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AN EXTENSION OF A THEOREM BY CHEEGER AND MÜLLER

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This volume is dedicated to Jeff Cheeger and Werner Müller

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Introduction

Let M be a compact manifold of dimension n. Let F be a flat vector bundle on M. Let $H^\bullet(M,F)=\bigoplus_{i=0}^n H^i(M,F)$ be the cohomology of the sheaf of locally flat sections of F.

If E is a finite dimensional vector space, set $\det E=\Lambda^{\max}(E)$. Following an established tradition in algebraic geometry, we define the determinant of the cohomology of F to be the real line $\det H^\bullet(M,F)$ given by

$$\det H^\bullet(M,F)=\bigotimes_{i=0}^{n}\left(\det H^i(M,F)\right)^{(-1)^i}. \tag{0.1}$$

Let g^F be a metric on the flat vector bundle F. Assume temporarily that g^F is flat, so that F can be obtained through a representation of $\pi_1(M)$ into $O(\dim F)$. If $H^\bullet(M,F)=\{0\}$, Franz [F], Reidemeister [Re] and de Rham [Rh1] have shown how to associate to (F,g^F) a positive number, the torsion of F.

In fact let F^* be the dual of F. Let K be a smooth triangulation of M. Then the cohomology of the simplicial complex $(C_\bullet(K,F^*),\partial)$ is canonically isomorphic to $H^\bullet(M,F)$. It is then a standard fact that there is a canonical isomorphism of real lines

$$\det H^\bullet(M,F)\simeq\left(\det C_\bullet\,(K,F^*)\right)^{-1}. \tag{0.2}$$

Let B be the set of barycenters of the simplexes $\sigma\in K$. For $x\in B$, let g^{F_x} be a metric on F_x. Then $C_\bullet(K,F^*)$ is a $\mathbb{Z}$-graded Euclidean vector space. We define the Reidemeister metric $\|\ \|^{R,K}_{\det H^\bullet(M,F)}$ to be the metric on the line $\det H^\bullet(M,F)$ corresponding to the obvious metric on $(\det C_\bullet(K,F^*))^{-1}$ via the canonical isomorphism (0.2). The metric $\|\ \|^{R,K}_{\det H^\bullet(M,F)}$ depends on K, B, and on the g^{F_x} 's $(x \in B)$. If $H^{\bullet}(M,F) = \{0\}$, then $\det H^{\bullet}(M,F) \simeq \mathbb{R}$, and the metric $\| \ \|^{R,K}_{\det H^{\bullet}(M,F)}$ on the trivial line $\det H^{\bullet}(M,F)$ is now defined by a positive number, which is the norm of the canonical section $\mathbb{1} \in \mathbb{R}$. This number is called the torsion of the complex $(C_{\bullet}(K,F^*),\partial)$.

Let g^F be a flat metric on F, and assume that the $g^{F_x}(x \in B)$ are obtained by restricting g^F to B. Then if $H^{\bullet}(M,F) = \{0\}$, it is a basic result of Franz, Reidemeister and de Rham that the torsion does not depend on B or on K. It is a topological invariant of the flat Euclidean vector bundle F. More generally, even if $H^{\bullet}(M,F)$ is not reduced to 0, one can show that the metrics $\| \ \|^{R,K}_{\det H^{\bullet}(M,F)}$ do not depend on B or on K. The metric $\| \ \|^{R,K}_{\det H^{\bullet}(M,F)}$ on $\det H^{\bullet}(M,F)$ is then a topological invariant of F, which we denote by $\| \ \|^{R}_{\det H^{\bullet}(M,F)}$.

Suppose that the metric $\| \ \|_{\det F}$ induced by g^F on the line $\det F$ is flat. Assume that the metrics $g^{F_x}(x \in B)$ are still obtained by restricting g^F to F_x $(x \in B)$. Then in [Mü2], Müller has shown that the Reidemeister metric $\| \ \|^{R,K}_{\det H^{\bullet}(M,F)}$ is also a topological invariant, which we still denote $\| \ \|^{R}_{\det H^{\bullet}(M,F)}$.

Let now g^{TM} and g^F be smooth metrics on TM and F. Let $(\mathbb{F},d^F)$ be the de Rham complex of smooth sections of $\Lambda(T^*M)\otimes F$ over M. Then the de Rham theorem asserts that

$$H^{\bullet}\left(\mathbb{F},d^F\right) \simeq H^{\bullet}(M,F). \tag{0.3}$$

By Hodge theory, the harmonic forms in $(\mathbb{F},d^F)$ with respect to the metrics g^{TM} and g^F represent canonically the cohomology of $(\mathbb{F},d^F)$.

In [RS1], Ray and Singer constructed the logarithm of the analytic torsion of $(\mathbb{F},d^F)$, as a combination of derivatives at 0 of the zeta functions of the Laplacian acting on forms in $\mathbb{F}$ of various degrees. By following a well-known recipe indicated by Quillen [Q2] for Dolbeault complexes, to g^{TM} and g^F, we can associate a metric on the line $\det H^{\bullet}(M,F)$, which is the product of the standard L_2 metric on $\det H^{\bullet}(M,F)$ (obtained by identifying $H^{\bullet}(M,F)$ with the harmonic elements of $(\mathbb{F},d^F)$), by the Ray-Singer analytic torsion of [RS1]. This metric is called the Ray-Singer metric on $\det H^{\bullet}(M,F)$, and is denoted $\| \ \|^{RS}_{\det H^{\bullet}(M,F)}$. Ray and Singer showed that if $\dim M$ is odd, then $\| \ \|^{RS}_{\det H^{\bullet}(M,F)}$ does not depend on g^{TM} and g^F, i.e. it is a topological invariant of F.

Assume that g^F is a flat metric on F. Then the real line $\det H^\bullet(M,F)$ can be equipped with two natural invariant metrics, the Reidemeister metric $\| \ \|^R_{\det H^\bullet(M,F)}$, and the Ray-Singer metric $\| \ \|^{RS}_{\det H^\bullet(M,F)}$. Ray and Singer [RS1] made the conjecture that in this case,

$$\| \ \|^R_{\det H^\bullet(M,F)} = \| \ \|^{RS}_{\det H^\bullet(M,F)} . \tag{0.4}$$

They based this conjecture on previous computations by Ray [R] of the torsion of lens spaces. In celebrated independent papers, Cheeger [C] and Müller [Mü] proved that this is indeed the case. The proofs of Cheeger and Müller are very interesting in themselves and are based on entirely different principles.

In [C], Cheeger proves that under surgery, the Ray-Singer metric behaves in the same way as the Reidemeister metric. Then he shows how to pass from $M \times S^6$ to $M \times S^3 \times S^3$ by a sequence of surgeries. Using trivial identities for Reidemeister and Ray-Singer metrics on product spaces, Cheeger [C] finally obtains (0.4).

In [Mü1], by using the invariance of the Reidemeister metrics under subdivision of a triangulation and combinatorial parametrices, Müller shows first that the ratio of the Ray-Singer metric to the Reidemeister metric does not depend on the orthogonally flat bundle F. Then Müller [Mü1] uses surgery to reduce the proof of (0.4) to the case of the trivial bundle on the sphere, for which the result was already known.

Assume now that M is odd dimensional, and that only the metric $\| \cdot \|_{\det F}$ induced by g^F on $\det F$ is flat. Then the metrics $\| \ \|^R_{\det H^\bullet(M,F)}$ and $\| \ \|^{RS}_{\det H^\bullet(M,F)}$ are still topological invariants. By using the methods of Cheeger [C], Müller [Mü2] has shown that equality (0.4) still holds.

The purpose of this paper is to extend the results of Cheeger [C] and Müller [Mü1,2] to the general case, where the metric $\| \ \|_{\det F}$ on $\det F$ is not necessarily flat.

As an important intermediary step, we prove first anomaly formulas for the Ray-Singer metrics $\| \ \|^{RS}_{\det H^\bullet(M,F)}$. In fact, let (g^{TM}, g^F) and (g'^{TM}, g'^F) be two couples Euclidean metrics on (TM, F). Let $\| \ \|_{\det F}$ and $\| \ \|'_{\det F}$ be the associated metrics on the line bundle $\det F$. Let ∇^{TM} and ∇'^{TM} be the corresponding Levi-Civita connections on TM, and let $e(TM, \nabla^{TM})$ and $e(TM, \nabla'^{TM})$ be the associated representatives of the Euler class of TM in Chern-

Weil theory. Let $\widetilde{e}(TM, \nabla^{TM}, \nabla'^{TM})$ be the class of Chern-Simons $n-1$ forms on TM such that

$$(0.5) \qquad d\widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right) = e\left(TM, \nabla'^{TM}\right) - e\left(TM, \nabla^{TM}\right).$$

Let $\theta(F, g'^F)$ be the closed 1-form, defined in Definition 4.5, which measures the variation of the metric $\| \ \|'_{\det F}$ on $\det F$ with respect to the obvious flat connection on $\det F$. The cohomology class of $\theta(F, g'^F)$ does not depend on g'^F, and $\theta(F, g'^F)$ vanishes if and only if the metric $\| \ \|'_{\det F}$ is flat.

Let $\| \ \|^{RS}_{\det H^\bullet(M,F)}$ and $\| \ \|'^{RS}_{\det H^\bullet(M,F)}$ be the Ray-Singer metrics on $\det H^\bullet(M, F)$ associated to the metrics (g^{TM}, g^F) and (g'^{TM}, g'^F).

A first result which is proved in this paper is as follows.

Theorem 0.1. *The following identity holds,*

$$(0.6) \quad \operatorname{Log}\left(\frac{\| \ \|'^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{RS}_{\det H^\bullet(M,F)}}\right)^2 = \int_M \operatorname{Log}\left(\frac{\| \ \|'_{\det F}}{\| \ \|_{\det F}}\right)^2 e\left(TM, \nabla^{TM}\right) - \int_M \theta\left(F, g'^F\right) \widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right).$$

Of course if $\dim M$ is odd, the right-hand side of (0.6) is zero.

Let $f : M \to \mathbb{R}$ be a Morse function. Let X be the gradient vector field of f with respect to a given metric on M. Let B be the finite set of zeroes of X. If $x \in B$, let $W^s(x)$ and $W^u(x)$ be the stable and unstable cells of $-X$ at x. We assume that X verifies the Smale transversality conditions [Sm1, 2]. The Thom-Smale complex $(C_\bullet(W^u, F^*), \partial)$ is a finite dimensional complex whose homology is canonically isomorphic to $H_\bullet(M, F^*)$. As in (0.2), we still have

$$(0.7) \qquad \det H^\bullet(M, F) \simeq \left(\det C_\bullet\left(W^u, F^*\right)\right)^{-1}.$$

Let g^F be a smooth metric on F. As above, the metrics $g^{F_x} (x \in B)$ determine a metric on $\det H^\bullet(M, F)$ via the canonical isomorphism (0.7) which we call the Milnor metric, and which we denote by $\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}$.

By Milnor [Mi1, Theorem 9.3], if g^F is a flat metric on F, and if the metrics $g^{F_x} (x \in B)$ are the restriction of g^F to $F_x (x \in B)$, then the Milnor metric $\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}$ coincides with the Reidemeister metric associated to g^F.

Let now g^{TM} and g^F be smooth metrics on TM and F. Let X be a gradient vector field verifying the Smale transversality conditions. Let B the set of zeroes of X. The metric g^F induces metrics g^{F_x} on the F_x 's $(x \in B)$. Let $\| \quad \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}$ be the corresponding Milnor metric on $\det H^\bullet(M,F)$. Let $\| \quad \|^{RS}_{\det H^\bullet(M,F)}$ be the Ray-Singer metric attached to the metrics g^{TM}, g^F on TM, F.

Let $\psi(TM, \nabla^{TM})$ be the $n-1$ current on TM which is constructed in [MQ] and in [BGS4, Section 3], whose restriction to $TM \backslash \{0\}$ is induced by a smooth form on the sphere bundle which transgresses the form $e(TM, \nabla^{TM})$.

The main purpose of this paper is to prove the following extension of the Cheeger-Müller theorem.

Theorem 0.2. *The following identity holds,*

$$\text{(0.8)} \qquad \operatorname{Log}\left(\frac{\| \quad \|^{RS}_{\det H^\bullet(M,F)}}{\| \quad \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2 = -\int_M \theta\left(F, g^F\right) X^* \psi\left(TM, \nabla^{TM}\right).$$

The arch-typical application of Theorem 0.2 is the case where $M = S_1 \simeq \mathbb{R}/\mathbb{Z}$ and where F is the trivial vector bundle $\mathbb{R}$, such that for a given $\alpha \in \mathbb{R}^*$, the flat parallel transport operator τ on F from 0 to $t \in [0,1[$ is given by $e^{t\alpha}$. In this case $H^\bullet(M,F) = \{0\}$ and so $\det H^\bullet(M,F)$ has a canonical section $\mathbb{1}$.

A simple calculation shows that

$$\text{(0.9)} \qquad \operatorname{Log}\left(\|\mathbb{1}\|^{RS}_{\det H^\bullet(M,F)}\right)^2 = -\operatorname{Log}\left|2\sinh\left(\frac{\alpha}{2}\right)\right|^2.$$

Let g^F be the constant metric on $F \simeq \mathbb{R}$. Let $f : M \to \mathbb{R}$ be a Morse function, having only two critical points, a maximum at 0, and a minimum at $\beta \in]0,1[$. Let $\| \quad \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}$ denote the corresponding Milnor metric on $\det H^\bullet(M,F)$. Then one verifies easily that

$$\text{(0.10)} \qquad \operatorname{Log}\left(\|\mathbb{1}\|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}\right)^2 = -\operatorname{Log}\left|2\sinh\left(\frac{\alpha}{2}\right)\right|^2 + \alpha(2\beta - 1).$$

On the other hand, $(\nabla f)^* \psi(TM, \nabla^{TM})$ is a section of $o(TM)$. In fact on $M \backslash \{0, \beta\}, -2\psi(TM, \nabla^{TM})$ defines the orientation given by ∇f. Moreover

$\theta(F, g^F) = 2\alpha dt$. So we find that

(0.11)
$$-\int_M \theta\left(F, g^F\right) (\nabla f)^* \psi\left(TM, \nabla^{TM}\right) = -\int_0^\beta \alpha dt + \int_\beta^1 \alpha dt = -\alpha(2\beta - 1).$$

So (0.9)-(0.11) fit with (0.8).

Although Theorem 0.1 can be obtained as a consequence of Theorem 0.2, establishing first Theorem 0.1 is essential in our proof of Theorem 0.2.

Let

$$(F^\bullet, v) : 0 \to F^0 \underset{v}{\to} F^1 \to \cdots \underset{v}{\to} F^m \to 0. \tag{0.12}$$

be a flat exact sequence of flat vector bundles on M. Let σ be the canonical nonzero section of the flat line bundle $\det F^\bullet = \bigotimes_{j=0}^m (\det F^j)^{(-1)^j}$ defined in [KMu], [BGS1].

By [KMu], to the exact sequence (0.12), one can associate a canonical nonzero section τ of the line $\det H^\bullet(M, F^\bullet) = \bigotimes_{j=0}^m (\det H^\bullet(M, F^j))^{(-1)^j}$.

Let $g^{F^0}, \cdots, g^{F^m}$ be Euclidean metrics on $F^0, \cdots, F^m$. Let $\|\ \|_{\det F^\bullet}$ be the corresponding metric on $\det F^\bullet$. Let g^{TM} be an Euclidean metric on TM. Let $\|\ \|^{RS}_{\det H^\bullet(M,F^0)}, \cdots, \|\ \|^{RS}_{\det H^\bullet(M,F^m)}$ denote the associated Ray-Singer metrics on $\det H^\bullet(M, F^0), \cdots, \det H^\bullet(M, F^m)$, and let $\|\ \|^{RS}_{\det H^\bullet(M,F^\bullet)}$ be the corresponding metric on the line $\det H^\bullet(M, F^\bullet)$.

As an easy consequence of Theorem 0.2, we also obtain the following result.

Theorem 0.3. *The following identity holds,*

$$\operatorname{Log}\left(\|\tau\|^{RS,2}_{\det H^\bullet(M,F^\bullet)}\right) = \int_M \operatorname{Log}\left(\|\sigma\|^2_{\det F^\bullet}\right) e(TM, \nabla^{TM}). \tag{0.13}$$

Now, we will briefly describe the general strategy of our proofs of Theorems 0.1 and 0.2, and also the techniques which we use in this paper.

1. Ray-Singer metrics and Quillen metrics

In [BL1, 2], Bismut and Lebeau have considered a problem which is formally related to the problem which we solve here. In fact let $i : Y \to X$ be an embedding of complex manifolds. Let η be a holomorphic vector bundle which resolves the sheaf $i_* \mathcal{O}_Y(\eta)$. Let $\lambda(\xi)$ and $\lambda(\eta)$ be the inverses of the determinants of the Dolbeault cohomology of η and ξ. Then by [KMu], the lines $\lambda(\xi)$ and $\lambda(\eta)$ are canonically isomorphic. If metrics are introduced on TX, TY, ξ, η, let $\| \ \|_{\lambda(\xi)}$ and $\| \ \|_{\lambda(\eta)}$ be the corresponding Quillen metrics on the lines $\lambda(\xi)$ and $\lambda(\eta)$ [Q2], [BGS3]. In [BL1,2], an explicit formula was obtained for $\mathrm{Log}(\frac{\| \ \|_{\lambda(\xi)}}{\| \ \|_{\lambda(\eta)}})^2$ in terms of integrals of certain locally computable currents. One of the ideas of the proof of the main result of [BL2] is to deform the Hodge theory of (X, ξ) to the Hodge theory of (Y, η) by scaling the considered metrics on ξ.

Here, at a formal level, X is replaced by M, Y by B, and the current appearing in (0.7) replaces the currents of [BL2]. This essential analogy will be further explained.

For a detailed review of various results concerning Quillen metrics and complex immersions, we refer to the survey [B3].

2. A fundamental closed form

Let g^{TM}, g^F be smooth metrics on TM, F. Let $f : M \to \mathbb{R}$ be a smooth function. For $T \geq 0$, let g_T^F be the metric on F, $g_T^F = e^{-2Tf} g^F$. Let d_T^{F*} be the adjoint of the de Rham operator d^F with respect to the L_2 scalar product associated to the metrics g^{TM}, g_T^F. Set $D_T = d^F + d_T^{F*}$. Let N be the number operator defining the $\mathbb{Z}$-grading of F.

Let $\alpha_{t,T}$ be the 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+$,

$$\alpha_{t,T} = \frac{dt}{2t} \operatorname{Tr_s} \left[N \exp\left(-tD_T^2\right) \right] - dT \operatorname{Tr_s} \left[f \exp\left(-tD_T^2\right) \right] \ . \tag{0.14}$$

In (0.14), $\operatorname{Tr_s}$ is our notation for supertrace. Then we prove in Theorem 5.6 that the form $\alpha_{t,T}$ is closed. If Γ is a closed rectangle in $\mathbb{R}_+^* \times \mathbb{R}_+$, we obtain in Theorem 5.8 the basic identity

$$\int_\Gamma \alpha = 0 \ . \tag{0.15}$$

Theorem 0.2 will be ultimately obtained by taking f to be a Morse function such that the gradient field ∇f associated to the metric g^{TM} verifies the Smale transversality conditions, and by deforming the contour Γ to the boundary of $\mathbb{R}_+^* \times \mathbb{R}_+$. In this process, the contribution of each side of the rectangle diverges. Once divergences are substracted off, we will obtain an identity which is equivalent to Theorem 0.2.

3. The Witten complex and the Helffer-Sjöstrand calculus

Observe that

$$D_T = e^{Tf}\left(e^{-Tf}d^F e^{Tf} + e^{Tf}d^{F*}e^{-Tf}\right)e^{-Tf}. \tag{0.16}$$

When $F = \mathbb{R}$, the operator $e^{-Tf}d^F e^{Tf}$ is exactly the twisted de Rham operator introduced by Witten [W], in his proof of the Morse inequalities.

Set $\widetilde{D}_T = e^{-Tf}D_T^F e^{Tf}$. Let $\widetilde{\mathbb{F}}_T^{[0,1]}$ be the direct sum of the eigenspaces of the operator $\widetilde{D}_T^2$, corresponding to eigenvalues $\lambda \in [0,1]$. Then $(\widetilde{\mathbb{F}}_T^{[0,1]}, e^{-Tf}d^F e^{Tf})$ is a complex, whose cohomology is canonically isomorphic to $H^\bullet(M,F)$. In [W], Witten suggested that as $T \to +\infty$, this complex is "asymptotic" to the Thom-Smale complex associated to the vector field $-\nabla f$.

In [HSj4], when $F = \mathbb{R}$ and when ∇f verifies the Smale transversality conditions, Helffer and Sjöstrand established the precise asymptotics as $T \to +\infty$ of the complex $(\widetilde{\mathbb{F}}_T^{[0,1]}, e^{-Tf}d^F e^{Tf})$, in order to give an analytic proof of the fact that the Betti numbers of the Thom-Smale complex are the same as the Betti numbers of the de Rham complex. To calculate the asymptotics of the complex $(\widetilde{\mathbb{F}}_T^{[0,1]}, e^{-Tf}d^F e^{Tf})$, Helffer and Sjöstrand used their fundamental results [HSj1,2,3] on the semi-classical analysis of Schrödinger operators with multiple wells, to calculate the tunelling effects between these potential wells. An essential consequence of [HSj1,2,3] is in fact that the eigenvectors of such Schrödinger operators associated to small eigenvalues are approximated by the WKB solutions of certain transport equations on adequate regions of M. When $F = \mathbb{R}$, Helffer and Sjöstrand [HSj4] used in fact the results of [HSj1,2,3] to approximate the eigenvectors of the operator $\widetilde{D}_T^2$ associated to eigenvalues $\lambda \in [0,1]$, by solutions of WKB transport equations, which are themselves closely related to the Thom-Smale complex of $-\nabla f$.

Let $\mathbb{F}_T^{[0,1]}$ be the direct sum of the eigenvectors of D_T^2 corresponding to eigenvalues $\lambda \in [0,1]$. Then $(\mathbb{F}_T^{[0,1]}, d^F)$ is a complex, whose cohomology is canonically isomorphic to $H^\bullet(M,F)$. Now $\mathbb{F}_T^{[0,1]}$ is naturally equipped with the L_2 metric associated to the metrics g^{TM}, g_T^F. Let $\| \ \|^{\sim}_{\det H^\bullet(M,F),T}$ be the corresponding metric on $\det H^\bullet(M,F)$. In our proof of Theorem 0.2, a crucial role is played by Theorem 7.6, where we calculate the asymptotics of the metric $\| \ \|^{\sim}_{\det H^\bullet(M,F),T}$ as $T \to +\infty$ in terms of the Milnor metric on $\det H^\bullet(M,F)$. Roughly speaking, to calculate this asymptotics, we need informations on :

— the eigenspaces of D_T^2 associated to eigenvalues $\lambda \in]0,1]$.

— the kernel of D_T^2, i.e. the harmonic forms in $\mathbb{F}$ associated to the metrics g^{TM} and g_T^F.

When $F = \mathbb{R}$, what is needed concerning the nonzero eigenspaces of D_T^2 is essentially contained in the asymptotic description by Helffer-Sjöstrand [HSj4, Proposition 3.3] of the complex $(\widetilde{\mathbb{F}}_T^{[0,1]}, e^{-Tf} d^F e^{Tf})$. Here instead F is a vector bundle, and moreover the metric g^F is in general not flat, so that the operator $\widetilde{D}_T^2$ contains extra terms with respect to the corresponding operator considered in [HSj4]. Still, the results of [HSj1,2,3] and the techniques of [HSj4] can be adequately adapted to treat the more complicate problem which is considered here. Nevertheless, we have been forced to devote the whole Section 8 to summarize some of the essential results of Helffer-Sjöstrand [HSj1, 2,3], and to adapt the techniques of [HSj4] to our problem. Unsurprisingly, one important result of Section 8 is contained in Theorem 8.30, where we show that still in this case, as $T \to +\infty$, the complex $(\widetilde{\mathbb{F}}_T^{[0,1]}, e^{-Tf} d^F e^{Tf})$ can be asymptotically described in terms of the Thom-Smale complex $(C_\bullet(W^u, F^*), \partial)$.

Let us finally point out that if the metric g^F is flat, the results of [HSj4] can be directly adapted, since in this case, the operator $\widetilde{D}_T^2$ is essentially the one considered in [HSj4].

The potential which appears in the Schrödinger analysis of [HSj4] is exactly $|df|^2$. As shown by Witten [W], this explains the localization of the eigenvectors of $\widetilde{D}_T^2$ as $T \to +\infty$ near the potential wells for $|df|^2$, i.e. on the critical points of f. In [BL2], the submanifold Y described before is exactly the locus where a nonnegative operator V^2 has a nonzero kernel. This explains partly the analogy between [BL2] and our work, where Y is in fact replaced by B. Nevertheless, there is a fundamental difference : in [BL2], because of algebraic geometry considerations, there exists $c > 0$ such that for T large enough, the analogue of $\widetilde{D}_T^2$ has no eigenvalue in $[0,1]$ other than 0. To the contrary, the small eigenvalues play here an essential role. In fact in [BL2], the Morse inequalities are in fact equalities, and this explains why no 'instanton' analysis is needed, the difficulty being concentrated in the geometry of Y. Here B is simply a collection of points, and the analytic difficulties come in fact from the tunelling effects.

4. The de Rham map, and its extension by Laudenbach to Thom-Smale complexes

Our main result, in Theorem 0.2, compares two different metrics on the line $\det H^\bullet(M,F)$. This implies in particular that the cohomology groups of the de Rham complex $(\mathbb{F}, d^F)$ and of the Thom-Smale complex $(C_\bullet(W^u, F^*), \partial)$ have been canonically identified, and besides that this canonical identification appears explicitly in the analytic process of deformation of the de Rham complex to the Thom-Smale complex.

If K is a smooth triangulation of M, the de Rham map, which one obtains by integrating smooth forms on the simplexes $\sigma \in K$ provides the canonical identification of the cohomology groups of $(\mathbb{F}, d^F)$ with the cohomology groups of $(C_\bullet(K, F^*), \partial)$.

For general Thom-Smale complexes, it is more difficult to identify explicitly the de Rham cohomology with the cohomology of the Thom-Smale complex. In the Appendix, for gradient vector fields X which have a standard form near their zero set B, Laudenbach provides us with a complete answer to this question. In this case, the closure of the stable and unstable cells of the gradient vector field are in fact manifolds with conical singularities, on which smooth forms can be integrated, and the obvious analogue of the de Rham theorem still holds.

As explained before, the canonical identification of the de Rham cohomology with the Thom-Smale cohomology should appear explicitly in the analytic deformations process itself. This is shown to be the case in Section 9, as a consequence of our extension of the results of Helffer-Sjöstrand [HSj4] established in Section 8.

Let us point out that in [BL2, Section 10], the quasi-isomorphism of certain Dolbeault complexes on X and Y appears also explicitly in the analytic deformation process.

5. Local index theory and Berezin integrals

As in [BL2], local index theory techniques play an important role in the paper. In fact the term

$$-\int_M \theta\left(F, g^F\right) X^* \psi\left(TM, \nabla^{TM}\right)$$

in the right-hand side of (0.8) appears through local index theory techniques. Let us here just point out that in the case where the metric g^F is flat, it is easy to see that the local index contribution is identically zero, essentially because of Poincaré duality. In general,we need more sophisticate local index techniques. In principle, the Clifford rescaling techniques of Getzler [G] could be used in the whole paper. However, it is much more convenient to use a different local index theoretic technique, associated to the Berezin integral formalism. As explained in [BL2], standard index theoretic techniques produce in principle local Quillen's superconnection forms [Q1]. Here we obtain instead Berezin integrals. While, by Mathai-Quillen [MQ], we know that the forms produced by the superconnection formalism or the Berezin integral formalism are equivalent, it is here much more convenient to manipulate Berezin integrals, if only because they exhibit natural symmetry properties which are difficult to see in the superconnection formalism. Section 3 is entirely devoted to develop the Berezin integral formalism in the context of Morse theory, and also to establish a mysterious identity of differential forms, which is in fact also a consequence of the proof of Theorem 0.2.

