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Astérisque, tome 191 (1990), p. 97-108
<http://www.numdam.org/item?id=AST_1990__191__97_0>

Spaces of Null Homotopic Maps

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§1. INTRODUCTION

In 1983 Haynes Miller [7] proved a conjecture of Sullivan and used it to show that if π is a locally finite group and X is a simply connected finite dimensional CW-complex then the space of pointed maps from the classifying space $B\pi$ to X is weakly contractible, ie. $\mathrm{Map}_*(B\pi, X) \simeq *$. This result had immediate applications. Alex Zabrodsky [11] used it to study maps between classifying spaces of compact Lie groups. McGibbon and Neisendorfer [6] applied Miller's theorem to answer a question of Serre; they proved that if X is a simply connected finite dimensional CW-complex with $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ then there are infinitely many dimensions in which $\pi_*(X)$ has p-torsion.

The goal of this note is to use the functor T^V of [2] to generalize Miller's theorem and some of its corollaries to a large class of infinite dimensional spaces (see [5] for closely related earlier work in this direction). This generalization comes at the expense of working with one component of the function complex $\mathrm{Map}_*(B\pi, X)$ at a time.

Fix a prime number p.

THEOREM 1.1. *Let π be a locally finite group and X a simply connected p-complete space. Assume that $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra. Then the component of* $\mathrm{Map}_*(B\pi, X)$ *which contains the constant map is weakly contractible.*

REMARK: There is a standard way [7, 1.5] to relax the assumption in 1.1 that X is p-complete.

Theorem 1.1 is actually a special case of a more general assertion. Recall that an unstable module M over the mod p Steenrod Algebra $\mathbf{A}_p$ is said to be *locally finite* [4] if each element $x \in M$ is contained in a finite $\mathbf{A}_p$ submodule. If R is a connected unstable algebra over $\mathbf{A}_p$ then the *augmentation ideal* $I(R)$ is by definition the ideal of positive-dimensional elements and the *module of indecomposables* $Q(R)$ is the unstable $\mathbf{A}_p$ module $I(R)/I(R)^2$. An unstable algebra R over $\mathbf{A}_p$ is of *finite type* if each R^k is finite-dimensional as an $\mathbf{F}_p$ vector space.

Both authors were supported in part by the National Science Foundation. The first author would like to thank the University of Chicago Mathematics Department for its hospitality during the course of this work.

THEOREM 1.2. *Let π be a locally finite group and X a simply connected p-complete space such that $H^*(X, \mathbf{F}_p)$ is of finite type. Assume that the module of indecomposables $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then the component of $\mathrm{Map}_*(B\pi, X)$ which contains the constant map is weakly contractible.*

REMARK: Theorem 1.1 does in fact follow from Theorem 1.2, since if $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra then $Q(H^*(X, \mathbf{F}_p))$ is a *finite* $\mathbf{A}_p$ module.

REMARK: Theorem 1.2 has a converse, at least if $p = 2$ (see Theorem 3.2). There is also a generalization of 1.2 that deals with other components of the mapping space $\mathrm{Map}_*(B\pi, X)$ (see Theorem 4.1) but for this generalization it is necessary to assume that π is an elementary abelian p-group.

Given 1.2, the arguments of [**6**] go over more or less directly and lead to the following result. A CW-complex is of *finite type* if it has a finite number of cells in each dimension.

THEOREM 1.3. *Suppose that X is a two-connected CW-complex of finite type. Assume that $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ and that $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then there exist infinitely many k such that $\pi_k(X)$ has p-torsion.*

REMARK: The example of CP^∞ shows that it would not be enough in Theorem 1.3 to assume that X is 1-connected.

Some instances of 1.3 were previously known; for instance, if $X = BG$ for G a suitable compact Lie group then the conclusion of 1.3 can be obtained by applying [**6**] to the loop space on X. However, Theorem 1.3 applies in many previously inaccessible cases; for example, it applies if X is the Borel construction $EG \times_G Y$ of the action of a compact Lie group G on a finite complex Y or if X is a quotient space obtained from such a Borel construction by collapsing out a skeleton.

