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ON THE STABLE NORM OF SLIT TORI AND THE FAREY SEQUENCE

by Pablo Montealegre

Abstract. — Let M be a compact manifold endowed with a possibly singular Riemannian metric. The metric induces a norm on the homology of M, called the stable norm. We provide explicit computations of the stable norm of flat slit tori using the Farey sequence. We then glue several slit tori together to produce half-translation surfaces whose unit ball of the stable norm has faces of maximal dimension. Furthermore, we give a subquadratic estimate for the asymptotic counting of simple homology classes on these surfaces.

Résumé (*Sur la norme stable des tores fendus et la suite de Farey*). — Soit M une variété compacte équipée d'une métrique riemannienne que l'on autorise à être singulière. Cette métrique induit une norme sur l'homologie de M, appelée la norme stable. Nous calculons explicitement la norme stable de tores plats fendus à l'aide de la suite de Farey. Ensuite, nous recollons plusieurs tores fendus ensemble pour construire des surfaces de demi-translation dont la boule unité de la norme stable a des faces de dimension maximale. Enfin, nous calculons une estimation asymptotique sous-quadratique du nombre de classes d'homologies simples sur ces surfaces.

Introduction

The stable norm. — Let M be a connected compact manifold endowed with a smooth Riemannian metric. The metric induces a norm on the first homology

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group $H_1(M,\mathbb{R})$, named the stable norm by Gromov in [10]. More precisely, following Massart in [15], let $h \in H_1(M,\mathbb{Z})$ be an integral homology class. We define

$$f(h) := \inf_{\gamma} l(\gamma),$$

where the infimum is taken over all the closed rectifiable multicurves γ on M representing the class h, and where $l(\gamma)$ denotes the length of γ. According to [15, Proposition 1.1.3] the limit

$$\lim_{N\to\infty} \frac{f(Nh)}{N}$$

exists for any integral class h; we denote this limit $\|h\|_M$. One can check that for any two integral homology classes h_1 and h_2, the triangle inequality $\|h_1 + h_2\|_M \leq \|h_1\|_M + \|h_2\|_M$ holds. Moreover, for any $N \in \mathbb{Z}$ we have $\|Nh\|_M = |N|\|h\|_M$ so by homogeneity we can naturally extend the function $h \mapsto \|h\|_M$ to the rational homology classes $H_1(M,\mathbb{Q})$. Finally, by a density argument we can extend this function to the whole real homology $H_1(M,\mathbb{R})$, and [15, Proposition 1.1.5] tells us that the extension is a norm.

To this day, little is known about the stable norm of Riemannian manifolds. In particular, there are very few known explicit examples; see Babenko [2] on flat two-dimensional tori, Burago–Ivanov–Kleiner [6] for flat n-dimensional tori, or McShane–Rivin [20] on any punctured hyperbolic torus. The definition of the stable norm remains valid for Riemannian metrics with conical singularities such as translation and half-translation surfaces. These surfaces enjoy very nice geometrical properties; in particular, they are endowed with a flat metric with a finite number of conical singularities (see for instance Zorich's survey [24]) whose cone angles are all integral multiples of π. This greatly simplifies the task of finding geodesics and computing their length, so one can hope to be able to compute explicitly the stable norm on some of these surfaces. Unfortunately, it is still a difficult problem, so the idea of the present paper is to compute the stable norm on flat slit tori that we can later on glue together to obtain translation and half-translation surfaces. This is not new. Since slit tori are the building blocks of surfaces of main interest in modern geometry and dynamical systems they are often a preferential testing ground; see for example [8] or [22].

The slit torus. — A flat slit torus is a compact surface that is homeomorphic to a torus with one boundary component, equipped with a flat Riemannian metric relative to which the boundary of the surface is made of two parallel straight line segments of the same length ρ. Formally, the boundary of the surface can be written as a union $S_1 \cup S_2$, where S_1 and S_2 are homeomorphic to a segment, and $S_1 \cap S_2$ is equal to two distinct points $\{P, P'\}$, such that there exists two local charts φ_1 and φ_2 that map isometrically S_1 and S_2 to the *same* segment $[(0,0),(0,\rho)] \subset \mathbb{R}^2$ of length ρ. More precisely, we have

$\varphi_1(S_1) = \varphi_2(S_2) = [(0,0),(0,\rho)]$, with $\varphi_1(P) = \varphi_2(P) = (0,0)$ and $\varphi_1(P') = \varphi_2(P') = (0,\rho)$. The boundary of the surface is called the slit, the two segments S_1, S_2 that compose the slit are called the sides of the slit, and their common endpoints P, P' are the endpoints of the slit.

Throughout this article we will be interested in *the* flat slit torus, denoted X, whose metric comes from a *square* of area 1 (i.e., $X\backslash\{S_1, S_2\}$ is isometric to an open subset of a flat square torus) and whose slit of length $0 < \rho < 1$ is *vertical*, as these assumptions make the computations much easier. We fix an orientation on X so that it makes sense to talk about the left and right-hand sides of the slit and the upper and lower endpoint of the slit. Informally, the surface X can be visualized with the following polygonal model. Take a square of area 1 and cut it open along a vertical open interval of length ρ, then glue the opposites edges of the square by translation. Finally, glue one segment to each side of the cut part, so that the final surface is a compact genus 1 surface with boundary homeomorphic to a circle. The metric on the surface then simply comes from the canonical Euclidean metric on the square.

Since the slit torus is homotopy equivalent to the wedge sum of two circles, we have $\pi_1(X) \cong \mathbb{F}_2$, where $\mathbb{F}_2 = \mathbb{F}(s,t) = \langle s,t\rangle$ is the free group on the two generators s and t. More importantly, we also have $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^2$. From now on we will identify the elements h of $H_1(X,\mathbb{Z})$ with the integer couples (m,n) in $\mathbb{Z}^2$.

The Farey sequence. — Our first goal is to compute explicitly the stable norm of any given homology class on X. We also want to describe the structure of the unit ball $\mathcal{B}$ of the stable norm of X, which is a convex set in $\mathbb{R}^2$; more precisely, we want to know whether there are any segments contained in its boundary $\partial\mathcal{B}$, what its extreme points are, if there are any vertices, etc. It turns out that the stable norm of an integral homology class $(m,n) \in H_1(X,\mathbb{Z})$ depends on arithmetic properties of the rational number n/m. More precisely, it depends on its position in the Farey sequence.

The *Farey sequence of order* $k \in \mathbb{N}^*$ is the ordered sequence $\mathcal{F}_k$ of irreducible fractions in $\mathbb{Q}$ whose denominator is less than or equal to k. For example,

$$\mathcal{F}_1 = \left\{\dots, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \dots\right\} \text{ and } \mathcal{F}_2 = \left\{\dots, \frac{-1}{1}, \frac{-1}{2}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \dots\right\}.$$

The Farey sequence is omnipresent in mathematics, mainly due to its remarkable combinatorial properties. Here are the main properties that we will need, which can be found in [12]. Let $a/b < c/d$ be two irreducible fractions that are consecutive in some Farey sequence $\mathcal{F}_k$ of order k. These fractions are called a Farey pair and are said to be Farey neighbours. They satisfy the following properties:

1. $bc - ad = 1$. Note that this property is actually equivalent to being Farey neighbours.

2. If p/q is an irreducible fraction such that a/b, p/q, and c/d are consecutive in some Farey sequence, then
$$\frac{p}{q} = \frac{a+c}{b+d}.$$
We denote by $p/q = a/b \oplus c/d$ and we say that p/q is the mediant of a/b and c/d.

Thus if a/b and c/d are Farey neighbours, the first fraction p/q that appears between them as the order of the sequence increases is their mediant; we say that p/q is the *Farey child* of a/b and c/d, and that a/b and c/d are the *Farey parents* of p/q.

3. Let $p/q = a/b \oplus c/d$ be an irreducible fraction, with $a/b < c/d$ its Farey parents. The Farey parents are the best rational approximations of p/q with denominator less than q. More precisely, $a/b < p/q < c/d$, and there are no other rational number than p/q with denominator less than q in the interval $[a/b, c/d]$. In other words, the last two convergents in the continued fraction expansion of p/q are its Farey parents.

Note that if a/b and c/d are the Farey parents of p/q, then the mediant $a/b \oplus p/q$ is also a Farey child of a/b. In fact, the children of a/b are all the rationals one of whose Farey parents is a/b; this is of great importance for our main result. Interestingly, the integers are the Farey ancestors of all the rationals numbers; indeed, starting from the integers $n/1$ and taking the successive mediants one can obtain any rational number.

Main result. — The main result of this paper is the explicit computation of the stable norm of the slit torus X, along with a complete description of the unit ball of the stable norm of X.

Theorem A. — *Let X be the square flat torus with a vertical slit of length ρ. Let $L = \lfloor 1/\rho \rfloor$ be the integral part of $1/\rho$. Let $h = (m, n) \in H_1(X, \mathbb{Z})$ be a primitive integral homology class and let a/b and c/d be the Farey parents of n/m, with $a/b < c/d$. The unit ball $\mathcal{B}$ of the stable norm of X has a vertex in the direction h if and only if either*

- $\frac{n}{m} \in \mathcal{F}_L$, *that is to say $1 \le |m| \le L$. In that case*
$$\|(m, n)\|_X = \sqrt{m^2 + n^2},$$

or

- $\frac{n}{m}$ *does not belong to $\mathcal{F}_L$, but $\frac{a}{b} \in \mathcal{F}_L$ or $\frac{c}{d} \in \mathcal{F}_L$. In that case, we have*
$$\|(m, n)\|_X = \sqrt{b^2 + (a+\rho)^2} + \sqrt{d^2 + (c-\rho)^2}.$$

Moreover, the stable norm is strictly convex in the directions $\pm(0, 1)$, and $\|(0, 1)\|_X = 1$. Finally, the point of the unit sphere $\partial\mathcal{B}$ of the stable norm *in every other direction of* $H_1(X,\mathbb{R})$ *lies in the interior of a segment. More precisely, if* $|m|, |b|, |d| > L$*, we have* $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.

Note that since the unit ball $\mathcal{B}$ of the stable norm has infinitely many vertices it is not a polygon, although it very much looks like one at first glance. Indeed, the angles at the vertices are so close to being flat that $\mathcal{B}$ only seems to have a finite number of vertices. Sections 1 to 5 are devoted to proving this theorem.

Stable norm of a half-translation surface. — A question one might ask is which convex bodies can be realized as the unit ball of the stable norm of a manifold. In dimension $n \geq 3$, Babenko and Balacheff [3] showed that given a closed smooth manifold M, any centrally symmetric polytope with vertices in rational direction can be realized as the unit ball of the stable norm of a Riemannian metric on M. However, in dimension 2, Massart showed in [16] that the dimension of the flats of the unit ball of a closed orientable *smooth* surface of genus g is at most $g-1$; in particular, the unit ball of the stable norm is far from being a polytope. But what about the stable norm of singular metrics? In Section 6, we glue two slit tori X_i, $i = 1, 2$, along a long flat cylinder to obtain a genus 2 half-translation surface Σ on which we are able to compute the stable norm thanks to Theorem A.

Theorem B. — *The unit ball of the stable norm of* Σ *is the convex hull of the set*

$$\left\{ \frac{(m,n,0,0)}{\|(m,n,0,0\|_\Sigma} \text{ with } (m,n) \in V(X_1) \right\} \cup \left\{ \frac{(0,0,p,q)}{\|(0,0,p,q)\|_\Sigma} \text{ with } (p,q) \in V(X_2) \right\},$$

where $V(X_i)$ *is the set of directions of vertices of the stable norm of* X_i.

In particular, the unit ball of the stable norm of Σ has faces of dimension 3, which is maximal since $H_1(\Sigma,\mathbb{R})$ has dimension 4. We will then use a similar construction to obtain a half-translation surface of genus g whose flats of the unit ball of its stable norm are of maximal dimension, that is $2g-1$. Again, the unit ball of the stable norm of Σ is not a polytope as it has infinitely many vertices, but it looks a lot like one.

Asymptotic growth of the number of simple homology classes. — The classical simple curve counting problem on surfaces is the following: how many *simple* closed curves of length less than a positive number x on a given surface are there? This question, along with many related problems, has seen significant progress during the last two decades. Now, one might ask which homology classes these curves represent. We say that an integral homology class on a (possibly singular) Riemannian surface S is *simple* if its stable norm is realized by the length of a simple closed curve. We then ask the following question: how many are there simple homology classes of stable norm less than x on S?

In dimension 2, it is known (see Balacheff–Massart [4]) that it is possible to find closed non-orientable surfaces with only a finite number of simple homology classes. On the other hand, Babenko [2] showed that on flat two-dimensional tori there are infinitely many simple homology classes, as in the case of the punctured hyperbolic torus (McShane–Rivin [20]). Thus, can we find asymptotic estimates? Gutkin and Massart [11] showed, based on previous work by Masur [18], [19] on the growth rate of closed orbits for quadratic differentials, that the number of simple homology classes of stable norm less than x on translation surfaces grows quadratically in x. Since any half-translation surface is doubly covered by a translation surface, these two classes of surfaces share many geometrical properties. So, is the asymptotic growth of simple homology classes on half-translation surfaces quadratic? In Section 7 we use the example of the half-translation surface Σ constructed in Section 6 to show that the answer to this question is negative.

THEOREM C. — *Let $p(x)$ be the number of simple homology classes on Σ whose stable norm is less than $x \geq 0$. Then*

$$p(x) = 8\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b}\right) x \ln x + O(x),$$

where φ is Euler's totient function.

We would like to point out that this result agrees with [17, Conjecture 1.2]. Roughly speaking, this conjecture states that for any closed orientable surface M of genus 2 equipped with a reasonable Finsler metric, the number $p(x)$ of simple homology classes of M with stable norm less than x satisfies

$$\lim_{x\to\infty} \frac{p(x)}{x^2} = \Omega,$$

where Ω is the area of the closure of the set of vertices of the unit ball of the stable norm of M. In the case of our surface Σ, the closure of the set of vertices of the unit ball has measure zero, and Theorem C confirms that $p(x)$ is indeed subquadratic.

1. Preliminaries

The Abelian covering space of the slit torus. — Recall that X denotes the square torus with a vertical slit of length $\rho \in]0,1[$. Let $p : \tilde{X} \longrightarrow X$ be the Abelian covering space of X, i.e., the normal cover of X whose transformation group is isomorphic to $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^2$. One convenient way to picture $\tilde{X}$ is the Euclidean plane where a slit is made along $\mathbb{Z}^2$-translates of the vertical segment $[(0,0),(0,\rho)]$ of length ρ, with a similar definition as for the slit of X. This way, the local isometry between $\tilde{X}$ and X is the obvious projection.