Another difficulty in the application of local index techniques is that the usual 'fantastic cancellations' conjectured by McKean-Singer [McKS] do not occur here. Part of the difficulty is often to calculate the second term in an asymptotic expansion of the supertrace of heat kernel. This difficulty ressembles superficially a similar difficulty already considered in Bismut-Gillet-Soulé [BGS2] and also in [BL2]. Again, the Berezin integral formalism is very useful to make the required calculations, which are very different from the ones in [BGS2] or [BL2].

6. The asymptotics of two parameters supertraces

Set $D = d + d^*, \widehat{c}(\nabla f) = df \wedge + i_{\nabla f}$. In the course of the proof, it is essential to calculate the asymptotics as $t \to 0$ of $\mathrm{Tr_s}[f \exp(-(tD + T\widehat{c}(\nabla f))^2)]$ for $T \leq \frac{1}{t}$, for $T \simeq \frac{1}{t}$, and for $T \geq \frac{1}{t}$. In a different context, this problem was already encountered in [BL2]. In fact for $T \leq \frac{1}{t}$, this term explains the appearance of $-\int_M \theta(F, g^F) X^* \psi(TM, \nabla^{TM})$, in the right-hand side of (0.8). For $T \simeq \frac{1}{t}$, the harmonic oscillators near the critical points of f are ultimately responsible for a modest term $\mathrm{Log}(\pi)$, whose role is ultimately to cancel another $\mathrm{Log}(\pi)$ coming from the asymptotics of the complex $(\mathbb{F}_T^{[0,1]}, d^F)$. We hope to show in a forthcoming paper that, as in [BL2], harmonic oscillators may express themselves in a more forceful way.

As in [BL2], the difficulty is to establish estimates which take into account the painful transition from the region $T \leq \frac{1}{t}$ to the region $T \geq \frac{1}{t}$. Although here, the geometry of B is trivial (while in [BL2], the geometry of the embedding $i : Y \to X$ played an essential role), the fact that one needs to go beyond the first term in the asymptotics introduces new difficulties with respect to [BL2].

7. Some simplifying assumptions on the metrics

As we already explained, we prove first the anomaly formulas of Theorem 0.1, by using the local index techniques and the Berezin integral formalism, which we described before. This allows us to reduce the proof of Theorem 0.2 to the case of one single couple of metrics (g^{TM}, g^F), which we choose to be as simple as possible near the critical points of f. Incidently, note that using the techniques of this paper, a direct proof of Theorem 0.2 with arbitrary metrics would break down.

8. From Milnor metrics to Milnor metrics : Cerf's theory and Laudenbach's description of a one parameter deformation of the Thom-Smale complex

By Theorem 0.2, we deduce a formula which compares the Milnor metrics associated to two gradient vector fields.

It is natural to expect that a formula comparing two Milnor metrics could be established directly, without comparing first these metrics to the Ray-Singer metric. Now, given two Morse functions f and g, Cerf's theory [Ce] allows us to connect f and g by a one parameter smooth path of smooth functions, which are Morse except at a finite number of values of the parameter, corresponding to the birth or the death of critical points. In the Appendix, over such a path, Laudenbach constructs a smooth path of gradient fields, which verify the Smale transversality conditions [Sm1], except at a finite number of values of the parameter, where he describes explicitly the bifurcation of the Thom-Smale complex. In Section 16, this allows us to give a direct proof of the formula comparing two Milnor metrics, which does not use Theorem 0.2. Thus, if the reader is willing to take for granted the results of the Appendix and of Section 16, we only need to prove Theorem 0.2 for one single gradient vector field X.

This paper is organized as follows. In Section 1, we construct the Reidemeister and Milnor metrics and in Section 2, the Ray-Singer metrics.

In Section 3, we describe the Berezin integral formalism in connection with Morse theory, which we apply in Section 4 to the proof of the anomaly formulas of Theorem 0.1 for Ray-Singer metrics.

In Section 5, we construct the closed form $\alpha_{t,T}$.

In Section 6, we give various properties of the integral

$$-\int_M \theta\left(F, g^F\right) X^* \psi\left(TM, \nabla^{TM}\right) .$$

In Section 7, we state nine intermediary results whose proofs are delayed to Sections 8-15, and we prove Theorem 0.2.

In Section 8, we describe the results of Helffer-Sjöstrand [HSj1–4], and we extend their results on the asymptotics as $T \to +\infty$ of the complex $(\widetilde{\mathbb{F}}_T^{[0,1]}, e^{-Tf} d^F e^{Tf})$.

In Section 9, we calculate the asymptotics of the metric $\| \ \|^{\sim}_{\det H^\bullet(M,F),T}$ as $T \to +\infty$.

Sections 10-15 are devoted to the proofs of the remaining intermediary results stated in Section 7, which concern in particular the two parameter supertraces described before.

Finally, in Section 16, we compare two Milnor metrics directly, by using results of Laudenbach proved in the Appendix.

We now say a few words concerning our notation. If $\mathbb{A}$ is a $\mathbb{Z}_2$-graded algebra, if $A, B \in \mathbb{A}$, we define the supercommutator [A, B] by the formula

$$[A, B] = AB - (-1)^{\deg A \deg B} BA. \tag{0.17}$$

It is now time to describe our debts. We first owe a special mention to Tangerman [Ta] who announced some five years ago that he was trying to give a new proof of the Cheeger and Müller theorem using Helffer and Sjöstrand's results [HSj4] on the Witten complex. As far as we know, his program has not been terminated. Apparently, Tangerman's idea was to use a combination of Helffer-Sjöstrand results and of surgery techniques, which should make his program very different from ours.

We have had many discussions with F. Laudenbach, whose contribution to the success of our program has been essential.

We owe our hearty thanks to J. Sjöstrand. He helped us to orient ourselves in his papers with Helffer, and patiently answered our many questions.

Also we are very much indebted to J. Cheeger for many discussions, for the encouragement he gave us in our study of nonorthogonally flat metrics, and also for his friendly questioning of our final formula.

The results contained in this paper were announced in Bismut-Zhang [BZ].

I. Reidemeister metrics and Milnor metrics

In this Section, we construct the Reidemeister metrics and the Milnor metrics on the determinant of the cohomology of a flat vector bundle.

This Section is organized as follows. In a), we recall some elementary properties of the determinant of a finite dimensional complex, and of the corresponding metrics.

In b), we construct the Reidemeister metrics on the determinant of the cohomology of a flat vector bundle associated to a smooth triangulation.

In c), we describe the Thom-Smale complex associated to the gradient vector field of a Morse function.

Finally in d), we construct the Milnor metrics on the determinant of the cohomology of a flat vector bundle, associated to a gradient vector field.

a) A metric on the determinant of the cohomology of a finite dimensional chain complex

If λ is a real line, let λ^{-1} be the dual line. If E is a finite dimensional real vector space, set

$$\det E = \Lambda^{\max}(E). \tag{1.1}$$

Let

$$(V^\bullet, \partial) : 0 \to V^0 \underset{\partial}{\to} \cdots \underset{\partial}{\to} V^n \to 0 \tag{1.2}$$

be a chain complex of finite dimensional real vector spaces, so that $V^\bullet = \bigoplus_{i=0}^n V^i$. Let $H^\bullet(V) = \bigoplus_{i=0}^n H^i(V)$ be the cohomology of $(V^\bullet, \partial)$.

Set

$$\det V^\bullet = \bigotimes_{i=0}^{n} \left(\det V^i\right)^{(-1)^i}, \tag{1.3}$$

$$\det H^\bullet(V) = \bigotimes_{i=0}^{n} \left(\det H^i(V)\right)^{(-1)^i}.$$

Then by [KMu], [BGS1, Section 1a)], there is a canonical isomorphism of real lines

$$\det V^\bullet \simeq \det H^\bullet(V). \tag{1.4}$$

Let $\| \ \|_{\det V^0}, \cdots, \| \ \|_{\det V^n}$ be metrics on the lines $\det V^0, \cdots, \det V^n$. We equip the dual lines $(\det V^0)^{-1}, \cdots, (\det V^n)^{-1}$ with the dual metrics

$$\| \ \|_{(\det V^0)^{-1}}, \cdots, \| \ \|_{(\det V^n)^{-1}}.$$

Let $\| \ \|_{\det V^\bullet}$ be the metric on the line $\det(V^\bullet)$,

$$\| \ \|_{\det V^\bullet} = \bigotimes_{i=0}^{n} \| \ \|_{(\det V^i)^{(-1)^i}}. \tag{1.5}$$

Let $\| \ \|_{\det H^\bullet(V)}$ be the metric on the line $\det H^\bullet(V)$ corresponding to the metric $\| \ \|_{\det V^\bullet}$ via the canonical isomorphism (1.4).

Let $g^{V^0}, \cdots, g^{V^n}$ be Euclidean metrics on $V^0, \cdots, V^n$, inducing the metrics $\| \ \|_{\det V^0}, \ldots, \| \ \|_{\det V^n}$ on $\det V^0, \cdots, \det V^n$. We equip $V = \bigoplus_{i=0}^n V^i$ with the metric $g^V = \bigoplus_{i=0}^n g^{V^i}$, which is the orthogonal sum of the metrics $g^{V^0}, \cdots, g^{V^n}$.

Let ∂^* be the adjoint of ∂ with respect to the metric g^V. Using finite dimensional Hodge theory, we have the canonical identifications

$$H^i(V) \simeq \left\{v \in V^i; \partial v = 0, \partial^* v = 0\right\}, \quad 0 \leq i \leq n. \tag{1.6}$$

As a vector subspace of V^i, the vector space in the right-hand side of (1.6) inherits an Euclidean metric from the metric g^{V^i}. Let $g^{H^i(V)}$ be the corresponding metric on $H^i(V)$ via the identification (1.6). Then the line $\det H^\bullet(V)$ inherits a metric $| \ |_{\det H^\bullet(V)}$.

The metrics $\| \ \|_{\det H^\bullet(V)}$ and $| \ |_{\det H^\bullet(V)}$ do not coincide in general. We describe the discrepancy. Set

$$D = \partial + \partial^*. \tag{1.7}$$

The Laplacian $D^2 = \partial\partial^* + \partial^*\partial$ preserves the splitting $V^\bullet = \bigoplus_{i=0}^n V^i$. Let P be the orthogonal projection operator from V on $\operatorname{Ker} D^2 \simeq H^\bullet(V)$. Set $P^\perp = 1 - P$.

Let $N \in \mathrm{End}(V)$ be the number operator of the complex $(V^\bullet, \partial)$, i.e. N acts on $V^i (0 \le i \le n)$ by multiplication by i.

Set

(1.8)
$$V^+ = \bigoplus_{i\,\mathrm{even}} V^i \; V^- = \bigoplus_{i\,\mathrm{odd}} V^i.$$

Then $V = V^+ \oplus V^-$ is a $\mathbb{Z}_2$-graded vector space. Let $\tau = \pm 1$ on $V^\pm$. If $A \in \mathrm{End}(V^\bullet)$, we define the supertrace $\mathrm{Tr}_s[A]$ by the formula

(1.9)
$$\mathrm{Tr}_s[A] = \mathrm{Tr}[\tau A].$$

For $s \in \mathbb{C}$, set

(1.10)
$$\theta^V(s) = -\,\mathrm{Tr}_s\left[N\left(D^2\right)^{-s} P^\perp\right].$$

Let $D^{2,>0}$ be the restriction of the operator D^2 to the orthogonal space to $\mathrm{Ker}\, D^2$ in $V^\bullet$. Then

(1.11)
$$\theta^{V\prime}(0) = \mathrm{Tr}_s\left[N \,\mathrm{Log}\left(D^{2,>0}\right)\right].$$

The following result is proved in [BGS1, Proposition 1.5].

Theorem 1.1. *The following identity holds,*

(1.12)
$$\|\ \|_{\det H^\bullet(V)} = |\ |_{\det H^\bullet(V)} \exp\left\{\frac{1}{2}\theta^{V\prime}(0)\right\}.$$

Remark 1.2. It should be pointed out that the metric $\|\ \|_{\det H^\bullet(V)}$ only depends on the metrics $\|\ \|_{\det V^0}, \cdots, \|\ \|_{\det V^n}$, while the metric $|\ |_{\det H^\bullet(V)}$ and also $\theta^{V\prime}(0)$ depend in general on the metrics $g^{V^0}, \cdots, g^{V^n}$.

b) The Reidemeister metric on the determinant of the cohomology of a simplicial complex

Let M be a compact manifold of dimension n. Let F be a real flat vector bundle on M, and let F^* be its dual.

Let $\mathcal{F}$ be the locally constant sheaf of flat sections of F. For $0 \leq i \leq n$, let $H^i(M,F)$ be the i-th cohomology group of $\mathcal{F}$. Set

$$H^\bullet(M,F) = \bigoplus_{i=0}^{n} H^i(M,F). \tag{1.13}$$

Definition 1.3. Let $\det H^\bullet(M,F)$ be the real line

$$\det H^\bullet(M,F) = \bigotimes_{i=0}^{n} \left(\det H^i(M,F)\right)^{(-1)^i}. \tag{1.14}$$

Let $H_\bullet(M,F^*) = \bigoplus_{i=0}^{n} H_i(M,F^*)$ denote the singular homology of sections of the flat vector bundle F^*. Then

$$H^i(M,F) = \left(H_i\left(M,F^*\right)\right)^* \quad 0 \leq i \leq n. \tag{1.15}$$

Let K be a smooth triangulation of M. Then K consists of a finite set of simplexes σ whose orientation is *fixed* once and for all. Let B be the finite subset of M of the barycenters of the simplexes in K. Let $b: K \to B$ and $\sigma: B \to K$ denote the obvious one-to-one maps.

For $0 \leq i \leq n$, let K^i be the union of the simplexes in K of dimension $\leq i$. For $0 \leq i \leq n$, $K^i \backslash K^{i-1}$ is the union of simplexes of dimension i.

If $\sigma \in K$, let $[\sigma]$ be the real line generated by σ. Let $(C_\bullet(K,F^*),\partial)$ be the complex of simplicial chains in K with values in F^*. For $0 \leq i \leq n$, we have the identity

$$C_i\left(K,F^*\right) = \bigoplus_{\sigma \in K^i \backslash K^{i-1}} [\sigma] \otimes_{\mathbb{R}} F^*_{b(\sigma)}. \tag{1.16}$$

The chain map ∂ maps $C_i(K,F^*)$ into $C_{i-1}(K,F^*)$. Also the homology of the complex $(C_\bullet(K,F^*),\partial)$ can be canonically identified with the singular homology $H_\bullet(M,F^*)$.

If $\sigma \in K$, let $[\sigma]^*$ be the line dual to the line $[\sigma]$. Let $(C^\bullet(K,F),\widetilde{\partial})$ be the complex dual to the complex $(C_\bullet(K,F^*),\partial)$. In particular, for $0 \leq i \leq n$, we have the identity

$$C^i\left(K,F\right) = \bigoplus_{\sigma \in K^i \backslash K^{i-1}} [\sigma]^* \otimes_{\mathbb{R}} F_{b(\sigma)}. \tag{1.17}$$

The cohomology of the complex $(C^\bullet(K,F),\widetilde{\partial})$ can be canonically identified to the dual $(H_\bullet(M,F^*))^*$ of $H_\bullet(M,F^*)$. In view of (1.15), the cohomology of $(C^\bullet(K,F),\widetilde{\partial})$ can be identified with $H^\bullet(M,F)$.

The complex $(C^\bullet(K,F),\widetilde{\partial})$ can be described more explicitly. In fact, let K^* be a smooth cell polyhedral decomposition of M which is dual to the triangulation K. Then B is also the set of barycenters of the polyhedra in K^*. Again, we fix once and for all the orientation of the polyhedra of K^*.

Let $o(TM)$ be the orientation bundle of TM. Then if $\sigma \in K$ and if $\sigma^* \in K^*$ is the dual polyhedron, there is a canonical identification of lines

$$[\sigma]^* \simeq [\sigma^*] \otimes o(TM)_{|b(\sigma)}. \tag{1.18}$$

From (1.18), we deduce the canonical identification of complexes

$$\left(C^\bullet(K,F),\widetilde{\partial}\right) \simeq \left(C_{n-\bullet}(K^*,F\otimes o(TM)),\partial(-1)^{\bullet+1}\right). \tag{1.19}$$

Using (1.19), we obtain the Poincaré duality isomorphism

$$(H^\bullet(M,F))^* = H^{n-\bullet}(M,F^*\otimes o(TM)). \tag{1.20}$$

Set

$$\det C_\bullet(K,F^*) = \bigotimes_{i=0}^{n} (\det C_i(K,F^*))^{(-1)^i}, \tag{1.21}$$

$$\det C^\bullet(K,F) = \bigotimes_{i=0}^{n} \left(\det C^i(K,F)\right)^{(-1)^i}.$$

Then

$$(\det C^\bullet(K,F)) = (\det C_\bullet(K,F^*))^{-1}. \tag{1.22}$$

Using (1.4), we get a canonical isomorphism of real lines

$$\det C^\bullet(K,F) \simeq \det H^\bullet(M,F). \tag{1.23}$$

For every $x \in B$, we equip the line $\det F_x$ with a metric $\det F_x$. For every $\sigma \in K$, we equip the line $[\sigma]$ with the trivial metric $\|\ \|_{[\sigma]}$ such that $\|\sigma\|_{[\sigma]} = 1$. For every $x \in B$, the line $\det([\sigma(x)]^* \otimes F_x)$ inherits a metric $\|\ \|_{\det([\sigma(x)]^*\otimes F_x)}$. For $0 \le i \le n$, we equip the line $\det C^i(K,F)$ with the metric $\|\ \|_{\det C^i(K,F)}$ which is the tensor product of the metrics $\|\ \|_{\det([\sigma]^*\otimes F_{b(\sigma)})}$ $(\sigma \in K^i \backslash K^{i-1})$.

Let $\| \ \|_{\det C^\bullet(K,F)}$ be the metric on the line $\det C^\bullet(K,F)$ associated to the metrics $\| \ \|_{\det C^i(K,F)}$ as in (1.5).

Definition 1.4. The Reidemeister metric $\| \ \|^{R,K}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M,F)$ is the metric corresponding to the metric $\| \ \|_{\det C^\bullet(K,F)}$ via the canonical isomorphism (1.23).

We equip the line $o(TM)$ with its canonical trivial metric. For $x \in B$, let $\| \ \|_{\det(F^* \otimes o(TM))_x}$ be the metric on the line $\det(F^* \otimes o(TM))_x$ associated to the metric $\| \ \|_{\det F_x}$ on $\det F_x$. Let $\| \ \|^{R,K^*}_{\det H^\bullet(M,F^*\otimes o(TM))}$ be the Reidemeister metric on the line $\det H^\bullet(M, F^* \otimes o(TM))$ asssociated to the cell decomposition K^* and to the metrics $\| \ \|_{\det(F^*\otimes o(TM))_x}, x \in B$.

By (1.20), we obtain the canonical isomorphism

$$\det H^\bullet\,(M, F^* \otimes o(TM)) \simeq (\det H^\bullet(M,F))^{(-1)^{n-1}}. \tag{1.24}$$

The identification (1.24) also identifies the Reidemeister metrics $\| \ \|^{R,K^*}_{\det H^\bullet(M,F^*\otimes o(TM))}$ and $(\| \ \|^{R,K}_{\det H^\bullet(M,F)})^{(-1)^{n-1}}$. This is a result of Milnor [Mi2].

Remark 1.5. Assume that F can be equipped with a flat metric g^F. This metric induces metrics $\| \ \|_{\det F_x}$ on the lines $\det F_x (x \in B)$. The associated Reidemeister metric $\| \ \|^{R,K}_{\det H^\bullet(M,F)}$ was constructed by Franz [F], Reidemeister [Re], and de Rham [Rh1] (see [Mi1, Section 8]). They showed that the Reidemeister metric $\| \ \|^{R,K}_{\det H^\bullet(M,F)}$ is invariant by simplicial subdivision. We thus obtain a metric $\| \ \|^{R}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M,F)$ which is a topological invariant. Recently, Müller [Mü2] extended this result to the case where the line $\det F$ posseses a flat metric $\| \ \|_{\det F}$, and where the lines $\det F_x (x \in B)$ are equipped with the corresponding metrics $\| \ \|_{\det F_x}$.

c) The Thom-Smale complex of the gradient field of a Morse function

Let M be a compact manifold. Let $f : M \to \mathbb{R}$ be a Morse function. Let B be the set of critical points of f, i.e.

$$B = \{x \in M; df(x) = 0\}. \tag{1.25}$$

If $x \in B$, recall that the index $\operatorname{ind}(x)$ is the number of negative eigenvalues of the quadratic form $d^2 f(x)$ on $T_x M$.

Let g^{TM} be a metric on TM, and let $\nabla f \in TM$ be the corresponding gradient vector field of f. Consider the differential equation

$$\frac{dy}{dt} = -\nabla f(y). \tag{1.26}$$

Equation (1.26) defines a group of diffeomorphism $(\psi_t)_{t \in \mathbb{R}}$ of M.

If $x \in B$, set

$$\begin{aligned} W^u(x) &= \left\{ y \in M; \lim_{t \to -\infty} \psi_t(y) = x \right\}, \\ W^s(x) &= \left\{ y \in M; \lim_{t \to +\infty} \psi_t(y) = x \right\}. \end{aligned} \tag{1.27}$$

The cells $W^u(x)$ and $W^s(x)$ will be called the unstable and stable cells at x.

We assume that the vector field ∇f verifies the Smale transversality conditions [Sm1,2]. Namely, we suppose that if $x, y \in B$, $x \neq y$, $W^u(x)$ and $W^s(y)$ intersect transversally. In particular if $\operatorname{ind}(y) = \operatorname{ind}(x) - 1, W^u(x) \cap W^s(y)$ consists of a finite set $\Gamma(x, y)$ of integral curves γ of the vector field $-\nabla f$, with $\gamma_{-\infty} = x, \gamma_{+\infty} = y$, along which $W^u(x)$ and $W^s(y)$ intersect transversally.

By [Sm1, Theorem A], given a Morse function f, there exists a metric g^{TM} on TM such that ∇f verifies the transversality conditions.

We fix an orientation on each $W^u(x), x \in B$.

Let $x, y \in B$ with $\operatorname{ind}(y) = \operatorname{ind}(x) - 1$. Take $\gamma \in \Gamma(x, y)$. Then $T_y W^u(y)$ is orthogonal to $T_y W^s(y)$ and is oriented. So for $t \in]-\infty, +\infty]$, the orthogonal space $T^{\perp}_{\gamma_t} W^s(y)$ to $T_{\gamma_t} W^s(y)$ in $T_{\gamma_t} M$ carries a natural orientation. Also for $t \in]-\infty, +\infty[$, the orthogonal space $T'_{\gamma_t} W^s(x)$ to $-\nabla f(\gamma_t)$ in $T_{\gamma_t} W^u(x)$ can be oriented in such a way that s is an oriented base of $T'_{\gamma_t} W^u(x)$ if $(-\nabla f(\gamma_t), s)$ is an oriented base of $T_{\gamma_t} W^u(x)$. Finally since $W^u(x)$ and $W^s(y)$ are transversal along γ, for $t \in]-\infty, +\infty[$, $T^{\perp}_{\gamma_t} W^s(y)$ and $T'_{\gamma_t} W^u(x)$ can be identified, and their orientations can be compared. Set

$$\begin{aligned} n_\gamma(x, y) &= +1 \quad \text{if the orientations are the same,} \\ &= -1 \quad \text{if the orientations differ.} \end{aligned} \tag{1.28}$$

If $x \in B$, let $[W^u(x)]$ be the real line generated by $W^u(x)$. Let F be a flat vector bundle on M, and let F^* be its dual. Set

$$
\begin{aligned}
C_\bullet(W^u, F^*) &= \bigoplus_{x \in B} [W^u(x)] \otimes_{\mathbb{R}} F_x^*, \\
C_i(W^u, F^*) &= \bigoplus_{\substack{x \in B \\ \operatorname{ind}(x)=i}} [W^u(x)] \otimes_{\mathbb{R}} F_x^*.
\end{aligned} \tag{1.29}
$$

If $x \in B$, the flat vector bundle F^* is canonically trivialized on $W^u(x)$. In particular, if $x, y \in B$ are such that $\operatorname{ind}(y) = \operatorname{ind}(x) - 1$, and if $\gamma \in \Gamma(x, y)$, $f^* \in F_x^*$, let $\tau_\gamma(f^*) \in F_y^*$ be the parallel transport of $f \in F_x^*$ into F_y^* along γ with respect to the flat connection of F^*.

If $x \in B, f^* \in F_x^*$, set

$$
\partial(W^u(x) \otimes f^*) = \sum_{\substack{y \in B \\ \operatorname{ind}(y)=\operatorname{ind}(x)-1}} \sum_{\gamma \in \Gamma(x,y)} n_\gamma(x, y) W^u(y) \otimes \tau_\gamma(f^*). \tag{1.30}
$$

Then ∂ maps $C_i(W^u, F^*)$ into $C_{i-1}(W^u, F^*)$.

We now recall a basic result of Thom [T], Smale [Sm2].

Theorem 1.6. *$(C_\bullet(W^u, F^*), \partial)$ is a chain complex. Moreover, we have a canonical identification of $\mathbb{Z}$-graded vector spaces*

$$
H_\bullet(C_\bullet(W^u, F^*), \partial) \simeq H_\bullet(M, F^*). \tag{1.31}
$$

Remark 1.7. In the Appendix, if X has a canonical form near B, Laudenbach gives a proof of Theorem 1.6, and he constructs the CW complex associated to the cells $W^u(x)(x \in B)$. Moreover he shows that the closures of the $W^u(x)$'s are manifolds with conical singularities.

Remark 1.8. If ∇f verifies the Smale transversality conditions, $\nabla(-f)$ verifies also the Smale transversality conditions. Let $W'^u(x), W'^s(x)(x \in B)$ be the corresponding unstable and stable cells. Clearly, if $x \in B$,

$$
\begin{aligned}
W'^u(x) &= W^s(x), \\
W'^s(x) &= W^u(x).
\end{aligned} \tag{1.32}
$$

If $x \in B$, let $[W^u(x)]^*$ be the line dual to the line $[W^u(x)]$. Let $(C^\bullet(W^u, F), \widetilde{\partial})$ be the complex which is dual to $(C_\bullet(W^u, F^*), \partial)$. For $0 \le i \le n$, we have the identity

$$C^i(W^u, F) = \bigoplus_{\substack{x \in B \\ \operatorname{ind}(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x. \tag{1.33}$$

Then by Theorem 1.6,

$$H^\bullet\left(C^\bullet(W^u, F), \widetilde{\partial}\right) \simeq H^\bullet(M, F). \tag{1.34}$$

Fix an orientation on each $W^s(x)$. Then one easily verifies that

$$\left(C^\bullet(W^u, F), \widetilde{\partial}\right) \simeq \left(C_{n-\bullet}(W^s, F \otimes o(TM)), \partial(-1)^{\bullet+1}\right). \tag{1.35}$$

Using (1.35), we recover Poincaré duality

$$(H^\bullet(M, F))^* = H^{n-\bullet}(M, F^* \otimes o(TM)). \tag{1.36}$$

We will make more explicit the canonical identification (1.31). Here we follow Milnor [Mi1, Section 9].

By a result of Smale [Sm1, Theorem B], we may and we will assume that f is a nice Morse function, i.e. f takes the value i on the critical points of index i. For $i \in \mathrm{N}$, set

$$V^i = f^{-1}\left[0, i + \frac{1}{2}\right]. \tag{1.37}$$

Let $S(F^*)$ be the complex of singular chains in M with value in F^*. For $0 \le i \le n$, let $S^i(F^*)$ be the complex of singular chains in V^i with value in F^*. Then the $S^i(F^*)$ define a filtration of $S(F^*)$,

$$0 \subset S^0(F^*) \ldots \subset S^n(F^*) = S(F^*). \tag{1.38}$$

By Morse theory, we know that for $0 \le i,\ p \le n$, $H_p(V^i, V^{i-1}, F^*)$ is nonzero only for $p = i$, and moreover

$$H_i(V^i, V^{i-1}, F^*) = C_i(W^u, F^*). \tag{1.39}$$

Set

$$E^0_{(p,q)} = \frac{S^p_{p-q}(F^*)}{S^{p-1}_{p-q}(F^*)}. \tag{1.40}$$

Then $(E^0_{(p,q)}, d^0)$ is the first term of the spectral sequence $(E^r_{(p,q)}, d^r)$ associated to the filtration (1.38). By definition

$$E^1_{(p,q)} = H_{p-q}(V^p, V^{p-1}, F^*). \tag{1.41}$$

The previous considerations show that

$$\begin{aligned} E^1_{(p,q)} &= C_p(W^u, F^*), \qquad \text{if } q = 0 \\ &= \{0\}, \qquad\qquad \text{if } q \neq 0. \end{aligned} \tag{1.42}$$

Then, (E^1, d^1) is a chain complex. In view of (1.42), one verifies easily that the complexes $(E^1_{(\bullet,0)}, d^1)$ and $(C_\bullet(W^u, F^*), \partial)$ are identical.

