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M. C. CRABB

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THE FULLER INDEX AND $\mathbb{T}$-EQUIVARIANT STABLE HOMOTOPY THEORY

by M.C. CRABB

0. Introduction

In a remarkable paper [8], published more than twenty years ago, Fuller introduced an index which counts periodic orbits of smooth flows. Let w be a smooth vector field defined on a (finite-dimensional) closed manifold X and $\theta_t : X \to X$, $(t \in \mathbb{R})$, the corresponding flow (so that $\theta_0 = 1$ and $\dot{\theta}_t = w(\theta_t)$, where the dot denotes differentiation). Suppose that U_1 is an open subspace of $(0,\infty) \times X$ such that the set

$$(0.1) \qquad F = \{(T,x) \in U_1 \mid \theta_T x = x\}$$

is compact. To such a field w and open set U_1, Fuller associates a $\mathbb{Q}$-valued index, which vanishes if F is empty.

In 1985, Ize [10] and Dancer [6] observed, independently, that the natural setting for Fuller's index is $\mathbb{T}$-equivariant homotopy theory, $\mathbb{T}$ being the circle group $\mathbb{R}/\mathbb{Z}$. My purpose here is to describe their work from the viewpoint of algebraic topology using the standard methods of equivariant fixed-point theory over a base.

The relevance of the $\mathbb{T}$-equivariant theory is not hard to see. Indeed, if $(T,x) \in F$, (0.1), then the compactness of F implies that $(T,\theta_t x) \in F$ for all $t \in \mathbb{R}$ and, also, that x is not a stationary point of the flow $(w(x) \neq 0)$. So we can define a fixed-point-free circle action on F by:

$$(0.2) \qquad [t].(T,x) = (T,\theta_{tT}x),$$

for $t \in \mathbb{R}$, $[t] = t + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. The Fuller index is, in a sense to be made precise, a count of this set F, with the fixed-point-free $\mathbb{T}$-action, over the base $(0,\infty)$.

Each point $(T,x) \in F$ determines a periodic solution $\gamma(t) = \theta_t x$, of period T, of the differential equation:

$$\dot{\gamma} - w(\gamma) = 0, \tag{0.3}$$

or, by re-scaling, a solution $\alpha: \mathbb{R} \to X$, $\alpha(t) = \theta_{tT}x$, of period 1 of:

$$\dot{\alpha} - Tw(\alpha) = 0. \tag{0.4}$$

It is convenient to make no distinction in notation between a map $\alpha: \mathbb{R} \to X$ of period 1 and the corresponding loop $\alpha: \mathbb{R}/\mathbb{Z} = \mathbb{T} \to X$. Then we can think of solutions of (0.4) as zeros of a vector field on the infinite-dimensional manifold $M = LX$ of smooth loops $\mathbb{T} \to X$ in the following way. (See, for example, Atiyah [1] and Bismut [3].)

Recall that the tangent space $\tau_\alpha M$ at a point $\alpha \in M$, $\alpha: \mathbb{T} \to X$, can be identified with the space of smooth sections of $\alpha^*\tau X$ over $\mathbb{T}$. So we can regard $t \mapsto w(\alpha(t))$ as a tangent vector $w(\alpha) \in \tau_\alpha M$, and the vector field w on X thus defines a vector field, of the same name, on M. The circle acts on M by rotating loops: $([t].\alpha)(u) = \alpha(t+u)$, for $t,u \in \mathbb{R}$. This $\mathbb{T}$-action has a generating vector field, s say, given by differentiation:

$$s(\alpha) = \dot{\alpha}. \tag{0.5}$$

The zero-set of s, or the fixed subspace $M^{\mathbb{T}}$, is the space X of constant loops.

Now we have a family $v_T = s - Tw$, $T > 0$, of $\mathbb{T}$-equivariant vector fields on M, parametrized by $(0,\infty)$, and the zero-set of v_T is precisely the set of solutions of (0.4). Let U_∞ be the open subset $\{(T,\alpha) \in (0,\infty) \times M \mid (T,\alpha(t)) \in U_1 \text{ for all } t \in \mathbb{R}\}$ of $(0,\infty) \times M$. Then the zero-set

$$\{(T,\alpha) \in U_\infty \mid v_T(\alpha) = 0\} \tag{0.6}$$

is equivariantly homeomorphic to F, (0.1) and (0.2), and so compact.

The problem is to define an index for such a family of vector fields v_T with compact zero-set in some open subspace of $(0,\infty) \times M$. There are technical difficulties in infinite-dimensions: in order to apply the Leray-Schauder theory (as described in [9], for example) it is necessary to replace v_T by a "renormalized" field satisfying a certain compactness condition. This analysis, which is joint work with A.J.B. Potter, will appear elsewhere. In this paper, following Dancer [6], I shall concentrate on the analogous finite-dimensional problem, which illustrates all the algebraic topological features of the Fuller index. This is done in Section

2. Section 1 reviews the, now standard, equivariant index theory over a base for zeros of vector fields and fixed-points of maps, developed by Dold, Becker and Gottlieb in the mid seventies.

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1. The vector-field index

This section contains an outline, in a form tailored to the applications, of the Poincaré-Hopf index theory for vector fields. Whilst this theory can be viewed as a special case of the Lefschetz fixed-point theory, it seems worth maintaining a conceptual distinction. We confine the discussion to the non-equivariant theory. The modifications needed to produce the G-equivariant index theory, for a compact Lie group G (acting smoothly on manifolds), are technical rather than conceptual. The treatment here is strongly influenced by the work of Dold (as in [7] and the references there). A detailed account can be found in [12].

Consider first a (continuous) vector field v defined on an open subset U of a (finite-dimensional) Euclidean space V, and suppose that the zero-set

(1.1) $$\mathrm{Zero}(v) = \{x \in U \mid v(x) = 0\}$$

is compact. The basic index, $\tilde{I}(v,U)$ say, is a stable map $S^0 \to U_+$ (where the subscript "+" denotes adjunction of a disjoint basepoint). It is defined by an explicit geometric construction in the style of Pontrjagin-Thom as follows.