A more practical way of computing the stable norm of X. — Let $h \in H_1(X, \mathbb{Z})$ be an integral homology class. By definition, the stable norm $\|h\|_X$ is the limit of the sequence $f(Nh)/N$ as N goes to infinity, where

$$f(h) = \inf_{\gamma}\{l(\gamma), \text{ where } \gamma \text{ is a closed multicurve such that } [\gamma] = h\}.$$

This is not very practical as the limit could be difficult to handle; however, in our case this is not as bad as it seems.

Proposition 1.1. — *The stable norm coincides with the functional f on integral homology classes. More precisely, if $h = (m, n) \in H_1(X, \mathbb{Z})$ is an integral homology class on X then $\|h\|_X = f(h)$. Moreover, we have $f(h) = \bar{f}(h) = \tilde{f}(h)$, where*

$$\bar{f}(h) := \inf_{\gamma}\{l(\gamma), \gamma \subset X \text{ closed loop based at the bottom of the slit such that } [\gamma] = h\}$$

and

$$\tilde{f}(h) := \inf_{\tilde{\gamma}}\{l(\tilde{\gamma}), \tilde{\gamma} \text{ path from } (0,0) \text{ to } (m,n) \text{ in } \tilde{X}\}.$$

Depending on the context, throughout this paper it will sometimes be more convenient to think about the stable norm of h as $\bar{f}(h)$ or as $\tilde{f}(h)$.

Proof. — Let $h = (m, n) \in H_1(X, \mathbb{Z})$ be a primitive homology class of X, i.e., $\gcd(m, n) = 1$, since because of the homogeneity of the stable norm we only need to compute it for primitive integral classes. Let us start with an observation: when computing $f(h)$, it is enough to consider closed curves on X passing through the lower endpoint of the slit (that is, the projection of $(0, 0)$ on X) instead of multicurves. Indeed, if γ is a closed multicurve on X that represents the homology class h then we can translate each connected component of γ that does not already pass through the lower endpoint of the slit upwards until it encounters the lower end of the slit. Since translations preserve the metric on X and act trivially on $H_1(X, \mathbb{Z})$, we then obtain a new representative γ' of h that is connected, passes through the lower endpoint of the slit, and has the same length as γ. Thus, the functional f coincides with $\bar{f}$ as claimed. Moreover, a curve such as γ' can be lifted isometrically to the Abelian cover $\tilde{X}$ in a unique way as a path from the origin $(0, 0)$ to the point (m, n), so $f(h) = \tilde{f}(h)$.

We will now show that the sequence $(f(Nh)/N)_{N\in\mathbb{N}^*}$ is constant, thus

$$\|h\|_X = \lim_{N\longrightarrow\infty} \frac{f(Nh)}{N} = f(h).$$

We reproduce an old argument from Hedlund [14], originally stated on the torus. Let $N \in \mathbb{N}^*$ and let γ be a rectifiable path on $\tilde{X}$ from $(0, 0)$ to (Nm, Nn). We will show by induction of N that γ projects onto X as the union of N closed curves representing h, thus $f(Nh) \geq Nf(h)$, and since the converse inequality

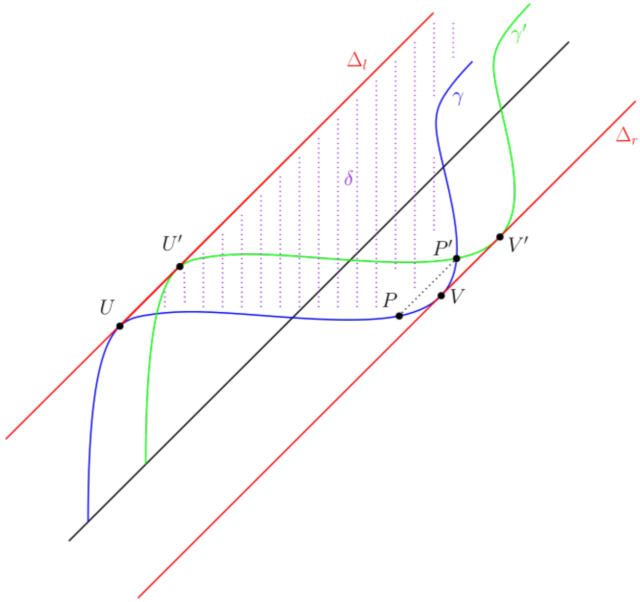

FIGURE 1.1. The geometric construction from the proof

is trivial we will finally get $f(Nh) = Nf(h)$. To do this, we apply the following algorithm.

Let Δ be the line from the origin to (m, n) in $\tilde{X}$ (it might encounter the slit) and let Δ_l (resp. Δ_r) the leftmost (resp. rightmost) parallel to Δ that meets γ. If $\Delta_l = \Delta = \Delta_r$ then $\gamma \subset \Delta$, and there is nothing left to prove, so let us assume that $\Delta_l \neq \Delta_r$. Moreover, to fix notations let us assume that γ hits Δ_l first. Fix a point $U \in \Delta_l \cap \gamma$, and let $U' = T(m, n).U$, where $T(m, n)$ is the translation by the (m, n) vector.

If $U' \in \gamma$ then we are done. Since U and U' project onto the same point in X, the arc of γ from U to U' projects onto a closed curve in X representing $h = (m, n)$. We then remove this arc from γ to obtain two subpaths of γ. By concatenation of the subpath of γ from $(0, 0)$ to U with the subpath from U' to (Nm, Nn) translated by $T(-m, -n)$ we obtain a new (and shorter) path from $(0, 0)$ to $((N-1)m, (N-1)n)$ and we conclude by induction.

If U' does not belong to γ, then U' lies on the boundary of some connected compact region δ, bounded by Δ_l, γ and the line orthogonal to Δ_l passing through (Nm, Nn). Let $V \in \Delta_r \cap \gamma$ and assume that $V' = T(m, n).V$ does not belong to γ (just like before, if $V' \in \gamma$ then we are done). See the figure below. By construction, V' lies in the exterior of δ; the only way it could belong to δ would be if it lied on γ, which we assumed is not the case. Consider the path

$\gamma' = T(m,n)\gamma$; it is well defined because γ does not encounter the slit as the copies of the slit are all related by integral translations. By construction, the path γ' passes through U' and V', one in the interior of δ and the other on its exterior; since γ' is continuous, it must cross transversally the boundary of δ at some point P'. Since γ' cannot intersect transversally the line Δ_l, the point P' must lie on γ instead. Thus, the arc of γ from $P := T(-m,-n)P'$ to P' projects onto a closed curve on C representing h. We then refer to the previous case; as before, we remove this arc from γ to get by concatenation a shorter path from $(0,0)$ to $((N-1)m,(N-1)n)$ and we conclude by induction.

This process is repeated N times until we have decomposed γ as N arcs, each projecting onto representatives of h in X. □

Thus $\|.\|_X$ coincides on $H_1(X,\mathbb{Z})$ with the functional $\bar{f}$ defined as the shortest length of a loop in the corresponding homology class based at the bottom of the slit. As we have already mentioned, since a closed curve on X and its lift to $\tilde{X}$ have the same length, computing $\|(m,n)\|_X$ comes down to finding short paths in $\tilde{X}$ from the origin to (m,n). This is a path-finding problem in a grid with vertical obstacles commonly known in computer sciences as an *any-angle path planning* problem, on which there already exist many efficient numerical methods (e.g., see [7]).

Minimizing curves. — We have shown that for an integral homology class $h \in H_1(X,\mathbb{Z})$ we have $\|h\|_X = \bar{f}(h)$, where $\bar{f}(h)$ is the infimum of the lengths of closed loops in X based at the bottom of the slit that represent h. But since the set of closed curves in X parametrized by arc length whose length does not exceed a fixed constant $C > 0$ is compact, a classical Arzela–Ascoli argument yields that the infimum in the definition of f is in fact a minimum. Thus if γ is a closed curve in X that represents h and realizes the minimum, i.e., if $l(\gamma) = \|h\|_X$, we say that γ is a *minimizing curve* for h, or *minimizing* for short. Such a minimizing representative might not be unique.

Symmetry of the problem. — Our goal is to compute the stable norm of an integral homology class $(m,n) \in H_1(X,\mathbb{Z})$. We claim that we can assume without loss of generality that $m,n \geq 0$. Indeed, since the stable norm is a norm we have $\|(m,n)\|_X = \|(-m,-n)\|_X$, so the stable norm is centrally symmetric with respect to the origin in $H_1(X,\mathbb{Z})$. Moreover, since the metric on the slit torus X (and on its Abelian covering space $\tilde{X}$) is symmetric with respect to vertical reflections we have $\|(m,n)\|_X = \|(-m,n)\|_X$, so the stable norm is symmetric with respect to the vertical axis in $H_1(X,\mathbb{Z})$. Composing those two symmetries we obtain the announced claim; thus from now on, we will always assume that $m,n \geq 0$ when doing computations.

Other homology classes. — So far we have only discussed the case of integral homology classes. What about the other ones? It turns out that we can ignore them for now. Indeed, if $h \in H_1(X,\mathbb{Z})$ is an integral homology class then by homogeneity of the norm for any rational number $p/q \in \mathbb{Q}$ we have $\|(p/q).h\|_X = |p/q|\,\|h\|_X$, so we can compute the stable norm on $H_1(X,\mathbb{Q})$ provided that we already know how to compute it on $H_1(X,\mathbb{Z})$. Then by density of $H_1(X,\mathbb{Q})$ inside $H_1(X,\mathbb{R})$ the stable norm of X is completely determined by its value on rational homology classes.

2. Visible homology classes

The trivial case of $(0,1)$. — Since the slit is vertical the geodesics representing $(0,1)$ are parallel to the slit, they never intersect it. Thus any vertical geodesic on X is a minimizing representative of $(0,1)$, so $\|(0,1)\|_X = 1$. From now on we will assume $m > 0$.

Straight line path. — Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be an integral homology class on X and assume that h is primitive, i.e., $\gcd(m,n) = 1$. We want to compute its stable norm $\|h\|_X$. By the previous section this comes down to finding the length of the shortest path from $(0,0)$ to (m,n) in the Abelian cover $\tilde{X}$. The obvious candidate for being such a path is the straight line segment $[(0,0),(m,n)]$; if it does not intersect the slit then it projects onto a closed curve on X that is minimizing for h. So, does it intersect the slit?

The segment $[(0,0),(m,n)]$ satisfies the equation $my = nx$, with $x \in [0,m]$. It crosses the vertical of the slits at the points $(k, kn/m)$, $k = 1,\dots,m-1$. So those points are where an intersection with the slit may occur. Recall that the slit has length ρ. Since the slits are the open segments $\{(p, q+s)$ with $s \in]0,\rho[\}$ with $(p,q) \in \mathbb{Z}^2$, we have to check how the fractional part $\{kn/m\}$ of kn/m compares with ρ for all $k = 1,\dots,m-1$. More precisely, there is an intersection with the slit at the vertical $x = k$ if and only if

$$\left\{k\frac{n}{m}\right\} < \rho.$$

Note that $\{kn/m\} = r/m$, where r is the remainder in the Euclidean division of kn by m, i.e., $kn = r$ in $\mathbb{Z}/m\mathbb{Z}$. Since m and n are coprime n is invertible in $\mathbb{Z}/m\mathbb{Z}$, and multiplication by n is an automorphism of $\mathbb{Z}/m\mathbb{Z}$, so r takes all the possible values $1,2,\dots,m-1$ when k varies from 1 to $m-1$. Hence, the smallest value $\{kn/m\}$ takes for k running from 1 to $m-1$ is always $1/m$. Now,

$$\frac{1}{m} < \rho \iff m\rho > 1.$$

So if $m\rho \le 1$, the segment $[(0,0),(m,n)]$ does not encounter the slit and projects onto a minimizing representative of (m,n) on X of length $\sqrt{m^2+n^2}$. If instead we assume $m < 0$ at the beginning the condition becomes $-m\rho \le 1$. Thus, we have shown the following.

Proposition 2.1. — *Let $h = (m, n) \in H_1(X, \mathbb{Z})$ be a primitive integral homology class. Let ρ be the length of the slit. The stable norm coincides with the Euclidean norm in the direction (m, n) if and only if $|m|\rho \leq 1$. Moreover, if $|m|\rho > 1$ then $\|(m, n)\|_X > \|(m, n)\|_2$.*

By the previous proposition, if $|m|\rho \leq 1$ then a light ray emitted from $(0, 0)$ and travelling in a straight line would illuminate the point (m, n) on $\tilde{X}$. Hence the following terminology, borrowed from [12]: we say that a primitive integral homology class (m, n) is *visible* if $|m|\rho \leq 1$, or equivalently if its stable norm coincides with the Euclidean norm. A visible direction in $H_1(X, \mathbb{R})$ is the direction of a visible homology class. Note that a primitive homology class (m, n) with $m \neq 0$ is visible if and only if n/m belongs to $\mathcal{F}_L$ the Farey sequence of order L, where $L = \lfloor 1/\rho \rfloor$ is the integral part of $1/\rho$.

There are infinitely many visible directions. Indeed, since $\rho < 1$ the classes of the form $(1, n)$ are always visible, whatever value ρ may take. This is simply because the segment from the origin to $(1, n)$ can never intersect any of the slits, as all the slits are contained in the vertical lines $x = k$ with $k \in \mathbb{Z}$, and the segment never crosses such a line.

Strictly convex directions of the stable norm. — We get immediate information on the structure of the unit ball of the stable norm.

