour l'avoir mise en évidence, il s'est lui-même fermement opposé à l'idée d'une création conceptuelle libre en mathématiques et a privilégié les calculs explicites et étendus sur les formes algébriques et les fonctions classiques. La manière dont Hermite témoignait de l'unité des mathématiques doit donc être reconstituée par une lecture attentive de ses articles, ce que nous ferons ici en suivant les indications de quelques mots-clés. Le résultat apparaît protéiforme, Hermite opérant tantôt en construisant des ponts à l'intérieur des mathématiques par le biais de formules, tantôt en recyclant et en adaptant des expressions algébriques bien connues, et même occasionnellement en fournissant des preuves alternatives d'un théorème. La cohérence de ces pratiques avec le point de vue général d'Hermite sur les mathématiques nous conduit à plaider pour une histoire plus riche du problème de l'unité des mathématiques.

The theme of the unity of science became classic in the philosophy of science during the twentieth century, mathematics being from the start a key feature in this edifice. One of the most famous testimonies of this trend is, of course, the First International Congress for the Unity of Science, which took place in 1935 in Paris, in parallel with the long-term project of the *International Encyclopedia of Unified Science*. Besides Otto Neurath and Rudolf Carnap, the leading figures of the Vienna Circle at the origin of the project, the Congress gathered several representatives of the cream of the mathematical crop of the time: Elie Cartan, Jacques Hadamard, Federigo Enriques, Bertrand Russell, and Richard von Mises, among others. [1] "Recent years have witnessed a striking growth of interest in scientific enterprise and especially in the unity of science," says the front flap of the first volume of the *Encyclopedia*. Boosted in part by the successes and hopes of general relativity and its developments, the theme of the unity of science was often at the time associated with that of the unity of the world—both human and natural—on one side and, on the other, that of the unity of mathematics which was supposed to reflect and to warrant it.

In an address fittingly entitled "The Unity of Mathematics" at the American Association for the Advancement of Science in 1937, the mathematician James Byrnie Shaw, professor at the university of Illinois, claimed for

1 [Neurath et al. 1938]. For a brief history of this movement, see [Morris 1960]; see also [Kremer-Marietti 2003] and [Bourdeau et al. 2018].

instance that "there is unity in architecture, in sculpture, in painting, in poetry, in music, in drama, in dancing, in mathematics. This unity is due to the central ideas which permeate the whole work [...] Mathematics was so interwoven with life that its central ideas are also those of life" [Shaw 1937, p. 402]. Describing these ideas as those of form, identity, invariance, dependence and ideality, Shaw concluded on an epic tone:

> The primeval gods were born of chaos, but their immense power is hurrying the particles of chaos and the ripples of its ocean, its intense fields and its creative spirits, into the unity of a universe. Through the ages of human life mathematics has come to be the screen upon which we may glimpse this unity.
> [Shaw 1937, p. 411]

Besides the spiritual, and even sometimes theological, component illustrated in Shaw's quote, the theme of the unity of mathematics then operates in several ways. One is material: confronted with a potentially discouraging proliferation of results, some argued in favor of new classifications of knowledge embodied in appropriate textual tools, from reviewing journals, such as the *Jahrbuch über die Fortschritte der Mathematik* or the *Répertoire bibliographique des sciences mathématiques*, to all-encompassing encyclopedias, such as Felix Klein and Wilhelm Meyer's *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* or, of course, general books encapsulating new principles of unity, from Giuseppe Peano's *Formulario mathematico* to Nicolas Bourbaki's *Éléments de mathématique*, in whose title the singular "mathématique" emphasizes the unity of the domain.

But the two most well-known and well-studied components of this striving for unity in mathematics are methodological and conceptual. At the beginning of the twentieth century, the unity of mathematics was usually attached to a reduction process (often including a kind of axiomatization, as in Hilbert's program). At a conceptual level, this fostered the emergence of mathematical structures that were supposed to capture the bare bones of various objects or theories at the forefront of research. For the mathematicians promoting them, structures themselves possessed an intrinsic character of unity and at the same time helped to warrant the unity of mathematics, as they can be recognized and used in various mathematical subdomains and situations [Corry 2004].

Tenuous threads link together these different components, with an emphasis depending on the author, the genre of the texts or the time. In 1894, Richard Dedekind, introducing what will be soon considered a key structure, that of a field—the (unfortunate for our purpose here) translation of the word *Körper* in the original German, that is, “body”—already emphasized how it conceptually conveys unity:

> This name [“body”], similarly to that in the natural sciences, in geometry, and in the life of human society, is also intended here to designate a system that possesses a certain completeness, perfection, closure, whereby it appears as an organic whole, as a natural unity. [2]

That this “natural unity” at a conceptual level paves the road to a more global view of the unity of mathematics was elaborated by a number of mathematicians after Dedekind, from Hilbert to Emmy Noether to, of course, the Bourbaki group. [3] Charles Ehresmann (a co-disciple of Jean Dieudonné at the École normale supérieure and also a member of the Bourbaki group) would be very explicit a few decades later:

> This is a time of proliferation of mathematics; however, we can recognize also significant trends toward unity. [...] Considering the similarities of all theories, a unification is obtained by giving a general definition of the notion of a structure, or more precisely of a species of structures over sets. [Ehresmann 1966]

A decisive piece on the theme of the unity of mathematics and science—at least for the French scene, as witnessed for instance by a search of these terms on Gallica—is the double thesis of Albert Lautman, a philosopher close to Charles Ehresmann and other members of the Bourbaki group: the main thesis is devoted to structure and existence, the complementary one to the unity of mathematics itself, which contributes to strengthening the association between the various components of the theme. Lautman concludes:

> The unity of mathematics is essentially that of the logical patterns which govern the organization of its edifices. … The analogies of structure and adaptations of existences … have no other purpose than to help highlight the existence within mathematics of logical patterns, which are only knowable through mathematics itself, and ensure both its intellectual unity and its spiritual interest. [4]

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2 [Dedekind 1894, p. 452, footnote]: “Dieser Name soll, ähnlich wie in den Naturwissenschaften, in der Geometrie und im Leben der menschlichen Gesellschaft, auch hier ein System bezeichnen, das eine gewisse Vollständigkeit, Vollkommenheit, Abgeschlossenheit besitzt, wodurch es als ein organisches Ganzes, als eine natürliche Einheit erscheint.”

3 A huge literature has now been devoted to more tightly link Dedekind’s work and viewpoint to this trend that identifies Dedekind as one of its main precursors, see for instance [Ferreirós & Reck 2020; Sieg & Schlimm 2017].

4 [Lautman 1938, p. 198]: “L’unité des mathématiques est essentiellement celle des schémas logiques qui président à l’organisation de leurs édifices … Les analogies de

The same features, reorganized and illustrated in a variety of ways, were subsequently used in numerous texts, as for instance in this preface of a book aimed at a more general audience that the mathematician Georges Bouligand wrote with the engineer and teacher Jean Desbats just after World War II:

> We first notice the obstacles that mathematical activity constantly comes up against; the share of the plural and the diversified imposes a struggle at all times to reduce, simplify and encompass. [...] The unity of Mathematics requires an integral struggle. The diversity of objects subjected to reasoning, and also the plurality of hypothetico-deductive systems, come to an arrangement, and this is the essential point, with a *unitary structure* of Mathematics, whose fundamental terms are the two notions of *invariance* and *group*. This unification in the structure facilitates methodological unity. [5]

The theme of the unity of science and the key position of mathematics in it predate of course the 1930s (going back for some to Antiquity), the description of its relevant features and components varying greatly according to the period. [6] In the nineteenth century, one may think of Martin Ohm's *Versuch eines vollkommen consequenten Systems der Mathematik*, George Boole's *Laws of Thought* or of course Auguste Comte. Charles-Ange Laisant, whose 1898 book *La Mathématique: Philosophie, Enseignement* was reedited in 1907, envisioned the unification of all sciences, following Auguste Comte, through a universal method, leading from the concrete to the abstract and reciprocally, "which draws its inspiration from Mathematics only because Mathematics expresses the intrinsic traits of the

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structure et les adaptations d'existences ... n'ont pas d'autre but que de contribuer à mettre en lumière l'existence au sein des mathématiques de schémas logiques, qui ne sont connaissables qu'à travers les mathématiques elles-mêmes et en assurent à la fois l'unité intellectuelle et l'intérêt spirituel."

5 [Bouligand & Desbats 1947, p. 8]: "On constate d'abord les obstacles auquel se heurte constamment l'activité mathématique; la part du plural et du diversifié impose une lutte de tous les instants, en vue de réduire, de simplifier et d'englober. [...] L'unité de la Mathématique exige une lutte intégrale [...] La diversité des objets soumis au raisonnement, et aussi bien, la pluralité des systèmes hypothético-déductifs pactisent, c'est là le point essentiel, avec une *structure unitaire* de la Mathématique dont les termes fondamentaux sont les deux notions d'*invariance* et de *groupe*. Cette unification dans la structure facilite l'unité méthodologique."

6 See for a sample of this variety [Kremer-Marietti 2003; Krömer 2007; Maronne 2014; Stump 1997]. Still, it should be noted that there is no mention of this theme in, for instance, the eighteenth-century *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, edited by Denis Diderot, Jean Le Rond d'Alembert and Louis de Jaucourt. For an example of a different approach which can be retroactively linked to the theme of the unity of mathematics, see [Rabouin 2009].

positivist spirit, [and] nonetheless possesses a profound social and moral extension”. [7]

The tensions between the effective work of the mathematicians and the explicit discourses on the components of the unity of mathematics, whether they come from philosophical studies or from general speeches made by mathematicians in the course of their mathematical life, are well-known. This situation poses a particular challenge for historians of mathematics: how to identify, and to account for, mathematical practices that, for their author, contribute to the unity of mathematics, without relying too hastily on a later (or simply a different) vision of what the unity of mathematics is or should be. And how not to consider the theme of unity as endowed with a synthetic explanatory capacity to more easily describe the work of a mathematician, but rather to perceive some modalities of this work which shape its specificity.

With these questions in mind, the case of Charles Hermite seems a particularly interesting one to tackle. His numerous mathematical heirs often emphasized his striving towards unity. At Hermite’s 1892 Jubilee, for instance, Henri Poincaré commented:

> It can be said that the value of your discoveries is further enhanced by the care you have always taken to highlight the mutual support that all these apparently diverse sciences lend to each other. [8]

And Hermite’s son-in-law and the editor of his complete works, the mathematician Émile Picard, introduced them along the same lines:

> These strange *rapprochements*, between questions of such different natures, exerted a sort of fascination on his mind. [...] Thus he wrote once about the work of Legendre and Gauss on the decomposition of numbers into squares: ‘These illustrious mathematicians, by pursuing at the cost of so much effort their profound researches in this part of Higher Arithmetic, thus tended unwittingly towards another area of Science and gave a memorable example of this mysterious unity, which manifests itself sometimes in the most seemingly remote analytical works’. [9]

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7 I am here quoting Hamdi Mlika: “L’unité des sciences ne s’inspire de la Mathématique que parce que cette dernière exprime les traits intrinsèques de l’esprit positiviste, possède néanmoins une extension sociale et morale profonde” [Mlika 2009]. We again notice the use of the singular “mathématique”.

8 [Jubilé 1893, p. 7]: “On peut dire en effet que le prix de vos découvertes est encore rehaussé par le soin que vous avez toujours eu de mettre en évidence l’appui mutuel que se prêtent les unes aux autres toutes ces sciences en apparence si diverses.”

9 [Hermite 1905-1917, vol. 1, p. xxix]: “Ces rapprochements étranges, entre des questions de natures si différentes, exerçaient sur son esprit une sorte de fascination. [. . .] Aussi écrivait-il un jour à propos des travaux de Legendre et de Gauss sur

Although, to my knowledge, Hermite's work has never been used, nor even mentioned, by philosophers discussing the unity of mathematics, it would also be easy to spot aspects of it that would make Hermite a precursor of the versions of unity, particularly structuralist ones, mentioned above: the importance he attached to the observation of algebraic transformations and their effects [Goldstein 2019], his insistence on questions of invariance [Parshall 2024], his application of Galois's ideas to the monodromy of complex functions, prompting their later use in the study of differential equations [10], or his participation in a unified vision of what Norbert Schappacher and I have called with deliberate anachronism "arithmetic algebraic analysis" [Goldstein 2015; Goldstein & Schappacher 2007].

