Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 153
Fascicule 1

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 213-235

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 153, mars 2025

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 165 € ($ 247),
avec supplément papier : Europe 251 €, hors Europe 289 € ($ 433)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
153 (1), 2025, p. 213-235

NAKANO POSITIVITY OF DIRECT IMAGE SHEAVES OF ADJOINT LINE BUNDLES WITH MILD SINGULARITIES

BY YONGPAN ZOU

ABSTRACT. — In this paper, we explore the Nakano positivity of direct image sheaves of twisted relative canonical bundles when the metric of the twisted line bundle has mild singularities. We address this problem using two methods: L^2 estimates and curvature computations within the framework of L^2 Hodge theory.

RÉSUMÉ (*Positivité de Nakano des faisceaux images directes des fibrés en droites adjoints avec des singularités modérées*). — Dans cet article, nous explorons la positivité de Nakano des faisceaux images directs de faisceaux canoniques relatifs tordus lorsque la métrique du fibré en droites tordu présente des singularités modérées. Nous abordons ce problème en utilisant deux méthodes : des estimations L^2 et des calculs de courbure dans le cadre de la théorie de Hodge L^2.

1. Introduction

This note aims to study the Nakano positivity properties of direct image sheaves of relative canonical bundle twisted by holomorphic line bundle with a possible singular metric. More specifically, let $p : \mathcal{X} \to D$ be a holomorphic proper morphism from a Kähler manifold $\mathcal{X}$ onto the polydisk D and let L be a holomorphic line bundle endowed with a possibly singular hermitian metric h_L.

Texte reçu le 1er septembre 2023, modifié le 18 septembre 2024, accepté le 9 octobre 2024.

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Mathematical subject classification (2010). — 32L99.

Key words and phrases. — Nakano positivity, direct image sheaves, L^2 estimate, singular metric.

doi:10.24033/bsmf.2900

In general, we assume that the metric h_L is positively curved, i.e., its curvature current $i\Theta_{h_L}(L) \geq 0$. An important object of study is the direct image sheaf

$$\mathcal{G} := p_*((K_{\mathcal{X}/D} + L) \otimes \mathcal{I}(h_L))$$

endowed with the metric which induced by h_L, here $\mathcal{I}(h_L)$ be the multiplier ideal sheaf associated with the metric h_L. Given the numerous important applications in algebraic and complex geometry, this topic has attracted significant attention from researchers.

Griffiths positivity and Nakano positivity of holomorphic vector bundles are two major positive concepts in complex geometry. Between the two, Nakano positivity is stronger than Griffiths positivity. In [1], Berndtsson proved the induced metric on $\mathcal{G}$ has non-negative curvature in the sense of Nakano when the metric h_L on L is smooth. Since then, researchers have begun to investigate positivity properties in the singular case. Regarding Griffiths positivity, when the metric h is smooth, it is well known that a holomorphic vector bundle (E, h) is Griffiths semi-positive if and only if its dual (E^*, h^*) is Griffiths semi-negative. This equivalence allows for the generalization of Griffiths positivity to the singular case. Consequently, the theory of Griffiths positivity for direct image sheaves of adjoint line bundles with singular metrics has been established (see [12], [11], [10], [9] for various results and generalizations). Since Griffiths positivity in a singular setting has been thoroughly studied, it is natural to explore Nakano positivity in this context. However, the duality property is no longer valid, which complicates the analysis of Nakano positivity (for one definition of Nakano positivity of singular Hermitian metrics on holomorphic vector bundles, consult [13]). Recently, Cao–Guenancia–Păun generalized Berndtsson's curvature formulas to cases where the metric h_L has analytic singularities, and the base space is one-dimensional ([4]). By using the horizontal lift, they successfully obtained a representative of the holomorphic section and established the general curvature formula through this representative.

Meanwhile, in [8], the authors provided a characterization of the Nakano positivity of holomorphic vector bundle with smooth metric via the optimal L^2-estimate condition. Due to the fact that in our main case Z below, the metric h_L of the line bundle is singular, but it is smooth concerning the t variables. It is possible to use the optimal L^2-estimate condition to study the Nakano positivity in these types of singular cases, as I introduce the settings of my study below.

1.1. Set-up of case Z. — Let $p : \mathcal{X} \to D$ be a holomorphic proper fibration (i.e., submersion) from an $(n+m)$-dimensional Kähler manifold $\mathcal{X}$ onto the bounded pseudoconvex domain $D \subset \mathbb{C}^m$, and let (L, h_L) be a holomorphic line bundle endowed with a possibly singular hermitian metric h_L. Let $\Omega \subset \mathcal{X}$ be a coordinate subset on $\mathcal{X}$. We take $(z_1, \ldots z_n, t_1, \ldots, t_m)$ as a coordinate system on Ω such that the last m variables $t_1, \ldots, t_m$ corresponds to the map p itself.

(Z.1) The metric $h_L = e^{-\psi_L}$ and the local weights ψ_L have **Poincaré type singularities**, or **logarithmic type singularities**, or **klt type singularities** along E, as illustrated in Example 1.3 below.
(Z.2) The Chern curvature of (L, h_L) satisfies $i\Theta_{h_L}(L) := i\partial\bar{\partial}\psi_L \geq 0$ in the sense of currents on $\mathcal{X}$.
(Z.3) The multiplier ideal sheaf $\mathcal{I}(h_{L_t}) = \mathcal{O}_{X_t}$ for each $t \in D$.
(Z.4) The Kähler manifold $\mathcal{X}$ contains a Stein Zariski open subset.

With these assumptions we set

$$\mathcal{F} := p_*((K_{\mathcal{X}/D} + L) \otimes \mathcal{I}(h_L)) = p_*(K_{\mathcal{X}/D} + L). \tag{1}$$

By assumption (Z.2) and the Kähler version of Ohsawa–Takegoshi theorem (cf. [3]), $\mathcal{F}$ is, indeed, a vector bundle, and $\mathcal{F}_t = H^0(X_t, K_{X_t} + L_t)$ for every $t \in D$. There is a Hermitian metric $\|\cdot\|$ on $\mathcal{F}$ induced by h_L, i.e., for any $u_t \in \mathcal{F}_t$,

$$\|u_t\|^2 = \int_{X_t} c_n u \wedge \bar{u} e^{-\psi_L} \text{ with } c_n = (\sqrt{-1})^{n^2}.$$

By assumption (Z.1), the metric $\|\cdot\|$ on the direct image sheaf $\mathcal{F}$ is well defined and smooth. Indeed, we can use partitions of unity to reduce to checking that integrals of the form $\int_{\Omega\cap X_t} |u_t|^2 e^{-\psi_L}$ vary smoothly with t, where ψ_L is given by the expression in (Z.1). The reader can consult Lemma 2.2 in [4] for more details. We now introduce the main theorem of this paper.

Theorem 1.1. — *Under the set-up of case Z, the Hermitian holomorphic vector bundle $(\mathcal{F}, \|\cdot\|)$ over D defined in* (1) *is semi-positive in the sense of Nakano.*

Remark 1.2. — To solve the $\bar{\partial}$-equation with a singular weight on $\mathcal{X}$, we must assume that the Kähler manifold $\mathcal{X}$ contains a Stein Zariski open subset, which is why assumption (Z.4) is required. An important example is when $p : \mathcal{X} \to D$ is a projective morphism, in which case assumption (Z.4) is naturally satisfied.

Example 1.3. — We provide some important examples. We assume that there exists a divisor $E = E_1 + \cdots + E_N$ whose support is contained in the total space $\mathcal{X}$ of p such that the following requirements are fulfilled. The divisor E intersects each fiber transversally, i.e., for every $t \in D$ the restriction divisor $E_t := E|_{X_t}$ of E on each fiber X_t has simple normal crossings. Let $\Omega \subset \mathcal{X}$ be a coordinate subset on $\mathcal{X}$. We take $(z_1, \ldots z_n, t_1, \ldots, t_m)$ a coordinate system on Ω such that the last m variables $t_1, \ldots, t_m$ corresponds to the map p itself and such that $z_1 \ldots z_r = 0$ is the local equation of $E \cap \Omega$.

1. The metric h_L has **Poincaré type singularities** along E, i.e., its local weights ψ_L on Ω can be written as

$$\psi_L \equiv -\sum_I b_I \log\left(\big(\prod_{i\in I} |z_i|^{2m_i}\big)\big(\phi_I(z) - \log\big(\prod_{i\in I} |z_i|^{2k_i}\big)\big)\right)$$

modulo $\mathcal{C}^\infty$ functions, where b_I are positive real numbers for all I, and m_i, k_i are positive integers. All $(\phi_I)_I$ are smooth functions on Ω. The set of indexes in the sum coincides with the non-empty subsets of $\{1, \dots, r\}$.

2. The metric h_L has **logarithmic type singularities** along E, i.e., its local weights ψ_L on Ω can be written as

$$\psi_L \equiv -\sum_I b_I \log \left(\phi_I(z) - \log(\prod_{i \in I} |z_i|^{2k_i}) \right)$$

modulo $\mathcal{C}^\infty$ functions, where b_I are positive real numbers satisfying that $b_I < 1$ for all I, all k_i are positive integers, and $(\phi_I)_I$ are smooth functions on Ω. The set of indexes in the sum coincides with the non-empty subsets of $\{1, \dots, r\}$.