Corollary 2.2. — *Let $\mathcal{B}$ be the unit ball of the stable norm of X and let $\mathbb{D}$ be the unit Euclidean disk in $\mathbb{R}^2$, with boundary $\mathbb{S}^1$. Then $\mathcal{B} \subset \mathbb{D}$ and $\partial\mathcal{B} \cap \mathbb{S}^1$ is non-empty, with intersections occurring exactly in the visible directions.*

Proof. — The only limit points of slopes of visible directions are $\pm 1/0$, associated to the vertical classes $\pm(0, 1)$. In particular, an irrational number cannot be a limit of elements belonging in the Farey sequence $\mathcal{F}_L$ of order $L = \lfloor 1/\rho \rfloor$, so the unit sphere $\partial\mathcal{B}$ of the stable norm cannot intersect $\mathbb{S}^1$ in a direction with irrational slope. □

The unit ball of the stable norm $\mathcal{B}$ is a convex set of the plane, symmetric with respect to the origin. We say that $\mathcal{B}$ has *a flat* in a direction (m, n) if the unit vector $(m, n)/\|(m, n)\|_X$ lies in the interior of a segment contained in the boundary $\partial\mathcal{B}$ of the unit ball. If $\mathcal{B}$ does not have a flat in the direction (m, n) we say that $\mathcal{B}$ is strictly convex in the direction (m, n), and the corresponding point on the boundary of $\mathcal{B}$ is called an extreme point. Since the unit circle $\mathbb{S}^1$ is strictly convex in every direction, we get the following.

Corollary 2.3. — *The unit ball of the stable norm is strictly convex in every visible direction.*

3. Shortest non-simple closed curves

Let $h = (m, n) \in H_1(X, \mathbb{Z})$ be a primitive homology class on X. Assume h is not visible, i.e., $|m|\rho > 1$, where ρ is the length of the slit. By Proposition 2.1 we know that a minimizing representative of h cannot be the projection of the straight line segment from $(0, 0)$ to (m, n). To find exactly what a minimizing representative of h looks like, we will search separately for the shortest simple closed curve and for the shortest non-simple closed curve representing h, as those two cases need to be dealt with differently. We will start with the latter, as it is the easier case. Note that when we say simple curves we actually mean *injective* curves; there are no self-intersection points, even if the intersections are not transverse.

Intuition of the link with the Farey sequence. — Here is an informal idea of why the Farey sequence appears in the computation of the stable norm of the slit torus. Given a non-visible homology class (m, n), we want to find the shortest path from the point $(0, 0)$ to the point (m, n) in $\tilde{X}$, but since the class is not visible the straight line path is forbidden as it eventually intersects the slit. Hence we have to deviate from this direction at some point. Since the straight line segment would be the shortest path to the point (m, n) if it did not intersect the slit, we want to deviate from it as little as possible in order to minimize the total length of the path.

Say, for example, that it is possible to go from the origin to (m, n) not in a straight line but in a path made of two straight line segments, with the change of direction occurring at a point $(p, q) \in \mathbb{Z}^2$. This is far from being anecdotal; we will see later that this situation actually happens quite often. The first segment has slope q/p, and the second segment has slope $(n - q)/(m - p)$; the direction (m, n) corresponds to the slope n/m. Here, we notice that

$$\frac{n}{m} = \frac{q}{p} \oplus \frac{n-q}{m-p},$$

the mediant of the slopes of each segment. Thus in some sense "the sum of two directions is their mediant". Since we want our path from the origin to (m, n) to deviate as little as possible from the straight line, the above observation suggests to consider the best rational approximations of n/m, that is to say its Farey parents.

Shortest non-simple curves and Farey parents. — We formalize the above intuition in the homology space of the slit torus. Note that the Farey parents of a rational n/m are not defined if $m = 1$. This is no concern to us as we have pointed out that homology classes of the form $(1, n)$ are always visible, so we already know what their stable norm is. Thus we are now only interested in classes (m, n) with $m \geq 2$. Remember that by symmetry of the problem we assume that $m, n \geq 0$.

PROPOSITION 3.1. — *Let $h = (m, n)$ be a non-visible primitive integral homology class with $m \geq 2$. Let a/b and c/d be the Farey parents of the rational n/m. Then the shortest non-simple closed curve representing h has length $\|(b, a)\|_X + \|(d, c)\|_X$.*

Proof. — Let γ be a non-simple closed curve representing $h = (m, n)$. Since γ has a self-intersection we can see γ as the union of two closed curves γ_1 and γ_2 representing two non-trivial integral homology classes (p, q) and $(m-p, n-q)$ with $p, q \in \mathbb{Z}$. Recall that γ lifts to a path $\tilde{\gamma}$ from $(0, 0)$ to (m, n) in the Abelian covering $\tilde{X}$ of the slit torus, which is the concatenation of a path from $(0, 0)$ to (p, q) and of another path from (p, q) to (m, n).

Moreover, since we are looking for *short* curves representing $h = (m, n)$, we can assume $0 \leq p \leq m$, $0 \leq q \leq n$. Indeed, if it was not the case, then at least one of the classes (p, q) and $(m-p, n-q)$ would have a negative coordinate. This means that $\tilde{\gamma}$ would eventually escape the $m \times n$ box having the segment $[(0, 0), (m, n)]$ as one of its diagonals. Then, cutting away the part of $\tilde{\gamma}$ outside of that box and replacing it by the straight line segment between its exit and re-entry points in that box one would produce a new representative of h that would be *shorter* than $\tilde{\gamma}$.

Now, if the homology classes of γ_1 and γ_2 are fixed, i.e., if (p, q) is fixed, the length of γ is minimal if both the lengths of these two curves are minimal in their homology class, so we get the following inequality:

$$l(\gamma) \geq \|(p, q)\|_X + \|(m-p, n-q)\|_X$$

with equality if and only if γ is the union of two minimizing curves for (p, q) and $(m-p, n-q)$, respectively (each of those possibly translated upwards so it starts at the bottom of the slit). Hence, the minimal length of a non-simple closed curve representing h is

$$\min_{(p,q)} \|(p, q)\|_X + \|(m-p, n-q)\|_X$$

with $0 \leq p \leq m$ and $0 \leq q \leq n$ and (p, q) different from $(0, 0)$ or (m, n).

Let C be the convex hull of

$$\left\{ \frac{(p, q)}{\|(p, q)\|_X} | (p, q) \in [\![0, m]\!] \times [\![0, n]\!] \setminus \{(m, n), (0, 0)\} \right\} \cup (0, 0).$$

Since $\mathcal{B}$ is convex, it contains the convex hull of any of its subsets, so $C \subset \mathcal{B}$. Thus, since $(m, n)/\|(m, n)\|_X$ belongs to $\partial\mathcal{B}$, either $(m, n)/\|(m, n)\|_X$ belongs to ∂C or it lies outside of C. For $(p, q) \in \mathbb{N}^2$ with $p \leq m$ and $q \leq n$ with $(p, q) \neq (0, 0), (m, n)$, denote by $z_{p,q}$ the intersection point of the segment of endpoints $(p, q)/\|(p, q)\|_X$ and $(m-p, n-q)/\|(m-p, n-q)\|_X$ and of the line Δ from the origin passing through (m, n), which is the line of slope n/m.

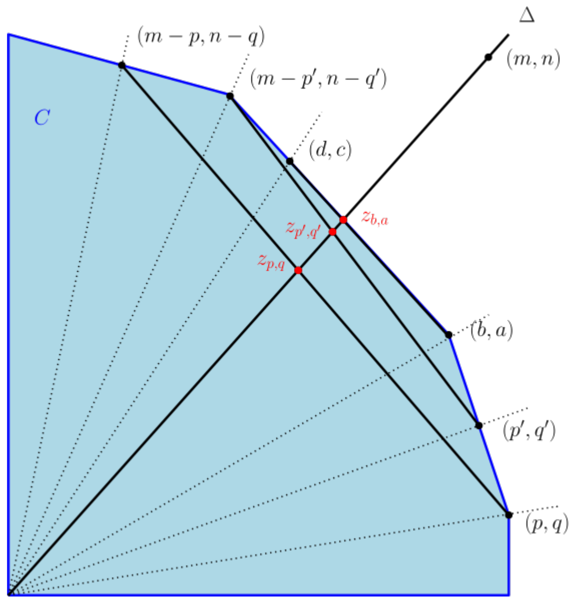

Figure 3.1. The construction from the proof

A direct computation yields

$$z_{p,q} = \frac{(m,n)}{\|(p,q)\|_X + \|(m-p,n-q)\|_X}.$$

Denote by $|z_{p,q}|$ the Euclidean length of the segment from the origin to $z_{p,q}$. By construction,

$$\big(|z_{p,q}| \le |z_{p',q'}|\big) \iff \big(\|(p',q')\|_X + \|(m-p',n-q')\|_X \le \|(p,q)\|_X + \|(m-p,n-q)\|_X\big).$$

Thus finding (p,q) such that $\|(p,q)\|_X + \|(m-p,n-q)\|_X$ is minimal is equivalent to finding he rightmost point of the set $\{z_{a,b}\}_{a,b}$ on Δ, where Δ is oriented positively from the origin to (m,n). Up to exchanging (p,q) and $(m-p,n-q)$ we can assume $q/p < n/m$ (thus $(n-q)/(m-p) > n/m$) and we finally have:

$$|z_{p,q}| \le |z_{p',q'}| \iff \frac{q}{p} \le \frac{q'}{p'}.$$

This means that we have to find the greatest rational $q/p < n/m$ with $p < m$. This is precisely the characterization of a Farey parent of n/m, hence $(p,q) =$

(b,a) and $(m-p,n-q)=(d,c)$. We have shown:

$$\min_{(p,q)} \|(p,q)\|_X + \|(m-p,n-q)\|_X = \|(b,a)\|_X + \|(d,c)\|_X. \qquad \square$$

4. Shortest simple closed curves

Given a non-visible homology class $h \in H_1(X,\mathbb{Z})$ we now want to compute the length of the shortest simple curve that represents h.

Simple closed curves up to free homotopy. — We will need an algebraic result due to Osborne and Zieschang. Recall that $\mathbb{F}_2$ denotes the free group on two elements and that an element $w \in \mathbb{F}_2$ is primitive if there exists $w' \in \mathbb{F}_2$ such that $\langle w, w' \rangle = \mathbb{F}_2$. Equivalently, given a system of generators (s,t) of $\mathbb{F}_2$, an element w is primitive if and only if there exists $\phi \in \mathrm{Aut}(\mathbb{F}_2)$ such that $\phi(s)=w$. Recall that (m,n) is called a primitive element of $\mathbb{Z}^2$ if $\gcd(m,n)=1$; note that even if they are named using the same word, primitive elements of a free group and primitive elements of a lattice have different definitions.

Osborne and Zieschang perform the following geometric construction. Let (s,t) be a basis of $\mathbb{F}_2$. On a square grid, draw the open segment from $(0,0)$ to (m,n). Following this segment from the origin to (m,n), write s (resp. t) whenever it crosses a vertical (resp. horizontal) line of the grid. We obtain a word $V'_{m,n} \in \mathbb{F}_2$ in the two letters s and t. Note that by taking an *open* segment we do not take into account the intersections with the lines of the grid at the points $(0,0)$ and (m,n) when writing this word.

THEOREM 4.1 ([21, Corollary 3.2, Proposition 2.3]). — *Let (m,n) be a primitive element of $\mathbb{Z}^2$ and let $\psi : \mathbb{F}_2 \longrightarrow \mathbb{Z}^2$ be the canonical abelianizing homomorphism. The word $V_{m,n}$ satisfies the following assertions.*

1. *$V_{m,n}$ is primitive in $\mathbb{F}_2$.*
2. *$\psi(V_{m,n}) = (m,n)$.*
3. *If $w \in \mathbb{F}_2$ is primitive and $\psi(w)=(m,n)$ then w and $V_{m,n}$ are conjugate in $\mathbb{F}_2$.*

Thus on the slit torus, where for all $x \in X$, $\pi_1(X,x) \cong \mathbb{F}_2$ and $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^2$, up to conjugacy there is a one-to-one correspondence between primitive homotopy classes and primitive integral homology classes. In other words, since the conjugacy classes of the fundamental group can be identified with free homotopy classes, there is a one-to-one correspondence between primitive free homotopy classes on X and primitive elements of $H_1(X,\mathbb{Z})$.

COROLLARY 4.2. — *If $h \in H_1(X,\mathbb{Z})$ is a primitive integral homology class on X, then there exist simple closed curves that represent h, and these curves are all freely homotopic to one another.*

Proof. — It remains to prove that a free homotopy class is primitive if and only if it contains simple closed curves. Denote by (s,t) a basis of $\mathbb{F}_2 \cong \pi_1(X)$ such that $\psi(s) = (1,0)$ and $\psi(t) = (0,1)$, and let a be a simple closed curve representing s.

If γ is a simple closed curve on X then by the theorem of classification of (non-separating) simple closed curves on surfaces (see [9, Paragraph 1.3.1]) there exists a homeomorphism Φ of X such that $\Phi(a) = \gamma$. This homeomorphism induces an automorphism φ of $\pi_1(X)$ such that $\varphi(s) = \bar{\gamma}$, hence the free homotopy class $\bar{\gamma}$ is primitive.

Conversely, if $\bar{\gamma} \in \pi_1(X)$ is primitive there exists $\varphi \in \mathrm{Aut}(\pi_1(X))$ such that $\varphi(s) = \bar{\gamma}$. Since X is a $K(\mathbb{F}_2, 1)$, by [13, Proposition 1B.9] φ is induced by a homeomorphism Φ of X. Then, since a is simple and Φ is a homeomorphism, $\Phi(a)$ is a simple closed curve that represents the free homotopy class $\bar{\gamma}$. □

Constraints on simple representatives of a homology class. — Osborne and Zieschang also proved a remarkable property of the word $V_{m,n}$, which we rephrase here using the Farey sequence.

THEOREM 4.3 ([21, Lemma 2.2]). — *Let a/b and c/d be the Farey parents of n/m, with $a/b < c/d$. Then*

$$V_{m,n} = V_{d,c}V_{b,a} = V_{b,a}tsV'_{d,c}.$$

This theorem contains a lot of information that can be extracted to provide geometric constraints on the simple representatives of a primitive homology class (m,n) on the slit torus. The first equality $V_{m,n} = V_{d,c}V_{b,a}$ tells us that in order to avoid the slits the lift to $\tilde{X}$ of a simple curve representing (m,n) on X must cross the vertical line $x = d$ between the points $(d, c-1+\rho)$ and (d,c), as illustrated by Figure 4.1.

Similarly, the second equality $V_{m,n} = V_{b,a}tsV'_{d,c}$ tells us that the lift of a simple curve representing (m,n) on X must cross the vertical line $x = b$ between the points $(b, a+\rho)$ and $(b, a+1)$.