However, confronted in the second half of the nineteenth century with the first manifestations of this structuralist movement, Hermite repeatedly stated his skepticism towards it. He preferred, for instance, to compute with explicit, down-to-earth, representatives, the so-called reduced forms, rather than to deal with intrinsically defined classes of quadratic forms (one of the first steps in the advent of mathematical structures and of their operations). He would insist on the need to complete the study of the relations among roots of an algebraic equation, according to Galois's ideas—that is, for us, its Galois group—with an explicit, analytic, parametrization of these roots, in particular by elliptic functions in the quintic case. Moreover, he claimed to be indifferent, if not hostile, to philosophical issues. [11]

The main questions the present text addresses are thus: how can we get some access to Charles Hermite's viewpoint on the "mysterious unity of mathematics"? When and how was it expressed and constructed by a mathematician (of great influence in his time) who was hostile to systematic reflections on foundations, to axiomatization, to infinite sets considered as a whole, as well as to structural reductionism?

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la décomposition des nombres en carrés: 'Ces illustres géomètres, en poursuivant au prix de tant d'efforts leurs profondes recherches sur cette partie de l'Arithmétique supérieure, tendaient ainsi à leur insu vers une autre région de la Science et donnaient un mémorable exemple de cette mystérieuse unité, qui se manifeste parfois dans les travaux analytiques en apparence les plus éloignés'."

10 On this point, see [Archibald 2011] and, for Hermite himself, my own study of Hermite's underestimated importance in the reception of Galois [Goldstein 2011b].

11 Hermite's preferences and their effect on the creation of mathematical objects and his views on computations are discussed in more detail in [Goldstein 2011a].

1. THE WORDS OF UNITY

Tracking the word "unity" itself in the Hermitian corpus is quite misleading: "unity" ("unité" in French) also designates the number 1 and its surrogates, such as the various "roots of unity" which routinely appear in Hermite's algebraic works. In the published research works of Hermite, one single mention of "unity" in the sense we are interested in is to be found, that quoted above by Picard, which originally occurred in [Hermite 1864b]. [12] However, Picard's quote suggests to follow other words, such as "rapprochements".

Thanks to François Lê's seminal approach, [13] it is possible to locate and contextualize all words in Hermite's work. I have thus selected a number of them belonging to the semantic field of "rapprochements" and which are used several times in Hermite's articles: analogie (analogy), analogue (analogous), rapprochement, rapprocher (to bring closer), lien (link), liaison (connection), relation (relation). [14]

In the following table, the number of instances of several of these words is given: [15]

analogie	40
analogue(s)	157
liaison(s)	10
lien(s)	13
rapprochement(s)	10
rapprocher, rapproché(e,es)	35
relation(s)	990

These words also have a very local use, that is, as a substitute or a complement to one of the most frequent words in Hermite's articles, the

12 I have not found it either in the letters written by Hermite I have been able to consult, for instance to Thomas Stieltjes, Gösta Mittag-Leffler, Andreï Andreïevitch Markov, Ernesto Cesàro, Rudolf Lipschitz or Leopold Kronecker.

13 See in particular [Lê 2023; 2024].

14 Other words might play occasionally a similar role. For instance, Hermite once mentions "a sort of junction between the theory of elliptic sines and those of Göpel and Rosenhain's functions, [Hermite 1905-1917, vol. 3, p. 251]: "une sorte de jonction entre la théorie des sinus d'amplitude et celles des fonctions de Göpel and Rosenhain". For the sake of space, I have left aside here words with only one or very few occurrences and I will follow only those occurring more frequently and mentioned in the table.

15 I owe these data to François Lê whom I warmly thank. My table gathers singular and plural forms for the nouns and the conjugated forms for the verb. Let me also note that "unité," as in "roots of unity," appears c. 300 times in Hermite's published papers.

word "equation" [Lê 2024] ; it may be misleading for us. It is obvious with "relation," which Hermite routinely used to announce an algebraic or differential equation: "the quantities satisfy the relation so and so" (as for "unity" used as a surrogate of "1," this explains the high number of its instances). But it is also true of the word "liaison" (connection), again employed in exactly the same local context: "There exists between these periods, drawn from integral calculus, a *connection* expressed by the following equation ..." [16], or even of "rapprochement". [17]

What particularly interests us here is their occurrence, with a larger meaning, in comments about how parts of mathematics or concepts arising from different subdomains are interrelated. It is then necessary to consider these words as markers and to analyze the mathematics they are attached to through a direct and systematic reading of the texts where they appear. This presents the usual difficulties in reading closely Hermitian texts: sudden change of notations, allusive remarks, missing assumptions, etc. However, such an examination brings to light several interesting phenomena. For reasons of space, I will not report here on each occurrence, but only illustrate by one or two examples the main results of this enquiry.

This down-to-earth, but systematic, selection of texts based on explicit, verifiable criteria, has for me the advantage of somewhat stripping this selection of tacit presuppositions about what does or doesn't count as a contribution to the unity of mathematics. [18] A side benefit is to draw attention on little-studied texts which are nonetheless relevant and instructive for our purpose.

Hermite does not work in isolation. On the contrary, as this will appear here incidentally, he often reacts very quickly to writings of his predecessors or contemporaries. Here, I will not try and establish the genesis or mathematical environment of all Hermitian results included in the selected texts, and only a few necessary contextual elements will be provided: my limited purpose is not to study the collective dynamics involving a concept or a theorem, but only to assess what practices the chosen words designate within Hermitian mathematics. [19]

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16 [Hermite 1855, p. 251]: "Il existe entre ces périodes, telles qu'on les tire du calcul intégral, une *liaison* exprimée par l'équation suivante" ... My emphasis.

17 See, for instance, [Hermite 1871, p. 21].

18 For instance, Hermite's 1851 article interpreting Puiseux's previous work on monodromy in a Galois setting does not appear in the texts selected with our criteria.

19 For a global study of all the authors quoted by Hermite, see [Goldstein 2012]. Almost all sources used by Hermite, tacitly or explicitly, have now been included in

2. FROM WORDS TO MATHEMATICS

2.1. *Recognizing expressions*

A first noticeable feature that this enquiry brings to light is precisely the shifts from one operating level of relationships and rapprochements to another. Let us illustrate this point with a short and rather elementary article—an extract of a letter—Hermite sent to Ernst Leonard Lindelöf only a year before his death, in December 1899 [Hermite 1899-1900]. Lindelöf had answered a question on discrete probability coming from the *Intermédiaire des mathématiciens* [Lindelöf 1899-1900]: One has n sets of p balls, numbered from 1 to p in each set. One draws the balls one by one and randomly, while counting aloud from 1 to p, n times. What is the probability that the number called during the drawing coincides with the number written on the ball?

Lindelöf's solution is purely combinatorial: noticing first that the question is the same if one counts aloud n times 1, then n times 2, etc., and finally n times p, he sets up to establish the probability that there is no coincidence for the balls marked with the number 1 (that is, none of these balls is drawn in the first n draws); the repetition of the same procedure for the other marks $(2, \ldots, p)$ then provides him with the final solution of the initial question. Lindelöf computes the first step of his procedure, the probability that there is no coincidence for the balls marked with the number 1, in two different ways. One way is to deduce it from the probabilities to have one coincidence at least, then two coincidences at least, etc. An alternative path relies on a direct computation: dividing the number of drawings which *avoid* the balls marked with 1 in the first n places by the total number $(pn)!$ of possibilities. Because the two paths should arrive at the same result, Lindelöf deduces from these two computations the following identity, which he describes as "curious enough to be mentioned":

$$\frac{(r-n-1)(r-n-2)\ldots(r-2n)}{r(r-1)\ldots(r-n)} = \frac{1}{r} - \frac{n^2}{r(r-1)} + \frac{1}{1\cdot 2}\cdot\frac{n^2(n-1)^2}{r(r-1)(r-2)} - \cdots,$$

where $r = pn$ is the total number of balls.[20]

the database Thamous. References to historical studies of some results, their sources, and their developments will of course be given when they exist.

[20] I give this identity in the form provided in Hermite's article (equation (A)); it is in fact derived from Lindelöf's original one by changing r into $r-1$ and by dividing by r.

The right hand side of the equation can also be written as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\frac{k!}{r(r-1)\cdots(r-k)}.$$

It is this "curious" equation that triggers Hermite's interest. He immediately recognizes that the terms of the last sum are easily linked to special values of Eulerian integrals (here the beta functions):

$$\frac{k!}{r(r-1)\cdots(r-k)}=\int_0^1 x^{r-k-1}(1-x)^k dx=B(r-k,k+1)$$

and thus, with $x=\frac{y+1}{2}$,

$$\frac{k!}{r(r-1)\cdots(r-k)}=\frac{1}{2^r}\int_{-1}^{1}(1+y)^{r-k-1}(1-y)^k dy.$$

Hermite then uses a well-known (at the time) expression of Legendre polynomials P_n:[21]

$$2^nP_n(y)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2(1+y)^{n-k}(1-y)^k.$$

Lindelöf's identity thus becomes (with $s=r-n-1$):

$$\frac{s(s-1)\cdots(s-n+1)}{(s+1)(s+2)\cdots(s+n+1)}2^{s+1}=\int_{-1}^{1}P_n(y)(1+y)^s dy.$$

The integral on the right is thus 0 for $s=0,1,\ldots,n-1$. This provides, in particular, a key property of orthogonality for the Legendre polynomials.

Hermite comments: "[Your identity] is tightly *linked* to the theory of polynomials $P_n(x)$ of Legendre and opens up a new path to their fundamental properties. … I thought you might be interested to see a *rapprochement*, a close *link* I might say, between the problem of probability that you solved and an important theory of analysis, the theory of spherical functions."[22]

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21 Legendre polynomials are for instance defined as the coefficients of the powers of t in the expansion $\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$. This polynomial system is orthogonal for the inner product $\langle P,Q\rangle=\int_1^1 P(t)Q(t)dt$, among many other properties. Through another change of variables, Hermite gets the expression he needs from a 1837 paper of Peter Gustav Lejeune-Dirichlet, also mentioned in Hermite's 1878 presentation of the second edition of Eduard Heine's *Handbuch der Kugelfunctionen*, see [Dirichlet 1837; Hermite 1878].

22 [Hermite 1899-1900, p. 88, p. 90]: "[Votre identité] est *liée* étroitement à la théorie des polynômes $P_n(x)$ de Legendre, et ouvre une nouvelle voie pour parvenir à leurs propriétés fondamentales … J'ai pensé que vous ne verriez pas sans quelque intérêt un *rapprochement*, une étroite *liaison* je puis dire, entre le problème du calcul des

This rather marginal and late example nevertheless seems to me to highlight a way of working that we often find in in Hermite's work, including in his important achievements such as the creation of Hermitian forms [Goldstein 2019]: the starting point is a symbolic expression, a formula, which he observes and transforms, for instance by means of successive, often quite intricate, changes of variables. This allows him to locate them inside a stock of familiar expressions, sometimes in far-away theories, or to generalize them to other cases. The *rapprochement* on the domain level, as here between combinatorial probability theory and the theory of spherical functions, is constructed on the basis of *rapprochements* at the very local level of the expressions themselves, .

2.2. *Discovering the right analogy*

Legendre polynomials regularly appear in Hermite's work. As we will see, they also appear in association with the word "analogy," and in this context, they serve both as a mould and as a model for building bridges between fields or mathematical questions that may seem a priori far apart. They are not the only mathematical objects to play these roles, but I have chosen this example to illustrate how analogies are handled by Hermite, because it seems to me one of the simplest and shortest to present.

In 1864, struck by the importance of e^{-x^2} (or more generally of $e^{-\phi(x,y,z,\ldots)}$ for a quadratic form ϕ) in the representation of elliptic and Abelian functions, Hermite notices that these exponentials "give rise, as the radical

$$(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}[\ldots],$$

to a system of integral polynomials which may be used for the expansion of functions of any number of variables. [...] One will see, by the way, the most complete *analogy* between the properties of such expressions coming from so different origins". [23]

probabilités dont vous avez donné la solution, et une grande théorie de l'analyse, celle des fonctions sphériques." I have emphasized here our selected words, as well as "liée," the adjective derived from "lier" and "lien". Spherical functions (at the time) are functions on the sphere satisfying Laplace-type differential equations, among which are Legendre polynomials. They are the subject of a book by Eduard Heine, which Hermite praised and presented to the Academy of sciences [Hermite 1878].

