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A CONVERGENCE OF PATHS: CAYLEY, HERMITE, SYLVESTER, AND EARLY INVARIANT THEORY

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In loving memory of Brian J. Parshall
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Abstract. — This paper considers the beginnings of a *theory* of invariants in the early 1850s in the broader contexts of individual pathways toward the establishment of reputation and of the professionalization of mathematics in the nineteenth century. In particular, it treats the different, but intersecting, mathematical paths by which two Englishmen, Arthur Cayley and James Joseph Sylvester, and one Frenchman, Charles Hermite, came to focus on an analysis *per se* of the transformation of homogeneous forms by linear substitutions. It then looks at the intense mathematical exchanges in the first half of the 1850s that resulted in their early invariant-theoretic results. Although by the close of the 1850s, Cayley, Hermite, and Sylvester had largely gone their own separate mathematical ways, the three remained united in their sense of having created what they called the "New Algebra."

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Mots clefs. — Arthur Cayley, Charles Hermite, James Joseph Sylvester, history of invariant theory, nineteenth century, professionalization and internationalization of mathematics.

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Résumé. — Cet article considère les origines d'une *théorie* d'invariants au début des années 1850 dans les contextes plus larges des chemins suivis pour établir une réputation mathématique et de la professionnalisation des mathématiques au XIX^e siècle. En particulier, il s'occupe des différentes voies mathématiques, mais néanmoins voies croisées, par lesquelles deux Anglais, Arthur Cayley et James Joseph Sylvester, et un Français, Charles Hermite, en vinrent à se concentrer sur une analyse de la transformation de formes homogènes par les substitutions linéaires. Il se penche ensuite sur les échanges mathématiques intenses dans la première moitié des années 1850, échanges qui ont abouti aux premiers résultats proprement dits invariant-théoriques. Bien qu'à la fin des années 1850, les chemins mathématiques de Cayley, Hermite, et Sylvester avaient largement divergés, les trois mathématiciens sont restés unis dans le sentiment d'avoir créé ce qu'ils ont appellé la « nouvelle algèbre. »

Arthur Cayley was a twenty-two-year-old assistant tutor at and fellow of Trinity College, Cambridge in June 1844 when he wrote to George Boole, a Lincolnshire schoolteacher more than six year's his senior. The 1842 Cambridge Senior Wrangler had been reading a paper published by the largely self-taught Boole in the *Cambridge Mathematical Journal* and had produced "a few formulae relative to it" that he hoped would spark Boole's interest. [1] Following on his reading primarily of Joseph-Louis Lagrange's *Méchanique analitique* and the *Mécanique céleste* of Pierre Simon de Laplace, Boole had been intrigued by what he styled an "important and oft recurring problem of analysis," namely, "[t]he transformation of homogeneous functions by linear substitutions" [Boole 1841-1843a, p. 1]. Cayley, spurred to a large extent by "analytical geometry, his growing passion in mathematics" [Crilly 2006, p. 86], carried Boole's ideas further with the "formulae" that he published in 1845 and that marked, in some sense, his entrée into what would later become a theory of invariants [Cayley 1845].

Instances of the transformation of homogeneous functions by linear substitutions also cropped up in settings other than the mathematization of mechanics. Carl Friedrich Gauss had explored, in the number-theoretic context of his *Disquisitiones arithmeticae* of 1801, the question of how a binary quadratic form with integer coefficients was affected by a linear transformation [Gauss 1966, pp. 111-112]. It was Charles Hermite's independent reading of that source, as well as of Lagrange's *Traité de la résolution des équations numériques de tous les degrés* (first published in 1789 and revised by the author in 1808), that had exposed the *collégien* to both higher algebra and number theory while at Paris's Collège Louis-le-Grand

1 The letter is quoted, among other places, in [Crilly 2006, p. 86] and [Wolfson 2008, p. 43].

in the early 1840s [Picard 1905, p. viii]. By 1848, Hermite, who was one year Cayley's junior, had passed his *baccalauréat* and *licence*, had become a *répétiteur* and admissions examiner at the École polytechnique, and had published a note in which he, too, focused on the problem of transformation, but from a number-theoretic point of view [Hermite 1848].

A year earlier, yet another mathematician, James Joseph Sylvester, had also come to the matter of transformation from number theory [Sylvester 1847a], although he had cut his mathematical eyeteeth in the 1830s on a purely algebraic approach to the theory of elimination, that is, the theory involved, for example, in finding when two polynomial equations of degrees m and n in one variable have a common root or in determining the real roots of an algebraic equation $f(x) = 0$ of degree n. [2] Almost seven years Cayley's senior, Sylvester, a Jew and the Second Wrangler in 1837, had had a checkered career as he had tried, ultimately unsuccessfully, to establish himself as a mathematician in academe both in England and the United States over the course of the late 1830s and early 1840s. Back in England by the close of 1843, he took a position in 1844 as an actuary at the Equity and Law Life Assurance Society in London. This soon put him in close proximity to Cayley, who had left Cambridge in 1846 to prepare for a career at the Bar. By 1847, the two had met and started up what would become the mathematical correspondence they would maintain for essentially the rest of their lives. [3]

It is not clear exactly when Sylvester and Cayley met Hermite. Both Englishmen were, however, intent on making mathematical reputations for themselves in England and beyond. They had both participated in the French mathematical scene in the 1840s, and both would publish regularly in European journals. It is clear that Sylvester united the three of them in print in 1851 under the common rubric of "transformation" in his paper, "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms" [Sylvester 1851e]. There, he gave an early statement of "invariance" as it had emerged in the work of Boole and Cayley, at the same time that he referred to his "admirable friend M. Hermite" [Sylvester 1851e, pp. 185 and 190, resp.]. [4] Over the course of the first half of the 1850s, these three young mathematicians—separated by the English Channel—made common cause in the development of a new

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2 See the discussion in [Parshall 2006, pp. 59-62].

3 This early period in Sylvester's life is treated in [Parshall 2006, pp. 49-94]. For a glimpse of Cayley and Sylvester's correspondence, see [Parshall 1998].

4 Page references given for papers by Sylvester, Cayley, and Hermite in what follows refer to the pagination in their respective collected works.

theory—the theory of invariants—or what they would unabashedly term "the New Algebra" [Sylvester 1851a, p. 252]. At the same time, they worked to establish their respective careers in mathematics.

Although much has been written on Cayley and Sylvester's roles in the early development of invariant theory, [5] neither Hermite's part in that development nor his relationship with Cayley and Sylvester and with what became a British school of invariant theory has received particular historical scrutiny. What mathematical paths led them to focus on an analysis *per se* of the transformation of homogeneous forms by linear substitutions? What was the dynamic of the mathematical interchange between them—two in England and one in France—in the first half of the 1850s that resulted in their early invariant-theoretic results? In addressing these questions, this paper not only highlights Hermite's participation in the early development of what was later recognized as the British strain of invariant theory but also provides an interesting case study of how new mathematical ideas could develop in the mid-nineteenth century.

CAYLEY'S ANALYTICAL-GEOMETRICAL PATH

Boole's 1841 paper, "Exposition of a General Theory of Linear Transformations," focused on the determination of "the relations by which" the coefficients of a homogeneous polynomial of degree n in m unknowns "are held in mutual dependence" before and after a linear transformation of its variables is applied. [6] The general technique that he developed to address this problem involved the elimination of the variables from the given polynomial via partial differentiation with respect to each of its unknowns, and he illustrated it in a number of specific examples before stating a general result.

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5 See, for example, [Crilly 1986], [Parshall 1989], [Parshall 1998], [Crilly 2006], and [Parshall 2006]. The latter three works also provide details on the subsequent evolution of invariant theory at the hands of Cayley and Sylvester as does Crilly [1988], while Parshall [1989] analyzes and compares the contemporaneous British and German approaches to the field. And, although David Hilbert purportedly struck a near fatal blow to invariant theory in 1890 with the publication of his paper "Über die Theorie der algebraischen Formen" [Hilbert 1890] and actually "killed" it in 1893 [Hilbert 1893], sociologist of science Charles Fischer pronounced the field "dead" only by the 1920s in [Fischer 1966] and [Fischer 1967]. See [Parshall 1990], however, for more on the "death" argument.

6 See [Boole 1841-1843a, p. 3], as quoted in [Parshall 1989, p. 161]. In what follows, I have massaged the notation in the original papers in order to generate a notation for this paper more consistent across the various works discussed.

