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THÉORIE DE L'HOMOTOPIE

Colloque CNRS-NSF-SMF au C.I.R.M.
du 11 au 15 juillet 1988
organisé par

Haynes R. MILLER (M.I.T.), Jean-Michel LEMAIRE (Nice)
Lionel SCHWARTZ (Paris-sud), Michel ZISMAN (Paris VII)

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INTRODUCTION

On trouvera dans ce volume les articles issus du colloque international sur la théorie de l'homotopie qui s'est tenu au Centre International de Rencontres Mathématiques de Luminy du 11 au 15 juillet 1988.

La théorie de l'homotopie a connu dans les années 80 des développements spectaculaires; le but du colloque était de faire le point sur un certain nombre de ces développements et leurs perspectives: deux thèmes principaux ont été particulièrement évoqués pendant le colloque et se retrouvent dans la plupart des articles de ce volume:

- l'étude de l'homotopie des espaces fonctionnels, notamment lorsque la source est l'espace classifiant d'un groupe fini ou de Lie, dont la solution des conjectures de Segal et de Sullivan a montré la richesse. Au plan des méthodes, l'accent a été mis sur les suites spectrales d'Adams instables et sur les modules instables sur l'algèbre de Steenrod.
- l'utilisation et la construction de modèles algébriques de l'homotopie, particulièrement performants en homotopie rationnelle, mais aussi à présent avec des coefficients plus généraux, à savoir des corps quelconques ou des sous-anneaux appropriés de **Q**.

Ce colloque a permis de réunir une bonne part des spécialistes américains et européens du sujet, et a bénéficié à ce titre d'un soutien important de la National Science Foundation et du Centre National de la Recherche Scientifique dans le cadre de leur accord de coopération; il a également bénéficié de l'aide financière des Universités de Paris VII, Paris-Sud et Nice, et bien entendu de celle de la SMF sous la forme de la subvention accordée par le Conseil Scientifique du CIRM et de la publication du présent volume.

Les organisateurs tiennent à exprimer leur reconnaissance à toutes ces institutions, ainsi qu'au personnel du C.I.R.M., grâce auquel cette semaine a été - comme toujours - aussi riche en échanges scientifiques et humains qu'agréable sur le plan matériel.

INTRODUCTION

This volume contains papers submitted at the international conference on Homotopy Theory held at the C.I.R.M. in Marseille-Luminy from July 11 to 15, 1988.

Striking progress has been achieved in homotopy theory during the eighties. The purpose of this conference was to survey some of these achievements and their prospects: two main subjects were especially discussed during the conference and can be found in most contributions to this volume:

-The study of the homotopy type of function spaces, in particular when the domain is the classifying space of a finite group or a Lie group, whose richness has been revealed by the proofs of Segal's and Sullivan's conjectures. Among the methods used in this field, unstable Adams spectral sequences and unstable modules over the Steenrod algebra were emphazised.

-The use and construction of algebraic models of homotopy types, which have proven themselves especially fruitful in rational homotopy theory, but also now with more general coefficient rings, namely arbitrary fields and suitable subrings of **Q**.

A fair number of the experts in the field from America and Europe had the opportunity to meet at this conference, which was indeed largely supported by the NSF-CNRS cooperation agreement. Financial support was also provided by the Universities of Paris VII, Paris-Sud and Nice, and by the Mathematical Society of France through the grant distributed by the Scientific Committee of the C.I.R.M. and through the publication of this volume of Astérisque.

The organizers of this conference are pleased to express their warmest thanks to all sponsoring institutions, and to the C.I.R.M. staff who did their usual best to make that week a most profitable and pleasant one.

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AVRAMOV Luchezar & FÉLIX Yves - *Espaces de Golod.*

Nous considérons des fibrations nilpotentes $F \to E \to B$, où E et B sont des CW complexes finis simplement connexes. Un espace X est dit *de Golod* s'il existe un entier n tel que le revêtement n-connexe de X ait le type d'homotopie rationnelle d'un bouquet de sphères. Cette notion topologique correspond à celle des anneaux de Golod en algèbre locale. Nous montrons que si la base B de la fibration est un espace de Golod, la série de Poincaré de la fibre F est rationnelle.

BAKER Andrew - *Exotic multiplications on Morava K-theories and their liftings.*

For each prime p and (finite) integer n>0, there is a ring spectrum K(n) called the n-th Morava K-theory at p. We discuss exotic multiplications upon K(n) and their liftings to certain characteristic zero spectra $\hat{E}(n)$.

BROWN Edward H. & SZCZARBA R.H. - *Continuous homology and real homotopy type II.*

In our earlier paper "Continuous homology and real homotopy type", we studied localization of simplicial spaces at the reals and established an equivalence between the category of free nilpotent differential graded commutative algebras of finite type over the reals, and nilpotent simplicial spaces of finite type localized at the reals. In this paper, we extend these results by eliminating the nilpotent condition on the algebraic side, thus proving a conjecture of Sullivan. The main technical work consists in introducing local coefficients into continuous cohomology, continuous de Rham cohomology, the Serre spectral sequence and the constructions involved in real homotopy type.

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In 1967, F.B. Fuller introduced a remarkable index for counting periodic orbits of smooth flows. It has become apparent in recent work of J. Ize and E.N. Dancer that the natural setting for Fuller's index is

$\mathbb{T}$-equivariant homotopy theory, where $\mathbb{T}$ is the circle group. This paper describes their work in the conventional framework of equivariant stable homotopy theory over a base and index theory for fixed-points of maps and zeroes of vector fields.

DROR-FARJOUN Emmanuel & SMITH Jeffrey - *A geometric interpretation of Lannes' functor T.*

In this paper we prove a version of a conjecture of Lannes concerning the mod. p cohomology of the space of maps from $B\mathbb{Z}/p\mathbb{Z}$ to a rather general space X. This gives a topological meaning to an algebraic functor for modules over the Steenrod algebra, defined by Lannes. That functor has proven very useful in understanding spaces of maps from classifying spaces. As a corollary we get new proofs of several results of Lannes.

DWYER William G. & WILKERSON Clarence - *Spaces of null homotopic maps.*

We study the null component of the space of pointed maps from $B\pi$ to X when π is a locally finite group, and other components of the mapping space when π is elementary abelian. Results about the null component are used to give a general criterion for the existence of torsion in arbitrary high dimensions in the homotopy of X.

GOERSS Paul G.- *André-Quillen cohomology and the Bousfield-Kan spectral sequence.*

This paper undertakes to exploit the observation that the non-abelian homological algebra of Quillen and, in particular, the commutative algebra cohomology of André and Quillen provides a framework for discussing the unstable Adams spectral sequence of Bousfield and Kan. We take this observation in a variety of directions; for instance, we show that the long exact "transitivity sequence" in André-Quillen cohomology is related to the homotopy long exact sequence of a fibration, and we show that a product in André-Quillen cohomology can be used to compute the Whitehead product in homotopy.

HENN Hans-Werner. - *Cohomological p-nilpotence criteria for compact Lie groups.*

We introduce the concept of a p-nilpotent compact Lie group and discuss various group theoretical characterisations of such groups. These characterizations are then used to generalize cohomological p-nilpotence criteria for finite groups due to Atiyah and Quillen to the case of compact Lie groups.

MARKL Martin - *The rigidity of Poincaré duality algebras and classification of homotopy types of manifolds.*

We prove that Poincaré duality algebras are characterized by a certain rigidity property. As a consequence of this fact, we show that the **k**-isomorphism class of a Poincaré duality algebra H^* of top dimension n is uniquely determined by the factor H^*/H^n, provided **k** is algebraically closed. Using this and usual methods of descent theory, we obtain a description of the set of **k**-isomorphism classes of Poincaré duality algebras with the same given isomorphism class of H^*/H^n, for any field **k** of characteristic zero. These results are then applied to the study of homotopy types of simply connected Poincaré duality spaces.

MAY J. Peter - *Some remarks on equivariant bundles and classifying spaces.*

A number of results are given on the relationship between equivariant and non-equivariant bundles and their classification. The bundles dealt with are the projections to orbits $E \longrightarrow E/\Pi$, where Π is a normal subgroup of a compact Lie group Γ and E is a Π-free Γ-space.The base space has an action by $G = \Gamma/\Pi$, and such bundles are classified by a G-space $B(\Pi, \Gamma)$. Information about the homotopy type of this G-space gives information about the set of equivalence classes of such bundles with base a given G-space X . The bundle theory considerably simplifies when G acts freely on X, and the main theme is the study of the transformation on bundle theories induced by the natural projection $EG \times X \longrightarrow X$.

VIGUÉ-POIRRIER Micheline - *Homologie de Hochschild et homologie cyclique des algèbres différentielles graduées.*

Pour toute algèbre différentielle graduée libre (T(V), d) sur un corps commutatif quelconque, nous donnons une description explicite de deux complexes: l'homologie du premier est l'homologie de Hochschild de (T(V), d) et celle du second est l'homologie cyclique de (T(V), d). Ces complexes servent aussi de modèles pour calculer l'homologie (resp. l'homologie équivariante) de l'espace des lacets libres sur un espace simplement connexe.

WOJTKOWIAK Zdzisław - *Maps between p-completions of the Clark-Ewing spaces.*

Let $\mathbf{Z}_p$ denote the ring of p-adic integers. Let $W \subset GL(n,\mathbf{Z}_p)$ be a finite group such that p does not divide the order of W. The group W acts on $K((\mathbf{Z}_p)^n, 2)$. Let $X(W,p,n)_p$ be the p-completion of the space $K((\mathbf{Z}_p)^n, 2) \times_W EW$. We classify homotopy classes of maps between spaces $X(W,p,n)_p$.

ZARATI Saïd - *Derived functors of the destabilization and the Adams spectral sequence.*

In this note we prove the following

Theorem - Let X and Y be two pointed CW complexes such that

(i) $\bar{H}^*(X; \mathbb{F}_2) \cong \Sigma^2 I$, where I is an injective unstable module

(ii) $\bar{H}^*(Y; \mathbb{F}_2)$ is gradually finite and nil-closed.

Then the Adams spectral sequence for the group $[S^\infty X, S^\infty Y]$ degenerates at the E_2 term.

This theorem is deduced from the theory of the higher Hopf invariant introduced by Lannes and the author, and from the relationship between the Ext groups and the derived functors of the destabilization.

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On the space of maps between R-local CW complexes

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ON THE SPACE OF MAPS BETWEEN R-LOCAL CW COMPLEXES

by

D.J. Anick[1] and E. Dror Farjoun

1. Summary of Results and Notations

The papers [A1,A2] introduced and studied a differential graded Lie algebra (dgL) associated as a model to certain spaces. Building on that work, we construct in this note a simplicial skeleton for the space of pointed maps between two R-local simply-connected CW complexes ($R \subset \mathbb{Q}$). The construction entails two steps. First is the construction, in the category of dgL's, of a cosimplicial resolution and an associated "function complex" valid in a range of dimensions; and second is the connection with the topological mapping space via the above-mentioned models.

1.1. A function complex for dgL's. Let $R = \mathbb{Z}[(p-1)!]^{-1} \subset \mathbb{Q}$ for a prime p, and let L, M be free r-reduced dgL's over R having all generators in dimensions below rp ($r \geq 1$). We will construct a simplicial set, to be denoted $\underline{\hom}(L,M)$, which serves in a range of dimensions as a function complex in the sense of Dwyer and Kan [DK]. Our construction is explicit, in terms of generators and differentials; it is something which could be implemented on a computer. When L and M arise as models for finite spaces X and Y, this means that a simplicial model for the pointed mapping space Y^X is computable in a range of dimensions.

1.2. The range of dimensions. When X and Y are R-local r-connected CW complexes ($r \geq 1$), whose dimensions m_X and m_Y are bounded above by m and by rp respectively ($m < rp$), we may associate to them the dgL models L_X and L_Y. Then Y^X has the d-type of $\hom(L_X, L_Y)$, where

$$d = \min(rp - 1,\ r + 2p - 3) - m\ .$$

[1]Partially supported by a National Science Foundation grant.

Beyond dimension d, $\underline{\text{hom}}(L_X, L_Y)$ is still defined, but its connection with the geometry becomes much hazier.

1.3. Relation to tame homotopy. In view of [D] and [DK], one may associate to a pair of tame spaces (S,T) a function complex in the category of simplicial Lazard algebras. This function complex is homotopy equivalent (as a simplicial set) with the pointed mapping space T^S. When T is not tame, however, it is not obvious how one would obtain information about T^S through this technique. The desire to handle the non-tame case motivated this paper. Instead of requiring spaces to be tame, we require them to be R-local, and we restrict the dimensions where their cells may occur.

(The referee has proposed that Dwyer's functor may be able to be specialized suitably to the category of r-connected simplicial sets generated in dimension $\leq m$. This specialization, call it $\mathcal{S}$, might yield information about T^S when S belongs to CW_r^m. To accomplish this, one would attempt to use $\mathcal{S}$ in largely the same way that we have used $\mathcal{L}$ in this paper.)

1.4. Notations. We work over a fixed subring R of the rationals, and we denote by p the least non-inverted prime, i.e., $p = \inf\{n \in \mathbb{Z}_+ \mid n^{-1} \notin R\}$. In general, then, $\mathbb{Z}[(p-1)!]^{-1} \subseteq R \subseteq \mathbb{Q}$. As in tame homotopy, the relevant dimension ranges vary with a connectivity parameter r, where $r \geq 1$. Following [A1,A2] we introduce several categories.

- SS denotes the category of simplicial sets.
- TOP is the category of pointed topological spaces and pointed continuous maps.
- $CW_r^n(R)$ denotes the full subcategory of TOP, consisting of r-connected R-local CW complexes of dimension $\leq n$. "Dimension" means as an R-local cell complex, e.g., the local n-sphere belongs to $\mathcal{O}b CW_r^n(R)$ even though it has topological dimension $n + 1$.
- $\mathcal{H}o CW_r^n(R)$ is the category obtained from $CW_r^n(R)$ by collapsing (pointed) homotopy classes of maps.
- DGL(R) is the category of connected dgL's over R. A dgL is <u>free</u> if it is free as a Lie algebra (ignoring the differential); in this case we write it as $(\mathbb{L}(V), \delta)$, where the <u>R-module of</u>

<u>generators</u> $V = \oplus_{i=1}^{\infty} V_i$ is free and positively graded, and the differential δ has degree -1.

- $DGL_r^m(R)$ denotes the full subcategory of $DGL(R)$ whose objects have the form $(\mathbb{L}(V),\delta)$ where $V = \oplus_{i=r}^{m} V_i$, i.e., they are free with all generators occurring in dimensions r through m, inclusive.
- $\mathcal{L}$ denotes the model, introduced in [A1], which carries $CW_r^{m+1}(R)$ to $DGL_r^m(R)$ when $m < rp$.

<u>1.5. Distinguished morphisms in $DGL_r^m(R)$</u>. The category $DGL_r^m(R)$ cannot be made into a closed model category, but we will find it convenient to distinguish three classes of morphisms anyway. Call $f \in Mor DGL_r^m(R)$ a <u>weak equivalence</u> if it induces an isomorphism on homology of universal enveloping algebras. It is a <u>cofibration</u> if it splits as an inclusion of free Lie algebras (ignoring the differential), and it is a <u>fibration</u> if it is surjective in dimensions above r. <u>Trivial fibrations</u> are simultaneously fibrations and weak equivalences.

<u>2. Function Complexes in $DGL_r^m(R)$</u>

We will now investigate the possibility of doing homotopy theory in $DGL_r^m(R)$. The dimension limitation, viz., the "m" in $DGL_r^m(R)$, spoils our hope of doing so in the sense of Quillen [Q] or even Baues [B]. We cannot dispense entirely with the bound m, because dgL's exhibit a variety of undesirable behaviors when generator dimensions are permitted to exceed rp. On the other hand, the canonical constructions of turning a map into a fibration or cofibration tend to increase the dimensions of generators, and thus they eventually bump us out of any fixed $DGL_r^m(R)$.

An alternate approach is suggested in [T] and [A1]. We may define for $m < rp$ a homotopy relation on morphisms by utilizing a certain cylinder construction, which raises by one the maximum generator dimension. The gap between m and rp then offers us a "breathing space" in which we can perform the standard constructions approximately $(rp - m)$ times, and thus higher homotopy information is obtainable up to dimension (approximately) $rp - m$. This cylinder construction, known as the <u>Tanré cylinder</u>, is recalled next.

2.1. The Tanré cylinder. This is developed in [T] and [A1] so we provide here only a brief overview. Given a dgL $L = (\mathbb{L}(V),\delta)$ in $DGL_r^m(R)$, where $m < rp$, Tanré associates to it another dgL in $DGL_r^{m+1}(R)$, denoted $IL = (I\mathbb{L}(V), I\delta)$. Taking the set of weak equivalences to be as in 1.5, the dgL IL is a valid cylinder object on L in the sense of [Q] or [B]. In particular, I comes with natural weak equivalences j_0, j_1: id $\to I$, and if $L \overset{f}{\underset{g}{\rightrightarrows}} M$ are two morphisms in $DGL_r^m(R)$, then f and g are homotopic if and only if $f \amalg g$ factors through IL. Collapsing homotopy classes gives us a category which we denote by $\mathit{Ho}DGL_r^m(R)$.

We remark that I is not a functor, although $If\colon IL \to IM$ exists non-canonically for each $f\colon L \to M$ in $\mathit{Mor}DGL_r^m(R)$. However, I does satisfy the weak naturality condition $If \circ j_0(L) = j_0(M) \circ f$, $If \circ j_1(L) = j_1(M) \circ f$.

2.2. Constructing the cosimplicial resolution. We construct next an initial segment of a cosimplicial resolution for objects in $DGL_r^m(R)$. We shall use it to define a function complex between two such dgL's. We follow as closely as possible the standard procedure, due to Dwyer and Kan [DK], for constructing cosimplicial resolutions in any closed model category. By a <u>cosimplicial resolution</u> for an object A we mean a (not necessarily functorial) diagram

$$A \rightleftarrows \underset{\sim}{\Delta}^1 A \;\;\underset{\sim}{\Delta}^2 A \;\ldots\; \underset{\sim}{\Delta}^n A \;\ldots \tag{1}$$

satisfying the usual cosimplicial identities. In (1), each arrow is a weak equivalence; the coface maps are cofibrations, while the codegeneracies are fibrations. (See [DK, Section 4.3] for a precise definition.)

Let us review the Dwyer-Kan construction for a closed model category $\mathcal{C}$. Given an object A, a <u>cylinder</u> on A is an object IA which provides the first stage of a cosimplicial resolution for A. That is, IA fits into a diagram

$$A \overset{i_0}{\underset{i_1}{\rightrightarrows}} A \amalg A \xrightarrow{c} IA \xrightarrow{q} A \tag{2}$$

such that c is a cofibration, q is a trivial fibration, and both composites are the identity on A. This $I(\ )$ need not be a functor, but we do assume the compatibility of $j_0 = ci_0$ and $j_1 = ci_1$ with any If. Typically I arises by factoring the folding morphism $A \amalg A \xrightarrow{\nabla} A$ into a cofibration followed by a trivial fibration.

Assuming one has such an I, let $\underset{\sim}{\Delta}^0$ be the identity functor and let $\dot{\underset{\sim}{\Delta}}^1$ be the functor $\dot{\underset{\sim}{\Delta}}^1 A = A \amalg A$. Then let $\underset{\sim}{\Delta}^1$ be the push-out of $A \xleftarrow{\nabla} \dot{\underset{\sim}{\Delta}}^1 A \xrightarrow{j_0} I\dot{\underset{\sim}{\Delta}}^1 A$. It is obvious how $\underset{\sim}{\Delta}^1 A$ serves as the first stage in the cosimplicial resolution (1).

Inductively, suppose the first $(n - 1)$ stages of (1) have been constructed. Let $\mathcal{F}_A$ be the functor from the category of faces of the simplicial complex $\dot{\Delta}^n$ and inclusions among them (see 3.2) to $\mathcal{C}$, which takes a k-simplex to $\underset{\sim}{\Delta}^k A$, and an inclusion to the appropriate arrow of (1). Let $\dot{\underset{\sim}{\Delta}}^n A$ be $\operatorname{colim}(\mathcal{F}_A)$ and let $\underset{\sim}{\Delta}^n A$ be the push-out of

(3) $$A \longleftarrow \dot{\underset{\sim}{\Delta}}^n A \longrightarrow I\dot{\underset{\sim}{\Delta}}^n A .$$

We wish to perform the Dwyer-Kan construction in the category $DGL_r^m(R)$, which is not a closed model category. Let us check precisely which axioms are used. Assuming the existence of I, we need: closure under finite colimits for diagrams of cofibrations; that the push-out of a (resp. trivial) cofibration exists and is a (resp. trivial) cofibration; that two out of three of f and g and gf being weak equivalences makes the third a weak equivalence; and the left lifting property for cofibrations with respect to trivial fibrations. When we take I to be $\mathcal{I}$, the category $DGL_r^m(R)$ satisfies these four axioms, for $m \le rp$.

However, as we have noted, the Tanré cylinder construction $\mathcal{I}$ applied to a dgL L having some m-dimensional generators will have some (m+1)-dimensional generators. Inductively, $\underset{\sim}{\Delta}^n L$ lies in $DGL_r^{m+n}(R)$. This dimension shift, along with the constraint $m + n \le rp$, is what confines us to an initial segment of a cosimplicial resolution (1).

We have actually verified

LEMMA 2.3. When $m + n \leq rp$, there are constructions

$$\underset{\sim}{\Delta}^n,\ \dot{\underset{\sim}{\Delta}}^{n+1}: DGL_r^m(R) \to DGL_r^{m+n}(R).$$

Applied to a dgL $L \in Ob DGL_r^m(R)$, they come with homomorphisms that provide the first $rp-m$ stages of a cosimplicial resolution (1) for L.

Definition 2.4. For $L \in Ob DGL_r^m(R)$, $M \in Ob DGL(R)$, let $\underset{\sim}{\Delta}^n$ be as in Lemma 2.3 for $n \leq rp - m$. Define the <u>function complex</u> between L and M, denoted <u>hom</u>(L,M), to be the simplicial set consisting of $Hom_{DGL(R)}(\underset{\sim}{\Delta}^n L, M)$ in dimension n when $n \leq rp - m$, and consisting of degeneracies only, above dimension $rp - m$.

Remark 2.5. Definition 2.4 may depend upon choices made during the construction of $\underset{\sim}{\Delta}^n L$. The results that we are interested in will hold regardless of which choices were made. More importantly, the definition depends upon m and r, in the sense that the relevant dimension range will vary according to which $DGL_r^m(R)$ we view a given L as lying in. In practice, of course, we will want to use the largest possible r and the smallest possible m. In this paper, the intended r and m will always be apparent from the context.

3. Constructing the Simplicial Map

Having constructed <u>hom</u>(L,M) for dgL's, we turn our attention to its connection with the pointed mapping space Y^X. We have mentioned the dgL model $\mathcal{L}$ for pointed R-local CW complexes. We will define a simplicial map $\hat{\mathcal{L}}$ from a skeleton of Y^X to <u>hom</u>($\mathcal{L}(X), \mathcal{L}(Y)$).

3.1. The model $\mathcal{L}$. In [A1] the first author showed that for any $X \in Ob CW_r^{m+1}(R)$ with $m < rp$ there exists $L \in Ob DGL_r^m(R)$ such that UL is an Adams-Hilton model for X. We write $\mathcal{L}(X)$ for this L. One has a similar assertion and notation for maps. The passage from X to L is not functorial, since X does not canonically determine

L; nor does a map $f: X \to Y$ uniquely determine $L(f)$, even after $L(X)$ and $L(Y)$ have been fixed. However, $L(f)$ is determined up to homotopy, and hence $L(X)$ is determined up to homotopy type. In spite of this indeterminacy, the function complex between such models always does the right thing up to a certain dimension.

The main advantage of L as a model for X is that it is built directly from a cellular decomposition of X, so it is fairly small and accessible to computations.

<u>3.2. Review of Y^X</u>. The pointed mapping space Y^X may be viewed as the simplicial set

$$Y^X = \{\mathrm{Hom}_{TOP}(|\Delta^n| \ltimes X, Y)\}_{n\geq 0} \ . \tag{4}$$

Here Δ^n is the standard simplicial complex whose geometric realization is the standard n-simplex, and $\ltimes$ denotes the half-smash. The subcomplex of Δ^n obtained by removing the n-simplex is denoted, as usual, by $\dot{\Delta}^n$.

Denote by $sd(\Delta^n)$ (resp. $sd(\dot{\Delta}^n)$) the first barycentric subdivision of Δ^n (resp. $\dot{\Delta}^n$). Whenever $X \in ObCW^m_r(R)$, then an easy Kunneth formula argument shows that $|sd(\Delta^n)| \ltimes X$ and $|sd(\dot{\Delta}^n)| \ltimes X$ belong to $ObCW^{m+n}_r(R)$ (cf. 4.4 for a discussion of CW structures). As long as $m + n \leq rp$, a model $L(|sd(\Delta^n)| \ltimes X)$ exists for $|\Delta^n| \ltimes X$.

<u>LEMMA 3.3</u>. For $X \in ObCW^m_r(R)$, $m + n \leq rp$, one can choose models such that there are isomorphisms

$$L(|sd(\Delta^n)| \ltimes X) \approx \underset{\sim}{\Delta}^n L(X) \ , \quad \text{and} \tag{5}$$
$$L(|sd(\dot{\Delta}^{n+1})| \ltimes X) \approx \underset{\sim}{\dot{\Delta}}^{n+1} L(X) \ .$$

Furthermore, the model L applied to the coface and codegeneracy maps

$$|sd(\Delta^n)| \ltimes X \leftrightarrows |sd(\Delta^{n+1})| \ltimes X$$

may be taken to be the coface and codegeneracy homomorphisms mentioned in Lemma 2.3, for $L = L(X)$.

<u>Proof</u>. This is easily deduced by induction on n. At each stage, L can be chosen to commute with colimits of inclusions of CW complexes [A1, Theorem 8.5i], with cylinders [A2, Lemma 5], and with push-outs in which one map is CW and the other is an inclusion into a cylinder [A2, Lemma 6].

PROPOSITION 3.4. Let $X \in ObCW_r^m(R)$ where $m \leq rp$, and let $Y \in ObCW_r^{rp}(R)$. There is a homomorphism of simplicial sets

$$(6) \qquad \hat{L}: (Y^X)^{rp-m} \to \underline{\mathrm{hom}}(L(X), L(Y)).$$

The source of (6) is the (rp-m)-skeleton of the simplicial set (4). For each $f \in (Y^X)^{rp-m}$, $\hat{L}(f)$ may be interpreted as a valid L-model for f.

Proof. We build $\hat{L}$ dimension by dimension. Assume we have the simplicial map

$$\hat{L}^{n-1}: (Y^X)^{n-1} \to \underline{\mathrm{hom}}(L(X), L(Y)).$$

For each element $f: |\Delta^n| \ltimes X \to Y$, view f as a map from the CW complex $|sd(\Delta^n)| \ltimes X$ to Y. Consider

$$(7) \qquad \underset{\sim}{\Delta}^n L(X) \xrightarrow[\text{by (5)}]{\approx} L(|sd(\Delta^n)| \ltimes X) \xrightarrow[L(f)]{} L(Y).$$

This composite belongs to the dimension n part of $\underline{\mathrm{hom}}(L(X), L(Y))$ if $n \leq rp - m$. Thus we may extend $\hat{L}^{n-1}$ to $\hat{L}^n: (Y^X)^n \to \underline{\mathrm{hom}}(L(X), L(Y))$ by defining $\hat{L}^n(f)$ to be the composite (7). The only subtlety is the requirement that $\hat{L}^n$ is to be a simplicial map, i.e., compatible with faces and degeneracies. This in turn requires that we utilize the flexibility inherent in our choices for $L(f)$.

We are supposing that $\hat{L}^{n-1}$ is simplicial, i.e., these choices have been made compatibly below dimension n. Given $f: |\Delta^n| \ltimes X \to Y$, let $\dot{f}$ denote the restriction $\dot{f}: |sd(\dot{\Delta}^n)| \ltimes X \to Y$, and for $0 \leq i \leq n$ let $f_i: |sd(\Delta^{n-1})| \ltimes X \to Y$ denote the further restriction to the i^{th} face of $|\Delta^n|$ half-smashed with X. By our inductive assumption, the $L(f_i)$ are compatible with faces; by [A1, theorem 8.5j] their colimit serves as a valid choice for $L(\dot{f})$. Lastly, use [A1, theorem 8.5h] to extend this choice for $L(\dot{f})$ to some valid model $L(f)$. By Lemma 3.3, the resulting choice for $\hat{L}(f)$ remains compatible with faces and degeneracies.

PROPOSITION 3.5. Let $X \in \mathit{Ob}\mathrm{CW}_r^t(R)$, $Y \in \mathit{Ob}\mathrm{CW}_r^{rp}(R)$, where $t = \min(rp - 1, r + 2p - 3)$. Then $\hat{L}$ induces a bijection

$$\pi_0(\hat{L}) \colon \pi_0(Y^X) \to \pi_0(\underline{\mathrm{hom}}(L(X), L(Y))).$$

If instead $X \in \mathit{Ob}\mathrm{CW}_r^{t+1}(R)$, then $\pi_0(\hat{L})$ is a surjection .

__Proof__. For $L, M \in \mathit{Ob}\mathrm{DGL}_r^{rp-1}(R)$, $f \sqcup g \colon L \sqcup L \to M$ extends over IL if and only if it extends over $\underset{\sim}{\Delta}^1 L$. Thus $\pi_0(\underline{\mathrm{hom}}(L,M))$ coincides with the (Tanré-induced) set of homotopy classes $[L;M]$. Also, this diagram commutes:

$$(8) \qquad \begin{tikzcd} \pi_0(Y^X) \arrow[d, "\approx"] & \pi_0((Y^X)^1) \arrow[l, "\approx"'] \arrow[r, "\pi_0(\hat{L})"] \arrow[d] & \pi_0(\underline{\mathrm{hom}}(L(X), L(Y))) \arrow[d, "\approx"] \\ {[X;Y]} & {[X;Y]} \arrow[l, "="'] \arrow[r, "(L)_\#"] & {[L(X);L(Y)]} \end{tikzcd} ,$$

where we have put $m = rp - 1$. By [A2, Theorem 3] the arrow $(L)_\#$ of (8) is a bijection. When $\dim(X) = t + 1$, use $(Y^X)^0$ in place of $(Y^X)^1$ in (8); then the upper left arrow and $(L)_\#$ are surjections, hence so is $\pi_0(\hat{L})$.

__4. The d-type of Y^X__

We conclude by showing that the simplicial map $\hat{L}$ of (6) is a homotopy equivalence in a range of dimensions. We fix the notation

(9) $\qquad t = \min(rp - 1, r + 2p - 3).$

__4.1. Simplicial d-type__. Let A and B denote simplicial sets, and let $d \geq 0$. A __d-equivalence__ is a simplicial map $g \colon A \to B$ such that, for every choice of base point $a_0 \in (A)_0$, g induces a bijection on π_n for $n < d$ and a surjection on π_d. We say that B and B' have the __same (d-1)-type__ if and only if there is a simplicial set A which comes with d-equivalences $B \xleftarrow{g} A \xrightarrow{g'} B'$. "Same (d-1)-type" is an equivalence relation because, if $B'' \longleftarrow A \xrightarrow{g} B \xleftarrow{g'} A' \longrightarrow B'$ are d-equivalences, letting A" be the fiber-homotopy pull-back of g

and g' leads to d-equivalences $A \leftarrow A'' \rightarrow A'$. (For an alternate approach to (d-1)-type, see [B, p. 364].) Note that the skeleton inclusion $A^d \rightarrow A$ is always a d-equivalence. Lastly, the condition on π_0 amounts to the requirement that g induce a bijection on path-components (resp., a surjection, if $d = 0$).

Two spaces having the same d-type tells us that their homotopy groups $\pi_n(\ )$ are isomorphic for $n \leq d$, but it tells us much more than this. For instance, the spaces S^2 and $CP^\infty \times S^3$ have isomorphic π_n for all n; they have the same 2-type $(S^2 \leftarrow S^2 \vee S^3 \rightarrow CP^\infty \times S^3)$ but not the same 3-type.

We assert (see 4.7) that Y^X and $\underline{\mathrm{hom}}(\mathcal{L}(X),\mathcal{L}(Y))$ have the same d-type, for a certain d.

<u>4.2. Relative homotopy in $DGL^m_r(R)$</u>. We need the concept of a relative homotopy, for dgL's. First let us review the concept for spaces. Let W be a pointed space and let X be a subspace; we fix a pointed map $\phi: X \rightarrow Y$. Denote by $\mathrm{Hom}_{TOP}(W,Y)_\phi$ the set of all extensions of ϕ over W. Two maps in $\mathrm{Hom}_{TOP}(W,Y)_\phi$ are <u>homotopic rel X</u>, denoted $f \underset{\phi}{\simeq} g$, if and only if there is a homotopy $F: W\times[0,1] \rightarrow Y$ such that $F|_{W\times 0} = f$, $F|_{W\times 1} = g$, and $F(w,s) = \phi(w)$ for $w \in X$. Denote by $[W;Y]_\phi$ the set of $\underset{\phi}{\simeq}$-equivalence classes. We will be especially interested in the case where W is a CW complex and X is a subcomplex.

Let $L \rightarrow K$ be a cofibration in $DGL^m_r(R)$, $m < rp$; we identify L with a sub-dgL of K. Let $M \in Ob\,DGL(R)$, and fix a dgL homomorphism $\lambda: L \rightarrow M$. Denote by $\mathrm{Hom}_{DGL(R)}(K,M)_\lambda$ the set of all extensions of λ over K.

Although we have stressed that the Tanré cylinder I is not natural, there is a cofibration $IL \rightarrow IK$ which extends the given cofibration $L \amalg L \rightarrow K \amalg K$. Let $q_L: IL \rightarrow L$ denote the trivial fibration which extends the fold map $L \amalg L \xrightarrow{\nabla} L$. Two dgL homomorphisms in $\mathrm{Hom}_{DGL(R)}(K,M)_\lambda$ are <u>homotopic rel L</u>, denoted $f \underset{\lambda}{\simeq} g$, if and only if there exists $F: IK \rightarrow M$ whose restriction to $K \amalg K$ is $f \amalg g$ and whose restriction to IL is λq_L. Denote the set of $\underset{\lambda}{\simeq}$-equivalence classes by $[K;M]_\lambda$.

PROPOSITION 4.3. Let $W \in Ob\mathrm{CW}_r^t(R)$, let X be a subcomplex, and let $Y \in Ob\mathrm{CW}_r^{rp}(R)$. Fix a map ϕ: $X \to Y$ and fix a model $\lambda = L(\phi)$: $L(X) \to L(Y)$. Then L induces a bijection

(10) $$Ho(L)\colon [W;Y]_\phi \to [L(W);L(Y)]_\lambda ,$$

in which a $\simeq_\phi$-class $[f]$ is sent to the $\simeq_\lambda$-class $[L(f)]$. If instead $W \in Ob\mathrm{CW}_r^{t+1}(R)$, then (10) is a surjection.

Proof. One may easily adapt the proof of [A2, Theorem 3] to cover this situation as well. One needs only to be careful always to choose $L(f)$ for f: $W \to Y$ so as to extend the model λ for $f|_X$.

4.4. Homomorphisms induced by $\hat{L}$. We intend to study the homomorphisms induced by the $\hat{L}$ of (6) on homotopy groups. Let $X \in Ob\mathrm{CW}_r^m(R)$, $m \le rp$, and $Y \in Ob\mathrm{CW}_r^{rp}(R)$. Fix a map ϕ: $X \to Y$ and view Y^X as the simplicial set (4); thus $\phi \in (Y^X)_0$. Fix $n \ge 0$ and take as base point the 0^{th} vertex $v_0 \in |sd(\dot{\Delta}^{n+1})|$. Henceforth, when we write S^n, we will intend S^n to be viewed as the CW realization $|sd(\dot{\Delta}^{n+1})|$ with base point v_0 (i.e., as a CW complex, S^n has one cell for each non-degenerate simplex of $sd(\dot{\Delta}^{n+1})$). Let $W = S^n \ltimes X$. The CW structures on S^n and on X give us a CW structure on W; note that $W \in Ob\mathrm{CW}_r^{m+n}(R)$. We identify X with the subcomplex $v_0 \times X$ of W. Clearly, $[W;Y]_\phi$ makes sense.

We consider the same setup in $\mathrm{DGL}_r^{rp}(R)$. Let $L \in Ob\mathrm{DGL}_r^m(R)$, $m < rp$, $M \in Ob\mathrm{DGL}(R)$. When $m + n \le rp$, $\underset{\sim}{\Delta}^n L$ is defined, and we may include L into $\underset{\sim}{\Delta}^n L$ "at the 0^{th} vertex" (see (1)). Thus L is viewed as a sub-dgL of $K = \underset{\sim}{\dot{\Delta}}^{n+1}L$, and $[K;M]_\lambda$ makes sense for any given λ: $L \to M$. When $L = L(X)$, we may by Lemma 3.3 identify K with $L(W)$. Then the inclusion of the sub-dgL L into K is a valid L-model for the subcomplex inclusion $X \to W$ described above.

Now let $X \in Ob\mathrm{CW}_r^m(R)$, $m \le rp$, $Y \in Ob\mathrm{CW}_r^{rp}(R)$, as above. Choose an $\hat{L}$ as in Proposition 3.4. Let $\lambda = \hat{L}(\phi)$, which is a valid L-model for ϕ. For $n \le rp - m$, consider the diagram

$$(11)\qquad \begin{array}{ccc}
\mathrm{Hom}_{SS}(\dot{\Delta}^{n+1}, v_0; (Y^X)^{rp-m}, \phi) & \xrightarrow{(\hat{L})_*} & \mathrm{Hom}_{SS}(\dot{\Delta}^{n+1}, v_0;\ \underline{\mathrm{hom}}(L(X), L(Y)), \lambda) \\
\downarrow \alpha & & \\
\mathrm{Hom}_{SS}(\dot{\Delta}^{n+1}, v_0;\ Y^X, \phi) & & \downarrow \approx \\
\downarrow \approx & & \\
\mathrm{Hom}_{TOP}(|\dot{\Delta}^{n+1}| \ltimes X, Y)_\phi & & \mathrm{Hom}_{DGL(R)}(\underset{\sim}{\dot{\Delta}}^{n+1} L(X), L(Y))_\lambda \\
\downarrow = & & \downarrow \approx \\
\mathrm{Hom}_{TOP}(W, Y)_\phi & \overset{L'}{\dashrightarrow} & \mathrm{Hom}_{DGL(R)}(L(W), L(Y))_\lambda
\end{array} .$$

Because all the vertical arrows in (11) are bijections, there is a unique L' which makes the diagram commute. The following lemma follows easily from the construction of $\hat{L}$.

<u>LEMMA 4.5</u>. For any choice of $\hat{L}$ as in Proposition 3.4, the function L' of (11) satisfies this: for any $f \in \mathrm{Hom}_{TOP}(W, Y)_\phi$, $L'(f)$ is a valid L-model for f.

The reader may now check that the equivalence relations that we have on the various sets in (11) are compatible with the arrows, and lead to the diagram

$$(12)\qquad \begin{array}{ccc}
\pi_n((Y^X)^{rp-m}, \phi) & \xrightarrow{(\hat{L})_\#} & \pi_n(\underline{\mathrm{hom}}(L(X), L(Y)), \lambda) \\
\downarrow \alpha_\# & & \\
\pi_n(Y^X, \phi) & & \downarrow \approx \\
\downarrow \approx & & \\
\mathrm{Hom}_{TOP}(W, Y)_\phi / (\underset{\phi}{\simeq}) & & \mathrm{Hom}_{DGL(R)}(\underset{\sim}{\dot{\Delta}}^{n+1} L(X), L(Y))_\lambda / (\underset{\lambda}{\simeq}) \\
\downarrow = & & \downarrow \approx \\
[W; Y]_\phi & \xrightarrow{(L')_\#} & [L(W); L(Y)]_\lambda
\end{array} .$$

The following two facts are also clear.

<u>LEMMA 4.6</u>. (a) In (12), $(L')_\#$ coincides with $Ho(L)$ of (10). (b) In (12), $\alpha_\#$ is bijective if $m + n < rp$ and surjective if $m + n = rp$.

<u>THEOREM 4.7</u>. Let $X \in ObCW_r^m(R)$, $m \le t + 1$, $Y \in ObCW_r^{rp}(R)$. Put $d = t - m$ (cf. (9)). The simplicial map $\hat{L}$ of (6) is a (d+1)-equivalence. Consequently, the simplicial sets Y^X and $\underline{\mathrm{hom}}(L(X), L(Y))$ have the same d-type.

<u>Proof</u>. The condition on π_0 is actually given by Proposition 3.5. When $t - m \geq n > 0$, $(\hat{L})_\#$ of (12) is bijective, by 4.3 and 4.6. When $n = t - m + 1$, $(\hat{L})_\#$ of (12) is surjective, again by 4.3 and 4.6.

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Espaces de Golod

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Espaces de Golod

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Un c.w. complexe fini 1-connexe X est appelé un *espace de Golod* si pour un certain n le revêtement n-connexe X_n de X a le type d'homotopie rationnelle d'un bouquet de sphères. La terminologie provient de l'analogie algèbre locale-homotopie rationnelle ([2]) : la notion constitue en fait la transposition en topologie de celle d'*anneau de Golod généralisé*, introduite dans [1]. Les espaces de Golod X jouissent de nombreuses propriétés :

- l'algèbre $H_*(\Omega X,\mathbb{Q})$ est cohérente (résulte de [5, Théorème 3]);
- tout $H_*(\Omega X,\mathbb{Q})$-module, qui admet une résolution finie par des $H_*(\Omega X,\mathbb{Q})$-modules libres, en possède une de longueur $1 + \sum_{i=1}^{[\frac{n-1}{2}]} \mathrm{rang}_{\mathbb{Z}} (\pi_{2i+1}(X))$ (résulte de [5, Théorème 2]);
- la série de Poincaré $P_{\Omega X}(t)$ est rationnelle [3, Corollaire (4.2)].

Pour tout espace vectoriel gradué de type fini V on désigne par $|V|(t)$ la série formelle $\Sigma_i \dim(V_i)t^i$, et pour tout c.w. complexe de type fini Y on pose $P_Y(t) = |H_*(Y,\mathbb{Q})|(t)$.

Le but du présent texte est la démonstration de la propriété suivante des fibrations à base un espace de Golod, qui généralise la dernière propriété énoncée ci-dessus.

THEOREME. *Soit* $F \to E \xrightarrow{p} B$ *une fibration, avec* E *un c.w. complexe fini* 1-*connexe. Supposons que* B *soit un espace de Golod alors la série de Poincaré de* F *est rationnelle.*

Plus précisément, il existe un polynôme $\mathrm{Den}_B(t)$ *à coefficients entiers, tel que* $\mathrm{Den}_B(t)$ $P_F(t)$ *soit un polynôme pour tout* E. *En plus,* $\mathrm{Den}_B(t)$ *peut être calculé à partir de l'égalité*

$$P_{\Omega B}(t) = \Big[\prod_{2\leq 2i\leq n} (1+t^{2i-1})^{\mathrm{rank}\,\pi_{2i}(B)}\Big] / \mathrm{Den}_B(t),$$

où n *est tel que le revêtement n-connexe* B' *de* B *a le type d'homotopie rationnel d'un bouquet de sphères.*

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Exemples : Une première famille d'espaces de Golod est fournie par les espaces formels de dimension finie dont la cohomologie est une algèbre graduée de Golod généralisée. Parmi ceux-ci on trouve tous les squelettes finis de produits d'espaces d'Eilenberg-MacLane, cf. [6].

Une seconde famille est fournie par les algèbres de Lie graduées de dimension finie, L. On note $\mathbb{L}\langle V\rangle$ (respectivement $\mathbb{L}\langle\{x_i\}_{i\in I}\rangle$) l'algèbre de Lie libre sur un espace vectoriel gradué V (respectivement sur un ensemble de générateurs $\{x_i\}_{i\in I}$). A une présentation $L = \mathbb{L}\langle\{x_i\}_{i\in I}\rangle/(\{y_j\}_{j\in J})$ on associe l'application

$$\vee_{j\in J} S^{|y_j|+2} \xrightarrow{g} \vee_{i\in I} S^{|x_i|+1}$$

qui envoie la classe fondamentale de $S^{|y_j|+2}$ sur le produit de Whitehead des classes fondamentales des $S^{|x_i|+1}$ correspondant à y_j. Notant par X la cofibre de g, on a par [4, Théorème 2] la suite exacte d'algèbres de Lie

$$0 \to \mathbb{L}\langle V\rangle \to \pi_*(\Omega X) \otimes \mathbb{Q} \to L \to 0$$

qui montre bien que X est de Golod.

La démonstration repose sur les deux propositions suivantes extraites de [3]. Pour rester self-contained nous en donnerons une démonstration.

PROPOSITION 1 [3, Théorème (4.1)]. *Soit* $F \to E \to B$ *une fibration avec* $\pi_*(B) \otimes \mathbb{Q}$ *et* $H_*(E;\mathbb{Q})$ *de dimension finie, alors*

(1) $H_*(F;\mathbb{Q})$ *est un* $H_*(\Omega B;\mathbb{Q})$*-module noethérien.*

(2) *La série de Poincaré de* F *est une fonction rationnelle de la forme*

$p(t) / \prod_{i\geq 1} (1-t^{2i})^{\operatorname{rank} \pi_{2i+1}(B)}$, *où* $p(t)$ *est un polynôme.*

PROPOSITION 2 [3, §4, Lemme 3]. *Soit* L *une algèbre de Lie graduée connexe de dimension finie sur un corps et* V *un* UL*-module gradué de type fini* ($V = \bigoplus_{p\geq 0} V_p$), *alors la série de Hilbert de* V

$$|V|(t) = \sum_{n\geq 0} \dim V_n t^n$$

est une fonction rationnelle de la forme $p(t) / \prod_{i\geq 1} (1-t^{2i})^{\dim L_{2i}}$, *où* $p(t)$ *est un polynôme.*

Démonstration proposition 1 : (1) La suite spectrale de Serre de la fibration $\Omega B \to F \to E$ est une suite spectrale de $H_*(\Omega B, \mathbb{Q})$-modules. Le terme ${}^2E = H_*(\Omega B) \otimes H_*(E)$ est un module noethérien. Il en est de même de chaque ${}^pE$. Comme $H_*(E)$ est de dimension finie, ${}^pE = {}^\infty E$ pour p assez grand. Il en résulte que $H_*(F, \mathbb{Q})$ est un module de type fini. (2) résulte de la proposition 2. ■

Démonstration proposition 2 : elle calque de près la démonstration classique de Hilbert de la rationalité de $|V|(t)$ lorsque V est un module gradué de type fini sur un anneau commutatif de polynômes.

Nous travaillons par récurrence sur dim L. Si dim L = 0, V est de dimension finie et le résultat s'en déduit. Supposons le résultat vrai pour les algèbres de Lie de dimension p et soit L une algèbre de Lie graduée de dimension p + 1 et V un UL-module de type fini. Le générateur x de degré maximal est donc dans le centre et on a une suite exacte de L-modules gradués

$$(*) \qquad 0 \to K \to V \xrightarrow{x} V \to C \to 0.$$

(1) Si x est de degré impair, on pose $K' = xV$. Par Poincaré-Birkhoff-Witt, K' est un L-module contenu dans K, donc un L/(x) module lui aussi. L'hypothèse de récurrence appliquée à la suite exacte

$$0 \to K \to V \to K' \to 0$$

montre que $|V|(t) = |K|(t) + t^{-|x|}\,|K'|(t)$ est rationnel de la forme souhaitée.

(2) Si x est de degré pair, la suite exacte $(*)$ donne l'égalité

$$(1-t^{|x|})\,|V|(t) = |C|(t) - t^{|x|}\,|K|(t).$$

Comme $xK = 0$ et $xC = 0$, K et C sont en fait des modules de type fini sur L/(x), et on conclut par l'hypothèse de récurrence. ■

Démonstration du théorème : Comme B est un espace de Golod, nous avons un diagramme commutatif de fibrations, avec $\pi_*(B'') \otimes \mathbb{Q} < \infty$ où B' désigne le recouvrement n-connexe de B :

$$\begin{array}{ccccc} F & = & F & & \\ r\downarrow & & \downarrow & & \\ E' & \to & E & \xrightarrow{p'} & B'' \\ \downarrow & & \downarrow q & & \| \\ B' & \to & B & \underset{p}{\to} & B''. \end{array}$$

Par la proposition 1, $H_*(E',\mathbb{Q})$ et $H_*(B';\mathbb{Q})$ ont des séries de Poincaré rationnelles de dénominateur $q(t) = \prod_{3\leq 2i+1\leq n} (1-t^{2i})^{\mathrm{rank}\,\pi_{2i+1}(B)}$. Le morphisme de fibrations

$$\begin{array}{ccc} F & \xrightarrow{r} & E' \\ \downarrow & & \downarrow \\ E & = & E \\ q\downarrow & & \downarrow p \\ B & \underset{p}{\to} & B'' \end{array}$$

induit un diagramme commutatif

$$\begin{array}{ccc} \Omega B\times F & \xrightarrow{v} & F \\ \Omega p\times r\downarrow & & \downarrow r \\ \Omega B''\times E' & \underset{v'}{\to} & E' \end{array}$$

où v et v' désignent respectivement les opérations d'holonomie de l'espace des lacets de la base sur la fibre dans les deux fibrations. Il en résulte que $I = \mathrm{Im}\, H_*(r;\mathbb{Q})$ est un sous $H_*(\Omega B'';\mathbb{Q})$-module de $H_*(E';\mathbb{Q})$. Il est donc de type fini. $|I|(t)$ est donc, par la proposition 2, une fraction rationnelle de dénominateur $q(t)$.

Revenons à la fibration $F \to E' \to B'$. Par [3, Proposition (8.1)] elle détermine une suite spectrale

$$^2E_{p,q} = \mathrm{Tor}_p^{H_*(\Omega B',\mathbb{Q})}(\mathbb{Q},H_*(F;\mathbb{Q}))_q \Rightarrow H_{p+q}(E,\mathbb{Q}).$$

Comme B' a le type d'homotopie rationnelle d'un bouquet de sphères, $H_*(\Omega B',\mathbb{Q})$ est l'algèbre associative libre sur la désuspension $s^{-1}\,\widetilde{H}_*(B',\mathbb{Q})$ de l'homologie réduite de B. Donc pour calculer le terme ${}^2E$ de la suite spectrale, on peut utiliser la résolution bien connue de $\mathbb{Q} = H_*(\Omega B',\mathbb{Q})/\widetilde{H}_*(\Omega B',\mathbb{Q})$, en tant que $H_*(\Omega B',\mathbb{Q})$-module à droite.

$$0 \to s^{-1}\,\widetilde{H}_*(B',\mathbb{Q}) \otimes H_*(\Omega B',\mathbb{Q}) \xrightarrow{\partial_1} H_*(\Omega B',\mathbb{Q})$$

Ceci donne la suite exacte d'espaces vectoriels gradués

$$0 \to {}^{\infty}E_{1,*} \to s^{-1}\,\widetilde{H}_*(B',\mathbb{Q}) \otimes H_*(F,\mathbb{Q}) \xrightarrow{\bar{\partial}_1} H_*(F,\mathbb{Q}) \to {}^{\infty}E_{0,*} \to 0$$

et les égalités ${}^{\infty}E_{p,*} = 0$ pour $p \neq 0,1$. En d'autres termes, on a le triangle exact d'espaces vectoriels gradués

$$\begin{tikzcd}[column sep=small]
H_*(F,\mathbb{Q}) \arrow[rr, "H_*(r,\mathbb{Q})"] & & H_*(E',\mathbb{Q}) \arrow[dl, "\delta"] \\
& s^{-1}\,\widetilde{H}_*(B',\mathbb{Q}) \otimes H_*(F,\mathbb{Q}), \arrow[ul, "\partial_1"] &
\end{tikzcd}$$

où δ est un homomorphisme de degré -1.

Un simple jeu sur les séries de Poincaré donne alors

$$(**) \qquad P_F(t) = \frac{(1+t^{-1})\,|\,I\,|(t) - t^{-1}\,P_{E'}(t)}{1 - t^{-1}\,(P_{B'}(t)-1)}.$$

Comme $|\,I\,|(t)$, $P_{E'}(t)$, et $P_{B'}(t)$ sont toutes des fonctions rationnelles de dénominateur $q(t)$, on trouve bien $P_F(t)$ sous la forme $p(t)/\mathrm{Den}_B(t)$, avec $p(t)$ un polynôme à coefficients entiers, et $\mathrm{Den}_B(t) = q(t)\,(1-t^{-1}\,(P_{B'}(t)-1)\,) \in \mathbb{Z}[t]$.

Reste à calculer $P_{\Omega B}(t)$. Dans ce cas la suite exacte d'homotopie rationnelle de la fibration $\Omega B' \to \Omega B=F \xrightarrow{r} E'=\Omega B''$ a un connectant nul, puisque B' est rationnellement un bouquet de sphères.

Ceci implique la surjectivité de $H_*(r,\mathbb{Q})$. La formule (**) devient

$$P_{\Omega B}(t) = \frac{P_{\Omega B''}(t)}{1 - t^{-1}(P_{B'}(t)-1)}.$$

Comme $P_{\Omega B''}(t) = \prod_{2\leq 2i\leq n} (1+t^{2i-1})^{\operatorname{rank} \pi_{2i}(B)} / q(t)$ par Poincaré-Birkhoff-Witt, on a bien la formule voulue. ■

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Exotic multiplications on Morava K-theories and their liftings

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EXOTIC MULTIPLICATIONS ON MORAVA K-THEORIES AND THEIR LIFTINGS

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Abstract. For each prime p and integer n satisfying $0 < n < \infty$, there is a ring spectrum $K(n)$ called the n th Morava K-theory at p. We discuss exotic multiplications upon $K(n)$ and their liftings to certain characteristic zero spectra $\widehat{E(n)}$.

Introduction.

The purpose of this paper is to describe exotic multiplications on Morava's spectrum $K(n)$ and certain "liftings" to spectra whose coefficient rings are of characteristic 0. Many of the results we describe are probably familiar to other topologists and indeed it seems likely that they date back to foundational work of Jack Morava in unpublished preprints. not now easily available. A published source for some of this is the paper of Urs Würgler [**12**]. We only give sketches of the proofs, most of which are straightforward modifications of existing arguments or to be found in [**12**]. For all background information and much notation that we take for granted, the reader is referred to [**1**] and [**7**].

I would like to express my thanks to the organisers of the Luminy Conference for providing such an enjoyable event.

Convention: Throughout this paper we assume that p is an *odd* prime.

§1 Exotic Morava K-theories.

Morava K-theory is usually defined to be a multiplicative complex oriented cohomology theory $K(n)^*(\ )$ which has for its coefficient ring

$$K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$$

where $v_n \in K(n)_{2p^n-2}$, and is canonically complex oriented by a morphism of ring spectra

$$\sigma^{K(n)}: BP \longrightarrow K(n)$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX which on coefficients induces the ring homomorphism

$$\sigma_*^{K(n)} \colon BP_* \longrightarrow K(n)_*$$
$$\sigma_*^{K(n)}(v_k) = \begin{cases} v_n & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here we have $BP_* = \mathbf{Z}_{(p)}[v_k : k \geq 1]$ with $v_k \in BP_{2p^k-2}$ being the k th *Araki generator*, defined using the formal group sum

$$[p]_{BP}X = \sum_{0 \leq k}^{BP} \left(v_k X^{p^k}\right).$$

As a homomorphism of graded rings, we can regard $\sigma_*^{K(n)}$ as a quotient homomorphism

$$\sigma_*^{K(n)} \colon v_n^{-1}BP_* \longrightarrow v_n^{-1}BP_*/\mathcal{M}_n \cong K(n)_*$$

where $\mathcal{M}_n = (v_k : 0 \leq k \neq n) \lhd v_n^{-1}BP_*$ is a maximal graded ideal of the ring $v_n^{-1}BP_*$. Thus we can interpret $K(n)_*$ as a (graded) residue field for this maximal ideal.

Clearly this ideal $\mathcal{M}_n$ is not the only such maximal ideal and we can reasonably look at other examples and ask if the associated quotient (graded) fields occur as coefficient rings for cohomology theories is an analogous fashion. Notice that $\mathcal{M}_n$ contains the invariant prime ideal $I_n = (v_k : 0 \leq k \leq n-1)$ and the formal group law $F^{K(n)}$ therefore has height n. One way to construct $K(n)$-theory is by using Landweber's Exact Functor Theorem (LEFT) [**6**] in its modulo I_n version [**14**]; this allows us to make the definition

$$K(n)^*(\ ) = K(n)_* \otimes_{P(n)_*} P(n)^*(\ )$$

on the category of finite CW spectra $\mathbf{CW^f}$, where $P(n)$ is the spectrum for which

$$P(n)_* = BP_*/I_n.$$

We thus concentrate on maximal ideals $\mathcal{M}' \lhd v_n^{-1}BP_*$ containing the ideal I_n. We then have

THEOREM (1.1). *Let $\mathcal{M}' \lhd v_n^{-1}BP_*$ be a maximal (graded) ideal containing I_n. Then there is a unique multiplicative cohomology theory*

$K(\mathcal{M}')^*(\ )$, *defined on* $\mathbf{CW^f}$, *for which there is a multiplicative natural isomorphism*

$$K(\mathcal{M}')^*(\ ) \cong K(\mathcal{M}')_* \otimes_{P(n)_*} P(n)^*(\ )$$

and where $K(\mathcal{M}')_* = v_n^{-1}BP_*/\mathcal{M}'$ *is the coefficient ring.*

The proof is immediate using LEFT.

Of course, if there is an isomorphism of graded rings, $K(\mathcal{M}')_* \cong K(\mathcal{M}'')_*$, then we need to decide if the two theories arising from $\mathcal{M}'$ and $\mathcal{M}''$ can be naturally equivalent.

THEOREM (1.2). *Let* $\mathcal{M}', \mathcal{M}'' \lhd v_n^{-1}BP_*$ *be graded maximal ideals containing* I_n *and let* $f\colon K(\mathcal{M}')_* \longrightarrow K(\mathcal{M}'')_*$ *be an isomorphism of graded rings. Then there is a natural isomorphism of multiplicative cohomology theories on* $\mathbf{CW^f}$,

$$\widetilde{f}\colon K(\mathcal{M}')^*(\ ) \longrightarrow K(\mathcal{M}'')^*(\ )$$

extending f *if and only if the the formal group laws* $f_*F^{v_n^{-1}BP_*/\mathcal{M}'}$ *and* $F^{v_n^{-1}BP_*/\mathcal{M}''}$ *are strictly isomorphic over the ring* $K(\mathcal{M}'')_*$.

The main observation required to prove this result is that these two formal group laws are associated to two complex orientations induced by the composite of the morphisms of ring spectra $BP \longrightarrow v_n^{-1}BP \longrightarrow K(\mathcal{M}'')$.

COROLLARY (1.3). *The theories* $K(\mathcal{M}')^*(\ )$ *and* $K(\mathcal{M}'')^*(\ )$ *are representable by ring spectra* $K(\mathcal{M}')$ *and* $K(\mathcal{M}'')$, *which are unique up to canonical equivalence in the stable category. Moreover,* $K(\mathcal{M}')$ *and* $K(\mathcal{M}'')$ *are equivalent as ring spectra if and only if the formal group laws* $f_*F^{v_n^{-1}BP_*/\mathcal{M}'}$ *and* $F^{v_n^{-1}BP_*/\mathcal{M}''}$ *are strictly isomorphic over the ring* $K(\mathcal{M}'')_*$.

Let us now consider such ring spectra $K(\mathcal{M}')$ where $K(\mathcal{M}')_* \cong K(n)_*$ as graded rings. By a result from **[12]** (see also **[5]**) these are precisely the ring spectra having the homotopy type of $K(n)$ (not necessarily multiplicatively). Thus, such ring spectra are classified to within equivalence as ring spectra by the set of maximal ideals $\mathcal{M}'$ modulo strict isomorphism of the associated formal group laws over $K(\mathcal{M}')_*$. We call the multiplicative cohomology theory associated to such a ring spectrum an *exotic Morava K-theory*.

Let us consider such a spectrum $K(\mathcal{M}')$, where $K(\mathcal{M}')_* \cong K(n)_*$ as graded rings. Then we have the following modification of a result of **[13]**, THEOREM (1.4). *As an algebra over* $K(n)_*$, *we have*

$$K(\mathcal{M}')_*(K(\mathcal{M}')) \cong K(\mathcal{M}')_*(t'_k : k \geq 1) \otimes \Lambda_{K(n)_*}(a'_0, \dots, a'_{n-1})$$

where $|t'_k| = 2p^k - 2$, $|a'_k| = 2p^k - 1$, *and there are polynomial relations of the form*

$$t'^{p^n}_k - v_n^{(p^k-1)/(p-1)} t'_k = h_k(t'_1, \dots, t'_{k-1})$$

over $K(n)_*$.

The symbol $\Lambda_{K(n)_*}$ denotes an exterior algebra over $K(n)_*$ on the indicated generators.

To prove this result, we rework the proof for the case of $K(n)$ (see [13], [7]) and define the generators t'_k by using the identity

$$\sum_{\substack{r \geq 0 \\ s \geq n}}^{K(\mathcal{M}')} (v'_s t'^{p^s}_r X^{p^{r+s}}) = \sum_{\substack{r \geq 0 \\ s \geq n}}^{K(\mathcal{M}')} (t'_r v'^{p^r}_s X^{p^{r+s}})$$

where

$$[p]_{F^{K(\mathcal{M}')}} X = \sum_{s \geq n}^{K(\mathcal{M}')} (v'_s X^{p^s}).$$

The exterior generators are similarly derived.

We can interpret the algebra

$$K(\mathcal{M}')_*(t'_k : k \geq 1)$$

as representing the strict automorphisms of the group law $F^{K(\mathcal{M}')}$, in a way analogous to the case of $K(n)$ (see [7]).

§2 Liftings of exotic Morava *K*-theories.

Recall that there is a ring spectrum $E(n)$ for which

$$E(n)^*(\ ) \cong E(n)_* \otimes_{BP_*} BP^*(\ )$$

on $\mathbf{CW^f}$. Here we have

$$E(n)_* = v_n^{-1} BP_*/(v_{n+k} : k \geq 1).$$

We showed in joint work with Urs Würgler (see [4]) that the *Noetherian completion* $\widehat{E(n)}$ of $E(n)$, characterised by the formula

$$\widehat{E(n)}^*(\ ) = \varprojlim_k \left(E(n)^*(\ )/I_n^k E(n)^*(\ )\right)$$

on $\mathbf{CW^f}$, is a summand of the *Artinian completion* $\widehat{v_n^{-1}BP}$ of $v_n^{-1}BP$ and indeed there is a product splitting

$$\widehat{v_n^{-1}BP} \simeq \prod_v \Sigma^{2d(v)} \widehat{E(n)}$$

of (topological) ring spectra for v ranging over a suitable indexing set and d a suitable numerical function. The algebra underlying the proof is intimately related to liftings of *Lubin-Tate* group laws, i.e. group laws over $\mathbf{F}_p$ algebras classified by homomorphisms from $K(n)_*$. In this section we describe the analogous situation for liftings of exotic Morava K-theories of the form $K(\mathcal{M}')$ as in §1.

Now if $K(\mathcal{M}')_* \cong K(n)_*$ as a ring, then the natural homomorphism

$$\theta_{\mathcal{M}'} \colon v_n^{-1}BP \longrightarrow K(\mathcal{M}')_*$$

given by

$$\theta_{\mathcal{M}'}(v_k) = \begin{cases} c_k v_n^{r(k/n)} & \text{if } n \mid k \\ 0 & \text{otherwise} \end{cases}$$

for $k \geq n$ and integers c_k. Here, the numerical function r is given by

$$r(m) = \frac{(p^{mn}-1)}{(p^n-1)}.$$

and we set $c_k = 0 = r(k/n)$ whenever k is not divisible by n or $k = 0$.

Now consider an ideal of the form

$$J = (v_k - c_k v_n^{r(k/n)} + g_k : k > n) \subset \mathcal{M}' \lhd v_n^{-1}BP_*$$

and satisfying

$$J + I_n = \mathcal{M}'.$$

Here $g_k \in \mathcal{M}'$ are certain elements chosen so that the last condition holds. Set $E(J)_* = v_n^{-1}BP_*/J$.

We now define a cohomology theory

$$E(J)^*(\ ) = E(J)_* \otimes_{BP_*} BP^*(\ )$$

on $\mathbf{CW^f}$. This *is* a cohomology theory by Landweber's Exact Functor Theorem, and is moreover multiplicative and canonically complex oriented by the obvious natural transformation

$$BP^*(\ ) \longrightarrow E(J)^*(\ ).$$

Furthermore there is a canonical multiplicative natural transformation

$$E(J)^*(\ ) \longrightarrow K(\mathcal{M}')^*(\ ).$$

We can form the Noetherian completion

$$\widehat{E(J)}^*(\ ) = \lim_{\substack{\longleftarrow \\ k}} (E(J)^*(\ )/I_n^k E(J)^*(\ ))$$

and also the Artinian completion of $v_n^{-1}BP$ with respect to the maximal ideal $\mathcal{M}'$, $v_n^{-1}\widehat{BP(\mathcal{M}')}$ (see [**4**]). Then we have

THEOREM (2.1). *There is a splitting of topological ring spectra*

$$v_n^{-1}\widehat{BP(\mathcal{M}')} \simeq \prod_w \Sigma^{2e(w)}\widehat{E(J)}$$

where w ranges over an appropriate indexing set and e is a numerical function.

The proof is a modification of that in [**4**] which rests on the fact that in the ring $v_n^{-1}BP_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} v_n^{-1}BP_*$, the generators t_k satisfy relations modulo $\mathcal{M}'$ of the form given in the statement of (1.4). Of course, in the case where $\mathcal{M}' = \mathcal{M} = (v_k : 0 \leq k \neq n)$, this shows that $\widehat{E(n)}$ is just one amongst many ring spectra splitting off of $v_n^{-1}\widehat{BP}$ in this way.

In [**12**] and [**5**] it was proved that any ring spectrum whose homotopy ring is isomorphic to $K(n)_*$ agrees with $K(n)$ up to equivalence as a spectrum. In fact we can lift such results to show

THEOREM (2.2). *Let F be a complex oriented topological ring spectrum such that as graded topological groups*

$$\pi_*(F) \cong \widehat{E(n)}_*,$$

and there is a maximal ideal $\mathcal{M}' \lhd v_n^{-1}BP_*$ for which there is a morphism of ring spectra $F \longrightarrow K(\mathcal{M}')$ which is surjective in homotopy. Then there is an ideal $J \subset \mathcal{M}'$ such that there is an equivalence of topological ring spectra $F \simeq \widehat{E(J)}$.

The proof of this makes use of a tower

$$\cdots \longrightarrow E(J)/I_n^{k+1} \longrightarrow E(J)/I_n^k \longrightarrow \cdots \longrightarrow E(J)/I_n = K(\mathcal{M}')$$

of A_∞ module spectra over $\widehat{E(J)}$ generalising that constructed in [**3**], together with the existence of an A_∞ structure on $\widehat{E(J)}$ (see §3).

§3 A_∞ structures on exotic Morava K-theories.

In [8] it was shown that for any odd prime p, the standard ring spectrum structure on $K(n)$ admits uncountably many distinct A_∞ structures in the sense of [9], [10] and [11]. One can similarly ask if this is true for any of the exotic structures discussed in our earlier sections. In fact, by (1.4), the arguments of [8] can be used in the more general context. Indeed, this is also true for the results of [3] and the liftings $\widehat{E(J)}$ which have unique topological A_∞ structures, and the natural morphisms of ring spectra $\widehat{E(J)} \longrightarrow K(\mathcal{M}')$ admit A_∞ structures whichever of the A_∞ structures is put on $K(\mathcal{M}')$.

One consequence of the existence of A_∞ structures is that there are Künneth and Universal Coefficient spectral sequences for A_∞ module theories over these ring spectra. For example, if M is a (topological) A_∞ module spectrum over $\widehat{E(J)}$, then for any spectrum X, there is a spectral sequence

$$\mathrm{E}_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{\widehat{E(J)}}(\widehat{E(J)}_*(X), M_*) \Longrightarrow M^{s+t}(X).$$

Such spectral sequences promise to be of great use in calculations.

§4 Some examples.

We end by considering two examples of cohomology theories which are related to exotic Morava K-theories as discussed in the earlier sections. These are essentially the only known periodic theories which have (or are suspected to have) geometric descriptions, and remarkably they both appear to be naturally related to the original versions of Morava $K(1)$ and $K(2)$ and its liftings, rather than truly exotic versions.

K-theory. Consider the case of complex K-theory localised at a prime p, $K^*(\ ) = KU^*_{(p)}(\ )$. Then reduction modulo p gives a theory $K/p^*(\ )$ satisfying

$$K/p^*(\ ) \cong \bigoplus_{0\leq k\leq (p-1)} K(1)^{*+2k}(\ )$$

as multiplicative theories where we define the product on the direct sum by requiring that there be an isomorphism of rings

$$K/p_* \cong K(1)_*[u]/(u^{p-1} - v_1)$$

when evaluated on a point. Thus

$$K/p \simeq \bigvee_{0\leq k\leq (p-1)} \Sigma^{2k} K(1)$$

as ring spectra, where the wedge is given an appropriate algebra spectrum structure over $K(1)$. Lifting this result gives an equivalence $\widehat{K} \simeq \widehat{E(n)}$ which is known to arise before p-adic completion.

Elliptic cohomology. Let $E\ell\ell$ be the spectrum representing the version of elliptic cohomology whose coefficient ring is the ring of modular forms for $SL_2(\mathbb{Z})$ meromorphic at infinity (see [**2**]), localised at a prime $p > 3$. Then in [**2**] we showed that if E_{p-1} denotes the $(p-1)$ st Eisenstein function, then there is an equivalence of ring spectra

$$E\ell\ell/(p, E_{p-1}) \simeq \bigvee_{\alpha} \Sigma^{2f(\alpha)} K(2)$$

where $E\ell\ell/(p, E_{p-1})$ is the reduction of $E\ell\ell$ modulo the ideal (p, E_{p-1}) in an appropriate sense. This lifts to a splitting of topological ring spectra

$$E\ell\ell\widehat{_{(p,E_{p-1})}} \simeq \bigvee_{\alpha'} \Sigma^{2f'(\alpha')} \widehat{E(2)}.$$

In both cases the wedge is finite, and we need to impose appropriate algebra spectra structures over the bottom summands.

It would be of interest to find "naturally" occurring examples involving truly exotic versions of Morava K-theories. Of course, for the examples given we can take an exotic $K(1)$ or $K(2)$ and use this to impose an exotic multiplication upon either mod p K-theory or elliptic cohomology, but it is then unclear whether the resulting multiplicative theories have any geometric descriptions.

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CONTINUOUS COHOMOLOGY AND REAL HOMOTOPY TYPE II

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Introduction.

In our earlier paper "Continuous Cohomology and Real Homotopy Type" [**3**], we studied localization of simplicial spaces at the reals and established an equivalence between the category of free nilpotent differential graded commutative algebras of finite type over the reals and nilpotent simplicial spaces of finite type localized at the reals. In this paper, we extend these results by eliminating the nilpotent condition on the algebraic side, thus proving a conjecture of Sullivan [**8**]. (See Theorem 1.2, Part (iv), below.) The main technical work consists in introducing local coefficients into continuous cohomology, continuous de Rham cohomology, the Serre Spectral Sequence, and the constructions involved in real homotopy type.

We also obtain information about secondary characteristic classes of G foliations in the sense of Haefliger [**1**,3,4,6], namely that when G is compact, the continuous cohomology of the appropriate classifying space injects into the ordinary cohomology. This result is stated and proved at the end of Section 2. (See Proposition 2.5).

Our main results are stated in Section 1. The remainder of the paper is devoted to proving these results.

1. Statements of Results.

We begin by recalling some of the notation and definitions from [**3**].

Let $\mathcal{CA}$ denote the category of differential (degree $+1$), graded, commutative (in the graded sense), locally convex topological algebras with unit over R and $\Delta\mathcal{T}$ the category of compactly generated simplicial spaces. Let Ω_q^p denote the space of C^∞ differential p-forms on the standard q-simplex Δ^q in the C^∞ topology. Then $\Omega^p = \{\Omega_q^p\}$ is in $\Delta\mathcal{T}$, $\Omega_q = \{\Omega_q^p\}$ is in $\mathcal{CA}$, and $\Omega = \{\Omega_q^p\}$ is in $\Delta\mathcal{CA}$. Define contravariant functors $\Delta : \mathcal{CA} \longrightarrow \Delta\mathcal{T}$ and $\mathcal{A} : \Delta\mathcal{T} \longrightarrow \mathcal{CA}$ by

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$\Delta(A)_q = (A, \Omega_q) =$ the simplicial space of algebra mappings $A \to \Omega_q$,

$\mathcal{A}(X)^p = (X, \Omega^p) =$ the vector spaces of continuous simplicial mappings $X \to \Omega^p$.

The simplicial structure on Ω gives $\Delta(A)$ a simplicial structure and the algebra structure on Ω gives one on $\mathcal{A}(X)$. We view $\Delta(A)$ as the *simplicial realization* of A and $\mathcal{A}(X)$ as the *algebra of differential forms* on X.

For $X \in \Delta\mathcal{T}$ and any topological abelian group G, let $C^q(X;G)$ be the space of continuous mappings $u : X_q \to G$ with $u \circ s_i = 0, 0 \le i \le q-1$, and define $\delta : C^q(X;G) \to C^{q+1}(X;G)$ by

$$\delta u = \sum_{j=0}^{q+1} (-1)^j u \circ \partial_j$$

Here, s_i, ∂_j denotes the face and degeneracy mappings of X. *The continuous cohomology of X with coefficients in G* is defined by

$$H^*(X;G) = H_*(C^*(X;G);\delta).$$

The usual deRham mapping defines an isomorphism

$$\psi : H^*(\mathcal{A}(X);d) \longrightarrow H^*(X;R) = H^*(X).$$

(See Theorem 2.4 of [**3**].)

We next describe homology of $A \in \mathcal{CA}$ with local coefficients. Suppose L is a finite dimensional Lie algebra which acts on a finite dimensional vector space V via a Lie algebra homomorphism $\gamma : L \to g\ell(V) = \mathrm{Hom}(V,V)$. Let $C^*(L)$ denote the usual cochain algebra on L, that is, $C^p(L)$ is the space of alternating, multilinear functions

$$u : L^p = L \times L \times \cdots \times L \to R$$

with $d : C^p(L) \to C^{p+1}(L)$ given by

$$du(\ell_1, \dots, \ell_{p+1}) = \sum_{i<j} (-1)^{i+j} u([\ell_i, \ell_j], \ell_1, \dots, \hat{\ell}_i, \dots, \hat{\ell}_j, \dots, \ell_{p+1})$$

For $A \in \mathcal{CA}$, we define L-local coefficients on A as follows. Let $\ell_1, \dots, \ell_n$ be a basis for L, $\ell_1^*, \dots, \ell_n^*$ the dual basis for L^*, and suppose $\lambda : C^*(L) \to A$ is a $\mathcal{CA}$ mapping. Define $d_\lambda : A \otimes V \to A \otimes V$ by

$$d_\lambda(a \otimes v) = da \otimes v + (-1)^p \sum_{i=1}^{n} a\lambda(\ell_i^*) \otimes \ell_i v.$$

where $\ell_i v = \gamma(\ell_i)(v)$. It is easy to check that d_λ is independent of the choice of basis, that $d_\lambda^2 = 0$, and that d_λ is functorial in both A and V. Let $H_*(A;V_\lambda) = H_*(A \otimes V, d_\lambda)$.

Remark 1.1. If $A = C^*(L), \lambda =$ identity, $\gamma : L \to g\ell(V)$, and $J : C^*(L) \otimes V \to C^*(L; V)$ is the standard isomorphism, then $Jd_\lambda = d_\gamma J$ where $d_\gamma : C^p(L; V) \to C^{p+1}(L; V)$ is given by

$$d_\gamma \omega = d\omega + \gamma \wedge \omega.$$

Here, γ is considered as a $g\ell(V)$-valued 1-form on L and the wedge product $\gamma \wedge \omega$ is defined using the action of $g\ell(V)$ on V.

Suppose now that $A \in \mathcal{CA}$ is free and of finite type; that is, A is the tensor product of a polynomial algebra on even dimensional generators with an exterior algebra on odd dimensional generators and each A^j is a finite dimensional vector space, $j \geq 0$. According to Proposition 7.11 of [**2**], we can find a basis $t_1, \dots, t_n$ for A^1 such that, for $1 \leq i \leq m$,

$$dt_i = \sum_{\substack{1 \leq i < j \leq m \\ j < k}} a_i^{jk} t_j t_k$$

and for $m < i \leq n, dt_i$ is a polynormal generator for A. One easily sees that, if A and B are free and of finite type, then $\Delta(A \otimes B) = \Delta(A) \times \Delta(B)$ and if $A = R[x, y]$ with $dx = y$, then ΔA is contractible in $\Delta\mathcal{T}$. Hence, up to homotopy type, $\Delta(A)$ is unchanged by dividing A by the ideal generated by $\{t_i, dt_i \mid i > m\}$. Henceforth, we include the condition $n = m$ in the notion of free and of finite type.

Given A as above, let L be the dual vector space to A^1 and let $\alpha_1, \dots, \alpha_m$ be the basis for L dual to $t_1, \dots, t_m$. Then L is a Lie algebra with

$$[\alpha_j, \alpha_k] = 2 \sum_{i=1}^{m} a_i^{jk} \alpha_i.$$

The inclusion

$$\lambda : C^*(L) \simeq R[t_1, \dots, t_m] \subset A$$

defines L-local coefficients on A. As in [**3**], we define $i : A \to \mathcal{A}(\Delta(A))$ by $i(a)(u) = u(a)$. Then $i\lambda : C^*(L) \to \mathcal{A}(\Delta(A))$ defines L-local coefficients on $\mathcal{A}(\Delta(A))$. Finally, if $A^{(1)}$ denotes the subalgebra of A generated by $t_1, \dots, t_m$, then $C^*(L)$ is naturally isomorphic to $A^{(1)}$.

The following result is stated in [**8**] as "Theorem" 8.1.

THEOREM 1.2. *Suppose $A \in \mathcal{CA}$ is free of finite type, and that $A^{(1)} = C^*(L)$ as above.*

(i) Let $G = G_A$ be the connected, simply connected Lie group with $L(G) = L$. Then

$$\begin{aligned} \pi_i(\Delta A^{(1)}) &\simeq G \quad \text{for } i = 1, \\ &\simeq \pi_i(G) \quad \text{for } i > 1. \end{aligned}$$

(ii) Let V be a finite dimensional vector space on which L acts and $\lambda : C^*(L) \to A$ the inclusion map. Then the mapping $i : A^{(1)} \to \mathcal{A}(\Delta(A^{(1)}))$ induces an isomorphism

$$i_* : H_*(A^{(1)}; V_\lambda) \to H_*(\mathcal{A}(\Delta(A^{(1)})); V_{i\lambda}) \simeq H^*(\Delta A^{(1)}; V_{i\lambda}).$$

*(iii) Let $\tilde{A}$ be the quotient algebra of A by the ideal generated by $A^{(1)}$. Then $A^{(1)} \subset A$ induces a fibre map $\Delta(A) \to \Delta(A^{(1)})$ with fibre $\Delta(\tilde{A}) \subset \Delta(A)$. (Note that $\tilde{A}$ is free, nilpotent, and of finite type and hence the homotopy type of $\Delta(\tilde{A})$ is described in [**3**].)*

(iv) For any action of L on V, the mapping $i : A \to \mathcal{A}(\Delta(A))$ induces isomorphisms

$$i_* : H_*(A; V_\lambda) \longrightarrow H_*(\mathcal{A}(\Delta(A)); V_{i\lambda}) \simeq H^*(\Delta(A); V_{i\lambda}).$$

In [**8**], Sullivan gives a very brief sketch of (i) and (iii) and, asserts that (ii) is "a reformulation of the theorem of Van Est". No proof is given for (iv). We give a detailed proof of (iv) in general and of (ii) when $G = G_A$ in the universal cover of a compact group. Actually (i) follows from Proposition 2.4 and (iii) follows from results of Section 5. In Section 2, we give an analysis of $\Delta(C^*(L))$ (see Theorem 2.3 and Proposition 2.4). Section 3 deals with local coefficients and the de Rham theorem. Proposition 2.4, the de Rham theorem with local coefficients, and an unpublished result of Graeme Segal are used in Section 4 to prove (ii) when G is the universal cover of a compact group. The result of Segal is that the continuous cohomology and the ordinary cohomology of the singular complex of a CW complex are isomorphic. We give Segals proof in Section 7.