23 [Hermite 1864b, p. 94]: "... les exponentielles e^{-x^2} et $e^{-\phi(x,y,z,\ldots)}$ donnent naissance, comme le radical $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$ ... à un système de polynômes entiers, pouvant servir au développement des fonctions d'un nombre quelconque de variables. ... On verra, du reste, entre les propriétés d'expressions d'origine si différente, l'*analogie* la plus complète". My emphasis.

The coefficients of the radical $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$ developed into the powers of α indeed give rise to the Legendre polynomials. Hermite defines, for any non-zero real a, the polynomials U_n (in one variable x) such that

$$e^{-\frac{a}{2}(x-h)^2} = e^{-\frac{x^2 a}{2}}(U_0 + \frac{h}{1}U_1 + \frac{h^2}{1\cdot 2}U_2 + \cdots),$$

then states and proves for his U_n the same type of properties as for the classical Legendre polynomials, in particular:

— the U_n satisfy a recursion formula: $U_{n+1} - axU_n + anU_{n-1} = 0$,

— the U_n satisfy a second-order differential equation: $\frac{d^2U}{dx^2} - ax\frac{dU}{dx} + anU = 0$,

— the U_n form an orthogonal system, that is $\int_{-\infty}^{\infty} e^{\frac{ax^2}{2}} U_n U_{n'} = n!a^n\sqrt{\frac{2\pi}{a}}$ if $n = n'$ and 0 otherwise [Hermite 1864b, p. 267].

It is to be noticed that while the analogous elements in both situations are polynomials endowed with specific properties—and Hermite mentions several other families of such polynomials—, they come from quite distinct horizons: they arise from algebraic functions $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$ in Legendre's case, from transcendental exponential functions in Hermite's new case. The analytical similarities—development in series, differential equations, values of integrals—are again much more important for Hermite than the partition into algebra and analysis we have come to consider as decisive.

Hermite also extends his construction to several variables, beginning by the original Legendre polynomials themselves. For these polynomials, and two variables instead of one, a 1865 letter to Carl Borchardt (who published it in his journal [Hermite 1865a]) gives us a rare insight on the way rapprochements and analogies also guide Hermite's work to generalization. To begin with, and in order to mimic the construction of Legendre polynomials as the coefficients of the expansion with respect to a of $\frac{1}{\sqrt{1-2ax+a^2}}$, Hermite simply tries to replace $\frac{1}{\sqrt{1-2ax+a^2}}$ by $\frac{1}{\sqrt{1-2ax-2by+a^2+b^2}}$. However, the polynomials he thus obtains do not form an orthogonal family. "In order to re-establish the *analogy* with the functions of one variable which originates from the development of $\frac{1}{1-2ax+a^2}$ and seems to be lost here," Hermite explains,[24] he then replaces his original choice of the quadratic

[24] [Hermite 1865a, p. 295]: "Pour rétablir l'*analogie* avec les fonctions d'une variable ayant pour origine le développement de $\frac{1}{1-2ax+a^2}$ et qui me semble ici se perdre...". My emphasis.

form $1 - 2ax - 2by + a^2 + b^2$ in the variables a, b by its adjunct, that is, the quadratic form $(1 - ax - by)^2 - (a^2 + b^2)(x^2 + y^2 - 1)$.

This original idea of the role of the adjunct comes from Hermite's early work on Carl Friedrich Gauss's *Disquisitiones arithmeticae* and on the theory of quadratic forms, where the use of the adjunct form is a key ingredient. [25] The polynomials obtained with this adjunct form satisfy a formula close to the so-called Rodrigues' formula [26] for the Legendre polynomials X_n:

$$X_n = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n},$$

and Hermite can then use it to prove the other wanted properties and display "the *analogy* as complete as possible with Legendre functions" ([Hermite 1865a, p. 295], my emphasis). Again, the bridge is constructed by careful handling of symbolic expressions, the behavior of which is tested through various computations and the exact form corrected until it fits the desired scheme.

Transfers by analogy from arithmetic to algebra to analysis are frequent in Hermite's work. For instance, Hermite stresses in a letter to Paul Gordan the analogy between the arithmetical study of successive minima of $x + ay + bz$ (with a, b integers) and the study of $U \sin x + V \cos x + W$, where U, V, W are polynomials in x [Hermite 1873]: in this situation, the analogy is based on a continued fraction expansion (numerical or algebraic, according to which case is considered), a technique that Hermite will also use at the same date in his celebrated proof of the transcendence of e. [27]

Other examples involve the discovery of Hermitian forms in n variables, first introduced as a subclass of real quadratic forms in $2n$ variables [Goldstein 2019]: again, this is the concrete display of the coefficients of a specific family of quadratic forms which suggests to Hermite that a study using complex numbers may bring light to their characteristics, as it allows their arithmetic study to mimic that of quadratic forms with n variables and integral real coefficients. This, he concludes, is "adding new characters of similitude between real integers and complex numbers," at the time still a controversial issue [Hermite 1905-1917, vol. 1, p. 477]. The same kind of

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25 [Brechenmacher 2011; Goldstein 2007]. In 1865, Hermite uses both Gauss's terminology of "adjunct form" and the terminology of the then blossoming invariant theory, "quadratic contravariant".

26 In his 1865 letter to Borchardt, dated from January 27, Hermite only attributes the formula to Carl Jacobi. He emphasizes Olinde Rodrigues's priority in a subsequent communication to the French Academy of Sciences on March 13, 1865 [Hermite 1865b, p. 517].

27 On this point, see [Goldstein 2015; Serfati 1992; Waldschmidt 1983].

construction is transferred by Hermite to forms associated to the theory of transformation of Abelian functions (now called symplectic forms): "This analytic theory of transformation is tightly *linked* to the arithmetical theory of quadratic forms I spoke about...," he writes. [28]

In some cases, links operate in several ways for the same question. Hermite's use of elliptic functions to solve general quintic equations is well-known, but it is perhaps less known that it perfectly illustrates Hermite's practice of analogy (and links) in mathematics. [29] His point of departure here is a quartic equation associated to the flex points of cubic curves: these nine flex points are aligned three by three on 12 lines, which in turn form 4 triangles (such that each of them contains the nine flex points), giving rise to a quartic equation. [30] Hermite comments:

> From the study of these equations, I noticed that they have the closest affinity with those found in the third-order transformation of elliptic functions, and so I thought it would not be useless to look into this *rapprochement*... This *analogy*, indeed, opened for me the way of representing by elliptic transcendental functions the roots of the general quartic equation. [31]

And so we are on familiar ground here: this is again by a direct comparison of two specific quartic equations (the quartic equation linked to the flex points and the multiplier equations for the transformation of elliptic functions of degree 3, whose roots are known in terms of special values of elliptic functions) that Hermite weaves links between two separate areas of mathematics. One consequence of this rapprochement is to express the roots of all these quartic equations as values of modular elliptic functions, that is, values of analytic, non-algebraic functions.

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28 [Hermite 1855, p. 784]: "cette théorie analytique de la transformation se trouve étroitement *liée* à la théorie arithmétique des formes quadratiques dont j'ai parlé". My emphasis. This case is studied in [Brouzet 2004].

29 On Hermite's place in the history of algebraic equations, see for instance [Gray 2018; Houzel 2002; Zappa 1995]. The context of Hermite's work on this topic, in particular its relation with Galois's viewpoint and the modular equations, is discussed in [Goldstein 2011b].

30 On the historical role of these equations arising from geometry, see [Lê 2015]. Here, as at other times, Hermite neglects the potential of a link to the advanced geometry of his time.

31 [Hermite 1905-1917, vol. 2, p. 22]: "L'étude de ces équations m'ayant fait remarquer qu'elles offrent la plus étroite affinité avec celles qu'on rencontre dans la transformation du troisième ordre des fonctions elliptiques, il ne m'a pas paru inutile de m'arrêter à ce *rapprochement* ... Cette *analogie*, d'ailleurs, m'a ouvert la voie pour représenter par les transcendantes elliptiques les racines de l'équation générale du quatrième degré." My emphasis. Let me remark that "affinité" (affinity) which belongs to the same semantic field as "rapprochement," is an hapax in Hermite's work.

This approach will be exported with success by Hermite to quintic equations, a case which, contrarily to the quartic one, is not in general solvable by radicals. This is the case that mobilized the attention of historians (and of contemporaries), and in particular the expression of the roots as explicit values of elliptic functions: depending on the authors, this analytic approach to an algebraic problem has been both praised, as establishing new bridges between far-away domains of mathematics and as allowing a concrete computation of the roots, and criticized, as introducing foreign elements to the solution. [32] Hermite's elliptic study of the quartic equation, for which algebraic expressions of the roots were already available, seems to have escaped to attention. However, for our issue, it is worth underlining how a famous Hermitian achievement again began with a close comparison between particular, well-known, equations. And that Hermite, here as before, forsakes the problem of the algebraic resolution of an algebraic equation for the constitution of a unique domain, with analysis both at the basis and at the lead.

2.3. *Rapprochement by the proofs*

There is still another instance of the word "rapprochement": it occurs when several proofs or methods of resolution of a problem are known. In the case of the quintic equation we just mentioned, "the theory of elliptic function leads to two methods for the resolution of the equation of the fifth degree [33]: the one proposed by Hermite in 1858 is based, as said earlier, on the possibility to link any quintic general equation, through adequate changes of variables, to the reduced form of a modular equation.

Jacobi had introduced his doubly-periodic functions (the so-called Jacobian elliptic functions) through integrals of the type $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, with $0 < k < 1$ a real number. Their periods are 2 or 4 times (depending on the function) the so-called complete integrals:

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}},$$

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32 On this point, illustrated by Felix Klein's opinion, see [Goldstein 2011b, 46-51].

33 [Hermite 1905-1917, vol. 2, p. 347]: "La théorie des fonctions elliptiques conduit à deux méthodes pour la résolution de l'équation du cinquième degré." Hermite devoted to this issue 12 communications to the French Academy of Sciences in 1865 and 1866, later gathered in a booklet. To simplify, I will quote them from the second volume of his collected works where they are all reproduced.

with $k' = \sqrt{1-k^2}$. If one puts $\omega = \sqrt{-1}\frac{K'}{K}$, $\sqrt[4]{k}$ is a uniform function of ω, say $\phi(\omega)$, and there exists an algebraic equation of degree $n+1$ between $\phi(\omega)$ and $\phi(n\omega)$, for each integer $n > 1$. This is the modular equation of level n.[34]

Another method, due to Leopold Kronecker and completed by Francesco Brioschi, also in 1858, is based on a direct construction of cyclic functions of the roots of the quintic equation; this method finally links it to the so-called multiplier equation, which appears in the transformation theory of elliptic functions.[35]

In his work of 1865-1866, Hermite's aim is "to take M. Kronecker's method a step further and *bring it closer* to the previous one, using as a basis the remarkable and inventive work in which M. Brioschi set out its principles."[36] As in the rapprochements of the mathematical expressions, this work on the different proofs should provide alternative paths to a variety of properties. For instance, in the case of the quintic equation, Hermite comments: "I will add to this *rapprochement* between the two methods of solving the equation of the fifth degree, by deducing from the second the conditions for the reality of the roots," which he had previously studied by his own approach.[37]

More generally, and against the common idea of finding *the* right proof, Hermite advocates for multiple proofs of a result in order to fully understand it.[38] But reciprocally the very fact that there exist several available proofs is for him an incentive to clarify the links that this suggests.

34 See [McKean & Moll 1999] for a clear presentation for a modern reader.

35 Definitions and details on this method are given in [Zappa 1995].

36 [Hermite 1905-1917, vol. 2, p. 348]: "approfondir la méthode de M. Kronecker et la *rapprocher* de la précédente, en prenant pour base le travail remarquable et plein d'invention dans lequel M. Brioschi en a exposé les principes".

37 [Hermite 1905-1917, vol. 2, p. 392]: "J'ajouterai encore à ce *rapprochement* entre les deux méthodes de résolution de l'équation du cinquième degré, en déduisant de la seconde les conditions de réalité des racines." My emphasis.

