

Astérisque

SAID ZARATI

**Derived functors of the destabilization and the
Adams spectral sequence**

Astérisque, tome 191 (1990), p. 285-298

<http://www.numdam.org/item?id=AST_1990__191__285_0>

© Société mathématique de France, 1990, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

DERIVED FUNCTORS OF THE DESTABILIZATION and THE ADAMS SPECTRAL SEQUENCE

by Said ZARATI

Introduction

Let A be the modulo 2 Steenrod algebra, $\mathcal{A}t$ the category of graded A -modules and A -linear maps of degree zero, and \mathcal{U} the full sub-category of $\mathcal{A}t$ whose objects are unstable A -modules. We denote by $D : \mathcal{A}t \rightarrow \mathcal{U}$ the destabilization functor and by $D_s, s \geq 0$, its derived functors. We have a natural transformation $\eta : D_s \rightarrow D_{s-1}$, $s \geq 1$, induced by the adjoint of the identity $\Omega D = D \Sigma^{-1}$ where $\Sigma^m, \mathcal{A}t \rightarrow \mathcal{A}t, m \in \mathbb{Z}$, is the m^{th} suspension functor and Ω is the left adjoint of $\Sigma : \mathcal{U} \rightarrow \mathcal{U}$.

In this note we prove the following theorem which will be more precise in section 2.3.

Theorem 1.1. Let M be a nil-closed unstable A -module. Then the natural map $\Omega D_s \Sigma^{-s} M \rightarrow D_s \Sigma^{-s-1} M$ is an isomorphism for every $s \geq 0$.

Using the higher Hopf invariants introduced in [7] we prove the following property of the Adams spectral sequence, in the modulo 2 cohomology, for the group $\{X, Y\}$ of homotopy classes of stable maps from X to Y , in certain cases.

Theorem 1.2. : Let X and Y two pointed CW-complexes such that

- (i) $\bar{H}^*(X; \mathbb{F}_2) \simeq \Sigma^2 I$ where ΣI is an injective unstable A -module.
- (ii) $\bar{H}^*(Y; \mathbb{F}_2)$ is gradually finite and nil-closed.

Then, the Adams spectral sequence for the group $\{X, Y\}$ degenerates at the E_2 -term : $E_2^{s, s} \simeq E_r^{s, s}$ for every $r \geq 2$ and $s \geq 0$.

S.M.F.

Astérisque 191 (1990)

The infinite real projective space $\mathbb{R}P^\infty$ is an example of a space Y satisfying the hypotheses of theorem 1.2.

The organization of the rest of this note is as follows. In section 2 we give a characterization of nil-closed A -modules which allows us to prove the theorem 1.1 (see theorem 2.3.3). Section 3 gives the proof of theorem 1.2 and an application. We finish this note by a remark concerning the case $p > 2$.

All cohomology is taken with \mathbb{F}_2 coefficients. We write $H^*()$ for $H^*(; \mathbb{F}_2)$ and we denote by $\bar{H}^*()$ the reduced modulo 2 cohomology.

2. Derived functors of the destabilization

2.1. Let A be the modulo 2 Steenrod algebra. We denote by \mathcal{A} the category whose objects are graded A -modules ($M = \{M^n, n \in \mathbb{Z}\}$) and whose morphisms are A -linear maps of degree zero. We denote by \mathcal{U} the full sub-category of \mathcal{A} whose objects are unstable A -modules (an A -module M is called unstable if $Sq^i x = 0$ for every x in M^n and every $i > n$; in particular $M^n = 0$ if $n < 0$).

The forgetful functor $\mathcal{U} \rightarrow \mathcal{A}$ has a left adjoint functor $D : \mathcal{A} \rightarrow \mathcal{U}$, called the destabilization functor, which satisfies : $\text{Hom}_{\mathcal{A}}(M, N) = \text{Hom}_{\mathcal{U}}(DM, N)$ for every A -module M and every unstable A -module N . The functor $D : \mathcal{A} \rightarrow \mathcal{U}$ is right exact, we denote $D_s : \mathcal{A} \rightarrow \mathcal{U}$, $s \geq 0$, its derived functors. One of the motivations for the study of the derived functors of the destabilization is the following isomorphism :

$$(2.1) \quad \text{Ext}_{\mathcal{A}}^s(M, I) \simeq \text{Hom}_{\mathcal{U}}(D_s M, I)$$

for every A -module M and every unstable injective A -module I .