3. The metric h_L has **Kawamata (klt)-type singularities** along E, i.e., its local weights ψ_L on Ω can be written as

$$\psi_L \equiv \sum_{i \in I} a_i \log |z_i|^2$$

modulo $\mathcal{C}^\infty$ functions, where a_i are real numbers satisfying that $a_i < 1$ for all i. The set of indexes in the sum coincides with the non-empty subsets of $\{1, \dots, r\}$.

In Section 3, we aim to remove assumption (Z.4) by investigating the case where the metric h_L has Kawamata-type singularities. We utilize the strong decomposition in L^2 Hodge theory. In other words, we attempt to generalize Berndtsson's seminal result in [1] to this singular setting.

2. Nakano positivity via the optimal L^2-estimate

In this section, we will prove Theorem 1.1. In [8], the authors investigate the positivity properties of Hermitian holomorphic vector bundles using L^p-estimates of the $\bar\partial$ operator. They introduce several L^p-estimate (extension) conditions, among which one is referred to as the optimal L^2-estimate condition.

Definition 2.1. — Let (X, ω) be a Kähler manifold of dimension n, which admits a positive Hermitian holomorphic line bundle and (E, h) be a (singular) Hermitian vector bundle over X. The vector bundle (E, h) satisfies *the optimal L^2-estimate condition* if for any positive Hermitian holomorphic line bundle (A, h_A) on X, for any $f \in \mathcal{C}^\infty_c(X, \wedge^{n,1} T^*_X \otimes E \otimes A)$ with $\bar\partial f = 0$, there is $u \in L^p(X, \wedge^{n,0} T^*_X \otimes E \otimes A)$, satisfying $\bar\partial u = f$ and

$$\int_X |u|^2_{h \otimes h_A} dV_\omega \le \int_X \langle B^{-1}_{A,h_A} f, f \rangle dV_\omega,$$

provided that the right-hand side is finite, where $B_{A,h_A} = [i\Theta_{A,h_A} \otimes \mathrm{Id}_E, \Lambda_\omega]$.

One of the main results in [8] was the following characterization of Nakano positivity in terms of optimal L^2-estimate condition.

THEOREM 2.2 ([8, Theorem 1.1]). — *Let (X,ω) be a Kähler manifold of dimension n with a Kähler metric ω, which admits a positive Hermitian holomorphic line bundle, (E,h) be a smooth Hermitian vector bundle over X, and $\theta \in \mathcal{C}^0(X, \wedge^{1,1}T_X^* \otimes End(E))$ such that $\theta^* = \theta$. If for any $f \in \mathcal{C}_c^\infty(X, \wedge^{n,1}T_X^* \otimes E \otimes A)$ with $\bar\partial f = 0$, and any positive Hermitian line bundle (A, h_A) on X with $i\Theta_{A,h_A} \otimes Id_E + \theta > 0$ on $suppf$, there is $u \in L^2(X, \wedge^{n,0}T_X^* \otimes E \otimes A)$, satisfying $\bar\partial u = f$ and*

$$\int_X |u|^2_{h\otimes h_A} dV_\omega \leq \int_X \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A} dV_\omega,$$

provided that the right-hand side is finite, where $B_{h_A,\theta} = [i\Theta_{A,h_A} \otimes Id_E + \theta, \Lambda_\omega]$, then $i\Theta_{E,h} \geq \theta$ in the sense of Nakano. On the other hand, if in addition X is assumed to have a complete Kähler metric, the above condition is also necessary for that $i\Theta_{E,h} \geq \theta$ in the sense of Nakano. In particular, if (E,h) satisfies the optimal L^2-estimate condition, then (E,h) is Nakano semi-positive.

REMARK 2.3. — As remark 1.2 in [8] said, if X admits a strictly plurisubharmonic function, we can take A to be the trivial bundle (with nontrivial metrics).

We aim to use this theorem to study Nakano positivity in the singular setting. In general, the direct image sheaf was not the smooth vector bundle, and, therefore, we can not use Theorem 2.2, but if the singular metric h_L of the line bundle L satisfies the Assumption (Z.1), then the direct image sheaf has a smooth L^2 canonical metric. The following lemmas are crucial for solving the $\bar\partial$-equation with L^2 estimate, which is essential for establishing the desired positivity results in this setting.

LEMMA 2.4 ([6, Lemma 3.2], [8, Appendix]). — *Let X be a complex manifold with dimension n, assume that $\theta \in \wedge^{1,1}T_X^*$ be a positive $(1,1)$-form, and fix an integer $q \geq 1$.*

1. *For each form $u \in \wedge^{n,q}T_X^*$, $\langle [\theta, \Lambda_\omega]^{-1} u, u\rangle dV_\omega$ is non-increasing with respect to θ and ω.*
2. *For each form $u \in \wedge^{n,1}T_X^*$, $\langle [\theta, \Lambda_\omega]^{-1} u, u\rangle dV_\omega$ is independent with respect to ω.*

We need the Richberg-type global regularization result for unbounded quasi-plurisubharmonic functions. Recall that an upper semi-continuous function $\phi : X \to [-\infty, +\infty)$ on a complex manifold X is quasi-psh if it is locally of the form $\phi = u + f$, where u is plurisubharmonic(psh) function, and f is a smooth function.

Lemma 2.5 ([2, Theorem 3.8]). — *Let ϕ be a quasi-psh function on a complex X and assume given finitely many closed, real $(1,1)$-forms θ_α such that $\theta_\alpha + i\partial\bar\partial\phi \geq 0$ for all α. Suppose that either X is strongly pseudoconvex or that $\theta_\alpha > 0$ for all α. Then we can find a sequence $\phi_j \in \mathcal{C}^\infty(X)$ with the following properties.*

1. *ϕ_j converges point-wise to ϕ.*
2. *For each relatively compact open subset $U \Subset X$, there exists $j_U \gg 1$ such that the sequence (ϕ_j) becomes decreasing with $\theta_\alpha + i\partial\bar\partial\phi_j > 0$ for each α when $j \geq j_U$.*

The following L^2-estimate for the $\bar\partial$ equation is fundamental in complex geometry.

Lemma 2.6 ([7, Theorem 4.5]). — *Let (X,ω) be a complete Kähler manifold, with a Kähler metric that is not necessarily complete. Let (E,h) be a Hermitian vector bundle of rank r over X and assume that the curvature operator $B := [i\Theta_{E,h}, \Lambda_\omega]$ is semi-positive definite everywhere on $\wedge^{n,q}T_X^* \otimes E$, for some $q \geq 1$. Then for any form $g \in L^2(X, \wedge^{n,q}T_X^* \otimes E)$ satisfying $\bar\partial g = 0$ and $\int_X \langle B^{-1}g, g\rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \wedge^{n,q-1}T_X^* \otimes E)$ such that $\bar\partial f = g$, and*

$$\int_X |f|^2 dV_\omega \leq \int_X \langle B^{-1}g, g\rangle dV_\omega.$$

Theorem 2.7. — *Let $p : \mathcal{X} \to D$ be a holomorphic proper fibration from an $(n+m)$-dimensional Kähler manifold $\mathcal{X}$ onto the bounded pseudoconvex domain $D \subset \mathbb{C}^m$ and let (L, h_L) be a holomorphic line bundle endowed with a possibly singular hermitian metric h_L with local weight ψ and curvature current $i\Theta_{h_L}(L) \geq 0$. We assume that the Kähler manifold $\mathcal{X}$ contains a Stein Zariski open subset and ϕ is any smooth strictly plurisubharmonic function on D. If*

$$v \in L^2_{loc}(\mathcal{X}, \wedge^{n,1}T_{\mathcal{X}}^* \otimes L)$$

satisfying $\bar\partial v = 0$ and

$$\int_{\mathcal{X}} \langle [i\partial\bar\partial p^*\phi, \Lambda_\omega]^{-1} v, v\rangle_\psi e^{-p^*\phi} dV_\omega < \infty.$$

Then $v = \bar\partial u$ for some $u \in L^2(\mathcal{X}, \wedge^{n,0}T_{\mathcal{X}}^* \otimes L)$ such that

$$\int_{\mathcal{X}} |u|^2_\psi e^{-p^*\phi} dV_\omega \leq \int_{\mathcal{X}} \langle [i\partial\bar\partial p^*\phi, \Lambda_\omega]^{-1} v, v\rangle_\psi e^{-p^*\phi} dV_\omega. \tag{2}$$

Here, the subscript $|\cdot|^2_\psi, \langle\cdot\rangle_\psi$ means the inner product with respect to metric weight ψ of L.

Proof. — Firstly, we note $h_L e^{-p^*\phi}$ is also the singular metric of L because $p^*\phi$ is a globally function on $\mathcal{X}$. Therefore, we have $i\Theta_{h_L}(L) + i\partial\bar\partial p^*\phi \geq i\partial\bar\partial p^*\phi$ in the sense of currents. To prove the claim, we need an L^2-version of the

Riemann extension principal. This is to say, if $\alpha \in L^2_{loc}$ be an L-valued form on a complex X such that $\bar\partial\alpha = \beta$ outside a closed analytic subset $A \subset X$, then $\bar\partial\alpha = \beta$ holds on the whole X. On the other hand, if X is a Stein manifold and L a line bundle on X, there exists a hypersurface $H \subset X$ such that $X \setminus H$ is Stein and L is trivial on $X \setminus H$. Thanks to this, we can assume that $\mathcal{X}$ is Stein, and L is trivial on $\mathcal{X}$. Then the metric $h_L = e^{-\psi}$ and its local weight ψ is globally defined on $\mathcal{X}$. Now we can use the global regularization of unbounded quasi-psh functions.