The development of the proof of (iv) is as follows: Suppose $A \in \mathcal{CA}$ is free and of finite type. Then $A = UA^{(n)}$ where $A^{(0)} = R$ and $A^{(n)} = A^{(n-1)}[x_1^{(n)}, \ldots, x_k^{(n)}]$ and the $x_i^{(n)}$ have dimension n. We compute $H^*(\Delta(A); V_{i\lambda})$ by computing $H^*(\Delta(A^{(n)}); V_{i\lambda})$ using induction on n. In Section 6, we use Proposition 2.4 to prove Theorem 5.3, namely that

$$\Delta(A[x_1, \ldots, x_k]) \to \Delta(A)$$

is a fibration with fibre $\Delta(R[x_1, \ldots, x_k])$. In Section 6 we develop the Serre Spectral Sequence for continuous cohomology with local coefficients and apply it to the above fibration to prove the inductive step in the proof of (iv).

Recall that, for $A, B \in \mathcal{CA}$, a function complex $\mathcal{F}(A, B) \in \Delta\mathcal{T}$ was defined in [**3**] (following [**2**]) by $\mathcal{F}(A, B)_q = (A, \Omega_q \otimes B)$, the space of continuous differential graded algebra mappings from A to $\Omega_q \otimes B$. If A and $B \in \mathcal{CA}$ are free and of finite type and $h : A^{(1)} \to B^{(1)}$ is a map in $\mathcal{CA}$, define $\mathcal{F}(A, B; h)$ to be the simplicial subspace of $\mathcal{F}(A, B)$ whose q simplicies are maps $u : A \to \Omega_q \otimes B$ which give a commutative diagram

$$\begin{array}{ccc} A^{(1)} & \xrightarrow{h} & B^{(1)} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{u} & \Omega_q \otimes B \end{array}$$

where $i : A^{(1)} \to A$ is the inclusion and $j(b) = 1 \otimes b$.

Similarly, for $X, Y \in \Delta\mathcal{T}$, $\mathcal{F}(X,Y) \in \Delta\mathcal{T}$ is given by $\mathcal{F}(X,Y)_q = (X \times \Delta[q], Y)$, the space of simplicial mappings from $X \times \Delta[q]$ to Y where $\Delta[q]$ is the simplicial model for the standard q-simplex. If $h : A^{(1)} \to B^{(1)}$ is as above, let $\mathcal{F}(\Delta B, \Delta A; \Delta(h))$ be the simplicial subspace of $\mathcal{F}(\Delta B, \Delta A)$ whose q-simplicies are mappings $f : \Delta[B] \times \Delta[q] \to \Delta(A)$ for which the diagram

$$\begin{array}{ccc} \Delta[B] \times \Delta[q] & \xrightarrow{f} & \Delta(A) \\ {\scriptstyle \bar{j}}\downarrow & & \downarrow{\scriptstyle \Delta(i)} \\ \Delta(B^{(1)}) & \xrightarrow{\Delta(h)} & \Delta(A^{(1)}), \end{array}$$

is commutative, where $\bar{j}(s,u) = u \mid \Delta(B^{(1)})$. Just as in [**3**], Theorem 1.20, we prove

THEOREM 1.3. *Suppose $A, B \in \mathcal{CA}$ are free and of finite type and $h : A^{(1)} \to B^{(1)}$ is a mapping in $\mathcal{CA}$. Then $\Delta : \mathcal{F}(A,B) \to \mathcal{F}(\Delta B, \Delta A)$ defines a weak equivalence*

$$\Delta : \mathcal{F}(A, B; h) \to \mathcal{F}(\Delta B, \Delta A; \Delta h).$$

The proof of this result is given at the end of Section 5.

2. The Simplicial Space $\Delta(C^\star(L))$.

We give here an analysis of the simplicial space $\Delta(C^\star(L))$ and prove an independence result for characteristic classes of G-foliations. Although the results of this section are stated for finite dimensional Lie groups, they do hold more generally for infinite dimensional Lie groups which are regular (in the sense of Milnor [**7**]) and for which the Lie algebra $L(G)$ is reflexive. In particular, they hold for $G = \mathrm{Diff}(M)$, M compact, where $L(G)$ is the Lie algebra of vector fields on M.

Let X be a manifold, G a Lie group, and let (X, G) be the space of C^∞ mappings $f : X \to G$. Let G act on (X, G) by $(gf)(x) = gf(x)$ and let $\hat{\Omega}^1(X; L) \subset \Omega^1(X, L)$ be given by

$$\hat{\Omega}^1(X; L) = \{w \in \Omega^1(X; L) \mid dw - w \wedge w = 0\}.$$

Define $\hat{\rho} : (X, G) \to \Omega^1(X; L)$ by $\hat{\rho}(f) = -f^{-1}df$ where

$$(f^{-1}df)(v) = dL^{-1}_{f(x)} df(v) \in TG_e = L(G)$$

for $v \in TX_x$. It is easily checked that $\hat{\rho}(g \cdot f) = \hat{\rho}(f)$ for any $f \in (X, G), g \in G$, and that $d(\hat{\rho}(f)) - \hat{\rho}(f) \wedge \hat{\rho}(f) = 0$. Thus, $\hat{\rho}$ defines

$$\rho : (X, G)/G \to \hat{\Omega}^1(X; L).$$

Similarly, we define $\tilde{\sigma} : (X, G) \to (C^*(L), \Omega^*(X))$ by $\tilde{\sigma}(f)(\alpha) = f^*(\bar{\alpha})$ for $\alpha \in C^1(L) = L^*$ where $\bar{\alpha} \in \Omega^1(G)$ is the invariant 1-form defined by α. Then $\tilde{\sigma}(g \cdot f) = \tilde{\sigma}(f)$ so $\tilde{\sigma}$ defines

$$\sigma : (X, G)/G \to (C^*(L), \Omega^*(X)).$$

Finally, we define $\psi : \Omega^1(X, L) \to \mathrm{Hom}(L^*; \Omega^1(X))$ by $\psi(w)(\alpha)(v) = \alpha(w(v))$. One easily checks (see **[5]**) that ψ defines a bijection

$$\psi : \tilde{\Omega}^1(X; L) \to (C^*(L), \Omega^*(X)).$$

THEOREM 2.1. *Suppose X is simply connected. Then each of the mappings ρ and σ defined above are bijections and the diagram*

$$\begin{tikzcd} (X,G)/G \arrow[r, "\rho"] \arrow[dr, "\sigma"'] & \hat{\Omega}^1(X;L) \arrow[d, "\varsigma"] \\ & (C^*(L), \Omega^*(X)) \end{tikzcd}$$

is commutative where $\zeta(w) = \psi(-w)$.

Remark. If the spaces above are given the C^∞ topologies, then each of the mappings in the diagram is a homeomorphism.

PROOF: The fact that the diagram of Theorem 2.1 commutes is an immediate consequence of the definitions. Since $\psi : \tilde{\Omega}(X; L) \to (C^*(L), \Omega^*(X))$ is a bijection, it follows that $\zeta : \tilde{\Omega}(X; L) \to (C^*(L), \Omega^*(X))$ is a bijection so Theorem 2.1 will be proved if we can show that $\rho : (X, G)/G \to \hat{\Omega}^1(X; L)$ is a bijection. This is an immediate consequence of the following.

LEMMA 2.2. *Let U be a neighborhood of $x \in X$ and suppose $w \in \Omega^1(U, L)$ satisfies $dw = w \wedge w$. Then there is a neighborhood $U_0 \subset U$ of x and a unique C^∞ function $f : U_0 \to G$ such that $f(x) = e$ and $w = f^{-1} df$.*

For a proof of this lemma, see **[9]**.

Let ΔG denote the simplicial space of C^∞ singular simplices of G. Setting $X = \Delta^q, q = 0, 1, 2, \ldots$, in the previous discussion yields simplicial mappings

$$\tilde{\sigma} : \Delta G \to \Delta C^*(L)$$
$$\sigma : \Delta G/G \to \Delta C^*(L)$$

As an immediate consequence of Theorem 2.1, we have

THEOREM 2.3. *The mapping $\sigma : \Delta G/G \to \Delta C^*(L)$ is a simplicial homeomorphism.*

The next result proves part (i) of Theorem 1.2.

PROPOSITION 2.4. *The mapping $\tilde{\sigma} : \Delta G \to \Delta C^*(L)$ is a twisted cartesian product with fibre and group $\tilde{G}$ where $\tilde{G}$ is the simplicial group with $\tilde{G}_q = G$ for all q and with the identity as face and degeneracy mappings.*

By Theorem 2.3, it is sufficient to show that the natural mapping $\pi : \Delta G \to \Delta G/G$ is a twisted cartesian product with fibre and group $\tilde{G}$. To accomplish this, define $\tau : \Delta G/G \to \tilde{G}$ by $\tau([T]) = T(v_1)T(v_0)^{-1}$ and $h : \Delta G \to (\Delta G/G) \times_\tau \tilde{G}$ by $h(T) = ([T], T(v_0))$ where $T \in (\Delta G/G)_q$ and $v_0, v_1, \dots, v_q$ are the vertices of Δ^q. Then τ is a twisting function and h is a simplicial homeomorphism such that the diagram

$$\begin{array}{ccc} \Delta G & \xrightarrow{h} & (\Delta G/G) \times_\tau \tilde{G} \\ \pi \downarrow & & \downarrow \pi_1 \\ \Delta G/G & \xrightarrow{id} & \Delta G/G \end{array}$$

is commutative.

We conclude this section with a result concerning characteristic classes of G foliations (in the sense of [5]). According to Haefliger [5], if L is the Lie algebra of a Lie group G, then $\Delta C^*(L)$ is a classifying space for G-foliations transverse to fibres of a product. The following can be interpreted as an independence statement for the continuous cohomology characteristic classes of these foliations.

PROPOSITION 2.5. *Let G be a compact Lie group with Lie algebra L. Then the homomorphism $H^*(\Delta C^*(L)) \to H^*(\Delta C^*(L)^\delta)$ is injective.*

Here, $\Delta C^*(L)^\delta$ is the simplicial space $\Delta C^*(L)$ in the discrete topology.

PROOF: By Theorem 2.3, it is enough to prove that $i^* : H^*(\Delta G/G) \to H^*((\Delta G/G)^\delta)$ is injective. To do this, we consider the commutative diagram

$$\begin{array}{ccccc} C^*(\Delta G/G) & \xrightarrow{j} & C^*(\Delta G)^G & \xrightarrow{\ell} & C^*(\Delta G) \\ i^* \downarrow & & \downarrow & & \downarrow k \\ C^*((\Delta G/G)^\delta) & \longrightarrow & C^*(\Delta G^\delta)^G & \longrightarrow & C^*(\Delta G^\delta). \end{array}$$

The mapping j is an isomorphism and k is a homology isomorphism by Theorem 4.9. Using the Haar integral, we can construct a cochain mapping $r : C^*(\Delta G) \to C^*(\Delta G)^G$ with $r\ell = id$. (See Proposition 4.4.) It follows that the composite $k\ell j$ is injective on homology so i^* is injective on homology.

REMARK: The analogue of Proposition 2.5 with local coefficients can be proved using the techniques developed in Section 4.

3. Local Coefficients and the de Rham Theorem.

In this section, we describe local coefficients systems in several different ways. We also prove a local coefficient version of the continuous cohomology de Rham theorem.

Let G be a Lie group with Lie algebra L and let V be a finite dimensional vector space. Suppose G acts on V via a representation $\Gamma : G \to GL(V)$ so that L acts on V via a representation $\gamma : L \to g\ell(V)$. In Section 1, we defined a local L system on $A \in \mathcal{CA}$ to be a $\mathcal{CA}$ map $\lambda : C^*(L) \to A$ and a differential

$$d_\lambda : A^p \otimes V \to A^{p+1} \otimes V$$

given by the formula

$$d_\lambda(a \otimes v) = (da) \otimes v + (-1)^p \sum_i a\lambda(\ell_i^*) \otimes \ell_i v$$

where $\{\ell_i\}$ is a basis for L, $\{\ell_i^*\}$ the dual basis for $L^*, a \in A^p$ and $\ell_i v = \gamma(\ell_i)(v)$. We now translate this into a more familiar form.

Let V be as above and define $\Omega(V)$ to be the simplicial topological differential graded vector space given by

$$\Omega_q^p(V) = \Omega^p(\Delta^q; V),$$

the smooth differential p-forms on Δ^q with values in V. For $X \in \Delta\mathcal{T}$, let $\mathcal{A}(X, V)$ be the differential topological graded vector space with

$$\mathcal{A}^p(X; V) = (X, \Omega^p(V)),$$

the space of simplicial mappings from X to $\Omega^p(V)$. It is easy to see that $\mathcal{A}(X, V) = \mathcal{A}(X) \otimes V$. Note that $\Delta C^*(L)$ can be considered to be contained in $\Omega^1(L)$.

Suppose $\lambda : C^*(L) \to \mathcal{A}(X)$ is an $\mathcal{CA}$ map and let $\phi = \phi_\lambda$ be the composite

$$X \xrightarrow{j} \Delta(\mathcal{A}(X)) \xrightarrow{\Delta(\lambda)} \Delta(C^*(L)) \subset \Omega^1(L)$$

where $j(x)(u) = u(x)$. Then $\phi \in \mathcal{A}^1(X; L)$ and one easily checks that $d\phi(x) + \phi(x)_\wedge\phi(x) = 0$ for all $x \in X_q$. We define $d_\phi : \mathcal{A}(X; V) \to \mathcal{A}(X, V)$ by

$$d_\phi\omega = d\omega + \phi_\wedge\omega.$$

Then $d_\phi^2 = 0$ and we have

PROPOSITION 3.1. *Let $\iota : \mathcal{A}(X) \otimes V \to \mathcal{A}(X, V)$ be the isomorphism defined by $\iota(\omega \otimes v)(x) = \omega(x)v$. Then $\iota d_\lambda = d_\phi \iota$.*

PROOF: If $\omega \in \mathcal{A}^p(X), v \in V$, and $x \in X_q$, then

$$\begin{aligned} \iota d_\lambda(\omega \otimes v)(x) &= \iota((d\omega) \otimes v + (-1)^p \sum_i \omega\lambda(\ell_i^*) \otimes \ell_i v)(x) \\ &= (d\omega)(x)v + (-1)^p \sum_i \omega(x)\lambda(\ell_i^*)(x)\ell_i v \\ &= (d\omega)(x)v + (-1)^p \omega(x)\phi(x) \\ &= d_\phi \iota(\omega \otimes v)(x). \end{aligned}$$

since $\phi(x) = \sum_i \lambda(\ell_i^*)(x)\ell_i$.

We next reformulate these notions into an equivariant setting. Let $\tilde{\lambda}$ be the composite

$$X \xrightarrow{j} \Delta(\mathcal{A}(X)) \xrightarrow{\Delta(\lambda)} \Delta C^*(L)$$

and let $\tilde{X}$ be the pullback

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\tilde{p}"'] & \Delta G \arrow[d, "\tilde{\sigma}"] \\ X \arrow[r, "\tilde{\lambda}"] & \Delta C^*(L) \end{tikzcd}$$

where $\tilde{\sigma}$ is defined in Section 2. Let G act on $\mathcal{A}(\tilde{X}, V)$ by

$$(g\omega)(x, T) = g\omega(x, g^{-1}T)$$

where $x \in X_q, g \in G$, and $T \in (\Delta G)_q$ and let $H : \mathcal{A}(X, V) \to \mathcal{A}(\tilde{X}, V)$ be given by

$$H(\omega)(x, T) = T \cdot \omega(x)$$

for $\omega \in \mathcal{A}(X, V)$ and $(x, T) \in \tilde{X}$. Here

$$T \cdot \omega(x)(w_1, \dots, w_p) = (\Gamma T)(y) \cdot (\omega(x)(w_1, \dots, w_p))$$

where $\Gamma : G \to GL(V)$ is the homomorphism defined by the action of G on V, $y \in \Delta^q, w_1, \dots, w_p \in T\Delta^q_y$, and $(\Gamma T)(y)$ acts on $\omega(x)(w_1, \dots, w_p) \in V$.

PROPOSITION 3.2. *The map H defines an isomorphism of $\mathcal{A}(X, V)$ onto $\mathcal{A}(\tilde{X}, V)^G$ with $Hd_\phi = dH$.*

PROOF: The verification that $gH(\omega) = H(\omega)$ is straightforward. To see that H is an isomorphism, let $\mathcal{O} : \Delta C^*(L) \to \Delta G$ be the composite

$$\Delta C^*(L) \xrightarrow{\sigma^{-1}} \Delta G/G \xrightarrow{\beta} \Delta G$$

where $\sigma : \Delta G/G \to \Delta C^*(L)$ is the simplicial homeomorphism defined in Section 2, $\beta[T] = T(v_0)^{-1}T$, and v_0 is the initial vertex of Δ^q. Then $H^{-1} : \mathcal{A}(\tilde{X}, V)^G \to \mathcal{A}(X, V)$ is given by

$$H^{-1}(\omega)(x) = \mathcal{O}(\tilde{\lambda}(x))^{-1}\omega(x, \mathcal{O}(\tilde{\lambda}(x))).$$

In order to prove that $Hd_\phi = dH$, we need the following, which will also be useful in the next section.

LEMMA 3.3. *Let M be a manifold, V a finite dimensional vector space, $f \in \Omega^0(M; GL(V))$*
$\subset \Omega^0(M; g\ell(V))$, and $\omega \in \Omega^1(M; V)$. Define $\Delta f \in \Omega^1(M; g\ell(V))$ by $\Delta f = f^{-1}df$. Then

$$d(fw) = f(dw + \Delta f \wedge w).$$

The proof is straightforward.

To prove $Hd_\phi = dH$, consider $\omega \in \mathcal{A}(X, V), (x, T) \in \tilde{X}_q \subset X_q \times \Delta G_q$, and let $f : \Delta^q \to GL(V)$ be the composite

$$\Delta^q \xrightarrow{T} G \xrightarrow{\Gamma} GL(V).$$

Then $\phi(x) = \Delta f$ since $\tilde{\lambda}(x) = \sigma(T) = T^{-1}dT$ and we have

$$\begin{aligned}(Hd_\phi\omega)(x, T) &= (\beta T) \cdot d_\phi\omega(x)\\ &= f \cdot (\cdot\omega(x) + \Delta f \wedge \omega(x))\\ &= d(f \cdot \omega(x)) = dH(\omega)(x, T).\end{aligned}$$

by Lemma 3.3.

We conclude this section by reviewing the usual definition of local coefficients.

Let $t : X_1 \to G$ be a continuous function satisfying $t(\partial_1 x) = t(\partial_2 x)t(\partial_0 x)$ for $x \in X_2$. Define $\delta_t \cdot C^p(X; V) \to C^{p+1}(X; V)$ by

$$(\delta_t u)(x) = t(\partial_2^{p-1}x)u(\partial_0 x) + \sum_{i=1}^{p+1}(-1)^i u(\partial_j x)$$

for $x \in X_{p+1}$. Then $\delta_t^2 = 0$ and we define

$$H^*(X; V_t) = H_*(C^*(X; V); \delta_t).$$

Suppose $\lambda : C^*(L) \to \mathcal{A}(X)$ with $\mathcal{O}, \phi$, and $\tilde{X}$ as above. Define $t = t_\lambda$ by

$$t_\lambda(x) = \mathcal{O}(\phi(x))(v_1)$$

where $v_0, v_1, \ldots, v_q$ are the vertices of Δ^q and let G act on $C^*(\tilde{X}; V)$ by

$$(gu)(x, T) = gu(x, g^{-1}T).$$

Define $K : C^*(X; V) \to C^*(\tilde{X}; V)$ by $K(u)(x, T) = T(v_0)u(x)$.

PROPOSITION 3.4. *The function K maps $C^*(X,V)$ isomorphically onto $C^*(\tilde{X},V)^G$ with $Kd_t = dK$ where $t = t_\lambda$.*

PROOF: It is easy to see that K^{-1} is given by

$$K^{-1}(u)(x) = u(x, \mathcal{O}(\tilde{\lambda}(x))).$$

The remainder of the proof is similar to the proof of Proposition 3.2 and we omit it.

Let $\tilde{\Psi} = \Psi \otimes \mathrm{id} : \mathcal{A}(X,V) \to C^*(X,V)$ where $\Psi : \mathcal{A}(X) \to C^*(X)$ is defined in [3]. We now have the following local coefficient version of the de Rham Theorem.

THEOREM 3.5. *The map $\tilde{\psi}$ induces an isomorphism*

$$\tilde{\psi}_* : H_*(\mathcal{A}(\tilde{X};V)^G) \to H_*(C^*(\tilde{X};V)^G)$$

and hence an isomorphism

$$(K^{-1}\tilde{\psi}H)_* : H_*(\mathcal{A}(X,V), d_\phi) \to H_*(C^*(X,V), d_t)$$

where $\phi = \phi_\lambda$ and $t = t_\lambda$.

PROOF: In the proof of Theorem 2.4 of [**3**], natural mappings $\phi : C^*(X) \to \mathcal{A}(X)$ and $\gamma : \mathcal{A}^p(X) \to \mathcal{A}^{p-1}(X)$ were constructed satisfying $\psi\phi = \mathrm{id}$ and $d\gamma + \gamma d = \phi\psi - \mathrm{id}$. Tensoring everything in sight with V gives the desired result.

4. The Proof of Theorem 1.2 (ii).

We now prove part (ii) of Theorem 1.2 in the case where $G = G_A$ is compact.

Let G be a connected, simply connected Lie group G with Lie algebra L. Suppose L acts on a finite dimensional vector space V via a homomorphism $\gamma : L \to g\ell(V)$. Viewing γ in $C^1(L; g\ell V)$, define a differential d_γ on $C^*(L;V)$ by

$$d_\gamma(\alpha) = d\alpha + \gamma_\wedge \alpha$$

as in Remark 1.1. Similarly, we define a differential $d_{i\gamma}$ on $\mathcal{A}(\Delta C^*(L); V)$ by

$$d_{i\gamma}(\omega) = d\omega + (i\gamma) \wedge \omega$$

where $i : C^\star(L; g\ell(V)) \to \mathcal{A}(\Delta(C^\star(L)); g\ell(V))$ is the canonical map. The wedge product $(i\gamma) \wedge \omega$ is defined using the pairing

$$\mathcal{A}(\Delta C^*(L); g\ell(V)) \otimes \mathcal{A}(\Delta C^*(L); V) \to \mathcal{A}(\Delta C^*(L); V).$$

According to Remark 1.1, part (ii) of Theorem 1.2 is a consequence of the following.

THEOREM 4.1. *If G is compact, then the mapping*

$$i : (C^*(L;V); d_\gamma) \to (\mathcal{A}(\Delta C^*(L); V); d_{i\gamma})$$

induces an isomorphism on homology.

The remainder of this section will be devoted to proving this result.

Define $\tilde{F} : C^*(L;V) \longrightarrow \mathcal{A}(\Delta G/G; V)$ by $\tilde{F}(\alpha \otimes v)(T) = (T^*\bar{\alpha}) \otimes v$ where $v \in V$, $T \in \Delta G, \alpha \in C^p(L)$, and $\bar{\alpha} \in \Omega^p(G)$ is the left invariant p-form defined by α. Define $\tilde{\gamma} = \tilde{F}\gamma \in \mathcal{A}^1(\Delta G/G; g\ell(V))$ where $\tilde{F}$ is defined as above with V replaced by $g\ell(V)$. Let

$$d_{\tilde{\gamma}} : \mathcal{A}^p(\Delta G/G; V) \to \mathcal{A}^{p+1}(\Delta G/G; V)$$

be given by $d_{\tilde{\gamma}}(u) = du + \tilde{\gamma} \wedge u$ where d is the usual differential on $\mathcal{A}^*(\Delta G/G)$ and $\tilde{\gamma} \wedge v$ is the wedge product defined using the pairing

$$\Omega^*(\Delta^q; g\ell(V)) \otimes \Omega^*(\Delta^q; V) \to \Omega^*(\Delta^q; V).$$

LEMMA 4.2. *For any $\alpha \in C^*(L;V), \tilde{F}d_\gamma(\alpha) = d_{\tilde{\gamma}}\tilde{F}(\alpha)$ and the diagram*

$$\begin{tikzcd} & \mathcal{A}(\Delta C^*(L); V) \arrow[dr, "\sigma^*"] & \\ C^*(L;V) \arrow[ur, "i"] \arrow[rr, "\tilde{F}"] & & \mathcal{A}(\Delta G/G; V) \end{tikzcd}$$

commutes.

PROOF: We first verify that the diagram commutes. It is enough to do this when $V = R$ in which case each of the mappings is an algebra homomorphism. Since $C^*(L)$ is generated by one dimensional elements, we need only show that $\sigma^* i(\alpha) = \tilde{F}(\alpha)$ for $\alpha \in L^* = C^1(L)$. If $T : \Delta^q \to G$, we have

$$\begin{aligned} \sigma^* i(\alpha)(T) &= i(\alpha)(\sigma(T)) \\ &= \sigma(T)(\alpha) = \alpha(T^{-1}dT) \end{aligned}$$

Now $T^{-1}dT$ is the L-valued 1-form on Δ^q given by $(T^{-1}dT)(u) = dL_{T(t)^{-1}}dT(u)$ where u is a tangent vector to Δ^q at $t \in \Delta^q$ and $L_{T(t)^{-1}} : G \to G$ is left translation by $T(t)^{-1}$. Thus, $\alpha(dL_{T(t)^{-1}}dT(u)) = \bar{\alpha}(dT(u))$ so that $\alpha(T^{-1}dT) = T^*(\bar{\alpha})$ and the diagram commutes.

To prove $\tilde{F}d_\gamma\alpha = d_{\tilde{\gamma}}\tilde{F}(\alpha)$, we note that $d\tilde{F} = \tilde{F}d$ so it is enough to show that $\tilde{F}(\gamma \wedge \alpha) = \tilde{\gamma} \wedge \tilde{F}(\alpha)$. If $\gamma = \beta \otimes A$ as an element of $C^1(L) \otimes g\ell(V) \simeq C^1(L; g\ell(V))$ and $\alpha = \alpha_1 \otimes v$ as an element of $C * (L) \otimes V \simeq C^*(L;V)$, then

$$\begin{aligned} \tilde{F}(\gamma \wedge \alpha)(T) &= \tilde{F}(\beta \wedge \alpha_1 \otimes A(v))(T) \\ &= T^*(\bar{\beta} \wedge \bar{\alpha}_1) \otimes A(v) \\ &= (\tilde{F}(\gamma) \wedge \tilde{F}(\alpha))(T) = \tilde{\gamma} \wedge \tilde{F}(\alpha)(T). \end{aligned}$$

The general case now follows from the fact that any $\gamma \in C^1(L; g\ell(V))$ and $\alpha \in C^*(L; V)$ are sums of elements of the form considered above.

Since σ is a simplicial homeomorphism, σ^* is an isomorphism of differential graded vector spaces and i will be a homology isomorphism if and only if $\tilde{F}$ is a homology isomorphism. To prove $\tilde{F}$ a homology isomorphism, we will define mappings which give the following commutative diagram of differential graded vector spaces:

$$\begin{tikzcd}
(C^*(L;V);d_\gamma) \arrow[r, "\tilde{F}"] \arrow[d, "\mu"'] & (\mathcal{A}(\Delta G/G;V);d_{\tilde{\gamma}}) \arrow[d, "\tilde{\mu}"] \\
(\Omega^*(G;V)^G;d) \arrow[r, "\bar{F}"] \arrow[d, shift left, "j"] & (\mathcal{A}(\Delta G;V)^G;d) \arrow[d, shift left, "\tilde{j}"] \\
(\Omega^*(G;V);d) \arrow[r, "F"] \arrow[u, shift left, "r"] & (\mathcal{A}(\Delta G;V);d) \arrow[u, shift left, "\tilde{r}"]
\end{tikzcd} \tag{4.3}$$

Here, j and $\tilde{j}$ are inclusion mappings.

PROPOSITION 4.4. *Suppose that, in the commutative diagram (4.3), μ and $\tilde{\mu}$ are isomorphisms, F is a homology isomorphism, $rj = id$, and $\tilde{r}\tilde{j} = id$. Then $\tilde{F}$ is a homology isomorphism.*

PROOF: It is clearly enough to prove $\bar{F}$ a homology isomorphism. Now, $rj = id$ implies that j_* is injective. Thus $(Fj)_*^{-1}(\tilde{j}\bar{F})_*$ is injective and it follows that $\bar{F}_*$ is injective. To prove $\bar{F}_*$ surjective, consider $u \in H_*(\mathcal{A}(\Delta G;V)^G;d)$ and let $v = r_*F_*^{-1}\tilde{j}_*u$. Then $\bar{F}_*(v) = u$ so $\bar{F}_*$ is surjective and thus an isomorphism.

We now proceed to define the mappings in diagram (4.3) and prove that the hypotheses of Proposition 4.4 are satisfied. We begin by defining an action of G on $\Omega^*(G;V)$ and on $\mathcal{A}(\Delta G;V)$ which give the middle row of (4.3).

The Lie algebra homomorphism $\gamma : L \to g\ell(V)$ determines a unique Lie group homomorphism $\Gamma : G \to GL(V)$ with $\gamma = d\Gamma : L = TG_e \to TGL(V)_e = g\ell(V)$. (Recall that G is assumed simply connected.) Define actions of G on $\Omega^*(G;V)$ and on $\mathcal{A}(\Delta G;V)$ as follows. For $g \in G, w \in \Omega^*(G;V)$, let $gw \in \Omega^p(G;V)$ be given by $gw = \Gamma(g)(L^*_{g^{-1}}w)$. Similarly, for $u : \Delta G \to \Omega^p(\Delta^q;V) \in \mathcal{A}^p(\Delta G;V), g \in G$, and $T \in (\Delta G)_q$, let gu be the element of $\mathcal{A}^p(\Delta G;V)$ given by $(gu)(T) = \Gamma(g)u(g^{-1}T)$. Then $\Omega^*(G;V)^G$ and $\mathcal{A}(\Delta G;L)^G$ denote the cochain complexes of elements fixed under the action of G with the standard differential d. It is straightforward to show that $d(gu) = g(du)$ for $u \in \Omega^*(G;V)$ or $u \in \mathcal{A}(\Delta G; V)$.

Define $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G; V)$ by $F(w)(T) = T^*w$ for $w \in \Omega^*(G;V), T \in \Delta G$. Then $F(dw) = dF(w)$ is immediate and

$$\begin{aligned} F(gw)(T) &= T^*(gw) \\ &= T^*\Gamma(g)L^*_{g^{-1}}w \\ &= (id \otimes \Gamma(g))(T^* \otimes id)L^*_{g^{-1}}w \\ &= \Gamma(g)(L_{g^{-1}}T)^*w = (gF(w))(T). \end{aligned}$$

Thus F induces $\bar{F} : (\Omega^*(G,V)^G, d) \to (\mathcal{A}(\Delta G; V)^G, d)$.

Define mappings

$$r : \Omega^*(G;V) \to \Omega^*(G;V) \quad \tilde{r} : \mathcal{A}(\Delta G; V) \to \mathcal{A}(\Delta G; V)$$

by

$$r(w) = \int_G gw, \quad \tilde{r}(u)(T) = \int_G (gu)(T)$$

for $w \in \Omega^*(G;V), u \in \mathcal{A}(\Delta G; V), T \in \Delta G$, and the integral is the Haar integral on G normalized so that the volume of G is one.

PROPOSITION 4.5. *For any $g \in G, w \in \Omega^*(G;V), u \in \mathcal{A}^1(\Delta G; V)$, we have $g(r(w)) = r(w)$ and $g(\tilde{r}(u)) = \tilde{r}(u)$. Furthermore, $dr = rd, d\tilde{r} = \tilde{r}d$, and $\tilde{r}F = \bar{F}r$.*

The proof of this proposition is straightforward. For example, to prove that $\bar{F}r = \tilde{r}F$, we simply use the definitions:

$$\begin{aligned} (\bar{F}r(w))(T) &= T^*r(w) \\ &= T^* \int_G \Gamma(g)L^*_{g^{-1}}w \\ &= \int_G \Gamma(g)T^*L^*_{g^{-1}}w \\ &= \int_G \Gamma(g)(L_{g^{-1}}T)^*w \\ &= \int \Gamma(g)F(w)(L_{g^{-1}}T) = (\tilde{r}F(w))(T). \end{aligned}$$

It follows from Proposition 4.5 that

$$r(\Omega^*(G;V)) \subset \Omega^*(G;V)^G, \quad \tilde{r}(\mathcal{A}(\Delta G; V)) \subset \mathcal{A}(\Delta G; V)^G.$$

Thus, we have established the existence of the lower rectangle of mappings in (4.3).

To obtain the upper rectangle in (4.3), we need first of all to define a second action of G on $\Omega^*(G;V)$ and $\mathcal{A}(\Delta G;V)$. For $w \in \Omega^*(G,V)$ and $g \in G$, let $g * w$ be the element of $\Omega^*(G;V)$ given by

$$g * w = L^*_{g^{-1}} w.$$

Similarly, for $u \in (\Delta G;V)$, let $g * u \in \mathcal{A}(\Delta G;V)$ be defined by

$$(g * u)(T) = u(g^{-1}T)$$

for $T \in \Delta G$. Then $d(g * w) = g * dw$ for $w \in \Omega^*(G;V)$ or $w \in \mathcal{A}(\Delta G;V)$ and we let $\Omega^*(G;V)^{G*}$ and $\mathcal{A}(\Delta G;V)^{G*}$ denote the subspaces of elements fixed under these actions of G. Of course, $C^*(L;V)$ can be identified with $\Omega^*(G;V)^{G*}$. The next result gives the corresponding identification for $\mathcal{A}(\Delta G;V)$.

LEMMA 4.6. *The natural mapping $p : \Delta G \to \Delta G/G$ induces an isomorphism*

$$p^* : \mathcal{A}(\Delta G/G;V), d_{\tilde{\gamma}}) \to (\mathcal{A}(\Delta G;V)^{G*}, d_{\tilde{\gamma}})$$

of differential graded vector spaces.

The proof is trivial.

In order to define the mappings μ and $\tilde{\mu}$ of diagram (4.3), we first define related mappings. Let

$$\eta : \Omega^*(G;V) \to \Omega^*(G;V), \quad \tilde{\eta} : \mathcal{A}(\Delta G;V) \to \mathcal{A}(\Delta G;V)$$

be defined by $\eta(\omega) = \Gamma \cdot \omega$, $\tilde{\eta}(u)(T) = (\Gamma \circ T) \cdot u(T)$.

LEMMA 4.7. *The mappings η and $\tilde{\eta}$ are bijective. Furthermore, we have $\eta(g * \omega) = g\eta(\omega)$ and $\tilde{\eta}(g * u) = g\tilde{\eta}(u)$ for any $g \in G$, $\omega \in \Omega^*(G;V)$, and $u \in \mathcal{A}(\Delta G;V)$.*

PROOF: The inverses to η and $\tilde{\eta}$ are defined just as η and $\tilde{\eta}$ are defined using $\Gamma^{-1} : G \to GL(V), \Gamma^{-1}(g) = \Gamma(g)^{-1}$, in place of Γ. The verification of the two equations of Lemma 4.7 are similar; we do only the second, leaving the first to the reader. Thus, for $g \in G, u \in \mathcal{A}(\Delta G;V)$, and $T \in \Delta G$, we have

$$\begin{aligned}(g\tilde{\eta}(u))(T) &= \Gamma(g)\tilde{\eta}(u)(g^{-1}T)\\ &= \Gamma(g)(\Gamma \circ g^{-1}T)u(g^{-1}T)\\ &= (\Gamma(g)\Gamma(g^{-1})\Gamma \circ T)u(g^{-1}T)\\ &= (\Gamma \circ T)u(g^{-1}T) = \tilde{\eta}(g * u)(T).\end{aligned}$$

As indicated above, any element $\alpha \in C^*(L;V)$ can be considered as a left invariant form $\bar{\alpha} \in \Omega^*(G;V)$. In particular, $\gamma \in C^1(L;g\ell(V))$ determines $\bar{\gamma} \in \Omega^1(G;g\ell(V))$ where $\gamma(X) = \gamma(dL_{g^{-1}}X)$ for $X \in TG_g$. Define

$$d_{\bar{\gamma}} : \Omega^*(G;V) \to \Omega^*(G;V)$$

by $d_{\bar{\gamma}}w = dw + \bar{\gamma} \wedge w$. Define $\tilde{\gamma} \in \mathcal{A}(\Delta G; g\ell(V))$ by $\tilde{\gamma} = F\bar{\gamma}$ where $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G; V)$ is given by $F(w)(T) = T^*w$, (see Proposition 4.5), and let

$$d_{\tilde{\gamma}} : \mathcal{A}(\Delta G; V) \to \mathcal{A}(\Delta G; V)$$

be defined by $d_{\tilde{\gamma}}u = du + \tilde{\gamma} \wedge u$. Then $d_{\bar{\gamma}}^2 = 0$ and $d_{\tilde{\gamma}}^2 = 0$ and we have

LEMMA 4.8. *For $w \in \Omega^*(G;V)$ and $u \in \mathcal{A}(\Delta G; V)$, we have $d\eta(w) = \eta(d_{\bar{\gamma}}w)$ and $d\tilde{\eta}(u) = \tilde{\eta}(d_{\tilde{\gamma}}u)$*

PROOF: Again, the verifications of the two equations are similar so we carry out only the proof of the second equation. Thus,

$$\begin{aligned}(d\tilde{\eta}(u))(T) &= d(\tilde{\eta}(u)(T)) \\ &= d((\Gamma \circ T)u(T)) \\ &= \Gamma \circ T(du(T) + \Delta(\Gamma \circ T) \wedge u(T))\end{aligned}$$

by Lemma 3.3. Now, if $X \in T\Delta_t^q$, we have $\Delta(\Gamma \circ T)(X) = dL_{\Gamma(T(t)^{-1})} d\Gamma dT(X)$. Identifying $\gamma : L \to g\ell(V)$ with $d\Gamma : TG_e \to TGL(V)_e$, we have $d\Gamma = dL_{\Gamma(T(t))}\gamma dL_{T(t)^{-1}}$ (since Γ is a homomorphism) and

$$\begin{aligned}\Delta(\Gamma \circ T)(X) &= \gamma dL_{T(t)^{-1}} dT(X) \\ &= \bar{\gamma}(dT(X)) \\ &= F(\bar{\gamma})(X) = \tilde{\gamma}(X).\end{aligned}$$

Then

$$\begin{aligned}d(\tilde{\eta}(u))(T) &= (\Gamma \circ T)(du(T) + \tilde{\gamma} \wedge u(T)) \\ &= (\tilde{\eta}(d_{\tilde{\gamma}})u)(T).\end{aligned}$$

According to Lemmas 4.7 and 4.8, η and $\tilde{\eta}$ induce isomorphisms

$$\begin{aligned}\eta &: (\Omega^*(G;V)^{G*}; d_{\bar{\gamma}}) \to (\Omega^*(G;V)^G; d) \\ \tilde{\eta} &: (\mathcal{A}(\Delta G; V)^{G*}; d_{\tilde{\gamma}}) \to (\mathcal{A}(\Delta G; V)^G; d)\end{aligned}$$

of differential graded vector spaces. Let μ and $\tilde{\mu}$ be the composites

$$\begin{aligned}\mu &: (C^*(L;V); d_\gamma) \simeq (\Omega^*(G;V)^{G*}; d_{\bar{\gamma}}) \xrightarrow{\eta} (\Omega^*(G;V)^G; d) \\ \tilde{\mu} &: (\mathcal{A}(\Delta G/G; V); d_{\tilde{\gamma}}) \xrightarrow{p^*} (\mathcal{A}(\Delta G; V)^{G*}; d_{\tilde{\gamma}}) \xrightarrow{\tilde{\eta}} (\mathcal{A}(\Delta G; V)^G; d)\end{aligned}$$

If $\bar{F} : \Omega^*(G;V)^G \to \mathcal{A}(\Delta G; V)^G$ is the restriction of $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G; V)$, then diagram (4.3) is easily seen to be commutative.

We have now established the commutative diagram (4.3) of differential, graded vector spaces. Moreover, the mappings μ and $\tilde{\mu}$ are isomorphisms. Thus, according to Proposition 4.4, the proof of Theorem 1.2, part (ii), will be complete if we can show that F is a homology isomorphism. For this we need the following.

THEOREM 4.9. *Let X be a CW complex, ΔX the singular complex of X in the compact open topology, and ΔX^δ the singular complex of X in the discrete topology. Then the natural mapping $j : \Delta X^\delta \to \Delta X$ induces an isomorphism*

$$H^*(\Delta X) \xrightarrow{j^*} H^*(\Delta X^\delta) = H^*(X).$$

We are indebted to Graeme Segal for the proof of this result which we give in Section 7.

To see that $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G;V)$ induces an isomorphism on homology, consider the following commutative diagram

$$\begin{array}{ccc} \Omega^*(G;V) & \xrightarrow{F} & \mathcal{A}(\Delta G;V) \\ \tilde{\psi}\big\downarrow & & \big\downarrow\psi \\ C^*(\Delta G^\delta;V) & \xrightarrow{j^*} & C^*(\Delta G;V) \end{array}$$

Here all differentials are the ordinary untwisted differentials, ψ is defined in [**3**], Section 5, and $\tilde{\psi}$ is the usual deRham mapping. Now, ψ is a homology isomorphism by Theorem 2.4 of [**3**], j^* is a homology isomorphism by Theorem 4.9 above, and $\tilde{\psi}$ is well known to be a homology isomorphism. Thus F is a homology isomorphism and Theorem 1.2, part (ii) follows from Proposition 4.4.

5. Fibrations.

Suppose G is a connected, simply connected Lie group with Lie algebra L, $A \in \mathcal{CA}$, and $\lambda : C^*(L) \to A$ is a map in $\mathcal{CA}$. Let X be a graded vector space with basis $x_1, \dots, x_k$, deg $x_j = n, j = 1, \dots, n$, and let $\{\ell_i\}$ be a basis for L. Let $A[X] = A[x_1, \dots, x_k]$ be the free algebra over A on $x_1, \dots, x_k$ and suppose $A[X]$ has a differential d with $dA \subset A$ and such that

$$dx_i = \sum b_i^{jm} \lambda(\ell_j^*) x_m + c_i \tag{5.1}$$

where $b_i^{jm} \in R$, $c_i \in A^{n+1}$ and $\{\ell_i^*\}$ is the basis for L^* dual to $\{\ell_i\}$. The relation $d^2 x_i = 0$ yields

$$dc_i = \sum b_i^{jm} \lambda(\ell_j^*) c_m \tag{5.2}$$

Let X^* be the dual space of X, $\{x_i^*\}$ the basis for X^* dual to $\{x_i\}$, and define $\mu : L \otimes X^* \to X^*$ by

$$\mu(\ell_j \otimes x_m^*) = \sum b_i^{jm} x_i^*.$$

The equation $d^2 = 0$ implies that μ defines an action of L on X^* *as a Lie algebra.* Therefore, we have a corresponding action of G on X^*.

Let $\tilde{G} \in \Delta\mathcal{T}$ be given by $\tilde{G}_q = G$, $\delta_i = s_i = id$ for all q, i and set $P = \Delta R[X] \times \tilde{G}$. (Here, $R[X] \in \mathcal{CA}$ with $dX = 0$.) We make P into a simplicial topological group and define an action of P on $\Delta(R[X])$ by

$$(v, g)(v', g') = (v + g \cdot v', gg')$$
$$(v, g)\omega = v + g\omega$$

for $g, g' \in \tilde{G}_q = G$, $v, v', \omega \in \Delta(R[X])_q \subset \Omega^n(\Delta^q; X^*)$.

In [2], Section 5, we defined a map $\mu_0 : \Omega^p(\Delta^q) \to \Omega^{p-1}(\Delta^q)$ satisfying $d\mu_0 + \mu_0 d = id$, $\mu_0 s_j = s_j \mu_0$ for $j \geq 0$, and $\mu_0 \partial_i = \partial_i \mu_0$ for $i > 0$. We extend this map to a mapping

$$\mu_0 : \Omega^p(\Delta^q; X^*) \to \Omega^{p-1}(\Delta^q; X^*)$$

with these same properties by $\mu_0(\omega \otimes x) = \mu_0(\omega) \otimes x$ and define $c : X \to A^{n+1}$ by $c(x_i) = c_i$.

THEOREM 5.3. *The simplicial space $\Delta(A[X])$ is a twisted cartesian product $\Delta(A) \times_\tau \Delta(R[X])$ with group P and twisting function $\tau : \Delta A_q \to P_{q-1}$ given by*

$$\tau(u) = (\mathcal{O}(u \circ \lambda)(v_1)^{-1}((\partial_0\mu_0 - \mu_0\partial_0)(\mathcal{O}(u \circ \lambda)u \circ c),\ \mathcal{O}(u \circ \lambda)(v_1)^{-1})$$

for $u \in \Delta A_q = (A, \Omega_q)$ and v_1 is the second vertex of Δ^q. Here $\mathcal{O}$ is defined in Section 3 to be the composite $\Delta C^*(L) \xrightarrow{\sigma^{-1}} \Delta G/G \xrightarrow{\beta} \Delta G$ where $\beta(T) = T(v_0)^{-1}T$.

PROOF: We identify (X, Ω_q^n), the space of linear mappings from X to Ω_q^n, with $\Omega^n(\Delta^q; X^*)$ by $v \longmapsto \sum v(x_i)x_i^*$, $v : X \to \Omega_q^n$. If $u \in \Delta(A)_q$, then $u \circ c \in (X, \Omega_q^{n+1}) = \Omega^{n+1}(\Delta^q; X^*)$ and $u \circ \lambda \in \Delta(C^*(L)) \subset \Omega^1(\Delta^q; L)$. Thus

$$\begin{aligned}
\Delta(A[X])_q &= (A[X], \Omega_q) \\
&= \{(u, v) \in \Delta(A)_q \times (X, \Omega_q^n) \mid dv(x_i) = \Sigma b_i^{jm}(u \circ \lambda(\ell_j^*))v(x_m) + u(c_i)\} \\
&= \{(u, v) \in \Delta(A)_q \times \Omega^n(\Delta^q; X^*) \mid dv = (u \circ \lambda) \wedge v + u \circ c\}
\end{aligned}$$

Note that

$$\begin{aligned}
\Delta(R[X])_q &= (R[X], \Omega_q) \\
&= \{v : X \to \Omega_q \mid dv = 0\} \\
&= \{v \in \Omega^n(\Delta^q; X^*) \mid dv = 0\}.
\end{aligned}$$

Define $f : \Delta(A)_q \times \Delta(R[X])_q \to \Delta(A[X])_q$ by $f(u, v) = (u, v')$ where

$$v' = \mathcal{O}(u \circ \lambda)^{-1}(\mu_0(\mathcal{O}(u \circ \lambda)u \circ c) + v)$$

where $\beta([T]) = T(v_0)^{-1}T$. In order to insure that $(u, v') \in \Delta(A[X])_q$, we need to show that $dv' = (u \circ \lambda) \wedge v' + u \circ c$. If $\mathcal{O}_0 = \mathcal{O}(u \circ \lambda) \in (\Delta G)_q$, then

$$\begin{aligned} dv' &= d\mathcal{O}_0^{-1} \cdot (\mu_0(\mathcal{O}_0 \cdot u \circ c) + v) + \mathcal{O}_0^{-1} \cdot (d\mu_0(\mathcal{O}_0 \cdot u \circ c) + dv) \\ &= -\mathcal{O}_0^{-1} \cdot d\mathcal{O}_0 \cdot \mathcal{O}_0^{-1}(\mu_0(\mathcal{O} \cdot u \circ c) + v) + \mathcal{O}_0^{-1} \cdot (\mathcal{O}_0 \cdot u \cdot c - \mu_0(d(\mathcal{O}_0 \cdot u \circ c))) \\ &= -\mathcal{O}_0^{-1} d\mathcal{O}_0 \wedge v' + u \circ c - \mathcal{O}_0^{-1} \cdot \mu_0(d(\mathcal{O}_0 \cdot u \circ c)) \end{aligned}$$

since $d\mathcal{O}_0^{-1} = -\mathcal{O}_0^{-1} d\mathcal{O}_0 \mathcal{O}_0^{-1}$.

We now need the following results.

If $u \circ \lambda \in \Delta C^*(L)_q$ is considered an element of $\Omega_q^1(L)$ (as in the discussion following Theorem 2.10), we have

LEMMA 5.4. $d\mathcal{O}_0 = -\mathcal{O}_0 \cdot u \circ \lambda$.

PROOF: Identifying $\Delta C^*(L)_q$ with $\hat{\Omega}^1(\Delta^q; L)$, we have

$$\begin{aligned} -\mathcal{O}_0^{-1} d\mathcal{O} &= \rho(\mathcal{O}_0) \\ &= \rho(\mathcal{O}(u \circ \lambda)) = u \circ \lambda \end{aligned}$$

by the definition of the mappings involved.

COROLLARY. $\mathcal{O}_0^{-1} d\mathcal{O}_0 = -u \circ \lambda$.

LEMMA 5.5. $d(u \circ c) = u \circ \lambda \wedge u \circ c$.

PROOF: The element in $\Omega^{n+1}(\Delta^q; X^*)$ corresponding to $u \circ c$ is $\Sigma(u \circ c)(x_i)x_i^* = \Sigma u(c_i)x_i^*$. Thus

$$\begin{aligned} d(u \circ c) &= d\Sigma u(c_i)x_i^* \\ &= \Sigma u(dc_i)x_i^* \\ &= \Sigma b_i^{jm} u \circ \lambda(\ell_j^*)u(c_m)x_i^* \\ &= (\Sigma u \circ \lambda(\ell_j^*)\ell_j)(\Sigma u \circ c(x_m)x_m^*) \\ &= u \circ \lambda \wedge u \circ c. \end{aligned}$$

COROLLARY. $d(\mathcal{O}_0 \cdot u \circ c) = 0$.

PROOF:

$$\begin{aligned} d(\mathcal{O}_0 \cdot u \circ c) &= (d\mathcal{O}_0) \wedge u \circ c + \mathcal{O}_0 \cdot du \circ c \\ &= -\mathcal{O}_0 \cdot u \circ \lambda \wedge u \circ c + \mathcal{O}_0 \cdot u \circ \lambda \wedge u \circ c = 0. \end{aligned}$$

It follows from the two corollaries above that dv' has the required form.

Define $f^{-1} : \Delta(A[X])_q \to \Delta(A)_q \times \Delta R[X]_q$ by $f^{-1}(u, v') = (u, v)$ where

$$v = \mathcal{O}(u \circ \lambda)v' - \mu_0(\mathcal{O}(u \circ \lambda) \cdot u \circ c).$$

It is easy to check that f^{-1} is actually an inverse for f. We now determine the twisting function τ.

For $(u, v) \in \Delta A_q \times \Delta R[X]_q$, set $\partial_0(u, v) = (\partial_0 u, \bar{v})$. Then $f(\partial_0(u, v)) = (\partial_0 u, \bar{v}')$ where

$$\bar{v}' = \mathcal{O}(\partial_0(u \circ \lambda))^{-1}(\mu_0(\mathcal{O}(\partial_0(u \circ \lambda)) \cdot \partial_0(u \circ c)) + \bar{v})$$

Furthermore, $\partial_0 f(u, v) = (\partial_0 u, \partial_0 v')$ where

$$\partial_0 v' = \partial_0 \mathcal{O}(u \circ \lambda)^{-1}(\partial_0 \mu_0(\mathcal{O}(u \circ \lambda) \cdot u \circ c) + \partial_0 v)$$

Thus, if $\partial_0 v' = \bar{v}$, we have

$$\bar{v} = \mathcal{O}(\partial_0(u \circ \lambda)) \cdot \partial_0 \mathcal{O}(u \circ \lambda)^{-1}(\partial_0 \mu_0(\mathcal{O}(u \circ \lambda) \cdot u \circ c) + \partial_0 v) - \mu_0(\mathcal{O}(\partial_0(u \circ \lambda)) \cdot \partial_0(u \circ c))$$

It is easy to see that $g \cdot \mu_0(\omega) = \mu_0(g \cdot \omega)$ and $\mathcal{O}(\partial_0 \alpha) = \mathcal{O}(\alpha)(v_1)^{-1}\partial_0 \mathcal{O}(\alpha)$ for $g \in G$, $\omega \in \Omega^*(\Delta^q; X^*)$, and $\alpha \in \Delta C^*(L)$. It follows that

$$\bar{v} = \mathcal{O}(u \circ \lambda)(v_1)^{-1}((\partial_0 \mu_0 - \mu_0 \partial_0)(\mathcal{O}(u \circ \lambda) \cdot u \circ c) + \mathcal{O}(u \circ \lambda)(v_1)^{-1}\partial_0 v)$$

so that

$$\tau(u) = (\mathcal{O}(u \circ \lambda)(v_1)^{-1}((\partial_0 \mu_0 - \mu_0 \partial_0)(\mathcal{O}(u \circ \lambda) \cdot u \circ c), \mathcal{O}(u \circ \lambda)(v_1)^{-1}).$$

The verification that $\partial_i f = f \partial_i$ for $i > 0$ and $s_i f = f s_i$ is routine and left to the reader.

We conclude this section with a proof of Theorem 1.3. The proof of Theorem 5.3 of this paper can easily be extended to show that

$$\mathcal{F}(R[X], B) \to \mathcal{F}(A[X], B) \to \mathcal{F}(A, B) \tag{5.6}$$

is a twisted cartesian product and hence a fibration in $\Delta\mathcal{T}$. (Theorem 5.3 corresponds to the case $B = R$.) For example, identifying $(X, \Omega_q \otimes B)$ with $\Omega(\Delta^q; X^*) \otimes B)^n$, we have

$$\mathcal{F}(A[X, B)_q = \{(u, v) \in \mathcal{F}(A[X], B) \times (\Omega(\Delta^q; X^*) \otimes B)^n \mid dv = u\lambda v + uc\}.$$

The pullback of (5.6) to $\mathcal{F}(A, B; h) \subset \mathcal{F}(A, B)$ yields a fibration and a commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}(R[X], B) & \longrightarrow & \mathcal{F}(A[X], B; h) & \longrightarrow & \mathcal{F}(A, B; h) \\
\Big\downarrow{\scriptstyle \Delta_1} & & \Big\downarrow & & \Big\downarrow \\
\mathcal{F}(\Delta B, \Delta R[X]) & \longrightarrow & \mathcal{F}(\Delta B, \Delta(A[X]); \Delta h) & \longrightarrow & \mathcal{F}(\Delta B, \Delta A; \Delta h)
\end{array}$$

If we write $A = \cup A_n$, then induction, the diagram above, and the fact that Δ_1 is a weak equivalence gives the theorem for $A = A_n$. A limit argument as in the proof of Theorem 2.20 of **[3]** yields the desired result.

6. The Continuous Cohomology Serre Spectral Sequence with Local Coefficients

Suppose that $E = B \times_\tau F$ is a twisted cartesian product in $\Delta\mathcal{T}$ with group P and suppose $\pi_0(P) = P_0$. In **[3]**, Section 8, we constructed a local system $\tau = \tau \mid B_1 \to P_0$, on action of P_0 on $C^*(F; R)$, and a map

$$\Delta^* : C^*(B; C^*(F; R)) \to C^*(B \times_\tau F; R)$$

which was filtration preserving with respect to the obvious filtrations. In general, Δ^* is not a cochain mapping (relative to the usual differential δ on $C^*(B \times_\tau F; R)$ and the twisted differential δ_τ on $C^*(B; C^*(F; R))$) but it does in fact induce an isomorphism on $E_r^{p,q}$ for $r \leq 2$. (See **[3]**, Section 8.) Furthermore, if F is splittable, this map gives an isomorphism

$$E_2^{p,q}(B \times_\tau F) \simeq H^*(B; H^*(F; R)).$$

Suppose now that L, G, and V are as in Section 3 and $t : B_1 \to G$ is a local system. If, in the above paragraph, one replaces R by V, δ on $C^*(E; V)$ by $\delta_{tp}, p : E \to B$, and δ_τ by $\delta_{\bar{\tau}}, \bar{\tau} : B_1 \to P_0 \times G, \bar{\tau}(b) = (\tau(b), t(b))$, then the statements remain true with the same proofs as in Section 8 of **[3]**. Hence we have

THEOREM 6.1. *If F is splittable, then the Serre spectral sequence for $H^*(B \times_\tau F; V_{tp})$ converges in the usual way and Δ^* induces an isomorphism*

$$E_2^{p,q} \simeq H^p(B; H^q(F; V)_{\bar{\tau}}).$$

We next apply Theorem 6.1 to $\Delta(A[X]) = \Delta(A) \times_\tau \Delta(R[X])$. Let $\lambda : C^*(L) \to A$ be a map in $\mathcal{CA}$ and recall that

$$i = i_A : A \otimes V \to (\Delta(A)) \otimes V$$

is given by $i(a \otimes v) = i(a) \otimes v$ where $i(a)(f) = f(a)$.

LEMMA 6.2. *If i_A induces an isomorphism*

$$i_A : H_*(A \otimes V, d_\lambda) \xrightarrow{\simeq} H_*(\mathcal{A}(\Delta(A)), d_\phi);$$

then the same is true for $i_{A[X]} : A[X] \otimes V \to \mathcal{A}^*(\Delta(A[X])) \otimes V$.

PROOF: Let $\bar{\psi} = K^{-1}\tilde{\psi}H : \mathcal{A}(\Delta(A); V) \to C^*(\Delta A; V)$ be the mapping defined in Section 3 (see Theorem 3.5) and let

$$\bar{i} = \bar{\psi} i : A \otimes V \to C^*(\Delta A; V).$$

It is sufficient to prove Lemma 6.2 with i replaced by $\bar{i}$ and d_ϕ by d_t. Define a filtration on $A[X] \otimes V = A \otimes R[X] \otimes V$ by

$$F^p = \{a \otimes w \otimes v \mid \dim a \geq p\}.$$

Exactly as in [**3**], Lemma 9.4, one checks that $\bar{i}$ is filtration preserving and hence induces a mapping of the corresponding spectral sequences.

We prove Lemma 6.2 by showing that $\bar{i}$ for $A[X]$ induces an isomorphism at the E_2 level. As in [**3**], Section 9, the $\hat{E}_1$ term for $A \otimes R[X] \otimes V$ is

$$\hat{E}_1^{p,q} = A^p \otimes R[X]^q \otimes V$$

and for $\Delta(A[X])$,

$$E_1^{p,q} = C^p(\Delta(A), H^q(\Delta(R[X]; V))).$$

In both cases, d_2 is the appropriate local coefficient differential. Furthermore,

$$\bar{i} : R[X] \otimes V \to C^*(\Delta(R[X]); V)$$

induces an isomorphism on homology, this being the untwisted version of our theorem which we proved as Proposition 2.8 in [**3**]. By hypothesis

$$\bar{i}_{R[X]} \otimes \bar{i}_A : \hat{E}_2 \to E_2$$

induces an isomorphism. Thus we must show that $\bar{i}_{A[X]}$ induces this map. The map induced by $\bar{i}_{A[X]}$ is the composition

$$\begin{aligned} A[X] \otimes V &\xrightarrow{\bar{i}} C^*(\Delta(A[X]), V) \xrightarrow{f^*} C^*(\Delta(A) \times_\tau \Delta(R[X])) \\ &\xrightarrow{\eta} \sum C^p(\Delta(A), C^{*-p}(\Delta(R[X]), V)) \end{aligned}$$

where η is induced by the usual Eilenberg-Zilber map $C_p(X) \otimes C_q(X) \to C_{p+q}(X)$ involving shuffles of degeneracy maps.

For any $X, x \in X$, and $w \in \Omega^n(X;V)$,

$$\begin{aligned}\bar{\psi}\omega(x) &= K^{-1}\mathcal{O}\psi H\omega(x)\\ &= \tilde{\psi}H\omega(x,\mathcal{O}(\phi(x)))\\ &= \int_{\Delta^n} H\omega(x,\mathcal{O}(\phi(x)))\\ &= \int_{\Delta^n}\mathcal{O}(\phi(x)\omega(x)).\end{aligned}$$

For $u \in \Delta^n(A), v \in \Delta_q(R[X])$, and (α,β) a (p,q) shuffle,

$$f(s_\alpha u, s_\beta v) = (s_\alpha u, s_\alpha(\mathcal{O}(u\lambda)^{-1}\mu_0\mathcal{O}(u\lambda)uw) + s_\alpha\mathcal{O}(u\lambda)^{-1}s_\beta v$$

The first terms in each factor are s_α degenerate, thus will be s_α degenerate when evaluated on an element of $A[X]\otimes V$ and hence will drop out when we integrate. Suppose $a \in A^p, e \in R[X]^q$ and $z \in V$. Then

$$\bar{i}f^*\eta(a\otimes e\otimes z)(u)(v) = \sum_{\alpha,\beta}\pm\int_{\Delta^{p+q}} s_\alpha\mathcal{O}(u\lambda)(s_\alpha u(a)(s_\alpha\mathcal{O}(u\lambda)^{-1}s_\beta\hat{v}(e))z$$
$$= \int_{\Delta^p}\mathcal{O}(u\lambda)(z)u(a)\mathcal{O}(u\lambda)^{-1}\int_{\Delta^q}\hat{v}(e)$$

where $\hat{v} \in (R[X],\Omega)$ is defined by $v \in \mathcal{A}(\Delta_q;X)$. Similarly

$$i\bar{\psi}i_A : A\otimes R[X]\otimes V \to C^*(\Delta(A), C^*(\Delta R[X],V))$$

is given as follows: Note first that the group in question is $\pi_0(G\times P) = G\times G$ which acts on $R[X]\otimes V$ by

$$(g_1,g_2)(e\otimes z) = (eg_2, g_1z).$$

Since G acts on the left of X^*, it acts on the right of $R[X]$ and on the left of $\Delta(R[X])$. For $u \in \Delta(A)$ the $\mathcal{O}$ in this case is $(\mathcal{O}(u\lambda),\mathcal{O}(u\lambda)^{-1})$ (see Section 5). Hence

$$\bar{\psi}i_A(a\otimes e\otimes z)(u) = \int_{\Delta^p} e\mathcal{O}(u\lambda)^{-1}\otimes\mathcal{O}(u\lambda)u(a)z$$

and

$$i\bar{\psi}i_A(a\otimes e\otimes z)(u)(v) = \int_{\Delta^q}\int_{\Delta^p}(\mathcal{O}(u\lambda)^{-1}v)(e)\mathcal{O}(u\lambda)u(a)z$$

where $\mathcal{O}(u\lambda) \in \Omega^0(\Delta^p;G), u(a) \in \Omega^p(\Delta^p), v(e) \in \Omega^q(\Delta^q)$. Comparing this with
$\bar{i}f^*\eta(a\otimes e\otimes z)(u)(v)$ we see they are equal and the lemma is proved.

We conclude this section with a proof of Theorem 1.2, (iv). Let $A \in \mathcal{CA}$ be free and finite type so that $A = \cup A^{(n)}, A^{(n)} = A^{(n-1)}[X_n], A_0 = R$. By Theorem 1.2, (ii), i induces an isomorphism

$$H_*(A^{(1)}; V_\lambda) \simeq H_*(\mathcal{A}(\Delta(A^{(1)}) \otimes V_{i\lambda}).$$

By Lemma 6.2 and induction on n, i induces an isomorphism

$$H_*(A^{(n)} \otimes V, d_\lambda) \approx H_*(\mathcal{A}(\Delta(A^{(n)})) \otimes V, d_\lambda)$$

and hence the same holds for A.

7. The Proof of Theorem 4.9.

We give here the proof of Theorem 4.9 which was communicated to us by G. Segal.

Let X be a paracompact space and $\Delta(X)$ the singular complex of X in the compact open topology. Thus, $H^*(\Delta X)$ is the continuous cohomology of the simplicial space $\Delta(X)$ and $H^*(\Delta(X)^\delta)$ is the singular cohomology of X. Let $\mathcal{U}$ be a covering of X which has a partition of unity $\{\lambda_U; U \in \mathcal{U}\}$ subordinate to it. Let $\Delta(X,\mathcal{U}) \subset \Delta(X)$ be the simplicial subspace consisting of those $T : \Delta^q \to X$ with $T(\Delta^q) \subset U$ for some $U \in \mathcal{U}$.

LEMMA 7.1. *The inclusion mapping $\Delta(X,\mathcal{U}) \subset \Delta(X)$ induces an isomorphism*

$$H^*(\Delta(X,\mathcal{U})) \xrightarrow{\simeq} H^*(\Delta(X))$$

on continuous cohomology.

PROOF: Let $C_q(X)$ be the singular chains on X with integer coefficients, $sd : C_q(X) \to C_q(X)$ be the usual subdivision mapping, and $D : C_q(X) \to C_{q+1}(X)$ the chain homotopy with $\partial D + D\partial = sd - id$. The maps sd and D are natural and obtained from the first barycentric subdivision of Δ^q. Let sd^n be the n^{th} iterate of $sd, id_q : \Delta^q \to \Delta^q$ the identity map considered as an element of $\Delta_q(\Delta^q)$, and write

$$sd^n(id_q) = \sum_i \pm\tau_i^n$$

where $\tau_i^n \in \Delta_q(\Delta^q)$. Note that the diameter of $\tau_i^n(\Delta^q)$ approaches 0 as n approaches infinity.

Let $\rho : R \to R$ be a smooth non decreasing function with $\rho(x) = 0$ for $x \le 0$ and $\rho(x) = 1$ for $x \ge \frac{1}{2}$. Define continuous functions $\varphi^n : \Delta(X) \to R$ by

$$\varphi^n(T) = \rho(\min_i \sum_{U \in \mathcal{U}} \min_{t \in \Delta^q}(\lambda_U(T(\tau_i^n(t)))))$$

The following may be verified by inspection.

LEMMA 7.2. *The functions $\varphi^n : \Delta(X) \to R$ satisfy the following.*

(i) *φ^n is continuous.*

(ii) *For each $T \in \Delta(X)$, there is an integer $N = N_T$ with $\varphi^n(T) = 1$ for $n \geq N$.*

(iii) *If $\varphi^n(T) \neq 0$, then there is a $U \in \mathcal{U}$ with*

$$T(\tau_i^n(\Delta^q)) \subset supp\ \lambda_U \subset U$$

for all i.

Remark. The functions $\psi_n = \varphi_i - \varphi_{n-1}, \psi_0 = \varphi_0$, define a partition of unity subordinate to the covering $\{\Delta^n(X,\mathcal{U}), n \geq 0\}$ where

$$\Delta^n(X,\mathcal{U}) = \{T \in \Delta(X) \mid sd^n T \in C_q(\Delta(X,\mathcal{U}))\}.$$

Let $sd^* : C^q(\Delta(X)) \to C^q(\Delta(X))$ and $D^* : C^q(\Delta(X)) \to C^{q-1}(\Delta(X))$ be induced by sd and D. We show $H^q(\Delta(X)) \approx H^q(\Delta(X;\mathcal{U}))$. Suppose $u \in C^q(\Delta(X)), \delta u = 0$ and $v \in C^{q-1}(\Delta(X;\mathcal{U}))$ with $\delta v = u$ on $\Delta(X;\mathcal{U})$. We define $v_n \in C^{q-1}(\Delta^n(X;\mathcal{U}))$ with $\delta v_n = u$ on $\Delta^n(X;\mathcal{U})$ by induction on n. Let $v_0 = v$ and define

$$v_{n+1} = sd^* v_n - D^* u.$$

Then, if $\delta v_n = u$,

$$\begin{aligned}\delta v_{n+1} &= sd^* u - (sd^* u - u - D^* \delta u) \\ &= u\end{aligned}$$

We modify the v_n so that they fit together to give an element of $C^{q-1}(\Delta(X))$. If $w \in C^q(\Delta^n(X;\mathcal{U}))$, define $\varphi^n w$ by $(\varphi^n w)(T) = \varphi^n(T) w(T)$. Since $\sup \varphi^n \subset \Delta^n(X,\mathcal{U})$, we have $\varphi^n w \in C^q(\Delta(X))$. Let $t \in C^{q-1}(\Delta(X))$ be defined by

$$t \mid \Delta^n(X;\mathcal{U}) = t_n = v_n - \delta(\sum_{i<n} \varphi^i D^* v_i - \sum_{i \geq n} (1 - \varphi^i) D^* v_i)$$

Since $\varphi^i(T) = 1$ for i large, the above sum makes sense. Note that $\delta t_n = u$ and that

$$\begin{aligned}\delta D^* v_i &= sdv_i - v_i - D^* U \\ &= v_{i+1} - v_i\end{aligned}$$

Hence, for $T \in \Delta^n(X;\mathcal{U})$ and N large,

$$t_n(T) = (v_{N+1} + \delta \sum_{i \leq N} \varphi^i D^* v_i)(T).$$

It follows that $t_{n+1}(T) = t_n(T)$, t is well defined and $\delta t = 0$.

Suppose $v \in C^q(\Delta(X;\mathcal{U}))$ and $\delta v = 0$. Let $U \in C^q(\Delta(X))$ be defined by

$$u \mid \Delta^n(X,\mathcal{U}) = u_n = sd^n v - \delta(\sum_{i<n} \varphi^i D^* sd^i v - \sum_{i\geq n}(1-\varphi^i)Dsd^i v)$$

Then for $T \in \Delta^n(X,\mathcal{U})$ and N large

$$u_n(T) = sd^{N+1}v - \delta(\sum_{i\leq N} \varphi^i D^* sd^i v)$$

Thus u is well defined and $u = u_0 = v - \delta z$ on $\Delta(X;\mathcal{U})$. This completes the proof of Lemma 7.1.

PROOF OF THEOREM 4.9: We verify that $H^*(\Delta(X))$ satisfies additivity and the

Eilenberg-Steenrod axioms on pairs of CW complexes. Homotopy, additivity and the dimension axiom are obvious. Excision follows from Lemma 7.1. To verify exactness one needs to show that if (X, A) is a CW pair, then $u \in C^q(\Delta(A))$ can be extended to $v \in C^q(\Delta(A))$. Let U be a neighborhood of A, $r : U \to A$ a retraction, and $\sigma : X \to [0,1]$ a mapping with support in U and with $f(a) = 1$ for $a \in A$. Define $v \in C^q(\Delta(X))$ by

$$v(T) = (\min_{y\in\Delta^q} \sigma(T(s)))u(r \otimes T)$$

This completes the proof of Theorem 4.9.

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THE FULLER INDEX AND $\mathbb{T}$-EQUIVARIANT STABLE HOMOTOPY THEORY

by M.C. CRABB

0. Introduction

In a remarkable paper [8], published more than twenty years ago, Fuller introduced an index which counts periodic orbits of smooth flows. Let w be a smooth vector field defined on a (finite-dimensional) closed manifold X and $\theta_t : X \to X$, $(t \in \mathbb{R})$, the corresponding flow (so that $\theta_0 = 1$ and $\dot{\theta}_t = w(\theta_t)$, where the dot denotes differentiation). Suppose that U_1 is an open subspace of $(0,\infty) \times X$ such that the set

$$(0.1) \qquad F = \{(T,x) \in U_1 \mid \theta_T x = x\}$$

is compact. To such a field w and open set U_1, Fuller associates a $\mathbb{Q}$-valued index, which vanishes if F is empty.

In 1985, Ize [10] and Dancer [6] observed, independently, that the natural setting for Fuller's index is $\mathbb{T}$-equivariant homotopy theory, $\mathbb{T}$ being the circle group $\mathbb{R}/\mathbb{Z}$. My purpose here is to describe their work from the viewpoint of algebraic topology using the standard methods of equivariant fixed-point theory over a base.

The relevance of the $\mathbb{T}$-equivariant theory is not hard to see. Indeed, if $(T,x) \in F$, (0.1), then the compactness of F implies that $(T,\theta_t x) \in F$ for all $t \in \mathbb{R}$ and, also, that x is not a stationary point of the flow ($w(x) \neq 0$). So we can define a fixed-point-free circle action on F by:

$$(0.2) \qquad [t].(T,x) = (T,\theta_{tT}x),$$

for $t \in \mathbb{R}$, $[t] = t + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. The Fuller index is, in a sense to be made precise, a count of this set F, with the fixed-point-free $\mathbb{T}$-action, over the base $(0,\infty)$.

Each point $(T,x) \in F$ determines a periodic solution $\gamma(t) = \theta_t x$, of period T, of the differential equation:

$$\dot{\gamma} - w(\gamma) = 0, \tag{0.3}$$

or, by re-scaling, a solution $\alpha: \mathbb{R} \to X$, $\alpha(t) = \theta_{tT}x$, of period 1 of:

$$\dot{\alpha} - Tw(\alpha) = 0. \tag{0.4}$$

It is convenient to make no distinction in notation between a map $\alpha: \mathbb{R} \to X$ of period 1 and the corresponding loop $\alpha: \mathbb{R}/\mathbb{Z} = \mathbb{T} \to X$. Then we can think of solutions of (0.4) as zeros of a vector field on the infinite-dimensional manifold $M = LX$ of smooth loops $\mathbb{T} \to X$ in the following way. (See, for example, Atiyah [1] and Bismut [3].)

Recall that the tangent space $\tau_\alpha M$ at a point $\alpha \in M$, $\alpha: \mathbb{T} \to X$, can be identified with the space of smooth sections of $\alpha^*\tau X$ over $\mathbb{T}$. So we can regard $t \mapsto w(\alpha(t))$ as a tangent vector $w(\alpha) \in \tau_\alpha M$, and the vector field w on X thus defines a vector field, of the same name, on M. The circle acts on M by rotating loops: $([t].\alpha)(u) = \alpha(t+u)$, for $t,u \in \mathbb{R}$. This $\mathbb{T}$-action has a generating vector field, s say, given by differentiation:

$$s(\alpha) = \dot{\alpha}. \tag{0.5}$$

The zero-set of s, or the fixed subspace $M^{\mathbb{T}}$, is the space X of constant loops.

Now we have a family $v_T = s - Tw$, $T > 0$, of $\mathbb{T}$-equivariant vector fields on M, parametrized by $(0,\infty)$, and the zero-set of v_T is precisely the set of solutions of (0.4). Let U_∞ be the open subset $\{(T,\alpha) \in (0,\infty) \times M \mid (T,\alpha(t)) \in U_1 \text{ for all } t \in \mathbb{R}\}$ of $(0,\infty) \times M$. Then the zero-set

$$\{(T,\alpha) \in U_\infty \mid v_T(\alpha) = 0\} \tag{0.6}$$

is equivariantly homeomorphic to F, (0.1) and (0.2), and so compact.

The problem is to define an index for such a family of vector fields v_T with compact zero-set in some open subspace of $(0,\infty) \times M$. There are technical difficulties in infinite-dimensions: in order to apply the Leray-Schauder theory (as described in [9], for example) it is necessary to replace v_T by a "renormalized" field satisfying a certain compactness condition. This analysis, which is joint work with A.J.B. Potter, will appear elsewhere. In this paper, following Dancer [6], I shall concentrate on the analogous finite-dimensional problem, which illustrates all the algebraic topological features of the Fuller index. This is done in Section

2. Section 1 reviews the, now standard, equivariant index theory over a base for zeros of vector fields and fixed-points of maps, developed by Dold, Becker and Gottlieb in the mid seventies.

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1. The vector-field index

This section contains an outline, in a form tailored to the applications, of the Poincaré-Hopf index theory for vector fields. Whilst this theory can be viewed as a special case of the Lefschetz fixed-point theory, it seems worth maintaining a conceptual distinction. We confine the discussion to the non-equivariant theory. The modifications needed to produce the G-equivariant index theory, for a compact Lie group G (acting smoothly on manifolds), are technical rather than conceptual. The treatment here is strongly influenced by the work of Dold (as in [7] and the references there). A detailed account can be found in [12].

Consider first a (continuous) vector field v defined on an open subset U of a (finite-dimensional) Euclidean space V, and suppose that the zero-set

$$\mathrm{Zero}(v) = \{x \in U \mid v(x) = 0\} \tag{1.1}$$

is compact. The basic index, $\tilde{I}(v,U)$ say, is a stable map $S^0 \to U_+$ (where the subscript "+" denotes adjunction of a disjoint basepoint). It is defined by an explicit geometric construction in the style of Pontrjagin-Thom as follows.

We can regard the vector field v simply as a map $v\colon U \to V$. Let $N \subseteq V$ be an open neighbourhood of Zero(v) such that $\overline{N}$ is compact and $\overline{N} \subseteq U$, and choose a (finite) open ball B, centre O, in V so small that $v(x) \notin B$ for all $x \in \overline{N} - N$. Using a superscript "+" for one-point-compactification, we define a map $q\colon V^+ \to (V/(V-B)) \wedge U_+$, by $q(x) = [v(x),x]$ if $x \in \overline{N}$, $q(x) = *$ (basepoint) if $x \notin N$. Then, identifying $V/(V-B) = B^+$ with V^+ by radial extension, we obtain a well-defined homotopy class $V^+ \to V^+ \wedge U_+$, which represents the stable map $\tilde{I}(v,U)\colon S^0 \to U_+$.

1.2 REMARK. At this level the vector-field and fixed-point problems are indistinguishable. The construction just described defines the Lefschetz fixed-point index of the map $f\colon U \to V$ given by $f(x) = x - v(x)$. The zeros of v are the fixed-points of f.

Two fundamental properties of the index are evident from the construction.

1.3 PROPERTIES OF THE INDEX.

(a) <u>Suppose that</u> U' <u>is an open subset of</u> U <u>containing</u> $\mathrm{Zero}(v)$. <u>Then</u> $\tilde{I}(v,U) = i_+ \circ \tilde{I}(v,U')$, <u>where</u> i_+ <u>is the inclusion</u>.

(b) <u>Suppose that</u> U <u>is a disjoint union of open subsets</u> U_1 <u>and</u> U_2. <u>Then</u> $\tilde{I}(v,U) = i_+^1 \circ \tilde{I}(v,U_1) + i_+^2 \circ \tilde{I}(v,U_2)$, <u>where</u> i_+^1 <u>and</u> i_+^2 <u>are the respective inclusions of</u> U_1 <u>and</u> U_2 <u>in</u> U.

Composing $\tilde{I}(v,U)$ with the map $S^0 \to U_+$ which collapses U to a point, we obtain a stable map $S^0 \to S^0$ or, in other words, an element, $I(v,U)$ say, of the stable cohomotopy ring $\omega^0(*)$. (The symbol "ω" is used for unreduced stable homotopy.) This class $I(v,U)$ is the traditional Poincaré-Hopf index. Of course, in this case it is just an integer and determined by $\mathbb{Z}$-cohomology. The definitions have been formulated in this way so as to generalize directly to the equivariant bundle theory.

Next we recall the computation of the index for a field with isolated zeros. Suppose that $\mathrm{Zero}(v)$ lies in the interior of the unit disc $D(V)$ in V and that $D(V) \subseteq U$. Then $I(v,U) \in \omega^0(*)$ is the stable homotopy class represented by the map of spheres:

$$S(V) \to S(V) : x \mapsto \frac{1}{|v(x)|} v(x), \tag{1.4}$$

(so in this case the classical degree). With the additivity of the index, (1.3)(b), this determines $I(v,U)$ when $\mathrm{Zero}(v)$ is discrete.