38 For instance, Hermite urges Thomas Stieltjes: "Il sera utile de donner pour parvenir aux mêmes conclusions deux procédés très différents," [Hermite & Stieltjes 1905, vol. 1, p. 41]. Or in a 1873 article,"On reconnaîtra volontiers que, dans le domaine mathématique, la possession d'une vérité importante ne devient complète et définitive qu'autant qu'on a réussi à l'établir par plus d'une méthode," [Hermite 1905-1917, vol. 3, p. 128]. Respectively: "It will be useful to consider two very different procedures to arrive at the same conclusions," and "It can be readily admitted that, in mathematics, the possession of an important truth only becomes complete and definitive when we have succeeded in establishing it by more than one method."

A good illustration is provided by Hermite's work on the number of classes of binary quadratic forms $ax^2 + 2bxy + cy^2$, with integer coefficients a, b, c. Two forms are said to be equivalent, or in the same class, when they can be deduced from each other by an invertible, linear change of variables with integer coefficients. The discriminant $b^2 - ac$ (called "determinant" by Gauss in the *Disquisitiones arithmeticae*, and by Hermite following him) is the same for equivalent forms; on the other hand, for a given determinant, there are a finite number of classes of equivalent forms. These class numbers had been computed by analytic means by Peter Gustav Lejeune-Dirichlet in a celebrated application of Fourier analysis to number theory. From the 1860s on, Kronecker established recurrence formulas for these class numbers by using elliptic modular functions.[39] In the wake of his involvement with the fifth-degree equation and the publication of Kronecker's article in French in 1860, Hermite writes to Joseph Liouville:

> M. Kronecker's beautiful theorems on the class numbers of quadratic forms [...] remained, however, isolated and belonging to a very distinct order of ideas to which only the theory of complex multiplication in elliptic functions seemed able to give access.[40]

Besides the doubly-periodic elliptic functions arising from the integrals

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

alluded to above, Jacobi had introduced theta-functions, that is, special quasi-periodic functions, whose quotients also provide these elliptic functions and which can be developed as series involving trigonometric functions and rational powers of $q = e^{-\pi\frac{K'}{K}}$. Following then Jacobi's model, Hermite sets out to derive some of Kronecker's results from the developments of quotients of theta functions in series of sinus and cosinus, such as:[41]

$$\frac{4\sqrt{q}\sin x}{1+q} - \frac{4\sqrt{q^3}\sin 3x}{1+q^3} + \frac{4\sqrt{q^5}\sin 5x}{1+q^5} + \cdots \qquad (1)$$

39 On these developments, see [Dickson 1919, ch. VI] and [Goldstein & Schappacher 2007].

40 [Hermite 1862, p. 25]: "[Les beaux théorèmes de M. Kronecker sur les nombres de classes de formes quadratiques ...] restaient cependant comme isolés et appartenant à un ordre d'idées très distinct où la théorie de la multiplication complexe dans les fonctions elliptiques paraissait seule pouvoir donner accès."

41 [Hermite 1864b, p. 146-147], formulas 5 and 15. Hermite provides forty-six formulas of this type, which he then combines.

or

$$\cot x + \frac{4q \sin 2x}{1-q} - \frac{4q^2 \sin 4x}{1+q^2} + \frac{4q^3 \sin 6x}{1-q^3} + \cdots . \tag{2}$$

In particular,[42] Hermite finds that $\sqrt{\frac{2K}{\pi}}$ can be developed as

$$\theta_1 =: \sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \cdots .$$

Multiplying two by two such developments adequately and integrating the result between 0 and $\frac{\pi}{2}$, Hermite obtains new series in powers of q such that the coefficient of the term in q^{Δ} (or $q^{\frac{\Delta}{4}}$) depends on the class numbers of binary quadratic forms $ax^2 + 2bxy + cy^2$, with integer coefficients a, b, c and discriminant $b^2 - ac = -\Delta$.

For instance, multiplying (1) and (2) above and integrating the result provides:

$$\theta_1^3 = 1 + 4 \sum_{n>0} \frac{q^n}{1+(-q)^n} - 2 \sum_{n>0} (-1)^n q^{n^2} \mathcal{B}_n + 4 \sum_{n>0} \frac{q^{n^2+n} \mathcal{B}_n}{1+(-q)^n},$$

with $\mathcal{B}_n = 1 + 2q^{-1} + 2q^{-4} + \cdots + 2q^{-(n-1)^2}$.

Hermite then carefully studies the development into the powers of q of the three series composing θ_1^3. The third one, for instance, becomes after development $\sum_{n>0} (-1)^{a(n+1)} q^{n^2+n+an-b^2}$, displaying as the power of q the opposite Δ of the discriminant of the quadratic form $nx^2 + 2bxy + (n+1+a)y^2$, with $0 \leq\mid b \mid< n$. A careful study of the forms finally shows that

$$\theta_1^3 = 1 + 12 \sum_{n>0} (\mathcal{H}_0 - \mathcal{H}_1) q^{\Delta},$$

with complementary terms if Δ is a square or the triple of a square.[43] Here $\mathcal{H}_0$ is the number of reduced binary quadratic forms with one of the extreme coefficients odd and $\mathcal{H}_1$ is the number of such forms with even extreme coefficients.[44]

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42 The notations in this domain change according to the authors and, in the case of Hermite, according to the date of the paper. A concordance table is given in [Dickson 1919, p. 93].

43 There is misprint or miscalculation in the original publication, the 1 is missing. It is corrected in the complete works, [Hermite 1905-1917, vol. 2, p 253].

44 A binary quadratic form $ax^2 + 2bxy + cy^2$ with integer coefficients and negative determinant is reduced when $2 \mid b \mid\leq a \leq c$, and $b \geq 0$ if $a = 2 \mid b \mid$ or $a = c$. There is in this case a unique reduced form in each class—the reduced form is a representative of all the forms of the class—and thus counting the reduced forms amounts to computing the class number. The extreme coefficients are a and c.

This corresponds (if one expresses it in the language of classes of forms) to one of Kronecker's recurrence formulas:

$$E(n) + 2E(n-1) + 2E(n-4) + 2E(n-9) + \cdots = \frac{2}{3}[2 + (-1)^n]X(n),$$

where $E(n) = 2F(n) - G(n)$, $G(n)$ is the number of classes of quadratic forms with determinant $-n$, $F(n)$ is the number of quadratic forms with determinant $-n$ and one odd extreme coefficient, $X(n)$ is the sum of the odd divisors of n. [45]

This number of classes, in special cases, had been linked by Legendre and Gauss to the number of decomposition of a number into the sum of three squares. Computing the cube of an expression made of exponentials of squares leads to exponentials of sums of three squares and thus Hermite's previous developments also provide, in particular, the number of representations of an integer as sums of three squares. This is indeed in such a context that Hermite used the word "unity" in our sense for the first and unique time in his published papers:

> We have two absolutely distinct methods which connect by a double *link* Legendre's and Gauss's propositions on the decomposition of numbers into three squares to the theory of elliptic functions. In so doing, these illustrious mathematicians were unknowingly reaching out to another region of science, and providing a memorable example of the mysterious *unity* that sometimes manifests itself in analytical work that is seemingly the furthest removed from the realm of science. [46]

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45 [Dickson 1919, p. 109]. Kronecker himself was particularly pleased to be able to "draw *only* from the theory of elliptic functions the beautiful propositions of higher arithmetic that until now were based on the deep considerations included in Gauss's *Disquisitiones arithmeticae*," [Kronecker 1860, p. 297]: "De cette manière, on peut tirer de la *seule* théorie des fonctions elliptiques les belles propositions d'arithmétique supérieure, qui jusqu'ici étaient fondées sur les profondes considérations que renferment les Disq. Arithm. de Gauss." Kronecker's paper does not include the detailed proofs, which were completed by Henry Smith in his Report to the British Association, [Smith 1865].

46 [Hermite 1905-1917, vol. 2, p. 254]: "On a deux méthodes absolument distinctes qui rattachent par un double *lien* à la théorie des fonctions elliptiques les propositions de Legendre et de Gauss sur la décomposition des nombres en trois carrés. Ces illustres géomètres ... tendaient ainsi à leur insu vers une autre région de la science et donnaient un mémorable exemple de cette mystérieuse *unité* qui se manifeste parfois dans les travaux analytiques en apparence les plus éloignés." My emphasis. This issue is not explored further here, but let us note the theological undertone of this passage, in line with Hermite's views that human beings are not the masters of a free mathematical creation: unity "manifests itself," the mathematicians "unknowingly" reach out etc. See [Goldstein 2011a].

In July 1862, Hermite again emphasizes that "the theory of elliptic functions presents two main points where it is *linked* to arithmetic and particularly to the theory of binary forms with negative determinant" [47], the first one being his own development in series, explained above, which he describes as elementary and the second one being Kronecker's use of complex multiplication.

Obtaining new proofs not only serves to solidify the links between the arithmetic of quadratic forms and elliptic functions, but also to extend their now common territory. In his letter to Liouville, Hermite explains:

> In arriving at these theorems of Mr. Kronecker by another route, I have attached them in the most direct way, i think, to the order of ideas which belongs to you, and, if I am not mistaken, in the very sense of your predictions, for the arithmetical notion of class is replaced by the much simpler and elementary idea of reduced forms. [... I would like to indicate], in conclusion, how I see the *connection* between the theory of elliptic functions, in its applications to arithmetic, and your general research on numerical functions. [48]

And a few months later, in November of the same year, he would again extend his viewpoint, this time to sew onto it Dirichlet's research on class numbers:

> M. Dirichlet's method for the determination of the class numbers of quadratic forms with the same determinant and those recently drawn from the consideration of elliptic functions in the case of negative determinants lead for the same question to such different solutions that it seems as difficult to find any *link* between their results as between the principles on which they are based. [49]

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47 This series of communications to the Academy is published again later under various titles and groupings and, in order to simplify, I quote them here from Hermite's complete works, [Hermite 1905-1917, vol. 2, p. 241]: "La théorie des fonctions elliptiques présente deux points principaux où elle vient *se lier* à l'Arithmétique et spécialement à la théorie des formes quadratiques à deux indéterminées de déterminant négatif." My emphasis. Again, let me note the grammatical choice of "se lier" which seems to exclude a human intervention.

48 [Hermite 1862, p. 25-26, p. 38]: "En parvenant par une autre voie à ces théorèmes de M. Kronecker, c'est à l'ordre d'idées qui vous appartient que je pense les avoir rattachés de la manière la plus directe, et, si je ne me trompe, dans le sens même de vos prévisions, car la notion arithmétique de classe se trouve remplacée par l'idée beaucoup plus simple et élémentaire des formes réduites. [... Je vous indique], en terminant, de quelle manière je conçois la *liaison* de la théorie des fonctions elliptiques, dans ses applications à l'arithmétique, avec vos recherches générales sur les fonctions numériques." My emphasis.

49 [Hermite 1905-1917, vol. 2, p. 255]: "La méthode de M. Dirichlet pour la détermination du nombre des classes de formes quadratiques de même déterminant, et celles qu'on a tirées récemment de la considération des fonctions elliptiques dans le cas des

To do this, Hermite proposes either to find a purely arithmetical proof of Kronecker's results or an elliptic-based one of Dirichlet's results, which he prepares by revisiting Dirichlet's original one. [50]

Hermite's involvement with Kronecker's formulas did not stop there. In 1884, for instance, he would tackle them again. His new procedure gives in particular an alternative for the quantity $(\sqrt{\frac{2kK}{\pi}})^3$, and a rapprochement between his formula and Kronecker's provides a new approach to some simple arithmetical relations involving class numbers and established by Gauss in the *Disquisitiones arithmeticae*. Once more, at this occasion, Hermite insists on the fact that they "reveal a tight *connection* between the arithmetical theory of quadratic forms and the analytic theory of elliptic transcendental functions" [51].

3. CIRCULATING ALGEBRAIC EXPRESSIONS: A DETAILED EXAMPLE

The preceding exemples clearly suggest that Hermite works on analogies or constructs links and rapprochements at the level of the symbolic expressions themselves, even when he states them in terms of connections between fields or mathematical sub-disciplines. For example, he doesn't try to derive from Legendre polynomials a list of properties and criteria that would define a general class of objects to be studied; he adapts, through transformations and calculations, explicit expressions to new situations, thus weaving, thread by thread, the new, associated, connection between algebraic and analytic functions.

This feature has some serious historiographical consequences, if only in terms of the writing adapted to discuss this type of practice. To gain a deeper understanding of how expressions circulate, creating the unity Hermite's successors would enthusiastically praise, we need to take a closer

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déterminants négatifs, conduisent pour la même question à des solutions tellement différentes, qu'il semble aussi difficile de trouver un *lien* quelconque entre leurs résultats qu'entre les principes sur lesquels elles se fondent." My emphasis.

