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CRYSTALLINE COHOMOLOGY OF RIGID ANALYTIC SPACES

by Haoyang Guo

Abstract. — In this article, we introduce the infinitesimal cohomology for rigid analytic spaces that are not necessarily smooth, with coefficients in a p-adic field or in Fontaine's de Rham period ring $\mathrm{B}_{\mathrm{dR}}^+$.

Résumé (*Cohomologie cristalline des espaces*). — Dans cet article, nous introduisons la cohomologie infinitésimale pour les espaces analytiques rigides qui ne sont pas nécessairement lisses, avec des coefficients dans un corps p-adique ou dans l'anneau des périodes de de Rham de Fontaine $\mathrm{B}_{\mathrm{dR}}^+$.

1. Introduction

1.1. Background. — Let X be a complex algebraic variety. Attached to the set of $\mathbb{C}$-points of X, there is a natural analytic structure which makes $X(\mathbb{C})$ a complex analytic space. This allows us to obtain a topological invariant of X via singular cohomology $\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C})$, which is computed transcendentally.

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As the topological space $X(\mathbb{C})$ comes from an algebraic variety, it is natural to ask whether we could compute this singular cohomology algebraically. When X is a smooth algebraic variety over $\mathbb{C}$, it is a result of Grothendieck ([19]) that singular cohomology is isomorphic to algebraic de Rham cohomology. Thus, there exists a natural isomorphism

$$\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \mathrm{H}^i(X,\Omega^\bullet_{X/\mathbb{C}}),$$

where $\Omega^i_{X/\mathbb{C}}$ is the sheaf of the i-th algebraic Kähler differentials over the variety X, and $\Omega^\bullet_{X/\mathbb{C}}$ is the algebraic de Rham complex. As a consequence, we get a purely algebraic way to compute singular cohomology group.

However, if X is non-smooth, the cohomology of the usual algebraic de Rham complex may fail to compute singular cohomology of $X(\mathbb{C})$ (c.f. [2, Example 4.6]). To get the correct answer, in particular to get an algebraic cohomology theory which computes singular cohomology, there are several methods modifying algebraic de Rham cohomology in the non-smooth setting:

(1) In [24], Hartshorne discovered that if X admits a closed immersion into a smooth variety Y, then the formal completion $\widehat{\Omega^\bullet_{Y/\mathbb{C}}}$ of the de Rham complex $\Omega^\bullet_{Y/\mathbb{C}}$ along $X \to Y$ computes singular cohomology. Precisely, there exists the following isomorphism:

$$\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \mathrm{H}^i(X,\widehat{\Omega^\bullet_{Y/\mathbb{C}}}).$$

The result was obtained independently by Deligne (unpublished) and by Herrera–Lieberman [25].

In the general case when X is not necessarily embeddable, there exists a ringed *infinitesimal site* $(X/\mathbb{C}_{\mathrm{inf}},\mathcal{O}_{X/\mathbb{C}})$ (or the crystalline site in characteristic zero) introduced by Grothendieck [20]. It can be shown that its cohomology $\mathrm{H}^i(X/\mathbb{C}_{\mathrm{inf}},\mathcal{O}_{X/\mathbb{C}})$ coincides with $\mathrm{H}^i(X,\widehat{\Omega^\bullet_{Y/\mathbb{C}}})$ whenever $X \to Y$ is a closed immersion into a smooth variety as above. In particular, we obtain a conceptual cohomology theory that is independent of immersions. Moreover, the method allows us to compute cohomology with nontrivial coefficients, where we could replace $\mathcal{O}_{X/\mathbb{C}}$ by vector bundles with flat connections (or in other words, *crystals*).

(2) Extending the de Rham complex of smooth $\mathbb{C}$-algebras via simplicial resolutions, one obtains the *(Hodge-completed) derived de Rham complex*, first invented by Illusie [28]. To any scheme X over $\mathbb{C}$, we can associate a filtered derived algebra $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$. It was shown by Illusie in loc. cit. that the cohomology of the derived de Rham complex $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$ is isomorphic to the Hartshorne's cohomology, assuming X is a local complete intersection. Later on, using Adams completion from the algebraic topology, Bhatt [6] shows that the comparison is true for any finite type scheme in characteristic zero, without the l.c.i condition. In particular, for an

arbitrary variety $X/\mathbb{C}$, we get the isomorphism

$$\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \mathrm{H}^i(X, \widehat{\mathrm{dR}}_{X/\mathbb{C}}).$$

Here we mention that the first graded piece of $\widehat{\mathrm{dR}_{X/\mathbb{C}}}$ is the cotangent complex $\mathbb{L}_{X/\mathbb{C}}$ up to a shift, which plays an important role in the deformation theory of schemes. Moreover, similarly to the universal property of the algebraic de Rham complex, it could be shown that the derived de Rham complex is the initial object among all filtered (derived) algebras $\mathcal{A}$ over X that is equipped with a homomorphism $\mathcal{O}_X \to \mathrm{gr}^0\mathcal{A}$ ([37]).

(3) Another modification of the de Rham complex is called the *Deligne–Du Bois complex*, introduced by Deligne and studied by Du Bois ([12]). The Deligne–Du Bois complex is defined via the cohomological descent for resolution of singularities. More precisely, the Deligne–Du Bois complex for X is defined as the limit of the de Rham complex of X_n

$$R\lim_{[n]\in\Delta} Rf_{n*}\Omega^\bullet_{X_n/\mathbb{C}},$$

where $f_\bullet : X_\bullet \to X$ is a simplicial variety constructed using blowups at smooth nowhere dense centers, such that each X_n is smooth over $\mathbb{C}$. It could be shown that singular cohomology of X is isomorphic to the cohomology of the Deligne–Du Bois complex. Moreover, the Deligne–Du Bois complex admits a finite *Hodge–Deligne filtration* where each graded piece is a bounded complex of coherent sheaves in the derived category. The induced filtration on cohomology is the Hodge filtration for the underlying mixed Hodge structure. Furthermore, the Deligne–Du Bois complex together with its filtration also admits a site-theoretical interpretation via the h-topology, where the latter is introduced by Voevodsky in [42]. The theory of h-cohomology of X is studied for example in [26] and [32].

The above provides three algebraic methods computing singular cohomology of a complex variety that is not necessarily smooth. It is then natural to ask whether we have an analogous picture in the non-archimedean geometry. The goal of our article is to study the theory of cohomology for non-smooth rigid spaces in non-Archimedean geometry, analogous to the three modifications for complex algebraic varieties as above.

To start, let us fix some notations. Let K be a p-adic extension of $\mathbb{Q}_p$; i.e., K is a field that is complete with respect to a non-Archimedean valuation extending that of $\mathbb{Q}_p$. In the 1960s, Tate introduced the notion of the *rigid analytic space* that forms a natural p-adic analogue of the complex analytic space. Here, similarly to complex analytic spaces, examples of rigid analytic spaces include analytifications of algebraic varieties over K.

As a p-adic field is totally disconnected, singular cohomology of a rigid space over K is not an interesting object. However, we could still define a de Rham complex $\Omega^\bullet_{X/K}$ of X, where each $\Omega^i_{X/K}$ is the sheaf of Kähler differentials that are continuous under p-adic topology. When X is smooth and proper over K, it can be shown that the cohomology of its de Rham complex behaves very well: it lives within cohomological degrees $[0, 2\dim(X)]$, such that each cohomology group is a finite dimensional K-vector space. In particular, when X is the analytification of a smooth proper algebraic variety over K, the p-adic analytic de Rham cohomology of X and the algebraic de Rham cohomology of the variety match up, so we get the correct Betti numbers.

In the following, we consider general rigid spaces over K that may not be smooth.

1.2. Main results. — Let X be a rigid space over K. We introduce the *infinitesimal site* X/K_{inf}, defined on the category of all pairs of rigid spaces (U,T) for nil closed immersions $U \to T$ over K, such that U is an open subset in X. Here a collection of maps $\{(U_i,T_i) \to (U,T)\}$ is a covering in this site if $\{T_i \to T\}$ is an open covering for the rigid space T. The infinitesimal site X/K_{inf} naturally admits an *infinitesimal structure sheaf* $\mathcal{O}_{X/K}$, sending a thickening (U,T) to the ring of global sections $\mathcal{O}_T(T)$. Moreover, the infinitesimal structure sheaf admits a surjection onto $\mathcal{O}_X$, and the kernel ideal $\mathcal{J}_{X/K}$ defines a natural filtration on $\mathcal{O}_{X/K}$. The induced filtration on the cohomology of $\mathcal{O}_{X/K}$ is called the *infinitesimal filtration.*

Now we can formulate our first main result.

Theorem 1.2.1. — *There is a K-linear cohomology theory*

$$X \mapsto R\Gamma_{\mathrm{inf}}(X/K) := R\Gamma(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K})$$

together with the filtration defined by $R\Gamma(X/K_{\mathrm{inf}}, \mathcal{J}^*_{X/K})$ for rigid spaces X over K, and it takes values in the filtered complete derived category of K-vector spaces. It satisfies the following properties:

(i) Explicit formula (Theorem 4.2.2): *Assume $X \to Y$ is a closed immersion into a smooth rigid space Y and J is the ideal sheaf defining X. Then $R\Gamma_{\mathrm{inf}}(X/K)$ is filtered isomorphic to the cohomology of the formal completion of the de Rham complex $\Omega^\bullet_{Y/K}$ along $X \to Y$:*

$$R\Gamma_{\mathrm{inf}}(X/K) \longrightarrow R\Gamma(X, \widehat{\Omega^\bullet_{Y/K}}),$$

where the j-th filtration on the right side is $R\Gamma(X, J^{j-\bullet}\widehat{\Omega^\bullet_{Y/K}})$.

In particular, when X itself is smooth over k, the infinitesimal cohomology coincides with the de Rham cohomology with its Hodge filtration.

(ii) Derived de Rham comparison (Theorem 5.5.5): *There exists a natural filtered morphism from cohomology of the analytic derived de Rham complex to $R\Gamma_{\mathrm{inf}}(X/K)$:*

$$R\Gamma(X, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}) \longrightarrow R\Gamma_{\mathrm{inf}}(X/K).$$

The map induces an isomorphism on their underlying complexes and is a filtered isomorphism if X is a local complete intersection.

(iii) Éh comparison (Theorem 6.2.2): *The cohomology $R\Gamma_{\mathrm{inf}}(X/K)$ admits a natural filtered morphism to éh de Rham cohomology introduced in* [22]*, inducing an isomorphism on their underlying complexes:*

$$R\Gamma_{\mathrm{inf}}(X/K) \longrightarrow R\Gamma(X_{\text{éh}}, \Omega^{\bullet}_{\text{éh}}).$$

In particular, the underlying complex of $R\Gamma_{\mathrm{inf}}(X/K)$ satisfies the éh-hyperdescent for rigid spaces X over K.

(iv) Finiteness (Theorem 6.3.1): *When X is proper of dimension n over K, the underlying complex of $R\Gamma_{\mathrm{inf}}(X/K)$ is a perfect K-complex that lives in cohomological degree* $[0, 2\dim(X)]$.

(v) Base extension (Corollary 6.4.4): *Assume K_0 is a subfield of K and X is a proper rigid space over K_0. Then the natural base extension map is a filtered isomorphism*

$$R\Gamma_{\mathrm{inf}}(X/K_0) \otimes_{K_0} K \longrightarrow R\Gamma_{\mathrm{inf}}(X_K/K).$$

(vi) Comparison with singular cohomology (Corollary 6.4.3): *Assume there exists an abstract isomorphism of fields $K \to \mathbb{C}$ and X is the analytification of a proper algebraic variety $\mathfrak{X}$ over K. Then we have a filtered isomorphism*

$$\mathrm{H}^i_{\mathrm{inf}}(X/K) \otimes_K \mathbb{C} \simeq \mathrm{H}^i_{Sing}(\mathfrak{X}(\mathbb{C}), \mathbb{C}),$$

where the latter is filtered by the algebraic infinitesimal filtration (cf. [6] *and* [24]*).*

Here the *underlying complex* of a filtered object is defined as the complex forgetting its filtration.

Remark 1.2.2 (Analytic derived de Rham complex). — In Theorem 1.2.1.(ii), the usual notion of the derived de Rham complex of Illusie is not suitable when we are working with rigid analytic spaces; instead, we modify the construction so that it is continuous under the p-adic topology. Our strategy is to first apply the (derived) p-adic completion to the algebraic derived de Rham complex for the rings of definitions over $\mathcal{O}_K$, then to consider the filtered completion of its generic fiber. This produces a filtered $\mathbb{E}_\infty$-algebra $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}$ in the derived (∞) category of sheaves of K-modules over X, whose graded pieces are wedge products of the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/K}$ introduced by Gabber–Ramero in [18, Section 7]. We refer readers to Subsection 5.3 for details.

This notion has been considered in [23], where Shizhang Li and the author show that applying at affinoid perfectoid algebras, the analytic derived de Rham complex can recover the de Rham period sheaves $\mathbb{B}^+_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ over the pro-étale site, previously introduced in [14] and [39]. We also want to mention that a version of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}$ for derived analytic stacks X has been considered independently by Jorge António in [1].

Remark 1.2.3 (Éh topology). — The éh cohomology in Theorem 1.2.1.(iii) is the cohomology of the éh-sheafification of the de Rham complex over the éh site, where the latter is analogous to Voevodsky's h-topology for algebraic schemes in the method (3) in Subsection 1.1. It is designed as the minimal refinement of étale topology that is locally smooth (thanks to resolution of singularities of rigid spaces in, for example, [11] and [40]) and is defined by adding universal homeomorphisms and coverings associated with blowups. It can be shown that when X is the analytification of a proper algebraic variety, the éh de Rham cohomology in Theorem 1.2.1.(iii) is filtered isomorphic to the cohomology of (analytification of) the Deligne–Du Bois complex ([22, Corollary 5.2.2]). So we recover the algebraic Deligne–Du Bois complex in the p-adic analytic setting.

Remark 1.2.4 (Éh de Rham complex). — Let $\pi : X_{\text{éh}} \to X_{\mathrm{rig}}$ be the natural map from the éh site to the rigid site introduced in [22]. The éh de Rham complex in Theorem 1.2.1.(iii) is obtained by applying the derived global section to the complex

$$R\pi_*\Omega^\bullet_{\text{éh}}.$$

Here, each $\Omega^i_{\text{éh}}$ is the éh-sheafification of the continuous sheaf of differentials for rigid spaces, and it is shown in [22, Section 6] that each $R\pi_*\Omega^i_{\text{éh}}$ is a complex of coherent $\mathcal{O}_X$-modules that lives in cohomological degree $[0, \dim(X)]$ and vanishes for $i > \dim(X)$. As a consequence, analogous to the classical Hodge–de Rham filtration on de Rham cohomology of complex varieties, (the underlying complex of) the infinitesimal cohomology $R\Gamma_{\mathrm{inf}}(X/K)$ can be computed via cohomology of (a finite number of) coherent sheaves and Theorem 1.2.1.(iii).

Remark 1.2.5 (Various filtrations). — Infinitesimal cohomology $R\Gamma_{\mathrm{inf}}(X/K)$ admits a natural filtered map to the usual de Rham cohomology of X, and the latter maps to éh de Rham cohomology. So summarizing the various filtrations in Theorem 1.2.1, we get the following sequence of maps in the filtered derived category:

$$R\Gamma(X, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}) \xrightarrow{(1)} R\Gamma_{\mathrm{inf}}(X/K) \xrightarrow{(2)} R\Gamma(X, \Omega^\bullet_{X/K}) \xrightarrow{(3)} R\Gamma(X_{\text{éh}}, \Omega^\bullet_{\text{éh}}),$$

enhancing the canonical maps on the underlying complexes. In particular, thanks to Theorem 1.2.1.(ii, iii), both the map (1) and the composition of (2) and (3) induce isomorphisms on the underlying complexes. As a consequence, (the underlying complex of) the infinitesimal cohomology forms a direct summand of the usual continuous de Rham cohomology.

Remark 1.2.6 (Crystals). — Similarly to the schematic crystalline theory, we also have a theory of crystals over the infinitesimal site. Moreover, it could be shown that analogous statements in Theorem 1.2.1.(i), (iii), (iv) and (v) hold true for cohomology of crystals. It is expected that Theorem 1.2.1.(ii) for derived de Rham complex also admits a generalization with coefficients via an approximate extension of classical crystals to the simplicial world. We leave this question to a future investigation.

Now let K be a complete and algebraically closed p-adic field. Let $\mathrm{B}_{\mathrm{dR}}^+$ be Fontaine's de Rham period ring of K in p-adic Hodge theory ([17]), and let ξ be a fixed generator of the kernel for the canonical surjection $\mathrm{B}_{\mathrm{dR}}^+ \to K$ (see Section 2 for definition). Recall for a smooth rigid space X over K that Bhatt–Morrow–Scholze introduced the *crystalline cohomology of X over* $\mathrm{B}_{\mathrm{dR}}^+$ ([8, Section 13]). It is defined locally via the inverse limit

$$\varprojlim_{e\in\mathbb{N}} \widehat{\Omega^\bullet_{Y_e/\Sigma_e}}.$$

Here, $\{Y_e\}$ is a compatible family of smooth adic spaces over $\Sigma_e := \mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^+/\xi^e)$ that admit a closed immersion from X, and $\widehat{\Omega^\bullet_{Y_e/\Sigma_e}}$ is the formal completion of the continuous de Rham complex $\Omega^\bullet_{Y_e/\Sigma_e}$ along $X \to Y_e$. It is shown in loc. cit. that the cohomology is independent of choices of closed immersions. Moreover, after inverting ξ, there exists a natural isomorphism between the crystalline cohomology of X in [8], and pro-étale cohomology of the de Rham period sheaves $\mathbb{B}_{\mathrm{dR}}$ for quasi-compact quasi-separated rigid spaces.

As the construction is analogous to the computation of the crystalline cohomology of schemes (see Subsection 1.1), in loc. cit., Bhatt–Morrow–Scholze expect that there is a conceptual crystalline theory for rigid spaces, defined similarly to the infinitesimal cohomology in the schematic theory, whose cohomology is isomorphic to the crystalline cohomology in [8] (see [8, Remark 13.2]). Our next goal is to answer this question. In fact, our project was initiated after the author read the question from [8].

Let us consider the infinitesimal site X/Σ_{inf}, which is defined on the category of nil closed immersions (U,T) such that U is open in X, $U \to T$ is a nil closed immersion, and T is a locally topological finite-type adic space over $\mathrm{B}_{\mathrm{dR}}^+/\xi^e$ for some $e \in \mathbb{N}$. The covering structure of X/Σ_{inf} is defined by the open coverings of adic spaces as the one in X/K_{inf}, and we can equip X/Σ_{inf} with a natural infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma}$ and, analogously, with an infinitesimal ideal sheaf $\mathcal{J}_{X/\Sigma}$.

Now we can state the next main result.

Theorem 1.2.7. — *There is a* $\mathrm{B}_{\mathrm{dR}}^+$*-linear cohomology theory*

$$X \mapsto R\Gamma_{\mathrm{inf}}(X/\Sigma) := R\Gamma(X/\Sigma_{\mathrm{inf}}\mathcal{O}_{X/\Sigma}),$$

together with the filtration defined by $R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{J}^*_{X/\Sigma})$ *for rigid spaces* X *over* K*, and it takes values in the filtered complete derived category over* $\mathrm{B}^+_{\mathrm{dR}}$ *(where the latter is equipped with* ξ*-adic filtration). It satisfies the following properties:*

(i) Reduction to K (Theorem 7.2.3.(iii)): *There exists a natural filtered base change isomorphism*

$$R\Gamma_{\mathrm{inf}}(X/\Sigma) \otimes^L_{\mathrm{B}^+_{\mathrm{dR}}} K \longrightarrow R\Gamma_{\mathrm{inf}}(X/K),$$

where $R\Gamma_{\mathrm{inf}}(X/K)$ *is the infinitesimal cohomology with its infinitesimal filtration in Theorem* 1.2.1*.*

(ii) Explicit formula (Theorem 7.2.3.(ii)): *Assume* $\{X \to Y_e\}_e$ *is a system of closed immersions from* X *to smooth adic spaces* Y_e *over* Σ_e*, such that they are compatible via isomorphisms* $Y_{e+1} \times_{\Sigma_{e+1}} \Sigma_e \simeq Y_e$*. Then there is a natural filtered isomorphism*

$$R\Gamma_{\mathrm{inf}}(X/\Sigma) \longrightarrow R\Gamma(X, \varprojlim_{e\in\mathbb{N}} \widehat{\Omega^\bullet_{Y_e/\Sigma_e}}),$$

where $\widehat{\Omega^\bullet_{Y_e/\Sigma_e}}$ *is the formal completion of the de Rham complex* $\Omega^\bullet_{Y_e/\Sigma_e}$ *along* $X \to Y_e$*.*

(iii) Derived de Rham comparison (Corollary 7.2.6): *There exists a natural filtered morphism inducing an isomorphism on underlying complexes*

$$R\Gamma_{\mathrm{inf}}(X/\Sigma) \longrightarrow R\Gamma(X, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma}),$$

where $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma}$ *is defined as the derived inverse limit of the underlying complexes of* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma_e}$ *over* $e \in \mathbb{N}$*.*

(iv) Éh hyperdescent (Corollary 7.2.8): *The underlying complex of* $R\Gamma_{\mathrm{inf}}(X/\Sigma)$ *satisfies the éh-hyperdescent for rigid spaces* X *over* K*.*

(v) Pro-étale comparison (Theorem 7.3.2): *There exists a natural* $\mathrm{B}^+_{\mathrm{dR}}$*-linear map from the underlying complex of* $R\Gamma_{\mathrm{inf}}(X/\Sigma)$ *to the pro-étale cohomology*

$$R\Gamma_{\mathrm{inf}}(X/\Sigma) \longrightarrow R\Gamma(X_{\mathrm{pro\acute{e}t}}, \mathbb{B}^+_{\mathrm{dR}}).$$

For quasi-compact quasi-separated rigid spaces X*, the above induces an isomorphism after inverting* ξ*:*

$$R\Gamma_{\mathrm{inf}}(X/\Sigma)[\frac{1}{\xi}] \simeq R\Gamma(X_{\mathrm{pro\acute{e}t}}, \mathbb{B}_{\mathrm{dR}}).$$

The map is $\mathrm{Gal}(K/K_0)$*-equivariant when* X *is isomorphic to* $X_0 \times_{K_0} K$ *for a rigid space* X_0 *over* K_0 *and a subfield* K_0 *of* K*.*

(vi) Finiteness (Proposition 7.2.9) and Torsion-freeness (Theorem 7.3.5): *When* X *is proper of dimension* n *over* K*, each cohomology group* $\mathrm{H}^i_{\mathrm{inf}}(X/\Sigma)$ *is a finite free* $\mathrm{B}^+_{\mathrm{dR}}$*-module, and it vanishes for* $i \notin [0, 2\dim(X)]$*.*

Remark 1.2.8. — The comparison with the derived de Rham complex over $\mathrm{B}_{\mathrm{dR}}^+$ in Theorem 1.2.7.(iii) is compatible with the one in Theorem 1.2.1.(ii) over K under the natural base change map in Theorem 1.2.7.(i).

Remark 1.2.9 (Torsion-freeness). — Theorem 1.2.7.(vi) shows that each cohomology group $\mathrm{H}^i_{\mathrm{inf}}(X/\Sigma)$ is finite free over $\mathrm{B}_{\mathrm{dR}}^+$ for X proper over K. This could be surprising at first sight; however, in the special case when X is defined over K_0 as in Corollary 1.2.13, the result follows easily from the base change formula, as in Corollary 1.2.13.

Remark 1.2.10 (Degeneracy of Hodge-éh de Rham). — As a byproduct of proving Theorem 1.2.7.(vi), we also show (in Theorem 7.3.5) that the Hodge-éh de Rham spectral sequence for X/K splits, where X is a proper rigid space. This strengthens the result of [22, Proposition 8.0.8], where we assumed X to be defined over K_0.

A consequence of the explicit computation in Theorem 1.2.7.(ii) is the following:

Corollary 1.2.11. — *The infinitesimal cohomology $R\Gamma_{\mathrm{inf}}(X/\Sigma)$ is isomorphic to the crystalline cohomology of X over $\mathrm{B}_{\mathrm{dR}}^+$ in the sense of* [8, Section 13].

Remark 1.2.12. — A variant of the infinitesimal site for smooth rigid spaces has been considered by Zijian Yao in [43, Section 5], where the crystalline cohomology of [8, Section 13] is reconstructed conceptually and compared with the pro-étale cohomology of the de Rham period sheaf. Using the Čech–Alexander complex, it can be shown that our $R\Gamma_{\mathrm{inf}}(X/\mathrm{B}_{\mathrm{dR}}^+)$ for smooth rigid spaces coincides with the crystalline cohomology of [43].

Finally, we comment on the case when X is defined over a discretely valued subfield. By the Primitive Comparison Theorem of Scholze [38] and the éh-hyperdescent in Theorem 1.2.7.(iv), we can compare our infinitesimal cohomology with étale cohomology.

Corollary 1.2.13. — *Let K_0 be a discretely valued subfield of K that has a perfect residue field, and let X be a proper rigid space over K_0.*

(i) *The cohomology $R\Gamma_{\mathrm{inf}}(X_K/\Sigma)$ can be defined over K_0. Thus, there exists a natural base extension formula for underlying complexes*[1]

$$R\Gamma_{\mathrm{inf}}(X_K/\Sigma) \simeq R\Gamma_{\mathrm{inf}}(X/K_0) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}^+.$$

1. The curious reader might ask how the filtrations relate to each other under this isomorphism. In fact, using the recent advance of condensed mathematics of Clausen–Scholze, one can extend the tensor product formula to a filtered enhancement, as in the sequel [21, Cor. 8.2.5, Rmk. 8.2.6].

(ii) *For every $n \in \mathbb{N}$, there exists a natural* $\mathrm{Gal}(K/K_0)$*-equivariant isomorphism*

$$\mathrm{H}^n_{\mathrm{inf}}(X_K/\Sigma)\left[\frac{1}{\xi}\right] \simeq \mathrm{H}^n_{\text{ét}}(X_K, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}.$$

Proof. — For (i), there exists a natural map from the right to the left, via a natural functor of infinitesimal sites defined by the base extension, and both sides are complete with respect to ξ-adic topology. When X is smooth proper over K_0, the infinitesimal cohomology complex can be computed using de Rham cohomology by Theorem 1.2.1.(i), so after a base change along $\mathrm{B}^+_{\mathrm{dR}} \to K$, part (i) can be reduced to the following known base change formula for the continuous de Rham cohomology:

$$R\Gamma_{\mathrm{dR}}(X/K_0) \otimes_{K_0} K \simeq R\Gamma_{\mathrm{dR}}(X_K/K).$$

In general, part (i) follows from the éh-hyperdescent among rigid spaces over K_0 in Theorem 1.2.1.(iii) and Theorem 1.2.7.(iv). Part (ii) follows from the Primitive Comparison Theorem ([38, Thm. 3.17]) and the éh-proétale comparison in [22, Theorem 1.1.4]. □

Remark 1.2.14 (Hodge–Tate filtration). — The natural filtration of B_{dR} induces a $\mathrm{Gal}(K/K_0)$-equivariant filtration on étale cohomology $\mathrm{H}^n_{\text{ét}}(X_K, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$. In fact, it is shown in [22, Theorem 1.1.4] that this filtration is isomorphic (via Corollary 1.2.13) to the product filtration for the filtration on B_{dR} and the éh–Hodge filtration on the infinitesimal cohomology $R\Gamma_{\mathrm{inf}}(X/K_0)$, defined by extending the Hodge filtration of the de Rham cohomology of smooth rigid spaces via éh hyperdescent. In particular, the graded pieces of the filtration on étale cohomology can be understood via the Hodge–Tate decomposition and éh cohomology ([22, Theorem 1.1.4]):

$$\mathrm{H}^n_{\text{ét}}(X_K, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \bigoplus_{i+j=n} \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh},K_0}) \otimes_{K_0} K(-j).$$

1.3. Summary of sections. — We now give a summary of each section in this article.

We start in Section 2 by introducing the (small) infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$ and its big site version $X/\Sigma_{e\mathrm{INF}}$. Here the base space Σ_e is defined as the adic space $\mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR,e}})$ for the p-adic Huber ring $\mathrm{B}^+_{\mathrm{dR,e}} := \mathrm{B}^+_{\mathrm{dR}}/\xi^e$ and $e \in \mathbb{N}_{>0}$. The discussion in this section is analogous to the discussion of the crystalline site and its topos for a pair of schemes, and we verify various formal properties, including the relation with the rigid topos, the comparison between the big and the small topoi, and the functoriality of the infinitesimal topoi. We also introduce the envelope for an immersion of rigid spaces $X \to Y$, regarded as either a colimit of representable sheaves in the infinitesimal topos or as a locally ringed space defined over the underlying topology of X. Here we mention that it is slightly different from the crystalline theory of a scheme over a p-nilpotent basis that an envelope in the infinitesimal site is almost never representable (see discussions in Subsection 2.2).

In Section 3, we study the coherent crystals over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$. As the base $\mathrm{B}^+_{\mathrm{dR,e}}$ has nilpotent elements, a coherent crystal $\mathcal{F}$ over $X/\Sigma_{e\,\mathrm{inf}}$ may not always be locally free. We show that $\mathcal{F}$ is a crystal in vector bundles if and only if it is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$ (see Definition 3.2.1 and Theorem 3.2.2) for details). Moreover, we prove that the category of coherent crystals is equivalent to the category of coherent sheaves with integrable connections over an envelope (Theorem 3.3.1).

Section 4 is devoted to a sheafified version of Theorem 1.2.1.(i) for general coherent crystals over $X/\Sigma_{e\,\mathrm{inf}}$ (cf. Theorem 4.2.2). Here, we adapt the idea from [7] for the computation: first we relate the Čech–Alexander complex and the de Rham complex of the envelope via a bicomplex, and then we show that the associated total complex converges to both of them. In particular, we improve the loc. cit. to a filtered isomorphism for the infinitesimal filtration via a finer and concrete study on graded pieces of the completed de Rham complex. We also obtain a base change formula for the cohomology sheaf over $\mathrm{B}^+_{\mathrm{dR,e}}$ for different $e \in \mathbb{N}$ (cf. Proposition 4.2.3).

In Section 5 we develop the foundations of the analytic derived de Rham complex, for a map of locally topological finite-type adic spaces over Σ_e. We first recall the basics of the analytic cotangent complex introduced by Gabber–Ramero in [18], of which we make a slight generalization from K-affinoid algebras to $\mathrm{B}^+_{\mathrm{dR,e}}$-affinoid algebras. We then introduce the analytic derived de Rham complex for affinoid algebras and show various properties thereof. We use the technique of hypersheaves and hyperdescent to extend the affinoid construction to the global setting, which we explain in Subsection 5.4. Finally, we compare the cohomology of the analytic derived de Rham complex to the infinitesimal cohomology in Subsection 5.5 and thus prove Theorem 1.2.1.(ii). Here we mention that we will use mildly the language of ∞-category in this section, which is mainly to incorporate the use of hypersheaves via a procedure of unfolding.

After that, we prove the éh-hyperdescent for the infinitesimal cohomology in Section 6. Here we first show the descent along a blowup square of rigid spaces over K in Theorem 6.1.2, following the strategy in [24, Chapter II]. The hyperdescent along an éh-hypercovering for the cohomology of a general crystal is then shown in Theorem 6.2.5, where we use the base change formula to reduce to the K-linear case in Theorem 6.2.2. This in particular implies Theorem 1.2.1.(iii). The rest of the section is devoted to the finiteness of the infinitesimal cohomology, the comparison with the algebraic cohomology, and a base field extension result, namely items (iv), (v), and (vi) in Theorem 1.2.1. We want to mention that as an object $X' \in X_{\text{éh}}$ is not necessarily an open subset of X, in order to make sense of éh descent, we need to use crystals over the big infinitesimal site X/K_{INF} in this section. However, we will not lose anything, for crystals and their cohomology are independent of working over big or small sites, thanks to Proposition 3.1.7 and Corollary 2.2.8.

At the end of the article, in Section 7, we consider the infinitesimal cohomology of a rigid space with coefficients in the de Rham period ring $\mathrm{B}_{\mathrm{dR}}^+$. Though the topology on the ring $\mathrm{B}_{\mathrm{dR}}^+$ is not p-adic, we could still define the infinitesimal site of X over $\mathrm{B}_{\mathrm{dR}}^+$, denoted as X/Σ_{inf}, by taking the union of all $X/\Sigma_{e\,\mathrm{inf}}$. We show that the cohomology of a crystal over X/Σ_{inf} is in fact the limit of the cohomology of its restrictions onto $X/\Sigma_{e\,\mathrm{inf}}$ in Theorem 7.2.3. In this way, we could apply results of previous sections to study the cohomology of X/Σ_{inf} and thus prove the Theorem 1.2.7 in the last two subsections.

As a convention, we will use the language of adic spaces throughout the article. We refer the reader to Huber's book [27] for basics of the theory.

2. Infinitesimal geometry over $\mathbf{B}_{\mathbf{dR,e}}^+$

In this section, we introduce the basics around the infinitesimal geometry over the de Rham period ring $\mathrm{B}_{\mathrm{dR,e}}^+ := \mathrm{B}_{\mathrm{dR}}^+/\xi^e$ and over a p-adic extension of $\mathbb{Q}_p$.

2.1. de Rham period ring and infinitesimal sites. — We first introduce the big and the small infinitesimal sites of a rigid space over them and study two natural maps between their topoi.

de Rham period rings. As a setup, we recall the basics of the de Rham period ring. A more detailed introduction of the de Rham period ring can be found in [17].

Let K be a p-adic valuation extension of $\mathbb{Q}_p$ that is complete and algebraically closed. Denote by $\mathcal{O}_K$ the ring of integers of K. Then we can define the p-adic ring $\mathrm{A}_{\mathrm{inf}}(\mathcal{O}_K)$ as

$$\mathrm{A}_{\mathrm{inf}} := W(\varprojlim_{x\mapsto x^p} \mathcal{O}_K).$$

There exists a canonical continuous surjection $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$, where the kernel $\ker(\theta)$ is principal. Fix a compatible system of p^n-th root of unity $\{\zeta_{p^n}\}_n$ in K. Then the element $\xi := \frac{[\epsilon]-1}{[\epsilon]^{\frac{1}{p}}-1}$ generates the ideal $\ker(\theta)$, where $[\epsilon]$ is the Teichmüller lift of the element $(\zeta_1, \zeta_p, \ldots)$ in $\mathrm{A}_{\mathrm{inf}}$.

The *de Rham period ring* $\mathrm{B}_{\mathrm{dR}}^+$ is defined as the ξ-adic completion of the ring $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$. By abuse of the notation, we write $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to K$ as the canonical continuous surjection induced from $\mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$. Note that for each $n \in \mathbb{N}$, we have

$$\mathrm{B}_{\mathrm{dR}}^{+}/\xi^{e} = \mathrm{A}_{\inf}\left[\frac{1}{p}\right]/\xi^{e},$$

which is a p-adic Tate ring with a canonical ring of definition $\mathrm{A}_{\inf}/\xi^e$ in it. So we can form a Huber pair $(\mathrm{B}_{\mathrm{dR}}^{+}/\xi^{e}, (\mathrm{B}_{\mathrm{dR}}^{+}/\xi^{e})^{\circ})$ over $(\mathbb{Q}_p, \mathbb{Z}_p)$ for $n \in \mathbb{N}$. The adic space $\Sigma_e := \mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^{+}/\xi^{e}, (\mathrm{B}_{\mathrm{dR}}^{+}/\xi^{e})^{\circ})$ is a nilpotent extension of $\mathrm{Spa}(K, \mathcal{O}_K)$.

In the rest of the article, we often use $\mathrm{A}_{\mathrm{inf,e}}$ and $\mathrm{B}_{\mathrm{dR,e}}^{+}$ to denote quotient rings $\mathrm{A}_{\inf}/\xi^e$ and $\mathrm{B}_{\mathrm{dR}}^{+}/\xi^{e}$, respectively, in order to simplify the notations.

Infinitesimal topology. We now introduce the infinitesimal site for rigid spaces.

Definition 2.1.1. — Let e be a positive integer. A *rigid space over* $\mathrm{B}_{\mathrm{dR,e}}^{+}$ is defined as an adic space of topological finite presentation over Σ_e. Thus, X can be covered by affinoid open subspaces which are of the form

$$\mathrm{Spa}(\mathrm{B}_{\mathrm{dR,e}}^{+}\langle t_1, \ldots, t_n\rangle/I),$$

where I is a (finitely generated) ideal in $\mathrm{B}_{\mathrm{dR,e}}^{+}\langle t_1, \ldots, t_n\rangle$.

The category of rigid spaces over Σ_e is denoted by $\mathrm{Rig}_{\Sigma_{\mathrm{e}}}$.

Recall that for a map of rigid spaces $f : U \to T$, it is called a *nil closed immersion* if f is a closed immersion (defined by the vanishing of a coherent ideal $\mathcal{I}$ in $\mathcal{O}_T$), such that T admits an open covering $\{T_i,\ i\}$, with $\mathcal{I}|_{T_i}$ being nilpotent. The closed immersion is called *nilpotent* if there is an integer $n \in \mathbb{N}$, such that $I^n = 0$. Note that a nilpotent closed immersion is always a nil closed immersion. The converse is true locally or assuming the quasi-compactness of the target space.

Definition 2.1.2. — (a) Let X be a rigid space over Σ_e. The *(small) infinitesimal site* $X/\Sigma_{e\,\inf}$ is the site defined as follows:

- The underlying category of $X/\Sigma_{e\,\inf}$ is the collection of pairs (U, T), called the *infinitesimal thickening*, where T is a rigid space over Σ_e and U is an open subspace of X and a closed analytic subspace of T, such that $U \to T$ is a nil closed immersion.
  Here, morphisms between (U_1, T_1) and (U_2, T_2) are defined as maps of pairs over Σ_e such that $U_1 \to U_2$ is an open immersion inside X.
- A collection of morphisms $(U_i, T_i) \to (U, T)$ in $X/\Sigma_{e\,\inf}$ is a covering if both $\{T_i \to T,\ i\}$ and $\{U_i \to U,\ i\}$ are open coverings for the rigid spaces T and U, respectively.

(b) The *big infinitesimal site* $\mathrm{Rig}_{\Sigma_{\mathrm{e}},\mathrm{INF}}$ over Σ_e is defined on the category of all of the pairs (U, T) for $U \to T$ being a nil closed immersion of rigid spaces over Σ_e, with the same covering structure as above.

(c) The *big infinitesimal site* $X/\Sigma_{e\mathrm{INF}}$ *of* X is defined as the localization $\mathrm{Rig}_{\Sigma_{\mathrm{e}},\mathrm{INF}}|_X$ of the big site $\mathrm{Rig}_{\Sigma_{\mathrm{e}},\mathrm{INF}}$ at X, namely it is defined on the category of all of the tuples $\{(U,T), f: U \to X\}$, where (U,T) is an object in $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}$, and $f: U \to X$ is a map of rigid spaces over Σ_e. The covering structure is induced from that of $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}$.

By the definition of the infinitesimal site above, a sheaf $\mathcal{F}$ over the infinitesimal site is equivalent to the data of $\mathcal{F}_T$ and φ_g as below:

- a sheaf $\mathcal{F}_T$ over the rigid space T for each infinitesimal thickening $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$;
- a map of sheaves over T_1,

$$\varphi_g : g^{-1}\mathcal{F}_{T_2} \longrightarrow \mathcal{F}_{T_1},$$

  for a given morphism of infinitesimal thickenings $(i,g) : (U_1,T_1) \to (U_2,T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$;

together with the natural cocycle condition, namely for given morphisms $(U_1,T_1) \xrightarrow{(i,g)} (U_2,T_2) \xrightarrow{(j,h)} (U_3,T_3)$ of infinitesimal thickenings, we have equalities of maps

$$\varphi_{h\circ g} = g^{-1}\varphi_h \circ \varphi_g.$$

The same holds for a sheaf over the big infinitesimal site. We call the category of sheaves on $X/\Sigma_{e\,\mathrm{inf}}$ (or $X/\Sigma_{e\mathrm{INF}}$) the *infinitesimal topos* and denote it by $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ (or $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$).

For two sheaves $\mathcal{F}$ and $\mathcal{G}$ over the infinitesimal site, we sometimes use the notation $\mathcal{F}(\mathcal{G})$ to denote the set of homomorphisms

$$\mathrm{Hom}(\mathcal{G},\mathcal{F}).$$

In the case where $\mathcal{G}$ is a representable sheaf h_T for an infinitesimal thickening (U,T), the above hom set is the set of sections

$$\mathrm{Hom}(h_T,\mathcal{F}) = \mathcal{F}(U,T).$$

There is a natural *structure sheaf* $\mathcal{O}_{X/\Sigma_e}$ over the big or small infinitesimal site, which is defined as

$$\mathcal{O}_{X/\Sigma_e}(U,T) := \mathcal{O}_T(T),\ (U,T) \in X/\Sigma_{e\,\mathrm{inf}}.$$

Here we note that by the equivalent description right below Definition 2.1.2, the above formula is naturally a sheaf. On the other hand, one can define the *analytic structure sheaf* $\mathcal{O}_X$ on the infinitesimal sites via the formula

$$\mathcal{O}_X(U,T) := \mathcal{O}_U(U),\ (U,T) \in X/\Sigma_{e\,\mathrm{inf}}.$$

By construction, there is a natural surjection of sheaves $\mathcal{O}_{X/\Sigma_e} \to \mathcal{O}_X$, and we define the *infinitesimal ideal sheaf* $\mathcal{J}_{X/\Sigma_e}$ to be the sheaf of kernel ideals for the surjection.

Remark 2.1.3. — It is clear from the above definition that the infinitesimal site can be defined for any pair of analytic adic spaces $X \to Z$, not just for $X \to \Sigma_e$. In particular, when $Z = \mathrm{Spa}(K_0)$ is a discretely valued field and X is a rigid space over K_0, we get the analogous version of the infinitesimal site of X over K_0. Moreover, there exists a natural map of sites $X_K/K_{\mathrm{inf}} \to X/K_{0,\mathrm{inf}}$, defined by the base field extension.

Here are some basic properties of the infinitesimal sites:

Lemma 2.1.4. — *Let X be a rigid space over Σ_e. Then we have*

(i) *The fiber product exists in the big and the small infinitesimal site of X over Σ_e and is compatible with the inclusion functor between the big and the small sites.*

(ii) *The equalizer exists in the big and the small infinitesimal site of X over Σ_e and is compatible with the inclusion functor as in (i).*

(iii) *The nonempty finite product is ind-representable in the big and the small infinitesimal site of X over Σ_e and is compatible with the inclusion functor as in (i).*

Proof. — (i) Let (V_i, T_i) for $i = 0, 1, 2$ be three objects in the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$, with arrows $g_i : (V_i, T_i) \to (V_0, T_0)$ for $i = 1, 2$. Thus, each V_i admits a map to X, and $V_i \to T_i$ is a closed immersion that has a nil defining ideal. Then we can form the fiber products of rigid spaces $V_3 := V_1 \times_{V_0} V_2$ and $T_3 := T_1 \times_{T_0} T_2$ over Σ_e, together with a natural map $V_3 \to T_3$. Here the existence of fiber products is guaranteed in [27, Prop. 1.2.2]. Any infinitesimal thickening (V, T) that admits a compatible family of maps $(V, T) \to (V_i, T_i)$ for $i = 0, 1, 2$ would produce a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & T \\ \downarrow & & \downarrow \\ V_3 & \longrightarrow & T_3. \end{array}$$

So it is left to show that $V_3 \to T_3$ is a nil closed immersion, which can be checked locally by choosing affinoid open subsets of $T_i, i = 0, 1, 2$, as we shall explain.

For $i = 0, 1, 2$, we let T_i be the affinoid adic space $\mathrm{Spa}(B_i)$ and let each closed subspace V_i be defined by a nilpotent ideal $I_i \subset B_i$. We let A_i for each $i = 0, 1, 2$ be a compatible choice of subrings of definition of B_i and let J_i be the intersection $I_i \cap A_i$, which satisfies the equality that $J_i[1/p] = I_i$ by construction and in particular is nilpotent. Under the assumption, the fiber product T_3 is $\mathrm{Spa}((A_1 \otimes_{A_0} A_2)^\wedge_p[1/p])$, and the fiber product V_3 is $\mathrm{Spa}((A_1/J_1 \otimes_{A_0/J_0} A_2/J_2)^\wedge_p[1/p])$. Notice that as the kernel ideal J of the surjection $A_1 \otimes_{A_0} A_2 \to A_1/J_1 \otimes_{A_0/J_0} A_2/J_2$ is generated by the image of the nilpotent ideals J_0, J_1, and J_2, the ideal J in particular is nilpotent itself. As a consequence, since the kernel for the p-completed surjection $(A_1 \otimes_{A_0} A_2)^\wedge_p \to (A_1/J_1 \otimes_{A_0/J_0} A_2/J_2)^\wedge_p$ is generated by the image of J, we see that the closed immersion $V_3 \to T_3$ is indeed a nilpotent closed immersion.