We first noticed Theorem 1.1 as part of our work [**1**] on calculating fragments of T^V with Smith theory techniques. The proof of 1.1 given here does not use the localization approach of [**1**]; it is partly for this reason that the proof generalizes to give 1.2.

Organization of the paper. Section 2 recalls some properties of the functor T^V. In §3 there is a proof of 1.2 and in §4 a generalization of 1.2 to other components of the mapping space. Section 5 uses the ideas of [**6**] to deduce 1.3 from 1.2.

Notation and terminology. The prime p is fixed for the rest of the paper; all unspecified cohomology is taken with $\mathbf{F}_p$ coefficients. The symbol $\mathcal{U}$ (resp. $\mathcal{K}$) will denote the category of unstable modules (resp. algebras) [**2**] over $\mathbf{A}_p$. If $R \in \mathcal{K}$ then $\mathcal{U}(R)$ (resp. $\mathcal{K}(R)$) will stand for the category of objects of $\mathcal{U}$ (resp. $\mathcal{K}$) which are also R-modules (resp. R-algebras) in a compatible way [**1**].

For a pointed map $f : K \to X$ of spaces we will let $\mathrm{Map}_*(K, X)_f$ denote the component of the pointed mapping space $\mathrm{Map}_*(K, X)$ containing f. The component of the *unpointed* mapping space containing f is $\mathrm{Map}(K, X)_f$.

§2 THE FUNCTOR T^V

Let V be an elementary abelian p-group, ie., a finite-dimensional vector space over $\mathbf{F}_p$, and H^V the classifying space cohomology H^*BV. Lannes [**2**] has constructed a functor $T^V : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor given by tensor product (over $\mathbf{F}_p$) with H^V and has shown that T^V lifts to a functor $\mathcal{K} \to \mathcal{K}$ which is similarly left adjoint to tensoring with H^V.

PROPOSITION 2.1 [**2**]. *For any object R of $\mathcal{K}$ the functor T^V induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V(R))$. The functor T^V is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V(M \otimes_R N) \cong T^V(M) \otimes_{T^V(R)} T^V(N)$$

Now suppose that $\gamma : R \to H^V$ is a $\mathcal{K}$-map. The adjoint of γ is a map $T^V(R) \to \mathbf{F}_p$ or in other words a ring homomorphism $\hat{\gamma} : T^V(R)^0 \to \mathbf{F}_p$. For $M \in \mathcal{U}(R)$, let $T^V_\gamma(M)$ be the tensor product $T^V(M) \otimes_{T^V(R)^0} \mathbf{F}_p$, where the action of $T^V(R)^0$ on $\mathbf{F}_p$ is given by $\hat{\gamma}$. Note that $T^V_\gamma(R) \in \mathcal{K}$.

PROPOSITION 2.2 [**1**, 2.1]. *For any $\mathcal{K}$-map $\gamma : R \to H^V$ the functor $T^V_\gamma(-)$ induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V_\gamma(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V_\gamma(R))$. The functor T^V_γ is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V_\gamma(M \otimes_R N) \cong T^V_\gamma(M) \otimes_{T^V_\gamma(R)} T^V_\gamma(N) .$$

The following proposition is a straightforward consequence of the above two.