By Lemma 2.5, we may find an exhaustion of $\mathcal{X}$ by weakly pseudoconvex open subsets Ω_j such that $\psi_j = \psi|_{\Omega_j}$ is the decreasing limit of sequence $\psi_{j,k} \in \mathcal{C}^\infty(\Omega_j)$ with

$$i\partial\bar\partial\psi_{j,k} \geq 0 \Longrightarrow i\partial\bar\partial\psi_{j,k} + i\partial\bar\partial p^*\phi \geq i\partial\bar\partial p^*\phi.$$

Because weakly pseudoconvex manifold admits a complete Kähler metric, on Ω_j we can solve the classical $\bar\partial$ equation with the L^2-estimate as Lemma 2.6, i.e., there exist $u_{j,k} \in L^2(\Omega_j, \wedge^{n,0}T^*_{\Omega_j} \otimes L)$ such that $\bar\partial u_{j,k} = v$ on Ω_j and

$$\begin{aligned}
\int_{\Omega_j} |u_{j,k}|^2_{\psi_{j,k}} e^{-p^*\phi} dV_\omega &= \int_{\Omega_j} |u_{j,k}|^2 e^{-\psi_{j,k}} e^{-p^*\phi} dV_\omega \\
&\leq \int_{\Omega_j} \langle [i\partial\bar\partial p^*\phi, \Lambda_\omega]^{-1} v, v\rangle_{\psi_{j,k}} e^{-p^*\phi} dV_\omega \\
&\leq \int_{\Omega_j} \langle [i\partial\bar\partial p^*\phi, \Lambda_\omega]^{-1} v, v\rangle_{\psi_j} e^{-p^*\phi} dV_\omega \\
&\leq \int_{\mathcal{X}} \langle [i\partial\bar\partial p^*\phi, \Lambda_\omega]^{-1} v, v\rangle_{\psi} e^{-p^*\phi} dV_\omega \\
&= M(constant).
\end{aligned}$$

The second inequality because of ψ_j is the decreasing limit of a sequence $\psi_{j,k}$. By monotonicity of $(\psi_{j,k})_k$, we know the integration $\int_{\Omega_j} |u_{j,k}|^2 e^{-\psi_{j,l}} e^{-p^*\phi} dV_\omega \leq M$ for $k \geq l$; and this shows, in particular, that $(u_{j,k})_k$ is bounded in $L^2(\Omega_j, e^{-p^*\phi-\psi_{j,l}})$. After passing to the subsequence, we thus assume that $u_{j,k}$ converges weakly in $L^2(\Omega_j, e^{-p^*\phi-\psi_{j,l}})$ to u_j, which may further be assumed to be the same for all l, by a diagonal argument. Now we have $\bar\partial u_j = v$, and $\int_{\Omega_j} |u_j|^2 e^{-\psi_{j,l}} e^{-p^*\phi} dV_\omega \leq M$ for all l, and, therefore, $\int_{\Omega_j} |u_j|^2 e^{-\psi} e^{-p^*\phi} dV_\omega \leq M$ by monotone convergence of $\psi_{j,l} \to \psi$. Once again by a diagonal argument, we may arrange that $u_j \to u$ weakly in $L^2(K, e^{-\psi})$ for each compact subset $K \subset X$, and finally we are led to the desired conclusion. □

We can now prove Theorem 1.1 by following the approach of Deng, Ning, Wang, and Zhou.

Theorem 2.8. — *Under the set-up of case Z, the Hermitian holomorphic vector bundle* $(\mathcal{F}, \|\cdot\|)$ *over* D *defined in* (1) *satisfies the optimal* L^2*-estimate condition.*

Proof. — According to Theorem 2.2, it suffices to prove that $(\mathcal{F} = p_*(K_{\mathcal{X}/D} + L), \|\cdot\|)$ satisfies the optimal L^2-estimate condition with the standard Kähler metric ω_0 on $D \subset \mathbb{C}^n$. Let ω be an arbitrary Kähler metric on $\mathcal{X}$.

Let f be a $\bar{\partial}$-closed compact supported smooth $(m,1)$-form with values in $\mathcal{F}$ and let ϕ be any smooth strictly plurisubharmonic function on D. We can write $f(t) = dt \wedge (f_1(t)d\bar{t}_1 + \cdots + f_n(t)d\bar{t}_n)$, with $f_i(t) \in \mathcal{F}_t = H^0(X_t, K_{X_t} \otimes L)$. One can identify f as a smooth compact supported $(n+m,1)$-form $\tilde{f}(t,z) := dt \wedge (f_1(t,z)d\bar{t}_1 + \cdots + f_n(t,z)d\bar{t}_n)$ on $\mathcal{X}$, with $f_i(t,z)$ being holomorphic section of $K_{X_t} \otimes L|_{X_t}$. We have two observations as follows. The first is that $\bar{\partial}_z f_i(t,z) = 0$ for any fixed $t \in D$, since $f_i(t,z)$ are holomorphic sections $K_{X_t} \otimes L|_{X_t}$. The second is that $\bar{\partial}_t f = 0$, since f is a $\bar{\partial}$-closed form on D. It follows that $\tilde{f}$ is a $\bar{\partial}$-closed compact supported $(n+m,1)$-form on $\mathcal{X}$ with values in L. We want to solve the equation $\bar{\partial}\tilde{u} = \tilde{f}$ on X by using Theorem 2.7. Now, we equipped L with the metric $\tilde{h} := he^{-\pi^*\phi}$, and then $i\Theta_{L,\tilde{h}} = i\Theta_{L,h} + i\partial\bar{\partial}\pi^*\phi$, which is also semi-positive in the sense of currents. Hence there is $\tilde{u} \in \wedge^{m+n,0}T^*_{\mathcal{X}} \otimes L$, such that $\bar{\partial}\tilde{u} = \tilde{f}$ and satisfies the following estimate

$$
\begin{aligned}
(3) \qquad \int_{\mathcal{X}} c_{m+n}\tilde{u} \wedge \bar{\tilde{u}} e^{-p^*\phi} &= \int_{\mathcal{X}} |\tilde{u}|^2 e^{-p^*\phi} dV_\omega \\
&\leq \int_{\mathcal{X}} \langle [i\partial\bar{\partial}p^*\phi, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle e^{-p^*\phi} dV_\omega \\
&= \int_{\mathcal{X}} \langle [i\partial\bar{\partial}p^*\phi, \Lambda_{\omega'}]^{-1}\tilde{f}, \tilde{f}\rangle e^{-p^*\phi} dV_{\omega'} \\
&= \int_D \langle [i\partial\bar{\partial}\phi, \Lambda_{\omega_0}]^{-1} f, f\rangle_t e^{-\psi} dV_{\omega_0}.
\end{aligned}
$$

The first inequality is due to (2), the second equality holds because $\tilde{f}$ is $(n+m,1)$-form, and therefore $\langle [i\partial\bar{\partial}p^*\phi, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle dV_\omega$ are independent to ω in view of Lemma 2.4. The last equality is valid because here we choose $\omega' = i\Sigma_{j=1}^m dt_j \wedge d\bar{t}_j + i\Sigma_{j=1}^n dz_j \wedge d\bar{z}_j$. The notation $\langle\cdot,\cdot\rangle_t$ here means a pointwise inner product concerning the Hermitian metric $\|\cdot\|$ on $\mathcal{F}$.

Set $\tilde{u}_t := \tilde{u}(t,\cdot)$; we observe that $\bar{\partial}\tilde{u}_t = 0$ for any fixed $t \in D$, since $\bar{\partial}\tilde{u} = \tilde{f}$ and the $(n+m,1)$-form $\tilde{f}$ contains only the terms of $d\bar{t}_i$. This means that $\tilde{u}_t \in \mathcal{F}_t$, and hence we may view $\tilde{u}$ as a section u of $\mathcal{F}$. It is obvious that $\bar{\partial}u = f$. Due to Fubini's theorem, we have

$$
(4) \qquad \int_{\mathcal{X}} c_{m+n}\tilde{u} \wedge \bar{\tilde{u}} e^{-p^*\phi} = \int_D \|u_t\|^2 e^{-\phi} dV_{\omega_0}.
$$

Combining (3) with (4), we obtain

$$\int_D \|u_t\|^2 e^{-\phi} dV_{\omega_0} \leq \int_D \langle [i\partial\bar{\partial}\phi, \Lambda_{\omega_0}]^{-1} f, f\rangle_t e^{-\phi} dV_{\omega_0}.$$

So we have proved that $\mathcal{F}$ satisfies the optimal L^2-estimate condition, thus due to Theorem 2.2 (and Remark 2.3), $\mathcal{F}$ is semi-positive in the sense of Nakano. $\square$

3. Nakano positivity via curvature computation

In this section, we will investigate the Nakano positivity of direct image sheaves under the Kähler condition alone, thereby eliminating the need for Assumption (Z.4). We adopt Berndtsson's approach from [1].

