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THE RIGIDITY OF POINCARÉ DUALITY ALGEBRAS AND CLASSIFICATION OF HOMOTOPY TYPES OF MANIFOLDS

MARTIN MARKL

INTRODUCTION

This paper is devoted to the study of homotopy types of simply connected rational Poincaré duality spaces. We will frequently use the language and results of rational homotopy theory, a good common reference is the book [14].

So, let X be a rational Poincaré duality space of the (top) dimension n, i.e. a simply connected space, whose rational cohomology algebra is a Poincaré duality algebra of the formal dimension n; see §3. It is well-known (see also §3) that X has the rational homotopy type of a space of the form $Y \cup_h e^n$, where Y is a simply connected CW-complex of dimension $< n$ and $h : S^{n-l} = \partial e^n \to Y$ is a continuous map. The space Y, defined uniquely up to rational homotopy type, will be called (with some inaccuracy) the skeleton of X and will be denoted by $X_{<n}$. If X is a simply connected n-dimensional manifold, the construction above can be described even more geometrically: take $X \setminus B^n$, where B^n is a (sufficiently small) n-dimensional open disc. It is easy to remark that the n-dimensional manifold with boundary, $X \setminus B^n$, has the same rational homotopy type as the skeleton $X_{<n}$, constructed above.

Recall that two simply connected spaces X and Y are said to have the same $\mathbf{k}$-homotopy type, where $\mathbf{k}$ is a field of characteristic zero, if their Quillen minimal models [14; III.3.(1)] are isomorphic over $\mathbf{k}$; this fact will be denoted by $X \sim_{\mathbf{k}} Y$. Of course, for $\mathbf{k} = \mathbf{Q}$ we get the usual definition of the rational homotopy equivalence.

Fix an n-dimensional rational Poincaré duality space X (simply connected by definition). The aim of this paper is to give a description of the set $PDS_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of rational Poincaré duality spaces Y whose skeleta $Y_{<n}$ have the same rational homotopy type as the skeleton $X_{<n}$ of X, when X is formal. It is interesting to point out that the set $PDS_{\mathbf{k}}(X)$ is, according to rational surgery results [3],

for $n \not\equiv 0 \pmod 4$ naturally isomorphic to the set $Man_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of n-dimensional compact simply connected manifolds M with $M_{<n} \sim_{\mathbf{Q}} X_{<n}$.

The first attempt towards the description of $PDS_{\mathbf{k}}(X)$ was made in [12], where it is stated [12; Theorem 1] that the rational homotopy type of a rational Poincaré duality space is uniquely determined by the rational homotopy type of its skeleton, if the cohomology algebra of X is fixed. Here we will always suppose that X is formal, the hypothesis taken by M. Aubry [1,2].

We give here a complete description of the set $PDS_{\mathbf{k}}(X)$ in terms of usual algebraic objects – Galois cohomology and induced maps – when X is formal. Using this description, we are able to prove, for example, that the $\mathbf{k}$-homotopy type of a rational Poincaré duality space is uniquely determined by its skeleton provided that $\mathbf{k}$ is algebraically closed. We prove also that the set $PDS_{\mathbf{k}}(X)$ (and hence also $Man_{\mathbf{k}}(X)$) is finite for fields satisfying $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ (for example for $\mathbf{k} = \mathbf{R}$, the case of real homotopy types). As an example of explicit calculations we construct a large class of Poincaré duality spaces X for which the set $PDS_{\mathbf{k}}(X)$ consists of the $\mathbf{k}$-homotopy type of X only, $\mathbf{k}$ arbitrary. On the other hand, we give an example of a manifold M, for which the set $PDS_{\mathbf{Q}}(M)$ is infinite.

The algebraic counterpart of the description of $PDS_{\mathbf{k}}(X)$ is the following classification problem: let H^* be a Poincaré duality algebra of formal dimension n, how to describe the set $PDA_{\mathbf{k}}(H^*)$ of all isomorphism classes of Poincaré duality algebras H'^* with $H'^*/H'^n \cong H^*/H^n$. Our approach to the study of the set $PDA_{\mathbf{k}}(H^*)$ is based on a rigidity property of Poincaré duality algebras over an algebraically closed field and on the usual method of descent. We hope that this approach can be used even in more general situation – for the classification of all Gorenstein rings R having the "skeleton" $R/Socle(R)$ fixed (see [15]).

Our paper is organized as follows. In the first paragraph we prove a rigidity theorem for Poincaré duality algebras. The proof of this statement is based on a deliberate use of the deformation theory; note that this machinery has already been systematically used in rational homotopy theory in [4]. As a by-product we obtain a characterization of Poincaré duality in terms of Harrison cohomology. These results are in the next paragraph applied to the solution of our classification problem for Poincaré duality algebras. The main result of this section is Theorem 2.7. In the third paragraph the algebraic theory is applied to the study of the set $PDS_{\mathbf{k}}(X)$ as introduced above, a description is given in Theorem 3.2. Notice that both Theorem 3.2 and the forthcoming examples explicitely describe the effect of the ground field $\mathbf{k}$ on the structure of $PDS_{\mathbf{k}}(X)$, hence all the material of this paragraph can be considered as a contribution to the study of descent and non-descent phenomena in rational homotopy theory in the spirit of [10].

I would like to express here my thanks to Ştefan Papadima for drawing my attention to the possible use of descent methods. Also the formulation of the condition iii) of Theorem 1.5 is due to him. I wish also to acknowledge my indebtedness to the referee for useful comments and references.

1. Rigidity of Poincaré duality algebras

As usually, by a Poincaré duality algebra (over a field $\mathbf{k}$) of the formal dimension n is meant a (finite dimensional) graded commutative $\mathbf{k}$-algebra $H^* = \bigoplus_{i\geq 0} H^i$ such that H^n is isomorphic to k, $H^i = 0$ for $i > n$ and the bilinear form $B : H^* \bigotimes H^* \to \mathbf{k}$ of degree $-n$ defined by

$$B(x,y) = \begin{cases} x.y \in \mathbf{k} \cong H^n & \text{for } deg(x) + deg(y) = n \\ 0 & \text{otherwise} \end{cases}$$

is nondegenerate in the usual sense. All Poincaré duality algebras (and Poincaré duality spaces) in this paper are supposed to have the same formal dimension equal to a given natural number n.