LEMMA 2.3. *Suppose that $\alpha : R_1 \to R_2$ and $\beta : R_2 \to H^V$ are morphisms of $\mathcal{K}$, and let $\gamma : R_1 \to H^V$ denote the composite $\beta \cdot \alpha$.*

(1) *If α is a surjection and $M \in \mathcal{U}(R_2)$ is treated via α as an object of $\mathcal{U}(R_1)$, then the natural map $T^V_\gamma(M) \to T^V_\beta(M)$ is an isomorphism.*

(2) *If $M \in \mathcal{U}(R_1)$ then the natural map $T^V_\beta(R_2) \otimes_{T^V_\gamma(R_1)} T^V_\gamma(M) \to T^V_\beta(R_2 \otimes_{R_1} M)$ is an isomorphism.*

There is a natural map $\lambda_X : T^V(H^*X) \to H^* \operatorname{Map}(BV, X)$ for any space X. If $g : BV \to X$ is a map which induces the cohomology homomorphism $\gamma : H^*X \to H^V$ then λ_X passes to a quotient map

$$\lambda_{X,g} : T^V_\gamma(H^*X) \to H^* \operatorname{Map}(BV, X)_g \,.$$

A lot of the geometric usefulness of T^V is explained by the following theorem.

THEOREM 2.4 [**3**]. *Let X be a 1-connected space, $g : BV \to X$ a map, and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Assume that H^*X is of finite type, that $T^V_\gamma H^*X$ is of finite type, and that $T^V_\gamma H^*X$ vanishes in dimension 1. Then $\lambda_{X,g}$ is an isomorphism.*

For any object M of $\mathcal{U}$ the adjunction map $M \to H^V \otimes_{\mathbf{F}_p} T^V(M)$ can be combined with the unique algebra map $H^V \to \mathbf{F}_p$ to give a map $M \to T^V(M)$; call this map ϵ. (If $M = H^*X$ for some space X, then ϵ fits into a commutative diagram involving λ_X and the cohomology homomorphism induced by the basepoint evaluation map $\operatorname{Map}(BV, X) \to X$.)

THEOREM 2.5 [**4**, 6.3.2]. *The map $\epsilon : M \to T^V(M)$ is an isomorphism iff M is locally finite as a module over $\mathbf{A}_p$.*

If $R \in \mathcal{K}$, $M \in \mathcal{U}(R)$ and $\gamma : R \to H^V$ is a $\mathcal{K}$-map, we will denote the composite $M \xrightarrow{\epsilon} T^V(M) \to T^V_\gamma(M)$ by ϵ_γ. Theorem 2.5 leads to the following result, which we will need in §4.

PROPOSITION 2.6. *Let M be an object of $\mathcal{U}(H^V)$ and $\iota : H^V \to H^V$ the identity map. Then $\epsilon_\iota : M \to T^V_\iota(M)$ is an isomorphism iff M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$.*

PROOF: The fact that ϵ_ι is an isomorphism if M has the stated tensor product decompositon follows directly from 2.3(2), 2.5 and [**2**, 4.2]. Conversely, under the assumption that ϵ_ι is an isomorphism Proposition 2.4 of [**1**] guarantees that M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$; the fact that N is locally finite is again a consequence of 2.3(2) and 2.5.

§3 The null component

In this section we will prove Theorem 1.2. Before doing this we will recast the conclusion of the theorem in a slightly different form.

Lemma 3.1. *Let K be a finite pointed CW-complex, X a 1-connected space, and $f : K \to X$ a pointed map. Then $\mathrm{Map}_*(K, X)_f$ is weakly contractible if and only if the inclusion of the basepoint in K induces a weak equivalence $\mathrm{Map}(K, X)_f \to X$.*

Proof: As in [**7**, 9.1] the inclusion $* \to K$ gives rise to a fibration sequence $\mathrm{Map}_*(K, X)_f \to \mathrm{Map}(K, X)_f \to X$.

The arguments of [**7**, §9] now show that Theorem 1.2 follows directly from the following result.

Theorem 3.2. *Let V be an elementary abelian p-group and X a 1-connected p-complete space such that H^*X is of finite type. Let $f : BV \to X$ be a constant map and $\phi : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:*

(1) *QH^*X is locally finite as an $\mathbf{A}_p$ module*
(2) *the map $\epsilon_\phi : H^*X \to T^V_\phi H^*X$ is an isomorphism*
(3) *the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\mathrm{Map}(BV, X)_f \to X$.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Moreover, if $p = 2$ then (3) $\Longrightarrow$ (1).