In the differentiable case, the index of a non-degenerate zero lies in the image of the J-homomorphism. Suppose that the vector field v is continuously differentiable (C^1) with $\mathrm{Zero}(v) = \{0\}$ and the derivative $(Dv)(0)\colon V \to V$ invertible. Then $(Dv)(0)$ defines an element "sign det" of $KO^{-1}(*) = \mathbb{Z}/2$, and $I(v,U)$ is the image of this class under the J-homomorphism

$$J : KO^{-1}(*) \to \omega^0(*)^{\cdot} \subseteq \omega^0(*) \tag{1.5}$$

to the group of units $\omega^0(*)^{\cdot} = \{\pm 1\}$ in the stable cohomotopy ring.

The first extension of the theory is from Euclidean space to a (finite-dimensional, smooth) manifold. Let v now be a vector field, with compact zero-set, on an open subset U of a closed manifold M. The index $\tilde{I}(v,U)$, a stable map $S^0 \to U_+$, is defined by embedding M in Euclidean space V. Let ν be the normal bundle of the embedding and choose an open tubular neighbourhood $M \subseteq N \subseteq V$, where N is an open disc-bundle in ν. Write $r: N \to M$ for the projection. Then we can identify the tangent-bundle τN with $r^*(\tau M \oplus \nu)$ and extend v to a field $\bar{v}$ on $r^{-1}U$, with the same zeros, by: $\bar{v}(x) = (v(rx),x) \in \tau_{rx}M \oplus \nu_{rx}$. The index $\tilde{I}(v,U)$ is defined as the composition $r_+ \circ \tilde{I}(\bar{v},r^{-1}U): S^0 \to (r^{-1}U)_+ \to U_+$.

1.6 REMARK. Let $A \subseteq U$ be a compact manifold of codimension zero with $\mathrm{Zero}(v) \subseteq A - \partial A$. (Such a manifold can always be obtained as $\psi^{-1}[c,\infty)$, where c is a regular value, $0 < c < 1$, of a smooth function $\psi: U \to \mathbb{R}$ which is 1 on a neighbourhood of $\mathrm{Zero}(v)$ and 0 outside a compact set.) Then we can form the relative, stable cohomotopy, Euler class of τA with respect to the nowhere-zero section v on ∂A. This is an element of the stable cohomotopy of the relative Thom space $(A,\partial A)^{-\tau A}$: $\gamma(\tau A, v|\partial A) \in \omega^0(A,\partial A;-\tau A)$ in the notation of [5:1]. By duality this group is identified with $\omega_0(A)$ and the relative Euler class gives a stable map $S^0 \to A_+$. Its composition with the inclusion $A_+ \to U_+$ is equal to the index $\tilde{I}(v,U)$. (This can be established by arguing from the definitions: the duality between $(A,\partial A)^{-\tau A}$ and A_+ is itself defined using Gysin maps and so, ultimately, by the Pontrjagin-Thom construction. Compare the proof of (2.5).)

From $\tilde{I}(v,U)$ we again obtain a Poincaré-Hopf index $I(v,U) \in \omega^0(*)$ by mapping U_+ to S^0. (By including U_+ in M_+ one also obtains an intermediate index, sometimes called a transfer, in $\omega_0(M)$.)

1.7 REMARK. In this case the vector-field index is related to the fixed-point index as follows. Choose a Riemannian metric on M. Then, for all sufficiently small $\varepsilon > 0$, the fixed points of the map $x \mapsto \exp_x(-\varepsilon v(x)) : U \to M$ are the zeros of v and its index is $\tilde{I}(v,U)$.

We begin the bundle theory by considering a trivial bundle $p: B \times V \to B$, where B is a compact ENR and V an Euclidean space. Write $\tau(p)$ for the bundle of tangents along the fibres of p. (Here it is simply the trivial bundle with fibre V.) Suppose that v is a section of $\tau(p)$, defined on an open subset $U \subseteq B \times V$, with compact zero-set. Thus v is a family of vector fields v_b, parametrized by $b \in B$, defined on open subsets $U_b = \{x \in V \mid (b,x) \in U\}$ of V. Carrying out the construction of the basic index fibrewise, we obtain a stable map over $B : B \times S^0 \to U_{+B}$, where $U_{+B} = U \amalg B$ is obtained by adjoining a disjoint basepoint in each fibre. We denote this index over B by $\tilde{I}_B(v,U)$. Composition with the map $U_{+B} \to B \times S^0$, induced by p, which collapses each fibre of U to a point, gives a stable map over $B : B \times S^0 \to B \times S^0$, that is, an element, $I_B(v,U)$ say, of $\omega^0(B)$.

Again, we can easily treat families of isolated zeros. If $\mathrm{Zero}(v) \subseteq B \times (D(V) - S(V)) \subseteq B \times D(V) \subseteq U$, $I_B(v,U)$ is represented by a self-map, given on fibres by (1.4), of the (trivial) sphere-bundle $B \times S(V)$. When v is C^1 (in the sense that it is differentiable on fibres with its derivative Dv continuous on U), if $\mathrm{Zero}(v) = B \times \{0\}$ and each $(Dv_b)(0) : V \to V$ is invertible, then $I_B(v,U)$ is the image under

$$J : KO^{-1}(B) \to \omega^0(B)^{\cdot} \subseteq \omega^0(B) \tag{1.8}$$

of the K-theory class determined by the automorphism $(Dv)(0)$ of the (trivial) vector bundle $B \times V$ over B.

From the vector bundle we can proceed to a trivial bundle $p: B \times M \to B$ with fibre a closed manifold M. If v is a family of vector fields defined on an open set $U \subseteq B \times M$, (that is, a section of the pull-back $\tau(p)$ of τM), with $\mathrm{Zero}(v)$ compact, indices $\tilde{I}_B(v,U): B \times S^0 \to U_{+B}$ over B and $I_B(v,U) \in \omega^0(B)$ are defined by embedding the bundle of manifolds in a vector bundle (such as $B \times V \to B$).

REMARK 1.9. By including U in M we get a stable map over B: $B \times S^0 \to B \times M_+$ or, equivalently, a stable map $B_+ \to M_+$ lifting $I_B(v,U): B_+ \to S^0$.

We shall need a form of relative index. Suppose that $A \subseteq B$ is a closed sub-ENR such that there are no zeros of v over A: $p^{-1}A \cap \mathrm{Zero}(v) = \emptyset$. Then we can replace U by the smaller open neighbourhood $U \cap p^{-1}(B-A)$ of $\mathrm{Zero}(v)$. This gives us representatives of $I_B(v,U) : B \times S^0 \to B \times S^0$ which are trivial (not just null-homotopic) over A and so a relative index $I_{(B,A)}(v,U) \in \omega^0(B,A)$. (As in (1.9) we get a stable lift $B/A \to M_+$, too.)

These constructions are functorial in the base B. If $a\colon B' \to B$ is a map from a compact ENR B', v lifts to a vector field v' on $U' = (a \times 1)^{-1}U \subseteq B' \times M$. Then $\tilde{I}_{B'}(v',U')$ is the pull-back of $\tilde{I}_B(v,U)$ and $I_{B'}(v',U') = a^* I_B(v,U) \in \omega^0(B')$. This includes, as a special case, the homotopy invariance of the index.

The final generalization is from a trivial bundle to an arbitrary manifold over a compact ENR B. Let $p\colon E \to B$ be such a manifold over B, with fibre a closed manifold. (The usual examples are trivial bundles $B \times M \to B$ as above and locally trivial smooth fibre-bundles.) If v is a section of $\tau(p)$ on an open set $U \subseteq E$, the indices $\tilde{I}_B(v,U) : B \times S^0 \to U_{+B}$ and $I_B(v,U) \in \omega^0(B)$ are defined whenever Zero(v) is compact. (We also have stable transfer maps: $B \times S^0 \to E_{+B}$ over B and, factoring out basepoints, the induced map $B_+ \to E_+$.)

The index theory over a base provides a natural framework for discussion of the global bifurcation theory of Rabinowitz [11]. (Developments and variants of the original result abound; see [2] and references there.) Suppose that B is a compact (smooth) n-manifold and consider, to be definite, a trivial bundle $p\colon B \times M \to B$, with M closed. Take a collar neighbourhood of the boundary $\partial B = \partial B \times \{0\}\colon \partial B \times [0,\infty) \subseteq B$, and let $j\colon \partial B \to B - \partial B$ be the embedding $x \mapsto (x,1)$. One of the fundamental lemmas of cobordism theory asserts that the coboundary map $\delta\colon \omega^0(\partial B) \to \omega^1(B,\partial B)$ coincides (up to sign) with the Gysin map $j_!$.

Let v be a family of vector fields (on M) defined on an open subset $U \subseteq B \times M$. If that part of the zero-set of v over ∂B is compact, we can form the index $I_{\partial B}(v, U \cap p^{-1}\partial B) \in \omega^0(B)$.

1.10 LEMMA. <u>If</u> Zero(v) <u>is compact, then</u>

$$j_! I_{\partial B}(v, U \cap p^{-1}\partial B) = 0 \in \omega^1(B,\partial B).$$

This is clear from the identification of $j_!$ with $\pm\delta$. The class $j_! I_{\partial B}(v, U \cap p^{-1}\partial B)$ is, essentially, the bifurcation invariant of Bartsch [2]. (If B is a submanifold of $\mathbb{R}^n$ or, more generally, is framed, then we can map $\omega^1(B,\partial B)$ to $\omega_{n-1}(*)$ by the Gysin map.)

Now suppose that $A \subseteq B - \partial B$ is a compact submanifold of co-dimension zero and write $i\colon \partial A \to B$ for the inclusion. Assume that $\mathrm{Zero}(v) \cap p^{-1}(\overline{B-A})$ is compact. Then (1.10) applied to the manifold $\overline{B-A}$ yields, by transitivity of Gysin maps:

$$(1.11) \qquad i_! I_{\partial A}(v, U \cap p^{-1}\partial A) = j_! I_{\partial B}(v, U \cap p^{-1}\partial B) \in \omega^1(B, \partial B).$$

Repeated application of (1.10) and (1.11) establishes:

1.12 LEMMA. *Suppose that* Zero(v) *is compact and that* $U \cap p^{-1}(\overline{B-A})$ *is a disjoint union of open subsets* P *and* Q *of* $(\overline{B-A}) \times M$. *Then*

$$i_! I_{\partial A}(v, P \cap p^{-1}\partial A) = j_! I_{\partial B}(v, Q \cap p^{-1}\partial B).$$

2. A finite-dimensional analogue

Some familiarity with $\mathbb{T}$-equivariant homotopy will be assumed. Background and notation can be found in [4], to which frequent reference will be made.

Throughout this section M will be a finite-dimensional closed $\mathbb{T}$-manifold and s will denote the generating vector field of the circle action. Let $\Omega \subseteq M$ be an open $\mathbb{T}$-subset on which $\mathbb{T}$ acts without fixed points, and suppose that w is an equivariant vector field on Ω such that the set $\Xi = \{x \in \Omega \mid w(x) \in \mathbb{R}s(x)\}$, of points where w is parallel to the flow, is compact. Building on the work of Dancer [6], we shall construct an index $\mathcal{E}(w, \Omega) \in \omega_1^{\mathbb{T}}(E\mathfrak{F})$, where $E\mathfrak{F}$ is the classifying space of the family $\mathfrak{F}$ of finite subgroups of $\mathbb{T}$, [4: 1.13].

Consider the family of vector fields v_μ ($\mu \in \mathbb{R}$) on Ω given at $x \in \Omega$ by:

$$(2.1) \qquad v_\mu(x) = \mu s(x) + w(x).$$

The zero-set of v_μ is compact, and, for large $\rho > 0$, is empty if $|\mu| \geqslant \rho$. So we have a fibre-bundle $\mathbb{R} \times M \to \mathbb{R}$ and a vector field v, along the fibres on the subspace $U = \mathbb{R} \times \Omega$, with compact zero-set. Restricting to the compact subspace $B = [-\rho, \rho] \subseteq \mathbb{R}$, we can form the $\mathbb{T}$-equivariant relative Poincaré-Hopf index $I_{(B,\partial B)}(v, B \times \Omega)$ in the group $\omega^0_{\mathbb{T}}(B, \partial B)$, which is canonically identified with $\omega^{-1}_{\mathbb{T}}(*) = \omega_1^{\mathbb{T}}(*)$. The resultant class is clearly independent of ρ and should be regarded as an index with compact supports over the base $\mathbb{R}$. Since $\Omega^{\mathbb{T}} = \emptyset$, we can use the classifying map $\Omega \to E\mathfrak{F}$ to lift the index, as in (1.9), to an element

$$(2.2) \qquad \mathcal{E}(w, \Omega) \in \omega_1^{\mathbb{T}}(E\mathfrak{F}).$$

(In fact, we have $\omega_1^{\mathbb{T}}(*) = \omega_1^{\mathbb{T}}(E\mathfrak{F}) \oplus \omega_1(*)$.)

The group $\omega_1^{\mathbb{T}}(E\mathfrak{F})$ is a direct sum $\oplus_{n\geq 1} \mathbb{Z}\sigma_n$, [4:2.10]. So the index $\mathcal{E}(w,\Omega)$ is given by a sequence of integers. (These integer invariants are implicit in [10].)

A weaker index is obtained by mapping, via the Hurewicz homomorphism, to integral homology: $\omega_1^{\mathbb{T}}(E\mathfrak{F}) \to H_1^{\mathbb{T}}(E\mathfrak{F}) = \mathbb{Q}$, $\Sigma a_n\sigma_n \mapsto \Sigma a_n/n$, [4:2.11]. (This gives Fuller's original $\mathbb{Q}$-valued index, [8].)

One can also simply forget the $\mathbb{T}$-equivariance, mapping $\omega_1^{\mathbb{T}}(E\mathfrak{F}) \to \omega_1(E\mathfrak{F}) = \omega_1(*) = \mathbb{Z}/2 : \Sigma a_n\sigma_n \mapsto \Sigma a_n \pmod 2$. (Such mod 2-indices are standard tools in bifurcation theory; see [2] for a recent account.)

2.3 REMARK. It follows from (2.7) and (2.10) below that the homology Hurewicz image of $\mathcal{E}(w,\Omega)$ agrees with Dancer's index [6]. However, he restricts attention to gradient vector fields. Thus M has a $\mathbb{T}$-invariant Riemannian metric g and $w = \operatorname{grad}\psi$ for some $\mathbb{T}$-invariant C^1-function $\psi: M \to \mathbb{R}$. Since ψ is constant on orbits, we have $g(s,w) = (d\psi)(s) = 0$. So Ξ is just $\mathrm{Zero}(w)$, and $\mathrm{Zero}(v_\mu) = \emptyset$ if $\mu \neq 0$.

The index $\mathcal{E}$ has the following properties, which it inherits from the vector-field index.

2.4 LEMMA. (i) If w^λ $(\lambda \in [0,1])$ is a continuous family of vector fields on Ω such that $\{(\lambda,x) \in [0,1]\times\Omega \mid w^\lambda(x) \in \mathbb{R}s(x)\}$ is compact, then $\mathcal{E}(w^0,\Omega) = \mathcal{E}(w^1,\Omega)$.

(ii) If Ω' is an open subset of Ω with $\Xi \subseteq \Omega'$, then $\mathcal{E}(w,\Omega') = \mathcal{E}(w,\Omega)$.

(iii) If Ω is a disjoint union of open sets Ω_1 and Ω_2, then $\mathcal{E}(w,\Omega) = \mathcal{E}(w,\Omega_1) + \mathcal{E}(w,\Omega_2)$.

If the $\mathbb{T}$-action on M is fixed-point-free ($M^{\mathbb{T}} = \emptyset$), we may take $\Omega = M$. The index $\mathcal{E}(w,M)$ is, by (2.4)(i), independent of w and can be expressed in terms of Euler characteristics as follows. We write $\mathbb{T}(n)$ for the subgroup $\mathbb{Z}\frac{1}{n}/\mathbb{Z}$ of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ of order $n \geq 1$.

2.5 PROPOSITION. If $M^{\mathbb{T}} = \emptyset$, we have

$$\mathcal{E}(0,M) = \sum \chi_c(M_{(\mathbb{T}(n))}/\mathbb{T}).\sigma_n,$$

where $M_{(\mathbb{T}(n))}$ is the set of points in M with stabilizer of order n and χ_c denotes the Euler characteristic with compact supports.

Outline proof. An invariant $\mathcal{E}(M)$ is introduced in [4] as the Euler characteristic of the normal bundle, $\hat{\tau}$, to the orbits in M, and $\mathcal{E}(M)$ is calculated, [4:5.2], as the righthand expression in (2.5). It is, therefore, sufficient to show that $\mathcal{E}(M) = \mathcal{E}(M,0)$. This is done by direct inspection. We adopt the notation of [4:5].

The dual in $\omega_1^{\mathbb{T}}(M)$ of the Euler class $\gamma(\hat{\tau}) \in \tilde{\omega}_{\mathbb{T}}^0(M^{-\hat{\tau}})$ can be described as follows. Recall that duality is defined by Gysin maps. In particular, if M is embedded in a $\mathbb{T}$-module V with normal bundle ν, the Pontrjagin-Thom construction gives a map $V^+ \to M^{\nu}$, which represents the fundamental class in $\tilde{\omega}_0^{\mathbb{T}}(M^{-\tau M})$, dual to $1 \in \omega_{\mathbb{T}}^0(M) = \tilde{\omega}_{\mathbb{T}}^0(M_+)$. More generally, if ζ is a vector bundle over M, the composition $V^+ \to M^{\nu} \to M^{\nu\oplus\zeta}$ with the inclusion gives the dual in $\tilde{\omega}_0^{\mathbb{T}}(M^{\zeta-\tau M})$ of the Euler class $\gamma(\zeta) \in \tilde{\omega}_{\mathbb{T}}^0(M^{-\zeta})$. For $\zeta = \hat{\tau}$ we take the smash product with the identity on $\mathbb{R}^+$ to get a map $(\mathbb{R}\oplus V)^+ \to V^+ \wedge M_+$. Using the same embedding of M in V to construct the index of the vector field v, as in (1.9), we obtain a second map $(\mathbb{R}\oplus V)^+ \to V^+ \wedge M_+$. One checks that the two are homotopic. □

In the classical Poincaré-Hopf theory it is easy, as we have seen, to compute the index of a vector field with isolated zeros. To treat the analogous case here of a field w for which the set Ξ is a finite union of isolated orbits, we begin with a slightly more general problem. Suppose that Ω' is an invariant open subset of a closed $\mathbb{T}(k)$-manifold M', $k \geqslant 1$, and w' an equivariant vector field on Ω' with compact zero-set. Put

(2.6) $$\Omega = \mathbb{T} \times_{\mathbb{T}(k)} \Omega' \subseteq M = \mathbb{T} \times_{\mathbb{T}(k)} M',$$

and let w be the vector field on Ω induced from w'. (Thus w lifts to $0 \oplus w'$ on the k-fold cover $\mathbb{T} \times \Omega'$ of Ω, and $\Xi = \mathbb{T} \times_{\mathbb{T}(k)} \mathrm{Zero}(w')$ is compact.)

2.7 PROPOSITION. <u>The index</u> $\mathcal{E}(w,\Omega)$ <u>of the field</u> w <u>on the mapping torus is the image under the induction map</u>:

$$\omega_0^{\mathbb{T}(k)}(*) \to \omega_1^{\mathbb{T}(k)}(E\mathfrak{F}) \subseteq \omega_1^{\mathbb{T}}(*)$$

<u>of the vector-field index</u> $I(w',\Omega')$ <u>of</u> w'.

Since group-theoretic induction from the subgroup $\mathbb{T}(k)$ to $\mathbb{T}$ is, in essence, the construction $\mathbb{T}\times_{\mathbb{T}(k)} -$, the result is no surprise. The induction map in stable homotopy sends σ_n' to σ_n:

(2.8) $$\omega_0^{\mathbb{T}(k)}(*) = \oplus_{n|k}\, \mathbb{Z}\sigma_n' \to \omega_1^{\mathbb{T}}(E\mathfrak{F}) = \oplus\, \mathbb{Z}\sigma_n,$$

where σ'_n is the class of $\mathbb{T}(k)/\mathbb{T}(n)$ in the Burnside ring.

Outline proof of (2.7). Write $B = [-\rho,\rho]$ for any $\rho > 0$ and $C = \mathbb{T}/\mathbb{T}(k)$. The construction of the field v on $B \times \Omega \subseteq B \times M$ over B can be described as follows. We have a smooth fibre-bundle $p: M \to C$ and the field w is a section, on $\Omega \subseteq M$, of the bundle $\tau(p)$ of tangents along the fibres. On the trivial bundle $B \times C \to B$ we have a field t: $t_\mu = \mu s$ at $\mu \in B$. There is a splitting $\tau M = p^*\tau B \oplus \tau(p)$ (in general defined up to homotopy, in this case given) and $v = t \oplus w$.

The zero-sets of t and w are compact (and, for t, disjoint from $\partial B \times C$). So we can form the indices $I_C(w,\Omega) \in \omega^0_{\mathbb{T}}(C)$ and $\tilde{I}_{(B,\partial B)}(t, B\times C)$. The latter may be regarded, (1.9), as a stable map, f say: $\mathbb{R}^+ \simeq B/\partial B \to C_+$. In this situation one can establish the generalized multiplicativity formula:

$$(2.9) \qquad I_{(B,\partial B)}(t\oplus w, B\times\Omega) = I_C(w,\Omega).\tilde{I}_{(B,\partial B)}(t,B\times C).$$

Now recall that induction is defined as the composition: $\omega^0_{\mathbb{T}(k)}(*) \xrightarrow{\cong} \omega^0_{\mathbb{T}}(\mathbb{T}/\mathbb{T}(k)) \to \omega^{-1}_{\mathbb{T}}(*)$ of the canonical identification and the Gysin map determined by the left-invariant framing of $\mathbb{T}/\mathbb{T}(k)$. The proof is completed by observing that the first map lifts $I(w',\Omega')$ to $I_C(w,\Omega)$ and by checking that the second is induced by f. □

The proposition (2.7) gives the following prescription for computing $\mathcal{E}(w,\Omega)$ when Ξ is a finite union of isolated orbits. A tubular neighbourhood of a component C of Ξ in Ω can be written in the form $\mathbb{T}\times_{\mathbb{T}(k)} \Omega'$, where Ω' is an open disc, with centre 0, in some $\mathbb{T}(k)$-module V and C corresponds to $\mathbb{T} \times_{\mathbb{T}(k)} 0$. On this neighbourhood, if it meets no other component of Ξ, w is, up to addition of a constant multiple of s (and permissible homotopy, (2.4)(i)), induced from a field w' on Ω' with a single zero at 0. The contribution of C to $\mathcal{E}(w,\Omega)$ is determined, according to (2.7), by the index of w'.

In homology the induction map:

$$(2.10) \qquad H_0^{\mathbb{T}(k)}(*) = \mathbb{Z} \to H_1^{\mathbb{T}}(E\,\mathfrak{T}) = \mathbb{Q}$$

is just multiplication by $1/k$. So the recipe above gives the contribution of the isolated orbit $C \subseteq \Xi$ to the $\mathbb{Q}$-valued index as $1/k$ times the non-equivariant index of w'. (This was Dancer's starting point in [6].)

If, further, w' is C^1 with $Dw'(0) = L$ (say): $V \to V$ non-singular, then $I(w',\Omega')$ is the image under the J-homomorphism, (1.5), $J\colon KO^{-1}_{\mathbb{T}(k)}(*) \to \omega^{0}_{\mathbb{T}(k)}(*)$ of the class determined by L. Write d and d' for the elements of the group $\{\pm 1\}$ $(=\mathbb{Z}/2)$ defined by: $dd' = \mathrm{sign}(\det L)$, $d = \mathrm{sign}(\det L^{\mathbb{T}(k)})$, where $L^{\mathbb{T}(k)}\colon V^{\mathbb{T}(k)} \to V^{\mathbb{T}(k)}$ is the restriction of L to the fixed submodule. The group $KO^{-1}_{\mathbb{T}(k)}(*)$ is isomorphic to $\mathbb{Z}/2$ if k is odd, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if k is even, and the class of L is given, respectively, by d and (d,d'). In each case J is injective, and straightforward calculation yields:

2.11 PROPOSITION. <u>The index</u> $\mathcal{E}(w,\Omega)$ <u>of a non-degenerate (isolated) orbit of a</u> C^1<u>-field</u> w <u>as described above is equal to</u>:

$d\sigma_k$ <u>when</u> k <u>is odd</u>;

$d\sigma_k$ <u>if</u> $d' = +1$, $d(\sigma_{k/2}-\sigma_k)$ <u>if</u> $d' = -1$, <u>when</u> k <u>is even</u>. □

As a final computation of the index we describe a basic bifurcation theorem, following Dancer [6: p.339] and Ize [10: p.759]. Consider a continuous family w^λ $(\lambda \in [0,1])$ of $\mathbb{T}$-equivariant vector fields defined on the whole of M, and write v for the family $v^\lambda_\mu = \mu s + w^\lambda$, $(\lambda,\mu) \in [0,1]\times\mathbb{R}$, on $p\colon [0,1]\times\mathbb{R}\times M \to [0,1]\times\mathbb{R}$. We impose the following conditions on the zero-set $Z = \mathrm{Zero}(v)$.

2.12 HYPOTHESES. (i) The closure $(Z - Z^{\mathbb{T}})^-$ is compact.

(ii) The "bifurcation set" $\Pi = ((Z - Z^{\mathbb{T}})^-)^{\mathbb{T}}$ is discrete and disjoint from $\partial[0,1]\times\mathbb{R}\times M^{\mathbb{T}}$.

(iii) For each point $(\lambda,\mu,x) \in \Pi$, x is an isolated zero on $M^{\mathbb{T}}$ of w^λ.

2.13 PROPOSITION. <u>Under the assumptions</u> (2.12), <u>we have</u>

$$\mathcal{E}(w^1, M-M^{\mathbb{T}}) - \mathcal{E}(w^0, M-M^{\mathbb{T}}) = \textstyle\sum_{\pi\in\Pi}\iota(\pi),$$

<u>where</u> $\iota(\pi) \in \omega^{\mathbb{T}}_1(E\mathcal{F})$ <u>is the local index described below</u>.

<u>Outline proof</u>. It will be convenient to label a point $\pi \in \Pi$ as $(\lambda_\pi,\mu_\pi,x_\pi)$, and to write V_π for the tangent space of M at x_π. Put $B = [0,1]\times[-\rho,\rho]$, where $\rho > 0$ is chosen to satisfy: $(Z-Z^{\mathbb{T}})^- \subseteq [0,1]\times(-\rho,\rho)$; and let $A(\pi)$, for $\pi \in \Pi$, be the closed disc of radius ε, centre (λ_π,μ_π), in $\mathbb{R}^2$ with the Euclidean norm. The radius $\varepsilon > 0$ is chosen such that: $A(\pi) \subseteq B - \partial B$, $A(\pi)\cap A(\pi') = \emptyset$ if $(\lambda_\pi,\mu_\pi) \neq (\lambda_{\pi'},\mu_{\pi'})$, for $\pi,\pi' \in \Pi$. Set $A = \cup A(\pi)$, $\pi \in \Pi$.

For ε sufficiently small, (2.12) guarantees that we can find tubular neighbourhoods: $V_\pi \hookrightarrow M$ of each point $x_\pi \in M$ such that, for appropriate inner products:

(2.14) (i) the closed unit discs $D(V_\pi)$ are disjoint in M ($D(V_\pi) \cap D(V_{\pi'}) = \emptyset$ if $x_\pi \neq x_{\pi'}$); (ii) $(A(\pi) \times S(V_\pi)) \cap Z^{\mathbb{T}} = \emptyset$; and (iii) $(\partial A(\pi) \times D(V_\pi)) \cap (Z - Z^{\mathbb{T}}) = \emptyset$.

The field v is then, by (ii) and (iii), nowhere zero on $\partial A(\pi) \times S(V_\pi)$ and gives, as in (1.4), a map: $\partial A(\pi) \times S(V_\pi) \to S(V_\pi)$. This determines a stable homotopy class in $\omega^0_{\mathbb{T}}(\partial A(\pi)) = \omega^0_{\mathbb{T}}(*) \oplus \omega^{-1}_{\mathbb{T}}(*)$, and we denote its second component by $\iota(\pi)$. From (2.14)(ii) we see, by considering fixed-points, that: $\iota(\pi) \in \omega^{\mathbb{T}}_1(E\mathcal{F}) \subseteq \omega^{-1}_{\mathbb{T}}(*)$.

Next we use (1.12), choosing open sets P and Q such that: $P \supseteq Z^{\mathbb{T}} \cap p^{-1}(\overline{B-A})$ and $P \supseteq \partial A_\pi \times D(V_\pi)$, $Q \supseteq (Z - Z^{\mathbb{T}}) \cap p^{-1}(\overline{B-A})$. (To fit the precise form of the lemma, we can replace B by a slightly smaller disc with smooth boundary.) The index $j_! I_{\partial B}(v, Q \cap p^{-1}\partial B)$ in $\omega^1_{\mathbb{T}}(B, \partial B) = \omega^{-1}_{\mathbb{T}}(*)$ is clearly $\mathcal{E}(w^1, M-M^{\mathbb{T}}) - \mathcal{E}(w^0, M-M^{\mathbb{T}})$. On the other hand, $I_{\partial A}(v, P \cap p^{-1}\partial A)$ can be expressed as a sum $I_{\partial A}(v, R \cap p^{-1}\partial A) + I_{\partial A}(v, S \cap p^{-1}\partial A)$, where $R = \bigcup (A(\pi) \times (D(V_\pi) - S(V_\pi)))$ and S is an open subset of $A \times M$ such that: $R \cap S = \emptyset$ and $S \cap Z$ is the compact set $\{z \in Z^{\mathbb{T}} \cap p^{-1}A \mid z \notin \overline{R}\}$. By (1.10), we have $i_! I_{\partial A}(v, S \cap p^{-1}\partial A) = 0$. But the term $i_! I_{\partial A}(v, R \cap p^{-1}\partial A)$ is exactly $\sum \iota(\pi)$. □

When the family w is C^1 (that is, differentiable on fibres with the derivative continuous on $[0,1] \times M$), there is an elegant description (to be found in [10],[6] and earlier work) of the local index $\iota(\pi)$ at a "non-degenerate" bifurcation point π in terms of spectral flow. To explain this, we need some notation. For $n \geq 1$, let E^n be the complex $\mathbb{T}$-module $\mathbb{C}$ with $[t] \in \mathbb{R}/\mathbb{Z}$ acting as multiplication by $e^{2\pi i n t}$. Recall that any real $\mathbb{T}$-module V splits functorially as a direct sum:

$$V = V^{\mathbb{T}} \oplus \bigoplus_{n \geq 1} E^n \otimes_{\mathbb{C}} V^{(n)}, \tag{2.15}$$

where $V^{(n)}$ is the $\mathbb{C}$-vector space of $\mathbb{R}$-linear $\mathbb{T}$-maps: $E^n \to V$. Now, at a zero $x \in M$ of w^λ the derivative of w^λ defines an endomorphism, $L(\lambda,x)$ say, of the tangent space $\tau_x M$. If $x \in M^{\mathbb{T}}$, we can split $L(\lambda,x)$ into components: $L(\lambda,x)^{\mathbb{T}}$ on $\tau_x M^{\mathbb{T}}$, $L(\lambda,x)^{(n)}$ on $(\tau_x M)^{(n)}$. The non-degeneracy conditions at $\pi \in \Pi$ are the following.

2.16 HYPOTHESES. (i) Put $\Delta = \det(L(\lambda_\pi, x_\pi)^{\mathbb{T}})$. We suppose that $\Delta \neq 0$ (which implies (2.11)(iii)).

(ii) By the implicit function theorem, for sufficiently small $\delta > 0$ there is a unique continuous path γ: $(\lambda_\pi - \delta, \lambda_\pi + \delta) \to M^{\mathbb{T}}$ such that: $\gamma(\lambda_\pi) = x_\pi$ and $w^\lambda(\gamma(\lambda)) = 0$. Write χ_n^λ for the characteristic polynomial: $\chi_n^\lambda(z) = \det(z - (2\pi n)^{-1}L(\lambda,\gamma(\lambda))^{(n)})$. We assume that there exists η, $0 < \eta < \delta$, such that, for all $n \geqslant 1$,

$$\chi_n^\lambda(-i\mu) \neq 0 \text{ when } 0 < |\lambda-\lambda_\pi|^2 + |\mu-\mu_\pi|^2 \leqslant \eta^2.$$

Let ν_n denote the net flow of roots of χ_n^λ through $-i\mu$ from the left to the right of the imaginary axis as λ increases through λ_π. (To be precise, choose a small closed disc D, centre $-i\mu$, in $\mathbb{C}$ such that $\chi_n^{\lambda_\pi}$ has no roots in $D - \{-i\mu\}$. Then the number of roots z of χ_n^λ with $z \in D$ and $\mathrm{Re}(z) > 0$, counted with multiplicity, jumps by ν_n as λ increases through λ_π.)

2.17 PROPOSITION. Under the assumptions (2.16), the local index $\iota(\pi) \in \omega_1^{\mathbb{T}}(E\mathfrak{F})$ is equal to

$$-\sum \operatorname{sign}(\Delta)\nu_n.\sigma_n \in \oplus\, \mathbb{Z}\sigma_n.$$

Outline proof. We continue the notation in (2.13). Taking $\varepsilon \leqslant \eta$, we find that the field v has a non-degenerate (so isolated) zero at $\gamma(\lambda)$ over $(\lambda,\mu) \in \partial A(\pi)$ and may assume, by making suitable choices, that v has no other zeros in $\partial A(\pi) \times D(V_\pi)$. The derivative of v at $(\lambda,\mu,\gamma(\lambda))$ is the automorphism $\mu S(\gamma(\lambda)) + L(\lambda,\gamma(\lambda)) = T(\lambda,\mu)$, say, of $\tau_{\gamma(\lambda)}M$, where S is given by the $\mathbb{T}$-action (that is, the derivative of s).

The index $I_{\partial A(\pi)}(v, D(V_\pi) - S(V_\pi))$, which defines $\iota(\pi)$, is the image under

$$J : KO_{\mathbb{T}}^{-1}(\partial A(\pi)) \to \omega_{\mathbb{T}}^0(\partial A(\pi)) \tag{2.18}$$

of the class ℓ determined by the vector-bundle automorphism: $T(\lambda,\mu)$ on $\tau_{\gamma(\lambda)}M$ at $(\lambda,\mu) \in \partial A(\pi)$.

Now we have $KO_{\mathbb{T}}^{-1}(\partial A(\pi)) = KO_{\mathbb{T}}^{-1}(S^1) = KO_{\mathbb{T}}^{-1}(*) \oplus KO_{\mathbb{T}}^{-2}(*)$. The component of ℓ in $KO_{\mathbb{T}}^{-1}(*) = \mathbb{Z}/2$ (= $\{\pm 1\}$) is easily seen to be $\operatorname{sign}(\Delta)$. Corresponding to the decomposition (2.15) there is a splitting:

$$KO_{\mathbb{T}}^{-2}(*) = KO^{-2}(*) \oplus \bigoplus_{n\geqslant 1} K^{-2}(*).[E^n]. \tag{2.19}$$

Here the component of ℓ in $KO^{-2}(*)$ is trivial. The remaining components are obtained from (2.20) below: if we identify $K^{-2}(*) = \pi_1(U(\infty))$ with $\mathbb{Z}$ by "degree (det)", the nth term is $\nu_n.[E^n]$. Finally, we can read off the result from (2.18), since $J[E^n] = 1 - \sigma_n$ (under the current sign conventions). □

2.20 APPENDIX. Suppose that p^x, $x \in [-1,1]$, is a continuous family of monic complex polynomials, with no roots on the unit circle $S^1 \subseteq \mathbb{C}$, such that: (i) $p^0(z) \neq 0$ if $0 < |z| \leqslant 1$, and (ii) $p^x(z) \neq 0$ if $x \neq 0$, $z \in i\mathbb{R}$, $|z| \leqslant 1$. Then the degree of the map $S^1 \to \mathbb{C} - \{0\}$: $x + iy \mapsto p^x(-iy)$ is equal to the difference of the number of roots of p^1 and of p^{-1} in the region $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ |z| < 1\}$.

Proof. One easily reduces to the case in which all the roots of p^x lie in the real interval $(-1,1)$. (First discard roots z with $|z| > 1$, then deform the remaining roots within the unit disc to the real axis using the homotopy: $h_t(a+ib) = a + ib(1-t)$, $0 \leqslant t \leqslant 1$.) Now one can order the roots and so reduce to the linear case: $p^x(z) = z - a^x$ with $a^x \in \mathbb{R}$.

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A Geometric Interpretation of Lannes' Functor T.

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1. Introduction. In this note we are concerned with a question raised by [Lannes 2.3]. In what follows R will denote a finite field of the form $\mathbf{Z}/p\mathbf{Z}$, homology and cohomology are always taken with coefficient in R and denoted by H_*X etc. For a space X let $\{R_sX\}_s$ denote the Bousfield-Kan localization tower. We denote by $B\tau$ the classifying space of the underlying abelian group of R. Let P_sX denote the s-Postnikov section of X. By a "space" we mean a Kan complex or a C.W. complex.

1.1 Theorem: *If $H^iX < \infty$ for all $i \geq 0$, then $TH^*X \cong \varinjlim_s H^*(P_sR_sX)^{B\tau}$, where T is Lannes' functor* (see below). *If, in addition, X is nilpotent then $TH^*X \cong \varinjlim$*

$H^*(P_sX)^{B\tau} \cong \varinjlim H^*(P_sR_\infty X)^{B\tau}$

The proof of this theorem yields a new proof for Lannes theorem 1.5 below that essentially asserts 1.1 for dimension zero and was a the motivation for his question [Lannes 2.3]. The proof of theorem is based on the following technical proposition:

1.2 Proposition: *Let $G \to E \to B$ be a principal fibration where G is a (topological or simplicial) group. Assume that in each dimension the R-cohomology of the mapping spaces E^{B_t} and B^{B_t} is finite. Then if the relation $TH^*W \cong H^*W^{B_t}$ is satisfied by $W = E$ and $W = B$ then it is also satisfied by $W = G$.*

Remark: The finiteness assumption, noted by the referee, is necessary in order to use cohomological Eilenberg-Moore spectral sequence.

Corollary: *If W is a nilpotent space of finite type with $\pi_iW = 0$ for $i >> 0$ or a R-localization thereof then*

$$TH^*W \cong H^*W^{B\tau}.$$

Further, as a direct corollary of 1.1 and 9.3 of [Bousfield] one gets the following interesting special case due to Lannes [4].

1.3 Corollary. *Let $H^iX < \infty$ for all $i \geq 0$ for nilpotent space X of finite type. Assume that a given algebraic component T_cH^*X of TH^*X is finite in all dimensions and vanishes in dimension one. Then* $T_cH^*X \cong H^*X_c^{B\tau} \cong H^*(R_\infty X)_c^{B\tau}$

where $X_c^{B\tau}$ is the corresponding component.

Another example where the main result (1.1) implies a result on $H^*map(B\tau, X)$ is when the latter has a finite homotopy group in each dimension.

1.4 Corollary: *Let X be nilpotent space of finite type with π_1X finite. Assume that for a given $f : B\tau \to X$ the groups $\pi_i map(B\tau, P_nX)_{f_*}$ are finite for all $i, n \geq 0$. Then $H^*map(B\tau, X) \cong T_cH^*X$ where T_c is algebraic component of T that corresponds to f.*

The referee also notes that theorem 1.1 gives a new proof of the following result [Lannes, 0.4].

1.5 Corollary: *If Y is a nilpotent space with $H^n(Y, R)$ finite for all n, then the natural map*

$$[Bt, Y] \cong [Bt, R_\infty Y] \to Hom_K(H^*Y, H^*B_t)$$

is an isomorphism of profinite sets.

Proof: This follows directly from 1.1 above in light of the algebraic fact [Lannes 3.5] and the old result of [Dror] about the tower R_sY.

The authors would like to thank W. Dwyer for his suggestion to consider the tower R_sX as a starting point for a geometric interpretation of T, and to H. Miller for several useful discussions. The authors would also like to thank the referee for his careful reading and for correcting a non-fatal error in an earlier version of this paper. The referee notes that if one considers the fibration $\Omega X \to * \to X$ for X being the infinite wedge of $\mathbb{R}P^\infty/\mathbb{R}P^{2n+2}$ over the integers, the formula in 1.2 holds for $W = \Omega X$ but not for X itself. Therefore one cannot turn around 1.2 to conclude that either E or B satisfy the property $TH^*W = H^*W^{B_t}$, assuming the other two spaces do.

2 First examples.

Let $\mathcal{U}$ denote the category of unstable modules over the algebra $\mathcal{A}$ of Steenrod reduced powers relative to a prime $p = char\ R$. Let $\mathcal{K}$ denote the category of unstable $\mathcal{A}$-algebra. In [Lannes] a left adjoint T is defined to the functor $- \otimes H^*B\tau$, where the latter is taken either as a functor from $\mathcal{U}$ to itself or from $\mathcal{K}$ to itself. If one regards an element $A \in \mathcal{K}$ as an element of $\mathcal{U}$, the value of T does not change. Thus the defining property of T is $hom_C(TM, N) = hom_C(M, N \otimes H^*B\tau)$ where C is either $\mathcal{U}$ or $\mathcal{K}$.

2.1 Three basic properties [Lannes]: (i) T is exact. (ii) T commutes with tensor products. (iii) T commutes with direct limits.

2.2 Motivation: It can be seen from 1.1, 1.2, 1.3 that the construction of T is motivated by attempts to describe the cohomology of $X^{B\tau} \equiv map(B\tau, X)$ in terms of H^*X, when the latter is given as an object in $\mathcal{K}$. Lannes proves the relation between the homotopy class $[B\tau, X]$ and $(TH^*X)^0$ and X as in 1.3, see [Lannes 7.1.1]. [Miller] proves it for $dimX < \infty$.

2.3 Example. It is easy to calculate directly from the adjointness relation that if V is a finite dimensional vector space over R then

$$TH^*K(V,n) \cong \bigotimes_{n \geq i \geq 0} H^*K(V,i) \cong H^*map(B\tau, K(V,n)).$$

Here $map(X,Y)$ denotes the space of maps $X \to Y$ otherwise denoted by Y^X. Similar calculation holds for a finite products of $K(V_i, n_i)$ with $dim_R V_i < \infty$. However it turns out that for homotopically large space one cannot, in general, interpret TH^*X as the cohomology of $map(B\tau, X)$, (see 2.5 below).

2.4. Example. An important class of spaces on which T behaves nicely are finite Postnikov pieces of nilpotent spaces. The prime examples of such spaces are $K(\mathbf{Z}, n)$ spaces for $n \geq 0$.

Proposition: *For any $n \geq 0$ there is an isomorphism $TH^*K(\mathbf{Z},n) \cong H^*map(B\tau, K(\mathbf{Z},n))$.*

Proof: For $p = 2$ we show by a direct computation that $TH^*K(\mathbf{Z},n) \cong H^* \prod_{i=1}^{[n/2]} K(\mathbf{Z}/2\mathbf{Z},\ 2i)$. For $p > 2$ the argument is similar. Now since $H^*K(\mathbf{Z},n) \cong P(S_q^I \mid I$ admissible with $e_1(I) \geq 2$ and $e\ (I) \leq n-1)$ a map of the algebra $H^*K(\mathbf{Z},n)$ over A is given by the image of the generator in dimension n. Thus

$$hom_{\mathcal{K}}(K(\mathbf{Z},n), K) = ker\beta : K_n \to K_{n+1}$$

where K is any object in $\mathcal{K}$ and β is The Bockstein operation. Now compute:

$$
\begin{aligned}
hom_K(TH^*K(\mathbf{Z},n),K) &\cong hom_K(H^*K(\mathbf{Z},n),K\otimes H^*B\tau)\\
&\cong ker\beta : K\otimes H^*B\tau)_n \to (K\otimes H^*B\tau)_{n+1}\\
&\cong ker\beta : \bigoplus_{i+j=n} K_i\otimes H^jB\tau \to \bigoplus_{i+j=n+1} K_i\otimes H^jB\tau\\
&\cong \bigoplus_{\substack{i+j=n\\ j\ \text{even}}} K_i =\\
&\cong hom_K(H^*\prod_{i=1}^{[n/2]} K(\mathbf{Z}(2\mathbf{Z},i),K).
\end{aligned}
$$

This together with the adjointness property of T completes the proof. Similarly let $\mathbf{Z}_p$ denotes the p-adic numbers $\mathbf{Z}_p = \text{invlim } \mathbf{Z}/p^k\mathbf{Z}$. Then [B - K VI 6.4] one has an R homology equivalence $K(\mathbf{Z},n) \to K(\mathbf{Z}_p,n)$ for all $n \geq 0$. There is a pro-isomorphism on R-homology of $K(\mathbf{Z},n) \to (K(\mathbf{Z}/p^k\mathbf{Z},n))_n$. Therefore

$$H^*K(\mathbf{Z},n) \cong H^*K(\mathbf{Z}_p,n) = \varinjlim_k H^*K(\mathbf{Z}/p^k\mathbf{Z},n)$$

Moreover it follows by a spectral sequence argument that the tower $\{map(B\tau,K(\mathbf{Z}/p^k\mathbf{Z},k))\}_s$ is an R completion tower for the function complex $map(B\tau,K(\mathbf{Z},n))$, since all function complexes involved here are R-nilpotent. Again using comparison of spectral sequences converging to homology we see that there is an R-homology (hence R-cohomology) equivalence $map(B\tau,K(\mathbf{Z},n)) \to map(B\tau,K(\mathbf{Z}_p,n))$. Therefore the R-cohomology of the last range is isomorphic to the limit of the R-cohomologies $\varinjlim_k H^*map(B\tau,K(\mathbf{Z}/p^k\mathbf{Z},n))$. But since the functor T commutes with direct limits we get the desired example:

$$TH^*K(\mathbf{Z}_p,n) \cong H^*map(B\tau,K(\mathbf{Z}_p,n)).$$

The second example of $K(\mathbf{Z}_p,n)$ is in reality equivalent to the first using the isomorphism of cohomologies $H^*(B\tau,\mathbf{Z}) \cong H^*(B\tau,\mathbf{Z}_p)$. Since the function complexes $hom(B\tau,K(\mathbf{Z},n))$ and $hom(B\tau,K(\mathbf{Z}_p,n))$ are built out of these cohomology groups, the map $\mathbf{Z} \to \mathbf{Z}_p$ induces a homotopy equivalence between them. Now since $TH^*K(\mathbf{Z},n) \cong TH^*K(\mathbf{Z}_p,n)$ one satisfies 2.4 if and only if the other does.

2.5 Example. It is not hard to check that if V is an infinite dimensional vector space over R then 2.3 fails to hold.

Similarly, let $RB\tau$ be the free (simplicial) R-module generated by $B\tau$, then R has non-trivial homotopy groups in all dimensions and $H^0 map(B\tau, RB\tau)$ is larger then $T^0 H^* B\tau$ which is countable.

3. **Proof of 1.2.** The main observation of this note is (1.2) from which (1.1) and (1.3) follow. We use the Eilenberg Moore spectral sequence (EMSS) to gain information on H^*W as an object in $\mathcal{U}$, i.e. as an unstable module over the Steenrod algebra $\mathcal{A}$. D. Rector, L. Smith, A. Heller and others showed that there is a natural action on the Eilenberg-Moore spectral sequence $E^r(W \to E \xrightarrow{u} B)$ by $\mathcal{A}$ making the differentials $\mathcal{A}$-linears and such that E_∞ is a graded $\mathcal{A}$-module associated to a filtration:

3.1. $H^*W \supseteq \cdots \supseteq F^{-2} \supseteq F^{-1} \supseteq F^0 \supseteq 0 \supseteq 0 \supseteq \cdots$ of H^*W by $\mathcal{A}$-submodules. We shall need the following result of [Dwyer] that gives a necessary and sufficient condition for a strong convergences of the spectral sequence: For every n the above filtration of H^nW is *finite* iff $\pi_1 B$ operates nilpotently on H^i (fibre) for all i.

3.2 Observe that if $p : E \to B$ is a fibre map with B not necessarily connected and with $\pi_1(B, *)$ operates nilpotently on $H^i(p^{-1}(*))$, then EMSS of $(* \to B \leftarrow E)$ will be identical to the one associated to the connected component of $* \in B$ and therefore will likewise converge strongly to $H^*(p^{-1}(*))$. This is because the functor Tor_{H^*B} appearing in E_2 'eliminates' all the components of H^*E not hitting the component of $* \in B$ in H^*B, due to the trivial action of off base point elements in H^*B on $H^*(b) = R$.

Claim: If L is any space of then the Eilenberg-Moore spectral sequence for the fibre square

$$\begin{array}{ccc} map(L,W) & \to & map(L,E) \\ \downarrow & & \downarrow \bar{u} \\ map(L,*) & \to & map(L,B) \end{array} \tag{3.3}$$

induced by the fibration in (3.1) converges strongly.

Since $\text{map}(L, *) = *$ the above pull back square is, in fact, a fibre map $\bar{u}$ with a non-connected space $\text{map}(L, b)$ as the base and with $\text{map}(L, G)$ as the fibre are the component of the null homotopic maps in the base.

3.4 Lemma: *If $G \to E \to B$ is a principal fibration of spaces where G is a group, then for any space L the map* $\mathrm{map}(L, E) \to \mathrm{map}(L, B; E)$, *where the range is the space of maps $L \to B$ liftable to E, is a principal fibration with the group being* $\mathrm{map}(L, G)$ *and the action is pointwise.*

Proof: One checks directly that the pointwise action is a transitive action of $\mathrm{map}(L, G)$ on the fibres of the above maps.

It follows from the above that the EMSS of (3.3) converges strongly, and as argued above the E^s-terms are the same for the fibrations $\mathrm{map}(L, E;$ null homolopic on $B) \to \mathrm{map}(L, B;$ null homotopic$)$.

Now we wish to compare the Eilenberg-Moore spectral sequence of (3.4) to the spectral sequence gotten by applying T to the Eilenberg-Moore sequence of the given fibration $W \to E \xrightarrow{u} B$. Let $E_r(u)$ be the spectral sequence of the fibration u.

For each $r \geq 1$ and $s \leq 0$ the $\mathbf{Z}$ - graded objects $E_1^{s,*}, E_2^{s,*}$ are unstable modules over the Steenrod algebra since the first one is, being the cohomology of the space

$B \times B \times \cdots \times B \times E$ (product taken s times). (Notice, however, that if we grade $\{E_r^{p,q}\}$ by the total degree $p+q$, we do not get an unstable module, but rather a stable one - e.g. Sq^i can operate non-trivially on elements of bi-degree $(-s, s)$, for any $s \geq i > 0$.). Therefore, we can form a spectral sequence $\{TE_r;\ Td_r\}$ by applying T to each $E_r^{-s,*}(u)$ as an object in U, to get another object in U namely $TE_r^{-s,*}$.

3.5. **Claim.** $TE_r(u)$ converges to TH^*W in the sense that $TE_\infty^{s,*}(u)$ is associated graded $\mathcal{A}$-module to the $\mathcal{A}$-filtration TF^i with $\varinjlim TF^i \cong TH^*W$.

To see why notice (2.1) that T is exact so it converts an exact couple to an exact couple, and since all the terms in spectral sequence $E_r^{s,*}(u)$ are $\mathcal{A}$- modules in $\mathcal{U}$ and all derivations $\mathcal{A}$-maps one can apply the functor T to get another spectral sequence. Since T is a left adjoint it commutes with direct limits so that $H^*W = \varinjlim F^i$ implies what we need.

Let $E_r(\bar{u})$ be the spectral sequence of the fibre-square (3.4) for $L = B\tau$. One can construct a comparison map $TE_r(u) \to E_r(\bar{u})$ using the adjointness properties of T: the evaluation map $B\tau \times map(B\tau, X) \to X$ induces [Lannes] a map

$$TH^*X \to H^*map(B\tau, X).$$

Therefore there is a natural map of $\mathcal{A}$-modules

$$TE_r(u) \to E_r(\bar{u}).$$

Claim: Under the assumption of lemma 1.2, this map is an isomorphism.

Proof: First notice that if $K \in \mathcal{K}$ and $M, N \in \mathcal{U}$ are K-modules then $T(A \underset{K}{\otimes} B) = TA \underset{TK}{\otimes} TB$. This is because $A \underset{K}{\otimes} B$ is the cokernel of a difference map $A \otimes K \otimes B \to A \otimes B$ of the two operation of K. Now T commutes with $\otimes$ in $\mathcal{U}$ so TN, TM are TK-modules and again by commutation and (right) exactness of T (2.3) we get the tensor product. Next notice that since the unstable $\mathcal{A}$-model $Tor_s(M, N)$ is the s - homology group of a chain complex consisting in degree s of $M \otimes K \otimes K \otimes \ldots \otimes K \otimes N$ and since T preserves tensor products one has for all $s \geq 0$.

$$T(Tor_K^s(M, N)) = Tor_{TK}^s(TM, TN).$$

By assumption on the space E and by (2.3) we get the desired result. Thus we have an isomorphism for $r = 2$ thus for all r.

It follows that one has an isomorphism $TE_\infty(u) \cong E_\infty(\bar{u})$. Now we get for each submodule in the filtration an isomorphism:

$$TF^i(u) \cong F^i(\bar{u})$$

and taking direct limits, noting again that T commutes with direct limits, we get the desired result by comparison of spectral sequence, namely

$$TH^*W \cong H^*map(B\tau, W).$$

Thus using the bar construction of the EMSS we saw that $E_1^{s,*}$ is an unstable module over A. This proves 1.2.

4. **Proof of 1.1 and 1.3.** If $H^iX < \infty$ for all i, then $\pi_i P_s R_s X < \infty$ for all $s, i \geq 0$ where P_s is the Postnikov section. This means that the space $P_s R_s X$ satisfies the conditions of (1.2) and we have $TH^*P_sR_sX = H^*map(B\tau, P_sR_sX)$. But since $H_iX \to \{H_iR_sX\}$ is a pro-isomorphism of towers [Dror] of finite groups, we have $H^*X \cong \underrightarrow{lim}\ H^*P_sR_sX$ therefore (2.3)(iii) $TH^*X \cong \underrightarrow{lim}\ TH^*P_sR_sX = \underrightarrow{lim}\ H^*map(B\tau, H^*P_sR_sX)$, This gives 1.1.

Now one uses the following lemma of [Bousfield 9.3]. Consider a tower of fibrations of pointed R-nilpotent spaces $\{X_m\}$

4.1. **Lemma.** *If $\{\tilde{H}_i(X_m, R)\}_m$ are pro-trivial for $i \leq 1$ and pro-constant for all i, then $\underleftarrow{lim}\ X_m = X_\infty$ is simply connected and the map $H_i(X_\infty, R) \to \{H_i(X_m, R)\}$ is a pro-isomorphism for all i.*

We use (4.1) with $X_m = P_mR_mX$.

Notice that if $\{H_m\}$ is an inverse tower of finite groups with $A_\infty = \underleftarrow{lim}\ A_m$ a finite group then the map $A_\infty \to \{A_m\}_m$ is a pro-isomorphism, because $\underleftarrow{lim}^1\ (-)$ vanishes on tower of finite groups and $\underleftarrow{lim}$ is left exact. Consider the tower $H_i(X_s^{B\tau})_c = (H^i(X_s^{B\tau})_c^*$ where $(-)^*$ denotes the R-dual. By 1.2 this is a tower of finite groups since one considers only a component $(X_s^{B\tau})_c$ for which T_c is finite in all dimension. Therefore $H_0(X_s^{B\tau})_c \cong R$ the tower $H_1(X_s^{B\tau})_c$ is pro-trivial, being pro-isomorphic to $(T_c^1H^*(X))^*$. Therefore by lemma 4.1 the tower $\{H_i(X_s^{B\tau})_c\}_{s\geq 0}$ is pro-isomorphic to $H_i(\underleftarrow{lim}\ (X_s^{B\tau})_c)$.

Since for any tower $\cdots \to Y_{m+1} \to Y_m \to \cdots Y_0$ of fibrations taking inverse limit commutes with taking function complex $map(L, -)$ the desired conclusion follow from $\underleftarrow{lim}\ (X_s)^{B\tau} \sim (\underleftarrow{lim}\ X_s)^{B\tau} = X^{B\tau}$, since X is assumed to be R-nilpotent.

4.2. **Proof of 1.4.** By [Lannes 7.1.1] we have again $H^0map(B\tau, X) \cong T^0H^*X$ so that as in 1.3. T_cH^*X is a well defined component corresponding to $[f] \in [B\tau, X]$. We have $X_f^{B\tau} = \underleftarrow{lim}\ map(B\tau, P_nX)_{f_n}$ where f_n is the obvious composition $B\tau \xrightarrow{f} X \to P_nX$. Since all the relevant homotopy groups are finite, one gets vanishing lim^1 - term and thus $\pi_iX_f^{B\tau} \cong \underleftarrow{lim}\ \pi_i(map\ \ (B\tau, P_nX),\ \ f_n)$. But, again this means that there is a pro-isomorphism $\tau_iX_f^{B\tau} \cong \{\pi_i(map(B\tau, P_nX), f_n)\}_n$ for each i, so that the constant tower $X_f^{B\tau}$ is pro-equivalent to the tower $\{P_nX)_{f_n}^{B\tau}\}_n$. Thus, they have the same R - cohomology.

But the R - cohomology of the latter is pro-isomorphic to TH^*X as needed.

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Spaces of Null Homotopic Maps

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§1. INTRODUCTION

In 1983 Haynes Miller [7] proved a conjecture of Sullivan and used it to show that if π is a locally finite group and X is a simply connected finite dimensional CW-complex then the space of pointed maps from the classifying space $B\pi$ to X is weakly contractible, ie. $\mathrm{Map}_*(B\pi, X) \simeq *$. This result had immediate applications. Alex Zabrodsky [11] used it to study maps between classifying spaces of compact Lie groups. McGibbon and Neisendorfer [6] applied Miller's theorem to answer a question of Serre; they proved that if X is a simply connected finite dimensional CW-complex with $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ then there are infinitely many dimensions in which $\pi_*(X)$ has p-torsion.

The goal of this note is to use the functor T^V of [2] to generalize Miller's theorem and some of its corollaries to a large class of infinite dimensional spaces (see [5] for closely related earlier work in this direction). This generalization comes at the expense of working with one component of the function complex $\mathrm{Map}_*(B\pi, X)$ at a time.

Fix a prime number p.

THEOREM 1.1. *Let π be a locally finite group and X a simply connected p-complete space. Assume that $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra. Then the component of* $\mathrm{Map}_*(B\pi, X)$ *which contains the constant map is weakly contractible.*

REMARK: There is a standard way [7, 1.5] to relax the assumption in 1.1 that X is p-complete.

Theorem 1.1 is actually a special case of a more general assertion. Recall that an unstable module M over the mod p Steenrod Algebra $\mathbf{A}_p$ is said to be *locally finite* [4] if each element $x \in M$ is contained in a finite $\mathbf{A}_p$ submodule. If R is a connected unstable algebra over $\mathbf{A}_p$ then the *augmentation ideal* $I(R)$ is by definition the ideal of positive-dimensional elements and the *module of indecomposables* $Q(R)$ is the unstable $\mathbf{A}_p$ module $I(R)/I(R)^2$. An unstable algebra R over $\mathbf{A}_p$ is of *finite type* if each R^k is finite-dimensional as an $\mathbf{F}_p$ vector space.

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THEOREM 1.2. *Let π be a locally finite group and X a simply connected p-complete space such that $H^*(X, \mathbf{F}_p)$ is of finite type. Assume that the module of indecomposables $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then the component of* $\mathrm{Map}_*(B\pi, X)$ *which contains the constant map is weakly contractible.*

REMARK: Theorem 1.1 does in fact follow from Theorem 1.2, since if $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra then $Q(H^*(X, \mathbf{F}_p))$ is a *finite* $\mathbf{A}_p$ module.

REMARK: Theorem 1.2 has a converse, at least if $p = 2$ (see Theorem 3.2). There is also a generalization of 1.2 that deals with other components of the mapping space $\mathrm{Map}_*(B\pi, X)$ (see Theorem 4.1) but for this generalization it is necessary to assume that π is an elementary abelian p-group.

Given 1.2, the arguments of [**6**] go over more or less directly and lead to the following result. A CW-complex is of *finite type* if it has a finite number of cells in each dimension.

THEOREM 1.3. *Suppose that X is a two-connected CW-complex of finite type. Assume that $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ and that $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then there exist infinitely many k such that $\pi_k(X)$ has p-torsion.*

REMARK: The example of CP^∞ shows that it would not be enough in Theorem 1.3 to assume that X is 1-connected.

Some instances of 1.3 were previously known; for instance, if $X = BG$ for G a suitable compact Lie group then the conclusion of 1.3 can be obtained by applying [**6**] to the loop space on X. However, Theorem 1.3 applies in many previously inaccessible cases; for example, it applies if X is the Borel construction $EG \times_G Y$ of the action of a compact Lie group G on a finite complex Y or if X is a quotient space obtained from such a Borel construction by collapsing out a skeleton.

We first noticed Theorem 1.1 as part of our work [**1**] on calculating fragments of T^V with Smith theory techniques. The proof of 1.1 given here does not use the localization approach of [**1**]; it is partly for this reason that the proof generalizes to give 1.2.

Organization of the paper. Section 2 recalls some properties of the functor T^V. In §3 there is a proof of 1.2 and in §4 a generalization of 1.2 to other components of the mapping space. Section 5 uses the ideas of [**6**] to deduce 1.3 from 1.2.

Notation and terminology. The prime p is fixed for the rest of the paper; all unspecified cohomology is taken with $\mathbf{F}_p$ coefficients. The symbol $\mathcal{U}$ (resp. $\mathcal{K}$) will denote the category of unstable modules (resp. algebras) [**2**] over $\mathbf{A}_p$. If $R \in \mathcal{K}$ then $\mathcal{U}(R)$ (resp. $\mathcal{K}(R)$) will stand for the category of objects of $\mathcal{U}$ (resp. $\mathcal{K}$) which are also R-modules (resp. R-algebras) in a compatible way [**1**].

For a pointed map $f : K \to X$ of spaces we will let $\mathrm{Map}_*(K, X)_f$ denote the component of the pointed mapping space $\mathrm{Map}_*(K, X)$ containing f. The component of the *unpointed* mapping space containing f is $\mathrm{Map}(K, X)_f$.

§2 The functor T^V

Let V be an elementary abelian p-group, ie., a finite-dimensional vector space over $\mathbf{F}_p$, and H^V the classifying space cohomology H^*BV. Lannes [**2**] has constructed a functor $T^V : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor given by tensor product (over $\mathbf{F}_p$) with H^V and has shown that T^V lifts to a functor $\mathcal{K} \to \mathcal{K}$ which is similarly left adjoint to tensoring with H^V.

Proposition 2.1 [**2**]. *For any object R of $\mathcal{K}$ the functor T^V induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V(R))$. The functor T^V is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V(M \otimes_R N) \cong T^V(M) \otimes_{T^V(R)} T^V(N)$$

Now suppose that $\gamma : R \to H^V$ is a $\mathcal{K}$-map. The adjoint of γ is a map $T^V(R) \to \mathbf{F}_p$ or in other words a ring homomorphism $\hat{\gamma} : T^V(R)^0 \to \mathbf{F}_p$. For $M \in \mathcal{U}(R)$, let $T^V_\gamma(M)$ be the tensor product $T^V(M) \otimes_{T^V(R)^0} \mathbf{F}_p$, where the action of $T^V(R)^0$ on $\mathbf{F}_p$ is given by $\hat{\gamma}$. Note that $T^V_\gamma(R) \in \mathcal{K}$.

Proposition 2.2 [**1**, 2.1]. *For any $\mathcal{K}$-map $\gamma : R \to H^V$ the functor $T^V_\gamma(-)$ induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V_\gamma(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V_\gamma(R))$. The functor T^V_γ is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V_\gamma(M \otimes_R N) \cong T^V_\gamma(M) \otimes_{T^V_\gamma(R)} T^V_\gamma(N).$$

The following proposition is a straightforward consequence of the above two.

LEMMA 2.3. *Suppose that $\alpha : R_1 \to R_2$ and $\beta : R_2 \to H^V$ are morphisms of $\mathcal{K}$, and let $\gamma : R_1 \to H^V$ denote the composite $\beta \cdot \alpha$.*

(1) *If α is a surjection and $M \in \mathcal{U}(R_2)$ is treated via α as an object of $\mathcal{U}(R_1)$, then the natural map $T^V_\gamma(M) \to T^V_\beta(M)$ is an isomorphism.*
(2) *If $M \in \mathcal{U}(R_1)$ then the natural map $T^V_\beta(R_2) \otimes_{T^V_\gamma(R_1)} T^V_\gamma(M) \to T^V_\beta(R_2 \otimes_{R_1} M)$ is an isomorphism.*

There is a natural map $\lambda_X : T^V(H^*X) \to H^*\operatorname{Map}(BV, X)$ for any space X. If $g : BV \to X$ is a map which induces the cohomology homomorphism $\gamma : H^*X \to H^V$ then λ_X passes to a quotient map

$$\lambda_{X,g} : T^V_\gamma(H^*X) \to H^*\operatorname{Map}(BV, X)_g\,.$$

A lot of the geometric usefulness of T^V is explained by the following theorem.

THEOREM 2.4 [3]. *Let X be a 1-connected space, $g : BV \to X$ a map, and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Assume that H^*X is of finite type, that $T^V_\gamma H^*X$ is of finite type, and that $T^V_\gamma H^*X$ vanishes in dimension 1. Then $\lambda_{X,g}$ is an isomorphism.*

For any object M of $\mathcal{U}$ the adjunction map $M \to H^V \otimes_{\mathbf{F}_p} T^V(M)$ can be combined with the unique algebra map $H^V \to \mathbf{F}_p$ to give a map $M \to T^V(M)$; call this map ϵ. (If $M = H^*X$ for some space X, then ϵ fits into a commutative diagram involving λ_X and the cohomology homomorphism induced by the basepoint evaluation map $\operatorname{Map}(BV, X) \to X$.)

THEOREM 2.5 [4, 6.3.2]. *The map $\epsilon : M \to T^V(M)$ is an isomorphism iff M is locally finite as a module over $\mathbf{A}_p$.*

If $R \in \mathcal{K}$, $M \in \mathcal{U}(R)$ and $\gamma : R \to H^V$ is a $\mathcal{K}$-map, we will denote the composite $M \xrightarrow{\epsilon} T^V(M) \to T^V_\gamma(M)$ by ϵ_γ. Theorem 2.5 leads to the following result, which we will need in §4.

PROPOSITION 2.6. *Let M be an object of $\mathcal{U}(H^V)$ and $\iota : H^V \to H^V$ the identity map. Then $\epsilon_\iota : M \to T^V_\iota(M)$ is an isomorphism iff M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$.*

PROOF: The fact that ϵ_ι is an isomorphism if M has the stated tensor product decompositon follows directly from 2.3(2), 2.5 and [**2**, 4.2]. Conversely, under the assumption that ϵ_ι is an isomorphism Proposition 2.4 of [**1**] guarantees that M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$; the fact that N is locally finite is again a consequence of 2.3(2) and 2.5.

§3 The null component

In this section we will prove Theorem 1.2. Before doing this we will recast the conclusion of the theorem in a slightly different form.

Lemma 3.1. *Let K be a finite pointed CW-complex, X a 1-connected space, and $f : K \to X$ a pointed map. Then* $\mathrm{Map}_*(K,X)_f$ *is weakly contractible if and only if the inclusion of the basepoint in K induces a weak equivalence* $\mathrm{Map}(K,X)_f \to X$.

Proof: As in [7, 9.1] the inclusion $* \to K$ gives rise to a fibration sequence $\mathrm{Map}_*(K,X)_f \to \mathrm{Map}(K,X)_f \to X$.

The arguments of [7, §9] now show that Theorem 1.2 follows directly from the following result.

Theorem 3.2. *Let V be an elementary abelian p-group and X a 1-connected p-complete space such that H^*X is of finite type. Let $f : BV \to X$ be a constant map and $\phi : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:*

1. *QH^*X is locally finite as an $\mathbf{A}_p$ module*
2. *the map $\epsilon_\phi : H^*X \to T^V_\phi H^*X$ is an isomorphism*
3. *the inclusion of the basepoint $* \to BV$ induces a weak equivalence* $\mathrm{Map}(BV,X)_f \to X$.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *Moreover, if $p=2$ then* (3) $\Longrightarrow$ (1).

Remark 3.3: It is likely that the three conditions of Theorem 1.2 are equivalent for *any* prime p; the proof would depend on the odd primary version of the results in [9].

Proof of 3.2: First consider the implication (1) $\Longrightarrow$ (2). Let $R = H^*X$ and let $I \subset R$ be the augmentation ideal. Pick $s \geq 0$. The fact that the action of R on I^s/I^{s+1} factors through the augmentation $R \to \mathbf{F}_p$ implies that the action of $T^V(R)$ on $T^V(I^s/I^{s+1})$ factors through the map $T^V(R) \to T^V(\mathbf{F}_p) \cong \mathbf{F}_p$ induced by augmentation; since this last map is adjoint to $\phi : R \to H^*(BV)$ it follows from 2.3(1) that the quotient map $T^V(I^s/I^{s+1}) \to T^V_\phi(I^s/I^{s+1})$ is an isomorphism. Moreover, I^s/I^{s+1}, as a quotient of $(I/I^2)^{\otimes s}$, is the union of its finite $\mathbf{A}_p$ submodules so by 2.5 the map $\epsilon : I^s/I^{s+1} \to T^V(I^s/I^{s+1})$ is an isomorphism. Putting these two facts together shows that $\epsilon_\phi : I^s/I^{s+1} \to T^V_\phi(I^s/I^{s+1})$ is an isomorphism. By induction and exactness, then, the map $\epsilon_\phi : R/I^{s+1} \to T^V_\phi(R/I^{s+1})$ is an isomorphism. The map $T^V_\phi(R) \to T^V_\phi(\mathbf{F}_p) \cong \mathbf{F}_p$ induced by augmentation is an epimorphism, so by exactness $T^V_\phi(I)$ vanishes in dimension 0.

By Lemma 2.2 and exactness, $T_\phi^V(I^{s+1})$ vanishes up to and including dimension s, and hence again by exactness the map $T_\phi^V(R) \to T_\phi^V(R/I^{s+1})$ induced by the quotient projection $R \to R/I^{s+1}$ is an isomorphism up through dimension s. It follows immediately that $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism.

The implication (2) $\implies$ (3) is an easy consequence of Theorem 2.4.

For (3) $\implies$ (1), assume $p = 2$. According to [**9**, proof of 3.1] condition (3) implies that the loop space cohomology $H^*(\Omega X)$ is locally finite as an $\mathbf{A}_p$ module, ie., in the terminology of [**9**], that $H^*(\Omega X) \in \mathcal{N}\mathrm{il}_k$ for all k. According to [**9**, 2.1(iii)], this implies that $\Sigma^{-1}QH^*X \in \mathcal{N}\mathrm{il}_k$ for all k. This amounts to the assertion that $\Sigma^{-1}QH^*X$ (or equivalently QH^*X) is locally finite [**9**, proof of 3.1].

§4 Other mapping space components

In this section we will give a generalization of Theorem 1.2 to mapping space components other than the component containing the constant map; this generalization is limited, however, in that it deals with elementary abelian p-groups rather than with arbitrary locally finite groups.

Given an elementary abelian p-group V, call an object M of $\mathcal{U}(H^V)$ *f-split* if M is isomorphic to $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$. Suppose that $\gamma : R \to H^V$ is a map in $\mathcal{K}$ with image $S \subset H^V$ and kernel $I \subset R$. Say that γ is *almost f-split* if

(*i*) S is a Hopf subalgebra of H^V, and

(*ii*) for each $s \geq 0$ the tensor product $H^V \otimes_S (I^s/I^{s+1})$ is f-split as an object of $\mathcal{U}(H^V)$.

Recall from 3.1 that $\mathrm{Map}_*(K, X)_f$ is weakly contractible iff evaluation at the basepoint gives an equivalence $\mathrm{Map}(K, X)_f \cong X$.

Theorem 4.1. *Let V be an elementary abelian p-group and X a 1-connected p-complete space such that H^*X is of finite type. Let $g : BV \to X$ be a map and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:*

(1) *γ is almost f-split*

(2) *the map $\epsilon_\gamma : H^*X \to T_\gamma^V H^*X$ is an isomorphism*

(3) *the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\mathrm{Map}(BV, X)_g \to X$.*

Then (1) $\implies$ (2) $\implies$ (3). *Morever, if $p = 2$ then* (3) $\implies$ (2) $\implies$ (1).

Remark 4.2: As in the case of Theorem 3.2, it is likely that the three conditions of Theorem 4.1 are equivalent for any prime p.

LEMMA 4.3. *Let K be a pointed CW-complex, X a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that there exists a map $m : K \times X \to X$ which is 1_X on the axis $* \times X$ and $g : K \to X$ on the axis $K \times *$. Then the basepoint evaluation map $e_f : \mathrm{Map}(K, X)_f \to X$ is a weak equivalence if and only if the corresponding map $e_g : \mathrm{Map}(K, X)_g \to X$ is a weak equivalence.*

PROOF: Construct a commutative diagram

$$\begin{tikzcd} K \arrow[r, "="] \arrow[d, "a"'] & K \arrow[d, "b"] \\ K \times X \arrow[r, "(pr_1, m)"'] & K \times X \end{tikzcd}$$

in which $a(k) = (k, *)$, $b(k) = (k, g(k))$ and pr_1 is projection on the first factor. Since the lower horizontal map is a weak equivalence, it follows that the induced map $c : \mathrm{Map}(K, K \times X)_a \to \mathrm{Map}(K, K \times X)_b$ is a weak equivalence. It is clear that c commutes with the natural projections from its domain and range to $\mathrm{Map}(K, K)_i$, where i is the identity map of K. The lemma follows from the fact that the domain of c is $\mathrm{Map}(K, K)_i \times \mathrm{Map}(K, X)_f$ while the range is $\mathrm{Map}(K, K)_i \times \mathrm{Map}(K, X)_g$.

LEMMA 4.4. *Let K be a pointed CW-complex, X a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that the basepoint evaluation map $e_g : \mathrm{Map}(K, X)_g \to X$ is a weak equivalence. Then the basepoint evaluation map $e_f : \mathrm{Map}(K, X)_f \to X$ is also a weak equivalence.*

PROOF: The map m required in 4.3 is provided up to weak equivalence by the evaluation map $K \times \mathrm{Map}(K, X)_g \to X$.

LEMMA 4.5. *Let V be an elementary abelian p-group, R a connected object of $\mathcal{K}$, $\gamma : R \to H^V$ a map, and $\phi : R \to H^V$ the trivial map (ie. the map which factors through the augmentation $R \to \mathbf{F}_p$). Assume there exists a map $\mu : R \to H^V \otimes_{\mathbf{F}_p} R$ which gives 1_R when combined with the augmentation map of H^V and $\gamma : R \to H^V$ when combined with the augmentation map of R. Then $\epsilon_\phi : R \to T^V_\phi(R)$ is an isomorphism if and only if $\epsilon_\gamma : R \to T^V_\gamma(R)$ is an isomorphism.*

PROOF: This is essentially the proof of 4.3 with the arrows reversed.

Construct a commutative diagram

$$\begin{tikzcd} H^V & H^V \arrow[l, "="'] \\ H^V \otimes_{\mathbf{F}_p} R \arrow[u, "\alpha"] & H^V \otimes_{\mathbf{F}_p} R \arrow[l, "in_1 \cdot \mu"'] \arrow[u, "\beta"'] \end{tikzcd}$$

in which α is the product of 1_{H^V} with the augmentation of R, β is $(1_{H^V}) \cdot \gamma$, and in_1 is the map from H^V to the tensor product obtained using the unit of R. Since the lower horizontal map is an isomorphism, it follows that the induced map $\chi : T^V_\beta(H^V \otimes_{\mathbf{F}_p} R) \to T^V_\alpha(H^V \otimes_{\mathbf{F}_p} R)$ is an isomorphism. It is clear that χ respects the natural structures of its domain and range as modules over $T^V_\iota(H^V)$, where ι the identity map of H^V. The lemma follows from the fact [**1**, 2.2] that the domain of χ is $T^V_\iota(H^V) \otimes_{\mathbf{F}_p} T^V_\gamma(R)$ while the range is $T^V_\iota(H^V) \otimes_{\mathbf{F}_p} T^V_\phi(R)$.

LEMMA 4.6. *Let V be an elementary abelian p-group, R a connected object of $\mathcal{K}$, $\gamma : R \to H^V$ a map and $\phi : R \to H^V$ the trivial map. Assume that $\epsilon_\gamma : R \to T^V_\gamma(R)$ is an isomorphism. Then $\epsilon_\phi : R \to T^V_\phi(R)$ is also an isomorphism.*

PROOF: The map μ required in 4.5 is provided by the map $R \to H^V \otimes_{\mathbf{F}_p} T^V_\gamma(R)$ which is adjoint to the identity map of $T^V_\gamma(R)$.

REMARK 4.7: It follows from 4.5, 4.6 and 3.2 that at least if $p = 2$ the three conditions of 4.1 are equivalent to a fourth, namely, that QH^*X is locally finite as an $\mathbf{A}_p$ module and there exists a $\mathcal{K}$ map $H^*X \to H^V \otimes_{\mathbf{F}_p} H^*X$ which satisfies the conditions of 4.5.

LEMMA 4.8. *Let V be an elementary abelian p-group and $\nu : S \to H^V$ the inclusion of a subalgebra over $\mathbf{A}_p$. Then $\epsilon_\nu : S \to T^V_\nu(S)$ is an isomorphism if and only if ν includes S as a Hopf subalgebra of H^V.*

PROOF: Suppose that ϵ_ν is an isomorphism. In this case the adjunction homomorphism $S \to H^V \otimes_{\mathbf{F}_p} T^V_\nu(S)$ provides a map $\Delta_S : S \to H^V \otimes_{\mathbf{F}_p} S$ which fits into a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\Delta_S"] \arrow[d, "\nu"'] & H^V \otimes_{\mathbf{F}_p} S \arrow[d, "\iota\otimes\nu"] \\ H^V \arrow[r, "\Delta_{H^V}"'] & H^V \otimes_{\mathbf{F}_p} H^V \end{tikzcd}$$

$T^V_\gamma(H^V)$ is injective, and it follows from naturality and the fact that $H^V \to T^V_\gamma(H^V)$ is injective [**2**, 4.2] that $S \to T^V_\gamma(S)$ is injective. By 2.3(1) the map $\epsilon_\nu : S \to T^V_\nu(S)$ is an isomorphism and hence (4.8) S is a Hopf subalgebra of H^V.

By exactness the map $I^s \to T^V_\gamma(I^s)$ is seen to be an isomorphism if $s = 1$ and a monomorphism if $s > 1$; this first fact, though, combines with the tensor product formula (2.2) and exactness to show that $I^s \to T^V_\gamma(I^s)$ is an epimorphism for $s \geq 1$. Thus by exactness and 2.3(1) the maps $\epsilon_\nu : I^s/I^{s+1} \to T^V_\nu(I^s/I^{s+1})$ are isomorphisms. The proof is finished by running in reverse the argument used above at the end of the proof of (1) $\Longrightarrow$ (2).

§5 Torsion in homotopy groups

In this section we will use a slight variation on the ideas of [**6**] to prove Theorem 1.3.

Let $\mathbf{Z}$ denote the ring of integers, $\mathbf{Z}\hat{_p}$ the additive group of p-adic integers, and $\mathbf{Z}/p^n$ the cyclic group of order p^n. The group $\mathbf{Z}/p^\infty$ is by definition the locally finite group obtained by taking the direct limit of the groups $\mathbf{Z}/p^n$ under the standard inclusion maps.