3.1. Set-up of case V. — Let $p : \mathcal{X} \to D$ be a holomorphic proper fibration from an $(n+m)$-dimensional Kähler manifold $\mathcal{X}$ onto the polydisk $D \subset \mathbb{C}^m$ and let (L, h_L) be a holomorphic line bundle endowed with a possibly singular hermitian metric h_L. We assume that there exists a divisor $E = E_1 + \cdots + E_N$ whose support is contained in the total space $\mathcal{X}$ of p such that the following requirements are fulfilled.

(V.1) The divisor E intersects each fiber transversally, i.e., for every $t \in D$ the restriction divisor $E_t := E|_{X_t}$ of E on each fiber X_t has simple normal crossings. Let $\Omega \subset \mathcal{X}$ be a coordinate subset on $\mathcal{X}$. We take $(z_1, \ldots z_n, t_1, \ldots, t_m)$ as a coordinate system on Ω such that the last m variables $t_1, \ldots, t_m$ corresponds to the map p itself and such that $z_1 \ldots z_r = 0$ is the local equation of $E \cap \Omega$.

(V.2) The metric h_L has **klt type singularities** along E, i.e., its local weights ψ_L on Ω can be written as

$$\psi_L \equiv \sum_{i \in I} a_i \log |z_i|^2$$

modulo $\mathcal{C}^\infty$ functions, where a_i are real numbers satisfying that $a_i < 1$ for all i. The set of indexes in the sum coincides with the non-empty subsets of $\{1, \ldots, r\}$.

(V.3) The Chern curvature of (L, h_L) satisfies

$$i\Theta_{h_L}(L) \geq 0$$

in the sense of currents on $\mathcal{X}$.

One wants to study some interesting objects such as adjoint bundles $K_{\mathcal{X}/D} + L$ or $(K_{\mathcal{X}/D} + L) \otimes \mathcal{I}(h_L)$ and their direct image sheaves. Under the setting of case V, we set

$$\mathcal{G} := p_*((K_{\mathcal{X}/D} + L) \otimes \mathcal{I}(h_L)) = p_*(K_{\mathcal{X}/D} + L). \tag{5}$$

The last equality holds because $\mathcal{I}(h_L) = \mathcal{O}_{\mathcal{X}}$, which is due to the conditions in Assumption (V.2). We first remark that $\mathcal{G}$ is, indeed, a vector bundle by the

Kähler version of Ohsawa–Takegoshi theorem [3] under the assumption (V.3), which implies that any element of $H^0(X_t, K_{X_t}+L)$ extends to $\mathcal{X}$. One obtains

$$\mathcal{G}_t = H^0(X_t, K_{X_t}+L)$$

for every $t \in D$. Therefore, there exists an induced canonical L^2 metric on $\mathcal{G}$. We will interchangeably denote by $\|\cdot\|$ or $h_{\mathcal{G}}$ the L^2 metric on $\mathcal{G}$. If u is the section of $\mathcal{G}$, $u_t := u|_{X_t} \in \mathcal{G}_t = H^0(X_t, K_{X_t}+L)$, then

$$\|u_t\|^2 := c_n \int_{X_t} u \wedge \bar{u} e^{-\psi_L} \text{ with } c_n = (\sqrt{-1})^{n^2}.$$

By assumption, we claim that the L^2 metric is smooth on D. Indeed, we can use partitions of unity to reduce to checking that integrals of the form $\int_{\Omega\cap X_t} |u_t|^2 e^{-\psi_L}$ vary smoothly with t, where ψ_L is given by the expression in (V.2). Now it is clear there that all derivatives in the $t, \bar{t}$ variables of ψ_L are bounded and smooth, so that the result follows from general smoothness results for integrals depending on a parameter. We denote by ∇ the Chern connection of $(\mathcal{G}, \|\cdot\|)$ on D, and $\nabla^{1,0}, \nabla^{0,1}$ represent the $(1,0)$ part and $(0,1)$ part of the Chern connection, respectively.

We set $\mathcal{X}^\circ := \mathcal{X} \setminus E$ and $X_t^\circ := X_t \cap \mathcal{X}^\circ$ an open fiber, $L_t := L|_{X_t}$, $h_{L_t} := h_L|_{X_t}$. When working on a trivializing coordinate chart of L, we will always denote by ψ_L the local weight of h_L. Under assumption (V.1), we will write $E := \sum_{i=1}^N E_i$ for the decomposition of E into its (smooth) irreducible components. Next, let s_i be a section of $\mathcal{O}_{\mathcal{X}}(E_i)$ that cuts out E_i and let h_i be a smooth hermitian metric on $\mathcal{O}_{\mathcal{X}}(E_i)$. In the following, $|s_i|^2$ stands for $|s_i|^2_{h_i}$, and we assume that $|s_i|^2 < e^{-1}$. Let ω be a fixed Kähler metric on $\mathcal{X}$ and let

$$\omega_E := C\omega + dd^c\Big[-\sum_{i=1}^N \log\log\frac{1}{|s_i|^2}\Big] \qquad \text{on } \mathcal{X}^\circ \tag{6}$$

be a metric with Poincaré singularities along E; here the constant C is large enough to make ω_E positive. Thanks to (V.1) we infer that $\omega_E|_{X_t^\circ}$ is a complete Kähler metric on X_t° with Poincaré singularities along $E \cap X_t$ for each $t \in D$.

On each compact Kähler fiber with dimension n, the section u is a smooth $(n,0)$-form, and we have the next equality, which is very important for our case,

$$\|u_t\|^2 = c_n \int_{X_t} u \wedge \bar{u} e^{-\psi_L} = c_n \int_{X_t^\circ} u \wedge \bar{u} e^{-\psi_L} = c_n \int_{X_t^\circ} |u|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E}. \tag{7}$$

Here, ω_E is the metric in (6). The second equality since the divisor has measure zero, and the third equality holds thanks to the integrand being independent to ω_E. Due to (7), we can use the L^2-Hodge theory on X_t° with respect to metric ω_E and ψ_L.

Theorem 3.1. — *Under the set-up of case V, the Hermitian holomorphic vector bundle $(\mathcal{G}, \|\cdot\|)$ over D defined in* (5) *is semi-positive in the sense of Nakano.*

3.2. The Chern connection. — As in the set-up of case V, let u be the section of direct image sheaf $\mathcal{G}$; it is an L-valued $(n,0)$ form on $\mathcal{X}$. By the canonical L^2 metric we have $\|u_t\|^2 = c_n \int_{X_t} u \wedge \bar{u} e^{-\psi_L}$.

One well-known fact is that for any complex manifold X with dimension n, two positive $(1,1)$-forms ω and $\widetilde{\omega}$ on X with relation $\omega \leq \widetilde{\omega}$, and an (n,q)-form α on X, $|\alpha|^2_\omega dV_\omega \geq |\alpha|^2_{\widetilde{\omega}} dV_{\widetilde{\omega}}$. In particular, if α is an $(n,0)$-form, then we have $|\alpha|^2_\omega dV_\omega = |\alpha|^2_{\widetilde{\omega}} dV_{\widetilde{\omega}}$.

Therefore, we have the next equality which is very important for our case,

$$\|u_t\|^2 = c_n \int_{X_t} u \wedge \bar{u} e^{-\psi_L} = c_n \int_{X_t^\circ} u \wedge \bar{u} e^{-\psi_L} = c_n \int_{X_t^\circ} |u|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E}. \tag{8}$$

Here, ω_E is the metric with Poincaré singularities along E as in (6). Due to this, we can use the Hodge decomposition on X_t° with respect to metric ω_E and ψ_L. For any two sections u, v of $\mathcal{G}$, put $[u,v] = c_n u \wedge \bar{v} e^{-\psi_L}$, then $\|u_t\|^2 = \int_{X_t} [u,u] = p_*[u_t, u_t]$, and consequently we define the inner product on $\mathcal{G}$ by

$$(u,v) = p_*[u,v] = p_*(c_n u \wedge \bar{u} e^{-\psi_L}).$$

Let $\mathcal{X}$ be a Kähler manifold of dimension $m+n$, which is smoothly fibered over connected complex m-dimensional polydisk D, then the local coordinates on $\mathcal{X}$ are denoted by $(z_1, \ldots, z_n, t_1, \ldots, t_m)$. We define a complex structure on $\mathcal{G}$ by saying that the smooth section u defines a holomorphic section of $\mathcal{G}$ if $u \wedge dt$ is a holomorphic local section of $K_{\mathcal{X}} \otimes L$. Let u be a smooth local section of $\mathcal{G}$, $u_t \in H^0(X_t, K_{X_t} + L_t)$; this is an L-valued $(n,0)$-form on X_t. For example, if locally we write $u = f dz + \sum g_i \widehat{dz_i} \wedge dt_i$, then both u and $f dz$ represent the same section of $\mathcal{G}$. It is obvious that $\bar{\partial} u$ is also a smooth form on $\mathcal{X}$. On account of $\bar{\partial} u \wedge dt \wedge d\bar{t} = 0$, thus

$$\bar{\partial} u = \sum \nu^j \wedge d\bar{t}_j + \sum \eta^j \wedge dt_j,$$

where the ν^j and η^j are smooth forms on $\mathcal{X}$, and ν^j defines a section to $\mathcal{G}$. We define the $(0,1)$-part of the connection ∇ on $\mathcal{G}$ by letting

$$\nabla^{0,1} u = \sum \nu^j \wedge d\bar{t}_j. \tag{9}$$

Therefore, u is a holomorphic section if and only if $\bar{\partial} u \wedge dt = 0$, or if and only if $\bar{\partial} u = \sum \eta^j \wedge dt_j$.