Remark 3.3: It is likely that the three conditions of Theorem 1.2 are equivalent for *any* prime p; the proof would depend on the odd primary version of the results in [**9**].

Proof of 3.2: First consider the implication (1) $\Longrightarrow$ (2). Let $R = H^*X$ and let $I \subset R$ be the augmentation ideal. Pick $s \geq 0$. The fact that the action of R on I^s/I^{s+1} factors through the augmentation $R \to \mathbf{F}_p$ implies that the action of $T^V(R)$ on $T^V(I^s/I^{s+1})$ factors through the map $T^V(R) \to T^V(\mathbf{F}_p) \cong \mathbf{F}_p$ induced by augmentation; since this last map is adjoint to $\phi : R \to H^*(BV)$ it follows from 2.3(1) that the quotient map $T^V(I^s/I^{s+1}) \to T^V_\phi(I^s/I^{s+1})$ is an isomorphism. Moreover, I^s/I^{s+1}, as a quotient of $(I/I^2)^{\otimes s}$, is the union of its finite $\mathbf{A}_p$ submodules so by 2.5 the map $\epsilon : I^s/I^{s+1} \to T^V(I^s/I^{s+1})$ is an isomorphism. Putting these two facts together shows that $\epsilon_\phi : I^s/I^{s+1} \to T^V_\phi(I^s/I^{s+1})$ is an isomorphism. By induction and exactness, then, the map $\epsilon_\phi : R/I^{s+1} \to T^V_\phi(R/I^{s+1})$ is an isomorphism. The map $T^V_\phi(R) \to T^V_\phi(\mathbf{F}_p) \cong \mathbf{F}_p$ induced by augmentation is an epimorphism, so by exactness $T^V_\phi(I)$ vanishes in dimension 0.

By Lemma 2.2 and exactness, $T_\phi^V(I^{s+1})$ vanishes up to and including dimension s, and hence again by exactness the map $T_\phi^V(R) \to T_\phi^V(R/I^{s+1})$ induced by the quotient projection $R \to R/I^{s+1}$ is an isomorphism up through dimension s. It follows immediately that $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism.

The implication (2) $\Longrightarrow$ (3) is an easy consequence of Theorem 2.4.

For (3) $\Longrightarrow$ (1), assume $p = 2$. According to [**9**, proof of 3.1] condition (3) implies that the loop space cohomology $H^*(\Omega X)$ is locally finite as an $\mathbf{A}_p$ module, ie., in the terminology of [**9**], that $H^*(\Omega X) \in \mathcal{N}\mathrm{il}_k$ for all k. According to [**9**, 2.1(iii)], this implies that $\Sigma^{-1}QH^*X \in \mathcal{N}\mathrm{il}_k$ for all k. This amounts to the assertion that $\Sigma^{-1}QH^*X$ (or equivalently QH^*X) is locally finite [**9**, proof of 3.1].

§4 Other mapping space components

In this section we will give a generalization of Theorem 1.2 to mapping space components other than the component containing the constant map; this generalization is limited, however, in that it deals with elementary abelian p-groups rather than with arbitrary locally finite groups.

Given an elementary abelian p-group V, call an object M of $\mathcal{U}(H^V)$ *f-split* if M is isomorphic to $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$. Suppose that $\gamma : R \to H^V$ is a map in $\mathcal{K}$ with image $S \subset H^V$ and kernel $I \subset R$. Say that γ is *almost f-split* if

(*i*) S is a Hopf subalgebra of H^V, and

(*ii*) for each $s \geq 0$ the tensor product $H^V \otimes_S (I^s/I^{s+1})$ is f-split as an object of $\mathcal{U}(H^V)$.

Recall from 3.1 that $\mathrm{Map}_*(K, X)_f$ is weakly contractible iff evaluation at the basepoint gives an equivalence $\mathrm{Map}(K, X)_f \cong X$.

