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of a given number field : a new generalization of  
Dirichlet approximation theorem**

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**ON DIOPHANTINE APPROXIMATION  
BY ALGEBRAIC NUMBERS OF A GIVEN NUMBER FIELD :  
A NEW GENERALIZATION OF  
DIRICHLET APPROXIMATION THEOREM**

by

Roland QUÊME

**Introduction**

It is well known that for all  $\alpha \in \mathbb{R}$ ,  $\alpha \notin \mathbb{Q}$  there are infinitely many  $p/q$ ,  $|p|, q \in \mathbb{N}$  such that  $|\alpha - p/q| < 1/q^2$  (Dirichlet theorem), and that for any real algebraic number  $\alpha \notin \mathbb{Q}$  and for any  $\varepsilon \in \mathbb{R}$ ,  $\varepsilon > 0$ , there exist only finitely many  $p/q$ ,  $|p|, q \in \mathbb{N}$  such that  $|\alpha - p/q| < 1/q^{2+\varepsilon}$  (Roth theorem).

Let  $K$  be a number field of degree  $n$ , signature  $(r, s)$  and absolute value of discriminant  $D$ .

Let  $B$  be the Minkowski constant of  $K$  ( $B = (4/\pi)^s \cdot (n!/n^n) \cdot \sqrt{D}$ ).

Let  $\sigma : K \rightarrow \mathbb{R}^r \times \mathbb{C}^s$  be the embedding defined by :

$$\sigma(\rho) = (\sigma_1(\rho), \dots, \sigma_r(\rho), \sigma_{r+1}(\rho), \dots, \sigma_{r+s}(\rho))$$

where, as usually,  $K = \sigma_1(K)$ .

For  $x, y \in \mathbb{R}^r \times \mathbb{C}^s$  we note  $x = (x_j, j = 1, \dots, r + s)$ . Then we note  $x + y = (x_j + y_j, j = 1, \dots, r + s)$  and  $x \cdot y = (x_j \cdot y_j, j = 1, \dots, r + s)$ .

We define, for  $x \in \mathbb{R}^r \times \mathbb{C}^s$ , the distance function and the norm function :

$$\begin{aligned} d(x) &= |x_1| + \dots + |x_r| + 2|x_{r+1}| + \dots + 2|x_{r+s}|, \\ N(x) &= |x_1| \cdots |x_r| \cdot |x_{r+1}|^2 \cdots |x_{r+s}|^2. \end{aligned}$$

Let  $A$  be the ring of integers of  $K$ .

Then we obtain the diophantine approximation theorems :

- (i) For  $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$ , there exist infinitely many  $\beta = p/q$ ,  $p, q \in A$  such that  $0 < d(\alpha\sigma(q) - \sigma(p)) < n^2 \cdot B^{2/n}/d(\sigma(q))$ , with arbitrary large distance  $d(\sigma(q))$ .
- (ii) For  $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$ ,  $\alpha_j \notin \sigma_j(K)$ ,  $j = 1, 2, \dots, r + s$ , there exist infinitely many  $\beta = p/q$ ,  $p, q \in A$  such that  $0 < N(\alpha - \sigma(p/q)) < (B/N_{K/\mathbb{Q}}(q))^2$ .

We first summarize the state of the art with three types of generalizations found in the quoted literature for diophantine approximation by numbers of a given number field  $K$ . Let  $K$  be a number field of degree  $n$ , signature  $(r, s)$ . For  $\beta \in K$ , let  $P(\beta)$  be the field polynomial of  $\beta$ ,

$$P(\beta) = (x - \sigma_1(\beta)) \cdots (x - \sigma_r(\beta))(x - \sigma_{r+1}(\beta)) \overline{(x - \sigma_{r+1}(\beta))} \cdots (x - \sigma_{r+s}(\beta)) \overline{(x - \sigma_{r+s}(\beta))} .$$

Let  $C \in \mathbb{N}$  such that  $P_1(\beta) = CP(\beta) = b_n\beta^n + \cdots + b_1\beta + b_0$  is a polynomial with integer coprime coefficients  $b_i$ ,  $i = 0, 1, \dots, n$ . Then we define the height of  $\beta \in K$  by  $H_K(\beta) = \sup_{i=0, \dots, n} |b_i|$ .