Finally, we note that when (V_1, T_i) comes from the small site for $i = 0, 1, 2$ (namely the map $V_i \to X$ is an open immersion for $i = 0, 1, 2$), then the fiber product $V_3 = V_1 \times_{V_0} V_2$ is also open in X. In particular, the fiber product in this case lies in the small site $X/\Sigma_{e\,\mathrm{inf}}$.

(ii) For the equalizer, consider the two arrows $\alpha, \beta : (V_1, T_1) \rightrightarrows (V_2, T_2)$ in $X/\Sigma_{e\mathrm{INF}}$. Here both V_1 and V_2 admit a map to X, and $V_i \to T_i$ are nil closed immersions. We can first form the equalizer V_3 of $V_1 \rightrightarrows V_2$ and T_3 of $T_1 \rightrightarrows T_2$ in the category of rigid spaces over Σ_e by the pullback diagram

$$\begin{array}{ccc} V_3 & \longrightarrow & V_1 \\ \downarrow & & \downarrow \\ V_2 & \longrightarrow & V_2 \times_{\Sigma_e} V_2, \end{array} \qquad \begin{array}{ccc} T_3 & \longrightarrow & T_1 \\ \downarrow & & \downarrow \\ T_2 & \longrightarrow & T_2 \times_{\Sigma_e} T_2, \end{array}$$

where the bottom horizontal maps in both diagrams are diagonal embeddings. The left diagram admits a natural map to the right. Moreover, we notice that $V_3 \to T_3$ is a nil closed immersion, as all of the other three terms in the diagram of V_3 are nil immersed into the diagram of T_3. Furthermore, as the map $V_1 \to V_2 \times_{\Sigma_e} V_2$ factors through $V_2 \times_X V_2 \to V_2 \times_{\Sigma_e} V_2$, the pullback V_3 is also isomorphic to the equalizer of $V_1 \rightrightarrows V_2$ in the category of rigid spaces over X. In this way, the object $(V_3, T_3) \in X/\Sigma_{e\mathrm{INF}}$ obtained above forms the equalizer of α, β in the category.

We last note that the case when α, β comes from the small site is exactly when both of the arrows $V_1 \rightrightarrows V_2$ are open immersions (hence they are the same), where the obtained base change $V_3 \simeq V_2 \underset{V_2 \times_X V_2}{\times} V_1 \simeq V_2 \times_{V_2} V_1 = V_1$ is also open in X. Thus the construction of the equalizer is compatible with the one in the small site.

(iii) Let (V_i, T_i) for $i = 1, 2$ be two objects in the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. Then we can form the fiber product $V_3 := V_1 \times_X V_2$ over X and the fiber product $T_1 \times_{\Sigma_e} T_2$ over Σ_e together with a natural map from V_3, such that any object $(V', T') \in X/\Sigma_{e\mathrm{INF}}$ that admits a map to (V_i, T_i) for $i = 1, 2$ will admit a unique map onto the pair of rigid spaces $(V_3, T_1 \times_{\Sigma_e} T_2)$.

Now the only problem is that the pair $(V_3, T_1 \times_{\Sigma_e} T_2)$ is almost never a pair of infinitesimal thickening. However, notice that the map $V_3 \to T_1 \times_{\Sigma_e} T_2$ can be written as the composition

$$V_3 = V_1 \times_X V_2 \longrightarrow V_1 \times_{\Sigma_e} V_2 \longrightarrow T_1 \times_{\Sigma_e} T_2,$$

where the first map is a locally closed immersion (a composition of a closed immersion and an open immersion), and the second map is a nil closed immersion. This allows us to form the direct limit[2] $\varinjlim_m Y_m$ of all infinitesimal neighborhoods of V_3 into $T_1 \times_{\Sigma_e} T_2$, where each Y_m is the m-th infinitesimal neighborhood of V_3 inside of $T_1 \times_{\Sigma_e} T_2$. In this way, the fiber product of (V_1, T_1) and (V_2, T_2) is ind-represented by the colimit of (V_3, Y_m), for locally, each map from an object (V', T') onto the pair $(V_3, T_1 \times_{\Sigma_e} T_2)$ factors through some (V_3, Y_m) by Lemma 2.2.5.

Finally, we note that the construction is independent of big or small infinitesimal sites. Moreover, when V_1 and V_2 are open in X, from the construction above, the rigid space V_3 is also open in X. Thus, the nonempty finite product is compatible between the big and the small sites. □

REMARK 2.1.5. — In fact, the ind-representable sheaf for the directed limit $\varinjlim_m Y_m$ is the *envelope* of the immersion $V_3 \to T_1 \times_{\Sigma_e} T_2$, which we will introduce in Definition 2.2.1 soon.

Relation between big and small sites/topoi. Given a rigid space X over Σ_e, there are two natural morphisms of topoi between the big infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ and the small infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ of X. To see this, we first notice that by constructions, there exists a natural inclusion functor

$$X/\Sigma_{e\,\mathrm{inf}} \longrightarrow X/\Sigma_{e\mathrm{INF}}.$$

The inclusion functor is *continuous* in the sense of [3, Tag 00WV] and thus induces two functors between their topoi ([3, Tag 00WU]):

- For a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ over the small site, there exists a preimage functor μ^{-1}, with $\mu^{-1}\mathcal{F}$ being the sheaf associated with the presheaf

$$X/\Sigma_{e\mathrm{INF}} \ni (V, S) \longmapsto \varinjlim_{\substack{(V,S)\to(U,T) \\ (U,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{F}(U, T).$$

  By Lemma 2.1.4, the functor μ^{-1} commutes with nonempty finite limits.
- The direct image functor μ_*, which is the right adjoint of μ^{-1} and is computed by the restriction. Thus, for a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ over the big site, we have $\mu_*\mathcal{G}(U, T) = \mathcal{G}(U, T)$.

This pair of adjoint functors in fact forms a morphism of topoi

$$\mu : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}).$$

To see this, we claim the following:

LEMMA 2.1.6. — *The left adjoint functor $\mu^{-1} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ commutes with any nonempty finite limit.*

2. The direct limit is called the *envelope* for the locally closed immersion and will be formally introduced in Section 2.2, to which we refer the reader for detailed discussions.

Proof. — To see this, we first notice that as a left adjoint functor commutes with any small colimit, by [3, Tag 0GLW], it suffices to show this for a finite diagram of representable sheaves. Moreover, as a nonempty finite limit can be formed by a finite number of nonempty finite products and equalizers ([3, Tag 04AS]), it suffices to show that μ^{-1} commutes with finite products and equalizers of representable sheaves, which is given by Lemma 2.1.4. So we are done. □

The above, by definition, means that the left adjoint functor μ^{-1} is exact, when we transition to the minimal enlargement of the infinitesimal sites by adding the final object (cf. [3, Tag 03A1])[3], and hence we get a morphism of topoi ([3, Tag 00X1])

$$\mu : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}).$$

On the other hand, the inclusion functor is *cocontinuous* in the sense of [3, Tag 00XJ]. This is because if a collection of thickenings $\{(U_i, T_i)\} \subset X/\Sigma_{e\mathrm{INF}}$ covers a given $(U, T) \in X/\Sigma_{e\,\mathrm{inf}}$ in the big site, then each (U_i, T_i) is also an object in the small site, which together form a covering of (U, T). So by [3, Tag 00XO], the inclusion functor induces another map of topoi

$$\iota : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}),$$

consisting of the following adjoint pairs of functors:

- The functor $\iota^{-1} = \mu_* : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ is the restriction functor, which commutes with any finite limits.
- The functor $\iota_* : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$, which is the right adjoint of the functor ι^{-1}, sending a sheaf $\mathcal{F}$ over the small site to the sheaf $\iota_*\mathcal{F}$ with the equality

$$X/\Sigma_{e\mathrm{INF}} \ni (V, S) \longmapsto \varprojlim_{\substack{(V,S)\to(U,T) \\ (U,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{F}(U, T).$$

Here, we notice that when the thickening (V, S) is an object coming from the small site $X/\Sigma_{e\,\mathrm{inf}}$ (namely $V \to X$ is an open immersion), from the description above, we then have

$$(\iota_*\mathcal{F})(V, S) = \mathcal{F}(V, S).$$

Furthermore, notice that given an arrow $(V_1, T_1) \to (V_2, T_2)$ in the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$, the associated morphism of rigid spaces $V_1 \to V_2$ is an X-morphism. This in particular implies that the inclusion functor $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e\mathrm{INF}}$ is fully faithful, as when (V_1, T_1) and (V_2, T_2) come from the small

3. Precisely, as both infinitesimal sites do not admit the final object (equivalently the empty product), it does not make sense to talk about the right exactness of the functor μ^{-1}. To remedy this, one can enlarge the sites by adding the final objects simultaneously, which will not change the corresponding topoi by loc. cit.. In particular, the induced functor of μ^{-1} on the enlarged sites preserves all finite limits.

site, the only X-morphism between V_1 and V_2 is the open immersion. So by [3, Tag 00XS, Tag 00XT] and Lemma 2.1.4, we have[4]

- The functor μ^{-1} commutes with fiber products and equalizers (so with all finite connected limits).
- The canonical natural transformations below are isomorphisms of functors:

$$\mathrm{id} \longrightarrow \mu_* \circ \mu^{-1};\ \iota^{-1} \circ \iota_* = \mu_* \circ \iota_* \longrightarrow \mathrm{id}\,.$$

2.2. Envelopes. — Analogous to the infinitesimal theory of complex varieties in [20] and the crystalline theory of schemes in positive characteristic in [5], we can define the envelope for a locally closed immersion $X \to Y$ of rigid spaces.

Definition 2.2.1. — Let Y be a rigid space over Σ_e and X be a locally closed analytic subspace in Y, defined by a coherent ideal I in $\mathcal{O}_U$ for U an open subset inside of Y. We denote by Y_n the n-th infinitesimal neighborhood of X in Y, which forms an object (X, Y_n) in $X/\Sigma_{e\,\mathrm{inf}}$ and is defined by the ideal I^{n+1}.

The *envelope* $D_X(Y)$ *of* X *in* Y is an object in the infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, defined by the colimit of the direct system of representable sheaves h_{Y_n} of (X, Y_n) in $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$:

$$D_X(Y) := \varinjlim_{n\in\mathbb{N}} h_{Y_n}.$$

Note that the definition also works for the big infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$, and under the natural inclusion functor $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e\mathrm{INF}}$, the notions of the envelopes coincide.

Remark 2.2.2. — In many situations, it is convenient to regard $D_X(Y)$ as an actual locally ringed space, instead of a direct limit of representable sheaves in the infinitesimal topos. Here the associated ringed space structure of the envelope $D_X(Y)$ has the same topological space as the adic space X, and the structure sheaf $\mathcal{D} = \varprojlim_n \mathcal{O}_{Y_n}$ is the inverse limit of structure sheaves of infinitesimal neighborhoods Y_n.

Remark 2.2.3. — The existence of the colimit in the topos is guaranteed by [3, Tag 00WI].

Remark 2.2.4. — Here we want to mention that different from the crystalline theory of a scheme over $\mathbb{Z}_p/p^e$, the envelope is almost *never* representable. In

4. In the notation of [3, Tag 00XR], the functor μ^{-1} is equal to the functor $\iota_!$.

the mixed-characteristic case, the divided-power structure enforces the defining ideal for a divided-power thickening to be nilpotent. However, in equal-characteristic zero such a condition is lost and the envelope is not an infinitesimal thickening. This in particular appears when we consider the crystalline theory of a scheme over $\mathbb{C}$.

Though the envelope fails to be representable, we do have a description of an envelope that is similar to a representable sheaf:

Lemma 2.2.5. — *For a closed immersion $X \to Y$ of rigid spaces over Σ_e, the envelope $D_X(Y)$ is isomorphic to the sheaf on $X/\Sigma_{e\,\mathrm{inf}}$ (and $X/\Sigma_{e\mathrm{INF}}$), defined by*

$$(U,T) \longmapsto \mathrm{Hom}((U,T),(X,Y)),$$

where $\mathrm{Hom}((U,T),(X,Y))$ *is the set of commutative diagrams of Σ_e-rigid spaces*

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \uparrow & & \uparrow \\ U & \longrightarrow & X \end{array},$$

with $U \to X$ being the structure morphism for the object (U,T).

Proof. — We first notice that we have a natural map

$$D_X(Y)((U,T)) = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}((U,T),(X,Y_n)) \longrightarrow \mathrm{Hom}((U,T),(X,Y)),$$

induced by closed immersions $Y_n \to Y_{n+1} \to Y$. So it suffices to check that for a pair of affinoid rigid spaces $(U,T) = (\mathrm{Spa}(R/J), \mathrm{Spa}(R))$ in the infinitesimal site, the above is an isomorphism.

For the surjection, we notice that since (U,T) is an affinoid rigid space over Σ_e, the ring R is Noetherian and J is nilpotent. In particular, there exists an $n \in \mathbb{N}$, such that $J^{n+1} = 0$. So the map $\mathrm{Spa}(R) \to Y$ factors through a map $\mathrm{Spa}(R) \to Y_n$.

For the injection, assume there are two maps $\alpha, \beta : T \to Y_n$ of rigid spaces over Σ_e whose compositions with $Y_n \to Y$ are equal. Note that since $Y_n \to Y$ is a closed immersion, by restricting to an affinoid open covering of Y (thus Y_n), the compositions can be translated into the following maps of $\mathrm{B}^+_{\mathrm{dR,e}}$-algebras:

$$A \to A/I^{n+1} \to R.$$

So the equality of the maps $A \to R$ implies that the maps $A/I^{n+1} \to R$ are equal and hence implies the equality of α, β. □

The following simple observations justify this name of the envelope:

Lemma 2.2.6. — *Assume Y is smooth over Σ_e. Then the envelope $D_X(Y)$ for a closed immersion of X in Y covers the final object in the infinitesimal topoi $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ and $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$. In other words, the map from $D_X(Y)$ onto the final object in the infinitesimal topoi is an epimorphism of sheaves.*

Proof. — We denote by 1 the final object in $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ or $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$. Then, to show the surjection of the map of sheaves

$$D_X(Y) \longrightarrow 1,$$

it suffices to show that any object (U,T) in the infinitesimal site locally admits a morphism to $D_X(Y)$, namely there is an open cover (U_i,T_i) of (U,T) such that each (U_i,T_i) admits a map to $D_X(Y)$.

For an affinoid thickening $(U,T) = (\mathrm{Spa}(R/I), \mathrm{Spa}(R))$ with an open immersion $U \to X$, since $U \to T$ is a nil closed immersion and R is Noetherian, there exists an integer m such that $I^{m+1} = 0$ in R. By assumption that Y is smooth, locally there exists a morphism from $\mathrm{Spa}(R)$ to Y that makes the following diagram commute (cf. [27, Def. 1.6.5]):

$$\begin{array}{ccc} \mathrm{Spa}(R/I) & \longrightarrow & \mathrm{Spa}(R) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

By the nilpotence of the ideal I, the map $\mathrm{Spa}(R) \to Y$ factors canonically through Y_n for $n \geq m$. Thus the map $\mathrm{Spa}(R) \to Y$ factors through the direct limit $D_X(Y) = \varinjlim_{n\in\mathbb{N}} h_{Y_n} \to Y$. □

The above allows us to give a very general formula to compute the cohomology over the infinitesimal site, using the Čech nerve for an envelope.

Proposition 2.2.7. — *Let $X \to Y$ be a closed immersion into a smooth rigid space Y over Σ_e. For $n \in \Delta$, we denote $D(n)$ to be the simplicial space where each $D(n)$ is the envelope of X in $Y(n) := Y^{\times_{\Sigma_e} n+1}$. There is then a natural isomorphism of cohomology for a sheaf $\mathcal{F}$ over the small infinitesimal site*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}) \longrightarrow R \lim_{[n]\in\Delta} R\Gamma(D(n), \mathcal{F}).$$

It similarly holds for the big infinitesimal site.

Here we want to mention that for each $n \in \Delta$, the derived section functor $R\Gamma(D(n), \mathcal{F})$ is computed via the inverse limit

$$R \varprojlim_{m\in\mathbb{N}} R\Gamma((X, Y(n)_m), \mathcal{F}),$$

where each $Y(n)_m$ is the m-th infinitesimal neighborhood of X in $Y(n)$.

Proof. — We first notice that $D(n)$ is in fact the $(n+1)$-fold self-product of $D_X(Y)$ in the infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ (or $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$, respectively). This is because by Lemma 2.2.5, we know that

$$D(n) = \mathrm{Hom}(-, (X, Y^{n+1})),$$

which is the same as the contravariant functor $\mathrm{Hom}(-, (X,Y))^{n+1}$ on the infinitesimal site. So the simplicial object $D(\bullet)$ is in fact the coskeleton $\mathrm{cosk}_0(D_X(Y))$ over the final object (in other words, the Čech nerve for the map of sheaves $D_X(Y) \to 1$). In this way, since $D_X(Y) \to 1$ is an effective epimorphism (Lemma 2.2.6), by [3, Tag 09VU], $D(\bullet) \to 1$ is a hypercovering, and we get a natural equivalence of derived functors

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, -) \simeq R\Gamma(D(\bullet), -) = R \lim_{[n]\in\Delta} R\Gamma(D(n), -). \qquad \square$$

As an upshot, we see that the restriction functor from the big infinitesimal topos to the small one preserves the cohomology.

Corollary 2.2.8. — *Let $\mathcal{F}$ be an object in the derived category of sheaves over $X/\Sigma_{e\mathrm{INF}}$. Then the restriction functor $\iota^{-1} = \mu_* : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ (cf. Paragraph 2.1) induces the following isomorphism:*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mu_*\mathcal{F}) \longrightarrow R\Gamma(X/\Sigma_{e\mathrm{INF}}, \mathcal{F}).$$

Proof. — We first assume X admits a closed immersion into a smooth rigid space over Σ_e. The claim in this case then follows from Proposition 2.2.7, as an envelope is a direct limit $\varinjlim_{m\in\mathbb{N}} h_{Y(n)_m}$ of representable objects in the big and the small sites, and the restriction functor produces the natural equivalence

$$R\Gamma(D(n), \mathcal{F}) \longrightarrow R\Gamma(D(n), \mu_*\mathcal{F}).$$

In general, we may take a hypercovering by affinoid open spaces of X first to reduce to the above special cases, since an affinoid rigid space $\mathrm{Spa}(A)$ is topologically of finite type and the ring A thus admits a surjection from $\mathrm{B}^+_{\mathrm{dR,e}}\langle T_1, \ldots, T_n\rangle$ for some $n \in \mathbb{N}$. $\square$

2.3. Infinitesimal and rigid topology. — In this subsection, we relate the infinitesimal topos and the rigid topos to one another.

Let X be a rigid space over Σ_e. Recall that there is a Grothendieck topology X_{rig} on the category of open subsets in X, called the *rigid site* X_{rig}.

Consider the following two functors:

$$\begin{aligned} u_{X/\Sigma_e *} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) &\longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}); \\ \mathcal{F} &\longmapsto (U \mapsto \Gamma(U/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}|_{U/\Sigma_{e\,\mathrm{inf}}})). \\ u^{-1}_{X/\Sigma_e} : \mathrm{Sh}(X_{\mathrm{rig}}) &\longrightarrow \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}); \\ \mathcal{E} &\longmapsto ((U,T) \mapsto \mathcal{E}(U)). \end{aligned}$$

For a given infinitesimal thickening (U,T), since $(u^{-1}_{X/\Sigma_e}\mathcal{E})_T$ is equal to the sheaf $\mathcal{E}|_U$ on $U_{\mathrm{rig}} \simeq T_{\mathrm{rig}}$, the functor u^{-1}_{X/Σ_e} commutes with the finite inverse limit. Notice that the pair $(u^{-1}_{X/\Sigma_e}, u_{X/\Sigma_e *})$ is adjoint. Thus we get a morphism of topoi ([3, Tag 00XA])

$$u_{X/\Sigma_e} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}),$$

which we follow [5] and call the *projection morphism*.

The projection morphism u_{X/Σ_e} admits a section. Consider the functor $X/\Sigma_{e\,\mathrm{inf}} \to X_{\mathrm{rig}}$, sending (U,T) onto the open subset U of X. By the definition of $X/\Sigma_{e\,\mathrm{inf}}$, a covering of (U,T) is mapped onto a covering of U. In particular, the map of sites is continuous in the sense of [3]. So we get a morphism of sites

$$i_{X/\Sigma_e} : X_{\mathrm{rig}} \longrightarrow X/\Sigma_{e\,\mathrm{inf}}.$$

The morphisms induces a map of topoi, in a way that for $\mathcal{E} \in \mathrm{Sh}(X_{\mathrm{rig}})$,

$$i_{X/\Sigma_e *}\mathcal{E}(U,T) = \mathcal{E}(U),$$

and for $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, we have

$$i^{-1}_{X/\Sigma_e}\mathcal{F}(U) = \varinjlim_{(U,U)\to(V,T)} \mathcal{F}(V,T) = \mathcal{F}(U,U).$$

From the description, we see that the functor i^{-1}_{X/Σ_e} is the restriction functor sending a sheaf $\mathcal{F}$ over $X/\Sigma_{e\,\mathrm{inf}}$ to its restriction $\mathcal{F}_X$ on the rigid space X.

Remark 2.3.1. — By the construction of i_{X/Σ_e} and u_{X/Σ_e}, on the rigid topos $\mathrm{Sh}(X_{\mathrm{rig}})$, we have

$$u_{X/\Sigma_e *} \circ i_{X/\Sigma_e *} = id,\ i^{-1}_{X/\Sigma_e} \circ u^{-1}_{X/\Sigma_e} = id,$$

which implies that those morphisms of topoi satisfy

$$u_{X/\Sigma_e} \circ i_{X/\Sigma_e} = id.$$

This justifies the name of the projection morphism.

Remark 2.3.2. — The construction here naturally generalizes to two morphisms between the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$ and the big rigid site $\mathrm{Rig}_{\Sigma_e}|_X$ for a given rigid space over X.

2.4. Functoriality. — In this subsection, we introduce natural maps of infinitesimal topoi associated with a map of rigid spaces similar to the construction in [3, Tag 07IC, 07IK].

Let $f : X \to Y$ be a map of rigid spaces over $\Sigma_{e'}$, and assume the structure map $X \to \Sigma_{e'}$ factors through Σ_e for non-negative integers $e \leq e'$. By the construction of the big infinitesimal site, the map f induces a natural functor between $X/\Sigma_{e\mathrm{INF}}$ and $Y/\Sigma_{e'\mathrm{INF}}$, satisfying

$$X/\Sigma_{e\mathrm{INF}} \ni ((U,T), U \to X) \longmapsto ((U,T), U \to X \to Y),$$

where the map $U \to X \to Y$ is the composition of the map f with the structure map of $(U,T) \in X/\Sigma_{e\mathrm{INF}}$. Then it is easy to check that this functor is both continuous and cocontinuous and commutes with fiber products and equalizers (cf. Lemma 2.1.4). This in particular implies that the functor above induces a morphism of topoi ([3, Tag 00XN, 00XR])

$$f_{\mathrm{INF}} : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \longrightarrow \mathrm{Sh}(Y/\Sigma_{e'\mathrm{INF}}),$$

such that

- The inverse image functor f_{INF}^{-1} commutes with arbitrary limits and colimits, such that for a sheaf $\mathcal{G}$ over $Y/\Sigma_{e'\mathrm{INF}}$, we have
$$f_{\mathrm{INF}}^{-1}\mathcal{G}(U,T) = \mathcal{G}(U,T),$$
where the second (U,T) is regarded as an object in $Y/\Sigma_{e'\mathrm{INF}}$ by $U \to X \to Y$.
- The direct image functor $f_{\mathrm{INF}*}$, which is the right adjoint to the functor f_{INF}^{-1}, sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ to the sheaf $f_{\mathrm{INF}*}\mathcal{F}$ such that the section is given by
$$Y/\Sigma_{e'\mathrm{INF}} \ni (V,S) \longmapsto \varprojlim_{\substack{(U,T)\to(U,T) \\ (V,S)\in X/\Sigma_{e\mathrm{INF}}, \\ V\to U \text{ compatible with } f}} \mathcal{F}(U,T).$$

Now we consider the small topoi $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ and $\mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$. Analogous to [3, Tag 07IK], we use the map of big topoi to connect them. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) & \xrightarrow{f_{\mathrm{INF}}} & \mathrm{Sh}(Y/\Sigma_{e'\mathrm{INF}}) \\ {\scriptstyle \iota_X}\uparrow & & \downarrow{\scriptstyle \mu_Y} \\ \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) & \xrightarrow[f_{\mathrm{inf}}]{\cdots\cdots} & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}). \end{array}$$

Here, we define the morphism of topoi $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$ to be the composition

$$f_{\mathrm{inf}} = \mu_Y \circ f_{\mathrm{INF}} \circ \iota_X.$$

Then by the definition of those functors, we have

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$, the inverse image $f_{\mathrm{inf}}^{-1}\mathcal{G}$ is given by the "restriction" of $\mu_Y^{-1}\mathcal{G}$ to the category $X/\Sigma_{e\,\mathrm{inf}}$ via the map f, and it is equal to the sheaf associated with the presheaf
$$X/\Sigma_{e\,\mathrm{inf}} \ni (U,T) \longmapsto \varinjlim_{\substack{(U,T)\to(V,S) \\ (V,S)\in Y/\Sigma_{e'\,\mathrm{inf}}, \\ U\to V \text{ compatible with } f}} \mathcal{G}(V,S).$$

- The direct image functor $f_{\inf *}$ sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\inf})$ to the sheaf

$$f_{\inf *}\mathcal{F}(V,S) = \varprojlim_{\substack{(U,T)\to(V,S)\\ (U,T)\in X/\Sigma_{e\mathrm{INF}}\\ U\to V \ \textit{compatible with } f}} \mathcal{F}(U,T).$$

REMARK 2.4.1. — In the special case when $\mathcal{G} = h_S$ is the representable sheaf of $(V,S) \in Y/\Sigma_{e'\inf}$, its inverse image $f_{\inf}^{-1}h_S$ has a simpler formula by

$$\begin{aligned} f_{\inf}^{-1}h_S(U,T) &= \mathrm{Hom}_Y((U,T),(V,S)) \\ &:= \left\{ \text{commutative diagrams } \begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{nil} & T \\ {\scriptstyle f}\downarrow & & \vdots & & \vdots \\ Y & \longleftarrow & V & \xrightarrow{nil} & S \end{array} \right\}. \end{aligned}$$

Here the notation "nil" means that the arrows below are nil closed immersions, and the map $T \to S$ is a map of rigid spaces over $\Sigma_{e'}$.

REMARK 2.4.2. — The functoriality of infinitesimal topoi is compatible with the projection morphism to the rigid topos and its section. Thus the following two diagrams are commutative:

$$\begin{array}{ccc} \mathrm{Sh}(X/\Sigma_{e\inf}) & \xrightarrow{f_{\inf}} & \mathrm{Sh}(Y/\Sigma_{e'\inf}) \\ {\scriptstyle u_{X/\Sigma_e}}\downarrow & & \downarrow{\scriptstyle u_{Y/\Sigma_{e'}}} \\ \mathrm{Sh}(X_{\mathrm{rig}}) & \longrightarrow & \mathrm{Sh}(Y_{\mathrm{rig}}) \end{array}, \qquad \begin{array}{ccc} \mathrm{Sh}(X_{\mathrm{rig}}) & \longrightarrow & \mathrm{Sh}(Y_{\mathrm{rig}}) \\ {\scriptstyle i_{X/\Sigma_e}}\downarrow & & \downarrow{\scriptstyle i_{Y/\Sigma_{e'}}} \\ \mathrm{Sh}(X/\Sigma_{e\inf}) & \xrightarrow{f_{\inf}} & \mathrm{Sh}(Y/\Sigma_{e'\inf}). \end{array}$$

The commutativity can be checked readily using explicit formulae above, and we will not expand but refer the reader to [3, Tag 07KL] for the analogue in classical crystalline theory.

We also want to mention that $f_{\inf}$ is naturally a map of ringed topoi under the infinitesimal structure sheaves, and we could define the *pullback functor* $f_{\inf}^*$ on the category of $\mathcal{O}_{Y/\Sigma'_e}$-sheaves similar to the scheme theory. Here, given a sheaf of $\mathcal{O}_{Y/\Sigma'_e}$-modules $\mathcal{G}$ and an object $(U,T) \in X/\Sigma_{e\inf}$, the restriction of the pullback $f_{\inf}^*\mathcal{G}$ at the rigid space T is equal to the colimit

$$\varinjlim_{\substack{h:(U,T)\to(V,S)\\ (V,S)\in Y/\Sigma_{e'\inf}}} h^*(\mathcal{G}_S).$$

REMARK 2.4.3. — Here we remark that when $f : X \to X$ is the identity map but e is strictly smaller than e', the transition morphism $f_{\inf} : \mathrm{Sh}(X/\Sigma_{e\inf}) \to \mathrm{Sh}(X/\Sigma_{e'\inf})$ is induced from the map of sites $f_{\inf} : X/\Sigma_{e\inf} \to X/\Sigma_{e'\inf}$, where the corresponding functor sends $(V,S) \in X/\Sigma_{e'\inf}$ onto the thickening $(V, S \times_{\Sigma_{e'}} \Sigma_e)$.

3. Crystals

In this section, we study the coherent crystal and its canonical connection.

Before we start, we mention that though stated for rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$, the results and proofs in this section hold true verbatim for rigid spaces over an arbitrary p-adic extension K of $\mathbb{Q}_p$, namely K is a field that is complete with respect to a non-Archimedean valuation extending that of $\mathbb{Q}_p$.

3.1. Crystals and their connections. —

We first introduce the coherent crystal and a canonical connection associated with it.

Sheaf of differentials.

DEFINITION 3.1.1. — The *infinitesimal sheaf of differentials* $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}$ is a sheaf of $\mathcal{O}_{X/\Sigma_e}$-modules on $X/\Sigma_{e\,\mathrm{inf}}$ defined as

$$\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(U,T) := \Omega^{i,cont}_{T/\Sigma_e}(T), \quad (U,T) \in X/\Sigma_{e\,\mathrm{inf}}$$

locally given by the continuous differentials over Σ_e.

Similarly, we could define the infinitesimal sheaf of differentials $\Omega^i_{X/\Sigma_{e\mathrm{INF}}}$ over the big site $X/\Sigma_{e\mathrm{INF}}$. It can be checked easily that the restriction $\iota^{-1}\Omega^i_{X/\Sigma_{e\mathrm{INF}}}$ at the small site is equal to $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}$.

Here we recall the definition of the sheaf of continuous differentials as follows: let T be a rigid space over Σ_e and $T(m)$ be the $m+1$-th self-product of T over Σ_e, which is equipped with $m+1$ projection maps onto T and the diagonal map from T. For each $m \in \mathbb{N}$, we denote $T(n)_m$ as the m-th infinitesimal neighborhood of T in $T(n)$. Then each infinitesimal thickening $(T, T(n)_m)$ is an object in $X/\Sigma_{e\,\mathrm{inf}}$.

Let I_T be the coherent sheaf of ideals in $\mathcal{O}_{T(1)}$, defined as the kernel of the map $\mathcal{O}_{T(1)} \to \mathcal{O}_T$ given by the diagonal $T \to T(1)_1$. Then the sheaf of continuous differentials $\Omega^{1,cont}_{T/\Sigma_e}$ is the coherent sheaf I_T/I_T^2 over T. It can be checked that the sheaf of continuous differentials satisfies the universal property among continuous $\mathrm{B}^+_{\mathrm{dR,e}}$-linear derivatives. Without mentioning, we will use Ω^i_{T/Σ_e} to denote the i-th continuous differentials to simplify the notation.

Crystals.

DEFINITION 3.1.2. — Let $\mathcal{F}$ be a coherent sheaf over $X/\Sigma_{e\,\mathrm{inf}}$ (resp. $X/\Sigma_{e\mathrm{INF}}$). Thus, $\mathcal{F}$ is a sheaf on the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$ (resp. $X/\Sigma_{e\mathrm{INF}}$) such that $\mathcal{F}_T$ is a coherent $\mathcal{O}_T$-module for each infinitesimal thickening (U,T) in $X/\Sigma_{e\,\mathrm{inf}}$ (resp. in $X/\Sigma_{e\mathrm{INF}}$). We call $\mathcal{F}$ a *coherent crystal* if for each morphism of thickenings $(i,g) : (U_1,T_1) \to (U_2,T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$ (resp. in $X/\Sigma_{e\mathrm{INF}}$), the natural map

$$g^*\mathcal{F}_{T_2} = \mathcal{O}_{T_1} \otimes_{g^{-1}\mathcal{O}_{T_2}} g^{-1}\mathcal{F}_{T_2} \longrightarrow \mathcal{F}_{T_1}$$

is an isomorphism of $\mathcal{O}_{T_1}$ modules.

EXAMPLE 3.1.3. — The easiest example of a coherent crystal is the infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma_e}$, defined either over the small or big infinitesimal sites of X over Σ_e.

REMARK 3.1.4. — The infinitesimal sheaf of differentials is not a crystal in general, though it is a coherent sheaf over $\mathcal{O}_{X/\Sigma_e}$.

Here it is not hard to see that the pullback of a coherent crystal is a crystal.

LEMMA 3.1.5. — *Let $f : X \to Y$ be a map of rigid spaces over $\Sigma_{e'}$, and assume the structure map $X \to \Sigma_{e'}$ factors through Σ_e for non-negative integers $e \leq e'$. Let $\mathcal{G}$ be a coherent crystal over $Y/\Sigma_{e'\,\mathrm{inf}}$. We denote f_{inf} to be the functoriality map of infinitesimal topoi $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$.*

(i) *Let $(g, h) : (U, T) \to (V, S)$ be a map of thickenings for $(U, T) \in X/\Sigma_{e\,\mathrm{inf}}$ and $(V, S) \in Y/\Sigma_{e'\,\mathrm{inf}}$, respectively, such that $g : U \to V$ is compatible with $f : X \to Y$. Then the restriction of $f^*_{\mathrm{inf}}\mathcal{G}$ at T is naturally isomorphic to the pullback $h^*(\mathcal{G}_S)$ of the coherent sheaf $\mathcal{G}_S$ over S along the map of rigid spaces $h : T \to S$.*

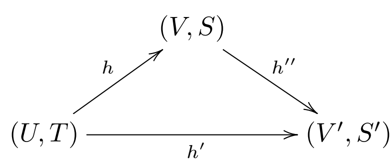
(ii) *The pullback $f^*_{\mathrm{inf}}\mathcal{G}$ is a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$.*

(iii) *Both (i) and (ii) hold true for $f_{\mathrm{INF}} : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(Y/\Sigma_{e'\mathrm{INF}})$ and a coherent crystal $\mathcal{G}$ over big infinitesimal sites.*

Proof. — Let (U, T) be an object in the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$. By the construction, we know that the restriction of the pullback $f^*_{\mathrm{inf}}\mathcal{G}$ at the rigid space T is equal to the colimit

$$\varinjlim_{\substack{h:(U,T)\to(V,S)\\ (V,S)\in Y/\Sigma_{e'\,\mathrm{inf}}}} h^*(\mathcal{G}_S),$$

where $h : T \to S$ is the map of rigid spaces over Σ'_e. On the one hand, by the definition of the coherent crystal, for a commutative diagram of infinitesimal thickenings



that is compatible with $f : X \to Y$, the pullback $(h'')^*(\mathcal{G}_{S'})$ is equal to the coherent sheaf $\mathcal{G}_S$ over S. On the other hand, as in Lemma 2.1.4, the finite products are ind-representable in the small site $Y/\Sigma_{e'\,\mathrm{inf}}$. In particular, given two maps of thickenings $h_i : (U, T) \rightrightarrows (V, S)$, where $U \to V$ is the compatible with f, both h_i locally factor through a thickening $(V, S(1)_m)$ for an infinitesimal neighborhood $S(1)_m$ of V in $S(1) = S \times_{\Sigma_{e'}} S$. As an upshot, the pullback $h^*\mathcal{G}_S$ is independent of the map h. In this way, the restriction of $f^*_{\mathrm{inf}}\mathcal{G}$ at T, which is equal to the colimit above, is naturally isomorphic to the coherent sheaf $h^*(\mathcal{G}_S)$ over T for any map of thickenings $h:(U,T)\to(V,S)$, where $(U,T)\in X/\Sigma_{e\,\mathrm{inf}}$ and $(V,S)\in Y/\Sigma_{e'\,\mathrm{inf}}$. This finishes the proof of (i).

To check the crystal condition of $f^*_{\mathrm{inf}}\mathcal{G}$, it suffices to note that given a map of objects $g:(U_1,T_1)\to(U_2,T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$ and a compatible map of infinitesimal thickenings $h:(U_2,T_2)\to(V,S)$ for $(V,S)\in Y/\Sigma_{e'\,\mathrm{inf}}$, we have

$$(f^*_{\mathrm{inf}}\mathcal{G})_{T_1}\simeq g^*h^*(\mathcal{G}_S)\simeq g^*(f^*_{\mathrm{inf}}\mathcal{G})_{T_2}.$$

Finally, notice that the proof is applicable no matter whether the structure maps $U\to X$ and $V\to Y$ are open immersions. So we are done. □

Example 3.1.6. — An example of a coherent crystal over the big site is the pullback of a crystal from the small site

$$\mu^*\mathcal{F}=\mu^{-1}\mathcal{F}\bigotimes_{\mu^{-1}\mathcal{O}_{X/\Sigma_{e\,\mathrm{inf}}}}\mathcal{O}_{X/\Sigma_{e\mathrm{INF}}},$$

where $\mu:\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})\to\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ is the canonical map from the big topos to the small topos, as in Subsection 2.1. Here the proof is identical to that of Lemma 3.1.5.

In particular, the pullback $\mu^*\mathcal{F}$ locally satisfies the same formula as in Lemma 3.1.5.(i). For a crystal $\mathcal{F}$ over the small site $X/\Sigma_{e\,\mathrm{inf}}$ and a thickening $(U,T)\in X/\Sigma_{e\mathrm{INF}}$ in the big site, the restriction of the infinitesimal sheaf $\mu^*\mathcal{F}$ on T is naturally isomorphic to the pullback $g^*(\mathcal{F}_S)$. Here the map $g:T\to S$ of rigid spaces comes from an arbitrary commutative diagram of objects $(i,g):(U,T)\to(V,S)$ in $X/\Sigma_{e\mathrm{INF}}$ that is compatible with their structure maps $U\to X$ and $V\to X$, such that $V\to X$ is an open immersion.

In fact, we have the following results about crystals over big and small infinitesimal sites:

Proposition 3.1.7. — *Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. There exists a natural equivalence as below:*

$$\{\textit{coherent crystals over } X/\Sigma_{e\,\mathrm{inf}}\}\Longleftrightarrow\{\textit{coherent crystals over } X/\Sigma_{e\mathrm{INF}}\};$$
$$\mathcal{F}\longmapsto\mu^*\mathcal{F};$$
$$\mu_*\mathcal{G}\longleftarrow\mathcal{G}.$$

Here we recall from Paragraph 2.1 that the functor μ_* is the restriction functor from $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ to $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$.

Proof. — It suffices to show the compositions are equivalences. Given a coherent crystal $\mathcal{F}$ over the small infinitesimal site and a thickening $(U,T)\in X/\Sigma_{e\,\mathrm{inf}}$, we have

$$\begin{aligned}(\mu_*\mu^*\mathcal{F})_T&=(\mu^*\mathcal{F})_T\\&=\mathcal{F}_T,\end{aligned}$$

where the second equality follows from the Example 3.1.6 for the identity map $(U,T) \to (U,T)$.

Conversely, let $\mathcal{G}$ be a coherent crystal over the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. For any object $(V,S) \in X/\Sigma_{e\mathrm{INF}}$, it can always be covered by open affinoid subsets (V_i, S_i) such that each (V_i, S_i) admits a map to a thickening $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$.[5] We denote $g : S_i \to T$ to be the associated map of rigid spaces. Then by the crystal condition of $\mathcal{G}$, we have $\mathcal{G}_{S_i} = g^*(\mathcal{G}_T)$. As an upshot, by Example 3.1.6 again, we get the following equalities:

$$\begin{aligned}(\mu^*\mu_*\mathcal{G})_{S_i} &= g^*\left((\mu_*\mathcal{G})_T\right)\\ &= g^*(\mathcal{G}_T)\\ &= \mathcal{G}_{S_i}.\end{aligned}$$

So we are done. □

Connection. Recall the definition of general connections for a coherent sheaf.

Definition 3.1.8. — Let $\mathcal{F}$ be a coherent sheaf over $X/\Sigma_{e\,\mathrm{inf}}$. A *connection* of $\mathcal{F}$ is an $\mathrm{B}^+_{\mathrm{dR,e}}$-linear morphism of sheaves

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_{e\,\mathrm{inf}}}} \Omega^1_{X/\Sigma_{e\,\mathrm{inf}}},$$

such that ∇ sends $f\cdot x$ onto $f\nabla(x) + x \otimes df$ for f and x being local sections of $\mathcal{O}_{X/\Sigma_{e\,\mathrm{inf}}}$ and $\mathcal{F}$, respectively.

Here we want to mention that similarly, we can define the connection for coherent sheaves over the big infinitesimal site.

Now let $\mathcal{F}$ be a coherent crystal on $X/\Sigma_{e\,\mathrm{inf}}$, and let (U,T) be an object in $X/\Sigma_{e\,\mathrm{inf}}$. Then, by the definition of crystals, the two projection maps $p_0, p_1 : T(1)_1 \to T$ induce an isomorphism of $\mathcal{O}_{T(1)_1}$-modules:

$$\varepsilon_T : p_0^*\mathcal{F}_T \simeq \mathcal{F}_{T(1)_1} \simeq p_1^*\mathcal{F}_T.$$

This induces a morphism of $\mathcal{O}_T$-modules given by

$$\begin{array}{ccccc}\mathcal{F}_T & \longrightarrow & \mathcal{O}_{T(1)_1}\otimes_{\mathcal{O}_T}\mathcal{F}_T & \xrightarrow{\varepsilon_T} & \mathcal{F}_T\otimes_{\mathcal{O}_T}\mathcal{O}_{T(1)_1},\\ x & \longmapsto & 1\otimes x & \longmapsto & \varepsilon_T(1\otimes x).\end{array}$$

Here we identify the sheaf of the $\mathcal{O}_{T(1)_1}$-module $p_1^*\mathcal{F}_T$ as $\mathcal{O}_{T(1)_1} \otimes_{\mathcal{O}_T} \mathcal{F}_T$ (similarly for $p_0^*\mathcal{F}_T = \mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{O}_{T(1)_1}$). Besides, the pullback of the above sequence along the diagonal map $T \to T(1)_1$ is the identity, so the image of $\varepsilon_T(1\otimes x)$ under this pullback map is exactly x.

5. To see this, we may assume that the structure map $V_i \to V \to X$ maps into an open affinoid subset U of X, where U admits a closed immersion into a smooth rigid space Y. Then, since $V_i \to S_i$ is a nilpotent thickening, by the smoothness of Y, the map $V_i \to U$ induces a map S_i to some Y_m, where Y_m is an infinitesimal neighborhood of U in Y. Thus the claim follows, as (U, Y_m) is in the small site $X/\Sigma_{e\,\mathrm{inf}}$.

The map in fact defines a *canonical connection* structure on the sheaf of the $\mathcal{O}_T$-module $\mathcal{F}_T$ by

$$\begin{aligned} \nabla_T:\ \mathcal{F}_T &\longrightarrow \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^1_{T/\Sigma_e}.\\ x &\longmapsto \varepsilon_T(1\otimes x) - x\otimes 1. \end{aligned}$$

Here $\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^1_{T/\Sigma_e} = \mathcal{F}_T \otimes_{\mathcal{O}_T} I_T/I_T^2$ can be identified as a subsheaf of $\mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{O}_{T(1)_1}$, since $\mathcal{O}_{T(1)_1}$ decomposes into the direct sum $\mathcal{O}_T \oplus \Omega^1_{T/\Sigma_e}$ as a left $\mathcal{O}_T$-module. Note that the map satisfies the axiom for the connection, in the sense that for a section f of $\mathcal{O}_T$ and x of $\mathcal{F}_T$, we have

$$\nabla_T(f\cdot x) = f\nabla_T(x) + x\otimes df,$$

where $df = 1\otimes f - f\otimes 1$ is in Ω^1_{T/Σ_e}.

