Astérisque

D. R. HEATH-BROWN

The Number of Abelian Groups of Order at Most x

Astérisque, tome 198-199-200 (1991), p. 153-163
<http://www.numdam.org/item?id=AST_1991__198-199-200__153_0>

THE NUMBER OF ABELIAN GROUPS OF ORDER AT MOST x

by

D.R. HEATH-BROWN

1. Introduction

Let $a(n)$ denote the number of isomorphism classes of Abelian groups of order n. The arithmetic function $a(n)$ is multiplicative, and has a generating series

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\cdots \quad .$$

We shall be concerned here with the counting function

$$A(x) = \sum_{n\leq x} a(n) \ ,$$

first considered by ERDŐS and SZEKERES [2]. One expects that $A(x)$ will be approximated by $\sum c_j x^{1/j}$, where

$$c_j = \prod_{\substack{k=1\\k\neq j}}^{\infty} \zeta(\frac{k}{j}).$$

Indeed, if we write

$$A(x) = \sum_{j=1}^{5} c_j x^{1/j} + \Delta(x), \tag{1.1}$$

then it is known on the one hand that

$$\Delta(x) \ll x^{97/381}(\log x)^{35}$$

(KOLESNIK [8]), and on the other, that

$$\int_1^X \Delta(x)^2 dx = \Omega(X^{4/3}\log X) \tag{1.2}$$

(Ivić [7] ; see also Balasubramanian and Ramachandra [1]). Thus

$$\Delta(x) = \Omega(x^{1/6}(\log x)^{1/2}),$$

so that the extra terms in the sum (1.1) that would correspond to $j \geq 6$, cannot be relevent. Note that

$$\frac{97}{381} = 0.25459\cdots > \ 0.16666\cdots = \frac{1}{6} \cdot$$

Our aim is to prove an upper bound corresponding to (1.2).

Theorem 1. *We have*

$$\int_1^X \Delta(x)^2 dx \ \ll X^{4/3}(\log X)^{89}$$

for $X \geq 2$.

Apart from the exponent of $\log X$ this is, of course, best possible. Ivić [6] has given a weaker estimate with exponent $\frac{39}{29}$ in place of $\frac{4}{3}$. A result similar to Theorem 1 was stated by Balasubramanian and Ramachandra [1], but it appears that their claim cannot be substantiated. We have made no attempt to obtain a good exponent for the power of $\log X$ in Theorem 1.

Our method is an elaboration of that used by the author [4] to estimate

$$\int_0^T \left|\zeta(\frac{5}{8} + it)\right|^8 dt \ .$$

We take this opportunity to point out that exactly the same technique yields :

Theorem 2. *We have*

$$\int_0^T \left|\zeta(\frac{11}{20} + it)\right|^{10} dt \ \ll T^{3/2}(\log T)^{52}$$

and

$$\int_0^T \left|\zeta(\frac{9}{20} + it)\right|^{10} dt \ \ll T^2(\log T)^{52}$$

for $T \geq 2$. *Hence, in the generalized divisor problem, one has* $\beta_5 \leq \frac{9}{20}$.

These results (with the exponent 52 replaced by 50) have been given without proof by Zhang [11].

Finally we observe that our method for proving Theorem 1 has a little to spare. An examination of the proof shows that the key estimate (3.1) can be obtained with a saving of a power of T, except when M and N differ only by a factor of a small power of T. In this latter case further arguments are available covering all possibilities except that in which M and N are both small powers of T. This argument suggests that one might actually hope to obtain an asymptotic formula for the integral in (1.2).

