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RELATIONS BETWEEN NUMERICAL DATA OF AN EMBEDDED RESOLUTION

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INTRODUCTION.

Let k be an algebraically closed field of charasteristic zero and let $f \in k[x,y]$.
Let (X,h) be an embedded resolution of $f=0$ in the affine plane $\mathbb{A}^2$, constructed by successive blowing-ups, and denote by $E_i, i \in T$, the irreducible components of $h^{-1}(f^{-1}\{0\})$.
We associate to each $E_i, i \in T$, a pair of numerical data (N_i,ν_i), where N_i and $\nu_i - 1$ are the multiplicities of E_i in the divisor of respectively $f \circ h$ and $h^*(dx \wedge dy)$ on X.

Fix one exceptional curve E with numerical data (N,ν) and say E intersects k times another irreducible component. Denote these components by $E_1,\ldots,E_k$. Then we have the relation

$$(*) \qquad \sum_{i=1}^{k}(\alpha_i - 1) + 2 = 0,$$

where $\alpha_i = \nu_i - \frac{\nu}{N}N_i$ for $i = 1,\ldots,k$.
When $f(x,y)$ is absolutely analytically irreducible, only $k = 1, 2$ or 3 occurs. The cases $k=1$ and $k=2$ were shown by Strauss [6, Th.1.] and Meuser [5, Lemma 1], and the case $k=3$ by Igusa [3, Lemma 2]. Loeser [4, Lemme II.2] proved the general relation.

Now we can obviously extend the definitions above to higher dimensions. Even if we only consider surfaces there are two essential differences compared with the situation for curves, causing extra difficulties in generalizing the relation $(*)$. In dimension one an exceptional curve E, when created by some blowing-up, is isomorphic to the projective line $\mathbb{P}^1$; and its strict transforms by the following blowing-ups of the resolution remain isomorphic to $\mathbb{P}^1$. Moreover the number of intersection points with other $E_i, i \in T$, remains the same

during the (canonical) resolution process.
In dimension two an exceptional surface E is created as the projective plane $\mathbb{P}^2$ or as some ruled surface. But its strict transform $\tilde{E}$ by the next blowing-up of the resolution can be either isomorphic to E or to E with some points blown-up. And moreover, in the latter case, there are more intersections of other $E_i, i \in T$, with $\tilde{E}$ than with E.

Our result is essentially the following. Let E be a fixed exceptional variety. There are *basic relations* (B1 and B2) associated to the creation of E in the resolution process, generalizing the relation $(*)$. And there are *additional relations* (A) associated to each blowing-up of the resolution that "changes" E.

§1. EMBEDDED RESOLUTION.

Let k be an algebraically closed field of characteristic zero and let $f \in k[x_1, \ldots, x_{n+1}]$ be a polynomial over k.
Let Y denote the zero set of f in affine $(n+1)$-space $\mathbb{A}^{n+1}$ over k and $Y_\ell, \ell \in I$, its reduced irreducible components. We exclude the trivial case $f \in k$, so Y is a subscheme of codimension one of $\mathbb{A}^{n+1}$.
We fix an embedded resolution (X, h) for Y in $\mathbb{A}^{n+1}$ in the sense of Hironaka's Main Theorem II [2, p.142] by means of *monoidal transformations* or *blowing-ups*. It consists of the following data.

Set $X_0 = \mathbb{A}^{n+1}, \quad Y^{(0)} = Y, \quad$ *and* $\quad Y_\ell^{(0)} = Y_\ell$ for all $\ell \in I$.
For $i = 0, \ldots, r-1$ we have a finite succession of monoidal transformations $g_i : X_{i+1} \longrightarrow X_i$ with irreducible nonsingular center $D_i \subset X_i$ and exceptional variety $E_{i+1}^{(i+1)} \subset X_{i+1}$ subject to the following conditions.
Let $E_j^{(i+1)}$, $Y^{(i+1)}$ and $Y_\ell^{(i+1)}$ denote the strict transform of respectively $E_j^{(i)}$, $Y^{(i)}$ and $Y_\ell^{(i)}$ in X_{i+1} by g_i for $j = 1, \ldots, i$ and all $\ell \in I$. Then