We also notice that the above is functorial with respect to $(U,T)\in X/\Sigma_{e\,\mathrm{inf}}$, in the sense that for a morphism $(i,g):(U_1,T_1)\to(U_2,T_2)$, we have the following commutative diagram:

$$\begin{array}{ccc} g^*(\mathcal{F}_{T_2}) & \xrightarrow{g^*\nabla_{T_2}} & g^*(\mathcal{F}_{T_2}\otimes_{\mathcal{O}_{T_2}}\Omega^1_{T_2/\Sigma_e}) \\ \downarrow & & \downarrow \\ \mathcal{F}_{T_1} & \xrightarrow{\nabla_{T_1}} & \mathcal{F}_{T_1}\otimes_{\mathcal{O}_{T_1}}\Omega^1_{T_1/\Sigma_e}. \end{array}$$

In particular, the functoriality leads to the morphism of sheaves over infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$

$$\nabla:\mathcal{F}\longrightarrow \mathcal{F}\otimes_{\mathcal{O}_{X/\Sigma_e}}\Omega^1_{X/\Sigma_{e\,\mathrm{inf}}}.$$

Definition 3.1.9. — Let $\mathcal{F}$ be a coherent crystal over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$. The *canonical connection of* $\mathcal{F}$ is defined as the morphism as above

$$\nabla:\mathcal{F}\longrightarrow \mathcal{F}\otimes_{\mathcal{O}_{X/\Sigma_e}}\Omega^1_{X/\Sigma_{e\,\mathrm{inf}}}.$$

Finally, assume that $X\to Y$ is a locally closed immersion of rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$ such that Y is smooth, and let $\mathcal{F}$ be a coherent crystal over X/Σ_e. By taking the evaluation of the canonical connection of $\mathcal{F}$ at the envelope $D=\varinjlim_{n\in\mathbb{N}} Y_n$, we get a natural $\mathrm{B}^+_{\mathrm{dR,e}}$-linear continuous connection over $\mathcal{D}$:

$$\nabla_D:\mathcal{F}_D\longrightarrow \mathcal{F}_D\otimes_{\mathcal{O}_Y}\Omega^1_{Y/\Sigma_e},$$

satisfying the local formula that $\nabla_D(fx) = x\otimes df + f\nabla_D(x)$, where f and x are sections of $\mathcal{O}_D=\varprojlim_{n\in\mathbb{N}}\mathcal{O}_{Y_m}$ and $\mathcal{F}_D$, respectively. More generally, given a coherent sheaf $\mathcal{M}$ over the envelope D, we define a *connection* $\nabla_{\mathcal{M}}$ on $\mathcal{M}$ to be a $\mathrm{B}^+_{\mathrm{dR,e}}$-linear continuous map

$$\nabla_{\mathcal{M}}:\mathcal{M}\longrightarrow \mathcal{M}\otimes_{\mathcal{O}_Y}\Omega^1_{Y/\Sigma_e}$$

that satisfies the same formula above.

de Rham complex of a crystal. Similar to the flat connection over schemes ([4], Chapter II, Section 3.2), we can associate a natural de Rham complex to a coherent crystal over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$ or $X/\Sigma_{e\mathrm{INF}}$ by the integrability of the canonical connection.

Let $\mathcal{F}$ be a coherent sheaf over $\mathcal{O}_{X/\Sigma_e}$ with a connection ∇. For each $k \in \mathbb{N}_+$, we can associate an $\mathcal{O}_{X/\Sigma_e}$-linear morphism

$$\nabla^k : \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^k_{X/\Sigma_e} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^{k+1}_{X/\Sigma_{e\,\mathrm{inf}}},$$

locally given by

$$x \otimes \omega \longmapsto \nabla(x) \wedge \omega + x \otimes d\omega.$$

This produces a chain of maps

$$(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}, \nabla) := \ 0 \longrightarrow \mathcal{F} \xrightarrow{\ \nabla\ } \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^1_{X/\Sigma_{e\,\mathrm{inf}}} \xrightarrow{\ \nabla^1\ } \cdots.$$

The connection is called *integrable* if the composition $\nabla^1 \circ \nabla$ is zero, under which assumption we call the above complex the *de Rham complex of $\mathcal{F}$*. Analogously, for a given envelope D of a locally closed immersion $X \to Y$ into a smooth rigid space Y and a coherent module with connection $(\mathcal{M}, \nabla_{\mathcal{M}})$ over D, we say $\nabla_{\mathcal{M}}$ is *integrable* if the composition $\nabla^1_{\mathcal{M}} \circ \nabla_{\mathcal{M}}$ is zero.

The following proposition justifies the name of the de Rham complex:

Proposition 3.1.10. — *Let $\mathcal{F}$ be a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$, and let ∇ be its canonical connection defined in the last subsection. Then for each $k \in \mathbb{N}$, we have*

$$\nabla^{k+1} \circ \nabla^k = 0.$$

In particular, the de Rham complex of $\mathcal{F}$ is in fact a complex.

Proof. — The proof is identical to that for a crystal over the crystalline site of a scheme, and we refer the reader to[3, Tag 07J6]. □

3.2. Crystal in vector bundles. — Given a coherent crystal $\mathcal{F}$ over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$, we say $\mathcal{F}$ is a *crystal in vector bundles* if the restriction $\mathcal{F}_T$ is locally free of finite rank over $\mathcal{O}_T$ for every object $(U, T) \in X/\Sigma_{e\,\mathrm{inf}}$. In this subsection, we provide a simple criterion when a coherent crystal is a crystal in vector bundles.

Definition 3.2.1. — A coherent crystal $\mathcal{F}$ is *flat over* $\mathrm{B}^+_{\mathrm{dR,e}}$ if for any thickening (U, T) in the infinitesimal site with T being flat over $\mathrm{B}^+_{\mathrm{dR,e}}$, the restriction $\mathcal{F}_T$ at T is also flat over $\mathrm{B}^+_{\mathrm{dR,e}}$.

Theorem 3.2.2. — *Let $\mathcal{F}$ be a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$ that is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$ in the sense of Definition* 3.2.1. *Then $\mathcal{F}$ is a crystal in vector bundles.*

Proof. —

Step 1. — We first consider the case when $e = 1$. Let us assume that X is defined over K and $\mathcal{F}$ is a coherent crystal over X/K_{inf} or X/K_{INF}, where the flatness of $\mathcal{F}$ over $\mathrm{B}^+_{\mathrm{dR,e}} = K$ is automatic.

For $m \in \mathbb{N}$, let $T(1)_m$ be the m-th infinitesimal neighborhood of T in $T \times_K T$. The projection map pr_0 to the first factor induces a map from $T(1)_m$ to T via

$$h:\ T(1)_m \longrightarrow T \times_K T \xrightarrow{\mathrm{pr}_0} T.$$

Moreover, as $T = T(1)_0$ admits a closed immersion into $T(1)_m$, we can form the following non-commutative diagram of thickenings:

$$(*)\qquad \begin{array}{ccccc} & & T & & \\ & \overset{h}{\nearrow} & & \searrow\!\!\hookrightarrow & \\ T(1)_m & & \xrightarrow[\mathrm{id}]{\qquad} & & T(1)_m \xrightarrow{h} T. \end{array}$$

Here we notice that the composition $T \hookrightarrow T(1)_m \xrightarrow{h} T$ above is the identity. We denote the composition of the map $h : T(1)_m \to T$ and the closed immersion $T \to T(1)_m$ by $g : T(1)_m \to T(1)_m$. Then we get two maps of thickenings of U as follows:

$$T(1)_m \underset{g}{\overset{\mathrm{id}}{\rightrightarrows}} T(1)_m.$$

Then, by the definition of the coherent crystal, pulling back along the above two arrows induces an isomorphism of coherent sheaves over $T(1)_m$:

$$g^*\mathcal{F}_{T(1)_m} \longrightarrow \mathcal{F}_{T(1)_m}.$$

Moreover, by the assumption on g, we have

$$(**)\qquad g^*\mathcal{F}_{T(1)_m} = h^*\mathcal{F}_T.$$

Now we base change the diagram $(*)$ above along the closed immersion $t : \mathrm{Spa}(K) \to T$ of any K-point t of T. By the construction of the map $h : T(1)_m \to T$, we get a non-commutative diagram

$$\begin{array}{ccccc} & & t = \mathrm{Spa}(K) & & \\ & \nearrow & & \searrow\!\!\hookrightarrow & \\ t_m & & \xrightarrow[\mathrm{id}]{\qquad} & & t_m \longrightarrow t, \end{array}$$

where t_m is the m-th infinitesimal neighborhood of t in T, and the base changed map $h : t_m \to t = \mathrm{Spa}(K)$ is the structure map of t_m over $\mathrm{Spa}(K)$. Furthermore, after the base change, the isomorphism $g^*\mathcal{F}_{T(1)_m} \simeq \mathcal{F}_{T(1)_m}$ in $(**)$ becomes the following isomorphism of torsion sheaves over T that are supported at t:

$$\mathcal{F}_T \otimes t_m \simeq h^*(\mathcal{F}_T \otimes t).$$

Notice that since the fiber $\mathcal{F}_T \otimes t$ is flat and finitely generated over K, the base change $\mathcal{F}_T \otimes t$ is, in particular, a finite free K-module. As a consequence, the pullback $h^*(\mathcal{F}_T \otimes t)$, which by the equality above is equal to $\mathcal{F}_T \otimes_{\mathcal{O}_T} t_m$, is finite free over t_m.

Finally, we take the inverse limit of $\mathcal{F}_T \otimes_{\mathcal{O}_T} t_m$ with respect to m. By the finite generatedness of $\mathcal{F}_T$ over $\mathcal{O}_T$ and the fact that T is locally Noetherian, we know that the completion $\varprojlim_{m\in\mathbb{N}} \mathcal{F}_T \otimes_{\mathcal{O}_T} t_m$ is equal to the tensor product $\mathcal{F}_{\mathcal{O}_T} \widehat{\mathcal{O}}_{T,t}$. So we see that $\mathcal{F}_T \otimes_{\mathcal{O}_T} \widehat{\mathcal{O}}_{T,t}$ is finite free over the formal completion $\widehat{\mathcal{O}}_{T,t}$ of the rigid space T at the K-point t. Since T is locally Noetherian, the formal completion $\widehat{\mathcal{O}}_{T,t}$ is isomorphic to the completion $\widehat{\mathcal{O}}_{T,\bar{t}}$, where the latter is the formal completion of T along its reduced K-valued point $\bar{t}$. In this way, by the faithful flatness of $\widehat{\mathcal{O}}_{T,t}$ over $\mathcal{O}_{T,t}$, the stalk of the coherent sheaf $\mathcal{F}_T$ at t is flat and finitely generated over the local ring, thus projective. Hence by the density of K-points in T, we get the local freeness of $\mathcal{F}_T$.

Step 2. — For general $e \in \mathbb{N}$, we make the following claim:

Claim 3.2.3. — *Let A be a flat Noetherian algebra over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let M be a finite A-module that is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$. Suppose M/ξ is free over A/ξ. Then M is free over A.*

Proof of the claim. — We prove the claim by induction on e. When $e = 1$, there is nothing to prove. Suppose $e \geq 2$. We choose a map of A-modules $f : A^{\oplus r} \to M$ whose reduction mod ξ is an isomorphism. Then, as f is a map of flat $\mathrm{B}^+_{\mathrm{dR,e}}$-modules, the short exact sequence $0 \to \mathrm{B}^+_{\mathrm{dR},e-1} \to \mathrm{B}^+_{\mathrm{dR,e}} \to K \to 0$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\oplus r} \otimes_{\mathrm{B}^+_{\mathrm{dR,e}}} \mathrm{B}^+_{\mathrm{dR},e-1} & \longrightarrow & A^{\oplus r} & \longrightarrow & A/\xi^{\oplus r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M \otimes_{\mathrm{B}^+_{\mathrm{dR,e}}} \mathrm{B}^+_{\mathrm{dR},e-1} & \longrightarrow & M & \longrightarrow & M/\xi & \longrightarrow & 0. \end{array}$$

Hence the map f is an isomorphism of A-modules by induction. □

Recall from Lemma 2.2.6 that any infinitesimal thickening $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$ locally admits a map to an envelope $D_U(Y)$, where Y is a smooth rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$ and, in particular, flat over $\mathrm{B}^+_{\mathrm{dR,e}}$. Notice that by construction, the structure sheaf of the envelope, which is the formal completion of $\mathcal{O}_Y$ along the closed immersion $U \to Y$, is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$ as well. As a consequence, thanks to the claim above, the sheaf $\mathcal{F}_{D_U(Y)}$ is a vector bundle over $D_U(Y)$, and hence its pullback along $T \to D_U(Y)$ is a vector bundle over $\mathcal{O}_T$. □

Corollary 3.2.4. — *Any coherent crystal over the infinitesimal site X/K_{inf} or X/K_{INF} is a crystal in vector bundles.*

3.3. Integrable connections over envelope. — As in the crystalline theory of schemes, there exists an equivalence between the category of coherent crystals over $X/\Sigma_{e\,\mathrm{inf}}$ and the category of coherent sheaves with integrable connections over the envelope.

Before we state the result, we recall from Remark 2.2.2 that given an envelope $D = \varinjlim_{n\in\mathbb{N}} Y_n$ of a locally closed immersion $X \to Y$, we can regard the envelope as a locally ringed space over the adic space X. The structure sheaf $\mathcal{D}$ of the envelope is defined as the inverse limit $\varprojlim_{n\in\mathbb{N}} \mathcal{O}_{Y_n}$ over D.

Theorem 3.3.1. — *Let $X \to Y$ be a closed immersion of rigid spaces with Y being smooth over Σ_e, and let $D = D_X(Y)$ be the envelope of X in Y. Then we have a natural equivalence of categories:*

$$\{\textit{coherent crystals over } \mathcal{O}_{X/\Sigma_e}\} \longrightarrow \{(M,\nabla) \mid M \in \mathrm{Coh}(\mathcal{D}),\ \nabla \textit{ integrable connection}\}$$
$$\mathcal{F} \longmapsto (\mathcal{F}_D, \nabla_D).$$

Here crystals are over either the big infinitesimal site or the small infinitesimal site.

Corollary 3.3.2. — *Let $X \to Y$ be a closed immersion of rigid spaces with Y being smooth over Σ_e, and let $D = D_X(Y)$ be the envelope of X in Y. Then the equivalence above induces the equivalence of the following three categories:*

- *{coherent crystals that are flat over $\mathrm{B}^+_{\mathrm{dR,e}}$}.*
- *{crystals in vector bundles}.*
- *$\{(M,\nabla) \mid M \in \mathrm{Vec}(\mathcal{D}),\ \nabla$ an integrable connection$\}$.*

Before the proof, we first give a description of the sheaf of differentials over the envelope. Below, for an affinoid rigid space, we slightly abuse notations for its sheaf of differentials and its global sections.

Lemma 3.3.3. — *Let $X = \mathrm{Spa}(A) \to Y = \mathrm{Spa}(P)$ be a closed immersion of affinoid rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$, with P a smooth affinoid algebra over $\mathrm{B}^+_{\mathrm{dR,e}}$.*

Then we have the following canonical isomorphism:

$$\Omega^1_D := \Omega^1_{X/\Sigma_{e\,\mathrm{inf}}}(D) \simeq \Omega^1_{P/\Sigma_e} \otimes_P \mathcal{D},$$

which is induced from the map $P \to \mathcal{D}$.

Moreover, the result is true for $\Omega^1_{X/\Sigma_{e\mathrm{INF}}}$ over the big infinitesimal site.

Proof. — Recall that Ω^1_D is defined as

$$\Omega^1_{X/\Sigma_{e\,\mathrm{inf}}}(\varinjlim_{m\in\mathbb{N}} Y_m),$$

which is equal to the inverse limit of the continuous differentials

$$\Gamma(X, \varprojlim_{m\in\mathbb{N}} \Omega^1_{Y_m/\Sigma_e}).$$

Denote t_i for $i = 1, \ldots, r$ to be the étale coordinates of P. This is guaranteed locally by the Jacobian criterion of smoothness, as in [27, 1.6.9]. Let $J = (f_1, \ldots, f_s)$ be the kernel of the surjection $P \to A$, with f_i being its generators. Then we have

$$\mathcal{O}(Y_m) = P/J^{m+1},\ \Omega^1_{Y_m/\Sigma_e} = (\bigoplus_i \mathcal{O}(Y_m)dt_i / \sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df)\check{}$$

So we get

$$\mathcal{D} = \varprojlim_{m\in\mathbb{N}} P/J^{m+1},\ \Omega^1_D = \varprojlim_{m\in\mathbb{N}} (\bigoplus_i \mathcal{O}(Y_m)dt_i / \sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df).$$

Now consider the natural map $\Omega^1_{P/\Sigma_e} \otimes_P \mathcal{D} \to \Omega^1_D$ sending the generator dt_i of Ω^1_{P/Σ_e} onto the dt_i in the limit.

Injectivity. — We first consider the injectivity. By writing each $f \in J^{m+1}$ as a finite sum of $af_{j_1}\cdots f_{j_{m+1}}$ for $1 \le j_l \le r$, each such df is contained in $\sum_j J^m\mathcal{O}(Y_m)df_j$. In particular, the submodule $\sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df$ of $\bigoplus \mathcal{O}(Y_m)dt_i$ is contained the submodule $\sum_j J^m\mathcal{O}(Y_m)df_j$. So it suffices to show the injectivity of

$$\Pi : \bigoplus_i \mathcal{D}dt_i \longrightarrow \varprojlim_{m\in\mathbb{N}} (\bigoplus_i \mathcal{O}(Y_m)dt_i / \sum_j J^m\mathcal{O}(Y_m)df_j).$$

However, the kernel of each $\bigoplus_i \mathcal{D}dt_i \to \bigoplus_i \mathcal{O}(Y_m)dt_i/\sum_j J^m\mathcal{O}(Y_m)df_j$ equals to $\sum J^{m+1}\mathcal{D}dt_i + \sum_j J^m df_j$, which is contained in $\bigoplus_i J^m\mathcal{D}dt_i$. In particular, any element $\sum_i g_i dt_i$ in the $\ker(\Pi)$ is contained in the ideal

$$\bigcap_m \bigoplus_i J^m\mathcal{D}dt_i.$$

But note that $\mathcal{D}$ is defined as the J-adic completion of P, which implies the above ideal is zero. So we get the injectivity.

Surjectivity. — We can write $\Omega^1_{P/\Sigma_e}\otimes_P\mathcal{D}$ as $\bigoplus_i \mathcal{D}dt_i = \varprojlim_{m\in\mathbb{N}}(\bigoplus_i \mathcal{O}(Y_m)dt_i)$. Then for each m, the map $\bigoplus_i \mathcal{O}(Y_m)dt_i \to \bigoplus_i \mathcal{O}(Y_m)dt_i/\sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df$ is surjective. For each m, the kernel of the map is $M_m := \sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df$, whose image in M_{m-1} is zero. Thus we get the surjectivity by the pro-acyclicity of the kernel. □

The local freeness of the differential sheaf over the envelope allows us to give a more explicit description of the connection associated with a crystal. We assume $X = \mathrm{Spa}(A) \to \mathrm{Spa}(P) = Y$ be a closed immersion of affinoid rigid spaces over Σ_e, such that Y is smooth over Σ_e with local coordinates $\{t_1, \ldots, t_r\}$. Let M be a coherent sheaf over $\mathcal{D}$ together with a connection ∇ over $\mathrm{B}^+_{\mathrm{dR,e}}$. By Lemma 3.3.3 above, the restriction of the infinitesimal differential over $D = D_X(Y)$ is free over $\mathcal{D} = \mathcal{O}_{X/\Sigma_e}(D) = \varprojlim \mathcal{O}(Y_m)$ with a basis dt_i for $i = 1, \ldots, r$. So for any section $x \in M$, we have

$$\nabla(x) = \sum_i \nabla_i(x) \otimes dt_i,$$

where $\nabla_i : M \to M$ is an $\mathrm{B}^+_{\mathrm{dR,e}}$-linear derivation map.

Now we assume that (M, ∇) is integrable. We compose ∇ with ∇^1 and get

$$\begin{aligned}\nabla^1(\nabla(x)) &= \sum_i \nabla^1(\nabla_i(x) \otimes dt_i)\\ &= \sum_j \sum_i \nabla_j(\nabla_i(x)) \otimes dt_j \wedge dt_i + \sum_i \nabla_i(x) \otimes d(dt_i)\\ &= \sum_j \sum_i \nabla_j(\nabla_i(x)) \otimes dt_j \wedge dt_i.\end{aligned}$$

By the local freeness of Ω^1_D, the element $dt_j \wedge dt_i$ for $j < i$ forms a basis of Ω^2_D. So we can rewrite the above as

$$\nabla^1(\nabla(x)) = \sum_{j<i} (\nabla_j(\nabla_i(x)) - \nabla_i(\nabla_j(x))) \otimes dt_j \wedge dt_i.$$

By the integrability condition of ∇, the above vanishes for any $x \in \mathcal{F}_D$. So we obtain the following equalities

$$\nabla_i \circ \nabla_j = \nabla_j \circ \nabla_i.$$

Here we note that the commutativity allows us to write the composition of a finite amount of ∇_i as

$$\prod_{E=(e_i)} \nabla_i^{e_i},$$

where $E = (e_i)$ is a tuple of non-negative integers parameterized by i.

Now we are ready for the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. — For a crystal $\mathcal{F}$ over $\mathcal{O}_{X/\Sigma_e}$, we can equip it with its canonical connection $\mathcal{F}$, which is integrable by Proposition 3.1.10. So by taking the associated coherent sheaf of $\mathcal{F}$ over D, we get a coherent sheaf $\mathcal{F}_D$ together with an integrable connection ∇_D.

Conversely, let M be a coherent sheaf over $\mathcal{D}$ with an integrable connection ∇. By the smoothness of Y over Σ_e, any object in $X/\Sigma_{e\,\mathrm{inf}}$ can be covered by an open affinoid covering where each piece admits a map to (X, Y). We assume that (U, T) is an affinoid thickening fitting in the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & T \\ \downarrow & & \downarrow g \\ X & \longrightarrow & Y. \end{array}$$

Since T is an nilpotent extension of U, the map $g : T \to Y$ factors through the envelope $D = \varinjlim_{m \in \mathbb{N}} Y_m$ of X in Y. We denote this map by $f : T \to D$. We then get a coherent sheaf $f^*M = M \otimes_{\mathcal{D}} \mathcal{O}_T$ over T.

Now we make the following claim:

Claim 3.3.4. — *The pullback f^*M over T is independent of the choice of $f : T \to D$.*

More precisely, let $f_1, f_2, f_3 : T \to D$ be any three maps induced produced as above. Then there exist natural isomorphisms of coherent sheaves $h_{ij} : f_i^*M \to f_j^*M$ over $\mathcal{O}_T$ such that

$$h_{23} \circ h_{12} = h_{13}.$$

We first grant the claim. For each thickening (U, T), we pick an arbitrary covering (U_i, T_i) of (U, T), where (U_i, T_i) admits a map to (X, Y). We then get the collection of coherent sheaves f_i^*M over each T_i. The claim allows us to produce a transition isomorphism for each restriction of f_i^*M on $T_i \cap T_j$, and they satisfy the cocycle condition when restricted at $T_i \cap T_j \cap T_k$. Hence, by gluing them together, we get a coherent sheaf $\mathcal{F}_T$ over (U, T). This produces a sheaf $\mathcal{F}$ over the infinitesimal site. Moreover, the coherent sheaf $\mathcal{F}$ is in fact a coherent crystal, namely the pullback $g^*\mathcal{F}_{T_2} \simeq \mathcal{F}_{T_1}$ for any map $(i, g) : (U_1, T_1) \to (U_2, T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$. This comes from the independence in the claim again by taking a composition with a map $T_2 \to D$.

Finally, we check that the functors are quasi-inverse to each other. We start with a coherent crystal $\mathcal{F}$ over the infinitesimal site, and let $(M = \mathcal{F}_D, \nabla_M)$ be the associated integrable connection over the envelope D defined in the first paragraph of the proof. Then the crystal $\mathcal{F}'$ induced by (M, ∇_M) is locally defined by assigning the module $M \otimes_{\mathcal{D}} \mathcal{O}_T$ to a thickening (U, T) for any morphism $(U, T) \to (X, D)$ in the infinitesimal site (which locally exists as the envelope covers the final object). This coincides with the value of coherent crystal $\mathcal{F}$ at (U, T) thanks to the equality $M = \mathcal{F}_D$. Conversely, we let (M, ∇_M) be an integrable connection over the envelope D and let $\mathcal{F}$ be the associated coherent crystal over the infinitesimal site defined in the second paragraph of the proof. The integrable connection over D that is induced by the crystal $\mathcal{F}$ is then defined on the module $M' = \mathcal{F}_D$, which by assumption is also equal to the module M. Moreover, the induced connection on M' also coincides with the input ∇_M thanks to the concrete calculation below Definition 3.1.8. So we are done.

Proof of the claim. — We finally deal with the claim. Let $\varphi_j : \mathcal{D} \to \mathcal{O}_T$ be maps of structure sheaves induced from $f_j : T \to D$. We define h_{jk} to be the $\mathcal{O}_T$-linear map given by

$$x \otimes 1 \longmapsto \sum_{E=(e_i)} \Big(\prod_i \nabla_i^{e_i}\Big)(x) \otimes \frac{(\varphi_j(t_i) - \varphi_k(t_i))^{e_i}}{e_i!}.$$

Since T is a nilpotent extension of U, for each $t \in \mathcal{D}$, the difference $\varphi_j(t) - \varphi_k(t)$ is nilpotent in $\mathcal{O}_T$. In particular, the above sum is only finite. Finally, by the general equality

$$\sum_{n=0}^{N} \frac{u^n}{n!} \cdot \frac{v^{N-n}}{(N-n)!} = \frac{(u+v)^N}{N!},$$

we have $h_{23} \circ h_{12} = h_{13}$. □

□

4. Cohomology over $\mathrm{B}^+_{\mathrm{dR},e}$

In this section we compute the cohomology of crystals over $X/\Sigma_{e\,\mathrm{inf}}$ using the de Rham complex over the envelope. Our strategy is to construct a double complex computing the Čech–Alexander complex and the de Rham complex in two separate directions, as in [7].

Remark 4.0.1. — Before we start, we mention that though our focus is rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$, the discussion in this section works alphabetically for cohomology of crystals over $X/K_{0,\mathrm{inf}}$, where K_0 is an arbitrary p-adic complete non-Archimedean field and X is a rigid space over K_0.

4.1. Cohomology of crystals over affinoid spaces. — We first compute the cohomology of crystals over $X/\Sigma_{e\,\mathrm{inf}}$ for X being an affinoid rigid space over Σ_e.

Let $X = \mathrm{Spa}(A)$ be an affinoid rigid space over Σ_e together with a closed immersion $X \to Y = \mathrm{Spa}(P)$ for a smooth affinoid rigid space Y over $\mathrm{B}^+_{\mathrm{dR,e}}$. Denote by D the envelope of X in Y (Definition 2.2.1), by $\mathcal{D}$ its structure sheaf $\varprojlim_m \mathcal{O}_{Y_m}$ (where Y_m is the m-th infinitesimal neighborhood of X in Y), and by J the kernel ideal for $\mathcal{D} \to \mathcal{O}_X$. By construction, the kernel ideal J is equal to the evaluation of the infinitesimal ideal sheaf $\mathcal{J}_{X/\Sigma_e}$ at the envelope D. We write Ω^i_D as the group of differentials $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D)$, which is equal to the inverse limit of continuous differentials

$$\varprojlim_m \Omega^i_{Y_m/\Sigma_e}.$$

By Lemma 3.3.3, Ω^i_D is isomorphic to the tensor product $\Omega^i_{Y/\Sigma_e} \otimes_{\mathcal{O}_Y} \mathcal{D}$ and is, in particular, locally free over $\mathcal{D}$.

We then take the section of the infinitesimal de Rham complex $(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}, \nabla)$ at D and get a chain complex of $\mathrm{B}^+_{\mathrm{dR,e}}$-modules

$$(M \otimes \Omega^\bullet_D, \nabla_D) := \ 0 \longrightarrow M \xrightarrow{\ \nabla\ } M \otimes_{\mathcal{D}} \Omega^1_D \xrightarrow{\ \nabla^1\ } \cdots,$$

where M is the evaluation $\mathcal{F}(D)$ of $\mathcal{F}$ at the envelope D. The complex is naturally filtered by the infinitesimal filtration, whose i-th filtration is the subcomplex

$$0 \longrightarrow J^i M \longrightarrow J^{i-1} M \otimes_{\mathcal{D}} \Omega^1_D \longrightarrow J^{i-2} M \otimes_{\mathcal{D}} \Omega^2_D \longrightarrow \cdots.$$

We also recall that by taking the powers of the infinitesimal ideal sheaf $\mathcal{J}_{X/\Sigma_e}$, we get a natural filtration on the structure sheaf $\mathcal{O}_{X/\Sigma_e}$ (cf. discussion before Remark 2.1.3) and hence on $\mathcal{F}$ by defining the i-th filtration as $\mathcal{J}^i_{X/\Sigma_e}\mathcal{F}$. This in particular induces a filtration on the cohomology complex $R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F})$, and we call the latter the *infinitesimal filtration* on the infinitesimal cohomology. Our main theorem in this subsection is the following:

Theorem 4.1.1. — *Let X, Y, $\mathcal{F}$, and M be as above. Then we have a natural filtered isomorphism in the filtered derived category of abelian groups:*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}) \longrightarrow (M \otimes \Omega^\bullet_D, \nabla_D),$$

where the left side is filtered by the infinitesimal filtration.

Remark 4.1.2. — Note that by Corollary 2.2.8, the above is also isomorphic to the cohomology of the crystal $\mathcal{G}$ over the big infinitesimal site, when $\mathcal{F} = \mu_*\mathcal{G}$ is the restriction of $\mathcal{G}$ defined over the big site $X/\Sigma_{e\mathrm{INF}}$.

The rest of this subsection will be devoted to the proof of the theorem.

Let us first fix some notations for this section. Denote by $D(n)$ the envelope of X in the $(n+1)$-fold self-product of Y over Σ_e. When $n = 0$, we write $D(0)$ as D. The simplicial object $D(\bullet)$ forms a hypercovering of the final object in $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, as in Proposition 2.2.7.

We fix a coherent crystal $\mathcal{F}$ on $X/\Sigma_{e\,\mathrm{inf}}$. Denote $M(n)$ to be the group of sections $\mathcal{F}(D(n))$ of $\mathcal{F}$ at $D(n)$, $\mathcal{D}(n)$ to be $\mathcal{O}_{X/\Sigma_e}(D(n))$, $J(n)$ to be the kernel for $\mathcal{D}(n) \to \mathcal{O}_X$, and $\Omega^i_{D(n)}$ to be $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D(n))$. When $n = 0$, we use M and Ω^i_D to abbreviate $M(0)$ and $\Omega^i_{D(0)}$. Here we recall that $\Omega^i_{D(n)} = \Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D(n))$ is isomorphic to the tensor product $\Omega^i_{Y(n)/\Sigma_e} \otimes_{\mathcal{O}_{Y(n)}} \mathcal{D}(n)$ (Lemma 3.3.3) and is, in particular, locally free over $\mathcal{D}(n)$.

Čech–Alexander complex. First we introduce the Čech–Alexander complex of a coherent $\mathcal{O}_{X/\Sigma_e}$ sheaf $\mathcal{F}$ (not necessarily a crystal).

We define $M(\bullet)$ to be the filtered cosimplicial cochain complex

$$M(\bullet) := (\mathcal{F}(D(0)) \longrightarrow \mathcal{F}(D(1)) \longrightarrow \cdots),$$

where the coboundary map is given by the alternating sum of degeneracy maps and the filtration is the infinitesimal filtration whose i-th filtration at $D(n)$ is $J(n)^i \cdot \mathcal{F}(D(n))$. It is called the *Čech–Alexander complex of* $\mathcal{F}$.

Proposition 4.1.3. — *Let $\mathcal{F}$ be a coherent infinitesimal sheaf of $\mathcal{O}_{X/\Sigma_e}$-modules as above. Then we have a functorial filtered isomorphism*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}) \simeq M(\bullet)$$

in the filtered derived category of abelian groups.

Proof. — We first notice that by the filtered enhancement of Proposition 2.2.7,[6] we get an isomorphism of filtered complexes

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}) = R\Gamma(D(\bullet), \mathcal{F}) = R \lim_{[n]\in\Delta} R\Gamma(D(n), \mathcal{F}).$$

Denote by $Y(n)_m$ the m-th infinitesimal neighborhood of X in $Y(n)$. Since X is the common closed analytic subspace of every $Y(n)_m$, $Y(\bullet)_m$ forms a simplicial object in $X/\Sigma_{e\,\mathrm{inf}}$ with $D(\bullet) = \varinjlim_{m\in\mathbb{N}} h_{Y(\bullet)_m}$. This leads to the equality

$$R\Gamma(D(\bullet), \mathcal{F}) = R \varprojlim_{m\in\mathbb{N}} R\Gamma(Y(\bullet)_m, \mathcal{F}).$$

Notice that for each n, the rigid space $Y(n)_m$ is affinoid, and the covering of a given infinitesimal thickening $(X, Y(n)_m)$ is defined by analytic covering of the rigid space $Y(n)_m$. As a consequence, by the vanishing of the analytic cohomology for coherent sheaves over affinoid rigid spaces in the positive degrees, we know that

$$R\Gamma(Y(\bullet)_m, \mathcal{F}) = \Gamma(Y(\bullet)_m, \mathcal{F}).$$

6. More precisely, the filtered complex $R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F})$ is defined as the derived limit of the filtered modules $\big((\mathcal{J}^i_{X/\Sigma_e}\mathcal{F})(U,T)\big)_i$ ranging over all the infinitesimal thickenings $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$. In particular, for each $[n] \in \Delta^{\mathrm{op}}$, by the limit presentation, there is a functorial map of filtered objects from $R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F})$ to $M(n)$. The functoriality of the construction (with respect to the simplicial diagram Δ^{op}) in particular induces a filtered map $R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}) \to M(\bullet)$. To check it is an isomorphism, it suffices to do so on the i-th filtration component for each $i \in \mathbb{Z}$, which is implied by op. cit..

Furthermore, by the coherence of $\mathcal{F}$ and the Noetherianity of $\mathcal{O}(Y(n)_m)$, for each $n \in \mathbb{N}$, the inverse system $\Gamma(Y(n)_m, \mathcal{F})$ satisfies the Mittag–Leffler condition. In this way, we get

$$\begin{aligned} R \varprojlim_{m\in\mathbb{N}} R\Gamma(Y(\bullet)_m, \mathcal{F}) &= R \varprojlim_{m\in\mathbb{N}} \Gamma(Y(\bullet)_m, \mathcal{F}) \\ &= \varprojlim_{m\in\mathbb{N}} \Gamma(Y(\bullet)_m, \mathcal{F}) \\ &= \Gamma(\varinjlim_{m\in\mathbb{N}} Y(\bullet)_m, \mathcal{F}) \\ &= M(\bullet). \end{aligned}$$

□

Čech–Alexander and the de Rham. We then connect the Čech–Alexander complex to the de Rham complex.

Consider the section of the de Rham complex $(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}, \nabla)$ at the simplicial space $(D(n))_{[n]\in\Delta^{\mathrm{op}}}$:

$$\Delta^{\mathrm{op}} \ni [n] \longmapsto (M(n) \otimes_{\mathcal{D}(n)} \Omega^\bullet_{D(n)}, \nabla).$$

This produces a double complex $M^{n,m} = M(n) \otimes_{\mathcal{D}(n)} \Omega^m_{D(n)}$ in the first quadrant, with the horizontal coboundary map given by the alternating sum of degeneracy maps for simplicial space $D(\bullet)$ and the vertical coboundary map being the de Rham differential ∇^m. Note that the first column $M^{0,\bullet}$ of this double complex is the de Rham complex $M \otimes_{\mathcal{D}} \Omega^\bullet_D$, while the first row $M^{\bullet,0}$ is the Čech–Alexander complex $M(\bullet)$. So this provides a natural framework for those two types of complexes that we care about.

Moreover, the double complex is naturally filtered via the infinitesimal filtration $\mathcal{O}_{X/\Sigma_e} \supset \mathcal{J}_{X/\Sigma_e} \supset \mathcal{J}^2_{X/\Sigma_e} \cdots$. This is a descending filtration on the double complex, compatible with the cosimplicial structure, such that the i-th filtration on the n-th column is the differential complex

$$J(n)^i \longrightarrow J(n)^{i-1}\Omega^1_{D(n)} \longrightarrow \cdots \longrightarrow J(n)^0\Omega^i_{D(n)} \longrightarrow \Omega^{i+1}_{D(n)} \longrightarrow \cdots,$$

as a subcomplex of $\Omega^\bullet_{D(n)}$. Here we recall that $J(n)$ is the kernel ideal of the surjection $\mathcal{D}(n) \to \mathcal{O}_X$, defined as $\mathcal{J}_{X/\Sigma_e}(D(n))$. Note that when $X = Y$ is smooth over $\mathrm{B}^+_{\mathrm{dR,e}}$, the filtration on $\Omega^\bullet_D = \Omega^\bullet_{X/\mathrm{B}^+_{\mathrm{dR,e}}}$ is the usual Hodge filtration.

Furthermore, there are two canonical E_1 spectral sequences associated with the double complex $M^{n,m}$ ([3, Tag 0130]), with the formations given by

$$\begin{aligned} {}'E_1^{p,q} &= \mathrm{H}^q(M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}); \\ {}''E_1^{p,q} &= \mathrm{H}^q(M(\bullet) \otimes_{\mathcal{D}(\bullet)} \Omega^p_{D(\bullet)}). \end{aligned}$$

Both of these two spectral sequences converge to the hypercohomology of the total complex ([3, Tag 0132]). The same applies when we replace the double complex by its i-th infinitesimal filtration.

Now we make the following two Lemmas about degeneracy of those two spectral sequences:

Lemma 4.1.4. — *For each $p > 0$, the filtered cochain complex associated with the cosimplicial complex with its infinitesimal filtration*

$$M(\bullet) \otimes_{\mathcal{D}(\bullet)} \Omega^p_{D(\bullet)}$$

is filtered acyclic.

Lemma 4.1.5. — *Any degeneracy map $D(p) \to D$ induces an filtered isomorphism of the following two de Rham complexes:*

$$M \otimes_{\mathcal{D}} \Omega^\bullet_D \longrightarrow M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}$$

that is functorial with respect to the crystal $\mathcal{F}$. Here, $M = \mathcal{F}_D$ is the evaluation of the crystal at the envelope D.

We first assume the two lemmas above. By Lemma 4.1.4, the spectral sequence $''E_1^{p,q}$ is filtered degenerated in its first page and is convergent to the cohomology of the Čech–Alexander complex $M(\bullet)$ with its infinitesimal filtration.

On the other hand, Lemma 4.1.5 implies that the horizontal coboundary map of $'E_1^{p,q}$ is given by

$$\mathrm{H}^q(M \otimes_{\mathcal{D}} \Omega^\bullet_D) \xrightarrow{0} \mathrm{H}^q(M(1) \otimes_{\mathcal{D}(1)} \Omega^\bullet_{D(1)}) \xrightarrow{1} \mathrm{H}^q(M(2) \otimes_{\mathcal{D}(2)} \Omega^\bullet_{D(2)}) \xrightarrow{0} \cdots.$$

From this, the second page of $'E_1^{p,q}$ vanishes everywhere except for the column $'E_2^{0,\bullet}$, which is exactly the infinitesimal filtered de Rham complex $M \otimes_{\mathcal{D}} \Omega^\bullet_D$.

In this way, since both of those two spectral sequences are convergent to the total complex in the filtered derived category, we get the filtered isomorphism between the de Rham complex $M \otimes_{\mathcal{D}} \Omega^\bullet_D$ and the Čech–Alexander complex $M(\bullet)$. So by Proposition 4.1.3, we get Theorem 4.1.1. Here the functoriality follows from that of Lemma 4.1.5 and Proposition 4.1.3.

Proof of Lemma 4.1.4. To complete the proof of Theorem 4.1.1, we first prove Lemma 4.1.4 in this paragraph.

We first give a proof for the special case where $\mathcal{F}$ is the structure sheaf and $p = 1$.

Lemma 4.1.6. — *The cosimplicial complex*

$$(*) \qquad \Omega^1_D \longrightarrow \Omega^1_{D(1)} \longrightarrow \Omega^1_{D(2)} \longrightarrow \cdots$$

is locally (with respect to the topology of X; cf. Remark 2.2.2) cosimplicial homotopic to zero, as filtered cosimplicial abelian groups.

Before the proof of this Lemma, we first recall that a *cosimplicial homotopic equivalence* of two maps $f, g : U \to V$ is defined as a cosimplicial morphism

$$h : U \to \mathrm{Hom}([1], V),$$

such that

$$h \circ s_0 = f,\ h \circ s_1 = g,$$

where $s_i : [0] \to [1]$ are two co-face maps.

A cosimplicial object U is called *cosimplicial homotopic to zero* if its identity map is cosimplicial homotopic to the zero map. Here we note that any additive functor F that sends cosimplicial objects to cosimplicial objects will preserve the cosimplicial homotopic equivalence.

We refer the reader to [3, Tag 019U] for the discussion about cosimplicial homotopic equivalence.

Proof. — We first recall that since $D(n)$ is the envelope of $X = \mathrm{Spa}(A)$ in the $n+1$-folded self-product of Y over Σ_e, by Lemma 3.3.3 above, we have

$$\Omega^1_{D(n)} = \Omega^1_{P^{\otimes n+1}/\Sigma_e} \otimes_{P^{\otimes n+1}} \mathcal{D}(n).$$

Besides, any cosimplicial boundaries map $P^{n+1} \to P^{l+1}$ induces a map $\Omega^1_{D(n)} \to \Omega^1_{D(l)}$. So the cosimplicial complex $(*)$ is the tensor product of the cosimplicial complex $\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}$ along the cosimplicial ring homomorphism

$$P^{\otimes \bullet+1} \longrightarrow \mathcal{D}(\bullet).$$

Moreover, the i-th filtration of the cosimplicial complex $(*)$ is

$$J^{i-1}\Omega^1_D \longrightarrow J(1)^{i-1}\Omega^1_{D(1)} \longrightarrow J(2)^{i-1}\Omega^1_{D(2)} \longrightarrow \cdots,$$

which is isomorphic to the fiber of a map between cosimplicial tensor products

$$\left(\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}\right) \otimes_{P^{\otimes \bullet+1}} \mathcal{D}(\bullet) \longrightarrow \left(\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}\right) \otimes_{P^{\otimes \bullet+1}} \left(\mathcal{D}(\bullet)/J(\bullet)^{i-1}\right).$$

Thus, to show the filtered acyclicity, it suffices to show that the cosimplicial module $\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}$ is homotopic equivalent to zero. Here we notice that when $P = \mathrm{B}^+_{\mathrm{dR,e}}\langle x_i \rangle$, each $P^{\otimes n+1}$ is a ring of convergent power series over $\mathrm{B}^+_{\mathrm{dR,e}}$, and the proof is totally identical to the case for polynomial rings, which is done in [7], Example 2.16. In general, when P is smooth over $\mathrm{B}^+_{\mathrm{dR,e}}$, it locally admits an étale morphism to an $\mathrm{B}^+_{\mathrm{dR,e}}\langle x_i \rangle$. So the exactness is true locally, hence globally by a Čech-complex argument associated with a covering. □

End of the proof for Lemma 4.1.4. — Consider the filtered complex $(*)$ as below:

$$(*) \qquad \Omega^1_D \longrightarrow \Omega^1_{D(1)} \longrightarrow \Omega^1_{D(2)} \longrightarrow \cdots.$$

As the statement is local, by shrinking to open subsets of X and Y if necessary, we could assume that the complex $(*)$ is filtered cosimplicial homotopic to zero as in Lemma 4.1.6. Then we apply the p-th wedge product functor and the tensor product functor $M(\bullet)\otimes_{\mathcal{D}(\bullet)} -$ successively to the cosimplicial complex $(*)$, and then the resulted cosimplicial complex is exactly the one in Lemma 4.1.4. But note that since any additive cosimplicial functor preserves the cosimplicial homotopic equivalence, the resulted complex is also filtered homotopic to zero. So we are done. □

Proof of Lemma 4.1.5. In this paragraph, we prove Lemma 4.1.5.

We first provide the following simpler description of the envelope $\mathcal{D}(p)$:

LEMMA 4.1.7. — *Assume that the $\mathrm{B}^+_{\mathrm{dR,e}}$-algebra P admits an étale map from a ring of convergent power series $\mathrm{B}^+_{\mathrm{dR,e}}\langle x_1,\dots,x_r\rangle$. Then the map of global sections of structure sheaves $\mathcal{D}\to\mathcal{D}(p)$ associated with the degeneracy map $D(p)\to D$ induces an isomorphism*

$$\mathcal{D}(p)\simeq\mathcal{D}[[\delta_{i,j},1\le i\le p,1\le j\le r]],$$

where the right side is a ring of formal power series over the topological ring $\mathcal{D}$.

The notation is explained as follows: the projection map $Y(p)\to Y$ of the $p+1$-th self-product onto the first copy induces the zero-th degeneracy map $D(p)\to D$. Then we can rewrite $P^{\otimes p+1}$ as $P\langle\delta_{i,j},\ 1\le i\le p,\ 1\le j\le r\rangle$, where $\delta_{i,j}$ is defined as $x_j\otimes 1\otimes\cdots\otimes 1-1\otimes\cdots\otimes x_j\otimes\cdots\otimes 1$, with x_j being in the i-th copy of P in the second term.

Proof. — We first consider the case when P is equal to the convergent power series ring.