2. Mean-Value Bounds

To estimate the average of $\Delta(x)^2$ we shall use the analysis of IVIĆ [6; pp.19-21]. After suitable modifications, this leads to

$$\int_{X/2}^{X} \Delta(x)^2 dx \ll X^{4/3}(\log X)^8 \max_{1\le T\le X} T^{-1} I_T , \tag{2.1}$$

where

$$I_T = \int_{T/2}^{T} |\zeta(1-\sigma+it)\zeta(1-2\sigma+2it)\zeta(3\sigma+3it)\zeta(4\sigma+4it)\zeta(5\sigma+5it)|^2 dt ,$$

and

$$\sigma = \frac{1}{6} + \frac{1}{\log X}.$$

In view of the inequality $2|ab| \le a^2+b^2$, we have

$$I_T \le \max(J_T, J_T') , \tag{2.2}$$

where

$$J_T = \int_{T/2}^{T} |\zeta(3\sigma+3it)^2\zeta(4\sigma+4it)^4\zeta(5\sigma+5it)^4| dt \tag{2.3}$$

and

$$J_T' = \int_{T/2}^{T} |\zeta(3\sigma+3it)^2\zeta(1-\sigma+it)^4\zeta(1-2\sigma+2it)^4| dt .$$

Since the estimation of J_T and J_T' is similar, we shall henceforth restrict our attention to J_T.

We replace the integral in (2.3) by a sum over well-spaced points $t_n \in [T/2, T]$ for which

$$|t_m - t_n| \ge 1 \quad (m \ne n) . \tag{2.4}$$

Since

$$\zeta(s) = \sum_{n\le K} n^{-s} + 0(1) \quad (T \le K \le 2T)$$

for

$$|\mathrm{Im}\,(s)| \le 5T\ ,\quad \frac{1}{2} \le \mathrm{Re}\,(s) \le \frac{7}{8}\ ,$$

by TITCHMARSH **[10**, Theorem 4.11**]**, we have, for example

$$\zeta(3\sigma + 3it) \ll (\log T) \max_{L \le T} |S_3(L, 3t)|\ , \tag{2.5}$$

where L runs over powers of 2, and

$$S_3(L, 3t) = \sum_{L<n\le 2L} n^{-3\sigma-3it}\ .$$

Of course, for the value of L giving the maximum in (2.5) we will clearly have

$$|S_3(L, 3t)| \ge |S_3(1, 3t)| \gg 1\ .$$

Similarly

$$\zeta(4\sigma + 4it) \ll (\log T) \max_{M \le T} M^{-1/6} |S_4(M, 4t)|$$

with

$$S_4(M, 4t) = \sum_{M<n\le 2M} M^{1/6} n^{-4\sigma-4it}\ , \tag{2.6}$$

and

$$\zeta(5\sigma + 5it) \ll (\log T) \max_{N \le T} N^{-1/3} |S_5(N, 5t)|\ ,$$

with

$$S_5(N, 5t) = \sum_{N<n\le 2N} N^{1/3} n^{-5\sigma-5it}\ . \tag{2.7}$$

It follows that

$$J_T \ll (\log T)^{13} M^{-2/3} N^{-4/3} \sum_n |S_3(L, 3t_n)^2 S_4(M, 4t_n)^4 S_5(N, 5t_n)^4|$$

for certain fixed L, M, N with

$$|S_3(L, 3t_n)|, |S_4(M, 4t_n)|, |S_5(N, 5t_n)| \gg 1\ .$$

We proceed to classify the points t_n according to the ranges

$$U < |S_3| \le 2U\ ,\quad V < |S_4| \le 2V$$

and

$$W < |S_5| \leq 2W$$

in which the relevent sums lie. Here U, V and W run over powers of 2 with

$$1 \ll U \ll L^{\frac{1}{2}} \ , \ 1 \ll V \ll M^{\frac{1}{2}} \ , \ \text{and} \ 1 \ll W \ll N^{\frac{1}{2}} \ . \qquad (2.8)$$

If there are $N(U,V,W)$ such points t_n for each triple (U,V,W) it follows that

$$\begin{aligned} J_T &\ll (\log T)^{13} M^{-2/3} N^{-4/3} \sum_{U,V,W} U^2 V^4 W^4 N(U,V,W) \\ &\ll (\log T)^{16} M^{-2/3} N^{-4/3} U^2 V^4 W^4 N(U,V,W) \ , \end{aligned} \qquad (2.9)$$

for some particular triple (U,V,W).