THEOREM 4.1. *Let V be an elementary abelian p-group and X a 1-connected p-complete space such that H^*X is of finite type. Let $g : BV \to X$ be a map and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:*

(1) *γ is almost f-split*

(2) *the map $\epsilon_\gamma : H^*X \to T_\gamma^V H^*X$ is an isomorphism*

(3) *the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\mathrm{Map}(BV, X)_g \to X$.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *Morever, if $p = 2$ then* (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

REMARK 4.2: As in the case of Theorem 3.2, it is likely that the three conditions of Theorem 4.1 are equivalent for any prime p.

LEMMA 4.3. *Let K be a pointed CW-complex, X a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that there exists a map $m : K \times X \to X$ which is 1_X on the axis $* \times X$ and $g : K \to X$ on the axis $K \times *$. Then the basepoint evaluation map $e_f : \mathrm{Map}(K, X)_f \to X$ is a weak equivalence if and only if the corresponding map $e_g : \mathrm{Map}(K, X)_g \to X$ is a weak equivalence.*

PROOF: Construct a commutative diagram

$$\begin{tikzcd} K \arrow[r, "="] \arrow[d, "a"'] & K \arrow[d, "b"] \\ K \times X \arrow[r, "(pr_1, m)"'] & K \times X \end{tikzcd}$$

in which $a(k) = (k, *)$, $b(k) = (k, g(k))$ and pr_1 is projection on the first factor. Since the lower horizontal map is a weak equivalence, it follows that the induced map $c : \mathrm{Map}(K, K \times X)_a \to \mathrm{Map}(K, K \times X)_b$ is a weak equivalence. It is clear that c commutes with the natural projections from its domain and range to $\mathrm{Map}(K, K)_i$, where i is the identity map of K. The lemma follows from the fact that the domain of c is $\mathrm{Map}(K, K)_i \times \mathrm{Map}(K, X)_f$ while the range is $\mathrm{Map}(K, K)_i \times \mathrm{Map}(K, X)_g$.

LEMMA 4.4. *Let K be a pointed CW-complex, X a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that the basepoint evaluation map $e_g : \mathrm{Map}(K, X)_g \to X$ is a weak equivalence. Then the basepoint evaluation map $e_f : \mathrm{Map}(K, X)_f \to X$ is also a weak equivalence.*

PROOF: The map m required in 4.3 is provided up to weak equivalence by the evaluation map $K \times \mathrm{Map}(K, X)_g \to X$.

LEMMA 4.5. *Let V be an elementary abelian p-group, R a connected object of $\mathcal{K}$, $\gamma : R \to H^V$ a map, and $\phi : R \to H^V$ the trivial map (ie. the map which factors through the augmentation $R \to \mathbf{F}_p$). Assume there exists a map $\mu : R \to H^V \otimes_{\mathbf{F}_p} R$ which gives 1_R when combined with the augmentation map of H^V and $\gamma : R \to H^V$ when combined with the augmentation map of R. Then $\epsilon_\phi : R \to T^V_\phi(R)$ is an isomorphism if and only if $\epsilon_\gamma : R \to T^V_\gamma(R)$ is an isomorphism.*

PROOF: This is essentially the proof of 4.3 with the arrows reversed.

Construct a commutative diagram

$$\begin{tikzcd} H^V & H^V \arrow[l, "="'] \\ H^V \otimes_{\mathbf{F}_p} R \arrow[u, "\alpha"] & H^V \otimes_{\mathbf{F}_p} R \arrow[l, "in_1 \cdot \mu"'] \arrow[u, "\beta"'] \end{tikzcd}$$