1.1. For a graded commutative algebra A^* denote:

$$\mathcal{B}(A^*) = \left\{ \begin{array}{c} \text{all bilinear forms } B : A^* \bigotimes A^* \to \mathbf{k} \text{ of degree } -n \text{ such} \\ \text{that } B(x,y) = (-1)^{deg(x)deg(y)} B(y,x) \text{ for } x, y \in A^* \end{array} \right\},$$

$$\mathcal{M}(A^*) = \{B \in \mathcal{B}(A^*); B(xy,z) = B(x,yz) \text{ for } x,y,z \in A^*\},$$

$$\mathcal{P}(A^*) = \left\{B \in \mathcal{M}(A^*); B \text{ is nondegenerate on } A^{>0} \bigotimes A^{>0}\right\} \quad \text{and}$$

$$G(A^*) = Aut(A^*) = \text{the group of graded automorphisms of } A^*.$$

Notice that all the sets above have the natural structure of a (not necessarily irreducible) algebraic variety. The geometry of $\mathcal{M}(A^*)$ is extremely simple—as all the defining equations are linear, it is isomorphic to an affine space. The set $\mathcal{P}(A^*)$ is plainly Zariski-open and dense in $\mathcal{M}(A^*)$. The group $G(A^*)$ acts naturaly from the left on $\mathcal{B}(A^*)$ by

$$\phi(B)(x,y) = B(\phi^{-1}(x), \phi^{-1}(y)).$$

Clearly $G(A^*)\mathcal{M}(A^*) \subset \mathcal{M}(A^*)$ and $G(A^*)\mathcal{P}(A^*) \subset \mathcal{P}(A^*)$. The action of $G(A^*)$ is plainly continuous in the Zariski topology.

We call an algebra A^* a fragment, if it is of the form

$$A^* = H^*_{<n} := H^*/H^n$$

for a Poincaré duality algebra H^*. The algebra $H^*_{<n}$ will be called the *skeleton* of H^*. Here $H^*_{<n}$ is defined as a quotient, but after having chosen a section, we may as well consider it as a subset of H^*.

It is interesting to remark that it is allways possible to decide in finitely many steps whether a given graded commutative algebra A^* is a fragment or not. To this end, find at first a basis of the affine space $\mathcal{M}(A^*)$. Our algebra A^* is then a fragment if and only if the polynomial function, representing the determinant, is not equal to zero on $\mathcal{M}(A^*)$ identically.

This characterization problem for fragments is the special case of the problem of deciding when a given local ring is a factor of a Gorenstein ring by the socle, see [15].

1.2. For a fragment A^* consider the set $\tilde{\mathcal{M}}(A^*)$ of all graded commutative algebras H^* with $H^i = 0$ for $i > n$, $H^n \cong \mathbf{k}$ and $H^*_{<n}$ isomorphic to A^*. For $H^* \in \tilde{\mathcal{M}}(A^*)$ choose an isomorphism $r : H^n \to \mathbf{k}$ and define $B \in \mathcal{M}(A^*)$ by $B(x,y) = r(x.y) \in \mathbf{k}$. The form B is defined canonically up to a nonzero multiple from $\mathbf{k}$. Keeping in mind this ambiguity, we can write $H^* = (A^*, B)$. Notice that H^* is a Poincaré duality algebra if and only if $B \in \mathcal{P}(A^*)$.

1.3. Let $A^* = H^*_{<n}$ be a fragment and denote by $PDA_{\mathbf{k}}(H^*)$ the set of all isomorphism classes of Poincaré duality $\mathbf{k}$-algebras having the skeleton isomorphic to A^*. We claim that the presentation 1.2 induces a bijection between $PDA_{\mathbf{k}}(H^*)$ and the orbit space $\mathcal{P}(A^*)/G(A^*)$ provided that $\mathbf{k}$ algebraically closed.

To verify this, notice at first that each algebra from $PDA_{\mathbf{k}}(H^*)$ is isomorphic to an algebra H'^* with $H'^*_{<n} = A^*$. Hence we can suppose immediately that $H'^*{}_{<n} = A^*$ for each $H'^* \in PDA_{\mathbf{k}}(H^*)$. Let $H'^* = (A^*, B')$ and $H''^* = (A^*, B'')$ be two algebras from

$PDA_{\mathbf{k}}(H^*)$ and suppose that they are isomorphic. This means that there exists an isomorphism $\phi : A^* \to A^*$ and a nonzero $\alpha \in \mathbf{k}$ such that $B''(\phi(x), \phi(y)) = \alpha B'(x, y)$. If we choose $\xi \in \mathbf{k}$ such that $\xi^n = \alpha$ and define $g \in Aut(A^*)$ by $g(x) = \xi^{-deg(x)}.x$, we see that $B''(\phi \circ g(x), \phi \circ g(y)) = B'(x, y)$, i.e. B' and B'' are in the same orbit of $G(A^*)$. On the other hand, it is easy to check that forms belonging to the same orbit define isomorphic algebras.