Denote by J the kernel of the surjection $P\to A$, and let I be the kernel of the map $P^{\otimes p+1}\to P$. By construction, the ring of sections $\mathcal{D}(p)=\mathcal{O}(D(p))$ is equal to the inverse limit

$$\varprojlim_{m\in\mathbb{N}} P^{\otimes p+1}/(J\otimes 1\otimes\cdots\otimes 1, I)^m,$$

while $\mathcal{D}=\mathcal{O}(D)$ is $\varprojlim_{m\in\mathbb{N}} P/J^m$. So to prove the lemma, it suffices to notice that the above inverse limit is the same as the inverse limit

$$\mathcal{D}(p)=\varprojlim_{n\in\mathbb{N}}(\varprojlim_{m\in\mathbb{N}} P^{\otimes p+1}/(J\otimes 1\otimes\cdots\otimes 1)^m)/\overline{I}^n,$$

where $\overline{I}$ is the image of I along the map $P^{\otimes p+1}\to\varprojlim_{m\in\mathbb{N}} P^{\otimes p+1}/(J\otimes\cdots\otimes 1)^m$.

In fact, we have the following more general result:

CLAIM 4.1.8. — *Let R be a Noetherian ring and I, J be two ideals in R. Then we have a canonical isomorphism*

$$\varprojlim_{m\in\mathbb{N}}(\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m\longrightarrow\varprojlim_{m\in\mathbb{N}} R/(I,J)^m,$$

where $\overline{J}$ is the ideal generated by the image of J along the map $R\to\varprojlim_{m\in\mathbb{N}} R/I^n$.

Proof of the claim. — First notice that the sequence of ideals $\{(I,J)^m\}$ and $\{(I^m,J^m)\}$ are cofinal to each other, since

$$(I^{2m},J^{2m})\subset (I,J)^{2m}=(I^iJ^{2m-i},\ 0\leq i\leq 2m)\subset (I^m,J^m).$$

So the right side $\varprojlim_{m\in\mathbb{N}} R/(I,J)^m$ can be replaced by $\varprojlim_{m\in\mathbb{N}} R/(I^m,J^m)$.

Then we notice that the R-algebra $A:=\varprojlim_{m\in\mathbb{N}}(\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m$ is (I,J)-adic complete over R. To show this, by [3, Tag 0DYC], it suffices to show that the ring $(\varprojlim_{n\in\mathbb{N}} R/I^n)/J$ is I-adic complete. We then note that $(\varprojlim_{n\in\mathbb{N}} R/I^n)/J=\widehat{R}\otimes_R R/J$, where $\widehat{R}$ is the I-adic completion of R. Since R/J is a finitely generated module over R, the tensor product $\widehat{R}\otimes_R R/J$ is the same as the I-adic completion of R/J. Thus the R-algebra A is (I,J)-adic complete. In particular, we have

$$A=\varprojlim_{l\in\mathbb{N}} A/(I^l,J^l).$$

Finally, the quotient ring $A/(I^m,J^m)$ is given as

$$\begin{aligned}A/(I^l,J^l)&=(\varprojlim_{m\in\mathbb{N}}(\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m)/(I^l,J^l)\\&=(\varprojlim_{n\in\mathbb{N}} R/I^n)/(\overline{I}^l,\overline{J}^l)\\&=R/(I^l,J^l).\end{aligned}$$

So we get

$$\begin{aligned}\varprojlim_{m\in\mathbb{N}}(\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m &=: A\\&\simeq \varprojlim_{l\in\mathbb{N}} A/(I^l,J^l)\\&=\varprojlim_{l\in\mathbb{N}} R/(I^l,J^l).\end{aligned}$$ □

Finally, let us assume P is a smooth affinoid algebra that admits an étale map to the ring of convergent power series. By the claim above and the Noetherianity of the envelope ([8, Lem. 13.4.(ii)]), the lemma is reduced to showing that the formal completion $\mathcal{D}(p)$ for $P^{\otimes p+1}\to P$ is isomorphic to $P[[\delta_{i,j}]]$, which is proved in [8, Lem. 13.12.(ii)]. □

Our next observation is about the Euler sequence for the degeneracy map $D(p)\to D$. Denote by $\Omega^1_{D(p)/D}$ the module of continuous differentials of $\mathcal{D}(p)$ over $\mathcal{D}$ under the $(\Delta(p))$-adic topology, where $\Delta(p)$ is the kernel ideal for the diagonal map $\mathcal{D}(p)\to\mathcal{D}$. Then we have

LEMMA 4.1.9. — *The Euler sequence for the projection map $Y(p) \to Y$ over Σ_e induces a natural exact sequence of free $\mathcal{D}(p)$-module:*

$$0 \longrightarrow \Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p) \longrightarrow \Omega^1_{D(p)} \longrightarrow \Omega^1_{D(p)/D} \longrightarrow 0,$$

where the map $\Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p) \to \Omega^1_{D(p)}$ sends $dx_i \otimes 1$ to dx_i.

Proof. — We consider the inverse limit of the Euler sequences of differentials for the triple $Y(p)_m \to Y_m \to \Sigma_e$, with $m \in \mathbb{N}$ (Proposition 5.2.12 and Corollary 5.2.18). Then by Lemma 3.3.3, we see that the inverse limit $\varprojlim_{m\in\mathbb{N}}(\Omega^1_{Y_m/\Sigma_e} \otimes_{\mathcal{O}(Y_m)} \mathcal{O}(Y(p)_m))$ is isomorphic to the $\mathcal{D}(p)$-module $\Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p)$. Similarly, the inverse limit $\varprojlim_{m\in\mathbb{N}} \Omega^1_{Y(p)_m/Y_m}$ is isomorphic to $\Omega^1_{D(p)/D}$. In particular, we get the following sequence of $\mathcal{D}(p)$-modules:

$$0 \longrightarrow \Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p) \longrightarrow \Omega^1_{D(p)} \longrightarrow \Omega^1_{D(p)/D} \longrightarrow 0.$$

To show that the sequence is an exact sequence, we may assume that P admits an étale map from the ring of convergent power series $\mathrm{B}^+_{\mathrm{dR,e}}\langle x_1, \dots, x_r\rangle$. We apply Lemma 3.3.3 to the immersion $X \to Y$ and $X \to Y(p) = Y \times \cdots \times Y$ separately. We then get an description of differentials as follows:

$$\Omega^1_D = \bigoplus_{j=1}^r \mathcal{D} dx_j,\ \Omega^1_{D(p)} = (\bigoplus_{j=1}^r \mathcal{D}(p) dx_j) \oplus (\bigoplus_{\substack{1\le i\le p\\ 1\le j\le r}} \mathcal{D}(p) d\delta_{i,j}),$$

Here the projection map $D(p) \to D$ induced from $Y(p) \to Y$ produces the natural monomorphism

$$\Omega^1_D \longrightarrow \Omega^1_{D(p)},$$

sending the generator dx_j onto dx_j in $\Omega^1_{D(p)}$. This gives the injectivity from $\Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p)$ into $\Omega^1_{D(p)}$.

Moreover, by the explicit formula in Lemma 4.1.7 for a ring of convergent power series, the $(\delta_{i,j})$-adic continuous differential of $\mathcal{D}(p)$ over $\mathcal{D}$ is the free $\mathcal{D}(p)$-module generated by $d\delta_{i,j}$ for $1 \le i \le p$ and $1 \le j \le r$. This is exactly the cokernel of the injection above and is the free $\mathcal{D}(p)$-module generated by $d\delta_{i,j}$. Thus we get the short exact sequence, as expected. □

We can construct the relative de Rham complex of $D(p)$ over D by taking wedge products of $\Omega^1_{D(p)/D}$ and considering the relative differential operator. Then we have the following filtered version of the Poincaré Lemma for infinitesimal differentials:

LEMMA 4.1.10 (Poincaré Lemma). — *There exists a natural quasi-isomorphism to the relative de Rham complex*

$$\mathcal{D} \longrightarrow \Omega^\bullet_{D(p)/D}.$$

Moreover, for each $m \in \mathbb{N}$, the natural induced map below is a quasi-isomorphism

$$\mathcal{D} \to \Omega^{\bullet}_{D(p)/D}/\Delta(p)^{m+1-\bullet}.$$

Proof. — We first assume that Y is a unit disc, and by Lemma 4.1.7, the ring $\mathcal{D}(p)$ is equal to the ring of formal power series over $\mathcal{D}$ with coordinates $\delta_{i,j}$. For the first argument, it suffices to show that the augmented complex

$$(*) \qquad 0 \to \mathcal{D} \to \mathcal{D}(p) \to \Omega^1_{D(p)/D} \to \Omega^2_{D(p)/D} \to \cdots \to \Omega^N_{D(p)/D} \to 0$$

is homotopic to 0, where $N = pr$. Using the coordinate interpretation, the complex $(*)$ is an N-th completed tensor product of the complex

$$0 \to \mathcal{D} \to \mathcal{D}[[x]] \to \mathcal{D}[[x]]dx \to 0$$

over $\mathcal{D}$, where the map $\mathcal{D}[[x]] \to \mathcal{D}[[x]]dx$ is the $\mathcal{D}$-linear relative differential. But since $\mathcal{D}$ contains $\mathbb{Q}$, the relative differential is surjective with kernel being $\mathcal{D}$, which proves the first statement in this case. Moreover, notice that by writing down the differentials $\Omega^i_{D(p)/D}$ in terms of coordinates $\delta_{i,j}$ by Lemma 4.1.7, the differential in the complex $(*)$ preserves the degree. In this way, since the quotient $\Omega^{\bullet}_{D(p)/D}/\Delta(p)^{m+1-\bullet}$ kills exactly elements of degrees higher than m, we get the statement about the quotient complex in this case.

In general, as the statement is étale local with respect to the smooth rigid space $Y = \mathrm{Spa}(P)$, we may assume Y admits an étale morphism to an unit disc. Then the claim follows from a term-wise base change formula in the complex $(*)$, thanks to Lemma 3.3.3. □

Here is another observation which we will need in order to compute the cohomology of infinitesimal filtration:

Lemma 4.1.11. — *Let $D(p) \to D$ be the degeneracy map of envelopes as before, and let $J(p)$, J, and $\Delta(p)$ be the kernel ideals for surjections $\mathcal{O}_{D(p)} \to \mathcal{O}_X$, $\mathcal{O}_D \to \mathcal{O}_X$, and $\mathcal{O}_{D(p)} \to \mathcal{O}_D$, respectively. Then, for $j \leq m$ in $\mathbb{N}$, the natural map below is an isomorphism of $\mathcal{O}_X$-modules*

$$J^{m-j}/J^{m-j+1} \cdot \Delta(p)^j/\Delta(p)^{j+1} \longrightarrow$$
$$(J^m, J^{m-1}\Delta(p), \ldots, J^{m-j}\Delta(p)^j, J(p)^{m+1})/(J^m, \ldots, J^{m-j+1}\Delta(p)^{j-1}, J(p)^{m+1}).$$

Proof. — As the statement is local with respect to Y, let us first assume $Y = \mathrm{Spa}(P)$ admits an étale map to a ring of convergent power series. By Lemma 4.1.7, $\mathcal{D}(p)$ is the formal power series ring $\mathcal{D}[[\delta_{i,j}]]$, and the ideal $\Delta(p)$ is generated by variables $(\delta_{i,j})$. Notice that as the map $\mathcal{D}[\delta_{i,j}] \to \mathcal{D}[[\delta_{i,j}]]$ is flat and the quotient ideals in the statement can be defined over $\mathcal{D}[\delta_{i,j}]$, it suffices to show the analogous statement for the polynomial ring $\mathcal{D}[\delta_{i,j}]$.

Then, as the ring $\mathcal{D}[\delta_{i,j}]$ is a free module over $\mathcal{D}$ with a basis given by monomials of $\delta_{i,j}$, we could express elements x in $\mathcal{D}[\delta_{i,j}]\cap(J^m, J^{m-1}\Delta(p),\ldots, J^{m-j}\Delta(p)^j, J(p)^{m+1})$ using the coordinates as below:

$$\begin{aligned} x &= a_{r_{l_0}} + \sum_{|r_{l_1}|=1} a_{r_{l_1}}\cdot\delta^{r_{l_1}} + \sum_{|r_{l_2}|=2} a_{r_{l_2}}\cdot\delta^{r_{l_2}} + \cdots, \\ a_{r_{l_n}} &\in J^{m-n},\ \text{for } 0\le n\le j; \\ a_{r_{l_n}} &\in J^{m+1-n},\ \text{for } j<n\le m+1; \\ a_{r_{l_n}} &\in \mathcal{D},\ \text{for } j>m+1. \end{aligned}$$

Here, $\delta^{r_{l_n}}$ are monomials in $\delta_{i,j}$ with multi-indexes. Similarly, we could do this for elements in $\mathcal{D}[\delta_{i,j}]\cap\left(J^m,\ldots,J^{m-j+1}\Delta(p)^{j-1}, J(p)^{m+1}\right)$, where in the obtained formula we replace j above by $j-1$. Comparing these expressions, we see that the statement in the lemma holds for $\mathcal{D}[\delta_{i,j}]$. So by extending this along the flat map $\mathcal{D}[\delta_{i,j}]\to\mathcal{D}[[\delta_{i,j}]]$, we get the result for $\mathcal{D}(p)\simeq\mathcal{D}[[\delta_{i,j}]]$. □

Now we are ready to prove Lemma 4.1.5.

Proof for Lemma 4.1.5. —

Step 1. — We first deal with the underlying complexes and forget the infinitesimal filtration. Our goal is to show that the natural map of complexes below is a quasi-isomorphism:

$$M\otimes_{\mathcal{D}}\Omega^\bullet_D \longrightarrow M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)}.$$

The de Rham complex $\Omega^\bullet_D$ is equipped with its Hodge filtration defined by $F^i\Omega^\bullet_D=\sigma^{\geq i}\Omega^\bullet_D$. By the Euler sequence in Lemma 4.1.9, the Hodge filtration of $\Omega^\bullet_D$ induces a natural descending filtration on the relative de Rham complex $\Omega^\bullet_{D(p)}$, whose graded piece is $gr^i\Omega^\bullet_{D(p)}=\Omega^i_D\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)/D}$.

Now consider the de Rham complex $(M\otimes_{\mathcal{D}}\Omega^\bullet_D,\nabla_D)$ and $(M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)},\nabla_{D(p)})$ of the crystal $\mathcal{F}$ at D and $D(p)$. The projection $D(p)\to D$ induces a map of complexes

$$M\otimes\Omega^\bullet_D\longrightarrow M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)}.$$

By the crystal condition, the base change of M along the map $D(p)\to D$ is exactly $M(p)$. Moreover, by the compatibility of the de Rham complexes, the filtration on $\Omega^\bullet_{D(p)}$ induces a filtration on $M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)}=M\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)}$, given by

$$F^i(M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)})=M\otimes_{\mathcal{D}}F^i\Omega^\bullet_{D(p)}.$$

Each $F^i(M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)})$ is a subcomplex of $M\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)}$, since ∇^i sends elements in M into $M\otimes\Omega^1_D\subset M(p)\otimes\Omega^1_{D(p)}$. Moreover, the i-th graded factor of this filtration is

$$M\otimes_{\mathcal{D}}\Omega^i_D\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)/D},$$

which by Lemma 4.1.10 is isomorphic to the $M\otimes_{\mathcal{D}}\Omega^i_D$ via the degeneracy map. In this way, the projection $D(p)\to D$ induces a map of filtered complexes

$$(*)\qquad M\otimes_{\mathcal{D}}\Omega^\bullet_D \longrightarrow M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)},$$

which is an isomorphism on each graded factor. Hence the map $(*)$ itself is an isomorphism by the spectral sequence associated with a finite filtration as in [3, Tag 012K].

Step 2. — We then show that the above quasi-isomorphism is filtered under the infinitesimal filtration. More precisely, we claim that the graded piece of the following map in Step 1 is a filtered quasi-isomorphism under their infinitesimal filtrations:

$$M\otimes_{\mathcal{D}}\Omega^i_D \longrightarrow M\otimes_{\mathcal{D}}\Omega^i_D\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)/D}.$$

Consider the $(m+i)$-th graded piece for $m\in\mathbb{N}$. On the one hand, the $(m+i)$-th graded piece for the infinitesimal filtration on $M\otimes_{\mathcal{D}}\Omega^\bullet_D$ induces a subquotient $J^m\cdot M\otimes\Omega^i_D/J^{m+1}$ of the left-hand side of the above. On the other hand, the $(m+i)$-th graded piece for infinitesimal filtration on $M(p)\otimes_{\mathcal{D}(p)}\Omega^\bullet_{D(p)}$ induces the following subquotient of the right-hand side:

$$J(p)^{m-\bullet}\cdot M\otimes_{\mathcal{D}}\Omega^i_D\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)/D}/J(p)^{m+1-\bullet}.$$

So we get the map of graded pieces as below:

$$(**)\quad J^m\cdot M\otimes\Omega^i_D/J^{m+1}\longrightarrow J(p)^{m-\bullet}\cdot M\otimes_{\mathcal{D}}\Omega^i_D\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)/D}/J(p)^{m+1-\bullet}.$$

Here we note that as the ideal J maps into $J(p)$, the right-hand side is an $\mathcal{O}_D/J=\mathcal{O}_X$-linear complex.

To show $(**)$ is a quasi-isomorphism, we need to subdivide the right-hand side in a finer way. We introduce a finite increasing filtration on the right hand-side of $(**)$, whose j-th filtration is

$$\begin{aligned}
&(J^{m-\bullet},J^{m-1-\bullet}\Delta(p),\cdots J^{m-j}\Delta(p)^{j-\bullet},J(p)^{m+1-\bullet})\\
&\quad\cdot M\otimes_{\mathcal{D}}\Omega^i_{\mathcal{D}}\otimes_{\mathcal{D}}\Omega^\bullet_{D(p)/D}/J(p)^{m+1-\bullet}\\
&=\mathit{complex}\Big((J^m,\dots,J^{m-j}\Delta(p)^j,J(p)^{m+1})\cdot M\otimes\Omega^i_{\mathcal{D}}\otimes\mathcal{O}_{D(p)}/J(p)^{m+1}\\
&\qquad\longrightarrow(J^{m-1},\dots,J^{m-j}\Delta(p)^{j-1},J(p)^m)\cdot M\otimes\Omega^i_{\mathcal{D}}\otimes\Omega^1_{D(p)/D}/J(p)^m\\
&\qquad\longrightarrow\cdots\\
&\qquad\longrightarrow(J^{m-j},J(p)^{m+1-j})\cdot M\otimes\Omega^i_{\mathcal{D}}\otimes\Omega^j_{D(p)/D}/J(p)^{m+1-j}\Big).
\end{aligned}$$

The graded piece of this filtration is the $\mathcal{O}_X$-linear complex

$$\begin{aligned}
&(J^{m-\bullet},\dots,J^{m-j}\Delta(p)^{j-\bullet},J(p)^{m+1-\bullet})\\
&\quad\cdot M\otimes\Omega^i_{\mathcal{D}}\otimes\Omega^\bullet_{D(p)/D}/(J^{m-\bullet},\dots,J^{m-j+1}\Delta(p)^{j-1-\bullet},J(p)^{m+1-\bullet}).
\end{aligned}$$

We apply Lemma 4.1.11 to these complexes, then the graded piece above can be rewritten as

$$\begin{aligned} &J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Delta(p)^{j-\bullet}\Omega^\bullet_{D(p)/D}/\Delta(p)^{j-\bullet+1} \\ &= \mathit{complex}\Big(J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Delta(p)^j/\Delta(p)^{j+1} \\ &\qquad\qquad \longrightarrow J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Delta(p)^{j-1}\Omega^1_{D(p)/D}/\Delta(p)^j \\ &\qquad\qquad \longrightarrow \cdots \\ &\qquad\qquad \longrightarrow J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Omega^j_{D(p)/D}/\Delta(p)\Big) \\ &\simeq (J^{m-j}M \otimes \Omega^i_D/J^{m-j+1}) \otimes_{\mathcal{D}} (\Delta(p)^{j-\bullet}\Omega^\bullet_{D(p)/D}/\Delta(p)^{j+1-\bullet}). \end{aligned}$$

Finally, by the graded version of relative Poincaré Lemma in Lemma 4.1.10, we have

$$\Delta(p)^{j-\bullet}\Omega^\bullet_{D(p)/D}/\Delta(p)^{j+1-\bullet} \simeq \begin{cases} 0, & j \geq 1; \\ \mathcal{D}, & j = 0. \end{cases}$$

In this way, the graded pieces of the right-hand side of $(**)$ are zero, except for the zero-th graded piece which is naturally isomorphic to $J^m M \otimes \Omega^i_D/J^{m+1}$. Hence $(**)$ is an isomorphism, and we finish the proof. □

Remark 4.1.12. — Here we mention that the same study of the infinitesimal filtration works with minor changes for schemes. In particular, the schematic analogue of the proof in [7, Theorem 2.12] can be improved into a filtered version, and we thus obtain the expected filtered isomorphism in the crystalline theory, which is proved by different methods in [5, Theorem 7.23].

4.2. Global result. — We now generalize the computation of cohomology to the global situation, without assuming that X is affinoid.

We recall that the infinitesimal ideal sheaf $\mathcal{J}_{X/\Sigma_e} := \ker(\mathcal{O}_{X/\Sigma_e} \to \mathcal{O}_X)$ naturally defines a filtration on a coherent crystal $\mathcal{F}$, so that the j-th filtration is $\mathcal{J}^j_{X/\Sigma_e}\mathcal{F}$. Moreover, one can naturally extend the filtration to each individual $\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^i_{X/\Sigma_e}$, the entire de Rham complex of $\mathcal{F}$, so that

$$\mathrm{Fil}^j\left(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^i_{X/\Sigma_e}\right) = \begin{cases} \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^i_{X/\Sigma_e}, & \text{if } j < i; \\ \mathcal{J}^{j-i}_{X/\Sigma_e}\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^i_{X/\Sigma_e} & \text{if } j \geq i, \end{cases}$$

and the j-th filtration of the de Rham complex of $\mathcal{F}$ is

$$\begin{aligned} &\mathcal{J}^j_{X/\Sigma_e}\mathcal{F} \longrightarrow \mathcal{J}^{j-1}_{X/\Sigma_e}\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^1_{X/\Sigma_e} \longrightarrow \cdots \\ &\qquad \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^j_{X/\Sigma_e} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^{j+1}_{X/\Sigma_e} \longrightarrow \cdots . \end{aligned}$$

By taking the derived direct image, we in particular get a filtration on $Ru_{X/\Sigma_e *}(\mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}})$.

Our first result in this subsection shows that the above direct image vanishes in higher cohomological degrees.

Proposition 4.2.1. — *Let X be a rigid space over Σ_e, and let $\mathcal{F}$ be a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$. Then for each $i>0$ and $j\in\mathbb{N}$, we have*

$$Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F}\otimes_{\mathcal{O}_{X/\Sigma_e}}\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}})=0.$$

In particular, after applying the derived direct image $Ru_{X/\Sigma_e *}$, the truncation map of the de Rham complex induces a filtered quasi-isomorphism:

$$Ru_{X/\Sigma_e *}(\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}})\longrightarrow Ru_{X/\Sigma_e *}\mathcal{F}.$$

Proof. — Recall from Subsection 2.3 that $\Gamma(U,u_{X/\Sigma_{e\,\mathrm{inf}}*}\mathcal{G})$ is defined as the 0-th cohomology $\Gamma(U/\Sigma_{e\,\mathrm{inf}},\mathcal{G})$, and similarly for its filtered analogue. So to show the vanishing of $Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}})$, it suffices to do this locally and assume that X is affinoid together with a closed immersion into a smooth rigid space Y over $\Sigma_{e\,\mathrm{inf}}$. We then notice that by Proposition 4.1.3, $Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}})$ is computed by the following cosimplicial complex:

$$\begin{aligned}J^j\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D)&\longrightarrow J(1)^j\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D(1))\\&\qquad\longrightarrow J(2)^j\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D(2))\longrightarrow\cdots,\end{aligned}$$

which by Lemma 4.1.4 is homotopic to zero when $i>0$. So we get the vanishing of $Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}})$ for each $i>0$ and $j\in\mathbb{N}$; in other words, the filtered sheaf of complexes $Ru_{X/\Sigma_e *}(\mathcal{F}\otimes\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}})$ vanishes for $i>0$. By induction, the latter in particular shows that the truncation map

$$Ru_{X/\Sigma_e *}(\mathcal{F}\otimes\Omega^{\leq i}_{X/\Sigma_{e\,\mathrm{inf}}})\longrightarrow Ru_{X/\Sigma_e *}\mathcal{F}$$

is a filtered quasi-isomorphism for each $i>0$. As a consequence, since the filtered complex $Ru_{X/\Sigma_e *}(\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}})$ is the derived limit of $Ru_{X/\Sigma_e *}(\mathcal{F}\otimes\Omega^{\leq i}_{X/\Sigma_{e\,\mathrm{inf}}})$, we get the isomorphism as claimed in the second half of the statement. □

Now we can generalize Theorem 4.1.1 to the global case without assuming the affinoid condition:

Theorem 4.2.2. — *Let $X\to Y$ be a closed immersion of X into a smooth rigid space Y over Σ_e. Let $\mathcal{F}$ be a coherent crystal over $\mathcal{O}_{X/\Sigma_e}$, and let $\mathcal{F}_D=\varprojlim_{m\in\mathbb{N}}\mathcal{F}_{Y_m}$ be the restriction of $\mathcal{F}$ at the envelope $D=D_X(Y)=\varinjlim_{m\in\mathbb{N}}Y_m$, with its de Rham complex $\mathcal{F}_D\otimes\Omega^\bullet_D$. Then there exists a natural isomorphism in the filtered derived category of sheaves of abelian groups over X*

$$Ru_{X/\Sigma_e *}\mathcal{F}\longrightarrow\mathcal{F}_D\otimes\Omega^\bullet_D.$$

Before the proof, we want to mention that the strategy is to produce a natural map between those two complexes of sheaves of abelian groups, where the isomorphism will follow from the affinoid computation.

Proof. — By Proposition 4.2.1, the truncation map of the de Rham complex $\mathcal{F}\otimes_{\mathcal{O}_{X/\Sigma_e}}\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}\to\mathcal{F}[0]$ produces a canonical filtered isomorphism in the derived category of $\mathcal{O}_X$-modules

$$Ru_{X/\Sigma_e*}(\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}})\to Ru_{X/\Sigma_e*}\mathcal{F}.$$

On the other hand, we recall that the envelope $D=D_X(Y)$ is defined as the direct limit $\varinjlim_{m\in\mathbb{N}}h_{Y_m}$ of representable sheaves, where Y_m is the m-th infinitesimal neighborhood of X into Y. In the infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, the map from the envelope D to the final object 1 induces a map of derived functors

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}},-)\longrightarrow R\Gamma(D,-)=R\varprojlim_m R\Gamma(Y_m,-).$$

Similarly for its filtered analogue.

We apply the natural transformation to the filtered de Rham complex $\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}$ and get

$$\begin{aligned}R\Gamma(X/\Sigma_{e\,\mathrm{inf}},\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}})&\to R\varprojlim_m R\Gamma(Y_m,\mathcal{F}_{Y_m}\otimes_{\mathcal{O}_{Y_m}}\Omega^\bullet_{Y_m/\Sigma_e})\\&=R\Gamma(X,R\varprojlim_{m\in\mathbb{N}}\mathcal{F}_{Y_m}\otimes_{\mathcal{O}_{Y_m}}\Omega^\bullet_{Y_m/\Sigma_e})\\&=R\Gamma(X,\mathcal{F}_D\otimes_{\mathcal{O}_D}\Omega^\bullet_D),\end{aligned}$$

where the last equality follows from the observation that the inverse system $\{\mathcal{F}_{Y_m}\otimes_{\mathcal{O}_{Y_m}}\Omega^\bullet_{Y_m/\Sigma_e}\}_m$ admits a finite filtration, where each graded piece $\{\mathcal{F}_{Y_m}\otimes_{\mathcal{O}_{Y_m}}\Omega^i_{Y_m/\Sigma_e}\}_m$ is a pro-coherent system satisfying the sheaf version Mittag–Leffler condition ([5, Lemma 7.20]). Similarly for the subcomplex $\mathcal{J}^{m-\bullet}_{X/\Sigma_e}\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}$. Notice that the map is functorial with respect to all locally closed immersions (X,Y) into smooth rigid spaces. In particular, by varying X among all open subsets U of X and considering the above map for locally closed immersions (U,Y), we could enhance the above into the sheaf version filtered morphism

$$Ru_{X/\Sigma_e*}(\mathcal{F}\otimes\Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}})\longrightarrow\mathcal{F}_D\otimes_{\mathcal{D}}\Omega^\bullet_D.$$

Thus, by composing with (the inverse of) the filtered isomorphism at the beginning, we get a natural map in the filtered derived category of sheaves of abelian groups over X:

$$Ru_{X/\Sigma_e*}\mathcal{F}\longrightarrow\mathcal{F}_D\otimes_{\mathcal{D}}\Omega^\bullet_D.$$

Finally, to show the filtered isomorphism, we note that the evaluation functor defined in the second paragraph of the proof is compatible with restriction onto open affinoid subspaces of Y and X. In particular, as the map is analytic local with respect to X, the compatibility allows us to reduce to the case when both X and Y are affinoid. In the latter case, we know by Theorem 4.1.1 that the map is a filtered isomorphism, which finishes the proof. □

As a consequence, we get a change of bases formula quite easily.

PROPOSITION 4.2.3. — *Let X be a rigid space over Σ_e and $e' \geq e$ be an integer. Let $\mathcal{F}'$ be a crystal in vector bundles over $X/\Sigma_{e'\,\mathrm{inf}}$ and $\mathcal{F}$ be the pullback of $\mathcal{F}'$ along the map of infinitesimal topoi $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e'\,\mathrm{inf}})$. Then there exists a natural isomorphism of complexes of sheaves of $\mathrm{B}^+_{\mathrm{dR},e'}$-modules as below:*

$$(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} \longrightarrow Ru_{X/\Sigma_e *}\mathcal{F}.$$

Proof. — We first notice that the natural morphism of infinitesimal sites $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e'\,\mathrm{inf}}$ induces a canonical map in the derived category of sheaves over X

$$Ru_{X/\Sigma_{e'}*}\mathcal{F}' \longrightarrow Ru_{X/\Sigma_e *}\mathcal{F}.$$

Moreover, as the target is $\mathrm{B}^+_{\mathrm{dR,e}}$-linear, by the adjunction for the forgetful functor (from $\mathrm{B}^+_{\mathrm{dR,e}}$-modules to $\mathrm{B}^+_{\mathrm{dR},e'}$-modules), we get a natural map of complexes

$$(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} \longrightarrow Ru_{X/\Sigma_e *}\mathcal{F}.$$

So it suffices to show this adjunction map is an isomorphism.

As the statement is analytic local with respect to X, by shrinking X if necessary, we can assume that there exists a closed immersion $X \to Y'$ of X into a smooth rigid space over $\Sigma_{e'}$. By Theorem 4.2.2, we have the following natural isomorphisms:

$$Ru_{X/\Sigma_{e'}*}\mathcal{F}' \longrightarrow \mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'};$$
$$Ru_{X/\Sigma_e *}\mathcal{F} \longrightarrow \mathcal{F}_D \otimes \Omega^\bullet_D,$$

where D' is the envelope of X in Y', and D is the envelope of X in $Y = Y' \times_{\Sigma_{e'}} \Sigma_e$, where the latter is smooth over Σ_e. Notice that $\mathcal{O}_{Y'}$ is flat over $\Sigma_{e'}$, and the structure sheaves $\mathcal{O}_{D'}$ is flat over $\mathcal{O}_{Y'}$ (for it is defined as the formal completion of $\mathcal{O}_{Y'}$ along $X \to Y'$). In this way, by the assumption that $\mathcal{F}'$ is a crystal in vector bundles, the complex $\mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'}$ is a $\mathrm{B}^+_{\mathrm{dR},e'}$-linear bounded complex of sheaves of flat $\mathrm{B}^+_{\mathrm{dR},e'}$-modules. Thus we get the isomorphisms

$$\begin{aligned}(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} &\simeq (\mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'}) \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} \\ &\simeq (\mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'})/\xi^e,\end{aligned}$$

which are then isomorphic to the complex $\mathcal{F}_D \otimes \Omega_D^\bullet$, as the envelope $D = D_X(Y)$ is equal to the pullback of $D' = D_X(Y')$ along the surjection $\mathrm{B}_{\mathrm{dR},e'}^+ \to \mathrm{B}_{\mathrm{dR,e}}^+ = \mathrm{B}_{\mathrm{dR},e'}^+/\xi^e$. □

5. Derived de Rham cohomology over $\mathrm{B}_{\mathrm{dR,e}}^+$

In this section, we introduce the derived de Rham cohomology of a rigid space over $\mathrm{B}_{\mathrm{dR,e}}^+$, and prove the comparison between the derived de Rham cohomology and the infinitesimal cohomology.

Before we start, we want to mention that we will use mildly the language of ∞-category throughout this section. The main reason is to globalize the affinoid constructions and get a good theory of "sheaf of derived objects" using the ∞-categorical cohomological descent.

Remark 5.0.1. — The construction of the analytic derived de Rham complex in this section can be applied to the more general class of analytic Huber rings, which includes, for example, rigid spaces over an arbitrary p-adic non-Archimedean field and perfectoid spaces.

The results of this section hold true for rigid spaces over a general p-adic field. Moreover, in an upcoming work [23] by Shizhang Li and the author, we show that the analytic derived de Rham complex of perfectoid rings is isomorphic to the de Rham period sheaves in [39].

Derived ∞-category and filtered ∞-category. We first setup the convention of derived ∞-category and its filtered version in this section.

Let $\mathcal{A}$ be a Grothendieck abelian category ([3, Tag 079A]). We can associate to $\mathcal{A}$ a natural ∞-category $\mathcal{D}(\mathcal{A})$, called the *derived ∞-category of $\mathcal{A}$* ([33], 1.3.5). This is the ∞-categorical enhancement of the classical derived ∞-category, and the homotopy category $\mathrm{hCh}(\mathcal{A})$ of $\mathcal{D}(\mathcal{A})$ is the usual derived category $D(\mathcal{A})$. Here we want to mention that the derived ∞-category $\mathcal{D}(\mathcal{A})$ is a stable presentable ∞-category. In the special case where $\mathcal{A}$ is the category of modules over an ring R, we use $\mathcal{D}(R)$ to denote $\mathcal{D}(\mathcal{A})$, which is equipped with a symmetric monoidal structure by the derived tensor product of complexes. As a convention in this section, we will call $\mathcal{D}(R)$ the derived category.

For a presentable ∞-category $\mathcal{C}$, we recall the *filtered ∞-category in* $\mathcal{C}$ is defined as the ∞-category

$$\mathcal{DF}(\mathcal{C}) := \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{C}).$$

Moreover, $\mathcal{DF}(\mathcal{C})$ admits a full sub-∞-category $\widehat{\mathcal{DF}}(\mathcal{C})$, called the *filtered complete ∞-category in* $\mathcal{C}$, consisting of objects $C_\bullet$ such that $\lim C_\bullet \simeq 0$. The natural inclusion functor $\widehat{\mathcal{DF}}(\mathcal{C}) \to \mathcal{DF}(\mathcal{C})$ admits a left adjoint, called the *filtered completion*. When $\mathcal{C} = \mathcal{D}(R)$ is the derived ∞-category of R-modules, we use $\mathcal{DF}(R)$ and $\widehat{\mathcal{DF}}(R)$ to denote $\mathcal{DF}(\mathcal{C})$ and $\widehat{\mathcal{DF}}(R)$, respectively. Here we note that by their homotopy categories (and induced functors), we recover the ordinary filtered derived category.

Hypersheaves. We then give a quick review about sheaves of complexes.

Let X be a site, and let $\mathcal{C}$ be a presentable ∞-category. The ∞-category of presheaves in $\mathcal{C}$, denoted as $\mathrm{PSh}(X, \mathcal{C})$, is defined to be the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$ of contravariant functors from X to $\mathcal{C}$. The ∞-category $\mathrm{PSh}(X, \mathcal{C})$ admits a full sub-∞-category $\mathrm{Sh}(X, \mathcal{C})$ of *(infinity) sheaves in* $\mathcal{C}$, consisting of functors $\mathcal{F} : X^{\mathrm{op}} \to \mathcal{C}$ that send coproducts to products and satisfy the descent along Čech nerves: for any covering $U' \to U$ in X, the natural morphism to the limit below is required to be a weak equivalence

$$(*) \qquad \mathcal{F}(U) \longrightarrow \lim_{[n]\in\Delta} \mathcal{F}(U'_n),$$

where $U'_\bullet \to U$ is the Čech nerve associated with the covering $U' \to U$. Here we note that this is the ∞-categorical analogue of the classical sheaf condition in ordinary categories.

There is a stronger descent condition which requires $(*)$ above to hold with respect to all *hypercovers* $U'_\bullet \to U$ in the site X. Sheaves satisfying such a stronger condition are called *hypersheaves*. For example, given any bounded below complex C of ordinary sheaves on a site X, the assignment $U \mapsto \mathrm{R}\Gamma(U, C)$ gives rise to a hypersheaf. The collection of hypersheaves in $\mathcal{C}$ forms a full sub-∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C})$ inside $\mathrm{Sh}(X, \mathcal{C})$.

Let $\mathcal{C} = \mathcal{D}(R)$ be the derived ∞-category of R-modules. Then the ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C})$ of hypersheaves over X is in fact equivalent to the derived ∞-category $\mathcal{D}(X, R)$ of classical sheaves of R-modules over X by [34, Corollary 2.1.2.3]. As an upshot, the underlying homotopy category of $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C})$ is the classical derived category of sheaves of R-modules over X. In particular, given a hypersheaf $\mathcal{F}$ of R-modules over X, we can always represent it by an actual complex of sheaves of R-modules.

5.1. Topological algebras over $\mathbf{A}_{\mathbf{inf,e}}$.

— As a preparation, we first setup basics around the topologically finite type algebras over $\mathrm{A}_{\mathrm{inf,e}} := \mathrm{A}_{\mathrm{inf}}/\xi^e$ and the construction of the analytic cotangent complex, generalizing the discussion for $e = 1$ in [18] Section 7.

In this subsection only, we make the convention that $M^\wedge$ is the classical p-adic completion of M, where M is a $\mathbb{Z}_p$-module.

Definition 5.1.1. — Let R be an $\mathrm{A}_{\mathrm{inf,e}}$-algebra.

(i) We call R *topologically finite type over* $\mathrm{A}_{\mathrm{inf,e}}$ if there exists a surjection of $\mathrm{A}_{\mathrm{inf,e}}$-algebras $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \dots, T_m\rangle \to R$ for some $m \in \mathbb{N}$.

(ii) We call R *topologically of finite presentation over* $\mathrm{A}_{\mathrm{inf,e}}$ if R admits a surjection from $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle \to R$ with kernel being a finitely generated ideal.

We denote $\mathrm{Alg}_{\mathrm{tfp},e}$ the category of p-adically complete p-torsion-free algebras R over $\mathrm{A}_{\mathrm{inf,e}}$ that are of topologically finite presentation, where the morphisms are defined as maps of $\mathrm{A}_{\mathrm{inf,e}}$-algebras.

Similarly, we can extend these notions to the relative situation, replacing $\mathrm{A}_{\mathrm{inf,e}}$ by any $\mathrm{A}_{\mathrm{inf,e}}$-algebra.

Here we list some basic properties about modules over a given $R \in \mathrm{Alg}_{\mathrm{tfp},e}$.

Lemma 5.1.2 (cf. [18], 7.1.1). — *Let R be an algebra in $\mathrm{Alg}_{\mathrm{tfp},e}$. Then we have*

(i) *Every finitely generated p-torsion-free R-module is finitely presented.*
(ii) *The ring R is coherent.*
(iii) *Let N be a finitely generated R-module, $N' \subset N$ a submodule. Then there exists an integer $c \geq 0$, such that*

$$p^k N \cap N' \subset p^{k-c} N'$$

for every $k \geq c$. In particular, the subspace topology on N' induced from the p-adic topology on N agrees with the p-adic topology of N'.
(iv) *Every finitely generated R-module M is p-adically complete and separated; i.e., every such M is isomorphic to its p-adic completion $M^{\wedge}$.*
(v) *Every submodule of a finite type free R-module F is closed for the p-adic topology of F.*

Proof. — (i) This is proved in the proof of [8, 13.4. (iii.b)]; for completeness, we record it here. We do this by induction, and note that for $n = 1$ the case is given in [13] 1.2.

Let $\overline{M}$ be the image of M in $M/\xi[\frac{1}{p}]$, and let N be the kernel of $M \to \overline{M}$. The image $\overline{M}$ is a finitely generated p-torsion-free R/ξ-module, which by induction is a finitely presented R/ξ-module. Note that this also implies that the R/ξ-module $\overline{M}$ is a finitely presented R-module. So by [3, Tag 0519], N is finitely generated over R, and to show the finite presentedness of M, it suffices to show the finite presentedness of N. But note that for $x \in N$, there exists some $k \in \mathbb{N}$ such that $p^k x \in \xi M$. This implies that $p^k \xi^{n-1} x = 0$ in M, as the element is contained in $\xi^n M = 0$, and by the p-torsion-freeness of M, we have $\xi^{n-1} x = 0$. So N is a finitely generated p-torsion-free R/ξ^{n-1}-module, and by induction we get the result.

(ii) By definition, a ring R is coherent if every finitely generated ideal of R is finitely presented. So by the p-torsion-freeness of R and (i), we get the result.

(iii) Let M be the kernel of the map $N \to N/N'[\frac{1}{p}]$, and we have the following short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow N/N'\left[\frac{1}{p}\right].$$

Then, since the image of N in $N/N'[\frac{1}{p}]$ is finitely generated and p-torsion-free, by (i) we know the image is finitely presented, and thus M is finitely generated ([3, Tag 0519]). Note that the quotient M/N' is p^∞-torsion, so by the finitely generatedness there exists some $c \in \mathbb{N}$ such that $p^c M \subset N'$. Besides, for $x \in N$ such that $p^k x \in M$, the image of x in $N/N'[\frac{1}{p}]$ is also zero. So the definition of M implies that $x \in M$ and $p^k x \in p^k M$. In this way, for $k \geq c$, we have

$$\begin{aligned} p^k N \cap N' &\subset p^k N \cap M \\ &\subset p^k M \\ &\subset p^{k-c} N'. \end{aligned}$$

(iv) We can fit M into the following short exact sequence of R-modules:

$$0 \longrightarrow N \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

We apply the p-adic completion to the sequence. Then note that since the subspace topology on N is the isomorphic to the p-adic topology by (iii), while the quotient topology on M is the same as the p-adic topology, by [36] Theorem 8.1, we get an exact sequence of p-adically complete R-modules with continuous maps

$$0 \longrightarrow N^\wedge \longrightarrow R^{\oplus n} \longrightarrow M^\wedge \longrightarrow 0.$$

Compared with the above two exact sequences, we see the natural map $N \to N^\wedge$ is injective while $M \to M^\wedge$ is surjective.

We then assume that the R-module M is finitely presented. By [3, Tag 0519], we know that N is finitely generated. In this way, since the surjection of $M \to M^\wedge$ is true for any finitely generated R-module, we see $N \to N^\wedge$ is an isomorphism. In particular, we get $M \simeq M^\wedge$. This finishes (iv) for M being finitely presented over R.

In general, let M be any finitely generated module over R. Take $\overline{M}$ to be the image of M in $M[\frac{1}{p}]$. Then, since $\overline{M}$ is finitely generated and p-torsion-free, we know that $\overline{M}$ is finitely presented and hence the kernel $N = \ker(M \to \overline{M})$ is finitely generated by loc. cit. Notice that by definition N is p^∞-torsion. So there exists some $m \in \mathbb{N}$ such that $p^m N = 0$. Now, by the p-torsion-freeness of $\overline{M}$, the base change of the exact sequence $0 \to N \to M \to \overline{M} \to 0$ along $R \to R/p^s$ is exact. Moreover, since the inverse system $\{N \otimes_R R/p^s\}_s$ is essentially constant, the inverse limit of the short exact sequence of inverse system is exact, and we get

$$0 \longrightarrow N^\wedge = N \longrightarrow M^\wedge \longrightarrow \overline{M}^\wedge \longrightarrow 0,$$

and by the isomorphism $\overline{M} \simeq \overline{M}^\wedge$ we get the result

$$M \simeq M^\wedge.$$

So we are done.

(v) Let N be a submodule of a finite free R-module F, and let $M := F/N$ be the quotient. By (iv), since M is finitely generated, we have the canonical isomorphism $M \simeq M^\wedge$. As in the proof of (iv), the p-adic completion induces the following short exact sequence:

$$0 \longrightarrow N^\wedge \longrightarrow F \longrightarrow M^\wedge \simeq M \longrightarrow 0.$$

Hence we get the isomorphism $N \simeq N^\wedge$. In particular, since N is complete and its p-adic topology is isomorphic to its subspace topology, we get the closedness of N in F by standard topological argument. □

Corollary 5.1.3. — *Let R be a topologically finite-type algebra over* $\mathrm{A}_{\mathrm{inf,e}}$.