in which α is the product of 1_{H^V} with the augmentation of R, β is $(1_{H^V}) \cdot \gamma$, and in_1 is the map from H^V to the tensor product obtained using the unit of R. Since the lower horizontal map is an isomorphism, it follows that the induced map $\chi : T_\beta^V(H^V \otimes_{\mathbf{F}_p} R) \to T_\alpha^V(H^V \otimes_{\mathbf{F}_p} R)$ is an isomorphism. It is clear that χ respects the natural structures of its domain and range as modules over $T_\iota^V(H^V)$, where ι the identity map of H^V. The lemma follows from the fact [**1**, 2.2] that the domain of χ is $T_\iota^V(H^V) \otimes_{\mathbf{F}_p} T_\gamma^V(R)$ while the range is $T_\iota^V(H^V) \otimes_{\mathbf{F}_p} T_\phi^V(R)$.

LEMMA 4.6. *Let V be an elementary abelian p-group, R a connected object of $\mathcal{K}$, $\gamma : R \to H^V$ a map and $\phi : R \to H^V$ the trivial map. Assume that $\epsilon_\gamma : R \to T_\gamma^V(R)$ is an isomorphism. Then $\epsilon_\phi : R \to T_\phi^V(R)$ is also an isomorphism.*

PROOF: The map μ required in 4.5 is provided by the map $R \to H^V \otimes_{\mathbf{F}_p} T_\gamma^V(R)$ which is adjoint to the identity map of $T_\gamma^V(R)$.

REMARK 4.7: It follows from 4.5, 4.6 and 3.2 that at least if $p = 2$ the three conditions of 4.1 are equivalent to a fourth, namely, that QH^*X is locally finite as an $\mathbf{A}_p$ module and there exists a $\mathcal{K}$ map $H^*X \to H^V \otimes_{\mathbf{F}_p} H^*X$ which satisfies the conditions of 4.5.

LEMMA 4.8. *Let V be an elementary abelian p-group and $\nu : S \to H^V$ the inclusion of a subalgebra over $\mathbf{A}_p$. Then $\epsilon_\nu : S \to T_\nu^V(S)$ is an isomorphism if and only if ν includes S as a Hopf subalgebra of H^V.*

PROOF: Suppose that ϵ_ν is an isomorphism. In this case the adjunction homomorphism $S \to H^V \otimes_{\mathbf{F}_p} T_\nu^V(S)$ provides a map $\Delta_S : S \to H^V \otimes_{\mathbf{F}_p} S$ which fits into a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\Delta_S"] \arrow[d, "\nu"'] & H^V \otimes_{\mathbf{F}_p} S \arrow[d, "\iota\otimes\nu"] \\ H^V \arrow[r, "\Delta_{H^V}"'] & H^V \otimes_{\mathbf{F}_p} H^V \end{tikzcd}$$

$T_\gamma^V(H^V)$ is injective, and it follows from naturality and the fact that $H^V \to T_\gamma^V(H^V)$ is injective [**2**, 4.2] that $S \to T_\gamma^V(S)$ is injective. By 2.3(1) the map $\epsilon_\nu : S \to T_\nu^V(S)$ is an isomorphism and hence (4.8) S is a Hopf subalgebra of H^V.

By exactness the map $I^s \to T_\gamma^V(I^s)$ is seen to be an isomorphism if $s = 1$ and a monomorphism if $s > 1$; this first fact, though, combines with the tensor product formula (2.2) and exactness to show that $I^s \to T_\gamma^V(I^s)$ is an epimorphism for $s \geq 1$. Thus by exactness and 2.3(1) the maps $\epsilon_\nu : I^s/I^{s+1} \to T_\nu^V(I^s/I^{s+1})$ are isomorphisms. The proof is finished by running in reverse the argument used above at the end of the proof of (1) $\Longrightarrow$ (2).

§5 Torsion in homotopy groups

In this section we will use a slight variation on the ideas of [**6**] to prove Theorem 1.3.

Let $\mathbf{Z}$ denote the ring of integers, $\mathbf{Z}_p\hat{}$ the additive group of p-adic integers, and $\mathbf{Z}/p^n$ the cyclic group of order p^n. The group $\mathbf{Z}/p^\infty$ is by definition the locally finite group obtained by taking the direct limit of the groups $\mathbf{Z}/p^n$ under the standard inclusion maps.