The first generalization of Dirichlet theorem found in bibliography is :

Assume that  $r > 0$  and choose a real embedding  $\sigma_1 : K \rightarrow \mathbb{R}$ . For every  $\alpha \in \mathbb{R} - \sigma_1(K)$ , then there exist infinitely many  $\beta \in K$  such that  $|\alpha - \sigma_1(\beta)| < C_1(K) \max(1, \alpha^2)/H_K(\beta)^2$  where  $C_1(K)$  is a constant depending only on  $K$  (see SCHMIDT [8] p.253).

The second generalization of Dirichlet theorem is :

Assume that  $s > 0$  and choose a complex embedding  $\sigma_2 : K \rightarrow \mathbb{C}$ . For every  $\alpha \in \mathbb{C} - \sigma_2(K)$ , then there exist infinitely many  $\beta \in K$  such that  $|\alpha - \sigma_2(\beta)| < C_2(K)/H_K(\beta)$  where  $C_2(K)$  is a constant depending only on  $K$  (see SCHMIDT [6] p.206).

The third generalization is :

Let  $\beta_1, \dots, \beta_\ell \in K$ ; let  $\mathfrak{b}$  be the fractional ideal of  $K$  generated by  $(1, \beta_1, \dots, \beta_\ell)$ .

We define the generalized height of the  $\ell$ -tuple  $(\beta_1, \dots, \beta_\ell)$  by :

$$h_K(\beta_1, \dots, \beta_\ell) = N_{K/\mathbf{Q}}(\mathfrak{b}) \prod_{j=1}^r \max(1, |\sigma_j(\beta_1)|, \dots, |\sigma_j(\beta_\ell)|) \prod_{j=r+1}^{r+s} \max(1, |\sigma_j(\beta_1)|, \dots, |\sigma_j(\beta_\ell)|)^2.$$

- (i) if  $r > 0$ , let  $\sigma_3 : K \rightarrow \mathbf{R}$  be a real embedding and  $\alpha_1, \dots, \alpha_\ell \in \mathbf{R}$ , not all in  $\sigma_3(K)$  ; put in that case  $\nu = 1$  ;
- (ii) if  $s > 0$ , let  $\sigma_3 : K \rightarrow \mathbf{C}$  be a complex embedding and  $\alpha_1, \dots, \alpha_\ell \in \mathbf{C}$ , not all in  $\sigma_3(K)$  ; put in that case  $\nu = 2$  ;

then there is a constant  $C_3(K, \alpha_1, \dots, \alpha_\ell)$  depending only on  $K, \alpha_1, \dots, \alpha_\ell$  such that there exist infinitely many  $\beta = (\beta_1, \dots, \beta_\ell), \beta_i \in K$ , with

$$|\alpha_i - \sigma_3(\beta_i)|^\nu < C_3(K, \alpha_1, \dots, \alpha_\ell) \cdot h_K(\beta_1, \dots, \beta_\ell)^{-1-1/\ell}, \quad i = 1, 2, \dots, \ell \quad (1)$$

(see SCHMIDT [7] p.2).

The main difference between the quoted formulation and our theorem are summarized in the four next points :

- 1) In classical approximations above,  $|\alpha - \beta|$  is obtained for *one* of the conjugates  $\beta = \sigma_1(\beta)$ . On the other hand, our estimate involves simultaneously *all* the conjugates of the same  $\beta \in K$ ,

for the distance function,

$$d(\alpha\sigma(q) - \sigma(p)) = |\alpha_1\sigma_1(q) - \sigma_1(p)| + \dots + |\alpha_r\sigma_r(q) - \sigma_r(p)| + 2|\alpha_{r+1}\sigma_{r+1}(q) - \sigma_{r+1}(p)| + \dots + 2|\alpha_{r+s}\sigma_{r+s}(q) - \sigma_{r+s}(p)|$$

for the norm function,

$$N(\alpha - \sigma(p/q)) = |\alpha_1 - \sigma_1(p/q)| \dots |\alpha_r - \sigma_r(p/q)| \cdot |\alpha_{r+1} - \sigma_{r+1}(p/q)|^2 \dots |\alpha_{r+s} - \sigma_{r+s}(p/q)|^2.$$