(i) *The ring R is p-adically complete and separated.*

(ii) *The ring R is topologically finitely presented over* $\mathrm{A}_{\mathrm{inf,e}}$ *if it is p-torsion-free.*

(iii) *Assume R is in* $\mathrm{Alg}_{\mathrm{tfp,e}}$ *and I is an ideal of R. Then I is finitely presented over R if R/I is p-torsion-free.*

Proof. — (i) Note that R is the quotient of $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle$ for some m, with the latter being in $\mathrm{Alg}_{\mathrm{tfp},e}$. In particular, R is a finitely generated $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle$-module. So the result follows from Lemma 5.1.2.(iv).

(ii) By (i), we know R is p-adically complete and p-torsion-free. So it suffices to check that for a surjection $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle \to R$, the kernel is finitely generated. This then follows from Lemma 5.1.2.(i), since R is a finitely generated $\mathrm{A}_{\mathrm{inf,e}}\langle T_i\rangle$ module that is p-torsion-free.

(iii) By Lemma 5.1.2.(i) applied at the p-torsion-free R-module R/I, we know that I is a finitely generated ideal in R. So thanks to 5.1.2.(ii), we know that I is in fact finitely presented. □

Lemma 5.1.4. — *Let R be in* $\mathrm{Alg}_{\mathrm{tfp},e}$ *and F be a flat R-module*

(i) *The functor $M \mapsto (M \otimes_R F)^\wedge$ is exact on the category of finitely presented R-modules.*

(ii) *Given a finitely presented R-module M, the following canonical map is an isomorphism:*

$$M \otimes_R F^\wedge \longrightarrow (M \otimes_R F)^\wedge.$$

(iii) *The R-module $F^\wedge$ is flat over R and is p-torsion-free.*

Proof. — (i) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely presented R-modules. By assumption, the tensor product with F over R is exact, so it suffices to show that the p-adic completion is flat on $0 \to M' \otimes F \to M \otimes F \to M'' \otimes F \to 0$. By Lemma 5.1.2.(iii), there exists an integer $c \geq 0$ such that $p^k M \cap M' \subset p^{k-c} M'$. Applying this inclusion with the tensor product functor $- \otimes_R F$ and noticing that the flatness of F implies $(p^k M \cap M') \otimes F = (p^k M \otimes F) \cap (M' \otimes F)$, we see that the p-adic topology on $M' \otimes F$ is isomorphic to the subspace topology induced from $M \otimes F$. In particular, by [36] Theorem 8.1, we get the exactness

$$0 \longrightarrow (M' \otimes F)^{\wedge} \longrightarrow (M \otimes F)^{\wedge} \longrightarrow (M'' \otimes F)^{\wedge} \longrightarrow 0.$$

(ii) Assume that M has the following presentation:

$$R^{\oplus n} \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0.$$

The tensor product of this with F gives

$$F^{\oplus n} \longrightarrow F^{\oplus m} \longrightarrow M \otimes F \longrightarrow 0.$$

We then take the p-adic completion. By (i), we get an exact sequence

$$(F^{\wedge})^{\oplus n} \longrightarrow (F^{\wedge})^{\oplus m} \longrightarrow (M \otimes F)^{\wedge} \longrightarrow 0. \tag{1}$$

On the other hand, we replace F by $F^{\wedge}$ in the second exact sequence above and get

$$(F^{\wedge})^{\oplus n} \longrightarrow (F^{\wedge})^{\oplus m} \longrightarrow M \otimes F^{\wedge} \longrightarrow 0. \tag{2}$$

The canonical maps from (2) to (1) are identities on $F^{\wedge,\oplus n}$ and $F^{\wedge,\oplus m}$. Thus we get the isomorphism.

(iii) It suffices to show that for any injective map of finitely presented modules $M' \to M$, the tensor product with $F^{\wedge}$ is still injective. This then follows from (ii) and (i). □

Corollary 5.1.5. — *Let $f : A \to B$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp,e}}$. Then the kernel of any surjective A-homomorphism $\rho : A\langle T_i \rangle \to B$ is finitely generated over A. In particular, B is a topologically finitely presented A-algebra.*

Proof. — By assumption, we can write A as $\mathrm{A}_{\mathrm{inf,e}}\langle U_j \rangle / I$ for some finitely presented ideal I. This allows us to write the surjection ρ as $\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I \to B$, and it suffices to show that the ideal $J := \ker(\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I \to B)$ is finitely presented. Notice that the quotient ring B by assumption is p-torsion free. So by Corollary 5.1.3.(iii), the finite presentedness of the ideal J follows if we can show that the ring $\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I$ is in $\mathrm{Alg}_{\mathrm{tfp},e}$. The latter by Corollary 5.1.3.(ii) is equivalent to showing that the ring $\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I$ is topologically finite type over $\mathrm{A}_{\mathrm{inf,e}}$ and is p-torsion-free.

Finally, to check the latter conditions, we notice that the ring $\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I$ is by construction topologically finite type over $\mathrm{A}_{\mathrm{inf,e}}$. On the other hand, since

$\mathrm{A}_{\mathrm{inf,e}}\langle U_j\rangle/I = A\langle T_i\rangle$ is the p-adic completion of the flat A-module $A[T_i]$, by Lemma 5.1.4.(iii), we know it is p-torsion-free. □

5.2. Analytic cotangent complex: affinoid case. — We then introduce the analytic cotangent complex for algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$ and affinoid rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$ in this subsection.

Derived p-adic completion. We recall the basics of derived p-adic completion.

Let R be a $\mathbb{Z}_p$-algebra. For a complex $C = C^\bullet$ of R-modules, recall that the *derived p-adic completion* of C is defined as

$$\mathrm{R}\varprojlim_{m\in\mathbb{N}}(C \otimes^L_R \mathrm{cofib}(\ R \xrightarrow{p^m} R\ )),$$

as an object in the derived category $\mathcal{D}(R)$ of R-modules. Here the object $\mathrm{cofib}(\ R \xrightarrow{p^m} R\ )$ is the cone of the map $p^m : R \to R$. An object $C \in \mathcal{D}(R)$ is called *derived p-complete* if C is isomorphic to its derived p-adic completion. The subcategory $\mathcal{D}_p(R)$ of derived p-complete objects is a full subcategory ([3, Tag 091U]) of $\mathcal{D}(R)$, and the derived p-adic completion forms a left adjoint functor to the inclusion functor $\mathcal{D}_p(R) \to \mathcal{D}(R)$ ([3, Tag 091V]).

There exists a natural isomorphism of complexes of R-modules

$$R \otimes^L_{\mathbb{Z}_p} \mathrm{cofib}(\ \mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p\ ) \simeq \mathrm{cofib}(\ R \xrightarrow{p^m} R\ ).$$

From this, the derived functor $C \mapsto C \otimes^L_R \mathrm{cofib}(\ R \xrightarrow{p^m} R\ )$ in $\mathcal{D}(R)$ can be rewritten as

$$\begin{aligned} C \longmapsto\ & C \otimes^L_R R \otimes^L_{\mathbb{Z}_p} \mathrm{cofib}(\ \mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p\ ) \\ \simeq\ & C \otimes^L_{\mathbb{Z}_p} \mathrm{cofib}(\ \mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p\ ). \end{aligned}$$

Here we note that since $\mathbb{Z}_p$ is p-torsion-free, the complex $\mathrm{cofib}(\ \mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p\ )$ is isomorphic to the $\mathbb{Z}_p$-module $\mathbb{Z}_p/p^m[0]$ living at degree 0. In the case where C is a p-torsion-free R-module, by the flatness of C over $\mathbb{Z}_p$, its derived p-adic completion is exactly its classical p-adic completion $\varprojlim_m C/p^m$. This in fact holds true in full generality for complexes as follows:

LEMMA 5.2.1. — *Let C be a cochain complex of p-torsion-free $\mathbb{Z}_p$-modules. Then the derived p-completion of C can be represented by a cochain complex $\widetilde{C} \in \mathrm{Ch}(\mathbb{Z}_p)$, which is obtained by the term-wise classical p-completion of C.*

Proof. — We first notice that $\widetilde{C}$ is derived p-complete, as the derived p-completeness can be checked by cohomology ([3, Tag 091N]) and each $\mathrm{H}^i(\widetilde{C})$ is derived p-complete.

When C is bounded to the right, the derived tensor product $C \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$ is represented by the cochain complex $C/p^m \in \mathrm{Ch}(\mathbb{Z}_p)$, obtained via term-wise quotient by p^m. In this case, the claim follows from via [3, Tag 09AU] and can be checked by taking the mod p^n reductions, as $\widetilde{C} \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m = C/p^m$.

In general, consider the naive truncation

$$\sigma^{>n} C \longrightarrow C \longrightarrow \sigma^{\leq n} C.$$

By $\mathbb{Z}_p/p^m$ is quasi-isomorphic to a perfect complex over $\mathbb{Z}_p$, the derived tensor product functor $-\otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$ commutes with derived limit functor. In particular, we have

$$C \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m \simeq R \varprojlim_n \left((\sigma^{\leq n} C) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m \right).$$

Hence we have

$$\begin{aligned} R \varprojlim_m C \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m &\simeq R \varprojlim_m R \varprojlim_n \left((\sigma^{\leq n} C) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m \right) \\ &\simeq R \varprojlim_m R \varprojlim_n \left((\sigma^{\leq n} C)/p^m \right) \\ &\simeq R \varprojlim_n R \varprojlim_m \left((\sigma^{\leq n} C)/p^m \right) \\ &\simeq R \varprojlim_n \sigma^{\leq n} \widetilde{C} \\ &= \widetilde{C}. \end{aligned}$$

□

Analytic cotangent complex for affine formal schemes. Now we introduce the definition and the basic properties of analytic cotangent complexes, for a map of algebras over $\mathrm{A}_{\mathrm{inf,e}}$. The analogous discussion for topologically finite type algebras over K can be found in [18, Section 7.1].

Construction 5.2.2. — Let $f : A \to B$ ba a map of $\mathrm{A}_{\mathrm{inf,e}}$-algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$. Both A and B are p-adically complete p-torsion-free algebras over $\mathrm{A}_{\mathrm{inf,e}}$ that are quotients of $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m \rangle$ for some $m \in \mathbb{N}$. As an A-algebra, the ring B admits a standard simplicial resolution

$$P_\bullet \to B,$$

where each P_i is a polynomial over A ([3, Tag 08PM]). This allows us to give simplicial $P_\bullet$-modules $\Omega^1_{P_\bullet/A}$, where each $\Omega^1_{P_i/A}$ is the algebraic differential of P_i over A. Recall that the *algebraic cotangent complex* $\mathbb{L}_{B/A}$ is the image of the cochain complex $\Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B$ in the derived category over A. The *analytic cotangent complexes* $\mathbb{L}^{\mathrm{an}}_{B/A}$ *for the* $\mathrm{A}_{\mathrm{inf,e}}$*-algebras* $B \to A$ is then defined as the image of the derived p-adic completion of the $\Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B$ in the derived category of A-modules.

REMARK 5.2.3. — As the polynomial resolution is functorial with respect to the pair (A,B), by Lemma 5.2.1, the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ can be represented functorially by the cochain complex in $\mathrm{Ch}(B)$ produced by the term-wise p-adic completion of $\Omega^1_{P_\bullet/A}\otimes_{P_\bullet} B$.

There exists a canonical map from the algebraic cotangent complex $\mathbb{L}_{B/A}$ to the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$. This is given by the counit map of the adjoint pair for the derived p-completion and the inclusion functor $\mathcal{D}_p(A)\to\mathcal{D}(A)$.

Here are some useful results for the analytic cotangent complex of $\mathrm{A}_{\mathrm{inf,e}}$-algebras:

PROPOSITION 5.2.4. — *Let $f: A\to B$ be a map of $\mathrm{A}_{\mathrm{inf,e}}$-algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$. Assume f that is formally smooth. Then we have a canonical isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A}\longrightarrow\Omega^{1,\mathrm{an}}_{B/A}[0],$$

where the right-hand side is the module of (p-adic) continuous differential forms.

Proof. — Since f is formally smooth, by Elkik's algebraization result of formally smooth adic algebras (for this specific situation, see [10, Page 11, Footnote 6], where the Noetherian assumption is not needed), B is isomorphic to the p-adic completion of a smooth A-algebra. In particular, f is flat (Lemma 5.1.4) and the map $f_n := (A/p^n\to B/p^n)$ is smooth. So by the derived base change formula for the algebraic cotangent complex ([3, Tag 08QQ]), since $B/p^n = B\otimes^L_A A/p^n$, we have

$$\mathbb{L}_{B/A}\otimes^L_A A/p^n = \mathbb{L}_{(B/p^n)/(A/p^n)}.$$

Moreover, the smoothness of f_n gives a canonical isomorphism

$$\mathbb{L}_{(B/p^n)/(A/p^n)}\simeq\Omega^1_{(B/p^n)/(A/p^n)}[0]=\Omega^1_{B/A}\otimes_A A/p^n[0].$$

In this way, by taking the derived p-adic completion of $\mathbb{L}_{B/A}$ and noticing the p-torsion-freeness of B and A, we get

$$\begin{aligned}\mathbb{L}^{\mathrm{an}}_{B/A} &= \mathrm{R}\varprojlim_{n\in\mathbb{N}}\mathbb{L}_{B/A}\otimes^L_A A/p^n\\ &\simeq \mathrm{R}\varprojlim_{n\in\mathbb{N}}\Omega^1_{B/A}/p^n[0]\\ &= \Omega^{1,\mathrm{an}}_{B/A}[0].\end{aligned}$$

□

In the next result, we show that the analytic cotangent complex for a finite morphism coincides with the associated algebraic cotangent complex. Recall that for an object $L^\bullet$ in the derived category of R-modules, it is called *pseudo-coherent* if it is isomorphic to an upper-bounded complex of finite free R-modules.

Proposition 5.2.5. — *Let $A \to B$ be a map of two topologically finitely presented* $\mathrm{A}_{\mathrm{inf,e}}$*-algebras in* $\mathrm{Alg}_{\mathrm{tfp},e}$*, such that B is a finitely presented A-module. Then the algebraic cotangent complex $\mathbb{L}_{B/A}$ is pseudo-coherent. In particular, $\mathbb{L}_{B/A}$ is derived p-complete and we have a canonical isomorphism*

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Proof. — We first show that it suffices to assume that $A \to B$ is a surjection. Too see this, we first pick a polynomial algebra $A[x_1, \ldots, x_r]$ that maps surjectively onto B. By the finite presentedness assumption of B over A, each x_i satisfies a monic polynomial $f_i(x_i)$ of x_i in A, and the induced map $B' = A[x_1, \ldots, x_r]/(f_1, \ldots, f_r) \to B$ is also surjective. Here we note that the ring B', as a finite algebra over A that is p-torsion-free, is automatically p-complete and is also in $\mathrm{Alg}_{\mathrm{tfp},e}$. Moreover, notice that since the sequence $\{f_1, \ldots, f_r\}$ is a regular sequence in $A[x_i]$, by the distinguished triangle of algebraic cotangent complexes for $A \to B' \to B$, we get

$$\mathbb{L}_{B'/A} \otimes^L_{B'} B \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/B'}.$$

Here we note that as B' is relatively of local complete intersection over A, by [3, Tag 08SL] we know that $\mathbb{L}_{B'/A}$ is a perfect complex over B'. Thus to show the pseudo-coherence of $\mathbb{L}_{B/A}$, it suffices to show this for $\mathbb{L}_{B/B'}$, where $B' \to B$ is a surjective map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$.

Recall that by assumption, B is a finite A-module that is p-torsion-free. So Lemma 5.1.2 implies that B is a finitely presented A-module and there exists an exact sequence of A-modules as below:

$$A^{\oplus m} \xrightarrow{f} A^{\oplus r} \longrightarrow B \longrightarrow 0.$$

Moreover, as the image of f is a submodule of $A^{\oplus r}$, which by Lemma 5.1.2.(i) is finitely presented, we know that $\ker(f)$ is also finitely generated (and hence finitely presented as it is inside of $A^{\oplus m}$). This procedure allows us to give a finite free A-module resolution of B. In particular, this shows that B is pseudo-coherent over A.

We then take P to be a simplicial polynomial resolution of B over A, and let J be the kernel of the map $P \otimes_A B \to B$. Then by the finite presentedness of B over A, the simplicial A-algebra P is also pseudo-coherent over A. So by taking a base change along $A \to B$, we see that $P \otimes_A B$ is a simplicial B-algebra that is pseudo-coherent over B. Moreover, since the map $P \otimes_A B \to B$ has a natural section, the kernel J is also pseudo-coherent ([3, Tag 064X]). Notice that the cotangent complex $\mathbb{L}_{B/A}$ fits into the distinguished triangle (cf. [28, Chap. III. 3.3.2]

$$J \longrightarrow \mathbb{L}_{B/A} \longrightarrow J^2[1].$$

To show the pseudo-coherence of $\mathbb{L}_{B/A}$, by [3, Tag 064U] it suffices to show by decreasing induction that $\mathbb{L}_{B/A}$ is n-pseudo-coherent for each $n \leq 1$. When

$n=1$, the result is clear, as $\mathbb{L}_{B/A}$ has cohomological degree ≤ 0. Suppose the result is true for $n \leq 0$. Since $A \to B$ is a surjection, the induced surjective map $P \otimes_A B \to B$ is isomorphic on π_0 with kernel living in cohomological degree ≤ -1. Thus by [28] Chap III, 3.3, we have

$$\mathrm{H}^m(J^i) = 0, \ \textit{for } i > -m.$$

This implies that when $i > -n$, J^i is n-pseudo-coherent. On the other hand, by [28] Chap III, 3.3.2, there exists an isomorphism

$$J^j/J^{j+1} \longrightarrow L\mathrm{Sym}^j_B(\mathbb{L}_{B/A}), \ j \geq 0.$$

The derived symmetric product preserves the n-pseudo-coherence ([18] 7.1.18), so J^j/J^{j+1} is n-pseudo-coherent for any $j \geq 0$. Thus the fiber sequence for the quotient $J^i \to J^i/J^{i+1}$ allows us to deduce the n-pseudo-coherent of every J^i. In particular, when $i=2$, by taking the cohomological twist we know that $J^2[1]$ is $(n-1)$-pseudo-coherent. So combining with the quasi-coherence of J, we get the $(n-1)$-quasi-coherence of $I/I^2 = \mathbb{L}_{B/A}$ by [3, Tag 064V]. □

Corollary 5.2.6. — *Let A be a p-torsion-free topologically finitely presented $\mathrm{A}_{\mathrm{inf,e}}$-algebra and I be a finitely generated regular ideal in A such that $B := A/I$ is p-torsion-free. Then we have a canonical isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow I/I^2[1].$$

Proof. — Since B is p-torsion-free, by Corollary 5.1.3, we know that B is in $\mathrm{Alg}_{\mathrm{tfp},e}$. So the result follows from Proposition 5.2.5 and the case for algebraic cotangent complex. □

Here is another useful result about the distinguished triangles for triples:

Proposition 5.2.7. — *Let $A \to B \to C$ be maps in $\mathrm{Alg}_{\mathrm{tfp},e}$. Then we have*

(i) *The analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a pseudo-coherent object in the derived category of B-modules.*
(ii) *there exists a natural distinguished triangle of pseudo-coherent objects in the derived category of C-modules*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B C \longrightarrow \mathbb{L}^{\mathrm{an}}_{C/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{C/B}.$$

Before the proof, we make the following claim:

Lemma 5.2.8. — *Let $A \to B$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$. Let K be a bounded above complex of A-modules, and K' be its derived p-completion. Then the derived p-completion of $K \otimes^L_A B$ is isomorphic to the derived p-completion of $K' \otimes^L_A B$.*

Proof of Lemma. — It suffices to check by applying the derived tensor product functor $- \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$, which is then clear, as $K \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p \simeq K' \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$. □

Proof of Proposition 5.2.7. — (i) By Corollary 5.1.5, we may write B as the quotient P/I, where $P = A\langle T_i\rangle$ is a convergent power series ring over A, and I is a finitely generated ideal by Corollary 5.1.3.(iii). We take the distinguished triangle of algebraic cotangent complexes for $A \to P \to B$ and get

$$\mathbb{L}_{P/A} \otimes_P^L B \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/P}.$$

Note that $\mathbb{L}_{P/A}^{\mathrm{an}}$ is isomorphic to the finite free P-module $\Omega_{P/A}^{1,\mathrm{an}}$ (Proposition 5.2.4), so after applying the derived p-completion and using the lemma above, we get a distinguished triangle

$$\Omega_{P/A}^{1,\mathrm{an}} \otimes_P B[0] \longrightarrow \mathbb{L}_{B/A}^{\mathrm{an}} \longrightarrow \mathbb{L}_{B/P}^{\mathrm{an}}.$$

Here the $\mathbb{L}_{B/P}^{\mathrm{an}}$ is pseudo-coherent by Proposition 5.2.5. Thus we are done.

(ii) For (ii), take the distinguished triangle for algebraic cotangent complexes. We get

$$\mathbb{L}_{B/A} \otimes_B^L C \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{C/B}.$$

So the result follows from the lemma above and the pseudo-coherence of each analytic cotangent complex. □

Analytic cotangent complex for affinoid rigid spaces. We then introduce the basics of the analytic cotangent complex for a map of affinoid algebras over $\mathrm{B}_{\mathrm{dR,e}}^+$ using the integral construction given in the last paragraph. The analogous discussion for topologically finite type algebras over K can be found in [18, Section 7.2].

Construction 5.2.9. — Let $f : A \to B$ be a map of topologically finite type affinoid algebras over $\mathrm{B}_{\mathrm{dR,e}}^+$, namely both A and B are quotients of $\mathrm{B}_{\mathrm{dR,e}}^+\langle T_1, \ldots, T_m\rangle$ for some $m \in \mathbb{N}$. Denote by $\mathcal{C}_{B/A}$ the category of pairs of rings (B_0, A_0), where A_0 and B_0 are rings of definition of A and B, respectively, such that both of them are in $\mathrm{Alg}_{\mathrm{tfp},e}$, and $f(A_0) \subset B_0$. The morphism among pairs is defined by inclusion maps on each entry.

Assume (B_0, A_0) is an object of $\mathcal{C}_{B/A}$. By the construction of the last paragraph, we can construct the analytic cotangent complex $\mathbb{L}_{B_0/A_0}^{\mathrm{an}}$ for B_0/A_0 as the derived p-completion of the algebraic cotangent complex $\mathbb{L}_{B_0/A_0}$.

Definition 5.2.10. — The *analytic cotangent complexes for affinoid algebras* B/A are defined as the colimit

$$\mathbb{L}_{B/A}^{\mathrm{an}} := \operatorname*{colim}_{(B_0,A_0)\in\mathcal{C}_{B/A}} \left(\mathbb{L}_{B_0/A_0}^{\mathrm{an}}\left[\frac{1}{p}\right]\right)$$

as an object in the derived category of B-modules.

REMARK 5.2.11. — As there exists a canonical actual complex representing $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$ (by the term-wise p-adic completion of $\Omega^1_{P/A_0} \otimes_P B$, where P is the standard polynomial resolution of B_0 over A_0), the analytic cotangent complexes can also be represented by a canonical actual complex, defined by taking the colimit of the actual term-wise complete complexes and then inverting by p.

Here we note that there exists a canonical map from the algebraic cotangent complex $\mathbb{L}_{B_0/A_0}$ to $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$, induced by the adjoint pair for the derived completion. By inverting p we also have a canonical map from the algebraic cotangent complex $\mathbb{L}_{B/A}$ to the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ for a map of algebras $A \to B$ over $\mathrm{B}^+_{\mathrm{dR,e}}$.

We then give a simple description of the analytic cotangent complex for a smooth morphism.

PROPOSITION 5.2.12. — *Let $A_0 \to B_0$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$, and let $A = A_0[\frac{1}{p}]$ and $B = B_0[\frac{1}{p}]$ be their generic fibers, respectively, with the induced map $f : A \to B$. Assume the corresponding map of affinoid rigid spaces $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$ is smooth. Then we have a natural isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right] \longrightarrow \left(\Omega^{1,\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right]\right).$$

Proof. — By Corollary 5.1.5, B_0 is a topologically finitely presented A_0-algebra. So we can write B_0 as the quotient ring of the relative convergent power series ring $P_0 = A_0\langle T_1, \ldots, T_m\rangle$ by some finitely generated ideal $I_0 \subset P_0$. Denote by P and I the ring $P_0[\frac{1}{p}]$ and the ideal $I_0[\frac{1}{p}]$, respectively. Then the surjection $P \to B$ induces a closed immersion of $\mathrm{Spa}(B)$ into the m-dimensional unit disc $\mathrm{Spa}(P)$ over $\mathrm{Spa}(A)$. Since both $\mathrm{Spa}(B)$ and $\mathrm{Spa}(P)$ are smooth over $\mathrm{Spa}(A)$, by the Jacobian criterion for the smoothness of adic spaces ([27], 1.6.9), for each maximal ideal $\mathfrak{P}$ of P that contains I, we can always find generators $s_1, \ldots, s_l$ of $I_{\mathfrak{P}}$ such that their derivatives $ds_1, \ldots ds_l$ can be extended to a basis of the continuous differential $\Omega^1_{P/A,\mathfrak{P}}$ at $\mathfrak{P}$. We denote by $\mathfrak{p}$ the intersection $\mathfrak{P} \cap A$. Then the above implies that the image of s_i in $P_{\mathfrak{P}} \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ forms a regular sequence. So by the flatness of $P_{\mathfrak{P}}$ over $A_{\mathfrak{p}}$ and [16, Chap. 0, Proposition 15.1.16], (s_i) forms a regular sequence in $P_{\mathfrak{P}}$. Since this is true for every maximal ideal $\mathfrak{P}$ of P containing I, we see B is a local complete intersection of P.

Now thanks to the surjectivity of $P_0 \to B_0$, Proposition 5.2.5 implies that

$$\mathbb{L}^{\mathrm{an}}_{B_0/P_0} \simeq \mathbb{L}_{B_0/P_0}.$$

Moreover, by the flat base change formula for the algebraic cotangent complex ([3, Tag 08QQ]), there exits a canonical isomorphism of algebraic cotangent complexes

$$\mathbb{L}_{B_0/P_0}\left[\frac{1}{p}\right] \simeq \mathbb{L}_{B/P},$$

which, by the local complete intersection of $P \to B$, is isomorphic to $I/I^2[1]$. On the other hand, by Proposition 5.2.7 and Proposition 5.2.4, we have a natural distinguished triangle

$$\left(\Omega^{1,\mathrm{an}}_{P_0/A_0} \otimes_{P_0} B_0\left[\frac{1}{p}\right]\right) \longrightarrow \mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B_0/P_0}\left[\frac{1}{p}\right].$$

Replacing the right side by the ideal $I/I^2[1]$ in degree 1, we get an isomorphism

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right] \simeq \mathrm{cofib}\left(I/I^2 \longrightarrow \Omega^{1,\mathrm{an}}_{P_0/A_0} \otimes_{P_0} B_0\left[\frac{1}{p}\right]\right).$$

Notice that by [27, Prop. 1.6.9.(ii)] and the smoothness of $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$, the above map on the right-hand side is injective. Hence the cofiber complex above lives in cohomological degree zero and is by definition equal to

$$\left(\Omega^{1,\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right]\right)[0]. \qquad \square$$

As a quite useful upshot, to compute the analytic cotangent complex for affinoid rings, it suffices to use one single pair of rings of definition.

Proposition 5.2.13. — *Let $A_0 \to B_0$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$, and let $A = A_0[\frac{1}{p}]$ and $B = B_0[\frac{1}{p}]$ be their generic fibers, respectively, with the induced map $f : A \to B$. Then the map below is an isomorphism:*

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Proof. — It suffices to show that for any commutative diagram of topologically finitely presented rings of definition

$$\begin{array}{ccc} A_0' & \longrightarrow & B_0' \\ \uparrow & & \uparrow \\ A_0 & \longrightarrow & B_0 \end{array},$$

the induced morphism $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}] \to \mathbb{L}^{\mathrm{an}}_{B_0'/A_0'}[\frac{1}{p}]$ is an isomorphism. Moreover, using the distinguished triangles for $A_0 \to A_0' \to B_0'$ and $A_0 \to B_0 \to B_0'$, respectively, we can reduce to show that

$$\mathbb{L}^{\mathrm{an}}_{A_0'/A_0}\left[\frac{1}{p}\right] = 0.$$

Then we notice that since $A_0 \to A_0'$ is an isomorphism after inverting by p, this satisfies the assumption of Proposition 5.2.12. So it suffices to prove that

$\Omega^{1,\mathrm{an}}_{A'_0/A_0}[\frac{1}{p}] = 0$. By Corollary 5.1.5, A'_0 is a topologically finitely presented algebra over A_0. We pick a set of generators x_i of A'_0 over A_0. Then by $A'_0[\frac{1}{p}] = A_0[\frac{1}{p}]$, there exists a positive integer N such that $p^N x_i \in A_0$. Note that the continuous differential form $\Omega^{1,\mathrm{an}}_{A'_0/A_0}$ is generated by the dx_i. So we get

$$p^N dx_i = d(p^N x_i) = 0.$$

This implies that $\Omega^{1,\mathrm{an}}_{A'_0/A_0}$ is p^∞-torsion. In particular, we have

$$\Omega^{1,\mathrm{an}}_{A'_0/A_0}\left[\frac{1}{p}\right] = 0.$$

So we are done. □

Here are some applications of the above result.

Corollary 5.2.14. — *Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$ is smooth. Then the projection onto the zero-th homotopy group induces natural isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow \Omega^1_{B/A}[0],$$

where the right side is the modules of continuous differential forms.

Proof. — This follows from Proposition 5.2.13 and Proposition 5.2.12. □

Similar to the integral case, when B is a quotient ring of A, the analytic cotangent complex coincides with the algebraic cotangent complex.

Corollary 5.2.15. — *Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$ such that B is a finite A-module. Then the natural map from the algebraic cotangent complex to the analytic cotangent complex below is an isomorphism:*

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Proof. — Pick a ring of definition A_0 of A that is topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$. We first notice that under the assumption of $A \to B$, we can find a ring of definition B_0 of B that contains $f(A_0)$ and is finite over A_0. To find such B_0, we pick a set of A-module generators x_i of B over A. Then each x_i satisfies a monic polynomial $f_i(X) = \sum_{j=0}^{r_i} a_{ij} X^j$ with coefficients in A. Since $A = A_0[\frac{1}{p}]$, we can pick a common integer $N \in \mathbb{N}$, such that coefficients $p^N a_{ij}$ are inside of A_0 for each i and j. From this, we see that the element $p^N x_i$ satisfies a monic polynomial with coefficient in A_0. In other words, the subring $B_0 = f(A_0)[p^N x_i]$ of B is finite over A_0.

Now the corollary follows easily from Proposition 5.2.13 and Proposition 5.2.5, since $\mathbb{L}^{\mathrm{an}}_{B/A}$ is isomorphic to $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]$, while the latter is computed by inverting p at the algebraic cotangent complex $\mathbb{L}_{B_0/A_0}$. Notice that thanks to the flat base change of the algebraic cotangent complex, $\mathbb{L}_{B_0/A_0}[\frac{1}{p}]$ is exactly the algebraic cotangent complex of B over A. So we get the result. □

As expected, we have the following simple description of the analytic cotangent complex for regular immersion:

Corollary 5.2.16. — *Let $f : A \to B$ ba a surjective map of topologically finite type algebras over $\mathrm{B}_{\mathrm{dR,e}}^+$, such that the kernel I is a regular ideal in A. Then we have a natural isomorphism*

$$\mathbb{L}_{B/A}^{\mathrm{an}} \longrightarrow I/I^2[1].$$

Proof. — This follows from Corollary 5.2.15 and [3, Tag 08SJ]. □

Another quick upshot is the pseudo-coherence of the analytic cotangent complex.

Corollary 5.2.17. — *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}_{\mathrm{dR,e}}^+$. Then the analytic cotangent complex $\mathbb{L}_{B/A}^{\mathrm{an}}$ is a pseudo-coherent complex of B-modules.*

Proof. — This follows from Proposition 5.2.13 and the integral version of the result of Proposition 5.2.7. □

We also obtain the distinguished triangle for triples as follows:

Corollary 5.2.18. — *Let $A \to B \to C$ be maps of topologically finite type algebras over $\mathrm{B}_{\mathrm{dR,e}}^+$. Then there exists a distinguished triangle of analytic cotangent complexes of affinoid algebras*

$$\mathbb{L}_{B/A}^{\mathrm{an}} \otimes_B^L C \longrightarrow \mathbb{L}_{C/A}^{\mathrm{an}} \longrightarrow \mathbb{L}_{C/B}^{\mathrm{an}}.$$

Proof. — Let $A_0 \to B_0 \to C_0$ be an arbitrary choice of rings of definition of $A \to B \to C$ that are topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$. By Proposition 5.2.7, we have a distinguished triangle

$$\mathbb{L}_{B_0/A_0}^{\mathrm{an}} \otimes_{B_0}^L C_0 \longrightarrow \mathbb{L}_{C_0/A_0}^{\mathrm{an}} \longrightarrow \mathbb{L}_{C_0/B_0}^{\mathrm{an}}.$$

Note that for the first term above we have the isomorphism

$$\begin{aligned}(\mathbb{L}_{B_0/A_0}^{\mathrm{an}} \otimes_{B_0} C_0) \otimes_{\mathbb{Z}_p}^L \mathbb{Q}_p &\simeq (\mathbb{L}_{B_0/A_0}^{\mathrm{an}} \otimes_{\mathbb{Z}_p}^L \mathbb{Q}_p) \otimes_{B_0 \otimes_{\mathbb{Z}_p}^L \mathbb{Q}_p}^L (C_0 \otimes_{\mathbb{Z}_p}^L \mathbb{Q}_p) \\ &\simeq \mathbb{L}_{B/A}^{\mathrm{an}} \otimes_B^L C.\end{aligned}$$

So the corollary follows from Proposition 5.2.13 by the base change along $\mathbb{Z}_p \to \mathbb{Q}_p$. □

A quick consequence is the following change of base equality:

Corollary 5.2.19. — *Let $A \to A' \to B$ be maps of topologically finite algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, with $A \to A'$ being étale. Then the natural map below is an isomorphism:*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A'}.$$

Proof. — By the distinguished triangle of the triple in Corollary 5.2.18, it suffices to show that $\mathbb{L}^{\mathrm{an}}_{A'/A}$ vanishes. So this follows from the assumption and Corollary 5.2.14. □

Finally, we have the étale base change formula as below.

Corollary 5.2.20. — *Let $A \to B \to B'$ be maps of topologically finite algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, with B'/B being étale. Then we have the following natural isomorphism:*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B' \longrightarrow \mathbb{L}^{\mathrm{an}}_{B'/A}.$$

In particular, when there exists an étale morphism $A \to A'$ such that $B' = A' \otimes_A B$, we get the base change formula

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_A A' \simeq \mathbb{L}^{\mathrm{an}}_{B'/A'}.$$

Proof. — The first isomorphism follows from the distinguished triangle (Corollary 5.2.18) for $A \to B \to B'$ and the étaleness of B'/B (Corollary 5.2.14). The second isomorphism follows from Corollary 5.2.19 and the isomorphism

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B' = \mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B \otimes_A A'.$$ □

5.3. Derived de Rham complex: affinoid case. — In this subsection, we construct the derived de Rham cohomology for topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$.

As a preparatory step, we first construct the rational analytic derived de Rham complex for a map $A_0 \to B_0$ of $\mathrm{A}_{\mathrm{inf}\,e}$-algebras, which is a complete filtered complex over $A_0[\frac{1}{p}]$, namely an object in $\widehat{\mathcal{DF}}(A_0[\frac{1}{p}])$. We also refer the reader to [6, Const. 4.1] for basics on the algebraic derived de Rham complex of algebras.

Construction 5.3.1. — Let A_0 be an $\mathrm{A}_{\mathrm{inf,e}}$-algebra in $\mathrm{Alg}_{\mathrm{tfp},e}$. We want to build a functor $F : \mathrm{Alg}_{\mathrm{tfp},e/A_0} \to \widehat{\mathcal{DF}}(\mathrm{B}^+_{\mathrm{dR,e}})$, sending $A_0 \to B_0$ to a filtered complete $\mathbb{E}_\infty$-algebra over $A_0[\frac{1}{p}]$.

Step 1. — Let P be the standard polynomial resolution of B_0 over A_0. The de Rham complex $\Omega^{\bullet}_{P/A_0}$ of P over A_0 is then a simplicial complex of P-modules. Moreover, the (direct sum) totalization $\mathrm{Tot}(\Omega^{\bullet}_{P/A_0})$ is a cochain complex of A_0-modules that comes with a canonical decreasing filtration defined by $\mathrm{Fil}^i = \mathrm{Tot}(\Omega^{\geq i}_{P/A_0})$.

Step 2. — Now we take the derived p-adic completion of the filtered cochain complex $(\mathrm{Tot}(\Omega^{\bullet}_{P/A_0}), \mathrm{Fil}^i)$ to get an object $(E, \mathrm{Fil}^i E)$ in the filtered derived category, such that E and $\mathrm{Fil}^i E$ are all derived p-adic complete. Then we invert p in $(E, \mathrm{Fil}^i E)$ to get an object $(E[\frac{1}{p}], \mathrm{Fil}^i E[\frac{1}{p}])$ in the filtered derived category of $A_0[\frac{1}{p}]$-modules.

Step 3. — Finally, we denote $F(B_0/A_0)$ to be the filtered completion of $(E[\frac{1}{p}], \mathrm{Fil}^i E[\frac{1}{p}])$.[7] Thus we get a functor from maps in $\mathrm{Alg}_{\mathrm{tfp},e}$ to the filtered complete derived category of $\mathrm{B}^+_{\mathrm{dR,e}}$-modules (even $A_0[\frac{1}{p}]$-modules), sending $A_0 \to B_0$ to $F(B_0/A_0)$.

Remark 5.3.2. — From the construction above, it is clear that the i-th graded piece of $F(B_0/A_0)$ is isomorphic to

$$\left(L \wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right]\right)[-i].$$

Remark 5.3.3. — Recall given a complex of $\mathbb{Z}_p$-modules C. It admits a natural map onto its derived p-adic completion $\widetilde{C}$. Applying this to the construction above (Step 2), we see there exists a natural filtered map from the algebraic derived de Rham complex $\widehat{\mathrm{dR}}_{B_0[\frac{1}{p}]/A_0[\frac{1}{p}]}$ to $F(B_0, A_0)$.

Remark 5.3.4. — From the construction above, the natural map from P to B_0 induces a filtered map from $F(B_0, A_0)$ to the continuous de Rham complex $\Omega^{\bullet,\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]$, which is compatible with the differentials. Here the filtration on the latter is the usual Hodge filtration.

Remark 5.3.5. — As the de Rham complex is equipped with a structure of commutative differential graded algebra and the above constructions are all lax-symmetric monoidal, the filtered complex $F(B_0/A_0)$ is also naturally a filtered $\mathbb{E}_\infty$-algebra in $\mathrm{B}^+_{\mathrm{dR,e}}$-modules.

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7. Roughly speaking, the filtered completion here means we are taking the derived inverse limit $R\varprojlim_j E[\frac{1}{p}]/\mathrm{Fil}^j E[\frac{1}{p}]$, together with the induced filtration with the i-th filtration being $R\varprojlim_{j\geq i} \mathrm{Fil}^i E[\frac{1}{p}]/\mathrm{Fil}^j E[\frac{1}{p}]$. This can be understood more canonically as the image of $(E[\frac{1}{p}], \mathrm{Fil}^i E[\frac{1}{p}])$ along the symmetric monoidal *filtered completion functor* $\mathcal{DF}(A_0[\frac{1}{p}]) \to \widehat{\mathcal{DF}}(A_0[\frac{1}{p}])$.

Now we consider the constructions for affinoid rigid spaces. Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Recall that the category $\mathcal{C}_{B/A}$ is defined as pairs of rings (B_0, A_0), where A_0 and B_0 are rings of definition of A and B separately, such that both of them are topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$ and $f(A_0) \subset B_0$. The morphism among pairs is defined by the natural inclusion map of pairs. Here we note that by Corollary 5.1.5, B_0 is a topologically finitely presented algebra over A_0 automatically.

Definition 5.3.6. — Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let $\mathcal{C}_{B/A}$ be the category of pairs of their rings of definitions as above. The *analytic derived de Rham complex of B over A*, denoted by $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$, is an object in $\widehat{\mathcal{DF}}(\mathrm{B}^+_{\mathrm{dR,e}})$ defined as

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} := \text{filtered completion of } \operatorname*{colim}_{(B_0,A_0)\in\mathcal{C}_{B/A}} F(B_0/A_0).$$

The filtration of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ is called the *algebraic Hodge filtration.* If we forget the filtered structure, we get the *underlying complex of* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$. It is denoted as $\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{B/A}$ and is defined as the 0-th filtration of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$, which is the image under the natural projection functor

$$\begin{aligned}\widehat{\mathcal{DF}}(\mathrm{B}^+_{\mathrm{dR,e}}) \subset \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{D}(\mathrm{B}^+_{\mathrm{dR,e}})) &\longrightarrow \mathcal{D}(\mathrm{B}^+_{\mathrm{dR,e}});\\ C_\bullet &\longmapsto C_0.\end{aligned}$$

Corollary 5.3.7. — *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the i-th graded piece of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ is naturally isomorphic to $L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i]$. In particular, for any pair of rings of definition $(B_0, A_0) \in \mathcal{C}_{B/A}$, the natural map below is a filtered isomorphism*

$$F(B_0, A_0) \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

Proof. — By the construction above, the algebraic Hodge filtration $\mathrm{Fil}^i\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ has the graded factor

$$\mathrm{gr}^i\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} = \operatorname*{colim}_{(B_0,A_0)\in\mathcal{C}_{B/A}} L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right][-i].$$

By Proposition 5.2.13 and the assumption on (B_0, A_0), each $L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]$ is isomorphic to the i-th derived wedge product of the analytic cotangent complex for the affinoid algebras B/A. In particular, the transition maps in the colimit above are all isomorphisms, and we can replace them by one single term. Thus we get

$$\mathrm{gr}^i\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \simeq L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i].$$

As a consequence, thanks to the filtered completeness, the filtered isomorphism can be checked on the graded pieces, and hence the natural map $F(B_0, A_0) \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ is a filtered isomorphism. □

REMARK 5.3.8. — By taking the colimit for the natural filtered map

$$\widehat{\mathrm{dR}}_{B_0[\frac{1}{p}]/A_0[\frac{1}{p}]} \longrightarrow F(B_0, A_0),$$

we get a canonical filtered map from the algebraic derived de Rham complex $\widehat{\mathrm{dR}}_{B/A}$ to the analytic derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$. This is compatible with the canonical map of each graded factor

$$L \wedge^i \mathbb{L}_{B/A} \longrightarrow L \wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}.$$

REMARK 5.3.9. — As the colimit is a lax-symmetric monoidal functor, the analytic derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ is naturally a filtered $\mathbb{E}_\infty$-algebra in $\mathcal{D}(\mathrm{B}^+_{\mathrm{dR,e}})$.

Here we provide a simple description of the analytic derived de Rham complex for two special cases: the smooth case and the complete intersections.

PROPOSITION 5.3.10. — *Let $A \to B$ be a smooth map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the natural morphism below from the analytic derived de Rham complex to the continuous de Rham complex is a filtered isomorphism:*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \longrightarrow \Omega^\bullet_{B/A}.$$

Proof. — By Remark 5.3.4, there exists a natural filtered map from $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ to the continuous de Rham complex $\Omega^\bullet_{B/A}$, which is compatible with the differential maps. By the assumption and Corollary 5.2.14, the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is isomorphic to the module of continuous differential forms $\Omega^1_{B/A}[0]$, which is a free B-module whose rank is equal to the relative dimension $\dim_A(B)$. On the other hand, the de Rham complex of affinoid algebras B over A is bounded above by the relative dimension and is thus complete under the Hodge filtration. The derived wedge product $L \wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}$ is isomorphic to $\wedge^i \Omega^1_{B/A}[0] = \Omega^i_{B/A}[0]$, which vanishes when $i > \dim_A(B)$. So by the Construction 5.3.1 above, the natural map from the analytic derived de Rham complex to the de Rham complex of B/A induces an isomorphism from the i-th graded factor $\mathrm{gr}^i \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} = L \wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i]$ to the i-th continuous differential $\Omega^i_{B/A}[-i]$. In this way, we get a filtered isomorphism from $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ to the de Rham complex $\Omega^\bullet_{B/A}$. □

Proposition 5.3.11. — *Let $A \to B$ be a surjective map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the canonical map below is a filtered isomorphism*

$$\widehat{\mathrm{dR}}_{B/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

As an upshot, the underlying complex $\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{B/A}$ is isomorphic to the formal completion $\widehat{A}$ for the surjection $A \to B$.

Proof. — As both $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ and $\widehat{\mathrm{dR}}_{B/A}$ are filtered complete, it suffices to show that the induced map on each graded factor is an isomorphism. For each $i \in \mathbb{N}$, the induced map $\mathrm{gr}^i\widehat{\mathrm{dR}}_{B/A} \to \mathrm{gr}^i\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ is exactly the natural map induced from the derived p-completion integrally (Construction 5.2.9). So the first claim follows from the assumption and Corollary 5.2.15.

