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Astérisque, tome 192 (1990), p. 17-33
<http://www.numdam.org/item?id=AST_1990__192__17_0>

COMPUTATION OF THE TOPOLOGY OF A REAL CURVE

M.-F. ROY

The problem of giving an algorithm for the computation of the topological type of a real curve from its equation has been considered by several authors ([GT] and more recently in the particular case of non singular curves by [AM].) The approach presented here relies on a basic result in real algebraic geometry, Thom's lemma; it can be viewed as an illustration of the philosophy in [CR]. Our algorithm runs in polynomial-time, needs no regularity hypothesis on the curve or on the projection and seems better adapted to situations where the connected components of the curve are small.

1. TERMINOLOGY AND GENERAL DISCUSSION

Let us introduce first some definitions.

1. 1. SEMI-ALGEBRAIC SETS

Let $F=(P_1(X_1,\ldots,X_n),\ldots,P_m(X_1,\ldots,X_n))$ be a family of polynomials with integer coefficients. An *F-semi-algebraic set* S of $\mathbb{R}^n$ is a semi-algebraic set contained in $\mathbb{R}^n$, described by a boolean combination of sign conditions on the polynomials of F. A *F-basic semi-algebraic set* S of $\mathbb{R}^n$ is a semi-algebraic set contained in $\mathbb{R}^n$, described by a conjunction of sign conditions on the polynomials of F. One can know from basic results of real algebraic geometry ([B C R]) that a set S is semi-algebraic (because it is described by a first order formula of the language of ordered fields, using Tarski-Seidenberg, or because it is a connected component of a semi-algebraic set) without knowing polynomials F such that S is F-semi-algebraic.

1. 2. CYLINDRICAL DECOMPOSITIONS AND STRATIFICATIONS

Depending on the problem one wants to solve about semi-algebraic sets, one can use several kinds of cylindrical decompositions ([C], [Co]).

Let S be a F-semi-algebraic set, Π the canonical projection of $\mathbb{R}^n$ on $\mathbb{R}^{n-1}$.

(D_1) A *cylindrical decomposition* of F with respect to Π is given by a partition of $\mathbb{R}^{n-1}$ in a finite number of semi-algebraic sets A_i , such that above each A_i

(a) the real roots of the non-identically nul P_i are in constant number and given by continuous semi-algebraic functions $\zeta_{i,1} < \ldots < \zeta_{i,l_i}$;
(b) for all $x = (x_1,\ldots,x_{n-1})$ of A_i and all $j = 0,\ldots, l_i$ the sign of $P_i(x_1,\ldots,x_{n-1}, X_n)$ between $\zeta_{i,j}(x)$ and $\zeta_{i,j+1}(x)$ (with the convention $\zeta_{i,0}(x) = -\infty$ and $\zeta_{i,l_i+1}(x) = +\infty$) is fixed.

It is then clear that S is a finite union of *cells of the cylindrical decomposition* that is of graphs of $\zeta_{i,j}$, of slices between two $\zeta_{i,j}$ and $\zeta_{i,j+1}$ and of cylinders of the form $A_i \times \mathbb{R}$.

Let us remark that we do not ask for the A_i to be connected. If they are connected (b) is a consequence of (a).

(D_2) An *explicit cylindrical decomposition* of F is given by a family of polynomials F' (containing the polynomials F) and a cylindrical decomposition of F such that the sets A_i and the graphs of the $\zeta_{i,j}$ are F'-semi-algebraic sets.

Cylindrical decompositions give no information on adjacency relations: is A_i contained in the closure of $A_{i'}$? if it is the case, how do the functions $\zeta_{i',j}$ glue above A_i i.e. in which case is the graph of $\zeta_{i,j}$ contained in the closure of the graph of $\zeta_{i',j}$?

From now on, we shall suppose the polynomials of F monic with respect to X_n (which means that considered as polynomials in X_n, their leading coefficient belongs to $\mathbb{R}$). This can always be done by a linear change of coordinates.

(D_3) A *semi-algebraic stratification of F* is given by a cylindrical decomposition of F such that
(a') for all i, A_i is connected
(b') for all i and i' (i) either $A_i \cap A_{i'}$ is empty
(ii) or $A_i \subset \mathrm{adh}(A_{i'})$
(iii) or $A_{i'} \subset \mathrm{adh}(A_i)$
and given a pair *(i, i')* one explicitely knows which is the case
(c') one knows in case (ii) the adjacency relations between the $\zeta_{i,j}$ and the $\zeta_{i',j}$, i.e. for each pair *(i, i')* such that (ii) and for all *(j, j')* with $j \in \{1,\ldots,l_i\}$, $j' \in \{1,\ldots, l_{i'}\}$ one explicitely knows whether the graph of $\zeta_{i,j}$ is contained in the closure of the graph of $\zeta_{i',j'}$ or not.

(D_4) An *explicit semi-algebraic stratification* of F is a semi-algebraic stratification which is an explicit cylindrical decomposition.

1. 3. DIFFERENT SORT OF PROBLEMS

Let us consider now the following problems (S is a semi-algebraic set contained in $\mathbb{R}^n$:

(P_1) is S empty?

(P_2) what is the dimension of S?

(P_3) what is the number of connected components of S?

(P_4) what are the topological invariants of S (homology groups)?

(P_5) do two points of S belong to the same connected component of S?

(P_6) does a point belong to the projection of S?

(P_7) what is the explicit semi-algebraic description of the projection of S on $\mathbb{R}^{n-1}$?