We will start by considering this second constraint on simple curves, illustrated by Figure 4.2; as we will see, it already provides plenty of information. Satisfying this constraint is a necessary condition for a curve representing a non-visible class (m,n) to be simple. Amongst all the curves (simple or not) satisfying this constraint, the obvious candidate to be the shortest is the path γ made by concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$. Thus, provided that γ projects onto a closed curve on X, that is to say provided that it does not encounter the slit, it will be the shortest curve satisfying the constraint illustrated by Figure 4.2. Its length will then provide a lower bound on the length of any curve satisfying the constraint of Figure 4.2, and in particular on the length of simple curves representing (m,n).

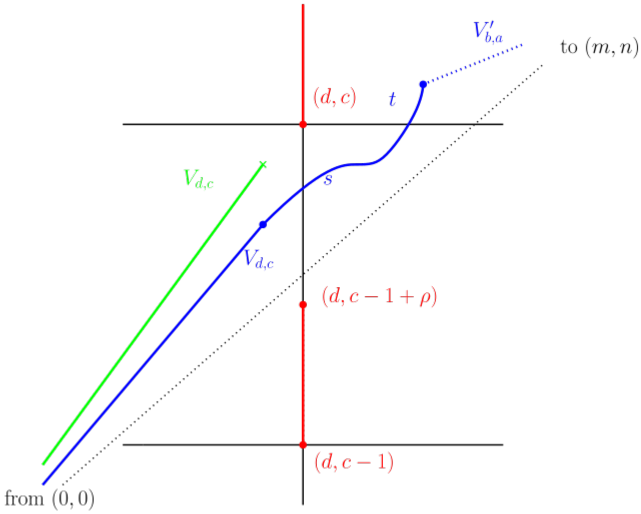

FIGURE 4.1. First constraint for a curve in $V_{m,n}$

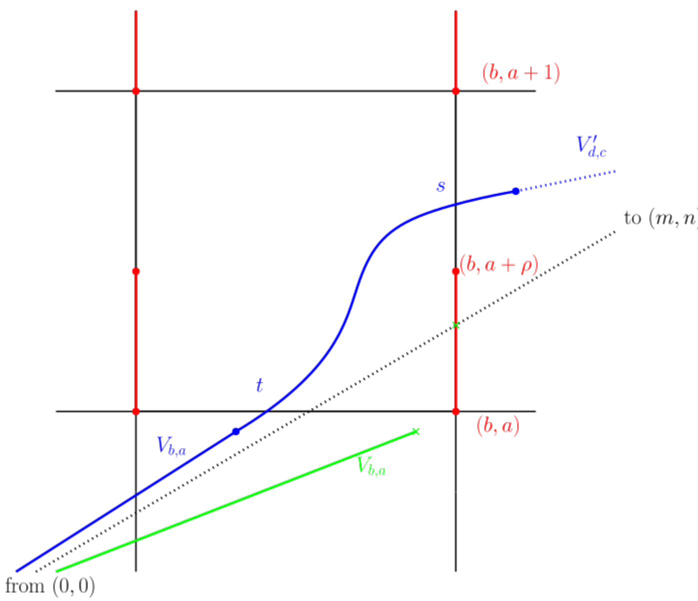

FIGURE 4.2. Second constraint for a curve in $V_{m,n}$

Moreover, if γ projects onto a simple curve this lower bound would be a minimum, and we would have found the shortest simple curve representing h. Hence we have to check if this path γ satisfies the required properties, that is not to intersect the slit and to project onto a simple curve on X. As we will see, the answers to these questions depend on the position of n/m in the Farey sequence. There are two distinct cases to consider, corresponding to whether or not the Farey parents of n/m are associated to visible homology classes.

If one of the Farey parents is visible. — Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be a primitive homology class on X that is not visible. Let $a/b < c/d$ be the Farey parents of the rational n/m, and assume that at least one of the homology classes (b,a), (d,c) is visible. In that sense, we say that one of the Farey parents of n/m is visible.

LEMMA 4.4. — *Let γ be the path in $\tilde{X}$ from $(0,0)$ to (m,n), obtained by the concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$. If at least one of the Farey parents of n/m is visible then the path γ does not transversally intersect the slit.*

Proof. — Assume that at least one of the classes (b,a) and (d,c) is visible. We need to fix the visible class in order to be able to conduct computations, so let us assume that (d,c) is visible. The computations are similar if (b,a) is visible instead. We check that each segment composing γ avoids the slits.

- The segment $[(0,0),(b,a+\rho)]$ crosses the vertical of the slit at points $(k,k(a+\rho)/b)$ with $k = 1,\ldots,b-1$, with intersection with the slit if and only if $\{k(a+\rho)/b\} < \rho$ where $\{.\}$ denotes the fractional part. Since a/b and c/d are a Farey pair, we have
$$\frac{a}{b} = \frac{c}{d} - \frac{1}{bd}$$
so
$$\left\{k\frac{a+\rho}{b}\right\} = \left\{k\left(\frac{c}{d} - \frac{1}{bd} + \frac{\rho}{b}\right)\right\} = \left\{\frac{r}{d} + k\frac{\rho - 1/d}{b}\right\},$$
where r is the remainder in the Euclidean division of kc by d. Remark that since (d,c) is visible we have $d\rho \leq 1$ so $\rho - 1/d \leq 0$. On the one hand
$$\frac{r}{d} + k\frac{\rho - 1/d}{b} \leq \frac{r}{d} \leq \frac{d-1}{d} < 1$$
and on the other hand
$$\frac{r}{d} + k\frac{\rho - 1/d}{b} \geq \frac{1}{d} + \frac{k(\rho - 1/d)}{b} \geq \frac{1}{d} + \frac{(b-1)(\rho - 1/d)}{b},$$
so
$$\frac{r}{d} + k\frac{\rho - 1/d}{b} \geq \rho + \frac{1/d - \rho}{b} \geq \rho.$$

Thus $\{\frac{r}{d}+k\frac{\rho-1/d}{b}\} = \frac{r}{d}+k\frac{\rho-1/d}{b} \geq \rho$ and the segment $[(0,0),(b,a+\rho)]$ does not intersect the slit.

- For the case of the segment from $(b, a+\rho)$ to (m,n), the computations become easier when we translate the problem by $(-b,-a-\rho)$. Since $m = b+d$ and $n = a+c$, the problem is equivalent to determining whether or not the segment $[(0,0),(d,c-\rho)]$ intersects the slit, with the copies of the slit this time being the sets $\{(p,q-s), s \in [0,\rho]\}$. The segment $[(0,0),(d,c-\rho)]$ has slope $(c-\rho)/d$ and crosses the vertical of the slit at the points $(k, k(c-\rho)/d)$, $k = 1,\dots,d-1$, with intersection with the slit if and only if

$$\left\{k\frac{c-\rho}{d}\right\} > 1-\rho.$$

Let r be the remainder in the Euclidean division of kc by d. We have $\{k(c-\rho)/= d\} = \{(r-k\rho)/d\}$. Moreover,

$$\frac{r-k\rho}{d} > \frac{1-k\rho}{d} \geq \frac{1-(d-1)\rho}{d} > 0$$

because $(d-1)\rho \leq 1-\rho < 1$, and since $1/d > \rho$, we have

$$\frac{r-k\rho}{d} < \frac{d-1-k\rho}{d} = 1 - \frac{1+k\rho}{d} < 1 - \frac{1}{d} \leq 1-\rho.$$

Hence the segment $[(b,a+\rho),(m,n)]$ does not intersect the slit. □

Thus the path γ, defined as the concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$, projects onto a closed curve on X that represents h that has the same length as γ. Is this curve simple? We start with the easier special case where *both* the classes associated to the Farey parents of n/m are visible.

LEMMA 4.5. — *With the previous notations, if both the homology classes (b,a) and (d,c) are visible, then γ projects onto a simple closed curve on X. Moreover, this curve is minimizing in the homology class (m,n).*

Proof. — Remember that $m,n \geq 0$. Assume that both the classes (b,a) and (d,c) are visible, where $a/b < c/d$ are the Farey parents of n/m. By Proposition 3.1 we know that the shortest non-simple curve representing h has length $\|(b,a)\|_X + \|(d,c)\|_X$, and since we assumed that both (b,a) and (d,c) are visible by Proposition 2.1 we know that $\|(b,a)\|_X = \sqrt{b^2+a^2}$ and $\|(d,c)\|_X = \sqrt{d^2+c^2}$. Thus, the shortest non-simple curve representing h on X has length

$$\|(b,a)\|_X + \|(d,c)\|_X = \sqrt{b^2+a^2} + \sqrt{d^2+c^2},$$

and we only need to check that γ is shorter. Indeed, if so, the projected curve is automatically simple as it represents the homology class (m,n).

Since a/b and c/d are Farey neighbours they satisfy the relation $bc - ad = 1$, so

$$\frac{n}{m} - \frac{a+\rho}{b} = \frac{a+c}{b+d} - \frac{a}{b} - \frac{\rho}{b} = \frac{1}{b}\left(\frac{1}{b+d} - \rho\right) < 0$$

because $b + d = m > 1/\rho$. Since $d\rho \leq 1$ we have $\rho \leq 1/d$ and

$$\frac{a+\rho}{b} = \frac{a}{b} + \frac{\rho}{b} \leq \frac{a}{b} + \frac{1}{bd} = \frac{c}{d},$$

and hence we have

$$\frac{a}{b} \leq \frac{c-\rho}{d} < \frac{n}{m} < \frac{a+\rho}{b} \leq \frac{c}{d}.$$

Thus the triangle formed by the points $(0,0)$, (d,c), and (m,n) contains the triangle formed by the points $(0,0), (b, a+\rho)$ and (m,n) (see Figure 4.3). We deduce that the perimeter of the first triangle is greater than the perimeter of the second. But their perimeters are, respectively,

$$\|(b,a)\|_X + \|(d,c)\|_X + \sqrt{m^2+n^2}$$

and

$$\sqrt{b^2 + (a+\rho)^2} + \sqrt{d^2 + (c-\rho)^2} + \sqrt{m^2+n^2} = l(\gamma) + \sqrt{m^2+n^2},$$

so we finally have $l(\gamma) < \|(b,a)\|_X + \|(d,c)\|_X$, and hence γ projects onto a curve that is shorter than the shortest non-simple curve that represents h. Thus, γ projects onto a simple curve on X.

By construction γ is the shortest path satisfying the constraint illustrated by Figure 4.2. Since all (lifts to $\tilde{X}$ of) simple curves representing the class (m,n)

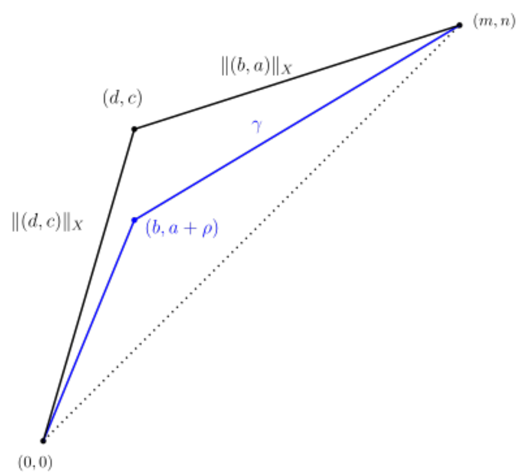

FIGURE 4.3. The path γ is shorter

on X must satisfy this constraint, the path γ projects onto is the shortest simple closed curve that represents (m, n). Since we have already shown it is shorter than the shortest non-simple closed curve representing (m, n) on X, we conclude that it is minimizing in the homology class (m, n). □

We can now compute the stable norm of the class (m, n).

PROPOSITION 4.6. — *Let $h = (m, n)$ be a primitive non-visible integral homology class on X. Let $a/b < c/d$ be the Farey parents of n/m. If at least one of the classes (b, a) ad (d, c) is visible then*

$$\|(m, n)\|_X = \sqrt{b^2 + (a + \rho)^2} + \sqrt{d^2 + (c - \rho)^2}.$$

We will need the following *ad hoc* definition of the distance to a couple of Farey ancestors of a rational number to conduct the proof. Recall that starting from any pair of Farey neighbours $p_1/q_1 < p_2/q_2$ and by taking mediants with p_1/q_1 and p_2/q_2 one can obtain every rational number in the interval $]p_1/q_1, p_2/q_2[$. In other words, for any rational $r/s \in]p_1/q_1, p_2/q_2[$ there exist $k_1, k_2 \in \mathbb{N}^*$ such that $r/s = (k_1 p_1 + k_2 p_2)/(k_1 q_1 + k_2 q_2)$. We say that the distance from r/s to the couple $(p_1/q_1, p_2/q_2)$ of its Farey ancestors is $k_1 + k_2 - 1$, as it is the number of steps that one has to take to obtain r/s from p_1/q_1 and p_2/q_2 with the operations "taking the mediant with p_1/q_1" and "taking the mediant with p_2/q_2". The expert reader will recognize this as the distance between r/s and $p_1/q_1 \oplus p_2/q_2$ in the Stern–Brocot tree (see [5], for instance).

Proof. — Let γ be the path in $\tilde{X}$ obtained by concatenation of the segments $[(0, 0), (b, a + \rho)]$ and $[(b, a + \rho), (m, n)]$. We follow the same strategy as in Lemma 4.5. We will show that γ projects on X onto a curve that is shorter than the shortest non-simple closed curve on X that represents the homology class (m, n). More precisely, we will show that

$$l(\gamma) < \|(b, a)\|_X + \|(d, c)\|_X.$$

Then the curve that γ projects onto will automatically be simple and, because it is the projection of the shortest path satisfying the constraint illustrated in Figure 4.2, minimizing in the homology class (m, n), so $\|(m, n)\|_X = l(\gamma)$. We proceed by induction on $\kappa \geq 1$, where κ is the distance from n/m to its latest couple of visible Farey ancestors, that is, the smallest possible distance from n/m to a couple of its Farey ancestors that are both visible.