In estimating $N(U,V,W)$ we shall illustrate our methods by examining S_3. We begin by using the mean-value theorem for Dirichlet polynomials due to MONTGOMERY [**9** ; Theorem 7.3], with $Q = 1, \chi = 1, \delta = 1$. When applied to $S_3(L,t)^k$ this yields

$$\begin{aligned} U^{2k} N(U,V,W) &\ll (L^k + T)(\log T) \sum_{L^k < n \leq (2L)^k} d_k(n)^2 n^{-6\sigma} \\ &\ll (L^k + T)(\log T)^{1+k^2}. \end{aligned}$$

Similarly we have

$$V^{2k} N(U,V,W) \ll (M^k + T)(\log T)^{1+k^2} \qquad (2.10)$$

and

$$W^{2k} N(U,V,W) \ll (N^k + T)(\log T)^{1+k^2} \ . \qquad (2.11)$$

Notice that our purpose in making the somewhat peculiar definitions (2.6) and (2.7) was to produce bounds for $N(U,V,W)$ which are symmetric in S_3, S_4 and S_5. Our second estimate uses the Halász method, in the form due to HUXLEY [**5** ; p.171] (with a trivial modification to allow for the weaker spacing condition (2.4)). When applied to $S_3(L,t)^2$ this yields

$$\begin{aligned} N(U,V,W) &\ll L^2 U^{-4} \left(\sum_{L^2 < n \leq 4L^2} d(n)^2 n^{-6\sigma} \right) (\log T) \\ &\qquad + T L^2 U^{-12} \left(\sum_{L^2 < n \leq 4L^2} d(n)^2 n^{-6\sigma} \right)^3 (\log T)^5 \\ &\ll (L^2 U^{-4} + T L^2 U^{-12})(\log T)^{17} \ . \end{aligned}$$

Similarly one finds

$$N(U,V,W) \ll (M^2V^{-4} + TM^2V^{-12})(\log T)^{17} \qquad (2.12)$$

and

$$N(U,V,W) \ll (N^2W^{-4} + TN^2W^{-12})(\log T)^{17} .$$

For our remaining estimates we start from Perron's formula (see TITCHMARSH [**10** ; Lemma 3.19]), which yields

$$\begin{aligned} S_3(L,3t) &= \frac{1}{2\pi i}\int_{\frac12-4iT}^{\frac12+4iT} \zeta(s+3\sigma+3it)\,\frac{(2L)^s-L^s}{s}\,ds + 0(\log X) \\ &= \frac{1}{2\pi i}\int_{\frac12-3\sigma-4iT}^{\frac12-3\sigma+4iT} \zeta(s+3\sigma+3it)\,\frac{(2L)^s-L^s}{s}\,ds + 0(\log X) \\ &\ll \int_{-7T}^{7T} \left|\zeta(\frac12+i\tau)\right| \frac{d\tau}{\frac{1}{\log X}+|\tau-3t|} + \log X \ . \end{aligned}$$

Thus

$$\begin{aligned} U^4N(U,V,W) &\le \sum_n |S_3(L,3t_n)|^4 \\ &\ll (\log X)^4 \sum_n \left(1+\left\{\int_{-7T}^{7T}\left|\zeta(\frac12+i\tau)\right| \frac{d\tau}{1+|\tau-3t_n|}\right\}^4\right) \\ &\ll (\log X)^4 \left(T+\sum_n \int_{-7T}^{7T}\left\{\frac{d\tau}{1+|\tau-3t_n|}\right\}^3\right. \\ &\qquad \left.\left\{\int_{-7T}^{7T}\left|\zeta(\frac12+i\tau)\right|^4 \frac{d\tau}{1+|\tau-3t_n|}\right\}\right) \\ &\ll (\log X)^7 \left(T+\int_{-7T}^{7T}\left|\zeta(\frac12+i\tau)\right|^4 \left\{\sum_n \frac{1}{1+|\tau-3t_n|}\right\} d\tau\right) \\ &\ll (\log X)^8 \int_{-7T}^{7T}\left|\zeta(\frac12+i\tau)\right|^4 d\tau + T(\log X)^7 \ . \end{aligned}$$