Now we define the $(1,0)$-part of the Chern connection. If u is a smooth section of $\mathcal{G}$, let $u^\circ := u|_{\mathcal{X}^\circ}$, We eliminate all terms containing dt_j in u°, as they represent the same section of $\mathcal{G}$. Then we have

$$\partial^{\psi_L} u^\circ := e^{\psi_L}\partial(e^{-\psi_L}u^\circ) = (-\partial\psi_L)u^\circ + \partial u^\circ = \sum \mu^j \wedge dt \tag{10}$$

for some L-valued $(n,0)$-forms μ^j that contain terms of $\partial\psi_L$ on $\mathcal{X}^\circ$. As we saw in assumption (V.2), the weight ψ_L has singularities along the divisor E, so ψ_L is smooth outside E. And we have μ^j is L^2 integrable with respect to $(\omega_E, e^{-\psi_L})$, since here $\partial\psi_L$ takes the derivative with respect to the t_j variables, not the z_i variables. (We choose u° such that it does not contain any dt_j terms.)

The restrictions of μ^j defined in (10) on each open fiber X_t° are, in general, not holomorphic. So we let $P(\mu^j)$ be the orthogonal projection of μ^j onto the space of holomorphic forms on each open fiber. Similarly to Remark 3.5 in [4], since $P(\mu^j)$ is holomorphic on X_t° and L^2-integrable, it extends holomorphically to the compact fiber X_t. We denote by $P'(\mu^j)$ the extension of $P(\mu^j)$. Similarly to [1, Lemmma 4.1], we define the $(1,0)$-part of the Chern connection by

$$\nabla^{1,0}u = \sum P'(\mu^j)dt_j. \tag{11}$$

We also need to verify that

$$\partial_{t_j}(u,v) = (P'(\mu^j), v) + (u, \bar\partial_{t_j} v)$$

for any smooth section u and v to $\mathcal{G}$, and here $\bar\partial v = \sum v^j \wedge d\bar t_j + \sum w^j \wedge dt_j$. We have defined that $\nabla^{0,1}v = \sum v^j \wedge d\bar t_j$ and sometimes we write $v^j = \bar\partial_{t_j} v$. On the other hand, by the commutativity of p_* and ∂, we have $\partial(u,v) = \partial p_*([u,v]) = \partial\int_{X_t} c_n u\wedge \bar v e^{-\psi_L} = \partial \int_{X_t^\circ} c_n u \wedge \bar v e^{-\psi_L}$. Hence, one obtains

$$\begin{aligned}
\partial(u,v) &= \int_{X_t^\circ} c_n \partial^{\psi_L} u \wedge \bar v e^{-\psi_L} + (-1)^n \int_{X_t^\circ} c_n u \wedge \overline{\bar\partial v} e^{-\psi_L} \\
&= \int_{X_t^\circ} c_n \sum \mu^j \wedge dt_j \wedge \bar v e^{-\psi_L} + (-1)^n \int_{X_t^\circ} c_n u \wedge \sum \bar v^j \wedge dt_j e^{-\psi_L} \\
&\quad + (-1)^n \int_{X_t^\circ} c_n u \wedge \sum \bar w^j \wedge d\bar t_j e^{-\psi_L}.
\end{aligned}$$

The last term above vanishes owing to the degree of forms, and, thus, we obtain

$$\begin{aligned}
\partial(u,v) &= \int_{X_t} c_n \sum \mu^j \wedge dt_j \wedge \bar v e^{-\psi_L} + (-1)^n \int_{X_t} c_n u \wedge \sum \bar v^j \wedge dt_j e^{-\psi_L} \\
&= \sum((\mu^j, v) + (u, v^j))dt_j \\
&= \sum((P'(\mu^j), v) + (u, \bar\partial_{t_j} v))dt_j,
\end{aligned}$$

and so we are led to the conclusion that the connection is compatible with the inner product.

3.3. Nakano positivity. — To study the Nakano positivity, we will use the so-called $\partial\bar{\partial}$-Bochner–Kodaira technique illustrated in Section 2 in [1]. Let E be a holomorphic vector bundle with connection D. Let u_j be an m-tuple of holomorphic sections to E, satisfying $Du_j = 0$ at fixed point 0. Put $T_u = \sum(u_j, u_k)\widehat{dt_j \wedge d\bar{t}_k}$; here $\widehat{dt_j \wedge d\bar{t}_k}$ denotes the product of all differential dt_i and $d\bar{t}_i$, except dt_j, $d\bar{t}_k$, and is multiplied by a number of modulus 1, so that T_u is non-negative. Then by calculation, we obtain $i\partial\bar{\partial}T_u = -\sum(\Theta^E_{j,k}u_j, u_k)dV_t$. So E is Nakano positivity at the given point if and only if this expression $i\partial\bar{\partial}T_u$ is negative for any choice of holomorphic sections u_j satisfying $Du_j = 0$ at the chosen point.

So, we let u_j be an m-tuple of holomorphic sections of $\mathcal{G}$ satisfying $\nabla^{1,0}u_j = 0$ at a given point $t = 0$. Let

$$T_u = \sum(u_j, u_k)\widehat{dt_j \wedge d\bar{t}_k}$$

as above, so that T_u is non-negative. We put $\widehat{u} = \sum u_j \wedge \widehat{dt_j}$ be an L-valued $(N, 0)$-form with $N = n + m - 1$, and thus

$$T_u = c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} e^{-\psi_L}).$$

Then, we obtain

$$\begin{aligned}
\bar{\partial}T_u &= c_N p_*(\bar{\partial}\widehat{u} \wedge \overline{\widehat{u}} e^{-\psi_L}) + (-1)^N c_N p_*(\widehat{u} \wedge \overline{\partial^{\psi_L}\widehat{u}} e^{-\psi_L}) \\
&= (-1)^N c_N p_*(\widehat{u} \wedge \overline{\partial^{\psi_L}\widehat{u}} e^{-\psi_L}).
\end{aligned} \tag{12}$$

Therefore,

$$\partial\bar{\partial}T_u = (-1)^N c_N p_*(\partial^{\psi_L}\widehat{u} \wedge \overline{\partial^{\psi_L}\widehat{u}} e^{-\psi_L}) + c_N p_*(\widehat{u} \wedge \overline{\bar{\partial}\partial^{\psi_L}\widehat{u}} e^{-\psi_L}).$$

Using the identity $\bar{\partial}\partial^{\psi_L} + \partial^{\psi_L}\bar{\partial} = \partial\bar{\partial}\psi_L$, we can change it to the next equation.

$$\begin{aligned}
\partial\bar{\partial}T_u &= (-1)^N c_N p_*(\partial^{\psi_L}\widehat{u} \wedge \overline{\partial^{\psi_L}\widehat{u}} e^{-\psi_L}) - c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \partial\bar{\partial}\psi_L e^{-\psi_L}) \\
&\quad + (-1)^N c_N p_*(\bar{\partial}\widehat{u} \wedge \overline{\bar{\partial}\widehat{u}} e^{-\psi_L}).
\end{aligned} \tag{13}$$

This formula make sense since $\bar{\partial}\widehat{u}$, $\partial^{\psi_L}\widehat{u}$, $\bar{\partial}\partial^{\psi_L}\widehat{u}$ and $\partial\bar{\partial}\psi_L$ are L^2-integrable with respect to ω_E and $e^{-\psi_L}$. Note that we have $\partial\bar{\partial}\sum_{i\in I} a_i \log|z_i|^2 = 0$ on $\mathcal{X}^\circ$. The next two lemmas help us to simplify the formula (13), which is the corollary of Hodge theory in Section 4.

Lemma 3.2 ([1, Lemma 4.3]). — *Let u be an $(n, 0)$-form on $\mathcal{X}$, representing a holomorphic section of $\mathcal{G}$; we can write*

$$\bar{\partial}u = \sum \eta^j \wedge dt_j.$$

Then, $\eta^j \wedge \omega$ are $\bar{\partial}$-exact on each fiber.