Also by (1.42), the spectral sequence degenerates at E^2, i.e. the chain map d^2, vanishes. Tautologically

$$\begin{aligned} E^2_{(\bullet,q)} = H_\bullet(C_\bullet(W^u, F^*), \partial) \qquad &\text{if } q = 0, \\ \{0\} \qquad &\text{if } q \neq 0. \end{aligned} \tag{1.43}$$

Let

$$0 \subset G^0 H_\bullet(M, F^*) \subset \ldots \subset G^n H_\bullet(M, F^*) = H_\bullet(M, F^*) \tag{1.44}$$

be the filtration on $H_\bullet(M, F^*)$ induced by the filtration (1.38). Then a basic result on spectral sequences asserts that

$$E^2_{(p,q)} = \frac{G^p H_{p-q}(M, F^*)}{G^{p-1} H_{p-q}(M, F^*)} \tag{1.45}$$

By (1.43), (1.45), we see that for $0 \leq i \leq n$,

$$\begin{aligned} H_i(M, F^*) &= G^i H_i(M, F^*), \\ G^{i-1} H_i(M, F^*) &= 0. \end{aligned} \tag{1.46}$$

By (1.45), (1.46), we get

$$E^2_{(p,0)} = H_p(M, F^*). \tag{1.47}$$

By (1.43), (1.47), we obtain (1.31).

d) Milnor metrics on the determinant of the cohomology of a flat vector bundle.

We make the same assumptions and we use the same notation as in Section 1c). By (1.4) and by Theorem 1.6, we know that

$$\det C^{\bullet}(W^u, F) \simeq \det H^{\bullet}(M, F). \tag{1.48}$$

For $x \in B$, let $\| \ \|_{\det F_x}$ be a metric on the line $\det F_x$. As in Section 1 b), the metrics $\| \ \|_{\det F_x} (x \in B)$ induce a metric $\| \ \|_{\det C^{\bullet}(W^u, F)}$ on $\det C^{\bullet}(W^u, F)$.

Definition 1.9. The Milnor metric $\| \ \|^{\mathcal{M}, \nabla f}_{\det H^{\bullet}(M,F)}$ on the line $\det H^{\bullet}(M, F)$ is the metric corresponding to the metric $\| \ \|_{\det C^{\bullet}(W^u,F)}$ via the canonical isomorphism (1.48).

Remark 1.10. Assume that F can be equipped with a flat metric g^F. This metric induces metrics $\| \ \|_{\det F_x}$ on the lines $\det F_x (x \in B)$. The corresponding metrics $\| \ \|^{\mathcal{M}, \nabla f}_{\det H^{\bullet}(M,F)}$ was constructed in Milnor [Mi1, Section 9]. It was shown in [Mi1, Theorem 9.3] that the metric $\| \ \|^{\mathcal{M}, \nabla f}_{\det H^{\bullet}(M,F)}$ does not depend on ∇f, and coincides with the Reidemeister metric $\| \ \|^{R}_{\det H^{\bullet}(M,F)}$. More generally, assume that g^F is a metric on F, such that the induced metric $\| \ \|_{\det F}$ on $\det F$ is flat. The same arguments as in [Mi1, Theorem 9.3] show that the corresponding Milnor metric $\| \ \|^{\mathcal{M}, \nabla f}_{\det H^{\bullet}(M,F)}$ coincides with the Reidemeister metric $\| \ \|^{R}_{\det H^{\bullet}(M,F)}$.

II. Ray-Singer metrics and the de Rham map

In this Section we construct the Ray-Singer metrics on the determinant of the cohomology of a flat vector bundle. Also we describe the de Rham map, which identifies the cohomology of the de Rham complex and the cohomology of the simplicial complex associated to a smooth triangulation. We also explain the extension of this result by Laudenbach in the Appendix to certain Thom-Smale complexes.

This Section is organized as follows. In a), we introduce the Ray-Singer metrics. In b), we construct the de Rham map for simplicial complexes and in c), we describe the de Rham map for Thom-Smale complexes.

a) The Ray-Singer metric on $\det H^\bullet(M,F)$

Let M be a compact manifold, let F be a flat vector bundle and let F^* be its dual. Let g^{TM}, g^F be smooth metrics on TM, F. Let $\langle\ \rangle_F$ and $\langle\ \rangle_{\Lambda(T^*M)\otimes F}$ be the corresponding scalar products on F and $\Lambda(T^*M)\otimes F$.

Let $\mathbb{F} = \bigoplus_{i=0}^n \mathbb{F}^i$ be the vector space of smooth sections over M of $\Lambda(T^*M)\otimes F = \bigoplus_{i=0}^n (\Lambda^i(T^*M)\otimes F)$.

Let ∇^F denote the flat connection on F. Let d^F denote the obvious action of ∇^F on $\mathbb{F}$. Then

$$d^{F,2} = 0. \tag{2.1}$$

By the de Rham theorem, we know that the cohomology groups of the complex $(\mathbb{F}, d^F)$ are canonically isomorphic to $H^\bullet(M,F)$.

Let dv_M be the volume form on M associated to the metric g^{TM}. Let $*$ be the Hodge operator associated to g^{TM} acting on $\Lambda(T^*M)$. The operator $*$ also acts on $\Lambda(T^*M) \otimes F$.

If $\alpha, \alpha' \in \mathbb{F}$, set

$$\langle \alpha, \alpha' \rangle_{\mathbb{F}} = \int_M \langle \alpha \wedge *\alpha' \rangle_F . \tag{2.2}$$

Equivalently

$$\langle \alpha, \alpha' \rangle_{\mathbb{F}} = \int_M \langle \alpha, \alpha' \rangle_{\Lambda(T^*M)\otimes F}(x) dv_M(x) \tag{2.3}$$

The $\mathbb{F}^i$'s $(0 \leq i \leq n)$ are mutually orthogonal in $\mathbb{F}$ with respect to the scalar product $\langle \ , \ \rangle_{\mathbb{F}}$. Let d^{F*} be the formal adjoint of d^F with respect to the scalar product $\langle \ , \ \rangle_{\mathbb{F}}$. For $0 \leq i \leq n$, set

$$\begin{aligned} \mathbb{F}^{\{0\},i} &= \{ f \in \mathbb{F}^i, d^F f = 0, d^{F*} f = 0 \} , \\ \mathbb{F}^{\{0\}} &= \bigoplus_{i=0}^{n} \mathbb{F}^{\{0\},i}. \end{aligned} \tag{2.4}$$

By Hodge theory, we know that for $0 \leq i \leq n$, $H^i(M,F)$ and $\mathbb{F}^{\{0\},i}$ are canonically isomorphic. As finite dimensional vector subspaces of the $\mathbb{F}^i$ ' s, the $\mathbb{F}^{\{0\},i}$ ' s inherit the scalar product $\langle \ , \ \rangle_{\mathbb{F}}$. Let $g^{H^\bullet(M,F)}$ denote the corresponding metric on $H^\bullet(M,F)$. Thus the line $\det H^\bullet(M,F)$ inherits a metric $| \ |^{RS}_{\det H^\bullet(M,F)}$, which is also called the L_2 metric.

Set

$$D = d^F + d^{F*}. \tag{2.5}$$

Then $D^2 = d^F d^{F*} + d^{F*} d^F$ is the Hodge Laplacian associated to the metrics g^{TM}, g^F. Let $\mathbb{F}^{\{0\},\perp}$ denote the orthogonal space to $\mathbb{F}^{\{0\}}$ in $\mathbb{F}$ with respect to the scalar product $\langle \ , \ \rangle_{\mathbb{F}}$. Let $P, P^\perp$ denote the orthogonal projection operators from $\mathbb{F}$ on $\mathbb{F}^{\{0\}}, \mathbb{F}^{\{0\},\perp}$. The Hodge Laplacian D^2 acts as an invertible operator on $\mathbb{F}^{\{0\},\perp}$, and its inverse is denoted $(D^2)^{-1}$.

Let N be the operator defining the $\mathbb{Z}$-grading of $\mathbb{F}$, i.e. N acts on $\mathbb{F}^i$ by multiplication by i.

If $A \in \mathrm{End}(\mathbb{F})$ is trace class, we define its supertrace $\mathrm{Tr}_s[A]$ as in (1.9).

Definition 2.1. For $s \in \mathbb{C}, \mathrm{Re}(s) > n/2$, set

$$\theta^{\mathbb{F}}(s) = -\operatorname{Tr}_s\left[N\left(D^2\right)^{-s} P^{\perp}\right]. \tag{2.6}$$

By a result of Seeley [Se], $\theta^{\mathbb{F}}(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$.

Definition 2.2. Let $\| \ \|^{RS}_{\det H^\bullet(M,F)}$ be the Ray-Singer metric on the line $\det H^\bullet(M,F)$

$$\| \ \|^{RS}_{\det H^\bullet(M,F)} = | \ |^{RS}_{\det H^\bullet(M,F)} \exp\left\{\frac{1}{2}\frac{\partial\theta^{\mathbb{F}}}{\partial s}(0)\right\}. \tag{2.7}$$

Remark 2.3. The quantity $\exp\{\frac{1}{2}\frac{\partial\theta^{\mathbb{F}}}{\partial s}(0)\}$ was originally called by Ray and Singer [RS1] the analytic torsion of the complex $(\mathbb{F}, d^F)$. The holomorphic analogue for Dolbeault complexes was introduced by Ray and Singer [RS2]. Quillen [Q2] constructed the corresponding Quillen metric on the determinant of the holomorphic cohomology. Quillen metrics have been the object of several recent developments [BGS1, 2, 3], [BL1, 2], some of which will be central to our understanding of the Ray-Singer metric.

Let g^{F^*} be the metric on F^* induced by the metric g^F on F. We equip the orientation line $o(TM)$ with the trivial metric. The vector bundle $F^* \otimes o(TM)$ is then equipped with a metric $g^{F^*\otimes o(TM)}$. Let $\| \ \|^{RS}_{\det H^\bullet(M,F^*\otimes o(TM))}$ be the Ray-Singer metric on $\det H^\bullet(M, F^* \otimes o(TM))$ attached to the metric g^{TM} on TM and the metric $g^{F^*\otimes o(TM)}$ on $F^* \otimes o(TM)$. It is easy to see that under the isomorphism (1.24), the metrics $\| \ \|^{RS}_{\det H^\bullet(M,F^*\otimes o(TM))}$ and $(\| \ \|_{\det H^\bullet(M,F)})^{(-1)^{n-1}}$ correspond.

Remark 2.4. When M is odd dimensional, Ray and Singer [RS1, Theorem 2.1] proved that the metric $\| \ \|_{\det H^\bullet(M,F)}$ is a topological invariant, i.e. does not depend on the metrics g^{TM} or g^F.

When M is even dimensional and oriented, if the metric g^F is flat, it follows from Ray and Singer [RS1, Theorem 2.3] that

$$\| \ \|^{RS}_{\det H^\bullet(M,F)} = | \ |^{RS}_{\det H^\bullet(M,F)}. \tag{2.8}$$

Remark 2.5. Assume that the metric g^F is flat. Let $\| \ \|^{R}_{\det H^\bullet(M,F)}$ denote the corresponding Reidemeister metric on the line $\det H^\bullet(M,F)$, which is constructed in Remark 1.5. It was conjectured by Ray and Singer [RS1] that if M is odd dimensional, the Ray-Singer metric $\| \ \|^{RS}_{\det H^\bullet(M,F)}$ and the Reidemeister metric $\| \ \|^{R}_{\det H^\bullet(M,F)}$, which are both topological invariants, are equal. This was proved in celebrated papers of Cheeger [C] and Müller [Mü1]. Müller [Mü2] recently extended this result to the case where the metric $\| \ \|_{\det F}$ on the line $\det F$ is flat.

b) A quasi-isomorphism of complexes : the de Rham map for smooth triangulations

Take a smooth triangulation K of M as in Section 1b). The flat vector bundle F is canonically trivialized over each simplex $\sigma \in K$ by using the flat connection ∇^F.

The line $[\sigma]$ has non zero a canonical section σ. Let $\sigma^* \in [\sigma]^*$ be dual to $\sigma \in [\sigma]$, so that $\langle \sigma, \sigma^* \rangle = 1$. If $\alpha \in \mathbb{F}$, the integral $\sigma^* \otimes \int_\sigma \alpha$ lies in $[\sigma]^* \otimes F_{b(\sigma)}$. Of course if $\alpha \in \mathbb{F}^i$, $\int_\sigma \alpha$ is nonzero only if $\sigma \in K^i \backslash K^{i-1}$.

Definition 2.6. Let P_∞ be the map

$$\alpha \in \mathbb{F} \to P_\infty \alpha = \sum_{\sigma \in K} \sigma^* \otimes \int_\sigma \alpha \in C^\bullet(K, F). \tag{2.9}$$

Theorem 2.7. *The map P_∞ is a quasi-isomorphism of the $\mathbb{Z}$-graded complexes $(\mathbb{F}, d^{\mathbb{F}})$ and $(C^\bullet(K,F), \widetilde{\partial})$, which provides the canonical identification of the cohomology groups of both complexes.*

Proof. Clearly P_∞ maps $\mathbb{F}^i$ into $C^i(K,F)$. Take $\sigma \in K, f^* \in F^*_{b(\sigma)}$. By definition, if $\alpha \in \mathbb{F}$, then

$$\langle P_\infty \alpha, \sigma \otimes f^* \rangle = \left\langle f^*, \int_\sigma \alpha \right\rangle. \tag{2.10}$$

Then

(2.11)

$$\begin{aligned} \langle P_\infty d^F \alpha, \sigma \otimes f^* \rangle &= \left\langle f^*, \int_\sigma d^F \alpha \right\rangle = \left\langle f^*, \int_{\partial\sigma} \alpha \right\rangle = \langle P_\infty \alpha, \partial(\sigma \otimes f^*) \rangle \\ &= \left\langle \widetilde{\partial} P_\infty \alpha, \sigma \otimes f^* \right\rangle. \end{aligned}$$

From (2.11), we see that P_∞ is a homomorphism of complexes. The de Rham theorem asserts that P_∞ is a quasi-isomorphism, i.e. it identifies canonically the cohomology groups of $(\mathbb{F}, d^F)$ and of $(C^\bullet(K,F), \widetilde{\partial})$. □

c) A quasi-isomorphism of complexes : the de Rham map for Thom-Smale complexes

We use the same notation as in Section 1c).

Let $f : M \to \mathbb{R}$ be a Morse function, let g^{TM} be a metric on TM. Let B be the set of critical points of f. If $x \in B$, let $\operatorname{ind}(x)$ be the index of f at x. We assume that for any $x \in B$, there exists a coordinate system $y = (y^1, \dots, y^n)$ near x such that 0 represents x, and moreover, near x,

$$
\begin{aligned}
g^{TM} &= \sum_1^n |dy^i|^2, \\
f(y) &= f(x) - \frac{1}{2} \sum_1^{\operatorname{ind}(x)} |y^i|^2 + \frac{1}{2} \sum_{\operatorname{ind}(x)+1}^{n} |y^i|^2.
\end{aligned}
\tag{2.12}
$$

Let ∇f be the gradient vector field of f. We assume that ∇f verifies the Smale transversality conditions.

In the Appendix, Laudenbach proves that the closed cells $\overline{W^u(x)}$ $(x \in B)$ are submanifolds of M with conical singularities. Therefore smooth forms can be integrated on the $\overline{W^u(x)}$'s $(x \in B)$.

The vector bundle F is canonically trivialized over each cell $W^u(x)$.

If $x \in B$, the line $[W^u(x)]$ has a canonical nonzero section $W^u(x)$. Let $W^u(x)^* \in [W^u(x)]^*$ be dual to $W^u(x) \in [W^u(x)]$, so that $\langle W^u(x), W^u(x)^* \rangle = 1$. If $\alpha \in \mathbb{F}$, the integral $W^u(x)^* \otimes \int_{W^u(x)} \alpha$ lies $[W^u x)]^* \otimes F_x$. Clearly if $\alpha \in \mathbb{F}^i$, $\int_{W^u(x)} \alpha$ is nonzero only if $\operatorname{ind}(x) = i$.

Definition 2.8. Let P_∞ be the map

$$
\alpha \in \mathbb{F} \to P_\infty \alpha = \sum_{x \in B} [W^u(x)]^* \otimes \int_{\overline{W^u(x)}} \alpha \in C^\bullet(W^u, F). \tag{2.13}
$$

Theorem 2.9. *The map P_∞ is a quasi-isomorphism of the $\mathbb{Z}$-graded complexes $(\mathbb{F}, d^{\mathbb{F}})$ and $(C^\bullet(W^u, F), \partial)$, which provides the canonical identification of the cohomology groups of both complexes.*

Proof. We use the notation of Section 1 c). Let $(\mathcal{D}'(M, F^*), d^{F^*})$ be the complex of currents on M with values in F^*. If $x \in B$, let $\delta_{\overline{W}^u(x)}$ be the current of integration on $\overline{W}^u(x)$.

Take $\beta \in C_\bullet(W^u, F^*)$. Then β can be written in the form

$$\beta = \sum_{x \in B} \beta_x [W^u(x)] \otimes f_x^*, \quad \beta_x \in \mathbb{R}, \quad f_x^* \in F_x^*. \tag{2.14}$$

If $f_x^* \in F_x^*$, we extend f_x^* to a flat section of F^* on $\overline{W}^u(x)$, which we still note f_x^*. Set

$$I(\beta) = \sum_{x \in B} \beta_x f_x^* \delta_{\overline{W}^u(x)}. \tag{2.15}$$

Then $I(\beta) \in \mathcal{D}'(M, F^*)$. By a result of Laudenbach [Appendix, Proposition 7], I is a quasi-isomorphism from $(C_\bullet(W^u, F^*), \partial)$ into $(\mathcal{D}'(M, F^*), d^{F^*})$. Let $\overset{\circ}{I}$: $H_\bullet(C(W^u, F^*), \partial) \to H_\bullet(M, F^*)$ be the induced isomorphism.

Take $i, 0 \leq i \leq n$, $\beta \in C_i(W^u, F^*)$. Then $I(\beta)$ vanishes near ∂V^i, and $dI(\beta) = I(\partial\beta)$ is supported in V^{i-1}. So $I(\beta)$ defines a homology class in $H_i(V^i, V^{i-1}, F^*) = C_i(W^u, F^*)$ which coincides tautologically with β.

It follows from the previous considerations that $\overset{\circ}{I}$ is indeed the canonical isomorphism $H_\bullet(C(W^u, F^*), \partial) \simeq H_\bullet(M, F^*)$. Also if $\alpha \in \Omega^\bullet(M, F)$, $\beta \in C_\bullet(W_u, F^*)$, then

$$\langle P_\infty \alpha, \beta \rangle = \langle \alpha, I(\beta) \rangle. \tag{2.16}$$

Therefore P_∞ is the transpose of I. Theorem 2.9 follows. $\square$

III. Berezin integrals and Morse functions

In this Section, we recall the construction by Mathai-Quillen [MQ] of Thom forms and of the transgressed Euler forms for Euclidean vector bundles in the Berezin integral formalism. Also we establish certain identities on Berezin integrals involving the gradient vector field of a smooth function. Finally when this function is a Morse function, we prove certain mysterious identities involving currents which are constructed using Berezin integrals.

This Section is organized as follows. In a), we introduce the Berezin integral. In b), we construct the Thom forms of Mathai-Quillen [MQ] on the total space of an Euclidean vector bundle with connection. In c), we recall results of [BGS4] on the convergence of the Mathai-Quillen Thom forms, as a parameter T tends to $+\infty$. In d) we construct a transgressed Euler class, which is a current on the total space of a vector bundle.

In e), we specialize the previous considerations to the case of the tangent bundle. In f), we establish a crucial symmetry property for a Berezin integral involving a gradient vector field. In g), we introduce a canonical section of an exterior algebra. In h), we establish transgression formulas for currents which are expressed as Berezin integrals. In i) and j), we take the limit, as a parameter T tends to $+\infty$, of certain identities of currents associated to a Morse function. Finally, in k), we consider the case where the metric on the tangent space is flat near the critical points of the Morse function.

As we will see in Section 7e), the identity established in Section 3j) is in fact a consequence of the proof of Theorem 0.2. It has seemed convenient to us to give a direct proof of these identities. Also the symmetry property of Section 3f) will be of constant use in the sequel.

For an introduction to Berezin integrals and their application to the construction of Thom forms and of Euler forms, we also refer to Berline-Getzler-Vergne [BeGV, Chapter 1].

This Section is self-contained.

a) The Berezin integral

Let E and V be real finite dimensional vector spaces of dimension n and m.

Let g^E be an Euclidean metric on E. Let $e_1, \cdots, e_n$ be an orthonormal base of E, and let $e^1, \cdots, e^n$ be the corresponding dual base of E^*.

Assume temporarily that E is oriented and that $e_1, \cdots, e_n$ is an oriented base of E. Let $\int^B$ be the linear map from $\Lambda(V^*)\widehat{\otimes}\Lambda(E^*)$ into $\Lambda(V^*)$ which is such that if $\alpha \in \Lambda(V^*), \beta \in \Lambda(E^*)$, then

$$\int^B \alpha\beta = 0 \qquad \text{if} \quad \deg \beta < \dim E, \tag{3.1}$$
$$\int^B \alpha e^1 \wedge \cdots \wedge e^n = \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^{\frac{n}{2}}} \alpha.$$

More generally, let $o(E)$ be the orientation line of E. Then $\int^B$ defines a linear map from $\Lambda(V^*)\widehat{\otimes}\Lambda(E^*)$ into $\Lambda(V^*) \otimes o(E)$. The linear map $\int^B$ is called a Berezin integral.

In the sequel, we do not assume any more that E is oriented. Let A be an antisymmetric endomorphism of E. We identify A with the element of $\Lambda(E^*)$,

$$A = \frac{1}{2} \sum_{1 \leq i,j \leq n} \langle e_i, Ae_j \rangle \, e^i \wedge e^j. \tag{3.2}$$

By definition, the Pfaffian $\mathrm{Pf}[\frac{A}{2\pi}]$ of $\frac{A}{2\pi}$ is defined by the formula

$$\int^B \exp\left(\frac{-A}{2}\right) = \mathrm{Pf}\left[\frac{A}{2\pi}\right]. \tag{3.3}$$

Then $\mathrm{Pf}[\frac{A}{2\pi}]$ lies in $o(E)$. Clearly $\mathrm{Pf}[\frac{A}{2\pi}]$ vanishes if n is odd.

b) Vector bundles and Berezin integrals : the Mathai-Quillen Thom forms

Let M be a real manifold of dimension m. Let $\pi : E \to M$ be a real vector bundle of dimension n. Let g^E be an Euclidean metric on E.

Let ∇^E be an Euclidean connection on (E, g^E) and let $R^E = (\nabla^E)^2$ be the curvature of ∇^E. Then R^E is a smooth section of $\Lambda^2(T^*M) \otimes \mathrm{End}(E)$.

Also $\pi^*\nabla^E$ is an Euclidean connection on $\pi^*(E, g^E)$ and π^*R^E is the curvature of $\pi^*\nabla^E$. Moreover π^*R^E is a smooth section of $\Lambda^2(T^*E) \otimes \mathrm{End}(\pi^*E)$.

Let $e_1, \cdots, e_n$ be an orthonormal base of E and let $e^1, \cdots, e^n$ be the corresponding dual base of E^*. Let $f_1, \cdots, f_m$ be a base of TM, and let $f^1, \cdots, f^m$ be the corresponding dual base of T^*M. We identify R^E with the section $\dot{R}^E$ of $\Lambda^2(T^*M)\widehat{\otimes}\Lambda^2(E^*)$

$$\dot{R}^E = \frac{1}{4} \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq \alpha,\beta \leq n}} \left\langle e_\alpha, R^E(f_i, f_j) e_\beta \right\rangle f^i \wedge f^j \wedge e^\alpha \wedge e^\beta. \tag{3.4}$$

Equivalently

$$\dot{R}^E = \frac{1}{2} \sum_{1 \leq \alpha,\beta \leq n} \left\langle e_\alpha, R^E e_\beta \right\rangle e^\alpha \wedge e^\beta. \tag{3.5}$$

The connection ∇^E defines a horizontal subspace T^HE of TE such that $TE = T^HE \oplus E$. Let P^E be the projection $TE \to E$ and let $P^{E*} : E^* \to T^*E$ be the transpose of P^E. Then P^E is a section of $T^*E \otimes E$. If we identify E with E^* by the metric g^E, P^E can be considered as a section of $T^*E \otimes E^*$. Clearly

$$P^E = \sum_1^n \left(P^{E*}e^i\right) e^i. \tag{3.6}$$

Let Y be the generic element of E.

Definition 3.1. For $T \geq 0$, let A_T be the element of $(\Lambda(T^*E)\widehat{\otimes}\pi^*\Lambda(E^*))^{\mathrm{even}}$,

$$A_T = \frac{\pi^*\dot{R}^E}{2} + \sqrt{T}P^E + T|Y|^2. \tag{3.7}$$

Recall that we identify E with E^*. If $e \in E$, we will often write $\widehat{e}$ when e is considered as an element of $\Lambda(E^*)$, and we still denote $P^{E*}e$ the corresponding element of $\Lambda(T^*E)$.

The connection $\pi^*\nabla^E$ acts as a differential operator on smooth sections of $\Lambda(T^*E)\widehat{\otimes}\pi^*\Lambda(E^*)$. Also if $e \in E$, the interior multiplication i_e acts naturally on $\Lambda(E^*)$, and also as a derivation of the graded algebra $\Lambda(T^*E)\widehat{\otimes}\pi^*\Lambda(E^*)$. To indicate clearly that i_e only acts on the second factor $\pi^*\Lambda(E^*)$ of $\Lambda(T^*E)\widehat{\otimes}\pi^*\Lambda(E^*)$, we will write $i_{\widehat{e}}$ instead of i_e. In particular we have

$$\pi^*\dot{R}^E = \frac{1}{2} \sum_{1\leq\alpha,\beta\leq n} \left\langle e_\alpha, \left(\pi^*R^E\right) e_\beta\right\rangle \widehat{e^\alpha} \wedge \widehat{e^\beta}, \tag{3.8}$$

$$P^E = \sum_1^n P^{E*}e^i \wedge \widehat{e^i}.$$

The following result is proved in [MQ, Section 6] and [BeGV, Lemma 1.85 and Propositions 1.87 and 1.88].