(1) for $i = 0, \ldots, r-1$ we have $D_i \subset Y^{(i)}, codim(D_i, X_i) \geqslant 2$, and the multiplicity on $Y^{(i)}$ of all $x \in D_i$ equals the maximal multiplicity on $Y^{(i)}$;

(2) $\bigcup\limits_{1 \leqslant j \leqslant i} E_j^{(i)}$ has only normal crossings and only normal crossings with D_i (in X_i) for $i = 1, \ldots, r-1$; and

(3) $(\bigcup\limits_{1 \leqslant j \leqslant r} E_j^{(r)}) \bigcup (\bigcup\limits_{\ell \in I} Y_\ell^{(r)}) = [(g_{r-1} \circ \cdots \circ g_0)^{-1}(Y)]_{red}$ has only normal crossings in X_r . In particular all $Y_\ell^{(r)}, \ell \in I$, are nonsingular.

Now we set $X = X_r$ and $h = g_{r-1} \circ \cdots \circ g_0$.
The *numerical data of the resolution* (X, h) *for* Y are defined as follows.

For all irreducible components E of $(h^{-1}Y)_{red}$ (i.e. for all $E_j^{(r)}, 1 \leqslant j \leqslant r$, and all $Y_\ell^{(r)}$), let N be the multiplicity of E in the divisor of $f \circ h$ on X, and let $\nu - 1$ be the multiplicity of E in the divisor of $h^*(dx_1 \wedge \cdots \wedge dx_{n+1})$ on X. We have $N, \nu \in \mathbb{N}_0$; and if Y is reduced, then all $Y_\ell^{(r)}$ have numerical data $(N, \nu) = (1, 1)$.

§2. CHANGES ON AN EXCEPTIONAL VARIETY DURING THE RESOLUTION PROCESS.

From now on we fix one $j \in \{1, \ldots, r\}$ and drop the j-indices, i.e. we set $E^{(i)} = E_j^{(i)}$ for all $i = j, \ldots, r$ and $(N, \nu) = (N_j, \nu_j)$.

We describe how the exceptional variety E_j and its intersections with other exceptional varieties and with the strict transform of Y change by the blowing-ups g_i, $j \leqslant i < r$. So we fix one such $g_i : X_{i+1} \longrightarrow X_i$ and set during this section $g = g_i$ and $D = D_i$.

Since $E^{(i)}$ has normal crossings with D we have the following important fact (see e.g. [1,p.605]).

(1) The restriction $g' : E^{(i+1)} \longrightarrow E^{(i)}$ of g to $E^{(i+1)}$ is the blowing-up of $E^{(i)}$ with (nonsingular) center $D \cap E^{(i)}$.

Note that $D \cap E^{(i)}$ can eventually be reducible. The total blow-up of $E^{(i)}$ with center $D \cap E^{(i)}$ can then be considered as the result of consecutive blowing-ups of $E^{(i)}$ with centers the irreducible components of $D \cap E^{(i)}$.

Let E^* denote the exceptional divisor of the blowing-up g' and $\bar{Z}$ the strict transform in $E^{(i+1)}$ of any subscheme Z of $E^{(i)}$ by g'. Then

$$E^* = E_{i+1}^{(i+1)} \cap E^{(i+1)}, \tag{2}$$

and if $\operatorname{codim}(D \cap E^{(i)}, E^{(i)}) \geqslant 2$, we have

$$\begin{aligned} \overline{E_k^{(i)} \cap E^{(i)}} &= E_k^{(i+1)} \cap E^{(i+1)} \qquad \text{and} \\ \overline{(Y_k^{(i)} \cap E^{(i)})_{red}} &= (Y_k^{(i+1)} \cap E^{(i+1)})_{red}. \end{aligned} \tag{3}$$

The remaining situation $\operatorname{codim}(D \cap E^{(i)}, E^{(i)}) = 1$ occurs if and only if $D \subset E^{(i)}$ and $dim\, D = n - 1$. In this case we have that g' is an isomorphism making E^* correspond to D.