Lemma 5.1. *For any finitely-generated abelian group A the cohomology group $H^k(B\mathbf{Z}/p^\infty, A)$ is isomorphic to $\mathbf{Z}\hat{_p} \otimes A$ if $k > 0$ is even and is zero if k is odd. The natural map $A \to \mathbf{Z}\hat{_p} \otimes A$ induces isomorphisms $H^k(B\mathbf{Z}/p^\infty, A) \cong H^k(B\mathbf{Z}/p^\infty, \mathbf{Z}\hat{_p} \otimes A)$ for all $k > 0$.*

Sketch of proof: One way to do this is to calculate the homology $H_*(B\mathbf{Z}/p^\infty, \mathbf{Z})$ as a direct limit $\varinjlim_n H_*(B\mathbf{Z}/p^n, \mathbf{Z})$ and then pass to cohomology by using the universal coefficient theorem. The key algebraic ingredient is the fact that

$$\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/p^\infty, \mathbf{Z}) \cong \mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/p^\infty, \mathbf{Z}\hat{_p}) \cong \mathbf{Z}\hat{_p}\,.$$

Let P_nX stand for the n'th Postnikov stage of the space X and $k^{n+1}(X)$ for the Postnikov invariant of X which lies in $H^{n+1}(P_{n-1}X, \pi_nX)$(see [**10**, IX]).

Lemma 5.2. *If Y is a loop space ΩX and Y has finitely-generated homotopy groups, then the Postnikov invariants of Y are torsion cohomology classes.*

Proof: This follow from [**8**, p. 263]. In effect, Milnor and Moore show that the rationalized Postnikov invariants

$$k^{n+1}(Y) \otimes \mathbf{Q} \in H^{n+1}(P_{n-1}Y, \pi_n(Y) \otimes \mathbf{Q})$$

where ι is the identity map of H^V and we have used the fact [**2**, 4.2] that $\epsilon_\iota : H^V \to H^V$ is an isomorphism. It is easy to see that Δ_{H^V} is the Hopf algebra comultiplication map on H^V. It now follows from the fact that the comultiplication on H^V is cocommutative that $\Delta_S(S) \subset S \otimes_{\mathbf{F}_p} S$ and thus that S is a Hopf subalgebra of H^V.

Suppose conversely that S is a Hopf subalgebra of H^V, and let $\phi : S \to H^V$ be the trivial map which factors through the augmentation $S \to \mathbf{F}_p$. The Hopf algebra H^V is primitively generated, and the associated restricted Lie algebra of primitives [8, 6.7] is a free abelian restricted Lie algebra on a finite collection of generators (in dimensions 1 and 2). It follows from [**8**, 6.13–6.16] that S is primitively generated and is isomorphic as an algebra to a finite tensor product of exterior and polynomial algebras; in particular, $Q(S)$ is a finite unstable $\mathbf{A}_p$ module. By the proof of (1) $\Longrightarrow$ (2) in Theorem 3.2 the map $\epsilon_\phi : S \to T_\phi^V(S)$ is an isomorphism. Since the comultiplication of S produces the map μ required for Lemma 4.5, an application of this lemma finishes the proof.

Proof of 4.1: Let R denote H^*X, I the kernel of $\gamma : R \to H^V$, S the image of γ and $\nu : S \to H^V$ the inclusion map. We will use f to stand for a constant map $BV \to X$ and ϕ for the cohomology homomorphism induced by f.

(1) $\Longrightarrow$ (2). The assumption that S is a Hopf subalgeba of H^V implies by 4.8 that $\epsilon_\nu : S \to T_\nu^V(S)$ and hence (2.3(1)) $\epsilon_\gamma : S \to T_\gamma^V(S)$ are isomorphisms. Pick $s \geq 1$ and let $M = I^s/I^{s+1}$. If we can show that $\epsilon_\gamma : M \cong T_\gamma^V(M)$ we will be able to finish up by imitating the proof of (1) $\Longrightarrow$ (2) in Theorem 3.2. By 2.3(1) it is enough to show that $\epsilon_\nu : M \cong T_\nu^V(M)$. Proposition 2.6 ensures that $\epsilon_\iota : H^V \otimes_S M \to T_\iota^V(H^V \otimes_S M)$ is an isomorphism, where ι is the identity map of H^V. By 2.3(2) and [**2**, 4.2] , however, the map ϵ_ι is $\iota \otimes_S \epsilon_\nu$, so the desired result follows from the fact that H^V is free [8, 4.4] and therefore faithfully flat as a module over S.

(2) $\Longrightarrow$ (3). This is an immediate consequence of 2.4.

(3) $\Longrightarrow$ (2). By Lemma 4.4 and Theorem 3.2 the map $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism. The evaluation map $m : BV \times \mathrm{Map}(BV, X)_g \to X$ induces a cohomology homomorphism $\mu : R \to H^V \otimes_{\mathbf{F}_p} R$ which satisfies the conditions of 4.5, so the implication follows from the conclusion of 4.5.

(2) $\Longrightarrow$ (1). This implication does not in fact require the assumption that $p = 2$. The map $T_\gamma^V(R) \to T_\gamma^V(S)$ is surjective and it follows immediately from naturality that $\epsilon_\gamma : S \to T_\gamma^V(S)$ is surjective. The map $T_\gamma^V(S) \to$

are zero. Under the stated finite generation assumption this implies that the Postnikov invariants themselves are torsion.

Proof of 1.3: Let S_1 be the set of all k such that $\pi_k(X) \otimes \mathbf{Z}_p^\wedge \neq 0$ and S_2 the set of all k such that $\pi_k X$ contains p-torsion. The set S_1 is non-empty (because $H^*(X, \mathbf{F}_p) \neq 0$) and clearly contains S_2. Suppose that S_2 is finite. In that case we can find an integer k in S_1 such that no integer j greater than k belongs to S_2. Let $Y = \Omega^{k-2} X$. (Note that because X is 2-connected the integer k is greater than 2 and Y is a loop space.) By Lemma 5.1 the space $\mathrm{Map}_*(B\mathbf{Z}/p^\infty, P_1 Y)$ is contractible and hence $\mathrm{Map}_*(B\mathbf{Z}/p^\infty, P_2 Y) \cong \mathrm{Map}_*(B\mathbf{Z}/p^\infty, K(\pi_2 Y, 2))$. Because of the way in which k was chosen we can thus, by Lemma 5.1 again, find an essential map $f : B\mathbf{Z}/p^\infty \to P_2 Y$ which remains essential in the p-completion $(P_2 Y)_p^\wedge$. The obstructions to lifting f to a map $g : B\mathbf{Z}/p^\infty \to Y$ are the pullbacks to $B\mathbf{Z}/p^\infty$ of the Postnikov invariants of Y [**10**, p. 450]; by Lemma 5.2 these obstructions are torsion, but by Lemma 5.1 and the choice of k they lie in torsion-free abelian groups. Therefore the obstructions vanish, and the lift g exists. The composite h of g with the completion map $Y \to Y_p^\wedge$ is non-trivial because the composite of h with the projection map $Y_p^\wedge \to P_2(Y_p^\wedge) \cong (P_2 Y)_p^\wedge$ is essential. The adjoint of h is then non-zero element of $\pi_{k-2} \mathrm{Map}_*(B\mathbf{Z}/p^\infty, X)$, an element which by Theorem 1.2 cannot exist. This contradiction shows that S_2 is infinite and proves the theorem.

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Dedicated to the memory of Alex Zabrodsky

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André-Quillen cohomology and the Bousfield-Kan spectral sequence

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André-Quillen Cohomology and the Bousfield-Kan Spectral Sequence

by Paul G. Goerss*

In this paper we investigate the Bousfield-Kan spectral sequence [6], [8]

$$Ext^s_{\mathcal{UA}}(H^*X, H^*S^t) \Rightarrow \pi_{t-s}X_p \tag{1}$$

where $H^*X = H^*(X, \mathbb{F}_p)$ is cohomology with coefficients in the prime field $\mathbb{F}_p$, $\mathcal{UA}$ is the category of augmented unstable algebras over the Steenrod algebra, S^t is the t-sphere and, X_p is the p-completion of the pointed space X. This spectral sequence, an unstable Adams spectral sequence, is a major tool in attempts to compute or understand the homotopy groups of spaces.

Because $\mathcal{UA}$ is not an abelian category, $Ext^*_{\mathcal{UA}}(H^*X, H^*S^t)$ must be defined using a cotriple and, hence, a simplicial resolution of the algebra H^*X. Our first point is to notice that if $s\mathcal{UA}$ is the category of simplicial objects in $\mathcal{UA}$, then there is a contravariant functor

$$H^*_{\mathcal{QA}} : s\mathcal{UA} \to nn\mathbb{F}_p \tag{2}$$

to the category of bigraded $\mathbb{F}_p$ vector spaces that generalizes $Ext_{\mathcal{UA}}$ in the following sense: if Λ is an object in $\mathcal{UA}$, then we may regard Λ as the constant simplicial object that is Λ in every simplicial degree and every face and degeneracy operator is the identity. Then we will have the equation

$$[H^s_{\mathcal{QA}}\Lambda]_t \cong Ext^s_{\mathcal{UA}}(\Lambda, H^*S^t) \tag{3}$$

where $[H^s_{\mathcal{QA}}\Lambda]_t$ denotes the elements of degree t in the graded vector space $H^s_{\mathcal{QA}}\Lambda$.

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For $A \in s\mathcal{UA}$, $H^*_{Q\mathcal{A}}A$ is a cohomology of A in the sense of Quillen [17]; in fact, this is the sort of cohomology of algebras that has been studied extensively by André and Quillen [1], [18]. Hence the title of the paper.

We will take the observation of the existence of $H^*_{Q\mathcal{A}}$ in two directions. The first is this: the category of simplicial objects $s\mathcal{UA}$ has a structure of a closed model category and, as such, we can do the usual homotopy theoretic constructions. In particular, if $f : A \to B$ is a morphism in $s\mathcal{UA}$, then f has a homotopy cofiber $M(f)$ and there is a long exact sequence in cohomology similar to the one that Quillen called a transitivity sequence [18]:

$$\cdots \to H^{n-1}_{Q\mathcal{A}}A \to H^n_{Q\mathcal{A}}M(f) \to H^n_{Q\mathcal{A}}B \xrightarrow{H^*_{Q\mathcal{A}}f} H^n_{Q\mathcal{A}}A \to \cdots. \tag{4}$$

Here we need the full generality of $s\mathcal{UA}$. For, even if $f : \Lambda \to \Gamma$ is a morphism of constant simplicial algebras of the type considered in equation (3), $M(f)$ is not necessarily such an object. In fact, if we define the homotopy of an object $A \in s\mathcal{UA}$ by

$$\pi_*A = H_*(A, \partial)$$

where ∂ is the alternating sum of the face operators in A, then for a morphism $f : \Lambda \to \Gamma$ of constant simplicial algebras

$$\pi_*M(f) \cong Tor^\Lambda_*(\mathbf{F}_p, \Gamma).$$

We extend the long exact sequence in cohomology to the homotopy spectral sequence. The Bousfield-Kan spectral sequence is an example of the homotopy spectral sequence of a cosimplicial space. Given a fibrant cosimplicial space Z, there is a spectral sequence [8]

$$\pi^s\pi_tZ \Rightarrow \pi_{t-s}Tot(Z)$$

where $Tot(Z)$ is a kind of "codiagonal" of Z given by the mapping space of cosimplicial spaces

$$Tot(Z) = map(\Delta, Z)$$

where Δ is the cosimplicial space that in cosimplicial degree s is the standard s-simplex $\Delta[s]$. Now if Z is a cosimplicial space, then H^*Z is a simplicial object in $\mathcal{UA}$ and we will notice that we can extend the Bousfield-Kan spectral sequence to a spectral sequence

$$[H^s_{Q\mathcal{A}}H^*Z]_t \Rightarrow \pi_{t-s}Tot(Z)_p. \tag{5}$$

If X is a space and $Z = X$ is the constant simplicial space, then $H^*Z = H^*X$ is a constant simplicial algebra we can combine equation (3) with equation (5) to obtain the spectral sequence (1).

In particular, if $f : Z \to Y$ is a morphism of cosimplicial spaces, we will define a new cosimplicial space F so that $H^*F \cong M(f^*)$ — the homotopy cofiber of f^* in $s\mathcal{UA}$ — and so that there is a homotopy fibration sequence of spaces

$$Tot(F) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p.$$

Further, there will be a diagram of spectral sequences

$$\begin{array}{ccccccccc}
\to & [H^s_{Q\mathcal{A}}H^*Z]_t & \to & [H^s_{Q\mathcal{A}}H^*Y]_t & \xrightarrow{\partial} & [H^{s+1}_{Q\mathcal{A}}H^*F]_t & \to \\
 & \Downarrow & & \Downarrow & & \Downarrow & \\
\to & \pi_{t-s}Tot(Z)_p & \to & Tot_{t-s}Tot(Y)_p & \xrightarrow{\delta} & \pi_{t-s-1}Tot(F) & \to
\end{array}$$

where the top row comes from the long exact sequence (4) and the bottom row is the long exact sequence of the fibration sequence. The hard work here is to produce the commutative diagram of spectral sequences

$$\begin{array}{ccc}
[H^s_{Q\mathcal{A}}H^*Y]_t & \xrightarrow{\partial} & [H^{s+1}_{Q\mathcal{A}}H^*F]_t \\
\Downarrow & & \Downarrow \\
\pi_{t-s}Tot(Y)_p & \xrightarrow{\delta} & \pi_{t-s-1}Tot(F).
\end{array}$$

This is done in section 5.

There are other ramifications to the idea that $s\mathcal{UA}$ is a closed model category. Among them are the notions of universal infinite cycles and universal differentials for the spectral sequence (5). Although these were noted by Bousfield and Kan [6] and have been extended by the work of Bousfield [3], and are related to Barratt's desuspension spectral sequence, as rediscovered by Hopkins, they have not been systematically studied from our point of view. We undertake this study, beginning in section 3, but extending our computations into further sections.

The universal cycle, for example, is a cosimplicial space $\bar{\mathbf{F}}_p^{\cdot}S(s,t)$ whose cohomology is relevant to the Quillen cohomology of equation (2) in the following way. There is an object $K(s,t)_+ \in s\mathcal{UA}$ that corepresents cohomology in the usual way: there is a universal class $\iota \in [H^s_{Q\mathcal{A}}K(s,t)_+]_t$ and an isomorphism

$$[A, K(s,t)_+]_{s\mathcal{UA}} \xrightarrow{\cong} [H^s_{Q\mathcal{A}}A]_t$$

given by

$$f \longmapsto f^*(\iota).$$

$[\ ,\ ]_{s\mathcal{UA}}$ denotes the morphisms in the homotopy category associated to the closed model category on $s\mathcal{UA}$. Then, we have for the universal cycle $\bar{\mathbf{F}}_p^{\cdot}S(s,t)$, a weak equivalence in $s\mathcal{UA}$

$$H^*\bar{\mathbf{F}}_p^{\cdot}S(s,t) \xrightarrow{\simeq} K(s,t)_+$$

and, hence, an equation

$$H^*_{Q\mathcal{A}}H^*\bar{\mathbf{F}}_p^{\cdot}S(s,t) \cong H^*_{Q\mathcal{A}}K(s,t)_+.$$

Furthermore $Tot(\bar{\mathbf{F}}_p^{\cdot}S(s,t)) \simeq S_p^{t-s}$. Hence we get a spectral sequence

$$H^*_{Q\mathcal{A}}K(s,t)_+ \Rightarrow \pi_* S_p^{t-s}$$

that is universal in the following sense. Suppose that, for a fibrant cosimplicial space Z,

$$\alpha \in [H^s_{Q\mathcal{A}}H^*Z]_t$$

survives to E_∞ in the spectral sequence (5), and detects

$$x \in \pi_{t-s}Tot(Z)_p.$$

Then there is a morphism in $s\mathcal{UA}$

$$f : H^*Z \to K(p,q)_+$$

corepresenting α and the resulting map

$$H^*_{Q\mathcal{A}}f : H^*_{Q\mathcal{A}}K(s,t)_+ \to H^*_{Q\mathcal{A}}H^*Z$$

will fit into a diagram of spectral sequences, at least when $t-s>1$:

$$\begin{array}{ccc} H^*K(s,t)_+ & \Rightarrow & \pi_*S_p^{t-s} \\ \downarrow {\scriptstyle H^*_{\mathcal{QA}}f} & & \downarrow \\ H^*_{\mathcal{QA}}H^*Z & \Rightarrow & \pi_*Tot(Z)_p \end{array}$$

where, under the map $\pi_*S^{t-s} \to \pi_*Tot(Z)_p$ the identity map passes to x. This is discussed in section 3 and the computation of $H^*_{\mathcal{QA}}K(s,t)_+$ is considered in section 9.

The second direction we take the existence of the Quillen cohomology functor $H^*_{\mathcal{QA}}$ is this: if $H^*_{\mathcal{QA}}$ is truly a cohomology theory, it should support products and operations. This is, in fact, the case. The work of Bousfield and Kan [7] can be interpreted to prove the existence of a commutative bilinear product

$$[\ ,\ ] : H^s_{\mathcal{QA}}A \otimes H^{s'}_{\mathcal{QA}}A \to H^{s+s'+1}_{\mathcal{QA}}A$$

satisfying the Jacobi identity and adding internal degree. In the spectral sequence this product will converge to the Whitehead product; that is, there is a commutative diagram of spectral sequences

$$\begin{array}{ccc} [H^s_{\mathcal{QA}}H^*Z]_t \otimes [H^{s'}_{\mathcal{QA}}H^*Z]_{t'} & \Rightarrow & \pi_{t-s}Tot(Z) \otimes \pi_{t'-s'}Tot(Z) \\ \downarrow {\scriptstyle [\ ,\ ]} & & \downarrow {\scriptstyle [\ ,\ ]} \\ [H^{s+s'+1}_{\mathcal{QA}}H^*Z]_{t+t'} & \Rightarrow & \pi_{t+t'-(s+s')-1}Tot(Z) \end{array}$$

where the right vertical map is the Whitehead product in homotopy.

If we specialize to the prime 2, there are also operations. These are homomorphisms

$$P^i : H^s_{\mathcal{QA}}A \to H^{s+i+1}_{\mathcal{QA}}A$$

doubling internal degree, and satisfying an unstable condition, a formula relating the Whitehead product to the operations, and a set of relations among themselves. In particular, we might call these operations "divided Whitehead products" because $P^i = 0$ if $i > s$ and

$$P^s(x) = [x,x].$$

The exact statement of the various relations is given at the beginning of section 7.

An interesting fact about these operations is that they do not, in general, commute with the differentials in the Bousfield-Kan spectral sequence. We prove this by examining the universal example mentioned above; in fact, portions of sections 7, 8 and 9 are devoted to investigating the operations in the spectral sequence of the universal infinite cycle:

$$H^*_{Q\mathcal{A}}K(s,t)_+ \Rightarrow \pi_* S^{t-s}_p.$$

At this point the reader might think that the work to be done here is highly theoretical, and this is largely the case. However, the last two sections of this paper are devoted to tools for computation, and there we make a serious attempt to compute and understand in detail the functor $H^*_{Q\mathcal{A}}$. We will use a composite functor spectral sequence due, in principal, to Haynes Miller [19] to begin computing the cohomology of the universal examples mentioned above and to undertake other projects, including trying to understand to what extent the Bousfield-Kan spectral sequence (1) satisfies the Hilton-Milnor Theorem. It turns out that we need the full generality of $H^*_{Q\mathcal{A}}$ to address this question, even if we are only trying to understand $Ext_{\mathcal{UA}}$. There are other tools available for computation, among them the work of André [1], and [12], which owes a debt to the work of Miller [19, Section 4].

There are also numerous examples scattered throughout, and clearly marked as such. These are intended to provide some concreteness to our work and to explain the relevance on the project to the work of others. In particular, see section 4.

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3. The Bousfield-Kan spectral sequence II: we generalize the spectral sequence and explore various examples, including universal infinite cycles and differentials.
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9. The cohomology of abelian objects: we compute $H^*_{Q\mathcal{A}}$ applied to some universal objects, including those of section 3, and show that the operations P^i don't commute with differentials.

Notation and conventions: Because cohomology algebras are more intuitive than homology coalgebras, we work with the former. However, we sacrifice generality, especially in convergence statements about homotopy spectral sequences. Therefore, we often make *finite type* hypotheses. A graded $\mathbb{F}_p$ vector space is of finite type if, for every n, the elements of degree n form a finite vector space.

A *space* is a simplicial set, usually pointed; that is, the space comes equipped with a chosen basepoint. If we are in a situation where the spaces are not pointed, we say so.

$\mathbf{F}_p$ is the field with p elements for some prime p, $\mathcal{A}$ is the mod p Steenrod algebra, and all homology and cohomology of spaces is with $\mathbf{F}_p$ coefficients.

If $\mathbf{C}$ is a category, then $s\mathbf{C}$ will denote the category of simplicial objects in $\mathbf{C}$ and $n\mathbf{C}$ will denote the category of graded objects in $\mathbf{C}$. In particular, $n\mathbf{F}_p$ will denote the category of graded $\mathbf{F}_p$ vector spaces and $nn\mathbf{F}_p$ the category of bigraded vector spaces.

If V is a simplicial vector space, we define

$$\pi_* V = H_*(V, \partial)$$

where

$$\partial = \sum_{i=0}^{s} (-1)^i d_i : V_s \to V_{s-1}$$

is the alternating sum of the face operators. If V is a cosimplicial vector space, we set

$$\pi^* V = H^*(V, \partial^*)$$

where ∂^* is the alternating sum of the coface operators. These definitions can be extended to any category with a forgetful functor to the category of vector spaces, graded vector spaces, or abelian groups. If V is a simplicial graded vector space, the $\pi_* V$ is a bigraded vector space. We refer to the elements of $[\pi_s V]^t$ as being of external degree s and internal degree t.

Part I: Quillen Cohomology and the Bousfield-Kan Spectral Sequence

1. The Bousfield-Kan Spectral Sequence I

This preliminary section is devoted to the definition of the basic object of study and to establishing notation. The Bousfield-Kan spectral sequence [6, 8] is an Adams-type spectral sequence passing from the homology of a space X to the homotopy of its p-completion X_p. A good introduction to this spectral sequence is given in Section 1 of Miller's paper [19].

We begin by defining some categories. Fix a prime p and let $\mathbb{F}_p$ be the field with p elements. We let $\mathcal{UA}$ be the category of unstable algebras over the Steenrod algebra. Thus $H \in \mathcal{UA}$ is a graded, commutative, supplemented $\mathbb{F}_p$ algebra that supports an action by the Steenrod algebra and so that the two structures are related by the Cartan formula and by the *unstable* condition: if $p > 2$, then

$$\mathrm{P}^n(x) = \begin{cases} 0, & \text{if } deg(x) < 2n; \\ x^p, & \text{if } deg(x) = 2n. \end{cases}$$

and if p = 2, then

$$\mathrm{Sq}^n(x) = \begin{cases} 0, & \text{if } deg(x) < n; \\ x^2, & \text{if } deg(x) = n. \end{cases}$$

The symbol $deg(x)$ means the degree of x as an element of the graded algebra H; the vector space of elements of degree n will be denoted by H^n.

If X is a pointed (based) space, then $H^*\mathrm{X} = H^*(X, \mathbb{F}_p)$ is an object of $\mathcal{UA}$.

There is a simpler, associated category $\mathcal{U}$ – the category of unstable modules over the Steenrod algebra $\mathcal{A}$. This is the full sub-category of the category of modules over $\mathcal{A}$ specified by the conditions that $M \in \mathcal{U}$ if

$$\beta^\epsilon \mathrm{P}^n(x) = 0 \qquad \text{if } deg(x) < 2n + \epsilon$$

$$\mathrm{Sq}^n(x) = 0 \qquad \text{if } deg(x) < n.$$

$\mathcal{U}$ is an abelian category.

The augmentation ideal functor $I : \mathcal{UA} \to \mathcal{U}$ has a left adjoint U; for example,

$$H^*S^{2k+1} \cong U(\Sigma^{2k+1}\mathbb{F}_p)$$

where $\Sigma^{2k+1}\mathbb{F}_p$ is the trivial $\mathcal{A}$ module of dimension 1 over $\mathbb{F}_p$ concentrated in degree $2k+1$.

Next consider the forgetful functor $J : \mathcal{U} \to n\mathbb{F}_p$ where $n\mathbb{F}_p$ is the category of graded $\mathbb{F}_p$ vector spaces. This, too has a left adjoint $P : n\mathbb{F}_p \to \mathcal{U}$; indeed, if V is of finite type, then

$$P(V) = PH^*K(V^*)$$

where the right hand side is the primitives in the indicated Hopf algebra, V^* is the graded vector space dual, and $K(V^*)$ is a generalized Eilenberg-MacLane space with $\pi_*K(V^*) \cong V^*$

As a consequence of the existence of P, the augmentation ideal functor $I : \mathcal{UA} \to n\mathbb{F}_p$ has a left adjoint G; namely

$$G = U \circ P \text{ or } G(V) = U(P(V)).$$

If V is of finite type, then $G(V) \cong H^*K(V^*)$. The composite functors

$$\overline{G} = G \circ I : \mathcal{UA} \to \mathcal{UA}$$

$$\overline{P} = P \circ J : \mathcal{U} \to \mathcal{U}$$

both have the structure of a cotriple on the respective category; that is, there are natural transformations

$$\epsilon_H : \overline{G}(H) \to H \text{ and } \epsilon_M : \overline{P}(M) \to M$$

$$\eta_H : \overline{G}(H) \to \overline{G}^2(H) \text{ and } \eta_M : \overline{P}(M) \to \overline{P}^2(M)$$

and these are related in such a manner that we may form the simplicial objects $\overline{G}.(H) \in s\mathcal{UA}$ and $\overline{P}.(M) \in s\mathrm{U}$. For example,

$$\overline{G}_n(H) = \overline{G}^{n+1}(H)$$

and

$$d_i : \overline{G}_n \to \overline{G}_{n-1}$$

is defined by

$$d_i = \overline{G}^i \epsilon \overline{G}^{n-i}, \qquad 0 \le i \le n$$

and

$$s_i : \overline{G}_n(H) \to \overline{G}_{n+1}(H)$$

is given by

$$s_i = \overline{G}^i \eta \overline{G}^{n-i}, \qquad 0 \le i \le n.$$

Both $\overline{G}.(H)$ and $\overline{P}.(M)$ are *augmented* simplicial objects in the sense that ϵ induces maps

$$\epsilon_H : \overline{G}_0(H) \to H \text{ and } \epsilon_M : \overline{P}_0(M) \to M$$

such that $\epsilon d_0 = \epsilon d_1$. More than this ϵ induces isomorphisms

$$\pi_* \overline{G}.(H) \cong H \text{ and } \pi_* \overline{P}.(M) \cong M$$

concentrated in external degree 0. The retraction that guarantees these isomorphisms is given by the inclusions in $n\mathbb{F}_p$

$$IH \to I\overline{G}(H) \qquad \text{and} \qquad M \to \overline{P}(M)$$

adjoint to the identity.

Thus $\epsilon : \overline{G}.(H) \to H$ and $\epsilon : \overline{P}.(M) \to M$ may be regarded as acyclic resolutions in the relevant category and we may define Ext – the right derived functors of Hom in the category – by

$$Ext^s_{\mathcal{UA}}(H, K) = \pi^s Hom_{\mathcal{UA}}(\overline{G}.(H), K)$$

and

$$Ext^s_{\mathcal{U}}(M, N) = \pi^s Hom_{\mathcal{U}}(\overline{P}.(M), N).$$

We need π^* because these Hom functors are contravariant.

To obtain a spectral sequence with E_2-term of the form $Ext_{\mathcal{UA}}$, Bousfield and Kan proceed as follows. If X is a space (that is, a pointed simplicial set), let $\mathsf{F}_p(X)$ denote the simplicial vector space on the simplicial set X and let $\bar{\mathsf{F}}_pX = \mathsf{F}_p(\mathrm{X})/\mathsf{F}_p(*)$ where $* \in X$ is the basepoint. Then $\bar{\mathsf{F}}_p(\ )$ has the structure of a triple on the category of spaces and one obtains (in a manner dual to the process above) an augmented cosimplicial space

$$X \to \bar{\mathsf{F}}_p^{\cdot}X$$

where $\bar{\mathsf{F}}_p^{(s)}X = \bar{\mathsf{F}}_p \circ \ldots \circ \bar{\mathsf{F}}_pX$ with the composition taken s+1 times. Since

$$\pi_*\bar{\mathsf{F}}_pX = \bar{H}_*X$$

and $\bar{\mathsf{F}}_pX$ is a simplicial vector space, we have that

$$H^*\bar{\mathsf{F}}_pX \cong \bar{G}(H^*X)$$

as an unstable algebra. Thus for any space Y and H^*X of finite type, we have isomorphisms

$$\pi^s\pi_t map_*(Y, \bar{\mathsf{F}}_p^{\cdot}X) \cong \pi^s[\Sigma^tY, \bar{\mathsf{F}}_p^{\cdot}X]$$
$$= \pi^s Hom_{\mathcal{UA}}(H^*\bar{\mathsf{F}}_p^{\cdot}X, H^*\Sigma^tY)$$

since simplicial vector spaces are Eilenberg-MacLane spaces. So

$$\pi^s\pi_t map_*(Y, \bar{\mathsf{F}}_p^{\cdot}X) \cong Ext^s_{\mathcal{UA}}(H^*X, H^*\Sigma^tY) \tag{1.1}$$

wherever this makes sense; that is, for t > 0 if s $\geq$ 1.

Now, Bousfield and Kan noticed that given a fibrant cosimplicial space $Z^{\cdot}$, there is a spectral sequence

$$\pi^s\pi_tZ^{\cdot} \Rightarrow \pi_{t-s}Tot(Z^{\cdot}) \tag{1.2}$$

where Tot($Z^{\cdot}$) is the simplicial set of cosimplicial maps

$$\mathrm{Tot}(Z^{\cdot}) = \mathrm{map}(\Delta, Z^{\cdot})$$

where Δ is the cosimplicial space with $\Delta^s = \Delta$[s], the standard s-simplex. The definition of this spectral sequence will be spelled out in section 5. If we define the p-completion of a space X by the equation

$$X_p = Tot(\bar{\mathsf{F}}_p^{\cdot}X)$$

and set $Z^{\cdot} = map_*(Y, \bar{\mathbb{F}}_p^{\cdot} Z)$, we have that

$$Tot\ map_*(Y, \bar{\mathbb{F}}_p^{\cdot} X) = map_*(Y, Tot(\bar{\mathbb{F}}_p^{\cdot} X)) = map_*(Y, X_p). \tag{1.3}$$

Combining (1.1), (1.2), and (1.3) we obtain the Bousfield-Kan spectral sequence:

$$Ext^s_{\mathcal{UA}}(H^*X, H^*\Sigma^t Y) \Rightarrow \pi_{t-s} map_*(Y, X_p) \tag{1.4}$$

We insist, to make the conclusions above, that H^*X and H^*Y be of finite type. Convergence of the spectral sequence of (1.4) is not automatic, but follows when H^*Y is finite. See [8].

The relationship between X and X_p is not evident either. There is a natural map

$$\eta : X \to X_p$$

and under various hypotheses on the fundamental group of X, η is an isomorphism in homology with $\mathbb{F}_p$ coefficients and the induced map

$$\pi_n \eta : \pi_n X \to \pi_n X_p$$

is a suitably defined $\mathbb{F}_p$ completion. This will be true if, for example, X is simply connected or nilpotent. See [8] for details.

One of the purposes of this paper is to explore $Ext_{\mathcal{UA}}$. An initial step is the following result, deceptive in its simplicity:

Proposition 1.5: Let $M \in \mathcal{U}$ and $K \in \mathcal{UA}$. Then there is a natural isomorphism

$$Ext^s_{\mathcal{UA}}(U(M), K) \cong Ext^s_{\mathcal{U}}(M, IK)$$

where $IK \in \mathcal{U}$ is the augmentation ideal of K.

This is proved in [5], among other places.

2. The Quillen cohomology of unstable algebras

The purpose of this section is to extend the definition of $Ext_{\mathcal{UA}}$ to a larger category and, therfore, obtain greater flexibility for calculation.

Let $s\mathcal{UA}$ be the category of simplicial unstable algebras over the Steenrod algebra. We already have an example of an object of this category: $\bar{G}.(H)$ with $H \in \mathcal{UA}$. Another example — admittedly a trivial one — is a constant simplicial object: if $H \in \mathcal{UA}$, then we may regard H as an object of $s\mathcal{UA}$ by letting $H_n = H$ for all n and setting all face a degeneracy operators to be the identity.

The initial observation is that $s\mathcal{UA}$ has a structure of a closed model category in the sense of Quillen. There are weak equivalences, fibrations and cofibrations satisfying the axioms CM1–CM5 of [17]. We now supply the definitions. Notice that for $A \in s\mathcal{UA}$, we have that $\pi_* A$ is a bigraded, supplemented, commutative $\mathbb{F}_p$-algebra, that $\pi_0 A \in s\mathcal{UA}$, and that for each $n > 0, \pi_n A \in \mathcal{U}$. Furthermore, $\pi_0 A$ is a quotient of A_0 and the quotient map

$$A_0 \to \pi_0 A$$

defines a map of simplicial algebras

$$\epsilon : A \to \pi_0 A$$

where $\pi_0 A$ is regraded as a constant simplicial algebra. If $A = \bar{G}.H$ as in the previous section, then this augmentation is the one given there:

$$\bar{G}.H \to \pi_0 \bar{G}.H \cong H.$$

If $f : A \to B$ is a morphism in $s\mathcal{UA}$, we obtain a diagram

$$\begin{array}{ccc} A & \xrightarrow{\epsilon} & \pi_0 A \\ \downarrow f & & \downarrow \pi_0 f \\ B & \xrightarrow{\epsilon} & \pi_0 B \end{array}$$

and hence a canonical map in $s\mathcal{UA}$

$$(f, \epsilon) : A \to B \times_{\pi_0 B} \pi_0 A$$

where the target is the evident pullback. The morphism f will be called *surjective on components* if this map is a surjection.

Definition 2.1:1.) A morphism $f : A \to B$ in $s\mathcal{UA}$ is a *weak equivalence* if

$$\pi_* f : \pi_* A \to \pi_* B$$

is an isomorphism.

2.) $f : A \to B$ is a *fibration* if it is a surjection on components; f is an *acyclic fibration* if it is a fibration and a weak equivalence.

3.) $f : A \to B$ is a *cofibration* if for every acyclic fibration $p : X \to Y$ in $s\mathcal{UA}$, there is a morphism $B \to X$ so that is the following diagram both triangles commute:

$$\begin{array}{ccc} A & \to & X \\ \downarrow f & \nearrow & \downarrow p \\ B & \to & Y. \end{array}$$

As specializations of these ideas we have *fibrant* and *cofibrant* objects. We write $\mathbb{F}_p$ for the terminal and the initial object of $s\mathcal{UA}$. Then we say that $A \in s\mathcal{UA}$ is cofibrant if the unit map $\eta : \mathbb{F}_p \to A$ is a cofibration. Similarly, we say that A is fibrant if the augmentation $\epsilon : A \to \mathbb{F}_p$ is a fibration. Every object in $s\mathcal{UA}$ is fibrant, so we say no more about this concept.

The following now follows from Theorem 4, pII.4.1 of [17].

Proposition 2.2: With the notions of weak equivalence, fibration, and cofibration defined above, $s\mathcal{UA}$ is a closed model category.

Of course, cofibrations are somewhat mysterious objects and difficult to recognize at this point. We will now be more concrete.

Let $G : n\mathbb{F}_p \to \mathcal{UA}$ be the left adjoint to augmentation ideal functor I. This functor was discussed in the previous section. We will call a morphism $f : A \to B$ in $s\mathcal{UA}$ *almost-free* if, for every n ≥ 0, there is a sub-vector space $V_n \subseteq IB_n$ and maps of vector spaces

$$\delta_i : V_n \to V_{n-1}, \qquad 1 \leq i \leq n$$

$$\sigma_i : V_n \to V_{n+1}, \qquad 0 \leq i \leq n$$

so that the evident extension

$$A_n \otimes G(V_n) \to B_n$$

is an isomorphism for each n and there are commutative diagrams, with the horizontal maps isomorphisms:

$$\begin{array}{ccc} A_n \otimes G(V_n) & \xrightarrow{\cong} & B_n \\ \downarrow d_i \otimes G\delta_i & & \downarrow d_i \\ A_{n-1} \otimes G(V_{n-1}) & \xrightarrow{\cong} & B_{n-1} \end{array}$$

for i $\geq$ 1 and

$$\begin{array}{ccc} A_n \otimes G(V_n) & \xrightarrow{\cong} & B_n \\ \downarrow s_i \otimes G\sigma_i & & \downarrow s_i \\ A_{n+1} \otimes G(V_{n+1}) & \xrightarrow{\cong} & B_{n+1} \end{array}$$

for i $\geq$ 0. Only d_0 is *not* induced up from n$\mathbf{F}_p$. The following result (which is implicit in Quillen, section II.4) can be proved exactly as the corresponding result in section 3 of [19,20].

Theorem 2.3: Almost-free morphisms are cofibrations.

Proposition 2.4: Any morphism $f : A \to B$ in $s\mathcal{UA}$ may be factored canonically as

$$A \xrightarrow{i} X \xrightarrow{p} B$$

with i almost-free and p an acyclic fibration.

We will prove Proposition 2.4, as the construction will prove useful in the later discussion. To begin, let $H \in \mathcal{UA}$. Then we may define the category $H/\mathcal{UA}$ to be the category of objects under H; that is, objects $K \in \mathcal{UA}$ equipped with a morphism $H \to K$ in $\mathcal{UA}$ making K into an H-algebra. The augmentation ideal functor $I : H/\mathcal{UA} \to n\mathbf{F}_p$ has a left adjoint

$$G^H(V) = H \otimes G(V).$$

This pair of adjoint functors yields a cotriple $\bar{G}^H : H/\mathcal{UA} \to H/\mathcal{UA}$ and, as in the previous section, this yields an augmented simplicial object

$$\bar{G}^H_{\cdot} K \to K$$

for any object $K \in H/\mathcal{UA}$. If $H = \mathbf{F}_p$, this is exactly the situation of the previous section.

Now, let $f : A \to B$ be a morphism in $s\mathcal{UA}$. Then the last paragraph yields an augmented bisimplicial algebra

$$\bar{G}^A_{\cdot,\cdot} B \to B \tag{2.5}$$

with

$$\bar{G}^A_{p,q} B = (\bar{G}^{A_q})^{p+1} B_q.$$

Let

$$\bar{G}^A B = diag(\bar{G}^A_{\cdot,\cdot} B)$$

be the resulting diagonal simplicial algebra. Thus, we have factored $f : A \to B$ as

$$A \to \bar{G}^A B \to B. \tag{2.6}$$

The first map is almost-free, the second map is a fibration, and the construction is canonical and functorial in f. We need only show that $\bar{G}^A B \to$ B is an acyclic fibration. But, since $\bar{G}^A B$ is the diagonal simplicial algebra of $\bar{G}^A_{\cdot,\cdot} B$, we may filter $\bar{G}^A_{\cdot,\cdot} B$ by degree in q to obtain a spectral sequence converging to $\pi_* \bar{G}^A B$. But since $\pi_* \bar{G}^{A_q}_{\cdot} B_q \cong B_q$, and the isomorphism is induced by the augmentation, the result follows.

The great strength of the construction of (2.6) is precisely that $\bar{G}^A B$ is the diagonal of a bisimplicial algebra. This allows the construction of many spectral sequences.

As a bit of notation, if $f = \eta : \mathbf{F}_p \to B$ we abbreviate $\bar{G}^{\mathbf{F}_p} B$ as $\bar{G}_{\cdot} B$ in keeping with the conventions of the previous section.

Indeed, consider the case where $H \in \mathcal{UA}$ is regarded as a constant simplcial algebra and we take the morphism f to be the be the unit map $\mathbf{F}_p \to H$. Then the construction of (2.6) yields an acyclic fibration $X \to H$

with X cofibrant. The reader should note that $X = \bar{G}.(H)$, as in the previous section and that the acyclic fibration is the augmentation

$$\bar{G}.(H) \to H.$$

The next obvious subject to bring up is the definition of the homology of an object in the model category $s\mathcal{UA}$ — after all, if $s\mathcal{UA}$ is supposed to be a good place to do homotopy theory, it must have a good notion of homology. However, in order to make sure that our constructions are well-defined, we need technical lemma on homotopies. For this, of course, we need the definition of homotopy. Notice that in $s\mathcal{UA}$, tensor product is the coproduct and if $A \in s\mathcal{UA}$, then the algebra multiplication

$$\mu : A \otimes A \to A$$

is the "fold" map; that is, multiplication supplies the canonical map from the coproduct from A to itself. Factor μ as a cofibration followed by an acyclic fibration

$$A \otimes A \xrightarrow{i} Cy(A) \xrightarrow{p} A.$$

By Proposition 2.4 this may be done functorially in A. Cy(A) is a cylinder object on A. Then two morphisms $f, g : A \to B$ in $s\mathcal{UA}$ are homotopic if there is a morphism H making the following diagram commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{i} & Cy(A) \\ \downarrow{\scriptstyle f \vee g} & & \downarrow{\scriptstyle H} \\ B & \xrightarrow{=} & B \end{array}$$

where $f \vee g = \mu(f \otimes g)$. If $f = g$ and we let H be the composite

$$Cy(A) \xrightarrow{p} A \xrightarrow{f} B$$

we obtain the constant homotopy from f to itself. The reader is invited to prove that homotopy defines an equivalence relation on the set of maps from an object A to an object B.

We can specialize these notions somewhat. If $h : C \to A$ is another morphism in $s\mathcal{UA}$ and $f, g : A \to B$ are two maps, then we say that f and g are *homotopic under* C if $fh = gh$ and there is some homotopy from f to

g which restricts to the constant homotopy on fh. If $q : B \to D$ is a map, then there is a corresponding notion of a *homotopy over* D.

The following, then, is the lemma that we need to show that our definitions of homology and cohomology will be well-defined.The proof is in [18] as Proposition 1.3.

Lemma 2.7: Let $f : A \to B$ be a cofibration and $p : X \to Y$ be an acyclic fibration. Then any two solutions $B \to X$ in the diagram

$$\begin{array}{ccc} A & \to & X \\ \downarrow f & \nearrow & \downarrow p \\ B & \to & Y \end{array}$$

are homotopic under A and over Y.

In the following $\mathcal{A}$ denotes the Steenrod algebra.

Definition 2.8: Let $A \in s\mathcal{U}\mathcal{A}$. Define $H_*^{Q\mathcal{A}}A$ as follows. Choose an acyclic fibration

$$p : X \to A$$

with X cofibrant in $s\mathcal{U}\mathcal{A}$ and set

$$H_*^{Q\mathcal{A}}A = \pi_*(\mathbb{F}_p \otimes_{\mathcal{A}} QX).$$

Define $H^*_{Q\mathcal{A}}A$ by

$$H^*_{Q\mathcal{A}}A = (H_*^{Q\mathcal{A}}A)^* = Hom_{\mathbf{F}_p}(H_*^{Q\mathcal{A}}A, \mathbb{F}_p).$$

Remark 2.9: It is a consequence of Lemma 2.7 that $H_*^{Q\mathcal{A}}A$ is well-defined and functorial in A. It is also a consequence of Lemma 2.7 that if

$$f : A \to B$$

is a weak equivalence in $s\mathcal{U}\mathcal{A}$ then

$$H_*^{Q\mathcal{A}}f{:}H_*^{Q\mathcal{A}}A \to H_*^{Q\mathcal{A}}B$$

is an isomorphism.

Example 2.10.1.) Let X be space. Then we may regard H^*X as a constant simplicial algebra in $s\mathcal{UA}$. Then, as mentioned above, the augmented simplicial algebra

$$\bar{G}.H^*X \to H^*X$$

is an acylic fibration in $s\mathcal{UA}$ and $\bar{G}.H^*X$ is an almost-free and, hence cofibrant object in $s\mathcal{UA}$. Then we have

$$\begin{aligned} H^*_{Q\mathcal{A}}H^*X &\cong Hom_{\mathbf{F}_p}(H_*^{Q\mathcal{A}}H^*X, \mathbf{F}_p) \\ &\cong \pi^* Hom_{\mathbf{F}_p}(\mathbf{F}_p \otimes_{\mathcal{A}} \bar{G}.H^*X, \mathbf{F}_p) \end{aligned}$$

by the universal coefficient theorem for fields. Therefore, we have, in internal degree t,

$$\begin{aligned} [H^*_{Q\mathcal{A}}H^*X]_t &\cong \pi^* Hom_{\mathbf{F}_p}(\mathbf{F}_p \otimes_{\mathcal{A}} Q\bar{G}.H^*X, \Sigma^t\mathbf{F}_p) \\ &\cong \pi^* Hom_{\mathcal{UA}}(\bar{G}.H^*X, H^*S^t) \\ &\cong Ext^*_{\mathcal{UA}}(H^*X, H^*S^t) \end{aligned}$$

Thus $H^*_{Q\mathcal{A}}$ is one way to generalize $Ext_{\mathcal{UA}}$.

Example 2.10.2.) As an example of a simplicial algebra with interesting higher homotopy, we offer the bar construction. Let $H \in \mathcal{UA}$ and let $\bar{B}(H)$ be the bar construction on H. Then $\bar{B}(H) \in s\mathcal{UA}$ and

$$\pi_*\bar{B}(H) \cong Tor_*^H(\mathbf{F}_p, \mathbf{F}_p).$$

This bigraded algebra is a Hopf algebra, a divided power algebra, and more. We will see below in (2.15) that

$$H^s_{Q\mathcal{A}}\bar{B}(H)_t \cong Ext^{s-1}_{\mathcal{UA}}(H, H^*S^t).$$

This offers a new perspective for computing $Ext_{\mathcal{UA}}$.

We end this section with an example of the flexibility that general objects in $s\mathcal{UA}$ supplies. This is the long exact sequence of a cofibration in

$s\mathcal{UA}$ – a long exact sequence related to Quillen's transitivity sequence [18]. Let $f : A \to B$ be a morphism in $s\mathcal{UA}$. Using the construction of (2.6), form the commutative square

$$\begin{array}{ccc} \bar{G}.A & \xrightarrow{\bar{G}.f} & \bar{G}.B \\ \downarrow p_A & & \downarrow p_B \\ A & \xrightarrow{f} & B \end{array}$$

and factor $\bar{G}.f$ as an almost-free map followed by an acyclic fibration

$$\bar{G}.A \xrightarrow{i} X \xrightarrow{p} \bar{G}.B.$$

Then define the *mapping cone* of the morphism f by the equation

$$M(f) = \mathbb{F}_p \otimes_{\bar{G}.A} X.$$

$M(f)$ is almost-free and, hence, cofibrant. Lemma 2.7 implies that $M(f)$ is unique up to homotopy equivalence and functorial in f up to homotopy. (A homotopy equivalence is a weak equivalence with a homotopy inverse.) We could use the construction of (2.6) to make $M(f)$ strictly functorial.

Proposition 2.11: There is a long exact sequence in homology

$$\cdots \to H_n^{Q\mathcal{A}} A \xrightarrow{H_*^{Q\mathcal{A}} f} H_n^{Q\mathcal{A}} B \to H_n^{Q\mathcal{A}} M(f) \to H_{n-1}^{Q\mathcal{A}} A \to \cdots$$

and a long exact sequence in cohomology

$$\cdots \to H_{Q\mathcal{A}}^{n-1} A \to H_{Q\mathcal{A}}^n M(f) \to H_{Q\mathcal{A}}^n B \xrightarrow{H^*_{Q\mathcal{A}} f} H_{Q\mathcal{A}}^n A \to \cdots.$$

Proof: The cohomology result is obtained from the homology result by dualizing. To prove the homology result, notice that since $\bar{G}.A$ is almost-free and i is an almost-free morphsim, the sequence of simplicial algebras

$$\bar{G}.A \xrightarrow{i} X \to \mathbb{F}_p \otimes_{\bar{G}.A} X$$

yields a short exact sequence of simplicial vector spaces

$$0 \to \mathbb{F}_p \otimes_{\mathcal{A}} Q\bar{G}.a \to \mathbb{F}_p \otimes_{\mathcal{A}} QX \to \mathbb{F}_p \otimes_{\mathcal{A}} Q(\mathbb{F}_p \otimes_{\bar{G}.A} X) \to 0$$

Since $p : X \to \bar{G}_.B$ is an acyclic fibration and the composition of cofibrations is a cofibration, we have that

$$\pi_*\mathbb{F}_p \otimes_{\mathcal{A}} X \cong H_*^{Q\mathcal{A}} B$$

and the result follows.

The higher homotopy of $M(f)$ is often non-trivial, even if π_*A and π_*B are concentrated in degree 0. For computational purposes, we have the following result, from [17, Theorem II.6.b)]. Let $f : A \to B$ be a morphism in $s\mathcal{UA}$.

Proposition 2.12: There is a first quadrant spectral sequence of algebras

$$Tor_p^{\pi_*A}(\mathbb{F}_p, \pi_*B)_q \Rightarrow \pi_{p+q}M(f).$$

Notice that if $f : H \to K$ is a map of constant simplicial objects in $s\mathcal{UA}$, then this result implies that

$$\pi_*M(f) \cong Tor_*^H(\mathbb{F}_p, K).$$

Of particular interest is the case where B $= \mathbb{F}_p$ is the terminal object in $s\mathcal{UA}$ and $f = \epsilon : A \to \mathbb{F}_p$ is the augmentation. Because the cofiber of a the map to the terminal object deserves to be called a suspension, we define the *suspension* of A by the equation

$$\Sigma A = M(\epsilon).$$

Since $H_*^{Q\mathcal{A}}\mathbb{F}_p = 0$, 2.11 says that there are isomorphisms

$$H_n^{Q\mathcal{A}}\Sigma A \cong H_{n-1}^{Q\mathcal{A}} A \qquad n \geq 1$$

(2.13) $$H^n_{Q\mathcal{A}}\Sigma A \cong H^{n-1}_{Q\mathcal{A}} A \qquad n \geq 1$$

and

$$H_0^{Q\mathcal{A}}\Sigma A = 0 = H^0_{Q\mathcal{A}}\Sigma A.$$

The suspension has other properties that are worth recording here. For example, from [12] we have that there is a homotopy associative coproduct

$$\psi : \Sigma A \to \Sigma A \otimes \Sigma A$$

that gives $\pi_*\Sigma A$ the structure of a Hopf algebra that is connected in the sense that $\pi_0\Sigma A = \mathbb{F}_p$. This coproduct can be used to turn the spectral sequence, obtained as a corollary to Proposition 2.12

$$Tor_*^{\pi_* A}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*\Sigma A \tag{2.14}$$

into a spectral sequence of Hopf algebras.

To specialize even further, if we regard $H \in \mathcal{UA}$ as a constant simplicial algebra, then the spectral sequence of (2.14) collapses and we obtain an isomorphism of Hopf algebras

$$\pi_*\Sigma H \cong Tor_*^H(\mathbb{F}_p, \mathbb{F}_p).$$

Finally, the work of Miller [19,Section 5;20] implies that if $\bar{B}(H)$ is the bar construction, then there is a weak equivalence is $s\mathcal{UA}$

$$\Sigma H \to \bar{B}(H). \tag{2.15}$$

Thus (2.13) and (2.10.1) sustain the claims of Example 2.10.2.

2.16: The homotopy category. Associated to $s\mathcal{UA}$ and the closed model category structure we have on $s\mathcal{UA}$ there is an associated homotopy category. This category has the same objects as $s\mathcal{UA}$ and morphisms

$$[A, B]_{s\mathcal{UA}} = Hom_{s\mathcal{UA}}(X, B)/\sim$$

where $\sim$ denotes the equivalence relation generated by homotopy and $p : X \to A$ is an acyclic fibration with X cofibrant. Lemma 2.7 implies that $[A, B]_{s\mathcal{UA}}$ is well-defined. A morphism in the homotopy category may be represented by a diagram

$$A \xleftarrow{p} X \xrightarrow{f} B$$

and an *isomorphism* in the homotopy category is such a diagram where f is a weak equivalence. This homotopy category is relatively simple because every object in $s\mathcal{UA}$ is fibrant.

Notice that for $f : A \to B$, the mapping cone $M(f)$ is well-defined in the homotopy category and that ΣA is co-group object in the homotopy category.

3. The Bousfield-Kan Spectral Sequence II

In this section we show that the Quillen cohomology of the previous section is the E_2 term of a more general spectral sequence than that described in section 1. This spectral sequence will converge to the homotopy groups of the total space of a cosimplicial space that is often interesting in applications. We end the section with some examples: a universal infinite cycle and a universal r-cycle.

If X is a (pointed) space, let $X \to \bar{\mathbb{F}}_p^{\cdot} X$ be the augmented cosimplicial space of the first section. Then if we let $Z = Z^{\cdot}$ be a fibrant cosimplicial space. Then we may use the functor $\bar{\mathbb{F}}_p^{\cdot}(\ )$ to define an augmented bi-cosimplicial space

$$Z \to \bar{\mathbb{F}}_p^{\cdot} Z^{\cdot} \tag{3.1}$$

by setting

$$(\bar{\mathbb{F}}_p^{\cdot} Z^{\cdot})^{(s,t)} = \bar{\mathbb{F}}_p^{s+1} Z^t$$

and letting the augmentation $Z^t \to \bar{\mathbb{F}}_p^{\cdot} Z^t$ define the augmentation for (3.1). Define $\bar{\mathbb{F}}_p^{\cdot} Z$ by the equation

$$\bar{\mathbb{F}}_p^{\cdot} Z = diag(\bar{\mathbb{F}}_p^{\cdot} Z^{\cdot}).$$

Thus

$$(\bar{\mathbb{F}}_p^{\cdot} Z)^s = \bar{\mathbb{F}}_p^{s+1} Z^s.$$

The augmentation of (3.1) induces a canonical map of cosimplicial spaces

$$\eta : Z \to \bar{\mathbb{F}}_p^{\cdot} Z.$$

Now let us consider the induced map of simplicial algebras

$$H^* \eta : H^* \bar{\mathbb{F}}_p^{\cdot} Z \to H^* Z.$$

An examination of the definitions of (2.5) and (2.6) demonstrate that we have a natural commutative square with the vertical maps isomorphisms:

$$\begin{array}{ccc} H^* \bar{\mathbb{F}}_p^{\cdot} Z & \xrightarrow{H^* \eta} & H^* Z \\ \downarrow \cong & & \downarrow = \\ \bar{G}_{.} H^* Z & \xrightarrow{p} & H^* Z \end{array} \tag{3.3}$$

In particular, we have proven the following result.

Lemma 3.4: In the category $s\mathcal{UA}$

$$H^*\eta : H^*\bar{\mathbf{F}}_p^{\cdot} Z \to H^* Z$$

is an acylic fibration with $H^*\bar{\mathbf{F}}_p^{\cdot} Z$ almost-free.

The following result now delineates the affect of the construction (3.2) in homotopy.

Lemma 3.5: Let Z be a fibrant cosimplicial space. Suppose that

$$\pi^s H_t Z = 0, \qquad t - s \leq 1$$

and, for all n and sufficiently large s,

$$\pi^s H_{s+n} Z = 0.$$

Then

$$Tot(\eta) : Tot(Z) \to Tot(\bar{\mathbf{F}}_p^{\cdot} Z)$$

is the Bousfield-Kan $\mathbf{F}_p$-completion of $Tot(Z)$.

We postpone the proof to record a corollary of the previous two lemmas.

Corollary 3.6: Let Z be a fibrant cosimplicial space so that $\pi_s H^t Z$ is finite for all s and t, $\pi_s H^t Z = 0$ or all $t - s \leq 1$ and $\pi_s H^{s+n} Z = 0$ for all n and sufficiently large s. Then there is a convergent spectral sequence

$$[H^s_{Q\mathcal{A}}(H^* Z)]_t \Rightarrow \pi_{t-s} Tot(Z)_p.$$

Proof: This is the homotopy spectral sequence of the cosimplicial space $\bar{\mathbf{F}}_p^{\cdot} Z$:

$$\pi^s \pi_t \bar{\mathbf{F}}_p^{\cdot} Z \Rightarrow \pi_{t-s} Tot(Z)_p.$$

We must notice that under the finiteness hypotheses listed, we have

$$\begin{aligned} \pi^s \pi_t \bar{\mathbf{F}}_p^{\cdot} Z &\cong \pi^s Hom_{\mathbf{F}_p}(\mathbf{F}_p \otimes_{\mathcal{A}} H^* \bar{\mathbf{F}}_p^{\cdot} Z, H^* S^t) \\ &\cong [H^s_{Q\mathcal{A}}(H^* Z)]_t. \end{aligned}$$

The result now follows from Lemma 3.5.

Remark: If Z is *not* a fibrant cosimplicial space, we still get a spectral sequence

$$H^*_{Q\mathcal{A}}H^*Z \Rightarrow \pi_*Tot(\bar{\mathbb{F}}_p^{\cdot}Z)$$

because $\bar{\mathbb{F}}_p^{\cdot}Z$ is fibrant, being group-like in the sense of Bousfield and Kan [8, X.4.9]. But we are not able to identify the the abuttment with the $\mathbb{F}_p$-completion of $Tot(Z)$. Indeed, $Tot(Z)$ may be unintersting, but $Tot(\bar{\mathbb{F}}_p^{\cdot}Z)$ might be of great interest. We will give some examples below where this generality is of importance.

To prove Lemma 3.5, we need the following result of Bousfield [2, Theorem 3.5].

Theorem 3.7: Let Z be a fibrant cosimplicial space. Then there is a natural second quadrant spectral sequence

$$\pi^s H_t Z \Rightarrow H_{t-s}Tot(Z)$$

If $\pi^s H_t Z = 0$ for $t - s \leq 1$ and $\pi^s H_{s+n}Z = 0$ for all n and sufficiently large s, the spectral sequence converges and $Tot(Z)$ is simply connected.

Proof of Lemma 3.5: By the tower lemmas of Bousfield and Kan [8, III.6.2] $Tot(\bar{\mathbb{F}}_p^{\cdot}Z)$ is $\mathbb{F}_p$-complete. Theorem 3.7 and Lemma 3.4 imply that

$$H_*Tot(Z) \to H_*Tot(\bar{\mathbb{F}}_p^{\cdot}Z)$$

is an isomorphism. The result now follows from the universal property of $\mathbb{F}_p$-completion.

We complete this section with a sequence of examples to justify the generality.

Example 3.8: Let X be a pointed, fibrant space and let $Z^{\cdot} = X$ be the constant cosimplicial space on X; that is, $Z^s = X$ for all s and every coface and codegeneracy map is the identity. Then the construction of (3.2) and the spectral sequence of Corollary 3.6 yield the Bousfield-Kan spectral sequence of the first section. To identify the E_2 terms of these two spectral sequences we use Example 2.10.1.

Example 3.9: This example contructs a universal infinite cycle for the spectral sequence of Corollary 3.6.

We begin with some remarks on simplicial unstable algebras. If V is a simplicial graded $\mathbf{F}_p$-vector space, then we may define a trivial simplicial algbra V_+ as follows. For each n, give V_n the structure of a trivial A-module and let

$$[V_+]_n = V_n \oplus \mathbf{F}_p$$

be the trivial algbera; that is, the augmentation ideal of $[V_+]_n$ is V_n and $(V_n)^2 = 0$. The face and degeneracy maps of V_+ are the obvious ones and a moment's thought will demonstrate that

$$\pi_*(V_+) \cong (\pi_* V)_+$$

where $(\pi_* V)_+$ is the evident bigraded trivial algebra.

In particular, we let $K(p,q)$ be the simplicial graded vector space with

$$\pi_* K(p,q) \cong \Sigma^q \mathbf{F}_p$$

concentrated in π_p — we will say that the non-zero bidegree is in *external* degree p and *internal* degree q. For any object $A \in s\mathcal{UA}$, choose an acyclic fibration $p : X \to A$ with X cofibrant. Then, in the language of 2.15, we have

$$\begin{aligned}[A, K(p,q)_+]_{s\mathcal{UA}} &\cong [\mathbf{F}_p \otimes_{\mathcal{A}} QX, K(p,q)]_{sn\mathbf{F}_p} \\ &\cong Hom_{nn\mathbf{F}_p}(\pi_*(\mathbf{F}_p \otimes_{\mathcal{A}} QX), \pi_* K(p,q)) \\ &\cong (H^p_{Q\mathcal{A}} A)_q\end{aligned} \tag{3.10}$$

where $[A, B]_C$ means the homotopy classes of maps in the relevant category. The second isomorphism in (3.10) follows from the fact that a homotopy class of maps in the category of simplicial vector spaces is completely determined by the map on homotopy.

The conclusion to be drawn from (3.10) is that the functor $H^p_{Q\mathcal{A}}(\ )_q$ is a corepresentable functor — as any functor we label cohomology should be — and that $K(p,q)_+ \in s\mathcal{UA}$ acts as an Eilenberg-MacLane space in this category. Therefore, $H^*_{Q\mathcal{A}}K(p,q)_+$ is a good thing to compute. If we can do the computation for all p and q we will have computed the "algebra" of cohomology operations.

The next point of this example is that if $p \leq q$ (or $q - p \geq 0$), then there is a cosimplicial space $S(p,q)$ so that $H^*S(p,q) \cong K(p,q)_+$ in $s\mathcal{UA}$. Let Δ be the cosimplicial space with $\Delta[s]$ the standard s-simplex and let $sk_n(\ )$ be the n-skeleton functor. Then, for $q - p \geq 0$, let

$$S(p,p) = \Delta / sk_{p-1}\Delta$$

and

$$S(p,q) = \Sigma^{q-p}S(p,p).$$

In [6] it was shown that $H^*S(p,q) \cong K(p,q)_+$ and that S(p,q) has the following universal property. There is a class $\iota \in \pi^p\pi_q S(p,q)$ that is the *universal* infinite cycle in the sense that if Z is a fibrant cosimplicial space and $z \in \pi^p\pi_q Z$ survives to E_∞ in the homotopy spectral sequence

$$\pi^s\pi_t Z \Rightarrow \pi_{t-s}Tot(Z)$$

then there is a morphism of cosimplicial spaces

$$f : S(p,q) \to Z$$

so that

$$\pi^*\pi_* f(\iota) = z.$$

We will see this in section 5. Now $S(p,q)$ is not evidently fibrant; however, we can preform the construction (3.2) nonetheless and obtain

$$\eta : S(p,q) \to \bar{\mathbb{F}}^{\cdot}_p S(p,q).$$

Lemma 3.5 will no longer be valid, however. But, $\bar{\mathbb{F}}^{\cdot}_p S(p,q)$ is a fibrant cosimplicial space — being group-like — and Theorem 3.7 and the fact that $Tot(\bar{\mathbf{F}}_p^{\cdot}S(p,q))$ is p-complete imply that if $q-p>1$, then there is a homology equivalence

$$S^{q-p} \to Tot(\bar{\mathbf{F}}_p^{\cdot}S(p,q));$$

that is, $Tot(\bar{\mathbf{F}}_p^{\cdot}S(p,q))$ is the $\mathbf{F}_p$ completion of the sphere S^{q-p}. Therefore, we obtain a spectral sequence

$$[H^s_{\mathcal{QA}}K(p,q)_+]_t \Rightarrow \pi_{t-s}S^{q-p} \tag{3.11}$$

where we, as is customary, confuse the sphere with its $\mathbf{F}_p$-completion. This spectral sequence is related to Barratt's desuspension spectral sequence [see 2, Section 4] and also [15, Section 3]. This spectral sequence is also universal in the following sense. Let Z be a fibrant cosimplicial space and, with $q-p>1$,

$$z \in \pi^p\pi_q\bar{\mathbf{F}}_p^{\cdot}Z \cong [H^p_{\mathcal{QA}}H^*Z]_q$$

a permanent cycle in the Bousfield-Kan spectral sequence. Then, by the remarks made on $S(p,q)$ above, there is a morphism of cosimplicial spaces

$$f : S(p,q) \to \bar{\mathbf{F}}_p^{\cdot}Z$$

so that $\pi^*\pi_*f(\iota) = z$. Then there is a commutative diagram

$$\begin{array}{ccc} S(p,q) & \xrightarrow{f} & \bar{\mathbf{F}}_p^{\cdot}Z \\ \downarrow \eta & & \downarrow \eta \\ \bar{\mathbf{F}}_p^{\cdot}S(p,q) & \xrightarrow{\bar{\mathbf{F}}_p^{\cdot}f} & \bar{\mathbf{F}}_p^{\cdot}\bar{\mathbf{F}}_p^{\cdot}Z. \end{array}$$

Since we have that

$$H^*\eta : H^*\bar{\mathbf{F}}_p^{\cdot}\bar{\mathbf{F}}_p^{\cdot}Z \to H^*\bar{\mathbf{F}}_p^{\cdot}Z$$

is a weak equivalence in $s\mathcal{UA}$

$$\eta : \bar{\mathbf{F}}_p^{\cdot}Z \to \bar{\mathbf{F}}_p^{\cdot}\bar{\mathbf{F}}_p^{\cdot}Z$$

induces an isomorphism of spectral sequences. Thus, if we confuse

$$\iota \in \pi^p\pi_qS(p,q)$$

with its image under $\pi^*\pi_*\eta$ in

$$\pi^p\pi_q\bar{\mathsf{F}}^{\cdot}_pS(p,q) \cong [H^p_{\mathcal{Q}\mathcal{A}}K(p,q)_+]_q$$

we obtain a diagram of spectral sequences

$$\begin{array}{lccc} & [H^s_{\mathcal{Q}\mathcal{A}}K(p,q)_+]_t & \Rightarrow & \pi_{t-s}S^{q-p} \\ (3.12) & \downarrow H^*_{\mathcal{Q}\mathcal{A}}f & & \downarrow \pi_*Tot(\bar{\mathsf{F}}^{\cdot}_p f) \\ & [H^s_{\mathcal{Q}\mathcal{A}}H^*Z]_t & \Rightarrow & \pi_{t-s}Tot(Z)_p \end{array}$$

so that $H^*_{\mathcal{Q}\mathcal{A}}f(\iota) = z$.

Thus we conclude that not only does $K(p,q)_+$ corepresent cohomology, but that this phenomenon extends in a precise way to the Bousfield-Kan spectral sequence as well.

Example 3.13: There is also a universal r-cycle. Let Δ and $sk_n(\ )$ be as in the previous example and set, for $r \geq 2$,

$$D(r,p,p) = sk_{p+r-1}\Delta/sk_{p-1}\Delta$$

and for $q-p \geq 0$

$$D(r,p,q) = \Sigma^{q-p}D(r,p,p).$$

These cosimplicial spaces have the following universal property: there are classes

$$\iota \in \pi^p\pi_qD(r,p,q)$$

and

$$\vartheta \in \pi^{p+r}\pi_{q+r-1}D(r,p,q)$$

so that ι survives to E_r in the homotopy spectral sequence and

$$d_r(\iota) = \vartheta.$$

This differential is universal in this sense: if Z is a fibrant cosimplicial space and $x \in \pi^p\pi_qZ$ survives to E_r in the homotopy spectral sequence for Z, and if

$$d_r(x) = y,$$

then there exists a morphism of cosimplicial spaces $f : D(r,p,q) \to Z$ so that

$$\pi^*\pi_* f(\iota) = x \qquad \textit{and} \qquad \pi^*\pi_* f(\vartheta) = y.$$

$D(r,p,q)$ may not be fibrant, so the homotopy spectral sequence for this cosimplicial space must be adjusted as follows: there is a fibrant cosimplicial space $\hat{D}(r,p,q)$ and a homotopy spectral sequence

$$\pi^s\pi_t D(r,p,q) \Rightarrow \pi_{t-s} Tot(\hat{D}(r,p,q)).$$

However, this technicality will be avoided completely below.

Bousfield and Kan [6] have computed the homology spectral sequence

$$\pi^s H_t D(r,p,q) \Rightarrow H_{t-s}\hat{D}(r,p,q).$$

Let

$$h : \pi^*\pi_* D(r,p,q) \to \pi^* H_* D(r,p,q)$$

be the map induced by the Hurewicz homomorphism. Then $\pi^* H_* D(r,p,q)$ is of dimension 3 over $\mathbb{F}_p$ with basis

$$1 \in \pi^0 H_0 D(r,p,q)$$

$$h(\iota), h(\vartheta) \in \pi^* H_* D(r,p,q).$$

Since h induces a map of spectral sequences

$$d_r h(\iota) = h(\vartheta).$$

Thus the homology spectral sequence

$$\pi^* H_* D(r,p,q) \Rightarrow H_* Tot(\bar{\mathbb{F}}_p^{\cdot} D(r,p,q))$$

implies that $Tot(\bar{\mathbb{F}}_p^{\cdot} D(r,p,q))$ is contractible — if $q-p>1$.

Therefore, in the homotopy spectral sequence, $q-p>1$,

$$[H^s_{Q\mathcal{A}} H^* D(r,p,q)]_t \Rightarrow \pi_{t-s} Tot(\bar{\mathbb{F}}_p^{\cdot} D(r,p,q))$$

we have $E_\infty = 0$. Incidentally,

$$H^*D(r,p,q) \cong (\bar{H}^*D(r,p,q))_+.$$

Finally, arguing as for 3.12, we see that if Z is a fibrant cosimplicial space and $x, y \in H^*_{Q\mathcal{A}}H^*Z$ are so that $d_r x = y$, then there is a map of spectral sequences

$$\begin{array}{ccc} H^*_{Q\mathcal{A}}H^*D(r,p,q) & \Rightarrow & \pi_*Tot(\bar{\mathsf{F}}^{\bullet}_p D(r,p,q)) = 0 \\ \downarrow H^*_{Q\mathcal{A}}f & & \downarrow \\ H^*_{Q\mathcal{A}}H^*Z & \Rightarrow & \pi_*Tot(Z)_p \end{array}$$

This should imply the existence of many formal differentials.

4. Fibrations and the Bousfield-Kan Spectral Sequence

In this section we discuss certain fibration sequences of cosimplicial spaces, demonstrate the relationship between these and fibrations of spaces, and show how these behave with respect to the spectral sequence of the previous section. We close with some examples from the work of Mahowald. This section is one of the major justifications for the generality of the previous two sections.

The first remark to make is that there is another way to generalize the construction, for a space X

$$X \to \bar{\mathsf{F}}^{\bullet}_p X$$

of the first section. This will produce a relative version of this cosimplicial space. We do this by producing a triple on the category of spaces over a fixed space Y. This construction is the object used in [11] to define a fibre-wise completion of X.

Fix a pointed space Y and let $f : X \to Y$ be a map of pointed spaces. Define

$$(\bar{\mathsf{F}}_p)_Y X = Y \times \bar{\mathsf{F}}_p X \tag{4.1}$$

and give $(\bar{\mathsf{F}}_p)_Y(\ )$ the structure of a triple with the following structure maps. Define

$$\eta : X \to (\bar{\mathsf{F}}_p)_Y X$$

by

$$\eta = f \times \eta : X \to Y \times \bar{\mathsf{F}}_p X$$

where the second η is $\eta : X \to \bar{\mathsf{F}}_p X$ — the unit of the triple $\bar{\mathsf{F}}_p(\ )$. Define

$$\epsilon : (\bar{\mathsf{F}}_p)^2_Y X \to (\bar{\mathsf{F}}_p)_Y X$$

to be the composite

$$Y \times \bar{\mathsf{F}}_p(Y \times \bar{\mathsf{F}}_p X) \xrightarrow{1 \times \bar{\mathsf{F}}_p \pi_2} Y \times \bar{\mathsf{F}}_p^2 X \xrightarrow{1 \times \epsilon} Y \times \bar{\mathsf{F}}_p X$$

where π_2 is the projection onto the second factor and $\epsilon : \bar{\mathsf{F}}_p^2 X \to \bar{\mathsf{F}}_p X$ is the structure map for the triple $\bar{\mathsf{F}}_p(\ )$.

One easily checks that there are commutative diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & Y \times \bar{\mathsf{F}}_p X & = & (\bar{\mathsf{F}}_p)_Y X \\ \downarrow f & & \downarrow \pi_1 & & \\ Y & \xrightarrow{=} & Y & & \end{array}$$

where π_1 is projection onto the first factor and

$$\begin{array}{ccccccc} (\bar{\mathsf{F}}_p)^2_Y X & = & Y \times \bar{\mathsf{F}}_p(Y \times \bar{\mathsf{F}}_p X) & \xrightarrow{\epsilon} & Y \times \bar{\mathsf{F}}_p X & = & (\bar{\mathsf{F}}_p)_Y X \\ & & \downarrow \pi_1 & & \downarrow \pi_1 & & \\ & & Y & \xrightarrow{=} & Y & & \end{array}$$

and, thus, $((\bar{\mathsf{F}}_p)_Y, \eta, \epsilon)$ is a triple on the category of spaces over Y. This is the category whose objects are maps $f : X \to Y$ and whose morphisms are commutative diagrams. Let

$$X \to (\bar{\mathsf{F}}_p)^{\cdot}_Y X \tag{4.2}$$

be the resulting augmented cosimplicial space over Y. Notice that if we prefer, we could say that there is a map of cosimplicial spaces

$$(\bar{\mathsf{F}}_p)^{\cdot}_Y X \to Y \tag{4.3}$$

where Y is regarded as a constant cosimplicial space. Notice that the construction (4.2) is natural in the map $f : X \to Y$.

Therefore, we can generalize this relative construction to cosimplicial spaces. Suppose that

$$f : Z \to Y$$

is a map of cosimplicial spaces, with Y not necessarily a constant cosimplicial space. Then we can form the bi-cosimplicial space $(\bar{\mathsf{F}}_p)_Y^{\cdot\cdot} Z$ with

$$(\bar{\mathsf{F}}_p)_Y^{(s,t)} Z = (\bar{\mathsf{F}}_p)_{Y^t}^{s+1} Z^t$$

with the obvious vertical and horizontal coface and codegeneracy maps. Define

$$(\bar{\mathsf{F}}_p)_Y^{\cdot} Z = diag(\bar{\mathsf{F}}_p)_Y^{\cdot\cdot} Z. \tag{4.4}$$

The augmentation of (4.2) yields an augmentation

$$i : Z \to (\bar{\mathsf{F}}_p)_Y^{\cdot} Z$$

and the projection of (4.3) yields a natural projection

$$p : (\bar{\mathsf{F}}_p)_Y^{\cdot} Z \to Y$$

so that the composite

$$Z \xrightarrow{i} (\bar{\mathsf{F}}_p)_Y^{\cdot} Z \xrightarrow{p} Y$$

is the original map $f : Z \to Y$.

Lemma 4.5: There is an isomorphism

$$i_* : \pi^* H_* Z \to \pi^* H_* (\bar{\mathsf{F}}_p)_Y^{\cdot} Z.$$

Proof: By (4.4), there is a spectral sequence

$$\pi^t \pi^s H_* (\bar{\mathsf{F}}_p)_Y^{\cdot\cdot} Z \Rightarrow \pi^{s+t} H_* (\bar{\mathsf{F}}_p)_Y^{\cdot} Z.$$

But

$$\pi^s H_* (\bar{\mathsf{F}}_p)_Y^{\cdot,t} Z = \pi^s H_* (\bar{\mathsf{F}}_p)_{Y^t}^{\cdot} Z^t = \begin{cases} H_* Z^t, & \text{if } s = 0; \\ 0, & \text{if } s > 0. \end{cases}$$

and the result follows.

Remark 4.6: In actual fact, much more is true. There is a commutative diagram

$$\begin{array}{ccccc} H^*Y & \xrightarrow{p^*} & H^*(\bar{\mathbb{F}}_p)^{\cdot}_Y Z & \xrightarrow{i^*} & H^*Z \\ \downarrow = & & \downarrow & & \downarrow = \\ H^*Y & \xrightarrow{i} & \bar{G}^{H^*Y}_{\cdot} H^*Z & \xrightarrow{p} & H^*Z \end{array}$$

where the bottom row is the factoring of the morphism $f^* : H^*Y \to H^*Z$ as an almost-free map followed by an acyclic fibration constructed in Section 2. Thus p^* is almost free in $s\mathcal{U}\mathcal{A}$.

Now suppose that Y is a group-like cosimplicial space. Then for any cosimplicial space Z and any map $f : Z \to Y$, one easily checks that $(\bar{\mathbb{F}}_p)^{\cdot}_Y Z$ is group-like and that

$$p : (\bar{\mathbb{F}}_p)^{\cdot}_Y Z \to Y$$

is a (level-wise) surjection of group-like objects in the category of cosimplicial spaces. Any such is a fibration in the category of cosimplicial spaces. If we let $*$ denote the initial object in the category of cosimplicial spaces, then we may define the fiber $F(p)$ of p by the pull-back diagram

$$\begin{array}{ccc} F(p) & \to & (\bar{\mathbb{F}}_p)^{\cdot}_Y Z \\ \downarrow & & \downarrow p \\ * & \to & Y. \end{array}$$

Lemma 4.7: If $F(p)$ is the fiber of $p : (\bar{\mathbb{F}}_p)^{\cdot}_Y Z \to Y$ with Y group-like, then $F(p)$ is group-like and there is a natural isomorphism

$$H^*F(p) \cong \mathbb{F}_p \otimes_{H^*Y} H^*(\bar{\mathbb{F}}_p)^{\cdot}_Y Z.$$

Proof: For each s,

$$(\bar{\mathbb{F}}_p)^s_Y Z = (\bar{\mathbb{F}}_p)^{s+1}_{Y^s} Z^s = Y^s \times \bar{\mathbb{F}}_p((\bar{\mathbb{F}}_p)^s_{Y^s} Z^s).$$

Thus, for each s, there is a fibration sequence induced by p:

$$F(p)^s \to Y^s \times \bar{\mathbb{F}}_p((\bar{\mathbb{F}}_p)^s_{A^s} S^s) \xrightarrow{\pi_1} Y^s.$$

In particular,

$$F(p)^s = \bar{\mathbb{F}}_p((\bar{\mathbb{F}}_p)^s_{Y^s} Z^s)$$

and the result follows.

Because $F(p)$ is fibrant, $Tot(F(p))$ is a meaningful object from the point of view of homotopy theory. In particular, there is a fibration sequence in homotopy

$$Tot(F(p)) \to Tot((\bar{\mathbb{F}}_p)^{\cdot}_Y Z) \xrightarrow{Tot(p)} Tot(Y). \tag{4.8}$$

This follows from [8,p.277].

Now consider the case of an arbitrary map of fibrant cosimplicial spaces $f : Z \to Y$. Applying the functor $\bar{\mathbb{F}}^{\cdot}_p(\ )$ to this map, we obtain a map of group like cosimplicial spaces

$$\bar{\mathbb{F}}^{\cdot}_p f : \bar{\mathbb{F}}^{\cdot}_p Z \to \bar{\mathbb{F}}^{\cdot}_p Y.$$

If we apply the construction of (4.4) we obtain a factoring of $\bar{\mathbb{F}}^{\cdot}_p f$:

$$\bar{\mathbb{F}}^{\cdot}_p Z \xrightarrow{i} X = (\bar{\mathbb{F}}_p)^{\cdot}_{\bar{\mathbb{F}}^{\cdot}_p Y} \bar{\mathbb{F}}^{\cdot}_p Z \xrightarrow{p} \bar{\mathbb{F}}^{\cdot}_p Y$$

where p is a fibration and

$$i_* : \pi^* H_* \bar{\mathbb{F}}^{\cdot}_p Z \to \pi^* H_* X$$

is an isomorphism. This last implies that there is a homotopy equivalence

$$Tot(\bar{\mathbb{F}}^{\cdot}_p Z) \simeq Tot(X).$$

Let $F(p)$ be the fiber of $p : X \to \bar{\mathbb{F}}^{\cdot}_p Y$. Then, in light of Remark 4.6, Lemma 4.7, and the material before Proposition 2.12, we have that

$$\pi^* \pi_* F(p) \cong H^*_{Q\mathcal{A}} M(f^*)$$

where $M(f^*)$ is the mapping cone of $f^* : H^*Y \to H^*Z$ in $s\mathcal{UA}$. Therefore there is a spectral sequence

$$[H^s_{Q\mathcal{A}} M(f^*)]_t \Rightarrow \pi_{t-s} F(p).$$

Thus we obtain a fiber sequence up to homotopy

$$Tot(F(p)) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p$$

and a long exact sequence of E_2-terms:

$$\to [H^s_{\mathcal{QA}} M(f^*)]_t \to [H^s_{\mathcal{QA}} H^* Z]_t \xrightarrow{f^*} [H^s_{\mathcal{QA}} H^* Y]_t \xrightarrow{\partial} [H^{s+1}_{\mathcal{QA}} M(f^*)]_t \to$$

We would like the morphism ∂ to be induced by a morphism of spectral sequences. In the next section we will prove the following result.