Proof. — We include the proof for the reader's convenience. Since $u \wedge \omega$ is of bi-degree $(n+1,1)$, we can write locally

$$u \wedge \omega = \sum u^j \wedge dt_j.$$

The coefficients u^j are not unique, but their restrictions on fibers are unique. Indeed, since $\sum u^j \wedge dt_j = 0$ implies $\sum u^j \wedge dt_j \wedge \widehat{dt_j} = u^j \wedge dt = 0$, the latter shows that u^j vanishes when restricted to any fiber. Thus, u^j are well-defined global forms on any fiber. Moreover,

$$\sum \eta^j \wedge \omega \wedge dt_j = \overline{\partial} u \wedge \omega = \sum \overline{\partial} u^j \wedge dt_j,$$

so $\sum (\eta^j \wedge \omega - \overline{\partial} u^j) \wedge dt_j = 0$. Again wedging with $\widehat{dt_j}$, we see that $\eta^j \wedge \omega = \overline{\partial} u^j$ on each fiber. □

LEMMA 3.3. — *Let u be a holomorphic section of $\mathcal{G}$ on D such that $\nabla u = 0$ at $t = 0$. Then u can be represented by an L^2-form $\tilde{u}$ on $\mathcal{X}^\circ$ such that $\overline{\partial}\tilde{u} = \eta^j \wedge dt_j$ for some L^2-form η^j, which are primitive on central open fiber X_0° and $\partial^{\psi_L} \tilde{u} = \mu^j \wedge dt_j$ for some L^2-form μ^j satisfying $\mu^j|_{X_0^\circ} = 0$.*

Proof. — We have $\partial^{\psi_L} u = \sum \mu^j \wedge dt_j$ and because $\nabla u(0) = 0$, each $\mu_0^j = \mu^j|_{X_0^\circ}$ is orthogonal to the space of L^2 holomorphic sections on X_0°. We write $\overline{\partial} u = \sum \eta^j \wedge dt_j$, according to the above Lemma 3.2 and we know that $\eta^j \wedge \omega = \overline{\partial} u^j$ on the center fiber, since $\mu_0^j - \overline{\partial}^* u^j$ is orthogonal to the space of L^2 holomorphic sections on X_0°. Hodge theory, especially Theorem 4.1, shows that $\mu_0^j - \overline{\partial}^* u^j$ is $\overline{\partial}^*$-exact, i.e., there exists a $\overline{\partial}$-closed L^2-form β_0^j on X_0° such that $\overline{\partial}^* \beta_0^j = \mu_0^j - \overline{\partial}^* u^j$. We set $\gamma_0^j = *(\beta_0^j + u^j)$, and then it easy to see

$$\partial^{\psi_L} \gamma_0^j = \mu_0^j,$$
$$\overline{\partial} \gamma_0^j \wedge \omega = \eta^j \wedge \omega.$$

Let γ^j be an arbitrary global L^2 extension of γ_0^j on $\mathcal{X}^\circ$. Then $\tilde{u} = u - \sum \gamma^j \wedge dt_j$ is the representative that we are looking for. We have $\overline{\partial}\tilde{u} = \overline{\partial} u - \sum \overline{\partial} \gamma^j \wedge dt_j = \sum (\eta^j - \overline{\partial}\gamma^j) \wedge dt_j$, and, thus, on the center fiber

$$(\eta^j - \overline{\partial}\gamma_0^j) \wedge \omega = \eta^j \wedge \omega - \overline{\partial}\gamma_0^j \wedge \omega = 0.$$

On the other hand, $\partial^{\psi_L} \tilde{u} = \partial^{\psi_L} u - \partial^{\psi_L} \gamma^j \wedge dt_j = \sum (\mu^j - \partial^{\psi_L} \gamma^j) \wedge dt_j$. When we restrict it on X_0°, we know that $\mu^j - \partial^{\psi_L} \gamma^j = 0$, and, therefore, the claim is proved. □

Note that we can assume that $\bar{\partial}\gamma^j$, $\partial^{\psi_L}\gamma^j$ and $\bar{\partial}\partial^{\psi_L}\gamma^j$ are all L^2 integrable on $\mathcal{X}^\circ$ with respect to ω_E and $e^{-\psi_L}$. Therefore, $\tilde{u}$ also satisfies these regularity properties. Indeed, since γ_0^j possesses good regularity properties on the fibers, we work locally around X_0° (a tubular neighborhood), employing a coordinate cover and a partition of unity subordinate to it. On each coordinate patch,

γ_0^j can be extended locally by setting $\gamma^j(z,t) := \gamma_0^j(z)$. Using the partition of unity, we obtain a global extension over the tubular neighborhood of X_0°. To extend this to $\mathcal{X}^\circ$, we multiply by a smooth function supported on this tubular neighborhood.

With the help of Lemma 3.3, we can replace u by $\widetilde{u}$ in the definition of T_u,

$$T_u = c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} e^{-\psi_L}) = c_N p_*(\widehat{\widetilde{u}} \wedge \overline{\widehat{\widetilde{u}}} e^{-\psi_L}).$$

Note that $T_u = T_{\tilde{u}}$ according to the construction of $\tilde{u}$. For ease of notation, we still denote by u the section satisfying the conclusion in the above lemma. Then we can simplify the formula (13) at the point $t = 0$.

$$\begin{aligned}(14)\qquad \partial\bar{\partial} T_u &= (-1)^N c_N p_*(\partial^{\psi_L}\widehat{u} \wedge \overline{\partial^{\psi_L}\widehat{u}} e^{-\psi_L}) - c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \partial\bar{\partial}\psi_L e^{-\psi_L}) \\ &\quad + (-1)^N c_N p_*(\bar{\partial}\widehat{u} \wedge \overline{\bar{\partial}\widehat{u}} e^{-\psi_L}) \\ &= -c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \partial\bar{\partial}\psi_L e^{-\psi}) + (-1)^N c_N p_*(\bar{\partial}\widehat{u} \wedge \overline{\bar{\partial}\widehat{u}} e^{-\psi_L}).\end{aligned}$$

For further simplification, the next lemma will be useful.

Lemma 3.4. — *Let X be a complex manifold with complex dimension n, and if α is a form of bidegree $(n-1,1)$. Then*

$$\sqrt{-1} c_{n-1} \alpha \wedge \bar{\alpha} = (\|\alpha\|^2 - \|\alpha \wedge \omega\|^2) dV_\omega.$$

Since we know that η is primitive on central open fiber X_0°, we can further turn formula (14) into

$$\sqrt{-1}\partial\bar{\partial} T_u = -c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \sqrt{-1}\partial\bar{\partial}\psi_L e^{-\psi_L}) - \int_{X_0} \|\eta\|^2 dV_t.$$

Because $\sqrt{-1}\partial\bar{\partial}\psi_L$ is a positive current, and $c_N \widehat{u} \wedge \overline{\widehat{u}}$ is a strongly positive form. We obtain that $c_N \widehat{u} \wedge \overline{\widehat{u}} \wedge \sqrt{-1}\partial\bar{\partial}\psi_L e^{-\psi_L}$ is a positive current. The pushforward of positive current under proper morphism is again positive current. For details, we refer the reader to [7, Chapter III. §1]. It is obvious that $\sqrt{-1}\partial\bar{\partial} T_u$ is a smooth (m,m)-form on D, and hence $c_N p_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \sqrt{-1}\partial\bar{\partial}\psi_L e^{-\psi_L})$ is the positive form. This formula means that $\sqrt{-1}\partial\bar{\partial} T_u \leq 0$, and so $\mathcal{G}$ is positive in the sense of Nakano. Now the proof of Theorem 3.1 is completed.

4. Some L^2 Hodge theory

We introduce a few results of L^2-Hodge theory for L-valued forms on a complete manifold endowed with a Poincaré type metric, following closely [5] and [4].

Let X be an n-dimensional compact Kähler manifold and let (L, h_L) be a line bundle endowed with a (singular) metric $h_L = e^{-\psi_L}$ such that

(R.1) h_L has klt type singularities along divisor E with simple normal crossing supports as Example 1.3.

(R.2) ω_E be a complete Kähler metric on $X^\circ := X\backslash E$ with Poincaré singularities along E as (6).

(R.3) Its Chern curvature satisfies $i\Theta_{h_L}(L) \geq 0$ in the sense of currents.

We denote by D' the $(1,0)$-part of the Chern connection on (L, h_L). The famous Bochner–Kodaira–Nakano formula links up two Laplace $\Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $\Delta' = D'D'^* + D'^*D'$ as follows:

$$\Delta'' = \Delta' + [i\Theta_{h_L}(L), \Lambda_{\omega_E}]. \tag{15}$$

Let us also recall the well-known fact that the self-adjoint operator

$$A := [i\Theta_{h_L}(L), \Lambda_{\omega_E}]$$

is semi-positive when acting on (n,q) forms, for any $0 \leq q \leq n$, as long as $i\Theta_{h_L}(L) \geq 0$. An immediate consequence of (15) is that for an L^2-integrable form u with values in L of any type in the domains of Δ' and Δ'', we have

$$\|\bar\partial u\|^2_{L^2} + \|\bar\partial^* u\|^2_{L^2} = \|D'u\|^2_{L^2} + \|D'^* u\|^2_{L^2} + \int_{X^\circ} \langle Au, u\rangle dV_{\omega_E}, \tag{16}$$

where $\|\cdot\|_{L^2}$ (resp., $\langle,\rangle$) denotes the L^2-norm (resp., pointwise hermitian product) taken with respect to (h_L, ω_E). Let $\star : \Lambda^{p,q}T^*_{X^\circ} \to \Lambda^{n-q,n-p}T^*_{X^\circ}$ be the Hodge star with respect to ω_E;

Theorem 4.1 ([4, Theorem 3.6]). — *Let X be an n-dimensional compact Kähler manifold, and let (L, h_L) be a line bundle endowed with a (singular) metric $h_L = e^{-\psi_L}$ such that the assumptions (R.1), (R.2), (R.3) are satisfied. Then we have the following Hodge decomposition:*

$$L^2_{n,1}(X^\circ, L) = \mathcal{H}_{n,1}(X^\circ, L) \oplus \mathrm{Im}\bar\partial \oplus \mathrm{Im}\bar\partial^*,$$

where $\mathcal{H}_{n,1}(X^\circ, L)$ is the space of L^2 Δ''-harmonic $(n,1)$-forms.