Let us describe the Farey ancestors of n/m more precisely. By assumption, one of the classes $(b, a), (d, c)$ is visible. To fix notations assume that (b, a) is visible, as (b, a) and (d, c) play similar roles in this proof. Since (b, a) is visible the number a/b, which is a Farey parent of n/m, is part of the latest couple of visible Farey ancestors of n/m. Let α/β be the other one; possibly $\alpha/\beta = c/d$ if both the Farey parents of n/m are visible. By definition, there exist $k_1, k_2 \in \mathbb{N}^*$ such that $n/m = (k_1 a + k_2 \alpha)/(k_1 b + k_2 \beta)$ and $\kappa = k_1 + k_2 - 1$. Since a/b and

n/m (resp. a/b and α/β) are Farey neighbours we have $bn - am = 1$ (resp. $b\alpha - a\beta = 1$). Then $bn - am = b(k_1 a + k_2\alpha) - a(b + k_2\beta) = k_2(b\alpha - a\beta) = k_2$. Hence $k_2 = 1$ and

$$\frac{n}{m} = \frac{\kappa a + \alpha}{\kappa b + \beta} = \left(\left(\left(\frac{a}{b} \oplus \frac{\alpha}{\beta}\right) \oplus \frac{a}{b}\right) \oplus \ldots \oplus \frac{a}{b}\right),$$

i.e., n/m is obtained from its latest visible Farey ancestors by taking κ times the mediant with a/b. Thus, the Farey ancestors of n/m up to α/β in the Farey genealogy are of the form $(ka + \alpha)/(kb + \beta)$ with k running from $\kappa - 1$ down to 1. In particular, the second Farey parent of n/m is $c/d = ((\kappa - 1)a + \alpha)/((\kappa - 1)b + \beta)$.

If $\kappa = 1$, then $\alpha/\beta = c/d$, meaning both the classes (b, a) and (d, c) are visible. By Lemma 4.5 we know γ projects onto a minimizing simple closed curve for the class (m, n) on X, so

$$\|(m, n)\|_X = l(\gamma) = \sqrt{b^2 + (a + \rho)^2} + \sqrt{d^2 + (c - \rho)^2},$$

and we are done.

Assume the proposition is proven up to $\kappa - 1$ with $\kappa \geq 2$. We want to compare $l(\gamma)$ to $\|(b, a)\|_X + \|(d, c)\|_X$. Since (b, a) is visible, we have $\|(b, a)\|_X = \sqrt{b^2 + a^2}$. Since $\kappa \neq 1$ we have $c/d \neq \alpha/\beta$, so (d, c) is not visible. However, the distance from c/d to its latest couple of Farey ancestors $(a/b, \alpha/\beta)$ is $\kappa - 1$, so by the induction hypothesis we know how to express the stable norm of (d, c) using only its Farey parents. More precisely, the Farey parents of $c/d = ((\kappa - 1)a + \alpha)/((\kappa - 1)b + \beta)$ are a/b and $((\kappa - 2)a + \alpha)/((\kappa - 2)b + \beta)$, at least one of which being visible, so according to the induction hypothesis at rank $\kappa - 1$, we have

$$\|(d, c)\|_X = \sqrt{b^2 + (a + \rho)^2} + \sqrt{((\kappa - 2)b + \beta)^2 + ((\kappa - 2)a + \alpha - \rho)^2}.$$

We then have

$$\begin{aligned}
&\|(b, a)\|_X + \|(d, c)\|_X - l(\gamma) \\
&\quad = \sqrt{b^2 + a^2} + \sqrt{((\kappa - 2)b + \beta)^2 + ((\kappa - 2)a + \alpha - \rho)^2} - \sqrt{d^2 + (c - \rho)^2} \\
&\quad = \|(b, a)\|_2 + \|((\kappa - 2)b + \beta, (\kappa - 2)a + \alpha - \rho)\|_2 - \|(d, c - \rho)\|_2,
\end{aligned}$$

and since $(d, c - \rho) = (b, a) + ((\kappa - 2)b + \beta, (\kappa - 2)a + \alpha - \rho)$, by the triangle inequality of the Euclidean norm, we finally have $\|(b, a)\|_X + \|(d, c)\|_X - l(\gamma) > 0$.

Thus, γ projects onto a simple closed curve on X that represents the homology class (m, n). Since any simple curve representing (m, n) on X must lift to a path in $\tilde{X}$ that satisfies the constraint illustrated by Figure 4.2, and since among the paths that satisfy this constraint γ is the shortest, we deduce that the curve γ projects onto is the shortest simple closed curve on X that represent (m, n). We have seen that this curve is shorter than the shortest

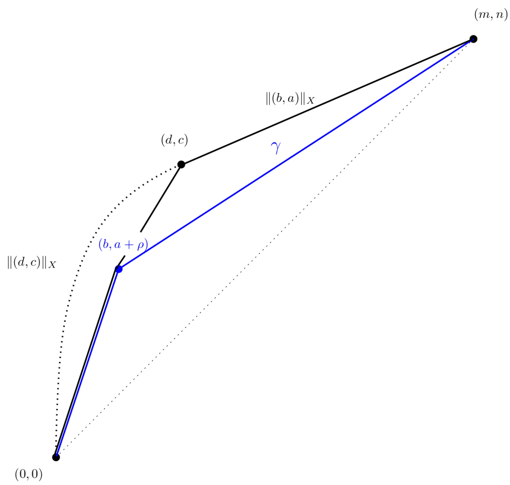

FIGURE 4.4. The path γ is shorter

non-simple closed curve representing (m, n), so it must be minimizing in its homology class. Hence we finally have

$$\|(m,n)\|_X = l(\gamma) = \sqrt{b^2 + (a+\rho)^2} + \sqrt{d^2 + (c-\rho)^2}. \qquad \square$$

If none of the Farey parents are visible. — Finally, we investigate what happens when none of the classes associated to the Farey parents a/b and c/d of n/m are visible. This time, the path that is the concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$ does not project onto a simple closed curve on the slit torus. Indeed, it intersects the slit, for instance at $x = d$, as one can check with a direct computation that we will not detail. Instead, we will prove a stronger statement.

PROPOSITION 4.7. — *If none of the Farey parents of n/m are visible, there is no simple geodesic in the free homotopy class $V_{m,n}$, where $V_{m,n}$ is the unique free homotopy class of simple curves that represent the homology class (m, n).*

By geodesic in the context of a singular metric we mean a curve that is locally distance-minimizing between any two of its points. First, we need the following lemma.

LEMMA 4.8. — *Let γ be the lift to $\tilde{X}$ of a geodesic in the free homotopy class $V_{m,n}$ starting at the origin. Then γ changes direction at least twice.*

Proof. — Recall that $m, n \geq 0$. Let $a/b < c/d$ be the Farey parents of n/m; by assumption (b, a) and (d, c) are not visible, so $b, d > 1/\rho$.

- If $d < b$, since $d > 1/\rho$ and $bc - ad = 1$, straightforward computations yield
$$\frac{c-1+\rho}{d} < \frac{a}{b} < \frac{n}{m} < \frac{c}{d} < \frac{a+\rho}{b},$$
so by putting together the constraints illustrated by Figure 4.1 and Figure 4.2, we obtain the following picture, in which it is clear that the lift of a curve in $V_{m,n}$ must change direction at least twice.

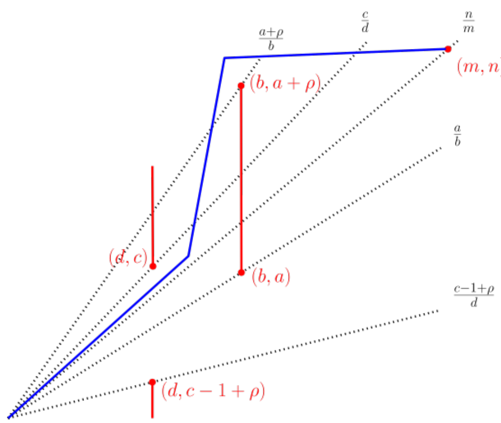

FIGURE 4.5. The blue curve in $V_{m,n}$ must change direction twice

- If $b < d$, we similarly get the following picture,

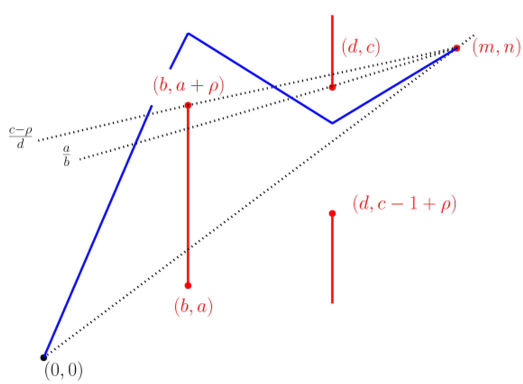

FIGURE 4.6. Again, the blue curve in $V_{m,n}$ must change direction twice

and since $b > 1/\rho$ we have $\frac{c-\rho}{d} < \frac{a}{b}$, so the lift to $\tilde{X}$ of a curve in $V_{m,n}$ has to change direction in order to pass under (d, c). □

We can now prove Proposition 4.7.

Proof. — Since the metric is flat, a geodesic on X must be a broken line with angles located at the slit's endpoints. Indeed, if a geodesic changes direction outside of an endpoint of the slit, then it has a corner, and the geodesic is not locally length-minimizing, which leads to a contradiction. Since we know that every lift of a geodesic in $V_{m,n}$ must change direction at least twice between $(0,0)$ and (m,n), this means that any geodesic in $V_{m,n}$ passes at least three times through an endpoint of the slit: the two times when changing direction and the third time corresponding to start and finish, i.e., the projection of the points $(0,0)$ and (m,n). The slit has only two endpoints, and hence any geodesic in $V_{m,n}$ meets a single endpoint of the slit at least twice, so it is not simple. □

Obviously, a minimizing curve for a homology class h has to be a geodesic. Thus, if there are no simple geodesics representing h, this means that h is minimized by a non-simple curve, and we know its length according to Proposition 3.1. We have shown the following.

Proposition 4.9. — *Let $h = (m,n)$ be a primitive integral homology class and let $a/b, c/d$ be the Farey parents of n/m. Assume that none of the Farey parents of n/m are visible. Then $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.*

Although this proposition does not provide an explicit formula for the stable norm of a class none of whose Farey parents are visible, it relates its stable norm to the stable of the classes associated to its Farey parents. These homology classes have smaller coordinates than (m,n), so following a recursive algorithm one can produce formulas for the stable norm of any integral homology class on X.

5. Structure of the unit ball

Non-simple minimizing curves and flats of the unit ball. — First, we prove the following lemma, linking homology classes minimized by non-simple closed curves to flats of the unit ball of the stable norm.

Lemma 5.1. — *Let $h \in H_1(X,\mathbb{Z})$. The unit ball of the stable norm $\mathcal{B}$ has a flat in the direction h if and only if there exists a connected minimizing curve for h with at least one self-intersection. Equivalently, the unit ball $\mathcal{B}$ is strictly convex in the direction h if and only if every minimizing curve for h is connected and simple.*

Proof. — Let $h = (m,n)$ be a primitive integral homology class and assume that the unit ball of the stable norm $\mathcal{B}$ has a flat in the direction h. By Proposition 5.1, the classes $h_1 = (b,a)$ and $h_2 = (d,c)$, where $a/b, c/d$ are the Farey parents of n/m, belong to the same flat of $\mathcal{B}$ as h. Indeed, since $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$, we have

$$\frac{(m,n)}{\|(m,n)\|_X} = \lambda \frac{(b,a)}{\|(b,a)\|_X} + (1-\lambda)\frac{(d,c)}{\|(d,c)\|_X},$$

where $\lambda = \|(b,a)\|_X / \|(m,n)\|_X$. Since $h = h_1 + h_2$, one can get a multicurve that is minimizing for h by taking the union of two multicurves, representing (and minimizing for), respectively, h_1 and h_2. Let γ be such a minimizing multicurve. By translating each of its component upwards until they encounter the lower endpoint of the slit, we get another representative of h that is also minimizing, because applying a translation does not change the homology class or the length of each curve. This representative is connected and has at least one self-intersection, located at the lower endpoint of the slit.

Conversely, if an integral class h is minimized by a closed curve γ with at least one self-intersection point $p \in X$, one can see γ as the union of two closed curves γ_1 and γ_2 based at p. These curves necessarily have non-trivial homology classes h_1 and h_2; otherwise, by simply deleting one of them from γ we would get another closed curve representing h that is strictly shorter than γ, which contradicts its minimality. Moreover, these curves must be minimizing in their homology class, simply because if one of them is not, we could replace it by a minimizing curve of its homology class to get another representative of h shorter than γ, again contradicting the minimality of γ. Hence $h = h_1 + h_2$ and $\|h\|_X = \|h_1\|_X + \|h_2\|_X$, so $h/\|h\|_X$ lies in the interior of the segment $\left[h_1/\|h_1\|_X, h_2/\|h_2\|_X\right]$, and the unit ball of the stable norm has a flat in the direction h. □

Proof of Theorem A. — We now have all the required elements to prove Theorem A. Putting together Proposition 2.1, 3.1, 4.6, and 4.9 we have determined the stable norm of the slit torus X on $H_1(X,\mathbb{Z})$; given a primitive homology class (m,n) we either have explicit formulas for its stable norm or we can produce some through an iterative process. To sum things up, we have shown that there are three different types of classes.

1. If (m,n) is visible then it is minimized by a simple curve and $\|(m,n)\|_X = \sqrt{m^2+n^2}$.
2. If (m,n) is not visible but (b,a) or (d,c) is visible, where $a/b < c/d$ are the Farey parents of n/m, then (m,n) is minimized by a simple curve and $\|(m,n)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2}$, where ρ denotes the length of the slit.
3. If $(m,n), (b,a)$ and (d,c) are not visible, then (m,n) is minimized by a non-simple curve and $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.

Note that for classes of type 1 and 2 we have found a simple minimizing curve that is strictly shorter than any non-simple curve representing (m,n), so *every* minimizing curve of such a class must be simple.

It remains to prove all the combinatorial statements contained in Theorem A, that is to say to describe the vertices and the flats of the unit ball $\mathcal{B}$ of the stable norm of X.

Let us start with the flats. By Lemma 5.1, the unit ball $\mathcal{B}$ of the stable norm has a flat in the direction of a primitive integral homology class (m, n) if and only if (m, n) is minimized by a non-simple curve. According to our computation of the stable norm, this only occurs for type 3 classes. Moreover, as we have already noted, the points of $\partial\mathcal{B}$ in the directions $(m, n), (b, a)$ and (d, c) belong to the same flat.

Thus, the other directions of $\mathcal{B}$, i.e., the directions of classes of type 1 and 2, are strictly convex directions of the stable norm. It remains to prove that the points p of $\partial\mathcal{B}$ in those directions are actually vertices, meaning there are infinitely many *supporting lines* at p, i.e., lines whose intersection with $\mathcal{B}$ is reduced to $\{p\}$. This is trivial for type 2 classes. The direction (m, n) is isolated from other strictly convex directions of the stable norm. Indeed, the slopes of strictly convex directions of the stable norm are the slopes of the visible directions and their successive Farey children, which are discrete and accumulate in the visible directions. Thus, the point of $\partial\mathcal{B}$ in the direction (m, n) lies on the edge of two flats of $\mathcal{B}$; since the stable norm is strictly convex in the direction (m, n) the only possibility is that $\mathcal{B}$ has a vertex in the direction (m, n).