(P_8) what is the explicit semi-algebraic description of $\{x \in \mathbb{R}^n \mid \Phi(x)\}$, where Φ is a first order formula of the language of ordered fields?

For the problems (P_1) to (P_5), one can choose the projection, that is one can make a linear change of coordinates, not for the problems (P_6) to (P_8).

A cylindrical decomposition allows to answer to (P_1), (P_2) and (P_6). An explicit cylindrical decomposition also to (P_7) and (P_8) (by induction). A semi-algebraic stratification to (P_3), (P_4) and (P_5). An explicit semi-algebraic stratification allows, in case the polynomials are monic with respect to the last variable (since in the problem (P_6) to (P_8) one cannot change the direction of the projection), to obtain (P_1) to (P_8).

A typical problem of robotics "à la Schwartz and Scharir ", the piano's mover problem [SS] is naturally formulated in term of (P_8) and (P_5) : one asks whether, semi-algebraic walls being explicitely given, there exists a movement allowing to pass from a position of an objet (the piano) to another without knocking the walls; one then considers the explicit semi-algebraic set S of allowed positions (problem (P_8)) and one answers yes if the initial and final positions belong to the same connected component of S.

1. 4. DIFFERENT FAMILIES OF POLYNOMIALS

It is not surprising that the computations to obtain these different sorts of cylindrical decompositions are different.

Different families of polynomials are to consider.

(F_1) A family of polynomials $F'=(P_{k,j}(X_1,\ldots,X_k)\ k=1,\ldots,n,\ j=1,\ldots,r_k)$ is *cylindrifying* for F if it contains F and is stable for the following operations:

-if $P(X_1,...,X_k)$ and $Q(X_1,...,X_k)$ are in F' the leading term of P, Q and of the subresultants of P and Q with respect to X_k are in F'
-if $P(X_1,...,X_k)$ is in F' the leading terms of the subdiscriminants of P with respect to X_k are in F'.

A family of polynomials $P_{1,j}(X_1)$, $j=1,..., l_1$, is always cylindrifying; the cells of the cylindrifying family are the real roots of the $P_{1,j}(X_1)$, $j=1,..., l_1$ and the intervals between such roots.

(F_2) A family of polynomials $P_{k,j}(X_1,...,X_k)$ $k=1,...,n$, $j=1,...,r_k$ is *glueing* for F if it contains a cylindrifying family for F, $Q_{k,j}(X_1,...,X_k)$ $k=1,...,n$, $j=1,...,r'_k$, and all the derivatives of the $Q_{k,j}(X_1,...,X_k)$ with respect to X_k.

(F_3) A family of polynomials $P_{k,j}(X_1,...,X_k)$ $k=1,...,n$, $j=1,...,r_k$ is *stratifying* for F if it is cylindrifying and stable under derivation (i.e. the family contains the derivative with respect to the variable X_k of the polynomials $P_{k,j}$).

Let us consider a real plane curve C of equation $P(X,Y)=0$, with P monic as polynomial in Y, squarefree and with coefficients in $\mathbb{Z}$, let D be the discriminant of P with respect to the variable Y.

Let us precise in this simple situation what are the different families of polynomials we have just defined:

(F_1) A cylindrifying family for P consists of P and D the discriminant of P with respect to Y.

(F_2) A glueing family for P consists of P and its derivatives with respect to Y, as well as D and its derivatives with respect to X.

(F_3) A stratifying family for P consists of P and its derivatives with respect to Y, of the discriminant D and the resultants obtained by eliminating Y between the different derivatives of P with respect to Y, then of the derivatives in X of D and of these resultants.

A cylindrifying family gives (D_1), a stratifying family (D_2) (in this case the cells of the cylindrical decomposition are basic semi-algebraic sets). If $P_1(X_1,...,X_n)$,..., $P_m(X_1,...,X_n)$ are monic with respect to X_n a glueing family gives (D_3) and a stratifying family (D_4).

The passage from (F_1) to (D_1) is done by Collins [C]: one takes the polynomials in X_1 belonging to the cylindrifying family, one isolates their roots on intervals with rational endpoints, one chooses a rational point in each interval between two roots and on the left and on the right of all the roots. Above each root and each chosen point, one computes by Sturm sequence the number of roots of the polynomials in the variables (X_1, X_2) of the family, one chooses a point on each graph and in each slice between two graphs (using rational numbers or interval of rational numbers) and one goes by induction to the polynomials $P_1(X_1,\ldots,X_n)$,..., $P_m(X_1,\ldots,X_n)$. The passage from (F_3) to (D_2) is similar (this corresponds to the augmented projection of Collins [C]) but there, because of properties of stratifying families ([BCR]), the choice of a point in each cell allows to give a basic semi-algebraic description of the cell: the cell is the set of points for which the sign conditions on the polynomials of the stratifying family coincide with the sign conditions realized on the chosen point.

The passage from (F_3) to (D_4) is given in [CR]: one uses Thom's lemma to get an algorithm giving adjacency relations between cells.

The aim of this paper is to discuss the passage from (F_2) to (D_3) in the case of plane curves.

2. THE CASE OF PLANE CURVES

Let us consider a real plane curve C of equation $P(X,Y)=0$, with P monic as polynomial in Y, squarefree and with coefficients in $\mathbb{Z}$, let D be the discriminant of P with respect to the variable Y.

