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Duffin-Schaeffer Theorem of Diophantine Approximation for Complex Numbers

by
Hitoshi Nakada and Gerold Wagner

Let $o(d)$ be the set of integers in $\mathbf{Q}(\sqrt{\mathbf{d}})$ for a square-free negative integer d, that is, $o(d)$ is the set

$$\{n + m\omega : n, m \in \mathbf{Z}\}$$

with

$$\omega = \begin{cases} (1+\sqrt{d})/2 & \text{if } d = 1 \pmod 4 \\ \sqrt{d} & \text{if } d = 2,3 \pmod 4). \end{cases}$$

We put $I := \{z : z = x + y\omega, 0 \leq x, y < 1\}$ and consider the inequality

$$|z - \frac{a}{r}| \;<\; \frac{f(r)}{|r|}, \;\; (a,r) = 1, \;\; a, r \in o(d) \tag{1}$$

for $z \in I$ with a given non-negative function f defined on $o(d)$. We assume that $f(r) = f(u \cdot r)$ for all units in $o(d)$. We denote by $\Phi(r)$ the Euler function of $\mathbf{Q}(\sqrt{\mathbf{d}})$, which is equal to the number of reduced residual classes mod r and is equal to the number of integers relatively prime to r in $r \cdot I$.

By the Borel-Cantelli lemma, it is easy to see that (1) has only finitely many solutions for almost all $z \in I$ (with respect to Lebesgue measure) if

$$\sum_{r \in o(d)} f^2(r) \cdot \Phi(r)/|r|^2 < \infty.$$

Here, we ask the converse of this : are there infinitely many solutions of (1) for almost all $z \in I$ whenever

$$\sum f^2(r) \cdot \Phi(r)/|r|^2$$

diverges? This is a complex version of Duffin-Schaeffer conjecture [1]. Although the one-dimensional (original) Duffin-Schaeffer conjecture remains unsolved, its higher dimensional analogues (which somehow correspond to our situation) have very recently been settled by Pollington and Vaughan (see

[4]). In the sequel, we give a sufficient condition on f to having infinitely many solutions of (1) for almost all z, which corresponds to Duffin-Schaeffer's theorem.

To prove our complex version, we do not follow the original Duffin and Schaeffer's proof but Sprindžuk's one [5]. Then we also have a complex version of Gallagher's theorem [2].

Theorem 1 *Let A_f be the set of $z(\in I)$, for which (1) has infinitely many solutions. Then we have*

$$\mu(A_f) = 0 \text{ or } 1$$

for any non-negative function f, where μ denotes the normalized Lebesgue measure on I.

By using this theorem, we prove the complex Duffin-Schaeffer theorem :

Theorem 2 *Suppose that*

$$\sum_{r \in o(d)} f^2(r) \; = \; \infty \tag{2}$$

and there exist infinitely many $R \in \mathbf{N}$ such that

$$\sum_{|r|<R\,,\, r \in o(d)} f^2(r) \; < \; c_1 \cdot \sum_{|r|<R\,,\, r \in o(d)} f^2(r) \cdot \Phi(r)/|r|^2 \tag{3}$$

for some constant $c_1 > 0$. Then (1) has infinitely many solutions for almost all $z \in I$

To prove Theorem1, first we note the following :

Lemma 1 *For any $z \in o(d)$ and $r \in o(d), r \neq 0$, there exists an integer $a \in o(d)$ such that*

$$\text{(i)} \quad (r, a) = 1$$

and

$$\text{(ii)} \quad |r \cdot z - a| < c(\epsilon) \cdot |r|^{\epsilon}.$$

Here $\epsilon > 0$ is arbitrary and $c(\epsilon)$ is a positive constant depending on ϵ only.

This lemma shows that if there exists a sequence of integers $\{r_n\}$ such that $f(r_n) \gg |r_n|^{\epsilon}$ for some $\epsilon > 0$, then (1) has always a solution for each sufficiently large r_n. Thus we can assume that

$$f(r) \ll |r|^{\epsilon}$$

for any $\epsilon > 0$.

It is easy to prove the following two lemmas (see [5]).