For the second claim, it follows from the isomorphism between the underlying complex $\underline{\widehat{\mathrm{dR}}}_{B/A}$ of the algebraic derived de Rham complex and the formal completion $\widehat{A}$, which is the main result in [6] (see [6, 4.14, 4.16]). □

Corollary 5.3.12. — *Let $A \to B$ be a surjective map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the kernel ideal I is regular in A. Then for each $i \in \mathbb{N}$, we have a natural isomorphism*

$$A/I^i[0] \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}/\mathrm{Fil}^i.$$

In particular, by taking the derived limit, we get a filtered isomorphism of algebras

$$\widehat{A} \simeq \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A},$$

where the left side is the (classical) I-adic completion of A.

Proof. — This follows from Proposition 5.3.11 and the case for algebraic cotangent complex explained in Example 4.5 in [6], originally proved in [29, Theorem 2.2.6]. □

5.4. Global constructions. — In this subsection, we construct the global analytic cotangent complexes and the global analytic derived de Rham complexes. Our strategy is to show that the affinoid constructions satisfy the hyperdescent for the rigid topology and thus can be extended to a complex of sheaves over the given rigid space.

Unfolding. We first recall the unfolding of a hypersheaf in ∞-category. We refer the reader to [35, §6.5.3] for a more detailed discussion on hypercoverings, hypersheaves, and the hypercompletion.

Let X be a site that admits fiber products, and let $\mathcal{B}$ be a basis of X, namely $\mathcal{B}$ is a subcategory of X such that for each object U in X, there exists an object

U' in $\mathcal{B}$ covering U. So any hypercovering of an object in X can be refined to a hypercovering with each term in $\mathcal{B}$.

Let $\mathcal{C}$ be a presentable ∞-category. Consider a hypersheaf $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$ over $\mathcal{B}$. We can then *unfold* the sheaf $\mathcal{F}$ to a hypersheaf $\mathcal{F}'$ on X, such that its evaluation at any $V \in X$ is given by

$$\mathcal{F}'(V) = \operatorname*{colim}_{U'_\bullet \to V} \varprojlim_{[n]\in\Delta} \mathcal{F}(U'_n),$$

where the colimit is indexed over all hypercoverings $U'_\bullet \to V$ with $U'_n \in \mathcal{B}$ for all n. It follows from [35, Thm. 6.5.3.12] (see also the discussion at the beginning of [35, §6.5.3]) that one hypercovering suffices to compute the value of $\mathcal{F}'(V)$ in the above formula; in other words, for a hypercovering $U'_\bullet \to V$ with each U'_n in the basis $\mathcal{B}$, we have a natural weak equivalence

$$R\lim_{[n]\in\Delta} \mathcal{F}(U'_n) \longrightarrow \mathcal{F}'(V).$$

In particular, for any $U \in \mathcal{B}$, the natural map $\mathcal{F}(U) \longrightarrow \mathcal{F}'(U)$ is a weak equivalence.

The above construction is functorial with respect to $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$, and we get a natural unfolding functor

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C}),$$

which is in fact an equivalence, with the inverse given by the restriction functor $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C}) \to \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$. Here we mention that the equivalence follows quickly from the limit formula in the last paragraph, which says that the restriction of $\mathcal{F}'$ onto the basis $\mathcal{B}$ naturally coincides with the input sheaf $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$.

Recall in the special case where $\mathcal{C} = \mathcal{D}(R)$ is the derived ∞-category of R-modules. We have a natural equivalence

$$\begin{aligned} \mathcal{D}(X, R) &\longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{D}(R)); \\ C &\longmapsto (V \mapsto R\Gamma(V, C)). \end{aligned}$$

As an upshot, to define a complex of sheaves of R-modules over X, it suffices to specify a contravariant functor from the basis $\mathcal{B}$ to $\mathcal{D}(R)$, such that it satisfies the hyperdescent condition within $\mathcal{B}$.

Hyperdescent of $\mathbb{L}^{\mathrm{an}}_{B/A}$ and $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$. We first consider the analytic cotangent complex.

Proposition 5.4.1. — *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let $B \to B_\bullet$ be a map from B to a cosimplicial algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the associated map of rigid spaces $\mathrm{Spa}(B_\bullet) \to \mathrm{Spa}(B)$ is a* *rigid open hypercovering. Then the induced map below is an isomorphism:*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow R \lim_{[n]\in\Delta} \mathbb{L}^{\mathrm{an}}_{B_n/A}.$$

Proof. — We first notice that by the étaleness of the map $B \to B_n$ and Corollary 5.2.20, $\mathbb{L}^{\mathrm{an}}_{B_n/A}$ is naturally isomorphic to the base change $\mathbb{L}_{B/A} \otimes^L_B B_n$. So it suffices to show that the map below is an isomorphism

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow R \lim_{[n]\in\Delta} \mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n.$$

Note that since each $\mathrm{Spa}(B_n) \to \mathrm{Spa}(B)$ is an open covering of the rigid space $\mathrm{Spa}(B)$, the induced map $B \to B_n$ is flat on structure sheaves ([27, Proposition 1.7.6]). In this way, by the surjectivity of $\mathrm{Spa}(B_n) \to \mathrm{Spa}(B)$, we see that the map of affine schemes $\mathrm{Spec}(B_\bullet) \to \mathrm{Spec}(B)$ is a faithfully flat hypercover.

Finally, we recall from the faithfully flat descent of quasi-coherent sheaves over the affine scheme $\mathrm{Spec}(B)$ that for a given B-module M, the canonical map $M \to R\lim_{[n]\in\Delta} M \otimes^L_B B_n$ is an isomorphism. This, by induction, implies that for any bounded complex of B-modules C, we have $C \simeq R\lim_{[n]\in\Delta} C \otimes^L_B B_n$. As a consequence, since $\mathbb{L}^{\mathrm{an}}_{B/A}$ is bounded above, by taking cohomological truncations of $\mathbb{L}^{\mathrm{an}}_{B/A}$, we get

$$\begin{aligned}
\mathbb{L}^{\mathrm{an}}_{B/A} &= R \varprojlim_{i\le 0} \tau^{\ge i}\mathbb{L}^{\mathrm{an}}_{B/A} \\
&\simeq R \varprojlim_{i\le 0} R \lim_{[n]\in\Delta} \left(\tau^{\ge i}\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n\right) \\
&\simeq R \lim_{[n]\in\Delta} R \varprojlim_{i\le 0} \left(\tau^{\ge i}\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n\right) \\
&\simeq R \lim_{[n]\in\Delta} R \varprojlim_{i\le 0} \tau^{\ge i}\left(\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n\right) \\
&= R \lim_{[n]\in\Delta} \mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n.
\end{aligned}$$

Here the second isomorphism uses the fact that limit functors commute to each other, and the third isomorphism uses the flatness of B_n over B. □

Using the unfolding technique in Paragraph 5.4, we can extend the affinoid construction of the analytic cotangent complex to the global case.

COROLLARY 5.4.2. — *Let $X \to Y = \mathrm{Spa}(A)$ be a map of rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a complex of sheaves of A-modules $\mathbb{L}^{\mathrm{an}}_{X/Y}$ over X, such that for any affinoid open subset $U = \mathrm{Spa}(B)$ of X, we have a natural isomorphism*

$$R\Gamma(U, \mathbb{L}^{\mathrm{an}}_{X/Y}) = \mathbb{L}^{\mathrm{an}}_{B/A}.$$

The complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is called the analytic cotangent complex of X over Y.

Similarly, we could unfold the construction of the analytic derived de Rham complex to an arbitrary rigid space.

Proposition 5.4.3. — *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let $B \to B_\bullet$ be a map from B to cosimplicial algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the associated map of rigid spaces $\mathrm{Spa}(B_\bullet) \to \mathrm{Spa}(B)$ is a rigid open hypercovering. Then the induced filtered map below is an isomorphism:*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \longrightarrow R \lim_{[n]\in\Delta} \widehat{\mathrm{dR}}^{\mathrm{an}}_{B_n/A}.$$

Proof. — As a limit functor preserves the filtered completeness, $R\lim_{[n]\in\Delta} \widehat{\mathrm{dR}}^{\mathrm{an}}_{B_n/A}$ is an object in $\widehat{\mathcal{DF}}(A)$ and, checking the isomorphism above, can be reduced to its graded pieces. Moreover, notice that the graded piece functor commutes with small limits and colimits (cf. [9, Lemma 5.2]). Thus we get

$$\begin{aligned}
\mathrm{gr}^i\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} = L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i] \longrightarrow & \ \mathrm{gr}^i(R \lim_{[n]\in\Delta} \widehat{\mathrm{dR}}^{\mathrm{an}}_{B_n/A}) \\
\simeq & \ R \lim_{[n]\in\Delta} \mathrm{gr}^i\widehat{\mathrm{dR}}^{\mathrm{an}}_{B_n/A} \\
= & \ R \lim_{[n]\in\Delta} L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_n/A}[-i].
\end{aligned}$$

Notice that the wedge product functor commutes with the tensor product, and for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_n/A} &\simeq L\wedge^i (\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n) \\
&\simeq (L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}) \otimes^L_B B_n.
\end{aligned}$$

In this way, the natural map of graded pieces above is an isomorphism by the similar fpqc hyperdescent for $B \to B_\bullet$, as in the proof of Proposition 5.4.1. So we are done. □

Corollary 5.4.4. — *Let $X \to Y = \mathrm{Spa}(A)$ be a map of rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a filtered complex of sheaves of A-modules $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/Y}$ over X, such that for any affinoid open subset $U = \mathrm{Spa}(B)$ of X, we have a natural isomorphism*

$$R\Gamma(U, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/Y}) = \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

The complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/Y}$ *is called the* analytic derived de Rham complex of X over Y.

5.5. Infinitesimal cohomology and derived de Rham complex. — In this subsection, we give a comparison theorem between the infinitesimal cohomology and (the underlying complex of) the analytic derived de Rham complex of a rigid space X over $\mathrm{B}^+_{\mathrm{dR,e}}$.

Affinoid comparison. We first consider the affinoid case. Our tools are the Čech–Alexander complex for the infinitesimal cohomology and the structure of the analytic derived de Rham complex for closed immersions.

THEOREM 5.5.1. — *Let* $X = \mathrm{Spa}(A)$ *be an affinoid rigid space over* $\mathrm{B}^+_{\mathrm{dR,e}}$*. Then there exists a natural isomorphism as below:*

$$\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}).$$

Here $\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}}$ *is the underlying complex of the analytic derived de Rham complex.*

Before the proof, we want to mention that in the proof below, we will see that the isomorphism in the statement is induced from a chosen closed immersion $X \to Y$, where Y is a smooth rigid space. Later on, we will use this observation to globalize a general comparison.

Proof. — Let $P = \mathrm{B}^+_{\mathrm{dR,e}}\langle T_1, \ldots, T_m\rangle \to A$ be a surjection of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. By Proposition 4.1.3 for the crystal $\mathcal{F} = \mathcal{O}_{X/\Sigma_e}$, the infinitesimal cohomology of $X/\Sigma_{e\,\mathrm{inf}}$ can be computed by the cosimplicial cochain complex

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}) \simeq (\mathcal{O}_{X/\Sigma_e}(D(0)) \longrightarrow \mathcal{O}_{X/\Sigma_e}(D(1)) \longrightarrow \cdots),$$

where $D(\bullet)$ is the cosimplicial object of sheaves over the infinitesimal site produced by the envelope of A in $P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}} \bullet+1}$ (see the discussion before Theorem 4.1.1). Here we recall that by the definition of an envelope (cf. Definition 2.2.1), the sheaf $D(m)$ is the direct limit of all infinitesimal neighborhoods of $\mathrm{Spa}(A)$ in $\mathrm{Spa}(P^{\widehat{\otimes} m+1})$. In particular, we have the following isomorphism:

$$\mathcal{O}_{X/\Sigma_e}(D(m)) = \varprojlim_i P^{\widehat{\otimes} m+1}/I(m)^i,$$

where $I(m)$ is the kernel of the surjection $P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}} m+1} \to A$.

Now, by Proposition 5.3.11, there exists a natural filtered morphism inducing an isomorphism of their underlying complexes

$$\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{A/P^{\widehat{\otimes} m+1}} \longrightarrow \varprojlim_i P^{\widehat{\otimes} m+1}/I(m)^i.$$

By taking the cosimplicial version of the above isomorphism, we get isomorphisms of cosimplicial complexes

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}) \longrightarrow \mathcal{O}_{X/\Sigma_e}(D(\bullet)) \longleftarrow \underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{A/P^{\widehat{\otimes}\bullet+1}}.$$

So in order to prove the theorem, it suffices to show that the natural map $\mathrm{B}^+_{\mathrm{dR,e}} \to P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}} \bullet+1} \to A$ induces an isomorphism on analytic derived de Rham complexes. Moreover, by the filtered completeness, it reduces to show the isomorphism

$$\mathbb{L}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow \mathbb{L}^{\mathrm{an}}_{A/P^{\widehat{\otimes}\bullet+1}}.$$

Note that as both A and $P^{\widehat{\otimes}\bullet+1}$ are term-wise topologically finite type over $\mathrm{B}^+_{\mathrm{dR,e}}$, by the distinguished triangle of cotangent complexes for triples (Proposition 5.2.18), it suffices to show the vanishing of the following:

$$\mathbb{L}^{\mathrm{an}}_{P^{\widehat{\otimes}\bullet+1}/\mathrm{B}^+_{\mathrm{dR,e}}}.$$

Finally, we notice that by Proposition 5.2.13, the complex $\mathbb{L}^{\mathrm{an}}_{P^{\widehat{\otimes}\bullet+1}/\mathrm{B}^+_{\mathrm{dR,e}}}$ can be computed by inverting p at the term-wise derived p-completion of the algebraic cotangent complex of the Čech nerve $\check{\mathrm{C}}\mathrm{ech}(\mathrm{A}_{\mathrm{inf,e}} \to \mathrm{A}_{\mathrm{inf,e}}[T_1, \dots, T_r])$. So the vanishing we want follows from the vanishing of the algebraic cotangent complex $\mathbb{L}_{\check{\mathrm{C}}\mathrm{ech}(\mathrm{A}_{\mathrm{inf,e}} \to \mathrm{A}_{\mathrm{inf,e}}[T_i])/(\mathrm{A}_{\mathrm{inf,e}})}$, which is proved in the first part of Corollary 2.7 in [6]. □

In the special case where A is a complete intersection, the above can be improved into a filtered isomorphism. Here we recall that the filtration structure on the infinitesimal cohomology, which is called *infinitesimal filtration*, is defined via $R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{J}^\bullet_{X/\Sigma_e})$, where $\mathcal{J}_{X/\Sigma_e}$ is the kernel of the surjection $\mathcal{O}_{X/\Sigma_e} \to \mathcal{O}_X$ over the infinitesimal site (cf. the discussion above Remark 2.1.3).

Corollary 5.5.2. — *Let $X = \mathrm{Spa}(A)$ be an affinoid rigid space that admits a regular closed immersion into a smooth affinoid rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a natural filtered isomorphism as below:*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}).$$

Proof. — The proof is identical to the proof of Theorem 5.5.1, with the use of Corollary 5.3.12. □

Comparison in general. We are now ready to prove the comparison between the infinitesimal cohomology and (the underlying complex of) the analytic derived de Rham complex for a general rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$.

We first introduce the category of smooth immersions.

Definition 5.5.3. — Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. We define the *site* SE_X *of smooth immersions of* X, where

- objects of SE_X consist of tuples $(U, Z, i : U \to Z)$, with U being an affinoid open subset of X, Z a smooth affinoid rigid space over Σ_e, and $i : U \to Z$ is closed immersion;
- morphisms from $(U_1, Z_1, i_1 : U_1 \to Z_1)$ and $(U_2, Z_2, i_2 : U_2 \to Z_2)$ consist of commutative diagrams as below:

$$\begin{array}{ccc} U_1 & \xrightarrow{i_1} & Z_1 \\ \downarrow & & \downarrow \\ U_2 & \xrightarrow[i_2]{} & Z_2, \end{array}$$

  where $U_1 \to U_2$ is an open immersion over X.
- A collection of maps $\{(U_j, Z_j, i_j) \to (U, Z, i)\}$ is a covering if $\{U_j \to U\}$ and $\{Z_j \to Z\}$ are coverings of rigid spaces, respectively.

There exists a natural projection functor from SE_X to the category of affinoid open subsets X_{aff} of X by sending an object $(U, Z, i : U \to Z)$ to the open subset U in X. This functor is continuous under their topology. Here the associated *push-forward functor* π_* is the constant functor; i.e., for an ordinary sheaf $\mathcal{F}$ in the topos $\mathrm{Sh}(X_{\mathrm{aff}})$, the push-forward $\pi_*\mathcal{F}$ satisfies

$$(\pi_*\mathcal{F})(U, Z, i) = \mathcal{F}(U).$$

The *pullback functor* π^{-1} sends a sheaf $\mathcal{G}$ in $\mathrm{Sh}(\mathrm{SE}_X)$ to the sheaf associated with the presheaf

$$(\pi^{-1}\mathcal{G})(U) = \operatorname*{colim}_{(U,Z,i)} \mathcal{G}(U, Z, i), \ U \in X_{\mathrm{rig}}.$$

The colimit above is a filtered colimit, as given any two closed immersions $i_1 : U \to Z_1$ and $i_2 : U \to Z_2$, we can find a common refinement of them by $i : U \to Z_1 \times_{\Sigma_e} Z_2$, with natural projection maps $Z_1 \times_{\Sigma_e} Z_2 \to Z_j$ for $j = 1, 2$. In particular, the colimit (thus the inverse image functor π^{-1}) is exact. So, by translating this into the language of sites ([3, Tag 00X1]), we get a natural morphism of sites

$$\pi : X_{\mathrm{aff}} \longrightarrow \mathrm{SE}_X.$$

Before we prove the main theorem, we first notice that to check that two objects in $\mathcal{D}(X_{\mathrm{aff}}) = \mathcal{D}(X_{\mathrm{aff}}, \mathbb{Z})$ are isomorphic, it suffices to do so by pulling back to the category of smooth immersions. Precisely, we have the following general lemma:

Lemma 5.5.4. — *Let $\mathcal{F}$ and $\mathcal{G}$ be two objects in the derived category $\mathcal{D}(X_{\mathrm{aff}})$ of sheaves of abelian groups over the site X_{aff}. Assume $f : R\pi_*\mathcal{F} \to R\pi_*\mathcal{G}$ is* *an isomorphism in $\mathcal{D}(\mathrm{SE}_X)$. Then the following natural arrows are all isomorphisms:*

$$\begin{CD} \pi^{-1}R\pi_*\mathcal{F} @>{\widetilde{f}}>> \mathcal{G} \\ @VVV \\ \mathcal{F} \end{CD} \quad ,$$

where the arrows are counit maps associated with $f : \mathcal{F} \to \mathcal{G}$ and the identity $\mathcal{F} \to \mathcal{F}$, respectively. In particular, there exists a natural isomorphism $\mathcal{F} \to \mathcal{G}$ in the derived category $\mathcal{D}(X_{\mathrm{aff}})$.

Proof. — For each $U \in X_{\mathrm{aff}}$, we have

$$\begin{aligned} R\Gamma(U, \pi^{-1}R\pi_*\mathcal{F}) &= \operatorname*{colim}_{(U,Z,i)} R\Gamma((U,Z,i), R\pi_*\mathcal{F}) \\ &= \operatorname*{colim}_{(U,Z,i)} R\Gamma(U, \mathcal{F}) \\ &\simeq R\Gamma(U, \mathcal{F}), \end{aligned}$$

where the last map is an isomorphism as the colimit above is filtered (thus the geometric realization of the index set is contractible). In particular, this implies that the counit maps $\pi^{-1}R\pi_*\mathcal{F} \to \mathcal{F}$ and $\pi^{-1}R\pi_*\mathcal{G} \to \mathcal{G}$ are isomorphisms. Thus the claim follows from the following diagram of natural isomorphisms:

$$\begin{CD} \pi^{-1}R\pi_*\mathcal{F} @>{\pi^{-1}f}>> \pi^{-1}R\pi_*\mathcal{G} \\ @VVV @VVV \\ \mathcal{F} @>{\widetilde{f}}>> \mathcal{G}. \end{CD}$$ □

Now we are able to prove the comparison theorem.

Theorem 5.5.5. — *Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a natural filtered morphism from the analytic derived de Rham complex to the Hodge-filtered infinitesimal cohomology sheaf as below:*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma_e} \longrightarrow Ru_{X/\Sigma_e *}\mathcal{J}^*_{X/\Sigma_e}.$$

Moreover, the induced map between their underlying complexes is an isomorphism

$$\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma_e} \longrightarrow Ru_{X/\Sigma_e *}\mathcal{O}_{X/\Sigma_e}.$$

Proof. — We first notice that the isomorphism can be constructed using smooth embeddings. To see this, we recall the equivalence of ∞-categories $\mathcal{D}(X,R) \simeq \mathrm{Sh}^{\mathrm{hyp}}(X,R)$ (cf. Paragraph 5), where the right side is the full sub-∞-category of contravariant functors from $U \in X_{\mathrm{rig}}$ to the derived category

$\mathcal{D}(R)$. As X_{aff} is a basis of the rigid site X_{rig}, we may use the equivalence $\text{Sh}^{\text{hyp}}(X, R) = \text{Sh}^{\text{hyp}}(X_{\text{aff}}, R)$ and regard objects in $\mathcal{D}(X, R)$ as contravariant functors on affinoid open subsets of X. So a map or an isomorphism of sheaves of complexes can be constructed via their evaluations at X_{aff}. Moreover, by Lemma 5.5.4, by applying the constant functor $R\pi_*$ to the given two objects, it suffices to show that there is an isomorphism between their images over the site of smooth immersions SE_X.

Now for each smooth immersion $i : U = \text{Spa}(B) \to Z = \text{Spa}(P)$ for affinoid open subset $U \subset X$, as in the proof of Theorem 5.5.1, we have the following maps of cosimplicial objects in $\widehat{\mathcal{DF}}(\mathrm{B}^+_{\text{dR,e}})$:

$$\widehat{\text{dR}}^{\text{an}}_{B/\mathrm{B}^+_{\text{dR,e}}} \longrightarrow \widehat{\text{dR}}^{\text{an}}_{B/P^{\widehat{\otimes}\bullet+1}} \longrightarrow \widehat{\text{dR}}_{B/P^{\widehat{\otimes}\bullet+1}} \longrightarrow \mathcal{O}_{D(\bullet)} \longleftarrow R\Gamma(U/\Sigma_{e\,\text{inf}}, \mathcal{O}_{X/\Sigma_e}),$$

where $D(n)$ is the envelope of the surjection

$$P^{\widehat{\otimes}(n+1)} = P^{\widehat{\otimes}_{\mathrm{B}^+_{\text{dR,e}}}(n+1)} \to B.$$

Here we recall that the first map above is induced from the map of rings $\mathrm{B}^+_{\text{dR,e}} \to P^{\widehat{\otimes}\bullet+1}$, the second map is the inverse of the algebraicity isomorphism in Theorem 5.5.1, the third map is the isomorphism in [6, Cor. 4.14], and the last map is induced from the limit presentation of the cohomology complex. In particular, the above arrows are all functorial with respect to the smooth embedding $i : U \to P$, and the induced maps of their underlying complexes above are all isomorphisms by the proof of the affinoid comparison in Theorem 5.5.1.

Finally, note that the above maps are functorial with respect to smooth immersions $i : U \to Z$, so we can improve the above map into the level of sheaves over SE_X

$$R\pi_*\widehat{\text{dR}}^{\text{an}}_{X/\Sigma_e} \longrightarrow \widehat{\text{dR}}^{\text{an}}_{\text{SE}_X} \longrightarrow \mathcal{O}_{\text{SE}_X} \longleftarrow R\pi_* Ru_{X/\Sigma_e *}, \mathcal{O}_{X/\Sigma_e}\ ,$$

where $\widehat{\text{dR}}^{\text{an}}_{\text{SE}_X}$ is the (cosimplicial) sheaf sending $i : U \to Z$ to the filtered complex $\widehat{\text{dR}}^{\text{an}}_{B/P^{\widehat{\otimes}\bullet+1}}$, and $\mathcal{O}_{\text{SE}_X}$ is the (cosimplicial) sheaf sending $i : U \to Z$ to the structure sheaves $\mathcal{O}_{D(\bullet)}$ of envelopes $D(\bullet)$. In this way, the isomorphism of the underlying complexes over the site SE_X of smooth immersions

$$R\pi_*\underline{\widehat{\text{dR}}}^{\text{an}}_{X/\mathrm{B}^+_{\text{dR,e}}} \longrightarrow R\pi_* Ru_{X/\Sigma_e *}, \mathcal{O}_{X/\Sigma_e}$$

follows from the above computation at the sections $i : U \to Z$, and thus we get the result by Lemma 5.5.4. So we are done. □

REMARK 5.5.6. — The above comparison map, though functorial with respect to the rigid space X, is constructed in an indirect way. It is natural to ask whether we can directly produce a natural morphism from the analytic derived de Rham complex to the infinitesimal cohomology sheaf. Here we want to mention that for algebraic schemes over $\mathbb{C}$, this could be achieved by the universal property of the derived de Rham complex in the ∞-category of filtered $\mathbb{E}_\infty$-algebras.

6. Éh descent

In this section, we prove the éh hyperdescent of the infinitesimal cohomology of crystals in vector bundles over X/K and $X/\mathrm{B}^+_{\mathrm{dR},\mathrm{e}_{\mathrm{inf}}}$ for a rigid space X. Our goal is to show the comparison between the éh de Rham cohomology and the infinitesimal cohomology of a crystal.

In order to extend a crystal over X to any rigid space that admits a map to X (which, in particular, is not necessarily an open immersion), we will work with coherent crystals over the big site. Here we note that this is only for technical convenience, as crystals and their cohomology over $X/\Sigma_{e\,\mathrm{inf}}$ or $X/\Sigma_{e\mathrm{INF}}$ are equivalent via pullback and restrictions (Proposition 3.1.7, Proposition 4.1.2).

6.1. Descent for blowup coverings. — We first deal with the descent for the blowup covering over the base $\Sigma_1 = \mathrm{Spa}(K)$ for an arbitrary complete non-Archimedean p-adic field K that is not necessarily algebraically closed. The essential idea follows from Hartshorne's proof for algebraic de Rham cohomology [24, Chap II, Section 4], where he provides a long exact sequence of the algebraic de Rham cohomology for a blowup square.

We first give a Mayer–Vietories sequence for the infinitesimal cohomology:

Proposition 6.1.1 (Mayer–Vietories sequence). — *Let X be a union of two closed analytic subspaces X_1 and X_2 over K, and let $\mathcal{F}$ be a coherent crystal over X/K_{INF}. Then the map of rigid spaces $X_1 \cap X_2 \to X_1 \cup X_2 \to X$ induces a natural distinguished triangle as below:*

$$Ru_{X/K*}\mathcal{F} \longrightarrow Ru_{X_1/K*}\mathcal{F} \bigoplus Ru_{X_2/K*}\mathcal{F} \longrightarrow Ru_{X_1\cap X_2/K*}\mathcal{F}.$$

Proof. — As the functor $u_{X/K*}$ is the sheaf version of the global section functor $\Gamma(X/K_{\mathrm{INF}}, -)$ (see Subsection 2.3), it suffices to show that the maps in the statement above produce a natural distinguished triangle after applying $R\Gamma(U, -)$ for every $U \subset X$ open affinoid. So we may assume there exists a smooth affinoid rigid space $Z = \mathrm{Spa}(P)$ over Σ_e together with a closed immersion of $X = \mathrm{Spa}(P/I)$ into Z, where I is the defining ideal. Let X_i be the closed analytic subspace defined by the ideal I_i in P. Then, by assumption, X is defined by the ideal $I_1 \cap I_2$, and the intersection $X_3 := X_1 \cap X_2$ is defined by $I_3 := I_1 + I_2$. We denote by D and D_i the envelope of X and X_i in Z, respectively. Here we regard D and D_i to be the ringed spaces, where the underlying topological spaces are X and X_i, and their (global sections of) structure sheaves are $\mathcal{O}_D = \varprojlim_n P/I^n$ and $\mathcal{O}_{D_i} = \varprojlim P/I_i^n$, respectively (cf. Remark 2.2.2).

Now by Theorem 4.2.2, the infinitesimal cohomology of the coherent crystal can be functorially identified as the derived global sections of the following sequence:

$$\mathcal{F}_D \otimes \Omega_D^\bullet \longrightarrow \mathcal{F}_{D_1} \otimes \Omega_{D_1}^\bullet \bigoplus \mathcal{F}_{D_2} \otimes \Omega_{D_2}^\bullet \longrightarrow \mathcal{F}_{D_3} \otimes \Omega_{D_3}^\bullet,$$

where $\mathcal{F}_D \otimes \Omega_D^\bullet$ (resp. $\mathcal{F}_{D_i} \otimes \Omega_{D_i}^\bullet$) is the restriction of the de Rham complex of the coherent crystal $\mathcal{F}$ on the envelope $D_X(Z)$ (resp. $D_{X_i}(Z)$). Moreover, by the crystal condition, the finite projective $\mathcal{O}_{D_i}$-module $\mathcal{F}_{D_i}$ is equal to the tensor product $\mathcal{F}_D \otimes_{\mathcal{O}_D} \mathcal{O}_{D_i}$. Combining with Lemma 3.3.3, we know that each term $\mathcal{F}_{D_i} \otimes \Omega_{D_i}^n$ of the de Rham complex, regarded as a module over $\mathcal{O}_{D_i}$, is equal to the pullback of $\mathcal{F}_D \otimes_{\mathcal{O}_Z} \Omega_{Z/K}^1$ along $D_i \to D$. As a consequence, by taking the naive truncations term-wisely, to show the sequence above is distinguished, it suffices to show that the following natural sequence of rings is a short exact sequence:

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D_1} \bigoplus \mathcal{O}_{D_2} \longrightarrow \mathcal{O}_{D_3} \longrightarrow 0.$$

And since each structure sheaf of envelopes is given by the formal completions of the ring P, we reduce the question to show that the following sequence of inverse systems is exact:

$$0 \longrightarrow \{P/(I_1 \cap I_2)^n\}_n \longrightarrow \{P/I_1^n\}_n \bigoplus \{P/I_2^n\}_n \longrightarrow \{P/(I_1 + I_2)^n\}_n \longrightarrow 0.$$

Notice that for fixed n, we always have the following short exact sequence:

$$0 \longrightarrow P/(I_1^n \cap I_2^n) \longrightarrow P/I_1^n \bigoplus P/I_2^n \longrightarrow P/(I_1^n + I_2^n) \longrightarrow 0.$$

The proof thus follows since the ring P is Noetherian, and the inverse systems below are canonically isomorphic:

$$\{P/(I_1^n + I_2^n)\}_n \longrightarrow \{P/(I_1 + I_2)^n\}_n, \; \{P/(I_1 \cap I_2)^n\}_n \longrightarrow \{P/(I_1^n \cap I_2^n)\}_n.$$

□

Here is our main theorem in this subsection.

THEOREM 6.1.2. — *Let X be a rigid space over K, and let Y be a smooth analytic closed subset of X over K, with the blowup map $f : X' := \mathrm{Bl}_X(Y) \to X$ and the preimage $Y' := Y \times_X X'$ in X'. Then for any coherent crystal $\mathcal{F}$ over X/K_{INF}, the blowup square for $X' \to X$ induces the following distinguished triangle:*

$$(*) \qquad Ru_{X/K*}\mathcal{F} \longrightarrow Rf_*Ru_{X'/K*}\mathcal{F} \bigoplus Ru_{Y/K*}\mathcal{F} \longrightarrow Rf_*Ru_{Y'/K*}\mathcal{F}.$$

In particular, by taking the derived global sections at X, we get a distinguished triangle of the infinitesimal cohomology

$$R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F}) \bigoplus R\Gamma(Y/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(Y'/K_{\mathrm{INF}}, \mathcal{F}).$$

Before we prove the result, first we recall the formal function theorem for a proper map of rigid spaces.

THEOREM 6.1.3. — [31, Thm. 3.7] *Let $f : X' \to X$ be a proper map of rigid space over K, and let $\mathcal{I}$ be a sheaf of ideal over X, with Y the analytic closed subset of X defined by $\mathcal{I}$. Assume $\mathcal{G}$ is a coherent sheaf over X'. Then the following natural map is an isomorphism:*

$$(R^j f_* \mathcal{G})^\wedge \longrightarrow R^j f_*(\widehat{\mathcal{G}}).$$

Here $(-)^\wedge$ is the classical completion of a sheaf of $\mathcal{O}_X$-modules (or $\mathcal{O}_{X'}$-modules) with respect to the ideal $\mathcal{I}$ (resp. $f^{-1}\mathcal{I} \cdot \mathcal{O}_{X'}$).

REMARK 6.1.4. — Here we note that we may get a more derived version of the above theorem using the derived completion, as in [3, Tag 0A0H]. For our uses, we do not jump into this generality.

The rest of this subsection is devoted to proving Theorem 6.1.2.

Special case: X is smooth. First we deal with the special case, assuming X itself is smooth over K.

When X is smooth, as the blowup center Y is assumed to be smooth, the blowup X' itself is also smooth over K.[8] By Theorem 4.2.2, the derived direct image of the coherent crystal over X/K_{inf} and X'/K_{inf} can be computed by their de Rham complexes $\mathcal{F}_X \otimes \Omega^\bullet_{X/K}$ and $\mathcal{F}_{X'} \otimes \Omega^\bullet_{X'/K} = f^*\mathcal{F}_X \otimes \Omega^\bullet_{X'/K}$, respectively. On the other hand, the derived direct images $Ru_{Y/K*}\mathcal{F}$ and $Ru_{Y'/K*}\mathcal{F}$ are naturally isomorphic to the de Rham complex over the envelopes $D_Y(X)$ and $D_{Y'}(X')$, namely the complexes

$$\mathcal{F}_D \otimes \Omega^\bullet_{D_Y(X)},\ \mathcal{F}_{D'} \otimes \Omega^\bullet_{D_{Y'}(X')},$$

which are compatible with the Hodge filtrations of $\Omega^\bullet_{X/K}$ and $\Omega^\bullet_{X'/K}$. So the sequence $(*)$ in Theorem 6.1.2 is naturally isomorphic to the following sequence of de Rham complexes:

$$\begin{aligned}\mathcal{F}_X \otimes \Omega^\bullet_{X/K} \longrightarrow Rf_*(f^*\mathcal{F}_X \otimes \Omega^\bullet_{X'/K}) \bigoplus \mathcal{F}_D \otimes \Omega^\bullet_{D_Y(X)} \\ \longrightarrow Rf_*(\mathcal{F}_{D'} \otimes \Omega^\bullet_{D_{Y'}(X')}).\end{aligned}$$

In fact, we want to show the following more general statement:

PROPOSITION 6.1.5. — *Let $f : X' \to X$ be a proper morphism of smooth connected rigid spaces over K, and let Y be a closed analytic subset of X, with $Y' = f^{-1}(Y)$. Let $\mathcal{G}'$ and $\mathcal{G}$ be coherent sheaves over X' and X separately, such that $f : X' \to X$ induces an injective map of $\mathcal{O}_X$-modules $\mathcal{G} \to f_*\mathcal{G}'$. Assume f induces an isomorphism between open subsets $X'\backslash Y'$ and $X\backslash Y$ and*

8. To see this, one can use the algebraization argument in the proof of [22, Prop. 5.1.1] to reduce to a blowup of a smooth closed subscheme in a smooth scheme, where the claim is clear.

also induces an isomorphism of the restriction of $\mathcal{G} \to f_*\mathcal{G}'$ *on* $X\backslash Y$*. Then the following natural sequence is a distinguished triangle:*

$$\mathcal{G} \otimes \Omega^i_{X/K} \longrightarrow Rf_*(\mathcal{G}' \otimes \Omega^i_{X'/K}) \bigoplus \mathcal{G} \otimes \Omega^i_{D_Y(X)/K} \longrightarrow Rf_*(\mathcal{G}' \otimes \Omega^i_{D_{Y'}(X')/K}),$$
$i \in \mathbb{N}$.

Proof. — Let $\mathcal{M}$ be the coherent sheaf $\mathcal{G} \otimes \Omega^i_{X/K}$ over X, and let $\mathcal{M}'$ be the coherent sheaf $\mathcal{G}' \otimes \Omega^i_{X'/K}$ over X'. By the assumption of smoothness, both $\Omega^i_{X/K}$ and $\Omega^i_{X'/K}$ are locally free, and the natural map $\mathcal{M} \to f_*\mathcal{M}'$ is injective. Furthermore, the restriction of the map $\mathcal{M} \to f_*\mathcal{M}'$ on the open subset $X\backslash Y$ is an isomorphism.

Recall that the i-th differential sheaf $\Omega^i_{D_Y(X)/K}$ of $D_Y(X)$, as a sheaf over X, is defined as the inverse limit $\varprojlim_m \Omega^i_{X_m/K}$, where X_m is the m-th infinitesimal neighborhood of Y in X. As is shown in Lemma 3.3.3, the sheaf $\Omega^i_{D_Y(X)/K}$ is naturally isomorphic to the formal completion of the coherent sheaf $\Omega^i_{X/K}$ along $Y \to X$, which is also equal to the tensor product $\Omega^i_{X/K} \otimes \mathcal{O}_{D_Y(X)}$. Moreover, since $\mathcal{G}$ is coherent over X, the tensor product $\mathcal{G} \otimes \Omega^i_{D_Y(X)}$ is isomorphic to the formal completion $\widehat{\mathcal{M}}$ of $\mathcal{M} = \mathcal{G} \otimes \Omega^i_{X/K}$ along $Y \to X$. The same also holds for X', Y' and $\mathcal{M}'$.

We denote by C_1 and C_2 cones of the map $\mathcal{M} \to Rf_*\mathcal{M}'$ and $\widehat{\mathcal{M}} \to Rf_*\widehat{\mathcal{M}}'$, respectively. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & Rf_*\mathcal{M}' & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{M}} & \longrightarrow & Rf_*\widehat{\mathcal{M}}' & \longrightarrow & C_2. \end{array}$$

Here both rows are distinguished.

Now we make the following claim.

Claim 6.1.6. — *The natural map* $C_1 \to C_2$ *of cones above is an isomorphism.*

Proof of the claim. — First we notice that since the map $f : X' \to X$ is an isomorphism on the open subsets $X'\backslash Y' \to X\backslash Y$ and both X' and X are smooth, the sheaves of differentials $\Omega^i_{X/K}$ and $\Omega^i_{X'/K}$ are vector bundles, and the induced map $\Omega^i_{X/K} \to f_*\Omega^i_{X'/K}$ is injective.[9] On the other hand, the

9. For an open immersion $i : U \to X$, thanks to the smoothness, the restriction map $\Omega^i_{X/K}(X) \to \Omega^i_{X/K}(U)$ is injective. When U is away from Y, the open immersion $U \to X$ factors through X', and we get the injectivity of the composition $\Omega^i_{X/K}(X) \to \Omega^i_{X'/K}(X') \to \Omega^i_{X/K}(U)$ as well, hence $\Omega^i_{X/K}(X) \to \Omega^i_{X'/K}(X')$ is injective. The same applies when X and X' are replaced by open subspaces, which justifies the injectivity at the level of sheaves.

map $\mathcal{G} \to f_*\mathcal{G}'$ is assumed to be an injective map of coherent sheaves in the proposition. Combining the above two conditions, we see the map $\mathcal{M} \to f_*\mathcal{M}'$ is injective, and the cone lives in cohomological degree $[0, +\infty)$.

By the Formal Function Theorem 6.1.3, the cohomology sheaf $R^j f_* \widehat{\mathcal{M}'}$ is naturally isomorphic to the formal completion $(R^j f_*\mathcal{M}')^\wedge$ of $R^j f_*\mathcal{M}'$ along $Y \to X$. Moreover, by the exactness of the formal completion on coherent sheaves, the natural map $\widehat{\mathcal{M}} \to (f_*\mathcal{M}')^\wedge$ is injective, and we have a short exact sequence

$$0 \longrightarrow \widehat{\mathcal{M}} \longrightarrow (f_*\mathcal{M}')^\wedge \longrightarrow \mathcal{H}^0(C_2) \longrightarrow 0.$$

This implies that C_2 also lives in cohomological degree no smaller than zero. Furthermore, by the exactness of the above sequence, the cohomology sheaf $\mathcal{H}^0(C_2)$ is isomorphic to the formal completion of $\mathcal{H}^0(C_1)$ at Y. But since $\mathcal{H}^0(C_1)$ is coherent and is already supported at Y, we have

$$\mathcal{H}^0(C_1) = \mathcal{H}^0(C_1)^\wedge \simeq \mathcal{H}^0(C_2).$$

This finishes the degree zero part.

For the higher cohomology, we consider the following diagram of cohomologies:

$$\begin{array}{ccc} R^j f_*\mathcal{M}' & \longrightarrow & \mathcal{H}^j(C_1) \\ \downarrow & & \downarrow \\ (R^j f_*\mathcal{M}')^\wedge & \longrightarrow & \mathcal{H}^j(C_2). \end{array}$$

As $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are living in cohomological degree zero, the horizontal maps above are isomorphisms, and it suffices to show for each $i > 0$, the left vertical map above is an isomorphism. But notice that since f induces an isomorphism between $\mathcal{M}$ and $\mathcal{M}'$ over $X \backslash Y$, the higher cohomology sheaf $R^j f_*\mathcal{M}'$ is coherent and is supported over Y. In particular, the formal completion of $R^j f_*\mathcal{M}'$ along $Y \to X$ is equal to itself; i.e., the natural map below is an isomorphism:

$$R^j f_*\mathcal{M}' \longrightarrow (R^j f_*\mathcal{M}')^\wedge.$$

This leads to the isomorphism

$$\mathcal{H}^j(C_1) \simeq \mathcal{H}^j(C_2), \ \forall j \geq 1,$$

and we finish the isomorphism between C_1 and C_2. □

We change the notation back to Proposition 6.1.5. Then we get two rows of distinguished triangles

$$\begin{array}{ccccc}
\mathcal{G}\otimes\Omega^i_{X/K} & \longrightarrow & Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K}) & \longrightarrow & C_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G}\otimes\Omega^i_{D_Y(X)/K} & \longrightarrow & Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K}) & \longrightarrow & C_2;
\end{array}$$

the third vertical map is an isomorphism.

Finally, we consider the following two maps:

$$\phi:\ \mathcal{G}\otimes\Omega^i_{X/K} \xrightarrow{(+,+)} Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K})\bigoplus(\mathcal{G}\otimes\Omega^i_{D_Y(X)/K});$$

$$\psi:\ Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K})\bigoplus(\mathcal{G}\otimes\Omega^i_{D_Y(X)/K}) \xrightarrow{(-,+)} Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K}),$$

where $+$ and $-$ indicate the signs of the map. As the composition of the above two maps is equal to zero, the map ψ factors though a morphism $\mathrm{Cone}(\phi)\to Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K})$ ([3, Tag 08RI]). In this way, by chasing diagrams and the claim above, the map $\mathrm{Cone}(\phi)\to Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K})$ is an isomorphism, and we get the distinguished triangle we want. □

General case. We then deal with the general case of Theorem 6.1.2.

Proof of Theorem 6.1.2.As the theorem is a local statement, by passing to an open covering if necessary, it suffices to assume that X is affinoid and admits a closed immersion into a smooth affinoid rigid space Z. Moreover, as any coherent crystal over X/K_{INF} is a crystal in vector bundles (Corollary 3.2.4), by further taking open subsets we may assume that $\mathcal{F}_X\simeq\mathcal{O}_X^{\oplus m}$ is trivial over X.

As Y is smooth over K, the blowup $Z'=\mathrm{Bl}_Z(Y)$ of the smooth rigid space Z at the center Y is also smooth. Moreover, as $X\to Z$ is a closed immersion, while X' is the blowup of X at Y, the natural map $X'\to Z'$ is also a closed immersion, which is equal to the preimage of X along the blowup map $f:Z'\to Z$. So we get the following commutative diagrams of rigid spaces over K with both of the squares being cartesian:

$$\begin{array}{ccccc}
Y' & \longrightarrow & X' & \longrightarrow & Z' \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & X & \longrightarrow & Z.
\end{array}$$

Since the restriction $\mathcal{F}_{D_X(Z)}$ is a vector bundle over $\mathcal{O}_{D_X(Z)}$ whose pullback along $X\to D_X(Z)$ is trivial, by Nakayama's lemma, we know that $\mathcal{F}_{D_X(Z)}$ is already a trivial bundle, and we may let $\mathcal{G}$ be a trivial vector bundle over Z

such that the tensor product $\mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{O}_{D_X(Z)}$ is equal to $\mathcal{F}_{D_X(Z)}$. Let $\mathcal{G}'$ be the pullback $f^*\mathcal{G}$ as a trivial bundle over Z'. Then by the crystal condition of $\mathcal{F}$ we have

$$\mathcal{G}' \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{D_{X'}(Z')} = \mathcal{F}_{D_{X'}(Z')}.$$

Here we note that by our choices and the diagram above, the map $\mathcal{G} \to f_*\mathcal{G}'$ is injective and is an isomorphism when restricted to open subsets $Z\backslash Y$ and $Z\backslash X$. Similar to the proof of Proposition 6.1.5, we let $\mathcal{M}$ and $\mathcal{M}'$ be the tensor products $\mathcal{G} \otimes \Omega^i_{Z/K}$ and $\mathcal{G}' \otimes \Omega^i_{Z'/K}$ over Z and Z', respectively.