The main goal of this section is to establish the following decomposition theorem, which is analogous to the corresponding result in [4]. For the reader's convenience, we will sketch the proof and emphasize the differences. Let X be an n-dimensional compact Kähler manifold and let (L, h_L) be a line bundle endowed with a (singular) metric $h_L = e^{-\psi_L}$ such that

(S.1) h_L has logarithmic type singularities along divisor E with simple normal crossing supports, i.e., its local weights ψ_L on Ω can be written as

$$\psi_L \equiv -\sum_I b_I \log\left(\phi_I(z) - \log(\prod_{i\in I} |z_i|^{2k_i})\right)$$

modulo $\mathcal{C}^\infty$ functions, where b_I are positive real numbers satisfying that $b_I < 1$ for all I, all k_i are non-negative integers and $(\phi_I)_I$ are smooth functions on Ω. The set of indexes in the sum coincides with the non-empty subsets of $\{1, \ldots, r\}$.

(S.2) ω_E be a complete Kähler metric on $X^\circ := X\backslash E$ with Poincaré singularities along E as in (6);
(S.3) Its Chern curvature satisfies $i\Theta_{h_L}(L) \geq 0$ in the sense of currents.

Theorem 4.2. — *Let X be an n-dimensional compact Kähler manifold and let (L, h_L) be a line bundle endowed with a (singular) metric $h_L = e^{-\psi_L}$ such that the assumptions (S.1), (S.2), (S.3) are satisfied. Then we have the following Hodge decomposition*

$$L^2_{n,1}(X^\circ, L) = \mathcal{H}_{n,1}(X^\circ, L) \oplus \mathrm{Im}\bar{\partial} \oplus \mathrm{Im}\bar{\partial}^*,$$

where $\mathcal{H}_{n,1}(X^\circ, L)$ is the space of L^2 Δ''-harmonic $(n,1)$-forms.

Firstly, we need a lemma that ensures that we can use the compact support approximation.

Lemma 4.3 ([4, Lemma 3.7]). — *There exists a family of smooth functions $(\mu_\varepsilon)_{\varepsilon>0}$ with the following properties.*

(1) *For each $\varepsilon > 0$, the function μ_ε has compact support in X°, and $0 \leq \mu_\varepsilon \leq 1$,*
(2) *The sets $(\mu_\varepsilon = 1)$ are providing an exhaustion of X°,*
(3) *There exists a positive constant $C > 0$ independent of ε such that we have*

$$\sup_{X^\circ} \left(|\partial\mu_\varepsilon|^2_{\omega_E} + |\partial\bar{\partial}\mu_\varepsilon|^2_{\omega_E}\right) \leq C.$$

We also need the local Poincaré type inequality for the $\bar{\partial}$-operator when acting on L-valued $(p,0)$-forms. The next proposition is very important for our proposal. This is about the Poincaré type inequality involving the metric of a line bundle, and the metric ψ_L is no longer with analytic singularities (as in [5]) but with Poincaré type singularities. Therefore, we need to reprove it.

Proposition 4.4. — *Let $(\Omega_j)_{j=1,\dots,N}$ be a finite union of coordinate sets of X covering E and let U be any open subset contained in their union. Let τ be a $(p,0)$-form with compact support in a set $U \setminus E \subset X$ and values in (L, h_L). Then we have*

$$\int_U |\tau|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} \leq C \int_U |\bar{\partial}\tau|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E}, \tag{17}$$

where C is a positive numerical constant.

Proof. — We consider the restriction of the form τ to some coordinate open set Ω whose intersection with E is of type $z_1 \dots z_r = 0$. It can be written as sum of forms of type $\tau_I := f_I dz_I$. Assume that $I \cap \{1, \ldots, r\} = \{1, \ldots, p\}$ for some p. We have

$$|\tau_I|_g^2 e^{-\psi_L} dV_g = \frac{|f_I|^2 e^{-\psi_L}}{\prod_{\alpha=p+1}^r |z_\alpha|^2 \log^2 |z_\alpha|^2} d\lambda. \tag{18}$$

In (18) we denote by g the model Poincaré metric

$$\sum_{i=1}^r \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \sum_{i=r+1}^n \sqrt{-1} dz_i \wedge d\bar{z}_i,$$

and $d\lambda = \frac{\omega^n}{n!}$, where $\omega = \sum_{i=1}^n \sqrt{-1} dz_i \wedge d\bar{z}_i$. By our assumption (R.2), we know

$$e^{-\psi_L} \equiv \prod_I \left(\phi_I(z) - \log \big(\prod_{i \in I} |z_i|^{2k_i} \big) \right)^{b_I}$$

modulo $\mathcal{C}^\infty$ function. Now, for the $\bar{\partial}\tau_I$, we have

$$\begin{aligned} |\bar{\partial}\tau_I|_g^2 e^{-\psi_L} dV_g = & \sum_{i=1}^r \left| \frac{\partial f_I}{\partial \bar{z}_i} \right|^2 \frac{|z_i|^2 \log^2 |z_i|^2 e^{-\psi_L}}{\prod_{\alpha=p+1}^r |z_\alpha|^2 \log^2 |z_\alpha|^2} d\lambda \\ & + \sum_{i=r+1}^n \left| \frac{\partial f_I}{\partial \bar{z}_i} \right|^2 \frac{e^{-\psi_L}}{\prod_{\alpha=p+1}^r |z_\alpha|^2 \log^2 |z_\alpha|^2} d\lambda. \end{aligned} \tag{19}$$

To prove the claim, we can reduce it to the one-dimensional case by Fubini's theorem, i.e., unit disk $\mathbb{D}$ in complex plane $\mathbb{C}$. Combining (18) and (19), it is easy to see that the next two inequalities are all we need. These are

$$\int_{\mathbb{D}} |f|^2 (-\log |z|)^b d\lambda(z) \leq C \int_{\mathbb{D}} \left| \frac{\partial f}{\partial \bar{z}} \right|^2 |z|^2 (-\log |z|)^{b+2} d\lambda(z), \tag{20}$$

as well as

$$\int_{\mathbb{D}} |f|^2 \frac{(-\log |z|)^b d\lambda(z)}{|z|^2 \log^2 |z|} \leq C \int_{\mathbb{D}} \left| \frac{\partial f}{\partial \bar{z}} \right|^2 (-\log |z|)^b d\lambda(z). \tag{21}$$

Here, b is positive real number, and we can arrange f has small compact support, i.e., $\mathrm{supp} f \subset \{z \mid 0 < |z| < \epsilon\}$, where ϵ is small enough. Furthermore, it is better to reduce this integration to real line $\mathbb{R}$, and hence we consider the Fourier series

$$f = \sum_{k \in \mathbb{Z}} a_k(t) e^{\sqrt{-1} k\theta}$$

of f, and, thus, we have

$$\frac{\partial f}{\bar{\partial} z} = \sum_{k \in \mathbb{Z}} \left(a_k'(t) - \frac{k}{t} a_k(t) \right) e^{\sqrt{-1} k\theta}.$$

The identity

$$t^k \frac{d}{dt}\left(\frac{a_k}{t^k}\right) = a_k'(t) - \frac{k}{t} a_k(t)$$

reduces the proof of our statement to the next inequalities. We have

$$\int_{\mathbb{D}} |f|^2 (-\log|z|)^b d\lambda(z) = \sum_k \int_0^1 |a_k(t)|^2 (-\log t)^b t dt$$

and

$$\int_{\mathbb{D}} \left|\frac{\partial f}{\partial \bar z}\right|^2 |z|^2 (-\log|z|)^{b+2} d\lambda(z) = \sum_k \int_0^1 |t^k \frac{d}{dt}(\frac{a_k}{t^k})|^2 t^3 (-\log t)^{b+2} dt.$$

Let $b_k = \frac{a_k}{t^k}$; then we turn the above two equalities into

$$\int_{\mathbb{D}} |f|^2 (-\log|z|)^b d\lambda(z) = \sum_k \int_0^1 |b_k(t)|^2 (-\log t)^b t^{2k+1} dt$$

and

$$\int_{\mathbb{D}} \left|\frac{\partial f}{\partial \bar z}\right|^2 |z|^2 (-\log|z|)^{b+2} d\lambda(z) = \sum_k \int_0^1 |b_k'(t)|^2 t^{2k+3} (-\log t)^{b+2} dt.$$

So the inequality (20) is equivalent to the next inequality

$$\sum_k \int_0^1 |b_k(t)|^2 (-\log t)^b t^{2k+1} dt \leq C \sum_k \int_0^1 |b_k'(t)|^2 t^{2k+3} (-\log t)^{b+2} dt. \tag{22}$$