Consider, as did Boole, the simplest case of the binary quadratic form

$$Q = ax^2 + 2bxy + cy^2, \tag{1}$$

that is, the homogeneous polynomial of degree two in two unknowns, where a, b, and c are implicitly real numbers [Boole 1841-1843a, p. 6]. Calculating the partial derivatives of Q with respect to x and y and setting the results equal to zero generated two equations,

$$\frac{\partial Q}{\partial x} = 2ax + 2by = 0$$
$$\frac{\partial Q}{\partial y} = 2bx + 2cy = 0,$$

from which Boole eliminated the variables to get the expression

$$\theta(Q) = b^2 - ac,$$

that is, one of the desired relations by which the coefficients of Q "are held in mutual dependence" or what was later termed by Sylvester the discriminant of Q.[7] He next applied the linear transformation

$$x = mx' + ny'$$
$$y = m'x' + n'y', \tag{2}$$

for m, n, m', $n' \in \mathbb{R}$ (and $mn' - m'n \neq 0$, although he assumed, but did not explicitly note, this restriction) to Q to get a new binary quadratic form

$$R = A(x')^2 + 2Bx'y' + C(y')^2.$$

Clearly, calculating the partial derivatives of R yielded

$$\theta(R) = B^2 - AC.$$

Later in his paper [Boole 1841-1843a, p. 19], Boole proved, by explicit calculation, that $\theta(R)$ and $\theta(Q)$ were equal up to a power of the determinant of the linear transformation (2).

Repeating the same analysis *mutatis mutandis* for the binary cubic $Q = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ shows that its discriminant

$$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$$

also remains unchanged up to a power of the determinant of the linear transformation. In language that would emerge only later (see below), but

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7 Sylvester coined the term "discriminant" in a letter to Cayley dated 25 August, 1851. As he put it, the purpose of that letter was to "submit" for Cayley's "approval & ratification" a number of terms in addition to "discriminant," among them, "invariant" and "resultant." See [Parshall 1998, pp. 35-37]. Sylvester used the term "discriminant" in print for the first time in [Sylvester 1851d, p. 280].

that will now be adopted here, Boole concluded in general that the discriminant is an invariant of the homogeneous binary form of degree n. His paper's sequel closed with the suggestive remark that "[a]n equally important subject of inquiry presents itself in the connection between linear transformations and an extensive class of theorems [i.e., expressions] depending on partial differentials, particularly such as are met with in Analytical Geometry. ...To those who may be disposed to engage in the investigation, it will, I believe, present an ample field of research and discovery." [8]

Enter Cayley. On reading Boole's paper, he had seen how to generalize the ideas presented there from homogeneous polynomials of degree n in m unknowns to multilinear forms, that is, to forms made up of "n sets of m variables," in which "the variables of each set [enter] linearly." [9] In so doing, he had developed a computationally nightmarish construct—he called it the hyperdeterminant—that allowed him to produce the "few formulae" about which he had written to Boole. His method, presented in "On the Theory of Linear Transformations" in the *Cambridge Mathematical Journal* in 1845, thus generated invariants for multilinear forms. By suitably identifying the coefficients and the variables in a multilinear form, however, a homogeneous polynomial in m variables results. Similar identification of the coefficients of an invariant of a multilinear form produces an invariant of the underlying polynomial. [10] It was general, but it was complicated, and even Cayley had to admit that he had "not yet succeeded in obtaining the general expression of a hyperdeterminant" and "[could] do so" only in three cases [Cayley 1845, p. 85].

Still, he had found a particularly interesting expression, $ae - 4bd + c^2$, in the coefficients of the binary quartic form

$$Q = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

that was an invariant different from its discriminant $\theta(Q)$ [Cayley 1845, p. 89]. He apparently communicated his new finding to Boole directly, for in an addendum to his paper, he noted that Boole rather quickly not only isolated a third such expression, namely, $ace - b^2e - ad^2 - c^3 + 2bdc$, but also showed by a brute force calculation that

$$\theta(Q) = (ae - 4bd + c^2)^3 - 27(ace - b^2e - ad^2 - c^3 + 2bdc)^2. \tag{3}$$

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8 See [Boole 1841-1843b, p. 119], as quoted in [Wolfson 2008, p. 43].

9 See [Cayley 1845, p. 80] for the quotation. In more familiar terms, Cayley considered monic polynomials in n sets of m variables each: $f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, \ldots, z_1, z_2, \ldots, z_m)$. If $x_i = y_i = \cdots = z_i$, for $1 = 1, 2, \ldots, m$, then f becomes an m-ary form of degree mn.

10 Tony Crilly discussed this method in [Crilly 1986, 243-244].

As Cayley remarked, "[h]ence the two functions on which the linear transformation of functions of the fourth order ultimately depend are the very simple ones $ae - 4bd + 3ca$, $ace - ad^2 - eb^2 - c^3 + 2bdc$, the [discriminant] being merely a derivative from these" [Cayley 1845, p. 94].[11] He viewed it as an "absolute necessity" to his calculational goals to understand this phenomenon completely [Cayley 1845, p. 94].

Cayley quickly followed his 1845 paper with a sequel "On Linear Transformations" early in 1846. There, he not only made explicit what he deemed the central problem of the emerging theory, namely, (a) "[t]o find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables" but also foresaw (b) the "very great difficulties" in "determining the *independent* derivatives, and the relation" as in (3) "between these and the remaining ones" [Cayley 1846b, p. 95 (his emphasis)]. Relative to (a), he introduced yet another calculationally unwieldy technique for generating invariants, this one called hyperdeterminant derivation and applicable only to homogeneous polynomials of degree n in 2 variables. "[M]ainly of theoretical significance and not particularly suited to calculation," however, not even the inveterate calculator Cayley was "overly fond of it as a calculating mechanism" [Crilly 2006, p. 110]. Real progress on (b) would only come in the 1850s in concert especially with Sylvester (see below).

This flurry of invariant-theoretic activity in 1845 and 1846, so exuberantly communicated by Cayley in his private correspondence to Boole,[12] came to naught in the short term. The papers generated no apparent research interest among mathematicians in the British Isles, and the French translation of them that Cayley published in Crelle's *Journal für die reine und angewandte Mathematik* in 1846 failed to elicit a Continental response [Cayley 1846a]. The young mathematician, intent on establishing his reputation at home and abroad, would have to bide his time. In 1846, Cayley not only moved from Cambridge to London to pursue a legal career but also put invariant theory temporarily aside.

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11 In equation (13) in [Parshall 1989, p. 164], I inadvertently gave not the discriminant, which is the invariant that Boole's method generated, but rather the new cubic invariant $ace - b^2e - ad^2 - c^3 + 2bdc$ that Boole later found and communicated to Cayley. Note that Cayley is asserting here that, associated to the *binary quartic*, is a set of two independent invariants. His argument *in this case* for what would later be termed a "finite basis" preceded the remark. Note, too, that for Cayley in this context and at this point in time, a "derivative" was what would later be called an "invariant."

12 See [Crilly 1986, pp. 243-245] and [Crilly 2006, pp. 86-92]. On Cayley's correspondence with Boole, see [MacHale 1985, p. 55-58].

HERMITE'S NUMBER-THEORETIC PATH

Forty years before Boole published his work, Gauss, in the exhaustive exploration of quadratic forms that made up the fifth section of his *Disquisitiones arithmeticae*, had considered how a linear transformation (2) affected the variables of a binary quadratic form (1), where, for him, $a, b, c, m, n, m', n' \in \mathbb{Z}$ not $\mathbb{R}$. Substituting (2) into (1), he produced a new binary quadratic form

$$a'(x')^2 + 2b'x'y' + c'(y')^2, \tag{4}$$

explicitly exhibiting the new coefficients:

$$\begin{aligned} a' &= am^2 + 2bmm' + c(m')^2 \\ b' &= amn + b(mn' + nm') + cm'n' \\ c' &= an^2 + 2bnn' + c(n')^2. \end{aligned}$$

He then noted, first, that "[m]ultiplying the second equation by itself, the first by the third, and subtracting we get

$$b'b' - a'c' = (b^2 - ac)(mn' - nm')^2,"$$

second, that the discriminant of (4) is divisible by the discriminant of (1), and, third, that their quotient is square [Gauss 1966, pp. 111-112].[13] As early as 1801, then, Gauss had, without the terminology and within a different theoretical framework, already discovered that the discriminant of the binary quadratic form is an invariant.