Now by the proof of Proposition 6.1.5, we have the following natural commutative diagrams, with each row being distinguished:

$$\begin{array}{ccccc}
\mathcal{M} & \longrightarrow & Rf_*\mathcal{M}' & \longrightarrow & C_1 \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\mathcal{M}_{/X}} & \longrightarrow & Rf_*\widehat{\mathcal{M}'_{/X'}} & \longrightarrow & C_2 \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\mathcal{M}_{/Y}} & \longrightarrow & Rf_*\widehat{\mathcal{M}'_{/Y'}} & \longrightarrow & C_3.
\end{array}$$

Here the sheaf $\widehat{\mathcal{M}_{/X}}$, and similarly for the others, is denoted to be the formal completion of $\mathcal{M}$ along $X \to Z$. Thanks to Claim 6.1.6, the map $C_1 \to C_2$ and the map $C_1 \to C_3$ are isomorphisms. In particular, the map $C_2 \to C_3$ is an isomorphism. In this way, as in the last part of the proof for Proposition 6.1.5, the second and third rows above produce the following distinguished triangle:

$$\widehat{\mathcal{M}_{/X}} \xrightarrow{(+,+)} Rf_*\widehat{\mathcal{M}'_{/X'}} \bigoplus \widehat{\mathcal{M}_{/Y}} \xrightarrow{(-,+)} Rf_*\widehat{\mathcal{M}'_{/Y'}}.$$

Hence by Lemma 3.3.3, we may replace those formal completions by their corresponding sheaves over envelopes and obtain the distinguished triangle below:

$$\begin{aligned}\mathcal{F}_{D_X(Z)} \otimes \Omega^i_{D_X(Z)} \longrightarrow Rf_*(\mathcal{F}_{D_{X'}(Z')} \otimes \Omega^i_{D_{X'}(Z')}) \bigoplus \mathcal{F}_{D_Y(Z)} \otimes \Omega^i_{D_Y(Z)} \\ \longrightarrow Rf_*(\mathcal{F}_{D_{Y'}(Z')} \otimes \Omega^i_{D_{Y'}(Z')}),\end{aligned}$$

which implies the theorem by taking different i and Theorem 4.2.2. □

REMARK 6.1.7. — With the help of the blowup triangle in Theorem 6.1.2, we could show the following: for a universal homeomorphism of rigid spaces $f : X' \to X$ over K and a coherent crystal $\mathcal{F}$ over X/K_{INF}, there exists a natural isomorphism of cohomology sheaves as below:

$$Ru_{X/K*}\mathcal{F} \longrightarrow Rf_*Ru_{X'/K*}\mathcal{F}.$$

6.2. Éh-hyperdescent in general. — Now we are ready to prove the éh descent for a crystal over the infinitesimal site. We first deal with the case for the infinitesimal cohomology over an arbitrary p-adic field K, where the strategy is to use the blowup square for the éh-topology in [22] and the descent results in the first subsection. After that, we generalize to the case over $\mathrm{B}^+_{\mathrm{dR,e}}$.

Recall that the éh site $X_{\text{éh}}$ is defined over the category $\mathrm{Rig}_K|_X$ of all K-rigid spaces over X and is equipped with the éh-topology (cf. [22, Section 2]). For an object $X' \to X$ in $\mathrm{Rig}_K|_X$, we denote by $\pi_{X'} : X_{\text{éh}} \to X'_{\mathrm{rig}}$ the map from the éh site of X to the small rigid site of X' and denote by $\pi : X_{\text{ét}} \to \mathrm{Rig}_K|_X$ the map from the éh site of X to the big rigid site over X.

We first associate an infinitesimal crystal together with its de Rham complex an analogous construction over the éh topology.

Construction 6.2.1 (éh de Rham complex). — Let $\mathcal{F}$ be an infinitesimal sheaf of $\mathcal{O}_{X/K}$-modules over the big infinitesimal site X/K_{INF}. We then associate a sheaf $\mathcal{F}_{\mathrm{rig}} := i_{X/K}^{-1}\mathcal{F}$ of $\mathcal{O}_X = i_{X/K}^{-1}\mathcal{O}_{X/K}$-modules over the big rigid site $\mathrm{Rig}_K|_X$, where $i_{X/K} : \mathrm{Sh}(\mathrm{Rig}_K|_X) \to \mathrm{Sh}(X/K_{\mathrm{INF}})$ is the morphism of topoi as in Subsection 2.3. Here the section of $\mathcal{F}_{\mathrm{rig}}$ at an object $f : X' \to X$ in $\mathrm{Rig}_K|_X$ is the $\mathcal{O}_{X'}(X')$-module

$$\mathcal{F}(X', X'),$$

where $(X', X') \in X/\Sigma_{e\mathrm{INF}}$ is the trivial thickening of X'. We could then sheafify it with the éh-topology and thus get an éh-sheaf $\mathcal{F}_{\text{éh}}$ over the éh site $X_{\text{éh}}$.

Now we specify $\mathcal{F}$ to be a coherent crystal over big the infinitesimal site. As in the discussion of Paragraph 3.1, we could associate $\mathcal{F}$ its de Rham complex over the big infinitesimal site

$$\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}.$$

This allows us to get a complex of sheaves over $\mathrm{Rig}_K|_X$ and $X_{\text{éh}}$, respectively:

$$\mathcal{F}_{\mathrm{rig}} \longrightarrow \mathcal{F}_{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathrm{rig}}} \Omega^1_{\mathrm{rig}} \longrightarrow \mathcal{F}_{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathrm{rig}}} \Omega^2_{\mathrm{rig}} \longrightarrow \cdots;$$
$$\mathcal{F}_{\text{éh}} \longrightarrow \mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^1_{\text{éh}} \longrightarrow \mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^2_{\text{éh}} \longrightarrow \cdots.$$

Here the sheaf Ω^i_{rig} is the usual continuous Kähler differential sheaf over the rigid site, and $\Omega^i_{\text{éh}}$ is the éh continuous Kähler differential, which is equal to the éh sheafification of the usual continuous differential. The complex $\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}$ is called the *éh de Rham complex associated with the crystal $\mathcal{F}$*.

Construction 6.2.1 produces two maps of objects in the derived category of the big rigid topos:

$$Ru_{X/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow \mathcal{F}_{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathrm{rig}}} \Omega^\bullet_{\mathrm{rig}} \longrightarrow R\pi_*(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}).$$

Formally, the first map is given by the sheafified version of the natural transformation

$$\mathrm{Rig}_K|_X \ni X' \longmapsto (R\Gamma(X'/\Sigma_{e\mathrm{INF}}, -) \to R\Gamma((X', X'), -))$$

at the complex of infinitesimal sheaves $\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}$. The second map above comes from the counit morphism for the adjoint pair (π^{-1}, π_*), where π^{-1} is the éh-sheafification functor. Moreover, the above map can be improved into the filtered derived category, where the left side is equipped with the infinitesimal filtration and the remaining two complexes are equipped with their Hodge filtrations.

We also note that the natural map from the de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}$ to the crystal $\mathcal{F}$ itself induces a natural isomorphism as below:

$$Ru_{X/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \xrightarrow{\sim} Ru_{X/K*}\mathcal{F},$$

which is proved in Proposition 4.2.1.

Now we can state the descent result.

Theorem 6.2.2. — *Let X be a rigid space over K, and let $\mathcal{F}$ be a coherent crystal over the big infinitesimal site X/K_{INF}. Then the natural map of K-linear complexes below is an isomorphism:*

$$Ru_{X/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X*}(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}).$$

In particular, the infinitesimal cohomology of the coherent crystal $\mathcal{F}$ satisfies the éh-hyperdescent.

Proof. —

Step 1. — In Step 1, we show that by restricting to a smooth rigid space X' over K that admits a map to X, the morphism in the statement is an isomorphism. Thus the natural morphism below is an isomorphism:

$$Ru_{X'/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X'*}(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}).$$

We first apply $R\Gamma(X', -)$ for the smooth rigid space X'. On the one hand, we apply Theorem 4.2.2 to the trivial closed immersion $X' \to X'$ to get

$$\begin{aligned} R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F}) &\simeq R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \\ &\simeq R\Gamma(X', \mathcal{F}_D \otimes \Omega^\bullet_D) \\ &= R\Gamma(X'_{\mathrm{rig}}, \mathcal{F}_{X'} \otimes \Omega^\bullet_{X'/K}), \end{aligned}$$

where the envelope D for the trivial closed immersion $X' \to X'$ is just X' itself.

On the other hand, the éh-cohomology $R\Gamma(X'_{\text{éh}}, \mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}})$ is in fact isomorphic to the cohomology of the de Rham complex of $\mathcal{F}_{X'}$ given by the restriction of $\mathcal{F}_{\mathrm{rig}} \otimes \Omega^\bullet_{\mathrm{rig}}$ at X'. To see this, we notice that as the natural map of complexes

$\mathcal{F}_{\mathrm{rig}}\otimes\Omega^\bullet_{\mathrm{rig}}\to R\pi_{X'*}(\mathcal{F}_{\mathrm{éh}}\otimes\Omega^\bullet_{\mathrm{éh}})$ is filtered with respect to the Hodge filtrations, it suffices to show for each $i\in\mathbb{N}$, the following map is an isomorphism:

$$R\Gamma(X'_{\mathrm{rig}},\mathcal{F}_{X'}\otimes\Omega^i_{X'/K})\longrightarrow R\Gamma(X'_{\mathrm{éh}},\mathcal{F}_{\mathrm{éh}}\otimes\Omega^i_{\mathrm{éh}}).$$

Here the restriction $\mathcal{F}_{\mathrm{éh}}|_{X'}$ at the éh site of X' can be given by the éh-sheafification of the rigid sheaf $\mathcal{F}_{\mathrm{rig}}|_{X'}$ over the big site $\mathrm{Rig}_K|_{X'}$. Moreover, as $\mathcal{F}$ is a crystal in vector bundles (Corollary 3.2.4) and the statement is local on X' (namely both of the above two complexes satisfies the hyperdescent for rigid topology), by passing to an open rigid subspace of X' if necessary, we may assume that the restriction $\mathcal{F}_{\mathrm{rig}}|_{X'}$ of $\mathcal{F}$ at X' is isomorphic to the direct sum $\mathcal{O}^m_{X'}$ of structure sheaves. Thus we reduce to show that the natural map of cohomology of differentials below for a smooth K-rigid space X' is an isomorphism

$$R\Gamma(X'_{\mathrm{rig}},\Omega^i_{X'/K})\longrightarrow R\Gamma(X'_{\mathrm{éh}},\Omega^i_{\mathrm{éh}}),$$

which is proved in [22, Theorem 4.0.2].

In this way, as the map in the statement is given by the composition

$$\begin{aligned}R\Gamma(X'/K_{\mathrm{INF}},\mathcal{F}\otimes_{\mathcal{O}_{X/K}}\Omega^\bullet_{X/K_{\mathrm{INF}}})&\longrightarrow R\Gamma(X'_{\mathrm{rig}},\mathcal{F}_{X'}\otimes\Omega^\bullet_{X'/K})\\&\longrightarrow R\Gamma(X'_{\mathrm{éh}},\mathcal{F}_{\mathrm{éh}}\otimes_{\mathcal{O}_{\mathrm{éh}}}\Omega^\bullet_{\mathrm{éh}}),\end{aligned}$$

and we see that both maps above are isomorphisms when X' is smooth over K.

Finally, notice that the cofiber C of the map $Ru_{X'/K*}(\mathcal{F}\otimes_{\mathcal{O}_{X/K}}\Omega^\bullet_{X/K_{\mathrm{INF}}})\longrightarrow R\pi_{X'*}(\mathcal{F}_{\mathrm{éh}}\otimes_{\mathcal{O}_{\mathrm{éh}}}\Omega^\bullet_{\mathrm{éh}})$ is a bounded below complex of sheaves over the small rigid site X'_{rig}. If C is not acyclic, then there would exist an open subspace U of X' such that the cohomology $R\Gamma(U_{\mathrm{rig}},C)$ does not vanish, which contradicts the computation above. So we get the isomorphism for smooth K-rigid space X' that admits a map to X:

$$Ru_{X'/K*}(\mathcal{F}\otimes_{\mathcal{O}_{X/K}}\Omega^\bullet_{X/K_{\mathrm{INF}}})\longrightarrow R\pi_{X'*}(\mathcal{F}_{\mathrm{éh}}\otimes_{\mathcal{O}_{\mathrm{éh}}}\Omega^\bullet_{\mathrm{éh}}).$$

Step 2. — Now we prove the isomorphism of the derived push-forwards as in the statement.

Let $i:X_{\mathrm{red}}\to X$ be the closed immersion by the reduced sub rigid space of X. We first notice that the natural map below is an isomorphism:

$$Ru_{X/K*}\mathcal{F}\longrightarrow i_*Ru_{X_{\mathrm{red}}/K*}(\mathcal{F}).$$

This is because locally, both maps are computed using the de Rham complex $\mathcal{F}_D\otimes\Omega^\bullet_D$, where D is the envelope of X in a smooth rigid space (Theorem 4.2.2). The same isomorphism holds for the derived direct image of the infinitesimal de Rham complex $\mathcal{F}\otimes\Omega^\bullet_{X_{\mathrm{red}}/K_{\mathrm{INF}}}$ by Proposition 4.2.1. On the other hand, We notice that as the closed immersion $i:X_{\mathrm{red}}\to X$ is an éh-covering ([22, Section

2.4]), which forms a constant éh-hypercovering as the product $X_{\mathrm{red}} \times_X X_{\mathrm{red}}$ is equal to X_{red} itself, we get

$$R\pi_{X*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}) \simeq i_* R\pi_{X_{\mathrm{red}}*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}).$$

Thus the above two isomorphisms allow us to assume that X is reduced, and by passing to an open subset if necessary, we may assume that X is quasi-compact.

Now we can do the induction on the dimension of X. When $\dim(X)$ is of dimension zero, as X is quasi-compact and reduced, it is then equal to a disjoint union of finite points and, in particular, is smooth over K, where the statement follows from Step 1.

In general, by Temkin's resolution of singularities for rigid spaces ([40, 1.2.1, 5.2.2]), we can find finite compositions $X_n \to \cdots X_1 \to X_0 = X$, where each $X_i \to X_{i-1}$ is a blowup at a smooth nowhere dense closed subspace $Y_i \subset X_{i-1}$, such that in the last step, X_n is smooth over K. We denote by Y_i' the preimage $Y_i \times_{X_{i-1}} X_i$ in X_i, which is of dimension strictly smaller than $\dim(X_i) = \dim(X)$, and we let f_i be the blowup map $X_i \to X_{i-1}$. Then for each $1 \leq i \leq n$, we get a natural distinguished triangle by Theorem 6.1.2:

(∗)
$$\begin{aligned} Ru_{X_{i-1}/K*}(\mathcal{F} \otimes \Omega^\bullet_{\mathrm{INF}}) &\longrightarrow Rf_{i*}Ru_{X_i/K*}(\mathcal{F} \otimes \Omega^\bullet_{\mathrm{INF}}) \bigoplus Ru_{Y_i/K*}(\mathcal{F} \otimes \Omega^\bullet_{\mathrm{INF}}) \\ &\longrightarrow Rf_{i*}Ru_{Y_i'/K*}(\mathcal{F} \otimes \Omega^\bullet_{\mathrm{INF}}). \end{aligned}$$

Again here we use Proposition 4.2.1 to replace $\mathcal{F}$ by its de Rham complex. On the other hand, by the sheafified version of the blowup square in the éh-topology ([22, Proposition 5.1.4]), we have a natural distinguished triangle

(∗∗)
$$\begin{aligned} R\pi_{X_{i-1}*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}) &\longrightarrow R\pi_{X_i*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}) \bigoplus R\pi_{Y_i*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}) \\ &\longrightarrow R\pi_{Y_i'*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}). \end{aligned}$$

The functoriality of the map in the statement allows us to produce a map from the triangle (∗) to the triangle (∗∗). Moreover, by the dimension assumption and the induction assumption, we know that the statement in the Theorem is true for X_n and all of Y_i and Y_i'. In this way, since $X_0 = X$, by a finite step of descending inductions via comparing the above two triangles (∗) and (∗∗), the natural map below is then an isomorphism

$$Ru_{X/K*}(\mathcal{F} \otimes \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}).$$

So we are done. □

Remark 6.2.3. — As in Remark 1.2.5, by the construction of the map in Theorem 6.2.2, we see that (the underlying complex of) the infinitesimal cohomology of a coherent crystal $\mathcal{F}$ over X/K_{INF} is a direct summand of the cohomology $R\Gamma(X_{\mathrm{rig}}, \mathcal{F}_X \otimes \Omega^\bullet_{X/K})$ of the usual de Rham complex over X_{rig}.

REMARK 6.2.4. — The isomorphism in the theorem above cannot always be improved into a filtered isomorphism. The discrepancy already appears in the schematic theory (see [6, Example 5.6]).

Now we are able to generalize the éh-hyperdescent to coherent crystals over $X/\Sigma_{e\mathrm{INF}}$ for general e, not just for K-linear crystals. We assume that K is complete and algebraically closed in the next theorem, so $\mathrm{B}^+_{\mathrm{dR,e}}$ is well defined for K.

THEOREM 6.2.5. — *Let X be a rigid space over X, and let $\mathcal{F}$ be a crystal in vector bundles over the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. Then the infinitesimal cohomology of $\mathcal{F}$ over $X/\Sigma_{e\mathrm{INF}}$ satisfies the éh-hyperdescent; i.e., for an éh-hypercovering $X'_\bullet \to X'$ of K-rigid spaces over X, the following natural map is an isomorphism:*

$$R\Gamma(X'/\Sigma_{e\mathrm{INF}}, \mathcal{F}) \longrightarrow R\lim_{\Delta}\left(R\Gamma(X'_\bullet/\Sigma_{e\mathrm{INF}}, \mathcal{F})\right).$$

Proof. — We prove the result by induction on e. For $e = 1$, it is Theorem 6.2.2. In general, we take the derived tensor product of short exact sequence $0 \to K \to \mathrm{B}^+_{\mathrm{dR,e}} \to \mathrm{B}^+_{\mathrm{dR},e-1} \to 0$ with the complex of sheaves $Ru_{X/\Sigma_e *}\mathcal{F}$. By the big site version of the base change formula in Proposition 4.2.3 (cf. Corollary 2.2.8), we get a natural distinguished triangle

$$Ru_{X/K*}\mathcal{F}_1 \longrightarrow Ru_{X/\Sigma_e *}\mathcal{F} \longrightarrow Ru_{X/\Sigma_{e-1}*}\mathcal{F}_{e-1},$$

where $\mathcal{F}_1$ and $\mathcal{F}_{e-1}$ are pullbacks of $\mathcal{F}$ along maps of sites $X/K_{\mathrm{INF}} \to X/\Sigma_{e\mathrm{INF}}$ and $X/\Sigma_{e-1,\mathrm{INF}} \to X/\Sigma_{e\mathrm{INF}}$, respectively. In this way, applying the natural transformation $R\Gamma(X', -) \to R\lim_\Delta R\Gamma(X'_\bullet, -)$ to the above triangle, we get the result by induction. □

6.3. Finiteness. — With the use of the éh-hyperdescent, we show in this subsection the finiteness and the cohomological boundedness of the infinitesimal cohomology, assuming the properness of the rigid space.

THEOREM 6.3.1. — *Let X be a proper rigid space over K, and let $\mathcal{F}$ be a crystal in vector bundles over the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. Then the infinitesimal cohomology $R\Gamma(X/\Sigma_{e\mathrm{INF}}, \mathcal{F})$ is a bounded complex supported in the cohomological degrees $[0, 2n]$, where each cohomology is a finite $\mathrm{B}^+_{\mathrm{dR,e}}$-module.*

Proof. —

Smooth case. — We first notice that when X is smooth, the infinitesimal cohomology $R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F})$ is computed by the cohomology of the de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/K}$ via Theorem 4.2.2. We then have

$$R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F}) \simeq R\Gamma(X, \mathcal{F} \otimes \Omega^\bullet_{X/K}).$$

Moreover, each term $\mathcal{F}\otimes\Omega^i_{X/K}$ of the de Rham complex is a coherent sheaf over X. Notice that the cohomology of a coherent sheaf over a quasi-compact rigid space vanishes when the degree is above the dimension ([30, Proposition 2.5.8]). Thus by the Hodge–de Rham spectral sequence for $\mathcal{F}\otimes\Omega^\bullet_{X/K}$, we get the result for $R\Gamma(X/K_{\mathrm{INF}},\mathcal{F})$ with smooth proper X.

In general, we use the base change formula in Proposition 4.2.3. By taking the derived tensor product of $Ru_{X/\Sigma_e*}\mathcal{F}$ with the short exact sequence $0\to K\to \mathrm{B}^+_{\mathrm{dR,e}}\to \mathrm{B}_{\mathrm{dR},e-1}\to 0$, we get a distinguished triangle

$$R\Gamma(X/K_{\mathrm{INF}},\mathcal{F})\longrightarrow R\Gamma(X/\Sigma_{e\mathrm{INF}},\mathcal{F})\longrightarrow R\Gamma(X/\Sigma_{e-1\mathrm{INF}},\mathcal{F}).$$

In this way, the claim for smooth proper X follows from the induction on e.

General case. — In general, we prove by induction on the dimension of X. When X is of dimension zero, it is equal to a nilpotent extension of several points $\mathrm{Spa}(K)$. So the result follows from the éh-hyperdescent along closed immersions by reduced subspaces in Theorem 6.2.5 (in other words, we apply to the Čech nerve at closed immersion of the reduced subspace).

Now assume that X is reduced of dimension n, and the claim is true for any rigid space of smaller dimension. By the resolution of singularities of rigid space in [41], there exists a finite sequence of maps $X_m\to\cdots X_1\to X_0=X$, where X_m is smooth, and each map $X_i\to X_{i-1}$ is a blowup at a closed analytic subspace Y_{i-1} of X_{i-1}, such that each Y_{i-1} is nowhere dense in X_{i-1}. We denote E_i to be the exceptional locus $Y_{i-1}\times_{X_{i-1}}X_i$ of the i-th blowup. We could then apply the éh-hyperdescent in Theorem 6.2.5 to the Čech nerve associated with the blowup covering $X_i\coprod Y_{i-1}\to X_{i-1}$. The limit $R\lim_\Delta$ of the infinitesimal cohomology for the hypercovering is isomorphic to the fiber of the blowup square

$$R\Gamma(X_i/K_{\mathrm{INF}},\mathcal{F})\bigoplus R\Gamma(Y_{i-1}/K_{\mathrm{INF}},\mathcal{F})\longrightarrow R\Gamma(Y_i/K_{\mathrm{INF}},\mathcal{F}),$$

and thus we get a long exact sequence

$$\cdots\longrightarrow \mathrm{H}^j(X_{i-1}/K_{\mathrm{INF}},\mathcal{F})\longrightarrow \mathrm{H}^j(X_i/K_{\mathrm{INF}},\mathcal{F})\bigoplus \mathrm{H}^j(Y_{i-1}/K_{\mathrm{INF}},\mathcal{F})\\ \longrightarrow \mathrm{H}^j(Y_i/K_{\mathrm{INF}},\mathcal{F})\longrightarrow\cdots.$$

In this way, with the help of the induction assumption for all Y_i, a further descending induction from X_m to $X_0=X$ finishes the proof. □

6.4. Algebraic and analytic infinitesimal cohomology. — At the end of the section, we prove the comparison between the algebraic infinitesimal cohomology and the analytic infinitesimal cohomology for a proper algebraic variety.

Recall that for an algebraic variety[10] $\mathfrak{X}$ over a p-adic field K, we can define its *(algebraic) infinitesimal site* $\mathfrak{X}/K_{\mathrm{inf}}$, whose objects are schematic infinitesimal

10. For our purposes, a *variety* is defined to be a locally finite type scheme over a field in the article.

thickenings $(\mathcal{U}, \mathcal{T})$, where $\mathcal{U}$ is a Zariski open subset of $\mathcal{X}$. The infinitesimal site $\mathcal{X}/K_{\text{inf}}$ is equipped with a structure sheaf $\mathcal{O}_{\mathcal{X}/K}$, and its cohomology is called the *algebraic infinitesimal cohomology.* Moreover, the infinitesimal structure sheaf admits a surjection $\mathcal{O}_{\mathcal{X}/K} \to \mathcal{O}_{\mathcal{X}}$ to the Zariski structure sheaf, whose kernel $\mathcal{J}_{\mathcal{X}/K}$ defines a natural filtration on $R\Gamma(\mathcal{X}/K_{\text{inf}}, \mathcal{O}_{\mathcal{X}/K})$. Similar to the analytic theory, we call this filtration the *(algebraic) infinitesimal filtration.*

Let $X = \mathcal{X}^{\text{an}}$ be the rigid space over K defined as the analytification of a variety $\mathcal{X}$. As the analytification functor $\text{Sch}_K \to \text{Rig}_K$ preserves open and closed immersions, it induces a natural map of ringed sites

$$(X/K_{\text{inf}}, \mathcal{O}_{X/K}) \longrightarrow (\mathcal{X}/K_{\text{inf}}, \mathcal{O}_{\mathcal{X}/K}).$$

Moreover, as the surjection $\mathcal{O}_{\mathcal{X}/K} \to \mathcal{O}_{\mathcal{X}}$ is compatible with $\mathcal{O}_{X/K} \to \mathcal{O}_X$, the natural map of infinitesimal structure sheaves above is then a filtered map. As a consequence, by passing to their cohomology, we get a natural filtered morphism in derived category

$$R\Gamma(\mathcal{X}/K_{\text{inf}}, \mathcal{O}_{\mathcal{X}/K}) \longrightarrow R\Gamma(X/K_{\text{inf}}, \mathcal{O}_{X/K}).$$

Our main result in this subsection is the following:

THEOREM 6.4.1. — *Let $\mathcal{X}$ be a proper algebraic variety over K, and let X be its analytification. Then the analytification functor induces a filtered isomorphism of the infinitesimal cohomology*

$$R\Gamma(\mathcal{X}/K_{\text{inf}}, \mathcal{O}_{\mathcal{X}/K}) \longrightarrow R\Gamma(X/K_{\text{inf}}, \mathcal{O}_{X/K}).$$

Before the proof, we first recall that there is a natural map of ringed sites $(X_{\text{rig}}, \mathcal{O}_X) \to (\mathcal{X}_{\text{Zar}}, \mathcal{O}_{\mathcal{X}})$. Here the rigid structure sheaf is flat over the Zariski structure sheaf, and the pullback along the map induces a fully faithful functor from coherent $\mathcal{O}_{\mathcal{X}}$-modules to coherent $\mathcal{O}_X$-modules.

Moreover, the above map of sites is compatible with the infinitesimal topos. Recall that there exists a natural map of topoi

$$\begin{aligned} u_{\mathcal{X}/K} : \text{Sh}(\mathcal{X}/K_{\text{inf}}) &\longrightarrow \text{Sh}(\mathcal{X}_{\text{Zar}}); \\ \mathcal{F} &\longmapsto (\mathcal{U} \mapsto \Gamma(\mathcal{U}/K_{\text{inf}}, \mathcal{F}|_{\mathcal{U}})). \end{aligned}$$

By construction, this functor is compatible with its rigid version $u_{X/K} : \text{Sh}(X/K_{\text{inf}}) \to \text{Sh}(X_{\text{rig}})$ (cf. Subsection 2.3). Thus, the following diagram is commutative:

$$\begin{array}{ccc} \text{Sh}(X/K_{\text{inf}}) & \longrightarrow & \text{Sh}(\mathcal{X}/K_{\text{inf}}) \\ \downarrow{\scriptstyle u_{X/K}} & & \downarrow{\scriptstyle u_{\mathcal{X}/K}} \\ \text{Sh}(X_{\text{rig}}) & \longrightarrow & \text{Sh}(\mathcal{X}_{\text{Zar}}). \end{array}$$

We then claim the following result:

Proposition 6.4.2. — *Let $\mathcal{X}$ be an algebraic variety over K, and let X be its analytification. Then the complex of coherent $\mathcal{O}_X$-modules $Ru_{X/K*}(\mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K})$ is naturally isomorphic to the analytification of the complex of coherent $\mathcal{O}_{\mathcal{X}}$-modules $Ru_{\mathcal{X}/K*}(\mathcal{J}^n_{\mathcal{X}/K}/\mathcal{J}^{n+1}_{\mathcal{X}/K})$.*

Proof. — We denote the complex of coherent $\mathcal{O}_{\mathcal{X}}$-modules $Ru_{\mathcal{X}/K*}(\mathcal{J}^n_{\mathcal{X}/K}/\mathcal{J}^{n+1}_{\mathcal{X}/K})$ by C and $Ru_{X/K*}(\mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K})$ by C'. Then it suffices to show that the natural map below induced from the pullback from $\mathcal{X}_{\mathrm{Zar}}$ to X_{rig} is an isomorphism of complexes of $\mathcal{O}_X$-modules:

$$C \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X \longrightarrow C'.$$

As the result is a local statement for $\mathcal{X}$, let us assume that $\mathcal{X} = \mathrm{Spec}(A)$ is a finite type affine scheme over K and $\mathcal{X} \to \mathcal{Y} = \mathrm{Spec}(A')$ be a closed immersion into an affine space over K. Moreover, notice that the isomorphism could be checked locally on X, so we may take an open affinoid disc of certain radius $\mathrm{Spa}(B')$ in $\mathcal{Y}^{\mathrm{an}}$, with the open subset $X \cap \mathrm{Spa}(B') = \mathrm{Spa}(B)$ in X. From our choices, we get a cartesian diagram as below, where horizontal maps are surjective and vertical maps are flat:

$$\begin{array}{ccc} B' & \twoheadrightarrow & B \\ \uparrow & & \uparrow \\ A' & \twoheadrightarrow & A \end{array}$$

So it suffices to show that

$$R\Gamma(\mathrm{Spec}(A)/K_{\mathrm{inf}}, \mathcal{J}^n_{\mathcal{X}/K}/\mathcal{J}^{n+1}_{\mathcal{X}/K}) \otimes_A B \simeq R\Gamma(\mathrm{Spa}(B)/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}).$$

We then recall from [7, Section 2] that the algebraic infinitesimal cohomology can be computed by the Čech–Alexander complex as below:

$$D \longrightarrow D(1) \longrightarrow D(2) \longrightarrow \cdots,$$

where $D(m)$ is the formal completion of $A'(m) := {A'}_K^{\otimes m+1}$ along the surjection $A'(m) \to A$. We take the n-th graded piece for the algebraic infinitesimal filtration, then the cohomology group $R\Gamma(\mathrm{Spec}(A)/K_{\mathrm{inf}}, \mathcal{J}^n_{\mathcal{X}/K}/\mathcal{J}^{n+1}_{\mathcal{X}/K})$ is isomorphic to the following map of A-linear cosimplicial complexes:

$$J^n_D/J^{n+1}_D \longrightarrow J^n_{D(1)}/J^{n+1}_{D(n)} \longrightarrow J^n_{D(2)}/J^{n+1}_{D(2)} \longrightarrow \cdots,$$

where $J_{D(m)}$ is the kernel of the surjection $A'(m) \to A$. On the other hand, by the Čech–Alexander complex for rigid spaces in Proposition 4.1.3, we have

$$\begin{aligned} &R\Gamma(\mathrm{Spa}(B)/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}) \\ &\simeq \left(J^n_{\mathcal{D}}/J^{n+1}_{\mathcal{D}} \longrightarrow J^n_{\mathcal{D}(1)}/J^{n+1}_{\mathcal{D}(1)} \longrightarrow J^n_{\mathcal{D}(2)}/J^{n+1}_{\mathcal{D}(2)} \longrightarrow \cdots\right), \end{aligned}$$

where $\mathcal{D}(m)$ is the formal completion for the surjection $B'(m) := B'^{\widehat{\otimes}_K m+1} \to B$, and $J_{\mathcal{D}(m)}$ is the kernel of the map $B'(m) \to B$. Thus it remains to show the quasi-isomorphism for the canonical map of B-linear cosimplicial complexes below:

$$\begin{aligned}&\big(J_D^n/J_D^{n+1} \to J_{D(1)}^n/J_{D(n)}^{n+1} \to J_{D(2)}^n/J_{D(2)}^{n+1} \to \cdots\big) \otimes_A B \\ &\longrightarrow \big(J_{\mathcal{D}}^n/J_{\mathcal{D}}^{n+1} \to J_{\mathcal{D}(1)}^n/J_{\mathcal{D}(1)}^{n+1} \to J_{\mathcal{D}(2)}^n/J_{\mathcal{D}(2)}^{n+1} \to \cdots\big).\end{aligned}$$

Finally, notice that by our choices, the rigid space $\mathrm{Spa}(B')$ is an open disc of some radius in the affine space $\mathrm{Spec}(A')^{\mathrm{an}}$. In particular, the following map of rings is a cartesian diagram such that vertical maps are flat:

$$\begin{array}{ccc} B'(m) & \twoheadrightarrow & B' \\ \uparrow & & \uparrow \\ A'(m) & \twoheadrightarrow & A' \end{array}$$

In this way, combining this with the cartesian diagram in the first paragraph, we see that the kernel $J_{\mathcal{D}(m)}$ of the surjection $B'(m) \to B$ is equal to the base change of $J_{D(m)}$ along the flat map $A'(m) \to B'(m)$. Hence we get the natural equalities

$$\begin{aligned}&J_{D(m)}B'(m) = J_{D(m)} \otimes_{A'(m)} B'(m) = J_{\mathcal{D}(m)};\\ &(J_{D(m)}^n/J_{D(m)}^{n+1}) \otimes_A B = (J_{D(m)}^n/J_{D(m)}^{n+1}) \otimes_{A'(m)} B'(m) = J_{\mathcal{D}(m)}^n/J_{\mathcal{D}(m)}^{n+1}.\end{aligned}$$

So we are done. □

Finally, we finish the proof of Theorem 6.4.1.

Proof of Theorem 6.4.1. — To show that the natural map in the statement is a filtered isomorphism, it suffices to show the isomorphisms for their underlying complexes and each graded piece separately, as both of them are filtered complete.[11]

For the underlying complexes, this follows from the éh descent. To see this, we first notice that when $\mathcal{X}$ is smooth and proper over K, then the algebraic and the analytic infinitesimal cohomology are isomorphic to the algebraic and the analytic de Rham cohomology, respectively ([20], Theorem 4.2.2), which are isomorphic to each other by applying the GAGA theorem to their Hodge filtrations (cf. [15, Appendix A.1]). In general, we may assume $\mathcal{X}_\bullet \to \mathcal{X}$ is a simplicial smooth variety by resolving singularities. Then its analytification

11. For each affinoid infinitesimal thickening (U, T), the kernel ideal $\mathcal{J}_T = \ker(\mathcal{O}_T \to \mathcal{O}_U)$ is nilpotent and hence $\mathcal{O}_T$ is complete under $\mathcal{J}_T$-adic topology. This in particular implies the inverse limit formula $\mathcal{O}_{X/K} = \varprojlim_i \mathcal{O}_{X/K}/\mathcal{J}_{X/K}^i$, and similarly for the algebraic version.

$X_\bullet \to X$ is an éh-hypercovering by smooth rigid spaces, and we get the isomorphism

$$\begin{aligned} R\Gamma(\mathfrak{X}/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}/K}) &\simeq R\lim_{[n]\in\Delta} R\Gamma(\mathfrak{X}_n/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}_n/K}) \\ &\simeq R\lim_{[n]\in\Delta} R\Gamma(X_n/K_{\mathrm{inf}}, \mathcal{O}_{X_n/K}) \\ &\simeq R\Gamma(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}), \end{aligned}$$

where the first equality is the h-hyperdescent of algebraic de Rham cohomology for blowups in [24] and the last is the éh-hyperdescent for the analytic infinitesimal cohomology in 6.2.2.

For the graded pieces, by Proposition 6.4.2, we have

$$\begin{aligned} R\Gamma(X/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}) &= R\Gamma(X_{\mathrm{rig}}, Ru_{X/K*}(\mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K})) \\ &\simeq R\Gamma\Big(X_{\mathrm{rig}}, \Big(Ru_{\mathfrak{X}/K*}(\mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})\Big)^{\mathrm{an}}\Big). \end{aligned}$$

We denote C to be the bounded below complex of coherent $\mathcal{O}_{\mathfrak{X}}$-modules $Ru_{\mathfrak{X}/K*}(\mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})$. As $R\Gamma(X_{\mathrm{Zar}}, \tau^{>n}C)$ lives in cohomological degree larger than n, we have the natural equalities

$$R\Gamma(X_{\mathrm{Zar}}, C) = \operatorname*{colim}_n R\Gamma(X_{\mathrm{Zar}}, \tau^{\leq n}C).$$

Similarly, we have

$$R\Gamma(X_{\mathrm{rig}}, C^{\mathrm{an}}) = \operatorname*{colim}_n R\Gamma(X_{\mathrm{rig}}, \tau^{\leq n}(C^{\mathrm{an}})).$$

On the other hand, as the rigid structure sheaf $\mathcal{O}_X$ is flat over $\mathcal{O}_{\mathfrak{X}}$, the analytification functor $(-)^{\mathrm{an}} = - \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_X$ on coherent complexes is an exact functor. So for each $n \in \mathbb{N}$, there exists a natural equality

$$\tau^{\leq n}C^{\mathrm{an}} = (\tau^{\leq n}C)^{\mathrm{an}}.$$

Notice that for each bounded complex $\tau^{\leq n}C$ of coherent sheaves, by the rigid GAGA theorem ([15, Appendix A.1]), we have

$$R\Gamma(X_{\mathrm{Zar}}, \tau^{\leq n}C) \simeq R\Gamma(X_{\mathrm{rig}}, (\tau^{\leq n}C)^{\mathrm{an}}).$$

In this way, combining all of the isomorphisms above, we get

$$\begin{aligned} R\Gamma(X_{\mathrm{Zar}}, C) &\simeq \operatorname*{colim}_n R\Gamma(X_{\mathrm{Zar}}, \tau^{\leq n}C) \\ &\simeq \operatorname*{colim}_n R\Gamma(X_{\mathrm{rig}}, (\tau^{\leq n}C)^{\mathrm{an}}) \\ &= \operatorname*{colim}_n R\Gamma(X_{\mathrm{rig}}, \tau^{\leq n}(C^{\mathrm{an}})) \\ &\simeq R\Gamma(X_{\mathrm{rig}}, C^{\mathrm{an}}). \end{aligned}$$

Finally, substituting back the definition of C and Proposition 6.4.2, we then obtain the formula for graded piece of infinitesimal filtrations:

$$R\Gamma(\mathfrak{X}/K_{\mathrm{inf}}, \mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K}) \simeq R\Gamma(X/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}). \qquad \square$$

As an application, we get the comparison with singular cohomology when K is abstractly isomorphic to the field of complex numbers, proving Theorem 1.2.1.(v).

COROLLARY 6.4.3. — *Assume there exists an abstract isomorphism of fields $K \to \mathbb{C}$. Then for any proper algebraic variety $\mathcal{X}/K$ with its analytification X, there exists a filtered isomorphism of cohomology*

$$\mathrm{H}^i(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}) \simeq \mathrm{H}^i_{Sing}(\mathcal{X}(\mathbb{C}), \mathbb{C}),$$

where singular cohomology of $\mathcal{X}(\mathbb{C})$ is filtered by the algebraic infinitesimal filtration.

Proof. — This follows from Theorem 6.4.1 and the classical result of Hartshorne in [24]. □

Using the same idea of the proof for Proposition 6.4.2 and Theorem 6.4.1, we can prove the base extension formula for the infinitesimal cohomology.

COROLLARY 6.4.4. — *Let K_0 be a complete p-adic extension of $\mathbb{Q}_p$, and let K be a complete extension of K_0. Assume X is a proper rigid space over K_0, and let $\mathcal{F}$ be a coherent crystal over $X/K_{0,\mathrm{inf}}$. Then the following natural map of filtered complexes is an isomorphism:*

$$R\Gamma(X/K_{0,\mathrm{inf}}, \mathcal{F}) \otimes_{K_0} K \longrightarrow R\Gamma(X_K/K_{\mathrm{inf}}, \mathcal{F}_K).$$

7. Cohomology over $\mathrm{B}^+_{\mathrm{dR}}$

In this section, we extend previous results to the infinitesimal site over $\mathrm{B}^+_{\mathrm{dR}}$ for a rigid space X over Σ_r for some fixed $r \in \mathbb{N}$. Our goal is to show Theorem 1.2.7 from the introduction.

7.1. Infinitesimal sites and topoi over $\mathrm{B}^+_{\mathrm{dR}}$. — We fix a complete algebraic closed p-adic field K. Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR}}/\xi^r$ for some fixed $r \in \mathbb{N}$. To build an infinitesimal cohomology theory with the coefficient being $\mathrm{B}^+_{\mathrm{dR}} = \varprojlim_{e\in\mathbb{N}} \mathrm{B}^+_{\mathrm{dR,e}}$, we construct an infinitesimal site X/Σ_{inf} as a union of all $X/\Sigma_{e\,\mathrm{inf}}$ for $e \in \mathbb{N}_{\geq r}$ and consider its relation to each infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$.

The site X/Σ_{inf}. We first give the definition of the infinitesimal site over $\Sigma = \varinjlim_{e\in\mathbb{N}} \Sigma_e$, where the latter is regarded as the ringed space whose underlying topological space is $\mathrm{Spa}(K)$, with the structure sheaf given by $\mathrm{B}^+_{\mathrm{dR}}$.

DEFINITION 7.1.1. — Let X be a rigid space over $\Sigma_r = \mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR}}/\xi^r)$ for some fixed $r \in \mathbb{N}$. The *infinitesimal site* X/Σ_{inf} *over* $\mathrm{B}^+_{\mathrm{dR}}$ is defined as follows:

- The underlying category of X/Σ_{inf} is the category of pairs (U,T) for (U,T) being a thickening in $X/\Sigma_{e\,\mathrm{inf}}$ for some $e \geq r$.
  A morphism between (U_1,T_1) and (U_2,T_2) is a morphism of objects in $X/\Sigma_{e\,\mathrm{inf}}$ for e large enough such that both pairs are objects in $X/\Sigma_{e\,\mathrm{inf}}$.
- A collection of morphisms $(U_i,T_i) \to (U,T)$ in X/Σ_{inf} is a covering if $\{T_i \to T\}$ is an open covering for the rigid space T.

As a category, X/Σ_{inf} is the union of $X/\Sigma_{e\,\mathrm{inf}}$ for all $e \geq r$. It is clear that the topology is locally rigid over each object in X/Σ_{inf}. Thus the description of a sheaf over X/Σ_{inf} is similar to that of a sheaf over $X/\Sigma_{e\,\mathrm{inf}}$, as in Section 2.

Remark 7.1.2. — Similarly to the discussion in Section 2, we could define the big version infinitesimal site X/Σ_{INF}, where the objects are infinitesimal thickenings (U,T) for U being a rigid space over X and $U \to T$ a nil-extension over $\mathrm{B}_{\mathrm{dR}}^+$. The relation between the big infinitesimal sites X/Σ_{INF} and the small one X/Σ_{inf}, including the constructions in the rest of the subsection, are exactly identical to the case over $\mathrm{B}_{\mathrm{dR,e}}^+$ in Paragraph 2.1, and we will not duplicate again here.

Functoriality of $\mathbf{Sh}(X/\Sigma_{\mathrm{inf}})$. The infinitesimal topos $\mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$ is functorial with respect to the rigid space X. Thus, for a map of $\mathrm{B}_{\mathrm{dR}}^+$-rigid spaces $f : X \to Y$ where ξ is nilpotent in both $\mathcal{O}_X$ and $\mathcal{O}_Y$, we have a natural map of topoi

$$f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}).$$

The corresponding adjoint pair of functors is given by the following:

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}})$, the inverse image $f_{\mathrm{inf}}^{-1}\mathcal{G}$ is given by the restriction of $\mu_Y^{-1}\mathcal{G}$ to the category X/Σ_{inf} along the map f and is equal to the sheaf associated with the presheaf

$$X/\Sigma_{\mathrm{inf}} \ni (U,T) \longmapsto \varinjlim_{\substack{(U,T)\to(V,S)\\ (V,S)\in Y/\Sigma_{\mathrm{inf}},\\ U\to V \text{ compatible with } f}} \mathcal{G}(V,S).$$

- The direct image functor $f_{\mathrm{inf}\,*}$ sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$ to the sheaf

$$f_{\mathrm{inf}\,*}\mathcal{F}(V,S) = \varprojlim_{\substack{(U,T)\to(V,S)\\ (U,T)\in X/\Sigma_{\mathrm{INF}}\\ U\to V \text{ compatible with } f}} \mathcal{F}(U,T).$$

We want to remind the reader that the construction of those two functors is identical with the construction of the functoriality morphism $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to$

$\mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$ for the map of rigid spaces

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Sigma_e & \longrightarrow & \Sigma_{e'}, \end{array}$$

as in Subsection 2.4.