We can regard the vector field v simply as a map $v\colon U \to V$. Let $N \subseteq V$ be an open neighbourhood of Zero(v) such that $\overline{N}$ is compact and $\overline{N} \subseteq U$, and choose a (finite) open ball B, centre O, in V so small that $v(x) \notin B$ for all $x \in \overline{N} - N$. Using a superscript "+" for one-point-compactification, we define a map $q\colon V^+ \to (V/(V-B)) \wedge U_+$, by $q(x) = [v(x),x]$ if $x \in \overline{N}$, $q(x) = *$ (basepoint) if $x \notin N$. Then, identifying $V/(V-B) = B^+$ with V^+ by radial extension, we obtain a well-defined homotopy class $V^+ \to V^+ \wedge U_+$, which represents the stable map $\tilde{I}(v,U)\colon S^0 \to U_+$.

1.2 REMARK. At this level the vector-field and fixed-point problems are indistinguishable. The construction just described defines the Lefschetz fixed-point index of the map $f: U \to V$ given by $f(x) = x - v(x)$. The zeros of v are the fixed-points of f.

Two fundamental properties of the index are evident from the construction.

1.3 PROPERTIES OF THE INDEX.

(a) *Suppose that* U' *is an open subset of* U *containing* Zero(v). *Then* $\tilde{I}(v,U) = i_+ \circ \tilde{I}(v,U')$, *where* i_+ *is the inclusion*.

(b) *Suppose that* U *is a disjoint union of open subsets* U_1 *and* U_2. *Then* $\tilde{I}(v,U) = i_+^1 \circ \tilde{I}(v,U_1) + i_+^2 \circ \tilde{I}(v,U_2)$, *where* i_+^1 *and* i_+^2 *are the respective inclusions of* U_1 *and* U_2 *in* U.

Composing $\tilde{I}(v,U)$ with the map $S^0 \to U_+$ which collapses U to a point, we obtain a stable map $S^0 \to S^0$ or, in other words, an element, $I(v,U)$ say, of the stable cohomotopy ring $\omega^0(*)$. (The symbol "ω" is used for unreduced stable homotopy.) This class $I(v,U)$ is the traditional Poincaré-Hopf index. Of course, in this case it is just an integer and determined by $\mathbb{Z}$-cohomology. The definitions have been formulated in this way so as to generalize directly to the equivariant bundle theory.

Next we recall the computation of the index for a field with isolated zeros. Suppose that Zero(v) lies in the interior of the unit disc $D(V)$ in V and that $D(V) \subseteq U$. Then $I(v,U) \in \omega^0(*)$ is the stable homotopy class represented by the map of spheres:

$$S(V) \to S(V) \; : \; x \mapsto \frac{1}{|v(x)|} v(x), \tag{1.4}$$

(so in this case the classical degree). With the additivity of the index, (1.3)(b), this determines $I(v,U)$ when Zero(v) is discrete.

In the differentiable case, the index of a non-degenerate zero lies in the image of the J-homomorphism. Suppose that the vector field v is continuously differentiable (C^1) with Zero(v) $= \{0\}$ and the derivative $(Dv)(0): V \to V$ invertible. Then $(Dv)(0)$ defines an element "sign det" of $KO^{-1}(*) = \mathbb{Z}/2$, and $I(v,U)$ is the image of this class under the J-homomorphism

$$J \; : \; KO^{-1}(*) \to \omega^0(*)^{\cdot} \subseteq \omega^0(*) \tag{1.5}$$

to the group of units $\omega^0(*)^{\cdot} = \{\pm 1\}$ in the stable cohomotopy ring.

The first extension of the theory is from Euclidean space to a (finite-dimensional, smooth) manifold. Let v now be a vector field, with compact zero-set, on an open subset U of a closed manifold M. The index $\tilde{I}(v,U)$, a stable map $S^0 \to U_+$, is defined by embedding M in Euclidean space V. Let ν be the normal bundle of the embedding and choose an open tubular neighbourhood $M \subseteq N \subseteq V$, where N is an open disc-bundle in ν. Write $r\colon N \to M$ for the projection. Then we can identify the tangent-bundle τN with $r^*(\tau M \oplus \nu)$ and extend v to a field $\bar{v}$ on $r^{-1}U$, with the same zeros, by: $\bar{v}(x) = (v(rx),x) \in \tau_{rx}M \oplus \nu_{rx}$. The index $\tilde{I}(v,U)$ is defined as the composition $r_+ \circ \tilde{I}(\bar{v},r^{-1}U)\colon S^0 \to (r^{-1}U)_+ \to U_+$.

1.6 REMARK. Let $A \subseteq U$ be a compact manifold of codimension zero with $\mathrm{Zero}(v) \subseteq A - \partial A$. (Such a manifold can always be obtained as $\psi^{-1}[c,\infty)$, where c is a regular value, $0 < c < 1$, of a smooth function $\psi\colon U \to \mathbb{R}$ which is 1 on a neighbourhood of Zero(v) and 0 outside a compact set.) Then we can form the relative, stable cohomotopy, Euler class of τA with respect to the nowhere-zero section v on ∂A. This is an element of the stable cohomotopy of the relative Thom space $(A,\partial A)^{-\tau A}$: $\gamma(\tau A, v|\partial A) \in \omega^0(A,\partial A;-\tau A)$ in the notation of [5:1]. By duality this group is identified with $\omega_0(A)$ and the relative Euler class gives a stable map $S^0 \to A_+$. Its composition with the inclusion $A_+ \to U_+$ is equal to the index $\tilde{I}(v,U)$. (This can be established by arguing from the definitions: the duality between $(A,\partial A)^{-\tau A}$ and A_+ is itself defined using Gysin maps and so, ultimately, by the Pontrjagin-Thom construction. Compare the proof of (2.5).)