50 It is here interesting to note that Dirichlet's approach is described as arithmetical (despite his use of real series). In his own version, Hermite emphasizes only his restructuration of the (arithmetical) formulas, neglecting to prove a (necessary) convergence result which allows the identification of his formulas with Dirichlet class number formula. For details on this point and references to the authors who corrected it, see [Dickson 1919, p. 114-115].

51 [Hermite 1905-1917, vol. 4, p. 138-139]: "Les théorèmes dont je vais m'occuper consistent dans les relations suivantes [...] qui révèlent une *liaison* étroite entre la théorie arithmétique des formes quadratiques et la théorie analytique des transcendantes elliptiques." My emphasis.

look at some technical nooks and crannies of Hermite's work. In this section, I would like to analyze an example of such a circulation, based on avatars of Lagrange's resolvents.

This example concerns expressions of the kind

$$\phi(\alpha)(x_0, x_1, \ldots, x_n) = x_0 + x_1\alpha + x_2\alpha^2 + \cdots x_n\alpha^n,$$

that is, linear forms in the x_i, where $x_0, \ldots, x_n$ are indeterminates or variables (usually set at integral values) and α a root of an algebraic equation of degree $n + 1$. Hermite usually takes into account, at the same time, all the $\phi(\xi)$, ξ running over all the roots of the algebraic equation.

To consider these expressions is of course not original with Hermite. For instance, they appear, for $n = 1$ or $n = 2$, in Joseph-Louis Lagrange's last Addition to Leonhard Euler's *Algebra*[52] or, for any degree, in a letter addressed to Joseph Liouville by Dirichlet, and published twice in French. [53] Dirichlet, in particular, uses the notation $\phi(\alpha)$ and considers the value at integers of the form obtained as the product of the $\phi(\xi)$, for all the roots ξ.

3.1. *Decomposition of prime numbers*

As for Hermite, these forms appear quite early in his work, in 1847, in a (successful) attempt to prove a statement of Jacobi. In a 1839 communication to the Berlin Academy, translated and published in French in 1843, Jacobi had commented on how Gauss used complex numbers $a + b\sqrt{-1}$, with a and b ordinary integers, and developed their arithmetic in order to establish quartic reciprocity laws; it led Jacobi to advocate for a study of the decomposition of ordinary prime numbers into (different types of) complex numbers. [54] If p is a prime of the form $4n + 1$, such as 5, 13 or

52 [Lagrange 1774, t. 2, p. 651, 655 in the French 1774 edition]. To solve a cubic equation, for instance, Lagrange uses the fact that $x_1 + x_2 j + x_3 j^2$, with j a primitive cubic root of 1, takes 6 values when the x_i are permuted, but that the 6th-degree equation of which these 6 values are the roots is in fact quadratic in x^3, and thus easily solvable.

53 See for instance [Dirichlet 1840]. Liouville's closeness to Hermite and the publication of the letter both in the *Comptes rendus* of the French Academy and in Liouville's *Journal de mathématiques pures et appliquées* suggests that Hermite was probably aware of these texts; an explicit reference to them was added by Jacobi, when he published some letters from Hermite to him in 1850. On the other hand, Hermite certainly knew Lagrange's algebraic texts very well.

54 [Jacobi 1839; 1843]. Despite its shortness and elementary character, this paper proved to give a key impulse to the development of algebraic number theory. It is already discussed within this context in [Goldstein 2007; Goldstein & Schappacher 2007]. For the convenience of the readers, I repeat here briefly this presentation, emphasizing the work on specific algebraic expressions.

17, it can be written as a sum of two squares of integers, $p = a^2 + b^2$; thus $p = a^2+b^2 = (a+b\sqrt{-1})(a-b\sqrt{-1})$. It is true in particular if p is a prime of the form $8n+1$, such as 17. However, in this case, p can also be represented by other quadratic forms besides $x^2 + y^2$: $p = c^2 + 2d^2$ and $p = e^2 - 2f^2$. For example, $17 = 16 + 1 = 9 + 2 \times 4 = 25 - 2 \times 4$.

Jacobi proved that a further decomposition can provide the needed coherence in these representations. More precisely, in his paper, he factorizes $a + b\sqrt{-1} = \phi(\alpha)\phi(-\alpha)$, where α a primitive 8th-root of unity (that is, as written at the time, "$\alpha = \sqrt[4]{-1}$") and $\phi(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3$, with integers a_0, a_1, a_2, a_3.

As $\alpha^4 + 1 = 0$, $\phi(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3$ is exactly an expression of the type we are interested in. One has also: $a + b\sqrt{-1} = \phi(\alpha)\phi(\alpha^5)$, and, similarly, $a - b\sqrt{-1} = \phi(\alpha^3)\phi(\alpha^7)$. Jacobi then obtains $p = 8n + 1 = a^2 + b^2 = \phi(\alpha)\phi(\alpha^5)\phi(\alpha^3)\phi(\alpha^7)$. And other groupings of the $\phi(\alpha^k)$ provide the other decompositions:

$$p = [\phi(\alpha)\phi(\alpha^3)][\phi(\alpha^5)\phi(\alpha^7)] = (c + d\sqrt{-2})(c - d\sqrt{-2}) = c^2 + 2d^2$$
$$p = [\phi(\alpha)\phi(\alpha^7)][\phi(\alpha^3)\phi(\alpha^5)] = (e + f\sqrt{2})(e - f\sqrt{2}) = e^2 - 2f^2.$$

To summarize, in order to get the various decompositions of a prime of the form $8n + 1$, Jacobi had to use not only what was considered at the time as "the" (only known, and even only thought of) complex integers, that is, Gaussian complex integers $a+b\sqrt{-1}$ with a, b ordinary integers, but also complex numbers built with 8th-roots of unity. A natural question then arises: what are, for each kind of primes, the relevant complex integers to consider and how? At the end of his paper, Jacobi asks the question for another case, that of the primes p of the form $5n + 1$ and states that in this case

$$p = N_1N_2N_3N_4,$$

where each N_i is of the form $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3$, with $\alpha^5 = 1, \alpha \neq 1$. However, he does not give any proof of this result.

A well-known development arising from Jacobi's paper is the arithmetical study of cyclotomic integers (that is, linear combinations of roots of unity, as above), which led to the introduction of ideal complex factors by Ernst Eduard Kummer, also c. 1847, and then to Dedekind's theory of ideals. Hermite, however, attacked the problem from a quite different angle. In a letter to Jacobi, he explains:

> It is in some very elementary properties of quadratic forms, with any number of variables, that I encountered the principles of analysis which I ask your permission to discuss. I have drawn from these principles a proof of your beautiful

theorem on the decomposition of prime numbers $5m + 1$, into four complex factors, built with the fifth roots of unity[55].

Hermite clearly had hopes to be able to handle a much more general case than the particular ones mentioned by Jacobi. In his letter, he considers any irreducible algebraic equation of degree n with integer coefficients

$$F(x) = x^n + Ax^{n-1} + \cdots + Kx + L = 0$$

the roots of which are designated by $\alpha, \beta, \ldots, \lambda$.

In particular, $F(x) = (x - \alpha)(x - \beta) \cdots (x - \lambda)$.

He then fixes an integer N and assumes that there exists an integer a such that $F(a) \equiv 0 \pmod N$ (in other terms, N divides $F(a)$). Let us notice that if N is a prime number of the form $pm+1$, for a prime number p and an integer m, Fermat's Little Theorem shows that $x^{N-1} \equiv 1 \pmod N$, for all integers x prime to N; thus the condition that there exists an integer a such that $F(a) \equiv 0 \pmod N$ is satisfied with $F(x) = \frac{x^p-1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x + 1$.[56]

A decisive move is now to introduce the quantities which will replace the $\phi(\alpha)$ of Jacobi's paper. Hermite *adapts* the linear forms we are interested in and defines:

$$\phi(\alpha) = Nx_0 + (\alpha - a)x_1 + (\alpha^2 - a^2)x_2 + \cdots + (\alpha^{n-1} - a^{n-1})x_{n-1}.$$

He also considers their product over all roots of F: $\mathcal{F} = \phi(\alpha)\phi(\beta) \cdots \phi(\lambda)$. A key point is that the coefficients of $\mathcal{F}$, which are already by construction symmetric functions of the roots $\alpha, \beta, \ldots, \lambda$ and thus integers, are now also all multiples of N. Indeed, these coefficients are obtained by products of terms which contain either N or a product $(\alpha - a)(\beta - a) \cdots (\lambda - a)$. Up to its sign, this last quantity is $F(a)$, which is, as explained, divisible by N.

However, the main object, as Hermite mentioned in the quote above, is not this product, but a (positive definite) quadratic form Hermite associated to it: If the roots $\alpha, \beta, \ldots, \lambda$ are real, he puts

$$f(x_0, x_1, \ldots, x_{n-1}) = \phi^2(\alpha) + \phi^2(\beta) + \cdots + \phi^2(\lambda).$$

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55 [Hermite 1850, p. 262]: "C'est dans quelques propriétés très élémentaires des formes quadratiques, à un nombre quelconque de variables, que j'ai rencontré les principes d'Analyse dont je vous demande la permission de vous entretenir. J'ai tiré de ces principes une démonstration de votre beau théorème sur la décomposition des nombres premiers $5m + 1$, en quatre facteurs complexes, formés des racines cinquièmes de l'unité."

56 In more modern terms, there exists an integer a whose image in $(\mathbb{Z}/N\mathbb{Z})^*$ is exactly of order p; such an element exists because this group is of order $N - 1$, which is divisible by p.

If β and γ, say, are complex conjugate roots, he uses the term $\phi(\beta)\phi(\gamma)$ instead of the squares. [57]

Generalizing Gauss's theory of reduction of binary quadratic forms to n-ary quadratic forms, Hermite had proved earlier that, for each quadratic form, there exist integers such that the value of the quadratic form at these integers is bounded, the bound depending only on the value of the determinant of the form and the number of variables. [58] More precisely, there exist integers $x_0, x_1, \ldots, x_{n-1}$ such that

$$0 < f(x_0, x_1, \ldots, x_{n-1}) < \left(\frac{4}{3}\right)^{\frac{n(n-1)}{2}} \sqrt[n]{|D|},$$

with D the determinant of the form f.

It is then easy to deduce bounds for the various terms $\phi^2(\alpha)$, $\phi^2(\beta)$, ... (or $\phi(\beta)\phi(\gamma)$) which compose the form, and thus for their product. Finally, taking into account that here the coefficients of $\mathcal{F}$ are multiples of N, as explained above, Hermite concludes that there exist integers $x_0, x_1, \ldots, x_{n-1}$ such that

$$\mathcal{F}(x_0, x_1, \ldots, x_{n-1}) = MN, \qquad M < \left(\frac{4}{3}\right)^{\frac{n(n-1)}{4}} (\Delta n^{-n})^{\frac{1}{2}},$$

where M is a non zero integer and Δ is the discriminant of the equation $F = 0$ (that is, the product of the difference of the roots, taken two by two).

To address Jacobi's statement, Hermite chooses

$$F(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1.$$

Let $n = 4$ and N a prime number of the form $5k + 1$. Then the construction of the $\phi, \mathcal{F}, \ldots$ above applies. One has $\Delta = 5^3$ and $M < 1.65$, thus $M = 1$. This provides:

$$N = \mathcal{F}(x_0, x_1, x_2, x_3) = \phi(\alpha)\phi(\beta)\phi(\gamma)\phi(\delta),$$

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57 Hermite introduces in fact a family of quadratic forms, built with terms $D_\alpha \phi^2(\alpha)$, or $D_{\beta,\gamma}\phi(\beta)\phi(\gamma)$, with $D_\alpha, D_{\beta,\gamma}$ positive real numbers. It provides infinitely many solutions, by varying the coefficients $D_\alpha, D_{\beta,\gamma}$, but as it is not relevant to our example, we will ignore this variant, see [Goldstein 2007] for details.