LEMMA 5.1. *For any finitely-generated abelian group A the cohomology group $H^k(B\mathbf{Z}/p^\infty, A)$ is isomorphic to $\mathbf{Z}_p\hat{} \otimes A$ if $k > 0$ is even and is zero if k is odd. The natural map $A \to \mathbf{Z}_p\hat{} \otimes A$ induces isomorphisms $H^k(B\mathbf{Z}/p^\infty, A) \cong H^k(B\mathbf{Z}/p^\infty, \mathbf{Z}_p\hat{} \otimes A)$ for all $k > 0$.*

SKETCH OF PROOF: One way to do this is to calculate the homology $H_*(B\mathbf{Z}/p^\infty, \mathbf{Z})$ as a direct limit $\varinjlim_n H_*(B\mathbf{Z}/p^n, \mathbf{Z})$ and then pass to cohomology by using the universal coefficient theorem. The key algebraic ingredient is the fact that

$$\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/p^\infty, \mathbf{Z}) \cong \mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/p^\infty, \mathbf{Z}_p\hat{}) \cong \mathbf{Z}_p\hat{}\,.$$

Let P_nX stand for the n'th Postnikov stage of the space X and $k^{n+1}(X)$ for the Postnikov invariant of X which lies in $H^{n+1}(P_{n-1}X, \pi_nX)$(see [**10**, IX]).

LEMMA 5.2. *If Y is a loop space ΩX and Y has finitely-generated homotopy groups, then the Postnikov invariants of Y are torsion cohomology classes.*

PROOF: This follow from [**8**, p. 263]. In effect, Milnor and Moore show that the rationalized Postnikov invariants

$$k^{n+1}(Y) \otimes \mathbf{Q} \in H^{n+1}(P_{n-1}Y, \pi_n(Y) \otimes \mathbf{Q})$$

where ι is the identity map of H^V and we have used the fact [2, 4.2] that $\epsilon_\iota : H^V \to H^V$ is an isomorphism. It is easy to see that Δ_{H^V} is the Hopf algebra comultiplication map on H^V. It now follows from the fact that the comultiplication on H^V is cocommutative that $\Delta_S(S) \subset S \otimes_{\mathbf{F}_p} S$ and thus that S is a Hopf subalgebra of H^V.

Suppose conversely that S is a Hopf subalgebra of H^V, and let $\phi : S \to H^V$ be the trivial map which factors through the augmentation $S \to \mathbf{F}_p$. The Hopf algebra H^V is primitively generated, and the associated restricted Lie algebra of primitives [8, 6.7] is a free abelian restricted Lie algebra on a finite collection of generators (in dimensions 1 and 2). It follows from [8, 6.13–6.16] that S is primitively generated and is isomorphic as an algebra to a finite tensor product of exterior and polynomial algebras; in particular, $Q(S)$ is a finite unstable $\mathbf{A}_p$ module. By the proof of (1) $\Longrightarrow$ (2) in Theorem 3.2 the map $\epsilon_\phi : S \to T^V_\phi(S)$ is an isomorphism. Since the comultiplication of S produces the map μ required for Lemma 4.5, an application of this lemma finishes the proof.

PROOF OF 4.1: Let R denote H^*X, I the kernel of $\gamma : R \to H^V$, S the image of γ and $\nu : S \to H^V$ the inclusion map. We will use f to stand for a constant map $BV \to X$ and ϕ for the cohomology homomorphism induced by f.