From $\tilde{I}(v,U)$ we again obtain a Poincaré-Hopf index $I(v,U) \in \omega^0(*)$ by mapping U_+ to S^0. (By including U_+ in M_+ one also obtains an intermediate index, sometimes called a transfer, in $\omega_0(M)$.)

1.7 REMARK. In this case the vector-field index is related to the fixed-point index as follows. Choose a Riemannian metric on M. Then, for all sufficiently small $\varepsilon > 0$, the fixed points of the map $x \mapsto \exp_x(-\varepsilon v(x))\colon U \to M$ are the zeros of v and its index is $\tilde{I}(v,U)$.

We begin the bundle theory by considering a trivial bundle $p\colon B \times V \to B$, where B is a compact ENR and V an Euclidean space. Write $\tau(p)$ for the bundle of tangents along the fibres of p. (Here it is simply the trivial bundle with fibre V.) Suppose that v is a section of $\tau(p)$, defined on an open subset $U \subseteq B \times V$, with compact zero-set. Thus v is a family of vector fields v_b, parametrized by $b \in B$, defined on open subsets $U_b = \{x \in V \mid (b,x) \in U\}$ of V. Carrying out the construction of the basic index fibrewise, we obtain a stable map over $B : B \times S^0 \to U_{+B}$, where $U_{+B} = U \amalg B$ is obtained by adjoining a disjoint basepoint in each fibre. We denote this index over B by $\tilde{I}_B(v,U)$. Composition with the map $U_{+B} \to B \times S^0$, induced by p, which collapses each fibre of U to a point, gives a stable map over $B : B \times S^0 \to B \times S^0$, that is, an element, $I_B(v,U)$ say, of $\omega^0(B)$.

Again, we can easily treat families of isolated zeros. If $\mathrm{Zero}(v) \subseteq B \times (D(V) - S(V)) \subseteq B \times D(V) \subseteq U$, $I_B(v,U)$ is represented by a self-map, given on fibres by (1.4), of the (trivial) sphere-bundle $B \times S(V)$. When v is C^1 (in the sense that it is differentiable on fibres with its derivative Dv continuous on U), if $\mathrm{Zero}(v) = B \times \{0\}$ and each $(Dv_b)(0) : V \to V$ is invertible, then $I_B(v,U)$ is the image under

$$(1.8) \qquad J : KO^{-1}(B) \to \omega^0(B)^{\cdot} \subseteq \omega^0(B)$$

of the K-theory class determined by the automorphism $(Dv)(0)$ of the (trivial) vector bundle $B \times V$ over B.

From the vector bundle we can proceed to a trivial bundle $p: B \times M \to B$ with fibre a closed manifold M. If v is a family of vector fields defined on an open set $U \subseteq B \times M$, (that is, a section of the pull-back $\tau(p)$ of τM), with $\mathrm{Zero}(v)$ compact, indices $\tilde{I}_B(v,U) : B \times S^0 \to U_{+B}$ over B and $I_B(v,U) \in \omega^0(B)$ are defined by embedding the bundle of manifolds in a vector bundle (such as $B \times V \to B$).

REMARK 1.9. By including U in M we get a stable map over B: $B \times S^0 \to B \times M_+$ or, equivalently, a stable map $B_+ \to M_+$ lifting $I_B(v,U) : B_+ \to S^0$.

We shall need a form of relative index. Suppose that $A \subseteq B$ is a closed sub-ENR such that there are no zeros of v over A: $p^{-1}A \cap \mathrm{Zero}(v) = \emptyset$. Then we can replace U by the smaller open neighbourhood $U \cap p^{-1}(B-A)$ of $\mathrm{Zero}(v)$. This gives us representatives of $I_B(v,U) : B \times S^0 \to B \times S^0$ which are trivial (not just null-homotopic) over A and so a relative index $I_{(B,A)}(v,U) \in \omega^0(B,A)$. (As in (1.9) we get a stable lift $B/A \to M_+$, too.)

These constructions are functorial in the base B. If $a\colon B' \to B$ is a map from a compact ENR B', v lifts to a vector field v' on $U' = (a \times 1)^{-1}U \subseteq B' \times M$. Then $\tilde{I}_{B'}(v',U')$ is the pull-back of $\tilde{I}_B(v,U)$ and $I_{B'}(v',U') = a^*I_B(v,U) \in \omega^0(B')$. This includes, as a special case, the homotopy invariance of the index.

The final generalization is from a trivial bundle to an arbitrary manifold over a compact ENR B. Let $p\colon E \to B$ be such a manifold over B, with fibre a closed manifold. (The usual examples are trivial bundles $B \times M \to B$ as above and locally trivial smooth fibre-bundles.) If v is a section of $\tau(p)$ on an open set $U \subseteq E$, the indices $\tilde{I}_B(v,U)\colon B \times S^0 \to U_{+B}$ and $I_B(v,U) \in \omega^0(B)$ are defined whenever Zero(v) is compact. (We also have stable transfer maps: $B \times S^0 \to E_{+B}$ over B and, factoring out basepoints, the induced map $B_+ \to E_+$.)

The index theory over a base provides a natural framework for discussion of the global bifurcation theory of Rabinowitz [11]. (Developments and variants of the original result abound; see [2] and references there.) Suppose that B is a compact (smooth) n-manifold and consider, to be definite, a trivial bundle $p\colon B \times M \to B$, with M closed. Take a collar neighbourhood of the boundary $\partial B = \partial B \times \{0\}\colon \partial B \times [0,\infty) \subseteq B$, and let $j\colon \partial B \to B - \partial B$ be the embedding $x \mapsto (x,1)$. One of the fundamental lemmas of cobordism theory asserts that the coboundary map $\delta\colon \omega^0(\partial B) \to \omega^1(B,\partial B)$ coincides (up to sign) with the Gysin map $j_!$.