Let l be the number of the real roots $\xi_1<\ldots<\xi_l$ of D, let A_i, $i=1,\ldots,2l+1$ be defined as

$A_i = \{\xi_i\}$, $i=1,\ldots,l$,

$A_{l+i} =]\xi_{i-1}, \xi_i[$, $i=1,\ldots, l+1$ (with the convention $\xi_0= -\infty$, $\xi_{l+1}= +\infty$),

let $\zeta_{i,j}$ be defined as the function associating to $x \in A_i$ the j'th real root of $P(x,Y)$. It is clear that $(A_i, \zeta_{i,j})$ is a cylindrical decomposition of C.

2. 1. THE ALGORITHM

The algorithm will compute the topology of the plane curve C, with the help of a glueing family.

2. 1. 1. THE DIFFERENT STEPS

A_1) Characterize the real roots of D.

A_2) Characterize above each ξ_i the roots $\zeta_{i,j}$ of $P(\xi_i,Y)$

A_3) Determine on each interval between the roots of D the number of branches of the curve C.

A_4) Determine how these different branches glue to the $\zeta_{i,j}$'s.

With the above information it is clear that one can completely determine the number of connected components, of singular points and the isotopy type of the curve C.

2. 1. 2. HOW TO CHARACTERIZE REAL ALGEBRAIC NUMBERS AND BRANCHES OF CURVES?

The computations will be based on the techniques proposed in [CR].

Let us recall that we characterize, using Thom's lemma, a real algebraic number by the signs it gives to the derivatives of a polynomial it is annulating. The algorithms for coding the roots of a polynomial with integer or real algebraic coefficients and evaluating the sign they give to other polynomials with integer or real algebraic coefficients are based on generalized Sturm sequences and are given in [CR] (algorithm b_5 to b_7).

Concerning the characterizing of the branches of a real algebraic curve we have the following results.

DEFINITIONS AND NOTATIONS

One calls strict sign condition > 0, < 0 or $= 0$.

One calls sign condition > 0, < 0, $= 0$, ≥ 0, ≤ 0.

If $\varepsilon = (\varepsilon_k)$, $k = 0,\ldots, n-1$ is an n-uple of sign conditions (> 0,< 0,$= 0$,≥ 0,≤ 0) one notes $\underline{\varepsilon}$ the n-uple obtained by relaxing the strict inequalities of ε, that is by replacing >0 (resp. <0) by ≥ 0 (resp. ≤ 0). One says that the n-uple of sign conditions $\varepsilon' = (\varepsilon'_k)$, $k = 0,\ldots, n-1$, is compatible with the n-uple of sign conditions $\varepsilon = (\varepsilon_k)$, $k = 0,\ldots, n-1$, if for all k $\varepsilon_k = \varepsilon'_k$ or, in the case where ε_k is >0 (resp. <0), $\varepsilon'_k = 0$.

If ξ is an element of $\mathbb{R}$ one calls *half-branch* $\zeta_{\xi+,j}$ (resp. $\zeta_{\xi-,j}$) *of* C *above* ξ_+ (resp. ξ_-) the (germ of the) graph of the function $\zeta_{i,j}$ on a small interval of the form $]\,\xi, \xi+\alpha[$ (resp. $]\,\xi-\alpha, \xi\,[$) where $\zeta_{i,j}$ is defined (so i is such that A_i contains an interval of the form $]\,\xi, \xi+\alpha\,[$ (resp. $]\,\xi-\alpha, \xi\,[$). One calls *sign taken by* $Q(X,Y)$ *on* $\zeta_{\xi+,j}$ (resp. $\zeta_{\xi-,j}$) the sign of $Q(x,\zeta_{i,j}(x))$ just to the right (resp. left) of ξ (i.e. on a small enough interval of the form $]\xi,\xi+\alpha'\,[$ (resp. $]\xi-\alpha', \xi\,[$) with $\alpha' \leq \alpha$ such that the sign of $Q(x,\zeta_{i,j}(x))$ for $x \in\,]\xi,\xi+\alpha'\,[$ (resp. $]\xi-\alpha',\xi\,[$) does not change).

REMARK:

The definition proposed here is the only reasonable one for the sign of a polynomial on 1-dimensional subsets of a curve: the sign of a polynomial is not fixed on the whole graph of $\zeta_{i,j}$ for example, so that signs on half-branches, which are a more local information, have to be considered. This is an illustration of the general theory of real spectra (see [BCR]).

PROPOSITION 1:

Let C be a real plane curve given by an equation $P(X,Y)=0$ with $P(X,Y)$ monic in Y and of degree n in Y. Let ξ be a real root of D, the discriminant of P (with respect to Y) and $\varepsilon = (\varepsilon_k)$, $k=0,\ldots, n-1$ the signs taken by $P(X,Y)$, $P'_Y(X,Y),\ldots, P^{(i)}_Y(X,Y),\ldots$ on an half-branch $\zeta_{\xi+,j}$ (resp. $\zeta_{\xi-,j}$) above ξ_+ (resp. ξ_-).

a) The sign conditions $\varepsilon = (\varepsilon_k)$ characterize $\zeta_{\xi+,j}$ (resp. $\zeta_{\xi-,j}$) above ξ_+ (resp. ξ_-).

b) There exists one and only one root ζ of $P(\xi,Y)$ such that the signs $\varepsilon' = (\varepsilon'_k)$, $k=0,\ldots,n-1$ taken by $P(X,Y)$, $P'_Y(X,Y),\ldots, P^{(i)}_Y(X,Y),\ldots$ in (ξ,ζ) are compatible with $\varepsilon = (\varepsilon_k)$, $k=0,\ldots, n-1$.

The proof of proposition 1 will be a consequence of Thom's lemma, that we recall now.