Here, $k \in \mathbb{Z}, b \in \mathbb{R}^+$, and $b_k(t)$ has small compact support. In fact, the inequality (21) also comes from (22). Now, we prove the inequality (22); according to integration by part, we have

$$\begin{aligned}\int_0^1 |b_k(t)|^2 (-\log t)^b t^{2k+1} dt &= -\int_0^1 |b_k(t)|^2 (-\log t)^{b+2} t^{2k+2} d\left(\frac{1}{\log t}\right)\\ &= \int_0^1 \frac{1}{\log t} d(|b_k|^2 t^{2k+2} (-\log t)^{b+2}).\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\int_0^1 |b_k(t)|^2 (-\log t)^b t^{2k+1} dt\\ &= -\int_0^1 (b_k' \bar b_k + b_k \bar b_k') t^{2k+2} (-\log t)^{b+1} dt\\ &\quad + \int_0^1 \frac{|b_k|^2 t^{2k+1}}{\log t} \big[(2k+2)(-\log t)^{b+2} - (b+2)(-\log t)^{b+1}\big] dt.\end{aligned} \tag{23}$$

Now we divide it into three cases, with the sign of $(2k+2)$:

(1) if $2k+2=0$, then the second term in (23) above is

$$\int_0^1 (b+2)|b_k|^2 t^{2k+1}(-\log t)^b dt.$$

Thus, we have

$$\begin{aligned}\int_0^1 (b+1)|b_k|^2 t^{2k+1}(-\log t)^b &= \int_0^1 (b_k'\bar{b}_k + b_k\bar{b}_k')t^{2k+2}(-\log t)^{b+1}dt \\ &\leq \sqrt{\int_0^1 |b_k|^2 t^{2k+1}(-\log t)^b dt}\sqrt{\int_0^1 |b_k'|^2 t^{2k+3}(-\log t)^{b+2}dt}.\end{aligned} \tag{24}$$

Therefore, we have the desired inequality

$$\int_0^1 |b_k|^2 t^{2k+1}(-\log t)^b dt \leq C\int_0^1 |b_k'|^2 t^{2k+3}(-\log t)^{b+2}dt.$$

(2) if $2k+2>0$, when t is small enough, we have

$$(2k+2)(-\log t)^{b+2} - (b+2)(-\log t)^{b+1} \geq (b+2)(-\log t)^{b+1}.$$

Then the second term in (23) is smaller than

$$\begin{aligned}&\int_0^1 \frac{|b_k|^2 t^{2k+1}}{\log t}\left[(2k+2)(-\log t)^{b+2} - (b+2)(-\log t)^{b+1}\right]dt \\ &\leq -(b+2)\int_0^1 |b_k|^2 t^{2k+1}(-\log t)^b.\end{aligned}$$

A similar calculation yields that inequality (22) holds.

(3) if $2k+2<0$, and we also make t small enough, we have

$$(2k+2)(-\log t)^{b+2} - (b+2)(-\log t)^{b+1} \leq -(b+2)(-\log t)^{b+1}.$$

Hence the second term in (23) is bigger than

$$\begin{aligned}&\int_0^1 \frac{|b_k|^2 t^{2k+1}}{\log t}\left[(2k+2)(-\log t)^{b+2} - (b+2)(-\log t)^{b+1}\right]dt \\ &\geq (b+2)\int_0^1 |b_k|^2 t^{2k+1}(-\log t)^b.\end{aligned}$$

Inequality (23) now becomes

$$\begin{aligned}&\int_0^1 |b_k|^2 t^{2k+1}(-\log t)^b dt \\ &\geq -\int_0^1 (b_k'\bar{b}_k + b_k\bar{b}_k')t^{2k+2}(-\log t)^{b+1}dt + \int_0^1 (b+2)|b_k|^2 t^{2k+1}(-\log t)^b dt.\end{aligned}$$

By Cauchy inequality as (24) above, we can also easily deduce that inequality (22) is valid in this case.

In summary, the proof of the proposition is completed. $\square$

Once the local Poincaré inequality has been established, as a corollary, we can obtain the following global Poincaré inequality for $\bar\partial$. In [4], the authors introduce for any integer $0 \leq p \leq n$ the space

$$H^{(p)} := \Big\{F \in H^0(X^\circ, \Omega^p_{X^\circ} \otimes L) \cap L^2;\ \int_{X^\circ} \langle A \star F, \star F\rangle dV_{\omega_E} = 0\Big\}, \tag{25}$$

and we can observe by Bochner formula that for an L^2 integrable, L-valued $(p,0)$-form F, one has

$$\begin{aligned}\Delta''(\star F) = 0 &\Longleftrightarrow \Delta'(\star F) = 0 \text{ and } \int_{X^\circ} \langle A \star F, \star F\rangle dV_{\omega_E} = 0 \\ &\Longleftrightarrow F \in H^{(p)}.\end{aligned} \tag{26}$$

Proposition 4.5 ([4, Proposition 3.9]). — *Let $p \leq n$ be an integer. There exists a positive constant $C > 0$ such that the following inequality holds*

$$\int_{X^\circ} |u|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} \leq C \left(\int_{X^\circ} |\bar\partial u|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} + \int_{X^\circ} \langle A \star u, \star u\rangle dV_{\omega_E}\right) \tag{27}$$

for any L-valued form u of type $(p,0)$, which belongs to the domain of $\bar\partial$ and which is orthogonal to the space $H^{(p)}$ defined by (25)*. Here, $\star$ is the Hodge star operator with respect to the metric ω_E.*

We have the following direct consequences of Proposition 4.5.

Corollary 4.6 ([4, Proposition 3.10]). — *There exists a positive constant $C > 0$ such that the following inequality holds*

$$\int_{X^\circ} |u|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} \leq C \left(\int_{X^\circ} |\bar\partial u|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E}\right) \tag{28}$$

for any L-valued form u of type $(n,0)$, which belongs to the domain of $\bar\partial$ and which is orthogonal to the kernel of $\bar\partial$.

Proof. — This follows immediately from Proposition 4.5 combined with the observation that the curvature operator A is equal to zero in bi-degree $(n,0)$. □

The next statement shows that in bi-degree (n,p), the image of the operator $\bar\partial^\star$ is closed.

Corollary 4.7 ([4, Proposition 3.11]). — *There exists a positive constant $C > 0$ such that the following holds. Let v be an L-valued form of type (n,p). We assume that v is L^2, in the domain of $\bar\partial$ and orthogonal to the kernel of the operator $\bar\partial^\star$. Then, we have*

$$\int_{X^\circ} |v|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} \leq C \int_{X^\circ} |\bar\partial^\star v|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E}. \tag{29}$$

Proof. — Let us first observe that the Hodge star $u := \star v$, of type $(n-p, 0)$, is orthogonal to $H^{(n-p)}$. Actually, let us pick $F \in H^{(n-p)}$. It follows from (26) that we have $\bar{\partial}^\star(\star F) = 0$. In other words, $\star F \in \ker \bar{\partial}^\star$. We thus have

$$\int_{X^\circ} \langle u, F\rangle dV_{\omega_E} = \int_{X^\circ} \langle v, \star F\rangle dV_{\omega_E} = 0.$$

Applying Bochner's formula (16) to v and using the facts that $\bar{\partial} v = 0$ (since v is orthogonal to $\ker \bar{\partial}^*$) and that $\bar{\partial}^* u = 0$ for degree reasons, we get

$$\|\bar{\partial}^* v\|^2_{L^2} = \|\bar{\partial} u\|^2_{L^2} + \int_{X^\circ} \langle A \star u, \star u\rangle dV_{\omega_E}. \tag{30}$$

This proves the corollary by applying Proposition 4.5. □

Now we can prove Theorem 4.2.

Proof of Theorem 4.2. — In the context of complete manifolds, one has the following decomposition

$$L^2_{n,1}(X^\circ, L) = \mathcal{H}_{n,1}(X^\circ, L) \oplus \overline{\mathrm{Im}\bar{\partial}} \oplus \overline{\mathrm{Im}\bar{\partial}^\star}.$$

The reader can consult [7, chapter VIII, pages 367-370] for details. We also know that the adjoint $\bar{\partial}^\star$ and $D'^\star$ in the sense of von Neumann coincide with the formal adjoint of $\bar{\partial}$ and D', respectively. It remains to show that the range of the $\bar{\partial}$ and $\bar{\partial}^\star$ operators are closed with respect to L^2 topology. We will utilize the inequalities (28) and (29); the former shows that the image of $\bar{\partial}$ is closed, and the latter does the same for $\bar{\partial}^\star$. Indeed, suppose that there exists a sequence $\bar{\partial} u_j \to u$ in the L^2 space. Since $\bar{\partial} u_j = \bar{\partial}(u_j^1 + u_j^2) = \bar{\partial} u_j^1$ (here, we set $u_j^1 \perp \ker \bar{\partial}$ and $u_j^2 \in \ker \bar{\partial}$), we can assume each u_j is orthogonal to the kernel space of $\bar{\partial}$. By (28), we obtain

$$\int_{X^\circ} |u_j - u_k|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} \leq C \left(\int_{X^\circ} |\bar{\partial} u_j - \bar{\partial} u_k|^2_{\omega_E} e^{-\psi_L} dV_{\omega_E} \right),$$

and, therefore, u_j is a Cauchy sequence. On the other hand, we know $\bar{\partial}$ is a closed operator. This yields that $u \in \mathrm{Im}\bar{\partial}$. Similarly, we can prove that the image of $\bar{\partial}^*$ is closed too. □

Acknowledgement. — The author would like to express his gratitude to Prof. Shigeharu Takayama for his guidance and warm encouragement. The author is also grateful to Prof. Junyan Cao for his comments on this manuscript.

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