In the final step above we have used the spacing condition (2.4). We can now apply the fourth power moment estimate for the Riemann Zeta-function (see TITCHMARSH [**10** ; (7.6.1)] for example) to give

$$U^4N(U,V,W) \ll T(\log X)^{12} \ . \qquad (2.13)$$

An entirely analogous argument based on twelfth power moments, and using the bound

$$\int_0^T \left|\zeta(\frac12+it)\right|^{12} dt \ll T^2(\log T)^{17}$$

of HEATH-BROWN [3], produces

$$U^{12}N(U,V,W) \ll T^2(\log X)^{41} . \tag{2.14}$$

Similarly we obtain

$$V^4N(U,V,W) \ll T(\log X)^{12} , \tag{2.15}$$
$$V^{12}N(U,V,W) \ll T^2(\log X)^{41} , \tag{2.16}$$
$$W^4N(U,V,W) \ll T(\log X)^{12} ,$$

and

$$W^{12}N(U,V,W) \ll T^2(\log X)^{41} .$$

3. Proof of Theorem 1

We now use our bounds for $N(U,V,W)$ to show that

$$U^2V^4W^4N(U,V,W) \ll TM^{2/3}N^{4/3}(\log X)^{65} . \tag{3.1}$$

From (2.9) we will conclude that $J_T \ll T(\log X)^{81}$, and similarly for J'_T. The theorem will then follow from (2.1) and (2.2). Because of the symmetry in our bounds for $N(U,V,W)$ it will suffice to prove (3.1) when $N \leq M$, since otherwise

$$M^{4/3}N^{2/3} \leq M^{2/3}N^{4/3} .$$

We shall therefore consider the following cases :

Case 1 $\quad N \leq M \leq T^{1/8}$,

Case 2 $\quad N \leq M^{1/4}$,

Case 3 $\quad M^4N^8 \geq T^3$,

and

Case 4 $\quad T^{1/32} \leq N \leq T^{1/4}$.

These are readily seen to exhaust all possibilities when $N \leq M$. In what follows we shall repeatedly use the principle that

$$\min(A_1,\ldots,A_k) \leq A_1^{\alpha_1}\cdots A_k^{\alpha_k}$$

for $A_i \geq 0$, $\alpha_i \geq 0$ and $\sum \alpha_i = 1$.

Case 1 : Here we use (2.10) and (2.11) with $k = 8$, together with (2.13). Thus

$$\begin{aligned}N(U,V,W) &\ll (\log X)^{65}\min((M^8+T)V^{-16},(N^8+T)W^{-16},TU^{-4})\\ &\ll (\log X)^{65}\min(TV^{-16},TW^{-16},TU^{-4})\\ &\ll (\log X)^{65}(TV^{-16})^{1/4}(TW^{-16})^{1/4}(TU^{-4})^{1/2}\\ &= TU^{-2}V^{-4}W^{-4}(\log X)^{65}\\ &\ll TM^{2/3}N^{4/3}U^{-2}V^{-4}W^{-4}(\log X)^{65}\ .\end{aligned}$$

The bound (3.1) follows.

Case 2 : Here it is convenient to consider two subcases, in which $V \geq T^{1/8}$ and $V \leq T^{1/8}$. If $V \geq T^{1/8}$ then (2.12) yields

$$N(U,V,W) \ll M^2V^{-4}(\log X)^{17}\ . \tag{3.2}$$

From (2.15), (2.16) and (2.14) we therefore have

$$\begin{aligned}N(U,V,W) &\ll (\log X)^{41}\min(M^2V^{-4},TV^{-4},T^2V^{-12},T^2U^{-12})\\ &\ll (\log X)^{41}(M^2V^{-4})^{1/4}(TV^{-4})^{1/2}(T^2V^{-12})^{1/12}(T^2U^{-12})^{1/6}\\ &= (\log X)^{41}TM^{1/2}U^{-2}V^{-4}\ .\end{aligned}$$