Theorem 3.2. *The following identities hold*

$$\left[\pi^*\nabla^E + 2\sqrt{T}\, i_{\widehat{Y}}, A_T\right] = 0, \qquad \frac{\partial A_T}{\partial T} = \left[\pi^*\nabla^E + 2\sqrt{T}\, i_{\widehat{Y}}, \frac{\widehat{Y}}{2\sqrt{T}}\right]. \tag{3.9}$$

Proof. The Bianchi identity asserts that

$$\left[\pi^*\nabla^E, \pi^*\dot{R}^E\right] = 0. \tag{3.10}$$

Also

$$\left[2\sqrt{T}\, i_{\widehat{Y}}, \pi^*\frac{\dot{R}^E}{2}\right] = -\sqrt{T} \sum_{1\leq\alpha\leq n} \left\langle \pi^*R^E Y, e_\alpha\right\rangle \widehat{e^\alpha}. \tag{3.11}$$

Moreover, one verifies easily that

$$\left[\pi^*\nabla^E, \sqrt{T}\, P^E\right] = \sqrt{T} \sum_{1\leq\alpha\leq n} \left\langle \pi^*R^E Y, e_\alpha\right\rangle \widehat{e^\alpha}. \tag{3.12}$$

From (3.11), (3.12), we get

$$\left[2\sqrt{T}\, i_{\widehat{Y}}, \pi^* \frac{\dot{R}^E}{2}\right] + \left[\nabla^E, \sqrt{T}\, P^E\right] = 0. \tag{3.13}$$

Moreover

$$\left[\pi^*\nabla^E, T|Y|^2\right] = 2TP^{E*}Y, \qquad \left[2\sqrt{T} i_{\widehat{Y}}, \sqrt{T} P^E\right] = -2TP^{E*}\, Y, \tag{3.14}$$

and so

$$\left[\pi^*\nabla^E, T|Y|^2\right] + \left[2\sqrt{T}\, i_{\widehat{Y}}, \sqrt{T}\, P^E\right] = 0. \tag{3.15}$$

From (3.10), (3.13), (3.15), we get the first identity in (3.9). Moreover

$$\frac{\partial}{\partial T} A_T = \frac{1}{2\sqrt{T}} P^E + |Y|^2. \tag{3.16}$$

Using (3.16), one obtains the second identity in (3.9). □

Let π_* denote the integral along the fibre of forms on E taking value in $\pi^* o(E)$.

We will apply the formalism of the Berezin integral developed in Section 3a), with $V = TE$. If ω is a smooth section of $\Lambda(T^*E)\widehat{\otimes}\pi^*\Lambda(E^*)$ over E, $\int^B \omega$ is a smooth section of $\Lambda(T^*E)\widehat{\otimes}\pi^* o(E)$, i.e. a smooth differential form over E with values in $\pi^* o(E)$.

Set

$$e\left(E, \nabla^E\right) = \mathrm{Pf}\left[\frac{R^E}{2\pi}\right]. \tag{3.17}$$

Then $e(E, \nabla^E)$ is a smooth closed section of $\Lambda^{\dim E}(T^*M) \otimes o(E)$. The form $e(E, \nabla^E)$ is a Chern-Weil representative of the rational Euler class of E. Of course, if $n = \dim E$ is odd, then

$$e\left(E, \nabla^E\right) = 0. \tag{3.18}$$

Definition 3.3. For $T \geq 0$ and $T > 0$, let α_T and β_T be the forms over E

$$\begin{aligned} \alpha_T &= \int^B \exp\left(-A_T\right), \\ \beta_T &= \int^B \frac{\widehat{Y}}{2\sqrt{T}} \exp\left(-A_T\right). \end{aligned} \tag{3.19}$$

We will establish a fundamental result which was first proved in Mathai-Quillen [MQ, Theorem 6.4].

Theorem 3.4. *For any $T \geq 0$, the forms α_T have degree n, are closed and their cohomology class does not depend on T. For $T > 0$, the forms α_T represent the Thom class of E, so that*

$$\pi_* \alpha_T = 1 \tag{3.20}$$

For $T > 0$, the forms β_T have degree $n-1$. Finally

$$\begin{aligned} \alpha_0 &= \pi^* e\left(E, \nabla^E\right), \\ \beta_T &= \frac{-i_Y \alpha_T}{2T},\ T > 0, \\ \frac{\partial \alpha_T}{\partial T} &= -d\beta_T,\ T > 0. \end{aligned} \tag{3.21}$$

Proof. Elements of $\Lambda(T^*E)\widehat{\otimes}\Lambda(E^*)$ have a partial degree in $\Lambda(T^*E)$ and also a partial degree in $\Lambda(E^*)$. Then A_T is a sum of forms of type (p,p), and so $\exp(-A_T)$ is also a sum of forms of type (p,p). Therefore the forms α_T have degree n, and the forms β_T have degree $n-1$.

If ω is a section of $\Lambda(T^*E)\widehat{\otimes}\Lambda(E^*)$, then

$$\int^B i_{\widehat{Y}}\, \omega = 0. \tag{3.22}$$

Using Theorem 3.2, we get

$$\left[\pi^*\nabla^E + 2\sqrt{T}\, i_{\widehat{Y}}, \exp\left(-A_T\right)\right] = 0. \tag{3.23}$$

Therefore, by (3.22), (3.23), we obtain

$$d\int^B \exp\left(-A_T\right) = \int^B \left[\pi^*\nabla^E + 2\sqrt{T}\, i_{\widehat{Y}}, \exp\left(-A_T\right)\right] = 0, \tag{3.24}$$

and so the forms α_T are closed.

By (3.3), we get the first identity in (3.21). Also

$$i_Y A_T = \sqrt{T}\, \widehat{Y}. \tag{3.25}$$

Therefore
(3.26)

$$i_Y \int^B \exp(-A_T) = \int^B (-i_Y A_T) \exp(-A_T) = \int^B \left(-\sqrt{T}\,\widehat{Y}\right) \exp(-A_T).$$

The second identity in (3.21) follows.

Moreover by using Theorem 3.2 and (3.22), we get

$$\frac{\partial \alpha_T}{\partial T} = -\int^B \frac{\partial A_T}{\partial T} \exp(-A_T) \tag{3.27}$$

$$= -\int^B \left[\pi^* \nabla^E + 2\sqrt{T}\, i_{\widehat{Y}}, \frac{\widehat{Y}}{2\sqrt{T}} \exp(-A_T)\right] = -d\,\beta_T.$$

Finally, for $T > 0$

$$\pi_* \alpha_T = \int_E \exp\left(-T|Y|^2\right) T^{n/2} \int^B (-1)^n P^{E*} e^1 \wedge \widehat{e^1} \wedge \cdots \wedge P^{E*} e^n \wedge \widehat{e^n} \tag{3.28}$$

$$= \int_E \exp(-T|Y|)^2\, T^{n/2} \int^B (-1)^{n + \frac{(n-1)n}{2}} P^{E*} e^1 \wedge \cdots \wedge P^{E*} e^n \wedge \widehat{e^1} \wedge \cdots \wedge \widehat{e^n} = 1.$$

The proof of Theorem 3.4 is completed. □

c) Convergence of the Mathai-Quillen currents over E

Let $o(TM)$ be the orientation bundle of TM. We identify M to the zero section of E. If $k \in \mathbf{N}$, and if K is a compact set in E, let $\|\ \|_{C^k_K(E)}$ be a natural norm on the Banach space $C^k_K(E)$ of forms in E with values in $\pi^* o(TM)$, which are continuous with k continuous derivatives, and whose support is included in K.

Let δ_M be the current of integration on M. If μ is a smooth compactly supported form on E with values in $\pi^* o(TM)$, then $\int_E \mu \delta_M = \int_M \mu$.

Theorem 3.5. *Let K be a compact subset of E. There exists a constant $C > 0$ such that for any smooth form μ on E with values in $\pi^* o(TM)$ whose support is included in K, for $T \geq 1$, then*

$$\left|\int_E \mu\,(\alpha_T - \delta_M)\right| \leq \frac{C}{\sqrt{T}} \|\mu\|_{C^1_K(E)},$$
$$\left|\int_E \mu \beta_T\right| \leq \frac{C}{T^{3/2}} \|\mu\|_{C^1_K(E)}. \tag{3.29}$$

Proof. The proof of Theorem 3.5 is essentially the same as the proof of [BGS4, Theorem 3.12]. It is left to the reader. □

d) A transgressed Euler class

Definition 3.6. Let $\psi(E,\nabla^E)$ be the current on E with values in $o(E)$,

$$\psi\left(E,\nabla^E\right)=\int_0^{+\infty}\beta_T dT. \tag{3.30}$$

The restriction of $\psi(E,\nabla^E)$ to the sphere bundle of E was first constructed in Mathai-Quillen [MQ, Section 7]. In view of Theorem 3.5, it is clear that the current $\psi(E,\nabla^E)$ is well-defined.

Recall that M is identified to the zero section of E. The normal bundle to M in E is exactly E.

Let g'^E be another metric on E, and let ∇'^E be an Euclidean connection on E with respect to g'^E. Let $\widetilde{e}(E,\nabla^E,\nabla'^E)$ denote the Chern-Simons class of forms of degree $n-1$ over M with values in $o(E)$, which is defined modulo exact forms, such that

$$d\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)=e\left(E,\nabla'^E\right)-e\left(E,\nabla^E\right). \tag{3.31}$$

If n is odd, then

$$\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)=0. \tag{3.32}$$

For the definition and properties of the wave front set of a current, we refer to [Ho, Chapter VIII].

Theorem 3.7. *The current $\psi(E,\nabla^E)$ has degree $n-1$. If λ is a smooth function on E with values in $\mathbb{R}^*$, under the map $e\in E\to\lambda e\in E,\psi(E,\nabla^E)$ is changed into $\psi(E,\nabla^E)$ for $\lambda>0$, into $(-1)^n\psi(E,\nabla^E)$ for $\lambda<0$. The current $\psi(E,\nabla^E)$ is locally integrable on E. The wave front set of $\psi(E,\nabla^E)$ is included in E^*. Also $\psi(E,\nabla^E)$ verifies the equation of currents over E*

$$d\psi\left(E,\nabla^E\right)=\pi^*e\left(E,\nabla^E\right)-\delta_M. \tag{3.33}$$

The restriction of $-\psi(E,\nabla^E)$ to the fibres of E coincides with the solid angle form of the fibre associated to the metric g^E.

If g'^E is another metric on E, and if ∇'^E is a connection on E which preserves the metric g'^E, then

$$\psi\left(E,\nabla'^E\right)-\psi\left(E,\nabla^E\right)=\pi^*\widetilde{e}\left(E,\nabla^E,\nabla'^E\right) \text{ modulo exact currents.} \tag{3.34}$$

Proof. By Theorem 3.4, $\psi(E,\nabla^E)$ has degree $n-1$. By proceeding as in [BGS4, Theorems 3.14 and 3.15], we see that $\psi(E,\nabla^E)$ is locally integrable, and that the wave front set of $\psi(E,\nabla^E)$ is included in E^*. Equation (3.33) follows from Theorems 3.4 and 3.5.

By (3.21) and (3.33), we know that $i_Y\psi=0, i_Yd\psi=0$. So if λ is a smooth function from E into $\mathbb{R}_+^*$, we see that $\psi(E,\nabla^E)$ is invariant under the map $Y\in E\to\lambda Y\in E$. Using the explicit formula (3.19), we find that under the map $Y\in E\to -Y\in E, \psi(E,\nabla^E)$ is changed into $(-1)^n\psi(E,\nabla^E)$.

Let ω be the volume form in the fibres E. Using (3.21), one verifies easily that the restriction of $-\psi(E,\nabla^E)$ to the fibres of E is given by

$$\frac{\Gamma(n/2)}{2\pi^{n/2}}\,\frac{i_Y\omega}{|Y|^n}, \tag{3.35}$$

which is the solid angle form of the fibres.

Finally equation (3.34) follows from equation (3.33) and from a simple deformation argument which is left to the reader. □

Remark 3.8. Assume that $\dim E\le\dim M$. Let s be a smooth section of E. Set

$$M'=\{x\in M; s(x)=0\}. \tag{3.36}$$

Suppose that over M', ds has maximal rank $\dim E$. Then M' is a smooth submanifold of M. Let $N_{M'/M}$ be the normal bundle to M' in M. Then $ds: N_{M'/M}\to E_{|M'}$ is an identification of vector bundles. Since the wave front set of $\psi(E,\nabla^E)$ is included in E^*, by [Ho, Theorem 8.2.4], the pulled-back current $s^*\psi(E,\nabla^E)$ on M is well-defined, and its wave front set is included in $N^*_{M'/M}$. Moreover

$$ds^*\psi\left(E,\nabla^E\right)=e\left(E,\nabla^E\right)-\delta_{M'}. \tag{3.37}$$

Also by proceeding as in [BGS4, Theorem 3.15], one verifies easily that the current $s^*\psi(E,\nabla^E)$ is locally integrable on M.

e) The Berezin integral formalism over the tangent space.

Let s be a smooth section of E over M. Recall that for $T \geq 0, A_T$ is a smooth section of E over $\Lambda(T^*E)\widehat{\otimes}\pi^*\Lambda(E^*)$. The pull-back s^*A_T, where the pull-back acts non trivially on the factor $\Lambda(T^*E)$, is now a smooth section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(E^*)$.

Let g^{TM} be a smooth metric on TM. Let ∇^{TM} be the Levi-Civita connection on (TM, g^{TM}), and let $R^{TM} = (\nabla^{TM})^2$ be its curvature. Let ∇^{T^*M} be the corresponding connection on T^*M.

We will apply the construction of Sections 3a)–3d) to (TM, g^{TM}) equipped with the connection ∇^{TM}. In particular π now denotes the projection $TM \to M$ and n is the dimension of M. Also, for $T \geq 0$, A_T is a smooth section of $\Lambda(T^*TM)\widehat{\otimes}\pi^*\Lambda(T^*M)$. If s is a smooth section of TM over M, s^*A_T is then a smooth section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$.

If ω is a smooth section of $\Lambda(T^*M)$, we identify ω with the section $\omega\widehat{\otimes}1$ of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$. Also $\widehat{\omega}$ will denote the corresponding section $1 \otimes \omega$ of $\Lambda(T^*M)\widehat{\otimes}\ \Lambda(T^*M)$.

Let $e_1, \cdots, e_n$ be an orthonormal base of TM, and let $e^1, \cdots, e^n$ be the corresponding dual base of T^*M. We identify R^{TM} to the smooth section $\dot{R}^{TM}$ of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$ given by

$$\dot{R}^{TM} = \frac{1}{4} \sum_{\substack{1 \leq i,j \leq n \\ 1 \leq, \alpha, \beta \leq n}} \left\langle e_\alpha, R^{TM}(e_i, e_j) e_\beta \right\rangle e^i \wedge e^j \wedge \widehat{e}^\alpha \wedge \widehat{e}^\beta. \tag{3.38}$$

Recall that we identify TM and T^*M by the metric g^{TM}.

Proposition 3.9. *Let s be a smooth section of TM. Then for $T \geq 0$, the following identity holds*

$$s^*A_T = \frac{\dot{R}^{TM}}{2} + \sqrt{T}\sum_1^n e^i \wedge \widehat{\nabla_{e_i}^{TM} s} + T|s|^2. \tag{3.39}$$

Proof. Formula (3.39) follows directly from Definition 3.1. □

f) Berezin integral and gradient vector fields : a symmetry property

We make the same assumptions as in Section 3 (e). Let f be a smooth function of M into $\mathbb{R}$. The differential df is a smooth section of T^*M. Let ∇f be the corresponding gradient vector field, which is a section of TM.

From Proposition 3.9, we get the following identity.

Proposition 3.10. *For* $T \geq 0$, *the following identity holds*

$$(\nabla f)^* A_T = \frac{\dot{R}^{TM}}{2} + \sqrt{T} \sum_1^n e^i \wedge \widehat{\nabla_{e_i}^{TM} \nabla} f + T|df|^2. \tag{3.40}$$

Let φ be the algebra homomorphism from $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$ into itself, which is such that if $\omega \in \Lambda(T^*M)$, then

$$\begin{aligned} \varphi(\omega) &= \widehat{\omega}, \\ \varphi(\widehat{\omega}) &= \omega. \end{aligned} \tag{3.41}$$

Proposition 3.11. *For* $T \geq 0$, *the following identity holds*

$$\varphi(\nabla f)^* A_T = (-\nabla f)^* A_T. \tag{3.42}$$

Proof. The basic symmetry property of the curvature tensor R^{TM} immediately shows that

$$\varphi\, \dot{R}^{TM} = \dot{R}^{TM}. \tag{3.43}$$

Also

$$\sum_1^n e^i \wedge \widehat{\nabla_{e_i}^{TM} \nabla} f = \sum_1^n \left\langle \nabla_{e_i}^{T^*M} df, e_j \right\rangle e^i \wedge \widehat{e^j}. \tag{3.44}$$

Since the connection ∇^{TM} is torsion free, we get from (3.44),

$$\sum_1^n e^i \wedge \widehat{\nabla_{e_i}^{TM} \nabla} f = \sum_1^n \left\langle \nabla_{e_i}^{T^*M} df, e_j \right\rangle e^j \wedge \widehat{e^i}, \tag{3.45}$$

and so

$$
\varphi\left(\sum_{1}^{n} e^{i}\wedge\widehat{\nabla_{e_i}^{TM}\nabla f}\right)=-\sum_{1}^{n}\left\langle\nabla_{e_i}^{T^*M}df,e_j\right\rangle e^{i}\wedge\widehat{e^{j}}
=-\sum_{1}^{n} e_i\wedge\widehat{\nabla_{e_i}^{TM}\nabla f}. \tag{3.46}
$$

Proposition 3.11 follows from (3.45), (3.46). □

The Berezin integral $\int^{B}$ maps smooth section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$ into smooth section of $\Lambda(T^*M)\otimes o(TM)$.

Definition 3.12. For $T\geq 0$, let B_T be the smooth section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$ over M,

$$
B_T=(\nabla f)^*\left(A_T\right). \tag{3.47}
$$

In the sequel, we will say that $\alpha\in\Lambda^{p}(T^*M)\widehat{\otimes}\Lambda^{q}(T^*M)$ is of type (p,q).

Theorem 3.13. *Let α be a smooth section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$ which is of type (p,p) $(0\leq p\leq n)$. Then,*

$$
\int^{B}\alpha\exp\left(-B_T\right)=(-1)^{p}\int^{B}\varphi(\alpha)\exp\left(-B_T\right). \tag{3.48}
$$

Proof. One has the easy identity

$$
\int^{B}\alpha=(-1)^{n}\int^{B}\varphi(\alpha). \tag{3.49}
$$

If we apply (3.49) to $\alpha\exp(-(\nabla f)^*A_T)$, using (3.42), we get

$$
\int^{B}\alpha\exp\left(-(\nabla f)^*\left(A_T\right)\right)=(-1)^{n}\int^{B}\varphi(\alpha)\exp\left(-(-\nabla f)^*\left(A_T\right)\right). \tag{3.50}
$$

Also one verifies easily that if α is of type (p,p), then

$$
\int^{B}\varphi(\alpha)\exp\left(-(-\nabla f)^*\left(A_T\right)\right)=(-1)^{n-p}\int^{B}\varphi(\alpha)\exp\left(-(\nabla f)^*\left(A_T\right)\right). \tag{3.51}
$$

From (3.49)–(3.51), we get (3.48). □

g) The canonical section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$

We make the same assumptions as in Sections 3e), 3f), and we use the same notation.

Definition 3.14. Let L be the smooth section of $\Lambda(T^*M)\widehat{\otimes}\Lambda(T^*M)$

$$L = \frac{1}{2}\sum_1^n e^i \wedge \widehat{e^i}. \tag{3.52}$$

Clearly L does not depend on the choice of the orthonormal base $e_1, \cdots, e_n$.

Proposition 3.15. *The following identity holds*

$$[\nabla^{TM}, L] = 0. \tag{3.53}$$

Proof. Since the connection ∇^{TM} is torsion free, we get (3.53). □

h) A variation formula for forms over M

We make the same assumptions as in Section 3f).

Proposition 3.16. *For any $T > 0$, the following identity of sections of $\Lambda^{\max}(T^*M)\otimes o(TM)$ holds*

$$\begin{aligned}\frac{\partial}{\partial T}\int^B L\exp(-B_T) = {} & -\sqrt{T}f\frac{\partial}{\partial T}\int^B \exp(-B_T) \\ & -\frac{d}{2}\int^B \left(\frac{L}{\sqrt{T}} + f\right)\widehat{df}\exp(-B_T).\end{aligned} \tag{3.54}$$

Proof. Using Theorem 3.2, we get

$$\begin{aligned}\frac{\partial}{\partial T}\int^B L\exp(-B_T) &= -\int^B L\frac{\partial B_T}{\partial T}\exp(-B_T) \\ &= -\int^B L\left[\nabla^{TM} + 2\sqrt{T}\, i_{\widehat{\nabla f}}, \frac{\widehat{df}}{2\sqrt{T}}\exp(-B_T)\right] \\ = -\frac{d}{2}\int^B \frac{L}{\sqrt{T}}\widehat{df}\exp(-B_T) &+ \int^B \left[\nabla^{TM} + 2\sqrt{T}\, i_{\widehat{\nabla f}}, L\right]\frac{\widehat{df}}{2\sqrt{T}}\exp(-B_T).\end{aligned} \tag{3.55}$$

By Proposition 3.15, we know that

$$\left[\nabla^{TM} + 2\sqrt{T}\, i_{\widehat{\nabla f}}, L\right] = -\sqrt{T}\, df. \tag{3.56}$$

So using (3.56) and Theorem 3.4, we get

$$\begin{aligned}
&\int^B \left[\nabla^{TM} + 2\sqrt{T}\, i_{\widehat{\nabla f}}, L\right] \frac{\widehat{df}}{2\sqrt{T}} \exp(-B_T) \\
&= -\frac{d}{2}\int^B f\widehat{df} \exp(-B_T) + f d\int^B \frac{\widehat{df}}{2} \exp(-B_T) \\
&= -\frac{d}{2}\int^B f\widehat{df} \exp(-B_T) - \sqrt{T} f \frac{\partial}{\partial T}\int^B \exp(-B_T).
\end{aligned} \tag{3.57}$$

From (3.55)–(3.57), we get (3.54). □

Theorem 3.17. *For any* $T_0 \geq 0$, *the following identity of smooth sections of* $\Lambda^{\max}(T^*M) \otimes o(TM)$ *holds*

$$\begin{aligned}
\int^B L\left(\exp(-B_{T_0}) - \exp(-B_0)\right) &= -\sqrt{T_0} f \int^B \exp(-B_{T_0}) \\
&+ \frac{f}{2}\int_0^{T_0} \left(\int^B \exp(-B_T)\right) \frac{dT}{\sqrt{T}} \\
&- \frac{d}{2}\int_0^{T_0} \left(\int^B \left(\frac{L}{\sqrt{T}} + f\right) \widehat{df} \exp(-B_T)\right) dT.
\end{aligned} \tag{3.58}$$

Proof. Using (3.54) and integrating by parts, we get (3.58). □

i) The limit as $T \to +\infty$ of certain currents over M

We now assume that f is a Morse function, i.e. f has isolated critical points $x_1, \cdots x_q, \cdots$ such that $d^2 f(x_1), \cdots, d^2 f(x_q), \cdots$ are nondegenerate quadratic forms over $T_{x_1}M, \cdots T_{x_q}M, \cdots$. For $i = 1, \cdots, q, \cdots$ let A_{x_i} be the self-adjoint element of $\mathrm{End}(T_{x_i}M)$ such that if $U, V \in T_{x_i}M$, then

$$\langle A_{x_i} U, V\rangle = d^2 f(x_i)(U, V). \tag{3.59}$$

Let $\mathrm{ind}(x_i)$ be the index of f at x_i, i.e. the number of negative eigenvalues of A_{x_i}.

Theorem 3.18. *Let K be a compact subset of M. There exists a constant $C > 0$ such that if g is smooth function from M into $\mathbb{R}$ whose support is included in K, and if μ is a smooth 1-form on M whose support is included in K, then*

$$\left|\int_M g\left(\int^B \exp(-B_T) - \sum(-1)^{\mathrm{ind}(x_p)}\delta_{x_p}\right)\right| \leq \frac{C}{T}\|g\|_{C^2_K(M)}, \tag{3.60}$$

$$\left|\int_M g \int^B L\exp(-B_T)\right| \leq \frac{C}{\sqrt{T}}\|g\|_{C^0_K(M)},$$

$$\left|\int_M \mu \int^B \widehat{df}\exp(-B_T)\right| \leq \frac{C}{T^{3/2}}\|\mu\|_{C^1_K(M)},$$

$$\left|\int_M \mu \int^B \frac{L}{\sqrt{T}}\widehat{df}\exp(-B_T)\right| \leq \frac{C}{T^{5/2}}\|\mu\|_{C^1_K(M)}.$$

Proof. For notational simplicity, we assume that M is compact, and that f has exactly q critical points. Let $a > 0$ be the injectivity radius of (M, g^{TM}). For $0 < \eta < a$, let $B^M(x_i, \eta)$ be the open ball of center x_i and radius η.

Take $\varepsilon > 0$ such that $0 < \varepsilon < a/2$ and that the balls $B^M(x_i, 2\varepsilon)$ do not intersect each other. Clearly, there exist $c > 0, C > 0$ such that for $T \geq 0$,

$$|\exp(-B_T)| \leq c\exp(-CT) \quad \text{on} \quad M \backslash \bigcup_1^q B^M(x_p, \varepsilon). \tag{3.61}$$

We fix $p, 1 \leq p \leq q$. Let $y = (y^1, \cdots, y^n) \in T_{x_p}M$ be a geodesic coordinate system centered at x_p such that $(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n})$ is an orthonormal base of $T_{x_p}M$, with respect to which the matrix A_{x_p} is diagonal with diagonal entries $\lambda_1, \cdots, \lambda_n$. Of course $0 \in T_{x_p}M$ is identified with $x_p \in M$.

For $T > 0$, let σ_T be the map $y \in T_{x_p}M \to \frac{y}{\sqrt{T}} \in T_{x_p}M$. Then

$$\int_{|y|\leq\varepsilon} g \int^B L\exp(-B_T) = \int_{|y|\leq\varepsilon\sqrt{T}} (\sigma_T^* g)\, \sigma_T^* \int^B L\exp(-B_T). \tag{3.62}$$

Now

$$\sigma_T^* L = \frac{1}{2\sqrt{T}}\sum_1^n e^i \wedge \widehat{e}^i. \tag{3.63}$$

Using (3.61)–(3.63), we easily obtain the second inequality in (3.60).

Also

$$(3.64)\quad \sigma_T^* B_T = \frac{1}{2T}\dot{R}^{TM}_{\frac{y}{\sqrt{T}}} + \sum_{1\le i,j\le n} \left\langle \nabla^{T^*M}_{e_i} df, e_j \right\rangle_{\frac{y}{\sqrt{T}}} e^i \wedge \widehat{e^j} + T \left| df\left(\frac{y}{\sqrt{T}}\right)\right|^2 .$$

Moreover

$$(3.65)\qquad \left(\sigma_T^* g\right)(y) = g(x_p) + g'(x_p)\frac{y}{\sqrt{T}} + \|g\|_{C^2(M)} O\left(\frac{|y|^2}{T}\right),$$

$$\sum_{1\le i,j\le n} \left\langle \nabla^{T^*M}_{e_i} df, e_j \right\rangle_{\frac{y}{\sqrt{T}}} e^i \wedge \widehat{e^j} + T \left| df\left(\frac{y}{\sqrt{T}}\right)\right|^2$$
$$= \sum_{1\le i\le n} \lambda_i e^i \wedge \widehat{e^i} + \sum_{1\le i\le n} \lambda_i^2 \left|y^i\right|^2$$
$$+\frac{1}{\sqrt{T}} \sum_{1\le i,j\le n} \left\langle \nabla^{T^*M}_{y} \nabla^{T^*M}_{e_i} df(x_p), e_j \right\rangle e^i \wedge \widehat{e^j}$$
$$+\frac{1}{\sqrt{T}} \left[|df|^2(x_p)\right]^{(3)}(y,y,y) + \frac{1}{T} O\left(|y|^2 + |y|^4\right).$$

The key fact is that in (3.65), the terms which appear with the weight $\frac{1}{\sqrt{T}}$ are odd polynomials in the variables $(y^1, \cdots, y^n)$, whose integral with respect to a Gaussian measure is 0. By proceeding as in (3.28), we obtain the first inequality in (3.60).

Clearly

(3.66)
$$\int_{|y|\le\varepsilon} \mu \int^B \widehat{df} \exp\left(-B_T\right) = \int_{|y|\le\varepsilon\sqrt{T}} \left(\sigma_T^*\mu\right) \int^B \widehat{df}\left(\frac{y}{\sqrt{T}}\right) \exp\left(-\sigma_T^* B_T\right).$$

By proceeding as before, we find easily that

$$(3.67)\qquad \lim_{T\to+\infty} T \int_{|y|\le\varepsilon\sqrt{T}} \sigma_T^*\mu \int^B \widehat{df}\left(\frac{y}{\sqrt{T}}\right) \exp\left(-\sigma_T^* B_T\right)$$
$$= \int_{T_{x_p}M} \mu(x_p) \int^B \widehat{A_{x_p} y} \exp\left(-\sum_1^n \lambda_i dy^i \wedge \widehat{dy^i} - \sum_1^n \lambda_i^2 \left|y^i\right|^2\right) = 0.$$

From (3.67), we easily deduce the third inequality in (3.60). To prove the last inequality, we use (3.63) and we proceed as before. □

j) An identity of currents over M

By Theorem 3.18, it is clear that the currents over M

$$\int_0^{+\infty}\left(\int^B \exp\left(-B_T\right)-\sum(-1)^{\operatorname{ind}(x_p)}\delta_{x_p}\right)\frac{dT}{\sqrt{T}}, \tag{3.68}$$
$$\int_0^{+\infty}\left(\int^B\left(\frac{L}{\sqrt{T}}+f\right)\widehat{df}\exp\left(-B_T\right)\right)dT,$$

are well-defined.