By 1843, Hermite had become an assiduous student of Gauss's (as well as Lagrange's) work and, as he would later put it, it was from these two masters that he "learned Algebra."[14] In the 1840s, however, this and other reading, especially of the work on elliptic functions of Carl Gustav Jacob Jacobi, had led Hermite "to study algebraic complex numbers in general and to classify them in the spirit of Lagrange's and Gauss's classification of binary quadratic forms," while "keep[ing] close to elliptic functions" [Goldstein & Schappacher 2007a, p. 42]. By the end of the decade, moreover, Hermite

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13 Gauss's observations are also discussed in [Parshall 1989, p. 159]. As Catherine Goldstein and Norbert Schappacher noted, Gauss treated quadratic forms in and of themselves and not just, as Lagrange and Legendre had done, as tools for studying the representation of integers [Goldstein & Schappacher 2007a, p. 8].

14 "C'est surtout dans ces deux livres ... que j'ai appris l'Algèbre" [Picard 1905, p. viii]. Compare also [Goldstein 2007, pp. 377-378].

had come more specifically to consider number theory *per se* both in correspondence with Jacobi and in a number of short papers. [15] It was in one of the latter, "Note sur la réduction des fonctions homogènes à coefficients entiers et à deux indéterminées," that the transformation of a binary form of degree n arose.

Hermite considered [Hermite 1848, pp. 84-86] a homogeneous form of degree n in two variables with integer coefficients

$$f(x,y) = Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 + \cdots + Kxy^{n-1} + Ly^n \tag{5}$$

and the transformation (2) (where, however, we have that m, n, m', $n' \in \mathbb{Z}$ and $mn' - m'n = \pm 1$) that takes (5) to

$$f(x',y') = A'x'^n + B'x'^{n-1}y' + C'x'^{n-2}y'^2 + \cdots + K'x'y'^{n-1} + L'y'^n.$$

Letting $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n (complex) roots (he then assumed they were all real) of the equation

$$Az^n + Bz^{n-1} + Cz^{n-2} + \cdots + Kz + L = 0,$$

he defined what he called the "determinant" of (5) to be a particular expression D in the differences of the roots. Writing $f(x,y)$ in terms of the roots as $A(x-\alpha_1 y)(x-\alpha_2 y)\cdots(x-\alpha_n y)$ and applying (2), he showed that "the value of D stays absolutely the same" under the action of (2). [16] Hermite's "determinant" was what Sylvester would later term the discriminant, but in a special, number-theoretically-oriented setting as opposed to that in which Boole had isolated it in 1841. Hermite's binary form had *integer* coefficients, and his linear transformation had determinant ± 1.

At roughly the same time that Hermite published this paper, he was writing to Jacobi about his emerging ideas. Excerpts from four of these (undated) letters, likely written at some point in 1847, [17] were published in Crelle's *Journal* in 1850. In the third, he confessed to Jacobi that although "the goal of [his] first researches had been to examine the new method of approximation that you have given in establishing the impossibility of a function with three imaginary periods," he had come to recognize that that project "paled in comparison with major issues in the theory of

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15 For a beautiful, finely grained account of Hermite's early number-theoretic work and its links to Gauss's *Disquisitiones arithmeticae*, see [Goldstein 2007].

16 "la valeur de D reste absolument la même" [Hermite 1848, p. 86].

17 Catherine Goldstein argued convincingly for this dating in [Goldstein 2007, p. 383 (note 22)].

forms, considered from a general point of view." [18] "In this vast expanse of research opened up for us by Mr. Gauss," he continued, "Algebra and the Theory of numbers seem to me to become merged into one [and the] same order of analytical ideas, of which our actual knowledge does not yet allow us to form an adequate idea." [19] Although he had touched on it in his first letter to Jacobi, in his third, he began to formulate just such an idea through the development of what, after Gauss, he termed "the notion of *formes adjointes*" [Hermite 1850, pp. 137ff] (see (7) below), that is, his construct for isolating what would later be termed contravariants. Given these developments, it is little wonder that the first half of the 1850s found Hermite both at work on the new field of research that Catherine Goldstein and Norbert Schappacher termed "arithmetic algebraic analysis" [Goldstein & Schappacher 2007a, p. 52] and, at the same time, engaged in the invariant-theoretic ideas coming out of England.

SYLVESTER'S ALGEBRAIC PATH

If Cayley was largely motivated in what would become his invariant-theoretic research by analytical-geometrical concerns in the 1840s and Hermite came to his via more number-theoretical interests, Sylvester tended to approach his mathematical research from a more purely algebraic point of view. This was made manifest in the series of papers he wrote on elimination as early as 1839 and 1840 from the post of professor of natural philosophy that he had secured in 1839 at University College, London. In particular, he had closely examined the auxiliary functions $f_i(x)$ that arise in the application of Sturm's theorem for finding the real roots α_i between any two real numbers a and b of an algebraic equation $f(x) = 0$. Inspired in this work by, as he put it, "happening to be present at a sitting of the French Institute" [Sylvester 1839, p. 44 (note ‡)], Sylvester succeeded not only in explicitly exhibiting the $f_i(x)$ as algebraic expressions in terms of the squares of the differences of the roots

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18 "le but de mes premières recherches avait été d'examiner le nouveau mode d'approximation que vous avez donné en établissant l'impossibilité d'une fonction à trois périodes imaginaires" and "les problèmes si vastes que j'avais cru me proposer m'ont semblé peu de chose à côté des grandes questions de la théorie des formes, considérée d'une manière générale" [Hermite 1850, p. 136].

19 "Dans cette immense étendu de recherches qui nous a été ouverte par M. Gauss, l'Algèbre et la Théorie de nombres me paraissent devoir se confondre dans un même ordre de notions analytiques, dont nos connaissances actuelles ne nous permettent pas encore de nous faire une juste idée" [Hermite 1850, p. 136].

but also in developing some of the basic properties of determinants.[20] Hourya Sinaceur was quite right when she remarked that "Sylvester was convinced, in effect, that all of analysis could be presented, finally, in the language of the theory of determinants, this 'algebra of algebra.' … At a time when the Paris school was dominated, a few exceptions aside, by the analytic spirit, Sylvester wanted to promote the algebraic spirit to the point of subjecting all of analysis to it!"[21]

Sylvester's early research activity, as well as his ability to participate directly in the Parisian mathematical scene, came to an end in the fall of 1841 when he made a bold transatlantic move by accepting what he hoped would be a more congenial professorship of *mathematics* at the University of Virginia. Those hopes, however, were quickly dashed. Sylvester quit the job after less than five months and spent the next year and a half trying, ultimately in vain, to find a new academic position in the United States.[22] Having returned to England by the beginning of 1844, he finally landed the actuarial job in London that had allowed him to recover, by 1846, what he described as his "footing in the world's slippery path" and to "secure a landing place whereon to breathe and calmly survey and determine upon [his] future course" [Parshall 1998, p. 15]. A year later, he had reengaged with both mathematics and mathematical Paris.

During "the course of a rapid tour of the continent," Sylvester was in the City of Lights and in conversation with his "illustrious friend M. Sturm" about new ideas he had had on cubic equations of the form

$$Ax^3 + By^3 + Cz^3 = Dxyz \tag{6}$$

(for integers A, B, C and D) and that he, in an effort to build his reputation in France, had asked Sturm to convey for him to the Institute [Sylvester 1847a, p. 109]. Sketches of his results subsequently appeared in print in three short papers published in the *Philosophical Magazine* in 1847.

Sylvester opened the first with the statement of what he styled "a general theorem of transformation." Assuming in (6) that "A and B are equal, or in the ratio of two cube numbers to one another" and that the expression $27ABC - D^3$, which he called the "determinant," satisfies certain specific conditions, he claimed (he gave no proofs in this paper)

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20 For more details and all of the pertinent references, see [Parshall 1998, pp. 5-9].

21 "Sylvester était convaincu, en effet, que toute l'Analyse pouvait être, en dernier ressort, présentée dans le langage de la théorie des déterminants, cette 'algèbre de l'algèbre.' … À une époque où l'école de Paris était, en dépit de quelques exceptions, dominée par l'esprit analytique, Sylvester veut promouvoir l'esprit algébrique jusqu'à y soumettre l'analyse toute entière!" [Sinaceur 1991, p. 126].