1.4. Before formulating the central result of this section, recall some necessary facts about the Harrison cohomology [13]. Let A^* be a graded commutative algebra and M^* a graded A^*-module. Define on $\bigotimes^m A^*$ a new grading, putting $deg(a_1 \otimes \cdots \otimes a_m) = 1 + \sum_1^m (deg(a_i) - 1)$ and denote by $C^{m,p}(A^*, M^*)$ the set of all linear maps $f : \bigotimes^m A^* \to M^*$ of degree p such that $f(a_1, \ldots, a_m) = 0$ whenever some $a_i = 1$, $1 \leq i \leq m$. The differential δ of bidegree $(1,1)$ on $C^{*,*}(A^*, M^*)$ is defined by the formula

$$\delta f(a_1, \ldots, a_{m+1}) = a_1 f(a_2, \ldots, a_{m+1}) + (-1)^{\nu(m+1)} f(a_1, \ldots, a_m) a_{m+1} + \\ + \sum_{j=1}^{m} (-1)^{\nu(j)} f(a_1, \ldots, a_j a_{j+1}, \ldots, a_{m+1}),$$

where $\nu(j) = \sum_{i=1}^{j} (deg(a_i) - 1)$. The cohomology of the complex $(C^{*,*}(A^*, M^*), \delta)$ is the usual Hochshild cohomology of A^* with coefficients in M^*. Consider the subspace $C^{m,p}_{\mathrm{Harr}}(A^*; M^*)$ of $C^{m,p}(A^*, M^*)$ consisting of all cochains of $C^{m,p}(A^*, M^*)$ which are zero on decomposable elements of the shuffle product in $\bigotimes A^*$ (see [14; p.18]). The subspace $C^{*,*}_{\mathrm{Harr}}(A^*, M^*)$ can be shown to be δ-stable and the associated cohomology

$$\mathrm{Harr}^{m,p}(A^*; M^*) := H^{m,p}(C^{*,*}_{\mathrm{Harr}}(A^*, M^*), \delta)$$

is called the *Harrison cohomology* of the graded commutative algebra A^* with coefficients in M^*.

For a given fragment A^* and an algebra $H^* \in \tilde{\mathcal{M}}(A^*)$ there are two natural A^*-modules: the "reduced" algebra $\tilde{H}^*$ (= the ideal of the natural augmentation $H^* \to \mathbf{k}$) with the action given simply by the multiplication and H^n with the trivial action ($1.h = h$ and $A^{>0}.H^n = 0$). The inclusion $\iota : H^n \to \tilde{H}^*$ is a morphism of A^*-modules and it induces the map

$$\iota_* : \mathrm{Harr}(A^*; H^n) \to \mathrm{Harr}(A^*; \tilde{H}^*)$$

in Harrison cohomology. In the following theorem we give three equivalent conditions on $H^* \in \tilde{\mathcal{M}}(A^*)$ to be a Poincaré duality algebra, where A^* is a fragment. Recall that it means by definition that we *a priori* assume the existence of a symmetric nondegenerate bilinear form on A^*.

THEOREM 1.5 (RIGIDITY THEOREM). *Suppose that the ground field* $\mathbf{k}$ *is algebraically closed of characteristic zero. Let* A^* *be a fragment and let* H^* *be a graded commutative algebra with* $H^i = 0$ *for* $i > n$, $H^n \cong \mathbf{k}$ *and* $H^*_{<n}$ *isomorphic to* A^* *(in other words,* $H^* \in \tilde{\mathcal{M}}(A^*)$*). Let* $B \in \mathcal{M}(A^*)$ *be the bilinear form corresponding to* H^* *as in 1.2. Then the following three conditions are equivalent:*

i) H^* *is a Poincaré duality algebra,*
ii) *the point* $B \in \mathcal{M}(A^*)$ *is rigid under the action of* $G(A^*)$†,
iii) *the map* $\iota : \mathrm{Harr}^{2,1}(A^*; H^n) \to \mathrm{Harr}^{2,1}(A^*; \tilde{H}^*)$ *is zero.*

Proof. Define $F_B : G(A^*) \to \mathcal{M}(A^*)$ by $F_B(g) = g(B)$. Let us try to describe the tangent map $T_eF_B : T_eG(A^*) \to T_B\mathcal{M}(A^*)$ at the unit e of $G(A^*)$. As the algebra A^* has finite dimension, we have $T_eG(A^*) \cong Der(A^*)$ (the set of derivations of the algebra A^* of degree 0). The set $\mathcal{M}(A^*)$ is isomorphic to an affine space (see 1.1), hence we can identify $T_B\mathcal{M}(A^*)$ with $\mathcal{M}(A^*)$ itself. Using these identifications, we can easily obtain

$$T_eF_B(\phi)(x,y) = -(B(\phi(x),y) + B(x,\phi(y))).$$

This means that the map T_eF_B is epic if and only if for each $f \in \mathcal{M}(A^*)$ there exists a derivation $\phi \in Der(A^*)$ with

$$f(x,y) = B(\phi(x),y) + B(x,\phi(y)) \tag{1.6}$$

for $deg(x) + deg(y) = n$. On the other hand, we can obtain immediately from the definitions that

$$Z^{2,1}_{\mathrm{Harr}}(A^*, H^n) = \left\{ \begin{array}{c} \text{bilinear forms } \tilde{f} : A^* \otimes A^* \to H^n \text{ of degree zero} \\ \text{such that } \tilde{f}(x,y) = (-1)^{deg(x)deg(y)+n}\tilde{f}(y,x) \\ \text{and } \tilde{f}(xy,z) = (-1)^{deg(y)}\tilde{f}(x,yz) \end{array} \right\}$$

†This means by definition that the orbit $G(A^*)(B)$ contains a Zariski-open neighbourhood of B, see [5] or [9]

and that

$$C^{1,0}_{\mathrm{Harr}}(A^*, \tilde{H}^*) = \left\{\text{linear maps } \tilde{\phi} : A^* \to \tilde{H}^* \text{ of degree zero}\right\},$$

while

$$\delta\tilde{\phi}(x,y) = x\tilde{\phi}(y) + (-1)^{deg(x)+deg(y)}\tilde{\phi}(x)y - (-1)^{deg(x)}\tilde{\phi}(xy).$$

Therefore $\iota_*(\tilde{f}) = \delta\tilde{\phi}$ in $C^{2,1}_{\mathrm{Harr}}(A^*, \tilde{H}^*)$ if and only if