- 2) Our approximation theorem cannot be immediately connected to usual simultaneous approximation theorems, because in simultaneous approximation  $|f(\alpha_1 - \beta_1)|, \dots, |f(\alpha_\ell - \beta_\ell)|$  the simultaneous approximations  $\beta_1, \dots, \beta_\ell$  are not conjugate of the same  $\beta \in K$  (see for instance (1)).

- 3) Our result contains not only effective but *explicit* constants with *simple* relationship to the structure of the number fields (the Minkowski constant for instance, with the distance function choosen).
- 4) Our proof is the exact generalization of the approximation by  $\mathbb{Q}$  to approximation by a given number field  $K$ , using geometry of numbers properties of number fields embedding in  $\mathbb{R}^n$ .

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### Prerequisites-Notations

$K$  : number field

$n$  : degree of  $K$

$(r, s)$  : signature of  $K$

$x$  :  $x \in \mathbb{R}^r \times \mathbb{C}^s$ ,  $x = (x_j \mid j = 1, \dots, r + s)$

$x + y$  :  $x + y = (x_j + y_j \mid j = 1, \dots, r + s)$

$x.y$  :  $x.y = (x_j.y_j \mid j = 1, \dots, r + s)$

$d(x)$  : for  $x \in \mathbb{R}^r \times \mathbb{C}^s$ , the distance function is defined by :

$$d(x) = |x_1| + \dots + |x_r| + 2|x_{r+1}| + \dots + 2|x_{r+s}|$$

$N(x)$  : for  $x \in \mathbb{R}^r \times \mathbb{C}^s$ , the norm form is defined by :

$$N(x) = |x_1| \cdots |x_r| \cdot |x_{r+1}|^2 \cdots |x_{r+s}|^2$$

$U(o, \tau)$  : for  $\tau \in \mathbb{R}_+$ , convex body of  $\mathbb{R}^n$  defined by

$$U(o, \tau) = \{x \mid x \in \mathbb{R}^r \times \mathbb{C}^s, d(x) < n\tau\}$$

where  $\mathbb{R}^r \times \mathbb{C}^s$  is isomorphically identified to  $\mathbb{R}^n$  by

$$x_{r+i} = (R(x_{r+i}), I(x_{r+i})), \quad i = 1, \dots, s,$$

where  $R$  and  $I$  are the real and imaginary part.

The volume of  $U(o, \tau)$  is  $v(U(o, \tau)) = 2^r (\pi/2)^s n^n \tau^n / n!$

(see for instance SAMUEL [5] p.70).

$A$  : ring of algebraic integers in  $K$ .

$\sigma(A)$  : embedding of  $A$  in  $\mathbb{R}^r \times \mathbb{C}^s$  defined, for  $a \in A$ , by

$$\sigma(a) = (\sigma_1(a), \dots, \sigma_r(a), \sigma_{r+1}(a), \dots, \sigma_{r+s}(a))$$

where  $\mathbb{R}^r \times \mathbb{C}^s$  is isomorphically identified to  $\mathbb{R}^n$  by

$$\sigma_{r+i}(a) = (R(\sigma_{r+i}(a)), I(\sigma_{r+i}(a))).$$

$\sigma(A)$  is a lattice.