Let $\Sigma^m : \mathcal{A} \rightarrow \mathcal{A}$, $m \in \mathbb{Z}$, the m^{th} suspension functor

which associates to a module $M = \{M^n, n \in \mathbb{Z}\}$ the module $\Sigma^m M = \{M^{n-m}, n \in \mathbb{Z}\}$. The A -module structure on $\Sigma^m M$ is given by $Sq^i(\Sigma^m x) = \Sigma^m Sq^i x$, x in M . The computation of $D_s \Sigma^{-t} M$, where M is an unstable A -module, is done by Lannes and Zarati [5] for $t \leq s$. In this paragraph we will compute $D_s \Sigma^{-(s+1)} M$ for a particular unstable A -modules called nil-closed. First let us recall the definition and some properties of nil-closed unstable A -modules.

2.2. Nil-closed unstable A -modules [1], [6]

Definition 2.2.1 An unstable A -module M is called reduced if the cup-square $Sq^n : M^n \rightarrow M^{2n}$, $x \rightarrow Sq^n x$, is injective for every $n \geq 0$.

Remark 2.2.2 We can verify easily that an unstable A -module is reduced if and only if it does not contain a non trivial nilpotent sub- A -module. An unstable A -module N is called nilpotent if for

$$\text{every } x \text{ in } M^n, \text{ there exist } r \geq 0 \text{ such that } Sq^{2^r n} \dots Sq^n x = 0.$$

Definition 2.2.3. An unstable A -module M is called nil-closed if
 (i) M is reduced
 (ii) An element x in M of even degree is in the image of the cup-square if and only if $Q_i x = 0$, for all $i \geq 0$, where Q_i is the i^{th} Milnor primitive in A .

Example 2.2.4 Let $B\mathbb{Z}/2$ denote a classifying space of the group $\mathbb{Z}/2$. The unstable A -module $H^*(B\mathbb{Z}/2)$ is nil-closed indeed, as a graded \mathbb{F}_2 -algebra $H^*(B\mathbb{Z}/2)$ is freely generated by one generator of degree one.

2.3. Computation of $D_s \Sigma^{-(s+1)} M$, M nil-closed and $s \geq 0$.

2.3.1 To state our result we use the functor $R_s : \mathcal{U} \rightarrow \mathcal{U}$, $s \geq 0$, introduction in [5] page 29 (see also [9]) whose main properties

are:

(i) The module $R_S M$ is a sub-A-module of $H^*(B(\mathbb{Z}/2)^S) \otimes M$. In particular $R_S M$ is an unstable A-module.

(ii) Let $H^*(B\mathbb{Z}/2) = \mathbb{F}_2[u]$ where u is of degree one. We denote by $L_S = H^*(B(\mathbb{Z}/2)^S)GL_S(\mathbb{Z}/2)$ the Dickson algebra, that is the sub-algebra of $H^*(B(\mathbb{Z}/2)^S)$ of invariants under the natural action of the general linear group $GL_S(\mathbb{Z}/2) = GL((\mathbb{Z}/2)^S)$. The module $R_S M$ is the L_S -module generated by the elements $St_S(x)$, x in M . These elements $St_S(x)$ are defined inductively by :

$$St_0(x) = x \quad , \quad x \in M.$$

$$St_1(x) = \sum_{i=0}^n u^{n-i} \otimes Sq^i x \quad , \quad x \in M^n.$$

$$St_s(x) = St_1(St_{s-1}(x)) \quad , \quad s \geq 1, x \in M$$

iii) Let $E_+ \mathfrak{S}_2^s$ be the disjoint union of a base point and a contractible space on which the symmetric group \mathfrak{S}_2^s acts freely. For any pointed space X , we denote by $\mathfrak{S}_2^s X$ the quotient of the space $E_+ \mathfrak{S}_2^s \wedge (X \wedge \dots \wedge X)$, X is smashed with itself 2^s times, by the diagonal action of \mathfrak{S}_2^s (\mathfrak{S}_2^s acts on $X \wedge \dots \wedge X$ by permutation of the factors). Let $\Delta_S : B_+(\mathbb{Z}/2)^S \wedge X \rightarrow \mathfrak{S}_2^s X$ be a "Steenrod diagonal" determined by a bijection between $(\mathbb{Z}/2)^S$ and $\{1,2,\dots,2^S\}$. The unstable A-module $R_S H^* X$ is the image of Δ_S^* in the modulo 2 cohomology.