Let v be a family of vector fields (on M) defined on an open subset $U \subseteq B \times M$. If that part of the zero-set of v over ∂B is compact, we can form the index $I_{\partial B}(v, U \cap p^{-1}\partial B) \in \omega^0(B)$.

1.10 LEMMA. <u>If</u> Zero(v) <u>is compact, then</u>

$$j_! I_{\partial B}(v, U \cap p^{-1}\partial B) = 0 \in \omega^1(B,\partial B).$$

This is clear from the identification of $j_!$ with $\pm\delta$. The class $j_! I_{\partial B}(v, U \cap p^{-1}\partial B)$ is, essentially, the bifurcation invariant of Bartsch [2]. (If B is a submanifold of $\mathbb{R}^n$ or, more generally, is framed, then we can map $\omega^1(B,\partial B)$ to $\omega_{n-1}(*)$ by the Gysin map.)

Now suppose that $A \subseteq B - \partial B$ is a compact submanifold of codimension zero and write $i\colon \partial A \to B$ for the inclusion. Assume that $\mathrm{Zero}(v) \cap p^{-1}(\overline{B-A})$ is compact. Then (1.10) applied to the manifold $\overline{B-A}$ yields, by transitivity of Gysin maps:

(1.11) $$i_! I_{\partial A}(v, U \cap p^{-1}\partial A) = j_! I_{\partial B}(v, U \cap p^{-1}\partial B) \in \omega^1(B, \partial B).$$

Repeated application of (1.10) and (1.11) establishes:

1.12 LEMMA. <u>Suppose that</u> Zero(v) <u>is compact and that</u> $U \cap p^{-1}(\overline{B-A})$ <u>is a disjoint union of open subsets</u> P <u>and</u> Q <u>of</u> $(\overline{B-A}) \times M$. <u>Then</u>

$$i_! I_{\partial A}(v, P \cap p^{-1}\partial A) = j_! I_{\partial B}(v, Q \cap p^{-1}\partial B).$$

2. A finite-dimensional analogue

Some familiarity with $\mathbb{T}$-equivariant homotopy will be assumed. Background and notation can be found in [4], to which frequent reference will be made.

Throughout this section M will be a finite-dimensional closed $\mathbb{T}$-manifold and s will denote the generating vector field of the circle action. Let $\Omega \subseteq M$ be an open $\mathbb{T}$-subset on which $\mathbb{T}$ acts without fixed points, and suppose that w is an equivariant vector field on Ω such that the set $\Xi = \{x \in \Omega \mid w(x) \in \mathbb{R}s(x)\}$, of points where w is parallel to the flow, is compact. Building on the work of Dancer [6], we shall construct an index $\mathcal{E}(w, \Omega) \in \omega_1^{\mathbb{T}}(E\mathfrak{F})$, where $E\mathfrak{F}$ is the classifying space of the family $\mathfrak{F}$ of finite subgroups of $\mathbb{T}$, [4: 1.13].

Consider the family of vector fields v_μ ($\mu \in \mathbb{R}$) on Ω given at $x \in \Omega$ by:

(2.1) $$v_\mu(x) = \mu s(x) + w(x).$$

The zero-set of v_μ is compact, and, for large $\rho > 0$, is empty if $|\mu| \geqslant \rho$. So we have a fibre-bundle $\mathbb{R} \times M \to \mathbb{R}$ and a vector field v, along the fibres on the subspace $U = \mathbb{R} \times \Omega$, with compact zero-set. Restricting to the compact subspace $B = [-\rho, \rho] \subseteq \mathbb{R}$, we can form the $\mathbb{T}$-equivariant relative Poincaré-Hopf index $I_{(B,\partial B)}(v, B \times \Omega)$ in the group $\omega^0_{\mathbb{T}}(B, \partial B)$, which is canonically identified with $\omega^{-1}_{\mathbb{T}}(*) = \omega_1^{\mathbb{T}}(*)$. The resultant class is clearly independent of ρ and should be regarded as an index with compact supports over the base $\mathbb{R}$. Since $\Omega^{\mathbb{T}} = \emptyset$, we can use the classifying map $\Omega \to E\mathfrak{F}$ to lift the index, as in (1.9), to an element

(2.2) $$\mathcal{E}(w, \Omega) \in \omega_1^{\mathbb{T}}(E\mathfrak{F}).$$

(In fact, we have $\omega_1^{\mathbb{T}}(*) = \omega_1^{\mathbb{T}}(E\mathfrak{F}) \oplus \omega_1(*)$.)

The group $\omega_1^{\mathbb{T}}(E\mathfrak{F})$ is a direct sum $\oplus_{n\geq 1}\mathbb{Z}\sigma_n$, [4:2.10]. So the index $\mathcal{E}(w,\Omega)$ is given by a sequence of integers. (These integer invariants are implicit in [10].)

A weaker index is obtained by mapping, via the Hurewicz homomorphism, to integral homology: $\omega_1^{\mathbb{T}}(E\mathfrak{F}) \to H_1^{\mathbb{T}}(E\mathfrak{F}) = \mathbb{Q}$, $\Sigma a_n\sigma_n \mapsto \Sigma a_n/n$, [4:2.11]. (This gives Fuller's original $\mathbb{Q}$-valued index, [8].)

One can also simply forget the $\mathbb{T}$-equivariance, mapping $\omega_1^{\mathbb{T}}(E\mathfrak{F}) \to \omega_1(E\mathfrak{F}) = \omega_1(*) = \mathbb{Z}/2 : \Sigma a_n\sigma_n \mapsto \Sigma a_n \pmod 2$. (Such mod 2-indices are standard tools in bifurcation theory; see [2] for a recent account.)