Observe that if n is even, then

$$\int^B L\exp\left(-\frac{\dot{R}^{TM}}{2}\right)=0. \tag{3.69}$$

Theorem 3.19. *The following identity of currents of degree n with values in $o(TM)$ holds*

$$\int^B L\exp\left(-\frac{\dot{R}^{TM}}{2}\right) \tag{3.70}$$
$$=-\frac{f}{2}\int_0^{+\infty}\left(\int^B \exp\left(-B_T\right)-\sum(-1)^{\operatorname{ind}(x_p)}\delta_{x_p}\right)\frac{dT}{\sqrt{T}}$$
$$+\frac{d}{2}\int_0^{+\infty}\left(\int^B\left(\frac{L}{\sqrt{T}}+f\right)\widehat{df}\exp\left(-B_T\right)\right)dT.$$

Proof. Clearly, for $T_0\geq 0$,

$$\sqrt{T_0}f\int^B \exp\left(-B_{T_0}\right)-\frac{1}{2}\int_0^{T_0} f\int^B\exp\left(-B_T\right)\frac{dT}{\sqrt{T}} \tag{3.71}$$
$$=\sqrt{T_0}f\left(\int^B\exp\left(-B_{T_0}\right)-\sum(-1)^{\operatorname{ind}(x_p)}\delta_{x_p}\right)$$
$$-\frac{f}{2}\int_0^{T_0}\left(\int^B\exp\left(-B_T\right)-\sum(-1)^{\operatorname{ind}(x_p)}\delta_{x_p}\right)\frac{dT}{\sqrt{T}}.$$

Then we use the estimates of Theorem 3.18, and we make $T_0\to+\infty$ in (3.58). We get (3.70). $\square$

k) The case where the metric g^{TM} is flat near the critical points

From now on, we assume that near any critical point x_p of f, there exists a system of coordinates $y = (y^1, \cdots, y^n)$ such that

— x_p is represented by 0.

— The metric g^{TM} is exactly $\sum_1^n |dy^i|^2$.

— There are non zero constants $\lambda_1, \cdots \lambda_n$ such that near x_p

$$f(y) = f(x_p) + \frac{1}{2} \sum_1^n \lambda_i \left|y^i\right|^2 . \tag{3.72}$$

Of course if f is a Morse function, there always exists a system of coordinates $(y^1, \cdots, y^n)$ near the x_p's and a metric g^{TM} on TM such that the previous assumptions are verified. Recall that A_{x_p} is the self-adjoint element of $T_{x_p}M$ associated to the quadratic form $d^2 f(x_p)$. Then the matrix of A_{x_p} with respect to the basis $\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}$ has diagonal entries $\lambda_1, \cdots, \lambda_n$.

Let g be a smooth function on M with values in $\mathbb{R}$. We calculate $g''(x_p)$ using the coordinates $(y^1, \cdots, y^n)$ near x_p. Then $g''(x_p)$ is a symmetric bilinear form on $T_{x_p}M$. We identify $g''(x_p)$ to a self-adjoint element of $T_{x_p}M$. Then $g \to \mathrm{Tr}[A_{x_p}^{-2} g''(x_p)]$ defines a current of degree n on M, which we note $\mathrm{Tr}[A_{x_p}^{-2} \delta''_{x_p}]$.

Similarly let μ be a smooth 1-form on M, which we write near x_p as

$$\mu = \sum_1^n \mu_i(y) dy^i. \tag{3.73}$$

Set

$$\mathrm{Tr}\left[A_{x_p}^{-2} \frac{\partial \mu}{\partial y}(x_p)\right] = \sum_{i=1}^n \frac{1}{\lambda_i^2} \frac{\partial \mu_i}{\partial y^i}(x_p). \tag{3.74}$$

Equation (3.74) defines a current of degree $n-1$ over M, which we note $\mathrm{Tr}[A_{x_p}^{-2} \frac{\partial}{\partial y}]$

Theorem 3.20. *Let K be a compact subset of M. There exist constants $c > 0$, $C > 0$ such that if g is a smooth real function whose support is included in K and if μ is a smooth 1-form on M whose support is included in K, then for $T \geq 1$,*

$$\left|\int_M g\left(\int^B L\exp(-B_T) + \frac{1}{2\sqrt{T}}\sum(-1)^{\mathrm{ind}(x_p)}\operatorname{Tr}\left[A_{x_p}^{-1}\right]\delta_{x_p}\right)\right| \leq \frac{C}{T}\|g\|_{C^1_K(M)}, \tag{3.75}$$

$$\left|\int_M g\Big(f\int^B \exp(-B_T) - \sum(-1)^{\mathrm{ind}(x_p)}f(x_p)\delta_{x_p}\right.$$
$$\left.-\frac{1}{4T}\left(\sum(-1)^{\mathrm{ind}(x_p)}\operatorname{Tr}\left[A_{x_p}^{-1}\right]\delta_{x_p} + \sum(-1)^{\mathrm{ind}(x_p)}f(x_p)\operatorname{Tr}\left[A_{x_p}^{-2}\delta''_{x_p}\right]\right)\Big)\right|$$
$$\leq \frac{C}{T^{3/2}}\|g\|_{C^3_K(M)},$$

$$\left|\int_M \mu\left(\int^B \widehat{df}\exp(-B_T) + \frac{1}{2T^{3/2}}\sum(-1)^{\mathrm{ind}(x_p)}\operatorname{Tr}\left[A_{x_p}^{-2}\frac{\partial}{\partial y}\right]\right)\right|$$
$$\leq \frac{C}{T^2}\|\mu\|_{C^2_K(M)}.$$

Proof. As in the proof of Theorem 3.18, we assume that M is compact. Also we use the notation in the proof of Theorem 3.18. Here $\varepsilon > 0$ will be chosen small enough so that for any p, over $B^M(x_p, 2\varepsilon)$, the assumptions which are stated at the beginning of this Section 3 k) hold.

Then over $B^M(x_p, 2\varepsilon)$, $R^{TM} = 0$. Therefore

$$\begin{aligned}
&\int_{|y|\leq\varepsilon} g\int^B L\exp(-B_T) \\
&= \int_{|y|\leq\varepsilon} g\int^B L\exp\left(-\sqrt{T}\sum_1^n \lambda^i dy^i\wedge\widehat{dy^i} - T\sum_1^n \lambda_i^2\left|y^i\right|^2\right) \\
&= \int_{|y|\leq\varepsilon\sqrt{T}} g\left(\frac{y}{\sqrt{T}}\right)\int^B \frac{L}{\sqrt{T}}\exp\left(-\sum_1^n \lambda^i dy^i\wedge\widehat{dy^i} - \sum_1^n \lambda_i^2\left|y^i\right|^2\right).
\end{aligned} \tag{3.76}$$

Also one finds easily that

$$\int_{T_{x_p}M}\int^B L\exp\left(-\sum_1^n \lambda^i dy^i\wedge\widehat{dy^i} - \sum_1^n \lambda_i^2\left|y^i\right|^2)\right) \tag{3.77}$$

$$= -\frac{(-1)^{\mathrm{ind}(x_p)}}{2}\sum_1^n \frac{1}{\lambda_i}.$$

From (3.28), (3.76), (3.77), we get easily the first inequality in (3.75).

Similarly,

$$(3.78)\quad \int_{|y|\le\varepsilon} gf\int^B \exp(-B_T) = \int_{|y|\le\varepsilon\sqrt{T}} g\left(\frac{y}{\sqrt{T}}\right)\left(f(x_p)+\frac{1}{2T}\sum_1^n \lambda_k\left|y^k\right|^2\right)$$
$$\exp\left(-\sum_1^n \lambda_i dy^i\wedge\widehat{dy^i}-\sum_1^n \lambda_i^2\left|y^i\right|^2\right).$$

Also

$$(3.79)\qquad g\left(\frac{y}{\sqrt{T}}\right) = g(x_p)+\frac{g'(x_p)y}{\sqrt{T}}+\frac{1}{2T}g''(x_p)(y,y)+\frac{1}{T^{3/2}}O\left(|y|^3\right).$$

We now use the trivial identities

$$(3.80)\qquad \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} xe^{-x^2}dx = 0\ ;\ \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} x^2e^{-x^2}dx = \frac{1}{2},$$

and we easily obtain the second inequality in (3.75).

Let μ be a smooth 1-form on M, which we write as in (3.73) near x_p. Then

$$(3.81)\quad \int_{|y|\le\varepsilon}\mu\int^B \widehat{df}\exp(-B_T) = \int_{|y|\le\varepsilon\sqrt{T}}\frac{1}{\sqrt{T}}\left(\sum_1^n \mu_i\left(\frac{y}{\sqrt{T}}\right)dy^i\right)$$
$$\int^B \frac{1}{\sqrt{T}}\left(\sum_1^n \lambda_k y^k\widehat{dy^k}\right)\exp\left(-\sum_1^n \lambda_i dy^i\wedge\widehat{dy^i}-\sum_1^n \lambda_i^2\left|y^i\right|^2\right).$$

Also

$$(3.82)\qquad \mu_i\left(\frac{y}{\sqrt{T}}\right) = \mu_i(x_p)+\mu_i'(x_p)\frac{y}{\sqrt{T}}+\frac{1}{T}O\left(|y|^2\right).$$

Using (3.28), (3.60), (3.81), (3.82), we obtain the third inequality in (3.75). The proof of our Theorem is completed. □

Remark 3.21. By adding (3.58) and (3.70), for any $T_0>0$, we obtain the identity

$$(3.83)\quad \int^B L\exp(-B_{T_0}) = -\sqrt{T_0}f\left(\int^B \exp(-B_{T_0})-\sum(-1)^{\mathrm{ind}(x_p)}\delta_{x_p}\right)$$

$$-\frac{f}{2}\int_{T_0}^{+\infty}\left(\int^B \exp\left(-B_T\right)-\sum(-1)^{\mathrm{ind}(x_p)}\delta_{x_p}\right)\frac{dT}{\sqrt{T}}$$
$$+\frac{d}{2}\int_{T_0}^{+\infty}\left(\int^B\left(\frac{L}{\sqrt{T}}+f\right)\widehat{df}\exp\left(-B_T\right)\right)dT.$$

Clearly both sides of (3.83) have asymptotic expansions as $T_0 \to +\infty$.

By Theorem 3.20, the coefficient of $\frac{1}{\sqrt{T_0}}$ in the asymptotic expansion of the left-hand side of (3.83) is given by

$$-\frac{1}{2}\sum(-1)^{\mathrm{ind}(x_p)}\operatorname{Tr}\left[A_{x_p}^{-1}\right]\delta_{x_p}. \tag{3.84}$$

By Theorems 3.18 and 3.20, the coefficient of $\frac{1}{\sqrt{T_0}}$ in the asymptotic expansion of the right-hand side of (3.83) is given by

$$\begin{aligned}
&-\frac{1}{2}\sum(-1)^{\mathrm{ind}(x_p)}\operatorname{Tr}\left[A_{x_p}^{-1}\right]\delta_{x_p}\\
&-\frac{1}{2}\sum(-1)^{\mathrm{ind}(x_p)}f(x_p)\operatorname{Tr}\left[A_{x_p}^{-2}\delta''_{x_p}\right]\\
&-\frac{1}{2}d\sum(-1)^{\mathrm{ind}(x_p)}f(x_p)\operatorname{Tr}\left[A^{-2}(x_p)\frac{\partial}{\partial y}\right].
\end{aligned} \tag{3.85}$$

Now the sum of the last two terms in (3.85) is trivially equal to 0. Then (3.84) and (3.85) effectively coincide.

IV. Anomaly formulas for Ray-Singer metrics

The purpose of this Section is to establish the anomaly formulas for Ray-Singer metrics, which were stated in Theorem 0.1 of the introduction. These anomaly formulas will play an important role in our proof of our main result stated in Theorem 0.2.

To establish these anomaly formulas, we use local index theory techniques, in combination with the Berezin integral formalism of Section 3. Our local index techniques are different from the techniques of Getzler [G], even if they have some obvious relation to them. They will be used again in Section 13.

This Section is organized as follows. In a), given a flat Euclidean vector bundle (F, g^F), we associate a connection $\nabla^{F,e}$ preserving the metric g^F. In b), we construct the closed 1-form $\theta(F, g^F)$, which plays a critical role in the whole paper. In c), we give the anomaly formulas, which compare the Ray-Singer metrics associated to two couples of metrics on TM and F.

In d), we introduce the Clifford algebra of an Euclidean vector space E, and its natural actions on $\Lambda(E^*)$.

In e), we establish a crucial Lichnerowicz formula for the Hodge Laplacian D^2.

In f), we state a classical formula evaluating the variation of the Ray-Singer metrics as the constant term in the asymptotic expansion of the supertrace of a heat kernel.

In g), we introduce an extra Clifford variable σ, which will considerably simplify our local index calculations. In h) using local index techniques, we obtain an explicit infinitesimal formula for the variation of the Ray-Singer metric. Finally in i), we establish the anomaly formulas.

In this Section, we use the assumptions and notation of Section 2a) and of Section 3.

a) A canonical connection on a flat Euclidean vector bundle

Let M be a compact manifold of dimension n. Let F be a real flat vector bundle of dimension m on M, and let ∇^F be the flat connection on F. Let F^* be the dual of F, and let ∇^{F^*} be the corresponding flat connection on F^*.

Let g^F be an Euclidean metric on F. Let g^{F^*} be the corresponding metric on F^*. Let i be the corresponding identification $F \to F^*$. The connection $\nabla^{F*} = i^{-1}\nabla^{F^*}$ is also a flat connection on F, which coincides with ∇^F if and only if g^F is flat. Once F and F^* are identified, it will often be convenient to view F as a vector bundle equipped with two flat connections ∇^F and ∇^{F*}.

Definition 4.1. Let $\omega(F,g^F)$ be the 1-form on M taking values in self-adjoint endomorphisms of F

$$\omega\left(F,g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{4.1}$$

Then

$$\nabla^{F*} = \nabla^F + \omega\left(F,g^F\right). \tag{4.2}$$

Definition 4.2. Let $\nabla^{F,e}$ be the connection on F

$$\nabla^{F,e} = \nabla^F + \frac{1}{2}\omega\left(F,g^F\right). \tag{4.3}$$

From (4.2), (4.3), we get

$$\nabla^{F,e} = \frac{1}{2}\left(\nabla^F + \nabla^{F*}\right). \tag{4.4}$$

One verifies easily that the connection $\nabla^{F,e}$ preserves the metric g^F. It is canonically determined by the metric g^F.

Let $\nabla^{F^*,e}$ be the connection on the flat vector bundle F^* which is associated to the metric g^{F^*}. Then

$$\nabla^{F,e} = i^{-1}\nabla^{F^*,e}. \tag{4.5}$$

Proposition 4.3. *The curvature* $(\nabla^{F,e})^2$ *of the connection* $\nabla^{F,e}$ *is given by*

$$(\nabla^{F,e})^2 = -\frac{1}{4}\left(\omega\left(F, g^F\right)\right)^2. \tag{4.6}$$

Proof. Clearly

$$\left[\nabla^F, \omega\left(F, g^F\right)\right] = -\left(\omega\left(F, g^F\right)\right)^2. \tag{4.7}$$

Equation (4.6) follows from (4.7). □

Remark 4.4. Let g^{TM} be a metric on TM. The metric g^{TM} determines a canonical connection ∇^{TM}, which is the Levi-Civita connection of TM. Then the metrics g^{TM}, g^F on TM, F determine canonical connections $\nabla^{TM}, \nabla^{F,e}$ on TM, F. This is very similar to what happens in the holomorphic category, where a metric canonically determines a connection. This formal analogy will play an essential role in our work.

b) A closed 1-form on M and its cohomology class

The homomorphism $u \in GL(m, \mathbb{R}) \to \operatorname{Log} |\det u|^2 \in \mathbb{R}$ permits us to construct an element c in the first Čech cohomology group of M, which measures the obstruction to the existence of a flat volume form on F.

Definition 4.5. Let $\theta(F, g^F)$ be the real 1-form on M

$$\theta\left(F, g^F\right) = \operatorname{Tr}\left[\omega\left(F, g^F\right)\right]. \tag{4.8}$$

One has the trivial result.

Proposition 4.6. *The form* $\theta(F, g^F)$ *is closed. Its cohomology class in* $H^1(M, \mathbb{R})$ *is equal to* c.

c) An anomaly formula for Ray-Singer metrics

Let g^{TM} be an Euclidean metric on TM. Let ∇^{TM} be the associated Levi-Civita connection on TM and let R^{TM} be its curvature. Recall that the Pfaffian of an antisymmetric matrix was defined in Section 3a).

Following (3.17), set

$$e\left(TM, \nabla^{TM}\right) = \operatorname{Pf}\left[\frac{R^{TM}}{2\pi}\right]. \tag{4.9}$$

Then $e(TM, \nabla^{TM})$ is a closed n-form on M with values in $o(TM)$. The form $e(TM, \nabla^{TM})$ represents the Euler class of TM in $H^n(M, o(TM))$.

If g^{TM}, g'^{TM} are two metrics on TM, and if ∇^{TM}, ∇'^{TM} are the corresponding Levi-Civita connections, let $\widetilde{e}(TM, \nabla^{TM}, \nabla'^{TM})$ be the Chern-Simons class of $n-1$ smooth forms on M valued in $o(TM)$, which is defined modulo exact $n-1$ forms, such that

$$d\,\widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right) = e\left(TM, \nabla'^{TM}\right) - e\left(TM, \nabla^{TM}\right). \tag{4.10}$$

Of course, if n is odd,

$$\widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right) = 0. \tag{4.11}$$

Let now g^{TM}, g'^{TM} be two Euclidean metrics on TM, and let g^F, g'^F be two Euclidean metrics on F. Let $\| \ \|_{\det F}$, $\| \ \|'_{\det F}$ be the metrics on the line bundle $\det F$ induced by the metrics g^F, g'^F . Observe that

$$d \operatorname{Log}\left(\frac{\| \ \|'_{\det F}}{\| \ \|_{\det F}}\right)^2 = \theta\left(F, g'^F\right) - \theta\left(F, g^F\right). \tag{4.12}$$

Let $\| \ \|^{RS}_{\det H^\bullet(M,F)}$ and $\| \ \|'^{RS}_{\det H^\bullet(M,F)}$ be the Ray-Singer metrics attached to the metrics (g^{TM}, g^F) and (g'^{TM}, g'^F).

The purpose of this Section is to establish Theorem 0.1, which we state again for convenience.

Theorem 4.7. *The following identity holds*

$$\begin{aligned}\operatorname{Log}\left(\frac{\| \ \|'^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{RS}_{\det H^\bullet(M,F)}}\right)^2 &= \int_M \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F}}{\| \ \|^2_{\det F}}\right) e\left(TM, \nabla^{TM}\right) \\ &\quad - \int_M \theta\left(F, g'^F\right) \widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right).\end{aligned} \tag{4.13}$$

In particular, if $\dim M = n$ *is odd, then*

$$\operatorname{Log}\left(\frac{\| \ \|'^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{RS}_{\det H^\bullet(M,F)}}\right)^2 = 0. \tag{4.14}$$

Proof. Theorem 4.7 will be proved in Sections 4d)–4i). □

Remark 4.8. Equation (4.14) is the well-known basic result of Ray and Singer [RS1, Theorem 7.3].

d) Clifford algebras and exterior algebras

Let E be a real finite dimensional vector space of dimension n. Let g^E be an Euclidean metric on E.

The exterior algebra $\Lambda(E^*)$ is $\mathbb{Z}$-graded, and so it posseses a natural $\mathbb{Z}_2$-grading. If $A \in \operatorname{End}(\Lambda(E^*))$, let $\operatorname{Tr_s}[A]$ be the supertrace of A, as defined in (1.9).

If $e \in E$, let $e^* \in E^*$ correspond to e by the metric g^E. Set

$$\begin{aligned} c(e) &= e^* \wedge - i_e, \\ \widehat{c}(e) &= e^* \wedge + i_e. \end{aligned} \tag{4.15}$$

The operators $c(e), \widehat{c}(e)$ act on $\Lambda(E^*)$. If $e, e' \in E$, then

$$\begin{aligned} c(e)c(e') + c(e')c(e) &= -2\langle e, e'\rangle, \\ \widehat{c}(e)\widehat{c}(e') + \widehat{c}(e')\widehat{c}(e) &= 2\langle e, e'\rangle, \\ c(e)\widehat{c}(e') + \widehat{c}(e')c(e) &= 0. \end{aligned} \tag{4.16}$$

From (4.16), we deduce that the maps $e \in E \to c(e), \widehat{c}(e)$ extend to representations of the Clifford algebra $c(E)$ of E. Also, $\operatorname{End}(\Lambda(E^*))$ is generated as an algebra by 1 and the $c(e), \widehat{c}(e)$'s.

Let $e_1, \cdots, e_n$ be an orthonormal base of E, let $e^1, \cdots, e^n$ be the dual base of E^*.

Proposition 4.9. *Among the monomials in the* $c(e_i), \widehat{c}(e_i)$ *'s, only* $c(e_1)\widehat{c}(e_1)\cdots c(e_n)$ $\widehat{c}(e_n)$ *has a nonzero supertrace. Moreover*

$$\operatorname{Tr_s}\left[c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)\right] = (-2)^n. \tag{4.17}$$

Proof. Assume that $n = 1$. Then 1, $c(e_1), \widehat{c}(e_1)$ have a supertrace equal to 0. Moreover

$$c(e_1)\widehat{c}(e_1) = 2e^1 \wedge i_{e_1} - 1, \tag{4.18}$$

and so

$$\mathrm{Tr_s}\left[c(e_1)\widehat{c}(e_1)\right] = -2. \tag{4.19}$$

Equation (4.19) immediately extends to (4.17). □

We consider the vector space $E \oplus E$. Then $e_1, \cdots, e_n$ still denotes an orthonormal base of the first copy of E in $E \oplus E$, and $\widehat{e}_1, \cdots, \widehat{e}_n$ the corresponding orthonormal base of the second copy of E. Also $e^1, \cdots, e^n$ and $\widehat{e}^1, \cdots, \widehat{e}^n$ denote the dual bases of the first and second copies of E^* in $E^* \oplus E^*$.

For $t > 0, e \in E$, if $e^* \in E^*$ corresponds to e by the metric g^E, set

$$\begin{aligned} c_t(e) &= \frac{e^*}{t^{1/4}} \wedge -t^{1/4} i_e, \\ \widehat{c}_t(e) &= \frac{\widehat{e^*}}{t^{1/4}} \wedge +t^{1/4} i_{\widehat{e}}. \end{aligned} \tag{4.20}$$

The operators $c_t(e), \widehat{c}_t(e)$ act on $\Lambda(E^* \oplus E^*) = \Lambda(E^*)\widehat{\otimes}\Lambda(E^*)$. Moreover if $e, e' \in E$,

$$\begin{aligned} c_t(e)c_t(e') + c_t(e')c_t(e) &= -2\langle e, e'\rangle, \\ \widehat{c}_t(e)\widehat{c}_t(e') + \widehat{c}_t(e')\widehat{c}_t(e) &= 2\langle e, e'\rangle, \\ c_t(e)\widehat{c}_t(e') + \widehat{c}_t(e')c_t(e) &= 0. \end{aligned} \tag{4.21}$$

Using (4.16), (4.21) we see that there is a homomorphism of algebras $\psi_t : \mathrm{End}(\Lambda(E^*)) \to \mathrm{End}(\Lambda(E^* \oplus E^*))$ which for $e \in E$, maps $c(e)$ in $c_t(e)$ and $\widehat{c}(e)$ in $\widehat{c}_t(e)$.

Now the operators $e^{i_1} \wedge \cdots e^{i_p} \wedge \widehat{e}^{j_1} \wedge \cdots \widehat{e}^{j_q} \wedge i_{e_{k_1}} \cdots i_{e_{k_{p'}}} i_{\widehat{e_{\ell_1}}} \cdots i_{\widehat{e_{\ell_{q'}}}}$ are linearly independent in $\mathrm{End}(\Lambda(E^*)\widehat{\otimes}\Lambda(E^*))$. Moreover, if $u \in \mathrm{End}(\Lambda(E^*)), \psi_t(u)$ is a linear combination of such operators.

Definition 4.10. For $u \in \mathrm{End}(\Lambda(E^*))$, let $\{\psi_t(u)\}^{\max} \in \mathbb{R}$ be the coefficient of the monomial $e^1 \wedge \cdots \wedge e^n \wedge \widehat{e}^1 \wedge \cdots \wedge \widehat{e}^n$ in the expansion of $\psi_t(u)$.

Proposition 4.11. *If* $u \in \mathrm{End}(\Lambda(E^*))$, *then for any* $t > 0$,

$$\mathrm{Tr}_s[u] = 2^n(-1)^{\frac{n(n+1)}{2}} t^{\frac{n}{2}} \left\{\psi_t(u)\right\}^{\max}. \tag{4.22}$$

Proof. Equation (4.22) follows from (4.17). □

e) A Lichnerowicz formula for the Hodge Laplacian

Recall that d^F denotes the natural action of ∇^F on $\mathbb{F}$. Also d^{F*} is the formal adjoint of d^F with respect to the scalar product $\langle\, , \rangle_{\mathbb{F}}$.

As in (2.5), set

$$D = d^F + d^{F*}. \tag{4.23}$$

The connection ∇^{TM} induces a connection $\nabla^{\Lambda(T^*M)}$ on $\Lambda(T^*M)$. Let ∇, ∇^e be the connections on $\Lambda(T^*M) \otimes F$

$$\begin{aligned} \nabla &= \nabla^{\Lambda(T^*M)} \otimes 1 + 1 \otimes \nabla^F, \\ \nabla^e &= \nabla^{\Lambda(T^*M)} \otimes 1 + 1 \otimes \nabla^{F,e}. \end{aligned} \tag{4.24}$$

Let $e_1, \cdots, e_n$ be an orthonormal base of TM, let $e^1, \cdots, e^n$ be the corresponding dual base of T^*M.

Proposition 4.12. *The following identity holds,*

$$D = \sum_1^n c(e_i)\nabla^e_{e_i} - \frac{1}{2}\sum_1^n \widehat{c}(e_i)\omega\left(F, g^F\right)(e_i). \tag{4.25}$$

Proof. Since ∇^{TM} is torsion free, it is clear that

$$d^F = \sum_1^n e^i \wedge \nabla_{e_i}. \tag{4.26}$$

Then a trivial computation shows that

$$d^{F*} = -\sum_1^n i_{e_i}\left(\nabla_{e_i} + \omega\left(F, g^F\right)(e_i)\right). \tag{4.27}$$

From (4.26), (4.27), we get (4.25). □

Let now $e_1,\cdots,e_n$ be a locally defined smooth section of the bundle of orthonormal frames of TM. Let Δ, Δ^e be the Bochner Laplacians

$$\Delta = \sum_1^n \left(\nabla_{e_i}^2 - \nabla_{\nabla_{e_i}^{TM} e_i}\right), \tag{4.28}$$
$$\Delta^e = \sum_1^n \left(\nabla_{e_i}^{e,2} - \nabla^e_{\nabla_{e_i}^{TM} e_i}\right).$$

The Laplacian Δ^e is self-adjoint with respect to the scalar product (2.2) on $\mathbb{F}$.

Let K be the scalar curvature of (M, g^{TM}). Now we prove the following extension of Lichnerowicz's formula [L].