On the other hand, if $V \leq T^{1/8}$, we deduce from (2.12) that

$$N(U,V,W) \ll (\log X)^{17}TM^2V^{-12}\ . \tag{3.3}$$

Now (2.15) and (2.13) yield

$$\begin{aligned}N(U,V,W) &\ll (\log X)^{17}\min(TM^2V^{-12},TV^{-4},TU^{-4})\\ &\ll (\log X)^{17}(TM^2V^{-12})^{1/4}(TV^{-4})^{1/4}(TU^{-4})^{1/2}\\ &= (\log X)^{17}TM^{1/2}U^{-2}V^{-4}\ .\end{aligned}$$

In either case we conclude that

$$\begin{aligned}U^2V^4W^4N(U,V,W) &\ll (\log X)^{41}TM^{1/2}W^4\\ &\ll (\log X)^{41}TM^{1/2}N^2\\ &\ll (\log X)^{41}TM^{2/3}N^{4/3}\ ,\end{aligned}$$

as required. Here we have used (2.8) together with the condition $N \leq M^{1/4}$.

Case 3 : Here we use (2.11) with $k = 4$, together with (2.14) and (2.16). Then

$$\begin{aligned} N(U,V,W) &\ll (\log X)^{41} \min(\max(T,N^4)W^{-8}, T^2U^{-12}, T^2V^{-12}) \\ &\ll (\log X)^{41}(\max(T,N^4)W^{-8})^{1/2}(T^2U^{-12})^{1/6}(T^2V^{-12})^{1/3} \\ &= (\log X)^{41} \max(T^{3/2}, TN^2)U^{-2}V^{-4}W^{-4} . \end{aligned}$$

However $T^{3/2} \leq TM^{2/3}N^{4/3}$ providing that $M^4N^8 \geq T^3$, and $TN^2 \leq TM^{2/3}N^{4/3}$, since $N \leq M$. The bound (3.1) therefore follows in this case.

Case 4 : Again we shall consider seperately the cases $V \geq T^{1/8}$ and $V \leq T^{1/8}$. If $V \geq T^{1/8}$ we have (3.2) just as in Case 2. Then (2.11), with $k = 8$, together with (2.14), (2.15) and (2.16), yield

$$\begin{aligned} N(U,V,W) &\ll (\log X)^{65} \min(M^2V^{-4}, \max(T,N^8)W^{-16}, \\ &\qquad T^2U^{-12}, TV^{-4}, T^2V^{-12}) \\ &\ll (\log X)^{65}(M^2V^{-4})^{1/3}(\max(T,N^8)W^{-16})^{1/4}(T^2U^{-12})^{1/6} \\ &\qquad \times (TV^{-4})^{1/24}(T^2V^{-12})^{5/24} \\ &= (\log X)^{65}M^{2/3} \max(T^{25/24}, T^{19/24}N^2)U^{-2}V^{-4}W^{-4} . \end{aligned}$$

On the other hand, if $V \leq T^{1/8}$, then (3.3) holds, as in Case 2. The bound (2.11), with $k = 8$, in conjunction with (2.13) and (2.14) now produces

$$\begin{aligned} N(U,V,W) &\ll (\log X)^{65} \min(TM^2V^{-12}, \max(T,N^8)W^{-16}, TU^{-4}, T^2U^{-12}) \\ &\ll (\log X)^{65}(TM^2V^{-12})^{1/3}(\max(T,N^8)W^{-16})^{1/4}(TU^{-4})^{3/8} \\ &\qquad \times (T^2U^{-12})^{1/24} \\ &= (\log X)^{65}M^{2/3} \max(T^{25/24}, T^{19/24}N^2)U^{-2}V^{-4}W^{-4} . \end{aligned}$$

We therefore get the same estimate whether $V \geq T^{1/8}$ or not. To prove (3.1) it remains to observe that

$$\max(T^{25/24}, T^{19/24}N^2) \leq TN^{4/3}$$

when $T^{1/32} \leq N \leq T^{1/4}$.

We have now proved (3.1) in each of the four cases. This completes the treatment of Theorem 1.