2.3 REMARK. It follows from (2.7) and (2.10) below that the homology Hurewicz image of $\mathcal{E}(w,\Omega)$ agrees with Dancer's index [6]. However, he restricts attention to gradient vector fields. Thus M has a $\mathbb{T}$-invariant Riemannian metric g and $w = \operatorname{grad}\psi$ for some $\mathbb{T}$-invariant C^1-function $\psi: M \to \mathbb{R}$. Since ψ is constant on orbits, we have $g(s,w) = (d\psi)(s) = 0$. So Ξ is just $\mathrm{Zero}(w)$, and $\mathrm{Zero}(v_\mu) = \emptyset$ if $\mu \neq 0$.

The index $\mathcal{E}$ has the following properties, which it inherits from the vector-field index.

2.4 LEMMA. (i) <u>If</u> w^λ $(\lambda \in [0,1])$ <u>is a continuous family of vector fields on</u> Ω <u>such that</u> $\{(\lambda,x) \in [0,1]\times\Omega \mid w^\lambda(x) \in \mathbb{R}s(x)\}$ <u>is compact, then</u> $\mathcal{E}(w^0,\Omega) = \mathcal{E}(w^1,\Omega)$.

(ii) <u>If</u> Ω' <u>is an open subset of</u> Ω <u>with</u> $\Xi \subseteq \Omega'$, <u>then</u> $\mathcal{E}(w,\Omega') = \mathcal{E}(w,\Omega)$.

(iii) <u>If</u> Ω <u>is a disjoint union of open sets</u> Ω_1 <u>and</u> Ω_2, <u>then</u> $\mathcal{E}(w,\Omega) = \mathcal{E}(w,\Omega_1) + \mathcal{E}(w,\Omega_2)$.

If the $\mathbb{T}$-action on M is fixed-point-free $(M^{\mathbb{T}} = \emptyset)$, we may take $\Omega = M$. The index $\mathcal{E}(w,M)$ is, by (2.4)(i), independent of w and can be expressed in terms of Euler characteristics as follows. We write $\mathbb{T}(n)$ for the subgroup $\mathbb{Z}\frac{1}{n}/\mathbb{Z}$ of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ of order $n \geq 1$.

2.5 PROPOSITION. <u>If</u> $M^{\mathbb{T}} = \emptyset$, <u>we have</u>

$$\mathcal{E}(0,M) = \sum \chi_c(M_{(\mathbb{T}(n))}/\mathbb{T})\cdot\sigma_n,$$

<u>where</u> $M_{(\mathbb{T}(n))}$ <u>is the set of points in</u> M <u>with stabilizer of order</u> n <u>and</u> χ_c <u>denotes the Euler characteristic with compact supports</u>.

Outline proof. An invariant $\mathcal{E}(M)$ is introduced in [4] as the Euler characteristic of the normal bundle, $\hat{\tau}$, to the orbits in M, and $\mathcal{E}(M)$ is calculated, [4:5.2], as the righthand expression in (2.5). It is, therefore, sufficient to show that $\mathcal{E}(M) = \mathcal{E}(M,0)$. This is done by direct inspection. We adopt the notation of [4:5].

The dual in $\omega_1^{\mathbb{T}}(M)$ of the Euler class $\gamma(\hat{\tau}) \in \tilde{\omega}^0_{\mathbb{T}}(M^{-\hat{\tau}})$ can be described as follows. Recall that duality is defined by Gysin maps. In particular, if M is embedded in a $\mathbb{T}$-module V with normal bundle ν, the Pontrjagin-Thom construction gives a map $V^+ \to M^\nu$, which represents the fundamental class in $\tilde{\omega}_0^{\mathbb{T}}(M^{-\tau M})$, dual to $1 \in \omega^0_{\mathbb{T}}(M) = \tilde{\omega}^0_{\mathbb{T}}(M_+)$. More generally, if ζ is a vector bundle over M, the composition $V^+ \to M^\nu \to M^{\nu\oplus\zeta}$ with the inclusion gives the dual in $\tilde{\omega}_0^{\mathbb{T}}(M^{\zeta-\tau M})$ of the Euler class $\gamma(\zeta) \in \tilde{\omega}^0_{\mathbb{T}}(M^{-\zeta})$. For $\zeta = \hat{\tau}$ we take the smash product with the identity on $\mathbb{R}^+$ to get a map $(\mathbb{R}\oplus V)^+ \to V^+ \wedge M_+$. Using the same embedding of M in V to construct the index of the vector field v, as in (1.9), we obtain a second map $(\mathbb{R}\oplus V)^+ \to V^+ \wedge M_+$. One checks that the two are homotopic. □

In the classical Poincaré-Hopf theory it is easy, as we have seen, to compute the index of a vector field with isolated zeros. To treat the analogous case here of a field w for which the set Ξ is a finite union of isolated orbits, we begin with a slightly more general problem. Suppose that Ω' is an invariant open subset of a closed $\mathbb{T}(k)$-manifold M', $k \geqslant 1$, and w' an equivariant vector field on Ω' with compact zero-set. Put

$$\Omega = \mathbb{T}\times_{\mathbb{T}(k)} \Omega' \subseteq M = \mathbb{T}\times_{\mathbb{T}(k)} M', \tag{2.6}$$

and let w be the vector field on Ω induced from w'. (Thus w lifts to $0 \oplus w'$ on the k-fold cover $\mathbb{T}\times\Omega'$ of Ω, and $\Xi = \mathbb{T}\times_{\mathbb{T}(k)} \mathrm{Zero}(w')$ is compact.)

2.7 PROPOSITION. The index $\mathcal{E}(w,\Omega)$ of the field w on the mapping torus is the image under the induction map:

$$\omega_0^{\mathbb{T}(k)}(*) \to \omega_1^{\mathbb{T}(k)}(E\mathfrak{F}) \subseteq \omega_1^{\mathbb{T}}(*)$$

of the vector-field index $I(w',\Omega')$ of w'.