Let (m, n) be a visible homology class with $m \geq 1$. According to the previous point, we know there are infinitely many vertices of $\mathcal{B}$ that accumulate to $p := (m, n)/\|(m, n)\|_X$; it might happen that there is only one supporting line at p, namely the line tangent to the circle at p. Those vertices accumulating on p correspond to the Farey children of n/m, so they are of the form

$$u_k := \frac{(\beta + km, \alpha + kn)}{\|(\beta + km, \alpha + kn)\|_X}$$

on the left-hand side of p and

$$v_k := \frac{(\delta + km, \gamma + kn)}{\|(\delta + km, \gamma + kn)\|_X}$$

on the right-hand side of p, where $k \in \mathbb{N}^*$ and (β, α) and (δ, γ) are two visible classes such that $\alpha/\beta, n/m$ and γ/δ are consecutive in the Farey sequence of some order. Indeed, the k-th Farey child of n/m with α/β is obtained by taking the k successive mediants with n/m, i.e., it is $n/m \oplus \ldots \oplus n/m \oplus \alpha/\beta$. Let $\Delta_k = p - u_k$. Since (m, n) is visible, we have $\|(m, n)\|_X = \sqrt{m^2 + n^2}$, and since u_k is the direction of a type 2 class, we have

$$\begin{aligned}\|(\beta + km, \alpha + kn)\|_X \\ &= \sqrt{m^2 + (n \pm \rho)^2} + \sqrt{(\beta + (k-1)m)^2 + (\alpha + (k-1)n \mp \rho)^2}.\end{aligned}$$

Injecting this formula back in Δ_k and taking the Taylor series expansion of each of its coordinates then yields

$$\Delta_k = (\frac{c_1}{k} + O(1/k^2), \frac{c_2}{k} + O(1/k^2)),$$

where c_1 and c_2 are non-zero constants depending on m, n, α, β and ρ. Thus the slopes of the vectors Δ_k go to the non-zero constant c_2/c_1 as k goes to infinity; in particular those slopes are bounded away from zero. Hence a line passing through p with a slope in between 0 and the slopes of the Δ_k's does not intersect $\mathcal{B}$ outside of p; it is a supporting line of $\mathcal{B}$ at p, thus p is a vertex of $\mathcal{B}$.

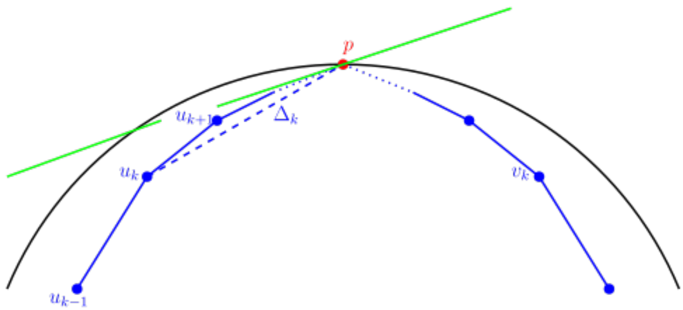

FIGURE 5.1. The green line is a supporting line of $\mathcal{B}$ at p

Finally, the classes $\pm(0,1)$ are visible but are not the direction of vertices of $\mathcal{B}$. Indeed, the other visible classes accumulate in those directions. But the points of $\partial\mathcal{B}$ in the direction of the visible classes lie on the unit circle $\mathbb{S}^1$, so any supporting line of $\mathcal{B}$ at $\pm(0,1)$ must not intersect the unit circle elsewhere; otherwise its intersection with $\mathcal{B}$ would not be reduced to a point. Thus, such a line must be tangent to $\mathbb{S}^1$ at $\pm(0,1)$, meaning there is a *unique* supporting line at $\mathcal{B}$ at $\pm(0,1)$.

What about the other directions, i.e., the directions of *irrational* homology classes? If (q,p) and (q',p') are two type 2 integral classes such that $p/q < p'/q'$ are Farey neighbours, we have seen that the class $(q+q', p+p')$ associated to their Farey child $p/q \oplus p'/q'$ lies inside a flat of $\mathcal{B}$. By convexity of $\mathcal{B}$, and since the three points $(q,p)/\|(q,p)\|_X$, $(q',p')/\|(q',p')\|_X$, and $(q+q',p+p')/\|(q+q',p+p')\|_X$ of $\partial\mathcal{B}$ are aligned, the entire segment $[(q,p)/\|(q,p)\|_X, (q',p')/\|(q',p')\|_X]$ is contained in $\partial\mathcal{B}$. Thus, the point of $\partial\mathcal{B}$ in the direction of any *real* (possibly irrational) homology class $(x,y) \in H_1(X,\mathbb{R})$ with $p/q < y/x < p'/q'$ lies in the interior of this flat. Since any irrational number falls between two slopes of type 2 Farey neighbour homology classes, that is to say between two Farey children of an element of $\mathcal{F}_L$ (the Farey sequence of order $L = \lfloor 1/\rho \rfloor$), the stable norm of X is flat in every irrational direction. Theorem A is now proved.

Computing the stable norm of a given homology class. — So far, we have provided explicit formulas for the stable norm of integral homology classes in strictly convex directions of $\mathcal{B}$ and said the other directions were flats of the stable norm. This data is sufficient to compute the stable norm of any other homology class $(x, y) \in H_1(X, \mathbb{R})$, even irrational ones. Note that x and y are non-zero, as we already know the stable norm of the classes $(1, 0), (0, 1)$. Here is how to proceed. First, find which flat of $\partial\mathcal{B}$ contains the point $(x, y)/\|(x, y)\|_X$. More precisely, find the directions (q, p) and (q', p') with $p/q < y/x < p'/q'$ of the endpoints of this flat, which are type 2 homology classes such that p/q and p'/q' are Farey neighbours. While rather easy since we know all type 2 homology classes are associated to the Farey children of the numbers in $\mathcal{F}_L$, this step has to be done numerically (note that for integral classes we simply have to look at their Farey ancestors). Then, there exists $\lambda \in [0, 1]$ such that

$$\frac{(x, y)}{\|(x, y)\|_X} = \lambda \frac{(q, p)}{\|(q, p)\|_X} + (1 - \lambda) \frac{(q', p')}{\|(q', p')\|_X},$$

so we can write the system

$$\begin{cases} \dfrac{x}{\|(x, y)\|_X} = \dfrac{\lambda q}{\|(q, p)\|_X} + \dfrac{(1 - \lambda) q'}{\|(q', p')\|_X} \\ \dfrac{y}{\|(x, y)\|_X} = \dfrac{\lambda p}{\|(q, p)\|_X} + \dfrac{(1 - \lambda) p'}{\|(q', p')\|_X} \end{cases},$$

where the unknown quantities are λ and $\|(x, y)\|_X$. Solving this system, which is always possible since x and y are non-zero, we then obtain a formula for the stable norm of the class (x, y).

Graphical representation of the unit ball. — It is difficult to produce a good picture of the unit ball of the stable norm of the slit torus. Indeed, it is so close to the unit circle that they are barely distinguishable from one another by eyesight; moreover, the angles at the vertices are nearly flat, so they appear to be almost invisible, making $\mathcal{B}$ look like a polygon. Hence it is pointless to try to represent $\mathcal{B}$ directly, as most of its interesting features will not be apparent.

In order to grasp the structure of $\mathcal{B}$, we flatten the arc of the unit circle between angles 0 and $\pi/4$ onto the segment $[0, 1]$, and to $x \in [0, 1]$ we associate the distance between the unit circle $\mathbb{S}^1$ and the unit ball of the stable norm $\mathcal{B}$ in the direction of slope x. The points of non-differentiability of the resulting curve correspond to the vertices of $\mathcal{B}$. In the left-hand figure, with $1/3 < \rho < 1/2$, we can clearly see the three visible directions where $\mathcal{B}$ intersects $\mathbb{S}^1$, namely $(1, 0), (2, 1)$ and $(1, 1)$. The right-hand figure, with $1/5 < \rho < 1/4$, highlights the accumulation of vertices on the visible direction $(1, 0)$.

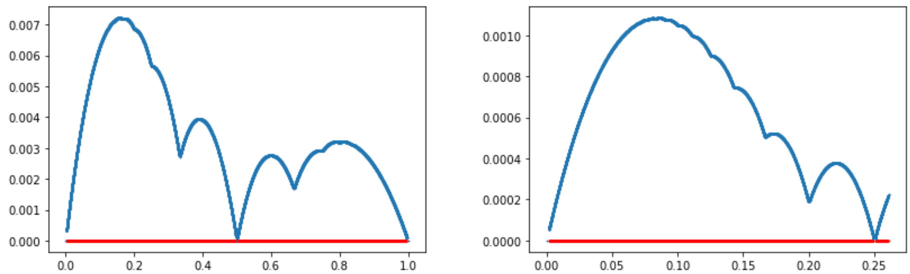

FIGURE 5.2. Unit ball of the stable norm (blue) near an arc of the unit circle (red)

6. Glueing slit tori

Now that we understand the stable norm of the slit torus we can move on to glueing together multiple copies of it to obtain flat surfaces on which we can compute the stable norm as well.

Glueing two slit tori along a flat cylinder. — Recall that X denotes the square torus of area 1 with a vertical slit of length $\rho \in]0,1[$. Let X_1, X_2 be two identical copies of X and let C be a flat cylinder of width w, with boundary components of length 2ρ. Let Σ be the closed genus 2 surface obtained by glueing C along the slits of X_1 and X_2. By construction, the surface Σ is endowed with a flat metric, obtained by glueing the flat metrics of the two slit tori and of the cylinder. By Riemann's uniformization theorem, we know that this metric cannot be smooth; indeed, a closer inspection shows the metric has four conical singularities of angle 3π located at each endpoint of the slits and is smooth elsewhere. One can check that Σ is a half-translation surface; see, for instance, Zorich's survey [24] for more on the geometry of these surfaces. We have

$$H_1(\Sigma, \mathbb{Z}) = H_1(X_1, \mathbb{Z}) \oplus H_1(X_2, \mathbb{Z}) \simeq \mathbb{Z}^4,$$

so if (e_i, f_i) is a basis of $H_1(X_i, \mathbb{Z})$, we have $H_1(\Sigma, \mathbb{Z}) = \langle e_1, f_1, e_2, f_2 \rangle$.

Stable norm of Σ. — The stable norm of a homology class h on Σ, denoted $\|h\|_\Sigma$, is defined as seen in the Introduction as the limit $f(Nh)/N$ as N goes to infinity, where $f(h)$ is the infimum of the lengths of representatives of h. Again, this is not a practical definition. However, Massart showed in [15] that for *closed* surfaces the stable norm can equivalently be defined as

$$\|h\|_\Sigma = \inf \left\{ \sum_i |a_i| l(\gamma_i), \text{ where } \sum_i a_i [\gamma_i] = h \right\}$$

and that for integral homology classes this infimum is realized by minimizing multicurves.

Restriction to the slit tori's homology planes. — We denote by $\|.\|_{X_i}$ the stable norm of the slit tori X_i.

LEMMA 6.1. — *The stable norm of Σ coincides with the stable norm of X_i on the plane Span(e_i, f_i).*

Proof. — Take, for instance, $i = 1$ and let $h = (m, n, 0, 0) \in H_1(\Sigma, \mathbb{Z}) \cap \mathrm{Span}(e_1, f_1)$ be an integral homology class and γ be a minimizing multicurve representing h. We will show that γ has to be fully contained in X_1. Thus, its length must be $\|(m,n)\|_{X_1}$ as it must be minimizing for the class (m,n) in the slit torus X_1, so $\|(m,n,0,0)\|_\Sigma = \|(m,n)\|_{X_1}$.

By contradiction, suppose γ goes outside of X_1 and enters X_2. Let $\gamma_2 := \gamma \cap X_2$ be the part of γ inside of X_2; γ_2 may have several connected components, one of which being an arc. Close this arc with a path μ_2 contained in X_2 that goes around the slit by following its boundary to obtain a closed curve $\tilde{\gamma}_2$ in X_2. Note that μ_2 has length at most ρ. Let $\tilde{\gamma}$ be the closed multicurve obtained by closing $\gamma \backslash \gamma_2$ with another copy of the path μ_2.

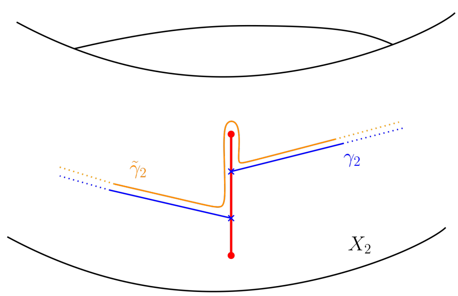

FIGURE 6.1. Closing γ_2 by going around the slit

By construction, $[\gamma] = [\tilde{\gamma}] + [\tilde{\gamma}_2] = (m, n, 0, 0)$. As $\tilde{\gamma}_2$ has intersection number zero with the canonical representatives of the homology classes $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$; it represents a homology class whose first two coordinates are zero. Similarly, the multicurve $\tilde{\gamma}$ represents a homology class whose last two coordinates are zero. Since $[\tilde{\gamma}] + [\tilde{\gamma}_2] = (m, n, 0, 0)$ we get $[\tilde{\gamma}_2] = 0$, so $\tilde{\gamma}$ represents the same homology class as γ. We have

$$l(\tilde{\gamma}) = l(\gamma) + l(\mu_2) - l(\gamma_2).$$

We assumed that $\gamma_2 = \gamma \cap X_2$ was non-trivial, i.e., $l(\gamma_2) > 0$. Necessarily, the connected components of $\tilde{\gamma}_2$ all have a non-trivial homotopy class in the slit torus X_2; otherwise we could shrink a connected component of $\tilde{\gamma}_2$ (and, thus, of γ_2 and γ) onto a point and reduce the length of γ, contradicting its minimality. Thus, any connected component of $\tilde{\gamma_2}$

- Either has to loop around the torus X_2 at least once, and since X_2 is obtained from a square of side of length 1 we have $l(\gamma_2) \geq 1$.
- Or is homotopic to a circle going around the slit so $l(\tilde{\gamma_2}) \geq 2\rho$, and since $l(\tilde{\gamma_2}) = l(\gamma_2) + l(\mu_2)$ and $l(\mu_2) \leq \rho$ we have $l(\gamma_2) \geq \rho$.