Lemma 2 *Let $I_k, k = 1, 2, \cdots$, be a sequence of disks with $\mu(I_k) \to 0$ as $k \to \infty$, and let A_k be a sequence of measurable subsets for which $A_k \subset I_k$ and*

$$\mu(A_k) > \delta \cdot \mu(I_k), \quad k = 1, 2, \cdots,$$

for some $\delta > 0$. Then we have

$$\mu(\cap_{l=1}^{\infty} \cup_{k=l}^{\infty} I_k) = \mu(\cap_{l=1}^{\infty} \cup_{k=l}^{\infty} A_k).$$

Lemma 3 *For every q and s in $o(d)$ with $|q| > 1$, the transformation $T : z \to q \cdot z + s/q \pmod{o(d)}$ of I onto itself is ergodic, that is, if a measurable subset of I is T-invariant, then its Lebesgue measure is 0 or 1 (in the normalized sense).*

Sketch of proof of Theorem 1 We consider a rational integer p which is prime in $o(d)$. Then we follow the proof of Theorem 7 of Sprindžuk[5]. Since there are infinitely many prime integers, we have the desired result.

Now, to prove Theorem 2, we only need to show that

$$\mu(A_f) > 0$$

when (2) and (3) hold. For this, we use the following Lamperti-Rényi's lemma.

Lemma 4 *Let (Ω, B, P) be a probability space. Suppose that $\{A_n\}$ be a sequence of measurable subsets with*

$$\sum P(A_n) = \infty \tag{4}$$

If there exists a subsequence $\{n_m\}$ such that

$$\sum_{k=1}^{n_m} \sum_{l=1}^{n_m} P(A_k \cap A_l) \ll \left[\sum_{k=1}^{n_m} P(A_k) \right]^2 \text{ as } m \to \infty, \tag{5}$$

then

$$P(\cap_{l=1}^{\infty} \cup_{k=l}^{\infty} A_k) > 0.$$

Sketch of proof of Theorem 2 We assume that f is bounded. We put

$$A_r := \{z \in I : \text{there is an integer } a \in o(d) \text{ such that}$$

$$(a, r) = 1 \text{ and } |z - a/r| < f(r)/r\}.$$

Since A_r is the union of $\Phi(r)$ open disks with the radii $f(r)/|r|$, intersected with I, we see that

$$c_2 \cdot \Phi(r) \cdot f^2(r)/|r|^2 < \mu(A_r) < c_3 \cdot \Phi(r) \cdot f^2(r)/|r|^2 \tag{6}$$

for some constants $c_2 > 0$ and $c_3 > 0$. Now we estimate $\mu(A_{r_1} \cap A_{r_2})$ for $|r_1| < |r_2|$. Then we can show that

$$\mu(A_{r_1} \cap A_{r_2}) < c_4 \cdot f^2(r_1) \cdot f^2(r_2)$$

for a constant $c_4 > 0$. From this inequality with (2),(3) and (6), we see that $\{A_r\}$ satisfies (4) and (5). This proves Theorem 2 when f is bounded. Because of the property (3), there exists a set A of integers such that on A $f(r) > 0$, $\Phi(r)/|r|^2 > c_5$ for a constant $c_5 > 0$ and $\sum f^2(r) = \infty$ where r runs over A. Then it is easy to see that the above method holds for unbounded case by cutting f and this completes the proof of Theorem 2.

Remark Let $\{a_n/r_n\}$ be the sequence of solutions of (1) with

$$|r_1| \leq |r_2| \leq \cdots .$$

We suppose that $f(r) = \theta/|r|$ for a positive constant θ. We put

$$\eta_n := r_n^2 \cdot |z - a_n/r_n|/\theta.$$

Then, by using a geometrical method (see [3]), we can show that $\{ \eta_n \}$ is uniformly distributed in the unit disk for almost all z. In general, it is not so hard to see that $\{ \arg (z - a_n/r_n) \}$ is dense (mod 2π) for almost all z when $\mu(A_f) > 0$. One may ask whether this is uniformly distributed. If we make some additional conditions on f (see [5]), then we may estimate the asymptotic number of solutions of (1) and this gives the answer of this question. We will discuss these facts in another occasion.

The first author would like to dedicate this paper to the memory of G. Wagner, who was killed in March 1990, after this paper had been submitted for publication, by an avalanche while skiing in the Alps.

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