2.3.2 Let $\Omega : \mathcal{U} \rightarrow \mathcal{U}$ be the left adjoint functor of $\Sigma : \mathcal{U} \rightarrow \mathcal{U}$, that is :

$$\text{Hom}_{\mathcal{U}}(M, \Sigma N) = \text{Hom}_{\mathcal{U}}(\Omega M, N)$$

for every unstable A-modules M and N .

We are now ready to state the main result of this paragraph which will be proved in 2.6

Theorem 2.3.3 : Let M be a nil-closed unstable A -module. There exist a natural isomorphism :

$$D_S \Sigma^{-(s+1)} M \simeq \Omega R_S M \quad , \quad s \geq 0$$

2.4. Some properties of nil-closed unstable A modules

In this paragraph we give two characterizations of nil-closed unstable A -modules which allow us to prove theorem 2.3.3

2.4.1. The first characterization of nil-closed unstable A -modules is given in [6] page 314.

Proposition 2.4.1.1. Let M be an unstable A -module. The following conditions are equivalent.

- (i) M is nil-closed
- (ii) $\text{Ext}_{\mathcal{U}}^i(N, M) = 0$ for every nilpotent N in \mathcal{U} and $i = 0, 1$.
- (iii) There exist an injective resolution of M starting

$$0 \longrightarrow M \longrightarrow K^0 \longrightarrow K^1$$

where K^0 and K^1 are reduced injective unstable A -modules.

Remark 2.4.1.2. The condition (iii) of the proposition 2.4.1.1 can be replaced by the following (see [4] page 163)

- (iii)' There exist an injective resolution of M starting

$$0 \longrightarrow M \longrightarrow \prod_{\alpha} H^*(BV_{\alpha}) \longrightarrow \prod_{\beta} H^*(BV_{\beta})$$

where V_{α} and V_{β} are elementary abelian 2-groups. We have the following easy corollary.

Corollary 2.4.1.3. Let M be an unstable A -module. The following conditions are equivalent.

- (i) M is nil-closed.
- (ii) There exist a nil-closed unstable A -module L containing M such that the quotient L/M is reduced.

2.4.2. Another characterization of nil-closed.

Proposition 2.4.2.1 Let M be an unstable A -module. The following properties are equivalent.

- (i) M is nil-closed
- (ii) M and ΩM are reduced

The proof of this proposition is based on the following technical lemma. Let $Q_i, i \geq 0$, the i^{th} Milnor primitive in A and Sq_k the cohomology operation defined by $Sq_k x = Sq^{n-k}x$ where x is an element of degree n of an A -module ($Sq^{n-k} = 0$ if $n < k$).

Lemma 2.4.2.2 Let M be an unstable A -module. We have the following formula :

$$(Q_{i+1} \circ Sq_1)(x) = (Sq_0 \circ Q_i)(x)$$

for every x in M and every $i \geq 0$.

Proof. The proof is done by induction on i using Adem's relations. Recall that the elements $Q_i, i \geq 0$, are defined by

$$Q_0 = Sq^1$$

$$Q_i = Q_{i-1} Sq^{2^i} + Sq^{2^i} Q_{i-1}, \quad i \geq 1$$

The case $i = 0$. Let x be an element of degree n of an unstable A -module, we have :

$$Sq^1 Sq_1(x) = Sq^1 Sq^{n-1}(x) = \begin{cases} 0 & \text{if } n \equiv 0(2). \\ Sq_0 x & \text{if } n \equiv 1(2). \end{cases}$$

$$Sq^2 Sq_1(x) = Sq^2 Sq^{n-1}(x) = \begin{cases} 0 & \text{if } 2 > 2n - 1. \\ Sq^2 Sq^1 x & \text{if } 2 = 2n - 2. \\ \sum_{c=0}^1 C_{n-2-c}^{2-2c} Sq^{n+1-c} Sq^c x & \text{if } 2 < 2n - 2. \end{cases}$$

$$= \begin{cases} 0 & \text{if } n = 1 \\ \text{Sq}^n \text{Sq}^1 x & \text{if } n \geq 2. \end{cases}$$