(1.7)

$$\begin{gathered} \tilde{f}(x,y) = x\tilde{\phi}(y) + (-1)^n\tilde{\phi}(x)y = B(x,\tilde{\phi}(y)) + (-1)^n B(\tilde{\phi}(x),y) \\ \text{for } deg(x)+deg(y)=n \text{ and} \\ (-1)^{deg(x)}\tilde{\phi}(xy) = x\tilde{\phi}(y) + (-1)^{deg(x)+deg(y)}\tilde{\phi}(x)y \\ \text{for } deg(x)+deg(y)<n \end{gathered}$$

The correspondence $\tilde{f} \longmapsto f$, $f(x,y) = (-1)^{deg(y)}\tilde{f}(x,y)$ clearly defines an identification of $Z^{2,1}_{\mathrm{Harr}}(A^*, H^n)$ and $\mathcal{M}(A^*)$. Writting in (1.7) $(-1)^{deg(y)}.f(x,y)$ instead of $\tilde{f}(x,y)$ and $(-1)^{deg(x)}\phi(x)$ instead of $\tilde{\phi}(x)$, we can easily verify that the map ι_* is zero if and only if for each $f \in \mathcal{M}(A^*)$ there exists a linear map $\phi : A^* \to \tilde{H}^*$ of degree zero such that

$$(1.7') \qquad \begin{cases} f(x,y) = B(x,\phi(y)) + B(y,\phi(x)) & \text{for } deg(x)+deg(y)=n \text{ and} \\ \phi(xy) = x\phi(y) + \phi(x)y & \text{for } deg(x)+deg(y)<n. \end{cases}$$

Notice that the second equation in (1.7') means that ϕ is a derivation of the algebra A^*, i.e. $\phi \in T_eG(A^*)$. Comparing (1.6) and (1.7') we see that the following statement is valid.

LEMMA 1.8. *The map T_eF_B is an epimorphism if and only if ι_* is zero.*

Proof of i) $\Longrightarrow$ iii). Suppose that H^* is a Poincaré duality algebra and prove (1.6). Let $f \in \mathcal{M}(A^*)$. As the form B is nondegenerate, the formula

$$(1.9) \qquad B(\phi(x),y) = \tfrac{deg(x)}{n} f(x,y)$$

defines a linear map $\phi : A^* \to A^*$ of degree zero. By (1.9) and the symmetry of B,

(1.10)

$$\begin{aligned} B(x,\phi(y)) &= (-1)^{deg(x)deg(y)} B(\phi(y),x) \\ &= \tfrac{deg(y)}{n}.(-1)^{deg(x)deg(y)}.f(y,x) \\ &= \tfrac{deg(y)}{n}.f(x,y). \end{aligned}$$

By (1.9) and (1.10), for $deg(x)+deg(y)=n$,

$$B(x,\phi(y)) + B(\phi(x),y) = (\tfrac{deg(x)}{n} + \tfrac{deg(y)}{n}).f(x,y) = f(x,y),$$

which is (1.6). It remains to show that ϕ is really a derivation of degree zero. This is, because B is nondegenerate, equivalent to

(1.11)

$$B(x\phi(y),z) + B(\phi(x)y,z) = B(\phi(xy),z)$$
$$\text{for each } x,y,z \in A^* \text{ with } deg(x)+deg(y)+deg(z)=n.$$

We easily deduce from (1.9) and (1.10) that

$$B(\phi(xy),z) = \tfrac{deg(x)+deg(y)}{n}.f(x,y)$$

and that

$$\begin{aligned} B(\phi(x)y,z) = B(\phi(x),yz) = \tfrac{deg(x)}{n}.f(x,yz) &= \tfrac{deg(x)}{n}.f(xy,z), \\ B(x\phi(y),z) &= (-1)^{deg(x)deg(y)}.B(\phi(y),xz) \\ &= \tfrac{deg(y)}{n}.(-1)^{deg(x)deg(y)} f(y,xz) \\ &= \tfrac{deg(y)}{n}.f(xy,z) \end{aligned}$$

Using these formulae, it is easy to verify (1.11), hence T_eF_B is epic and ι_* is zero by Lemma 1.8.

Proof of iii) $\Longrightarrow$ ii). Notice that the points $e \in G(A^*)$ and $B \in \mathcal{M}(A^*)$ are regular. If iii) is satisfied, the map T_eF_B is epic by Lemma 1.8 and $Im(F_B) = G(A^*)(B)$ contains an open neighbourhood by standard arguments of the algebraic geometry, see for example [9; Lemma 23.5].

Proof of ii) $\Longrightarrow$ i). Suppose that B is rigid and let $U \subset G(A^*)(B)$ be an open neighbourhood of B. Then both U and $\mathcal{P}(A^*)$ are nonempty open subsets in the affine space $\mathcal{M}(A^*)$, hence $\mathcal{P}(A^*) \cap U \neq 0$. They are both $G(A^*)$-invariant and $G(A^*)$ acts on U transitively, consequently, $U \subset \mathcal{P}(A^*)$, therefore $B \in \mathcal{P}(A^*)$, i.e. H^* is a Poincaré duality algebra.

2. Classification theorems for Poincaré duality algebras

Our classification is based on the fact that Poincaré duality algebras over an algebraically closed field are uniquely determined by their skeleton.

THEOREM 2.1. *Let A^* be a fragment and suppose that the ground field $\mathbf{k}$ is algebraically closed of characteristic zero. Then*

$$\#(\mathcal{P}(A^*)/G(A^*)) = 1,$$

in other words (see 1.3), the following statement is true:

Let H^* and H'^* be two Poincaré duality algebras over an algebraically closed field of characteristic zero such that $H^*_{<n}$ is isomorphic to $H'^*_{<n}$. Then the algebras H^* and H'^* are isomorphic, too.