22 On this period in Sylvester's life, see [Parshall 2006, pp. 64-80].

that (6) can "be made to depend upon another [expression] of the form $A'u^3 + B'v^3 + C'w^3 = D'uvw$, where $A'B'C' = ABC$, $D' = D$, [and] $uvw =$ some factor of z" [Sylvester 1847a, p. 107]. From this, he concluded (in [Sylvester 1847a] and [Sylvester 1847b]) that certain particular cases of (6) had no integer solutions, while in [Sylvester 1847c], he laid out, given a particular integral solution of (6), a process for generating the rest.[23]

Unfortunately, the "[p]ressing avocations" that had prevented him from providing actual proofs of his results on the cubic overwhelmed him thereafter [Sylvester 1847a, p. 109]. Sylvester was employed as an actuary; he had embarked on legal studies at the Inner Temple in London in 1846; he was instrumental in setting up the Institute of Actuaries, also in London, beginning in 1848. He did meet Cayley during this period of intense professional activity and, as their mathematical and personal friendship evolved, tried to think at least sporadically about mathematics. In a letter written with a formal air to "My dear Sir" and dated 24 November, 1847—the earliest known letter in the Sylvester-Cayley correspondence—Sylvester shared some number-theoretic thoughts with this other mathematician-turning-lawyer that had been sparked by his reading of Legendre's *Théorie des nombres* [Parshall 1998, pp. 18-19]. Still, three years would elapse before Sylvester, the mathematician, would be heard from again in print.

PATHS CONVERGE

The close proximity that Sylvester and Cayley enjoyed—the former at 26 Lincoln's Inn Fields, the latter at 2 Stone Buildings, both within the walls of the Inns of Court—as well as their shared mathematical interests made natural the convergence of their two paths. By the spring of 1850, they were in daily communication about ideas primarily of an algebraic-geometric cast that hearkened back to Cayley's work on linear transformations as well as to Sylvester's research on the theory of elimination. On Boxing Day in 1850, for example, Sylvester wrote to Cayley with both ideas and questions.

In his letter, he revealed his awareness of connections between Boole's 1841 "Exposition of a General Theory of Linear Transformations" and of work that the German mathematician, Otto Hesse, had published in 1844 [Parshall 1998, pp. 30-32]. As just noted, in 1847, Sylvester had considered the cubic Diophantine equation $Ax^3 + By^3 + Cz^3 = Dxyz$. In 1850, he was thinking about how to effect a transformation of an *arbitrary*

23 Compare [Dickson 1921–1923, 2: p. 589].

cubic form in three variables x, y and z into what, based on his reading of Hermite [Sylvester 1851e, p. 190], he termed the canonical form $a^3\xi^3 + b^3\eta^3 + c^3\zeta^3 + 6\epsilon abc\xi\eta\zeta$ in the variables ξ, η, and ζ. As Sylvester had apparently learned from Cayley [Parshall 1998, pp. 30 and 32], Hesse had undertaken an analysis of third-order plane curves in 1844 and, along the way, had considered exactly the same question about transformation [Hesse 1844, pp. 90-95]. For Sylvester, an application of Boole's technique gave him the desired result, but Sylvester asked Cayley "whether it is identical with Hesse's solution" [Parshall 1998, p. 30]. It was not, but by 1851, Sylvester had published a "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms" in which he had taken the first tentative steps toward uniting the work of Boole, Cayley, Hermite, himself, and others within a common framework. [24]

Sylvester was coming to appreciate that a number of very different techniques had led to similar results. As he vaguely put it, "from dependencies of equations, transition may be made to the relations of functional forms, and *vice versâ*" [Sylvester 1851e, p. 184]. One of those "dependencies" was the so-called resultant of which the discriminant is a special case. As noted, Boole and Cayley each had a method for producing the discriminant, and both had discovered that the discriminant so generated had the invariantive property. Moreover, Hermite, but "in a more restricted sense" and via his *formes adjointes*, had made a similar discovery [Sylvester 1851e, pp. 185-186]. In explicitly highlighting this confluence, Sylvester linked his own developing ideas with those of Cayley and Hermite for the first time in print. This marked at least the symbolic convergence of three distinct paths toward a theory of invariants.

At this juncture, Sylvester, ebullient about getting back into mathematical research after a long dry spell, served as the catalyst. In the two years from 1851 to 1853, he produced an impressive body of work that culminated in massive papers: "On the Principles of the Calculus of Forms" in 1852 [Sylvester 1852] and "On the Theory of Syzygetic Relations of Two

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24 Sylvester also included Hesse's technique in his overview of results, although he actually accused his German counterpart of having appropriated, without acknowledgment, the "method of finding the resultant of any set of three equations of degrees equal or differing only by a unit, one from those of the other two" that he had presented in work published four years earlier in 1851 [Sylvester 1851e, p. 189]. At this point, however, Sylvester seemed unaware of Gotthold Eisenstein's work on the binary cubic that also foreshadowed invariant-theoretic notions. See [Eisenstein 1844] and compare the discussion in [Parshall 1989, pp. 170-171]. Ultimately and contemporaneously, a parallel school of invariant theory developed in Germany around, in particular, Hesse, Siegfied Aronhold, Alfred Clebsch, and Paul Gordan (see below). Italians like Francesco Brioschi were also soon drawn into invariant-theoretic research.

Rational Integral Functions" in 1853 [Sylvester 1853b]. As the correspondence makes clear, Cayley was Sylvester's primary sounding board, but Sylvester's published works clearly show that he was also in regular contact with Hermite and was increasingly absorbing the work of his new French friend.

Sylvester followed his "Sketch" in the May 1851 issue of the *Cambridge and Dublin Mathematical Journal* with a short paper "On the General Theory of Associated Algebraical Forms" in the journal's November number. There, he began to roll out the evolving terminology of what was becoming a *theory* of invariants [Sylvester 1851b]. He also explicitly connected the work of Cayley and Hermite with those new concepts. In particular, he introduced the notions of "covariant" and "contravariant." Given a homogeneous form in n variables and a linear transformation of those variables, a covariant is an expression in the coefficients *and* variables of the form that remains unaltered by the linear transformation (up to a power of the linear transformation's determinant), while a contravariant is an expression in the coefficients and variables that remains so unaltered by that transformation's transpose.[25] As Sylvester noted, "[c]ovariants are Mr. Cayley's hyperdeterminants; contravariants, include, but are not coincident with, M. Hermite's formes-adjointes, if we understand by the last-named term such forms as may be derived by the process described by M. Hermite in the third of his letters to M. Jacobi" [Sylvester 1851b, p. 200]. Still, Hermite had also used his idea of a *forme adjointe* in that letter "in a sense as wide as that" of Sylvester's notion of contravariant in the following "most remarkable theorem" [Sylvester 1851b, p. 201]: "If we have a function of any number of letters, say of x, y, z as

$$ax^m + mbx^{m-1}y + mcx^{m-1}z + \frac{m(m-1)}{2}dx^{m-1}y^2 + \&c.,$$

and if I be any invariant of this function, then will

$$\left(x^m\frac{d}{da} + x^{m-1}y\frac{d}{db} + x^{m-1}z\frac{d}{dc} + x^{m-2}y^2\frac{d}{dd}\ \&c\right)^r I \tag{7}$$

be a '*forme adjointe*' of the given function." As Sylvester showed in the first (February 1852) installment of his paper "On the Principles of the Calculus of Forms," (7) is a contravariant [Sylvester 1852, p. 289].

Things were definitely heating up at the end of 1851. Sylvester and Hermite were in active conversation [Sylvester 1851c, p. 246]; Sylvester

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25 See [Sylvester 1851b, p. 200]. As noted, the terminology was in flux. Instead of the term "transpose," the early invariant-theorists used "inverse." See, for example, Salmon's definition of the contravariant in [Salmon 1859, pp. 117-118].

was writing almost daily to Cayley to fill him in on his latest ideas as he worked to write them up (see [Parshall 1998, pp. 30-37] for a sample of their correspondence); Cayley, actively engaged in the give-and-take, was separately preparing his own research for publication [Crilly 2006, pp. 168-179]. As Tony Crilly explained, however, "[a] modern arrangement such as a joint paper … was not the custom in Britain in the middle of the nineteenth century. Both young men [and Hermite, too] nursed mathematical ambition, but they were individualists" who "kept close guard on their own ideas" [Crilly 2006, p. 173].