These formulas imply the case $i = 0$ because we have :

$$\begin{aligned} Q_1 \text{Sq}_1 x &= \text{Sq}^3 \text{Sq}_1 x + \text{Sq}^2 \text{Sq}^1 \text{Sq}_1 x \\ &= \text{Sq}_0 \text{Sq}^1 x \\ &= \text{Sq}_0 Q_1 x. \end{aligned}$$

Suppose $Q_i \text{Sq}_1 x = \text{Sq}_0 Q_{i-1} x$ for every $i : 0 \leq i \leq j-1$ and for every element x (of degree n) of an unstable A -module. To prove this formula for $i = j$ we consider :

$$\begin{aligned} Q_j \text{Sq}_1(x) &= \text{Sq}^{2^j} Q_{j-1} \text{Sq}_1(x) + Q_{j-1} \text{Sq}^{2^j} \text{Sq}_1(x) \\ &= \text{Sq}^{2^j} \text{Sq}_0 Q_{j-2}(x) + Q_{j-1} \text{Sq}^{2^j} \text{Sq}_1(x) , \text{ (inductive assumption)} \\ &= \text{Sq}_0 \text{Sq}^{2^{j-1}} Q_{j-2}(x) + Q_{j-1} \text{Sq}^{2^j} \text{Sq}_1(x). \end{aligned}$$

In the last equality we have used the following easy formula :

$$\text{Sq}^k \text{Sq}_0 = \begin{cases} 0 & \text{if } k \equiv 1(2) . \\ \text{Sq}_0 \text{Sq}^{\frac{k}{2}} & \text{if } k \equiv 0(2) . \end{cases}$$

If remains to show :

$$Q_{j-1} \text{Sq}^{2^j} \text{Sq}_1(x) = \text{Sq}_0 Q_{j-2} \text{Sq}^{2^{j-1}}(x).$$

Using the unstability of M and Adem's relations we prove :

$$Sq^{2^j} Sq_1(x) = \begin{cases} 0 & \text{if } n \leq 2^{j-1} . \\ Sq_1 Sq^{2^{j-1}}(x) & \text{if } n \geq 2^{j-1} + 1 . \end{cases}$$

This formula gives :

$$Q_{j-1} Sq^{2^j} Sq_1(x) = \begin{cases} 0 & \text{if } n \leq 2^{j-1} . \\ Q_{j-1} Sq_1 Sq^{2^{j-1}}(x) & \text{if } n \geq 2^{j-1} + 1 . \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \leq 2^{j-1} . \\ Sq_0 Q_{j-2} Sq^{2^{j-1}}(x) & \text{if } n \geq 2^{j-1} + 1 , \text{ (inductive assumption).} \end{cases}$$

$$= Sq_0 Q_{j-2} Sq^{2^{j-1}}(x)$$

2.4.3. Functor R_S and nil-closed A -modules.

Proposition 2.4.3.1. Let M be an unstable A -module. If M is nil-closed then $R_S M$ is nil-closed.

Proof : Let (*) $0 \rightarrow M \rightarrow \prod_{\alpha} H^*(V_{\alpha}) \rightarrow \prod_{\beta} H^* V_{\beta}$ be the beginning of an injective resolution of the nil-closed unstable A -module M (see remark 2.4.1.2). The functor R_S is exact and comutes with products (see [6]) ; then, when we apply it to the exact sequence (*) we get the following exact sequence :

$$0 \rightarrow R_S M \rightarrow \prod_{\alpha} R_S H^* V_{\alpha} \rightarrow \prod_{\beta} R_S H^* V_{\beta} .$$

The computation of $R_s H^* V$, where V is an elementary abelian 2-group, is done by induction on s (see [6] page 321). Let $V_s = (\mathbb{Z}/2)^s \oplus V$, $R_s H^* V$ is the sub-module of $H^*(V_s)$ of invariants under the action of the sub-group of $GL(V_s)$, denoted $GL(V_s, V)$, of automorphisms of V_s which induces the identity on V . The proposition 2.4.3.1 is now a consequence of the corollary 2.4.1.3 and of the fact that the sub-A-module $H^*(V)^G$, $G < GL(V)$, of $H^*(V)$ is nil-closed (see [6] page 314).

Remark 2.4.3.2. A different proof of the proposition 2.4.3.1 for $s = 1$ is given in [3]

2.5.Proof of the proposition 2.4.2.1.