Proof. As $\mathcal{M}(A^*)$ is an affine space, there exists at most one open orbit of $G(A^*)$ in $\mathcal{M}(A^*)$. On the other hand, the orbit of every point $B \in \mathcal{P}(A^*)$ is open by Theorem 1.5 ii). Therefore all points of $\mathcal{P}(A^*)$ are in the same orbit, in other words, $G(A^*)$ acts on $\mathcal{P}(A^*)$ transitively and $\#(\mathcal{P}(A^*)/G(A^*)) = 1$.

2.2. Warning: Being $g : H^*_{<n} \cong H'^*_{<n}$ an isomorphism, then the isomorphism of H^* and H'^*, whose existence is guarranteed by Theorem 2.1, is not necessarily an extension of g.

Example 2.3. Let V be a $\mathbf{k}$-vector space and fix an even number $d > 0$. Let A^* be a graded algebra defined by $A^0 = \mathbf{k}$, $A^d = V$ and $A^i = 0$ otherwise, having the obvious product. Every nondegenerate symmetric bilinear form B on V defines a Poincaré duality algebra H^* of the formal dimension $n = 2d$ with $H^*_{<n} = A^*$ (see 1.2), this is the simplest nontrivial example of a Poincaré duality algebra.

Clearly, $\mathcal{P}(A^*)$ consists of all symmetric nondegenerate bilinear forms on V and the quotient $\mathcal{P}(A^*)/G(A^*)$ is the set of all equivalence classes of nondegenerate symmetric bilinear forms on V. If the field $\mathbf{k}$ is algebraically closed, Theorem 2.1 says that there exists exactly one equivalence class of nondegenerate bilinear forms on V. This result is classical.

2.4. Now, starting from the classification over algebraically closed fields, we can try to obtain a general result using the usual description of the descent by Galois cohomology, see [11]. Let us introduce the following notation and terminology.

Let $\mathbf{K}$ be an extension of $\mathbf{k}$ and let M be an object (vector space, algebra etc.) defined over $\mathbf{k}$. Denote $M_{\mathbf{K}} = M \otimes_{\mathbf{k}} K$. Two objects M and N, defined over $\mathbf{k}$, are said to be $\mathbf{K}$-isomorphic ($M \cong_{\mathbf{K}} N$), if there exists a $\mathbf{K}$-isomorphism between the $\mathbf{K}$-objects $M_{\mathbf{K}}$ and $N_{\mathbf{K}}$. Fix now a Poincaré duality $\mathbf{k}$-algebra H^* and let $A^* = H^*_{<n}$ be its skeleton. The central object of our study is the following set

$$PDA_{\mathbf{k}}(H^*) = \left\{ \begin{array}{c} \mathbf{k}\text{-isomorphism classes of all Poincaré} \\ \text{duality } \mathbf{k}\text{-algebras } H'^* \text{ with } H'^*_{<n} \cong_{\mathbf{k}} A^* \end{array} \right\}$$

Unfortunately, this set is not approachable to apply the descent method directly. We are led to consider also the following sets ($\bar{\mathbf{k}}$ denotes the algebraic closure of $\mathbf{k}$):

$$\tilde{E}_{\mathbf{k}} = \left\{ \begin{array}{c} \mathbf{k}\text{-isomorphism classes of Poincaré duality} \\ \mathbf{k}\text{-algebras } H'^* \text{ with } H'^* \cong_{\bar{\mathbf{k}}} H^* \end{array} \right\}$$

$$E_{\mathbf{k}} = \left\{ \begin{array}{c} \mathbf{k}\text{-isomorphism classes of graded commutative} \\ \mathbf{k}\text{-algebras } A'^* \text{ with } A'^* \cong_{\bar{\mathbf{k}}} A^* \end{array} \right\}$$

and define the map $F_{\mathbf{k}} : \tilde{E}_{\mathbf{k}} \to E_{\mathbf{k}}$ by $F_{\mathbf{k}}(H'^*) = H'^*{}_{<n}$. The sets $\tilde{E}_{\mathbf{k}}$ and $E_{\mathbf{k}}$ are related with $PDA_{\mathbf{k}}(H^*)$ as follows:

LEMMA 2.5. *Let $\mathbf{k}$ be a field of characteristic zero. Then there exists a natural correspondence between the elements of $PDA_{\mathbf{k}}(H^*)$ and algebras $H'^* \in \tilde{E}_{\mathbf{k}}$ satisfying $F_{\mathbf{k}}(H'^*) \cong_{\mathbf{k}} A^*$*

Proof. By Theorem 2.1, $H'^* \cong_{\bar{\mathbf{k}}} H^*$ for each $H'^* \in PDA_{\mathbf{k}}(H^*)$, the rest is trivial.

The following description of the descent for graded algebras can be obtained by a slight modification of the proof of [11; Proposition 1 in III.1.1], see also the comments to the proof given in the russian version of this book (Mir 1968).

PROPOSITION 2.6. *Suppose that* $\mathbf{K}$ *is a Galois extension of* $\mathbf{k}$. *Let* M^* *be a graded* $\mathbf{k}$*-algebra and denote by* $\mathcal{E}(\mathbf{K}/\mathbf{k})$ *the set of all* $\mathbf{k}$*-isomorphism classes of* $\mathbf{k}$*-algebras* M'^* *with* $M'^* \cong_{\mathbf{K}} M^*$.

Then the elements of the set $\mathcal{E}(\mathbf{K}/\mathbf{k})$ *are in a natural one-to-one correspondence with the Galois cohomology group* $H^1(G(\mathbf{K}/\mathbf{k}); Aut_{\mathbf{K}}(M^*_{\mathbf{K}}))$, *where the action of the Galois group* $G(\mathbf{K}/\mathbf{k})$ *on* $Aut_{\mathbf{K}}(M^*_{\mathbf{K}})$ *is defined by* $s(f) = (1 \otimes s) \circ f \circ (1 \otimes s)^{-1}$.

Using the explicit description of the correspondence in the previous proposition, we can infer easily from Lemma 2.5 the following classification result.