Since group-theoretic induction from the subgroup $\mathbb{T}(k)$ to $\mathbb{T}$ is, in essence, the construction $\mathbb{T}\times_{\mathbb{T}(k)}-$, the result is no surprise. The induction map in stable homotopy sends σ'_n to σ_n:

$$\omega_0^{\mathbb{T}(k)}(*) = \oplus_{n|k}\, \mathbb{Z}\sigma'_n \to \omega_1^{\mathbb{T}}(E\mathfrak{F}) = \oplus\, \mathbb{Z}\sigma_n, \tag{2.8}$$

where σ_n' is the class of $\mathbb{T}(k)/\mathbb{T}(n)$ in the Burnside ring.

<u>Outline proof of (2.7)</u>. Write $B = [-\rho,\rho]$ for any $\rho > 0$ and $C = \mathbb{T}/\mathbb{T}(k)$. The construction of the field v on $B \times \Omega \subseteq B \times M$ over B can be described as follows. We have a smooth fibre-bundle $p: M \to C$ and the field w is a section, on $\Omega \subseteq M$, of the bundle $\tau(p)$ of tangents along the fibres. On the trivial bundle $B \times C \to B$ we have a field t: $t_\mu = \mu s$ at $\mu \in B$. There is a splitting $\tau M = p^*\tau B \oplus \tau(p)$ (in general defined up to homotopy, in this case given) and $v = t \oplus w$.

The zero-sets of t and w are compact (and, for t, disjoint from $\partial B \times C$). So we can form the indices $I_C(w,\Omega) \in \omega^0_{\mathbb{T}}(C)$ and $\tilde{I}_{(B,\partial B)}(t, B\times C)$. The latter may be regarded, (1.9), as a stable map, f say: $\mathbb{R}^+ \simeq B/\partial B \to C_+$. In this situation one can establish the generalized multiplicativity formula:

$$I_{(B,\partial B)}(t\oplus w, B\times\Omega) = I_C(w,\Omega).\tilde{I}_{(B,\partial B)}(t,B\times C). \tag{2.9}$$

Now recall that induction is defined as the composition: $\omega^0_{\mathbb{T}(k)}(*) \xrightarrow{\cong} \omega^0_{\mathbb{T}}(\mathbb{T}/\mathbb{T}(k)) \to \omega^{-1}_{\mathbb{T}}(*)$ of the canonical identification and the Gysin map determined by the left-invariant framing of $\mathbb{T}/\mathbb{T}(k)$. The proof is completed by observing that the first map lifts $I(w',\Omega')$ to $I_C(w,\Omega)$ and by checking that the second is induced by f. □

The proposition (2.7) gives the following prescription for computing $\mathcal{E}(w,\Omega)$ when Ξ is a finite union of isolated orbits. A tubular neighbourhood of a component C of Ξ in Ω can be written in the form $\mathbb{T}\times_{\mathbb{T}(k)} \Omega'$, where Ω' is an open disc, with centre 0, in some $\mathbb{T}(k)$-module V and C corresponds to $\mathbb{T} \times_{\mathbb{T}(k)} 0$. On this neighbourhood, if it meets no other component of Ξ, w is, up to addition of a constant multiple of s (and permissible homotopy, (2.4)(i)), induced from a field w' on Ω' with a single zero at 0. The contribution of C to $\mathcal{E}(w,\Omega)$ is determined, according to (2.7), by the index of w'.

In homology the induction map:

$$H_0^{\mathbb{T}(k)}(*) = \mathbb{Z} \to H_1^{\mathbb{T}}(E\mathfrak{T}) = \mathbb{Q} \tag{2.10}$$

is just multiplication by $1/k$. So the recipe above gives the contribution of the isolated orbit $C \subseteq \Xi$ to the $\mathbb{Q}$-valued index as $1/k$ times the non-equivariant index of w'. (This was Dancer's starting point in [6].)

If, further, w' is C^1 with $Dw'(0) = L$ (say): $V \to V$ nonsingular, then $I(w',\Omega')$ is the image under the J-homomorphism, (1.5), $J: KO^{-1}_{\mathbb{T}(k)}(*) \to \omega^0_{\mathbb{T}(k)}(*)$ of the class determined by L. Write d and d' for the elements of the group $\{\pm 1\}$ $(= \mathbb{Z}/2)$ defined by: $dd' = \mathrm{sign}(\det L)$, $d = \mathrm{sign}(\det L^{\mathbb{T}(k)})$, where $L^{\mathbb{T}(k)}: V^{\mathbb{T}(k)} \to V^{\mathbb{T}(k)}$ is the restriction of L to the fixed submodule. The group $KO^{-1}_{\mathbb{T}(k)}(*)$ is isomorphic to $\mathbb{Z}/2$ if k is odd, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if k is even, and the class of L is given, respectively, by d and (d,d'). In each case J is injective, and straightforward calculation yields:

2.11 PROPOSITION. The index $\mathcal{E}(w,\Omega)$ of a non-degenerate (isolated) orbit of a C^1-field w as described above is equal to:

$d\sigma_k$ when k is odd;

$d\sigma_k$ if $d' = +1$, $d(\sigma_{k/2}-\sigma_k)$ if $d' = -1$, when k is even. □

As a final computation of the index we describe a basic bifurcation theorem, following Dancer [6: p.339] and Ize [10: p.759]. Consider a continuous family w^λ $(\lambda \in [0,1])$ of $\mathbb{T}$-equivariant vector fields defined on the whole of M, and write v for the family $v^\lambda_\mu = \mu s + w^\lambda$, $(\lambda,\mu) \in [0,1] \times \mathbb{R}$, on $p: [0,1] \times \mathbb{R} \times M \to [0,1] \times \mathbb{R}$. We impose the following conditions on the zero-set $Z = \mathrm{Zero}(v)$.

2.12 HYPOTHESES. (i) The closure $(Z - Z^{\mathbb{T}})^-$ is compact.

(ii) The "bifurcation set" $\Pi = ((Z - Z^{\mathbb{T}})^-)^{\mathbb{T}}$ is discrete and disjoint from $\partial[0,1] \times \mathbb{R} \times M^{\mathbb{T}}$.