2.5.1. First let us recall some properties of the functor Ω introduced in 2.3.2. Let $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ be the functor which associates to each unstable A-module A-module M , the "double of M ", denoted ΦM , defined by :

$$(\Phi M)^n = \begin{cases} 0 & \text{if } n \equiv 1(2). \\ M^{n/2} & \text{if } n \equiv 0(2). \end{cases} \quad \text{and } Sq^i(\Phi x) = \begin{cases} 0 & \text{if } i \equiv 1(2). \\ \Phi Sq^{i/2} x & \text{if } i \equiv 0(2). \end{cases}$$

we verify that the map $Sq_0 : \Phi M \rightarrow M$, $\Phi x \mapsto Sq_0 x$, is A-linear and that the kernel and the cokernel of Sq_0 are respectively $\Sigma \Omega_1 M$ and $\Sigma \Omega M$ where Ω_1 is the first and unique derived functor of Ω (see [5] page 30). We remark that an unstable A-module M is reduced if and only if $\Omega_1 M = 0$.

2.5.2. Proof the proposition 2.4.2.1. (i) \implies (ii). It suffices to prove that ΩM is reduced. Let y be an element of $(\Omega M)^k$ such that $Sq_0 y = 0$. To prove that $y = 0$ we envision two cases :

(*) The case $k \equiv 0(2)$. in this case $(\Omega M)^k = (\Sigma^{-1} M / \text{Im } Sq_0)^k = M^{k+1}$ then $y = \Sigma^{-1} x$ where x is an element of M^{k+1} . $Sq_0 y =$

$\Sigma^{-1}Sq_1x = 0$. This implies that $Sq_1Sq_1x = Sq_0x = 0$ and then $x = 0$ since M is reduced. This shows that $y = \Sigma^{-1}x = 0$.

(**) The case $k = 1(2)$. In this case $(\Omega M)^k = (\Sigma^{-1}M/\text{Im}Sq_0)^k = (M/\text{Im}Sq_0)^{k+1}$ then $y = \Sigma^{-1}[x]$ where x is an element of M^{k+1} . $Sq_0y = \Sigma^{-1}[Sq_1x] = \Sigma^{-1}Sq_1x = 0$ (Sq_1x is an element of M of odd degree) ; then, $Sq_1x = 0$. This implies that $Q_{i+1}Sq_1x = 0$ for every $i \geq 0$. Using the lemma 2.4.2.2 we get : $Sq_0Q_i x = 0$ for every $i \geq 0$ and then $Q_i(x) = 0, i \geq 0$, since M is reduced. Now x is an element of even degree of a nil-closed A -module M annihilated by all the $Q_i, i \geq 0$ then x is in the image of Sq_0 and then $y = \Sigma^{-1}[x] = 0$.

(ii) \implies (i). Since M is reduced then M embeds in a reduced injective unstable A -module K (see [6] page 313). To prove M nil-closed it suffices to prove that the quotient K/M is reduced and to use the corollary 2.4.1.3. If we apply the functor Ω to the exact sequence $0 \rightarrow M \rightarrow K \rightarrow K/M \rightarrow 0$ we get the following exact sequence : $0 \rightarrow \Omega_1(K/M) \rightarrow \Omega M \rightarrow \Omega K \rightarrow \Omega(K/M) \rightarrow 0$. The module $\Omega_1(K/M)$ is trivial because it is a nilpotent sub- A -module of the reduced unstable A -module ΩM . $\Omega_1(K/M)$ is nilpotent because, by definition, it is concentrated in odd degree. This shows that K/M is reduced and then M is nil-closed

2.6. Proof of the theorem 2.3.3

Let M be an unstable A -module. Consider the following exact sequence introduced in [5] page 32 :

$$(*) \quad 0 \rightarrow \Omega D_s \Sigma^{-s} M \rightarrow D_s \Sigma^{-(s+1)} M \rightarrow \Omega_1 D_{s-1} \Sigma^{-s} M \rightarrow 0$$

When M is reduced, the module $D_s \Sigma^{-s} M$ is naturally isomorphic to $R_s M$ ([5] proposition 4.6.2). The exact sequence becomes :

$$(**) \quad 0 \rightarrow \Omega R_s M \rightarrow D_s \Sigma^{-(s+1)} M \rightarrow \Omega_1 D_{s-1} \Sigma^{-s} M \rightarrow 0$$

The proof of the theorem 2.3.3 is done by induction on s . For $s = 0$ it is the identity $D \Sigma^{-1} = \Omega D$. Suppose that : $(H_k) D_k \Sigma^{-(k+1)} M \simeq \Omega R_k M$ for every $k : 0 \leq k \leq s-1$ and every nil-closed A -module M .