Theorem 4.9: There is a diagram of spectral sequences

$$\begin{array}{ccc} [H^s_{\mathcal{QA}} H^* Y]_t & \Rightarrow & \pi_{t-s} Tot(X)_p \\ \downarrow \partial & & \downarrow \delta \\ [H^{s+1}_{\mathcal{QA}} M(f^*)]_t & \Rightarrow & \pi_{t-s-1} Tot(F(p)). \end{array}$$

where $\delta : \pi_{t-s} Tot(X)_p \to \pi_{t-s-1} Tot(F(p))$ is the boundary map induced from the homotopy fibration sequence

$$Tot(F(p)) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p.$$

We close the section with a sequence of examples applying this technology.

Example 4.10: Let $Z = *$ be the initial object in the category of cosimplicial spaces. (Remember that all our spaces and morphisms are pointed.) Let Y be fibrant cosimplicial space so that $\pi^s H_t Y = 0$ for $t-s \leq 1$ and $\pi^s H_{s+n} Y = 0$ for all s and sufficiently large n. Then there is a natural weak equivalence

$$Tot(F(p)) \simeq \Omega Tot(Y)_p$$

because $Tot(Z)$ is contractible. On the other hand,

$$\epsilon = f^* : H^* Y \to H^* Z = \mathbf{F}_p$$

so $M(f^*) = \Sigma H^* Y$ in the terminology of 2.13. Therefore,

$$\partial : [H^s_{\mathcal{QA}} H^* Y]_t \to [H^{s+1}_{\mathcal{QA}} M(f^*)]_t \cong [H^{s+1}_{\mathcal{QA}} \Sigma H^* Y]_t$$

is an isomorphism for all s and t, and we get a commutative diagram of spectral sequences, where the vertical maps are isomorphisms:

$$\begin{array}{ccc} [H^s_{\mathcal{QA}}H^*Y]_t & \Rightarrow & \pi_{t-s}Tot(Y)_p \\ \downarrow \partial & & \downarrow \partial \\ [H^{s+1}_{\mathcal{QA}}\Sigma H^*Y]_t & \Rightarrow & \pi_{t-s-1}\Omega Tot(X)_p \end{array}$$

Also, the Hurewicz map induces a map of spectral sequences

$$\begin{array}{ccc} [H^s_{\mathcal{QA}}\Sigma H^*Y]_t & \Rightarrow & \pi_{t-s}\Omega Tot(Y)_p \\ \downarrow & & \downarrow \\ [\pi_s\Sigma H^*Y]^*_t & \Rightarrow & H_{t-s}\Omega Tot(Y)_p. \end{array}$$

This of particular importance if $Y = \bar{\mathbb{F}}^{\cdot}_p X$ for some pointed space X. Then (2.12) implies that

$$[\pi_s\Sigma H^*Y]_t \cong Tor_s^{H^*X}(\mathbb{F}_p, \mathbb{F}_p)_t$$

and, of course,

$$[H^s_{\mathcal{QA}}\Sigma H^*Y]_t \cong Ext^{s-1}_{\mathcal{UA}}(H^*X, H^*S^t).$$

The upshot, then, is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^{s-1}_{\mathcal{UA}}(H^*X, H^*S^t) & \Rightarrow & \pi_{t-s}\Omega X_p \\ \downarrow & & \downarrow \\ Cotor_s^{H_*X}(\mathbb{F}_p, \mathbb{F}_p)_t & \Rightarrow & H_{t-s}\Omega X_p. \end{array} \tag{4.11}$$

The homology spectral sequence is easily seen to be isomorphic to the Eilenberg-Moore spectral sequence. A similar construction has been used by Bousfield and Curtis [5] and Bousfield and Kan [6].

It is worth pointing out that the spectral sequence of (2.12)

$$Tor^{\pi_* H^*Y}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*\Sigma H^*Y$$

collapses for other examples than the example of $Y = \bar{\mathbb{F}}^{\cdot}_p X$; for example, it will collapse for either of the examples 3.8 or 3.13.

Example 4.12: In this example, we investigate the suspension homomorphism. Let $\bar{\mathsf{F}}_p(\ )$ be the underlying functor of the triple described in section 1. Then there is an evident natural map

$$\Sigma^k \bar{\mathsf{F}}_p X \to \bar{\mathsf{F}}_p \Sigma^k X$$

and this, in turn, induces a map of cosimplicial spaces

$$e_k : \bar{\mathsf{F}}_p^{\cdot} X \to \Omega^k \bar{\mathsf{F}}_p^{\cdot} \Sigma^k X.$$

Since $Tot(\Omega^k Z) \cong \Omega^k Tot(Z)$ for any fibrant cosimplicial space Z, and since

$$\pi^s \pi_t \Omega^k \bar{\mathsf{F}}_p^{\cdot} \Sigma^k X \cong \pi^s \pi_{t+k} \bar{\mathsf{F}}_p^{\cdot} \Sigma^k X$$

we obtain a diagram of spectral sequences

$$\begin{array}{ccc} Ext^s_{\mathcal{UA}}(H^*X, H^*S^t) & \Rightarrow & \pi_{t-s} X_p \\ \downarrow \pi^*\pi_* e_k & & \downarrow E_k \\ Ext^s_{\mathcal{UA}}(H^*\Sigma^k X, H^*S^{t+k}) & \Rightarrow & \pi_{t-s+k} \Sigma^k X_p. \end{array}$$

where E_k is the suspension homomorphism.

Now, from the work of Mark Mahowald, it is known that, for the case $X = S^n$, the algebraic suspension homomorphism $\pi^*\pi_* e_k$ fits into a long exact sequence. The work we have done here allows us to give name — from the point of view of homological algebra — to the third term in this long exact sequence and, perhaps, more flexibility for computation. The following lemma will help us to identify the E_2 term of various spectral sequences.

Lemma 4.13: Let Z be a cosimplicial space so that, for every s, Z^s is homotopy equivalent to an Eilenberg-MacLane space and $\pi_* Z^s$ is a graded F_p vector space. Then, for all s and t, we have that the homomorphism induced by the augmentation

$$\pi^s \pi_t Z \to \pi^s \pi_t \bar{\mathsf{F}}_p^{\cdot} Z$$

is an isomorphism.

Proof: $\bar{\mathsf{F}}^{\cdot}_p Z = diag \bar{\mathsf{F}}^{\cdot,\cdot}_p Z$ where $\bar{\mathsf{F}}^{p,q}_p Z = \bar{\mathsf{F}}^{p+1}_p Z^q$. If we filter $\pi_* \bar{\mathsf{F}}^{\cdot,\cdot}_p Z$ be degree in q, we obtain a spectral sequence

$$\pi^q \pi^p \pi_* \bar{\mathsf{F}}^{\cdot,\cdot}_p Z \Rightarrow \pi^{p+q} \pi_* \bar{\mathsf{F}}^{\cdot}_p Z.$$

Because of the hypotheses on Z^q, we have

$$\pi^p \pi_* \bar{\mathsf{F}}^{\cdot}_p Z^q \cong \begin{cases} \pi_* Z^q, & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

The result follows.

The hypothesis of Lemma 4.13 applies to both $Z = \bar{\mathsf{F}}^{\cdot}_p X$ and, especially, $Z = \Omega^k \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X$. Therefore,

$$[H^s_{\mathcal{QA}} H^* \bar{\mathsf{F}}^{\cdot}_p X]_t \cong Ext^s_{\mathcal{UA}}(H^*X, H^*S^t)$$

and

$$[H^s_{\mathcal{QA}} H^* \Omega^k \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X]_t \cong Ext^s_{\mathcal{UA}}(H^*\Sigma^k X, H^*S^{t+k})$$

and we obtain a long exact sequence

$$\begin{aligned} \to [H^s_{\mathcal{QA}} M(e_k^*)]_t &\to Ext^s_{\mathcal{UA}}(H^*X, H^*S^t) \\ &\xrightarrow{\pi^*\pi_* e_k} Ext^s_{\mathcal{UA}}(H^*\Sigma^k X, H^*S^{t+k}) \to [H^{s+1}_{\mathcal{QA}} M(e_k^*)]_t \to \end{aligned}$$

And if $C(E_k)$ is the homotopy fiber in the homotopy fibration sequence

$$C(E_k) \to X \xrightarrow{E_k} \Omega^k \Sigma^k X$$

then Theorem 4.9 implies that there is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^s_{\mathcal{UA}}(H^*\Sigma^k X, H^*S^{t+k}) & \Rightarrow & \pi_{t+k-s} \Sigma^k X_p \\ \downarrow & & \downarrow \\ [H^{s+1}_{\mathcal{QA}} M(e_k^*)]_t & \Rightarrow & \pi_{t-s-1} C(E_k)_p. \end{array}$$

Since there are techniques for computing $H^*_{\mathcal{QA}} A$ from knowledge of $\pi_* A$ (see [13]), it would be nice to know $\pi_* M(e_k^*)$. In principal, this can be done as follows. First of all,

$$\pi_* H^* \bar{\mathsf{F}}^{\cdot}_p X \cong H^* X$$

concentrated in external degree 0. On the other hand

$$\pi_* H^* \Omega^k \bar{\mathbf{F}}_p^{\cdot} \Sigma^k X$$

can (and this is the part that is only in principal) be computed using the derived functors of Lannes's mapping object functors [16]. For example — and here we offer only the prime 2, $k = 2$, and $X = S^n$:

$$\pi_* H^* \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2} \cong \Lambda(i_n) \otimes \Gamma[x_{2n+1}, x_{2n+2}, y_j]$$

where

$$i_n \in \pi_0 H^n \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2}$$

$$x_j \in \pi_1 H^j \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2}$$

and

$$y_j \in \pi_{2^j+1} H^{2^j(4n+5)} \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2}, \qquad j \geq 0.$$

Λ and Γ denote the exterior and divided power algebras respectively.

Once $\pi_* H^* \Omega^k \bar{\mathbf{F}}_p^{\cdot} \Sigma^k X$ is computed, one can appeal to the spectral sequence of (2.12) to compute $\pi_* M(e_k^*)$. In the case of

$$e_2^* : H^* \Omega^2 \bar{\mathbf{F}}_p^{\cdot} S^{n+2} \to H^* \bar{\mathbf{F}}_p^{\cdot} S^n$$

this spectral sequence will collapse.

5. The homotopy spectral sequence and twisted products

The purpose of this section is two-fold. First, we explain in detail how the homotopy spectral sequence of a fibrant cosimplicial space is constructed and, second, we use this explanation to prove Theorem 4.9. This theorem defines a boundary map in a "long-exact sequence" of spectral sequences. We begin with the first project.

Let Z be a fibrant cosimplicial space. If Y is a cosimplicial space, let $map(Y, Z)$ be the space of maps between Y and Z. The n-simplices of $map(Y, Z)$ are maps of cosimplicial spaces

$$\Delta[n] \times Y \to Z$$

where $\Delta[n]$ is the standard n-simplex. If Y, Z are pointed, let $map_*(Y, Z)$ denote the space of pointed maps between Y and Z. The n-simplices of this space are pointed maps of cosimplicial spaces

$$\Delta[n]_+ \wedge Y \to Z$$

where $+$ denotes a disjoint basepoint.

If Z is pointed and fibrant, there is a homotopy spectral sequence

$$\pi^s \pi_t Z \Rightarrow \pi_{t-s} Tot(Z)$$

where $Tot(Z) = map(\Delta, Z)$ and Δ is the cosimplicial space that is $\Delta[n]$ in cosimplicial degree n. This spectral sequence is a tower of fibrations

$$\begin{array}{ccccccccc} \cdots & \to & Tot_3(Z) & \to & Tot_2(Z) & \to & Tot_1(Z) & \to & Tot_0(Z) \\ & & \uparrow & & \uparrow & & \uparrow & & \| \\ & & F_3 Z & & F_2 Z & & F_1 Z & & F_0 Z \end{array} \tag{5.1}$$

Here

$$Tot_n(Z) = map(sk_n \Delta, Z)$$

where $sk_n(\ )$ is the n-skeleton functor and the fibrations

$$Tot_n(Z) \to Tot_{n-1}(Z)$$

are determined by the inclusion $sk_{n-1}\Delta \to sk_n\Delta$. Thus the fiber is the mapping space

$$F_nZ = map_*(sk_n\Delta/sk_{n-1}\Delta, Z).$$

Here and elsewhere we make the convention that

$$X/sk_{-1}Y = X_+.$$

Bousfield and Kan have given a description of F_nZ. Let

$$M^nZ \subseteq Z^n \times \cdots \times Z^n$$

be the *matching space* given by

$$M^nZ = \{(z^0, z^1, \ldots, z^n) \mid s^iz^j = s^{j-1}z^i,\ 0 \leq i < j \leq n\}$$

where the s^i are the codegeneracies in Z. There is a natural map

$$s : Z^n \to M^{n-1}Z$$

given by

$$s(z) = (s^0z, s^1z, ..., s^{n-1}z).$$

The condition that Z be fibrant is equivalent to the condition that s be a fibration for all n. Let N^nZ be defined by the fibration sequence

$$N^nZ \to Z \to M^{n-1}Z. \tag{5.2}$$

Bousfield and Kan now prove [8,X.6], using the fact that

$$[sk_n\Delta/sk_{n-1}\Delta]^n = S^n$$

that we have natural isomorphims

$$F_nZ \cong \Omega^nN^nZ$$

and

$$\pi_t\Omega^nN^nZ \cong \pi_{t+n}N^nZ \cong N^n\pi_{t+n}Z \tag{5.3}$$

where $N^n\pi_{t+n}Z$ is the n^{th} group in the normalized cochain complex of the cosimplicial group $\pi_{t+n}Z$. Furthermore Bousfield [10.4 of 3] shows that the composite

$$\pi_t F_n Z \to \pi_t Tot_n(Z) \to \pi_{t-1} F_{n+1} Z$$

induced by the fibrations of (5.1) is equivalent, under the isomorphisms of (5.3) to

$$(-1)^t \partial : N^n \pi_{t+n} Z \to N^{n+1} \pi_{t+n} Z \tag{5.4}$$

where ∂ is the boundary operator of $N\pi_{t+n}Z$. Thus if we use the tower (5.1) to define a spectral sequence with

$$E_1^{s,t} = \pi_{t-s} F_s Z \cong N^s \pi_t Z$$

then the spectral sequence reads, because of (5.4),

$$E_2^{s,t} \cong \pi^s \pi_t Z \Rightarrow \pi_{t-s} Tot(Z)$$

where we have used the identification

$$Tot(Z) = \varprojlim Tot_n(Z).$$

This is the spectral sequence *under* $Tot(Z)$. We can build the same spectral sequence from a tower *over* $Tot(Z)$. This is often more convenient, especially as it allows one to use pointed mapping spaces at all times. First notice that

$$Tot(Z) = map_*(\Delta_+, Z).$$

Call this $Tot^0 Z$. If $n \geq 1$, define

$$Tot^n(Z) = map_*(\Delta / sk_{n-1}\Delta, Z).$$

The fibration sequences

$$sk_{n-1}\Delta \to \Delta \to \Delta / sk_{n-1}\Delta$$

give rise to a diagram of fibration sequences

$$\begin{tikzcd}
 & & F_n Z \arrow[d] \\
Tot^{n+1}(Z) \arrow[r] \arrow[d] & Tot(Z) \arrow[r] \arrow[d, "="] & Tot_n(Z) \arrow[d] \\
Tot^{n}(Z) \arrow[r] \arrow[d] & Tot(Z) \arrow[r] & Tot_{n-1}(Z) \\
F_n Z & &
\end{tikzcd}$$

and, hence, to a tower of fibrations

$$\begin{tikzcd}
\cdots \arrow[r] & Tot^2(Z) \arrow[r, "p_1"] \arrow[d, "k_2"] & Tot^1(Z) \arrow[r, "p_0"] \arrow[d, "k_1"] & Tot^0(Z) \arrow[d, "k_0"] & = Tot(Z). \\
 & F_2 Z & F_1 Z & F_0 Z &
\end{tikzcd} \tag{5.5}$$

If we apply homotopy to this tower of fibrations, we obtain a spectral sequence with

$$E_1^{s,t} = \pi_{t-s} F_s Z$$

and standard arguments show that we have produced a spectral sequence isomorphic to the usual one.

The universal examples of section 3 are easily explained using the tower (5.5). Notice that, if $x \in \pi^s \pi_t Z$ is an infinite cycle detecting $\alpha \in \pi_{t-s} Tot(Z)$, then there is a diagram

$$\begin{tikzcd}
S^{t-s} \arrow[r, "f_s"] \arrow[d, equal] & Tot^s Z \arrow[r, "k_s"] \arrow[d] & F_s Z \\
S^{t-s} \arrow[r, "f"] & Tot(Z) &
\end{tikzcd}$$

so that the homotopy class of f is α and so that $k_s f_s$ represents x. The adjoint of the map

$$f_s : S^{t-s} \to map_*(\Delta / sk_{s-1}\Delta, Z)$$

yields a map

$$S(s,t) = S^{t-s} \wedge \Delta / sk_{s-1} \Delta \to Z$$

demonstrating the claim that $S(s,t)$ forms some sort of universal infinite cycle. The universal differential can be discussed in the same way.

We now turn to the discussion of the boundary maps between homotopy spectral sequences. To isolate the key point in the argument, we make the following definition.

Definition 5.6: A fibration sequence of pointed cosimplicial spaces

$$F \xrightarrow{i} Z \xrightarrow{p} Y$$

is called a *twisted product* if there are isomorphisms of pointed simplicial sets, $n > 0$,

$$\Theta^n : Z^n \to Y^n \times X^n$$

and commutative diagrams

$$\begin{array}{ccc} Z^n & \xrightarrow{\Theta^n} & Y^n \times F^n \\ \downarrow p & & \downarrow p_1 \\ Y^n & \xrightarrow{=} & Y^n \end{array}$$

where p_1 is the projection, and so that

$$(d^i \times d^i)\Theta^n = \Theta^{n+1} d^i, \qquad i > 0$$

and

$$(s^i \times s^i)\Theta^n = \Theta^{n-1} s^i, \qquad i \geq 0$$

where d^i and s^i are the appropriate coface and codegeneracy operators.
Only d^0 does not commute with the Θ^n and provides the twisting.

One easily checks that a twisted product is a fibrations sequence of cosimplicial spaces.

Lemma 5.7: Let $f : Z \to Y$ be any morphism of cosimplicial spaces. Then the fibration sequence

$$F(p) \to (\bar{\mathsf{F}}_p)_{\bar{\mathsf{F}}_p^{\cdot} Y} \bar{\mathsf{F}}_p^{\cdot} Z \to \bar{\mathsf{F}}_p^{\cdot} Y$$

of Section 4 is a twisted product.

Proof: This is a matter of examining the definitions. Indeed, if Y is a space, then $(\bar{\mathsf{F}}_p)_Y(\ ) = Y \times \bar{\mathsf{F}}_p(\ )$ and this splitting extends to the fibration sequence.

Remark: The class of twisted products contains more than the examples provided by this lemma. Indeed, if $F \to Z \to Y$ is a twisted product and X is a pointed space, than

$$map_*(X, F) \to map_*(X, Z) \to map_*(X, Y)$$

is also a twisted product.

Proposition 5.8: If $F \to Z \xrightarrow{p} Y$ is a twisted product, then there are isomorphisms of spaces

$$N^n Z \xrightarrow{\cong} N^n Y \times N^n F$$

for all $n \geq 0$ and a diagram

$$\begin{array}{ccc} N^n Z & \xrightarrow{\cong} & N^n Y \times N^n F \\ \downarrow Np & & \downarrow p_1 \\ N^n Y & \xrightarrow{=} & N^n Y. \end{array}$$

Proof: The matching spaces $M^n Z$ and the fibration sequences

$$N^n Z \to Z^n \xrightarrow{s} M^{n-1} Z$$

that define the spaces $N^n Z$ depend only on the codegeneracies of Z. Since $\Theta^n : Z^n \xrightarrow{\cong} Y^n \times F^n$ commutes with codegeneracies, the result follows.

Corollary 5.9: In the normalized cochain complex $N\pi_* Z$ there is an isomorphism

$$N^n \pi_* Z \cong N^n \pi_* Y \times N^n \pi_* F$$

and the isomorphism commute with the projection to $N\pi_* Y$. There is a long exact sequence

$$\cdots \to \pi^s \pi_t F \to \pi^s \pi_t Z \to \pi^s \pi_t Y \xrightarrow{\partial} \pi^{s+1} \pi_t F \to \cdots$$

Proof: The isomorphisms follow from Proposition 5.8 and the isomorphism of (5.3). The long exact sequence is now induced by the short exact sequence of cochain complexes

$$0 \to N\pi_*F \to N\pi_*Z \to N\pi_*Y \to 0.$$

Remark 5.10.1.) One easily checks that the long exact sequence obtained by combining Lemma 5.7 with Corollary 5.9 is the same as that obtained in Proposition 2.11 and used in Theorem 4.9.

2.) The map $\partial : \pi^s\pi_tY \to \pi^{s+1}\pi_tF$ has a canonical description on the cochain level given as follows. If $\alpha \in \pi^s\pi_tY$ is the residue class of the cocycle $y \in N^s\pi_tY$, then 5.9 identifies an element $z \in N^s\pi_tZ$ that passes, under the isomorphism, to $(y, 0)$. The coboundary ∂z passes to $(0, w)$ for some $w \in N^{s+1}\pi^tF$ and $\partial\alpha$ is the residue class of w.

In light of Lemma 5.7 and Remark 5.10.1, Theorem 4.9 is subsumed in the following result.

Theorem 5.11: Let $F \to Z \to Y$ be a twisted product of fibrant pointed cosimplicial spaces. Then there is a diagram of spectral sequences

$$\begin{array}{ccc} \pi^s\pi_tY & \Rightarrow & \pi_{t-s}Tot(Y) \\ \downarrow \partial & & \downarrow \delta \\ \pi^{s+1}\pi_tF & \Rightarrow & \pi_{t-s-1}Tot(F) \end{array}$$

where δ is induced by the fibration sequence of spaces

$$Tot(F) \to Tot(Z) \to Tot(Y).$$

This follows from the following omnibus lemma. We use the notation of (5.5).

Lemma 5.12.1.) There are maps

$$f^n : \pi_*\Omega Tot^n(Y) \to \pi_*Tot^{n+1}(F)$$

and

$$\partial^n : \pi_* \Omega^2 F_n Y \to \Omega F_{n+1} F$$

and a commutative diagram

$$\begin{array}{ccccccc}
\pi_*\Omega^2 F_n Y & \to & \pi_*\Omega Tot^n Y & \to & \pi_*\Omega Tot^{n-1} Y & \xrightarrow{\delta} & \pi_{*-1}\Omega^2 F_n Y \\
\downarrow \partial^n & & \downarrow f^n & & \downarrow f^{n-1} & & \downarrow \partial^n \\
\pi_*\Omega F_{n+1} F & \to & \pi_* Tot^{n+1}(F) & \to & \pi_* Tot^n(F) & \xrightarrow{\delta} & \pi_{*-1}\Omega F_{n+1} F
\end{array}$$

where the rows arise from the fibration sequences induced from (5.5).

2.) Under the isomorphisms

$$\pi_i \Omega^2 F_n Y \cong N^n \pi_{i+n+2} Y \qquad \text{and} \qquad \pi_i \Omega F_{n+1} F \cong N^{n+1} \pi_{i+n+2} F$$

the map ∂^n induces the map

$$\partial : \pi^n \pi_t Y \to \pi^{n+1} \pi_t F.$$

The proof will occupy the rest of the section. The delicate point is to produce f^n and ∂^n in a natural enough way to demonstrate the commutivity of the diagram of 5.12.1. We give the technique we will use, which exploits Proposition 5.8. Let $\mathcal{H}\wr(\nabla \mathcal{S}_*)$ be the homotopy category of pointed cosimplicial spaces [8,X].

Definition 5.13: An object $D \in \mathcal{H}\wr(\nabla \mathcal{S}_*)$ will be called a $d_1^{s,t}$ *model* if there is an isomorphism in $\mathcal{H}\wr(\nabla \mathcal{S}_*)$,

$$D(s,t,1) = \Sigma^{t-s} sk_s \Delta / sk_{s-1} \Delta \to D.$$

We can conclude that there is prefered generator

$$\iota_s \in N^s \pi_t D$$

and

$$0 \neq \partial \iota_s \in N^{s+1} \pi_t D.$$

We call D a d_1 *model* if there is an isomorphism in $\mathcal{H}\wr(\nabla \mathcal{S}_*)$

$$\vee_k \ D(s_k, t_k, 1) \to D$$

for some finite indexing set $\{k\}$. A map $f : D \to D'$ between d_1 models will be called a *projection/injection* if under the prefered isomorphisms in $\mathcal{H}o(\nabla\mathcal{S}_*)$ f corresponds to a projection onto one wedge summand followed by inclusion to another.

The key fact we will use use is this.

Lemma 5.14: Let $F \to Z \to Y$ be twisted product of pointed fibrant cosimplicial spaces and let D be a cofibrant d_1 model. Then the map

$$\pi_* map_*(D, Z) \to \pi_* map_*(D, Y)$$

is split surjective. Furthermore, this splitting is natural with respect to projection/inclusions $D \to D'$ of d_1 models.

Proof: Let

$$\vee_k\ D(s_k, t_k, 1) \to D$$

be the given isomorphism in $\mathcal{H}o(\nabla\mathcal{S}_*)$. Using this and [8,p.277] we obtain isomorphisms

$$\begin{aligned}\pi_* map_*(D, Z) &\cong \times_k\ \pi_* map_*(D(s_k, t_k, 1), Z)\\ &\cong \times_k\ \pi_* \Omega^{t_k} N^{s_k} Z.\end{aligned}$$

The result now follows from Proposition 5.8.

The lemma we use to construct the maps of 5.12.1 is the following.

Proposition 5.15: Let $A \to D \to C$ be a cofiber sequence of cofibrant pointed cosimplicial spaces and let D be a d_1 model. Let $F \to Z \to Y$ be a twisted product of pointed fibrant cosimplicial spaces. Then there is a map

$$\pi_* map_*(C, Y) \to \pi_* map_*(A, F).$$

This map is natural with respect to diagrams

$$\begin{array}{ccccc} A & \to & D & \to & C \\ \downarrow & & \downarrow g & & \downarrow \\ A' & \to & D' & \to & C' \end{array}$$

where g is a projection/inclusion of d_1 models.

Proof: Let $i : D \to C$ and $p : Z \to Y$ be the given maps. Then there is a pull-back diagram of fibrations

$$\begin{array}{ccc} map_*(i,p) & \to & map_*(D,Z) \\ \downarrow & & \downarrow \\ map_*(C,Y) & \to & map_*(D,Y) \end{array}$$

where $map_*(i,p)$ is the mapping space with n-simplices pointed commutative diagrams

$$\begin{array}{ccc} \Delta[n]_+ \wedge D & \to & Z \\ \downarrow {\scriptstyle 1\wedge i} & & \downarrow {\scriptstyle p} \\ \Delta[n]_+ \wedge C & \to & Y \end{array}$$

The splitting of the previous result yields a splitting

$$\pi_* map_*(C,Y) \to \pi_* map_*(i,p).$$

The result follows by composing with the natural map

$$\pi_* map_*(i,p) \to \pi_* map_*(A,F).$$

The naturality clause follows from the naturality clause of 5.14.

The proof of 5.12.1 now depends on making a good choice of cofibration sequences. Because the category of pointed cosimplicial spaces is a closed model category, we can make the following assertions. Let

$$\begin{array}{ccc} A_1 & \to & B_1 \\ \downarrow & & \downarrow \\ A_2 & \to & B_2 \end{array}$$

be a homotopy commutative diagram of cosimplicial spaces. Then there is a commutative diagram

$$\begin{array}{ccccc} A_1 & \to & X_1 & \to & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ Z_2 & \to & X_2 & \to & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ Z_3 & \to & X_3 & \to & Y_3 \end{array} \tag{5.16}$$

where the rows and columns are cofibration sequences and the square

$$\begin{array}{ccc} A_1 & \to & X_1 \\ \downarrow & & \downarrow \\ Z_2 & \to & X_2 \end{array}$$

is equivalent to the original square in the sense that there are weak equivalences

$$X_1 \to B_1 \qquad X_2 \to B_2 \qquad Z_2 \to A_2$$

and the diagrams

$$\begin{array}{ccc} A_1 & \xrightarrow{=} & A_1 \\ \downarrow & & \downarrow \\ Z_2 & \to & A_2 \end{array} \qquad \begin{array}{ccc} Z_2 & \to & X_2 \\ \downarrow & & \downarrow \\ A_2 & \to & B_2 \end{array}$$

$$\begin{array}{ccc} A_1 & \to & X_1 \\ \downarrow = & & \downarrow \\ A_1 & \to & B_1 \end{array}$$

commute and the following diagram commutes up to homotopy

$$\begin{array}{ccc} X_1 & \to & X_2 \\ \downarrow & & \downarrow \\ B_1 & \to & B_2. \end{array}$$

The diagram produced depends on the choice of homotopy. If the original square commutes exactly, then there is a canonical choice of homotopy: the constant homotopy.

We can now turn to the proof of Lemma 5.12.

Proof of Lemma 5.12.1: For every $n > 0$, there is a cofibration sequence

$$sk_n\Delta/sk_{n-1}\Delta \to \Delta/sk_{n-1}\Delta \to \Delta/sk_n\Delta.$$

By using a functorial mapping cylinder construction, this cofibration sequence yields a sequence

$$\Delta/sk_{n-1}\Delta \to \Delta/sk_n\Delta \xrightarrow{\lambda_n} B(n, n+1)$$

where $B(n,n+1) \simeq \Sigma sk_n\Delta/sk_{n-1}\Delta$ and is, hence, a $d_1^{n,n+1}$ model. In fact, if $\iota_{n+1} \in N^{n+1}\pi_{n+1}\Delta/sk_n\Delta$ is the generator, then under the induced map

$$N\lambda_n : N^{n+1}\pi_{n+1}\Delta/sk_n\Delta \to N^{n+1}\pi_{n+1}B(n,n+1)$$

we have that

$$N\lambda_n(\iota_{n+1}) = \partial\iota_n$$

where $\iota_n \in N^n\pi_{n+1}B(n,n+1)$ is the generator.

Now, inclusion of skeleta induces a diagram

$$\begin{array}{ccccc} sk_n\Delta/sk_{n-1}\Delta & \to & \Delta/sk_{n-1}\Delta & \to & \Delta/sk_n\Delta \\ \downarrow\gamma & & \downarrow & & \downarrow \\ sk_{n+1}\Delta/sk_n\Delta & \to & \Delta/sk_n\Delta & \to & \Delta/sk_{n+1}\Delta \end{array}$$

where γ is the constant map. Then, because the mapping cylinder construction is functorial, we get a commutative diagram with γ' constractable:

$$\begin{array}{ccc} \Delta/sk_n\Delta & \xrightarrow{\lambda_n} & B(n,n+1) \\ \downarrow & & \downarrow\gamma' \\ \Delta/sk_{n+1}\Delta & \xrightarrow{\lambda_{n+1}} & B(n+1,n+2) \end{array}.$$

Then, by applying the construction of (5.16) to this square, we obtain a diagram

$$\text{(5.17)} \quad \begin{array}{ccccc} \Delta/sk_n\Delta & \to & D(n,n+1) & \to & S(n,n+1) \\ \downarrow & & \downarrow\gamma'' & & \downarrow i \\ S(n+2,n+2) & \to & D(n+1,n+2) & \to & S(n+1,n+2) \\ \downarrow & & \downarrow j & & \downarrow \\ D'(n+1,n+2) & \xrightarrow{f} & Z & \xrightarrow{g} & D"(n,n+2) \end{array}$$

where

5.17.1) in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$ we have isomorphisms $S(s,t) \cong \Sigma^{t-s}\Delta/sk_{s-1}\Delta$ in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$ and, under these isomorphisms, the map i is isomorphic to the projection

$$\Sigma\Delta/sk_{n-1}\Delta \to \Sigma\Delta/sk_n\Delta;$$

5.17.2) $D(s,t)$ and $D'(s,t)$ are $d_1^{s,t}$ models; and

5.17.3) there is an isomorphism in $\mathcal{H}o(\nabla\mathcal{S}_*)$

$$Z \cong D(n+1,n+2) \vee \Sigma D(n,n+1)$$

and j is isomorphic to the inclusion.

Then, applying Proposition 5.15, we obtain, for any twisted product $F \to Z \to Y$ of fibrant cosimplicial spaces, a diagram

$$\begin{array}{ccc} \pi_* map_*(D'(n,n+2),Y) & \to & \pi_* map_*(D'(n+1,n+2),F) \\ \downarrow & & \downarrow \\ \pi_* map_*(S(n+1,n+2),Y) & \to & \pi_* map_*(S(n+2,n+2),F) \\ \downarrow & & \downarrow \\ \pi_* map*(S(n,n+1),Y) & \to & \pi_* map_*(\Delta/sk_n\Delta,F). \end{array} \tag{5.18}$$

Now we use 5.17.1-3) and the fact that for cosimplicial spaces A and B, we have

$$\pi_* map_*(\Sigma A, B) \cong \pi_* \Omega map_*(A,B)$$

to define the maps f_n and ∂_n. Indeed,

$$\begin{aligned} \pi_* map_*(D'(n,n+2),Y) &\cong \pi_* map_*(\Sigma^2 sk_n\Delta/sk_{n-1}\Delta, Y) \\ &\cong \pi_*\Omega^2 F_n Y \end{aligned}$$

and

$$\begin{aligned} \pi_* map_*(D'(n+1,n+2),F) &\cong \pi_* map_*(\Sigma sk_{n+1}\Delta/sk_n\Delta, F) \\ &\cong \pi_*\Omega F_{n+1} F. \end{aligned}$$

So the first row of (5.18) and these isomorphisms defines ∂_n. For f_n, use the fact that for a fibrant cosimplicial space W

$$\begin{aligned} \pi_* map_*(S(s,t),W) &\cong \pi_* map_*(\Sigma^{t-s}\Delta/sk_{s-1}\Delta, W) \\ &\cong \pi_*\Omega^{t-s} Tot^s W. \end{aligned}$$

and the bottom row of (5.18).

The diagram (5.18) now demonstrates the commutivity of the three of the four squares of the diagram in 5.12.1. To get the final commutative square, recapitulate this argument, beginning with the square

$$\begin{array}{ccc} S(n+2,n+2) & \to & D(n+1,n+2) \\ \downarrow & & \downarrow j \\ D'(n+1,n+2) & \xrightarrow{f} & Z. \end{array}$$

This completes the proof of 5.12.1 and leaves only the following.

Proof of 5.12.2: Let

$$D'(n+1,n+2) \xrightarrow{f} Z \xrightarrow{g} D'(n,n+2) \tag{5.19}$$

be the cofibration sequence of 5.17. We examine f and g in homotopy. Because of the commutative square

$$\begin{array}{ccc} D(n+1,n+2) & \rightarrow & S(n+1,n+2) \\ \downarrow j & & \downarrow \\ Z & \xrightarrow{g} & D'(n,n+2) \end{array}$$

we understand the composition gj: if $\iota_{n+1} \in N^{n+1}\pi_{n+2}D(n+1,n+2)$ is the generator, then, by (5.4)

$$N(gj)\iota_{n+1} = -\partial\iota_n$$

where $\iota_n \in N^n\pi_{n+2}D'(n,n+2)$ is the generator. Thus, if

$$j_n \in N^n\pi_{n+2}Z$$

and

$$j_{n+1} \in N^{n+1}\pi_{n+2}Z$$

are the generators obtained from the isomorphisms in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$

$$Z \cong D(n+1,n+2) \vee \Sigma D(n,n+1) \cong D(n+1,n+2) \vee D'(n,n+2)$$

and if $\iota_{n+1} \in N^{n+1}\pi_{n+2}D'(n+1,n+2)$ is the generator, then we have

$$Ng(j_n) = \iota_n$$

and

$$Ng(j_{n+1}) = -\partial\iota_n.$$

And, because the composition gf is constant, we conclude that

$$Nf(\iota_{n+1}) = j_{n+1} + \partial j_n.$$

The result now follows because ∂_n is obtained from 5.19, using 5.15, and because of the canonical description of ∂ given in Remark 5.10.2

Part II: Products and Operations in Quillen Cohomology

In the last four sections of this paper, we will define and explore the Whitehead product and the operations that appear in the E_2 term of the Bousfield-Kan spectral sequence. Then we will discuss to what extent these products and operations commute with the differentials and, hence, are reflected in the homotopy groups of spaces. Sections 8 and 9 are devoted to methods of computation and calculations in the universal examples of section 3. For these final sections we will restrict attention to the prime 2, although many of the results immediately generalize to other primes. What has not been generalized are the operations of section 7.

6. Products in Quillen cohomology

In this section, we expand on some work of Bousfield and Kan and show that there is a product in the spectral sequence

$$H^*_{\mathcal{QA}}H^*Z \Rightarrow \pi_*Tot(Z)_2.$$

This product will satisfy the Jacobi identity and abut to the Whitehead product in homotopy. In the next section we will show that there are Steenrod operations related to this product.

To begin with, it is useful to make the following definition:

Definition 6.1: A cosimplicial space Z is a $\mathbf{F}_2$-*like* if each Z^s is a simplicial $\mathbf{F}_2$ vector space for each s, and the coface and codegeneracy maps

$$d^i : Z^{s-1} \to Z^s, \qquad 1 \leq i \leq s$$

and

$$s^i : Z^{s+1} \to Z^s, \qquad 0 \leq i \leq s$$

are all maps of simplicial vector spaces. Only d^0 is not necessarily a map of simplicial vector spaces. In addition, a morphism of $\mathbf{F}_2$-like cosimplicial spaces is a map $f : Z \to Y$ of $\mathbf{F}_2$-like cosimplicial spaces so that each

$$f^s : Z^s \to Y^s$$

is a morphism of simplicial vector spaces.

Remark 6.2: In light of the constructions of Section 3, given any cosimplicial space Z, we can form the augmented cosimplicial space $Z \longrightarrow \bar{\mathsf{F}}_2^{\cdot} Z$ and $\bar{\mathsf{F}}_2^{\cdot} Z$ is F_2-like. Since

$$H^*_{Q\mathcal{A}} H^* Z \Rightarrow \pi_* Tot(Z)_2$$

is the homotopy spectral sequence of $\bar{\mathsf{F}}_2^{\cdot} Z$ all the subsequent results apply to the this generalization of the Bousfield-Kan spectral sequence. For example, see (6.6) below.

Also, if Z is F_2-like and $H_* Z^s$ is of finite type for each s, then

$$\pi_* Z \cong (\mathsf{F}_2 \otimes_{\mathcal{A}} QH^* Z)^*$$

so

$$\pi^* \pi_* Z \cong H^*_{Q\mathcal{A}} H^* Z.$$

In [7], Bousfield and Kan demonstrate how to put a Whitehead product into the homotopy spectral sequence of an F_2-like cosimplicial space. Define a map, for each s,

$$\zeta : Z^s \wedge Z^s \longrightarrow Z^{s+1}$$

by

$$\zeta(u \wedge v) = d^0(u+v) - (d^0(u) + d^0(v)). \tag{6.2}$$

Here we use the fact that Z^{s+1} is a vector space. Notice that ζ measures the deviation of d^0 from being a vector space homomorphism.This allows one to define a pairing

$$\omega_* : \pi_t Z^s \otimes \pi_{t'} Z^s \longrightarrow \pi_{t+t'} Z^{s+1}$$

as follows. Define a map

$$\wedge : \pi_t Z^s \otimes \pi_{t'} Z^s \longrightarrow \pi_{t+t'} Z^s \wedge Z^s$$

by the smash product pairing and let ω_* be the composition

$$\pi_t Z^s \otimes \pi_{t'} Z^s \xrightarrow{\wedge} \pi_{t+t'} Z^s \wedge Z^s \xrightarrow{\pi_* \zeta} \pi_{t+t'} Z^{s+1}.$$

The following can now be proved exactly as in Chapter III of [7].

Theorem 6.3: Let Z be an $\mathbf{F}_2$-like cosimplicial space. Then the pairing ω_* induces a product in the homotopy spectral sequence

$$\pi^s \pi_t Z \cong [H^s_{Q\mathcal{A}} H^* Z]_t \Rightarrow \pi_{t-s} Tot(Z)$$

abutting to the Whitehead product

$$[\ ,\ ] : \pi_* Tot(Z) \otimes \pi_* Tot(Z) \to \pi_* Tot(Z).$$

That is, there is a product

$$[\ ,\ ] : [H^s_{Q\mathcal{A}} H^* Z]_t \otimes [H^{s'}_{Q\mathcal{A}} H^* Z]_{t'} \to [H^{s+s'+1}_{Q\mathcal{A}} H^* Z]_{t+t'}$$

and a diagram of spectral sequences

$$\begin{array}{ccc} [H^s_{Q\mathcal{A}} H^* Z]_t \otimes [H^{s'}_{Q\mathcal{A}} H^* Z]_{t'} & \Rightarrow & \pi_{t-s} Tot(Z) \otimes \pi_{t'-s'} Tot(Z) \\ \downarrow [\ ,\ ] & & \downarrow [\ ,\ ] \\ [H^{s+s'+1}_{Q\mathcal{A}} H^* Z]_{t+t'} & \Rightarrow & \pi_{t+t'-(s+s')-1} Tot(Z) \end{array}$$

Remark 6.4: More technically, this should be phrased as follows. Let $\{E_r^{s,t} Z\}$ denote the homotopy spectral sequence. Then there exist natural products

$$[\ ,\ ] : E_r^{s,t} Z \otimes E_r^{s',t'} Z \to E_r^{s+s'+1,t+t'} Z$$

so that

1.) the product on $E_1 Z$ is induced by ω_*;

2.) $d_r[u, v] = [d_r u, v] + [u, d_r v]$;

3.) the product on $E_{r+1} Z$ is induced by the product on $E_r Z$ and the product on $E_\infty Z$ is induced by the product on $E_r Z$, $r < \infty$;

4.) the product on E_∞ is also induced by the Whitehead product on $\pi_* Tot(Z)$.

Bousfield and Kan also prove the following result.

Proposition 6.5: The product

$$[\ ,\ ] : [H^s_{Q\mathcal{A}}H^*Z]_t \otimes [H^{s'}_{Q\mathcal{A}}H^*Z]_{t'} \to [H^{s+s'+1}_{Q\mathcal{A}}H^*Z]_{t+t'}$$

is commutative, bilinear and satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

It follows that the product on E_rZ also satisfies the conclusion on Proposition 6.5.

In particular, let us consider the case where we are analyzing the cosimplicial space $\bar{\mathsf{F}}_2^{\cdot}Z$ where Z is a fibrant cosimplicial space. Then the homotopy spectral sequence reads

$$H^*_{Q\mathcal{A}}H^*Z \Rightarrow \pi_*Tot(Z)_2.$$

Therefore we obtain from the above results a commutative bilinear product

$$[\ ,\ ] : H^s_{Q\mathcal{A}}H^*Z \otimes H^{s'}_{Q\mathcal{A}}H^*Z \to H^{s+s'+1}_{Q\mathcal{A}}H^*Z. \tag{6.6}$$

This product adds the internal degree and abutts to the Whitehead product in the homotopy groups $\pi_*Tot(Z)_2$.

Notice that the product on $H^*_{Q\mathcal{A}}H^*Z$ is defined for *any* cosimplicial space Z because $\bar{\mathsf{F}}_2^{\cdot}Z$ is always F_2-like. We just have to be careful what the spectral sequence abuts to. See the examples at the end of section 3.

The rest of this section is devoted to studying the product (6.6).

The first thing to notice is that this product is actually intrinsic to $H^*_{Q\mathcal{A}}(\ )$ and does not depend on the existence of a cosimplicial space. To see this, let $A \in s\mathcal{U}\mathcal{A}$ be an almost-free unstable simplicial algebra. Then for all $s \geq 0$

$$A_s = G(V_s)$$

for some graded vector space V_s. Of course, $G : n\mathsf{F}_2 \to \mathcal{U}\mathcal{A}$ is the left adjoint to the augmentation ideal functor. The vector space diagonal

$$\Delta : V_s \to V_s \oplus V_s$$

yields, after applying G, a coproduct

$$\psi_s = G\Delta : A_s = G(V_s) \to G(V_s) \otimes G(V_s) = A_s \otimes A_s$$

that gives A_s the structure of a commutative, cocommutative Hopf algebra with conjugation in $\mathcal{UA}$. In particular, for any $\Lambda \in \mathcal{UA}$

$$Hom_{\mathcal{UA}}(A_s, \Lambda)$$

is a group; indeed

$$\begin{aligned} Hom_{\mathcal{UA}}(A_s, \Lambda) &\cong Hom_{\mathcal{UA}}(G(V_s), \Lambda) \\ &\cong Hom_{n\mathbf{F}_2}(V_s, I\Lambda) \end{aligned}$$

and all isomorphisms are group isomorphisms. Hence $Hom_{\mathcal{UA}}(A_s, \Lambda)$ is an $\mathbf{F}_2$ vector space. Now, because A is almost-free,

$$d_i : A_s \to A_{s-1}, \qquad 1 \leq i \leq s$$

and

$$s_i : A_s \to A_{s+1}, \qquad 0 \leq i \leq s$$

are maps of Hopf algebras. Only d_0 is not necessarily a map of Hopf alegbras; hence, it makes sense to measure the deviation of d_0 from being a Hopf algebra map. Define

$$\bar{\xi} : A_s \to A_{s-1} \otimes A_{s-1}$$

to be the product, in the group $Hom_{\mathcal{UA}}(A_s, A_{s-1} \otimes A_{s-1})$, of

$$(d_0 \otimes d_0)\psi_s : A_s \to A_{s-1} \otimes A_{s-1}$$

and

$$\psi_{s-1} d_0 : A_s \to A_{s-1} \otimes A_{s-1}.$$

The morphism $\bar{\xi}$ actually factors through a subalgebra of $A_{s-1} \otimes A_{s-1}$. For $\Lambda, \Gamma \in \mathcal{UA}$, define the product $\Lambda \times_{\mathbf{F}_2} \Gamma \in \mathcal{UA}$ by the pull-back diagram (of simplicial graded vector spaces)

$$\begin{array}{ccc} \Lambda \times_{\mathbf{F}_2} \Gamma & \to & \Gamma \\ \downarrow & & \downarrow \epsilon \\ \Lambda & \xrightarrow{\epsilon} & \mathbf{F}_2 \end{array}$$

If X and Y are pointed spaces, then

$$H^*(X \vee Y) \cong H^*X \times_{\mathbb{F}_2} H^*Y.$$

There is a natural map $\Lambda \otimes \Gamma \to \Lambda \times_{\mathbb{F}_2} \Gamma$ given by

$$u \otimes v \longmapsto (u\eta\epsilon(v), \eta\epsilon(u)v)$$

and we may define $\Lambda \wedge \Gamma$ by the pull-back diagram

$$\begin{array}{ccc} \Lambda \wedge \Gamma & \rightarrow & \Lambda \otimes \Gamma \\ \downarrow \epsilon & & \downarrow \\ \mathbb{F}_2 & \xrightarrow{\eta} & \Lambda \times_{\mathbb{F}_2} \Gamma. \end{array}$$

If X and Y are pointed spaces, then

$$H^*(X \wedge Y) \cong H^*X \wedge H^*Y.$$

Finally, notice for $A \in s\mathcal{UA}$ almost-free, there is a factoring

$$\begin{array}{ccc} A_s & \xrightarrow{\xi} & A_{s-1} \wedge A_{s-1} \\ \downarrow = & & \downarrow \\ A_s & \xrightarrow{\bar{\xi}} & A_{s-1} \otimes A_{s-1} \end{array} \tag{6.7}$$

To see this, one need only check that the two composites

$$A_s \xrightarrow{\bar{\xi}} A_{s-1} \otimes A_{s-1} \xrightarrow{\epsilon \otimes 1} A_{s-1}$$

and

$$A_s \xrightarrow{\bar{\xi}} A_{s-1} \otimes A_{s-1} \xrightarrow{1 \otimes \epsilon} A_{s-1}$$

are the trivial map

$$\eta\epsilon : A_s \to A_{s-1}.$$

For the morphism $\epsilon \otimes 1$, say, this is equivalent to showing that

$$(\epsilon \otimes 1)(d_0 \otimes d_0)\psi_s = (\epsilon \otimes 1)\psi_{s-1}d_0 : A_s \to A_{s-1}.$$

But this is obvious. A similiar argument can be given in the other case and that completes the definition of the map ξ of (6.7).

Notice that is $A = H^*Z$ where Z is an $\mathbf{F}_2$-like cosimplicial space, then

$$\xi = \zeta^* : H^*Z^s \to H^*Z^{s-1} \wedge H^*Z^{s-1}. \tag{6.8}$$

Thus we have algebraically copied the topological construction.

To define the product on cohomology of the simplicial unstable algebra $A \in s\mathcal{UA}$, we need the following lemmas. Let $Q(\ )$ denote the indecomposables functor.

Lemma 6.9: For $\Lambda, \Gamma \in \mathcal{UA}$, there are natural maps

$$Q(\Lambda \wedge \Gamma) \to Q\Lambda \otimes Q\Gamma$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q(\Lambda \wedge \Gamma) \to (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\Lambda) \otimes (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\Gamma).$$

Proof: The map $\Lambda \wedge \Gamma \to \Lambda \otimes \Gamma$ induces a map

$$I(\Lambda \wedge \Gamma) \to I\Lambda \otimes I\Gamma$$

where $I(\ )$ is the augmentation ideal functor. The result follows by investigating this map.

For the next lemma, we need some notation. If $f, g : A_s \to \Lambda$ with $A \in s\mathcal{UA}$ almost-free, let $f * g$ denote the product of f and g in the group $Hom_{\mathcal{UA}}(A_s, \Lambda)$; that is, $f * g$ is the composite

$$A_s \xrightarrow{\psi_s} A_s \otimes A_s \xrightarrow{f \otimes g} \Lambda \otimes \Lambda \to \Lambda$$

where the last map is multiplication. Notice that

$$Q(f * g) = Qf + Qg; QA_s \to Q\Lambda \tag{6.10.1}$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q(f * g) = \mathbf{F}_2 \otimes_{\mathcal{A}} Qf + \mathbf{F}_2 \otimes_{\mathcal{A}} Qg : \mathbf{F}_2 \otimes_{\mathcal{A}} QA_s \to \mathbf{F}_2 \otimes_{\mathcal{A}} Q\Lambda. \tag{6.10.2}$$

Thus, the next result will allow us to compute boundary homomorphisms in various chain complexes.

Lemma 6.11: Let $A \in s\mathcal{UA}$ be almost-free. Then if

$$\xi; A_s \to A_{s-1} \wedge A_{s-1}$$

is the map of (6.7), we have

1.) $(d_i \wedge d_i)\xi = \xi d_{i+1}, \qquad i \geq 1$; and

2.) $(d_0 \wedge d_0)\xi = [\xi d_0] * [\xi d_1]$.

Proof: These are simple consequences of the simplicial identities; we will do 2.)

It is sufficient to show that for

$$\bar{\xi} : A_s \to A_{s-1} \otimes A_{s-1}$$

we have the equation

$$(d_0 \otimes d_0)\bar{\xi} = [\bar{\xi} d_0] * [\bar{\xi} d_1].$$

This is because the map

$$Hom_{\mathcal{UA}}(A_s, A_{s-2} \wedge A_{s-2}) \to Hom_{\mathcal{UA}}(A_s, A_{s-2} \otimes A_{s-2})$$

is an injection. However,

$$\bar{\xi} = [(d_0 \otimes d_0)\psi] * [\psi d_0]$$

where the coproducts ψ_s and ψ_{s-1} are abbreviated to ψ. Now, since A is almost-free, the coproduct ψ commutes with d_i for $i \geq 1$:

$$(d_i \otimes d_i)\psi = \psi d_i, \qquad i \geq 1.$$

Thus we may compute, using the facts that $Hom_{\mathcal{UA}}(A_s, \Lambda)$ is an $\mathbf{F}_2$-vector space and that $d_0 d_1 = d_0 d_0$:

$$\begin{aligned}[(d_0 \otimes d_0)\bar{\xi}] * [\bar{\xi} d_1] &= [(d_0 \otimes d_0)^2\psi] * [(d_0 \otimes d_0 \psi d_0] * [(d_0 \otimes d_0)^2\psi] * [\psi d_0 d_0] \\ &= [(d_0 \otimes d_0)\psi d_0] * [\psi d_0 d_0] \\ &= \bar{\xi} d_0.\end{aligned}$$

The result follows.

We now use ξ to define a product in the cohomology of a simplicial unstable algebra. Let $A \in s\mathcal{UA}$. Since A is weakly equivalent to an almost-free object, we may assume that A is almost-free. Applying Lemma 6.9, we know that ξ induces maps of degree -1:

$$Q\xi : QA \to QA \otimes QA \tag{6.12.1}$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} \xi : \mathbf{F}_2 \otimes_{\mathcal{A}} QA \to \mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA. \tag{6.12.2}$$

By (6.10) and Lemma 6.11, we know that these are maps of chain complexes. We can use the second to define a product

$$[\ ,\ ] : H^p_{Q\mathcal{A}}A \otimes H^q_{Q\mathcal{A}}A \to H^{p+q+1}_{Q\mathcal{A}}A$$

as the map induced by the map of cochain complexes

$$(\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \otimes (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \to (\mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \xrightarrow{\mathbf{F}_2 \otimes_{\mathcal{A}} \xi} (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*.$$

The first map is the canonical homomorphism from $V^* \otimes W^* \to (V \otimes W)^*$ and we use the Eilenberg-Zilber Theorem to give a natural isomorphism

$$H^*[(\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \otimes (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*] \cong H^*_{Q\mathcal{A}}A \otimes H^*_{Q\mathcal{A}}A.$$

Notice that if $A = H^*Z$ for some $\mathbf{F}_2$-like cosimplicial space, then

$$\pi_* Z = (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*$$

and, thus, in light of 6.8, this product agrees with the one given by Theorem 6.3.

The following is obvious.

Proposition 6.13: The product

$$[\ ,\ ] : H^p_{Q\mathcal{A}}A \otimes H^q_{Q\mathcal{A}}A \to H^{p+q+1}_{Q\mathcal{A}}A$$

is bilinear, commutative, and adds internal degree.

More importantly, perhaps, the product is commutative on the chain level. We record this fact in the following result. If V is any vector space, let $T : V \otimes V \to V \otimes V$ be the switch map $T(u \otimes v) = v \otimes u$. The next result follows from the definitions.

Lemma 6.14: We have equality between the following morphisms:

$$Q\xi = TQ\xi : QA \to QA \otimes QA$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi = T\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi : \mathbf{F}_2 \otimes_{\mathcal{A}} QA \to \mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA.$$

7. Operations in Quillen Cohomology

Whenever one has a cohomology theory with a product that is commutative on the cochain level, then one has naturally defined Steenrod or divided product operations. Hence the results of the last section will yield "divided Whitehead squares." The purpose of this section is to define and explore the properties of these operations. In particular, we will note at the end of the section that these operations do not, in general, commute with the differentials in the Bousfield-Kan spectral sequence.

We will prove the following result.

Theorem 7.1: Let $A \in s\mathcal{UA}$ be a simplicial unstable algebra. Then there are natural homomorphisms

$$P^i : H^q_{Q\mathcal{A}} A \to H^{q+i+1}_{Q\mathcal{A}} A$$

so that

1.) there is an *unstable* condition:

$$P^i(x) = 0 \qquad if \quad i < 2 \quad \text{or} \quad i > q$$

and

$$P^q(x) = [x, x]$$

where [,] is the product of the previous section;

2.) for all $x, y \in H^*_{\mathcal{QA}}A$ and all i, there is a *Cartan Formula*:

$$[x, P^i(y)] = 0;$$

3.) there are *Adem Relations* for $j \geq 2i$:

$$P^j P^i = \sum_{s=j-i+1}^{j+i-2} \binom{2s-j-1}{s-i} P^{i+j-s} P^s.$$

Remark: If an element $x \in H^s_{\mathcal{QA}}H^*Z$ with $s = 0$ or 1 survives to E_∞ in the homotopy spectral sequence and detects an element $\alpha \in \pi_* Tot(Z)_2$ it is not immediately apparent what detects the Whitehead product $[\alpha, \alpha]$, since $[x, x] = P^s(x) = 0$. This will be the case, for example, if $Z = S^n$ regarded as a constant cosimplicial space and

$$\iota \in [H^0_{\mathcal{QA}}H^*S^n]_n \cong Ext^0_{\mathcal{UA}}(H^*S^n, H^*S^n)$$

detects the identity map. Since $[\iota, \iota] = 0$ in the E_2 term, the Whitehead product of the identity $1 \in \pi_n S^n$ with itself must be detected by an element in $Ext^s_{\mathcal{UA}}(H^*S^n, H^*S^t)$ with $t - s = 2n - 1$ and $s \geq 2$. If $n \neq 2^k - 1$ for some k, then it is known that $s = 2$.

There are several ways to define the operations P^i. The classical way is to appeal to the following lemma. If V is a simplicial $\mathbf{F}_2$-vector space, let $C(V)$ be the chain complex obtained by setting $C(V)_n = V_n$ and $\partial = \sum_{i=0}^n d_i$. Let T denote any switch map interchanging factors.

Lemma 7.2: Let V and W be simplicial $\mathbf{F}_2$-vector spaces. Then there are *higher Eilenberg-Zilber maps*:

$$D_i : C(V \otimes W) \to [C(V) \otimes C(W)]_{n+i}$$

so that

1.) D_0 is a chain map and a chain equivalence; and

2.) for $i \geq 1$

$$\partial D_i + D_i \partial = D_{i-1} + T D_{i-1} T.$$

These are standard and essentially unique. See [10].

We use these maps to define the operations. First assume $A \in s\mathcal{UA}$ almost-free,

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi : \mathbf{F}_2 \otimes_{\mathcal{A}} QA \to \mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA$$

be the chain coproduct of degree -1 defined in (6.12.1). Define a function

$$S^i : (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \to (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*$$

of degree $i+1$ by setting, for α of degree q

$$(7.3) \quad S^i(\alpha) = (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi)^* D^*_{q-i}(\alpha \otimes \alpha) + (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi)^* D^*_{q-i+1}(\alpha \otimes \partial\alpha).$$

Here we let $D_i = 0$ if $i < 0$. The one easily checks, using Lemma 6.14, that

$$(7.3.1) \qquad \partial S^i(\alpha) = (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi)^* D^*_{q+1-i}(\partial\alpha \otimes \partial\alpha) = S^i(\partial\alpha).$$

Let

$$P^i = \pi^* S^i : H^*_{Q\mathcal{A}} A \to H^*_{Q\mathcal{A}} A.$$

If $A \in s\mathcal{UA}$ is not almost-free, choose an acyclic fibration $X \to A$ and define the operations in $H^*_{Q\mathcal{A}} X \cong H^*_{Q\mathcal{A}} A$.

Since D_0 is the Eilenberg-Zilber chain equivalence, the following is clear

$$P^q(x) = [x, x].$$

This is part of Theorem 7.1.1.

Now, in order to establish the properties of the operations and to prove certain other facts about the structure of the functor $H^*_{Q\mathcal{A}}(\ )$, we establish the connection between this cohomology of simplicial unstable algebras and the ordinary André-Quillen cohomology of simplicial commutative algebras over a field. We will use only the field $\mathbf{F}_2$, but much of what say here will hold at other primes as well.

Let $\mathbf{A}$ be the category of graded, commutative, supplemented $\mathbf{F}_2$ algebras and let $s\mathbf{A}$ be the associated simplicial category. There is a forgetful functor $\mathcal{UA} \to \mathbf{A}$. As with $s\mathcal{UA}$, $s\mathbf{A}$ is a closed model category with a distinguished sub-catgeory of abelian objects and, hence, there is a notion of homology and cohomology. This goes back to André [1] and Quillen [18].

To be specific, we first say that weak equivalences, fibrations, and cofibrations are defined exactly as they were for $s\mathcal{UA}$ in Section 2. In particular, we have a notion of almost-free objects defined using the symmetric algebra functor

$$S : n\mathbf{F}_2 \to \mathbf{A}$$

left adjoint to the augmentation ideal functor I. Then for $\Lambda \in \mathbf{A}$, we obtain an augmented simplicial object

$$\bar{S}.\Lambda \to \Lambda$$

from the cotriple $\bar{S} = S \circ I$. Then, as in (2.5), we obtain, for $A \in s\mathbf{A}$, an augmented bisimplicial algebra

$$\bar{S}_{\cdot,\cdot}A \to A$$

and, if we set $\bar{S}.A = diag\bar{S}_{\cdot,\cdot}A$, then we have an acyclic fibration

$$\bar{S}.A \to A$$

in $s\mathbf{A}$ with $\bar{S}.A$ almost-free and, hence, cofibrant in $s\mathbf{A}$. This is all gone into in detail in [19] and [20]. Therefore, we define

$$H_*^Q A = \pi_* Q\bar{S}.A \tag{7.4}$$

and

$$H_Q^* A = \pi^*(Q\bar{S}.A)^*.$$

The appropriate analog of 2.7 implies that these are well-defined functors of A, independent of the choice of $\bar{S}.A$. Indeed, we may replace the acyclic fibration $\bar{S}.A \to A$ by any acyclic fibration $X \to A$ with X cofibrant in $s\mathbf{A}$. $H_Q^*(\ )$ supports a great deal of structure; indeed, H_Q^* is a functor from $s\mathbf{A}$ to the category $\mathcal{W}$ defined in the following definition.

Definition 7.5: Let $\mathcal{W}$ be the subcategory of bigraded $\mathbf{F}_2$-vector spaces defined as follows: $W = \{W_n^p\}$ is an object in $\mathcal{W}$ if

1.) there is a commutative bilinear product

$$[\ ,\ ] : W^p \otimes W^q \longrightarrow W^{p+q+1}$$

that adds internal degree and satisfies the Jacobi identity;

2.) there are homomorphisms

$$P^i : W^q \longrightarrow W^{q+i+1}$$

doubling internal degree, such that if $i < 2$ or $i > q$

$$P^i = 0$$

and if $i = q$

$$P^i(x) = [x, x]$$

and if $j \geq 2i$, then

$$P^j P^i = \sum_{s=j-i+1}^{j+i-2} \binom{2s-j-1}{s-i} P^{i+j-s} P^s$$

and for all x, y and i

$$[x, P^i(y)] = 0;$$

3.) there is a quadratic operation

$$\beta : W^0 \longrightarrow W^1$$

doubling internal degree and so that for all

$$x, y \in W^0$$

$$\beta(x+y) = \beta(x) + \beta(y) + [x, y]$$

and for all $x \in W$ and $y \in W^0$

$$[\beta(y), x] = [y, [y, x]].$$

A morphism in $\mathcal{W}$ preserves this structure.

Theorem 7.6:[14] André-Quillen cohomology defines a functor

$$H_Q^* : s\mathbf{A} \to \mathcal{W}.$$

The product and operations in $H_Q^*(\ )$ are defined in exactly the same fashion as the product and operations in $H_{Q\mathcal{A}}^*(\ )$. In fact, we note the following fact: if $A \in s\mathcal{U}\mathcal{A}$ is almost-free, then under the forgetful functor $s\mathcal{U}\mathcal{A} \to s\mathbf{A}$, A passes to an almost-free object in $s\mathbf{A}$. Thus we have, by the remarks after the definition of H_*^Q, that

$$\pi_* QA \cong H_*^Q A$$

and the quotient map

$$QA \to \mathbb{F}_2 \otimes_{\mathcal{A}} QA$$

induces a map

$$H_*^Q A \to H_*^{Q\mathcal{A}} A. \tag{7.7}$$

This fact will be exploited in the following sections.

Now, however, we wish to exploit Theorem 7.6. To do this, we define a functor $\mathcal{U}\mathcal{A} \to \mathbf{A}$ that kills all Steenrod operations except possibly the squaring (or top) operation. Let $\Lambda \in \mathcal{U}\mathcal{A}$. Define

$$J(\Lambda) \subseteq \Lambda$$

to be the ideal generated by all elements of the form

$$Sq^I(x) = Sq^{i_1} \cdots Sq^{i_s}(x)$$

so that

$$e(I) = i_1 - i_2 \cdots - i_s < deg(x)$$

and Sq^I in the augmentation ideal of $\mathcal{A}$. $J(\Lambda)$ is a functor of Λ and we may set

$$\Theta\Lambda = \Lambda / J(\Lambda).$$

Notice that $J(\Lambda)$ is not necessarily invariant under the action of the Steenrod algebra; hence Θ defines a functor $\Theta : \mathcal{UA} \longrightarrow \mathbf{A}$, but not a functor to $\mathcal{UA}$.

Let $G : n\mathbb{F}_2 \longrightarrow \mathcal{UA}$ and $S : n\mathbb{F}_2 \longrightarrow \mathbf{A}$ be the left adjoints to the augmentation ideal functors. The next result describes a few properties of the functor Θ.

Proposition 7.8:1.) For $V \in n\mathbb{F}_2$ there is a natural isomorphism

$$\Theta G(V) \cong S(V);$$

2.) for $V \in n\mathbb{F}_2$, there is a natural isomorphism

$$\mathbb{F}_2 \otimes_{\mathcal{A}} QG(V) \cong Q\Theta G(V);$$

3.) for $A \in s\mathcal{UA}$ almost-free

$$H^*_Q \Theta A \cong H^*_{Q\mathcal{A}} A;$$

4.) if $\wedge : \mathcal{UA} \times \mathcal{UA} \longrightarrow \mathcal{UA}$ is the smash product defined in the previous section, then for all $V, W \in n\mathbb{F}_2$ there is a ntural isomorphism

$$\Theta(G(V) \wedge G(W)) \cong \Theta G(V) \wedge \Theta G(W).$$

Proof: Parts 1.) and 2.) are obvious. For part 3.), if $A \in s\mathcal{UA}$ is almost-free, then part 1.) implies that ΘA is almost-free in $s\mathbf{A}$. Hence

$$\begin{aligned} H^*_{Q\mathcal{A}} A &\cong \pi^*(\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* \\ &\cong \pi^*(Q\Theta A)^* \qquad \text{by part 2.)} \\ &\cong H^*_Q \Theta A \end{aligned}$$

For part 4.), the definition of $\wedge$ implies that the following isomorphisms are sufficient to imply the result:

$$\begin{aligned} \Theta(G(V) \otimes G(W)) &\cong \Theta G(V \oplus W) \\ &\cong S(V \oplus W) \qquad \text{by part 1.)} \\ &\cong S(V) \otimes S(W) \cong \Theta G(V) \otimes \Theta G(W) \end{aligned}$$

and

$$\Theta(G(V) \times_{\mathbf{F}_2} G(W)) \cong \Theta G(V) \times_{\mathbf{F}_2} \Theta G(W)$$

by direct calculation. The result now follows.

We can now prove the result stated at the beginning of the section.

Proof of Theorem 7.1: We may assume that $A \in s\mathcal{UA}$ is almost-free. Let

$$\xi : A \to A \wedge A$$

be the comultiplication map used to define the product and operations in $H^*_{Q\mathcal{A}}A$. Then

$$\Theta\xi : \Theta A \to \Theta(A \wedge A) \cong \Theta A \wedge \Theta A$$

is used to define the product and coproduct in $H^*_Q\Theta A$. Here we use Propositon 7.8.4. The result now follows from Theorem 7.6 and Proposition 7.8.3.

A similar argument now proves the following result, independent of the work of Bousfield and Kan.

Proposition 7.9: The product

$$[\ ,\ ] : H^*_{Q\mathcal{A}}A \otimes H^*_{Q\mathcal{A}}A \to H^*_{Q\mathcal{A}}A$$

satisfies the Jacobi identity: for all $x, y, z \in H^*_{Q\mathcal{A}}A$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Now let $f : A \to B$ be morphism in $s\mathcal{UA}$ and

$$\partial : H^s_{Q\mathcal{A}}A \to H^{s+1}_{Q\mathcal{A}}M(f)$$

the boundary map in the long exact sequence of the resulting cofibration sequence, as in Proposition 2.11. Since one the f the focuses of this paper has been on how this map behaves with respect to the homotopy spectral sequence, we would like to know how it behaves with respect to the product and operations.

Lemma 7.10:1.) Let $x \in H^*_{Q\mathcal{A}}A$. Then, for all i

$$\partial P^i(x) = P^i(\partial x).$$

2.) For all $x \in H^*_{Q\mathcal{A}}A$ and $y \in H^*_{Q\mathcal{A}}M(f)$

$$[\partial x, y] = 0.$$

Proof: The map ∂ is the connecting homomorphism obtained from a short exact sequence of cochain complexes. See 2.11. Part 1.) follows from investigating formula (7.3.1) and part 2.) follows from the naturality of the homomorphism D_0 of Lemma 7.2.

Corollary 7.11: Let $A \in s\mathcal{UA}$ and let ΣA be the suspension of A. Then for all $x, y \in H^*_{Q\mathcal{A}}\Sigma A$

$$P^i(x) = 0 \qquad \text{for } i \geq deg(x)$$

and

$$[x, y] = 0.$$

Proof: This follows from the fact that

$$\partial : H^s_{Q\mathcal{A}}A \to H^{s+1}_{Q\mathcal{A}}\Sigma A$$

is an isomorphism and the previous lemma.

To obtain some initial understanding of how the operations behave in the homotopy spectral sequence, we consider the universal examples of section 3

$$H^*_{Q\mathcal{A}}K(p, q)_+ \Rightarrow \pi_* S^{q-p}$$

where the sphere is completed at 2 and we assume that $q - p > 1$. Let $\iota = \iota_{p,q} \in H^p_{Q\mathcal{A}}K(p, q)_+$ be the universal class. If $p = 1$, the Theorem 7.1.1 implies that

$$P^i(\iota) = 0$$

for all i. Thus, if

$$j \in H^p_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+$$

is the suspension of this class, Lemma 7.10 implies that

$$P^i(j) = 0 \tag{7.12}$$

for all i. By considering the results of section 4, we see that there is spectral sequence

$$H^*_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+ \Rightarrow \pi_*\Omega^{p-1}S^{q-1}.$$

Now, in the homotopy category associated to $s\mathcal{UA}$, let

$$e : \Sigma^{p-1}K(1,q)_+ \to K(p,q)_+$$

corepresent $j \in H^p_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+$. The results of sections 4 and 5 yield a diagram of spectral sequences which is long exact on the E_2 terms

(7.13)

$$\begin{array}{ccccccccc}
\to & H^*_{Q\mathcal{A}}M(e) & \xrightarrow{f} & H^*_{Q\mathcal{A}}K(p,q)_+ & \xrightarrow{e^*} & H^*_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+ & \to \\
 & \Downarrow & & \Downarrow & & \Downarrow & \\
\to & \pi_*C(p-1) & \to & \pi_*S^{q-p} & \xrightarrow{E_{p-1}} & \pi_*\Omega^{p-1}S^{q-1} & \to
\end{array}$$

where E_{p-1} is the suspension homomorphism and $C(p-1)$ is the homotopy fiber. Now (7.12) implies that

$$e^*P^i(\iota_{p,q}) = 0$$

for all i; hence, for each i, $2 \le i \le p$ there must exist a non-zero class

$$y_i \in [H^{p+i+1}_{Q\mathcal{A}}M(e)]_{2q}$$

so that

$$f(y_i) = P^i(\iota_{p,q})$$

where f is the map in (7.13). We now investigate the behavior of the class y_i in the diagram of spectral sequences induced by the Hurewicz homomorphism

$$\begin{array}{ccc}
H^*_{Q\mathcal{A}}M(e) & \Rightarrow & \pi_*C(p-1) \\
\downarrow h^* & & \downarrow h \\
\pi^*M(e)^* & \Rightarrow & H_*C(p-1)
\end{array}$$

where the lower spectral sequence is the homology spectral sequence of section 3. In fact, we will prove the following result. It is known from the calculations of [14, Appendix B] that $h^* y_i \neq 0$.

Proposition 7.14: The class $h^* y_i$ survives to E_∞ in the homology spectral sequence

$$\pi^* M(e)^* \Rightarrow H_* C(p-1)$$

and detects the unique non-zero class in

$$H_{2q-(p+i+1)} C(p-1).$$

What is more, we will identify this non-zero class and show that it often cannot be in the image of the Hurewicz homomorphism

$$h : \pi_* C(p-1) \to H_* C(p-1).$$

We will thus conclude the following corollary:

Corollary 7.15: If $p \geq 3$ and $q - p$ is an odd number, then y_{p-1} does not survive to E_∞ in the homotopy spectral sequence

$$H^*_{\mathcal{QA}} M(e) \Rightarrow \pi_* C(p-1).$$

Then we will use the calculations of the next two sections to prove that a similar statement can actually be made about $P^i(\iota_{p,q})$; in other words, since $\iota_{p,q}$ survives to E_∞ in the spectral sequence

$$H^*_{\mathcal{QA}} K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

and detects the identity, the operations P^i don't necessarily commute with the differentials.

The computation necessary for proving Proposition 7.14, begins with a familiar, but disguised calculation: most homotopy theorists have used the Eilenberg-Moore spectral sequence to compute $H_* W(p-1)$ and the same computation — bigraded, if you will — is used to compute $\pi^* M(e)^*$.

First we compute $\pi_*\Sigma^{p-1}K(1,q)_+$. This is done by double induction on the succesive spectral sequences, $0 \leq m \leq p-1$,

$$Tor^{\pi_*\Sigma^m K(1,q)_+}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \pi_*\Sigma^{m+1}K(1,q)_+ \Rightarrow H^*\Omega^{m+1}S^{q-1}.$$

Both these spectral sequences must collpase for every m for vector space dimension reasons. Here we use the fact that we know $H_*\Omega^{m+1}S^{q-1}$. Next we investigate the succesive spectral sequences

$$Tor^{\pi_*K(p,q)_+}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \pi_*M(e) \Rightarrow H^*C(p-1)$$

and conclude from our knowledge of $H_*W(p-1)$ that both these spectral sequences must collapse.

In particular, we can make the following conclusions. If

$$\mathbf{RP}^m_n = \mathbf{RP}^m/\mathbf{RP}^{n-1}$$

is the stunted real projective space with cells in dimensions n through m, then we know that there is map

$$\Sigma^{q-p-1}\mathbf{RP}^{q-2}_{q-p} \to W(p-1) \tag{7.16}$$

and for $2q-2p-1 \leq k \leq 2q-p-3$

$$H_kC(p-1) \cong \mathbb{F}_2$$

generated by the image of the non-zero class in $H_k\Sigma^{q-p-1}\mathbf{RP}^{q-1}_{q-p}$. Thus, the map of (7.14) induces an isomorphism on homotopy groups in a range of degrees.

Now, if $2 \leq i \leq p$, then $2q-2p-1 \leq 2q-(p+i+1) \leq 2q-p-3$ and there is a unique class, $2 \leq i \leq p$,

$$x_i \in \pi^{p+i+1}M(e)^*_{2q}$$

that detects the non-zero class in

$$H_{2q-(p+i+1)}W(p-1)$$

in the homology spectral sequence. Finally, in the diagram

$$\begin{array}{ccc} H^*_{Q\mathcal{A}}M(e) & \xrightarrow{f} & H^*_{Q\mathcal{A}}K(p,q)_+ \\ \downarrow h^* & & \\ \pi^*M(e)^* & & \end{array}$$

there is a class $y_i \in H^*_{Q\mathcal{A}}M(e)$ so that

$$f(y_i) = P^i(\iota_{p,q}) \in [H^{p+i+1}_{Q\mathcal{A}}K(p,q)_+]_{2q}$$

and

$$h^* y_i = x_i.$$

The first statement was asserted and proved above, and the second statement follows from the fact (proved in [14]) that $h^* y_i \neq 0$. This completes the proof of Proposition 7.14.

Proof of Corollary 7.15: Notice that the map

$$\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-2}_{q-p} \to W(p-1)$$

which is an equivalence in a range of degrees $k \leq 2q-p-3$, demonstrates that very often the non-zero class in $H_k W(p-1)$, $2p-2p-1 \leq k \leq 2q-p-3$ is not in the image of the Hurewicz map. In particular, if $p \geq 3$ and $q-p$ is odd, and if $z \in H^{2(q-p)-1}C(p-1)$ is the non-zero class, then $\mathrm{Sq}^1 z \neq 0$. Thus, the non-zero class in $H_{2(q-p)}C(p-1)$ is not in the image of the Hurewicz homomorphism. Hence there exists an r so that $d_r y_{p-1} \neq 0$.

In fact, we make the following remark, which is perhaps more philosophy than mathematics. If we regard

$$H^*_{Q\mathcal{A}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

as a desuspension spectral sequence — and we will explore this point more in the next few sections — we could say that the elements $P^i(\iota_{p,q})$ are the "cells" of $\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-2}_{q-p}$ in Toda's desuspension long exact sequence

$$\cdots \to \pi_n(\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-1}_{q-p}) \to \pi_n S^{q-p} \xrightarrow{E_{p-1}} \pi_{n+p-1}S^{q-1} \to \cdots$$

valid for $n \leq 2p-p-3$. This sequence is derived from the fiber sequence of (7.13) using the inclusion of (7.16).

8. Miller's Composite Functor Spectral Sequence

Because the functor $\mathbf{F}_2 \otimes_{\mathcal{A}} Q(\ )$ — the main object of study in this paper — can be written as the composite functor

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q(\ ) = \mathbf{F}_2 \otimes_{\mathcal{A}} (\ ) \circ Q(\ )$$

it is no surpise that there is a composite functor spectral sequence converging to $H^*_{Q\mathcal{A}}A$ for $A \in s\mathcal{UA}$. The purpose of this section is to explore this spectral sequence — due, in principal, to Haynes Miller — and to prepare the way for the computations of the next section.

If $A \in s\mathcal{UA}$, we may define H^Q_*A and H^*_QA — the homology and cohomology based on the indecomposables functor — in a manner analogous to $H^{Q\mathcal{A}}_*A$ and $H^*_{Q\mathcal{A}}A$. Let

$$p : X \to A$$

be an acyclic fibration in $s\mathcal{UA}$ with X almost-free. Then we let

$$H^Q_*A = \pi_*QX$$

and

$$H^*_QA = \pi^*(QX)^*. \tag{8.1}$$

Actually, we defined H^Q_*A and H^*_QA in the previous section, and we must show that this new defintion agrees with this one. This is proved by Miller in [19, Section 2]. In fact, the forgetful functor $s\mathcal{UA} \to s\mathbf{A}$ carries almost-free objects to almost-free objects, and the appropriate analog of Lemma 2.7 for the category $s\mathbf{A}$ implies that the two definitions agree with each other.

So let $X \to A$ be an acyclic fibration with X almost-free in $s\mathcal{UA}$. Since X_m is an unstable algebra for each m, QX_m is an unstable $\mathcal{A}$ module and, hence, H^Q_mA is an unstable $\mathcal{A}$ module for each m. More is true, however. Since for $x \in X^n_m$, $\mathrm{Sq}^n(x) = x^2$, we actually have that H^m_QA is a suspension in the category of unstable $\mathcal{A}$ modules. We now give this category a name. Let $\mathcal{U}_0 \subseteq \mathcal{U}$ be the full sub-catgeory specified by the condition that $M \in \mathcal{U}_0$ if an only if for all $x \in M^n$,

$$\mathrm{Sq}^i x = 0 \qquad \text{for } i \geq n.$$

The suspension functor defines an isomorphism of categories

$$\Sigma : \mathcal{U} \xrightarrow{\cong} \mathcal{U}_0. \tag{8.2}$$

The above remarks now imply that we have a functor for each m

$$H_m^Q(\ ) : s\mathcal{U}\mathcal{A} \to \mathcal{U}_0. \tag{8.3}$$

Because of the isomorphism of categories given in 8.2, homological algebra in $\mathcal{U}_0$ is basically the same as homological algebra in $\mathcal{U}$. Indeed, if $M \in \mathcal{U}_0$, then there is a natural isomorphism

$$Ext^s_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2) \cong Ext^s_{\mathcal{U}}(\Sigma^{-1} M, \Sigma^{t-1}\mathbb{F}_2) \tag{8.4}$$

for all s and t. To see this, note that the if $P : n\mathbb{F}_2 \to \mathcal{U}$ is left adjoint to the forgetful functor F, the composite

$$\bar{P} = P \circ F : \mathcal{U} \to \mathcal{U}$$

forms a cotriple and

$$Ext^s_{\mathcal{U}}(N, \Sigma^t \mathbb{F}_2) \cong \pi^s Hom_{\mathcal{U}}(\bar{P}_\cdot(N), \Sigma^t\mathbb{F}_2).$$

Furthermore,

$$\bar{P}' = \Sigma \circ \bar{P} \circ \Sigma^{-1} : \mathcal{U}_0 \to \mathcal{U}_0$$

forms a cotriple on $\mathcal{U}_0$ and

$$Ext^s_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2) \cong \pi^s Hom_{\mathcal{U}_0}(\bar{P}'_\cdot(M), \Sigma^t\mathbb{F}_2).$$

A simple comparison of the two defintions, using the fact that Σ is exact, now yields the equation (8.4) above.