(iii) For each point $(\lambda,\mu,x) \in \Pi$, x is an isolated zero on $M^{\mathbb{T}}$ of w^λ.

2.13 PROPOSITION. Under the assumptions (2.12), we have

$$\mathcal{E}(w^1, M-M^{\mathbb{T}}) - \mathcal{E}(w^0, M-M^{\mathbb{T}}) = \textstyle\sum_{\pi\in\Pi}\iota(\pi),$$

where $\iota(\pi) \in \omega_1^{\mathbb{T}}(E\mathfrak{F})$ is the local index described below.

Outline proof. It will be convenient to label a point $\pi \in \Pi$ as $(\lambda_\pi,\mu_\pi,x_\pi)$, and to write V_π for the tangent space of M at x_π. Put $B = [0,1] \times [-\rho,\rho]$, where $\rho > 0$ is chosen to satisfy: $(Z-Z^{\mathbb{T}})^- \subseteq [0,1] \times (-\rho,\rho)$; and let $A(\pi)$, for $\pi \in \Pi$, be the closed disc of radius ε, centre (λ_π,μ_π), in $\mathbb{R}^2$ with the Euclidean norm. The radius $\varepsilon > 0$ is chosen such that: $A(\pi) \subseteq B - \partial B$, $A(\pi) \cap A(\pi') = \emptyset$ if $(\lambda_\pi,\mu_\pi) \neq (\lambda_{\pi'},\mu_{\pi'})$, for $\pi,\pi' \in \Pi$. Set $A = \cup A(\pi)$, $\pi \in \Pi$.

For ε sufficiently small, (2.12) guarantees that we can find tubular neighbourhoods: $V_\pi \hookrightarrow M$ of each point $x_\pi \in M$ such that, for appropriate inner products:

(2.14) (i) the closed unit discs $D(V_\pi)$ are disjoint in M ($D(V_\pi) \cap D(V_{\pi'}) = \emptyset$ if $x_\pi \neq x_{\pi'}$); (ii) $(A(\pi) \times S(V_\pi)) \cap Z^{\mathbb{T}} = \emptyset$; and (iii) $(\partial A(\pi) \times D(V_\pi)) \cap (Z - Z^{\mathbb{T}}) = \emptyset$.

The field v is then, by (ii) and (iii), nowhere zero on $\partial A(\pi) \times S(V_\pi)$ and gives, as in (1.4), a map: $\partial A(\pi) \times S(V_\pi) \to S(V_\pi)$. This determines a stable homotopy class in $\omega^0_{\mathbb{T}}(\partial A(\pi)) = \omega^0_{\mathbb{T}}(*) \oplus \omega^{-1}_{\mathbb{T}}(*)$, and we denote its second component by $\iota(\pi)$. From (2.14)(ii) we see, by considering fixed-points, that: $\iota(\pi) \in \omega_1^{\mathbb{T}}(E\mathfrak{F}) \subseteq \omega^{-1}_{\mathbb{T}}(*)$.

Next we use (1.12), choosing open sets P and Q such that: $P \supseteq Z^{\mathbb{T}} \cap p^{-1}(\overline{B-A})$ and $P \supseteq \partial A_\pi \times D(V_\pi)$, $Q \supseteq (Z - Z^{\mathbb{T}}) \cap p^{-1}(\overline{B-A})$. (To fit the precise form of the lemma, we can replace B by a slightly smaller disc with smooth boundary.) The index $j_! I_{\partial B}(v, Q \cap p^{-1}\partial B)$ in $\omega^1_{\mathbb{T}}(B, \partial B) = \omega^{-1}_{\mathbb{T}}(*)$ is clearly $\mathcal{E}(w^1, M - M^{\mathbb{T}}) - \mathcal{E}(w^0, M - M^{\mathbb{T}})$. On the other hand, $I_{\partial A}(v, P \cap p^{-1}\partial A)$ can be expressed as a sum $I_{\partial A}(v, R \cap p^{-1}\partial A) + I_{\partial A}(v, S \cap p^{-1}\partial A)$, where $R = \bigcup (A(\pi) \times (D(V_\pi) - S(V_\pi)))$ and S is an open subset of $A \times M$ such that: $R \cap S = \emptyset$ and $S \cap Z$ is the compact set $\{z \in Z^{\mathbb{T}} \cap p^{-1}A \mid z \notin \overline{R}\}$. By (1.10), we have $i_! I_{\partial A}(v, S \cap p^{-1}\partial A) = 0$. But the term $i_! I_{\partial A}(v, R \cap p^{-1}\partial A)$ is exactly $\sum \iota(\pi)$. □

When the family w is C^1 (that is, differentiable on fibres with the derivative continuous on $[0,1] \times M$), there is an elegant description (to be found in [10],[6] and earlier work) of the local index $\iota(\pi)$ at a "non-degenerate" bifurcation point π in terms of spectral flow. To explain this, we need some notation. For $n \geqslant 1$, let E^n be the complex $\mathbb{T}$-module $\mathbb{C}$ with $[t] \in \mathbb{R}/\mathbb{Z}$ acting as multiplication by $e^{2\pi i n t}$. Recall that any real $\mathbb{T}$-module V splits functorially as a direct sum:

$$V = V^{\mathbb{T}} \oplus \bigoplus_{n \geqslant 1} E^n \otimes_{\mathbb{C}} V^{(n)}, \tag{2.15}$$

where $V^{(n)}$ is the $\mathbb{C}$-vector space of $\mathbb{R}$-linear $\mathbb{T}$-maps: $E^n \to V$. Now, at a zero $x \in M$ of w^λ the derivative of w^λ defines an endomorphism, $L(\lambda, x)$ say, of the tangent space $\tau_x M$. If $x \in M^{\mathbb{T}}$, we can split $L(\lambda, x)$ into components: $L(\lambda, x)^{\mathbb{T}}$ on $\tau_x M^{\mathbb{T}}$, $L(\lambda, x)^{(n)}$ on $(\tau_x M)^{(n)}$. The non-degeneracy conditions at $\pi \in \Pi$ are the following.