In both cases, we have $l(\gamma_2) \geq l(\mu_2)$. Hence $l(\tilde{\gamma}) = l(\gamma) + l(\mu_2) - l(\gamma_2) < l(\gamma)$, which contradicts the minimality of γ. Thus, $l(\gamma_2) = 0$ and the minimizing curve γ do not enter X_2 at all.

We repeat the previous process to show that γ does not enter the cylinder C. Let $\gamma_1 := \gamma \cap X_1$ and $\gamma_C := \gamma \cap C$. We get a closed curve $\tilde{\gamma_1}$ by closing γ_1 with the path $\mu_1 \subset X_1$ going around the slit in the shortest way. Again, $l(\mu_1) \leq \rho$. Set $\tilde{\gamma_C} = \gamma_C \cup \mu_1$. It is a closed curve contained in C with trivial homology class, and hence $\tilde{\gamma_1}$ represents the same homology class as γ because $[\gamma] = [\tilde{\gamma_1}] + [\tilde{\gamma_C}]$. If $l(\gamma_C) > 0$, we clearly have $l(\gamma_C) > l(\mu_1)$, so

$$l(\gamma_1) + l(\mu_1) = l(\tilde{\gamma_1}) < l(\gamma) = l(\gamma_1) + l(\gamma_C),$$

which contradicts the minimality of γ as $\tilde{\gamma_1}$ would be a shorter curve representing the same homology class (m, n). Thus, γ does not enter C and is fully contained in X_1. □

Note that Lemma 6.1 is true without any assumption of the width w of the cylinder. As a matter of fact, it is still true for $w = 0$.

Isolating the tori with a long cylinder. — Outside of the homology planes associated to each torus composing Σ, computing the stable norm is complicated. However, if the cylinder is long enough, then the problem becomes much easier. The idea is the following. If a connected curve represents a homology class (m, n, p, q) then at some point, it has to cross the cylinder, but if the cylinder is very long, so must be the curve. Hence it is much more length-efficient to have two connected components that avoid the cylinder by being fully contained inside the slit tori, representing the classes $(m, n, 0, 0)$ and $(0, 0, p, q)$, respectively.

Proposition 6.2. — *If the cylinder C is long enough, more precisely if $w > \rho$ with ρ the length of the slits of the X_i's, then for all $(m, n, p, q) \in H_1(\Sigma, \mathbb{Z})$ we have*

$$\|(m, n, p, q)\|_\Sigma = \|(m, n)\|_{X_1} + \|(p, q)\|_{X_2}.$$

Proof. — Let $h = (m, n, p, q) \in H_1(\Sigma, \mathbb{Z})$ be an integral class in Σ. If $(m, n) = (0, 0)$ or $(p, q) = (0, 0)$, we are in the setting of Lemma 6.1, and there is nothing left to prove, so let us assume $(m, n) \neq (0, 0)$ and $(p, q) \neq (0, 0)$. Let γ be a representative of h. Note that γ is not necessarily connected. We consider two distinct cases, depending on whether or not γ crosses the cylinder.

First, consider the case where γ does not cross the cylinder. More precisely, let us assume that there does not exist an arc of γ whose endpoints lie in X_1 and X_2, respectively. By the same argument as in the proof of Lemma 6.1 it is not length-efficient for γ to venture into the cylinder without fully crossing it, so for γ to be as short as possible, necessarily $\gamma \cap C = \emptyset$. Thus, $\gamma = \gamma_1 \cup \gamma_2$, with $\gamma_1 = \gamma \cap X_1$ (resp. $\gamma_2 = \gamma \cap X_2$) a multicurve contained in X_1 (resp. X_2) representing the homology class $(m, n, 0, 0)$ (resp. $(0, 0, p, q)$). The length of γ is minimal if and only if both γ_1 and γ_2 are minimizing in their homology class, so by Lemma 6.1 the minimal length for γ is $\|(m, n)\|_{X_1} + \|(p, q)\|_{X_2}$.

Now assume that γ crosses the cylinder at least once. For simplicity, assume that it crosses the cylinder exactly once; the same argument will hold if γ crosses the cylinder more than once. Denote by $\gamma_C := \gamma \cap C$ the part of γ that lies in the cylinder and $\gamma_i = \gamma \cap X_i$ the part of γ inside of X_i. We have

$$l(\gamma) = l(\gamma_C) + l(\gamma_1) + l(\gamma_2).$$

As we have cut some piece of γ away, γ_1 and γ_2 are composed of an arc and possibly some closed curves. Now close the arc in γ_1 by adding the shortest path between its endpoints (the points where γ exits and re-enters X_1) that follows the edges of the slit. Note that this path has length at most ρ. We obtain a union $\tilde{\gamma}_1$ of closed curves in X_1, with $l(\tilde{\gamma}_1) \leq l(\gamma_1) + \rho$. Moreover, $\tilde{\gamma}_1$ represents the homology class $(m, n, 0, 0)$, so $\|(m, n)\|_{X_1} \leq l(\tilde{\gamma}_1)$, and, finally,

$$l(\gamma_1) \geq \|(m, n)\|_{X_1} - \rho.$$

The same construction in X_2 yields

$$l(\gamma_2) \geq \|(p, q)\|_{X_2} - \rho.$$

Finally, since γ crosses the cylinder, there is an arc going from X_1 to X_2 through the cylinder; but this arc is part of a closed curve so γ has to go back to X_1 eventually. Hence γ_C in a union of *two* arcs whose endpoints lie in X_1 and X_2, so

$$l(\gamma_C) \geq 2w,$$

where w is the width of the cylinder. Putting everything together we finally get that in this case,

$$l(\gamma) \geq \|(m, n)\|_{X_1} + \|(p, q)\|_{X_2} + 2(w - \rho),$$

so if $w > \rho$ this is strictly longer than in the case of curves that do not cross the cylinder. Hence the shortest representative of (m, n, p, q) in Σ is a multicurve that does not encounter the cylinder, and it has length $\|(m, n)\|_{X_1} + \|(p, q)\|_{X_2}$. □

Proof of Theorem B. — Assume that $w > \rho$. By the previous proposition, if $(m,n) \neq (0,0)$ and $(p,q) \neq (0,0)$, the point $\dfrac{(m,n,p,q)}{\|(m,n,p,q)\|_\Sigma}$ of the unit sphere of the stable norm of Σ in the direction (m,n,p,q) lies in the interior of the segment $\left[\dfrac{(m,n,0,0)}{\|(m,n,0,0)\|_\Sigma}, \dfrac{(0,0,p,q)}{\|(0,0,p,q)\|_\Sigma}\right]$.

Thus, the strictly convex directions of the unit ball of the stable norm of Σ must lie in the homology planes associated to the slit tori X_1 and X_2. With the same argument as for Theorem A, one can check that these strictly convex directions are exactly the directions of the vertices of the stable norm of the X_i's, and that the stable norm of Σ has vertices in those directions. More precisely, $(m,n,0,0)$ (resp. $(0,0,p,q)$) is the direction of a vertex of the stable norm of Σ if and only if (m,n) (resp. (p,q)) is the direction of a vertex of the stable norm of X_1 (resp. X_2). Theorem B is proved. Observe that the unit ball of the stable norm of Σ has flats of dimension 3, which is maximal since $H_1(\Sigma,\mathbb{R})$ has dimension 4.

Genus g construction. — A construction similar to the one of Σ provides half-translation surfaces of genus g, with $g \geq 2$, for which an analogue of Theorem B holds. Let $X_1, \ldots, X_g$ be g identical copies of the slit torus X. On X_i, label the left (resp. right)-hand side of the slit A_i (resp. B_i). Take $2g$ identical $w \times \rho$ rectangles $r_1, \ldots, r_{2g}$, with $w > \rho$. Now glue as follows.

- The left-hand side of r_{2i-1} (resp. r_{2i}) to A_i (resp. B_i).
- The upper (resp. lower) sides of r_{2i-1} and r_{2i} together.
- For $i \leq g-1$ the right-hand side of r_{2i} to the right-hand side of r_{2i+1}, and the right-hand side of r_{2g} to the right-hand side of r_1.

The resulting surface Σ_g is a closed genus g surface equipped with a flat metric with 3π angle conical singularities at each endpoint of the slits and two $g\pi$ angle conical singularities at points where the top and bottom right-hand corners of the rectangles were glued together.

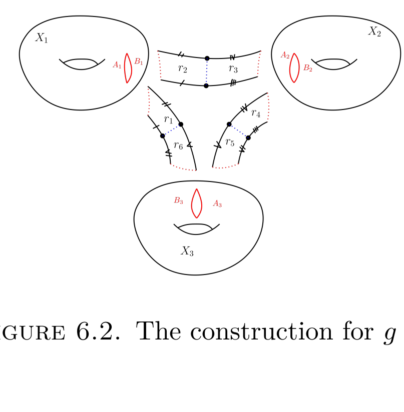

FIGURE 6.2. The construction for $g = 3$

The same arguments as in the case of Σ hold in this setting to prove an analogue of Theorem B. More precisely, if $(a_1, b_1, \dots, a_g, b_g)$ is an integral homology class on Σ_g, then

$$\|(a_1, b_1, \dots, a_g, b_g)\|_{\Sigma_g} = \sum_i \|(a_i, b_i)\|_{X_i}.$$

7. Counting simple homology classes

An integral homology class on a (possibly singular) Riemannian surface is *simple* if its stable norm is realized by a simple representative; in other words, there exists a minimizing representative of this class that is a simple closed curve. One natural question is a counting one: given a (possibly singular) Riemannian surface S and a positive number x, how many simple homology classes are there on S whose stable norm does not exceed x? It is known (see [11]) that on translation surfaces, this quantity grows quadratically in x. Since half-translation surfaces are so like translation surfaces, one would expect the number of simple homology classes to be quadratic in x as well on half-translation surfaces. But this is not true, and the aim of this section is to provide a counter-example.

Let X be the square torus with a vertical slit of length ρ. The surface that we are interested in is the genus 2 half-translation surface Σ from the previous section, obtained by glueing two copies of X along a flat cylinder. In Proposition 6.2 we saw that there are no connected minimizing representatives of a homology class $(m, n, p, q) \in H_1(\Sigma, \mathbb{Z})$, provided that $(m, n) \neq (0, 0)$ and $(p, q) \neq (0, 0)$, as any connected representative must cross the cylinder and, therefore, is too long to be minimizing. In particular, such a homology class cannot be simple; hence a necessary condition for a class (m, n, p, q) to be simple is $m = n = 0$ or $p = q = 0$. But by Lemma 6.1 the stable norm of Σ coincides with the stable norm of X on the homology planes $\mathrm{Span}((1, 0, 0, 0), (0, 1, 0, 0))$ and $\mathrm{Span}((0, 0, 1, 0), (0, 0, 0, 1))$. Thus, counting the simple classes on Σ comes down to counting twice the simple classes on X.

Proposition 7.1. — *Let $p(x) = \#\{h \in H_1(X, \mathbb{Z})$, with h simple and $\|h\|_X \leq x\}$. Then*

$$p(x) = 4 \left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b} \right) x \ln x + O(x),$$

where φ is Euler's totient function, and $\lfloor . \rfloor$ denotes the integral part.

Proof. — We use classical counting methods from analytic number theory; see, for instance, [1] or [23].

Expressing $p(x)$ as an integral. — Let $L := \{\|h\|_X \text{ with } h \in H_1(X,\mathbb{Z}) \text{ simple}\}$ be the simple length spectrum of X, where lengths are repeated with multiplicity. This is a countable set of positive numbers, and we order it by increasing order: $L = \{l_1 \leq l_2 \leq \ldots \leq l_n \leq \ldots\}$. For any complex number s with $\Re(s) > 1$, let

$$F(s) := \sum_{n=1}^{\infty} \frac{1}{l_n^s}.$$

This is a generalized Dirichlet series of the form $F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where for all $n, a_n = 1$ and $\lambda_n = \ln l_n$. Applying the Perron formula, for any $\sigma > 1$ and $y \in]\lambda_n, \lambda_{n+1}[$, we get

$$\sum_{k=1}^{n} a_k = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) \frac{e^{sy}}{s} ds.$$

The left-hand term is the number of λ_k such that $\lambda_k \leq y$, or equivalently this is the number of l_k such that $l_k \leq e^y$. By setting $x := e^y$, we get

$$p(x) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) \frac{x^s}{s} ds,$$

so all we have to do is to compute the right-hand side integral. To do this, we need to find a more practical expression of $F(s)$.

Rearranging the sum $F(s)$. — How are simple classes distributed in $H_1(X,\mathbb{Z})$? These classes correspond to the directions of the vertices of the unit ball of the stable norm of X, i.e., they are either visible classes or the Farey child of some visible class. Each visible class has infinitely many Farey children, which are all simple classes. Thus, we can rearrange the sum in $F(s)$:

$$F(s) = \sum_{(b,a) \text{ visible}} F_{b,a}(s) - E(s),$$

where, abusing the notations to say that (q,p) is the child of (b,a) if p/q is the Farey child of a/b,

$$F_{b,a}(s) = \frac{1}{\|(b,a)\|_X^s} + \sum_{(q,p) \text{ child of } (b,a)} \frac{1}{\|(q,p)\|_X^s},$$

and $E(s)$ is an excess correcting term, as we may count some classes multiple times.