Theorem 4.13. *The following identity holds*

$$\begin{aligned} D^2 = -\Delta^e + \frac{K}{4} + \frac{1}{8} \sum_{1\le i,j,k,\ell\le n} & \left\langle e_k, R^{TM}(e_i,e_j) e_\ell \right\rangle \\ & c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) + \frac{1}{4} \sum_{1\le i\le n} \left(\omega\left(F,g^F\right)(e_i)\right)^2 \\ & -\frac{1}{8} \sum_{1\le i,j\le n} \left(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\right)\left(\omega\left(F,g^F\right)\right)^2(e_i,e_j) \\ & -\frac{1}{4} \sum_{1\le i,j\le n} c(e_i)\widehat{c}(e_j)\left(\nabla^F_{e_i}\omega\left(F,g^F\right)(e_j) + \nabla^F_{e_j}\omega\left(F,g^F\right)(e_i)\right). \end{aligned} \tag{4.29}$$

Proof. Set

$$D^0 = \sum_1^n c(e_i)\nabla^e_{e_i}. \tag{4.30}$$

Then D^0 is an operator of Dirac type acting on $\mathbb{F}$.

If $A \in \operatorname{End}(TM)$ is antisymmetric, A acts on $\Lambda(T^*M)$ as a derivation, and its action is given by

$$\frac{1}{4} \sum_{1\le i,j\le n} \langle Ae_i, e_j\rangle \left(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\right). \tag{4.31}$$

Also $(\nabla^{F,e})^2$ is given by (4.6). By using an obvious extension of Lichnerowicz's formula [L] and also (4.31), we see that

$$(D^0)^2 = -\Delta^e + \frac{K}{4} + \frac{1}{8} \sum_{1\leq i,j,k,\ell\leq n} \langle e_k, R^{TM}(e_i,e_j)\, e_\ell\rangle \tag{4.32}$$

$$c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) - \frac{1}{8} \sum_{1\leq i,j\leq n} c(e_i)c(e_j)\left(\omega\left(F,g^F\right)\right)^2(e_i,e_j)\,.$$

Moreover by (4.16) and by Proposition 4.12, we get

$$D^2 = (D^0)^2 + \frac{1}{4} \sum_{1\leq i\leq n} \left(\omega\left(F,g^F\right)(e_i)\right)^2 + \frac{1}{8} \sum_{1\leq i,j\leq n} \widehat{c}(e_i)\widehat{c}(e_j) \tag{4.33}$$

$$\left(\omega\left(F,g^F\right)\right)^2(e_i,e_j) - \frac{1}{2} \sum_{1\leq i,j\leq n} c(e_i)\widehat{c}(e_j)\left(\nabla^{F,e}_{e_i}\omega\left(F,g^F\right)(e_j)\right).$$

Using (4.7), we obtain

$$\begin{aligned} \nabla^{F,e}_{e_i}\omega\left(F,g^F\right)(e_j) &= \nabla^{F}_{e_i}\omega\left(F,g^F\right)(e_j) + \frac{1}{2}\left(\omega\left(F,g^F\right)\right)^2(e_i,e_j) \\ &= \frac{1}{2}\left(\nabla^{F}_{e_i}\omega\left(F,g^F\right)(e_j) + \nabla^{F}_{e_j}\omega\left(F,g^F\right)(e_i)\right). \end{aligned} \tag{4.34}$$

From (4.32)–(4.34), we get (4.29). □

f) An infinitesimal variation formula for the Ray-Singer metric

Let $\ell \in \mathbb{R} \to (g_\ell^{TM}, g_\ell^F)$ be a smooth family of metrics on TM, F. Let $*_\ell$ be the Hodge operator associated to the metrics g_ℓ^{TM}. Let D_ℓ be the operator D defined in (4.23) attached to the metrics (g_ℓ^{TM}, g_ℓ^F). Let $\| \ \|^{RS}_{\det H^\bullet(M,F),\ell}$ be the corresponding Ray-Singer metric on $\det H^\bullet(M,F)$.

Theorem 4.14. *If n is even, as $t \to 0$, for any $k \in \mathbb{N}$, there is an asymptotic expansion*

$$\mathrm{Tr_s}\left[\left(*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell} + \left(g_\ell^F\right)^{-1}\frac{\partial g_\ell^F}{\partial \ell}\right)\exp\left(-tD_\ell^2\right)\right] = \sum_{j=-n/2}^{k} M_{j,\ell}t^j + o\left(t^k\right). \tag{4.35}$$

Also if n is even,

$$\frac{\partial}{\partial \ell}\operatorname{Log}\left(\| \ \|^{RS}_{\det H^\bullet(M,F),\ell}\right)^2 = M_{0,\ell}\,. \tag{4.36}$$

Moreover if n *is odd,*

$$\frac{\partial}{\partial \ell} \operatorname{Log}\left(\| \quad \|^{RS}_{\det H^\bullet(M,F),\ell}\right)^2 = 0. \tag{4.37}$$

Proof. Our Theorem follows from similar computations which are done in Ray-Singer [RS1, Theorems 2.1 and 7.3] and Bismut-Gillet-Soulé [BGS3, Theorem 1.18]. Note that in the case where n is odd, (4.37) is a consequence of the fact that there is no constant term in the asymptotic expansion of the left-hand side of (4.35). □

Let $e_1, \cdots, e_n$ be an orthonormal base of TM with respect to the metric g_ℓ^{TM}.

Proposition 4.15. *The following identity holds*

$$\left(*_\ell^{-1} \frac{\partial *_\ell}{\partial \ell} \right) = - \sum_{1 \le i,j \le n} \frac{1}{2} \left\langle \left(g_\ell^{TM}\right)^{-1} \frac{\partial g_\ell^{TM}}{\partial \ell} e_i, e_j \right\rangle_{g_\ell^{TM}} c(e_i)\widehat{c}(e_j). \tag{4.38}$$

Proof. Clearly

$$\begin{aligned}(*_\ell)^{-1} \frac{\partial *_\ell}{\partial \ell} = \frac{1}{2} \sum_1^n \left\langle \left(g_\ell^{TM}\right)^{-1} \frac{\partial g_\ell^{TM}}{\partial \ell} e_i, e_i \right\rangle_{g_\ell^{TM}} \\ - \sum_{1 \le i,j \le n} \left\langle \left(g_\ell^{TM}\right)^{-1} \frac{\partial g_\ell^{TM}}{\partial \ell} e_i, e_j \right\rangle_{g_\ell^{TM}} e_i \wedge i_{e_j}.\end{aligned} \tag{4.39}$$

Equation (4.38) follows. □

g) A Clifford algebra trick

Let σ be an auxiliary even Clifford variable, such that $\sigma^2 = 1$. So σ commutes with the $c(e_i)$'s, the $\widehat{c}(e_j)$'s and more generally with all the previously considered operators.

Let $A, B \in \operatorname{End}(\mathbb{F})$ be trace class. Then $A + \sigma B$ lies in $\operatorname{End}(\mathbb{F}) \widehat{\otimes} \mathbb{R}(\sigma)$. Set

$$\operatorname{Tr}_s^\sigma[A + \sigma B] = \operatorname{Tr}_s[B]. \tag{4.40}$$

Definition 4.16. Set

$$\begin{aligned}\left(D_\ell^2\right)^{\text{odd}} &= -\frac{1}{4} \sum_{1 \le i,j \le n} c(e_i)\widehat{c}(e_j) \left(\nabla^F_{e_i} \omega\left(F, g^F\right)(e_j) + \nabla^F_{e_j} \omega\left(F, g^F\right)(e_i) \right), \\ \left(D_\ell^2\right)^{\text{even}} &= D^2 - \left(D^2\right)^{\text{odd}}.\end{aligned} \tag{4.41}$$

The operator $(D_\ell^2)^{\mathrm{odd}}$ is in fact odd in the Clifford variables $c(e_i)$ or $\widehat{c}(e_i)$, while $(D_\ell^2)^{\mathrm{even}}$ is even in the Clifford variables $c(e_i)$ or $\widehat{c}(e_i)$.

Let $dv_{M,\ell}$ be the volume form on M with respect to the metric g_ℓ^{TM}.

Definition 4.17. Let $P_{t,\ell}(x,x')(\text{resp}\,.Q_{t,\ell}(x,x'))$ be the smooth kernel with respect to the volume form $dv_{M,\ell}(x')$ associated to the operator $\exp(-tD_\ell^2)$ (resp. the operator $\exp(-t((D_\ell^2)^{\mathrm{even}}+\sigma(D_\ell^2)^{\mathrm{odd}}))$).

Theorem 4.18. *If n is even, and if M is oriented, for any $x\in M, t>0$, the following identity holds*

$$\mathrm{Tr_s}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}P_{t,\ell}(x,x)\right]=\mathrm{Tr_s^\sigma}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}Q_{t,\ell}(x,x)\right]. \tag{4.42}$$

Proof. Since M is oriented, the operator $*_\ell$ maps $\mathbb{F}$ into itself. Also $*_\ell^2$ is a constant operator, and so

$$*_\ell\frac{\partial *_\ell}{\partial \ell}+\frac{\partial *_\ell}{\partial \ell}*_\ell=0. \tag{4.43}$$

Set

$$C=*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}. \tag{4.44}$$

From (4.43), we get

$$*_\ell C *_\ell^{-1}=-C. \tag{4.45}$$

In fact (4.45) can be directly verified by using (4.38).

Also $(D_\ell^2)^{\mathrm{even}}$ and $(D_\ell^2)^{\mathrm{odd}}$ preserve the $\mathbb{Z}$-grading in $\mathbb{F}$. Moreover one easily verifies that

$$\begin{aligned} *_\ell\left(D_\ell^2\right)^{\mathrm{even}}*_\ell^{-1}&=\left(D_\ell^2\right)^{\mathrm{even}},\\ *_\ell\left(D_\ell^2\right)^{\mathrm{odd}}*_\ell^{-1}&=-\left(D_\ell^2\right)^{\mathrm{odd}}. \end{aligned} \tag{4.46}$$

Let h be a smooth function from M into $\mathbb{R}$. Since $*_\ell$ is an even operator acting on $\mathbb{F}$ (i.e. it preserves the $\mathbb{Z}_2$-grading of $\mathbb{F}$), and since supertraces vanish on supercommutators [Q1], we see that

$$\mathrm{Tr_s}\left[*_\ell hC\exp\left(-t\left(\left(D_\ell^2\right)^{\mathrm{even}}+\sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)\right)*_\ell^{-1}\right] \tag{4.47}$$

$$= \mathrm{Tr_s}\left[hC\exp\left(-t\left(\left(D_\ell^2\right)^{\mathrm{even}} + \sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)\right)\right].$$

On the other hand, by using (4.45), (4.46), we get

$$\begin{aligned}(4.48)\qquad & \mathrm{Tr_s}\left[hC\exp\left(-t\left(\left(D_\ell^2\right)^{\mathrm{even}} + \sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)\right) *_\ell^{-1}\right]\\ &= -\mathrm{Tr_s}\left[hC\exp\left(-t\left(\left(D_\ell^2\right)^{\mathrm{even}} - \sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)\right)\right].\end{aligned}$$

From (4.47), (4.48), we conclude that

$$\begin{aligned}(4.49)\qquad & \mathrm{Tr_s}\left[hC\exp\left(-t\left(\left(D_\ell^2\right)^{\mathrm{even}} + \sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)\right)\right]\\ &= -\mathrm{Tr_s}\left[hC\exp\left(-t\left(\left(D_\ell^2\right)^{\mathrm{even}} - \sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)\right)\right].\end{aligned}$$

Since (4.49) holds for any smooth function $h : M \to \mathbb{R}$ we easily get (4.42). □

h) The small time asymptotics of the supertrace of certain heat kernels

We make the same assumptions as in Sections 4f) and 4g). Let ∇_ℓ^{TM} be the Levi-Civita connection on (TM, g_ℓ^{TM}), and let R_ℓ^{TM} be the curvature of ∇_ℓ^{TM}.

Let ρ be the projection $M \times \mathbb{R} \to M$. Let $g^{TM,\mathrm{tot}}$ be the metric on ρ^*TM which coincides with g_ℓ^{TM} over $M \times \{\ell\}$. Let $\nabla^{TM,\mathrm{tot}}$ be the connection over ρ^*TM

$$(4.50)\qquad \nabla^{TM,\mathrm{tot}} = \rho^*\nabla_\ell^{TM} + d\ell\left(\frac{\partial}{\partial \ell} + \frac{1}{2}\left(g_\ell^{TM}\right)^{-1}\frac{\partial g_\ell^{TM}}{\partial \ell}\right).$$

Then $\nabla^{TM,\mathrm{tot}}$ preserves the metric $g^{TM,\mathrm{tot}}$. The curvature $(\nabla^{TM,\mathrm{tot}})^2$ of $\nabla^{TM,\mathrm{tot}}$ is given by

$$(4.51)\quad \left(\nabla^{TM,\mathrm{tot}}\right)^2 = \rho^*R_\ell^{TM} + d\ell\left(\frac{\partial}{\partial \ell}\nabla_\ell^{TM} - \frac{1}{2}\left[\nabla_\ell^{TM}, \left(g_\ell^{TM}\right)^{-1}\frac{\partial g_\ell^{TM}}{\partial \ell}\right]\right).$$

Definition 4.19. Set

(4.52)

$$\tilde{e}'_\ell(TM) = \frac{\partial}{\partial b}\mathrm{Pf}\left[\frac{1}{2\pi}\left(R_\ell^{TM} + b\left(\frac{\partial}{\partial \ell}\nabla_\ell^{TM} - \frac{1}{2}\left[\nabla_\ell^{TM}, \left(g_\ell^{TM}\right)^{-1}\frac{\partial g_\ell^{TM}}{\partial \ell}\right]\right)\right)\right]_{b=0}$$

By a standard argument in Chern-Weil theory, we know that

$$(4.53)\qquad \frac{\partial}{\partial \ell}\tilde{e}\left(TM, \nabla_0^{TM}, \nabla_\ell^{TM}\right) = \tilde{e}'_\ell(TM).$$

For $x \in M, \varepsilon > 0$, let $B^M(x,\varepsilon)$ be the open ball of center x and radius ε in M with respect to the metric g_0^{TM}, and let $B^{T_xM}(0,\varepsilon)$ be the open ball of center 0 and radius ε in T_xM with respect to the metric $g_0^{T_xM}$.

Theorem 4.20. *Assume that n is even. Then*

$$M_{j,\ell} = 0 \quad \text{for } j < 0, \tag{4.54}$$

$$M_{0,\ell} = \int_M \operatorname{Tr}\left[(g_\ell^F)^{-1}\frac{\partial g_\ell^F}{\partial \ell}\right] e\left(TM, \nabla_\ell^{TM}\right) - \int_M \theta\left(F, g_\ell^F\right) \widetilde{e}_\ell'(TM).$$

Proof. In the whole proof, we will use the notation of Section 3 on the Berezin integral. We first calculate the asymptotics as $t \to 0$ of $\operatorname{Tr_s}[(g_\ell^F)^{-1}\frac{\partial g_\ell^F}{\partial \ell}\exp(-tD_\ell^2)]$. Here the metric g^{TM} will be fixed. Also we will often omit the subscript ℓ.

First we proceed as in Getzler [G]. Let $a > 0$ be the injectivity radius of (M, g^{TM}). Take ε such that $0 < \varepsilon \le a/2$. Take $x \in M$. Let $e_1, \cdots, e_n$ be an orthonormal base of T_xM. We identify the open ball $B^{T_xM}(0,\varepsilon)$ with the open ball $B^M(x,\varepsilon)$ in M using geodesic coordinates. Then $y \in T_xM, |y| \le \varepsilon$ represents an element of $B^M(x,\varepsilon)$. For $y \in T_xM$, $|y| \le \varepsilon$, we identify T_yM, F_y to T_xM, F_x by parallel transport along the geodesic $t \in [0,1] \to ty$ with respect to the connections $\nabla^{TM}, \nabla^{F,e}$.

Let $\Gamma^{TM,x}$ be the connection form for ∇^{TM} in the considered trivialization of TM. By [ABoP, Proposition 4.7], we know that

$$\Gamma_y^{TM,x} = \frac{1}{2}R_x^{TM}(y,\cdot) + O(|y|^2). \tag{4.55}$$

The induced connection form $\Gamma_y^{\Lambda(T_x^*M)}$ on $\Lambda(T_x^*M)$ is given by

(4.56)

$$\Gamma_y^{\Lambda(T^*M),x} = \frac{1}{8}\sum_{1\le i,j\le n}\left(\langle R_x^{TM}(y,\cdot)e_i, e_j\rangle + O(|y|^2)\right)\left(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\right).$$

The operator D^2 now acts on smooth sections of $(\Lambda(T^*M)\otimes F)_x$ over $B^{T_xM}(0,\varepsilon)$. If h is a smooth section of $(\Lambda(T^*M)\otimes F)_x$ over T_xM, set

$$T_t h(y) = h\left(\frac{y}{\sqrt{t}}\right). \tag{4.57}$$

Let K_t be the operator

$$K_t = T_t^{-1} t D^2 T_t. \tag{4.58}$$

Then K_t is a differential operator with coefficients in the algebra spanned by the $c(e_i)$'s, the $\widehat{c}(e_i)$'s and elements of $\mathrm{End}(F)_x$.

Let L_t be the operator obtained from K_t by replacing the Clifford variables $c(e_i), \widehat{c}(e_i)$ by $c_t(e_i), \widehat{c}_t(e_i)$ defined in (4.20). Let Δ^{T_xM} be the flat Laplacian over T_xM for the metric g^{T_xM} . Using (4.29), (4.56), one concludes easily that as $t \to 0$, the coefficients of L_t converge uniformly over compact sets together with their derivatives to the coefficients of the operator L_0 given by

$$L_0 = -\Delta^{T_xM} + \frac{1}{8} \sum_{1 \le i,j,k,\ell \le n} \left\langle e_k, R_\ell^{TM}(e_i, e_j) e_\ell \right\rangle e^i \wedge e^j \wedge \widehat{e}^k \wedge \widehat{e}^\ell. \tag{4.59}$$

If we use the notation in (3.38), we get

$$L_0 = -\Delta^{T_xM} + \frac{\dot{R}^{TM}}{2}. \tag{4.60}$$

Let dv_M be the volume element on TM with respect to the metric g^{TM}. Here dv_M is viewed as a section of $\Lambda^n(T^*M) \otimes o(TM)$. Using Proposition 4.11, equation (4.60), and proceeding as in Getzler [G], we see that as $t \to 0$,

$$\mathrm{Tr_s}\left[\left(g_\ell^F\right)^{-1} \frac{\partial g_\ell^F}{\partial \ell} P_t(x,x)\right] dv_M(x) \tag{4.61}$$

$$\to \left(\mathrm{Tr}\left[\left(g_\ell^F\right)^{-1} \frac{\partial g_\ell^F}{\partial \ell}\right] \int^B \exp\left(-\frac{\dot{R}^{TM}}{2}\right)\right)(x) \quad \text{uniformly on } M.$$

Moreover

$$\mathrm{Tr_s}\left[\left(g_\ell^F\right)^{-1} \frac{\partial g_\ell^F}{\partial \ell} \exp\left(-tD_\ell^2\right)\right] = \int_M \mathrm{Tr_s}\left[\left(g_\ell^F\right)^{-1} \frac{\partial g_\ell^F}{\partial \ell} P_t(x,x)\right] dv_M(x). \tag{4.62}$$

From (4.61), (4.62), we get

$$\lim_{t \to 0} \mathrm{Tr_s}\left[\left(g_\ell^F\right)^{-1} \frac{\partial g_\ell^F}{\partial \ell} \exp\left(-tD_\ell^2\right)\right] = \int_M \mathrm{Tr}\left[\left(g_\ell^F\right)^{-1} \frac{\partial g_\ell^F}{\partial \ell}\right] e\left(TM, g^{TM}\right). \tag{4.63}$$

Now we assume that the metric g^F on F is fixed, and that the metric g_ℓ^{TM} on TM depends on ℓ. We will calculate the asymptotics of $\mathrm{Tr_s}[(*_\ell^{-1})\frac{\partial *_\ell}{\partial \ell}\exp(-tD_\ell^2)]$. Clearly

$$\mathrm{Tr_s}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}\exp\left(-tD_\ell^2\right)\right] = \int_M \mathrm{Tr_s}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}P_{t,\ell}(x,x)\right] dv_{M,\ell}(x). \tag{4.64}$$

Take $x \in M$. We assume first that M is oriented. Then by Theorem 4.18, we get

$$\mathrm{Tr_s}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}P_{t,\ell}(x,x)\right] = \mathrm{Tr_s^\sigma}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}Q_{t,\ell}(x,x)\right]. \tag{4.65}$$

In the sequel, $e_1,\cdots,e_n$ is an orthonormal base of T_xM with respect to the metric g_ℓ^{TM}, and $e^1,\cdots,e^n$ is the corresponding dual base of T_x^*M .

We consider $\mathbb{R}$ equipped with its canonical Euclidean metric. Let $a = 1 \in \mathbb{R}$, let $a^* \in \mathbb{R}^*$ correspond to a by the metric of $\mathbb{R}$. For $t > 0$, set

$$\sigma_t = \frac{a^*\wedge}{\sqrt{t}} + \sqrt{t}\, i_a. \tag{4.66}$$

If $c + d\sigma \in \mathbb{R}[\sigma]$, then

$$c + d\sigma_t = c + \frac{da^*}{\sqrt{t}} \wedge + d\sqrt{t}\, i_a. \tag{4.67}$$

In the sequel, the operators $a^*\wedge$ and i_a will commute with all the other operators considered before.

Take $x \in M$. We trivialize TM and F on $B^M(x,\varepsilon)$ as before. Then the operators $(D_\ell^2)^{\mathrm{even}}, (D_\ell^2)^{\mathrm{odd}}$ act on smooth sections of $(\Lambda(T^*M)\otimes F)_x$ on $B^{T_xM}(0,\varepsilon)$. We define T_t as in (4.57). Set

$$K_t' = T_t^{-1} t\left(\left(D_\ell^2\right)^{\mathrm{even}} + \sigma\left(D_\ell^2\right)^{\mathrm{odd}}\right)T_t. \tag{4.68}$$

In K_t', we replace $c(e_i)$ by $c_t(e_i)$, $\widehat{c}(e_i)$ by $\widehat{c}_t(e_i)$ and σ by σ_t. So we obtain a new operator L_t'. Let $\Delta_\ell^{T_xM}$ be the Laplacian on T_xM with respect to the metric $g_\ell^{T_xM}$. Using (4.29) and (4.56), one verifies easily that as $t \to 0$, L_t' converges to L_0' given by

(4.69)

$$L_0' = -\Delta_\ell^{T_xM} + \frac{1}{2}\dot{R}_\ell^{TM} - \frac{1}{4}a^*\wedge \sum_{1\le i,j\le n} e^i\wedge\widehat{e^j}\left(\nabla_{e_i}\omega\left(F,g^F\right)(e_j) + \nabla_{e_j}\omega\left(F,g^F\right)(e_i)\right).$$

Let C_t be the operator obtained from $*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}$ by replacing $c(e_i)$ by $c_t(e_i)$, and $\widehat{c}(e_i)$ by $\widehat{c}_t(e_i)$. Using (4.38), we find that

$$\lim_{t\to 0}\sqrt{t}\,C_t = -\sum_{1\leq i,j\leq n}\frac{1}{2}\left\langle \left(g_\ell^{TM}\right)^{-1}\frac{\partial g_\ell^{TM}}{\partial \ell}e_i, e_j\right\rangle_{g_\ell^{TM}} e^i\wedge\widehat{e^j}. \tag{4.70}$$

By Proposition 4.11, by equations (4.69), (4.70), and by proceeding as before, we deduce easily that

$$\begin{aligned}&\lim_{t\to 0}\mathrm{Tr_s}\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}P_{t,\ell}(x,x)\right]dv_{M,\ell}(x) \\ &= -\left\{\int^B\left(\frac{1}{2}\sum_{1\leq i,j\leq n}\left\langle \left(g_\ell^{TM}\right)^{-1}\frac{\partial g_\ell^{TM}}{\partial \ell}e_i, e_j\right\rangle_{g_\ell^{TM}} e^i\wedge\widehat{e^j}\right)\right. \\ &\qquad \exp\left(-\frac{\dot{R}_\ell^{TM}}{2}\right)\wedge\frac{1}{4}\sum_{1\leq i,j\leq n}e^i\wedge\widehat{e^j} \\ &\qquad \left.\mathrm{Tr}\left[\nabla^F_{e_i}\omega\left(F,g^F\right)(e_j)+\nabla^F_{e_j}\omega\left(F,g^F\right)(e_i)\right]\right\}(x) \quad \text{uniformly on } M.\end{aligned} \tag{4.71}$$

When M is not orientable, equation (4.65) does not hold any more. However the evaluation of the asymptotics of the left-hand side of (4.71) is local near $x\in M$. By embedding the considered local neighborhood in an orientable manifold, we see that (4.71) remains valid in full generality.