58 This is what is now called the Hermite-Minkowski theorem, as Hermann Minkowski established later a better bound than Hermite's original one. It is a generalization of the bound given above for the first coefficient of a reduced form and thus a key result for the classification of quadratic forms at the time. To state this theorem, Hermite had first to define the determinant of a n-ary quadratic form (our modern discriminant).

where $\alpha, \beta, \gamma, \delta$ are the primitive 5th-roots of unity, and

$$\phi(\alpha) = Nx_0 + (\alpha - a)x_1 + (\alpha^2 - a^2)x_2 + (\alpha^3 - a^3)x_3,$$
$$\phi(\beta) = Nx_0 + (\beta - a)x_1 + (\beta^2 - a^2)x_2 + \cdots, \qquad \phi(\gamma) = \cdots,$$

that is, the decomposition of N into complex factors of the required type.

Hermite also handles in the same way new cases, that of prime numbers N of the form $7k \pm 1$. If $N = 7m + 1$, he puts:

$$\phi(\zeta^k)(x_0, \ldots, x_5) = Nx_0 + (\zeta^k - a)x_1 + \cdots + (\zeta^{5k} - a^5)x_5,$$

where ζ^k now runs over the roots of the cyclotomic equation $F(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$, that is, the primitive 7-th roots of unity and a is an integer such that $a^7 \equiv 1 \pmod N$ but $a \not\equiv 1 \pmod N$, and thus $F(a) \equiv 0 \pmod N$.

With the same construction as before, *mutatis mutandis*, the determinant D of the quadratic form $f = \phi(\zeta)\phi(\zeta^6) + \phi(\zeta^2)\phi(\zeta^5) + \phi(\zeta^3)\phi(\zeta^4)$ is, in absolute value, $\frac{7^5}{2^6}N^2$, and there exists integers such that the value of $\mathcal{F}$ at these integers is MN, with $M < (\frac{4}{3})^{\frac{15}{2}}(\frac{7^5}{6^6})^{\frac{1}{2}}$, that is, finally $M < 5{,}192$. Moreover, Hermite proves that the value at integers of $\mathcal{F}$ is also either divisible by 7 or $\equiv 1 \pmod 7$, thus again $M = 1$ and N is a product of six linear combinations of primitive 7th-roots of unity.

In the case $N = 7m - 1$, Hermite finds that N can be expressed as a product of three linear combinations of the roots of the cubic equation

$$x^3 + x^3 - 2x - 1 = 0.$$ [59]

At that time already, Hermite placed great hopes in the use of "forms whose coefficients depend on roots of algebraic equations with integral coefficients. Maybe we will succeed to deduce from them a complete system of characters for each species of this kind of quantities". [60] In the same series of letters to Jacobi, he uses the same construction, this time with the roots α, β, γ of a cubic equation, to study what we now call the units of the associated cubic field: for Hermite, the problem is to find the solutions of the equation

$$(x + \alpha y + \alpha^2 z)(x + \beta y + \beta^2 z)(x + \gamma y + \gamma^2 z) = 1.$$

59 These roots are the three periods of two roots of the 7-th cyclotomic equation, as explained in the last section of Gauss's *Disquisitiones arithmeticae* on cyclotomic equations.

60 [Hermite 1850, p. 286]: "... des formes dont les coefficients dépendent de racines d'équations algébriques à coefficients entiers. Peut-être parviendra-t-on à déduire de là un système complet de caractères pour chaque espèce de ce genre de quantités".

He shows in particular that, up to roots of unity, the solutions are the powers of one of them only (in the case of one real root) or the products of the powers of at most two of them, in the case of three real roots.[61] He also uses the relation between the product of the $x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots + \alpha^{n-1} x_{n-1}$, when α runs over the solutions of an algebraic equation with integral coefficients, and the associated quadratic forms (according to the procedure explained above between f and $\mathcal{F}$) to deduce from the theory of forms that the infinitely many algebraic equations with integral coefficients and a given discriminant define only a finite number of irrational numbers (their roots), up to a rational change of variables [Hermite 1905-1917, vol. 1, p. 225].

The results I just outlined announced (among other contemporary results, in particular by Dirichlet) what will be known later as "algebraic number theory". We are thus not surprised now that Hermite combined algebraic tools and notions to be able to use number-theoretical techniques (the theory of integral quadratic forms, in particular), in order to answers questions arising both in number theory and algebra (according to the classification of domains in his time). However, in the middle of the nineteenth century, this mixture, which Gauss used with spectacular effects in the last section of his *Disquisitiones arithmeticae*, was still worthy of note.[62] Hermite was very aware of this fusion and of its promises and challenges:

> In the immense range of research opened up by M. Gauss, algebra and number theory seem to me to merge into a same order of analytical notions, which our present knowledge does not yet allow us to form a correct idea of.[63]

3.2. *Sturm's theorem*

This merger will further extend its reach when Hermite will use his favorite quadratic forms to tackle a key result of algebra at the time, Sturm's

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61 Hermite will tackle later the general case, for algebraic equations of all degrees. However, he did not prove the *existence* of the independent basic solutions, a result known as Dirichlet's unit theorem.

62 On the place of number theory c. 1800 and the change brought about by the *Disquisitiones*, see [Goldstein et al. 2007].

63 [Hermite 1850, p. 291]: "Dans cette immense étendue de recherches qui nous a été ouverte par M. Gauss, l'Algèbre et la Théorie des nombres me paraissent devoir se confondre dans un même ordre de notions analytiques, dont nos connaissances actuelles ne nous permettent pas encore de nous faire une juste idée."

theorem, which provides the exact number of real solutions, located in a given interval, of an algebraic equation with real coefficients. [64]

Hermite has been interested in this result during all his professional life, both in research and teaching ; he called it still in 1890 "one of the most important propositions of the theory of algebraic equations ... which had the good fortune to become immediately classic." [65]

According to Sturm's theorem, the number of distinct real roots located between, say, x_1 and x_2 of a polynomial V in one variable with real coefficients is equal to the difference in the number of sign variations when one evaluates, at x_2 and at x_1 respectively, a finite sequence of polynomial functions V_i. In Charles Sturm's original memoir, the sequence V_i was obtained through successive Euclidean divisions, which required quite laborious computations, and the first proof of the theorem used the intermediate value theorem.

Around 1840, James Sylvester expressed Sturm's auxiliary polynomials V_i directly in function of the squares of differences of the roots of V ; if V is of degree m and its roots are $a, b, \ldots, k, l$, one has: [66]

$$\begin{aligned}
\frac{V_1}{V} &= \sum \frac{1}{x-a} \\
\frac{V_2}{V} &= \sum \frac{(a-b)^2}{(x-a)(x-b)} \\
\frac{V_3}{V} &= \sum \frac{(a-b)^2(a-c)^2(b-c)^2}{(x-a)(x-b)(x-c)} \\
&\vdots \\
\frac{V_m}{V} &= \frac{(a-b)^2(a-c)^2\cdots(k-l)^2}{(x-a)(x-b)\cdots(x-l)}.
\end{aligned}$$

Hermite brought several innovations to the theorem: he extended it to a system of several equations, he displayed infinitely many Sturm's series of auxiliary functions, he proved all the theorems without any recourse

64 This theorem, its complex genesis and its far-reaching reformulations have been studied in detail and in depth in [Sinaceur 1994].

65 [Hermite 1905-1917, vol. 4, p. 291]: ... un théorème qui est l'une des plus importantes propositions de la théorie des équations algébriques [... et qui] a eu le bonheur de devenir classique". On this point, see [Vincent 2020].

66 [Hermite 1853]. To simplify, I follow as much as possible here Hermite's notations, despite their ambiguity and the lack of precision on the summation range. We can assume here that the roots are distinct and ordered.

to continuity arguments (in particular, without the intermediate value theorem). Moreover, through the use of his favorite quadratic forms, he unified Sturm's theorem on the number of real roots of an equation in a given interval and Cauchy theorem on the number of imaginary roots in a bounded given domain. He also connected Sturm's theorem with the classification of algebraic forms, a topic at the forefront of research at the time. [67]

Here as elsewhere, working as closely as possible with formulas (and their multiple interpretations) is a decisive factor in Hermite's mathematical practice, a factor he himself often emphasizes. About his extension of Sturm's theorem to several variables, where he begins by replacing the factors of the type $\frac{1}{x-a}$ in the original Sturm's theorem by $\frac{1}{(x-a)(y-a')}$, he says for instance:

> I must first mention the beautiful expressions discovered by Mr. Sylvester for the auxiliary functions that appear in Mr. Sturm's theorem, and those deduced by Mr. Cayley, as having opened up a new path for me. These are, indeed, formulas *analogous* to those of these two learned geometers, which will be posited a priori for simultaneous equations, and from which properties all similar to those of Mr. Sturm's functions will be easily concluded. [68]

Let us just reconstruct briefly how the same quadratic forms we met above are used by Hermite in this other context. [69] Let us consider a polynomial V with real coefficients and m distinct roots a_i. Hermite associated to it (a family of) quadratic m-ary forms:

$$\sum \frac{1}{x-a_i}(x_0 + a_i x_1 + a_i^2 x_2 + \cdots + a_i^{m-1} x_{m-1})^2.$$

67 See in particular [Hermite 1905-1917, vol. 1, pp. 281-287, 397-414, 415-428, 479-481]. As for other topics, Hermite's work is tightly connected with the works of other mathematicians exactly at the same time, in this case Sturm himself, Carl Borchardt, Arthur Cayley, James Sylvester, and others [Sinaceur 1994]. They exchange letters, which are partially published, and complement each other's work, sometimes day by day. Given our focus on specific Hermite's tools here, this environment is left aside.

68 [Hermite 1905-1917, vol. 3, p. 2]: "Je dois indiquer d'abord les belles expressions découvertes par M. Sylvester pour les fonctions auxiliaires qui figurent dans le théorème de M. Sturm, et celles que M. Cayley en a déduites, comme m'ayant ouvert une voie nouvelle. Ce sont, en effet, des formules *analogues* à celles de ces deux savants géomètres qui seront posées a priori pour des équations simultanées et dont on conclut avec facilité des propriétés toutes semblables à celles des fonctions de M. Sturm." My emphasis.

69 Hermite's published papers on this topic are extracts of letters, with different notations and only allusive outlines of his arguments. For simplicity, I have standardized slightly the notation and detailed some calculations.

Here x is a real number, different from the roots a_i, and it is treated as a parameter; I will designate these expressions by $f_x(x_0, x_1, \ldots, x_{m-1})$. We recognize the type of forms, built with the roots of an algebraic equation, introduced before, except for the coefficients of the squares (here $\frac{1}{x-a_i}$) which may be here negative as well as positive. Negative coefficients occur when the root a_i is greater than x, positive ones when a_i is smaller than x, and thus the number of positive and negative coefficients keep track of the number of roots smaller and bigger than x (and by combination of f_x and $f_{x'}$ of the number of roots in the interval $]x, x'[$). Let us note that the forms f_x are rational symmetrical functions of the roots of V, thus are real quadratic forms when x is real.

The determinant $\Delta_{m-1,x}$ of f_x is

$$\Delta_{m-1,x} = \frac{1}{\prod(x-a_i)} \begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{m-1} \\ 1 & a_2 & a_2^2 & \ldots & a_1^{m-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & a_m & a_m^2 & \ldots & a_m^{m-1} \end{vmatrix}^2 .$$

The Vandermonde determinant is equal as usual to the product

$$\prod_{1 \le i < j \le m} (a_i - a_j)$$

and thus $\Delta_{m-1,x}$ is V_m/V. Let us designate by $\Delta_{0,x}$, $\Delta_{1,x}$, $\Delta_{2,x}$, etc., the determinants corresponding to the partial forms $\sum \frac{x_0^2}{x-a_i}$, $\sum \frac{(x_0+a_i x_1)^2}{x-a_i}$, $\sum \frac{(x_0+a_i x_1+a_i^2 x_2)^2}{x-a_i}$, etc. ; they similarly correspond to Sylvester's auxiliary functions for Sturm's theorem.

Hermite explains to Borchardt: "the reduction of a quadratic form to a sum of squares, which has been the topic of your memoir [...] plays the main role in my research". [70]

70 [Hermite 1856, p. 39]: "La réduction d'une forme quadratique à une somme de carrés, qui a été le sujet de votre Mémoire [...] joue le principal rôle dans mes recherches". The classification of real quadratic forms, more precisely the transformation by a linear, invertible change of variables, of any real quadratic form into a linear combination of squares with coefficients, say, +1 or -1, was a hot topic at the time. As well-known now, the number of coefficients of each sign is an invariant of the form: this is Sylvester law of inertia. On these developments, see [Brechenmacher 2007].