In 1851, as Sylvester pushed his research agenda, Cayley was flooding Crelle's *Journal* with a series of notes mostly on topics in analytic geometry. By the end of the year, however, Sylvester had apparently succeeded in drawing Cayley back into what was becoming an invariant-theoretic fray. On 21 November, Cayley returned to his earlier work on hyperdeterminants in a "Note sur la théorie des hyperdéterminants" that he sent to Crelle. There, he considered "the case of a homogeneous polynomial in two variables, using Mr. Sylvester's new terms" of "Covariant" and "Invariant" and essentially translated his earlier hyperdeterminant-theoretic work into this new language.[26] Two weeks later, he was writing to Sylvester with what may be seen as the birth certificate of nineteenth-century, British invariant theory. "Every invariant [U]," he wrote, "satisfies the partial diff[erentia]l equations

$$\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\cdots+nj\frac{d}{dk}\right)U=0$$
$$\left(b\frac{d}{db}+2c\frac{d}{dc}+3d\frac{d}{dd}+\cdots+nk\frac{d}{dk}\right)U=\frac{1}{2}nsU$$

(s the degree of the Invariant) & of course the two equations formed by taking the coeff[icien]ts in a reverse order. This will constitute the foundation of a new theory of Invariants."[27] Why? It gave a general calculational

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26 "le cas d'une fonction homogène à deux variables, et en me servant des nouveaux termes de M. Sylvester, je nomme 'Covariant' d'une fonction donnée, toute fonction qui ne change pas de forme en faisant subir aux variables des transformations linéaires quelconques, et 'Invariant' toute fonction des seuls coefficients qui a la propriété mentionnée" [Cayley 1851, p. 368].

27 For the quotation, see [Parshall 1998, p. 37]. The first of these operators is the functionally important one. To get a feel for the computation, let $U=(ad-bc)^2-4(b^2-ac)(c^2-bd)$, the discriminant of the binary quartic equation considered above. It is not hard to see that applying the operators $a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}$ and $d\frac{d}{dc}+2c\frac{d}{db}+3b\frac{d}{da}$ ("taking the coefficients in reverse order") to U yields 0 in both instances.

means for identifying invariants. The race was on, and the two friends were neck in neck.

The February 1852 number of the *Cambridge and Dublin*, for example, found Cayley's "On the Theory of Permutants" [Cayley 1852] immediately preceding the first installment of Sylvester's "On the Principles of the Calculus of Forms." In his paper, Cayley "was chiefly concerned with the organization of the algebra of forms," "assembl[ing] the various types of determinants he had dealt with previously and group[ing] them together under one umbrella concept, the algebraic form he called a permutant" [Crilly 2006, p. 173]. In his, Sylvester was, in addition to establishing much of the new language of a theory of invariants, presenting the technique for generating invariants that he called compound permutation. The two were actually publishing versions of the same technique, although a priority dispute, evidence of which may be found in both papers, was quickly diffused.[28] Sylvester's paper, moreover, also reflected his mathematical interactions—through publications, letters, and likely in person during trips across the Channel to Paris—with his "valued friend M. Hermite" in the early 1850s [Sylvester 1852, p. 296]. Sylvester was actively drawing Hermite not only into the fray but also into the British mathematical scene; by 1852 at least, Hermite had also been in correspondence with Cayley [Crilly 2006, p. 186].

Sylvester continued work on the next installment of "On the Principles of the Calculus of Forms" through the spring of 1852, made a concerted effort to build his mathematical reputation in France, became more fully aware of the parallel development of a theory of invariants in Germany at the hands initially of Hesse and Siegfried Aronhold and later of Alfred Clebsch and Paul Gordan, and nurtured a mathematical relationship with the Irish mathematician, George Salmon.[29] The summer of 1852, however, found him derailed by what he called the problem of syzygies, that is, the existence, that Boole and Cayley had recognized as early as 1845, of dependence relations like (3) among the invariants associated with a homogeneous polynomial of given degree in two variables. Sylvester mulled over that problem into 1853, ultimately presenting his enormous paper "On the Theory of Syzygetic Relations of Two Rational Integral Functions" to the Royal Society in June [Sylvester 1853b]. There, he explored questions like

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28 See [Sylvester 1852, p. 318] and [Cayley 1852, p. 26], and compare [Parshall 1998, pp. 33-35] and [Crilly 2006, p. 173].

29 On this period as a whole, see [Parshall 2006, pp. 112-122]. On Sylvester's reputation-building efforts, see [Parshall & Seneta 1997]. The non-British development of a theory of invariants is discussed in [Parshall 1989, pp. 170-180].

when do such relations exist among in- or covariants, and, if they exist, how can they be determined explicitly? He also used his new findings to reinterpret, in an invariant-theoretic context, some of his earlier findings on Sturm functions, incorporating then-recent work of Hermite. [30] It was a formidable paper and its completion, combined with the press of duties associated with his position at the Equity Law and Life, had left Sylvester exhausted by the close of 1853.

Cayley and Hermite had followed Sylvester's progress, while largely continuing to pursue their own respective lines of research. Cayley generated a number of short papers on a variety of mathematical topics from group theory to analytical geometry to the theory of probability, but by 1854, he had embarked on what would ultimately be a series of ten memoirs on quantics (Cayley's new term for homogeneous polynomials in two or more variables). The first of his papers appeared in the *Philosophical Transactions of the Royal Society* in 1854 [Cayley 1854]. For his part, Hermite continued to develop the new "arithmetic algebraic analysis" until 1854, when he, too, came out with a major invariant-theoretic study. Indicative of the fact that he had been drawn into Cayley and Sylvester's mathematical world just as much as they had been drawn into his, Hermite published "Sur la théorie des fonctions homogènes à deux indéterminées" [Hermite 1854] in the *Cambridge and Dublin.* [31]

Cayley's 1854 paper, "An Introductory Memoir upon Quantics," like Sylvester's "Sketch" three years earlier, laid out the vocabulary and basic definitions of the theory of invariants, introduced the general notation for quantics that he would continue to use in his invariant-theoretic work, and recast some of his earlier results in this new guise. It also reflected Cayley's knowledge of results that Hermite was only making known to the mathematical public at essentially the same time. [32]

In "On the Principles of the Calculus of Forms," Sylvester had alluded to a time- and calculation-saving theorem that he had discovered in 1852 "in the course of a most instructive and suggestive correspondence with Mr Salmon" and that he promised to publish "along with other very important matter, in the next number of the" *Cambridge and Dublin* [Sylvester

30 See [Sinaceur 1991, pp. 129-140] as well as, especially, [Hermite 1853].

31 Hermite also wrote up his findings for the broader European mathematical audience in 1854, although his two installments of that version only appeared in the *Journal für die reine und angewandte Mathematik* in 1856. See [Hermite 1856a] and [Hermite 1856b].

32 In a postscript to his paper dated 7 October, 1854. Cayley remarked that he had "deduced as a corollary, the law of reciprocity of MM. Sylvester and Hermite" [Cayley 1854, p. 139]. Hermite's result also appeared in print in 1854 [Hermite 1854].

1852, p. 339]. When that number appeared in February 1853, however, Sylvester had to confess that "[a]ccidental causes have prevented me from composing the additional sections on the Calculus of Forms, which I had destined for the present Number of this *Journal*" [Sylvester 1853a, p. 403]. At issue was the so-called law of reciprocity: "To every covariant of a form of degree m, which is, relative to the coefficients of this form, of degree p, corresponds a covariant of degree p relative to the coefficients of a form of degree m." [33] When the dust finally settled, it had been Hermite—as Sylvester acknowledged in 1853 followed by Cayley in 1854—who had actually given the theorem's proof. [34] With characteristic hyperbole, Sylvester declared that "[t]o M. Hermite, therefore, belongs the honor of reviving and establishing—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms" [Sylvester 1853a, p. 403]. "Our new calculus." Sylvester viewed the creation of a theory of invariants in the first half of the 1850s as a joint venture among the three friends, and Hermite echoed that sentiment in acknowledging how "encouraged" he was "by the very kind manner" in which his "friend Mr. Sylvester" had welcomed his new results. [35]

Although Hermite opened his 1854 paper "Sur la théorie des fonctions homogènes à deux indéterminées" with his proof of the law of reciprocity, that paper contained much more. In particular, he tackled the problem of the binary quintic form, that is, a homogeneous polynomial

$$g(x,y) = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5. \tag{8}$$

As was well-known at mid-century thanks to the work of Niels Henrik Abel, a general fifth-degree polynomial

$$h(x) = ax^5 + bx^4 + cx^3y + dx^2 + ex + f = 0 \tag{9}$$

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[33] "A tout covariant d'une forme de degré m, et qui par rapport aux coefficients de cette forme est du degré p, correspond un covariant de degré $[p]$ par rapport aux coefficients d'une forme du degré $[m]$" [Hermite 1854, p. 297]. (In Hermite's paper, the p and the m are incorrectly switched in the second clause; I have corrected them here.) As is evident in this quotation, the terminology was still in flux at this point, with "degree" referring to both the degree of a polynomial and the order of an invariant in the coefficients of its underlying homogeneous polynomial.