When D is not contained in respectively $(Y_k^{(i)} \cap E^{(i)})_{red}$ and $E_k^{(i)} \cap E^{(i)}$, the statement (3) above is still valid by the same argument.
Now if some irreducible component of $(Y_k^{(i)} \cap E^{(i)})_{red}$ is equal to D, then we can have *in a small enough neighbourhood of* E^* either

$$(4) \qquad Y_k^{(i+1)} \cap E^{(i+1)} = \emptyset \quad \text{or} \quad (Y_k^{(i+1)} \cap E^{(i+1)})_{red} = E^*.$$

If some irreducible component of $E_k^{(i)} \cap E^{(i)}$ is equal to D, then we have *in a small enough neighbourhood of* E^* always

$$(5) \qquad E_k^{(i+1)} \cap E^{(i+1)} = \emptyset.$$

§3. RELATIONS ASSOCIATED TO THE BLOWING-UPS OF AN EXCEPTIONAL VARIETY.

Fix again one blowing-up $g_i|_{E^{(i+1)}} : E^{(i+1)} \longrightarrow E^{(i)}$ with $D_i \cap E^{(i)} \neq \phi$ and $codim(D_i \cap E^{(i)}, E^{(i)}) \geqslant 2$, and one irreducible component D of $D_i \cap E^{(i)}$. We will associate a relation between numerical data to the blowing-up g of $E_{(i)}$ with center D, which can be considered as a composition factor of $g_i|_{E^{(i+1)}}$. (Here we suppose g to be the first blowing-up in the decomposition of $g_i|_{E^{(i+1)}}$ into such factors.)

Let $E'_k, k \in T$, be the reduced irreducible components of intersections of $E^{(i)}$ with other exceptional varieties $E_t^{(i)}, 1 \leqslant t < i$, or with components $Y_\ell^{(i)}, \ell \in I$, of the strict transform $Y^{(i)}$ of Y. According to the statements (2) - (5) of §4, the repeated strict transform of E'_k in $E^{(r)}$ by the consecutive $g_\ell|_{E^{(\ell+1)}} : E^{(\ell+1)} \longrightarrow E^{(\ell)}, i \leqslant \ell < r$, is equal to some irreducible component of the intersection of $E^{(r)}$ with another component of $(h^{-1}Y)_{red}$, say with $E_k^{(r)}$ or $Y_k^{(r)}$.
— Furthermore $E_k^{(r)}$ is different from the corresponding $E_t^{(r)}$ and/or $Y_\ell^{(r)}$ if and only if the center of some $g_\ell|_{E^{(\ell+1)}} : E^{(\ell+1)} \longrightarrow E^{(\ell)}, i \leqslant \ell < r$, contains the repeated strict transform of E'_k in $E^{(\ell)}$! —

Let E'_e denote the exceptional variety of the blowing-up g. Also the repeated strict transform of E'_e in $E^{(r)}$ by the other factors of $g_i|_{E^{(i+1)}}$ and the consecutive $g_\ell|_{E^{(\ell+1)}} : E^{(\ell+1)} \longrightarrow E^{(\ell)}, i+1 \leqslant \ell < r$, is an irreducible component of the intersection of $E^{(r)}$ with some other exceptional variety, say with $E_e^{(r)}$.
— Again we have that $E_e^{(r)}$ is different from $E_{i+1}^{(r)}$ if and only if the center of

some $g_\ell|_{E^{(\ell+1)}}$, $i+1 \leqslant \ell < r$, *contains the repeated strict transform of* E'_e *in* $E^{(\ell)}$ *!* —

We have the following relation between the numerical data of $E^{(r)}, E_e^{(r)}$ *and* $E_k^{(r)}$ *or* $Y_k^{(r)}, k \in T$.
Set $\alpha_e = \nu_e - \frac{\nu}{N}N_e$ *and* $\alpha_k = \nu_k - \frac{\nu}{N}N_k$ *for* $k \in T$. *Then*

RELATION A.

$$\alpha_e = \sum_{k\in T} \mu_k(\alpha_k - 1) + d,$$

where $d = codim(D, E^{(i)}) \geqslant 2$ *and* $\mu_k, k \in T$, *is the multiplicity of the generic point of* D *on* E'_k.