Recall that φ was defined in Section 3f). Then

$$\varphi\theta\left(F,g^F\right)=\sum_{i=1}^n\theta\left(F,g^F\right)(e_i)\widehat{e^i}. \tag{4.72}$$

By (4.7), (4.72), we get

$$\begin{aligned}&\frac{1}{2}\sum_{1\leq i,j\leq n}e^i\wedge\widehat{e^j}\,\mathrm{Tr}\left[\nabla^F_{e_i}\omega\left(F,g^F\right)(e_j)+\nabla^F_{e_j}\omega\left(F,g^F\right)(e_i)\right] \\ &= \sum_{1\leq i,j\leq n}e^i\wedge\widehat{e^j}\,\mathrm{Tr}\left[\nabla^F_{e_i}\omega\left(F,g^F\right)(e_j)\right]=\nabla^{TM}\varphi\theta\left(F,g^F\right).\end{aligned} \tag{4.73}$$

Using (4.64), (4.71), (4.73) and Stokes formula, we find that

$$\lim_{t\to 0}\mathrm{Tr_s}\left[*_\ell^{-1}\frac{\partial *_\ell^{TM}}{\partial \ell}\exp\left(-tD_\ell^2\right)\right] \tag{4.74}$$

$$= \int_M \left\{ \int^B \nabla_\ell^{TM} \left(\frac{1}{4} \sum_{1 \le i,j \le n} \left\langle \left(g_\ell^{TM}\right)^{-1} \frac{\partial g_\ell^{TM}}{\partial \ell} e_i, e_j \right\rangle_{g_\ell^{TM}} e^i \wedge \widehat{e^j} \right) \right.$$
$$\left. \exp\left(-\frac{\dot{R}_\ell^{TM}}{2} \right) \wedge \varphi\theta\left(F, g^F\right) \right\}.$$

Set $\nabla^{TM} = \nabla_0^{TM}$. Then the connection $\widetilde{\nabla}_\ell^{TM}$ given by

$$\widetilde{\nabla}_\ell^{TM} = \nabla^{TM} + \frac{1}{2}\left(g_\ell^{TM}\right)^{-1} \nabla^{TM} g_\ell^{TM} \tag{4.75}$$

preserves the metric g_ℓ^{TM}. Its torsion T_ℓ is such that if $X, Y \in TM$,

$$T_\ell(X,Y) = \frac{1}{2}\left(g_\ell^{TM}\right)^{-1}\left(\nabla_X^{TM} g_\ell^{TM}\right)Y - \frac{1}{2}\left(g_\ell^{TM}\right)^{-1}\left(\nabla_Y^{TM} g_\ell^{TM}\right)X. \tag{4.76}$$

From (4.76), we deduce that

$$\frac{\partial}{\partial \ell} T_\ell(X,Y)_{|\ell=0} = \left(\frac{1}{2} \nabla_X^{TM} \left(\left(g_\ell^{TM}\right)^{-1} \frac{\partial g_\ell^{TM}}{\partial \ell} \right) Y \right. \tag{4.77}$$
$$\left. - \frac{1}{2} \nabla_Y^{TM} \left(\left(g^{TM}\right)^{-1} \frac{\partial g_\ell^{TM}}{\partial \ell} \right) X \right)_{|\ell=0}.$$

Set

$$S_\ell = \nabla_\ell^{TM} - \widetilde{\nabla}_\ell^{TM}. \tag{4.78}$$

From (4.75), (4.78), we get

$$\frac{\partial}{\partial \ell} \nabla_{\ell|\ell=0}^{TM} - \frac{1}{2}\left[\nabla_\ell^{TM}, \left(g_\ell^{TM}\right)^{-1} \frac{\partial g^{TM}}{\partial \ell} \right]_{|\ell=0} = \frac{\partial S_\ell}{\partial \ell}_{|\ell=0}. \tag{4.79}$$

Let $\langle\ ,\ \rangle$ be the scalar product on TM for the metric g_0^{TM}. Since ∇_ℓ^{TM} is torsion free, one sees easily that if $X, Y, Z \in TM$,

$$2\left\langle \frac{\partial S_\ell}{\partial \ell}(X)Y, Z \right\rangle + \left\langle \frac{\partial T_\ell}{\partial \ell}_{|\ell=0}(X,Y), Z \right\rangle \tag{4.80}$$
$$+ \left\langle \frac{\partial T_\ell}{\partial \ell}_{|\ell=0}(Z,X), Y \right\rangle - \left\langle \frac{\partial T_\ell}{\partial \ell}_{|\ell=0}(Y,Z), X \right\rangle = 0.$$

Using (4.77), (4.80) we get

$$\left\langle Y, \frac{\partial S_\ell}{\partial \ell}_{|\ell=0}(X)Z \right\rangle = -\left\langle \frac{\partial T}{\partial \ell}_{|\ell=0}(Y,Z), X \right\rangle. \tag{4.81}$$

Set

$$\frac{\partial \dot{S}_\ell}{\partial \ell}_{|\ell=0} = \sum_{1\le i,j\le n} \frac{1}{2}\left\langle e_k, \frac{\partial S_\ell}{\partial \ell}_{|\ell=0}(e_i)e_\ell\right\rangle e^i \wedge \widehat{e^k} \wedge \widehat{e^\ell}. \tag{4.82}$$

Using (4.77), (4.81), (4.82), we see that

(4.83)

$$\varphi\left(\frac{\partial \dot{S}_\ell}{\partial \ell}_{|\ell=0}\right) = -\nabla^{TM}\left(\frac{1}{2}\sum_{1\le i,j\le n}\left\langle \left(g_\ell^{TM}\right)^{-1}\frac{\partial g_\ell^{TM}}{\partial \ell}e_i, e_j\right\rangle_{|\ell=0} e^i \wedge \widehat{e^j}\right).$$

So from (4.74), (4.83), we get

$$\begin{aligned}
&\lim_{t\to 0} \operatorname{Tr}_s\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}\exp\left(-tD_\ell^2\right)\right]_{|\ell=0} \\
&= \int_M\left\{\int^B -\frac{1}{2}\varphi\left(\frac{\partial \dot{S}_\ell}{\partial \ell}_{|\ell=0}\right)\exp\left(\frac{-\dot{R}_0^{TM}}{2}\right)\wedge \varphi\theta\left(F,g^F\right)\right\} \\
&= -\int_M\left\{\int^B \left(\varphi\theta\left(F,g^F\right)\right)\left(-\frac{1}{2}\varphi\left(\frac{\partial \dot{S}_\ell}{\partial \ell}_{|\ell=0}\right)\right)\exp\left(\frac{-\dot{R}_0^{TM}}{2}\right)\right\}.
\end{aligned} \tag{4.84}$$

Using now Theorem 3.13 and (4.84), we find that

$$\begin{aligned}
&\lim_{t\to 0} \operatorname{Tr}_s\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}\exp\left(-tD_\ell^2\right)\right]_{|\ell=0} \\
&= -\int_M \theta\left(F,g^F\right)\int^B \frac{\partial}{\partial b}\exp\left(-\left(\frac{\dot{R}_0^{TM} + b\frac{\partial \dot{S}_\ell}{\partial \ell}_{|\ell=0}}{2}\right)\right)_{b=0}.
\end{aligned} \tag{4.85}$$

From (3.3), (4.52), (4.79), (4.85), we finally get

$$\lim_{t\to 0} \operatorname{Tr}_s\left[*_\ell^{-1}\frac{\partial *_\ell}{\partial \ell}\exp\left(-tD_\ell^2\right)\right]_{|\ell=0} = -\int_M \theta\left(F,g^F\right)\widetilde{e}_0'\left(TM\right). \tag{4.86}$$

Of course (4.86) also holds for arbitrary ℓ. The proof of Theorem 4.20 is completed.

□

i) Proof of Theorem 4.7

By Theorems 4.14 and 4.19, we get

$$\frac{\partial}{\partial \ell} \operatorname{Log} \left(\| \quad \|^{RS}_{\det H^{\bullet}(M,F),\ell} \right)^2 \tag{4.87}$$
$$= \int_M \operatorname{Tr}\left[\left(g^F_\ell\right)^{-1} \frac{\partial g^F_\ell}{\partial \ell} \right] e\left(TM, \nabla^{TM}_\ell\right) - \int_M \theta\left(F, g^F_\ell\right) \widetilde{e}'_\ell\left(TM\right).$$

Using (4.53) and (4.87), we obtain (4.13). □

V. A closed 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+$

In this Section, given a smooth function $f : M \to \mathbb{R}$, we exhibit a closed 1-form $\alpha_{t,T}$ on $\mathbb{R}_+^* \times \mathbb{R}_+$ which is calculated in terms of the supertraces of certain two parameter heat kernels. This 1-form is very similar to a corresponding 1-form obtained in Bismut-Lebeau [BL2, Theorem 3.3] in a different context. By integrating $\alpha_{t,T}$ on a closed contour Γ, we will obtain an important identity. In the next Sections, by a suitable deformation of the contour Γ, we will ultimately derive Theorem 0.2 from this identity.

This Section is organized as follows. In a), we introduce the family of smooth metrics $e^{-2Tf}g^F$ on F. In b), we calculate the Witten Laplacian $\widetilde{D}_T^2$ [W] associated to the smooth function Tf. In c), we construct the 1-form $\alpha_{t,T}$. In d), by integrating $\alpha_{t,T}$ on a contour Γ, we obtain an identity, which is the main result of this Section.

Here we use the assumptions and notation of Section 2a) and of Sections 4a), 4b).

a) A family of smooth metrics on $\mathbb{F}$

Let M be a compact connected manifold. Let F be a real flat vector bundle on M. Let g^{TM} be a smooth metric on TM, let g^F be a smooth metric on F.

Recall that d^F denotes the natural action of the flat connection ∇^F on $\mathbb{F}$. Moreover $\langle\, ,\rangle_{\Lambda(T^*M)\otimes F}$ still denotes the scalar product on $\Lambda(T^*M)\otimes F$ which is attached to the metrics g^{TM} and g^F. Also $\omega(F,g^F)$, $\theta(F,g^F)$ are defined by (4.1), (4.8).

Let $f : M \to \mathbb{R}$ be a smooth function.

Definition 5.1. For $T \geq 0$, let g_T^F be the smooth metric on F

$$g_T^F = e^{-2Tf} g^F. \tag{5.1}$$

We equip $\mathbb{F}$ with the L^2 scalar product $\langle \ , \ \rangle_{\mathbb{F},T}$ attached to the metrics g^{TM}, g_T^F on TM, F. Namely, if $\alpha, \beta \in \mathbb{F}$, we have

$$\langle \alpha, \beta \rangle_{\mathbb{F},T} = \int_M \langle \alpha, \beta \rangle_{\Lambda(T^*M)\otimes F}(x) e^{-2Tf(x)} dv_M(x). \tag{5.2}$$

Let d_T^{F*} be the formal adjoint of d^F with respect to the scalar product $\langle \ \rangle_{\mathbb{F},T}$ on $\mathbb{F}$. Clearly

$$d_T^{F*} = e^{2Tf} d^{F*} e^{-2Tf}. \tag{5.3}$$

Set

$$D_T = d^F + d_T^{F*}. \tag{5.4}$$

The operator D_T is self-adjoint with respect to the scalar product $\langle \ \rangle_{\mathbb{F},T}$. Also $D_T^2 = d^F d_T^{F*} + d_T^{F*} d^F$ is the Hodge Laplacian associated to the metrics g^{TM}, g_T^F on TM, F.

Let $df \in T^*M$ be the differential of f. We identify T^*M to TM by the metric g^{TM}. Let $\nabla f \in TM$ be the corresponding gradient vector field.

Let $L_{\nabla f}$ be the Lie derivative acting on $\mathbb{F}$

$$L_{\nabla f} = d^F i_{\nabla f} + i_{\nabla f} d^F. \tag{5.5}$$

Proposition 5.2. *The following identities hold*

$$\begin{aligned} d_T^{F*} &= d^{F*} + 2T i_{\nabla f}, \\ D_T^2 &= D^2 + 2T L_{\nabla f}. \end{aligned} \tag{5.6}$$

Proof. The first identity is obvious. The second identity follows easily. □

b) The Witten Laplacian

Set

$$d_T^F = e^{-Tf} d^F e^{Tf}, \tag{5.7}$$

$$\delta_T^F = e^{Tf} d^{F*} e^{-Tf}.$$

The operators d_T^F, δ_T^F were introduced by Witten [W]. Clearly

$$\left(d_T^F\right)^2 = 0. \tag{5.8}$$

The complex $(\mathbb{F}, d_T^F)$ will be called the Witten complex.

Then δ_T^F is the adjoint of d_T^F with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F}} = \langle\ ,\ \rangle_{\mathbb{F},0}$.

Proposition 5.3. *The map*

$$\alpha \in \mathbb{F} \to e^{-Tf}\alpha \in \mathbb{F} \tag{5.9}$$

induces an isomorphism of the Euclidean complexes $(\mathbb{F}, d^F, \langle\ ,\ \rangle_{\mathbb{F},T})$ *and* $(\mathbb{F}, d_T^F, \langle\ ,\ \rangle_{\mathbb{F}})$.

Proof. This is obvious. □

Let $\widetilde{D}_T$ be the operator

$$\widetilde{D}_T = d_T^F + \delta_T^F. \tag{5.10}$$

Proposition 5.4. *The following identities hold*

$$\widetilde{D}_T = e^{-Tf} D_T e^{Tf}, \tag{5.11}$$

$$\widetilde{D}_T^2 = e^{-Tf} D_T^2 e^{Tf}.$$

Proof. This follows from (5.4), (5.10). □

Let $L_{\nabla f}^*$ be the adjoint of $L_{\nabla f}$ with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F}}$. Then $L_{\nabla f} + L_{\nabla f}^*$ is an operator of order 0 acting on $\mathbb{F}$. Also $\widehat{c}(\nabla f)$ is defined as in (4.15).

Proposition 5.5.

$$
\begin{aligned}
d_T^F &= d^F + T df \wedge, \\
\delta_T^F &= d^{F*} + T i_{\nabla f}, \qquad (5.12)\\
\widetilde{D}_T &= D + T\widehat{c}(\nabla f).
\end{aligned}
$$

Moreover

$$\widetilde{D}_T^2 = D^2 + T\left(L_{\nabla f} + L_{\nabla f}^*\right) + T^2|df|^2, \qquad (5.13)$$

$$\widetilde{D}_T^2 = D^2 - T\omega\left(F, g^F\right)(\nabla f) + T \sum_{1\le i,j\le n} \left\langle \nabla_{e_i}^{T^*M} df, e_j \right\rangle c(e_i)\widehat{c}(e_j) + T^2|df|^2.$$

Proof. The identities in (5.12) are obvious. Also

$$\widetilde{D}_T^2 = D^2 + T\left(d^F i_{\nabla f} + i_{\nabla f} d^F\right) + T\left(d^{F*} df \wedge + df \wedge d^{F*}\right) + T^2|df|^2. \qquad (5.14)$$

From (5.5), we get

$$d^{F*} df \wedge + df \wedge d^{F*} = L_{\nabla f}^*. \qquad (5.15)$$

The first identity in (5.13) follows from (5.14), (5.15). Using the last identity in (5.12), we obtain

$$\widetilde{D}_T^2 = D^2 + T\left[D, \widehat{c}(\nabla f)\right] + T^2|df|^2. \qquad (5.16)$$

By (4.16) and by Proposition 4.12, we find that

$$\left[D, \widehat{c}(\nabla f)\right] = \sum_{1\le i\le n} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} \nabla f\right) - \omega\left(F, g^F\right)(\nabla f). \qquad (5.17)$$

Using (5.16), (5.17) we get the last identity in (5.13). □

c) A basic closed 1-form

Here we prove an essential result, which is an analogue of a result of Bismut-Lebeau [BL2, Theorem 3.3].

Theorem 5.6. *Let $\alpha_{t,T}$ be the 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+$*

$$\alpha_{t,T} = \frac{dt}{2t} \operatorname{Tr_s}\left[N \exp\left(-t\widetilde{D}_T^2\right)\right] - dT \operatorname{Tr_s}\left[f \exp\left(-t\widetilde{D}_T^2\right)\right]. \qquad (5.18)$$

Then $\alpha_{t,T}$ is closed.

Proof. We proceed as in [BL2]. The vector space $\mathbb{F}$ is $\mathbb{Z}$-graded, and so it is $\mathbb{Z}_2$-graded. Let $\tau \in \mathrm{End}(\mathbb{F})$ be the operator defining the $\mathbb{Z}_2$-grading, i.e. $\tau = +1$ on $\mathbb{F}^{\mathrm{even}}, \tau = -1$ on $\mathbb{F}^{\mathrm{odd}}$. Then $\mathrm{End}(\mathbb{F})$ is a $\mathbb{Z}_2$-graded algebra, the even (resp. odd) elements of $\mathrm{End}(\mathbb{F})$ commuting (resp. anticommuting) with τ. Now the key fact is that d^F, d_T^{F*} and D_T are odd operators. Clearly

$$\frac{\partial}{\partial T}\frac{1}{2t}\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)\right] = \frac{1}{2}\frac{\partial}{\partial b}\left\{\operatorname{Tr_s}\left[N\exp\left(-tD_T^2 - b\left[D_T, \frac{\partial D_T}{\partial T}\right]\right)\right]\right\}_{b=0}. \tag{5.19}$$

Since the supertrace $\operatorname{Tr_s}$ vanishes on supercommutators [Q1], we get

$$\frac{\partial}{\partial T}\frac{1}{2t}\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)\right] = -\frac{1}{2}\frac{\partial}{\partial b}\left\{\operatorname{Tr_s}\left[[D_T, N]\exp\left(-tD_T^2 - b\frac{\partial D_T}{\partial T}\right)\right]\right\}_{b=0}. \tag{5.20}$$

Now

$$[D_T, N] = -d^F + d_T^{F*}. \tag{5.21}$$

Moreover, using (5.3), (5.4), we get

$$\frac{\partial D_T}{\partial T} = \left[2f, d_T^{F*}\right]. \tag{5.22}$$

So from (5.20)–(5.22), we obtain

$$\frac{\partial}{\partial T}\frac{1}{2t}\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)\right] = \frac{\partial}{\partial b}\left\{\operatorname{Tr_s}\left[\left(d^F - d_T^{F*}\right)\exp\left(-tD_T^2 + b\left[d_T^{F*}, f\right]\right)\right]\right\}_{b=0}. \tag{5.23}$$

Also

$$\left[d_T^{F*}, D_T^2\right] = 0. \tag{5.24}$$

Using again the fact that $\operatorname{Tr_s}$ vanishes on supercommutators, from (5.23), (5.24), we get

$$\frac{\partial}{\partial T}\frac{1}{2t}\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)\right] = \frac{\partial}{\partial b}\left\{\operatorname{Tr_s}\left[\left[d_T^{F*}, d^F - d_T^{F*}\right]\exp\left(-tD_T^2 + bf\right)\right]\right\}_{b=0} \tag{5.25}$$

$$= \frac{\partial}{\partial b}\left\{\mathrm{Tr}_s\left[D_T^2 \exp\left(-tD_T^2 + bf\right)\right]\right\}_{b=0}$$
$$= \frac{\partial}{\partial b}\left\{\mathrm{Tr}_s\left[f \exp\left(-tD_T^2 + bD_T^2\right)\right]\right\}_{b=0}$$
$$= -\frac{\partial}{\partial t}\mathrm{Tr}_s\left[f \exp\left(-tD_T^2\right)\right].$$

The proof of our Theorem is completed. □

Theorem 5.7. *For* $t > 0, T \geq 0$, *the following identity holds*

$$\alpha_{t,T} = \frac{dt}{2t}\mathrm{Tr}_s\left[N \exp\left(-t\widetilde{D}_T^2\right)\right] - dT\,\mathrm{Tr}_s\left[f \exp\left(-t\widetilde{D}_T^2\right)\right]. \tag{5.26}$$

Proof. Equation (5.26) follows from Proposition 5.4. □

d) A contour integral

We fix constants ε, A, T_0 such that $0 < \varepsilon < 1 < A < +\infty, 0 \leq T_0 < +\infty$.

Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the contour in $\mathbb{R}_+ \times \mathbb{R}_+^*$

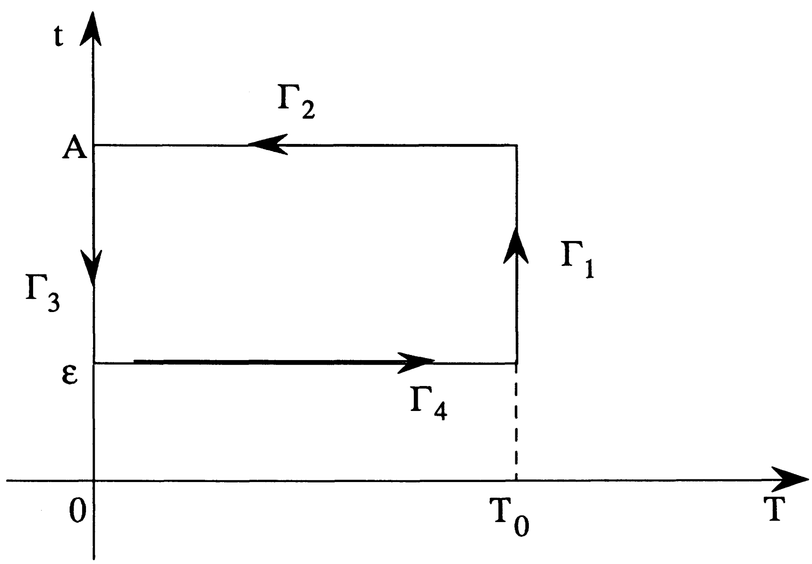

Figure 1

As shown in Figure 1, the contour Γ is made of four oriented pieces.

$$\begin{aligned} &\Gamma_1 : T = T_0, && \varepsilon \le t \le A; \\ &\Gamma_2 : 0 \le T \le T_0, && t = A; \\ &\Gamma_3 : T = 0, && \varepsilon \le t \le A; \\ &\Gamma_4 : 0 \le T \le T_0, && t = \varepsilon. \end{aligned} \tag{5.27}$$

The orientation of $\Gamma_1, \cdots, \Gamma_4$ is indicated on Figure 1.

For $1 \le k \le 4$, set

$$I_k^0 = \int_{\Gamma_k} \alpha. \tag{5.28}$$

Theorem 5.8. *The following identity holds*

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{5.29}$$

Proof. This follows from Theorem 5.6. □

Remark 5.9. The proof of Theorem 0.2 will now consist of two steps :

— A first step is to make an adequate choice of the function f, and of the metrics g^{TM} and g^F.

— A second step will be to make $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$ in this order in equality (5.29). Each term $I_k^0 (1 \le k \le 4)$ will diverge at one or several of these stages. Once the divergences will have been substracted off, we will ultimately obtain an identity which is exactly Theorem 0.2.

VI. Some properties of the integral $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$

Let $f : M \to \mathbb{R}$ be a Morse function, and let ∇f be the gradient field of f with respect to a given metric on TM.

In this Section, we show that when the metrics g^F, g^{TM} vary, or when the gradient field ∇f varies, the variation of $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$ is essentially the one which is predicted by the anomaly formulas for the Ray-Singer metric, which were stated in Theorem 4.7.

As explained in Section 7 b), this step permits us to reduce the proof of Theorem 0.2 to the case where the metrics g^{TM} and g^F are as simple as possible.

A by-product of Theorem 0.2 is that the integral $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$ only depends on the metrics g^{TM}, g^F and on the Thom-Smale complex associated to ∇f. In this Section, we give a more cohomological expression for this integral in terms of Chern-Simons forms and of the Euler number of a vector bundle on a cycle of codimension 1.

This Section is organized as follows. In a), we show that the integral $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$ is unchanged when replacing ∇f by another gradient field for f. In b), we give variation formulas for this integral. Finally in c), we express the integral in a more cohomological form.

a) Homotopy invariance of the integral

We make the same assumptions and we use the same notation as in Section 4. In particular M is a compact manifold and F is a flat vector bundle on M.

Let $f : M \to \mathbb{R}$ be Morse function. Let B be the set of critical points of f. If $x \in B$, let $\operatorname{ind}(x)$ be the index of f at x.

Let (g^{TM}, g^F) and (g'^{TM}, g'^F) be two couples of metrics on TM, F. We use the notation of Sections 4a) and 4b) for the couple (g^{TM}, g^F). The corresponding objects associated to (g'^{TM}, g'^F) will be denoted with a $'$. In particular, ∇f and $\nabla' f$ denote the gradient vector fields of f with respect to the metrics g^{TM} and g'^{TM}. Let $\| \ \|_{\det F}$ and $\| \ \|'_{\det F}$ be the metrics on the line bundle $\det F$ induced by g^F and g'^F .

Recall that the current $\psi(TM, \nabla^{TM})$ on TM was constructed in Section 3d). By Remark 3.8, $(\nabla f)^*\psi(TM, \nabla^{TM})$ and $(\nabla' f)^*\psi(TM, \nabla^{TM})$ are well-defined locally integrable currents on M with values in $o(TM)$, which are smooth on $M \backslash B$. Moreover they verify the equation of currents

$$d(\nabla f)^* \psi\left(TM, \nabla^{TM}\right) = e\left(TM, \nabla^{TM}\right) - \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \delta_x, \tag{6.1}$$

$$d(\nabla' f)^* \psi\left(TM, \nabla^{TM}\right) = e\left(TM, \nabla^{TM}\right) - \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \delta_x.$$

Proposition 6.1. *The following identity holds*

$$-\int_M \theta\left(F, g^F\right) (\nabla f)^* \psi\left(TM, \nabla^{TM}\right) = -\int_M \theta\left(F, g^F\right) (\nabla' f)^* \psi\left(TM, \nabla^{TM}\right). \tag{6.2}$$

Proof. For $\ell \in [0,1]$, set

$$g_\ell^{TM} = (1-\ell) g^{TM} + \ell g'^{TM}. \tag{6.3}$$

Let $\nabla_\ell f$ be the gradient of f with respect to the metric g_ℓ^{TM}. Then $\nabla_\ell f$ has the same zeroes as ∇f. Using the current equation (6.1) over $M \times [0,1]$, we deduce that the closed current $(\nabla f)^*\psi(TM, \nabla^{TM}) - (\nabla' f)^*\psi(TM, \nabla^{TM})$ is exact. Since the form $\theta(F, g^F)$ is closed, equation (6.2) follows. $\square$

Remark 6.2. The vector fields $\nabla' f$ are exactly the gradient vector fields for f in the sense of [Sm1]. Let $g : M \to \mathbb{R}$ be another Morse function having the same critical points as f with the same indexes. Laudenbach has shown to us that in general, the vector fields ∇f and ∇g are not homotopic in the class of vector fields which exactly vanish on B and are nondegenerate at B. Also in general the integrals $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$ and $-\int_M \theta(F, g^F)(\nabla g)^*\psi(TM, \nabla^{TM})$ take different values. The counterexample of Laudenbach is very simply constructed on the 2-dimensional torus.

b) Variation formulas for the integral $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$.

Here we study dependence of $-\int_M \theta(F, g^F)(\nabla f)^*\psi(TM, \nabla^{TM})$ in terms of g^F and ∇^{TM}.

Theorem 6.3. *The following identity holds*

$$(6.4)\quad -\int_M \theta\left(F, g'^F\right)\left(\nabla' f\right)^*\psi\left(TM, \nabla'^{TM}\right) + \int_M \theta\left(F, g^F\right)\left(\nabla f\right)^*\psi\left(TM, \nabla^{TM}\right)$$
$$= \int_M \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F}}{\| \ \|^2_{\det F}}\right) e\left(TM, \nabla^{TM}\right) - \int_M \theta\left(F, g'^F\right)\widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right)$$
$$- \sum_{x\in B}(-1)^{\operatorname{ind}(x)} \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F_x}}{\| \ \|^2_{\det F_x}}\right).$$

Proof. Clearly

$$(6.5)\qquad \theta\left(F, g'^F\right) - \theta\left(F, g^F\right) = d \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F}}{\| \ \|^2_{\det F}}\right).$$

Using the equation of currents (6.1), and (6.5), we get

$$(6.6)\qquad -\int_M \left(\theta\left(F, g'^F\right) - \theta\left(F, g^F\right)\right)\left(\nabla f\right)^*\psi\left(TM, \nabla^{TM}\right)$$
$$= \int_M \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F}}{\| \ \|^2_{\det F}}\right) e\left(TM, \nabla^{TM}\right) - \sum_{x\in B}(-1)^{\operatorname{ind}(x)} \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F_x}}{\| \ \|^2_{\det F_x}}\right).$$

Also by (3.34), we obtain

$$(6.7) \quad -\int_M \theta\left(F, g'^F\right)\left((\nabla f)^* \psi\left(TM, \nabla'^{TM}\right) - (\nabla f)^* \psi\left(TM, \nabla^{TM}\right)\right)$$
$$= -\int_M \theta\left(F, g'^F\right) \widetilde{e}\left(TM, \nabla^{TM}, \nabla'^{TM}\right).$$

Then (6.4) follows from (6.2), (6.6), (6.7). □

Let $x_1, \cdots, x_q$ be the elements of B.

Let $(\ell, x) \in \mathbb{R} \times M \to f_\ell(x) \in \mathbb{R}$ be a smooth function such that $f_0 = f$. Then there exists $\varepsilon > 0$ such that if $|\ell| \leq 2\varepsilon$, f_ℓ is a Morse function. Let B_ℓ be the set of critical points of f_ℓ. Then if $\varepsilon > 0$ is small enough, there are smooths maps $\ell \in]-\varepsilon, \varepsilon[\to x_{i,\ell} \in M$ $(1 \leq i \leq q)$ such that $x_{1,\ell}, \cdots, x_{q,\ell}$ are the critical points of f_ℓ, and their index does not depend on ℓ.

Proposition 6.4. *For* $|\ell| < \varepsilon$, *the following identity holds*

$$(6.8) \qquad \frac{\partial}{\partial \ell}\left(-\int_M \theta\left(F, g^F\right)\left(\nabla f_\ell\right)^* \psi\left(TM, \nabla^{TM}\right)\right)$$
$$= -\sum_{i=0}^{q} (-1)^{\operatorname{ind} x_{i,\ell}} \theta\left(F, g^F\right)\left(\frac{\partial x_{i,\ell}}{\partial \ell}\right).$$

Proof. Using again the fact that the form $\theta(F, g^F)$ is closed and the equation of currents (6.1), we get (6.8). □

Remark 6.5. A comparison of formulas (3.13) and (6.4) shows that they are not unrelated. Theorem 0.2 gives a precise content to their similarity.

In Section 16, by using Laudenbach's explicit description of the deformation of the Thom-Smale complex along a Cerf path [Ce] connecting two Morse functions, and also Proposition 6.4, we will give a direct proof of a formula calculating the ratio of two Milnor metrics, which does not rely on Theorem 0.2.

c) A cohomological expression for the integral $-\int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM})$.

Let K' be a smooth triangulation of M such that $K'^{n-1} \cap B = \emptyset$. Over each simplex $\sigma \in K'^n \backslash K'^{n-1}$, the 1-form θ has a primitive V_σ, i.e.

$$dV_\sigma = \theta\left(F, g^F\right) \quad \text{on } \sigma. \tag{6.9}$$

Of course V_σ is smooth on σ.

Let V be the locally integrable current of degree 0 on M, such that for any $\sigma \in K'^n \backslash K'^{n-1}$, V coincides with V_σ on σ. Obviously, there is a closed current γ of degree 1, whose support is included in K'^{n-1}, such that

$$dV = \theta\left(F, g^F\right) - \gamma. \tag{6.10}$$

In particular the support of γ is included in $M \backslash B$.