THEOREM 2.7. *Let* $\mathbf{k}$ *be a field of characteristic zero and denote by* $\bar{\mathbf{k}}$ *the algebraic closure of* $\mathbf{k}$. *Let* $A^* = H^*_{<n}$ *be a fragment defined over* $\mathbf{k}$. *Then there exists a natural one-to-one correspondence between the set* $PDA_{\mathbf{k}}(H^*)$ *of all* $\mathbf{k}$*-isomorphism classes of Poincaré duality* $\mathbf{k}$*-algebras* H'^* *with* $H'^*_{<n} \cong_{\mathbf{k}} A^*$ *and the set*

$$Ker(\iota_{\mathbf{k}} : H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}})) \to H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(A^*_{\bar{\mathbf{k}}})))$$

where the map $\iota_{\mathbf{k}}$ *is induced by the natural homomorphism*

$$j : Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}}) \to Aut_{\bar{\mathbf{k}}}(A^*_{\bar{\mathbf{k}}})$$

given by the restriction.

We close this section with the following corollary of Theorem 2.7.

COROLLARY 2.8. *Suppose that* $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$. *In this case there exists only finitely many isomorphism classes of Poincaré duality* $\mathbf{k}$*-algebras having a given skeleton (i.e.* $\#(PDA_{\mathbf{k}}(H^*)) < \infty$*).*

Proof. A field satisfying the condition $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ is of type (F) in the sense of [11; III.4], hence $H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}}))$ is finite [11; III.4.3] and the corollary follows from Theorem 2.7.

3. APPLICATIONS TO THE RATIONAL HOMOTOPY TYPE

By a rational Poincaré duality space (of the formal dimension n) is meant here a simply connected CW-complex X such that $H^*(X; \mathbf{Q})$ is a Poincaré duality algebra (of the formal dimension n). We show that X has the rational homotopy type of a space of the form $Y \cup_h e^n$, where Y is a CW-complex of the dimension $< n$ and $h\colon S^{n-1} = \partial e^n \to Y$ a continuous map, as promissed in the introduction.

To this end, suppose that $(\mathcal{L}(Z,\mu),\partial)$ is the Quillen minimal model of X, where the generator μ, $deg(\mu) = n-1$, corresponds to the "orientation class" of X in $H_n(X;\mathbf{Q})$. Let Y be a space corresponding to the minimal algebra $(\mathcal{L}(Z),\partial|\mathcal{L}(Z))$, we can clearly suppose that Y is a CW-complex of dimension $< n$. Let $h \in \pi_{n-1}(X)$ be an element corresponding (up to a nonzero rational multiple if necessary) to $[\partial(\mu)] \in H_{n-2}(\mathcal{L}(Z),\partial|\mathcal{L}(Z)) \cong \pi_{n-1}(X)\otimes\mathbf{Q}$. It follows easily from [14; III.3.(6)] that $Y \cup_h e^n \sim_{\mathbf{Q}} X$ and Y is what we call the skeleton of X and denote by $X_{<n}$. It is also clear that $H^*(X_{<n};\mathbf{k}) \cong H^*(X;\mathbf{k})_{<n}$, see [14; III.3(9)].

3.1. Observe that X is formal if and only if $X_{<n}$ is. Indeed, if X is formal, then the minimal model $(\mathcal{L}(Z,\mu),\partial)$ can be chosen so that ∂ is quadratic [14; II.7(5)]. Then also the minimal model $(\mathcal{L}(Z),\partial|\mathcal{L}(Z))$ of $X_{<n}$ is quadratic and $X_{<n}$ is formal again by [14; III.7(5)]. On the other hand, if $X_{<n}$ is formal, the formality of X follows easily from [12; Theorem 1], [1,2]. The central result of our paper now reads:

THEOREM 3.2. *Let $\mathbf{k}$ be a field of characteristic zero and let X be a formal rational Poincaré duality space of the top dimension n. Then there exists a natural one-to-one correspondence between the set $PDS_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of rational Poincaré duality spaces Y of the top dimension n, such that the skeletons $X_{<n}$ and $Y_{<n}$ have the same rational homotopy type, and the set $\phi_H(Ker(\iota_{\mathbf{Q}}))$, where the map*

$$\phi_H : H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}}))) \longrightarrow H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*(X;\bar{\mathbf{k}})))$$

is induced by the natural homomorphism $G(\bar{\mathbf{k}}/\mathbf{k}) \to G(\bar{\mathbf{Q}}/\mathbf{Q})$, and the map

$$\iota_{\mathbf{Q}} : H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}}))) \longrightarrow H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}})_{<n})))$$

is induced by the homomorphism $Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}})) \to Aut_{\bar{\mathbf{Q}}}(H^*(X;\bar{\mathbf{Q}})_{<n})$ given by the restriction.

The proof is postponed to the end of this section. Although the description in Theorem 3.2 seems to be unmanageable, it provides us with a few of corollaries.

COROLLARY 3.3. *If the field* $\mathbf{k}$ *is algebraically closed, the* $\mathbf{k}$*-homotopy type of a formal Poincaré duality space is uniquely determined by the rational homotopy type of its skeleton.*

Proof. If $\mathbf{k} = \bar{\mathbf{k}}$, then the group $G(\bar{\mathbf{k}}/\mathbf{k})$ is trivial, hence the map ϕ_H in 3.2 is trivial, too. Therefore $\#(PDS_{\mathbf{k}}(X)) = 1$.

COROLLARY 3.4. *If* $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ *and* X *is a formal rational Poincaré duality space, then the set* $PDS_{\mathbf{k}}(X)$ *is finite. Especially, there exists only finitely many real homotopy types of rational Poincaré duality spaces with a given (formal) rational homotopy type of the skeleton.*

Proof. By the same argument as in the proof of Corollary 2.8 we can easily see that the set $H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*(X;\bar{\mathbf{k}})))$ is finite. The rest follows from Theorem 3.2.