To prove (H_S) it suffices to remark that since M is nil-closed then, by the proposition 2.4.3.1, $R_{S-1}M$ is nil-closed. This implies that $\Omega R_{S-1}M$ is reduced (proposition 2.4.2.1), that is : $\Omega_1 \Omega R_{S-1}M = 0$. The exact sequence (**) and the inductive assumption give, for M nil-closed, the following natural isomorphism : $D_S \Sigma^{-(s+1)}M \simeq \Omega R_S M$.

3. Applications.

The topological applications of this note are based on the higher Hopf invariants introduced by Lannes and Zarati in [7]. Let X and Y be two pointed CW-complexes. We denote by $\{X, Y\}$ the group of homotopy classes of stable maps from X to Y . The Adams spectral sequence, in the modulo 2 cohomology, for the group $\{X, Y\}$ is denoted $\{E_r^{s,s} = E_r^{s,s}(X, Y), s \geq 0, d_r\}_{r \geq 2}$; $d_r : E_r^{s,s} \rightarrow E_r^{s+r, s+r-1}$ is the differential. We have the following theorem which will be proved in the section 3.4

Theorem 3.1 Let X and Y be two pointed CW-complexes such that :

- (i) $\bar{H}^*(X) \simeq \Sigma^2 I$ where ΣI is an injective unstable A -module.
- (ii) $\bar{H}^*(Y)$ is gradually finite ($\dim_{\mathbb{F}_2} H^n(Y) < +\infty, n \geq 0$) and nil-closed.

Then, the Adams spectral sequence, in the modulo 2 cohomology, for the group $\{X, Y\}$ degenerate at the E_2 -term $E_2^{s,s} \simeq E_r^{s,s}$ for every $r \geq 2$ and $s \geq 0$.

Remark 3.2 In [8] (see also [7]) there exist an analogous property of the Adams spectral sequence as in theorem 3.1 in the following two cases :

- (3.2.1) (i) $\bar{H}^*(X)$ is a reduced injective unstable A -module.
- (ii) $\bar{H}^*(Y)$ is gradually finite.

- (3.2.2) (i) $\Sigma \bar{H}^*(X)$ is an injective unstable A -module.
 (ii) $\bar{H}^*(Y)$ is a reduced gradually finite unstable A -module.

Corollary 3.3. Let X and Y be two pointed CW complexes which verify the hypothesis (i) and (ii) of theorem 3.1 and such that the Adams spectral sequence for the group $\{X, Y\}$ converges.

Then, the natural map :

$$h : \{S^1 X, Y\} \dashrightarrow \text{Hom}_{\mathcal{U}}(\bar{H}^* Y, \Sigma \bar{H}^* X)$$

is surjective.

Proof. Theorem 3.1 shows that the term $E_2^{0,1} \simeq \text{Hom}_{\mathcal{U}}(\bar{H}^* Y, \Sigma \bar{H}^* X)$ persists at the infinity. Since the Adams spectral sequences for $\{X, Y\}$ converges, then the natural map $h : \{S^1 X, Y\} \dashrightarrow \text{Hom}_{\mathcal{U}}(\bar{H}^* Y, \Sigma \bar{H}^* X)$ is surjective.

3.4. Proof of the theorem 3.1

Consider the following diagram whose commutativity is proved in [7], [8].

$$\begin{array}{ccc}
 & E_{\infty}^{s,s} & \xrightarrow{\mathcal{H}_{\infty}^{s,s}} \text{Hom}_{\mathcal{U}}(R_S \bar{H}^* Y, \Sigma^2 I) \\
 \nearrow & & \downarrow \wr \\
 Z_{2,\infty}^{s,s} & & \text{Hom}_{\mathcal{U}}(\Omega R_S \bar{H}^* Y, \Sigma I) \\
 \downarrow & & \textcircled{3} \uparrow \wr \\
 E_2^{s,s} \textcircled{1} \simeq \text{Ext}_{\mathcal{M}}^s(\Sigma^{-s-1} \bar{H}^* Y, \Sigma I) & \xrightarrow[\textcircled{2}]{\mathcal{H}_2^{s,s}} & \text{Hom}_{\mathcal{U}}(D_S \Sigma^{-s-1} \bar{H}^* Y, \Sigma I)
 \end{array}$$