[34] See [Sylvester 1853a, p. 403] and [Cayley 1854, p. 232], respectively. For Hermite's proof, see [Hermite 1854, pp. 297-299]. Both Sylvester and Cayley also noted in these papers that Cayley had come up with a proof different from Hermite's.

[35] "Depuis que la première Partie de ces recherches a été terminée, encouragé par la manière si bienveillante dont elles ont été accueilles par mon ami M. Sylvester ..." [Hermite 1854, p. 315].

is not solvable by radicals, that is, it is impossible to give a formula for finding the roots of (9) in terms of the binary operations of addition, subtraction, multiplication, and division together with the extraction of roots. But, was it possible to approach the quintic some other way and, in so doing, gain further insights into it? Cayley had thought there might be as early as the 1840s in light of his proto-invariant-theoretic work [Crilly 2006, p. 89]. By the 1850s, as a theory of invariants actually evolved and as Cayley and his partners worked toward, among other things, the early goal of finding "all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables" [Cayley 1846b, p. 95], the quintic had come increasingly under scrutiny.

In 1846, Cayley had already isolated a quintic invariant of order four in the coefficients of (8), namely, [36]

$$\begin{aligned} G = {} & a^2f^2 - 10abef + 4acdf + 16ace^2 + 16ace^2 \\ & - 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f \\ & - 76bcde + 48bd^3 + 48c^3e - 32c^2d^2. \end{aligned}$$

By 1854, fifteen different invariants and covariants of the quintic had been discovered and explicitly calculated by Cayley, Sylvester, Hermite, and others as they had proceeded with their agenda of collecting and classifying invariants and covariants—and of detecting the syzygies between them—for binary quantics of progressively higher degrees. [37] There had been three known invariants—of orders 4, 8, and 12—of the binary quintic until Hermite had stunned Cayley and Sylvester with his discovery of a new one of order 18 [Hermite 1854, pp. 305-306 and 310-314]. As Hermite put it in his published exposition of 1854, "Mr. Cayley, Mr. Sylvester and I had long thought that, in general, the invariants of forms of the m^{th} degree should be expressible as whole functions of $m - 2$ of them, and it is this that prevented Mr. Sylvester from seeking to prove the law of reciprocity, of which he, too, had presumed the existence, a necessary

36 See [Crilly 2006, p. 98] and compare [Cayley 1846b, pp. 107-108], where the invariant in question is denoted by D_{410}.

37 Tony Crilly noted the similarity between the collecting impulse of nineteenth-century invariant theorists and that of the nineteenth-century chemists and naturalists in his doctoral dissertation of 1981 [Crilly 1981] as well as in his biography of Cayley [Crilly 2006, pp. 193-196]. Catherine Goldstein focused on this aspect of Hermite's mathematical agenda, in particular, in [Goldstein 2016].

contradiction having emerged between this law and that of the number of fundamental invariants." [38] In his "Second Memoir on Quantics" submitted in the spring of 1855 but published in 1856, Cayley not only presented the combinatorial formula that he had discovered sometime prior to October 1854 for determining, for a homogeneous polynomial of given degree in two variables, the number of its linearly independent in- and covariants [Cayley 1854, p. 234] but also explicitly laid out those covariants for quantics up to and including the quintic [Cayley 1856]. [39]

PATHS DIVERGE

The first half of the 1850s had found Cayley, Hermite, and Sylvester in intense conversation as they developed their new theory of invariants. [40] It seems clear that, in the process, they had come to share the sense that being a mathematician in the mid-nineteenth century not only meant generating new mathematical ideas but also working to make those ideas known both at home and abroad. It was important to establish a reputation.

To the latter end, all three young men took care to give their work wide exposure. Cayley and Sylvester published their research in English-language journals like the *Cambridge and Dublin* and the *Philosophical Transactions* at the same time that they publicized it abroad, in Cayley's case through publication in Crelle's *Journal* and in Sylvester's through direct participation in the French mathematical scene and, after 1852, in foreign journals. [41]

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38 "M. Cayley, M. Sylvester et moi avions longtemps pensé qu'en général les invariants des formes de $m^{\text{ième}}$ degré devaient s'exprimer par les fonctions entières de $m-2$ d'entre eux, et c'est même ce qui a empêché M. Sylvester de chercher à démontrer la loi de réciprocité dont il avait aussi présumé l'existence, une contradiction nécessaire s'étant manifestée entre cette loi et celle du nombre des invariants fondamentaux" [Hermite 1854, p. 312].

39 As is well-known, though, Cayley mistakenly believed, on the basis of this work, that there were an infinite number of irreducible covariants associated with the binary quintic equation. For more on this and on Cayley's error, see [Hawkins 1987], [Parshall 1989, pp. 169-170], and [Parshall 1998, pp. 184-188]. Tony Crilly also discusses this in [Crilly 1986, pp. 248-249] and [Crilly 2006, pp. 204-207].

40 According to Hermite's biographer, Émile Picard, Sylvester said much later that "We thus formed, Cayley, Hermite, and I, an invariant trinity [Nous formions alors, Cayley, Hermite, et moi, une trinité invariantive]" [Picard 1905, p. xx]. Picard would not have found such a statement from Sylvester surprising, given the intense period of interaction the three men had had in the 1850s. The phrase "invariant trinity," however, has generally been used in reference to Cayley, Salmon, and Sylvester, and although its origins are unclear, that usage has often been attributed to Hermite.

41 Tony Crilly touched on this issue throughout his biography of Cayley [Crilly 2006], while I focused on it in Sylvester's case in [Parshall & Seneta 1997] and [Parshall 2006].

Similarly, Hermite published in the *Comptes rendus* of Paris's Académie des Sciences as well as in Crelle's *Journal* and in the *Cambridge and Dublin*.[42] When the latter journal ceased publication for financial reasons in 1854 and the new *Quarterly Journal of Pure and Applied Mathematics* took its place in 1855 under the co-editorship of Sylvester and Norman Ferrers, Hermite and Cayley joined the editorial team.[43] Despite this professional linkage, however, the mathematical paths of Sylvester, Hermite, and Cayley had already begun to diverge.

Sylvester's research push in 1852 and 1853, combined with his increasing sense that in order to pursue his mathematics most effectively he needed to be in an academic setting, left him at sea in 1854 and much of 1855 as he first tried and failed to leave the Equity Law and Life but then secured the professorship of mathematics at the Royal Military Academy in Woolwich. His mathematical malaise was only heightened when he realized that he had been a mere hair's breadth away of being the first to find the invariant of order 18 of the binary quintic. Looking back on things in 1870, he recounted that he had "discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. … The canonizant of the quartic (its cubic covariant) was the first thing to offer itself in the inquiry. I had but to think the words 'Resultant of Quintic and its Canonizant,' and the octodecadic skew invariant would have fallen spontaneously into my lap" [Sylvester 1904-1912, 2: 714].[44] With his assumption of the Woolwich professorship in the fall of 1855, moreover, Sylvester found himself distracted by his new duties and unfocused in his research, although as the second volume of his collected works attests, he did continue to produce new results. He would really only come back to invariant theory in the mid-1860s, and then, in particular, to the problem that he, Cayley, and Hermite had considered in the 1850s, namely, "a complete invariantive determination of the character of the roots of the general equation of the fifth degree."[45]

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42 As Tom Archibald argued [Archibald 2002], Hermite was particularly interested in establishing a reputation among the mathematicians in Germany.

43 On the history of this journal, see [Despeaux 2002, pp. 162-184] and [Crilly 2004]. Irish physicist and mathematician, George Gabriel Stokes, rounded out the editorial team.

44 Tony Crilly noted, and I missed, this cause (of several) of Sylvester's depression of the mid-1850s. See [Crilly 2006, pp. 205-206] and compare [Parshall 2006, pp. 131-138].

45 See [Sylvester 1864, p. 376] for the quotation and [Sylvester 1864, pp. 418-475] for the mathematics.

For his part, Hermite continued to work out various invariant-theoretic ramifications in a number of short papers into the 1860s, despite the setback brought on by the case of smallpox he contracted in 1856. While he produced a number of isolated results on forms—for example in three instead of two variables—his flashiest result reflected *his* continued interest in things quintic. In 1858, he succeeded in showing how to solve a general fifth-degree polynomial equation (9) using an interesting amalgam of invariant-theoretic notions and elliptic function theory [Hermite 1858]. [46] The end of the 1850s and into the 1860s also saw him return to more properly number-theoretic concerns as he took up the position of *maître de conférences* (in 1862) at the École normale supérieure. Hermite finally assumed an actual professorship in analysis at the École polytechnique and in the *Faculté des Sciences de Paris* in 1869 [Freudenthal 1970–1990].