Relation with $X/\Sigma_{e\,\mathrm{inf}}$. Topologically, the infinitesimal site X/Σ_{inf} is the limit of $X/\Sigma_{e\,\mathrm{inf}}$ for $e \geq r$. To make this precise, we consider the following morphism of sites:

$$u_e : X/\Sigma_{\mathrm{inf}} \longrightarrow X/\Sigma_{e\,\mathrm{inf}},$$

whose corresponding functor is the canonical inclusion functor that sends $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$ to the object $(U,T) \in X/\Sigma_{\mathrm{inf}}$. Note that by construction, this cocontinuous functor is a fully faithful embedding.

This morphism induces an adjoint pair of functors (u_e^{-1}, u_{e*}) given as follows:

- The functor u_{e*} is the restriction functor, in a way that for a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$, we have
$$(u_{e*}\mathcal{F})_T = \mathcal{F}_T.$$
- For a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, the sheaf $u_e^{-1}\mathcal{G}$ is the sheaf associated with the presheaf
$$\begin{aligned}(V,S) \mapsto & \varinjlim_{\substack{(V,S)\to(U,T)\\ (U,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{G}(U,T) \\ &= \begin{cases} \varnothing, & S \notin \mathrm{Rig}_{\Sigma_e}; \\ \mathcal{G}(V,S), & S \in \mathrm{Rig}_{\Sigma_e}. \end{cases}\end{aligned}$$

So by the definition of the site X/Σ_{inf}, the restriction of $u_e^{-1}\mathcal{G}$ at (V,S) is

$$(u_e^{-1}\mathcal{G})_S = \begin{cases} \varnothing, & S \notin \mathrm{Rig}_{\Sigma_e}; \\ \mathcal{G}_S, & S \in \mathrm{Rig}_{\Sigma_e}. \end{cases}$$

Here we notice that when $\mathcal{G} = h_{(U,T)}$ is the representable sheaf for some object $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$, the inverse image $u_e^{-1}h_{(U,T)}$ is nothing but the representable sheaf $h_{(U,T)}$ in $\mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$.

The morphism of site $u_e : X/\Sigma_{\mathrm{inf}} \to X/\Sigma_{e\,\mathrm{inf}}$ induces a map of topoi

$$u_e : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}).$$

It admits a section $i_e : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$, where the corresponding adjoint pair of functors is given as follows:

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{inf})$, the inverse image $i_e^{-1}\mathcal{G}$ is the sheaf associated with the presheaf

$$X/\Sigma_{e\,\mathrm{inf}} \ni (U,T) \longmapsto \varinjlim_{\substack{(U,T)\to(U,S)\\(U,S)\in X/\Sigma_{\mathrm{inf}}}} \mathcal{G}(U,S) = \mathcal{G}(U,T).$$

  Thus, $i_e^{-1} = u_{e*}$ is the restriction functor.
- The direct image functor i_{e*} sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ to the sheaf

$$i_{e*}\mathcal{F}(V,S) = \varprojlim_{\substack{(V,T)\to(V,S)\\(V,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{F}(V,T) = \mathcal{F}(V, S\times_\Sigma \Sigma_e).$$

It is clear that the composition $u_e \circ i_e$ is equal to the identity. We also note that the above functors are functorial with respect to e.

REMARK 7.1.3. — Here we notice that the map i_e is in fact induced from a natural map of sites

$$\begin{aligned} i_e : X/\Sigma_{e\,\mathrm{inf}} &\longrightarrow X/\Sigma_{\mathrm{inf}}; \\ (U, T\times_\Sigma \Sigma_e) &\longleftarrow\!\shortmid (U,T). \end{aligned}$$

This is analogous to the nilpotent bases situation, as in Remark 2.4.3

REMARK 7.1.4. — We also want to remind the reader that the construction of map i_e could be regarded as the functoriality morphism of infinitesimal topoi associated with the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow & & \downarrow \\ \Sigma_e & \longrightarrow & \Sigma. \end{array}$$

REMARK 7.1.5. — The construction of u_e and i_e is compatible with the functoriality morphism of infinitesimal topoi $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$ for a map of rigid spaces $f : X/\Sigma_e \to Y/\Sigma_{e'}$. Thus, we have the following commutative diagrams among infinitesimal topoi:

$$\begin{array}{ccc} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) & \xrightarrow{u_e} & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \\ {\scriptstyle f_{\mathrm{inf}}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{inf}}} \\ \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}) & \xrightarrow[u_{e'}]{} & \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}}); \end{array} \quad \begin{array}{ccc} \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) & \xrightarrow{i_e} & \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \\ {\scriptstyle f_{\mathrm{inf}}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{inf}}} \\ \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}}) & \xrightarrow[i_{e'}]{} & \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}). \end{array}$$

Relation to the rigid topos $\mathrm{Sh}(X_{\mathrm{rig}})$. Analogous to Subsection 2.3, there exists a natural map of topoi to the rigid site X_{rig} as below:

$$u_{X/\Sigma} : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}).$$

The corresponding preimage and the direct image functors are given as below:

- $u_{X/\Sigma *} : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{rig}});$
$$\mathcal{F} \longmapsto (U \mapsto \Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{F})).$$
- $u_{X/\Sigma}^{-1} : \mathrm{Sh}(X_{\mathrm{rig}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{\mathrm{inf}});$
$$\mathcal{E} \longmapsto ((U,T) \mapsto \mathcal{E}(U)).$$

The push-forward functor $u_{X/\Sigma *}$ is the sheafified version of the infinitesimal global section functor.

Remark 7.1.6. — The functor $u_{X/\Sigma}$ is functorial with respect to the rigid space X. Precisely, given a map of rigid spaces $f : X \to Y$ over Σ where ξ is nilpotent in both $\mathcal{O}_X$ and $\mathcal{O}_Y$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) & \xrightarrow{u_{X/\Sigma}} & \mathrm{Sh}(X_{\mathrm{rig}}) \\ \downarrow{\scriptstyle f_{\mathrm{inf}}} & & \downarrow{\scriptstyle f} \\ \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}) & \xrightarrow[u_{Y/\Sigma}]{} & \mathrm{Sh}(Y_{\mathrm{rig}}). \end{array}$$

Remark 7.1.7. — The functor $u_{X/\Sigma}$ is also compatible with $u_e : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ and $i_e : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$. Thus, the following diagrams commute:

$$\begin{array}{ccccc} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) & & \xrightarrow{\quad u_{X/\Sigma} \quad} & & \mathrm{Sh}(X_{\mathrm{rig}}); \\ & \searrow^{u_e} & & \nearrow_{u_{X/\Sigma_e}} & \\ & & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) & & \end{array}$$

$$\begin{array}{ccccc} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) & & \xrightarrow{\quad u_{X/\Sigma} \quad} & & \mathrm{Sh}(X_{\mathrm{rig}}). \\ & \nwarrow^{i_e} & & \nearrow_{u_{X/\Sigma_e}} & \\ & & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) & & \end{array}$$

Here $u_{X/\Sigma_e} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X_{\mathrm{rig}})$ is the analogous functor of $u_{X/\Sigma}$ onto the rigid site defined in Subsection 2.3.

7.2. Cohomology of crystals over X/Σ_{inf}. — In this section, we consider the cohomology of a crystal $\mathcal{F}$ over the infinitesimal site X/Σ_{inf}. Our strategy is to interpret the cohomology of $\mathcal{F}$ as the derived inverse limit of the cohomology of the pullback $i_e^*\mathcal{F}$, where $i_e^*\mathcal{F}$ is a crystal over the site $X/\Sigma_{e\,\mathrm{inf}}$.

To start with, we first describe a crystal over the infinitesimal site $X/\Sigma_{\inf}$.

Definition 7.2.1. — Let X be a rigid space over $\Sigma_{\inf}$ where ξ is nilpotent.

(i) The *infinitesimal structure sheaf* over $X/\Sigma_{\inf}$, denoted as $\mathcal{O}_{X/\Sigma}$, is a sheaf over $X/\Sigma_{\inf}$ sending a thickening $(U,T) \in X/\Sigma_{\inf}$ onto the global section of $\mathcal{O}_T$ at T as below:

$$\mathcal{O}_{X/\Sigma} : (U,T) \longmapsto \mathcal{O}_T(T).$$

(ii) The *infinitesimal ideal sheaf* over $X/\Sigma_{\inf}$, denoted as $\mathcal{J}_{X/\Sigma}$, is a sheaf over $X/\Sigma_{\inf}$ sending a thickening $(U,T) \in X/\Sigma_{\inf}$ onto the global section of $\ker(\mathcal{O}_T \to \mathcal{O}_U)$ at T as below:

$$\mathcal{O}_{X/\Sigma} : (U,T) \longmapsto \ker(\mathcal{O}_T(T) \to \mathcal{O}_U(U)).$$

(iii) A *coherent crystal* over $X/\Sigma_{\inf}$ is a $\mathcal{O}_{X/\Sigma}$-coherent sheaf $\mathcal{F}$ over $X/\Sigma_{\inf}$ that satisfies the crystal condition as in Definition 3.1.2. It is called *a crystal in vector bundle* if the restriction $\mathcal{F}_T$ at each infinitesimal thickening $(U,T) \in X/\Sigma_{\inf}$ is a vector bundle over $\mathcal{O}_T$.

Here we mention that similarly to Proposition 3.1.7, it can be shown that the categories of crystals over big and small sites are equivalent.

We notice that the morphism of sites $i_e : X/\Sigma_{e\inf} \to X/\Sigma_{\inf}$ in the last subsection is naturally a morphism of ringed sites for their structure sheaves. Moreover, since the preimage functor i_e^{-1} is equal to the restriction functor onto the subcategory $X/\Sigma_{e\inf}$, we get

$$i_e^{-1}\mathcal{O}_{X/\Sigma} = \mathcal{O}_{X/\Sigma_e},$$

and similarly for the infinitesimal ideal sheaves. So we can define the *pullback functor* $i_e^*\mathcal{F} := i_e^{-1}\mathcal{F} \otimes_{i_e^{-1}\mathcal{O}_{X/\Sigma}} \mathcal{O}_{X/\Sigma_e}$, which is the same as the restriction functor $i_e^{-1}\mathcal{F}$ itself; i.e., for an infinitesimal thickening $(U,T) \in X/\Sigma_{e\inf}$, we have

$$(i_e^*\mathcal{F})_T = (i_e^{-1}\mathcal{F})_T = \mathcal{F}_T.$$

Here we want to remark that the pullback functor $i_e^* = i_e^{-1}$ is compatible with the pullback functor $f_{\inf}^*$ of the morphism $f_{\inf} : \mathrm{Sh}(X/\Sigma_{e\inf}) \to \mathrm{Sh}(Y/\Sigma_{e'\inf})$ for a map of rigid spaces $f : X/\Sigma_e \to Y/\Sigma_{e'}$.

The main tool of the subsection is the following lemma relating a coherent crystal over $X/\Sigma_{\inf}$ with those over $X/\Sigma_{e\inf}$ of ξ-nilpotent coefficients:

Lemma 7.2.2. — *Let $\mathcal{F}$ be a coherent crystal over the infinitesimal site $X/\Sigma_{\inf}$ (resp. X/Σ_{INF}), and let X be defined over Σ_r for some $r \in \mathbb{N}$. Then we have the following:*

(i) *The pullback $i_e^*\mathcal{F}$ for each $e \in \mathbb{N}_{\geq r}$ is a crystal over $X/\Sigma_{e\inf}$. When $\mathcal{F}$ is a crystal in vector bundles, so is $\mathcal{F}$ over $X/\Sigma_{e\inf}$.*

(ii) *The counit map for the adjoint pairs* (i_e^*, i_{e*}) *induces the following isomorphism:*

$$\mathcal{F}/\xi^e \longrightarrow Ri_{e*}i_e^*\mathcal{F}.$$

In particular, we have the natural equivalences as below:

$$\mathcal{F} \longrightarrow R\varprojlim_{e\geq r} \mathcal{F}/\xi^e \longrightarrow R\varprojlim_{e\geq r} Ri_{e*}i_e^*\mathcal{F}.$$

Here the transition maps in the last limit are given by the map of infinitesimal sites $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e+1\,\mathrm{inf}}$ *(resp.* $X/\Sigma_{e\mathrm{INF}} \to X/\Sigma_{e+1\mathrm{INF}}$*) for the closed immersions of bases.*

Proof. — (i) The proof of (i) follows from the definition of the crystal condition.

(ii) We recall from the last subsection that the push-forward functor $i_{e*}\mathcal{G}$ is given by

$$(i_{e*}\mathcal{G})(U,T) = \mathcal{G}(U, T\times_\Sigma \Sigma_e)$$

for a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$. We denote the fiber product $T\times_\Sigma\Sigma_e$ by T_e, which is an infinitesimal thickening of U that is defined over Σ_e. Apply the above to the pullback $\mathcal{G} = i_e^*\mathcal{F}$ of the crystal $\mathcal{F}$, and notice that i_e^* is the restriction functor. We get

$$\begin{aligned}(Ri_{e*}i_e^*\mathcal{F})(U,T) &= R\Gamma((U,T_e),\mathcal{F}) \\ &= R\Gamma(T_e, \mathcal{F}_{T_e}) \\ &= R\Gamma(T_e, \mathcal{F}_T/\xi^e) \\ &= R\Gamma(T, \mathcal{F}_T/\xi^e),\end{aligned}$$

where the last equality follows from the observation that $T_e \to T$ has the same underlying topological spaces. Hence the cone of $\mathcal{F}/\xi^e \to Ri_{e*}i_e^*\mathcal{F}$, which is bounded below and has no cohomology, vanishes in the derived category.

Finally, notice that for a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{X/\Sigma}$-modules over X/Σ_{inf}, we always have

$$\mathcal{F} \simeq R\varprojlim_{e\geq r} \mathcal{F}/\xi^e \simeq \varprojlim_{e\geq r} \mathcal{F}/\xi^e.$$

So the last claim in (ii) follows. □

Now we are able to give the main result about the cohomology of crystals over the infinitesimal site X/Σ_{inf}. Analogous to the case over Σ_e, for a coherent crystal $\mathcal{F}$ over X/Σ_{inf}, we define a canonical filtration on it by $\mathrm{Fil}^i\mathcal{F} := \mathcal{J}^i_{X/\Sigma}\mathcal{F}$ for $i\in\mathbb{N}$. This then naturally induces a filtration on its derived direct image along the functor $u_{X/\Sigma}$ and on its derived global section, respectively.

THEOREM 7.2.3. — *Let X be a rigid space over some Σ_r, and let $\mathcal{F}$ be a coherent crystal over X/Σ_{inf}.*

(i) *There exists a natural filtered isomorphism of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules as below:*

$$Ru_{X/\Sigma *}\mathcal{F} \longrightarrow R\varprojlim_{e\geq r} Ru_{X/\Sigma_e *}(i_e^*\mathcal{F}).$$

In particular, by applying the derived global section functor, we get a filtered isomorphism

$$R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F}) \simeq R\varprojlim_{e\geq r} R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, i_e^*\mathcal{F}).$$

(ii) *Let $\{Y_e\}_{e\geq r}$ be a direct system of rigid spaces over Σ_e, such that each Y_e is smooth over Σ_e with $Y_{e+1}\times_{\Sigma_{e+1}}\Sigma_e \simeq Y_e$. Assume that X admits closed immersions into Y_r. Then we have natural filtered isomorphisms of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules as below:*

$$Ru_{X/\Sigma *}\mathcal{F} \longrightarrow \mathcal{F}_D\otimes\Omega_D^\bullet \simeq R\varprojlim_{e\geq r}(\mathcal{F}_{D_X(Y_e)}\otimes\Omega^\bullet_{D_X(Y_e)}),$$

where $D = \varinjlim_{e\geq r} D_X(Y_e)$ is the colimit[12] *of envelopes, and $\mathcal{F}_D\otimes\Omega_D^\bullet$ is the filtered de Rham complex of $\mathcal{F}$ over D*[13]*.*

(iii) *Suppose $\mathcal{F}$ is a crystal in vector bundles over X/Σ_{inf}. We equip the rings $\mathrm{B}_{\mathrm{dR}}^+$ and $\mathrm{B}_{\mathrm{dR,e}}^+$ with their ξ-adic filtrations. Then for each $e\geq r$, the natural maps below are filtered isomorphisms:*

$$(Ru_{X/\Sigma *}\mathcal{F})\otimes^L_{\mathrm{B}_{\mathrm{dR}}^+}\mathrm{B}_{\mathrm{dR,e}}^+ \longrightarrow Ru_{X/\Sigma_e *}(i_e^*\mathcal{F});$$

$$Ru_{X/\Sigma *}\mathcal{F} \longrightarrow R\varprojlim_{e}\left((Ru_{X/\Sigma *}\mathcal{F})\otimes^L_{\mathrm{B}_{\mathrm{dR}}^+}\mathrm{B}_{\mathrm{dR,e}}^+\right).$$

In particular, when X is quasi-compact quasi-separated, by applying the derived global section functor, we obtain the following canonical filtered equivalences:

$$R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F})\otimes^L_{\mathrm{B}_{\mathrm{dR}}^+}\mathrm{B}_{\mathrm{dR,e}}^+ \simeq R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, i_e^*\mathcal{F});$$

$$R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F}) \simeq R\varprojlim_{e}\left(R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F})\otimes_{\mathrm{B}_{\mathrm{dR}}^+}\mathrm{B}_{\mathrm{dR,e}}^+\right).$$

Before we prove, we want to remark that the result for crystals over the big site X/Σ_{INF} is true, and the proof is identical to the small site case.

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12. As in Definition 2.2.1, we again regard $D = \varinjlim_{e\geq r} D_X(Y_e)$ as an ind-representable sheaf in the infinitesimal topos, where the colimit always exits.

13. Here the filtration on the de Rham complex is defined analogously to the discussion above Lemma 4.1.4.

Proof. — (i) This follows from applying $Ru_{X/\Sigma *}$ to the equivalences $\mathcal{F} \longrightarrow R\varprojlim_{e\geq r} Ri_{e*}i_e^*\mathcal{F}$ in Lemma 7.2.2. Here we use the identity of maps of topoi in the last subsection

$$u_{X/\Sigma} \circ i_e = u_{X/\Sigma_e}.$$

(ii) For each $e \geq r$, by Theorem 4.2.2, there exists a natural filtered isomorphism of complexes of sheaves of $\mathrm{B}^+_{\mathrm{dR,e}}$-modules

$$Ru_{X/\Sigma_e *}i_e^*\mathcal{F} \longrightarrow \mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)}.$$

So the map of ringed sites $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e+1\,\mathrm{inf}}$ induced from the closed immersion of the bases $\Sigma_e \to \Sigma_{e+1}$ together with (i) produces the inverse limits

$$\begin{aligned} Ru_{X/\Sigma *}\mathcal{F} &\simeq R\varprojlim_{e\geq r}(\mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)}) \\ &\simeq \mathcal{F}_D \otimes \Omega^\bullet_D, \end{aligned}$$

where we use the compatibility of the de Rham complexes $\mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)}$ for different e by our choices of the direct system of smooth rigid spaces $\{Y_e\}_e$.

(iii) We first notice that the second half of the statement follows from its sheaf version by the following isomorphism:

$$R\Gamma(U, (Ru_{X/\Sigma *}\mathcal{F}) \otimes^L_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}) \simeq R\Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{F}) \otimes_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}.$$

Here the isomorphism follows by applying $R\Gamma(U, -)$ at the distinguished triangle resolving $\mathrm{B}^+_{\mathrm{dR,e}}$ over $\mathrm{B}^+_{\mathrm{dR}}$ as below:

$$Ru_{X/\Sigma *}\mathcal{F} \xrightarrow{\cdot \xi^e} Ru_{X/\Sigma *}\mathcal{F} \longrightarrow (Ru_{X/\Sigma *}\mathcal{F}) \otimes^L_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}.$$

On the other hand, to check the sheaf-level isomorphism, as the statement is rigid analytic local with respect to X, so it suffices to assume that X admits a closed immersion into a direct system of smooth rigid spaces $\{Y_e\}_e$ over Σ_e, where the results follow from the explicit calculation of the completed de Rham complexes as in part (ii) and Theorem 4.2.2. So we are done. □

REMARK 7.2.4. — Recall that for a smooth affinoid rigid space $X = \mathrm{Spa}(R)$ over K, the crystalline cohomology of X over $\mathrm{B}^+_{\mathrm{dR}}$, introduced in [8, Section 13], is defined as the inverse limit

$$\varprojlim_{e\in\mathbb{N}} \Omega^\bullet_{D_X(Y_e)},$$

where $X \to Y_e = \mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR,e}}\langle T_i^{\pm 1}\rangle)$ is a closed immersion. So Theorem 7.2.3 implies that the infinitesimal cohomology $R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$ coincides with the crystalline cohomology of X over $\mathrm{B}^+_{\mathrm{dR}}$ in the sense of [8].

With the help of Theorem 7.2.3, we can compare the infinitesimal cohomology of X over $\mathrm{B}^+_{\mathrm{dR}}$ with the derived de Rham complex.

Definition 7.2.5. — Let X be a rigid space over Σ_r. Then the *analytic derived de Rham complex of X over* $\mathrm{B}^+_{\mathrm{dR}}$, denoted as $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma}$, is defined to be the derived inverse limit of the filtered complexes

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma} := R \varprojlim_{e \geq r} \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma_e}.$$

Apply Theorem 7.2.3.(i) to the infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma}$ and the comparison in Theorem 5.5.5, and we get the following:

Corollary 7.2.6. — *Let X be a rigid space over Σ_r. There exists a natural filtered map between the analytic derived de Rham complex and the infinitesimal cohomology sheaves, inducing an isomorphism on the underlying complexes*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma} \longrightarrow Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma}.$$

In particular, applying the derived global section to the underlying complexes, we get the following comparison of cohomology:

$$R\Gamma(X, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/\Sigma}) \simeq R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma}).$$

The next result concerns the éh descent for cohomology of crystals over the big infinitesimal site X/Σ_{INF}, where X is a rigid space over K.

Proposition 7.2.7. — *Let X be a rigid space over K, and let $\mathcal{F}$ be a crystal in vector bundles over the big infinitesimal site X/Σ_{INF}. Then the cohomology sheaf $Ru_{X/\Sigma*}\mathcal{F}$ (without the filtration) satisfies the éh-hyperdescent. Thus for an éh-hypercovering $X'_\bullet \to X'$ of K-rigid spaces over X, the following natural map is an isomorphism:*

$$R\Gamma(X'/\Sigma_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R \lim_{[n]\in\Delta} \left(R\Gamma(X'_n/\Sigma_{\mathrm{INF}}, \mathcal{F})\right).$$

Proof. — By Lemma 7.2.2.(i), the pullback $i_e^*\mathcal{F}$ over $X/\Sigma_{e\mathrm{INF}}$ is a crystal in vector bundles. Thanks to Theorem 6.2.5, we know the natural map $X'_\bullet \to X'$ induces a natural isomorphism as below:

$$R\Gamma(X'/\Sigma_{\mathrm{INF}}, i_e^*\mathcal{F}) \longrightarrow R \lim_{[n]\in\Delta} \left(R\Gamma(X'_n/\Sigma_{\mathrm{INF}}, i_e^*\mathcal{F})\right).$$

Thus the result we want follows from taking the derived limit over all e by Theorem 7.2.3.(i). □

We want to mention that thanks to Corollary 2.2.8, it is safe to replace the cohomology of $\mathcal{F}$ over the big infinitesimal site by the cohomology $R\Gamma(X'/\Sigma_{\mathrm{inf}}, \iota^{-1}\mathcal{F})$ of the restriction $\iota^{-1}\mathcal{F}$ over the small infinitesimal site

X/Σ_{inf}. In particular, by applying the above result to the infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma}$, we see the infinitesimal cohomology over $\mathrm{B}_{\mathrm{dR}}^{+}$ satisfies the éh-hyperdescent.

COROLLARY 7.2.8. — *Let X be a rigid space over K. Then the infinitesimal cohomology $R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$ satisfies the éh-hyperdescent.*

Another quick upshot is the finiteness of the infinitesimal cohomology for a proper rigid space X.

PROPOSITION 7.2.9. — *Let X be a proper rigid space of dimension n over K, and let $\mathcal{F}$ be a coherent crystal. The infinitesimal cohomology $R\Gamma(X/\Sigma_{\mathrm{INF}}, \mathcal{F})$ is then a perfect $\mathrm{B}_{\mathrm{dR}}^{+}$-complex in cohomological degrees $[0, 2n]$.*

Proof. — Thanks to Theorem 7.2.3.(i), we can write $R\Gamma(X/\Sigma_{\mathrm{INF}}, \mathcal{F})$ as the derived limit of $R\Gamma(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})$. Here each $R\Gamma(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})$ is a bounded complex in cohomological degree $[0, 2n]$ such that each cohomology group is finite over $\mathrm{B}_{\mathrm{dR,e}}^{+}$ (Proposition 6.3.1). So the result then follows from the short exact sequence

$$\begin{aligned} 0 \longrightarrow R^1 \varprojlim_e \mathrm{H}^{i-1}(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F}) \longrightarrow \mathrm{H}^i(X/\Sigma_{\mathrm{INF}}, \mathcal{F}) \\ \longrightarrow \varprojlim_e \mathrm{H}^i(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F}) \longrightarrow 0. \end{aligned}$$

Here we note that the inverse system $\{\mathrm{H}^{2n}(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})\}_e$ satisfies the Mittag–Leffler condition by the finiteness of each $\mathrm{H}^{2n}(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})$ over $\mathrm{B}_{\mathrm{dR,e}}^{+}$. □

7.3. Comparison with pro-étale cohomology. — In this subsection, we compare the infinitesimal cohomology of X/Σ_{inf} with the pro-étale cohomology of the de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}$. As an application, we show the degeneracy of the Hodge–de Rham spectral sequence, together with a torsion-freeness of the infinitesimal cohomology $\mathrm{H}^i(X/\Sigma, \mathcal{O}_{X/\Sigma_{\mathrm{inf}}})$ over $\mathrm{B}_{\mathrm{dR}}^{+}$. Throughout the section, we will assume the basics of the pro-étale topology defined in [39].

Comparison theorem. Let X be a rigid space over K, and let $X_{\mathrm{proét}}$ be the pro-étale site of X. The pro-étale site admits a basis, which consists of affinoid adic spaces $U = \mathrm{Spa}(B, B^+)$ that are pro-étale over X and are *affinoid perfectoid* (i.e., the Huber pair (B, B^+) is a perfectoid algebra over K). Over the pro-étale site, we can associate the complete structure sheaf $\widehat{\mathcal{O}}_X$, whose section at an affinoid perfectoid space $U = \mathrm{Spa}(B, B^+)$ is the K-algebra B. Denote $\nu : X_{\mathrm{proét}} \to X_{\mathrm{rig}}$ to be the canonical morphism from the pro-étale site to the rigid site of X.

We recall from [39] that the *de Rham period sheaf* $\mathbb{B}_{\mathrm{dR}}^+$, defined as a sheaf of $\mathrm{B}_{\mathrm{dR}}^+$-algebras over $X_{\text{proét}}$, sending an affinoid perfectoid space $U = \mathrm{Spa}(B, B^+)$ onto the ring

$$\mathbb{B}_{\mathrm{dR}}^+(B, B^+) := \varprojlim_m \left(W(\varprojlim_{x \mapsto x^p} B^+/p)\left[\frac{1}{p}\right]/\xi^m \right).$$

The sheaf $\mathbb{B}_{\mathrm{dR}}^+$ admits a canonical surjection $\theta : \mathbb{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X$ that is compatible with the surjection map $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to K$ for the period ring $\mathrm{B}_{\mathrm{dR}}^+$. It can be shown that ξ is a nonzero-divisor in $\mathbb{B}_{\mathrm{dR}}^+$ locally, and the ideal $\ker(\theta) \subset \mathbb{B}_{\mathrm{dR}}^+$ is generated by $\xi \in \mathrm{B}_{\mathrm{dR}}^+$. So we could invert the element ξ to get a sheaf of $\mathrm{B}_{\mathrm{dR}} = \mathrm{B}_{\mathrm{dR}}^+[\frac{1}{\xi}]$-algebras over $X_{\text{proét}}$, which we denote by $\mathbb{B}_{\mathrm{dR}}$. The sheaf of rings $\mathbb{B}_{\mathrm{dR}}$ then admits a natural descending filtration where the i-th filtration for $\forall i \in \mathbb{Z}$ is defined by $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}} := \xi^i\mathbb{B}_{\mathrm{dR}}^+ \subset \mathbb{B}_{\mathrm{dR}}$. Each graded piece $\mathrm{gr}^i\mathbb{B}_{\mathrm{dR}}$, which are locally equal to $\widehat{\mathcal{O}}_X \cdot \xi^i$, is canonically isomorphic to the pro-étale structure sheaf up to a twist.

We first recall the comparison between the infinitesimal cohomology and the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}$ for smooth rigid spaces.

Theorem 7.3.1 ([8], Theorem 13.1). — *Let X be a smooth rigid space over K. Then there exists a natural map of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules over X*

$$Ru_{X/\Sigma*}\mathcal{O}_{X/\Sigma} \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}^+.$$

It is an isomorphism after inverting ξ.

Proof. — This is essentially proved in [8], Theorem 13.1, and we explain here the relation of their result to our statement.

Let X be a smooth rigid space over K of dimension d. Assume $U = \mathrm{Spa}(R)$ is a *very small* affinoid open subset in X; it admits an étale morphism onto a torus $\mathbb{T}_K^d$, where the map can be extended to a closed immersion into a larger torus $\mathbb{T}^n = \mathrm{Spa}(K\langle T_i^{\pm 1}\rangle)$. For any such closed immersion, we could associate the torus $\mathbb{T}^n$ with an affinoid perfectoid space $\mathbb{T}^{n,\infty} = \mathrm{Spa}(K\langle T_i^{\pm \frac{1}{p^\infty}}\rangle)$. The canonical map $\mathbb{T}^{n,\infty} \to \mathbb{T}^n$ is pro-étale, and its pullback along $U \to \mathbb{T}^n$ produces a pro-étale morphism from an affinoid perfectoid space $\mathrm{Spa}(R_\infty, R_\infty^+)$ over $U = \mathrm{Spa}(R)$.

We denote by D the envelope of U inside of the direct system $\{\mathbb{T}^n_{\mathrm{B}^+_{\mathrm{dR},e}}\}_e$ of tori over $\{\mathrm{B}^+_{\mathrm{dR},e}\}_e$. Then for any such choice of morphisms $(U \to \mathbb{T}^d \to \mathbb{T}^n)$, we could construct two $\mathrm{B}_{\mathrm{dR}}^+$-linear complexes

- The de Rham complex $\Omega_D^\bullet$ of U in $\{\mathbb{T}^n_{\mathrm{B}^+_{\mathrm{dR},e}}\}_e$ that computes the infinitesimal cohomology $R\Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$ by Theorem 7.2.3.

- The Koszul complex $K_{\mathbb{B}^+_{\mathrm{dR}}(R_\infty)} = K_{\mathbb{B}^+_{\mathrm{dR}}(R_\infty)}(\gamma_{u_i} - 1)$ that computes the pro-étale cohomology $R\Gamma(U_{\mathrm{pro\acute{e}t}}, \mathbb{B}^+_{\mathrm{dR}})$.

As in the proof of [8, Theorem 13.1], for any choice of $(U \to \mathbb{T}^d \to \mathbb{T}^n)$, there exists a natural map of actual complexes

$$\Omega^\bullet_D \longrightarrow K_{\mathbb{B}^+_{\mathrm{dR}}(R_\infty)},$$

which is functorial with respect to the choices of triples, such that it becomes an isomorphism after inverting ξ. Notice that the set of triples for a fixed U is filtered, and the transition maps of both complexes are isomorphisms. In this way, the induced isomorphism

$$R\Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})\left[\frac{1}{\xi}\right] \to R\Gamma(U_{\mathrm{pro\acute{e}t}}, \mathbb{B}_{\mathrm{dR}})$$

is independent of the triples $(U \to \mathbb{T}^d \to \mathbb{T}^n)$. Since the collection of very small open subsets of X form a basis in the rigid topology, we could then get a natural isomorphism as below:

$$Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma}\left[\frac{1}{\xi}\right] \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}. \qquad \square$$

Using the éh-hyperdescent, we could improve the above result into the non-smooth situation.

Theorem 7.3.2. — *Let X be a rigid space over K. Then there exists a natural map of complexes of sheaves of* $\mathrm{B}^+_{\mathrm{dR}}$*-modules over X as below:*

$$Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma} \longrightarrow R\nu_*\mathbb{B}^+_{\mathrm{dR}}.$$

It is an isomorphism after inverting ξ.

Proof. — In the proof, we use $\nu_X : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{rig}}$ to denote the natural map of sites associated with the rigid space X.

We first notice that the pro-étale cohomology sheaf $R\nu_{X*}\mathbb{B}^+_{\mathrm{dR}}$ and $R\nu_{X*}\mathbb{B}_{\mathrm{dR}}$ satisfy the éh-hyperdescent. To see this, we recall from [22, Section 4] that the derived push-forward $R\nu_{X*}\widehat{\mathcal{O}}_X$ is naturally isomorphic to $R\pi_{X*}C$, where $C = R\alpha_*\widehat{\mathcal{O}}_v \in D^{\geq 0}(X_{\mathrm{\acute{e}h}})$ is the derived push-forward of the completed v-structure sheaf (see [22, Section 3.2]), and $\pi_X : X_{\mathrm{\acute{e}h}} \to X_{\mathrm{rig}}$ is the natural map of sites. As an upshot, since C is a bounded below complex of éh-sheaves, its direct image $R\pi_{X*}C$ in the rigid site naturally satisfies the éh-hyperdescent; i.e., for an éh-hypercovering $\rho : X_\bullet \to X$ over K, the induced map below is an isomorphism:

$$R\pi_{X*}C \longrightarrow R\rho_*R\pi_{X_\bullet *}C.$$

We could then replace the above by the derived push-forward of the pro-étale structure sheaf to get a natural isomorphism

$$R\nu_{X*}\widehat{\mathcal{O}}_X \longrightarrow R\rho_* R\nu_{X_\bullet *}\widehat{\mathcal{O}}_{X_\bullet}.$$

On the other hand, notice that the de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}^+$ is completed under the ξ-adic topology such that the i-th graded piece is equal to the complete structure sheaf $\widehat{\mathcal{O}}_X \cdot \xi^i$ up to a twist. In this way, by the hyperdescent for graded pieces and the induction on e, we get

$$\begin{aligned} R\nu_{X*}\mathbb{B}_{\mathrm{dR}}^+ &= R\varprojlim_{e\in\mathbb{N}} R\nu_{X*}\mathbb{B}_{\mathrm{dR}}^+/\xi^e \\ &\simeq R\varprojlim_{e\in\mathbb{N}} R\rho_* R\nu_{X_\bullet *}\mathbb{B}_{\mathrm{dR}}^+/\xi^e \\ &\simeq R\rho_* R\varprojlim_{e\in\mathbb{N}} R\nu_{X_\bullet *}\mathbb{B}_{\mathrm{dR}}^+/\xi^e \\ &= R\rho_* R\nu_{X_\bullet *}\mathbb{B}_{\mathrm{dR}}^+. \end{aligned}$$

Thus the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}^+$ and hence $\mathbb{B}_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}^+[\frac{1}{\xi}]$ satisfies the éh-hyperdescent.

Finally, notice that the collection of maps $f : X' \to X$ for smooth rigid spaces X' forms a basis of the éh-site $X_{\text{éh}}$. In this way, the natural comparison map $Ru_{X'/\Sigma*}\mathcal{O}_{X'/\Sigma} \to R\nu_{X'*}\mathbb{B}_{\mathrm{dR}}^+$ for smooth X' extends to a map for X via the éh-hyperdescent (for the infinitesimal cohomology sheaf, this is Theorem 7.2.7), and by inverting ξ, we get the isomorphism

$$Ru_{X/\Sigma*}\mathcal{O}_{X/\Sigma}[\frac{1}{\xi}] \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}. \qquad \square$$

Remark 7.3.3. — The morphism between the infinitesimal cohomology and the pro-étale cohomology is constructed in an indirect way. In fact, by enlarging the infinitesimal site X/Σ_{inf} to a bigger site that allows all (adic spectra of) complete Huber rings as in [43, Construction 5.11], the de Rham period ring $\mathbb{B}_{\mathrm{dR}}^+(R_\infty)$ for a perfectoid algebra R_∞ can then be regarded as pro-thickening in this enlarged category. In this way, the arrow from the associated ind-object to the final object in the enlarged infinitesimal topos will induce a map on their cohomology, and it can be checked via computations in the smooth case and the éh-hyperdescent that this coincides with our morphism.

Below we consider a special case where X comes from a small subfield. Precisely, let K_0 be a discretely valued subfield of K such that the residue field of K_0 is perfect. Assume that Y is a proper rigid space over K_0 and $X = Y \times_{K_0} K$ is the base field extension of X_0. We recall from [22, Theorem 8.2.2] that there exists a $\mathrm{Gal}(K/K_0)$-equivariant filtered comparison between the pro-étale cohomology $R\Gamma(X_{\text{proét}}, \mathbb{B}_{\text{dR}})$ and the tensor product

$$R\Gamma(Y_{\text{éh}}, \Omega^{\bullet}_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\text{dR}}.$$

Here $\Omega^i_{\text{éh},/K_0}$ is the éh-differential for rigid spaces over K_0, and the filtration is defined by the product filtration, where the éh de Rham cohomology $R\Gamma(Y_{\text{éh}}, \Omega^{\bullet}_{\text{éh},/K_0})$ is equipped with a natural descending filtration by $\mathrm{Fil}^i = R\Gamma(Y_{\text{éh}}, \Omega^{\geq i}_{\text{éh},/K_0})$. Moreover, by taking the zero-th graded pieces, we get

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \simeq \bigoplus_i R\Gamma(Y_{\text{éh}}, \Omega^i_{\text{éh},/K_0}) \otimes_{K_0} K(-i).$$

From this, we get the following:

Corollary 7.3.4. — *Let Y be a proper rigid space over the discretely valued subfield K_0 of K as above, and let X be its base extension to K. Then we have a canonical isomorphism*

$$R\Gamma(X/\Sigma_{\text{inf}}, \mathcal{O}_{X/\Sigma})[\frac{1}{\xi}] \simeq R\Gamma(Y_{\text{éh}}, \Omega^{\bullet}_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\text{dR}}.$$

In particular, the infinitesimal cohomology of $Y \times_{K_0} K$ over B_{dR} admits a $\mathrm{Gal}(K/K_0)$-equivariant filtration such that the zero-th graded factor is equal to

$$\bigoplus_i R\Gamma(Y_{\text{éh}}, \Omega^i_{\text{éh},/K_0}) \otimes_{K_0} K(-i).$$

Torsion-freeness and Hodge–de Rham degeneracy. For the rest of the subsection, we prove that the infinitesimal cohomology $\mathrm{H}^n(X/\Sigma_{\text{inf}}, \mathcal{O}_{X/\Sigma})$ is torsion-free over B^+_{dR} (Theorem 1.2.7.(vi)) and show the degeneracy for the éh Hodge–de Rham spectral sequence.

Theorem 7.3.5. — *Let X be a proper rigid space over K. Then we have the following:*

(i) *the infinitesimal cohomology $\mathrm{H}^n(X/\Sigma_{\text{inf}}, \mathcal{O}_{X/\Sigma})$ is torsion-free over B^+_{dR} for each $n \in \mathbb{N}$.*

(ii) *the éh Hodge–de Rham spectral sequence over K below degenerates at its first page:*

$$E_1^{i,j} = \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh},X/K}) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{éh}}, \Omega^{\bullet}_{\text{éh},X/K}).$$

Remark 7.3.6. — Note that part (ii) generalizes the degeneracy result in [22, Proposition 8.0.8], where the latter needs the assumption for X to be defined over a discretely valued subfield.

Proof. — Let n be any integer. We first note that the pro-étale cohomology $\mathrm{H}^n(X_{\text{proét}}, \mathbb{B}^+_{\text{dR}})$ is finite free over B^+_{dR}: to see this, we recall the Primitive Comparison Theorem over B^+_{dR} as below ([38, Thm. 3.17])

$$\mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\text{dR}} \simeq \mathrm{H}^n(X_{\text{proét}}, \mathbb{B}^+_{\text{dR}}).$$

As the étale cohomology $\mathrm{H}^n(X_{\text{éh}},\mathbb{Q}_p)$ is a finite dimensional vector space over $\mathbb{Q}_p$, the above implies the finite freeness of $\mathrm{H}^n(X_{\text{proét}},\mathbb{B}_{\mathrm{dR}}^+)$ over $\mathrm{B}_{\mathrm{dR}}^+$. In particular, by Theorem 7.3.2 and the finiteness in Proposition 7.2.9, we get the following relations:

$$\begin{aligned}\dim_{\mathbb{Q}_p}\mathrm{H}^n(X_{\text{ét}},\mathbb{Q}_p) &= \mathrm{rank}_{\mathrm{B}_{\mathrm{dR}}^+}\mathrm{H}^n(X_{\text{proét}},\mathbb{B}_{\mathrm{dR}}^+)\\ &= \dim_{\mathrm{B}_{\mathrm{dR}}}\mathrm{H}^n(X_{\text{proét}},\mathbb{B}_{\mathrm{dR}})\\ &= \dim_{\mathrm{B}_{\mathrm{dR}}}\mathrm{H}^n(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})[\frac{1}{\xi}]\\ &= \mathrm{rank}_{\mathrm{B}_{\mathrm{dR}}^+}\mathrm{H}^n(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})/\mathit{torsion}\\ &\leq \dim_K\mathrm{H}^n(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})/\xi,\end{aligned}$$

where the last equality follows from the structure theorem of finite generated modules over the principal ideal domain $\mathrm{B}_{\mathrm{dR}}^+$.

On the other hand, by the base change formula in Theorem 7.2.3.(iii), for each $n\in\mathbb{Z}$, we get the following short exact sequence of K-vector spaces:

$$\begin{aligned}0\longrightarrow \mathrm{H}^n(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})/\xi &\longrightarrow \mathrm{H}^n(X/K_{\mathrm{inf}},\mathcal{O}_{X/K})\\ &\longrightarrow \mathrm{H}^{n+1}(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})[\xi]\longrightarrow 0,\end{aligned}$$

which implies the inequalities

$$\dim_K\mathrm{H}^n(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})/\xi\leq\dim_K\mathrm{H}^n(X/K_{\mathrm{inf}},\mathcal{O}_{X/K}),\ \forall n\in\mathbb{Z}.$$

In addition, notice that the cohomology complex of a sheaf of abelian groups always lives in the non-negative cohomological degrees. So both $\mathrm{H}^n(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})$ and $\mathrm{H}^n(X/K_{\mathrm{inf}},\mathcal{O}_{X/K})$ vanish for $n\leq -1$, and the short exact sequence above for $n=-1$ implies the vanishing of $\mathrm{H}^0(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})[\xi]$. In particular, we see the $\mathrm{B}_{\mathrm{dR}}^+$-module $\mathrm{H}^0(X/\Sigma_{\mathrm{inf}},\mathcal{O}_{X/\Sigma})$ is torsion-free.

Now using the comparison between the infinitesimal cohomology and the éh de Rham cohomology for the trivial crystal $\mathcal{F}=\mathcal{O}_{X/K}$ in Theorem 6.2.2, we have

$$\dim_K\mathrm{H}^n(X/K_{\mathrm{inf}},\mathcal{O}_{X/K})=\dim_K\mathrm{H}^n(X_{\text{éh}},\Omega_{\text{éh}}^\bullet).$$

Note that the natural Hodge filtration on the éh de Rham complex $\Omega_{\text{éh}}^\bullet$ induces the E_1 spectral sequence

$$E_1^{i,j}=\mathrm{H}^j(X_{\text{éh}},\Omega_{\text{éh}}^i)\Longrightarrow\mathrm{H}^{i+j}(X_{\text{éh}},\Omega_{\text{éh}}^\bullet).$$

As a consequence, we get

$$\dim_K\mathrm{H}^n(X/K_{\mathrm{inf}},\mathcal{O}_{X/K})\leq\sum_{i+j=n}\dim_K\mathrm{H}^j(X_{\text{éh}},\Omega_{\text{éh}}^i).$$

However, by Hodge–Tate decomposition in [22, Theorem 1.1.3], we have

$$\dim_{\mathbb{Q}_p}\mathrm{H}^n(X_{\text{ét}},\mathbb{Q}_p)=\sum_{i+j=n}\dim_K\mathrm{H}^j(X_{\text{éh}},\Omega_{\text{éh}}^i).$$

Hence combining all the relations of dimensions above, we see that all the inequalities should be equalities. The latter implies that the E_1 spectral sequence degenerates at the first page, and for any $n \in \mathbb{N}$ we have

$$\mathrm{H}^{n+1}(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})[\xi] = 0.$$
$$\mathrm{H}^{n}(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})/\xi = \mathrm{H}^{n}(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/K}).$$

These, together with the torsion-freeness of the $\mathrm{B}_{\mathrm{dR}}^{+}$-module $\mathrm{H}^0(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$, conclude the proof. □

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