Let us write $Hom_{\mathcal{U}_0}(M, \mathbb{F}_2)$ and $Ext^s_{\mathcal{U}_0}(M, \mathbb{F}_2)$ for the graded vector spaces with, in degree t,

$$Hom_{\mathcal{U}_0}(M, \mathbb{F}_2)_t = Hom_{\mathcal{U}_0}(M, \Sigma^t\mathbb{F}_2)$$

and

$$Ext^s_{\mathcal{U}_0}(M, \mathbb{F}_2)_t = Ext^s_{\mathcal{U}_0}(M, \Sigma^t\mathbb{F}_2).$$

The following is basically the spectral sequence of Miller's, found in section 2 of [19].

Theorem 8.5: For $A \in s\mathcal{U}\mathcal{A}$, there is a spectral sequence of graded vector spaces

$$Ext^p_{\mathcal{U}_0}(H^Q_q A, \mathbb{F}_2) \Rightarrow H^{p+q}_{Q\mathcal{A}} A.$$

Proof: Our proof is no different than Miller's, or the proof given for any composite functor spectral sequence. We may assume that A is amost free in $s\mathcal{U}\mathcal{A}$. Form the augmented bi-cosimplicial vector space

$$\lambda : Hom_{\mathcal{U}_0}(QA_q, \mathbb{F}_2) \to C^{p,q} = Hom_{\mathcal{U}_0}(\bar{P}'_p QA_q, \mathbb{F}_2) \tag{8.6}$$

and note that

$$Hom_{\mathcal{U}_0}(QA_q, \mathbb{F}_2) \cong Hom_{n\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\mathcal{A}} QA_q, \mathbb{F}_2) \cong (\mathbb{F}_2 \otimes_{\mathcal{A}} QA_q)^*.$$

Filtering $C^{\cdot,\cdot}$ by degree in q, we obtain a spectral sequence with

$$E_1^{p,q} \cong \begin{cases} Hom_{\mathcal{U}_0}(QA_q, \mathbb{F}_2), & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

This is because QA_m is projective in $\mathcal{U}_0$. Here we use the fact that, since A is almost-free, $A_m \cong G(V)$ for some graded vector space V and $QG(V) = P'(V)$. Therefore, the spectral sequence has the form

$$E_2^{p,q} \cong \begin{cases} H^q_{Q\mathcal{A}} A, & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

Thus λ, as in (8.6), induces an isomorphism

$$\lambda^* : H^*_{Q\mathcal{A}} A \to H^* C^{\cdot,\cdot}.$$

Therefore, filtering $C^{\cdot,\cdot}$ by degree in p, we obtain a spectral sequence abutting to $H^*_{Q\mathcal{A}} A$ with

$$E_1^{p,q} \cong Hom_{\mathcal{U}_0}(P'_p H^Q_q A, \mathbb{F}_2).$$

This is because

$$Hom_{\mathcal{U}_0}(P'(\ \ ), \mathbb{F}_2)$$

is an exact functor. Hence

$$E_2^{p,q} \cong Ext^p_{\mathcal{U}_0}(H_q^Q A, \mathbb{F}_2).$$

This finishes the proof.

Notice that the differentials in this spectral sequence are of the form

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

Before proceeding, let us do an example.

Example 8.7: Let us use this spectral sequence to begin to compute

$$Ext^*_{\mathcal{UA}}(H^*S^n, H^*S^t).$$

The spectral sequence in this case reads

$$Ext^p_{\mathcal{U}_0}(H_q^Q H^*S^n, \Sigma^t \mathbb{F}_2) \Rightarrow [H^{p+q}_{Q\mathcal{A}} H^*S^n]_t \cong Ext^{p+q}_{\mathcal{UA}}(H^*S^n, H^*S^t)$$

where H^*S^n is regarded as a constant simplicial algebra in $s\mathcal{UA}$. $H_*^Q H^*S^n$ has been known for years (see [4]):

$$H_0^Q H^*S^n \cong \mathbb{F}_2$$

concentrated in degree n,

$$H_0^Q H^*S^n \cong \mathbb{F}_2$$

concentrated in degree $2n$, and

$$H_q^Q H^*S^n = 0 \qquad \text{if } q \geq 2.$$

Thus the spectral sequence becomes a long exact sequence

$$\cdots \longrightarrow E_2^{p-1,1} \xrightarrow{d_2} E_2^{p+2,0} \longrightarrow Ext^{p+2}_{\mathcal{UA}}(H^*S^n, H^*S^n) \longrightarrow E_2^{p+1,1} \longrightarrow \cdots$$

which, using the relationship between $Ext_{\mathcal{U}_0}(\ ,\ )$ and $Ext_{\mathcal{U}}(\ ,\ )$ described above, yields a long exact sequence

$$\begin{aligned}\to Ext_{\mathcal{U}}^{p-1}(\Sigma^{2n-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2) \xrightarrow{d_2} & Ext_{\mathcal{U}}^{p+2}(\Sigma^{n-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2) \\ & \to Ext_{\mathcal{UA}}^{p+2}(H^*S^n, H^*S^t) \\ & \to Ext_{\mathcal{U}}^{p+1}(\Sigma^{2n-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2) \xrightarrow{d_2}\end{aligned}$$

This is easily seen to be the algebraic EHP sequence, much studied by Mahowald and others, perhaps most exhaustively in [9].

Next notice that there is an edge homomorphism

$$e : H^q_{Q\mathcal{A}}A \to E^{0,q} \subseteq Hom_{\mathcal{U}_0}(H_q^{Q\mathcal{A}}A, \mathbb{F}_2) \subseteq H^q_Q A. \tag{8.8}$$

Since

$$Hom_{\mathcal{U}_0}(QA, \mathbb{F}_2) \cong Hom_{n\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\mathcal{A}} QA, \mathbb{F}_2)$$

one easily sees that this edge homomorphism is given by the dual of the map

$$H_q^Q A \to H_q^{Q\mathcal{A}} A$$

induced by the quotient map

$$QA \to \mathbb{F}_2 \otimes_{\mathcal{A}} QA.$$

Now we know that $H^*_Q A$ and $H^*_{Q\mathcal{A}}A$ support a great deal of structure, including the product and operations as defined in the previous two sections. Furthermore, we will know that the edge homomorphism of (8.8) preserves the product and operations — see 8.14 below. Thus, it makes sense that the product and operations in $H^*_Q A$ should induce a product and operations in the composite functor spectral sequence and these should abut to the product and operations in $H^*_{Q\mathcal{A}}A$. In preparation for the computations of the next section, we show that this is in fact the case.

We will use the techniques of Singer [21], modified slightly. The modification is necessary, as our product is non-associative and induced by a chain map of degree -1, instead of degree 0. Let

$$C = C^{\cdot,\cdot} = Hom_{\mathcal{U}_0}(\bar{P}'_{\cdot} QA_{\cdot}, \mathbb{F}_2)$$

be the bi-cosimplicial vector space used to define the spectral sequence in (8.5). In order to apply Singer's line of argument, we must define a product

$$\xi'' : C^{\cdot,\cdot} \otimes C^{\cdot,\cdot} \to C^{\cdot,\cdot}$$

so that, if λ is the augmentation of (8.5), then there is a commutative diagram

$$\begin{array}{ccc} (\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* & \xrightarrow{\lambda\otimes\lambda} & C \otimes C \\ \downarrow \xi' & & \downarrow \xi'' \\ (\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* & \xrightarrow{\lambda} & C \end{array} \tag{8.9}$$

Here $\xi' = (\mathbb{F}_2 \otimes Q\xi)^*$ where $\mathbb{F}_2 \otimes Q\xi$ is as in Lemma 6.14. But this is simpy done. Let $\bar{P} : \mathcal{U} \to \mathcal{U}$ be the cotriple used above to define $Ext_{\mathcal{U}}$. Then, for $M, N \in \mathcal{U}$, there is a canonical map

$$\bar{P}(M \otimes N) \to \bar{P}(M) \otimes \bar{P}(N).$$

Since the cotriple $\bar{P}' : \mathcal{U}_0 \to \mathcal{U}_0$ is defined by the equations $\bar{P}' = \Sigma \circ \bar{P} \circ \Sigma^{-1}$, we can then define, for $A \in s\mathcal{UA}$ almost-free, a map

$$\zeta : \bar{P}'_{\cdot} QA \to \bar{P}' QA \otimes \bar{P}' QA$$

by the composition

$$\begin{aligned} \bar{P}'_p QA_q &\xrightarrow{\bar{P}'_q Q\xi} \bar{P}'_q(Q(A_{q-1} \wedge A_{q-1})) \\ &\to \bar{P}'_p(QA_{q-1} \otimes QA_{q-1}) \\ &\to \bar{P}'_p QA_{q-1} \otimes \bar{P}'_p QA_{q-1} \end{aligned}$$

where ξ is the as in (6.7) and we use (6.7) for the second map. Then we apply the functor $Hom_{\mathcal{U}_0}(\ ,\mathbb{F}_2)$ to obtain ξ'' satisfying the requirements of (8.9).

The following lemma is needed to use Singer's results. Let V be a bisimplicial vector space — for example, we could let $V = \bar{P}'_{\cdot} QA$ or $\bar{P}'_{\cdot} QA \otimes \bar{P}'_{\cdot} QA$, where we take the degree-wise tensor product. Let

$$\partial^h : V_{p,q} \to V_{p-1,q}$$

and

$$\partial^v : V_{p,q} \to V_{p,q-1}$$

be the horizontal and vertical boundary operators obtained by taking an appropriate sum of face maps. Let ζ be as above. Then

$$\zeta : (\bar{P}'_{\cdot} QA)_{p,q} \to (\bar{P}'_{\cdot} QA \otimes \bar{P}'_{\cdot} QA)_{p,q-1}.$$

Lemma 8.11: ζ commutes with the horizontal and vertical boundary operators:

$$\partial^h \zeta = \zeta \partial^h \qquad \text{and} \qquad \partial^v \zeta = \zeta \partial^v.$$

Proof: ζ actually commutes with the horizontal face maps, by the naturality of the construction of (8.10). That ζ commutes with vertical boundary operator is a consequence of 6.11.

The following results are now a direct consequence of Singer's work. Let $\{E_r A\}$ denote Miller's composite functor spectral sequence.

Proposition 8.11: For $2 \leq r \leq \infty$, there is a commutative, bilinear product

$$[\ ,\ ] : E_r^{p,q} A \otimes E_r^{p',q'} A \to E_r^{p+p',q+q'+1} A$$

so that 1.) $d_r[x,y] = [d_r x, y] + [x, d_r y]$:

2.) the product of $E_{r+1}A$ is induced from the product on $E_r A$:

3.) the product on $E_\infty A$ is induced by the product on $E_r A$, with $r < \infty$ and is induced by the product on $H^*_{Q\mathcal{A}} A$.

More informally, we can say that there is a diagram of spectral sequences

$$\begin{array}{ccc}
Ext^p_{\mathcal{U}_0}(H^Q_q A, \mathbb{F}_2) \otimes Ext^{p'}_{\mathcal{U}_0}(H^Q_{q'} A, \mathbb{F}_2) & \Rightarrow & H^{p+q}_{Q\mathcal{A}} A \otimes H^{p'+q'}_{Q\mathcal{A}} A \\
\downarrow [\ ,\ ] & & \downarrow [\ ,\ ] \\
Ext^{p+p'}_{\mathcal{U}_0}(H^Q_{q+q'+1} A, \mathbb{F}_2) & \Rightarrow & H^{p+p'+q+q'+1}_{Q\mathcal{A}} A
\end{array}$$

There are also operations. Some of these are only defined up to indeterminacy, which we now define. Let $\{E_r A\}$ be the composite functor spectral sequence. Define

$$B_s^{p,q}A \subseteq E_r^{p,q}A, \qquad s \geq r$$

to be the the vector space of elements that survive to $E_s^{p,q}A$ and have zero residue class in $E_s^{p,q}A$. An element $y \in E_r^{p,q}A$ is defined up to indeterminacy s if y is a coset representative for a particular element in $E_r^{p,q}A/B_s^{p,q}A$.

Proposition 8.13: For $2 \leq r \leq \infty$, there are operations

$$P^i : E_t^{p,q}A \to E_r^{p,q+i+1}A, \qquad 0 \leq i \leq q$$

and operations of indeterminacy $2r-2$

$$P^i : E_r^{p,q}A \to E_r^{p+i-q,2q+1}A, \qquad q \leq i \leq p+q$$

so that

1.) $P^{p+q}(x) = [x,x]$ modulo indeterminacy;

2.) if $d_r(x) = y$ then

$$\begin{aligned} d_r P^i(x) &= P^i(y), & 0 \leq i \leq q-r+1; \\ d_{i-q+2r-1}P^i(x) &= P^i(y), & q-r+1 \leq i \leq q; \\ d_{2r-1}P^i(y) &= P^i(y), & q \leq i \leq p+q \end{aligned}$$

modulo appropriate indeterminacy;

3.) the operations on $E_r A$ are determined by the operations on $E_{r'}A$ for $r' < r \leq \infty$, up to indeterminacy; and

4.) the operations on $E_\infty A$ are determined by the operations

$$P^i : H^q_{\mathcal{QA}}A \to H^q_{\mathcal{QA}}A.$$

In other words, for $0 \leq i \leq q$ there is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^p_{\mathcal{U}_0}(H^{\mathcal{Q}}_q A, \mathbb{F}_2) & \Rightarrow & H^{p+q}_{\mathcal{QA}}A \\ \downarrow P^i & & \downarrow P^i \\ Ext^p_{\mathcal{U}_0}(H^{\mathcal{Q}}_{q+i+1}A, \mathbb{F}_2) & \Rightarrow & H^{p+q+i+1}_{\mathcal{QA}}A \end{array}$$

and a similar diagram for $q \leq i \leq p+q$.

Remarks: 1.) Notice that there is never any indeterminacy at E_2A. Also notice that $P^i(x) \in E_rA$ determines a well-defined element in $E_{2r-1}A$. Hence $P^i : E_\infty A \to E_\infty A$ is well-defined.

2.) In 8.13.2 it is assumed that $P^i(x)$, $i > q - r + 1$ survives to an appropriate E_sA so that the statements make sense.

Singer's work implies the following result about the edge homomorphism

$$e : H^*_{Q\mathcal{A}}A \to H^*_QA$$

of (8.8).

Proposition 8.14: The edge homomorphism commutes with products and operations:

$$e[x,y] = [e(x), e(y)]$$

and

$$eP^i(x) = P^i(e(x)).$$

We now turn to understanding the operations at E_2A:

$$P^i : Ext^p_{\mathcal{U}_0}(H^Q_qA, \Sigma^t\mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(H^Q_{q+i+1}A, \mathbf{F}_2), \qquad 0 \leq i \leq q.$$

For this we need to understand how the operations commute with the action of the Steenrod algebra. Notice that for every q, H^q_QA is a right module over the Steenrod algebra; that is, the is an action of the Steenrod operations

$$(\cdot)\mathrm{Sq}^j : [H^q_QA]_n \to [H^q_QA]_{n-j}.$$

The following lemma relates the P^i to the Sq^j.

Proposition 8.15: For $A \in s\mathcal{UA}$ and $x, y \in H^q_QA$

$$\begin{aligned} [x,y]\mathrm{Sq}^j &= \sum_{a+b=j} [x\mathrm{Sq}^a, y\mathrm{Sq}^b] \\ P^i(x)\mathrm{Sq}^{2j} &= P^i(x\mathrm{Sq}^j) \\ P^i(x)\mathrm{Sq}^{2j+1} &= 0. \end{aligned}$$

Proof: We return to the definition of the operations given in section 7. Let $x \in H^q_Q A$ be the residue class of the the cycle $\alpha \in QA^*_q$. Then $P^i(x)$ is the residue class of $\xi' D^*_{q-i}(\alpha \otimes \alpha)$, where we write ξ' for $(\mathbf{F}_2 \otimes Q\xi)^*$. The naturality of ξ and the higher Eilenberg-Zilber maps, and the Cartan formula for Steenrod operations now imply that

$$\begin{aligned}
[\xi^* D^*_{q-i}(\alpha \otimes \alpha)]\mathrm{Sq}^j &= \xi^* D^*_{q-i}(\sum_{0 \leq a \leq j} \alpha \mathrm{Sq}^a \otimes \alpha \mathrm{Sq}^{j-a}) \\
&= \xi^* D^*_{q-i}(\alpha \mathrm{Sq}^{j/2} \otimes \alpha \mathrm{Sq}^{j/2}) \\
&\qquad + \partial \xi^* D^*_{q-i+1}(\sum_{0 \leq a < j/2} \alpha \mathrm{Sq}^a \otimes \alpha \mathrm{Sq}^{j-a})
\end{aligned}$$

where $\mathrm{Sq}^{j/2} = 0$ is j if odd. The formula involving the product is proved the same way, using D_0.

To apply this, we dualize the operations P^i and obtain operations acting on the right

$$(\cdot)P^i : H^Q_{q+i+1} A \to H^Q_q A$$

that halve the internal degree in the sense that they are identically zero on elements of odd internal degree. Interpreting Proposition 8.15 in this context, we have

$$(8.16) \qquad (x\mathrm{Sq}^{2j})P^i = \mathrm{Sq}^j(xP^i).$$

Thus $(\cdot)P^i$ is not quite a morphism in $\mathcal{U}_0$. This can be rectified as follows. Let

$$\Phi : \mathcal{U} \to \mathcal{U}$$

be the doubling functor. That is, for $M \in \mathcal{U}$

$$(\Phi M)^n = \begin{cases} M^m, & \text{if } n = 2m; \\ 0, & \text{if } n = 2m+1 \end{cases}$$

with the Steenrod algebra action given by

$$\begin{aligned}
\mathrm{Sq}^{2j}\phi(x) &= \phi(\mathrm{Sq}^j(x)) \\
\mathrm{Sq}^{2j+1}\phi(x) &= 0
\end{aligned}$$

where ϕ is the isomorphism between $(\Phi M)^{2m}$ and M^m. Notice that Φ restricts to a functor $\Phi : \mathcal{U}_0 \to \mathcal{U}_0$.

Then, equation (8.16) implies that $(\cdot)P^i$ induces a homomorphism in $\mathcal{U}_0$

$$\rho_i : H^{\mathbf{Q}}_{q+i+1}A \to \Phi H^{\mathbf{Q}}_q A$$

and, hence, a natural map

$$\rho_i^* : Ext^p_{\mathcal{U}_0}(\Phi H^{\mathbf{Q}}_q A, \Sigma^{2t}\mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(H^{\mathbf{Q}}_{q+i+1}A, \Sigma^{2t}\mathbf{F}_2).$$

We will define a canonical map, for $M \in \mathcal{U}_0$

$$\mathrm{Sq}_0 : Ext^p_{\mathcal{U}_0}(M, \Sigma^t\mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(\Phi M, \Sigma^{2t}\mathbf{F}_2)$$

and then appeal to Singer's work to claim that

$$P^i = \rho_i^* \circ \mathrm{Sq}_0$$

where P^i is as in (8.15).

The map Sq_0 is a familiar one to those working with stable Ext — there it is also known as Sq_0. Let $M \in \mathcal{U}_0$. The natural map of graded vaector spaces

$$M \to \bar{P}'M$$

adjoint to the identity $\bar{P}'M \to \bar{P}'M$ determines a map of vector spaces

$$\Phi M \to \Phi \bar{P}'M$$

and, hence, a natural morphism in $\mathcal{U}_0$

$$\bar{P}'\Phi M \to \Phi \bar{P}'M.$$

This, in turn, determines a map of simplicial objects in $\mathcal{U}_0$

$$\bar{P}'_{\cdot}\Phi M \to \Phi \bar{P}'_{\cdot}M.$$

Since Φ is an exact functor, we get a composite

$$\begin{aligned} Ext^p_{\mathcal{U}_0}(M, \Sigma^t\mathbf{F}_2) &\cong \pi^p Hom_{\mathcal{U}_0}(\Phi \bar{P}'_{\cdot}M, \Sigma^{2t}\mathbf{F}_2) \\ &\to \pi^p Hom_{\mathcal{U}_0}(\bar{P}'_{\cdot}\Phi M, \Sigma^{2t}\mathbf{F}_2) \\ &\cong Ext^p_{\mathcal{U}_0}(\Phi M, \Sigma^{2t}\mathbf{F}_2). \end{aligned} \tag{8.17}$$

This composite is Sq_0. The following is now immediately obvious from [Proposition 5.1 of 21].

Proposition 8.18: There is an equality of homomorphisms

$$P^i = \rho_i^* \circ \mathrm{Sq}_0 : Ext^p_{\mathcal{U}_0}(H^Q_q A, \Sigma^t \mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(H^Q_{q+i+1} A, \Sigma^{2t} \mathbf{F}_2)$$

for $0 \leq i \leq q$.

9. The cohomology of abelian objects

In this section we make some calculations with Miller's spectral sequence, including a computation — in terms of unstable Ext groups — of the E_2 term of the homotopy spectral sequence associated to the universal examples of section 3.

We begin by defining abelian obejects and recalling the Hilton-Milnor Theorem in the category $s\mathcal{UA}$. Let $A, B \in s\mathcal{UA}$. Then their tensor product $A \otimes B$ is the categorical coproduct of A and B; their product $A \times_{\mathbf{F}_2} B$ was defined in section 6. $A \in s\mathcal{UA}$ is an abelian group object if

$$Hom_{s\mathcal{UA}}(B, A)$$

is an abelian group for all $B \in s\mathcal{UA}$. This turns out to be equivalent to the following: A is an abelian group object if there is a morphism

$$\mu : A \times_{\mathbf{F}_2} A \to A$$

in $s\mathcal{UA}$ and a commutative diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f} & A \times_{\mathbf{F}_2} A \\ \downarrow m & & \downarrow \mu \\ A & \xrightarrow{=} & A \end{array}$$

where $f(a \otimes b) = (a\eta\epsilon(b), \eta\epsilon(a)b)$ and m is the algebra multiplication. One easily sees that this implies that

$$A \cong M_+$$

where $M \in s\mathcal{U}_0$ is a simplicial object in the category $\mathcal{U}_0$ and

$$(\ )_+ : s\mathcal{U}_0 \to s\mathcal{U}\mathcal{A}$$

is the functor that sets, for $N \in s\mathcal{U}_0$

$$N_+ = N \oplus \mathbb{F}_2$$

with $\mathbb{F}_2$ the unit, N the augmentation ideal and $N^2 = 0$. Thus, in particular, we have for an abelian object $A \cong M_+$

$$Hom_{s\mathcal{U}\mathcal{A}}(B, A) \cong Hom_{s\mathcal{U}_0}(QB, M)$$

where B is the indecomposables functor. Hence the functors $(\ )_+ : s\mathcal{U}_0 \to s\mathcal{U}\mathcal{A}$ and $Q : s\mathcal{U}\mathcal{A} \to s\mathcal{U}_0$ form an adjoint pair. Finally, notice that for $M_1, M_2 \in s\mathcal{U}_0$

$$(M_1)_+ \times_{\mathbb{F}_2} (M_2)_+ \cong (M_1 \times M_2)_+.$$

The Hilton-Milnor Theorem discusses the homotopy type of

$$\Sigma[(M_1)_+ \times_{\mathbb{F}_2} (M_2)_+] \cong \Sigma[(M_1 \times M_2)_+]$$

in $s\mathcal{U}\mathcal{A}$, where $\Sigma : s\mathcal{U}\mathcal{A} \to s\mathcal{U}\mathcal{A}$ is the suspension functor of section 3.

We need some further notation. The category $s\mathcal{U}_0$ is a category of modules and, as such, is equivalent to a category of chain complexes. Therefore, it is easy to construct a suspension functor

$$\sigma : s\mathcal{U}_0 \to s\mathcal{U}_0$$

so that there is a natural isomorphism

$$\pi_n \sigma M = \begin{cases} \pi_{n-1} M; & \text{if } n \geq 1; \\ 0; & \text{if } n = 0. \end{cases}$$

Now let L be the free algebra on two elements x_1, x_2. Let B be the Hall basis for L [22, p. 512]. Then $b \in B$ is an iterated Lie product in the elements x_1 and x_2. Let

$$\begin{aligned} i(b) &= \text{the number of appearances of } x_1 \text{ in } b \\ j(b) &= \text{the number of appearances of } x_2 \text{ in } b \\ \ell(b) &= i(b) + j(b) \end{aligned}$$

and if $M_1, M_2 \in s\mathcal{UA}$, define $M(b) \in s\mathcal{UA}$ by

$$M(b) = \sigma^{\ell(b)-1} M_1^{\otimes i(b)} \otimes M_2^{\otimes j(b)} \tag{9.1}$$

where $N^{\otimes k}$ means the tensor product of N with itself k times.

Theorem 9.2 (Hilton-Milnor)[12]: Let M_1 and M_2 be objects in $s\mathcal{U}_0$. Then there is a weak equivalence in $s\mathcal{UA}$

$$\Sigma[(M_1)_+ \times_{\mathbb{F}_2} (M_2)_+] \to \otimes_{b \in L} \Sigma[M(b)_+].$$

Remarks 9.3.1) The relationship to the usual Hilton-Milnor Theorem for spaces is given by the following: if X and Y are spaces, then $H^*\Sigma X \cong (\bar{H}^*\Sigma X)_+$ and

$$H^*\Sigma X \vee \Sigma Y \cong (\bar{H}^*\Sigma X)_+ \times_{\mathbb{F}_2} (\bar{H}^*\Sigma Y)_+.$$

Then, regarding $H^*\Sigma X \vee \Sigma Y$ as a constant simplicial algebra, we obtain a simplicial unstable algebra

$$\Sigma(H^*\Sigma X \vee \Sigma Y)$$

and, by example 4.8, a spectral sequence

$$H^*_{Q\mathcal{A}}\Sigma(H^*\Sigma X \vee \Sigma Y) \Rightarrow \pi_*\Omega(\Sigma X \vee \Sigma Y).$$

On the one hand

$$\pi_*\Omega(\Sigma X \vee \Sigma Y).$$

is computed using the classical Hilton-Milnor Theorem, and on the other hand

$$H^*_{Q\mathcal{A}}\Sigma(H^*\Sigma X \vee \Sigma Y)$$

is computed using Theorem 9.2 and the next remark.

9.3.2) Notice that if $A, B \in s\mathcal{UA}$, then there is a natural isomorphism

$$\mathbb{F}_2 \otimes_{\mathcal{A}} Q(A \otimes B) \cong \mathbb{F}_2 \otimes_{\mathcal{A}} QA \oplus \mathbb{F}_2 \otimes_{\mathcal{A}} QB$$

and, hence, a natural isomorphism

$$H^*_{Q\mathcal{A}} A \otimes B \cong H^*_{Q\mathcal{A}} A \times H^*_{Q\mathcal{A}} B.$$

Thus there is a sequence of isomorphisms

$$H^*_{Q\mathcal{A}}[(M_1)_+ \times_{\mathbf{F}_2} (M_2)_+] \cong H^{*+1}_{Q\mathcal{A}} \Sigma[(M_1)_+ \times_{\mathbf{F}_2} (M_2)_+]$$

and

$$H^*_{Q\mathcal{A}} \Sigma[(M_1)_+ \times_{\mathbf{F}_2} (M_2)_+] \cong \times_{b\in L} H^*_{Q\mathcal{A}} \Sigma M(b)_+$$

and

$$H^{*+1}_{Q\mathcal{A}} \Sigma M(b)_+ \cong H^*_{Q\mathcal{A}} M(b)_+.$$

In particular, this serves to compute

$$Ext_{\mathcal{UA}}(H^*\Sigma X \vee \Sigma Y, \mathbf{F}_2)$$

as a product of $H^*_{Q\mathcal{A}} M(b)_+$ where, in $s\mathcal{U}_0$

$$M(b) = \sigma^{\ell(b)-1}(\bar{H}^*\Sigma X)^{\otimes i(b)} \otimes (\bar{H}^*\Sigma Y)^{\otimes j(b)}$$

is $s\mathcal{U}_0$. *A priori*, one might have expected this *Ext* group to split as a product of *Ext* groups. This turns out not to be the case — $H^*_{Q\mathcal{A}}$ turns out to be the necessary generalization in this case. The reader is encouraged, as an example to consider the case where $X = Y = S^n$ for some n. Compare 9.3.4 below.

9.3.3) To compute $H^*_{Q\mathcal{A}} M_+$ for $M \in s\mathcal{U}_0$ — that is, to compute the cohomology of abelian objects in $s\mathcal{UA}$ — it is sufficient to compute $H^*_{Q\mathcal{A}} N_+$ for all $N \in s\mathcal{U}_0$ indecomposable in the sense that it has no non-trivial direct summands. Notice that one needs the full generality of the Hilton-Milnor Theorem and the preceeding remark even if one is only interested in $Ext_{\mathcal{UA}}(M_+, \mathbf{F}_2)$ with $M \in \mathcal{U}_0$.

9.3.4) In particular, if $D(r, p, q)$ is as is example 3.13, we have a weak equivalence in $s\mathcal{UA}$

$$H^*D(r,p,q) \simeq K(p,q)_+ \times_{\mathbf{F}_2} K(p+r, q+r-1)_+.$$

Thus, to compute $H^*_{Q\mathcal{A}}H^*D(r,p,q)$ — of interest because it is the E_2 term of the universal r-differential in the Bousfield-Kan spectral sequence — it is sufficient to compute

$$H^*_{Q\mathcal{A}}K(s,t)_+$$

for all s and t. This follows from Theorem 9.2 and the fact that there is a weak equivalence in $s\mathcal{U}_0$

$$K(s,t) \otimes K(s',t') \simeq K(s+s',t+t')$$

and

$$\sigma K(s,t) \simeq K(s+1,t).$$

This project will occupy the rest of the section.

So saying, let $M_+ \in s\mathcal{U}\mathcal{A}$ be an abelian object. We look to Miller' spectral sequence

$$Ext^s_{\mathcal{U}_0}(H^Q_r M_+, \mathbb{F}_2) \Rightarrow H^{s+r}_Q M_+$$

for our computations. This is plausible because $H^Q_* M_+$ is known. In fact, we will record $H^*_Q M_+$ and dualize. Let $\mathcal{W}$ be the category of Theorem 7.5. If $A \in s\mathcal{U}\mathcal{A}$, then $H^*_Q A \in \mathcal{W}$.

The forgetful functor $\mathcal{W} \to nn\mathbb{F}_2$ from $\mathcal{W}$ to the category of bigraded $\mathbb{F}_2$ vector spaces has a left adjoint

$$U : nn\mathbb{F}_2 \to \mathcal{W}$$

and one of the main theorem of [14] implies the following.

Theorem 9.4: If $M \in s\mathcal{U}_0$ with $\pi_* M$ of finite type. Then

$$H^*_Q M_+ \cong U(\pi_* M^*).$$

Now, $H^n_Q M_+$ is a right module over the Steenrod algebra and this structure is a consequence of Theorem 9.4 and the formulas

$$
\begin{aligned}
[x,y]\mathrm{Sq}^k &= \sum_{i+j=k} [x\mathrm{Sq}^i, y\mathrm{Sq}^j] \\
P^i(x)\mathrm{Sq}^{2k} &= P^i(x\mathrm{Sq}^k) \\
P^i(x)\mathrm{Sq}^{2k+1} &= 0 \\
\beta(x)\mathrm{Sq}^k &= \beta(x\mathrm{Sq}^{k/2}) + \sum_{i<k/2} [x\mathrm{Sq}^i, x\mathrm{Sq}^{k-i}]
\end{aligned}
\tag{9.5}
$$

where $\mathrm{Sq}^{k/2} = 0$ if k is odd. The first three formulas are in Proposition 8.15; the fourth is in [14, Section 4].

Example 9.6: Consider the example of $M = K(p,q)$. For $p = 0$ we have $K(0,q)_+ \cong H^*S^q$ regarded as a constant simplicial algebras and

$$H^0_Q K(0,q)_+ \cong \mathbf{F}_2$$

concentrated in degree q and generated by a class ι and

$$H^1_Q K(0,q)_+ \cong \mathbf{F}_2$$

concentrated in degree $2q$ and generated by $\beta(\iota)$. Miller's spectral sequence for this example was calculated in 8.7.

If $p > 0$ and $\iota \in H^p_Q K(p,q)_+$ is the non-zero class of degree q, then a basis for $H^*_Q K(p,q)_+$ as a bigraded $\mathbf{F}_2$ vector space is given by the elements

$$P^{i_1} P^{i_2} \cdots P^{i_s}(\iota) \tag{9.6.1}$$

where $2 \leq i_t < 2i_{t+1}$ for all t and $2 \leq i_s \leq p$. Furthermore, the equations of (9.5) imply that for each s $H^s_Q K(p,q)_+$ has the structure of a trivial module over the Steenrod algebra.

In particular

$$H^*_Q K(1,q)_+ \cong \mathbf{F}_2 \tag{9.6.2}$$

concentrated in bidegree $(1, q)$. In this case, then, Miller's spectral sequence must collapse and we have

$$\begin{aligned}[H^s_{\mathcal{QA}}K(1,q)_+]_t &\cong Ext^{s-1}_{\mathcal{U}_0}(\Sigma^q \mathbb{F}_2, \Sigma^t \mathbb{F}_2) \\ &\cong Ext^{s-1}_{\mathcal{U}}(\Sigma^{q-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2)\end{aligned}.$$

Thus the spectral sequence

$$H^*_{\mathcal{QA}}K(1,q)_+ \Rightarrow \pi_* S^{q-1}$$

quaranteed by 3.11 is, in fact, the same spectral sequence one obtains from Proposition 1.5 and the Bousfield-Kan spectral sequence.

The final result of this section expands on these computations.

Theorem 9.7: For all $p \geq 1$, the composite functor spectral sequence

$$Ext^s_{\mathcal{U}_0}(H^{\mathcal{Q}}_t K(p,q)_+, \mathbb{F}_2) \Rightarrow H^{s+t}_{\mathcal{Q}} K(p,q)_+$$

collapses.

Before proving this, we establish some notation and state a lemma. We have a prefered basis for $H^*_{\mathcal{Q}}K(p,q)_+$; namely, all, elements of the form

$$P^I(\iota) = P^{i_1} \cdots P^{i_k}(\iota)$$

with $2 \leq i_t < 2i_{t+1}$ for all t and $2 \leq i_k \leq p$. Call such I *allowable*, and let $\ell(I) = s$ and $e(I) = i_k$.

If M is a trivial module over the Steenrod algebra, then

$$Ext^s_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2) \cong \times_m \; Ext^s_{\mathcal{U}_0}(\Sigma^m \mathbb{F}_2, \Sigma^t \mathbb{F}_2) \otimes M^*_m$$

Where $M^m \subseteq M$ are the elements of degree m. Hence, since $(H^{\mathcal{Q}}_* A)^* \cong H^*_{\mathcal{Q}} A$,

$$Ext^s_{\mathcal{U}_0}(H^{\mathcal{Q}}_t K(p,q)_+, \mathbb{F}_2) \cong \times_m \; Ext^s_{\mathcal{U}_0}(\Sigma^m \mathbb{F}_2, \mathbb{F}_2) \otimes H^t_{\mathcal{Q}} K(p,q)_+.$$

If $\iota \in [H^p_Q K(,p,q)_+]_q$ is the fundamental class, then the properties of the operations P^i imply that

$$P^I(\iota) = P^{i_1} \cdots P^{i_k}(\iota) \in [H_Q^{p+i_1+\cdots+i_k+k} K(p,q)_+]_{2^{\ell(I)}q}.$$

Thus, if $\langle P^I(\iota)\rangle \subseteq H^*_Q K(p,q)_+$ is the sub-vector space generated by $P^I(\iota)$, we have

$$(9.8) \qquad Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathbf{F}_2) \cong \times_I \ Ext^s_{\mathcal{U}_0}(\Sigma^{2^{\ell(I)}q}\mathbf{F}_2, \mathbf{F}_2) \otimes \langle P^I(\iota)\rangle$$

where the product is taken over all allowable I so that

$$(9.8.1) \qquad e(I) \leq p$$

and

$$(9.8.2) \qquad p + i_1 + \cdots + i_k + k = t.$$

I can be empty, in which case $P^I(\iota) = \iota$ and $e(I) = 0$.

Proof of 9.7: First, since

$$(9.9) \qquad E_2^{s,t} \cong Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathbf{F}_2) = 0$$

for $t < p$ and the differentials have the form

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

all differentials vanish on $E_r^{s,p}$ for all $r \geq 2$ and all $p \geq 1$.

Second, we have

$$E_2^{0,a} \cong H^a_Q K(p,q)_+$$

and this vector space is spanned by elements of the form $P^I(\iota)$. Thus, from 8.13.2 and 8.14, we have that all differentials vanish on $E_r^{0,a}$ for $r \geq 2$. Also, if $0 \neq P^I(\iota) \in E_2^{0,a}$, then $P^I(\iota) \neq 0$ in $H^t_{Q\mathcal{A}} K(p,q)_+$. Let

$$f : K(p,q)_+ \to K(a,b)_+$$

be the morphism in the homotopy category of $s\mathcal{UA}$ (see 2.16) corepresenting

$$P^I(\iota) \in H^a_Q K(p,q)_+.$$

Of course

$$a = p + i_1 + i_1 + \cdots + i_k + k$$

and

$$b = 2^{\ell(I)} q.$$

The morphism f, then induces a diagram of spectral sequences

$$\begin{array}{ccc} Ext^s_{\mathcal{U}_0}(H^Q_t K(a,b)_+, \mathbb{F}_2) & \Rightarrow & H^{s+t}_{Q\mathcal{A}} K(a,b)_+ \\ \downarrow {\scriptstyle E_2 f} & & \downarrow {\scriptstyle H^*_{Q\mathcal{A}} f} \\ Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathbb{F}_2) & \Rightarrow & H^{s+t}_{Q\mathcal{A}} K(p,q)_+. \end{array}$$

Then (9.9) for (a,b) implies that all differentials vanish on

$$Ext^s_{\mathcal{U}_0}(\Sigma^{2^{\ell(I)}q}\mathbb{F}_2, \mathbb{F}_2) \otimes \langle P^I(\iota)\rangle \subseteq Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathbb{F}_2).$$

The result now follows from (9.8).

The following is a consequence of Theorem 9.7 and the equation (9.8).

Corollary 9.10: There is an isomorphism of vector spaces for $p \geq 1$

$$H^n_{Q\mathcal{A}} K(p,q)_+ \cong \times_I \; Ext^s_{\mathcal{U}_0}(\Sigma^{2^{\ell(I)}q}\mathbb{F}_2, \mathbb{F}_2) \otimes \langle P^I(\iota)\rangle$$

where the product is over all allowable I so that $e(I) \leq p$ and

$$s + p + i_1 + \cdots i_k + k = n.$$

Of course, by 8.4, we have

$$Ext^s_{\mathcal{U}_0}(\Sigma^m \mathbb{F}_2, \Sigma^t \mathbb{F}_2) \cong Ext^s_{\mathcal{U}}(\Sigma^{m-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2)$$

and the latter is a familiar, if somewhat intractable, object. Finally the action of the operations

$$P^i : H^n_{Q\mathcal{A}} K(p,q)_+ \to H^{n+i+1}_{Q\mathcal{A}} K(p,q)_+$$

can be computed up to filtration using 8.8 and 9.10.

We now say in what sense the spectral sequence

$$H^*_{Q\mathcal{A}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

is a desuspension spectral sequence. We assume that $p \geq 1$. Then , as in (7.13), we obtain a diagram of spectral sequences

$$\begin{array}{ccc} H^*_{Q\mathcal{A}}K(p,q)_+ & \Rightarrow & \pi_* S^{q-p} \\ \downarrow e^* & & \downarrow E_{p-1} \\ H^*_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+ & \Rightarrow & \pi_*\Omega^{p-1}S^{q-1} \end{array}$$

where E_{p-1} is the suspension homomorphism. Using Corollary 9.10 and the fact that

$$e^* P^I(\iota_{p,q}) = 0$$

— where $\iota_{p,q} \in [H^p_{Q\mathcal{A}}K(p,q)_+]_q$ is the generator — we see that e^* is surjective; indeed it is isomorphic (under the isomorphisms of 9.10) to projection onto the factor

$$Ext^*_{\mathcal{U}_0}(\Sigma^q \mathbf{F}_2, \mathbf{F}_2).$$

Since

$$Ext^*_{\mathcal{U}_0}(\Sigma^q \mathbf{F}_2, \mathbf{F}_2) \cong Ext^*_{\mathcal{U}}(\Sigma^{q-1}\mathbf{F}_2, \mathbf{F}_2)$$

and the latter is the E_2 term of a spectral sequence for computing $\pi_* S^{q-1}$, the other factors in $H^*_{Q\mathcal{A}}K(p,q)_+$ are present to correct the computation to a calculation of $\pi_* S^{q-p}$.

We end this paper with a calculation that demonstrates that not all the operations P^i commute with differentials in the Bousfield-Kan spectral sequence. Let $\alpha : S^{n-1} \to S^{n-1}$ be the identity map and let

$$h_0 \in Ext^1_{\mathcal{U}_0}(\Sigma^n \mathbf{F}_2, \Sigma^{n+1}\mathbf{F}_2)$$

be the element detecting $2\alpha \in \pi_{n-1}S^{n-1}$. Then we can let

$$P^p(\iota)h_0 \in Ext^1_{\mathcal{U}_0}(\Sigma 2q\mathbf{F}_2, \Sigma^{2q+1}\mathbf{F}_2) \otimes \langle P^p(\iota)\rangle$$

stand for the non-zero class.

Proposition 9.11: Let $p \geq 3$ and $q-p$ be an odd number. Then in the spectral sequence

$$H^*_{\mathcal{Q}\mathcal{A}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

there is a differential

$$d_2 P^{p-1}(\iota) = P^p(\iota)h_0.$$

Proof: We refer to the calculations of Corollary 7.15 and consider the diagram of spectral sequences of 7.13:

$$\begin{array}{ccc} H^*_{\mathcal{Q}\mathcal{A}}M(e^*) & \Rightarrow & \pi_* C(p-1) \\ \downarrow f & & \downarrow \\ H^*_{\mathcal{Q}\mathcal{A}}K(p,q)_+ & \Rightarrow & \pi_* S^{q-p}. \end{array}$$

Corollary 9.10 implies that there is a unique class $y_i \in H^*_{\mathcal{Q}\mathcal{A}}M(e^*)$ so that

$$f(y_i) = P^i(\iota).$$

Corollary 9.10 also implies that

$$0 \neq y_p h_0 \in [H^{2p+2}_{\mathcal{Q}\mathcal{A}}M(e^*)]_{2q+1}$$

and that

$$[H^s_{\mathcal{Q}\mathcal{A}}M(e^*)]_t = 0$$

for $t-s = 2(q-p)$ and $s < 2q$. Hence the calculation given in the proof of 7.15 implies that

$$d_2 y_{p-1} = y_p h_0.$$

The result follows.

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COHOMOLOGICAL p–NILPOTENCE CRITERIA FOR COMPACT LIE GROUPS

Hans–Werner Henn

Introduction

In [Q1] Quillen discussed cohomological criteria for p–nilpotence of finite groups. He proved that for odd primes p a finite group G is p–nilpotent if and only if the restriction map from the mod p cohomology $H^*(G;\mathbb{F}_p)$ to the mod p cohomology $H^*(G_p;\mathbb{F}_p)$ of a p–Sylow subgroup G_p is an F–isomorphism. Recall that a map $A \xrightarrow{\varphi} B$ of graded $\mathbb{F}_p$ algebras is called an F–isomorphism if and only if $a \in \mathrm{Kern}\varphi$ implies $a^n = 0$ for some n and for each $b \in B$ some power b^{p^n} is in the image of φ [Q2]. Furthermore Quillen sketched a proof of the following result which he attributed to Atiyah: If p is any prime and $H^i(G;\mathbb{F}_p) \to H^i(G_p;\mathbb{F}_p)$ is an isomorphism for all sufficiently large i, then G is p–nilpotent.

Quillen's main result in [Q2] can be interpreted as follows: For a compact Lie group G with classifying space BG the F–isomorphism type of $H^*(BG;\mathbb{F}_p)$ is determined by the sets $\mathrm{Rep}(V,G)$ of G–conjugacy classes of homomorphisms from elementary abelian p–groups V to G [HLS]. In particular, one can rephrase Quillen's p–nilpotence criterion in the following form: For an odd prime p a finite group G is p–nilpotent if and only if inclusion induces a bijection $\mathrm{Rep}(V,G_p) \xrightarrow{i} \mathrm{Rep}(V,G)$ for all elementary abelian p–groups V ([HLS; Prop. 4.2.3.]).

If G is a compact Lie group with maximal torus T, normalizer NT, Weyl group $W(G) = NT/T$, then G_p will denote the preimage of W_p in NT. In this case G_p will be called a p–Sylow normalizer and is known to be a good substitute for a p–Sylow subgroup.

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In this paper we give for odd primes a characterization of those compact Lie groups G for which $\mathrm{Rep}(V, G_p) \to \mathrm{Rep}(V, G)$ is a bijection for all V, or equivalently $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ is an F–isomorphism (Theorem 2.1.). The possibility of such a characterization was already mentioned in [HLS, Sect. 4.2.5.]. It seems appropriate to call such groups p–nilpotent compact Lie groups. We will also generalize Atiyah's criterion to the compact Lie group case (Theorem 2.5.). Our interest in such characterizations comes from the importance of BG_p for the study of the (stable) homotopy type of BG.

The paper is organized as follows. In section 1 we give the precise definition of a p–nilpotent compact Lie group and discuss some properties of such groups. We do not intend a systematic group theoretical study of this concept but will rather concentrate on properties which are relevant for our cohomological characterizations. These characterizations are stated and proved in section 2.

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1. p–nilpotent compact Lie groups

1.1 DEFINITION. *A compact Lie group G is called p–nilpotent if and only if there is a finite normal subgroup N of order prime to p which together with G_p generates G.*

1.2 REMARKS.

(a) For finite groups this reduces to the classical definition of p–nilpotence. Then N consists of all elements of order prime to p and G/N is isomorphic to G_p, i.e. G is a semidirect product $N \rtimes G_p$. In this case N is also called the normal p complement of G_p in G.

(b) In the compact Lie group case G is in general not a semidirect product. For example, if $G = \langle S^1,\ x, y \mid [x, S^1] = [y, S^1] = x^3 = y^3 = 1,$ $[x, y] = \zeta$ with ζ a primitive 3rd root of unity in $S^1\rangle$ and $p \neq 3$, then

$G_p = S^1$ and the normal subgroup $N = \langle x, y\rangle$ shows that G is p–nilpotent. However, $N \cap G_p \neq \{1\}$ and hence $G \not\cong N \rtimes G_p$. It is also obvious that G is not a semidirect product $\widetilde{N} \rtimes G_p$ for some other $\widetilde{N} \triangleleft G$.

Our definition of p–nilpotence above will be justified by the results below, which together with this example show that it would not be adequate to require the existence of a finite normal p–complement in the compact Lie group case.

1.3 PROPOSITION. *Let G be a compact Lie group and p be any prime. Then the following statements are equivalent.*

(a) *G is p–nilpotent.*

(b) $\mathrm{Rep}(Q, G_p) \xrightarrow{i} \mathrm{Rep}(Q, G)$ *is a bijection for all p–groups Q.*

(c) *If Q is any finite p–subgroup of G, then $N_G(Q)/C_G(Q)$, the quotient of the normalizer of Q in G by the centralizer of Q in G, is a finite p–group.*

(d) *Each finite subgroup H of G is p–nilpotent.*

(e) *G is a finite extension of a torus, i.e. there exists an exact sequence $T \hookrightarrow G \twoheadrightarrow \pi$ with π finite, and G has a finite p-nilpotent subgroup H with $H/H \cap T = \pi$ and $T_p = \{t \in T \mid t^p = 1\} \subset H$.*

(f) *G is an extension of a torus by a finite p–nilpotent group π and the conjugation action of the normal p–complement ν of π_p in π is trivial on T.*

<u>Proof. (a) $\Rightarrow$ (b):</u> Onto is equivalent to saying that any p–subgroup Q of G is conjugate to a subgroup of G_p, i.e. that the Q–set G/G_p has a nonempty Q–fixed point set $(G/G_p)^Q$. This follows from $\chi((G/G_p)^Q) \equiv \chi(G/G_p) \not\equiv 0 \bmod p$ where χ denotes Euler characteristic (cf. [HLS; Prop. 4.2.1.]).

To show that i is $1-1$ consider the projection $G_p \xrightarrow{\pi} G_p/G_p \cap N \cong G/N$. It suffices to show that π induces an injection on $\mathrm{Rep}(Q, ?)$. So let α_1, α_2 be two homomorphisms with $\pi\alpha_1 = g\pi\alpha_2 g^{-1}$ for some $g \in G_p$. By factoring out the kernel we may assume that $\pi\alpha_1$ is mono. Identify Q with its image in $G_p/G_p \cap N$. Then α_1 and $g\alpha_2 g^{-1}$ are sections of $\pi^{-1}(Q) \xrightarrow{\pi} Q$. Now $\mathrm{Kern}\,\pi = G_p \cap N$ is a subgroup of T of order prime to p and hence

$H^1(Q, G_p \cap N) = 0$, i.e. α_1 and $g\alpha_2 g^{-1}$ are even conjugate by an element in $G_p \cap N$ and we are done.

(b) $\Rightarrow$ (c): For any group G the automorphism group $\mathrm{Aut}(Q)$ acts on $\mathrm{Rep}(Q, G)$. If Q is a subgroup of G, then $N_G(Q)/C_G(Q)$ identifies naturally with the isotropy subgroup of the inclusion $Q \hookrightarrow G$, considered as an element in the $\mathrm{Aut}(Q)$–set $\mathrm{Rep}(Q, G)$.

Now (b) implies that we can assume that Q is a subgroup of G_p and that it suffices to show that $N_{G_p}(Q)/C_{G_p}(Q)$ is a p–group. So suppose that $x \in N_{G_p}(Q)$ has order prime to p in $N_{G_p}(Q)/C_{G_p}(Q)$. As in [HLS, sect. 4.3.] we may assume that x itself has order prime to p, i.e. $x \in T$. Then one sees as in [HLS, Lemma 4.3.3.] that x acts trivially on the quotient of Q by its Frattini–subgroup $\phi(Q)$ and hence trivially on Q (cf. [H, Satz III 3.18.]). Therefore x is in $C_{G_p}(Q)$ and we are done.

(c) $\Rightarrow$ (d): If Q is a subgroup of H, then $N_H(Q)/C_H(Q)$ is a subgroup of $N_G(Q)/C_G(Q)$ and hence the Frobenius criterion [H, Satz IV, 5.8.] implies that H is p–nilpotent.

For the remaining implications we need a Lemma. For a natural number ℓ let T_ℓ denote $\{t \in T \mid t^\ell = 1\}$.

1.4 LEMMA. *Let G be an extension of a torus T by a finite group π of order $|\pi|$. Then there is a finite subgroup F of G with $F/F \cap T = \pi$ and $F \cap T = T_{|\pi|}$.*

Proof. Interpret the (class of the) extension $T \hookrightarrow G \twoheadrightarrow \pi$ as an element $[e] \in H^2(\pi; T)$ and use that $|\pi| \cdot [e] = 0$ together with the long exact cohomology sequence arising from the short exact sequence $T_{|\pi|} \hookrightarrow T \xrightarrow{\bullet|\pi|} T$ of π–modules.

□

We continue with the proof of Proposition 1.3.

(d) $\Rightarrow$ (e): Assume that G is not a finite torus extension. Then $G_{(1)}$, the connected component of 1, is not abelian and hence contains a compact connected nonabelian Lie group of rank 1, i.e. either $SO(3)$ or $SU(2)$. Now $SO(3)$ contains A_4, the alternating group on four letters, as symmetry group of a regular tetrahedron. As neither A_4 nor its twofold cover in $SU(2)$ are 2–nilpotent, we may assume that p is odd. Next consider $\widetilde{G} := NT \cap G_{(1)}$. This is a finite torus extension, so there is a finite subgroup $\widetilde{F}$ as in Lemma 1.4. Let $\widetilde{H}$ be the finite subgroup of G, generated by $\widetilde{F}$ and T_p (finite because T_p is normal). If $G_{(1)} \neq T$, then the Weyl group $W(G_{(1)})$ is nontrivial. Pick a reflection in $W(G_{(1)})$ and represent it by an element $r \in \widetilde{H}$. Then r defines a nontrivial element of order 2 in $N_{\widetilde{H}}(T_p)/C_{\widetilde{H}}(T_p)$ and hence $\widetilde{H}$ is not p–nilpotent.

We conclude that $G_{(1)}$ is a torus and G is a finite torus extension. Now let $F \subset G$ be as in 1.4. Then $H = \langle F, T_p \rangle$ is the finite group with the desired properties.

<u>(e) $\Rightarrow$ (f):</u> If N is the normal p complement of H_p in H, then $N/N \cap T$ is the normal p complement of π_p in π. Therefore it suffices to show that N commutes with T. Now N and T_p are both normal in H and have trivial intersection, hence they commute. Finally, a smooth automorphism of T which fixes T_p is clearly trivial, if p is odd, or has order at most 2, if $p = 2$. Hence N commutes with T and we are done.

<u>(f) $\Rightarrow$ (a):</u> Let G' be the preimage in G of the normal p complement ν. Then Lemma 1.4 gives a subgroup F' of G' with $F'/F' \cap T = \nu$ and $F' \cap T = T_{|\nu|}$, where $|\nu|$ is the order of ν. Clearly, F' is a finite group of order prime to p which together with G_p generates G. However, F' need not be normal.

Therefore consider the subgroup $N = \langle F', T_{|\nu|^2} \rangle \subset G$. This is still a finite group of order prime to p. We claim that N is normal. For this it suffices to show that $gF'g^{-1} \subset N$ for all $g \in G$. So let x be in F'. Then $gxg^{-1} = yt$ for some $y \in F'$, $t \in T$, since ν is normal in π. It suffices to show that $t^{|\nu|^2} = 1$. This follows because the order of elements in F' clearly divides $|\nu|^2$ and because y commutes with t by assumption.

This finishes the proof of 1.3.

$\square$

2. Cohomological p–nilpotence criteria

Before we state our main result we recall that a subgroup V of G_p is said to be weakly closed in G_p with respect to G if $gVg^{-1} \subset G_p$, $g \in G$, implies $gVg^{-1} = V$.

2.1 THEOREM. *Let G be a compact Lie group and p be an odd prime. Then the following statements are equivalent.*

(a) *G is p–nilpotent.*

(b) $\mathrm{Rep}(V, G_p) \to \mathrm{Rep}(V, G)$ *is bijective for all elementary abelian p–groups V.*

(c) *Let V be any normal elementary abelian p–subgroup of G_p which contains T_p. Then V is weakly closed in G_p with respect to G and $N_G(V)/C_G(V)$ is a finite p–group.*

2.2 REMARKS.

(a) We recall that condition 2.1.(b) is equivalent to the map $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ being an F isomorphism. In fact, a transfer argument shows that this map is mono for all compact Lie groups G. If G is also p–nilpotent then the Leray–Serre spectral sequence of the fibration $B(N \cap G_p) \to BG_p \to B(G_p/G_p \cap N) = B(G/N)$ with mod p acyclic fibre shows that $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ is also onto and hence a genuine isomorphism.

(b) In the finite case condition 2.1.(c) above gives just Quillen's group theoretical version of his p–nilpotence criterion ([Q1, Thm. 1.5.]). The proof of implication (c) $\Rightarrow$ (a) below is essentially a careful modification of the proof of Theorem 1.5. in [Q1].

(c) For $p = 2$ there are examples of compact Lie groups G which satisfy conditions 2.1.(b) and 2.1.(c) but which are not 2–nilpotent. $G = SU(2)$ is an example of a connected group and $G = Q_8 \rtimes \mathbb{Z}/3$, the semidirect product of the quaternion group with $\mathbb{Z}/3$ (cf. [Q1]), is an example of a finite group.

A cohomological criterion for p–nilpotence that works for all primes will be given below in Theorem 2.5.

<u>Proof of Theorem 2.1.</u>

<u>(a) $\Rightarrow$ (b):</u> This follows from Proposition 1.3.

<u>(b) $\Rightarrow$ (c):</u> Clearly, (b) implies that a normal elementary abelian p–subgroup V of G_p is weakly closed with respect to G. The proof of Proposition 1.3. ((b) $\Rightarrow$ (c)) shows that $N_G(V)/C_G(V)$ is a p–group.

(c) $\Rightarrow$ (a): If G is not a finite torus extension, then we see as in the proof of Proposition 1.3. ((d) $\Rightarrow$ (e)) that $N_G(T_p)/C_G(T_p)$ contains a nontrivial element of order 2 in contradiction to our assumptions.

Therefore G is a finite torus extension. Denote G/T by π and let F be a finite subgroup of G with $T \cap F = T_{|\pi|}$ and $F/F \cap T = \pi$ as in Lemma 1.4. By criterion (e) of Proposition 1.3. it suffices to show that the finite group $H = \langle F, T_p \rangle$ is p–nilpotent.

We pick a p–Sylow subgroup H_p of H which is contained in G_p.

2.3 LEMMA. *Let V be any abelian subgroup of H (resp. H_p) which contains T_p. Then V is normal in H (resp. H_p) if and only if V is normal in G (resp. G_p), provided p is odd.*

Proof. Suppose V is abelian and contains T_p. Then V commutes with T_p and hence with T (p is odd!). Therefore, if H normalizes V, then $\langle H, T \rangle = G$ normalizes V. Similarly with H_p and G_p. The converse is trivial.

□

We return to the proof of 2.1. ((c) $\Rightarrow$ (a))

Lemma 2.3 implies that any normal elementary abelian p–subgroup V of H_p containing T_p is weakly closed in H_p with respect to H. Furthermore, $N_H(V)/C_H(V)$ is a subgroup of $N_G(V)/C_G(V)$, in particular a p–group.

Therefore, the p–nilpotence of H is a consequence of the following slight generalization of Quillen's Theorem 1.5. in [Q1].

2.4 PROPOSITION. *Let p be an odd prime and G be a finite group with p–Sylow subgroup G_p. Let U be a normal elementary abelian p–subgroup of G and assume that each normal elementary abelian p–subgroup V of G_p containing U is weakly closed in G_p with respect to G and that $N_G(V)/C_G(V)$ is a p–group for such V. Then G is p–nilpotent.*

Proof of 2.4. The proof is almost the same as in [Q1]. For the convenience of the reader we repeat the main steps.

The hypothesis of 2.4. are inherited by all subgroups of G which contain G_p. Therefore we can do induction on the order of such subgroups.

Let V be a subgroup of G_p which contains U and is maximal with respect to being elementary abelian and normal in G_p. Then V is a maximal elementary abelian subgroup of G (cf. [Q1, Prop. 4.1.]) and hence $C_G(V)$ is p–nilpotent by [H, Satz IV, 5.5.]. Now there are two cases:

Case 1: V is normal in G. Then G is p–nilpotent because $C_G(V)$ is p–nilpotent and $G/C_G(V) = N_G(V)/C_G(V)$ is a p–group.

Case 2: V is not normal in G. Then let W be a maximal G–normal subgroup of V which contains U. Define subgroups V_1 of V and N of G by

$$V_1/W = V/W \cap Z(G_p/W) \quad (Z \text{ denotes the center})$$
$$N = N_G(V_1).$$

Then everything works precisely as in [Q1].

- N contains G_p and is properly contained in G, hence N is p–nilpotent by induction.
- V_1/W is a central subgroup of G_p/W which is weakly closed with respect to G/W. Therefore, Grün's Theorem implies $H^1(G/W) \xrightarrow{\cong} H^1(N/W)$ and the cohomology 5–term exact sequences of the group extensions $W \hookrightarrow G \twoheadrightarrow G/W$, $W \hookrightarrow N \twoheadrightarrow N/W$ yield $H^1(G) \xrightarrow{\cong} H^1(N)$.
- Finally, Tate's H^1–criterion [T] implies that G is p–nilpotent.

□ □

The following result generalizes Atiyah's p–nilpotence criterion and is valid for all primes.

2.5 THEOREM. *Let G be a compact Lie group and suppose inclusion induces an isomorphism $H^i(BG;\mathbb{F}_p) \to H^i(BG_p;\mathbb{F}_p)$ for all sufficiently large i. Then G is p–nilpotent.*

Proof. By a transfer argument (cf. [Cl] for the existence of a stable transfer map) there is a p–local stable splitting $BG_p \underset{(p)}{\simeq} BG \vee X$ for some p–local connected X with bounded above and finite type mod p homology. Now G_p is a finite torus extension. Let F be a finite subgroup of G_p as in Lemma 1.4. If T_{p^∞} denotes the subgroup of T consisting of all torsion elements of order a power of p, then the inclusion $\langle T_{p^\infty}, F\rangle \hookrightarrow G_p$ induces a mod p homology equivalence and therefore there is for each n a finite p–subgroup F_n of $\langle T_{p^\infty}, F\rangle$ such that inclusion induces an epimorphism $H_i(BF_n; \mathbb{F}_p) \to H_i(BG_p; \mathbb{F}_p)$ for all $i \leq n$. In particular, there exists n such that there is a stable map $BF_n \to X$ (after localizing at p) which is onto in mod p homology. Now the solution of the Segal conjecture [Ca] forces X to be trivial because there are no nontrivial stable maps from BF_n to any positive dimensional sphere. We conclude that $H^i(BG; \mathbb{F}_p) \to H^i(BG_p; \mathbb{F}_p)$ is an isomorphism for all i.

For $i = 1$ we get

(2.6)

$$H^1(BG; \mathbb{F}_p) \cong \mathrm{Hom}(H_1(BG); \mathbb{F}_p) \cong \mathrm{Hom}(\pi_1(BG); \mathbb{F}_p) \cong \mathrm{Hom}(\pi_0(G); \mathbb{F}_p)$$

and therefore we have a bijection

$$\mathrm{Hom}(\pi_0(G); \mathbb{F}_p) \to \mathrm{Hom}(\pi_0(G_p); \mathbb{F}_p). \tag{2.7}$$

Because of Theorem 2.1 (cf. remark 2.2) we may assume $p = 2$. The determinant of the adjoint representation of a 2–Sylow normalizer G_2 on the Lie algebra LT defines a homomorphism $\pi_0(G_2) \xrightarrow{\varphi} \mathbb{F}_2$. If T is properly contained in $G_{(1)}$, the connected component of $1 \in G$, then the reflections in the Weyl group $W(G_{(1)})$ show that φ restricts nontrivially to $\pi_0(G_2 \cap G_{(1)})$ and can therefore not come from $\pi_0(G)$. It follows that $T = G_{(1)}$ and G is a finite torus extension.

Now (2.6), (2.7) and Tate's H^1–criterion imply that $\pi_0(G) = G/T$ is 2–nilpotent. By Proposition 1.3.(f) it suffices therefore to show that odd order elements of $\pi_0(G)$ act trivially on T.

Our hypothesis implies certainly that $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ is an F–isomorphism, hence $\mathrm{Rep}(V, G_p) \to \mathrm{Rep}(V, G)$ is bijective for all elementary abelian p–groups V and therefore $N_G(T_p)/C_G(T_p)$ is a p–group by the proof of Proposition 1.3.((b) $\Rightarrow$ (c)). For $p = 2$ it follows that odd order elements of $\pi_o(G)$ act trivially on T_2 and hence on T (cf. proof of Proposition 1.3. ((e) $\Rightarrow$ (f))). This finishes the proof of 2.5.

$\square$

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The rigidity of Poincaré duality algebras and classification of homotopy types of manifolds

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THE RIGIDITY OF POINCARÉ DUALITY ALGEBRAS AND CLASSIFICATION OF HOMOTOPY TYPES OF MANIFOLDS

MARTIN MARKL

INTRODUCTION

This paper is devoted to the study of homotopy types of simply connected rational Poincaré duality spaces. We will frequently use the language and results of rational homotopy theory, a good common reference is the book [14].

So, let X be a rational Poincaré duality space of the (top) dimension n, i.e. a simply connected space, whose rational cohomology algebra is a Poincaré duality algebra of the formal dimension n; see §3. It is well-known (see also §3) that X has the rational homotopy type of a space of the form $Y \cup_h e^n$, where Y is a simply connected CW-complex of dimension $< n$ and $h : S^{n-l} = \partial e^n \to Y$ is a continuous map. The space Y, defined uniquely up to rational homotopy type, will be called (with some inaccuracy) the skeleton of X and will be denoted by $X_{<n}$. If X is a simply connected n-dimensional manifold, the construction above can be described even more geometrically: take $X \setminus B^n$, where B^n is a (sufficiently small) n-dimensional open disc. It is easy to remark that the n-dimensional manifold with boundary, $X \setminus B^n$, has the same rational homotopy type as the skeleton $X_{<n}$, constructed above.

Recall that two simply connected spaces X and Y are said to have the same $\mathbf{k}$-homotopy type, where $\mathbf{k}$ is a field of characteristic zero, if their Quillen minimal models [14; III.3.(1)] are isomorphic over $\mathbf{k}$; this fact will be denoted by $X \sim_{\mathbf{k}} Y$. Of course, for $\mathbf{k} = \mathbf{Q}$ we get the usual definition of the rational homotopy equivalence.

Fix an n-dimensional rational Poincaré duality space X (simply connected by definition). The aim of this paper is to give a description of the set $PDS_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of rational Poincaré duality spaces Y whose skeleta $Y_{<n}$ have the same rational homotopy type as the skeleton $X_{<n}$ of X, when X is formal. It is interesting to point out that the set $PDS_{\mathbf{k}}(X)$ is, according to rational surgery results [3],

for $n \not\equiv 0 \pmod 4$ naturally isomorphic to the set $Man_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of n-dimensional compact simply connected manifolds M with $M_{<n} \sim_{\mathbf{Q}} X_{<n}$.

The first attempt towards the description of $PDS_{\mathbf{k}}(X)$ was made in [12], where it is stated [12; Theorem 1] that the rational homotopy type of a rational Poincaré duality space is uniquely determined by the rational homotopy type of its skeleton, if the cohomology algebra of X is fixed. Here we will always suppose that X is formal, the hypothesis taken by M. Aubry [1,2].

We give here a complete description of the set $PDS_{\mathbf{k}}(X)$ in terms of usual algebraic objects – Galois cohomology and induced maps – when X is formal. Using this description, we are able to prove, for example, that the $\mathbf{k}$-homotopy type of a rational Poincaré duality space is uniquely determined by its skeleton provided that $\mathbf{k}$ is algebraically closed. We prove also that the set $PDS_{\mathbf{k}}(X)$ (and hence also $Man_{\mathbf{k}}(X)$) is finite for fields satisfying $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ (for example for $\mathbf{k} = \mathbf{R}$, the case of real homotopy types). As an example of explicit calculations we construct a large class of Poincaré duality spaces X for which the set $PDS_{\mathbf{k}}(X)$ consists of the $\mathbf{k}$-homotopy type of X only, $\mathbf{k}$ arbitrary. On the other hand, we give an example of a manifold M, for which the set $PDS_{\mathbf{Q}}(M)$ is infinite.

The algebraic counterpart of the description of $PDS_{\mathbf{k}}(X)$ is the following classification problem: let H^* be a Poincaré duality algebra of formal dimension n, how to describe the set $PDA_{\mathbf{k}}(H^*)$ of all isomorphism classes of Poincaré duality algebras H'^* with $H'^*/H'^n \cong H^*/H^n$. Our approach to the study of the set $PDA_{\mathbf{k}}(H^*)$ is based on a rigidity property of Poincaré duality algebras over an algebraically closed field and on the usual method of descent. We hope that this approach can be used even in more general situation – for the classification of all Gorenstein rings R having the "skeleton" $R/Socle(R)$ fixed (see [15]).

Our paper is organized as follows. In the first paragraph we prove a rigidity theorem for Poincaré duality algebras. The proof of this statement is based on a deliberate use of the deformation theory; note that this machinery has already been systematically used in rational homotopy theory in [4]. As a by-product we obtain a characterization of Poincaré duality in terms of Harrison cohomology. These results are in the next paragraph applied to the solution of our classification problem for Poincaré duality algebras. The main result of this section is Theorem 2.7. In the third paragraph the algebraic theory is applied to the study of the set $PDS_{\mathbf{k}}(X)$ as introduced above, a description is given in Theorem 3.2. Notice that both Theorem 3.2 and the forthcoming examples explicitely describe the effect of the ground field $\mathbf{k}$ on the structure of $PDS_{\mathbf{k}}(X)$, hence all the material of this paragraph can be considered as a contribution to the study of descent and non-descent phenomena in rational homotopy theory in the spirit of [10].

I would like to express here my thanks to Ştefan Papadima for drawing my attention to the possible use of descent methods. Also the formulation of the condition iii) of Theorem 1.5 is due to him. I wish also to acknowledge my indebtedness to the referee for useful comments and references.

1. Rigidity of Poincaré duality algebras

As usually, by a Poincaré duality algebra (over a field $\mathbf{k}$) of the formal dimension n is meant a (finite dimensional) graded commutative $\mathbf{k}$-algebra $H^* = \bigoplus_{i\geq 0} H^i$ such that H^n is isomorphic to k, $H^i = 0$ for $i > n$ and the bilinear form $B : H^* \bigotimes H^* \to \mathbf{k}$ of degree $-n$ defined by

$$B(x,y) = \begin{cases} x.y \in \mathbf{k} \cong H^n & \text{for } deg(x) + deg(y) = n \\ 0 & \text{otherwise} \end{cases}$$

is nondegenerate in the usual sense. All Poincaré duality algebras (and Poincaré duality spaces) in this paper are supposed to have the same formal dimension equal to a given natural number n.

1.1. For a graded commutative algebra A^* denote:

$$\begin{aligned} \mathcal{B}(A^*) &= \left\{ \begin{array}{c} \text{all bilinear forms } B : A^* \bigotimes A^* \to \mathbf{k} \text{ of degree } -n \text{ such} \\ \text{that } B(x,y) = (-1)^{deg(x)deg(y)} B(y,x) \text{ for } x, y \in A^* \end{array} \right\}, \\ \mathcal{M}(A^*) &= \{B \in \mathcal{B}(A^*); B(xy,z) = B(x,yz) \text{ for } x,y,z \in A^*\}, \\ \mathcal{P}(A^*) &= \left\{B \in \mathcal{M}(A^*); B \text{ is nondegenerate on } A^{>0} \bigotimes A^{>0}\right\} \quad \text{and} \\ G(A^*) &= Aut(A^*) = \text{the group of graded automorphisms of } A^*. \end{aligned}$$

Notice that all the sets above have the natural structure of a (not necessarily irreducible) algebraic variety. The geometry of $\mathcal{M}(A^*)$ is extremely simple—as all the defining equations are linear, it is isomorphic to an affine space. The set $\mathcal{P}(A^*)$ is plainly Zariski-open and dense in $\mathcal{M}(A^*)$. The group $G(A^*)$ acts naturaly from the left on $\mathcal{B}(A^*)$ by

$$\phi(B)(x,y) = B(\phi^{-1}(x), \phi^{-1}(y)).$$

Clearly $G(A^*)\mathcal{M}(A^*) \subset \mathcal{M}(A^*)$ and $G(A^*)\mathcal{P}(A^*) \subset \mathcal{P}(A^*)$. The action of $G(A^*)$ is plainly continuous in the Zariski topology.

We call an algebra A^* a fragment, if it is of the form

$$A^* = H^*_{<n} := H^*/H^n$$

for a Poincaré duality algebra H^*. The algebra $H^*_{<n}$ will be called the *skeleton* of H^*. Here $H^*_{<n}$ is defined as a quotient, but after having chosen a section, we may as well consider it as a subset of H^*.

It is interesting to remark that it is allways possible to decide in finitely many steps whether a given graded commutative algebra A^* is a fragment or not. To this end, find at first a basis of the affine space $\mathcal{M}(A^*)$. Our algebra A^* is then a fragment if and only if the polynomial function, representing the determinant, is not equal to zero on $\mathcal{M}(A^*)$ identically.

This characterization problem for fragments is the special case of the problem of deciding when a given local ring is a factor of a Gorenstein ring by the socle, see [15].

1.2. For a fragment A^* consider the set $\tilde{\mathcal{M}}(A^*)$ of all graded commutative algebras H^* with $H^i = 0$ for $i > n$, $H^n \cong \mathbf{k}$ and $H^*_{<n}$ isomorphic to A^*. For $H^* \in \tilde{\mathcal{M}}(A^*)$ choose an isomorphism $r : H^n \to \mathbf{k}$ and define $B \in \mathcal{M}(A^*)$ by $B(x,y) = r(x.y) \in \mathbf{k}$. The form B is defined canonically up to a nonzero multiple from $\mathbf{k}$. Keeping in mind this ambiguity, we can write $H^* = (A^*, B)$. Notice that H^* is a Poincaré duality algebra if and only if $B \in \mathcal{P}(A^*)$.

1.3. Let $A^* = H^*_{<n}$ be a fragment and denote by $PDA_{\mathbf{k}}(H^*)$ the set of all isomorphism classes of Poincaré duality $\mathbf{k}$-algebras having the skeleton isomorphic to A^*. We claim that the presentation 1.2 induces a bijection between $PDA_{\mathbf{k}}(H^*)$ and the orbit space $\mathcal{P}(A^*)/G(A^*)$ provided that $\mathbf{k}$ algebraically closed.

To verify this, notice at first that each algebra from $PDA_{\mathbf{k}}(H^*)$ is isomorphic to an algebra H'^* with $H'^*_{<n} = A^*$. Hence we can suppose immediately that $H'^*{}_{<n} = A^*$ for each $H'^* \in PDA_{\mathbf{k}}(H^*)$. Let $H'^* = (A^*, B')$ and $H''^* = (A^*, B'')$ be two algebras from

$PDA_{\mathbf{k}}(H^*)$ and suppose that they are isomorphic. This means that there exists an isomorphism $\phi : A^* \to A^*$ and a nonzero $\alpha \in \mathbf{k}$ such that $B''(\phi(x), \phi(y)) = \alpha B'(x,y)$. If we choose $\xi \in \mathbf{k}$ such that $\xi^n = \alpha$ and define $g \in Aut(A^*)$ by $g(x) = \xi^{-deg(x)}.x$, we see that $B''(\phi \circ g(x), \phi \circ g(y)) = B'(x,y)$, i.e. B' and B'' are in the same orbit of $G(A^*)$. On the other hand, it is easy to check that forms belonging to the same orbit define isomorphic algebras.

1.4. Before formulating the central result of this section, recall some necessary facts about the Harrison cohomology [13]. Let A^* be a graded commutative algebra and M^* a graded A^*-module. Define on $\bigotimes^m A^*$ a new grading, putting $deg(a_1 \otimes \cdots \otimes a_m) = 1 + \sum_1^m (deg(a_i) - 1)$ and denote by $C^{m,p}(A^*, M^*)$ the set of all linear maps $f : \bigotimes^m A^* \to M^*$ of degree p such that $f(a_1, \ldots, a_m) = 0$ whenever some $a_i = 1$, $1 \le i \le m$. The differential δ of bidegree $(1,1)$ on $C^{*,*}(A^*, M^*)$ is defined by the formula

$$\delta f(a_1, \ldots, a_{m+1}) = a_1 f(a_2, \ldots, a_{m+1}) + (-1)^{\nu(m+1)} f(a_1, \ldots, a_m) a_{m+1} + \\ + \sum_{j=1}^{m} (-1)^{\nu(j)} f(a_1, \ldots, a_j a_{j+1}, \ldots, a_{m+1}),$$

where $\nu(j) = \sum_{i=1}^{j} (deg(a_i) - 1)$. The cohomology of the complex $(C^{*,*}(A^*, M^*), \delta)$ is the usual Hochshild cohomology of A^* with coefficients in M^*. Consider the subspace $C^{m,p}_{\mathrm{Harr}}(A^*; M^*)$ of $C^{m,p}(A^*, M^*)$ consisting of all cochains of $C^{m,p}(A^*, M^*)$ which are zero on decomposable elements of the shuffle product in $\bigotimes A^*$ (see [14; p.18]). The subspace $C^{*,*}_{\mathrm{Harr}}(A^*, M^*)$ can be shown to be δ-stable and the associated cohomology

$$\mathrm{Harr}^{m,p}(A^*; M^*) := H^{m,p}(C^{*,*}_{\mathrm{Harr}}(A^*, M^*), \delta)$$

is called the *Harrison cohomology* of the graded commutative algebra A^* with coefficients in M^*.

For a given fragment A^* and an algebra $H^* \in \tilde{\mathcal{M}}(A^*)$ there are two natural A^*-modules: the "reduced" algebra $\tilde{H}^*$ (= the ideal of the natural augmentation $H^* \to \mathbf{k}$) with the action given simply by the multiplication and H^n with the trivial action ($1.h = h$ and $A^{>0}.H^n = 0$). The inclusion $\iota : H^n \to \tilde{H}^*$ is a morphism of A^*-modules and it induces the map

$$\iota_* : \mathrm{Harr}(A^*; H^n) \to \mathrm{Harr}(A^*; \tilde{H}^*)$$

in Harrison cohomology. In the following theorem we give three equivalent conditions on $H^* \in \tilde{\mathcal{M}}(A^*)$ to be a Poincaré duality algebra, where A^* is a fragment. Recall that it means by definition that we *a priori* assume the existence of a symmetric nondegenerate bilinear form on A^*.