A more precise expression for $F_{b,a}(s)$. — The successive Farey children of (b,a) are of the form $(kb+\beta, ka+\alpha)$ and $(kb+\delta, ka+\gamma)$, where (β,α) and (δ,γ) are two other visible classes such that $\alpha/\beta, a/b$ and γ/δ are consecutive neighbours in some Farey sequence, and $k \in \mathbb{N}^*$. Since (b,a) is visible, we have

$$\|(b,a)\|_X = \|(b,a)\|_2.$$

For k big enough, $(kb+\beta, ka+\alpha)$ is not a visible class, so by Theorem A

$$\begin{aligned}\|(kb+\beta, ka+\alpha)\|_X \\ = \sqrt{((k-1)b+\beta)^2 + ((k-1)a+\alpha\pm\rho)^2} + \sqrt{b^2+(a\mp\rho)^2}.\end{aligned}$$

Taking the Taylor series expansion of this expression, we get

$$\|(kb+\beta, ka+\alpha)\|_X = k\|(b,a)\|_2 + \frac{c}{k} + O\left(\frac{1}{k^2}\right),$$

where c is a constant independent of k. Thus,

$$\frac{1}{\|(kb+\beta, ka+\alpha)\|_X^s} = \frac{1}{k^s\|(b,a)\|_2^s} - \frac{sc}{k^s} + r_{1,k}(s),$$

where $|r_{1,k}(s)/k^{s+2}| \longrightarrow 0$ when k goes to infinity. Similarly, we have

$$\frac{1}{\|(kb+\delta, ka+\gamma)\|_X^s} = \frac{1}{k^s\|(b,a)\|_2^s} + \frac{sc}{k^s} + r_{2,k}(s),$$

where $|r_{2,k}(s)/k^{s+2}| \longrightarrow 0$ when k goes to infinity. Injecting this back into $F_{b,a}(s)$, we obtain

$$F_{b,a}(s) = \frac{1}{\|(b,a)\|_2^s} + 2\sum_{k=1}^{\infty}\frac{1}{k^s\|(b,a)\|_2^s} + \sum_{k=1}^{\infty}(r_{1,k}(s)+r_{2,k}(s)),$$

so

$$F_{b,a}(s) = \frac{1}{\|(b,a)\|_2^s}\,(2\zeta(s)+1) + R_{b,a}(s),$$

where

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s}$$

is the Riemann zeta function, and $|R_{b,a}(s)/\zeta(s+2)|$ is bounded for $\Re(s) > 1$.

Summing the $F_{b,a}$'s. — Summing over all visible classes, we have

$$\sum_{(b,a)\text{ visible}} F_{b,a}(s) = (2\zeta(s)+1)\sum_{(b,a)\text{ visible}}\frac{1}{\|(b,a)\|_2^s} + \sum_{(b,a)\text{ visible}} R_{b,a}(s).$$

Since the terms of the Farey sequence between two integers n and $n+1$ are simply the terms of the Farey sequence between 0 and 1 translated by n, we can rewrite the first right-hand side sum as follows:

$$\sum_{(b,a)\text{ visible}} \frac{1}{\|(b,a)\|_2^s} = \sum_{\substack{(b,a)\text{ visible}\\ 0<a/b<1}} \sum_{k\in\mathbb{Z}} \frac{1}{\|(b,a+kb)\|_2^s}$$
$$= 2\sum_{\substack{(b,a)\text{ visible}\\ 0<a/b<1}} \sum_{k\geq 0} \frac{1}{\|(b,a+kb)\|_2^s}.$$

When k goes to infinity, we have

$$\frac{1}{\|(b,a+kb)\|_2^s} = \frac{1}{bk^s} + O\left(\frac{1}{k^{s+2}}\right),$$

so

$$\sum_{(b,a)\text{ visible}} \frac{1}{\|(b,a)\|_2^s} = 2\left(\sum_{\substack{(b,a)\text{ visible}\\ 0<a/b<1}} \frac{\zeta(s)}{b}\right) + R_1(s),$$

where $|R_1(s)/\zeta(s+2)|$ is bounded for $\Re(s) > 1$. Thus, we have

$$\sum_{(b,a)\text{ visible}} F_{b,a}(s) = 4\left(\sum_{\substack{b\leq\lfloor 1/\rho\rfloor\\ \gcd(b,a)=1}} \frac{1}{b}\right)\zeta^2(s) + c'\zeta(s) + R_2(s),$$

where c' is a constant, and $|R_2(s)/\zeta^2(s+2)|$ is bounded for $\Re(s) > 1$. Finally, we have

$$\sum_{(b,a)\text{ visible}} F_{b,a}(s) = 4\left(\sum_{b=1}^{\lfloor 1/\rho\rfloor} \frac{\varphi(b)}{b}\right)\zeta^2(s) + c'\zeta(s) + R_2(s),$$

where φ is Euler's totient function.

Estimating the error term. — Did we count classes multiple times? Yes we did: every visible class (m,n) has been counted as many times as it has Farey ancestors. Indeed, it has been counted as a Farey child of each of its ancestors, and since the first coordinate of the Farey parents are strictly less than the first coordinate of this class, this means (m,n) has been counted at most m times. Now, the first coordinate of visible classes is bounded above by $\lfloor 1/\rho\rfloor$, so any visible class that we counted multiple times has been counted at most $\lfloor 1/\rho\rfloor$ times. Moreover, the first Farey child of two visible classes has been counted twice, once for each of its visible parents. Thus, even if we cannot give a precise expression for $E(s)$, we deduce that $E(s)$ is dominated by

$$\lfloor 1/\rho\rfloor \sum_{(q,p)\text{ with both visible parents}} \frac{1}{\|(q,p)\|_X^s}.$$

This, in turn, is dominated by

$$c'' \sum_{(b,a)\text{visible}} \frac{1}{\|(b,a)\|_X^s},$$

where c'' is a positive constant. As we have already seen, this quantity is equal to $c''\zeta(s)$ plus a remainder that is small when $\Re(s) \longrightarrow 1$.

An expression of $F(s)$. — Putting all the formulas together, we finally get

$$F(s) = 4\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b}\right)\zeta^2(s) + C\zeta(s) + R(s),$$

where C is a non-zero constant, and $|R(s)/\zeta(s)|$ bounded for $\Re(s) > 1$.

Computing the integral. — Recall that by Perron formula for any $\sigma > 1$ we have

$$p(x) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\frac{x^s}{s}ds.$$

Replacing $F(s)$ in the integral by its expression from the previous paragraph, we obtain three terms that we can estimate separately.

- By the Perron formula,

$$\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta^2(s)\frac{x^s}{s}ds = \sum_{n\le x}\tau(n),$$

  where $\tau(n) = \#\{d \text{ such that } d \text{ divides } n\}$. This is due to a classical property of the Riemann zeta function: the Dirichlet series expansion of $\zeta^2(s)$ is known to be $\sum_{n\ge 1}\tau(n)n^{-s}$. Another classical number theory computation yields

$$\sum_{n\le x}\tau(n) = x\ln x + O(x).$$

- Again, by the Perron formula, since $\zeta(s) = \sum_{n\ge 1} n^{-s}$, we have

$$\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s)\frac{x^s}{s}ds = \sum_{n\le x} 1 = O(x).$$

- We cannot compute explicitly the term containing the remainder $R(s)$. However, since $R(s)$ is small compared to $\zeta(s)$ when $\Re(s)$ goes to 1, and since the Perron formula holds for *any* $\sigma > 1$, by letting σ go to 1 we are assured that the integral term containing $R(s)$ is small compared to the other integral terms. In particular, this integral term is at most linear in x, so for σ close to 1

$$\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} R(s)\frac{x^s}{s}ds = O(x).$$

Summing these three terms with the right multiplicative constants we finally get

$$p(x) = 4\left(\sum_{b=1}^{\lfloor 1/\rho\rfloor}\frac{\varphi(b)}{b}\right)x\ln x + O(x).$$

This concludes the proof of Proposition 7.1. □

Since the surface Σ from the previous section is made by glueing two identical slit tori we have $p_\Sigma(x) = 2p(x)$, so multiplying the above expression by 2 we have proved Theorem C.

Genus g version of Theorem C. — Since we have a genus g analogue of Theorem B on the surface Σ_g constructed in the previous section, we can repeat the previous argument on the surface Σ_g constructed in the previous section, as $p_{\Sigma_g}(x) = g.p(x)$. Thus, for every $g \geq 2$ we obtain explicit half-translation surfaces of genus g on which the counting function grows as

$$p_{\Sigma_g}(x) = 4g\left(\sum_{b=1}^{\lfloor 1/\rho\rfloor}\frac{\varphi(b)}{b}\right)x\ln x + O(x).$$

Translation vs half-translation surfaces. — To conclude we would like to comment on the difference between translation surfaces and half-translation surfaces with respect to the stable norm.

Formally, a translation surface is a pair (S,ω), where S is a Riemann surface and ω is a holomorphic 1-form on S. The surface is equipped with the flat metric $|\omega|$, with conical singularities located at the zeroes of ω. It is known (due to Masur) that the number of embedded flat cylinders of length x in S grows quadratically as x goes to infinity. Another way to define such a cylinder is as a maximal set of parallel closed geodesics of length less than x. If γ is a geodesic in a cylinder, it represents a non-trivial homology class. Indeed, since the length of γ is

$$l(\gamma) = \int_\gamma |\omega| > 0,$$

in particular the pairing $\langle[\gamma],\omega\rangle = \int_\gamma \omega$ is non-zero, so $[\gamma]$ is non-zero in $H_1(X,\mathbb{R})$. One can show that geodesics in a cylinder are minimizing in their homology class; thus, when counting simple homology classes one can obtain a (quadratic) lower bound by counting cylinders, and we get a quadratic estimate for simple classes on translation surfaces.

The situation is different on half-translation surfaces. Formally, a half-translation surface is a pair (S, q), where q is a holomorphic quadratic differential, i.e., a section of the symmetric square of the holomorphic cotangent bundle of S. Again, the flat metric on S is given by $|q|$, and the conical singularities correspond to the zeroes of q. Again, one can count cylinders of parallel closed geodesics on S of length less than x and show that this number grows quadratically as x goes to infinity. However, this time, if γ is closed geodesic in a cylinder it may happen that it represents the trivial homology class. Indeed, since q is not a differential form, the formula $l(\gamma) = \int_\gamma |q| > 0$ does not say anything about the pairing $\eta \in H^1(S, \mathbb{R}) \mapsto \langle [\gamma], \eta \rangle$ and the homology class $[\gamma]$. For instance, on our surface Σ, any closed geodesic looping around the central cylinder has trivial homology. Hence on half-translation surfaces, there are fewer simple homology classes than there are cylinders of parallel geodesics. Thus, because of this difference between translation surfaces and half-translation surfaces, it makes sense to find a subquadratic estimate on the number of simple closed geodesics on a half-translation surface such as Σ.

Remark. — Theorem C is a counterexample to the statement on singular surfaces in [11, Proposition 3]. Indeed, this proposition provides a quadratic lower bound on the number of simple homology classes on surfaces with conical singularities, such as half-translation surfaces. The quadratic lower bound is obtained first on non-singular surfaces and extended to singular surfaces by taking a limit on the metric; but this does not work, as the geodesics do not transform continuously when deforming the metric. More precisely, the argument that fails here in that the bound on the number of connected components of minimizing curves is not preserved when taking the limit; on a non-singular surface, this number is bounded above by the genus of the surface, but on our genus 2 surface Σ a minimizing curve can have up to 4 connected components.

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BIBLIOGRAPHY

[1] T. M. Apostol – *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.

[2] I. Babenko – "Asymptotic volume of tori and the geometry of convex bodies. (Russian)", *Mat. Zametki* **44** (1988), no. 2, p. 177–190, 286; translation in Math. Notes 44 (1988), no. 1-2, 579–586 (1989).

[3] I. Babenko & F. Balacheff – "Sur la forme de la boule unité de la norme stable unidimensionnelle (French. English summary) [On the form of the unit ball of the one-dimensional stable norm", *Manuscripta Math.* **3** (2006), no. 119, p. 347–358.
[4] F. Balacheff & D. Massart – "Stable norms of non-orientable surfaces", *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 4, p. 1337–1369.
[5] J. Berstel, A. Lauve, C. Reutenauer & F. Saliola – *Combinatorics on words. Christoffel words and repetitions in words*, CRM Monograph Series, no. 27, American Mathematical Society, 2009.
[6] D. Burago, S. Ivanov & B. Kleiner – "On the structure of the stable norm of periodic metrics", *Math. Res. Lett.* **4** (1997), no. 6, p. 791–808.
[7] K. Daniel, A. Nash, S. Koening & A. Felner – "Theta*: Any-angle path planning on grids", *J. Artif. Intell. Res. (JAIR)* **39** (2014).
[8] A. Eskin, H. Masur & M. Schmoll – "Billiards in rectangles with barriers", *Duke Math. J.* **118** (2003), no. 3, p. 427–463.
[9] B. Farb & D. Margalit – *A primer on mapping class groups*, Princeton Mathematical Series, no. 49, Princeton University Press, Princeton, NJ, 2012.
[10] M. Gromov – "Structures métriques pour les variétés riemanniennes", in *Textes Mathématiques, 1* (J. Lafontaine & P. Pansu, eds.), CEDIC, Paris, 1981.
[11] E. Gutkin & D. Massart – "Stable norm on homology and geodesics on translation surfaces", Comunicaciones del Cimat, Comunicacion Tecnica No-I-02-18-03-09-2002.
[12] G. H. Hardy & E. M. Wright – *An introduction to the theory of numbers*, 6 ed., Oxford University Press, Oxford, 2008, Revised by D. R. Heath-Brown and J. H. Silverman. With a foreword by Andrew Wiles.
[13] A. Hatcher – *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
[14] G. Hedlund – "Geodesics on a two-dimensional Riemannian manifold with periodic coefficients", *Ann. of Math. (2)* **33** (1932), no. 4, p. 719–739.
[15] D. Massart – "Normes stables des surfaces", 1996, Mathématiques [math]. Ecole normale supérieure de Lyon - ENS LYON. Français. tel-00589624.
[16] D. Massart – "Stable norms of surfaces: local structure of the unit ball of rational directions", *Geom. Funct. Anal.* **7** (1997), no. 6, p. 996–1010.
[17] D. Massart & H. Parlier – "On the homology length spectrum of surfaces", *Int. Math. Res. Not. IMRN* **2017** (2017), no. 8, p. 2367–2401.
[18] H. Masur – "Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential", Math. Sci. Res. Inst. Publ., no. 10, p. 215–228, Springer, New York, 1988.

[19] ———, “The growth rate of trajectories of a quadratic differential”, *Ergodic Theory Dynam. Systems* **10** (1990), no. 1, p. 151–176.
[20] G. McShane & I. Rivin – “Simple curves on hyperbolic tori”, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), no. 12, p. 1523–1528.
[21] R. P. Osborne & H. Zieschang – “Primitives in the free group on two generators”, *Invent. Math.* **63** (1981), no. 1, p. 17–24.
[22] A. Sanchez – “Gaps of saddle connection directions for some branched covers of tori”, *Ergodic Theory Dynam. Systems* **42** (2022), no. 10, p. 3191–3245.
[23] G. Tenenbaum – *Introduction à la théorie analytique et probabiliste des nombres*, collection SMF, 1995.
[24] A. Zorich – “Flat surfaces”, p. 437–583, Springer, Berlin, 2006.