THOM'S LEMMA:

Let P a polynomial of degree n with real coefficients and $\varepsilon = (\varepsilon_k)$, $k=0,\ldots, n-1$ a n-uple of sign conditions .

Let $A(\varepsilon) = \{x \in \mathbb{R} \mid P^{(k)}(x)\ \varepsilon_k\}$.

Then (i) $A(\varepsilon)$ is either empty or connected,

(ii) if $A(\varepsilon)$ is non empty the closure of $A(\varepsilon)$ is $A(\underline{\varepsilon})$.

proof: Easy, by induction on *n;* see [BCR] or [CR].

proof of proposition 1:

a) There exists $\zeta_{i,j}$ and $\alpha>0$ such that for all $x \in]\xi,\xi+\alpha[$ (resp. $]\xi-\alpha,\xi[$) the sign of $P^{(k)}(x,\zeta_{i,j}(x))$, $k = 0,\ldots, n-1$, coincides with ε_k. Choose $a \in]\xi,\xi+\alpha[$ (resp. $]\xi-\alpha,\xi[$). Apply then Thom's lemma to the sign conditions ε and the polynomial $P(a,Y)$.

b) The existence of such a root of $P(\xi,Y)$ comes from the fact that, P being monic in Y, the intersection of the closure of the graph of $\zeta_{i,j}$ above $]\xi, \xi+\alpha[$ (resp. $]\xi-\alpha,\xi[$) with the fiber $\{\xi\}\times\mathbb{R}$ is non empty. The set $A(\underline{\varepsilon}) = \{y \in \mathbb{R} \mid P^{(k)}(\xi, y)\ \underline{\varepsilon}_k\}$ is connected, non empty and contained in C, hence contains an unique point. It is clear that for all real root of $P(\xi,Y)$ ζ' such that the signs $\varepsilon'' = (\varepsilon''_k)$, $k=0,\ldots, n-1$ taken by $P(X,Y)$, $P'_Y(X,Y),\ldots, P^{(k)}_Y(X,Y),\ldots$ at (ξ,ζ') are compatible with $\varepsilon = (\varepsilon_k)$, $k=0,\ldots,n-1$, ζ' belongs to $A(\underline{\varepsilon})$

We are now able to give a more precise version of A_1 to A_4.

A'_1) Compute the number of real roots of D and determine at the different real roots $\xi_1,\ldots, \xi_l$, $\xi_1<\ldots<\xi_l$ of D the signs of the derivatives of D.

A'$_2$) Compute above each ξ_i the number l_i of roots of $P(\xi_i ,Y)$ and determine at the different roots $\zeta_{i,j}$ of $P(\xi_i ,Y)$, $\zeta_{i,1}<\ldots<\zeta_{i,l_i}$ the signs of the derivatives of $P(\xi_i ,Y)$ with respect to Y.

A'$_3$) Determine above ξ_{i-}, ξ_{i+}, $i=1,\ldots, l$ the number of half-branches of the curve C.

A'$_4$) Determine on each of the half-branches $\zeta_{\xi i+,j}$ (resp. $\zeta_{\xi i-,j}$) the sign of the derivatives in Y of P and deduce how the different $\zeta_{\xi i+,j'}$ (resp. $\zeta_{\xi i-,j'}$) glue to the $\zeta_{i,j}$'s, applying proposition 1 b).

The steps A'$_1$) andA'$_2$) consist in characterizing the roots of D and of the $P(\xi_i,Y)$. Step A'_3 can be done by deciding the signs of some polynomials with integer coefficients (precisely the leading coefficients of the Sturm sequence of $P(\xi_i ,Y)$) at ξ_i's. Step A'_4 can also be done by deciding the signs of some polynomials with integer coefficients at ξ_i's, using generalized Sturm sequences and proposition 1.

REMARKS:

1) Let us consider the following condition, called condition *(g)* :above every ξ_i there is at most one multiple real root $\zeta_{i,j}$ of $P(\xi_i,Y)$, that we note ζ_{i,j_0}. In this case, we know that, at each $\zeta_{i,j}$, $j\neq j_0$, there is one and only one $\zeta_{\xi i+,j'}$ (resp. $\zeta_{\xi i-,j}$) glueing to $\zeta_{i,j}$, and hence it is sufficient to know the total number of half-branches above ξ_{i-} (resp. ξ_{i+}) to know what happens in ζ_{i,j_0}, hence above ξ_i.

So if condition *(g)* is verified, the step A_4 can be replaced by $A'_{g\,4}$:

A'$_{g\,4}$) Determine how the different $\zeta_{\xi i+,j'}$ (resp. $\zeta_{\xi i-,j'}$) glue to the $\zeta_{i,j}$'s, using information about multiplicities of the $\zeta_{i,j}$.

2) It is always possible to realize condition *(g)* by a linear change of coordinates (see [GT]).

3) It is important to notice that in A'_4 one needs the signs of some polynomials just to the left and just to the right of real roots of D, and not anywhere on the interval between two roots of D. It can very well happen that the sign of these polynomials change between two roots of D. On the contrary we could choose any point in the interval between two roots of D to get the information needed in A_3.This is the difference between glueing families and stratifying families in the case of curves.

2. 1. 3. COMPLEXITY OF THE ALGORITHM

The algorithm runs in polynomial time (in n , defined as being equal to the maximum of the degree of P and the length of its coefficients). More details are given in [RS 2]. The steps to prove the result are the following.

1) The computation of D is polynomial-time.