THEOREM 1.5 (RIGIDITY THEOREM). *Suppose that the ground field* $\mathbf{k}$ *is algebraically closed of characteristic zero. Let* A^* *be a fragment and let* H^* *be a graded commutative algebra with* $H^i = 0$ *for* $i > n$, $H^n \cong \mathbf{k}$ *and* $H^*_{<n}$ *isomorphic to* A^* *(in other words,* $H^* \in \tilde{\mathcal{M}}(A^*)$*). Let* $B \in \mathcal{M}(A^*)$ *be the bilinear form corresponding to* H^* *as in 1.2. Then the following three conditions are equivalent:*

i) H^* *is a Poincaré duality algebra,*
ii) *the point* $B \in \mathcal{M}(A^*)$ *is rigid under the action of* $G(A^*)$†,
iii) *the map* $\iota : \mathrm{Harr}^{2,1}(A^*; H^n) \to \mathrm{Harr}^{2,1}(A^*; \tilde{H}^*)$ *is zero.*

Proof. Define $F_B : G(A^*) \to \mathcal{M}(A^*)$ by $F_B(g) = g(B)$. Let us try to describe the tangent map $T_eF_B : T_eG(A^*) \to T_B\mathcal{M}(A^*)$ at the unit e of $G(A^*)$. As the algebra A^* has finite dimension, we have $T_eG(A^*) \cong Der(A^*)$ (the set of derivations of the algebra A^* of degree 0). The set $\mathcal{M}(A^*)$ is isomorphic to an affine space (see 1.1), hence we can identify $T_B\mathcal{M}(A^*)$ with $\mathcal{M}(A^*)$ itself. Using these identifications, we can easily obtain

$$T_eF_B(\phi)(x,y) = -(B(\phi(x),y) + B(x,\phi(y))).$$

This means that the map T_eF_B is epic if and only if for each $f \in \mathcal{M}(A^*)$ there exists a derivation $\phi \in Der(A^*)$ with

$$f(x,y) = B(\phi(x),y) + B(x,\phi(y)) \tag{1.6}$$

for $deg(x) + deg(y) = n$. On the other hand, we can obtain immediately from the definitions that

$$Z^{2,1}_{\mathrm{Harr}}(A^*, H^n) = \left\{ \begin{array}{c} \text{bilinear forms } \tilde{f} : A^* \otimes A^* \to H^n \text{ of degree zero} \\ \text{such that } \tilde{f}(x,y) = (-1)^{deg(x)deg(y)+n}\tilde{f}(y,x) \\ \text{and } \tilde{f}(xy,z) = (-1)^{deg(y)}\tilde{f}(x,yz) \end{array} \right\}$$

†This means by definition that the orbit $G(A^*)(B)$ contains a Zariski-open neighbourhood of B, see [5] or [9]

and that

$$C^{1,0}_{\mathrm{Harr}}(A^*, \tilde{H}^*) = \left\{\text{linear maps } \tilde{\phi} : A^* \to \tilde{H}^* \text{ of degree zero}\right\},$$

while

$$\delta\tilde{\phi}(x,y) = x\tilde{\phi}(y) + (-1)^{deg(x)+deg(y)}\tilde{\phi}(x)y - (-1)^{deg(x)}\tilde{\phi}(xy).$$

Therefore $\iota_*(\tilde{f}) = \delta\tilde{\phi}$ in $C^{2,1}_{\mathrm{Harr}}(A^*, \tilde{H}^*)$ if and only if

(1.7)

$$\begin{gathered}\tilde{f}(x,y) = x\tilde{\phi}(y) + (-1)^n\tilde{\phi}(x)y = B(x,\tilde{\phi}(y)) + (-1)^n B(\tilde{\phi}(x),y) \\ \text{for } deg(x)+deg(y) = n \text{ and} \\ (-1)^{deg(x)}\tilde{\phi}(xy) = x\tilde{\phi}(y) + (-1)^{deg(x)+deg(y)}\tilde{\phi}(x)y \\ \text{for } deg(x)+deg(y) < n\end{gathered}$$

The correspondence $\tilde{f} \longmapsto f$, $f(x,y) = (-1)^{deg(y)}\tilde{f}(x,y)$ clearly defines an identification of $Z^{2,1}_{\mathrm{Harr}}(A^*, H^n)$ and $\mathcal{M}(A^*)$. Writting in (1.7) $(-1)^{deg(y)}.f(x,y)$ instead of $\tilde{f}(x,y)$ and $(-1)^{deg(x)}\phi(x)$ instead of $\tilde{\phi}(x)$, we can easily verify that the map ι_* is zero if and only if for each $f \in \mathcal{M}(A^*)$ there exists a linear map $\phi : A^* \to \tilde{H}^*$ of degree zero such that

$$\text{(1.7')} \qquad \begin{cases} f(x,y) = B(x,\phi(y)) + B(y,\phi(x)) & \text{for } deg(x)+deg(y) = n \text{ and} \\ \phi(xy) = x\phi(y) + \phi(x)y & \text{for } deg(x)+deg(y) < n. \end{cases}$$

Notice that the second equation in (1.7') means that ϕ is a derivation of the algebra A^*, i.e. $\phi \in T_eG(A^*)$. Comparing (1.6) and (1.7') we see that the following statement is valid.

LEMMA 1.8. *The map T_eF_B is an epimorphism if and only if ι_* is zero.*

Proof of i) $\Longrightarrow$ iii). Suppose that H^* is a Poincaré duality algebra and prove (1.6). Let $f \in \mathcal{M}(A^*)$. As the form B is nondegenerate, the formula

$$\text{(1.9)} \qquad B(\phi(x),y) = \tfrac{deg(x)}{n} f(x,y)$$

defines a linear map $\phi : A^* \to A^*$ of degree zero. By (1.9) and the symmetry of B,

(1.10)

$$\begin{aligned} B(x,\phi(y)) &= (-1)^{deg(x)deg(y)} B(\phi(y),x) \\ &= \tfrac{deg(y)}{n}.(-1)^{deg(x)deg(y)}.f(y,x) \\ &= \tfrac{deg(y)}{n}.f(x,y). \end{aligned}$$

By (1.9) and (1.10), for $deg(x) + deg(y) = n$,

$$B(x,\phi(y)) + B(\phi(x),y) = (\tfrac{deg(x)}{n} + \tfrac{deg(y)}{n}).f(x,y) = f(x,y),$$

which is (1.6). It remains to show that ϕ is really a derivation of degree zero. This is, because B is nondegenerate, equivalent to

(1.11)

$$B(x\phi(y),z) + B(\phi(x)y,z) = B(\phi(xy),z)$$
for each $x, y, z \in A^*$ with $deg(x) + deg(y) + deg(z) = n$.

We easily deduce from (1.9) and (1.10) that

$$B(\phi(xy),z) = \tfrac{deg(x)+deg(y)}{n}.f(x,y)$$

and that

$$\begin{aligned} B(\phi(x)y,z) = B(\phi(x),yz) = \tfrac{deg(x)}{n}.f(x,yz) &= \tfrac{deg(x)}{n}.f(xy,z), \\ B(x\phi(y),z) &= (-1)^{deg(x)deg(y)}.B(\phi(y),xz) \\ &= \tfrac{deg(y)}{n}.(-1)^{deg(x)deg(y)} f(y,xz) \\ &= \tfrac{deg(y)}{n}.f(xy,z) \end{aligned}$$

Using these formulae, it is easy to verify (1.11), hence $T_e F_B$ is epic and ι_* is zero by Lemma 1.8.

Proof of iii) $\Longrightarrow$ ii). Notice that the points $e \in G(A^*)$ and $B \in \mathcal{M}(A^*)$ are regular. If iii) is satisfied, the map $T_e F_B$ is epic by Lemma 1.8 and $Im(F_B) = G(A^*)(B)$ contains an open neighbourhood by standard arguments of the algebraic geometry, see for example [9; Lemma 23.5].

Proof of ii) $\Longrightarrow$ i). Suppose that B is rigid and let $U \subset G(A^*)(B)$ be an open neighbourhood of B. Then both U and $\mathcal{P}(A^*)$ are nonempty open subsets in the affine space $\mathcal{M}(A^*)$, hence $\mathcal{P}(A^*) \cap U \neq 0$. They are both $G(A^*)$-invariant and $G(A^*)$ acts on U transitively, consequently, $U \subset \mathcal{P}(A^*)$, therefore $B \in \mathcal{P}(A^*)$, i.e. H^* is a Poincaré duality algebra.

2. Classification theorems for Poincaré duality algebras

Our classification is based on the fact that Poincaré duality algebras over an algebraically closed field are uniquely determined by their skeleton.

THEOREM 2.1. *Let A^* be a fragment and suppose that the ground field $\mathbf{k}$ is algebraically closed of characteristic zero. Then*

$$\#(\mathcal{P}(A^*)/G(A^*)) = 1,$$

in other words (see 1.3), the following statement is true:

Let H^* and H'^* be two Poincaré duality algebras over an algebraically closed field of characteristic zero such that $H^*_{<n}$ is isomorphic to $H'^*_{<n}$. Then the algebras H^* and H'^* are isomorphic, too.

Proof. As $\mathcal{M}(A^*)$ is an affine space, there exists at most one open orbit of $G(A^*)$ in $\mathcal{M}(A^*)$. On the other hand, the orbit of every point $B \in \mathcal{P}(A^*)$ is open by Theorem 1.5 ii). Therefore all points of $\mathcal{P}(A^*)$ are in the same orbit, in other words, $G(A^*)$ acts on $\mathcal{P}(A^*)$ transitively and $\#(\mathcal{P}(A^*)/G(A^*)) = 1$.

2.2. Warning: Being $g : H^*_{<n} \cong H'^*_{<n}$ an isomorphism, then the isomorphism of H^* and H'^*, whose existence is guarranteed by Theorem 2.1, is not necessarily an extension of g.

Example 2.3. Let V be a $\mathbf{k}$-vector space and fix an even number $d > 0$. Let A^* be a graded algebra defined by $A^0 = \mathbf{k}$, $A^d = V$ and $A^i = 0$ otherwise, having the obvious product. Every nondegenerate symmetric bilinear form B on V defines a Poincaré duality algebra H^* of the formal dimension $n = 2d$ with $H^*_{<n} = A^*$ (see 1.2), this is the simplest nontrivial example of a Poincaré duality algebra.

Clearly, $\mathcal{P}(A^*)$ consists of all symmetric nondegenerate bilinear forms on V and the quotient $\mathcal{P}(A^*)/G(A^*)$ is the set of all equivalence classes of nondegenerate symmetric bilinear forms on V. If the field $\mathbf{k}$ is algebraically closed, Theorem 2.1 says that there exists exactly one equivalence class of nondegenerate bilinear forms on V. This result is classical.

2.4. Now, starting from the classification over algebraically closed fields, we can try to obtain a general result using the usual description of the descent by Galois cohomology, see [11]. Let us introduce the following notation and terminology.

Let $\mathbf{K}$ be an extension of $\mathbf{k}$ and let M be an object (vector space, algebra etc.) defined over $\mathbf{k}$. Denote $M_{\mathbf{K}} = M \otimes_{\mathbf{k}} K$. Two objects M and N, defined over $\mathbf{k}$, are said to be $\mathbf{K}$-isomorphic ($M \cong_{\mathbf{K}} N$), if there exists a $\mathbf{K}$-isomorphism between the $\mathbf{K}$-objects $M_{\mathbf{K}}$ and $N_{\mathbf{K}}$. Fix now a Poincaré duality $\mathbf{k}$-algebra H^* and let $A^* = H^*_{<n}$ be its skeleton. The central object of our study is the following set

$$PDA_{\mathbf{k}}(H^*) = \left\{ \begin{array}{c} \mathbf{k}\text{-isomorphism classes of all Poincaré} \\ \text{duality } \mathbf{k}\text{-algebras } H'^* \text{ with } H'^*_{<n} \cong_{\mathbf{k}} A^* \end{array} \right\}$$

Unfortunately, this set is not approachable to apply the descent method directly. We are led to consider also the following sets ($\bar{\mathbf{k}}$ denotes the algebraic closure of $\mathbf{k}$):

$$\tilde{E}_{\mathbf{k}} = \left\{ \begin{array}{c} \mathbf{k}\text{-isomorphism classes of Poincaré duality} \\ \mathbf{k}\text{-algebras } H'^* \text{ with } H'^* \cong_{\bar{\mathbf{k}}} H^* \end{array} \right\}$$
$$E_{\mathbf{k}} = \left\{ \begin{array}{c} \mathbf{k}\text{-isomorphism classes of graded commutative} \\ \mathbf{k}\text{-algebras } A'^* \text{ with } A'^* \cong_{\bar{\mathbf{k}}} A^* \end{array} \right\}$$

and define the map $F_{\mathbf{k}} : \tilde{E}_{\mathbf{k}} \to E_{\mathbf{k}}$ by $F_{\mathbf{k}}(H'^*) = H'^*{}_{<n}$. The sets $\tilde{E}_{\mathbf{k}}$ and $E_{\mathbf{k}}$ are related with $PDA_{\mathbf{k}}(H^*)$ as follows:

LEMMA 2.5. *Let $\mathbf{k}$ be a field of characteristic zero. Then there exists a natural correspondence between the elements of $PDA_{\mathbf{k}}(H^*)$ and algebras $H'^* \in \tilde{E}_{\mathbf{k}}$ satisfying $F_{\mathbf{k}}(H'^*) \cong_{\mathbf{k}} A^*$*

Proof. By Theorem 2.1, $H'^* \cong_{\bar{\mathbf{k}}} H^*$ for each $H'^* \in PDA_{\mathbf{k}}(H^*)$, the rest is trivial.

The following description of the descent for graded algebras can be obtained by a slight modification of the proof of [11; Proposition 1 in III.1.1], see also the comments to the proof given in the russian version of this book (Mir 1968).

PROPOSITION 2.6. *Suppose that $\mathbf{K}$ is a Galois extension of $\mathbf{k}$. Let M^* be a graded $\mathbf{k}$-algebra and denote by $\mathcal{E}(\mathbf{K}/\mathbf{k})$ the set of all $\mathbf{k}$-isomorphism classes of $\mathbf{k}$-algebras M'^* with $M'^* \cong_{\mathbf{K}} M^*$.*

Then the elements of the set $\mathcal{E}(\mathbf{K}/\mathbf{k})$ are in a natural one-to-one correspondence with the Galois cohomology group $H^1(G(\mathbf{K}/\mathbf{k}); Aut_{\mathbf{K}}(M^*_{\mathbf{K}}))$, where the action of the Galois group $G(\mathbf{K}/\mathbf{k})$ on $Aut_{\mathbf{K}}(M^*_{\mathbf{K}})$ is defined by $s(f) = (1 \otimes s) \circ f \circ (1 \otimes s)^{-1}$.

Using the explicit description of the correspondence in the previous proposition, we can infer easily from Lemma 2.5 the following classification result.

THEOREM 2.7. *Let $\mathbf{k}$ be a field of characteristic zero and denote by $\bar{\mathbf{k}}$ the algebraic closure of $\mathbf{k}$. Let $A^* = H^*_{<n}$ be a fragment defined over $\mathbf{k}$. Then there exists a natural one-to-one correspondence between the set $PDA_{\mathbf{k}}(H^*)$ of all $\mathbf{k}$-isomorphism classes of Poincaré duality $\mathbf{k}$-algebras H'^* with $H'^*_{<n} \cong_{\mathbf{k}} A^*$ and the set*

$$Ker(\iota_{\mathbf{k}} : H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}})) \to H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(A^*_{\bar{\mathbf{k}}})))$$

where the map $\iota_{\mathbf{k}}$ is induced by the natural homomorphism

$$j : Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}}) \to Aut_{\bar{\mathbf{k}}}(A^*_{\bar{\mathbf{k}}})$$

given by the restriction.

We close this section with the following corollary of Theorem 2.7.

COROLLARY 2.8. *Suppose that $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$. In this case there exists only finitely many isomorphism classes of Poincaré duality $\mathbf{k}$-algebras having a given skeleton (i.e. $\#(PDA_{\mathbf{k}}(H^*)) < \infty$).*

Proof. A field satisfying the condition $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ is of type (F) in the sense of [11; III.4], hence $H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}}))$ is finite [11; III.4.3] and the corollary follows from Theorem 2.7.

3. APPLICATIONS TO THE RATIONAL HOMOTOPY TYPE

By a rational Poincaré duality space (of the formal dimension n) is meant here a simply connected CW-complex X such that $H^*(X; \mathbf{Q})$ is a Poincaré duality algebra (of the formal dimension n). We show that X has the rational homotopy type of a space of the form $Y \cup_h e^n$, where Y is a CW-complex of the dimension $< n$ and $h\colon S^{n-1} = \partial e^n \to Y$ a continuous map, as promissed in the introduction.

To this end, suppose that $(\mathcal{L}(Z,\mu),\partial)$ is the Quillen minimal model of X, where the generator μ, $deg(\mu) = n-1$, corresponds to the "orientation class" of X in $H_n(X;\mathbf{Q})$. Let Y be a space corresponding to the minimal algebra $(\mathcal{L}(Z),\partial|\mathcal{L}(Z))$, we can clearly suppose that Y is a CW-complex of dimension $< n$. Let $h \in \pi_{n-1}(X)$ be an element corresponding (up to a nonzero rational multiple if necessary) to $[\partial(\mu)] \in H_{n-2}(\mathcal{L}(Z),\partial|\mathcal{L}(Z)) \cong \pi_{n-1}(X) \otimes \mathbf{Q}$. It follows easily from [14; III.3.(6)] that $Y \cup_h e^n \sim_{\mathbf{Q}} X$ and Y is what we call the skeleton of X and denote by $X_{<n}$. It is also clear that $H^*(X_{<n};\mathbf{k}) \cong H^*(X;\mathbf{k})_{<n}$, see [14; III.3(9)].

3.1. Observe that X is formal if and only if $X_{<n}$ is. Indeed, if X is formal, then the minimal model $(\mathcal{L}(Z,\mu),\partial)$ can be chosen so that ∂ is quadratic [14; II.7(5)]. Then also the minimal model $(\mathcal{L}(Z),\partial|\mathcal{L}(Z))$ of $X_{<n}$ is quadratic and $X_{<n}$ is formal again by [14; III.7(5)]. On the other hand, if $X_{<n}$ is formal, the formality of X follows easily from [12; Theorem 1], [1,2]. The central result of our paper now reads:

THEOREM 3.2. *Let $\mathbf{k}$ be a field of characteristic zero and let X be a formal rational Poincaré duality space of the top dimension n. Then there exists a natural one-to-one correspondence between the set $PDS_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of rational Poincaré duality spaces Y of the top dimension n, such that the skeletons $X_{<n}$ and $Y_{<n}$ have the same rational homotopy type, and the set $\phi_H(Ker(\iota_{\mathbf{Q}}))$, where the map*

$$\phi_H : H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}}))) \longrightarrow H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*(X;\bar{\mathbf{k}})))$$

is induced by the natural homomorphism $G(\bar{\mathbf{k}}/\mathbf{k}) \to G(\bar{\mathbf{Q}}/\mathbf{Q})$, and the map

$$\iota_{\mathbf{Q}} : H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}}))) \longrightarrow H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}})_{<n})))$$

is induced by the homomorphism $Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}})) \to Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}})_{<n})$ given by the restriction.

The proof is postponed to the end of this section. Although the description in Theorem 3.2 seems to be unmanageable, it provides us with a few of corollaries.

COROLLARY 3.3. *If the field $\mathbf{k}$ is algebraically closed, the $\mathbf{k}$-homotopy type of a formal Poincaré duality space is uniquely determined by the rational homotopy type of its skeleton.*

Proof. If $\mathbf{k} = \bar{\mathbf{k}}$, then the group $G(\bar{\mathbf{k}}/\mathbf{k})$ is trivial, hence the map ϕ_H in 3.2 is trivial, too. Therefore $\#(PDS_{\mathbf{k}}(X)) = 1$.

COROLLARY 3.4. *If $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ and X is a formal rational Poincaré duality space, then the set $PDS_{\mathbf{k}}(X)$ is finite. Especially, there exists only finitely many real homotopy types of rational Poincaré duality spaces with a given (formal) rational homotopy type of the skeleton.*

Proof. By the same argument as in the proof of Corollary 2.8 we can easily see that the set $H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*(X;\bar{\mathbf{k}})))$ is finite. The rest follows from Theorem 3.2.

Remember that, for $n \not\equiv 0 \pmod 4$, every rational Poincaré duality space (simply connected by definition) has the rational homotopy type of a compact simply connected manifold [3]. Consequently, Theorem 3.2 and the corollaries give in this case a description of the set $Man_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of simply connected compact *manifolds* M with $M_{<n} \sim_{\mathbf{Q}} X_{<n}$.

In the following example we construct a compact, simply connected manifold M for which the set $Man_{\mathbf{Q}}(M)$ is infinite.

Example 3.5. Let us denote by M the 6-dimensional simply connected compact manifold $\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C})$. Clearly, $H^*(M;\mathbf{Q}) \cong \mathbf{Q}[u,v]/(uv, u^3 - v^3)$, $deg(u) = deg(v) = 2$. It is also not hard to verify that $M_{<6}$ has the rational homotopy type of $\mathbf{P}^2(\mathbf{C}) \bigvee \mathbf{P}^2(\mathbf{C})$ (the one-point union) and that

$$H^*(M_{<6};\mathbf{Q}) \cong \mathbf{Q}[u,v]/(uv, u^3, v^3) = \left\{\mathbf{Q}[u,v]/(uv, u^3 - v^3)\right\}_{<6}.$$

As every 1-connected 6-dimensional manifold, the space M is formal [6].

Every rational Poincaré duality algebra H'^* with $H'^*_{<6} \cong_{\mathbf{Q}} H^*(M_{<6};\mathbf{Q})$ is of the form $\mathbf{Q}[u,v]/(uv, \alpha u^3 - \beta v^3)$ for some nonzero $\alpha, \beta \in \mathbf{Q}$. Notice that for such an algebra H'^* there always exists a manifold N with $H^*(N;\mathbf{Q}) \cong H'^*$ and $N_{<6} \sim_{\mathbf{Q}} M_{<6}$. Indeed, let N be a formal rational homotopy type corresponding to H'^*, as $6 \not\equiv 0 \pmod 4$ we can assume that N is a manifold. Then $N_{<6}$ is again formal by 3.1, and $N_{<6} \sim_{\mathbf{Q}} M_{<6}$ since $H^*(N_{<6};\mathbf{Q}) \cong_{\mathbf{Q}} H^*(M_{<6};\mathbf{Q})$ by the construction.

It can be easily verified that the algebra $\mathbf{Q}[u,v]/(uv, \alpha u^3 - \beta v^3)$ is isomorphic to the algebra $\mathbf{Q}[u,v]/(uv, \alpha' u^3 - \beta' v^3)$ if and only if either $\frac{\alpha\beta'}{\beta\alpha'}$ or $\frac{\alpha\alpha'}{\beta\beta'}$ is of the form q^3 for some rational number $q \in \mathbf{Q}$. Consequently, there exist infinitely many $\mathbf{Q}$-isomorphism classes of such algebras. According to the remarks above, the set $Man_{\mathbf{Q}}(\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C}))$ is infinite.

On the other hand it can be easily seen that $Man_{\mathbf{R}}(\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C}))$ consists of the real homotopy type of $\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C})$ only.

3.6. Now we aim to describe a family of manifolds (Poincaré duality spaces), whose rational homotopy type is uniquely determined by their skeleton. Let H^* be a Poincaré duality algebra and consider the canonical map $j : Aut(H^*) \to Aut(H^*_{<n})$ given by the restriction. Recall that this map plays an important role in our classification 2.7 and 3.2. As the subset $H^*_{<n} \subset H^*$ generates H^* as an algebra, every automorphism of H^* is uniquely determined by its restriction on $H^*_{<n}$, hence the map j is plainly a monomorphism. Suppose that the algebra H^* can be represented in the form

$$(*) \qquad H^* \cong \Lambda V^*/I, \text{where } I = (f_1, \ldots, f_s) \text{ and } deg(f_i) \neq n \text{ for } 1 \leq i \leq s.$$

We claim that in this case the map j is also an epimorphism.

To prove this, consider an element $g \in Aut(H^*_{<n})$. Since clearly

$$H^*_{<n} \cong \Lambda V^*/(I + (\Lambda V^*)^{\geq n}),$$

our map g lifts to some $\tilde{g} \in Aut(\Lambda V^*)$ with $\tilde{g}(I^{<n}) \subset I^{<n}$. Because of $(*)$ this implies plainly that $\tilde{g}(I^{\leq n}) \subset I^{\leq n}$. Since $H^{>n} = 0$, we know that $I^{>n} = (\Lambda V^*)^{>n}$, therefore in fact $\tilde{g}(I) \subset I$. This means that $\tilde{g}$ projects to an element $f \in Aut(H^*)$ which clearly satisfies $j(f) = g$.

Suppose now that X is a formal rational Poincaré duality space whose rational cohomology algebra satisfies $(*)$. Then the map $\iota_{\mathbf{Q}}$ in Theorem 3.2, induced by j, is an isomorphism and $\phi_H(Ker(\iota_{\mathbf{Q}}))$ consists of one element only. Thus we have proved:

PROPOSITION 3.7. *The rational homotopy type of a formal rational Poincaré duality space, whose rational cohomology algebra can be represented as in $(*)$, is uniquely determined by the rational homotopy type of its skeleton.*

The condition $(*)$ is clearly satisfied by all exterior algebras, hence the conclusion of 3.7 is valid for a product of odd-dimensional spheres.

Example 3.8. Consider the complex grassmannian $G(p,q)$ of complex p-planes in $\mathbf{C}^{p+q}$, $G(p,q) \cong \mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$. This is a formal one-connected compact manifold of dimension $2pq$. We claim that the cohomology of $G(p,q)$ can be represented as in (*) provided $(p,q) \neq (2,2)$. At first, clearly $G(n,1) \cong G(1,n) \cong \mathbf{P}^n(\mathbf{C})$ and the usual description

$$H^*(\mathbf{P}^n(\mathbf{C});\mathbf{Q}) \cong \mathbf{Q}[c_1]/(c_1^{n+1}), deg(c_1) = 2,$$

has the requisite form. It is not hard to see that the presentation

$$H^*(G(p,q);\mathbf{Q}) = \mathbf{Q}[c_1,\dots,c_p,c_1',\dots,c_q']/((1+\cdots+c_p)(1+\cdots+c_q') = 1),$$
$$deg(c_i) = 2i, deg(c_j') = 2j, \text{for } 1 \leq i \leq p, 1 \leq j \leq q,$$

satisfies (*) for $(p,q) \neq (n,1),(1,n)$ and $(2,2)$. Hence, by Proposition 3.7, the rational homotopy type of $G(p,q)$ is, for $(p,q) \neq (2,2)$, uniquely determined by $G_{<2pq}(p,q)$. On the other hand, the same method as in Example 3.5 can be used to show that there exist infinitely many rational homotopy types of 8-dimensional rational Poincaré duality spaces X with $X_{<8} \sim_{\mathbf{Q}} G(2,2)_{<8}$.

Proof of Theorem 3.2. Let $H^* = H^*(X;\mathbf{Q})$ and let us denote by $PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ the set of all $\mathbf{k}$-isomorphism classes of rational Poincaré duality algebras H'^* with $H'^*_{<n} \cong_{\mathbf{Q}} H^*_{<n}$. Consider now the map $\lambda : PDS_{\mathbf{k}}(X) \to PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ defined by $\lambda(Y) = H^*(Y;\mathbf{Q})$.

LEMMA 3.9. *The map $\lambda : PDS_{\mathbf{k}}(X) \to PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ defined above is an isomorphism.*

Proof of the lemma. Notice that every rational homotopy type $Y \in PDS_{\mathbf{k}}(X)$ is formal. Indeed, $Y_{<n}$ is formal as $Y_{<n} \sim_{\mathbf{Q}} X_{<n}$, hence Y is formal by 3.1.

λ is an epimorphism. For $H'^* \in PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ let Y be a formal rational homotopy type with $H^*(Y;\mathbf{Q}) \cong_{\mathbf{Q}} H'^*$. Since $H^*(Y_{<n};\mathbf{Q}) \cong_{\mathbf{Q}} H'^*{}_{<n} \cong_{\mathbf{Q}} H^*_{<n}$ and both $Y_{<n}$ and $X_{<n}$ are formal, $Y_{<n} \sim_{\mathbf{Q}} X_{<n}$, i.e. $Y \in PDS_{\mathbf{k}}(X)$. Plainly $\lambda(Y) = H'^*$.

λ is a monomorphism. Suppose $\lambda(Y) = \lambda(Z)$, i.e. $H^*(Y;\mathbf{Q}) \cong_{\mathbf{k}} H^*(Z;\mathbf{Q})$. As the spaces Y and Z are formal over $\mathbf{Q}$, they are formal also over $\mathbf{k}$ [7]. Since their cohomology algebras are isomorphic over $\mathbf{k}$, they have the same $\mathbf{k}$-homotopy type, in other words, $Y = Z$ considered as the elements of $PDS_{\mathbf{k}}(X)$. We point out that the Lemma 3.9 fails in general *without X being formal.*

To obtain a description of the set $PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$, consider the following commutative diagram (the notation is the same as in 2.4)

$$\begin{array}{ccc} \tilde{E}_{\mathbf{k}} & \xrightarrow{F_{\mathbf{k}}} & E_{\mathbf{k}} \\ {\scriptstyle \otimes\mathbf{k}}\uparrow & & \uparrow{\scriptstyle \otimes\mathbf{k}} \\ \tilde{E}_{\mathbf{Q}} & \xrightarrow{F_{\mathbf{Q}}} & E_{\mathbf{Q}}. \end{array} \tag{3.10}$$

Using the correspondence of Proposition 2.6 (compare also Theorem 2.7), it is easy to identify (3.10) with the diagram

$$\begin{array}{ccc} H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}})) & \xrightarrow{\iota_{\mathbf{k}}} & H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}((H^*_{<n})_{\bar{\mathbf{k}}}) \\ {\scriptstyle \phi_H}\uparrow & & \uparrow{\scriptstyle \phi_{H<n}} \\ H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*_{\bar{\mathbf{Q}}})) & \xrightarrow{\iota_{\mathbf{Q}}} & H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}((H^*_{<n})_{\bar{\mathbf{Q}}}) \end{array} \tag{3.11}$$

where all the maps are induced in the clear way. Our theorem now follows from (3.11) and the evident fact that

$$PDA_{\mathbf{k}/\mathbf{Q}}(H^*) = Im(\otimes\mathbf{k} : PDA_{\mathbf{Q}}(H^*) \to PDA_{\mathbf{k}}(H^*))$$

where, by Lemma 2.5, $PDA_{\mathbf{k}}(H^*) = F_{\mathbf{k}}^{-1}((H^*_{<n})_{\mathbf{k}})$ and $PDA_{\mathbf{Q}}(H^*) = F_{\mathbf{Q}}^{-1}(H^*_{<n})$.

Using the tools developed in the proof above, namely the diagrams (3.10) and (3.11), we can obtain also the following classification.

THEOREM 3.12. *Let $\mathbf{k}$ be a field of characteristic zero and let X be a formal rational Poincaré duality space. Then there exists a natural one-to-one correspondence between the set of all $\mathbf{k}$-homotopy types of rational Poincaré duality spaces Y with $Y_{<n} \sim_{\mathbf{k}} X_{<n}$ and the set $Im(\phi_H) \bigcap Ker(\iota_{\mathbf{k}})$.*

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Some remarks on equivariant bundles and classifying spaces

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Some remarks on equivariant bundles and classifying spaces

by J. P. May

Let Π be a normal subgroup of a topological group Γ with quotient group G; subgroups are understood to be closed. A principal $(\Pi;\Gamma)$-bundle is the projection to orbits $E \to E/\Pi$ of a Π-free Γ-space E. (Function spaces excepted, our Γ-spaces are to be of the homotopy type of Γ-CW complexes, and similarly for other groups.) For a G-space X, let $\mathcal{B}_G(\Pi;\Gamma)(X)$ denote the set of equivalence classes of principal $(\Pi;\Gamma)$-bundles over X. For a space X, let $\mathcal{B}(\Pi)(X)$ denote the set of equivalence classes of principal Π-bundles over X. Let X_G denote the Borel construction $EG \times_G X$ associated to a G-space X. We write

$$\mathcal{B}_G(\Pi;\Gamma)(EG \times X) = \mathcal{B}(\Pi;\Gamma)(X_G)$$

to emphasize that this set depends only on X_G as a space over BG. Equivalently, $\mathcal{B}(\Pi;\Gamma)(X_G)$ is the set of equivalence classes of free Γ-spaces P with a given equivalence $P/\Pi \cong EG \times X$ of G-bundles over $P/\Gamma \cong X_G$. We shall see that the calculation of this set reduces to a nonequivariant lifting problem, and we think of it as essentially a problem in ordinary nonequivariant bundle theory. In fact, in the classical case $\Gamma = G \times \Pi$, passage from P to P/G specifies a natural bijection

$$\Theta\colon \mathcal{B}(\Pi;G \times \Pi)(X_G) \to \mathcal{B}(\Pi)(X_G).$$

The projection $EG \times X \to X$ induces a natural map

$$\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G).$$

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In the classical case, $\Phi = \Theta\Psi$ is just the Borel construction on bundles. One of our goals is to determine how near the passage Ψ from equivariant bundle theory to ordinary bundle theory is to being an isomorphism. For example, we shall obtain the following result, which is essentially just an exercise in covering space theory.

THEOREM 1. *If* Γ *is discrete, then* $\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G)$ *is a bijection for any* G*-space* X*. If* Π *(but not necessarily* G*) is discrete, then* $\Phi\colon \mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ *is a bijection for any* G*-space* X*.*

We shall see that the following deeper result is a consequence of the Sullivan conjecture. The phrase "(strong) mod p equivalence" will be explained in due course.

THEOREM 2. *Let* G *be an extension of a torus by a finite* p*-group. If* Γ *is a compact Lie group, then the natural transformation* $\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G)$ *is represented by a mod p equivalence of classifying* G*-spaces. Therefore, if* Π *is a compact Lie group, then* $\Phi\colon \mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ *is represented by a mod p equivalence of classifying* G*-spaces. If* G *is a finite* p*-group, then the transformations* Ψ *and* Φ *are represented by strong mod p equivalences of classifying* G*-spaces.*

Restricting Π instead of G, we obtain the following theorem, which is the main result of [7].

THEOREM 3. *If* G *and* Π *are compact Lie groups with* Π *Abelian, then* $\Phi\colon \mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ *is a bijection for any* G*-space* X*.*

In a preprint version of this paper, the following assertion was claimed as a theorem.

ASSERTION 4. *Under the hypotheses of theorem 3, there is also a natural bijection*

$$\mathcal{B}G(\Pi;G \times \Pi)(X) \cong \mathcal{B}(\Pi)(X/G) \times Nat(\pi_0(X), R_\pi).$$

The fact that this assertion is false was discovered by John Wicks, a student at Chicago, who showed that, with $\Pi = S^1$ and $G = Z_2$, it implies an incorrect calculation of characteristic classes. Since the nature of the assertion and the mistake in its proof may be of interest, we shall discuss these matters in an Appendix.

The three theorems above are direct interpretations of results about equivariant classifying spaces, namely Theorems 5, 9, and 10 below. There is a universal example $E(\Pi;\Gamma) \to B(\Pi;\Gamma)$ of a principal $(\Pi;\Gamma)$-bundle. Up to Γ-homotopy type, the Γ-space $E(\Pi;\Gamma)$ is characterized by the requirement that, for $\Omega \subset \Gamma$, the fixed point space $E(\Pi;\Gamma)^\Omega$ be contractible if $\Omega \cap \Pi = e$ and empty otherwise.

By universality, we have a natural bijection

$$(*) \qquad \mathcal{B}G(\Pi;\Gamma)(X) \cong [X, B(\Pi;\Gamma)]_G,$$

where homotopy classes of unbased G-maps are understood. In particular, we have natural bijections

$$\mathcal{B}G(\Pi;\Gamma)(EG \times X) \cong [EG \times X, B(\Pi;\Gamma)]_G \cong [X, Map(EG, B(\Pi;\Gamma))]_G.$$

Let $p\colon X_G \to BG$ be the evident bundle and let $q\colon \Gamma \to G$ be the quotient homomorphism. Let $[X_G, B\Gamma]/BG$ be the set of homotopy classes of maps $f\colon X_G \to B\Gamma$ such that $Bq \circ f = p$ and define

$Sec(EG, B\Gamma)$ to be the G-space of maps $\varphi: EG \to B\Gamma$ such that $Bq \circ \varphi = p: EG \to BG$. A central idea in this paper is the modelling of classifying spaces by such spaces of sections. We introduce this idea by observing that the previous bijections are equivalent to

(✱) $$\mathcal{B}(\Pi;\Gamma)(X_G) \cong [X_G, B\Gamma]/BG \cong [X, Sec(EG, B\Gamma)]_G.$$

This should be clear from the equivalent bundle theoretic descriptions of the left sides already given, but we want to see it directly on the classifying space level. Since $E\Gamma$ is Π-free, the universal property of $E(\Pi;\Gamma)$ gives a Γ-map $\nu: E\Gamma \to E(\Pi;\Gamma)$, unique up to Γ-homotopy. The Γ-map $(Eq,\nu): E\Gamma \to EG \times E(\Pi;\Gamma)$ is clearly a Γ-homotopy equivalence, where Γ acts through q on EG, and it is a fiber Γ-homotopy equivalence provided we choose a model for $E\Gamma$ such that $Eq: E\Gamma \to EG$ is a Γ-fibration. Passing to orbits over Γ by first passing to orbits over Π and then over G, we obtain a homotopy equivalence

$$B\Gamma \to EG \times_G B(\Pi;\Gamma) = B(\Pi;\Gamma)_G$$

over BG. (Lemma 11 at the end will generalize this equivalence.) We have an evident G-homeomorphism $Sec(EG, X_G) \cong Map(EG, X)$ for any G-space X, and there results a G-homotopy equivalence

$$\xi: Sec(EG, B\Gamma) \to Sec(EG, B(\Pi;\Gamma)_G) \cong Map(EG, B(\Pi;\Gamma)).$$

Via the projection $EG \to pt$ and use of a chosen homotopy inverse to ξ, we obtain a G-map

$$\alpha: B(\Pi;\Gamma) \to Sec(EG, B\Gamma)$$

which induces the transformation Ψ under the isomorphisms (∗) and (✱). In order to prove Theorem 1, we model $E(\Pi;\Gamma)$ as a space of sections and use this model to obtain an explicit description of α. In the classical case $\Gamma = G \times \Pi$, we agree to abbreviate

$$E_G(\Pi) = E(\Pi;G \times \Pi) \quad \text{and} \quad B_G(\Pi) = B(\Pi;G \times \Pi);$$

here $B\Gamma = BG \times B\Pi$ and therefore $Sec(EG, B\Gamma) \cong Map(EG, B\Pi)$.

THEOREM 5. *Let Γ act through $q: \Gamma \to G$ on EG and by conjugation on the space $Sec(EG, E\Gamma)$ of maps $\varphi: EG \to E\Gamma$ such that $Eq \circ \varphi = id$. Then $Sec(EG, E\Gamma)$ satisfies the fixed point criteria characterizing $E(\Pi;\Gamma)$, hence the orbit space $Sec(EG, E\Gamma)/\Pi$ is a model for $B(\Pi;\Gamma)$. With this model, $\alpha: B(\Pi;\Gamma) \to Sec(EG, B\Gamma)$ is the G-map induced by $Sec(id, p)$, where $p: E\Gamma \to B\Gamma$ is the universal Γ-bundle. If Γ is discrete, then α is a homeomorphism. If $\Gamma = G \times \Pi$, then $Map(EG, E\Pi)$ is a model for $E_G(\Pi)$, $Map(EG, E\Pi)/\Pi$ is a model for $B_G\Pi$, $\alpha: B(\Pi;\Gamma) \to Map(EG, B\Pi)$ is induced by $p: E\Pi \to B\Pi$, and α is a homeomorphism if Π is discrete.*

When Γ is discrete, elementary covering space theory shows that any map $\varphi: EG \to B\Gamma$ such that $Bq \circ \varphi = p$ lifts to a section $\tilde{\varphi}: EG \to E\Gamma$ of Eq and that any two such lifts are in the same Π-orbit. The last homeomorphism is seen similarly, and Theorem 1 is an immediate consequence of these homeomorphisms.

To prove Theorem 5, we need a kind of topological analog of the standard comparison of projective and acyclic resolutions.

LEMMA 6. *Let G be a topological group, let X be a free G-CW complex, and let Y be a nonequivariantly contractible G-space. Then the space $Map_G(X,Y)$ of G-maps $X \to Y$ is contractible.*

PROOF. If $X = G \times K$ for a space K, then $\mathrm{Map}_G(X, Y) \cong \mathrm{Map}(K, Y)$ and the conclusion is clear. Since $\mathrm{Map}_G(?, Y)$ converts pushouts to pullbacks, G-cofibrations to fibrations, and colimits to limits, the conclusion follows in general by use of the cell structure on X.

PROOF OF THEOREM 5. Recall that we have a fiber Γ-homotopy equivalence $(Eq,\nu)\colon E\Gamma \to EG \times E(\Pi;\Gamma)$ over EG. Applying the functor $\mathrm{Map}(EG, ?)$ and restricting to the fiber over $\mathrm{id} \in \mathrm{Map}(EG, EG)$, we obtain a Γ-homotopy equivalence

$$\mathrm{Sec}(EG, E\Gamma) \to \mathrm{Map}(EG, E(\Pi;\Gamma)).$$

Let $\Omega \subset \Gamma$. Since EG is Π-trivial and $E(\Pi;\Gamma)$ is Π-free, there are no Ω-maps $EG \to E(\Pi;\Gamma)$ if $\Omega \cap \Pi \neq e$. If $\Omega \cap \Pi = e$, then Ω acts freely via q on EG while $E(\Pi;\Gamma)$ is Ω-contractible since $E(\Omega;\Gamma)^\Lambda$ is contractible for all $\Lambda \subset \Omega$. Therefore $\mathrm{Map}_\Omega(EG, E(\Pi;\Gamma))$ is contractible. The compatibility of $\mathrm{Sec}(\mathrm{id}, p)$ with the earlier map α is checked by an easy diagram chase.

To prove Theorem 2, we must first obtain a nonequivariant description of the fixed point maps α^H. At least if Γ is a Lie group, the fixed point structure of the G-space $B(\Pi;\Gamma)$ is given as follows [6, Thm 10]. Let $N_\Gamma\Omega$ and $Z_\Gamma\Omega$ be the normalizer and centralizer of Ω in Γ. If $\Omega \cap \Pi = e$, then an easy check shows that $\Pi \cap N_\Gamma\Omega = \Pi \cap Z_\Gamma\Omega$; we agree to write Π^Ω for this intersection.

THEOREM 7. *For $H \subset G$, $B(\Pi;\Gamma)^H = \coprod B\Pi^\Omega$, where the union runs over the Π-conjugacy classes of subgroups $\Omega \subset \Gamma$ such that $\Omega \cap \Pi = e$ and $q(\Omega) = H$; $B(\Pi;\Gamma)^H$ is empty if there are no such subgroups Ω.*

LEMMA 8. *For* $\Omega \subset \Gamma$ *such that* $\Omega \cap \Pi = e$ *and* $q(\Omega) = H$, *define* $\mu\colon H \times \Pi^{\Omega} \to \Gamma$ *by* $\mu(q(\lambda), \pi) = \lambda\pi$ *and note that* $q\circ\mu = i\circ\pi_1$. *The restriction of* α^H *to* $B\Pi^{\Omega}$ *is the adjoint of the classifying map*

$$B\mu\colon BH \times B\Pi^{\Omega} = B(H \times \Pi^{\Omega}) \to B\Gamma.$$

PROOF. Let $\tilde{\alpha}\colon EG \times E(\Pi;\Gamma) \to E\Gamma$ be a Γ-homotopy equivalence over EG inverse to (Eq,ν). Since the adjoint of α is obtained from $\tilde{\alpha}$ by passage to orbits and since $B\Pi^{\Omega} = E(\Pi;\Gamma)^{\Omega}/\Pi^{\Omega}$ as a subspace of $B(\Pi;\Gamma)$, it suffices to observe that the restriction of $\tilde{\alpha}$ to the free contractible $(H \times \Pi^{\Omega})$-space $EG \times E(\Pi;\Gamma)^{\Omega}$ is μ-equivariant:

$$\tilde{\alpha}(yq(\lambda), x\pi) = \tilde{\alpha}(yq(\pi\lambda), x\lambda\pi) = \tilde{\alpha}((y,x)\lambda\pi) = (\tilde{\alpha}(y,x))\lambda\pi$$

for $y \in EG$, $x \in E(\Pi;\Gamma)^{\Omega}$, $\lambda \in \Omega$, and $\pi \in \Pi^{\Omega}$.

Given this interpretation of α^H, Theorem 2 follows directly from the application of the Sullivan conjecture to the study of maps between classifying spaces given by Dwyer and Zabrodsky [3] and Notbohm [10]. We say that a map $f\colon X \to Y$ is a mod p equivalence if f induces an isomorphism on mod p homology. We say that f is a strong mod p equivalence if the following conditions hold.

(i) f induces an isomorphism $\pi_0(X) \to \pi_0(Y)$;

(ii) f induces an isomorphism $\pi_1(X,x) \to \pi_1(Y,f(x))$ for any $x \in X$;

(iii) f induces an isomorphism $H_*(\tilde{X}_x,Z_p) \to H_*(\tilde{Y}_{f(x)},Z_p)$ for any $x \in X$, where $\tilde{X}_x$ and $\tilde{Y}_{f(x)}$ are the universal covers of the components of X and Y containing x and $f(x)$.

We say that a G-map $f\colon X \to Y$ is a (strong) mod p equivalence if $f^H\colon X^H \to Y^H$ is a (strong) mod p equivalence for each $H \subset G$. The results of Dwyer and Zabrodsky and of Notbohm admit the following interpretation (their G and Π playing opposite roles from ours).

THEOREM 9. *If Γ is a compact Lie group and G is an extension of a torus by a finite p-group, then the G-map α: $B(\Pi;\Gamma) \to Sec(EG, B\Gamma)$ is a mod p equivalence. If G is a finite p-group, then α is a strong mod p equivalence.*

When $\Gamma = G \times \Pi$, $Sec(BH, B\Gamma) = Map(BH, B\Pi)$ and the second statement is Dwyer and Zabrodsky's [3, 1.1] while the first result is Notbohm's [10,1.1]. When $G = Z_p$, the result is [3, 4.5]. The result for general extensions follows from the result for trivial extensions exactly as in the deduction of [3, 4.5] from [3, 4.4]. Incidentally, as observed by Notbohm [private communication], the components of α^H induce injections but not surjections on the fundamental groups of corresponding components when G is an extension of a non-trivial torus by a finite p-group.

Of course, Theorem 2 is a restatement of Theorem 9. Some discussion of the significance of the represented form of the result is in order. For G-spaces Y, [9] constructs a functorial "fundamental groupoid G-space πY" and a natural G-map $\chi: Y \to \pi Y$. For $H \subset G$, $\chi^H: Y^H \to (\pi Y)^H$ induces a bijection on components and an isomorphism between the fundamental groups of corresponding components, while each component of $(\pi Y)^H$ has trivial higher homotopy groups. For $y \in Y^G$, let $\tilde{Y}_y$ be the homotopy fibre of χ regarded as a based map with respect to the basepoints y and $\chi(y)$. Then $(\tilde{Y}_y)^H$ is the homotopy fibre of the restriction of χ^H to the component of Y^H containing y. Clearly $\tilde{Y}_y$ is G-simply connected, in the sense that all of its fixed point spaces are simply connected. We can p-adically complete G-simply connected (or G-nilpotent) G-spaces and characterize the completion in terms of the usual homological characterization of completion on H-fixed point spaces for all H [8]. If $f\colon Y \to Z$ is a strong mod p equivalence, then the map $\pi f\colon \pi Y \to \pi Z$ and the p-adic completions $\hat{f_p}\colon (\tilde{Y}_y)^\wedge_p \to (\tilde{Z}_{f(y)})^\wedge_p$ for $y \in Y^G$ are all G-homotopy equivalences and so induce bijections on application of the functor $[X,?]_G$.

The following result is the represented equivalent of Theorem 3 and was proven in [7]. (The maps studied in [7] were defined a bit differently, but an easy diagram chase gives the conclusion in the form stated.) Recall that a G-map $f\colon Y \to Z$ is said to be a weak G-equivalence if each $f^H\colon Y^H \to Z^H$ is a weak equivalence and that $f_*\colon [X, Y]_G \to [X, Z]_G$ is then a bijection for any G-CW complex X.

THEOREM 10. *If Π and Γ are compact Lie groups with Π Abelian, then $\alpha\colon B_G(\Pi) \to Map(EG, B\Pi)$ is a weak G-equivalence.*

As a final remark, we give an equivariant generalization of the usual Borel construction model for the classifying space of an extension.

LEMMA 11. *Let $\Lambda \subset \Pi \subset \Gamma$, where Λ and Π are normal subgroups of the topological group Γ. Then, as (Γ/Π)-spaces,*

$$B(\Pi;\Gamma) \simeq E(\Pi/\Lambda;\ \Gamma/\Lambda) \times_{\Pi/\Lambda} B(\Lambda;\Gamma).$$

PROOF. For $\Omega \subset \Gamma$, $\Omega \cap \Pi = e$ if and only if both $\Omega \cap \Lambda = e$ and $\Theta \cap (\Pi/\Lambda) = e$, where Θ is the image of Ω in Π/Λ. Therefore, as Γ-spaces,

$$E(\Pi;\Gamma) \simeq E(\Pi/\Lambda;\ \Gamma/\Lambda) \times E(\Lambda;\Gamma)$$

by the characteristic behavior on fixed point sets. Now pass to Π-orbits by first passing to Λ-orbits and then to (Π/Λ)-orbits.

APPENDIX

Let G and Π be compact Lie groups. A Π-bundle over X/G may be regarded as a G-trivial $(\Pi;G \times \Pi)$-bundle, and it determines a $(\Pi;G \times \Pi)$-bundle over X by pullback. This gives a natural map

$$\zeta\colon \mathcal{B}(\Pi)(X/G) \to \mathcal{B}_G(\Pi;G \times \Pi)(X).$$

When Π is Abelian, the false proof of Assertion 4 to be described here would show that ζ is a naturally split injection.

The complementary factor would be $\mathrm{Nat}(\pi_0(X), R_\pi)$, which we proceed to define. Let $\mathcal{O}$ be the topological category of orbit G-spaces G/H and G-maps between them. Let $h\mathcal{O}$ be its homotopy category. For any n and any G-space X, there is an evident contravariant functor $\pi_n(X)\colon h\mathcal{O} \to \mathrm{Sets}$ which sends G/H to $\pi_n(X^H)$. There is also a contravariant functor $R_\pi\colon \mathcal{O} \to \mathrm{Sets}$ which sends G/H to the set of Π-conjugacy classes of Lie group homomorphisms $H \to \Pi$; R_π factors through $h\mathcal{O}$ since homotopic homomorphisms lie in the same Π-conjugacy class by the Montgomery-Zippin theorem [2, 38.1]. Let $\mathrm{Nat}(\pi_0(X), R_\pi)$ be the set of natural transformations $\pi_0(X) \to R_\pi$.

A principal $(\Pi;G \times \Pi)$-bundle over G/H determines and is determined by an element of $R_\pi(G/H)$. A principal $(\Pi;G \times \Pi)$-bundle over X determines a natural transformation $\pi_0(X) \to R_\pi$ by pulling the bundle back along G-maps $G/H \to X$ which represent elements of $\pi_0(X^H)$. This gives a natural map

$$\rho\colon \mathcal{B}(\Pi;G \times \Pi)(X) \to \mathrm{Nat}(\pi_0(X), R_\pi).$$

When Π is Abelian, the false proof of Assertion 4 would show that ρ is a naturally split surjection. A left inverse λ would construct a global bundle over X from compatible bundles over the domains of the representative G-maps $G/H \to X$. Given λ, a natural bijection

$$\mathcal{B}(\Pi)(X/G) \times \mathrm{Nat}(\pi_0(X), R_\pi) \to \mathcal{B}G(\Pi;G \times \Pi)(X)$$

would be obtained by using the Abelian structure of Π to add bundles in the images of the transformations ζ and λ.

The following is the represented equivalent of Assertion 4.

ASSERTION 12. *There is a weak G-equivalence*

$$B\Pi \times K(R_\pi,0) \to Map(EG, B\Pi),$$

where G *acts trivially on* $B\Pi$.

To explain this assertion, we must say a bit about diagrams of G-spaces and about Eilenberg-MacLane G-spaces $K(\pi,0)$. Define an $\mathcal{O}$-space to be a continuous contravariant functor from $\mathcal{O}$ to the category of spaces; a map of $\mathcal{O}$-spaces is a natural transformation. A G-space X determines the $\mathcal{O}$-space ΦX specified by $(\Phi X)(G/H) = X^H$. Conversely, by Elmendorf [4, Thm1], an $\mathcal{O}$-space T determines a G-space ΨT and an $\mathcal{O}$-map $\varepsilon\colon \Phi\Psi T \to T$ such that each component map $\varepsilon\colon (\Psi T)^H \to T(G/H)$ is a homotopy equivalence. In particular, with $H = e$ and $T = \Phi X$, the G-map $\varepsilon\colon \Psi\Phi X \to X$ is a weak G-equivalence. With the evident notion of homotopy in the category of $\mathcal{O}$-spaces, a slight refinement of [3, Thm 2] gives an adjunction on the level of homotopy classes of maps

$$(\perp) \qquad [X, \Psi T]_G \cong [\Phi X, T]_{\mathcal{O}}$$

when X has the homotopy type of a G-CW complex.

A space Y is homotopically discrete if each of its components is contractible, that is, if the discretization map $\delta\colon Y \to \pi_0 Y$ is a homotopy equivalence. A G-space Y is homotopically discrete if each Y^H is homotopically discrete. These are the $K(\pi,0)$'s referred to above, where π is a continuous functor from $\mathcal{O}$ to discrete spaces or, equivalently, a functor from the homotopy category $h\mathcal{O}$ to sets. Given such a functor π, we can construct $K(\pi,0)$ by setting $K(\pi,0) = \Psi\pi$; $(\perp)$ and the discreteness of π then give

$$[X, K(\pi,0)]_G \cong [\Phi X, \pi]_{\mathcal{O}} \cong \mathrm{Nat}(\pi_0(X), \pi).$$

Since we obviously have $[X, B\Pi]_G \cong [X/G, B\Pi]$, it is now clear that Assertion 12 implies Assertion 4.

For a G-space X, the discretization maps of fixed point spaces specify an $\mathcal{O}$-map $\delta\colon \Phi X \to \pi_0(X)$, and application of Ψ therefore gives a natural G-map $X \simeq \Psi\Phi X \to K(\pi_0(X),0)$. It seems reasonable to expect this map to admit a section, but it usually doesn't. To obtain a section, it would suffice to obtain a right inverse $\pi_0(X) \to \Phi X$ to δ, but there is usually no such natural choice of basepoints of components of fixed point spaces. This train of thought leads to a

"PROOF OF ASSERTION 12". The intuition is that there should be such a section of δ when $X = \mathrm{Map}(EG, B\Pi)$. With the standard functorial construction of EG, we have the two continuous covariant functors B and B' from $\mathcal{O}$ to spaces specified on objects by $B(G/H) = EH/H$ and $B'(G/H) = EG/H$. We may identify ΦX with the contravariant functor $\mathrm{Map}(B', B\Pi)$. Therefore

(A) $$\mathrm{Map}(EG, B\Pi) \simeq \Psi\Phi\mathrm{Map}(EG, B\Pi) = \Psi\mathrm{Map}(B', B\Pi).$$

On the other hand, passage to classifying maps defines an $\mathcal{O}$-map $\beta\colon R_\pi \to \mathrm{Map}(B, B\Pi)$. By [7, Prop. 4],

$$B\colon \mathrm{Hom}(G, \Pi) \to [BG, B\Pi]$$

is a bijection. Therefore $\pi_0\mathrm{Map}(B, B\Pi) = R_\pi$ and we have a map

(B) $$\Psi\beta\colon K(R_\pi,0) \to \Psi\mathrm{Map}(B, B\Pi).$$

It seems reasonable to expect there to be a weak G-equivalence

(C) $$\Psi\mathrm{Map}(B, B\Pi) \simeq \Psi\mathrm{Map}(B', B\Pi).$$

Given this, $\Psi\beta$ would transport under the equivalences (A) and (C) to give the desired section

$$\lambda\colon K(R_\pi,0) \to \mathrm{Map}(EG, B\Pi).$$

Letting $\varsigma\colon B\Pi \to \mathrm{Map}(EG, B\Pi)$ be induced by the projection $EG \to pt$ and φ be the product on $\mathrm{Map}(EG, B\Pi)$ induced by the product on the topological Abelian group $B\Pi$, the composite

$$\varphi\circ(\varsigma,\lambda)\colon B\Pi \times K(R_\pi,0) \to \mathrm{Map}(EG, B\Pi)$$

would then be a weak G-equivalence (compare [7, p.173]).

In fact, (C) fails. The obvious way to try to prove (C) would be to exploit the equivalences $B(G/H) = EH/H \to EG/H = B'(G/H)$ induced by the inclusions $EH \to EG$. However, these equivalences fail to define a map $B \to B'$ of $\mathcal{O}$-spaces. The requisite naturality fails, as we see by taking $H = e$ and observing that the map from the point Ee into EG cannot be a G-map.

Assertion 12 would imply an incorrect calculation of the characteristic classes of principal $(\Pi, G \times \Pi)$-bundles in Bredon cohomology. For a commutative ring k, a k-module valued coefficient system is a contravariant functor from $h\mathcal{O}$ to the category of k-modules. Write $\mathrm{Ext}^*_{h\mathcal{O}}$ for the Ext functor in the resulting Abelian category of $h\mathcal{O}$-k-modules. For a contravariant functor $\pi\colon h\mathcal{O} \to \mathrm{Sets}$, let $k\pi$ denote the $h\mathcal{O}$-k-module obtained by letting $k\pi(G/H)$ be the free k-module generated by $\pi(G/H)$. Let G and Π be compact Lie groups with Π Abelian and let M be an $h\mathcal{O}$-k-module, where k is a commutative ring such that $H^*(B\Pi; k)$ is k-free. Then there is a universal coefficients spectral sequence converging from $H^*(B\Pi; k) \otimes_k \mathrm{Ext}^*_{h\mathcal{O}}(kR_\pi, M)$ to $H^*_G(B_G(\Pi); M)$ [11]. Assertion 12 would imply that $E_2 = E_\infty$ in this spectral sequence, and this conclusion is usually false.

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Astérisque

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Homologie de Hochschild et homologie cyclique des algèbres différentielles graduées

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HOMOLOGIE DE HOCHSCHILD ET HOMOLOGIE CYCLIQUE DES ALGÈBRES DIFFÉRENTIELLES GRADUÉES

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La notion d'homologie de Hochschild pour une algèbre associative A sur un anneau commutatif unitaire k est bien connue, [Mc]. Elle est notée $HH_*(A)$ et définie par $HH_*(A) = \mathrm{Tor}^{A\otimes A^{op}}(A, A)$ où A^{op} est l'algèbre opposée de A .

Cette notion a été étendue à la catégorie des algèbres associatives différentielles graduées sur un anneau commutatif k (notée k-ADG) par plusieurs auteurs, [B1],[G].

La notion d'homologie cyclique, notée $HC_*(\cdot)$, est apparue plus récemment ; on trouvera un exposé complet dans [LQ] pour le cas des algèbres ; et pour la catégorie k-ADG dans [B1] ou [G].

Ce papier contient une généralisation à la catégorie k-ADG (où k est un corps quelconque) du résultat de Loday-Quillen du calcul explicite de l'homologie de Hochschild et de l'homologie cyclique pour une algèbre tensorielle. Nous fournissons un algorithme de calcul précis de l'homologie de Hochschild et de l'homologie cyclique pour une algèbre différentielle graduée libre. Le résultat général s'énonce ainsi :

Théorèmes 1.5 et 2.4 *Soit $(A,d) = (T(V),d)$ une algèbre différentielle graduée libre sur un corps commutatif k . Alors, on a des isomorphismes d'espaces vectoriels gradués :*

(1) $HH_*(A,d) = H_*(A \oplus (A\otimes \bar{V}), \delta)$ où $\bar{V}_n = V_{n-1}$, $\delta_{|A} = d$, $\delta(a\otimes \bar{v}) = da \otimes \bar{v} - S(a, dv) + (-1)^{|a|+|\bar{v}|}(av - (-1)^{|a|\cdot|v|}va)$ et S est l'application k-linéaire :

$$S(a, v_1\cdots v_p) = (-1)^{|a|}\sum_{i=1}^{p-1}(-1)^{\varepsilon_i} v_{i+1}\cdots v_p a v_1\cdots v_{i-1}\otimes \bar{v}_i + (-1)^{|a|} a v_1 \cdots v_{p-1}\otimes \bar{v}_p \, .$$

(2) $HC_*(A,d) = H_*(k[u]\otimes (A\oplus A\otimes \bar{V}), D)$ où $|u| = 2$, $D = 0$ sur $k[u]$, $D = \delta$ sur $A \oplus (k[u]\otimes (A\otimes \bar{V}))$ et

$$D(u^n\otimes v_1\cdots v_p) = u^n\otimes d(v_1\cdots v_p) + u^{n-1}\otimes [v_1\cdots v_{p-1}\otimes \bar{v}_p + \sum_{i=1}^{p-1}(-1)^{\mu_i} v_{i+1}\cdots v_p v_1\cdots v_{i-1}\otimes \bar{v}_i]$$

si $n \geq 1, p \geq 1$.

La motivation d'un tel travail est double : d'une part, il est facile de montrer que si (A,d) est une ADG quelconque, alors il existe une ADG libre $(T(V),d)$ et un morphisme $\rho : (T(V),d) \to$

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(A,d) qui induit un isomorphisme en homologie, et, par un résultat classique, [B1], les ADG (A,d) et $(T(V),d)$ ont des homologies de Hochschild et cycliques isomorphes ; le calcul, dans le cas général, se ramène donc, au calcul pour les ADG libres. D'autre part, en topologie algébrique, le calcul de l'homologie (resp. l'homologie équivariante)de l'espace des lacets libres sur un espace donné X se ramène à un calcul d'homologie de Hochschild (resp. cyclique) d'invariants topologiques liés à X, [BF],[G], [J]. Enfin, la théorie de Morse permet de montrer des résultats concernant la géométrie d'une variété riemannienne à partir uniquement de l'étude de la cohomologie de l'espace des lacets libres sur cette variété. Ceci explique notre recherche d'un "modèle" permettant de calculer l'homologie de l'espace des lacets libres sur un corps k quelconque. En caractéristique 0 , le problème a été complètement résolu dans [SV], en travaillant dans la catégorie des algèbres commutatives différentielles graduées. En caractéristique p non nulle, il est possible, dans certains cas, de travailler encore dans la catégorie des algèbres commutatives graduées, [HV].

Le plan de l'article est le suivant : Dans le § .1, nous donnons quelques rappels d'algèbre différentielle homologique, et nous définissons un complexe dont l'homologie calcule l'homologie de Hochschild. Dans le § .2, nous définissons sur le complexe précédent un opérateur β de degré $+1$; le complexe mixte ainsi obtenu permet de calculer l'homologie cyclique, cf. [K]. Dans le § .3, nous donnons une méthode de calcul de l'homologie (resp. de l'homologie équivariante) de l'espace des lacets libres sur un espace X , à valeurs dans un corps commutatif k , à partir de la donnée de l'algèbre $C_*(\Omega X,k)$ des chaînes sur l'espace des lacets ΩX ou de l'algèbre des cochaînes $C^*(X,k)$. Si X est un espace simplement connexe de L-S catégorie 1 , alors on a une formule explicite pour l'homologie (resp. l'homologie équivariante) de l'espace des lacets libres sur X . Elles coïncident avec celles données par Hsiang et Staffeldt [HS], et Burghelea [B1] pour X une suspension et k un corps de caractéristique 0. Cette formule figure aussi dans [CC]. Pour les sphères, un calcul analogue se trouve dans [H]. Notre modèle permet de calculer explicitement l'homologie de l'espace des lacets libres sur $X = \mathbf{C}P^2$.

Un problème reste ouvert. Peut-on montrer, en utilisant le modèle du § .3, la célèbre conjecture :

Conjecture : Soit X un espace simplement connexe tel que $H^*(X,k)$ soit de dimension finie (k corps quelconque).

Alors, la suite des nombres de Betti de l'homologie de l'espace des lacets libres à valeurs dans k n'est pas bornée si et seulement si $H^*(X,k)$ ne peut pas être engendrée par un seul élément en tant qu'algèbre commutative graduée.

Cette conjecture a été complètement résolue en caractéristique 0 , [SV]. En caractéristique $p \neq 0$, elle a été résolue dans certains cas, voir [HV].

§.1. Homologie de Hochschild d'une algèbre différentielle graduée libre

Les définitions de base en algèbre homologique différentielle se trouvent, par exemple, dans [HMS],[FHT1].

Tous les espaces vectoriels sont définis sur un corps commutatif k et sont $\mathbf{Z}$-gradués, avec la convention que $M^n = M_{-n}$ si $\oplus_n M^n$ est un k-espace vectoriel gradué. La valeur absolue du degré d'un élément x est notée $|x|$.

On dit que (A, d) est une k-algèbre différentielle graduée (en abrégé ADG), si $A = \oplus_{n\in\mathbf{Z}} A_n$ est un k-espace vectoriel gradué, muni d'une structure d'algèbre associative unitaire sur k telle que $A_n \cdot A_m \subset A_{n+m}$. De plus, d est une dérivation de k-algèbre de degré ± 1 vérifiant $d^2 = 0$. Dans la suite, on considérera uniquement, ou bien des algèbres différentielles graduées A_* avec $A_n = 0$ pour $n < 0$ et d de degré -1 , ou bien des algèbres différentielles graduées A^* avec $A^n = 0$ pour $n < 0$ et d de degré $+1$. Une ADG de ce dernier type sera étudiée comme une ADG (A_{-*}, d_{-*}) uniquement graduée en degrés négatifs et munie d'une différentielle de degré -1 . Si $V = \oplus_{n\in\mathbf{Z}} V_n$ est un k-espace vectoriel gradué, on note $T(V)$ l'algèbre associative libre construite sur V . Si (A, d) est une ADG, on définit l'ADG (A^{op}, d^{op}) par $A^{op} \simeq A$, $a^{op} \cdot b^{op} = (-1)^{|a|\cdot|b|}(ba)^{op}, d^{op}(a^{op}) = (da)^{op}$.

L'application $F_g : (A \otimes A^{op}) \otimes A \to A$ définie par $F_g(\alpha \otimes \beta^{op}, \gamma) = (-1)^{|\beta|\cdot|\gamma|}\alpha\gamma\beta$ munit A d'une structure de $(A \otimes A^{op})$-module différentiel gradué à gauche.

L'application $F_d : A \otimes (A \otimes A^{op}) \to A$ définie par $F_d(\gamma, \alpha \otimes \beta^{op}) = (-1)^{|\beta|(|\alpha|+|\gamma|)}\beta\gamma\alpha$ munit A d'une structure de $(A \otimes A^{op})$-module différentiel gradué à droite.

Définition 1.1 [B1],[G]. *Soit (A_*, d_*) une algèbre différentielle graduée telle que, ou bien $A_n = 0$ pour tout $n < 0$, ou bien $A_n = 0$ pour tout $n > 0$ et $A_o = k$, alors on définit l'homologie de Hochschild de (A_*, d_*) , notée $HH_*(A_*, d_*)$ par $HH_*(A_*, d_*) = \mathrm{Tor}^{A\otimes A^{op}}(A, A)$.*

Définition 1.1.' *Soit (A^*, d^*) une algèbre différentielle graduée telle que $A^o = k$, $A^* = \oplus_{n\geq 0} A^n$, et d^* de degré $+1$, on définit l'homologie de Hochschild $HH^*(A^*, d^*)$ par*

$$HH^*(A^*, d^*) = HH_{-*}(A_{-*}, d_{-*}) .$$

La définition du foncteur Tor dans la catégorie différentielle se trouve, par exemple, dans [FHT1] et ne sera pas rappelée.

Si (A, d) est une algèbre différentielle graduée vérifiant les hypothèses de la définition 1.1, on a alors $HH_*(A, d) = H_*(A \otimes_{A\otimes A^{op}} P)$ où $P \to A$ est une résolution quelconque semi-libre de A par des $(A \otimes A^{op})$-modules différentiels gradués à gauche. Rappelons que si R est une algèbre différentielle graduée, un R-module est dit libre si c'est un $R_{\#}$-module libre sur une base de cycles (où $R_{\#}$ est l'algèbre graduée sous-jacente à R), et un R-module P est dit semi-libre s'il est une réunion croissante de sous-modules $0 = P_{-1} \subset P_0 \subset \cdots$ tel que chaque P_i/P_{i-1} soit libre.

Dans le cas particulier où $A = T(V)$, avec V gradué en degré 0 et $d = 0$, rappelons le résultat de Loday-Quillen (lemme 5.1), on a une suite exacte de $(A \otimes A^{op})$-modules : $0 \to A \otimes V \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{m} A \to 0$ où $m(a \otimes a') = aa'$, $b'(a \otimes v \otimes a') = av \otimes a' - a \otimes va'$, ce qui leur permet de calculer facilement l'homologie de Hochschild et l'homologie cyclique de $T(V)$.

Nous supposons maintenant que $(A, d) = (T(V), d)$ est une ADG libre vérifiant les hypothèses de la définition 1.1. Nous utilisons une version graduée de la résolution de [L-Q] pour construire une résolution semi-libre de A par des $(A \otimes A^{op})$-modules différentiels gradués.

Il est immédiat de vérifier que, si $W = \oplus_n W_n$ est un k-espace vectoriel gradué, alors $A \otimes W \otimes A$ est un $(A \otimes A^{op})$-module différentiel gradué à gauche, si on pose $(\alpha \otimes \beta^{op}) \cdot (a \otimes w \otimes a') = (-1)^{|\beta| \cdot [|a|+|w|+|a'|]} \alpha a \otimes w \otimes a' \beta$, pour $\alpha, \beta, a, a' \in A, w \in W$. On a : $a \otimes w \otimes a' = (-1)^{|a'| \cdot |w|} (a \otimes a'^{op}) \cdot (1 \otimes w \otimes 1)$.

Soit $A = T(V)$ une algèbre tensorielle graduée, posons $\bar{V} = \oplus \bar{V}_n$ où $\bar{V}_n = V_{n-1}$. On définit $m : A \otimes A \to A$ par $m(a \otimes a') = aa'$, $\varepsilon : A \to A \otimes A$ par $\varepsilon(a) = a \otimes 1$, $\tilde{b}' : A \otimes \bar{V} \otimes A \to A \otimes A$ par $\tilde{b}'(a \otimes \bar{v} \otimes a') = (-1)^{|a'|}(av \otimes a' - a \otimes va')$ et $s : A \otimes A \to A \otimes \bar{V} \otimes A$ par $s(\alpha \otimes v_1 \cdots v_p) = \sum_{i=1}^{p-1} (-1)^{1+\sum_{k=i+1}^{p} |v_k|} \alpha v_1 \cdots v_{i-1} \otimes \bar{v}_i \otimes v_{i+1} \cdots v_p - \alpha v_1 \cdots v_{p-1} \otimes \bar{v}_p \otimes 1$, si $\alpha \in A$, $v_1 \cdots v_p \in T^p(V)$, et $s = 0$ sur $A \otimes k$.

Lemme 1.2 *Soit le diagramme suivant :* $A \otimes \bar{V} \otimes A \underset{s}{\overset{\tilde{b}'}{\rightleftarrows}} A \otimes A \underset{\varepsilon}{\overset{m}{\rightleftarrows}} A$ *où* $\tilde{b}', s, m, \varepsilon$ *ont été définis ci-dessus. Alors*

1) $m\varepsilon = \mathrm{Id}_A$, $\tilde{b}' s + \varepsilon m = \mathrm{Id}_{A \otimes A}$, $s \tilde{b}' = \mathrm{Id}_{A \otimes \bar{V} \otimes A}$

2) La suite $0 \to A \otimes \bar{V} \otimes A \xrightarrow{\tilde{b}'} A \otimes A \xrightarrow{m} A \to 0$ *est une résolution de* A *par des* $(A \otimes A^{op})$*-modules gradués.*

■ Il est facile de vérifier que $\tilde{b}', m, \varepsilon, s$ sont des morphismes de $(A \otimes A^{op})$-modules gradués. De plus, 1) se montre immédiatement et prouve que (ε, s) est une homotopie entre Id et 0 , ce qui donne 2). ■

Soit maintenant $(A, d) = (T(V), d)$ une ADG libre; nous allons définir une différentielle d_1 sur $A \otimes \bar{V} \otimes A$ qui fasse de $(A \otimes \bar{V} \otimes A, d_1)$ un $(A \otimes A^{op})$-module différentiel gradué et qui fasse de $\tilde{b}'$ un morphisme de modules différentiels gradués.

Si $a, a' \in A, \bar{v} \in \bar{V}$, on a : $a \otimes \bar{v} \otimes a' = (-1)^{|a'| \cdot |\bar{v}|} (a \otimes a'^{op}) \cdot (1 \otimes \bar{v} \otimes 1)$. Il suffit donc de définir $d_1(1 \otimes \bar{v} \otimes 1)$ et de l'étendre par :

$$(-1)^{|a'| \cdot |\bar{v}|} d_1(a \otimes \bar{v} \otimes a') = d(a \otimes a'^{op}) \cdot (1 \otimes \bar{v} \otimes 1) + (-1)^{|a|+|a'|} (a \otimes a'^{op}) \cdot d_1(1 \otimes \bar{v} \otimes 1) .$$

On a $d\tilde{b}'(1 \otimes \bar{v} \otimes 1) = d(v \otimes 1 - 1 \otimes v) = dv \otimes 1 - 1 \otimes dv$, mais $1 \otimes \bar{v} \otimes 1 = -s(1 \otimes v)$, donc $d\tilde{b}'(1 \otimes \bar{v} \otimes 1) = -d\tilde{b}' s(1 \otimes v)$. D'après le lemme 1.2, $d\tilde{b}'(1 \otimes \bar{v} \otimes 1) = -d(1 \otimes v) + d\varepsilon m(1 \otimes v) =$

$-1 \otimes dv + \varepsilon m(1 \otimes dv) = \tilde{b}'(-s(1 \otimes dv))$. Posons $d_1(1 \otimes \bar{v} \otimes 1) = -s(1 \otimes dv)$, soit $d_1 s(1 \otimes v) = s(1 \otimes dv)$.

$$(*) \qquad d_1(a \otimes \bar{v} \otimes a') = da \otimes \bar{v} \otimes a' + (-1)^{|a|+\bar{v}|} a \otimes \bar{v} \otimes da' + (-1)^{|a|+|a'|\cdot|v|+1} s(a \otimes (dv)a') .$$

Lemme 1.3 *(1) La formule (*) définit une différentielle d_1 sur $A \otimes \bar{V} \otimes A$ qui en fait un $(A \otimes A^{op})$-module différentiel.*

(2) On a $\tilde{b}' d_1 = d\tilde{b}'$ et $sd = d_1 s$ où d désigne la différentielle produit sur $A \otimes A$.

■ (1) est vrai par construction et (2) est vérifié sur $k \otimes \bar{V} \otimes k$, donc partout. ■

Posons $P_n = (A \otimes A)_n \oplus (A \otimes \bar{V} \otimes A)_n$, $P = \oplus_n P_n$, $D_{|A \otimes A} = d$, $D(\xi) = d_1(\xi) + (-1)^{|\xi|} \tilde{b}'(\xi)$ si $\xi \in A \otimes \bar{V} \otimes A$. On définit $\Phi : (P, D) \to (A, d)$ par $\Phi = m$ sur $A \otimes A$, $\Phi = 0$ sur $A \otimes \bar{V} \otimes A$.

Théorème 1.4 *Soit $(A, d) = (T(V), d)$ une ADG libre, et soit $\Phi : (P, D) \to (A, d)$ défini ci-dessus, alors $((P, D), \Phi)$ est une résolution semi-libre de (A, d) par des $(A \otimes A^{op})$-modules différentiels gradués.*

■ D est un morphisme de $(A \otimes A^{op})$-modules puisque $d_1, \tilde{b}'$, et d le sont. Si $\xi \in A \otimes \bar{V} \otimes A$, on a

$$D(D(\xi)) = D(d_1(\xi) + (-1)^{|\xi|} \tilde{b}'(\xi)) = d_1 \circ d_1(\xi) + (-1)^{|\xi|+1} \tilde{b}' d_1(\xi)$$
$$+ (-1)^{|\xi|} d\tilde{b}'(\xi) = (-1)^{|\xi|} [d\tilde{b}'(\xi) - \tilde{b}' d_1(\xi)] = 0 ,$$

donc $D^2 = 0$. Il est clair que (P, D) est un module semi-libre. Il reste à montrer que $\Phi_* : H_*(P, D) \to H_*(A, d)$ est un isomorphisme. Cela découle d'un argument classique de suites spectrales compte-tenu des lemmes 1.2 et 1.3. ■

Par définition du Tor différentiel, on a :

$$\mathrm{Tor}^{A \otimes A^{op}}(A, A) = H_*(A \otimes_{A \otimes A^{op}} P, d \otimes_{A \otimes A^{op}} D) .$$

Il est facile de vérifier que l'application $\theta : A \otimes_{A \otimes A^{op}} [(A \otimes A) \oplus (A \otimes \bar{V} \otimes A)] \to A \oplus (A \otimes \bar{V})$ définie ci-dessous est un isomorphisme de k-modules gradués.

$$\theta(\alpha \otimes_{A \otimes A^{op}} (b \otimes b')) = (-1)^{|b'|\cdot[|\alpha|+|b|]} b' \alpha b \text{ , si } b, b', \alpha \in A$$

$$\theta(\alpha \otimes_{A \otimes A^{op}} (b \otimes \bar{v} \otimes b')) = (-1)^{|b'|\cdot[|\alpha|+|b|+|\bar{v}|]} b' \alpha b \otimes \bar{v} \text{ si } \bar{v} \in \bar{V} .$$

Théorème 1.5 *Soit $(A, d) = (T(V), d)$ une algèbre différentielle graduée libre sur un corps commutatif k telle que, ou bien $V = \oplus_{n \geq 0} V_n$, ou bien $V = \oplus_{n \leq -1} V_n$, posons $\bar{V} = \oplus \bar{V}_n$ et $\bar{V}_n = V_{n-1}$. Alors il y a un isomorphisme de k-espaces vectoriels gradués : $HH_*(A, d) \simeq H_*(A \oplus (A \otimes \bar{V}), \delta)$ où la différentielle δ est donné par $\delta_{|A} = d$, $\delta(a \otimes \bar{v}) = da \otimes \bar{v} - S(a, dv) + (-1)^{|a|+|\bar{v}|}(av - (-1)^{|a||v|} va)$. L'application $S : A \otimes A \to A \otimes \bar{V}$ est k-linéaire et définie par*

$S(a, v_1 \dots v_p) = (-1)^{|a|} \sum_{i=1}^{p-1} (-1)^{\varepsilon_i} v_{i+1} \dots v_p a v_1 \dots v_{i-1} \otimes \bar{v}_i + (-1)^{|a|} a v_1 \dots v_{p-1} \otimes \bar{v}_p$ *et* $\varepsilon_i = (|v_{i+1}| + \dots + |v_p|)(|a| + |v_1| + \dots + |v_{i-1}|)$ *, si* $a \in A, v_i \in V$ *.*

■ Il suffit de transporter à $A \oplus (A \otimes \bar{V})$, à l'aide de l'isomorphisme θ , la formule de la différentielle $d \otimes_{A \otimes A^{op}} D$. En particulier, $S(a, dv) =$
$(-1)^{|a|} \theta(a \otimes_{A \otimes A^{op}} s(1 \otimes dv))$. ■

Pour tout $m \geq 1$, définissons la permutation cyclique $\tau_m : V^{\otimes m} \to V^{\otimes m}$ par $\tau_m(v_1 \otimes \dots \otimes v_m) = (-1)^{|v_m| \cdot [|v_1| + \dots + |v_{m-1}|]} v_m \otimes v_1 \otimes \dots \otimes v_{m-1}$ et on pose $\tau_0 = Id$.

Remarque 1.6 L'application $S : V^{\otimes m} \otimes V^{\otimes p} \to V^{\otimes m+p-1} \otimes \bar{V}$ définie pour $m \geq 0$, $p \geq 1$, est, au signe près, égale à :

$$\sigma \circ \left[Id + (\tau_{m+1} \otimes Id) \circ (Id \otimes \tau_p) + (\tau_{m+1}^2 \otimes Id) \circ (Id \otimes \tau_p^2) + \dots + (\tau_{m+1}^{p-1} \otimes Id) \circ (Id \otimes \tau_p^{p-1}) \right]$$

où σ est l'isomorphisme $V^{\otimes m+p} \to V^{\otimes (m+p-1)} \otimes \bar{V}$ qui envoie $v_1 \dots v_{m+p-1} v_{m+p}$ sur $v_1 \dots v_{m+p-1} \otimes \bar{v}_{m+p}$.

Le théorème 1.5 s'énonce ainsi dans le cas où $d = 0$.

Théorème 1.7 *Soit* $A = T(V)$ *une algèbre tensorielle graduée sur un corps commutatif* k *quelconque, avec, ou bien* $V_n = 0$ *pour* $n < 0$ *, ou bien* $V_n = 0$ $n \geq 0$ *, alors l'homologie de Hochschild* $HH_*(T(V))$ *se décompose en la somme directe de deux* k*-espaces vectoriels gradués :* $\bigoplus_{m \geq 0} (V^{\otimes m} / (\mathrm{Id} - \tau_m))$ *et* $\bigoplus_{m \geq 1} \overline{\mathrm{Ker}(\mathrm{Id} - \tau_m)}$ *, où* $\overline{\mathrm{Ker}(\ )_n} = \mathrm{Ker}(\ )_{n-1})$ *.*

§.2. Homologie cyclique d'une algèbre différentielle graduée libre [B1],[BV],[G],[J].