Remember that, for $n \not\equiv 0 \pmod 4$, every rational Poincaré duality space (simply connected by definition) has the rational homotopy type of a compact simply connected manifold [3]. Consequently, Theorem 3.2 and the corollaries give in this case a description of the set $Man_{\mathbf{k}}(X)$ of all $\mathbf{k}$-homotopy types of simply connected compact *manifolds* M with $M_{<n} \sim_{\mathbf{Q}} X_{<n}$.

In the following example we construct a compact, simply connected manifold M for which the set $Man_{\mathbf{Q}}(M)$ is infinite.

Example 3.5. Let us denote by M the 6-dimensional simply connected compact manifold $\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C})$. Clearly, $H^*(M;\mathbf{Q}) \cong \mathbf{Q}[u,v]/(uv, u^3 - v^3)$, $deg(u) = deg(v) = 2$. It is also not hard to verify that $M_{<6}$ has the rational homotopy type of $\mathbf{P}^2(\mathbf{C}) \bigvee \mathbf{P}^2(\mathbf{C})$ (the one-point union) and that

$$H^*(M_{<6};\mathbf{Q}) \cong \mathbf{Q}[u,v]/(uv, u^3, v^3) = \left\{\mathbf{Q}[u,v]/(uv, u^3 - v^3)\right\}_{<6}.$$

As every 1-connected 6-dimensional manifold, the space M is formal [6].

Every rational Poincaré duality algebra H'^* with $H'^*_{<6} \cong_{\mathbf{Q}} H^*(M_{<6};\mathbf{Q})$ is of the form $\mathbf{Q}[u,v]/(uv, \alpha u^3 - \beta v^3)$ for some nonzero $\alpha, \beta \in \mathbf{Q}$. Notice that for such an algebra H'^* there always exists a manifold N with $H^*(N;\mathbf{Q}) \cong H'^*$ and $N_{<6} \sim_{\mathbf{Q}} M_{<6}$. Indeed, let N be a formal rational homotopy type corresponding to H'^*, as $6 \not\equiv 0 \pmod 4$ we can assume that N is a manifold. Then $N_{<6}$ is again formal by 3.1, and $N_{<6} \sim_{\mathbf{Q}} M_{<6}$ since $H^*(N_{<6};\mathbf{Q}) \cong_{\mathbf{Q}} H^*(M_{<6};\mathbf{Q})$ by the construction.

It can be easily verified that the algebra $\mathbf{Q}[u,v]/(uv, \alpha u^3 - \beta v^3)$ is isomorphic to the algebra $\mathbf{Q}[u,v]/(uv, \alpha' u^3 - \beta' v^3)$ if and only if either $\frac{\alpha\beta'}{\beta\alpha'}$ or $\frac{\alpha\alpha'}{\beta\beta'}$ is of the form q^3 for some rational number $q \in \mathbf{Q}$. Consequently, there exist infinitely many $\mathbf{Q}$-isomorphism classes of such algebras. According to the remarks above, the set $Man_{\mathbf{Q}}(\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C}))$ is infinite.

On the other hand it can be easily seen that $Man_{\mathbf{R}}(\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C}))$ consists of the real homotopy type of $\mathbf{P}^3(\mathbf{C})\#\mathbf{P}^3(\mathbf{C})$ only.

3.6. Now we aim to describe a family of manifolds (Poincaré duality spaces), whose rational homotopy type is uniquely determined by their skeleton. Let H^* be a Poincaré duality algebra and consider the canonical map $j : Aut(H^*) \to Aut(H^*_{<n})$ given by the restriction. Recall that this map plays an important role in our classification 2.7 and 3.2. As the subset $H^*_{<n} \subset H^*$ generates H^* as an algebra, every automorphism of H^* is uniquely determined by its restriction on $H^*_{<n}$, hence the map j is plainly a monomorphism. Suppose that the algebra H^* can be represented in the form

(*) $\quad H^* \cong \Lambda V^*/I,$ where $I = (f_1, \ldots, f_s)$ and $deg(f_i) \neq n$ for $1 \leq i \leq s$.

We claim that in this case the map j is also an epimorphism.

To prove this, consider an element $g \in Aut(H^*_{<n})$. Since clearly

$$H^*_{<n} \cong \Lambda V^*/(I + (\Lambda V^*)^{\geq n}),$$

our map g lifts to some $\tilde{g} \in Aut(\Lambda V^*)$ with $\tilde{g}(I^{<n}) \subset I^{<n}$. Because of (*) this implies plainly that $\tilde{g}(I^{\leq n}) \subset I^{\leq n}$. Since $H^{>n} = 0$, we know that $I^{>n} = (\Lambda V^*)^{>n}$, therefore in fact $\tilde{g}(I) \subset I$. This means that $\tilde{g}$ projects to an element $f \in Aut(H^*)$ which clearly satisfies $j(f) = g$.

Suppose now that X is a formal rational Poincaré duality space whose rational cohomology algebra satisfies (*). Then the map $\iota_{\mathbf{Q}}$ in Theorem 3.2, induced by j, is an isomorphism and $\phi_H(Ker(\iota_{\mathbf{Q}}))$ consists of one element only. Thus we have proved:

PROPOSITION 3.7. *The rational homotopy type of a formal rational Poincaré duality space, whose rational cohomology algebra can be represented as in (*), is uniquely determined by the rational homotopy type of its skeleton.*

The condition (*) is clearly satisfied by all exterior algebras, hence the conclusion of 3.7 is valid for a product of odd-dimensional spheres.