2) The coding of the real roots of $D$is polynomial-time ([CR] or for more details [RS 1]).

3) To determine the information needed in A'_2 to A'_4 there is a polynomial number of sign evaluations at the roots of D.

4) Each of these sign evaluations can be made in polynomial time (using subresultants methods) ([RS 1]).

2. 2. EXPLICIT COMPUTATION OF AN EXAMPLE

Since we have for the moment no complete implementation of the algorithm, the computation of an explicit example (the choice of the example has been suggested by D. Duval whose work on algebraic numbers [DD] has influenced my philosophy) appeared to me as the best way of explaining how the algorithm works.

Let us first introduce some notations and remind some results. If R and $Q_1,\ldots,Q_k$ are polynomials in one variable (with real algebraic coefficients) and $\varepsilon = (\varepsilon_1,\ldots, \varepsilon_k)$ is a sequence of strict sign conditions ($> 0, < 0, = 0$) one notes $c_\varepsilon(R; Q_1,\ldots,Q_k)$ the number of real roots of R giving to $Q_1,\ldots,Q_k$the signs $\varepsilon_1,\ldots, \varepsilon_k$. In particular $c(R)$ is the number of real roots of R.

The generalized Sturm sequence associated to R and Q is defined by

$R_{Q,0}$ is R,

$R_{Q,1}$ is the remainder of the division of Q by R,

$R_{Q,i+1}$ the opposite of the remainder of the division of $R_{Q,\, i-1}$ by $R_{Q,i}$.

One notes $v_{P,Q}(-\infty)$ (resp. $v_{P,Q}(+\infty)$) the number of sign changes in the generalized Sturm sequence associated to P and Q at $-\infty$ (resp. $+\infty$). The property of the generalized Sturm sequence is the following: if P and Q are coprime, one has $c_{>0}(P; P'Q) - c_{<0}(P; P'Q) = v_{P,Q}(-\infty) - v_{P,Q}(+\infty)$ [B K R]. The Sturm sequence of P is the generalized Sturm sequence associated to P and P'.

Let us consider the curve C of equation $P(X,Y) = (Y^2+X^2-1)(Y-X-1)$, union of a circle and a line (so that, knowing in advance the topology we shall be able to control the result of our computation in each step).

E1) One computes the discriminant by taking Sturm sequence of $P(X,Y)$, using pseudo-remainders, (we shall need this Sturm sequence later):

$P_0 = P = Y^3 - (X+1)Y^2 + (X^2-1)\, Y - (X+1)(X^2-1)$

$P_1 = P'_Y = 3Y^2 - 2(X+1)Y + X^2-1$

$P_2 = -(X+1)[(X-2)Y - 2(X^3-1)]$
$P_3 = \boldsymbol{D = -9X^2(X^2-1)}$

One takes the squarefree part of D. One obtains $\boldsymbol{Q = X^3-X}$ with the same roots than D.

One computes the number of roots of X^3-X (by Sturm sequence !).
$Q_0 = Q = X^3 - X$
$Q_1 = Q' = 3X^2 - 1$
$Q_2 = X$
$Q_3 = 1$
One finds $v_{Q,Q'}(-\infty) = 3$ and $v_{Q,Q'}(+\infty) = 0$, hence $\boldsymbol{c(Q) = 3}$.

One characterizes the three roots of Q by the signs they give to Q' and Q''. This is done by generalized Sturm sequences.
It is clear that $\boldsymbol{c_{>0}(Q; Q') = 2}$ and $\boldsymbol{c_{<0}(Q; Q') = 1}$.
Let us compute now the generalized Sturm sequence associated to Q and Q''.
$Q_{Q'',0} = Q = X^3 - X$
$Q_{Q'',1} = 6X$
$Q_{Q'',2} = 0.$
One can see that Q and Q'' are not coprime, their GCD is X. In this case, one makes two different computations and one considers
a)the roots of $R = \mathrm{GCD}(Q,Q'')$ and the signs they give to Q'
b) the roots of $S = Q/\mathrm{GCD}(Q,Q'')$ and the signs they give to Q' and Q''.

For point a) one has in principle to compute the Sturm sequence of R, to determine the number of its real roots (here it is not too complicated to know that R has a unique root) . One computes then the generalized Sturm sequence of R and Q'.
$R_{Q',0} = X$
$R_{Q',1} = -1$
Here $R' = 1$ hence one has
$c_{>0}(R; Q') - c_{<0}(R; Q') = v_{R,Q'}(-\infty) - v_{R,Q'}(+\infty) = -1$
and $c_{>0}(R; Q') + c_{<0}(R; Q') = c(R) = 1,$
hence $c_{>0}(R; Q') = \boldsymbol{c_{>0,=0}(Q; Q',Q'') = 0}$
and $c_{<0}(R; Q') = \boldsymbol{c_{<0,=0}(Q; Q',Q'') = 1}.$

For point b) one has in principle to compute the generalized Sturm sequences associateds to S and Q'', S and $Q'Q''$. In fact in our example it is possible to conclude directly : since the root of R verifies $Q' < 0$, the two roots of S verify $Q' > 0$, and hence , after Thom's lemma, one verifies $Q'' > 0$ and the other $Q'' < 0$.

Let us resume the situation: Q has three roots ξ_1, ξ_2 and ξ_3, characterized by the following signs

ξ_1 **($S=X^2$-1=0, Q'>0, Q''<0)**

ξ_2 **($R=X$=0, Q'<0, Q''=0)**

ξ_3 **($S=X^2$-1=0, Q'>0, Q''>0)**.