(1) $\Longrightarrow$ (2). The assumption that S is a Hopf subalgeba of H^V implies by 4.8 that $\epsilon_\nu : S \to T^V_\nu(S)$ and hence (2.3(1)) $\epsilon_\gamma : S \to T^V_\gamma(S)$ are isomorphisms. Pick $s \geq 1$ and let $M = I^s/I^{s+1}$. If we can show that $\epsilon_\gamma : M \cong T^V_\gamma(M)$ we will be able to finish up by imitating the proof of (1) $\Longrightarrow$ (2) in Theorem 3.2. By 2.3(1) it is enough to show that $\epsilon_\nu : M \cong T^V_\nu(M)$. Proposition 2.6 ensures that $\epsilon_\iota : H^V \otimes_S M \to T^V_\iota(H^V \otimes_S M)$ is an isomorphism, where ι is the identity map of H^V. By 2.3(2) and [2, 4.2] , however, the map ϵ_ι is $\iota \otimes_S \epsilon_\nu$, so the desired result follows from the fact that H^V is free [8, 4.4] and therefore faithfully flat as a module over S.

(2) $\Longrightarrow$ (3). This is an immediate consequence of 2.4.

(3) $\Longrightarrow$ (2). By Lemma 4.4 and Theorem 3.2 the map $\epsilon_\phi : R \to T^V_\phi(R)$ is an isomorphism. The evaluation map $m : BV \times \mathrm{Map}(BV, X)_g \to X$ induces a cohomology homomorphism $\mu : R \to H^V \otimes_{\mathbf{F}_p} R$ which satisfies the conditions of 4.5, so the implication follows from the conclusion of 4.5.

(2) $\Longrightarrow$ (1). This implication does not in fact require the assumption that $p = 2$. The map $T^V_\gamma(R) \to T^V_\gamma(S)$ is surjective and it follows immediately from naturality that $\epsilon_\gamma : S \to T^V_\gamma(S)$ is surjective. The map $T^V_\gamma(S) \to$

are zero. Under the stated finite generation assumption this implies that the Postnikov invariants themselves are torsion.

Proof of 1.3: Let S_1 be the set of all k such that $\pi_k(X) \otimes \mathbf{Z}_p\hat{} \neq 0$ and S_2 the set of all k such that $\pi_k X$ contains p-torsion. The set S_1 is non-empty (because $H^*(X, \mathbf{F}_p) \neq 0$) and clearly contains S_2. Suppose that S_2 is finite. In that case we can find an integer k in S_1 such that no integer j greater than k belongs to S_2. Let $Y = \Omega^{k-2}X$. (Note that because X is 2-connected the integer k is greater than 2 and Y is a loop space.) By Lemma 5.1 the space $\mathrm{Map}_*(B\mathbf{Z}/p^\infty, P_1Y)$ is contractible and hence $\mathrm{Map}_*(B\mathbf{Z}/p^\infty, P_2Y) \cong \mathrm{Map}_*(B\mathbf{Z}/p^\infty, K(\pi_2Y, 2))$. Because of the way in which k was chosen we can thus, by Lemma 5.1 again, find an essential map $f : B\mathbf{Z}/p^\infty \to P_2Y$ which remains essential in the p-completion $(P_2Y)\hat{}_p$. The obstructions to lifting f to a map $g : B\mathbf{Z}/p^\infty \to Y$ are the pullbacks to $B\mathbf{Z}/p^\infty$ of the Postnikov invariants of Y [**10**, p. 450]; by Lemma 5.2 these obstructions are torsion, but by Lemma 5.1 and the choice of k they lie in torsion-free abelian groups. Therefore the obstructions vanish, and the lift g exists. The composite h of g with the completion map $Y \to Y\hat{}_p$ is non-trivial because the composite of h with the projection map $Y\hat{}_p \to P_2(Y\hat{}_p) \cong (P_2Y)\hat{}_p$ is essential. The adjoint of h is then non-zero element of $\pi_{k-2}\,\mathrm{Map}_*(B\mathbf{Z}/p^\infty, X)$, an element which by Theorem 1.2 cannot exist. This contradiction shows that S_2 is infinite and proves the theorem.

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Dedicated to the memory of Alex Zabrodsky

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