§4. RELATIONS ASSOCIATED TO THE CREATION OF AN EXCEPTIONAL VARIETY.

Set from now on $E = E^{(j)}$, $\quad D = D_{j-1}$, $\quad \Pi = g_{j-1}|_E : E \longrightarrow D$ and $k = codim(D, X_{j-1})$.

Let $E'_i, i \in T$, *be the irreducible components of intersections of* E *with other exceptional varieties or with the strict transform of* Y. *The strict transform of* E'_i *in* $E^{(r)}$ *by the consecutive* $g_\ell|_{E^{(\ell+1)}} : E^{(\ell+1)} \longrightarrow E^{(\ell)}, j \leqslant \ell < r$, *is equal to some irreducible component of the intersection of* $E^{(r)}$ *with another irreducible component of* $(h^{-1}Y)_{red}$, *say with* $E_i^{(r)}$ *or* $Y_i^{(r)}$, *having numerical data* (N_i, ν_i). *As usual* $\alpha_i = \nu_i - \frac{\nu}{N}N_i$ *for* $i \in T$, *where* (N, ν) *are the numerical data of* E.

RELATION B1. *We have*

$$\sum_{i\in T} d_i(\alpha_i - 1) + k = 0,$$

where $d_i, i \in T$, *is the degree of the intersection cycle* $E'_i \cdot F$ *on* F *for a general fibre* $F \cong \mathbb{P}^{k-1}$ *of* $\Pi : E \longrightarrow D$ *over a point of* D.

When $Pic\, D$ is not trivial we have also

RELATION B2. *Let $d_i, i \in T$, be the degree of the intersection cycle $E'_i \cdot F$ on F for a general fibre $F \cong \mathbb{P}^{k-1}$ of Π over a point of D. When $d_i = 0$, let $E'_i = \Pi^* B_i$ with $B_i \in Pic\, D$. Then we have*

$$\sum_{\substack{i \in T \\ d_i \neq 0}} \frac{1}{k d_i^{k-1}} (\alpha_i - 1) \Pi_*(E_i'^k) + \sum_{\substack{i \in T \\ d_i = 0}} (\alpha_i - 1) B_i = K_D$$

in $Pic\, D$, where K_D is the canonical divisor on D.

Remark. Relation $B2$ should not be seen as an expression in $Pic\, D \otimes \mathbb{Q}$ buth just as a more elegant notation for the expression with integer coefficients, obtained by reducing to the same denominator.

Example 1.

When Y is a curve ($n = 1$), only blowing-ups with a point as center occur. We have $E \cong \mathbb{P}^1$ and, since all E'_i are points on $E, d_i = 1$ for $i \in T$. So we obtain the familiar relation

$$\sum_{i \in T} (\alpha_i - 1) + 2 = 0.$$

Example 2.

When Y is a surface ($n = 2$), we only need blowing-ups with a point or a nonsingular curve as center. If D is a point, then $E \cong \mathbb{P}^2$ and relation B1 is

$$\sum_{i \in T} d_i (\alpha_i - 1) + 3 = 0,$$

where $d_i, i \in T$, is the degree of the curve E'_i in E.
If D is a nonsingular curve, then E is a projective space bundle over D with fibres isomorphic to $\mathbb{P}^1$. Relation B1 is in this case

$$\sum_{i \in T} d_i (\alpha_i - 1) + 2 = 0,$$

where $d_i, i \in T$, is the number of intersections of the curve E'_i with a "general" fibre F of Π.

If moreover D is projective (when Y has no other than isolated singularities only such curves occur as center of blowing-ups), then relation B2 becomes a numerical relation by taking degrees in $Pic\, D$.

Let g denote the genus of D and $\kappa_i = \deg E_i'^2, i \in T$, the *self-intersection number of* E_i' in E. Then we get

$$\sum_{\substack{i \in T \\ d_i \neq 0}} \frac{\kappa_i}{2d_i}(\alpha_i - 1) + \sum_{\substack{i \in T \\ d_i = 0}} (\alpha_i - 1) = 2g - 2.$$

(When $E_i' = \Pi^* B_i$ we must have $\deg B_i = 1$ since E_i' is irreducible.)

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