One knows that $\xi_1 < \xi_2 < \xi_3$ by looking at the signs of Q'' since $Q^{(3)}$ is constant (this is a particuliar case of the general algorithm allowing to compare the real roots of a polynomial P from the signs they give to the derivatives of P (see [CR])).

E$_2$) Let us determine now the number of real roots of $P(\xi_1,Y)$, $P(\xi_2,Y)$ and $P(\xi_3,Y)$. It is not necessary to compute again all the corresponding Sturm sequences, since we can use the Sturm sequence of $P(X,Y)$ computed in E$_1$) and replace in it X by ξ_i (in the case where ξ_i is not a root of a leading coefficient of the polynomials of the Sturm sequence).

No problem for P_0 and P_1 which are monic with respect to Y, nor for P_3 which is annulated by ξ_1, ξ_2 and ξ_3. Let us look at what happens to the coefficient $T=(X+1)(X-2)$ of P_2 and compute the generalized Sturm sequence associated to Q and T.

One has

$Q_{T,0} = Q = X^3-X$

$Q_{T,1} = T = X^2-X-2$

$Q_{T,2} = X+1$

$Q_{T,3} = 0.$

Polynomials Q and T are hence not coprime, and have a non-constant GCD $U = X+1$ which has only one root.

One has to determine at the root of U, the signs taken by Q' and Q'', which allows to know which root of Q is a root of U. One computes the generalized Sturm sequence of U and Q' and of U and Q'', which means exactly since it is of the first degree compute the signs taken by Q' and Q'' at -1. We get $Q'(-1) > O$ and $Q''(-1) < O$. By comparing with the signs taken by Q' and Q'' at ξ_1, ξ_2 and ξ_3 we know that the root of U is ξ_1.

One has now more information about ξ_1, ξ_2 and ξ_3:

ξ_1 **($U=X+1$=0,Q' >0, Q''<0,T=0)**

ξ_2 **($R=X$=0, Q'<0, Q''=0)**

ξ_3 **($S/U=X$-1=0,Q'>0, Q''>0)**.

It is clear that, the three roots of Q being now characterized as roots of polynomials of the first degree, the computations of generalized Sturm sequence are just now evaluations of polynomials at $\xi_1 (= -1)$, $\xi_2 (=0)$ and $\xi_3 (= +1)$. Hence one has $T(\xi_1) =0$, $T(\xi_2) <0$ and $T(\xi_3) <0$.

One has now the following information:
ξ_1 **$(U=X+1=0,\ Q' >0,\ Q''<0,\ T=0)$**
ξ_2 **$(R=X=0,\ Q'<0, Q''=0,\ T<0)$**
ξ_3 **$(S/U=X-1=0, Q' >0,\ Q''>0,\ T<0)$.**

One can then conclude since $v_{P(\xi_2,Y)}(-\infty) = v_{P(\xi_3,Y)}(-\infty) = 2$ and $v_{P(\xi_2,Y)}(+\infty) = v_{P(\xi_3,Y)}(+\infty) = 0$ that the number of roots of $P(\xi_2,Y)$ and $P(\xi_3,Y)$ is 2. One notes $\zeta_{2,1}$ and $\zeta_{2,2}$ (resp. $\zeta_{3,1}$ and $\zeta_{3,2}$) the first and second root of $P(\xi_2,Y)$ (resp. $P(\xi_3,Y)$).

Above ξ_2 (resp. ξ_3) the curve has two points $\zeta_{2,1}$ and $\zeta_{2,2}$ (resp. $\zeta_{3,1}$ and $\zeta_{2,2}$).

For $P(\xi_1,Y)$, we have to compute again Sturm sequence. One has (since ξ_1 is a root of $X+1$)

$P(\xi_1,Y)_0= P(\xi_1,Y) = Y^3$
$P(\xi_1,Y)_1= P'_Y(\xi_1,Y) = 3Y^2$
$P(\xi_1,Y)_2 = 0.$

Above ξ_1 the curve has a point $\zeta_{1,1}$.

One determines at the roots of $P(\xi_i ,Y)$ the signs of the derivatives of $P(\xi_i ,Y)$.

It is clear that $\zeta_{1,1}$ is characterized by the signs $(P=0,\ P'_Y =0,\ P''_Y =0)$:
$\zeta_{1,1}$ **$(Y=0,\ P'_Y(\xi_1,Y)=0,\ P''_Y(\xi_1,Y)=0)$**

At ξ_2 one replaces $P(\xi_2,Y)$ which is equal to Y^3-Y^2-Y+1(since ξ_2 is a root of X) by Y^2-1 which is squarefree. It is easy to see, by generalized Sturm sequences that

$c_{=0}(Y^2-1; P'_Y(\xi_2,Y)) = c(Y-1)=1$
$c_{>0}(Y^2-1; P'_Y(\xi_2,Y)) = c_{<0}(Y+1; P'_Y(\xi_2,Y))=1$
$c_{>0}(Y-1; P''_Y(\xi_2,Y))=1$

and $c_{<0}(Y+1; P''_Y(\xi_2,Y))=1$.