Example 3.8. Consider the complex grassmannian $G(p,q)$ of complex p-planes in $\mathbf{C}^{p+q}$, $G(p,q) \cong \mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$. This is a formal one-connected compact manifold of dimension $2pq$. We claim that the cohomology of $G(p,q)$ can be represented as in (*) provided $(p,q) \neq (2,2)$. At first, clearly $G(n,1) \cong G(1,n) \cong \mathbf{P}^n(\mathbf{C})$ and the usual description

$$H^*(\mathbf{P}^n(\mathbf{C}); \mathbf{Q}) \cong \mathbf{Q}[c_1]/(c_1^{n+1}), deg(c_1) = 2,$$

has the requisite form. It is not hard to see that the presentation

$$H^*(G(p,q); \mathbf{Q}) = \mathbf{Q}[c_1, \ldots, c_p, c_1', \ldots, c_q']/((1 + \cdots + c_p)(1 + \cdots + c_q') = 1),$$
$$deg(c_i) = 2i, deg(c_j') = 2j, \text{for } 1 \leq i \leq p, 1 \leq j \leq q,$$

satisfies (*) for $(p,q) \neq (n,1), (1,n)$ and $(2,2)$. Hence, by Proposition 3.7, the rational homotopy type of $G(p,q)$ is, for $(p,q) \neq (2,2)$, uniquely determined by $G_{<2pq}(p,q)$. On the other hand, the same method as in Example 3.5 can be used to show that there exist infinitely many rational homotopy types of 8-dimensional rational Poincaré duality spaces X with $X_{<8} \sim_{\mathbf{Q}} G(2,2)_{<8}$.

Proof of Theorem 3.2. Let $H^* = H^*(X;\mathbf{Q})$ and let us denote by $PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ the set of all $\mathbf{k}$-isomorphism classes of rational Poincaré duality algebras H'^* with $H'^*_{<n} \cong_{\mathbf{Q}} H^*_{<n}$. Consider now the map $\lambda : PDS_{\mathbf{k}}(X) \to PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ defined by $\lambda(Y) = H^*(Y;\mathbf{Q})$.

LEMMA 3.9. *The map $\lambda : PDS_{\mathbf{k}}(X) \to PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ defined above is an isomorphism.*

Proof of the lemma. Notice that every rational homotopy type $Y \in PDS_{\mathbf{k}}(X)$ is formal. Indeed, $Y_{<n}$ is formal as $Y_{<n} \sim_{\mathbf{Q}} X_{<n}$, hence Y is formal by 3.1.

λ is an epimorphism. For $H'^* \in PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$ let Y be a formal rational homotopy type with $H^*(Y;\mathbf{Q}) \cong_{\mathbf{Q}} H'^*$. Since $H^*(Y_{<n};\mathbf{Q}) \cong_{\mathbf{Q}} H'^*{}_{<n} \cong_{\mathbf{Q}} H^*_{<n}$ and both $Y_{<n}$ and $X_{<n}$ are formal, $Y_{<n} \sim_{\mathbf{Q}} X_{<n}$, i.e. $Y \in PDS_{\mathbf{k}}(X)$. Plainly $\lambda(Y) = H'^*$.

λ is a monomorphism. Suppose $\lambda(Y) = \lambda(Z)$, i.e. $H^*(Y;\mathbf{Q}) \cong_{\mathbf{k}} H^*(Z;\mathbf{Q})$. As the spaces Y and Z are formal over $\mathbf{Q}$, they are formal also over $\mathbf{k}$ [7]. Since their cohomology algebras are isomorphic over $\mathbf{k}$, they have the same $\mathbf{k}$-homotopy type, in other words, $Y = Z$ considered as the elements of $PDS_{\mathbf{k}}(X)$. We point out that the Lemma 3.9 fails in general *without X being formal.*

To obtain a description of the set $PDA_{\mathbf{k}/\mathbf{Q}}(H^*)$, consider the following commutative diagram (the notation is the same as in 2.4)

$$\begin{array}{ccc} \tilde{E}_{\mathbf{k}} & \xrightarrow{F_{\mathbf{k}}} & E_{\mathbf{k}} \\ {\scriptstyle\otimes\mathbf{k}}\uparrow & & \uparrow{\scriptstyle\otimes\mathbf{k}} \\ \tilde{E}_{\mathbf{Q}} & \xrightarrow{F_{\mathbf{Q}}} & E_{\mathbf{Q}}. \end{array} \tag{3.10}$$

Using the correspondence of Proposition 2.6 (compare also Theorem 2.7), it is easy to identify (3.10) with the diagram

$$\begin{array}{ccc} H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}(H^*_{\bar{\mathbf{k}}})) & \xrightarrow{\iota_{\mathbf{k}}} & H^1(G(\bar{\mathbf{k}}/\mathbf{k}); Aut_{\bar{\mathbf{k}}}((H^*_{<n})_{\bar{\mathbf{k}}}) \\ {\scriptstyle\phi_H}\uparrow & & \uparrow{\scriptstyle\phi_{H_{<n}}} \\ H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}(H^*_{\bar{\mathbf{Q}}})) & \xrightarrow{\iota_{\mathbf{Q}}} & H^1(G(\bar{\mathbf{Q}}/\mathbf{Q}); Aut_{\bar{\mathbf{Q}}}((H^*_{<n})_{\bar{\mathbf{Q}}}) \end{array} \tag{3.11}$$

where all the maps are induced in the clear way. Our theorem now follows from (3.11) and the evident fact that

$$PDA_{\mathbf{k}/\mathbf{Q}}(H^*) = Im(\otimes\mathbf{k} : PDA_{\mathbf{Q}}(H^*) \to PDA_{\mathbf{k}}(H^*))$$

where, by Lemma 2.5, $PDA_{\mathbf{k}}(H^*) = F_{\mathbf{k}}^{-1}((H^*_{<n})_{\mathbf{k}})$ and $PDA_{\mathbf{Q}}(H^*) = F_{\mathbf{Q}}^{-1}(H^*_{<n})$.

Using the tools developed in the proof above, namely the diagrams (3.10) and (3.11), we can obtain also the following classification.

THEOREM 3.12. *Let $\mathbf{k}$ be a field of characteristic zero and let X be a formal rational Poincaré duality space. Then there exists a natural one-to-one correspondence between the set of all $\mathbf{k}$-homotopy types of rational Poincaré duality spaces Y with $Y_{<n} \sim_{\mathbf{k}} X_{<n}$ and the set $Im(\phi_H) \bigcap Ker(\iota_{\mathbf{k}})$.*

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