Of the three, it was Cayley who remained the most steadfast to the theory of invariants after the mid-1850s. In quick succession, and in addition to his voluminous work particularly in geometry, he had followed his "Second Memoir on Quantics" with five more by 1861. The eighth, ninth, and tenth memoirs would appear in 1867, 1870, and 1878, respectively. He also continued to think seriously about the quintic as a cache of letters he wrote to Congregationalist minister, Robert Harley, between 1859 and 1863 makes evident [Crilly et al. 2017]. In addition to Harley, several other mathematicians took an interest in invariant theory in the aftermath of the intense early 1850s, but it was perhaps George Salmon who proved the most faithful, producing numerous editions of his *Lessons Introductory to the Modern Higher Algebra* following the first in 1859 [Salmon 1859]. And, like Sylvester and Hermite, Cayley, finally secured an academic post, his the new Sadleirian professorship of pure mathematics at Cambridge in 1863.

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The research paths of Cayley, Hermite, and Sylvester may have diverged by the end of the 1850s, but the three mathematicians remained friends for the rest of their lives. The bonds they had forged in the early 1850s had been strong. They had shared the desire for—and had succeeded in making—national and international reputations for themselves. Their examples, as well as the examples of others, can be seen as defining

46 By 1865, Hermite had seen how to unify his elliptic-function-theoretic approach to the solution of the quintic, which hinged on the modular equation, and that of Francesco Brioschi and Leopold Kronecker, which depended on the multiplier equation, via invariant theory. See [Hermite 1865–1866] and compare the discussion in [Goldstein 2011b, pp. 248-249]. See also [Goldstein 2011a].

what, over the course of the last half of the nineteenth century, became one of the new professions, the mathematician, namely, a person employed in academe—ideally as a professor of mathematics—who actively produced and published new mathematical results and who fostered and participated in accoutrements like journals—as authors, editors, and referees—and societies like the Royal Society of London and Paris's Académie des Sciences. [47] Yet, first and foremost at least in their minds, Cayley, Hermite, and Sylvester had been instrumental in the creation of a new area of mathematics, the invariant theory that they deemed the nineteenth century's "New Algebra."

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REFERENCES

Archibald (Thomas)

[2002] Charles Hermite and German Mathematics in France, in *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800–1945*, New York: American Mathematical Society and London: London Mathematical Society, 2002, pp. 122–137.

Boole (George)

[1841-1843a] Exposition of a General Theory of Linear Transformations: Part I, *Cambridge Mathematical Journal*, 3 (1841-1843), pp. 1–20.

[1841-1843b] Exposition of a General Theory of Linear Transformations: Part II, *Cambridge Mathematical Journal*, 3 (1841-1843), pp. 106–119.

47 Much has been written on the rise of professions, but see, for example, [Reader 1966] for the British context and [Geison 1984] for the French.

Cayley (Arthur)

[1845] On the Theory of Linear Transformations, *Cambridge and Dublin Mathematical Journal*, 4 (1845), pp. 193–209; or [Cayley 1889–1898, 1: 80-94].

[1846a] Mémoire sur les hyperdéterminants (traduction d'un mémoire anglais inséré dans le *Cambridge Mathematical Journal* avec quelques additions de l'auteur), *Journal für die reine und angewandte Mathematik*, 30 (1846), pp. 1–37; or [Cayley 1889–1898, 1: 117] (which does not give the translation).

[1846b] On Linear Transformations, *Cambridge and Dublin Mathematical Journal*, 1 (1846), pp. 104–122; or [Cayley 1889–1898, 1: 95-112].

[1851] Note sur la théorie des hyperdéterminants, *Journal für die reine und angewandte Mathematik*, 42 (1851), pp. 368–371; or [Cayley 1889–1898, 1: 577-580].

[1852] On the Theory of Permutants, *Cambridge and Dublin Mathematical Journal*, 7 (1852), pp. 40–51; or [Cayley 1889–1898, 2: 16-26].

[1854] An Introductory Memoir Upon Quantics, *Philosophical Transactions of the Royal Society of London*, 144 (1854), pp. 245–258; or [Cayley 1889–1898, 2: 221-234].

[1856] A Second Memoir on Quantics, *Philosophical Transactions of the Royal Society of London*, 146 (1856), pp. 101–126; or [Cayley 1889–1898, 2: 250-275].

[1889–1898] *The Collected Mathematical Papers of Arthur Cayley*, 14 vols., Cambridge: University Press, 1889–1898.

Crilly (Tony)

[1981] *The Mathematics of Arthur Cayley with Particular Reference to Linear Algebra*, Ph.D. Thesis, Middlesex Polytechnic, 1981.

[1986] The Rise of Cayley's Invariant Theory (1841–1862), *Historia Mathematica*, 13 (1986), pp. 241–254.

[1988] The Decline of Cayley's Invariant Theory (1863–1895), *Historia Mathematica*, 15 (1988), pp. 332–347.

[2004] 'The *Cambridge Mathematical Journal* and Its Descendants: The Linchpin of a Research Community in the Early and Mid-Victorian Age, *Historia Mathematica*, 31 (2004), pp. 455–497.

[2006] *Arthur Cayley: Mathematician Laureate of the Victorian Age*, Baltimore: Johns Hopkins University Press, 2006.

Crilly (Tony), Weintraub (Steven) & Wolfson (Paul)

[2017] Arthur Cayley, Robert Harley and the Quintic Equation: Newly Discovered Letters 1859–1863, *Historia Mathematica*, 44 (2017), pp. 150–169.

Despeaux (Sloan)

[2002] *The Development of a British Publication Community: Nineteenth-Century Mathematics in British Scientific Journals*, Ph.D. Thesis, University of Virginia, 2002.

Dickson (Leonard E.)

[1921–1923] *History of the Theory of Numbers*, Washington, D. C.: 3 vols. : Carnegie Institution of Washington, 1921-1923. Providence: AMS Chelsea Publishing, Reprint edition, 1921–1923.

Eisenstein (Gotthold)
[1844] Théorèmes sur les formes cubiques et solution d'une équation du quatrième degré à quatre indéterminées, *Journal für die reine und angewandte Mathematik*, 27 (1844), pp. 75–79.

Fischer (Charles S.)
[1966] The Death of a Mathematical Theory: A Study in the Sociology of Knowledge, *Archive for History of Exact Sciences*, 3 (1966), pp. 137–159.
[1967] 'The Last Invariant Theorists: A Sociological Study of the Collective Biographies of Mathematical Specialists, *Archives européennes de sociologie*, 8 (1967), pp. 216–244.

Freudenthal (Hans)
[1970–1990] Entry on "Hermite, Charles", in Gillispie (Charles C.), ed., *Dictionary of Scientific Biography*, New York: Charles Scribner's Sons, 1970–1990.

Gauss (Carl Friedrich)
[1966] *Disquisitiones arithmeticae*, Trans. Arthur A. Clarke, New Haven: Yale University Press, 1966.

Geison (Gerald L.), ed.
[1984] *Professions and the French State, 1700–1900*, Philadelphia: University of Pennsylvania Press, 1984.

Goldstein (Catherine)
[2007] The Hermitian Form of Reading the *Disquisitiones*, in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007, pp. 377–410.
[2011a] Un arithméticien contre l'arithmétisation: les principes de Charles Hermite, in Flament (Dominque) & Nabonnand (Philippe), eds., *Justifier en mathématiques*, Paris: Éditions de la Maison des sciences de l'homme, 2011, pp. 129–165.
[2011b] Charles Hermite's Stroll through the Galois fields, *Revue d'histoire des mathématiques*, 17 (2011), pp. 211–270.
[2016] 'Découvrir des principes en classant': La classification des formes quadratiques selon Charles Hermite., in Lê (François) & Paumier (Anne-Sandrine), eds., *La classification comme practique scientifique*, 2016, pp. 103–135; Cahiers François Viète, Ser. III. No. 1.

Goldstein (Catherine) & Schappacher (Norbert)
[2007a] A Book in Search of a Discipline (1801–1860), in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007, pp. 3–65.
[2007b] Several Disciplines and a Book (1860–1901), in Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds., *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007, pp. 67–103.