So:

$\zeta_{2,1}$ **$(Y+1=0,\ P'_Y(\xi_2,Y)<0,\ P''_Y(\xi_2,Y)<0)$**
$\zeta_{2,2}$ **$(Y-1=0,\ P'_Y(\xi_2,Y)=0,\ P''_Y(\xi_2,Y)>0)$.**

At ξ_3 one replaces $P(\xi_3,Y)$ which is equal to Y^3-$2Y^2$ (since ξ_3 is a root of X-1) by Y^2-$2Y$ which is squarefree. It is easy to see, by generalized Sturm sequences that

$c_{=0}\,(Y^2\text{-}2Y;\, P'_Y(\xi_3,Y)) = c(Y) = 1$

$c_{>0}\,(Y^2\text{-}2Y;\, P'_Y(\xi_3,Y)) = c_{<0}(Y\text{-}2;\, P'_Y(\xi_3,Y)) = 1$

$c_{<0}\,(Y;\, P''_Y(\xi_3,Y)) = 1$

and $c_{>0}\,(Y\text{-}2;\, P''_Y(\xi_3,Y)) = 1$.

So

$\boldsymbol{\zeta_{3,1}\ (Y=0, P'_Y(\xi_3,Y)=0, P''_Y(\xi_3,Y)<0)}$

$\boldsymbol{\zeta_{3,2}\ (Y\text{-}2=0, P'_Y\,(\xi_3,Y)>0,\ P''_Y\,(\xi_3,Y)>0)}$.

E$_3$**)** One determines above ξ_{1-}, ξ_{1+}, ξ_{2+}, ξ_{3+} the number of half-branches of C. One considers the Sturm sequence of P, computed in E$_1$, and one is interested in signs at ξ_{1-}, ξ_{1+}, ξ_{2+}, ξ_{3+} of leading monomials of P_0, P_1, P_2 and P_3. It remains to compute the sign of P_3 at ξ_{i+} for i=1, 2, 3 and the sign of T at ξ_{1+} (resp. ξ_{1-}) (the sign of T at ξ_{i+} for i=2, 3 is negative since T is strictly negative at ξ_i, for i=2,3). For this one computes the signs at ξ_i, for i=2, 3, of a number of derivatives of the discriminant sufficient to know the variation of D and the signs at ξ_1 of a number of derivatives sufficient to know the variation of T; one has $D'(\xi_1)>0$, $D'(\xi_2)=0$, $D''(\xi_2)>0$, $D'(\xi_3)<0$, $T'(\xi_1)<0$.

Hence at ξ_{1-} and ξ_{3+} the leading coefficient of the Sturm sequence have the following signs:

at $-\infty$ $\quad - + + -$

at $+\infty$ $\quad + + - -$.

Above ξ_{1-} and ξ_{3+} the curve has an half-branch.

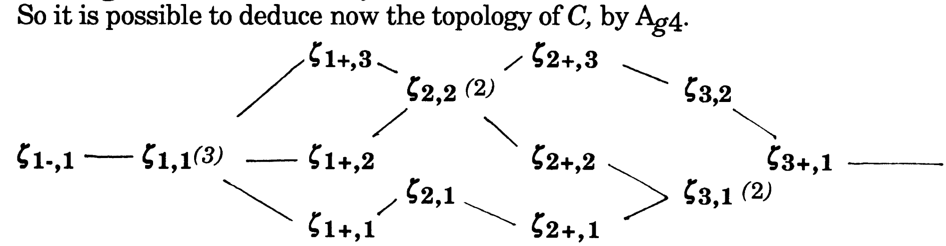
At ξ_{1+} and ξ_{2+} the leading coefficient of the Sturm sequence have the following signs

at $-\infty$ $\quad - + - +$

at $+\infty$ $\quad + + + +$.

Above ξ_{1+} and ξ_{2+} the curve has three half-branches.

E $_g$ 4)Looking at the $\zeta_{i,j}$ in E$_2$ we can see that condition *(g)* is satisfied. So it is possible to deduce now the topology of C, by A$_{g4}$.



Nevertheless, in order to illustrate our general algorithm, we shall make after the complete computation of A_4.

E_4) Let us hence describe step A_4 in the example. One determines on each half-branch the sign of P and of its derivatives with respect to Y. One already knows the signs of P'_Y on the differents half-branches of C:
-above ξ_{1-} (resp. ξ_{3+}) at $\zeta_{1-,1}$ (resp. $\zeta_{3+,1}$) $P'_Y > 0$
-above ξ_{1+}(resp. ξ_{2-}, ξ_{2+}, ξ_{3-}) at $\zeta_{1+,1}$ and $\zeta_{1+,3}$ (resp. $\zeta_{2-,1}$ and $\zeta_{2-,3}$, $\zeta_{2+,1}$ and $\zeta_{2+,3}$, $\zeta_{3-,1}$ and $\zeta_{3-,3}$)$P'_Y > 0$, at $\zeta_{1+,2}$ (resp. $\zeta_{2-,2}$, $_{2+,2}$ and $\zeta_{3-,2}$) $P'_Y < 0$.

Let us compute now the generalized Sturm sequence of P and P''_Y.

$P_{P''_Y,0} = P$

$P_{P''_Y,1} = P''_Y$

$P_{P''_Y,2} = 4 \,/\, 27(X+1)^2(5X-4)$.

At ξ_{1-}, ξ_{1+}, ξ_{2-}, ξ_{2+} $P_{P''_Y,2} < 0$.

At ξ_{3-}, ξ_{3+} $P_{P''_Y,2} > 0$.

At ξ_{1-}, ξ_{1+}, ξ_{2-}, ξ_{2+} the leading coefficient of the generalized Sturm sequence have the following signs

at $-\infty$	$- - -$
at $+\infty$	$+ + -$

hence above ξ_{1-}, ξ_{1+}, ξ_{2-}, ξ_{2+} $c_{>0}(P; P'_Y P''_Y) - c_{<0}(P; P'_Y P''_Y) = -1$.