Indeed, the form f_x can be reduced to a "sum of squares" (up to real constants) via a triangular change of variables of determinant 1. One has:

$$\begin{aligned} f_x = {} & \epsilon_0(x_0 + \alpha_{1,0}x_1 + \alpha_{2,0}x_2 + \ldots + \alpha_{m-1,0}x_{m-1})^2 \\ & + \epsilon_1(x_1 + \alpha_{2,1}x_2 + \alpha_{3,1}x_3 + \ldots + \alpha_{m-1,1}x_{m-1})^2 \\ & + \cdots \\ & + \epsilon_{m-2}(x_{m-2} + \alpha_{m-1,m-2}x_{m-1})^2 \\ & + \epsilon_{m-1}x_{m-1}^2. \end{aligned}$$

The ϵ_i are real coefficients. Thus $\Delta_{m-1,x} = \epsilon_0\epsilon_1 \ldots \epsilon_{m-1}$.

Moreover, putting $x_{m-1} = 0$ gives $\Delta_{m-2,x} = \epsilon_0 \ldots \epsilon_{m-2}$, putting $x_{m-2} = x_{m-1} = 0$ gives $\Delta_{m-3,x} = \epsilon_0 \ldots \epsilon_{m-3}$, etc. Finally, the real quadratic form f_x can be reduced by a linear transformation with real coefficients to a form of the type:

$$\Delta_{0,x}X_0^2 + \frac{\Delta_{1,x}}{\Delta_{0,x}}X_1^2 + \frac{\Delta_{2,x}}{\Delta_{1,x}}X_2^2 + \ldots + \frac{\Delta_{m-1,x}}{\Delta_{m-2,x}}X_{m-1}^2.$$

The signs of the coefficients provide the sign of the values of the Sturm's functions. In other terms, Sturm's theorem for a polynomial is now seen as an explicit version of the law of inertia for our special quadratic forms, built with the roots of this polynomial.

3.3. *Toward complex analysis*

Hermite then handles the complex case by a variant of the same expressions. Let F be any algebraic equation with complex coefficients, this time of degree n, $F(z) = Az^n + Bz^{n-1} + \cdots + Kz + L = 0$, with roots a, b, ..., k.[71] Hermite introduces the quadratic form:

$$\begin{aligned} \phi(x, y, \ldots, u) = {} & \frac{i}{F_0(a)F'(a)}(x + ay + \cdots + a^{n-1}u)^2 \\ & + \frac{i}{F_0(b)F'(b)}(x + by + \cdots + b^{n-1}u)^2 \\ & + \cdots \\ & + \frac{i}{F_0(k)F'(k)}(x + ky + \cdots + k^{n-1}u)^2, \end{aligned}$$

where F_0 is the polynomial the coefficients of which are the complex conjugate of those of F. The quadratic form ϕ is real and Hermite proves that its signature provides the number of roots in the upper, resp. lower,

71 In Hermite's fickle notations ; see [Hermite 1856, p. 40].

half plane. He deduces the number of roots of an algebraic equation contained in various domains of the complex plane, such as a rectangle or a hyperbole and, again by means of astute transformations, obtains particular cases of Cauchy's integral formula.

Hermite emphasizes the role of his special quadratic forms in his procedure:

> I came to this study largely by a search on arithmetic questions, which since the year 1847 have called my attention on the quadratic forms composed of a sum of squares of similar functions of the roots of the same equation.
>
> I have thus felt a real satisfaction to relate to these forms these magnificent theorems of M. Sturm and M. Cauchy, which open the new era of modern Algebra. [72]

3.4. *Toward invariant theory*

This account of the number of roots of an equation through the study of quadratic forms is used by Hermite in another work which leads this time to invariant theory. [73] The context is the resolution (by elliptic modular functions) of the fifth-degree equation which Hermite and Kronecker, and then Brioschi, approached as we have seen above, in different ways—triggering Hermite's attempt to link and re-interpret the various methods of resolution. His point of departure here is a general binary form of the fifth degree $f(x,y) = \alpha x^5 + 5\beta x^4 y + 10\gamma x^3 y^2 + 10\gamma' x^2 y^3 + 5\beta' x y^4 + \alpha' y^5$, and the associated fifth-degree equation $f(x,1) = 0$, with its roots $a, b, \ldots, k$. [74]

Hermite first shows that if, in this equation, one replaces x by $z = \frac{\phi(x,1)}{f'_x(x,1)}$, where $\phi(x,y)$ is a covariant of degree 3 in x, y of f, the coefficients of the transformed equation in z are invariants of $f(x,y)$. [75] Then, he adapts once

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72 [Hermite 1856, p. 50]: "J'ai été amené à cette étude en grande partie par des recherches sur des questions arithmétiques, qui, depuis l'année 1847, ont appelé mon attention sur les formes quadratiques composées d'une somme de carrés de fonctions semblables des racines d'une même équation. Aussi ai-je éprouvé une véritable satisfaction à rattacher à la considération de ces formes ces magnifiques théorèmes de M. Sturm et M. Cauchy, qui ouvrent l'ère nouvelle de l'Algèbre moderne."

73 On the history of this topic, see [Dieudonné 1971; Parshall 1989; Wolfson 2008]. On Hermite's contributions to it, see [Parshall 2024].

74 [Hermite 1866]. This booklet gathers 12 communications to the French Academy in 1865 and 1866. They are also reproduced in the second volume of Hermite's works.

75 Covariants (resp. invariants) are at the time polynomials in the coefficients of f and x, y (resp. in the coefficients of f), which are invariant under a linear change of variables x, y of determinant 1. A first example of an invariant is Gauss's determinant of a binary quadratic form, $b^2 - ac$.

more the expression of his favorite quadratic forms: he replaces the expression $(t_0 + at_1 + \cdots + a^4 t_4)^2$ in his special quadratic forms by

$$\left(t_0 + \frac{t_1\phi_1(a,1) + t_2\phi_2(a,1) + t_3\phi_3(a,1) + t_4\phi_4(a,1)}{f'_x(a,1)}\right)^2,$$

where the ϕ_i are four cubic covariants of f that Hermite provides explicitly.

Adding these new expressions for all roots $a, b, \ldots, k$, Hermite obtains a form whose coefficients are invariants of f. The signature of this form (that is, the number of positive and negative coefficients in its transformation as a linear combination of squares, according to Sylvester's law of inertia) provides, as previously, the number of real and imaginary roots of $f(x, 1) = 0$.

To go further, Hermite also adapts the expression of the functions V which intervene in Sturm's theorem.

First of all, he replaces simply

$$V_1 = (x-a)(x-b)\cdots(x-k)\sum \frac{1}{x-a}$$

by

$$\mathcal{V}_1 = (x-a)(x-b)\cdots(x-k)\sum \frac{x'-a}{x-a},$$

$$V_2 = (x-a)(x-b)\cdots(x-k)\sum \frac{(a-b)^2}{(x-a)(x-b)}$$

by

$$\mathcal{V}_2 = (x-a)(x-b)\cdots(x-k)\sum \frac{(x'-a)(x'-b)}{(x-a)(x-b)}(a-b)^2,$$

and so on. As in the original Sturm's theorem, where they were expressed as we have seen as determinants, the various terms

$$\frac{\mathcal{V}_{i+1}}{(x-a)(x-b)\cdots(x-k)}$$

are here invariants of the quadratic forms (which are again variants of the forms built with the roots of an equation, this time with parameters x and x'):

$$\sum \frac{x'-a}{x-a}(t_0 + at_1 + a^2 t_2 + \cdots + a^i t_i)^2.$$

More generally they share the basic properties of the Sturm's functions.

> It's easy to see how these new functions are *closely related* to those of Sturm's theorem, whose analytical properties they reproduce. They then serve as a

natural and easy transition to those [...] which are double covariants of the form $f(x,y)$, the proposed equation being $f(x,1)=0$.[76].

To obtain them, Hermite adapts once more his favorite quadratic forms, considering this time

$$\sum \frac{x'-ay'}{x-ay}\left(t_0 + \frac{t_1\phi_1(x,1)+t_2\phi_2(x,1)+t_3\phi_3(x,1)+t_4\phi_4(x,1)}{f'_x(x,1)}\right)^2.$$

These new functions, being composed of covariants in (x,y) and in (x',y') (the expressions $\frac{x'-ay'}{x-ay}$) and of invariants, are covariants which can replace Sturm's functions, as desired.

3.5. *Interpolation*

Perhaps more surprising is the use of the same kind of quadratic forms to approach problems of interpolation. More precisely, the objective is to approximate a (sufficiently regular) function F by a polynomial P of degree $m \leq n$, such that the distance between this polynomial P and F at $n+1$ given points x_i—that is, here, the sum of the squares of the differences between the value of P and F at the x_i—is minimum. Hermite was inspired to deal with this question through a communication of Pafnuti Tchebichef at the Academy of sciences of Saint-Petersburg on January 12, 1855, which was translated into French in 1858.[77]

Let us assume that the function F takes the values u_i at the given points x_i, $i = 0,1,\ldots,n$. Hermite's first step is to use Lagrange's interpolation formula to construct a polynomial Π of degree n with the same values u_i at the points x_i.

Let $f(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$ a polynomial of degree $n+1$ with the x_i as roots. Hermite puts:

$$\Pi(x) = \frac{f(x)}{(x-x_0)f'(x_0)}u_0 + \frac{f(x)}{(x-x_1)f'(x_1)}u_1 + \cdots + \frac{f(x)}{(x-x_N)f'(x_N)}u_n.$$

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76 [Hermite 1905-1917, vol .2, p. 382]: "On voit assez [...] l'étroite *liaison* de ces nouvelles fonctions avec celles du théorème de Sturm dont elles reproduisent les propriétés analytiques. Elles servent ensuite de transition naturelle et facile pour arriver à celles [...] qui sont des covariants doubles de la forme $f(x,y)$, l'équation proposée étant $f(x,1)=0$." My emphasis.

77 [Tchebichef & Bienaymé 1858]. Hermite's communication to the French Academy follows the French publication closely, taking place on January 10, 1859 [Hermite 1859]. Both Tchebichef and Hermite work in a slightly more general context than reported here as they multiply the squares by positive real numbers that may represent errors in the measures of the values $F(x_i)$. For the sake of simplicity, I shall ignore these factors here.

By a linear transformation of the u_i which preserves the Euclidean norm $\sum u_i^2$ (that is, by an orthogonal linear transformation), one can also write, with the new variables v_i:

$$\Pi(x) = \Phi_0(x)v_0 + \Phi_1(x)v_1 + \cdots + \Phi_n(x)v_n,$$

where the Φ_k are polynomials of degree n.

Hermite remarks that $\sum \Pi(x_i)^2 = \sum u_i^2 = \sum v_i^2$. And that this, in turn, implies that $\sum_i \Phi_k(x_i)^2 = 1$ and $\sum_i \Phi_k(x_i)\Phi_{k'}(x_i) = 0$, for all $0 \leq k \leq n$ and $0 \leq k' \leq n$, $k \neq k'$.

His second step implements the condition of minimality, that is, that the distance between F and Π should be the smallest possible. More precisely, Hermite explains how to choose coefficients $A, B, \ldots, H$ such that:

$$\sum_{i=0}^{n} [F(x_i) - A\Phi_0(x_i) - B\Phi_1(x_i) - \cdots - H\Phi_m(x_i)]^2$$

is minimum.

Finally, Hermite shows that the Φ_k can be constructed with $m = n$. It could be deduced from a counting argument on the number of variables and equations, but Hermite, as usual, looks for an explicit solution. Hermite says:

> It would be difficult in this way to explicitly express the new functions $[\Phi_i]$ by the quantities $x_0, x_1, \ldots, x_n$. It is [...] by making use of the properties of quadratic forms that we can achieve this." [78]

Once again, a variant of our usual quadratic forms, built this time with the roots x_i of f, will provide the expression of the Φ_i. Hermite puts:

$$\sum_{i=0}^{n} (x - x_i)(v_0 + x_i v_1 + x_i^2 v_2 + \cdots + x_i^{k-1} v_{k-1})^2.$$

For $1 < k \leq m$, let Δ_k be the determinant of the quadratic form ; it is a polynomial of degree k in x, and if δ_k is the coefficient of x^k in this polynomial, Hermite proves that $\Phi_k(x) = \frac{\Delta_k}{\sqrt{\delta_k \delta_{k+1}}}$ (with specific values for Φ_0 and Φ_1). He concludes:

> Finally, I would like to point out that the sequence of quantities $1, \Delta_1, \Delta_2, \ldots, \Delta_m$ has the properties of Sturm's functions with respect to the equation $f(x) = 0$. [79]

78 [Hermite 1859, pp. 65-66]: "Cependant il serait difficile par cette voie de parvenir à exprimer explicitement les nouvelles fonctions $[\Phi_i]$ par les quantités $x_0, x_1, \cdots x_n$. C'est [...] en faisant usage des propriétés des formes quadratiques qu'on y arrive."