Goldstein (Catherine), Schappacher (Norbert) & Schwermer (Joachim), eds.
[2007] *The Shaping of Arithmetic after C. F. Gausss Disquisitiones Arithmeticae*, Berlin, New York: Springer, 2007.

Hawkins (Thomas)
[1987] Cayley's Counting Problem and the Representation of Lie Algebras, in *Proceedings of the International Congress of Mathematicians—Berkeley* 2 vols., N.p. Amer. Math. Soc., 1987, pp. 1642–1656.

Hermite (Charles)
[1848] Note sur la réduction des fonctions homogènes à coefficients entiers et à deux indéterminées, *Journal für die reine und angewandte Mathematik*, 36 (1848), pp. 357–364; or [Hermite 1905–1917, 1: 84-93].
[1850] Extraits de lettres de M. Ch. Hermite à M. Jacobi sur différents objets de la théorie des nombres, *Journal für die reine und angewandte Mathematik*, 40 (1850), pp. 261–315; [Hermite 1905–1917, 1: 100-163].
[1853] Remarques sur le théorème de M. Sturm, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*, 36 (1853), pp. 294–297; or [Hermite 1905–1917, 1: 284-287].
[1854] Sur la théorie des fonctions homogènes à deux indéterminées, *Cambridge and Dublin Mathematical Journal*, 9 (1854), pp. 172–217; or [Hermite 1905–1917, 1: 296-349].
[1856a] Sur la théorie des fonctions homogènes à deux indéterminées: Premier mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 1–17; or [Hermite 1905–1917, 1: 350-371].
[1856b] Sur la théorie des fonctions homogènes à deux indéterminées: Second mémoire, *Journal für die reine und angewandte Mathematik*, 52 (1856), pp. 18–38; or [Hermite 1905–1917, 1: 372-396].
[1858] Sur la résolution de l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*, 46 (1858), pp. 508–515; or [Hermite 1905–1917, 2: 5-12].
[1865–1866] Sur l'équation du cinquième degré, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*, 61 (1865), 877–882, 965–972, 1073–1081 and 62 (1866), 65–72, 157–162, 245–252, 715–722, 919–924, 959–966, 1054–1059, 1161–1167, 1213–1215 (1865–1866), pp. 877–882; or [Hermite 1905–1917, 2: 347–424].
[1905–1917] *Œuvres de Charles Hermite*, Paris: Gauthier-Villars, 1905–1917; 4 vols.

Hesse (Otto)
[1844] Über die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln, *Journal für die reine und angewandte Mathematik*, 28 (1844), pp. 68–96.

Hilbert (David)
[1890] Über die Theorie der algebraischen Formen, *Mathematische Annalen*, 36 (1890), pp. 473–534.
[1893] Über die vollen Invariantensysteme, *Mathematische Annalen*, 42 (1893), pp. 313–373.

MacHale (Desmond)
[1985] *George Boole: His Life and Work*, Dublin: Boole Press Limited, 1985.

Parshall (Karen Hunger)
[1989] Toward a History of Nineteenth-Century Invariant Theory, in *The History of Modern Mathematics*, vol. 1, Boston: Academic Press, 1989, pp. 157–206.
[1990] The One-Hundredth Anniversary of the Death of Invariant Theory?, *The Mathematical Intelligencer*, 12(4) (1990), pp. 10–16.
[1998] *James Joseph Sylvester: Life and Work in Letters*, Oxford: Clarendon Press, 1998.
[2006] *James Joseph Sylvester: Jewish Mathematician in a Victorian World*, Baltimore: Johns Hopkins University Press, 2006.

Parshall (Karen Hunger) & Rice (Adrian), eds.
[2002] *Mathematics Unbound: The Evolution of an International Mathematical Research Community 1800–1945*, HMATH, vol. 23, Providence: Amer. Math. Soc. and London: London Mathematical Society, 2002.

Parshall (Karen Hunger) & Seneta (Eugene)
[1997] Building an International Mathematical Reputation: The Case of J. J. Sylvester (1814–1897), *The American Mathematical Monthly*, 104 (1997), pp. 210–222.

Picard (Émile)
[1905] Préface in [Hermite 1905–1917, 1: vii-xl], 1905.

Reader (William J.)
[1966] *Professional Men: The Rise of the Professional Classes in Nineteenth-Century England*, London and New York: Weidenfield and Nicolson and Basic Books, 1966.

Salmon (George)
[1859] *Lessons Introductory to the Modern Higher Algebra*, 1859; 5th edition, Dublin: Hodges, Figgis, and Co., 1885; Reprint Ed. Bronx: Chelsea Publishing Co., 1964.

Sinaceur (Hourya)
[1991] *Corps et modèles: Essai sur l'histoire de l'algèbre réelle*, Paris: Librairie philosophique J. Vrin, 1991.

Sylvester (James Joseph)
[1839] On Rational Derivation from Equations of Coexistence, That Is to Say, a New and Extended Theory of Elimination. Part I, *Philosophical Magazine*, 15 (1839), pp. 428–435; or [Sylvester 1904-1912, 1: 40-46].
[1847a] An Account of a Discovery in the Theory of Numbers Relative to the Equation $Ax^3 + By^3 + Cz^3 = Dxyz$, *Philosophical Magazine*, 31 (1847), pp. 189–191; [Sylvester 1904-1912, 1: 107-109].
[1847b] On the equation in numbers $Ax^3 + By^3 + Cz^3 + Dxyz$, and Its Associate System of Equations, *Philosophical Magazine*, 31 (1847), pp. 293–296; or [Sylvester 1904-1912, 1: 110-113].

[1847c] On the General Solution (in Certain Cases) of the Equation $x^3 + y^3 + Az^3 = Mxyz$, &c., *Philosophical Magazine*, 31 (1847), pp. 467–471; or [Sylvester 1904-1912, 1: 114-118].

[1851a] On a Certain Fundamental Theorem of Determinants, *Philosophical Magazine*, 2 (1851), pp. 142–145; or [Sylvester 1904-1912, 1: 252-255].

[1851b] On the General Theory of Associated Algebraical Forms, *Cambridge and Dublin Mathematical Journal*, 6 (1851), pp. 289–293; or [Sylvester 1904-1912, 1: pp. 198-202].

[1851c] On the Relation between the Minor Determinants of Linear Equivalent Quadratic Functions, *Philosophical Magazine*, 1 (1851), pp. 295–305; or [Sylvester 1904-1912, 1: 241-250].

[1851d] On a Remarkable Discovery in the Theory of Canonical Forms and of Hyperdeterminants, *Philosophical Magazine*, 2 (1851), pp. 391–410; or [Sylvester 1904-1912, 1: 265-283].

[1851e] Sketch of a Memoir on Elimination, Transformation, and Canonical Forms, *Cambridge and Dublin Mathematical Journal*, 6 (1851), pp. 186–200; or [Sylvester 1904-1912, 1: 184-197].

[1852] On the Principles of the Calculus of Forms, *Cambridge and Dublin Mathematical Journal*, 7 (1852), pp. 52–97, 179-217, and 8 (1852), pp. 256–269; or [Sylvester 1904-1912, 1: pp. 284-327, 328-363, and 411-422].

[1853a] Note on the Calculus of Forms, *Cambridge and Dublin Mathematical Journal*, 8 (1853), pp. 62–64; or [Sylvester 1904-1912, 1: pp. 402-403].

[1853b] On a Theory of the Syzygetic Relations of Two Rational Integral Functions, Comprising an Application to the Theory of Sturm's Functions, and That of the Greatest Algebraical Common Measure, *Philosophical Transactions of the Royal Society of London*, 143 (1853), pp. 407–548; or [Sylvester 1904-1912, 1: pp. 429-586].

[1864] Algebraical Researches, Containing a Disquisition on Newton's Rule for the Discovery of Imaginary Roots, and an Allied Rule Applicable to a Particular Class of Equations, Together with a Complete Invariantive Determination of the Character of the Roots of the General Equation of the Fifth Degree, &c., *Philosophical Transactions of the Royal Society of London*, 154 (1864), pp. 579–666; or [Sylvester 1904-1912, 2: pp. 376-479].

[1904-1912] *The Collected Mathematical Papers of James Joseph Sylvester*, Cambridge: Cambridge Univ. Press, 1904-1912; 4 vols., Reprint Ed. New York: Chelsea Publishing Co, 1973.

Wolfson (Paul)

[2008] George Boole and the Origins of Invariant Theory, *Historia Mathematica*, 35 (2008), pp. 37–46.