At ξ_{3-}, ξ_{3+} the leading coefficient of the generalized Sturm sequence have the following signs

at $-\infty$	$- - +$
at $+\infty$	$+ + +$

hence above ξ_{3-} and ξ_{3+} $c_{>0}(P;P'_Y P''_Y) - c_{<0}(P;P'_Y P''_Y) = 1$.

One has at last the following characterization of half-branches of C:

above ξ_{1-}	$\zeta_{1-,1}$	$(P'_Y > 0, P''_Y < 0)$
above ξ_{1+}	$\zeta_{1+,1}$	$(P'_Y > 0, P''_Y < 0)$
	$\zeta_{1+,2}$	$(P'_Y < 0, P''_Y > 0)$
	$\zeta_{1+,3}$	$(P'_Y > 0, P''_Y > 0)$
above ξ_{2-}	$\zeta_{2-,1}$	$(P'_Y > 0, P''_Y < 0)$
	$\zeta_{2-,2}$	$(P'_Y < 0, P''_Y > 0)$
	$\zeta_{2-,3}$	$(P'_Y > 0, P''_Y > 0)$
above ξ_{2+}	$\zeta_{2+,1}$	$(P'_Y > 0, P''_Y < 0)$
	$\zeta_{2+,2}$	$(P'_Y < 0, P''_Y > 0)$
	$\zeta_{2+,3}$	$(P'_Y > 0, P''_Y > 0)$
above ξ_{3-}	$\zeta_{3-,1}$	$(P'_Y > 0, P''_Y < 0)$
	$\zeta_{3-,2}$	$(P'_Y < 0, P''_Y < 0)$
	$\zeta_{3-,3}$	$(P'_Y > 0, P''_Y > 0)$
above ξ_{3+}	$\zeta_{3+,1}$	$(P'_Y > 0, P''_Y > 0)$

Using Thom's lemma and the preceeding characterizations of $\zeta_{i,j}$, $\zeta_{i-,j}$ and $\zeta_{i+,j}$ in E_4 we get:

$\zeta_{1+,3}$ — $\zeta_{2-,3}$ $\zeta_{2+,3}$ — $\zeta_{3-,3}$
$P'_Y>0$ $P'_Y>0$ $P'_Y>0$ $P'_Y>0$
$P''_Y>0$ $P''_Y>0$ $P''_Y>0$ $P''_Y>0$

$\zeta_{2,2}$ $\zeta_{3,2}$
$P'_Y=0$ $P'_Y>0$
$P''_Y>0$ $P''_Y>0$

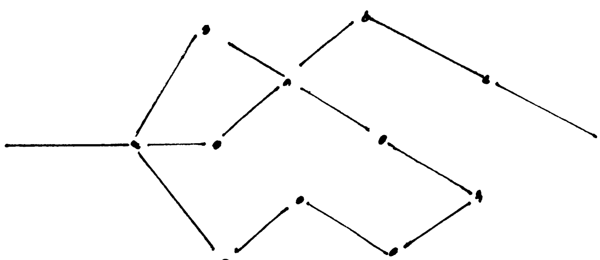
$\zeta_{1-,1} \rightarrow \zeta_{1,1} \leftarrow \zeta_{1+,2}$ — $\zeta_{2-,2}$ $\zeta_{2+,2}$ — $\zeta_{3-,2}$ $\zeta_{3+,1}$
$P'_Y>0$ $P'_Y=0$ $P'_Y<0$ $P'_Y<0$ $P'_Y<0$ $P'_Y<0$ $P'_Y>0$
$P''_Y<0$ $P''_Y=0$ $P''_Y>0$ $P''_Y>0$ $P''_Y>0$ $P''_Y<0$ $P''_Y>0$

$\zeta_{2,1}$ $\zeta_{3,1}$
$P'_Y>0$ $P'_Y=0$
$P''_Y<0$ $P''_Y<0$

$\zeta_{1+,1}$ — $\zeta_{2-,1}$ $\zeta_{2+,1} \rightarrow \zeta_{3-,1}$
$P'_Y>0$ $P'_Y>0$ $P'_Y>0$ $P'_Y>0$
$P''_Y<0$ $P''_Y<0$ $P''_Y<0$ $P''_Y<0$

Let us remark that Thom's lemma tells us how to glue the half-branches of C above ξ_2 and ξ_3. Let us remark also that between ξ_{2+} and ξ_{3-} the sign of P''_Y has changed on the second branch of C: one hence needs the sign of this polynomial both to the right of ξ_2 and to the left of ξ_3.

One gets finally the following drawing:



3. FINAL REMARKS

This paper has been ended in july 1987. Since this time several changes occured in the subject. Concerning general discussion in part 1, people are interested in new techniques for quantifier elimination, not so much on ideas based on general cylindrical algebraic decomposition ([GrV], [Gr], [Ca], [Re 1 or 2], [HRS 1,2 or 3]). These new techniques lead to algorithms doubly exponential in the number of alternations of quantifiers (and not in the number of variables as in [Co]).

Cylindrical decomposition are still important and useful in the case of curves (see the proof of [HRS 3]). The algorithm presented here has been studied in more details and improved in [R S 1 or 2], [CuGR]; algorithms for the analytic structure of curves have been studied on the same lines ([CuP3R]). In particular uniform techniques avoiding splittings and specialization problems have been introduced ([GLRR 1 or 2]). Implementations have been given by L. Gonzalez ([G]) in Reduce and F. Cucker in Maple and several examples are available.

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