$D_0$  : Let  $w_1, \dots, w_n \in A$  such that  $\sigma(w_1), \dots, \sigma(w_n)$  is a basis of the lattice  $\sigma(A)$ .

we define classically the fundamental domain  $D_0$  by :

$$D_0 = \{x \mid x \in \mathbb{R}^r \times \mathbb{C}^s, x = u_1 \sigma(w_1) + \dots + u_n \sigma(w_n), 0 \leq u_i < 1\}.$$

$D(\sigma(a))$  : fundamental domain of  $\sigma(A)$  deduced from the fundamental domain  $D_0$  by the translation  $0 \rightarrow \sigma(a)$  :

$$D(\sigma(a)) = \{(y_j) \in \mathbb{R}^r \times \mathbb{C}^s \mid (y_j - \sigma_j(a)) \mid j = 1, \dots, r + s) \in D_0\}.$$

## Results

**THEOREM 1.** *Let  $K$  be a number field of degree  $n$ , signature  $(r, s)$ , and absolute value of discriminant  $D$ . Let  $B$  be the Minkowski bound of  $K$  ( $B = (4/\pi)^s \cdot (n!/n^n) \cdot \sqrt{|D|}$ ). Let  $A$  be the ring of integers of  $K$ . Let  $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$ . Then, for any  $m \in \mathbb{R}$ ,  $m > 0$ , there are infinitely many different  $\beta = p/q$  where  $p, q \in A$ , such that  $d(\sigma(q)) > m$  and*

$$0 < d(\alpha \cdot \sigma(q) - \sigma(p)) < (n^2 \cdot B^{2/n}) / d(\sigma(q)).$$

*Proof :*

1) Let  $\varepsilon \in \mathbf{R}$ ,  $\varepsilon > 0$ ,

$$\lambda = (1 + 2\varepsilon)^{1/n} . B^{2/n} / 2 = (1 + 2\varepsilon)^{1/n} . (n! / n^n)^{2/n} . (4/\pi)^{(2s)/n} . D^{1/n} / 2.$$

Let  $m \in \mathbf{R}_+$ , arbitrary large and  $\mu = \lambda m^{-1/n}$ .

Consider the set  $E = U(o, m^{1/n}) \cap \sigma(A)$  where  $U$  and  $\sigma$  have the meaning of notations paragraph. From  $v(U(o, m^{1/n})) = 2^r (\pi/2)^s n^n m / n!$  and  $v(D(o)) = 2^{-s} \sqrt{D}$ , we deduce

$$t = \text{Card} (E) = (2^r (\pi/2)^s n^n m) / (n! 2^{-s} \sqrt{D}) + O(m^{1-1/n}).$$

Therefore, for  $m$  sufficiently large, we have  $t > \{2^r \pi^s n^n m / (n! \sqrt{D})\} . \{1 - \varepsilon\}$ . For any  $a \in A$ , for all  $q_i \in A$  with  $\sigma(q_i) \in E$ , it is possible to define  $p_i(a) \in A$  and  $\rho_i(a) \in \mathbf{R}^r \times \mathbf{C}^s$ ,  $i = 1, 2, \dots, t$ , such that  $\rho_i(a) = \alpha \sigma(q_i) - \sigma(p_i(a))$ ,  $i = 1, 2, \dots, t$  and  $\rho_i(a) \in D(\sigma(a))$ . Notice that the approximation function  $d(x)$  is meaningful because  $d(\alpha \sigma(q) - \sigma(p)) = 0$  leads to  $p = q = 0$  : from the definition of  $d(x)$ ,  $d(\alpha \sigma(q) - \sigma(p)) = 0$  implies  $\alpha_j \sigma_j(q) - \sigma_j(p) = 0$ ,  $j = 1, \dots, r + s$ , and thus  $\alpha_j = \sigma_j(p/q)$ ,  $j = 1, \dots, r + s$  and therefore  $\alpha \in \sigma(K)$ , which is in contradiction with hypothesis. Thus the  $\rho_i(a)$ ,  $i = 1, \dots, t$ , are different each others.