Soit (A, d) une algèbre associative sur un corps commutatif k vérifiant les hypothèses de la définition 1.1. On définit le complexe de Hochschild bigradué $(C_{p,q}(A), d, b)$, $p \geq 0$, par $C_{pq} = \bigoplus_{i_0 + i_1 \dots + i_p = q} A_{i_0} \otimes A_{i_1} \otimes \dots \otimes A_{i_p}$, $d(a_{i_0} \otimes \dots \otimes a_{i_p}) = (-1)^p \sum_{k=0}^p (-1)^{i_0 + \ + i_{k-1}} a_{i_0} \otimes a_{i_1} \dots \otimes da_{i_k} \otimes \dots \otimes a_{i_p}$, si $a_{i_j} \in A_{i_j}$.

$$b(a_{i_0} \otimes \dots \otimes a_{i_p}) = \sum_{k=0}^{p-1} (-1)^k a_{i_0} \otimes \dots \otimes a_{i_k} a_{i_{k+1}} \otimes \dots \otimes a_{i_p}$$
$$+ (-1)^{p + i_p(i_0 + \dots + i_{p-1})} a_{i_p} a_{i_0} \otimes \dots \otimes a_{i_{p-1}}$$

Proposition et Définition 2.1 $(C_{p,q}(A), d, b)$ *est un complexe bigradué appelé complexe de Hochschild bigradué. L'homologie du complexe total associé* $(C_* = \oplus_n C_n, d + b)$ *(où* $C_n = \bigoplus_{p+q=n} C_{p,q}$*) est isomorphe à l'homologie de Hochschild de* (A, d) *.*

Soit $N : C_{p,q} \to C_{p,q}$ défini par $N = \sum_{k+1}^p ((-1)^p \tau_{p+1})^k$, où $\tau_{p+1}(a_{i_0} \otimes a_{i_1} \dots \otimes a_{i_p}) = (-1)^{i_p(i_0 + \dots + i_{p-1})} a_{i_p} \otimes a_{i_0} \dots \otimes a_{p-1}$. Soit $s : C_{p,q} \to C_{p+1,q}$ définie par $s(a_0 \otimes \dots \otimes a_p) = 1 \otimes a_0 \otimes \dots \otimes a_p$. Posons $B : C_{p,q} \to C_{p+1,q}$, $B = (\mathrm{Id} - (-1)^{p+1} \tau_{p+2}) \circ s \circ N$.

On vérifie que $B^2 = Bb + bB = Bd + dB = 0$. On peut former le bicomplexe :

$$
\begin{tikzcd}
 & \vdots \arrow[d, "b+d"] & \vdots \arrow[d, "b+d"] & \vdots \arrow[d, "b+d"] \\
\cdots & \mathcal{C}_2 \arrow[l] \arrow[d, "b+d"] & \mathcal{C}_1 \arrow[l, "B"'] \arrow[d, "b+d"] & \mathcal{C}_0 \arrow[l, "B"'] \\
\cdots & \mathcal{C}_1 \arrow[l] \arrow[d, "b+d"] & \mathcal{C}_0 \arrow[l, "B"'] & \\
 & \mathcal{C}_0 & &
\end{tikzcd}
$$

dont le complexe total sera noté $k[u]\otimes_B \mathcal{C}$, ([K]), où $k[u]$ est l'algèbre de polynômes à un générateur de degré 2.

Définition 2.2 *L'homologie du complexe total $k[u]\otimes_B \mathcal{C}$ associé au complexe de Hochschild bigradué est appelée homologie cyclique de l'algèbre différentielle graduée (A,d) et notée $HC_*(A,d)$.*

Remarque : Si $A = \bigoplus_{n\geq 0} A_n$, on retrouve la définition de [B1] ou [G]. Si $A = (\bigoplus_{n\geq 0} A^n, d)$, $A_0 = k, d$ de degré $+1$, alors $HC_*(A_{-*}, d_{-*})$ est uniquement graduée en degrés négatifs ou nuls et coïncide avec l'homologie cyclique négative $HC^-(A,d)$ introduite par Jones, [J].

Rappelons brièvement la définition de la résolution standard de Hochschild de A comme $(A\otimes A^{op})$-module differentiel, (voir [Mo],§.6). Posons pour $n \geq 0$, $B'_{n,*} = A\otimes (A^{\otimes n})\otimes A$, et $b' : B'_{n,*} \to B'_{n-1,*}$ est défini par :

$$
\begin{aligned}
&b'(a\otimes(\lambda_1\otimes\cdots\otimes\lambda_n)\otimes a') = a\lambda_1\otimes\lambda_2\cdots\otimes\lambda_n\otimes a' \\
&+\sum_{k=1}^{n-1}(-1)^k a\otimes\lambda_1\otimes\cdots\otimes\lambda_k\lambda_{k+1}\otimes\cdots\otimes\lambda_n\otimes a' + (-1)^n a\otimes\lambda_1\cdots\lambda_{n-1}\otimes\lambda_n a' .
\end{aligned}
$$

Soit $\mathcal{B}'_* = \oplus_n \mathcal{B}'_n$ où $\mathcal{B}'_n = \oplus_{p+q=n} B'_{pq}$ et $\Phi' : \mathcal{B}'_* \to A$ définie par $\Phi'_{|B'_{p,*}} = 0$ si $p \geq 1$, $\Phi'_{|B'_0}(\alpha\otimes\beta) = \alpha\beta$ si $\alpha\in A, \beta\in A$; alors $((\mathcal{B}'_*, d+b'), \Phi')$ est une résolution semi-libre de A appelée résolution standard de Hochschild.

Revenons au cas où $(A,d) = (T(V),d)$ et généralisons, au cas différentiel gradué, les résultats de [K] §.3 : considérons le diagramme :

$$
\begin{tikzcd}
{} \arrow[r, "b'"] & B'_{1,*} = A \otimes A \otimes A \arrow[r, "b'"] \arrow[d, "j'"] & B'_{0,*} = A \otimes A \arrow[r, "m"] \arrow[d, "\mathrm{Id}"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
{} \arrow[r] & A \otimes \bar{V} \otimes A \arrow[r, shift left, "\tilde{b}'"] & A \otimes A \arrow[l, shift left, "s"] \arrow[r, shift left, "m"] & A \arrow[l, shift left, "\varepsilon"] \arrow[r] & 0
\end{tikzcd}
$$

où la ligne du bas est la résolution $((P, D), \Phi)$ du §.1.

Posons $j' = s \circ b'$, c'est, par définition, un morphisme de $(A \otimes A^{op})$-modules différentiels gradués, qui rend commutatif le diagramme ci-dessous. Il définit un morphisme explicite de la résolution standard de Hochschild vers la résolution définie au §.1. Tensorisons chacune des deux résolutions par $A \otimes_{A \otimes A^{op}}$, on obtient le diagramme commutatif suivant :

$$
\begin{tikzcd}
C_{2,*}(A) = A \otimes (A^{\otimes 2}) \arrow[r, "b"] \arrow[d] & C_{1,*} = A \otimes A \arrow[r, "b"] \arrow[d, "j"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & A \otimes \bar{V} \arrow[r, "\tilde{b}"] & A \arrow[r] & 0
\end{tikzcd}
$$

Le morphisme j n'est rien d'autre que $\mathrm{Id} \otimes_{A \otimes A^{op}} j'$, après avoir identifié $A \otimes_{A \otimes A^{op}} B'_{**}(A)$ au complexe de Hochschild bigradué et $A \otimes_{A \otimes A^{op}} P$ à $E_1 = (A \otimes \bar{V}, \delta_1) \xrightarrow{\tilde{b}} E_0 = (A, d)$ où $\tilde{b}(a \otimes \bar{v}) = (-1)^{|a|+|\bar{v}|}\left[av - (-1)^{|a|\cdot|v|}va\right]$ et $\delta_1(a \otimes \bar{v}) = da \otimes \bar{v} - S(a, dv)$.

On a donc construit un morphisme $\mathcal{I}$ entre le complexe de Hochschild bigradué $(C_{**}(A), d, b)$ et le complexe défini au théorème 1.5 : on a $\mathcal{I}(x) = 0$ si $x \in C_{p,*}$ et $p \geq 2$, $\mathcal{I}(x) = j(x)$ si $x \in C_{1,*}$ et $\mathcal{I}(x) = x$ si $x \in C_{0,*} = A$, et il est classique que $\mathcal{I}$ induit un isomorphisme en homologie. On a : $j(a \otimes 1) = 0$ et

$$
\begin{aligned}
&j(a \otimes v_1 \cdots v_p) = av_1 \cdots v_{p-1} \otimes \bar{v}_p \\
&+ \sum_{i=1}^{p-1} (-1)^{[|v_{i+1}|+\cdots+|v_p|]\cdot[|a|+|v_1|+\cdots+|v_i|]} v_{i+1} \cdots v_p a v_1 \cdots v_{i-1} \otimes \bar{v}_i
\end{aligned}
$$

si $a \in A$, $v_i \in V$.

Posons $\beta = j \circ B : A \to A \otimes \bar{V}$ et $\beta = 0$ sur $A \otimes \bar{V}$. On a $\beta(1) = 0$

$$
\begin{aligned}
&\beta(v_1 \cdots v_p) = v_1 \cdots v_{p-1} \otimes \bar{v}_p \\
&+ \sum_{i=1}^{p-1} (-1)^{[|v_{i+1}|\cdots+|v_p|][|v_1|+\cdots+|v_i|]} v_{i+1} \cdots v_p v_1 \cdots v_{i-1} \otimes \bar{v}_i
\end{aligned}
$$

Lemme 2.3 *L'application $\mathcal{I}$ de $(C_{**}(A), b, d, B)$ sur $(E_{1,*} = (A \otimes \bar{V}, \delta_1) \xrightarrow{\tilde{b}} E_{0,*} = (A, d), \beta)$ est un morphisme de complexes bigradués qui commute aux opérateurs B et β et qui induit un isomorphisme entre les homologies cycliques correspondantes. On a :*

$$HC_*(A, d) = H_*(k[u] \otimes_\beta (A \oplus A \otimes \bar{V}, \delta))$$

■ Démonstration identique à celle du lemme 3 de [K]. ■

On déduit du lemme 2.3, le théorème suivant :

Théorème 2.4 *Soit $(A,d) = (T(V),d)$ une algèbre différentielle graduée libre sur un corps commutatif k telle que ou bien $V = \oplus_{n\geq 0} V_n$, ou bien $V = \oplus_{n\leq -1} V_n$. Soit $(A \oplus (A \otimes \bar{V}), \delta)$ le complexe défini au théorème 1.5. Alors l'homologie cyclique de (A,d) est isomorphe à l'homologie du complexe $(k[u] \otimes (A \oplus A \otimes \bar{V}), D)$ où $|u| = 2$, $D = 0$ sur $k[u]$, $D = \delta$ sur $A \oplus (A \otimes \bar{V})$, $D(u^n \otimes (a \otimes \bar{v})) = u^n \otimes \delta(a \otimes \bar{v})$ si $n \geq 1$,*

$$D(u^n \otimes a) = u^n \otimes da + u^{n-1} \otimes (v_1 \cdots v_{p-1} \otimes \bar{v}_p$$
$$+ \sum_{i=1}^{p-1} (-1)^{[|v_{i+1}|+\cdots+|v_p|][|v_1|+\cdots+|v_i|]} v_{i+1} \cdots v_p v_1 \cdots v_{i-1} \otimes \bar{v}_i)$$

si $a = v_1 \cdots v_p$, et $n \geq 1$.

Rappelons que, si $V = \oplus_p V_p$ est un espace vectoriel gradué, alors pour tout $m \geq 1$, $V^{\otimes m}$ est un espace vectoriel gradué, $V^{\otimes m} = \oplus_p (V^{\otimes m})_p$, et les groupes d'homologie $H_n(\mathbf{Z}/m\mathbf{Z}, V^{\otimes m})$ où $\mathbf{Z}/m\mathbf{Z}$ agit sur $V^{\otimes m}$ via τ_m, sont des k-espaces vectoriels gradués; on a $H_n(\mathbf{Z}/m\mathbf{Z}, V^{\otimes m}) = \oplus_p H_{n,p}(m)$ où $H_{n,p}(m) = H_{n,p}(\mathbf{Z}/m\mathbf{Z}, V^{\otimes m})$ est le $n^{\text{ième}}$groupe d'homologie de $\mathbf{Z}/m\mathbf{Z}$ agissant sur $(V^{\otimes m})_p$ via τ_m. On peut alors énoncer :

Théorème 2.5. *Soit $A = T(V)$ une algèbre tensorielle graduée sur un corps commutatif quelconque k. Alors l'homologie cyclique réduite $H\tilde{C}_*(A,0) = HC_*(T(V),0)/HC_*(k)$ est donnée par :*

$$H\tilde{C}_n(T(V),0) = \bigoplus_{m\geq 1} \bigoplus_{p+q=n} H_{p,q}(m) .$$

■ Dans le cas particulier où la différentielle est nulle, le complexe défini au théorème 2.4 se décompose ainsi : $k[u] \otimes (A \oplus A \otimes \bar{V}) = k[u] \oplus \mathcal{F}^0_* \oplus \mathcal{F}^1_*$ où $\mathcal{F}^0_* = k[u] \otimes T^+(V)$, $\mathcal{F}^1_* = k[u] \otimes T(V) \otimes \bar{V}$, $D = 0$ sur $T^+(V)$, $D(u^n \otimes v_1 \cdots v_m) = u^{n-1} \beta(v_1 \cdots v_m)$ si $n \geq 1$ et $m \geq 1$, $D(u^n \otimes a \otimes \bar{v}) = u^n (-1)^{|a|} \times (av - (-1)^{|v||a|} va)$ si $n \geq 0$, $a \in A$, $v \in V$. On a $D(\mathcal{F}^0_*) \subset \mathcal{F}^1_*$ et $D(\mathcal{F}^1_*) \subset \mathcal{F}^0_*$. Reprenons les notations introduites dans la remarque 1.6, on a : $D(u^n \otimes \alpha) = u^{n-1} \sum_{i=0}^{m-1} \sigma_0 \tau^i_m(\alpha)$ si $\alpha \in V^{\otimes m}$ et $D(u^n \otimes \alpha \otimes \bar{v}) = (-1)^{|\alpha|} u^n (Id - \tau_{m+1}) \circ \sigma^{-1}(\alpha \otimes \bar{v})$ si $\alpha \in V^{\otimes m}$.

La formule du théorème 2.5 résulte du calcul classique des groupes d'homologie d'un G-module lorsque G est le groupe cyclique $\mathbf{Z}/m\mathbf{Z}$, [Mc]. ■

Corollaire 2.6 *Soit $A = T(V)$ une algèbre tensorielle graduée sur un corps commutatif de caractéristique 0, alors l'homologie cyclique réduite $H\tilde{C}_*(T(V),0)$ est isomorphe $T^+(V)/[T(V),T(V)]$ où $[T(V),T(V)]$ est le sous-espace engendré par les commutateurs gradués.*

■ Si le corps est de caractéristique 0 , les groupes d'homologie de $\mathbf{Z}/m\mathbf{Z}$ agissant sur $V^{\otimes m}$ sont nuls en degrés homologiques strictement positifs, d'où le corollaire à partir du théorème. Ce résultat figurait déjà dans [BF]. ■

Remarque. Si $A = T(V)$ est graduée uniquement en degré 0 , alors les groupes d'homologie $H_{n,p}(\mathbf{Z}/m\mathbf{Z}, V^{\otimes m})$ sont nuls si $p > 0$, et le théorème 2.5 redonne la proposition 5.4 de [LQ].

§.3. Homologie de l'espace des lacets libres sur un espace topologique

Le point de départ de cette étude est le résultat de Burghelea-Fiedorowicz [BF] et de Goodwillie [G] qui dit :

Théorème 3.0. *Soit X un espace connexe par arcs, pointé, et k un anneau commutatif, alors il existe des isomorphismes de k-modules gradués :*

(1) $\quad HH_*(C_*(\Omega X), k) \simeq H_*(X^{S^1}, k)$

(2) $\quad HC_*(C_*(\Omega X), k) \simeq H_*^{S^1}(X^{S^1}, k)$

Rappelons que X^{S^1} est l'espace des lacets libres sur X muni de la topologie compacte ouverte, et que $C_*(\Omega X, k)$ est la k-ADG des chaînes singulières sur l'espace des lacets de Moore de X . Enfin $H_*^{S^1}(X^{S^1}, k)$ désigne l'homologie équivariante de l'espace des lacets libres sur lequel le groupe S^1 agit par rotation des lacets; par définition, $H_*^{S^1}(X^{S^1}, k)$ est l'homologie, à coefficients dans k , de l'espace de Borel associé à cette action et noté $X^{S^1} \times_{S^1} ES^1$.

Dans toute la suite, on supposera que k est un corps commutatif et X est simplement connexe. Rappelons le résultat de Adams-Hilton [A-H], il existe une k-ADG libre $(\mathcal{A}_X = T(W), d_X)$ et un morphisme $\mathcal{O}_X : \mathcal{A}_X \to C_*(\Omega X)$ qui induit un isomorphisme en homologie, de plus $W_p \simeq H_{p+1}(X, k)$. Le calcul de l'homologie (resp. équivariante) de X^{S^1} se ramène donc au calcul de l'homologie de Hochschild (resp. cyclique) de l'ADG libre $(\mathcal{A}_X, d_X)$. Le point de vue dual de celui de [BF] et [G] consiste à considérer la k-ADG $C^*(X, k)$ des cochaînes singulières de X et à regarder son homologie de Hochschild (resp. cyclique). Dans [HV], Halperin et l'auteur montrent que $HH_*(C_*(\Omega X), k)$ et $HH^*(C^*(X, k))$ sont des espaces vectoriels duaux, et donc, d'après le théorème 3.0, l'homologie de Hochschild de $(C^*(X, k))$ est isomorphie à la cohomologie de X^{S^1}. Dans [J], Jones démontre que l'homologie cyclique négative HC^-_{-*} de $C^*(X, k)$ est isomorphe à la cohomologie équivariante de X^{S^1} , donc l'homologie cyclique de $C^*(X, k)$ définie au §.2 est isomorphe à la cohomologie équivariante de X^{S^1} .

La moralité de cette étude est que le calcul de l'homologie ou de la cohomologie (resp. équivariante) de X^{S^1} se ramène au calcul de l'homologie de Hochschild (resp. cyclique) de $C_*(\Omega X, k)$ ou de $C^*(X, k)$. D'après un résultat classique, (théorème I, [B1]), on peut remplacer $C_*(\Omega X, k)$ par son modèle de Adams-Hilton, ou la cobar construction de Adams [A], et $C^*(X, k)$ par un modèle libre dans la catégorie k-ADG. Alors les théorèmes 1.5 et 2.4 fournissent des complexes calculant l'homologie et l'homologie équivariante de l'espace des lacets libres.

Soit X un espace simplement connexe tel que $H_*(\Omega X, k)$ soit une algèbre tensorielle graduée $T(V)$, pour le produit induit par la composition des lacets, alors nécessairement V_p est isomorphe à $H_{p+1}(X,k)$ pour tout $p \geq 1$. Les théorèmes 1.7 et 2.5 s'énoncent ainsi :

Théorème 3.1. *Soit X un espace simplement connexe et k un corps commutatif tel que $H_*(\Omega X, k)$ soit isomorphe à une algèbre tensorielle graduée. Posons $V_* = H_{*+1}(X,k)$. Alors pour tout $n \geq 0$*

(1) $H_n(X^{S^1}, k) = HH_n(C_*(\Omega X, k))$ est la somme directe de $\oplus_{m\geq 0} V^{\otimes m}/(\mathrm{Id} - \tau_m)$ et $\oplus_{m \geq 1} \overline{\mathrm{Ker}(\mathrm{Id} - \tau_m)}$ où τ_m est la permutation cyclique agissant sur $V^{\otimes m}$.

(2) $H_n^{S^1}(X^{S^1}, k)/H_n(BS^1, k) = H\tilde{C}_n(C_*(\Omega X), k)$ est la somme directe

$$\bigoplus_{m \geq 1} \oplus_{i+j=n} H_{i,j}(m)$$

où $H_{i,j}(m)$ est la composante de degré j du $i^{\text{ème}}$ groupe d'homologie de $\mathbf{Z}/m\mathbf{Z}$ agissant sur $V^{\otimes m}$

Remarque. Le calcul de l'homologie cyclique des sphères est trivial à partir du résultat ci-dessus. Plus généralement, le théorème 3.1 permet de calculer l'homologie (resp. l'homologie équivariante) de l'espace des lacets libres sur n'importe quel espace simplement connexe de L-S-catégorie 1 , (L-S signifiant Lusternik-Schnirelman), comme le font remarquer les auteurs de [FHT2] dans l'introduction de leur article. Le premier exemple intéressant d'application des théorèmes 1.5 et 2.4 est l'espace $X = \mathbf{C}P^2$, en caractéristique $p > 2$. En effet, le modèle de Adams-Hilton est $(\mathcal{A}_X, d_X) = T(e_1, e_3)$ avec $|e_1| = 1$, $|e_3| = 3$, $de_1 = 0$, $de_3 = 2e_1^2$. On calcule d'abord $H_*(\mathcal{A}_X, d_X) = H_*(\Omega X)$: on montre que $c = e_3e_1 + e_1e_3$ est un cycle et n'est pas un bord, ainsi que toutes les puissances de c; de plus $e_1c^p - c^pe_1$ est un bord, pour tout $p \geq 0$; donc $H_*(\Omega X)$ contient l'algèbre commutative $k[e_1]/e_1^2 \otimes k[c]$. A l'aide de la suite spectrale de Serre du fibré $\cdots \to \Omega S^5 \to \Omega \mathbf{C}P^2 \to S^1 \to \cdots$ associée à $S^1 \to S^5 \to \mathbf{C}P^2$, on montre que $H_*(\Omega X) = k[e_1]/e_1^2 \otimes k[c]$. On utilise alors le modèle exhibé au théorème 1.5 pour calculer l'homologie de Hochschild de $C_*(\Omega X)$, on vérifie facilement que $\dim H_n((\mathbf{C}P^2)^{S^1}, k) = 1$ pour tout $n \geq 0$.

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Astérisque

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Maps between p-completions of the Clark-Ewing spaces $X(W, p, n)$

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MAPS BETWEEN p–COMPLETIONS OF THE CLARK–EWING SPACES X(W,p,n)

by

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Abstract. Let Z_p denote the ring of p–adic integers. Let $W \subset GL(n,Z_p)$ be a finite group such that p does not divide the order of W. The group W acts on $K((Z_p)^n,2)$. Let $X(W,p,n)_p$ be the p–completion of the space $K((Z_p)^n,2) \underset{W}{\times} EW$. We classified homotopy classes of maps between spaces $X(W,p,n)_p$.

0. INTRODUCTION

Let Z_p denote the ring of p–adic integers. Let Y_p denote the p–completion of a space Y.

Let T be a torus and let $W \subset GL(\pi_1(T) \otimes Z_p)$ be a fintie group. The group W acts on the space $(BT)_p$. Let

$$X(W,p,T) := ((BT)_p \times_W EW)_p$$

where EW is a contractible space equipped with a free action of W.

The aim of this paper is to apply the program from [1] to study maps between spaces X(W,p,T). The starting point was an attempt to generalize one result of Hubbuck (see [8] Theorem 1.1.). The plan of work will follow closely that of [3] and [13].

Example. *Let* G *be a connected, compact Lie group,* T *its maximal torus and* W *its Weyl group. If* p *does not divide the order of* W *then* $(BG)_p \approx (BT \times_W EW)_p$.

This example suggests the following defintion.

Definition. *Let us set* $X = X(W,p,T)$. *We shall call* T *a maximal torus of* X *and* W *a Weyl group of* X.

The projection $(BT)_p \times EW \longrightarrow (BT)_p \times_W EW$ *induces a map* $i : BT \longrightarrow X$. *We shall call* $i : BT \longrightarrow X$ *a structure map of* X.

We point out that in [5] A. Clark and J. Ewing studied cohomology algebras of spaces $(BT)_p \times_W EW$. We warn the reader that our notation is different from the notation used in [5]. The space $X(W,p,T)$ is the p–completion of the Clark–Ewing space $X(W,p,\mathrm{rank}\ T)$.

Through the whole paper we shall assume that p is an odd prime. We need this assumption to show Proposition 1.1. It is clear that this assumption is not essential, however we were not able to overcome technical difficulties for $p = 2$.

Now we shall state our main results.

Let us set $X = X(W,p,T)$ and $X' = X(W',p,T')$.

THEOREM 1. Assume that p *does not divide the orders of* W *and* W'. *Then for any map* $f : X \longrightarrow X'$ *there is a map* $\Upsilon : (BT)_p \longrightarrow (BT')_p$ *such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ (BT)_p & \xrightarrow{\quad \Upsilon \quad} & (BT')_p \end{array}$$

commutes up to homotopy. Moreover we have:

a) *if* $\Upsilon' : (BT)_p \longrightarrow (BT')_p$ *is such that* $f \circ i$ *is homotopic to* $i' \circ \Upsilon'$ *then there is* $w \in W'$ such that $w \circ \Upsilon'$ *is homotopic to* Υ,

b) *for any* $w \in W$ *there is* $w' \in W'$ *such that* $\Upsilon \circ w$ *is homotopic to* $w' \circ \Upsilon$.

The group W acts on $\pi_1(T) \otimes Z_p$, hence W acts on $\pi_1(T) \otimes R$ for any Z_p–module R.

DEFINITION 1. Let R *be a* Z_p*–algebra. We say that a homomorphism of* R*–modules*

$$\varphi : \pi_1(T) \otimes R \longrightarrow \pi_1(T') \otimes R$$

is admissible if for any $w \in W$ *there is* $w' \in W'$ *such that* $\varphi \circ w = w' \circ \varphi$.
We say that two admissible maps φ *and* ψ *from* $\pi_1(T) \otimes R$ *to* $\pi_1(T') \otimes R$ *are equivalent if there is* $w \in W'$ *such that* $w \circ \varphi = \psi$.

It is clear that the relation defined above is an equivalence relation on the set of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$. We shall denote by $\mathrm{Ahom}_R(T,T')$ the set of equivalence classes of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

Let us notice that the map $\pi_1(\Upsilon)$ induced by Υ from Theorem 1 on fundamental groups is admissible for $R = Z_p$. This map is unique up to the action of W', so any map $f : X \longrightarrow X'$ determines uniquely an equivalence class of $\pi_1(\Upsilon)$ in $\mathrm{Ahom}_{Z_p}(T,T')$ which we shall denote by $\chi(f)$.

THEOREM 2. Let us assume that p *does not divide the orders of* W *and* W'*. Then the natural map*

$$\chi : [X,X'] \longrightarrow \mathrm{Ahom}_{Z_p}(T,T')$$

is bijective.

For any space Y we set

$$H^*(Y,\mathbb{Q}_p) := H^*(Y,Z_p) \otimes \mathbb{Q} ,$$

where $\mathbb{Q}_p$ is a field of p–adic numbers.

THEOREM 3. Let us assume that p *does not divide the orders of* W *and* W'*. Then the natural map*

$$\phi : [X,X'] \longrightarrow \mathrm{Hom}(H^*(X',\mathbb{Q}_p), H^*(X,\mathbb{Q}_p))$$

is injective.

We denote by $K^0(\ ,R)$ the 0^{th}–term of complex K–theory with R–coefficients. Let $\mathcal{O}_R$ be the set of operations in $K^0(\ ,R)$. The functor $K^0(\ ,R)$ is equipped with the natural augmentation $K^0(\ ,R) \longrightarrow R$. Let $\mathrm{Hom}_{\mathcal{O}_R}(K^0(X',R),K^0(X,R))$ be the set of R–algebra homomorphisms which commute with the action of $\mathcal{O}_R$ and augmentations.

THEOREM 4. If p *does not divides the order of* W *and* W′, *then the natural map*

$$\psi : [X,X'] \longrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_p}}(K^0(X',Z_p),K^0(X,Z_p))$$

is bijective.

We can formulate our results in a nice categorical way.

We shall define a category $Z_p - \mathrm{Rep}$ in the following way. Objects of the category Z_p–Rep are representations $\rho : W \longrightarrow GL(M)$ where M is a free, finitely generated Z_p–module, W is a finite group and p does not divide the order of W. It remains to define morphisms in this category. If $\theta : W \longrightarrow GL(M)$ and $\theta' : W' \longrightarrow GL(M')$ are two objects of $Z_p - \mathrm{Rep}$, we say that a homomorphism of Z_p–modules $f : M \longrightarrow M'$ is admissible if for each $w \in W$ there is $w' \in W'$ such that $f \circ w = w' \circ f$. We say that two admissible homomorphisms f and g from M to M′ are equivalent if there is $w \in W'$ such that $f = w' \circ g$. We shall denote by $\mathrm{Ahom}(\theta,\theta')$ the set of equivalence classes of admissible homomorphisms from M to M′. The set $\mathrm{Ahom}(\theta,\theta')$ is the set of morphisms from θ to θ' in the category $Z_p -$ Rep. The category $Z_p - \mathrm{Rep}$ is equipped with the product defined in the following way:

$$(\theta{:}W \longrightarrow GL(M)) \oplus (\theta'{:}W' \longrightarrow GL(M')) = \theta\oplus\theta' : W \times W' \longrightarrow GL(M \oplus M').$$

The product of morphisms is defined in the obvious way.

We denote by Ht(p) the category whose objects are spaces X(W,p,T) such that p does not divide the order of W. Morphisms in Ht(p) are homotopy classes of maps. The category Ht(p) has products defined in the obvious way.

THEOREM 5. *There is an equivalence of categories*

$$R : Z_p - \mathrm{Rep} \longrightarrow \mathrm{Ht}(p)$$

with products.

THEOREM 6. *In Theorems* 1,2,3 *and* 4 *we can drop the assumption* "p *does not divide the order of* W′ " *if* $X' = (BG)_p$, *where* G *is a connected, compact Lie group.*

COROLLARY 7. *Let* $X = X(W,p,T)$ *and let* p *be a prime not dividing the order of* W. *Let us assume that the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$ *is irreducible. Then there is a finite number of self-maps* $I_1,\ldots,I_n$ *of* X *such that for any* $f : X \longrightarrow X$ *there is* k *for which* $f \circ I_k$ *is an Adams* ψ^α*—map i.e. the map induced by* $f \circ I_k$ *on* $H^{2i}(X,\mathbb{Q}_p)$ *is a multiplication by* α^i. *The number* n *is smaller or equal to the number of elements of* Aut(W)/Inn(W) *which preserve the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$.

Example. (see also [3])
Let $X = BSU(n)_p$. *The Weyl group of* SU(n) *is* Σ_n. *If* $n \neq 6$ *then* $\mathrm{Aut}\,\Sigma_n = \mathrm{Inn}\,\Sigma_n$ *and for* $n = 6$ *the outer automorphism does not preserve the natural representation of* Σ_6 *on* $\pi_1(T) \otimes \mathbb{Q}_p$. *This implies that the self-maps of* $BSU(n)_p$ *are Adams* ψ^α*—maps.*

We point out that Corollary 7 can be view as a generalization of a result of Hubbuck (see [8] Theorem 1.1.) The example is a special case of the result of Hubbuck. However, it concerns maps between p—completed spaces $BSU(n)_p$ while Hubbuck is dealing with classical spaces BG.

We would like to thank very much A. Zabrodsky who during the Barcelona conference on algebraic topology 1986 shared with us his unpublished papers and notes. We would like to express our gratitude to the referee for his patient readings of the manuscript, for his useful suggestions which allowed us to generalize substantially our results, and for pointing out several misprints in the manus-

cript.

1. THE LANNES T FUNCTOR FOR SPACES X(W,p,T)

Let $X = X(W,p,T)$. Let us assume that p does not divide the order of W. In this section we shall compute the cohomology of the mapping space map(BV,X) and its connected component $\mathrm{map}_f(BV,X)$ where V is an elementary abelian p–group and $f : BV \longrightarrow X$ is a map

It follows from [5] (see Proposition on p. 425) that

$$H^*(X,F_p) = H^*(BT,F_p)^W .$$

The map $f : BV \longrightarrow X$ induces a map $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$. Let us notice that $\mathrm{Im}\, f^*$ is contained in the kernel of the Bockstein homomorphism. Hence it suffices to look at the polynomial part of $H^*(BV,F_p)$ when extending f^* to $H^*(BT,F_p)$. It follows from [2] Proposition 1.10 that there is $g^* : H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ such that $f^* = g^* \circ i^*$ where $i^* : H^*(X,F_p) \longrightarrow H^*(BT,F_p)$ is the inclusion induced by a structure map $i : BT \longrightarrow X$.

For a torus T, the solutions in T of $t^p = 1$ make up a subgroup $T(1)$. The map g^* is induced by a homomorphism $\varphi : V \longrightarrow T(1)$. This follows from [9] Theorem 0.4. Let $\Lambda_f : V \otimes T(1)^* \longrightarrow F_p$ be an adjoint map of φ. The group W acts on $\mathrm{Hom}(V \otimes T(1)^*,F_p)$ through its action on $T(1)^*$. Let W_f be the isotropy subgroup of Λ_f.

PROPOSITION 1.1. *Let* $X = X(W,p,T)$. *Let us assume that* p *does not divide the order of* W. *Let* V *be an elementary abelian* p*–group and let* $f : BV \longrightarrow X$ *be any map. Then we have an isomorphism*

$$H^*(\mathrm{map}_f(BV,X);F_p) = H^*(BT,F_p)^{W_f} .$$

PROOF: For a vector space U over F_p let us denote by $P(U)$ the polynomial algebra on U, by $\Lambda(U)$ the exterior algebra on U and by $A(U)$ the symmetric algebra on U divided by the ideal generated by all polynomials $x^p - x$ for $x \in U$. The polynomial $x^p - x$ splits completely over F_p. Hence we have an isomorphism of F_p–algebras $A(U) = \bigoplus_{a \in U^*} F_p$. We point out that $A(U)$ is concentrated in degree zero.

Let us notice that we have the following natural identifications

$$H^*(BT,F_p) = P(T(1)^*)$$

and

$$H^*(BV,F_p) = P(V^*) \otimes \Lambda(\beta^{-1}V^*).$$

To simplify the notation let us set $A := A(V \otimes T(1)^*)$ and $H := H^*(BT,F_p) = P(T(1)^*)$. It follows from Corollary 2 in [4] that for any unstable A_p–algebra M and any A_p–algebra homomorphism $h : P((Z/p)^*) \longrightarrow M \otimes H^*(BZ/p,F_p)$ we have

$$h(t^*) = m_{t*} \otimes 1 + m_{v*} \otimes v^*.$$

This implies that we have a natural isomorphism

$$\Phi_M : \mathrm{Hom}(H;M \otimes H^*(BV)) \approx \mathrm{Hom}\,(A \otimes H;M).$$

where Hom(;) is in the category of unstable A_p–algebras. If $h(t^*) =$

$m_{t*} \otimes 1 + \sum_{v* \in V*} m_{v*} \otimes v^*$ then $\Phi_M(h)([v \otimes t^*] \otimes 1) = \sum_{v* \in V*} m_{v*} \cdot v^*(v)$

and $\Phi_M(h)(1 \otimes t^*) = m_{t*}$.

Hence it follows that

$$(*) \qquad T_V(H) = A \otimes H.$$

If $M = F_p$ then we have an isomorphism

$\Phi_{F_p} : \mathrm{Hom}(H;H^*(BV)) \approx \mathrm{Hom}(A \otimes H;F_p)$. The group W acts on H and A through its action on $T(1)^*$. The isomorphism $(*)$ and the fact that the functor $T_V(-)$ is exact implies that

$$(**) \qquad T_V(H^W) = (A \otimes H)^W$$

(see [4] Proposition 3).

Let $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$ be the map induced by f on cohomology. Let $\lambda : T_V(H^*(X,F_p)) \longrightarrow F_p$ be the adjoint map of f^* and let $\overline{\lambda} : T_V(H) \longrightarrow F_p$ be the adjoint map of g^* We recall that $g^* : H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ is such that $f^* = g^* \circ i^*$. The restriction of $\overline{\lambda}$ to $V \otimes T(1)^*$ is equal to Λ_f, where
$\Lambda_f : V \otimes T(1)^* \longrightarrow F_p$ is an adjoint map of $\varphi : V \longrightarrow T(1)$.

It follows from [6] 2.3 Theorem and the equality $(**)$ that

$$H^*(\mathrm{map}_f(BV,X),F_p) \approx T_V(H^*(X,F_p)) \underset{T_V^0(H^*(X,F_p))}{\otimes} F_p \approx (A \otimes H)^W \underset{A^W}{\otimes} F_p .$$

If $V^* \otimes T(1) = \coprod_{W'} W/W'$, as a W–set then $A \approx \bigoplus_{W'} F_p[W/W']$ as a W–module. This follows from the isomorphism $A(U) = \bigoplus_{a \in U^*} F_p$ mentioned at the beginning of the proof. For any $W' \subset W$, $F_p[W/W']^W \approx F_p$. The maps $\overline{\lambda}$ and λ induce $\tilde{\lambda} : A \longrightarrow F_p$ and $\tilde{\tilde{\lambda}} : A^W = \oplus F_p \longrightarrow F_p$. The algebra homomorphism $\tilde{\tilde{\lambda}}$ is the identity on one's of F_p's and it is zero on all others. We recall that the isotropy subgroup of Λ_f is W_f. The fact that $\tilde{\lambda}$ restricts to Λ_f on $V \otimes T(1)^*$ implies that $\tilde{\tilde{\lambda}}$ is the identity on $F_p[W/W_f]^W$. Hence we have the following isomorphisms

$$(A \otimes H)^W \underset{A^W}{\otimes} F_p \approx (F_p[W/W_f] \otimes H)^W \underset{F_p}{\otimes} F_p \approx H^{W_f}. \qquad \square$$

2. MAPS FROM BP TO X

Let T be a torus. For a torus T the solutions in T of $t^{p^n} = 1$ make up a subgroup $T(n)$; let $T(\infty) = \bigcup_n T(n)$. Let us set $M = \pi_1(T) \otimes Z_p$. Let $W \subset GL_{Z_p}(M)$ be a finite group. The action of W on M extends to the action of W on $M \otimes \mathbb{Q}$. The lattice M in $M \otimes \mathbb{Q}$ is invariant therefore W acts also on $M \otimes \mathbb{Q}/_M$. Observe that $M \otimes \mathbb{Q}/_M = T(\infty)$. From the action of W on $T(\infty)$ we can recover the original action of W on M if we take the induced action of W on $(H^2(BT(\infty);Z_p))^*$. Hence any finite subgroup of $GL_{Z_p}(M)$ can be realized as a subgroup of $\mathrm{Aut}(T(\infty))$.

PROPOSITION 2.1. *Let* W *be a finite subgroup of* $\mathrm{Aut}(T(\infty))$. *Let us assume that* p *does not divide the order of* W. *If* P *is a finite* p*–group then any map* $f : BP \longrightarrow (B(T(\infty)\tilde{\times}W))_p$ *is induced by a homomorphism* $\varphi : P \longrightarrow T(\infty)\tilde{\times}W$.

We were informed that a similar result was also known to W. Dwyer. This proposition is an analog of the theorem of Dwyer and Zabrodsky (see [7] 1.1. Theorem). The proof will follow closely the proof of the Dwyer and Zabrodsky theorem contained in [14], which depends very much on [10].

Let us set $G = T(\infty) \tilde{\times} W$.

LEMMA 2.2. *Let* $V = Z/p$, *let* $\varphi : V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\varphi$ *in* G *and let* $\varphi_0 : V \longrightarrow G_0$ *be the map induced by* φ. *Then the map*

$$\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p) \longrightarrow \mathrm{map}_{B\varphi}(BV,(BG)_p)$$

is a homotopy equivalence.

PROOF: It follows from Proposition 1.1 that

$H^*(\mathrm{map}_{B\varphi}(BV,(BG)_p),F_p) \approx P^{W_0}$ where $P \approx H^*(BT,F_p)$ and $W_0 = G_0/T(\infty)$ is the isotropy subgroup of $\varphi : V \longrightarrow T(\infty)$. In the same way we get

$H^*(\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p),F_p) = P^{W_0}$. Hence the map considered by us is a homotopy equivalence. □

LEMMA 2.3. *Let* P *be a p–group, let* $Z/p = V$ *be a subgroup of the center of* P. *Let* $\varphi : V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\,\varphi$ *in* G *and let* $\varphi_0 : V \longrightarrow G_0$ *be the induced homomorphism. Let*

$$[BP,(BG)_p](B\varphi) = \{f \in [BP,(BG)_p] : f_{|BV} \sim B\varphi\}$$

and let $[BP,(BG_0)_p](B\varphi_0)$ *be defined in an analogous way. Then the inclusion map* $i : G_0 \longrightarrow G$ *induces a bijection*

$$(*) \qquad [BP,(BG_0)_p](B\varphi_0) \longrightarrow [BP,(BG)_p](B\varphi) .$$

PROOF: We have a fibration $BV \longrightarrow BP \longrightarrow B(P/V)$. Let $BV \longrightarrow EP/V \longrightarrow E(P/V)$ be the fibration induced by pulling back over $pr : E(P/V) \longrightarrow B(P/V)$. The group P/V acts on EP/V through maps homotopics to the identity and the space EP/V is a model for BV. It follows from Lemma 2.2 that the map

$$\mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi_0}(EP/V,(BG_0)_p) \to \mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi}(EP/V,(BG)_p))$$

is a homotopy equivalence. There is a bijective correspondence between P/V–maps $E(P/V) \longrightarrow \mathrm{map}_{B\varphi_0}(EP/V,(BG_0)_p)$ and maps $E(P/V) \underset{(P/V)}{\times} EP/V \longrightarrow (BG_0)_p$ which composed with $E(P/V) \times EP/V \longrightarrow E(P/V) \underset{(P/V)}{\times} EP/V$ are homotopic to $B\varphi_0$. The same bijection holds if we replace φ_0 by φ and G_0 by G. This implies that the induced map on π_0 is the map $(*)$. This finishes the proof. □

LEMMA 2.4. (see [15] 1.5. Lemma) *Let* $\varphi : L \longrightarrow K$ *be a simplicial map. Let* $V_0^\varphi(L,X)$ *be the subspace of the space* $\mathrm{map}_.(L,X)$ *of pointed maps from* L *to* X *consisting of maps* $f : L \longrightarrow X$ *such that*

$f_{|\varphi^{-1}(k)} \sim *$ *for every* $k \in K$. *Let* $\mathrm{map}_*(\varphi^{-1}(k),X)$ *be the path component of the constant map in the space of pointed maps* $\mathrm{map.}(\varphi^{-1}(k),X)$. *Let us assume that for every* $k \in K$, *the space* $\mathrm{map}_*(\varphi^{-1}(k),X)$ *is weakly homotopy equivalent to* $*$. *Then* φ *induces a weak homotopy equivalence*

$$\varphi^* : \mathrm{map.}(K,X) \xrightarrow{\approx} V_0^{\varphi}(L,X) .$$

PROOF OF PROPOSITION 2.1: Let us assume that $P = Z/p$. It follows from [2] Proposition 1.10 that $f^* : H^*(BG,F_p) \longrightarrow H^*(BP,F_p)$ factors through $H^*(BT(\infty),F_p)$. But any morphism $H^*(BT(\infty),F_p) \longrightarrow H^*(BP,F_p)$ is of the form $B\varphi$ (see [9] Theorem 0.4). Hence f is induced by a homomorphism.

Let us suppose that any map $f : BP \longrightarrow (BG)_p$ is induced by a homomorphism if the order of P is less or equal to p^{n-1}.

Let the order of P be equal to p^n and let $f : BP \longrightarrow (BG)_p$ be a map. Let $V = Z/p$ be contained in the center of P and let $i : V \longrightarrow P$ be the inclusion.

Assume that the composition

$$BV \xrightarrow{Bi} BP \xrightarrow{f} X$$

is null homotopic. We want to show that f is homotopic to $f_1 \circ Bpr$ where $pr : P \longrightarrow P/_V$ is the natural homomorphism and $f_1 : B(P/_V) \longrightarrow X$ is a map. First we show that the space of pointed maps homotopic to $*$ $\mathrm{map}_*(BV,X)$ is weakly contractible. This space is p–local because BV and X are p–local. Let $\mathrm{map}_{\mathrm{const}}(BV,X)$ be the connected component containing a constant map of $\mathrm{map}\,(BV,X)$. It follows from Proposition 1.1 that

$$H^*(\mathrm{map}_{\mathrm{const}}(BV,X),F_p) = H^*(BT(\infty),F_p)^W .$$

The last group is of course $H^*(X,F_p)$. Hence the evaluation map $\mathrm{map}_{\mathrm{const}}(BV,X) \longrightarrow X$ is a weak homotopy equivalence and consequently the space $\mathrm{nap}_*(BV,X)$ is weakly contractible. Lemma 2.4 implies that f is homotopic to $f_1 \circ Bpr$. By the inductive assumption f_1 is induced by a homomorphism.

Let us suppose that $f \circ Bi$ is induced by a homomorphism $\varphi : V \longrightarrow G$ and $\varphi(V) \neq 0$. Let G_0 be the centralizer of $\varphi(V)$ in G . It follows from Lemma 2.3 that up to homotopy there is a unique map $f_0 : BP \longrightarrow (BG_0)_p$ such that $BP \xrightarrow{f_0} (BG_0)_p \longrightarrow (BG)_p$ is homotopic to f and f_0 restricted to BV is induced by φ. Let $\rho : G_0 \longrightarrow G_0/\varphi(V)$ be the natural projection. The composition

$$BV \longrightarrow BP \xrightarrow{f_0} (BG_0)_p \xrightarrow{(B\rho)_p} (BG_0/\varphi(V))_p$$

is null–homotopic hence $(B\rho)_p \circ f_0$ factors uniquely as

$$BP \xrightarrow{Bpr} B(P/V) \xrightarrow{f_1} B(G_0/\varphi(V))_p \ .$$

This follows from the previous discussion.

One has the homotopy pullback

$$\begin{tikzcd} BP \arrow[r, "f_0"] \arrow[d, "Bpr"] & (B(G_0)_p \arrow[d, "(B\rho)_p"] \\ B(P/V) \arrow[r, "f_1"] & (B(G_0/\varphi(V)))_p \end{tikzcd}$$

because $\varphi(V)$ is contained in the center of G_0 . By the inductive assumption f_1 is induced by a homomorphism $\varphi_1 : P/V \longrightarrow G_0/\varphi(V)$. We have a pullback of groups

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "pr"] & G_0 \arrow[d, "\rho"] \\ P/V \arrow[r, "\varphi_1"] & G_0/\varphi(V) \ . \end{tikzcd}$$

After applying the functor $(B\ )_p$ we get a homotopy pullback

$$\begin{tikzcd}
BP \arrow[r, "(B\psi)_p"] \arrow[d, "Bpr"] & (BG_0)_p \arrow[d, "(B\rho)_p"] \\
B(P/V) \arrow[r, "(B\varphi_1)_p"] & B(G_0/\varphi(V))_p
\end{tikzcd}$$

The map f_0 is homotopic to $(B\psi)_p$ hence f is homotopic to $(B\rho)_p \circ (B\psi)_p$. □

COROLLARY 2.5. *Let* T' *by any torus. Then any map* $g : BT'(\infty) \longrightarrow (BG)_p$ *is induced by a homomorphism* $\alpha : T'(\infty) \longrightarrow T(\infty)$.

PROOF. It follows from Proposition 2.1 that for any n the restriction of g to $BT'(n)$, $g_n : BT'(n) \longrightarrow (BG)_p$ is induced by a homomorphism. Let $S_n = \{\beta : T'(n) \longrightarrow G \mid (B\beta)_p \sim g_n\}$. The restriction of $\beta : T'(n) \longrightarrow G$ to $T'(n-1)$ maps S_n into S_{n-1}. Each set S_n is non-empty and finite. This implies that $\lim\limits_{\overleftarrow{n}} S_n$ is non-empty. Hence there is a homomorphism $\alpha : T'(\infty) \longrightarrow G$ such that α induces g and factorizes through $T(\infty)$. □

3. PROOFS.

We start with the following lemma.

Lemma 3.1 Let $X = X(W,p,T)$, let $i : BT(\infty) \longrightarrow X$ be a structure map of X and let $w : BT(\infty) \longrightarrow BT(\infty)$ be a map induced by $w \in W$. Then the maps i and $i \circ w$ are homotopic.

Proof. Let $\tilde{w} : BT(\infty) \times EW \longrightarrow BT(\infty) \times EW$ be w on $BT(\infty)$ and a translation by w^{-1} on EW. Observe that $\tilde{w}$ is a covering transformation of the projection $pr : BT(\infty) \times EW \longrightarrow BT(\infty) \times_W EW$. The composition $BT(\infty) \times EW \xrightarrow{pr} BT(\infty) \times_W EW \longrightarrow (BT(\infty) \times_W EW)_p$ is homotopic to i. Hence i and $i \circ w$ are homotopic. □

PROOF OF THEOREM 1:

It follows from Corollary 2.5 that $f\circ i$ is induced by a homomorphism $\varphi : T(\infty) \longrightarrow T'(\infty)$. We set $\Upsilon = (B\varphi)_p$. □

The proof of point a) is the same as the proof of Theorem 1.7 in [1]. Point b) follows from a) and Lemma 3.1. □

PROOF OF THEOREM 3:

Let $f,g : X \longrightarrow X'$ be two maps such that $H^*(f,\mathbb{Q}_p) = H^*(g,\mathbb{Q}_p)$. Let $i : BT_p \longrightarrow X$ be the map induced by a structure map $i : BT \longrightarrow X$. Corollary 2.5 implies that $f\circ i$ and $g\circ i$ are induced by two homomorphisms $\varphi,\Psi : T(\infty) \longrightarrow T'(\infty) \tilde{\times} W'$. We must show that φ and Ψ are conjugate.

For a finite group π let $R(\pi)$ be its complex representation ring. Let

$$R(T(\infty)) := \varprojlim_n R(T(n)) \text{ and } R(T'(\infty) \tilde{\times} W) := \varprojlim_n R(T'(n) \tilde{\times} W').$$

The Chern character $ch : K^0(\ ;Z_p) \longrightarrow \prod_i H^{2i}(\ ;\mathbb{Q}_p)$ is injective for spaces $BT(\infty)$ and $B(T'(\infty) \tilde{\times} W') = BT'(\infty) \underset{W}{\times} EW$. The group $R(T(\infty))$ is mapped injectively into $K^0(BT(\infty);Z_p)$. Hence we have

$$R(\varphi) = R(\Psi) : R(T'(\infty) \tilde{\times} W') \longrightarrow R(T(\infty)).$$

For each subgroup $S = Z/p^n$ of $T(\infty)$ the restrictions of φ and to S are conjugate by an element of W' because S is cyclic. The fact that W' is finite implies that the restrictions of φ and to any subgroup Z/p^∞ of $T(\infty)$ are conjugate by some element of W'. Once more the fact that W' is finite and the set of subgroups of the form Z/p^∞ in $T(\infty)$ is uncountable if rank $T > 1$ implies that φ and are conjugate by an element of W'. Hence $f\circ i$ and $g\circ i$ are homotopic. It follows from [12] Theorem 1 that f and g are homotopic. □

PROOF OF THEOREM 2:

We set $\chi(f) = \pi_1(\Upsilon)$ where Υ is the map from Theorem 1. The injectivity of χ follows from Theorem 3. Next one observe that $K^0(X';Z_p) = K^0((BT')_p;Z_p)^W$.

Then the proof of surjectivity is the same as in Theorem 1.5 in [13]. It is a standard application of Theorem 1 from [12]. □

PROOF OF THEOREM 4:
The fact that ψ is injective follows from Theorem 3 and the injectivity of Chern character. The proof of surjectivity is the same as in Theorem 1.5 in [13]. □

PROOF OF THEOREM 5:
Theorem 5 is a direct consequence of Theorem 2. □

PROOF OF THEOREM 6:
Let G be a connected, compact Lie group. Observe that any map $BT(\infty) \longrightarrow (BG)_p$ is induced by a homomorphism $T(\infty) \longrightarrow G$ what is an immediate consequence of [7] 1.1. Theorem. This was the crucial point to prove Theorems 1,2,3 and 4 for $X' = X(W',p,T')$. The proofs of Theorems 1,2 and 3 for $X' = (BG)_p$ are the same. Observe that $K^0((BG)_p;Z_p) = K^0((BT)_p;Z_p)^W$. Hence the proof of Theorem 4 carry over to the case $X' = (BG)_p$. □

PROOF OF COROLLARY 7: If the natural representation of W on $\pi_1(T) \otimes \mathbf{Q}_p$ is irreducible then $\pi_1(\Upsilon) : \pi_2((BT)_p) \longrightarrow \pi_2((BT)_p)$ is an isomorphism or a trivial map. The correspondence $w \longrightarrow w'$ from Theorem 7 point b) is then an isomorphism. The rest is obvious. □

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DERIVED FUNCTORS OF THE DESTABILIZATION
and
THE ADAMS SPECTRAL SEQUENCE

by Said ZARATI

Introduction

Let A be the modulo 2 Steenrod algebra, $\mathcal{M}$ the category of graded A-modules and A-linear maps of degree zero, and $\mathcal{U}$ the full sub-category of $\mathcal{M}$ whose objects are unstable A-modules. We denote by $D : \mathcal{M} \dashrightarrow \mathcal{U}$ the destabilization functor and by D_s, $s \geq 0$, its derived functors. We have a natural transformation : $D_s \dashrightarrow \Sigma D_s\Sigma^{-1}$, $s \geq 0$, induced by the adjoint of the identity $\Omega D = D\,\Sigma^{-1}$ where Σ^m, $\mathcal{M} \dashrightarrow \mathcal{M}$, $m \in \mathbb{Z}$, is the m^{th} suspension functor and Ω is the left adjoint of $\Sigma : \mathcal{U} \dashrightarrow \mathcal{U}$.

In this note we prove the following theorem wich will be more precise in section 2.3.

Theorem 1.1. Let M be a nil-closed unstable A-module. Then the natural map $\Omega D_s\Sigma^{-s} M \dashrightarrow D_s\Sigma^{-s-1}M$ is an isomorphism for every $s \geq 0$.

Using the higher Hopf invariants introduced in [7] we prove the following property of the Adams spectral sequence, in the modulo 2 cohomology, for the group {X,Y} of homotopy classes of stable maps from X to Y, in certain cases.

Theorem 1.2. : Let X and Y two pointed CW-complexes such that

(i) $\bar{H}^*(X,\mathbb{F}_2) \simeq \Sigma^2 I$ where ΣI is an injective unstable A-module.

(ii) $\bar{H}^*(Y;\mathbb{F}_2)$ is gradually finite and nil-closed.

Then, the Adams spectral sequence for the group {X,Y} degenerate at the E_2-term : $E_2^{s,s} \approx E_r^{s,s}$ for every $r \geq 2$ and $s \geq 0$.

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The infinite real projective space $\mathbb{R}P^{\infty}$ is an example of a space Y satisfying the hypotheses of theorem 1.2.

The organization of the rest of this note is as follows. In section 2 we give a characterization of nil-closed A-modules which allows us to prove the theorem 1.1 (see theorem 2.3.3). Section 3 gives the proof of theorem 1.2 and an application. We finish this note by a remark concerning the case $p > 2$.

All cohomology is taken with $\mathbb{F}_2$ coefficients. We write $H^*(\ )$ for $H^*(\ ;\mathbb{F}_2)$ and we denote by $\bar{H}^*(\ )$ the reduced modulo 2 cohomology.

2. Derived functors of the destabilization

2.1. Let A be the modulo 2 Steenrod algebra. We denote by $\mathcal{M}$ the category whose objects are graded A-modules ($M = \{M^n,\quad n \in \mathbb{Z}\}$) and whose morphisms are A-linear maps of degree zero. We denote by $\mathcal{U}$ the full sub-category of $\mathcal{M}$ whose objects are unstable A-modules (an A-module M is called unstable if $Sq^i x = 0$ for every x in M^n and every $i > n$; in particular $M^n = 0$ if $n < 0$).

The forgetful functor $\mathcal{U} \dashrightarrow \mathcal{M}$ has a left adjoint functor D : $\mathcal{M} \dashrightarrow \mathcal{U}$, called the destabilization functor, which satisfies : $\mathrm{Hom}_{\mathcal{M}}(M,N) = \mathrm{Hom}_{\mathcal{U}}(DM,N)$ for every A-module M and every unstable A-module N. The functor D : $\mathcal{M} \dashrightarrow \mathcal{U}$ is right exact, we denote $D_s : \mathcal{M} \dashrightarrow \mathcal{U}$, $s \geq 0$, its derived functors. One of the motivations for the study of the derived functors of the destabilization is the following isomorphism :

$$(2.1) \qquad \mathrm{Ext}^s_{\mathcal{M}}(M,I) \simeq \mathrm{Hom}_{\mathcal{U}}(D_s M,I)$$

for every A-module M and every unstable injective A-module I.

Let $\Sigma^m : \mathcal{M} \dashrightarrow \mathcal{M}$, $m \in \mathbb{Z}$, the m^{th} suspension functor

which associates to a module $M = \{M^n, n \in \mathbb{Z}\}$ the module $\Sigma^m M = \{M^{n-m}, n \in \mathbb{Z}\}$. The A-module structure on $\Sigma^m M$is given by $Sq^i(\Sigma^m x) = \Sigma^m Sq^i x$, x in M. The computation of $D_s\Sigma^{-t}M$, where M is an unstable A-module, is done by Lannes and Zarati [5] for $t \leq s$.
In this paragraph we will compute $D_s\Sigma^{-(s+1)}M$ for a particular unstable A-modules called nil-closed. First let us recall the definition and some properties of nil-closed unstable A-modules.

2.2. Nil-closed unstable A-modules [1], [6]

Definition 2.2.1 An unstable A-module M is called reduced if the cup-square $Sq^n : M^n \dashrightarrow M^{2n}$, $x \dashrightarrow Sq^n x$, is injective for every $n \geq 0$.

Remark 2.2.2 We can verify easily that an unstable A-module is reduced if and only if it does not contain a non trivial nilpotent sub-A-module. An unstable A-module N is called nilpotent if for every x in M^n, there exist $r \geq 0$ such that $Sq^{2^r n} \ldots\ldots Sq^n x = 0$.

Definition 2.2.3. An unstable A-module M is called nil-closed if
(i) M is reduced
(ii) An element x in M of even degree is in the image of the cup-square if and only if $Q_i x = 0$, for all $i \geq 0$, where Q_i is the i^{th} Milnor primitive in A.

Example 2.2.4 Let $B\mathbb{Z}/2$ denote a classifying space of the group $\mathbb{Z}/2$. The unstable A-module $H^*(B\mathbb{Z}/2)$ is nil-closed indeed, as a graded $\mathbb{F}_2$-algebra $H^*(B\mathbb{Z}/2)$ is freely generated by one generator of degree one.

2.3.Computation of $D_s\Sigma^{-(s+1)}M$, M nil-closed and $s \geq 0$.

2.3.1 To state our result we use the functor $R_s : \mathcal{U} \dashrightarrow \mathcal{U}$, $s \geq 0$, introduction in [5] page 29 (see also [9]) whose main properties are:

(i) The module R_sM is a sub-A-module of $H^*(B(\mathbb{Z}/2)^s) \otimes M$. In particular R_sM is an unstable A-module.

(ii) Let $H^*(B\mathbb{Z}/2) = \mathbb{F}_2[u]$ where u is of degree one. We denote by $L_s = H^*(B(\mathbb{Z}/2)^s)^{GL_s(\mathbb{Z}/2)}$ the Dickson algebra, that is the sub-algebra of $H^*(B(\mathbb{Z}/2)^s)$ of invariants under the natural action of the general linear group $GL_s(\mathbb{Z}/2) = GL((\mathbb{Z}/2)^s)$. The module R_sM is the L_s-module generated by the elements $St_s(x)$, x in M. These elements $St_s(x)$ are defined inductively by :

$$St_0(x) = x \quad , \quad x \in M.$$

$$St_1(x) = \sum_{i=0}^{n} u^{n-i} \otimes Sq^i x \quad , \; x \in M^n.$$

$$St_s(x) = St_1(St_{s-1}(x)) \quad , \; s \geq 1, \; x \in M$$

iii) Let $E_+\mathfrak{S}_{2^s}$ be the disjoint union of a base point and a contractible space on which the symmetric group $\mathfrak{S}_{2^s}$ acts freely. For any pointed space X, we denote by $\mathfrak{S}_{2^s}X$ the quotient of the space $E_+\mathfrak{S}_{2^s} \wedge (X \wedge .... \wedge X)$, X is smashed with itself 2^s times, by the diagonal action of $\mathfrak{S}_{2^s}$ ($\mathfrak{S}_{2^s}$ acts on $X \wedge ...... \wedge X$ by permutation of the factors). Let $\Delta_s : B_+(\mathbb{Z}/2)^s \wedge X \dashrightarrow \mathfrak{S}_{2^s} X$ be a "Steenrod diagonal" determined by a bijection between $(\mathbb{Z}/2)^s$and $\{1,2,....,2^s\}$. The unstable A-module R_sH^*X is the image of Δ_s^* in the modulo 2 cohomology.

2.3.2 Let $\Omega : \mathcal{U} \dashrightarrow \mathcal{U}$ be the left adjoint functor of $\Sigma : \mathcal{U} \dashrightarrow \mathcal{U}$, that is :

$$Hom_{\mathcal{U}}(M,\Sigma N) = Hom_{\mathcal{U}}(\Omega M,N)$$

for every unstable A-modules M and N.

We are now ready to state the main result of this paragraph which will be proved in 2.6

Theorem 2.3.3 : Let M be a nil-closed unstable A-module. There exist a natural isomorphism :

$$D_s \Sigma^{-(s+1)} M \simeq \Omega R_s M \quad , \quad s \geq 0$$

2.4. Some properties of nil-closed unstable A modules

In this paragraph we give two characterizations of nil-closed unstable A-modules which allow us to prove theorem 2.3.3

2.4.1. The first characterization of nil-closed unstable A-modules is given in [6] page 314.

Proposition 2.4.1.1. Let M be an unstable A-module. The following conditions are equivalent.

(i) M is nil-closed

(ii) $\mathrm{Ext}_{\mathcal{U}}^{i}(N,M) = 0$ for every nilpotent N in $\mathcal{U}$ and i = 0,1.

(iii) There exist an injective resolution of M starting

$$0 \dashrightarrow M \dashrightarrow K^o \dashrightarrow K^1$$

where K^o and K^1 are reduced injective unstable A-modules.

Remark 2.4.1.2. The condition (iii) of the proposition 2.4.1.1 can be replaced by the following (see [4] page 163)

(iii)' There exist an injective resolution of M starting

$$0 \dashrightarrow M \dashrightarrow \prod_{\alpha} H^*(BV_\alpha) \dashrightarrow \prod_{\beta} H^*(BV_\beta)$$

where V_α and V_β are elementary abelian 2-groups. We have the following easy corollary.

Corollary 2.4.1.3. Let M be an unstable A-module. The following conditions are equivalent.

(i) M is nil-closed.

(ii) There exist a nil-closed unstable A-module L containing M such that the quotient L/M is reduced.

2.4.2. Another characterization of nil-closed.

Proposition 2.4.2.1 Let M be an unstable A-module. The following properties are equivalent.

(i) M is nil-closed

(ii) M and ΩM are reduced

The proof of this proposition is based on the following technical lemma. Let Q_i, $i \geq 0$, the i^{th} Milnor primitive in A and Sq_k the cohomology operation defined by $Sq_k\, x = Sq^{n-k}x$ where x is an element of degree n of an A-module ($Sq^{n-k} = 0$ if $n < k$).

Lemma 2.4.2.2 Let M be an unstable A-module. We have the following formula :

$$(Q_{i+1} \circ Sq_1)(x) = (Sq_0 \circ Q_i)(x)$$

for every x in M and every $i \geq 0$.

Proof. The proof is done by induction on i using Adem's relations. Recall that the elements Q_i, $i \geq 0$, are defined by

$$Q_0 = Sq^1$$

$$Q_i = Q_{i-1}\, Sq^{2^i} + Sq^{2^i}\, Q_{i-1}\,, \quad i \geq 1$$

The case $i = 0$. Let x be an element of degree n of an unstable A-module, we have :

$$Sq^1 Sq_1(x) = Sq^1 Sq^{n-1}(x) = \begin{cases} 0 & \text{if } n \equiv 0(2). \\ Sq_0 x & \text{if } n \equiv 1(2). \end{cases}$$

$$Sq^2 Sq_1(x) = Sq^2 Sq^{n-1}(x) = \begin{cases} 0 & \text{if } 2 > 2n-1. \\ Sq^2 Sq^1 x & \text{if } 2 = 2n-2. \\ \displaystyle\sum_{c=0}^{1} C_{n-2-c}^{2-2c}\, Sq^{n+1-c}\, Sq^c x & \text{if } 2 < 2n-2. \end{cases}$$

$$= \begin{cases} 0 & \text{if } n = 1 \\ Sq^n Sq^1 x & \text{if } n \geq 2. \end{cases}$$

These formulas imply the case i = 0 because we have :

$$\begin{aligned} Q_1 Sq_1 x &= Sq^3 Sq_1 x + Sq^2 Sq^1 Sq_1 x \\ &= Sq_0 Sq^1 x \\ &= Sq_0 Q_1 x. \end{aligned}$$

Suppose $Q_i Sq_1 x = Sq_0 Q_{i-1} x$ for evry i : $0 \leq i \leq j-1$ and for every element x (of degree n) of an unstable A-module. To prove this formula for i = j we consider :

$$Q_j Sq_1(x) = Sq^{2^j} Q_{j-1} Sq_1(x) + Q_{j-1} Sq^{2^j} Sq_1(x)$$

$$= Sq^{2^j} Sq_0 Q_{j-2}(x) + Q_{j-1} Sq^{2^j} Sq_1(x) \text{ , (inductive assumption)}$$

$$= Sq_0 Sq^{2^{j-1}} Q_{j-2}(x) + Q_{j-1} Sq^{2^j} Sq_1(x).$$

In the last equality we have used the following easy formula :

$$Sq^k Sq_0 = \begin{cases} 0 & \text{if } k \equiv 1(2) \text{ .} \\ Sq_0 Sq^{\frac{k}{2}} & \text{if } k \equiv 0(2) \text{ .} \end{cases}$$

If remains to show :

$$Q_{j-1} Sq^{2^j} Sq_1(x) = Sq_0 Q_{j-2} Sq^{2^{j-1}} (x).$$

Using the unstability of M and Adem's relations we prove :

$$Sq^{2^j} Sq_1(x) = \begin{cases} 0 & \text{if } n \leq 2^{j-1} . \\ Sq_1 Sq^{2^{j-1}}(x) & \text{if } n \geq 2^{j-1} + 1 . \end{cases}$$

This formula gives :

$$Q_{j-1} Sq^{2^j} Sq_1(x) = \begin{cases} 0 & \text{if } n \leq 2^{j-1} . \\ Q_{j-1} Sq_1 Sq^{2^{j-1}}(x) & \text{if } n \geq 2^{j-1} + 1 . \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \leq 2^{j-1} . \\ Sq_0\, Q_{j-2} Sq^{2^{j-1}}(x) & \text{if } n \geq 2^{j-1} + 1 \text{ , (inductive assumption).} \end{cases}$$

$$= Sq_0 Q_{j-2} Sq^{2^{j-1}}(x)$$

2.4.3. Functor R_S and nil-closed A-modules.

Proposition 2.4.3.1. Let M be an unstable A-module. If M is nil-closed then $R_S M$ is nil-closed.

Proof : Let (*) $0 \dashrightarrow M \dashrightarrow \prod_\alpha H^*(V_\alpha) \dashrightarrow \prod_\beta H^* V_\beta$ be the beginning of an injective resolution of the nil-closed unstable A-module M (see remark 2.4.1.2). The functor R_S is exact and comutes with products (see [6]) ; then, when we apply it to the exact sequence (*) we get the following exact sequence :

$$0 \dashrightarrow R_s M \dashrightarrow \prod_\alpha R_s H^* V_\alpha \longrightarrow \prod_\beta R_s H^* V_\beta \ .$$

The computation of R_sH^*V, where V is an elementary abelian 2-group, is done by induction on s (see [6] page 321). Let $V_s = (\mathbb{Z}/2)^s \oplus V, R_sH^*V$ is the sub-module of $H^*(V_s)$ of invariants under the action of the sub-group of $GL(V_s)$, denoted $GL(V_s,V)$, of automorphisms of V_s which induces the identity on V. The proposition 2.4.3.1 is now a consequence of the corollary 2.4.1.3 and of the fact that the sub-A-module $H^*(V)^G$, $G < GL(V)$, of $H^*(V)$ is nil-closed (see [6] page 314).

Remark 2.4.3.2. A different proof of the proposition 2.4.3.1 for s = 1 is given in [3]

2.5.Proof of the proposition 2.4.2.1.

2.5.1. First let us recall some properties of the functor Ω introduced in 2.3.2. Let $\Phi : \mathcal{U} \dashrightarrow \mathcal{U}$ be the functor which associates to each unstable A-module A-module M, the "double of M", denoted ΦM, defined by :

$$(\Phi M)^n = \begin{cases} 0 & \text{if } n \equiv 1(2). \\ M^{n/2} & \text{if } n \equiv 0(2). \end{cases} \quad \text{and } Sq^i(\Phi x) = \begin{cases} 0 & \text{if } i \equiv 1(2). \\ \Phi Sq^{i/2}x & \text{if } i \equiv 0(2). \end{cases}$$

we verify that the map $Sq_0 : \Phi M \dashrightarrow M, \Phi x \mapsto Sq_0x$, is A-linear and that the kernel and the cokernel of Sq_0 are respectively $\Sigma\Omega_1 M$ and $\Sigma\Omega M$ where Ω_1 is the first and unique derived functor of Ω (see [5] page 30). We remark that an unstable A-module M is reduced if and only if $\Omega_1 M = 0$.

2.5.2. Proof the proposition 2.4.2.1. (i) ==> (ii). It suffices to prove that ΩM is reduced. Let y be an element of $(\Omega M)^k$ such that $Sq_0 y = 0$. To prove that y = 0 we envision two cases :

(*) The case $k \equiv 0(2)$. in this case $(\Omega M)^k = (\Sigma^{-1}M/\mathrm{Im}Sq_0)^k = M^{k+1}$ then $y = \Sigma^{-1}x$ where x is an element of M^{k+1}. $Sq_0 y =$

$\Sigma^{-1}Sq_1x = 0$. This implies that $Sq^1Sq_1x = Sq_0x = 0$ and then $x = o$ since M is reduced. This shows that $y = \Sigma^{-1}x = 0$.

(**) The case $k \equiv 1(2)$. In this case $(\Omega M)^k = (\Sigma^{-1}M/\mathrm{Im}Sq_0)^k = (M/\mathrm{Im}Sq_0)^{k+1}$ then $y = \Sigma^{-1}[x]$ where x is an element of M^{k+1}. $Sq_0y = \Sigma^{-1}[Sq_1x] = \Sigma^{-1}Sq_1x = 0$ (Sq_1x is an element of M of odd degree) ; then, $Sq_1x = 0$. This implies that $Q_{i+1}Sq_1x = 0$ for every $i \geq 0$. Using the lemma 2.4.2.2 we get : $Sq_0Q_ix = 0$ for every $i \geq 0$ and then $Q_i(x) = 0$, $i \geq 0$, since M is reduced. Now x is an element of even degree of a nil-closed A-module M annulated by all the Q_i, $i \geq 0$ then x is in the image of Sq_0 and then $y = \Sigma^{-1}[x] = 0$.

(ii) ==> (i). Since M is reduced then M embeds in a reduced injective unstable A-module K (see [6] page 313). To prove M nil-closed it suffices to prove that the quotient K/M is reduced and to use the corollary 2.4.1.3. If we apply the functor Ω to the exact sequence 0 ---> M ---> K ---> K/M ---> 0 we get the following exact sequence : 0 ---> Ω_1(K/M) ---> ΩM ---> ΩK ---> Ω(K/M) ---> 0. The module Ω_1(K/M) is trivial because it is a nilpotent sub-A-module of the reduced unstable A-module ΩM.Ω_1(K/M) is nilpotent because, by definition, it is concentrated in odd degree. This shows that K/M is reduced and then M is nil-closed

2.6. Proof of the theorem 2.3.3

Let M be an unstable A-module. Consider the following exact sequence introduced in [5] page 32 :

(*) $0 \to \Omega D_s\Sigma^{-s}M \to D_s\Sigma^{-(s+1)}M \to \Omega_1 D_{s-1}\Sigma^{-s}M \to 0$

When M is reduced, the module $D_s\Sigma^{-s}M$ is naturally isomorphic to R_sM ([5] proposition 4.6.2). The exact sequence becomes :

(**) $0 \to \Omega R_s M \to D_s\Sigma^{-(s+1)}M \to \Omega_1 D_{s-1}\Sigma^{-s}M \to 0$

The proof of the theorem 2.3.3 is done by induction on s.For s = 0 it is the identity $D\Sigma^{-1} = \Omega D$. Suppose that : (H_k) $D_k\Sigma^{-(k+1)}M \simeq \Omega R_kM$ for every k : $0 \leq k \leq s-1$ and every nil-closed A-module M.

To prove (H_S) it suffices to remark that since M is nil-closed then, by the proposition 2.4.3.1, $R_{S-1}M$ is nil-closed. This implies that $\Omega R_{S-1}M$ is reduced (proposition 2.4.2.1), that is : $\Omega_1 \Omega R_{S-1}M = 0$. The exact sequence (**) and the inductive assumption give, for M nil-closed, the following natural isomorphism : $D_S\Sigma^{-(s+1)}M \simeq \Omega R_S M$.

3. Applications.

The topological applications of this note are based on the higher Hopf invariants introduced by Lannes and Zarati in [7]. Let X and Y be two pointed CW-complexes. We donote by {X,Y} the group of homotopy classes of stable maps from X to Y. The Adams spectral sequencee, in the modulo 2 cohomology, for the group {X,Y} is denoted $\{E_r^{s,s} = E_r^{s,s}\ (X,Y),\ s \geq 0,\ d_r\}_{r\geq 2}$; $d_r : E_r^{s,s} \dashrightarrow E_r^{s+r,s+r-1}$ is the differential. We have the following theorem which will be proved in the section 3.4

Theorem 3.1 Let X and Y be two pointed CW-complexes such that :
(i) $\bar{H}^*(X) \simeq \Sigma^2 I$ where ΣI is an injective unstable A-module.
(ii) $\bar{H}^*(Y)$ is gradually finite ($\dim_{\mathbb{F}_2} H^n(Y) < +\infty$, $n \geq 0$) and nil-closed.
Then, the Adams spectral sequence, in the modulo 2 cohomology, for the group {X,Y} degenerate at the E_2-term $E_2^{s,s} \approx E_r^{s,s}$ for every $r \geq 2$ and $s \geq 0$.

Remark 3.2 In [8] (see also [7]) there exist an analogous property of the Adams spectral sequence as in theorem 3.1 in the following two cases :
(3.2.1) (i) $\bar{H}^*$ (X) is a reduced injective unstable A-module.
(ii) $\bar{H}^*$ (Y) is gradually finite.

(3.2.2) (i) $\Sigma\bar{H}^*(X)$ is an injective unstable A-module.

(ii) $\bar{H}^*(Y)$ is a reduced gradually finite unstable A-module.

Corollary 3.3. Let X and Y be two pointed CW complexes which verify the hypothesis (i) and (ii) of theorem 3.1 and such that the Adams spectral sequence for the group {X,Y} converges.
Then, the natural map :

$$h : \{S^1X, Y\} \dashrightarrow \mathrm{Hom}_{\mathcal{U}}(\bar{H}^*Y, \Sigma\bar{H}^*X)$$

is surjective.

Proof. Theorem 3.1 shows that the term $E_2^{0,1} \simeq \mathrm{Hom}_{\mathcal{U}}(\bar{H}^*Y, \Sigma\bar{H}^*X)$ persists at the infinity. Since the Adams spectral sequences for {X,Y} converges, then the natural map $h : \{S^1X, Y\} \dashrightarrow \mathrm{Hom}_{\mathcal{U}}(\bar{H}^*Y, \Sigma\bar{H}^*X)$ is surjective.

3.4. Proof of the theorem 3.1

Consider the following diagram whose commutativity is proved in [7], [8].

$$\begin{tikzcd}
 & E_\infty^{s,s} \arrow[rr, "\mathcal{H}_\infty^{s,s}"] & & \mathrm{Hom}_{\mathcal{U}}(R_s\bar{H}^*Y, \Sigma^2 I) \arrow[d, "\wr"] \\
Z_{2,\infty}^{s,s} \arrow[ur] \arrow[d, hook] & & & \mathrm{Hom}_{\mathcal{U}}(\Omega R_s\bar{H}^*Y, \Sigma I) \\
E_2^{s,s} \overset{1}{\approx} \mathrm{Ext}_{\mathcal{M}}^s(\Sigma^{-s-1}\bar{H}^*Y, \Sigma I) \arrow[rrr, "\mathcal{H}_2^{s,s}", "\sim"', "2"'] & & & \mathrm{Hom}_{\mathcal{U}}(D_s\Sigma^{-s-1}\bar{H}^*Y, \Sigma I) \arrow[u, "3\ \wr"']
\end{tikzcd}$$

($Z_{2,\infty}^{s,s}$ is the inverse image of $E_\infty^{s,s}$ in E_2, $\mathcal{H}_\infty^{s,s}$ and $\mathcal{H}_2^{s,s}$ are the Hopf invariants at the E_∞-level and the E_2-level respectively). The isomorphism 1 is clear since $E_2^{s,s} = \mathrm{Ext}^s_{\mathcal{M}}(\Sigma^{-s}\bar{H}^*Y, \Sigma^2 I) \simeq \mathrm{Ext}_{\mathcal{M}}^s(\Sigma^{-s-1}\bar{H}^*Y, \Sigma I)$. The isomorphism 2 follows from the fact that ΣI is an injective unstable A-module. The isomorphism 3 is a consequence of the theorem 2.3.3.

By definition of the differential $d_r : E_r^{s-r,s-r+1} \dashrightarrow E_r^{s,s}$ we have : $\mathrm{Im}\, d_r \subset Z_{r,\infty}^{s,s}$ and $Z_{r+1,\infty}^{s,s} = Z_{r,\infty}^{s,s}/\mathrm{Im}\, d_r$ (see, for example [2]). It follows from the commutativity of the previous diagram that $Z_{2,\infty}^{s,s} \xrightarrow{\simeq} E_\infty^{s,s}$ and then the differential $d_r : E_r^{s-r,s-r+1} \dashrightarrow E_r^{s,s}$, $r \geq 2$, is trivial. To prove that the differential $d_r : E_r^{s,s} \dashrightarrow E_r^{s+r,s+r-1}$, $r \geq 2$, is trivial we use the following isomorphism $E_2^{s,t}(X,Y) \approx E_2^{s,t+1}(X,SY)$ which allows us to use the results of [8] (see remark ?.2).

4. The case p > 2

In this note we can't replace 2 by an odd prime p since the proposition 2.4.2.1, which is the main algebraic result of this note, is false for $p > 2$. Here is an example ; the unstable A-module $H = H^*(B(\mathbb{Z}/p) ; \mathbb{F}_p)$ is the tensor product, $E(u) \otimes \mathbb{F}_p[v]$ of an exterior algebra on one generator u of degree one and of a polynomial algebra generated by v the Bockstein of u. We know that H is nil-closed (see [6]) but ΩH is not λ-projective (λ is the analog of Sq_0 for $p > 2$) ; the element $\Sigma^{-1} v^2$ of degree three of ΩH is such that : $\lambda(\Sigma^{-1}v^2) = \Sigma^{-1}\beta P^1 v^2 = 0$.

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