($Z_{2,\infty}^{s,s}$ is the inverse image of $E_{\infty}^{s,s}$ in E_2 , $\mathcal{H}_{\infty}^{s,s}$ and $\mathcal{H}_2^{s,s}$ are the Hopf invariants at the E_{∞} -level and the E_2 -level respectively). The isomorphism 1 is clear since $E_2^{s,s} = \text{Ext}_{\mathcal{M}}^s(\Sigma^{-s} \bar{H}^* Y, \Sigma^2 I) \simeq \text{Ext}_{\mathcal{M}}^s(\Sigma^{-s-1} \bar{H}^* Y, \Sigma I)$. The isomorphism 2 follows from the fact that ΣI is an injective unstable A -module. The isomorphism 3 is a consequence of the theorem 2.3.3.

By definition of the differential $d_r : E_r^{s-r, s-r+1} \rightarrow E_r^{s, s}$ we have : $\text{Im} d_r \subset Z_{r, \infty}^{s, s}$ and $Z_{r+1, \infty}^{s, s} = Z_{r, \infty}^{s, s} / \text{Im} d_r$ (see, for example [2]). It follows from the commutativity of the previous diagram that $Z_{2, \infty}^{s, s} \xrightarrow{\sim} E_{\infty}^{s, s}$ and then the differential $d_r : E_r^{s-r, s-r+1} \rightarrow E_r^{s, s}$, $r \geq 2$, is trivial. To prove that the differential $d_r : E_r^{s, s} \rightarrow E_r^{s+r, s+r-1}$, $r \geq 2$, is trivial we use the following isomorphism $E_2^{s, t}(X, Y) \approx E_2^{s, t+1}(X, SY)$ which allows us to use the results of [8] (see remark 3.2).

4. The case $p > 2$

In this note we can't replace 2 by an odd prime p since the proposition 2.4.2.1, which is the main algebraic result of this note, is false for $p > 2$. Here is an example ; the unstable A -module $H = H^*(B(\mathbb{Z}/p) ; \mathbb{F}_p)$ is the tensor product, $E(u) \otimes \mathbb{F}_p[v]$ of an exterior algebra on one generator u of degree one and of a polynomial algebra generated by v the Bockstein of u . We know that H is nil-closed (see [6]) but ΩH is not λ -projective (λ is the analog of Sq_0 for $p > 2$) ; the element $\Sigma^{-1} v^2$ of degree three of ΩH is such that : $\lambda(\Sigma^{-1} v^2) = \Sigma^{-1} \beta P^1 v^2 = 0$.

REFERENCES

- [1] **C.BROTO and S.ZARATI** : Nil-localization of unstable algebras over the Steenrod algebra, Math Z. 199, 525-537 (1988) .
- [2] **H.CARTAN and S.EILENBERG** : Homological algebra, Princeton Univ. Press 1956 .
- [3] **J.H.GUNAWARDENA, J.LANNES et S.ZARATI** : Cohomologie des groupes symétriques et application de Quillen, preprint 1986.
- [4] **J.LANNES and L.SCHWARTZ** : Sur la structure des A-modules instables injectifs ; Topology vol. 28, N°2, pp 153-169, 1989.
- [5] **J.LANNES et S.ZARATI** : Sur les foncteurs dérivés de la déstabilisation, Math. Z. 194, 25-59 (1987).
- [6] **J.LANNES et S.ZARATI** : Sur les \mathcal{U} -injectifs, Ann. Ec. Norm. Sup 4 e série, t. 19, 1986, p. 303 - 333 .
- [7] **J.LANNES et S.ZARATI** : Invariants de Hopf d'ordre supérieur et suite spectrale d'Adams, C.R.A.S t. 296 (1983) p. 695-698.
- [8] **J.LANNES et S.ZARATI** : Same title, to appear.
- [9] **W.M.SINGER** : The construction of certain algebras over the steenrod algebra, J. of pure and applied algebra, 11 (1977), 53-59.

S.ZARATI

Universite de Tunis
Faculté des Sciences
Département de Mathématiques
1060 TUNIS - TUNISIA -