Consider the set  $G = \{U(\rho_i(a), \mu/2) \mid i = 1, 2, \dots, t, \forall a \in A\}$ .  $G$  cannot be a packing of  $\mathbf{R}^n$  (for packing definition, see for instance LEKKERKERKER [2] p.169) because

$$\begin{aligned} tv(U(o, \mu/2)) &> \{(1 - \varepsilon) 2^r \pi^s n^n m / (n! \sqrt{D})\} . \\ &\quad \{2^r (\pi/2)^s n^n (1 + 2\varepsilon) (n! / n^n)^2 (4/\pi)^{2s} D m^{-1} / (2^n 2^n n!)\} \\ tv(U(o, \mu/2)) &> (1 - \varepsilon) (1 + 2\varepsilon) 2^{-s} \sqrt{D} > v(D(o)). \end{aligned}$$

Therefore, for  $m$  sufficiently large, there exist  $\rho_i(a)$  and  $\rho_{i'}(b)$  with

$$\rho_i(a) = \alpha \sigma(q_i) - \sigma(p_i(a)) \tag{1}$$

$$\rho_{i'}(b) = \alpha \sigma(q_{i'}) - \sigma(p_{i'}(b)) \tag{2}$$

such that  $U(\rho_i(a), \mu/2) \cap U(\rho_{i'}(b), \mu/2) \neq \emptyset$ .

Then  $d(\rho_i(a) - \rho_{i'}(b)) < n\mu$  from the definition of the convex set  $U(\rho(a), \mu/2)$ .

Let  $p = p_i(a) - p_{i'}(b)$ ,  $p \in A$  and  $q = q_i - q_{i'}$ ,  $q \in A$ . Then, from the value of  $\mu$ , we deduce

$$d(\alpha \sigma(q) - \sigma(p)) < n\mu = (n(1 + 2\varepsilon)^{1/n} . B^{2/n} / 2) m^{-1/n}. \tag{2'}$$

Consider the sequence of values of  $\varepsilon$  defined by  $\varepsilon_1 = 1, \varepsilon_2 = 1/2, \dots, \varepsilon_k = 1/k, \dots$ . Therefore, for  $m$  given, for any  $\varepsilon_k$  there exist  $p(\varepsilon_k), q(\varepsilon_k) \in A$  such that

$$d(\alpha\sigma(q(\varepsilon_k)) - \sigma(p(\varepsilon_k))) < (nB^{2/n}/2).m^{-1/n}.(1 + 2\varepsilon_k)^{1/n}. \quad (2'')$$

From  $\sigma(q(\varepsilon_k)) \in 2E$ , we deduce that  $d(\sigma(q(\varepsilon_k)))$  is bounded above independently of  $\varepsilon_k$ . From inequality (2''), we then deduce that  $d(\sigma(p(\varepsilon_k)))$  is also bounded above independently of  $\varepsilon_k$ . Like  $\sigma(A)$  is a lattice, it is possible to take out an infinite subsequence  $k_1, k_2, \dots, k_j$  such that  $p(\varepsilon_{k_1}) = p(\varepsilon_{k_2}) = \dots = p(\varepsilon_{k_j}) = p$  and  $q(\varepsilon_{k_1}) = q(\varepsilon_{k_2}) = \dots = q(\varepsilon_{k_j}) = q$  and then

$$d(\alpha\sigma(q) - \sigma(p)) \leq (nB^{2/n}/2)m^{-1/n}. \quad (3)$$

From  $\sigma(q_i) \in E$  in (1), we have  $d(\sigma(q_i)) < nm^{1/n}$ ,  
 From  $\sigma(q_{i'}) \in E$  in (2), we have  $d(\sigma(q_{i'})) < nm^{1/n}$ ,  
 and thus  $d(\sigma(q)) < 2nm^{1/n}$  or  $m^{-1/n} < 2n/d(\sigma(q))$ .