2.16 HYPOTHESES. (i) Put $\Delta = \det(L(\lambda_\pi, x_\pi)^{\mathbb{T}})$. We suppose that $\Delta \neq 0$ (which implies (2.11)(iii)).

(ii) By the implicit function theorem, for sufficiently small $\delta > 0$ there is a unique continuous path $\gamma\colon (\lambda_\pi - \delta, \lambda_\pi + \delta) \to M^{\mathbb{T}}$ such that: $\gamma(\lambda_\pi) = x_\pi$ and $w^\lambda(\gamma(\lambda)) = 0$. Write χ_n^λ for the characteristic polynomial: $\chi_n^\lambda(z) = \det(z - (2\pi n)^{-1}L(\lambda,\gamma(\lambda))^{(n)})$. We assume that there exists η, $0 < \eta < \delta$, such that, for all $n \geqslant 1$,

$$\chi_n^\lambda(-i\mu) \neq 0 \text{ when } 0 < |\lambda-\lambda_\pi|^2 + |\mu-\mu_\pi|^2 \leqslant \eta^2.$$

Let ν_n denote the net flow of roots of χ_n^λ through $-i\mu$ from the left to the right of the imaginary axis as λ increases through λ_π. (To be precise, choose a small closed disc D, centre $-i\mu$, in $\mathbb{C}$ such that $\chi_n^{\lambda_\pi}$ has no roots in $D - \{-i\mu\}$. Then the number of roots z of χ_n^λ with $z \in D$ and $\mathrm{Re}(z) > 0$, counted with multiplicity, jumps by ν_n as λ increases through λ_π.)

2.17 PROPOSITION. <u>Under the assumptions</u> (2.16), <u>the local index</u> $\iota(\pi) \in \omega_1^{\mathbb{T}}(E\mathfrak{F})$ <u>is equal to</u>

$$-\sum \mathrm{sign}(\Delta)\nu_n.\sigma_n \in \oplus\, \mathbb{Z}\sigma_n.$$

<u>Outline proof</u>. We continue the notation in (2.13). Taking $\varepsilon \leqslant \eta$, we find that the field v has a non-degenerate (so isolated) zero at $\gamma(\lambda)$ over $(\lambda,\mu) \in \partial A(\pi)$ and may assume, by making suitable choices, that v has no other zeros in $\partial A(\pi) \times D(V_\pi)$. The derivative of v at $(\lambda,\mu,\gamma(\lambda))$ is the automorphism $\mu S(\gamma(\lambda)) + L(\lambda,\gamma(\lambda)) = T(\lambda,\mu)$, say, of $\tau_{\gamma(\lambda)}M$, where S is given by the $\mathbb{T}$-action (that is, the derivative of s).

The index $I_{\partial A(\pi)}(v, D(V_\pi) - S(V_\pi))$, which defines $\iota(\pi)$, is the image under

(2.18) $$J : KO_{\mathbb{T}}^{-1}(\partial A(\pi)) \to \omega_{\mathbb{T}}^0(\partial A(\pi))$$

of the class ℓ determined by the vector-bundle automorphism: $T(\lambda,\mu)$ on $\tau_{\gamma(\lambda)}M$ at $(\lambda,\mu) \in \partial A(\pi)$.

Now we have $KO_{\mathbb{T}}^{-1}(\partial A(\pi)) = KO_{\mathbb{T}}^{-1}(S^1) = KO_{\mathbb{T}}^{-1}(*) \oplus KO_{\mathbb{T}}^{-2}(*)$. The component of ℓ in $KO_{\mathbb{T}}^{-1}(*) = \mathbb{Z}/2$ (= $\{\pm 1\}$) is easily seen to be $\mathrm{sign}(\Delta)$. Corresponding to the decomposition (2.15) there is a splitting:

(2.19) $$KO_{\mathbb{T}}^{-2}(*) = KO^{-2}(*) \oplus \bigoplus_{n\geqslant 1} K^{-2}(*).[E^n].$$

Here the component of ℓ in $KO^{-2}(*)$ is trivial. The remaining components are obtained from (2.20) below: if we identify $K^{-2}(*) = \pi_1(U(\infty))$ with $\mathbb{Z}$ by "degree (det)", the nth term is $\nu_n.[E^n]$. Finally, we can read off the result from (2.18), since $J[E^n] = 1 - \sigma_n$ (under the current sign conventions). □

2.20 APPENDIX. Suppose that p^x, $x \in [-1,1]$, is a continuous family of monic complex polynomials, with no roots on the unit circle $S^1 \subseteq \mathbb{C}$, such that: (i) $p^0(z) \neq 0$ if $0 < |z| \leqslant 1$, and (ii) $p^x(z) \neq 0$ if $x \neq 0$, $z \in i\mathbb{R}$, $|z| \leqslant 1$. Then the degree of the map $S^1 \to \mathbb{C} - \{0\}$: $x + iy \mapsto p^x(-iy)$ is equal to the difference of the number of roots of p^1 and of p^{-1} in the region $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ |z| < 1\}$.

Proof. One easily reduces to the case in which all the roots of p^x lie in the real interval $(-1,1)$. (First discard roots z with $|z| > 1$, then deform the remaining roots within the unit disc to the real axis using the homotopy: $h_t(a+ib) = a + ib(1-t)$, $0 \leqslant t \leqslant 1$.) Now one can order the roots and so reduce to the linear case: $p^x(z) = z - a^x$ with $a^x \in \mathbb{R}$.

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Michael CRABB
Department of Mathematics
University of Aberdeen
GB-ABERDEEN AB9 2TY