4. Proof of Theorem 2

To prove Theorem 2 we adopt the procedure of Section 2, using the sum

$$S(t) = \sum_{M<m\leq 2M} M^{1/20} m^{-11/20-it} \ , \quad (1 \ll M \ll T) \ .$$

We deduce that

$$\int_{T/2}^{T} \left|\zeta(\frac{11}{20}+it)\right|^{10} dt \ \ll (\log T)^{11} M^{-1/2} N(V) V^{10} \qquad (4.1)$$

for some V in the range $1 \ll V \ll M^{\frac{1}{2}}$, where $N(V)$ is the number of well spaced points $t_n \in [T/2, T]$ at which

$$V < |S(t)| \ \leq 2V \ .$$

If $V \geq \ T^{1/8}$ then (2.12) yields

$$N(V) \ll M^2 V^{-4} (\log T)^{17} \ .$$

From (2.16), adjusted by replacing $\log X$ by $\log T$, we therefore deduce that

$$\begin{aligned} N(V) &\ll (\log T)^{41} \min(M^2V^{-4}, T^2V^{-12}) \\ &\ll (\log T)^{41} (M^2V^{-4})^{1/4} (T^2V^{-12})^{3/4} \\ &= (\log T)^{41} T^{3/2} M^{1/2} V^{-10} \ . \end{aligned} \qquad (4.2)$$

Similarly, if $V \leq T^{1/8}$ then (2.12) produces

$$N(V) \ \ll TM^2V^{-12} (\log T)^{17} \ .$$

Hence (2.15) and (2.16) yield

$$\begin{aligned} N(V) &\ll (\log T)^{41} \min(TM^2V^{-12}, TV^{-4}, T^2V^{-12}) \\ &\ll (\log T)^{41} (TM^2V^{-12})^{1/4} (TV^{-4})^{1/4} (T^2V^{-12})^{1/2} \\ &= (\log T)^{41} T^{3/2} M^{1/2} V^{-10} \ . \end{aligned} \qquad (4.3)$$

The bounds (4.1), (4.2) and (4.3) lead to

$$\int_{T/2}^{T} \left|\zeta(\frac{11}{20}+it)\right|^{10} dt \ \ll T^{3/2} (\log T)^{52} \ ,$$

which gives the first statement of Theorem 2. The second part needs only an application of the functional equation, and the remark about β_5 follows from TITCHMARSH [**10** ; Theorem 12.5].

REFERENCES

[1] R. BALASUBRAMANIAN and K. RAMACHANDRA, Some problems of analytic number theory III, *Hardy-Ramanujan J.*, **4** (1981), 13-40.

[2] P. ERDŐS and G. SZEKERES, Über die Anzahl Abelscher Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem, *Acta Sci. Math. (Szeged)*, **7** (1935), 95-102.

[3] D.R. HEATH-BROWN, The twelfth power moment of the Riemann zeta-function, *Quart. J. Math. Oxford Ser.* (2), **29** (1978), 443-462.

[4] D.R. HEATH-BROWN, Mean values of the zeta-function and divisor problems, *Recent progress in analytic number theory,* 115-119, (Academic Press, London, 1981).

[5] M.N. HUXLEY, *The distribution of prime numbers*, (Oxford, 1972).

[6] A. IVIĆ, The number of finite non-isomorphic Abelian groups in mean square, *Hardy-Ramanujan J.*, **9** (1986), 17-23.

[7] A. IVIĆ, The general divisor problem, *J. Number Theory*, **26** (1987), 73-91.

[8] G. KOLESNIK, On the number of Abelian groups of a given order, *J. Reine Angew. Math.*, **329** (1981), 164-175.

[9] H.L. MONTGOMERY, *Topics in multiplicative number theory*, (Springer, Berlin, 1971).

[10] E.C. TITCHMARSH, *The theory of the Riemann zeta-function*, 2nd Edition (Oxford, 1986).

[11] W.-P. ZHANG, On the divisor problem, *Kexue Tongbao*, **33** (1988), 1484-1485.

Roger HEATH-BROWN
Magdalen College
Oxford OX1 4AU