We then have from (3)

$$\begin{aligned} d(\alpha\sigma(q) - \sigma(p)) &< (nB^{2/n}/2)(2n/d(\sigma(q))) \\ d(\alpha\sigma(q) - \sigma(p)) &< n^2 B^{2/n} / d(\sigma(q)). \end{aligned} \quad (3')$$

2) We shall now prove that there are infinitely many different  $\beta = p/q$  with

$$d(\alpha\sigma(q) - \sigma(p)) < n^2 B^{2/n} / d(\sigma(q)). \quad (4)$$

Let  $m_1, m_2 \in \mathbb{R}_+, m_1$  given,  $m_1 < m_2$  with  $m_2 \rightarrow +\infty$ . We have  $m_2 > m_1$  and  $\mu_1 > \mu_2$  with the meaning of  $m$  and  $\mu$  above.

From (3') inequality, we have

$$d(\alpha\sigma(q_1) - \sigma(p_1)) < n^2 B^{2/n} m_1^{-1/n}, \quad (5)$$

$$d(\alpha\sigma(q_2) - \sigma(p_2)) < n^2 B^{2/n} m_2^{-1/n}, \quad (6)$$

If  $\beta_2 = \beta_1$  then  $p_2/q_2 = p_1/q_1$  and  $\sigma_j(p_2/q_2) = \sigma_j(p_1/q_1), j = 1, \dots, r + s$ .  $\alpha_j \sigma_j(q_2) - \sigma_j(p_2) = \sigma_j(q_2)(\alpha_j - \sigma_j(p_1/q_1))$  and thus  $N(\alpha\sigma(q_2) - \sigma(p_2)) = N_{K/\mathbb{Q}}(q_2)N(\alpha - \sigma(p_1/q_1)), N(\alpha\sigma(q_2) - \sigma(p_2)) \geq N(\alpha - \sigma(p_1/q_1))$ . From the geometric mean inequality,  $d(\alpha\sigma(q_2) - \sigma(p_2)) \geq nN(\alpha - \sigma(p_1/q_1))^{1/n}$  and then  $\mu_2 > N(\alpha - \sigma(p_1/q_1))^{1/n}$ , which is possible only for  $m_2$  bounded above. Then, for any  $\beta_1 = p_1/q_1$  given which verify (5), there are finitely many couples  $(p_2, q_2)$  such that  $\beta_2 = p_2/q_2 = p_1/q_1$  and such that (2) is verified. Relation (4) is



verified by infinitely many couples  $(p, q)$ , because in (3)  $d(\alpha\sigma(q) - \sigma(p))$  can be made arbitrary small for  $m$  sufficiently large. Therefore there are infinitely many different  $\beta = p/q$  such that (4) is verified. 3) We shall prove that there are finitely many different  $\beta = p/q$  for one value of  $q$  given :

Let  $\beta_1 = p_1/q$  and  $\beta_2 = p_2/q$ . If  $q_1 = q_2 = q$  then

$$d(\alpha\sigma(q) - \sigma(p_1)) < n^2 B^{2/n} / d(\sigma(q)) \text{ and } d(\alpha\sigma(q) - \sigma(p_2)) < n^2 B^{2/n} / d(\sigma(q)).$$

Then, we deduce  $d(\sigma(p_1 - p_2)) < 2n^2 B^{2/n} / d(\sigma(q))$ , which is possible only, for  $p_1$  given, for a finite number of  $p_2$ .

4) From 2) and 3), there are infinitely many different  $q$ , thus with arbitrary large  $d(\sigma(q))$  such that

$$d(\alpha\sigma(q) - \sigma(p)) < (n^2 B^{2/n}) / d(\sigma(q)), \quad \text{Q.E.D.}$$

*Remark :* If  $\alpha$  is such that  $\alpha_1 = \alpha_2 = \dots = \alpha_{r+s}$ , then an immediate consequence of the Dirichlet approximation theorem is that there are infinitely many  $p/q$ ,  $p, q \in \mathbb{Z} \subset A$  such that  $d(\alpha\sigma(q) - \sigma(p)) < n/q = n^2 / d(\sigma(q)) < (n^2 B^{2/n}) / d(\sigma(q))$  : in that particular case, the theorem 1 is an immediate consequence of Dirichlet theorem.