79 [Hermite 1859, pp. 66-67]: "Je remarquerai enfin, en terminant, que la suite des quantités $1, \Delta_1, \Delta_2, \ldots, \Delta_m$ possède à l'égard de l'équation $f(x) = 0$ les propriétés des fonctions de Sturm."

This work, again, also illustrates how playing with explicit algebraic formulas is intertwined with relations at another level. In his article on interpolation, Tchebichef used expansions into continued fractions. Bienaymé had already interpreted this feature as a step to a unification, as he emphasized how this work "spreads a new light on the hidden links that unite the various parts of the analysis of series or of interpolation". [80]

In Hermite's youth, the expansion into continued fractions was the basic technique for approximation. [81] In his letters to Jacobi on the theory of numbers [Hermite 1850], Hermite proudly explains how to replace this tool with his own, that is, again, quadratic forms with appropriate coefficients. This context is relevant to understanding why and how Tchebichef's 1858 article (entitled "On continued fractions") would immediately attract Hermite's attention and direct him towards his favorite quadratic forms: the thematic level, here the theme of approximation, suggests him to use a specific process, and in turn this process which operates at the level of the algebraic expressions themselves reinforces a thematic rapprochement between mathematical subdomains, from arithmetic to algebra to analysis. [82]

4. REVISITING THE UNITY OF MATHEMATICS THROUGH HERMITE'S PRACTICES

The evolution of mathematics during the 19th century has sometimes been described as the replacement of an older approach, based on formulas, with a modern conceptual one. Such a description is often supported by some of the statements made by the mathematicians of the time, for instance that of Dirichlet noting: "the constantly increasing tendency of

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80 [Tchebichef & Bienaymé 1858, p. 290, note]: "Le travail de M. Tchebichef, en reliant aux fractions continues une classe au moins des fonctions ou des coefficients mis en évidence par Gauss dans la méthode des moindres carrés, répand une clarté nouvelle sur les liens cachés qui réunissent les diverses parties de l'analyse des séries ou de l'interpolation."

81 For Hermite, it will also be an important tool to adapt in a variety of contexts, either for numbers or for functions, launching a whole field of research for his students, in particular Henri Padé, see [Brezinski 1991].

82 Many years later—but again recycling some constructions and results of his letters to Jacobi on the theory of numbers—, Hermite tackles in the same way another result of Tchebichef, this time on the minima of $x - ay - b$, with a and b real constants and x and y integers to be found, and its algebraic counterpart, the best approximations of V by $X - UY$, where U and V are sufficiently regular given functions, and X and Y polynomials to be found, [Hermite 1880].

the new analysis is to put thoughts in the place of calculations."[83] Hermite offers of course a good counter-example to the relevance of such a dichotomy, if we wish to account for the actual work of mathematicians. In many of his articles, a stock of algebraic expressions and formulas, adapted to each specific mathematical situation, weaves like a garland through mathematical themes. They also serve as intermediary steps, enabling Hermite to arrive, by successive replacement, at the expressions needed to conclude, but also at new concepts, as though "naturally".[84] This point of view is consistent with Hermite's creative vision of computations and, more generally, his view of mathematics as a natural science, a nature guaranteed by an underlying divine order.[85] It is of course worth noting that other mathematicians, in the second half of the nineteenth century, extoll the role of formulas and computations too, as witnessed by a famous passage of a letter from Kronecker to Cantor in August 1884:

> I recognize true scientific value—in the field of mathematics—only in concrete mathematical truths, or to put it more sharply, "only in mathematical formulas". [...]The various theories for the foundations of mathematics have been blown away by time, but Lagrange's resolvent has remained.[86]

It is notable that Hermite's focus on explicit expressions extends beyond his articles, in particular to his teaching. If his son-in-law Émile Picard emphasizes how in his lectures, "on the most elementary of questions, [Hermite] suddenly opened up immense horizons, and alongside the Science

83 "Die immer mehr hervortretende Tendenz der neueren Analysis ist Gedanken an die Stelle der Rechnung zu setzen." See a synthetic reminder of this viewpoint, as well as relevant references, in the introduction of [Sørensen 2005]. Let us note the debatable substitution of "thought" by "concepts" in many interpretations of Dirichlet's quote. The opposition has already been criticized by historians for several decades, see [Gilain & Guilbaud 2015; Goldstein et al. 2007].

84 The importance of a construction presented as natural is essential for Hermite. For the example of Hermitian forms, see [Goldstein 2019].

85 See his criticism of Louis Poinsot's statement that "computation is an instrument that does not produce anything from itself and that somehow gives back only the ideas entrusted to it" in a letter to Leo Königsberger, [Goldstein 2011a, p. 147].

86 See [Jahnke 1987]: "Einen wahren wissenschaftlichen Werth erkenne ich auf dem Felde der Mathematik nur in concreten mathematischen Wahrheiten, oder schärfer ausgedrückt, 'nur in mathematischen Formeln'. [...] Die verschiedenen Theorien für die Grundlagen der Mathematik sind von der Zeit weggeweht, aber die Lagrange'sche Resolvente ist geblieben." For a deeper analysis of the role of formulas and computations in Kronecker's work, see [Edwards 2009; 1995; Vergnerie 2019]. Another interesting case, that of Dedekind, is studied in [Haffner 2021].

of today, we saw the Science of tomorrow,"[87] a more ironic and critical account of Hermite's courses testifies to the continuity between his teaching and his mathematical practice. Charles Rabut, a Ponts-et-Chaussées engineer, who had followed these courses at Polytechnique around 1871, complains:

> As for mathematical dreams, several of my schoolmates and I had them following Hermite's abominable lessons on Eulerian, elliptic, ultra-elliptic and other functions; they consisted of assimilations of algebraic symbols to things from real life. I remember a certain function $C(x)$, where C was a caravan. I believe that this phenomenon, which was quite widespread among our fellow students at the time, was a reflex protest by the cerebral organism against the inoculation of a veritable intellectual poison.[88]

With its caravan of embodied symbols, Rabut's nightmare seems remarkably resonant with what has been discussed here. As we have seen, Hermite's work on specific algebraic expressions constructs a unity from below, supported, in a less technical but decisive way, by a configuration of links and transfers operating at other levels. Whether in copying and adapting a tool developed in one branch or in reshaping a theorem in a new frame, the threads created by this very work on expressions is consubstantial with the identification of analogies and rapprochements between large swathes of mathematics, more specifically among arithmetic, algebra, and analysis (as defined in his time). While Hermite's proof of Sturm's theorem is often praised today for avoiding the use of analysis (in a perspective that privileges the purity of methods with respect to the statement or the disciplinary situation), we have seen that more important for Hermite was first the link between his number-theoretical research and a theorem of algebra, then the possibility of extending it to Cauchy's setting. In other cases, as in the resolution of the fifth-degree equation, Hermite would rejoice

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87 [Picard 1901, p. 29]: "à propos de la question la plus élémentaire il faisait surgir tout d'un coup d'immenses horizons, et où à côté de la Science d'aujourd'hui on apercevait la Science de demain".

88 [Maillet 1905, p. 26]: "Quant aux rêves mathématiques, plusieurs de mes camarades de salle à l'École et moi en avons fait à la suite des abominables leçons d'Hermite sur les fonctions eulériennes, elliptiques, ultra-elliptiques et autres; ils consistaient en des assimilations de symboles algébriques à des choses de la vie réelle. Ainsi je me souviens d'une certaine fonction $C(x)$, où C était une caravane. J'estime que ce phénomène, assez répandu parmi nos camarades à cette époque, est une protestation réflexe de l'organisme cérébral contre l'inoculation d'un véritable poison intellectuel".

about bringing analytic means into a number-theoretical question.[89] On this point, Paul Painlevé comments:

> The memoir [on the transformation of Abelian transcendents], in which the theory of functions is interwoven with Arithmetic and Algebra, is in some ways *representative* of Hermite's work. No one has shown more strikingly, through his methods and discoveries, the intimate relations that unite these three branches of Science, and the mutual support they can and must lend each other.[90]

While Hermite was particularly effective in building bridges between these domains, he was far from alone. Echoes of this can also be found for instance in Emile Borel's commentary on Edmond Laguerre's work:

> [O]ne is at first astonished to see Laguerre's research on whole transcendents, and others besides, appear among the memoirs of Algebra. This astonishment disappears when, instead of reading only the Table of Contents, we study the text itself; we then see that one of Laguerre's characteristic traits is the ease with which he solves many questions of Analysis by the methods of the most elementary Algebra, and we hardly notice the transition between the Algebra of polynomials and the Algebra of transcendental functions, if I may express myself in this fashion.[91]

However, simply highlighting the connection between integers in arithmetic, polynomials in algebra, and functions in analysis does not give access to the practices involved in establishing this connection, which vary widely depending on the moment and the author. Although Kronecker, Hermite, Laguerre, but also Dedekind and Heinrich Weber all inherited the mid-century research field of arithmetic algebraic analysis, they deployed it, interpreted it and worked within or beyond it in very different ways, prioritizing or not the various aspects. André Weil's trilingual dictionary, or

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89 On this role of analysis, taken in a very broad sense and not limited to continuous processes, see [Archibald 2024; Vincent 2024].

90 [Painlevé 1905, p. 50-51]: "Le mémoire [sur la transformation des transcendantes abéliennes], où la théorie des fonctions se mêle à l'Arithmétique et à l'Algèbre, est en quelques sorte *représentatif* de l'œuvre d'Hermite. Nul n'a montré, d'une façon plus éclatante, par ses méthodes et ses découvertes, les relations intimes qui unissent ces trois branches de la Science, l'appui mutuel qu'elles peuvent et doivent se prêter."

91 The quote comes from [Vincent 2024] to which I also refer for other relations between Hermite's and Laguerre's works: "[O]n est tout d'abord étonné de voir les recherches de Laguerre sur les transcendantes entières et d'autres encore, figurer parmi les mémoires d'Algèbre. Cet étonnement disparaît lorsque, au lieu de lire seulement la Table des matières, on étudie le texte même; on voit alors que l'un des traits caractéristiques de Laguerre est l'aisance avec laquelle il résout bien des questions d'Analyse par les méthodes de l'Algèbre la plus élémentaire, et l'on s'aperçoit à peine de la transition entre l'Algèbre des polynômes et l'Algèbre des fonctions transcendantes, si l'on peut ainsi s'exprimer."

Rosetta stone, half a century later, is yet another formulation of these possible analogies—even it would be tempting to see Hermite or Kronecker as its precursors.[92] Such a comparison, which would be necessary to understand the dynamics of unification between parts of these fields, goes far beyond the scope of this article, where I have only tried to better understand Hermite's related practice by following a few key words.

As the quotations in the introduction to this article testify, the question of the unity of mathematics has mainly been discussed by mathematicians or philosophers as a philosophical question, based on their epistemological priorities or on a global vision of mathematics, whether to describe its encompassing modalities or to define its present or future rules. More recently, the variety of mathematical activities in time and place has led others to criticize or to throw into doubt the very idea of a unity of mathematics [Booß-Bavnbek & Davis 2013]. Hermite's (counter-)example, however, suggests another conclusion than considering the unity of mathematics as a given to be studied, as an horizon the norms of which are to be established, or as an illusion to be dismissed. Examining the configuration of Hermite's writings closely, on a micro-scale,[93] reveals how the sense of unity in mathematics that drove him and was emphasized by his epigones was concretely put into practice and reinforced in his day-to-day mathematical work. It also suggests that the topic of the unity of mathematics can be associated with a richer set both of work practices and of representations than is usually taken into account, and which, interweaving together, also calls for a *historical* investigation.

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92 See [Dedekind & Weber 1882; Kronecker 1882; Weil 1979].

93 In the sense discussed in [Elias 1971; Grendi 1977; Lepetit 1995].

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