**COROLLARY 2 :** *Let  $K$  be a number field of degree  $n$ , signature  $(r, s)$  and absolute value of discriminant  $D$ . Let  $B$  be the Minkowski bound of  $K$  ( $B = (4/\pi)^s (n! / n^n) / \sqrt{|D|}$ ). Let  $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$ ,  $\alpha_j \notin \sigma_j(K)$ ,  $j = 1, \dots, r + s$ . Then there are infinitely many  $\beta = p/q$ ,  $p, q \in A$  such that*

$$0 < N(\alpha - \sigma(p/q)) < (B / N_{K/\mathbb{Q}}(q))^2.$$

*Proof :* From geometric mean inequality, we deduce from the theorem 1

$$n^n N(\alpha\sigma(q) - \sigma(p)) < n^{2n} B^2 / d(\sigma(q))^n.$$

From geometric mean inequality  $n^n N(\sigma(q)) < d(\sigma(q))^n$ , and then

$$N(\alpha - \sigma(p/q)) < (B / N(\sigma(q)))^2 = (B / N_{K/\mathbb{Q}}(q))^2.$$

From  $\alpha_j \notin \sigma_j(K)$  we deduce  $|\alpha_j \sigma_j(q) - \sigma_j(p)| > 0$ ,  $j = 1, \dots, r + s$ , and then

$$N(\alpha - \sigma(p/q)) > 0, \quad \text{Q.E.D.}$$

**COROLLARY 3 :** Let  $K$  be a number field of degree  $n$ , signature  $(r, s)$  and absolute value of discriminant  $D$ . Let  $A$  be the ring of integers of  $K$ . For  $x \in \mathbf{R}^r \times \mathbf{C}^s$ , let  $d_2(x)$  be the distance function defined by

$$d_2(x) = (|x_1|^2 + \dots + |x_r|^2 + 2|x_{r+1}|^2 + \dots + 2|x_{r+s}|^2)^{1/2}.$$

(i) then, for every  $m \in \mathbf{R}$ ,  $m > 0$  and every  $\alpha \in \mathbf{R}^r \times \mathbf{C}^s - \sigma(K)$ , there exist infinitely many different  $p/q$  with  $p, q \in A$  such that

$$0 < d_2(\alpha\sigma(q) - \sigma(p)) < n\{\Gamma(1 + n/2)(4/(\pi n))^{n/2}\sqrt{D}\}^{2/n}/d_2(\sigma(q))$$

with  $d_2(\sigma(q)) > m$ .

(ii) then, for  $\alpha \in \mathbf{R}^r \times \mathbf{C}^s$ ,  $\alpha_j \notin \sigma_j(K)$ ,  $j = 1, \dots, r + s$ , there exist infinitely many  $\beta = p/q$  where  $p, q \in A$  such that :

$$0 < N(\alpha - \sigma(p/q)) < \{\Gamma(1 + n/2)(4/(\pi n))^{n/2}\sqrt{D}/N_{K/\mathbf{Q}}(q)\}^2.$$

*Proof :* it is exactly of the same nature than the proofs of theorem 1 and corollary 2 with function  $d_2(x)$  instead of function  $d(x)$ .

### Some generalizations

It is possible to study some generalizations of preceding results : we mention some obtained generalizations or problems to solve.

1) In the corollaries 2 and 3, it would be possible to search for a proof that not only  $d(\sigma(q))$ , but also  $N_{K/\mathbf{Q}}(q)$ , can be chosen arbitrary large.

2) A "Roth type" theorem could have one of the formulations :

(i) Let  $\varepsilon \in \mathbf{R}$ ,  $\varepsilon > 0$ , for  $\alpha \in \mathbf{R}^r \times \mathbf{C}^s - \sigma(K)$ ,  $\alpha_j$ ,  $j = 1, \dots, r + s$  algebraic, then there would be only finitely many  $\beta = p/q$ ,  $p, q \in A$  such that  $d(\alpha\sigma(q) - \sigma(p)) < 1/d(\sigma(q))^{1+\varepsilon}$ .

(ii) if the assertion 1) is true (arbitrary large  $N_{K/\mathbf{Q}}(q)$ ), then for  $\varepsilon \in \mathbf{R}_+$ , for  $\alpha \in \mathbf{R}^r \times \mathbf{C}^s$ ,  $\alpha_j \notin \sigma_j(K)$   $j = 1, \dots, r + s$ ,  $\alpha_j$  algebraic  $j = 1, \dots, r + s$ , there would be only finitely many norms  $N_{K/\mathbf{Q}}(q)$  such that

$$0 < N(\alpha - \sigma(p/q)) < 1/N_{K/\mathbf{Q}}(q)^{2+\varepsilon}.$$

Compare to MAHLER [4] result (appendix C) : let  $\alpha \in \mathbf{C}^n$ , let  $\beta \in K$  and  $H_K(\beta)$  the height of  $\beta$  as previously defined.

$$\text{Let } f(\beta) = \prod_{j=1}^n \min(1, |\alpha_j - \sigma_j(\beta)|).$$

Let  $\delta \in \mathbf{R}$ ,  $\delta > 0$ . There are only finitely many  $\beta$  in  $K$  with

$$f(\beta) < H_K(\beta)^{-2-\delta}.$$

3) Let  $\alpha \in \mathbf{R}^r \times \mathbf{C}^s - \sigma(K)$ . It is always possible to find  $q_1 \in A$  such that

$$d(\alpha\sigma(q_1) - \sigma(p_1)) < n^2 B^{2/n} / d(\sigma(q_1))$$

and such that for all  $q' \neq q_1$ ,  $q' \in A$  with  $d(\alpha\sigma(q') - \sigma(p')) < n^2 B^{2/n} / d(\sigma(q'))$  then  $d(\sigma(q')) > d(\sigma(q_1))$  :  $\sigma(A)$  is a lattice, therefore

$$d(\sigma(q_1)) = \min\{d(\sigma(q)) \mid q \in A, \exists p, d(\alpha\sigma(q) - \sigma(p)) < n^2 B^{2/n} / d(\sigma(q))\}$$

exists. It is always possible to find in the same way  $q_2 \in A$  such that

$$\begin{aligned} d(\alpha\sigma(q_2) - \sigma(p_2)) &< d(\alpha\sigma(q_1) - \sigma(p_1)) \quad \text{with} \\ d(\sigma(q_2)) &= \min\{d(\sigma(q')) \mid d(\alpha\sigma(q') - \sigma(p')) < d(\alpha\sigma(q_1) - \sigma(p_1))\}. \end{aligned}$$

It is then possible to consider  $(\sigma(p_1), \sigma(q_1)), \dots, (\sigma(p_i), \sigma(q_i)), \dots$  as a sequence of best approximations of  $\alpha \in \mathbf{R}^r \times \mathbf{C}^s - \sigma(K)$  by elements of  $\sigma(K)$ , generalizing the concept of sequences of best approximations of elements  $\alpha \in \mathbf{R} - \mathbf{Q}$  by elements of  $\mathbf{Q}$ . This concept is studied in [10].

4) It is possible to generalize theorem 1 and corollaries 2 and 3 to simultaneous approximation. For instance, let  $(\alpha^1, \dots, \alpha^\ell) \in (\mathbf{R}^r \times \mathbf{C}^s)^\ell - \sigma(K)^\ell$ . Then, there exist infinitely many  $\ell$ -tuples  $(q_1, \dots, q_\ell) \in A^\ell$  and  $p \in A$  such that

$$0 < d(\alpha^1 \sigma(q_1) + \dots + \alpha^\ell \sigma(q_\ell) - \sigma(p)) < n^{\ell+1} B^{\ell+1} / d(\sigma(q_m))^\ell$$

where  $d(\sigma(q_m)) = \max_{i=1, \dots, \ell} (d(\sigma(q_i)))$ .

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