Over $M \backslash B$, the vector bundle TM has a nonzero section ∇f. By Chern-Simons theory, there is an unambiguously defined class $\widetilde{e}(TM, \nabla f, \nabla^{TM})$ of smooth forms of degree $n-1$ on $M \backslash B$, which is defined modulo exact smooth forms on $M \backslash B$, such that

$$d\widetilde{e}\left(TM, \nabla f, \nabla^{TM}\right) = e\left(TM, \nabla^{TM}\right) \quad \text{on } M \backslash B. \tag{6.11}$$

Of course,

$$\widetilde{e}\left(TM, \nabla f, \nabla^{TM}\right) = 0 \ \text{ if } n \text{ is odd}. \tag{6.12}$$

The quotient vector bundle $\frac{TM}{\{\nabla f\}}$ is well-defined on $M \backslash B$. Let $e(\frac{TM}{\{\nabla f\}})$ denote the corresponding Euler class. Then $e(\frac{TM}{\{\nabla f\}})$ is a cohomology class on $M \backslash B$, with values in the orientation bundle $o(\frac{TM}{\{\nabla f\}})$ of $\frac{TM}{\{\nabla f\}}$. Of course,

$$e\left(\frac{TM}{\{\nabla f\}}\right) = 0 \ \text{ if } n \text{ is even}. \tag{6.13}$$

Moreover it is clear that

$$o\left(\frac{TM}{\{\nabla f\}}\right) = o(TM) \ \text{ over } M \backslash B. \tag{6.14}$$

Therefore $\gamma e(\frac{TM}{\{\nabla f\}})$ is a cohomology class on M with values in $o(TM)$, and the integral $\int_M \gamma e(\frac{TM}{\{\nabla f\}})$ is well-defined.

Theorem 6.6. *The following identity holds*

$$(6.15) \qquad -\int_M \theta\left(F,g^F\right)(\nabla f)^* \psi\left(TM,\nabla^{TM}\right) = \int_M Ve\left(TM,\nabla^{TM}\right)$$
$$-\sum_{x\in B}(-1)^{\mathrm{ind}(x)}V(x) - \int_M \gamma\left(\widetilde{e}\left(TM,\nabla f,\nabla^{TM}\right) - \frac{1}{2}e\left(\frac{TM}{\{\nabla f\}}\right)\right).$$

Proof. Using (6.1), (6.10), it is clear that

$$(6.16) \qquad -\int_M \theta\left(F,g^F\right)(\nabla f)^* \psi\left(TM,\nabla^{TM}\right)$$
$$= \int_M Ve\left(TM,\nabla^{TM}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}V(x) - \int_M \gamma(\nabla f)^* \psi\left(TM,\nabla^{TM}\right).$$

Let $TM^{\perp}$ be the orthogonal bundle to ∇f in TM over $M\backslash B$. Then over $M\backslash B, TM = \{\nabla f\}\oplus TM^{\perp}$. Over $M\backslash B$, we can equip $TM = \{\nabla f\}\oplus TM^{\perp}$ with the connection $\widetilde{\nabla}^{TM} = \nabla^{\{\nabla f\}} \oplus \nabla^{TM^{\perp}}$ which is the direct sum of the projections of ∇^{TM} on $\{\nabla f\}$ and $TM^{\perp}$. The connection $\widetilde{\nabla}^{TM}$ still preserves the metric g^{TM}. Using (3.34), we find that

$$(6.17) \qquad (\nabla f)^* \psi\left(TM,\nabla^{TM}\right) - (\nabla f)^* \psi\left(TM,\widetilde{\nabla}^{TM}\right)$$
$$= -\widetilde{e}\left(TM,\nabla^{TM},\widetilde{\nabla}^{TM}\right) \quad \text{on } M\backslash B.$$

Also one sees easily that

$$(6.18) \qquad \widetilde{e}\left(TM,\nabla^{TM},\widetilde{\nabla}^{TM}\right) = -\widetilde{e}\left(TM,\nabla f,\nabla^{TM}\right)$$

Moreover by using the explicit formula (3.19), one finds that if $\widetilde{\beta}_T$ is the form β_T in associated to the connection $\widetilde{\nabla}^{TM}$, then

$$(6.19) \qquad (\nabla f)^*\widetilde{\beta}_T = -\frac{\exp\left(-T|\nabla f|^2\right)}{2\sqrt{\pi T}}|\nabla f| e\left(TM^{\perp},\nabla^{TM^{\perp}}\right),$$

and so

$$(6.20) \qquad (\nabla f)^* \psi\left(TM,\widetilde{\nabla}^{TM}\right) = -\frac{1}{2}e\left(TM^{\perp},\nabla^{TM^{\perp}}\right).$$

Using (6.16)–(6.20), we get (6.15). □

Remark 6.7. When n is odd, (6.15) takes the form

$$\begin{aligned}(6.21)\qquad & -\int_M \theta\left(F, g^F\right)(\nabla f)^* \psi\left(TM, \nabla^{TM}\right) \\ & = -\sum_{x\in B}(-1)^{\mathrm{ind}(x)} V(x) + \int_M \gamma \frac{1}{2} e\left(\frac{TM}{\{\nabla f\}}\right).\end{aligned}$$

Equations (6.15) and (6.21) exhibit clearly how the integral $-\int_M \theta(F, g^F)(\nabla f)^*$ $\psi(TM, \nabla^{TM})$ depends on the gradient field ∇f.

VII. An extension of a theorem of Cheeger and Müller

In this Section, we establish the main result of this paper, which was stated in Theorem 0.2. Namely we give an explicit formula relating Ray-Singer metrics to the Milnor metrics on the determinant of the cohomology of a flat vector bundle. This generalizes the basic result of Cheeger [C] and Müller [Mü 1,2]. Also, we establish Theorem 0.3.

This Section is organized as follows. In a), we restate for convenience the main result of this paper in Theorem 7.1. In b), by using the results of Sections 4 and 6, we show that we only need to establish Theorem 7.1 under simple assumptions on the metric g^{TM} on TM, on the Morse function f, and on the metric g^F on F. In c), we state without proof nine intermediary results, which will play a crucial role in establishing Theorem 7.1. The proofs of these results are delayed to Sections 8–15.

In d) starting from the crucial identity $\sum_{k=1}^{4} I_k^0 = 0$ established in (5.29), we study separately the terms $I_k^0 (1 \leq k \leq 4)$, by making in succession $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$. Each term diverges at one or several stages. In e), we verify that the divergences of the terms $I_k^0 (1 \leq k \leq 4)$ are compatible with our basic identity. We obtain in Theorem 7.19 an identity, which is shown in f) to be equivalent to Theorem 7.1. Finally, in g), we prove Theorem 0.3.

The organization of this Section is closely related to the organization of Section 6 in Bismut-Lebeau [BL2]. We have tried to make the resemblance as obvious as possible, although at many stages, the arguments are of an entirely different nature.

Throughout the Section, the assumptions and notation of Sections 1–6 will be in force.

a) An extension of the Cheeger-Müller theorem

We make the same assumptions as in Section 1.

Let g^{TM}, g^F be arbitrary smooth metrics on TM, F. Let $\| \ \|^{RS}_{\det H^\bullet(M,F)}$ be the corresponding Ray-Singer metric on the line $\det H^\bullet(M,F)$.

Let $f : M \to \mathbb{R}$ be a Morse function, and let B be the critical points of f. Let X be the gradient vector field of f with respect to a given smooth metric g_0^{TM} on TM (which does not necessarily coincide with the metric g^{TM}). We assume that the gradient vector field X verifies the Smale transversality conditions [Sm1,2].

The metric g^F on F induces metrics $\| \ \|_{\det F_x}$ on the lines $\det F_x$ $(x \in B)$. Let $\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}$ be the corresponding Milnor metric on $\det H^\bullet(M,F)$.

The main result of this paper is the extension of a theorem of Cheeger [C] and Müller [Mü 1,2], given in Theorem 0.2, which we restate for convenience.

Theorem 7.1. *The following identity holds*

$$\operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2 = -\int_M \theta\left(F, g^F\right) X^*\psi\left(TM, \nabla^{TM}\right). \tag{7.1}$$

Proof. The proof of Theorem 7.1 will occupy the rest of this Section. It relies on nine intermediary results stated in Theorems 7.6–7.14, whose proofs are delayed to Sections 8–15. □

Remark 7.2. Assume that the metric g^F is flat, or more generally that the metric $\| \ \|_{\det F}$ on the line bundle $\det F$ is flat. Then by Remark 1.10, $\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}$ coincides with the Reidemeister metric $\| \ \|^{R}_{\det H^\bullet(M,F)}$. Also $\theta(F, g^F) = 0$. From Theorem 7.1, we thus effectively recover the theorem of Cheeger [C] and Müller [Mü 1,2].

b) Some simplifying assumptions on the metrics g^{TM}, g^F

Let g'^{TM}, g'^F be another couple of metrics on TM, F. We denote with a $'$ all the objects associated to the metrics g'^{TM}, g'^F.

By Theorem 4.7, we know that

$$\text{(7.2)}\qquad \operatorname{Log}\left(\frac{\| \ \|'^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{RS}_{\det H^\bullet(M,F)}}\right)^2 = \int_M \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F}}{\| \ \|^2_{\det F}}\right) e\left(TM,\nabla^{TM}\right)$$
$$- \int_M \theta\left(F,g'^F\right)\widetilde{e}\left(TM,\nabla^{TM},\nabla'^{TM}\right).$$

If $x \in B$, let $\operatorname{ind}(x)$ be the index of f at x. By the very definition of Milnor metrics, it is clear that

$$\text{(7.3)}\qquad \operatorname{Log}\left(\frac{\| \ \|'^{\mathcal{M},X}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2 = \sum_{x\in B}(-1)^{\operatorname{ind}(x)}\operatorname{Log}\left(\frac{\| \ \|'^2_{\det F_x}}{\| \ \|^2_{\det F_x}}\right).$$

So from (7.2), (7.3), we get

$$\text{(7.4)}\qquad \operatorname{Log}\left(\frac{\| \ \|'^{RS}_{\det H^\bullet(M,F)}}{\| \ \|'^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2 - \operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2$$
$$= \int_M \operatorname{Log}\left(\frac{\| \ \|'^2_{\det F}}{\| \ \|^2_{\det F}}\right) e\left(TM,\nabla^{TM}\right) - \int_M \theta\left(F,g'^F\right)\widetilde{e}\left(TM,\nabla^{TM},\nabla'^{TM}\right)$$
$$- \sum_{x\in B}(-1)^{\operatorname{ind}(x)}\operatorname{Log}\left(\frac{\| \ \|'^2_{\det F_x}}{\| \ \|^2_{\det F_x}}\right).$$

Using Proposition 6.1, Theorem 6.3 and (7.4), we see that

$$\text{(7.5)}\qquad \operatorname{Log}\left(\frac{\| \ \|'^{RS}_{\det H^\bullet(M,F)}}{\| \ \|'^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2 - \operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2$$
$$= - \int_M \theta\left(F,g'^F\right) X^*\psi\left(TM,\nabla'^{TM}\right) + \int_M \theta\left(F,g^F\right) X^*\psi\left(TM,\nabla^{TM}\right).$$

By (7.5), it is clear that to establish Theorem 7.1 in full generality, we only need to establish (7.1) for one given couple g^{TM}, g^F of metrics on TM, F. So in the sequel, we may and we will assume that $g^{TM} = g_0^{TM}$, i.e. g^{TM} is exactly the metric from which the gradient vector field X is defined. Equivalently, we will suppose that $X = \nabla f$. Also we will assume that the metric g^F is flat near B.

For $z \in M, \alpha > 0$, let $B^M(z,\alpha)$ be the open ball of center z and radius α with respect to the Riemannian distance associated to g^{TM}.

By a simple argument of Helffer-Sjöstrand [HSj4, Proposition 5.1], for any $\alpha > 0$, there exists a Morse function $f_\alpha : M \to \mathbb{R}$, and a metric g_α^{TM} on TM, which have the following properties :

— f_α, g_α^{TM} coincide with f, g^{TM} on $M \backslash \bigcup_{x\in B} B^M(x,\alpha)$. Moreover f_α has the same critical points as f with the same indexes.

— Near $x \in B$, there is a coordinate system $y = (y^1, \cdots, y^n)$ on M centered at x, such that near x

$$g_\alpha^{TM} = \sum_1^n \left|dy^i\right|^2,$$

$$f_\alpha(y) = f(x) - \frac{1}{2}\sum_1^{\mathrm{ind}(x)} \left|y^i\right|^2 + \frac{1}{2}\sum_{\mathrm{ind}(x)+1}^{n} \left|y^i\right|^2. \tag{7.6}$$

— The gradient vector field $\nabla_\alpha f_\alpha$ of f_α with respect to the metric g_α^{TM} verifies the Smale transversality conditions. Also if $(C^\bullet(W^u,F),\partial)$ and $(C^\bullet(W_\alpha^u,F),\partial)$ are the Thom-Smale complexes associated to the gradient vector fields ∇f and $\nabla_\alpha f_\alpha$, the obvious map $C^\bullet(W^u,F) \to C^\bullet(W_\alpha^u,F)$ identifies the two Thom-Smale complexes.

Let $\| \ \|^{\mathcal{M},\nabla_\alpha f_\alpha}_{\det H^\bullet(M,F)}$ by the Milnor metric on the line $\det H^\bullet(M,F)$ associated to the gradient vector field $\nabla_\alpha f_\alpha$ and to the metrics $\| \ \|_{\det F_x}$ on the lines $\det F_x$ $(x \in B)$. Since the Milnor metric only depends on the associated Thom-Smale complex and on the metrics $\| \ \|_{\det F_x} (x \in B)$, it is clear

$$\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)} = \| \ \|^{\mathcal{M},\nabla_\alpha f_\alpha}_{\det H^\bullet(M,F)}. \tag{7.7}$$

Let ∇_α^{TM} be the Levi-Civita connection on (TM, g_α^{TM}). Let $\| \ \|^{RS}_{\det H^\bullet(M,F),\alpha}$ be the Ray-Singer metric associated to the metrics (g_α^{TM}, g^F) on (TM,F). By (7.2), (7.7), we see that

$$\begin{aligned} &\mathrm{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F),\alpha}}{\| \ \|^{\mathcal{M},\nabla_\alpha f_\alpha}_{\det H^\bullet(M,F)}}\right)^2 - \mathrm{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}\right)^2 \\ &= -\int_M \theta\left(F,g^F\right) \widetilde{e}\left(TM, \nabla^{TM}, \nabla_\alpha^{TM}\right). \end{aligned} \tag{7.8}$$

Using Theorem 6.3 and (7.8), we see that

$$(7.9)\qquad \operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},\nabla_\alpha f_\alpha}_{\det H^\bullet(M,F),\alpha}}\right)^2 - \operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}\right)^2$$

$$= -\int_M \theta\left(F,g^F\right)(\nabla_\alpha f_\alpha)^*\psi\left(TM,\nabla^{TM}_\alpha\right)+\int_M \theta\left(F,g^F\right)(\nabla f_\alpha)^*\psi\left(TM,\nabla^{TM}\right).$$

Since $\nabla f = \nabla_\alpha f_\alpha$ on $M\backslash\bigcup_{x\in B} B^M(x,\alpha)$, by using Theorem 6.6, it is clear that for $\alpha > 0$ small enough, then

(7.10)

$$-\int_M \theta\left(F,g^F\right)(\nabla f_\alpha)^*\psi\left(TM,\nabla^{TM}\right) = -\int_M \theta\left(F,g^F\right)(\nabla f)^*\psi\left(TM,\nabla^{TM}\right).$$

So from (7.9), (7.10), for $\alpha > 0$ small enough, we get

$$(7.11)\qquad \operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},\nabla_\alpha f_\alpha}_{\det H^\bullet(M,F),\alpha}}\right)^2 - \operatorname{Log}\left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}\right)^2$$

$$= -\int_M \theta\left(F,g^F\right)(\nabla_\alpha f_\alpha)^*\psi\left(TM,\nabla^{TM}_\alpha\right)+\int_M \theta\left(F,g^F\right)(\nabla f)^*\psi\left(TM,\nabla^{TM}\right).$$

So we deduce from (7.11) that, to establish (7.1) in full generality, we may and we will assume that :

— For any $x \in B$, the metric g^F is flat near B.

— For any $x \in B$, there is a system of coordinates $y = (y^1, \cdots y^n)$ centered at x such that near x

$$(7.12)\quad g^{TM} = \sum_1^n \left|dy^i\right|^2, \quad f(y) = f(x) - \frac{1}{2}\sum_1^{\operatorname{ind}(x)} \left|y^i\right|^2 + \frac{1}{2}\sum_{\operatorname{ind}(x)+1}^{n} \left|y^i\right|^2.$$

Remark 7.3. Recall that the vector field ∇f depends on the metric g^{TM}. Using Proposition 6.1 and Theorem 7.1, one deduces that the Milnor metric $\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}$ does not depend on the metric g^{TM}. A direct proof of this result is given in Section 16, by using the results of Laudenbach in the Appendix.

c) Nine intermediary results

For $1 \leq i \leq n$, let M^i be the number of $x \in B$ of index i. Set

$$\chi(F) = \sum_0^n (-1)^i \dim H^i(M,F), \tag{7.13}$$
$$\chi'(F) = \sum_0^n (-1)^i i \dim H^i(M,F).$$

Then $\chi(F)$ is the Euler characteristic of F, and $\chi'(F)$ is the derived Euler characteristic of F. Clearly,

$$\chi(F) = \operatorname{rk}(F) \sum_{x \in B} (-1)^{\operatorname{ind}(x)}. \tag{7.14}$$

Set

$$\widetilde{\chi}'(F) = \operatorname{rk}(F) \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \operatorname{ind}(x) = \operatorname{rk}(F) \sum_{i=0}^n (-1)^i i M^i, \tag{7.15}$$
$$\operatorname{Tr}_s^B[f] = \sum_{x \in B} (-1)^{\operatorname{ind}(x)} f(x).$$

We use the notation of Sections 3 and 5. In particular for $T \geq 0$, B_T is given by (3.47) and the scalar product $\langle\ ,\ \rangle_{\mathbb{F},T}$ on $\mathbb{F}$ is defined in (5.2).

Definition 7.4. For $T \geq 0$, let $\mathbb{F}_T^{[0,1]}$(resp. $\mathbb{F}_T^{]0,1]}$, resp. $\mathbb{F}_T^{\{0\}}$) be the direct sum of the eigenspaces of D_T^2 associated to eigenvalues $\lambda \in [0,1]$(resp. $\lambda \in]0,1]$, resp. $\lambda = 0$) Let $D_T^{2,[0,1]}$(resp. $D_T^{2,]0,1]}$) be the restriction of D_T^2 to $\mathbb{F}_T^{[0,1]}$(resp. to $\mathbb{F}_T^{]0,1]}$).

For $T \geq 0$, let $P_T^{[0,1]}$(resp. $P_T^{]0,1]}$, resp. P_T) be the orthogonal projection operator from $\mathbb{F}$ on $\mathbb{F}_T^{[0,1]}$(resp. $\mathbb{F}_T^{]0,1]}$, resp. $\mathbb{F}_T^{\{0\}}$) with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F},T}$. Set $P_T^{]1,+\infty[} = 1 - P_T^{[0,1]}$.

Definition 7.5. For $T \geq 0$, let $|\ |^{RS}_{\det H^\bullet(M,F),T}$ be the L_2 metric on the line $\det H^\bullet(M,F)$ constructed in Section 2a), which is associated to the metrics g^{TM}, g_T^F on TM, F.

In the sequel, we assume that the simplifying assumptions of Section 7 b) are verified.

We now state without proof nine intermediary results, which will play an essential role in the proofs of Theorem 7.1. The proofs of these results are delayed to Sections 8–15.

Theorem 7.6. *The following identity holds,*

$$\lim_{T\to+\infty}\left\{\operatorname{Tr_s}\left[N\operatorname{Log}\left(D_T^{2,]0,1]}\right)\right]+\operatorname{Log}\left(\frac{|\ |^{RS}_{\det H^\bullet(M,F),T}}{|\ |^{RS}_{\det H^\bullet(M,F)}}\right)^2\right.\tag{7.16}$$

$$\left.+2\operatorname{rk}(F)\operatorname{Tr_s^B}[f]T+\left(\frac{n}{2}\chi(F)-\widetilde{\chi}'(F)\right)\operatorname{Log}\left(\frac{T}{\pi}\right)\right\}=\operatorname{Log}\left(\frac{\|\ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}{|\ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 .$$

Theorem 7.7. *Given ε, A with $0<\varepsilon<A<+\infty$, there exists $C>0$ such that if $t\in[\varepsilon,A], T\geq 1$, then*

$$\left|\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)\right]-\widetilde{\chi}'(F)\right|\leq\frac{C}{\sqrt{T}}.\tag{7.17}$$

Theorem 7.8. *For any $t>0$,*

$$\lim_{T\to+\infty}\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)P_T^{]1,+\infty[}\right]=0.\tag{7.18}$$

Moreover there exist $c>0, C>0$ such that for $t\geq 1, T\geq 0$, then

$$\left|\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)P_T^{]1,+\infty[}\right]\right|\leq c\exp(-Ct).\tag{7.19}$$

Theorem 7.9. *For $T\geq 0$ large enough, then*

$$\dim\mathbb{F}_T^{[0,1],i}=\operatorname{rk}(F)M^i.\tag{7.20}$$

Also

$$\lim_{T\to+\infty}\operatorname{Tr}\left[D_T^{2,[0,1]}\right]=0.\tag{7.21}$$

Theorem 7.10. *As $t\to 0$, the following identity holds,*

$$\operatorname{Tr_s}\left[N\exp\left(-tD_T^2\right)\right]=\frac{n}{2}\chi(F)+O(t)\ \textit{if}\ n\ \text{is even},\tag{7.22}$$

$$=\operatorname{rk}(F)\int_M\int^B L\exp\left(-\frac{\dot{R}^{TM}}{2}\right)\frac{1}{\sqrt{t}}+O(\sqrt{t})\ \textit{if}\ n\ \text{is odd}.$$

Theorem 7.11. *For any $t > 0$, there is $c > 0$ such that as $T \to +\infty$,*
(7.23)
$$\mathrm{Tr}_s\left[f \exp\left(-tD_T^2\right)\right] = \mathrm{rk}(F)\,\mathrm{Tr}_s^B[f] + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T} + O\left(e^{-cT}\right).$$

Theorem 7.12. *For any $d > 0$, there exists $C > 0$ such that for $0 < t \leq 1$, $0 \leq T \leq \frac{d}{t}$, then*

$$\left|\frac{1}{t^2}\left\{\mathrm{Tr}_s\left[f \exp\left(-\left(tD + T\widehat{c}(\nabla f)\right)^2\right)\right] - \right.\right. \tag{7.24}$$
$$\left.\left. \mathrm{rk}(F)\int_M f \int^B \exp\left(-B_{T^2}\right) + t\int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)\right\}\right| \leq C.$$

Theorem 7.13. *For any $T > 0$, the following identity holds,*

$$\lim_{t\to 0}\frac{1}{t^2}\left(\mathrm{Tr}_s\left[f \exp\left(-\left(tD + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr}_s^B[f]\right) \tag{7.25}$$
$$= \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T\tanh(T)}.$$

Theorem 7.14. *There exist $c > 0, C > 0$ such that for $t \in]0,1], T \geq 1$, then*

$$\left|\frac{1}{t^2}\left(\mathrm{Tr}_s\left[f \exp\left(-\left(tD + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr}_s^B[f]\right.\right. \tag{7.26}$$
$$\left.\left. - \frac{t^2}{T}\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\right)\right| \quad \leq \quad c\exp\left(-CT\right).$$

Remark 7.15. Sections 8 and 9 are devoted to the proof of the crucial Theorem 7.6, Section 10 to the proof of Theorems 7.7, 7.8 and 7.9. Each of the Sections 11–15 is devoted to the proof of one of the Theorems 7.10–7.14.

d) The asymptotics of the I_k^0 's

Here we use the notation of Section 5. We start from the identity (5.29)

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{7.27}$$

We now will study individually each I_k^0 $(1 \leq k \leq 4)$, by making in succession $A \to +\infty$, $T_0 \to +\infty, \varepsilon \to 0$.

1) The term I_1^0. Clearly

$$I_1^0 = \int_\varepsilon^A \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right)\right] \frac{dt}{2t}\,. \tag{7.28}$$

$\alpha)$ $\underline{A \to +\infty}$

As $A \to +\infty$,

$$I_1^0 - \frac{1}{2}\chi'(F)\,\mathrm{Log}(A) \to I_1^1 = \int_\varepsilon^1 \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right)\right] \frac{dt}{2t} + \int_1^{+\infty} \left(\mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right)\right] - \chi'(F)\right) \frac{dt}{2t}\,. \tag{7.29}$$

$\beta)$ $\underline{T_0 \to +\infty}$

By Theorem 7.7, we see that as $T_0 \to +\infty$,

$$\int_\varepsilon^1 \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right)\right] \frac{dt}{2t} \to -\frac{1}{2}\widetilde{\chi}'(F)\,\mathrm{Log}(\varepsilon). \tag{7.30}$$

Moreover we have the obvious identity

$$\int_1^{+\infty} \left(\mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right)\right] - \chi'(F)\right) \frac{dt}{2t} \tag{7.31}$$
$$= \int_1^{+\infty} \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right) P_{T_0}^{]0,1]}\right] \frac{dt}{2t} + \int_1^{+\infty} \mathrm{Tr_s}\left[\dot{N} \exp\left(-tD_{T_0}^2\right) P_{T_0}^{]1,+\infty[}\right] \frac{dt}{2t}\,.$$

By definition,

(7.32)
$$\int_1^{+\infty} \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right) P_{T_0}^{]0,1]}\right] \frac{dt}{2t} = \int_1^{+\infty} \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^{2,]0,1]}\right)\right] \frac{dt}{2t}\,,$$

and so

$$\int_1^{+\infty} \mathrm{Tr_s}\left[N \exp\left(-tD_{T_0}^2\right) P_{T_0}^{]0,1]}\right] \frac{dt}{2t} \tag{7.33}$$
$$= \mathrm{Tr_s}\left[\int_{D_{T_0}^{2,[0,1]}}^1 N\left(e^{-t}-1\right) \frac{dt}{2t} P_{T_0}^{]0,1]}\right]$$
$$+ \mathrm{Tr_s}\left[N P_{T_0}^{]0,1]}\right] \int_1^{+\infty} e^{-t} \frac{dt}{2t} - \frac{1}{2}\mathrm{Tr_s}\left[N\,\mathrm{Log}\left(D_{T_0}^{2,]0,1]}\right)\right].$$

Moreover,

(7.34)
$$\operatorname{Tr}_s\left[\int_{D_{T_0}^{2,[0,1]}}^{1} N\left(e^{-t}-1\right)\frac{dt}{2t}P_{T_0}^{]0,1]}\right]$$
$$=\operatorname{Tr}_s\left[\int_{D_{T_0}^{2,[0,1]}}^{1} N\left(e^{-t}-1\right)\frac{dt}{2t}P_{T_0}^{[0,1]}\right]-\frac{1}{2}\chi'(F)\int_0^1\left(e^{-t}-1\right)\frac{dt}{t}\,.$$

Using Theorem 7.9 and (7.34), we see that as $T_0\to+\infty$,

(7.35)
$$\operatorname{Tr}_s\left[\int_{D_{T_0}^{2,[0,1]}}^{1} N\left(e^{-t}-1\right)\frac{dt}{2t}P_{T_0}^{]0,1]}\right]\to\frac{1}{2}\left(\widetilde{\chi}'(F)-\chi'(F)\right)\int_0^1\left(e^{-t}-1\right)\frac{dt}{t}\,.$$

Similarly,

(7.36)
$$\operatorname{Tr}_s\left[NP_{T_0}^{]0,1]}\right]=\operatorname{Tr}_s\left[NP_{T_0}^{[0,1]}\right]-\chi'(F)\,.$$

From Theorem 7.9 and (7.36), we find that as $T_0\to+\infty$,

(7.37)
$$\operatorname{Tr}_s\left[NP_{T_0}^{]0,1]}\right]\int_1^{+\infty}e^{-t}\frac{dt}{2t}\to\frac{1}{2}\left(\widetilde{\chi}'(F)-\chi'(F)\right)\int_1^{+\infty}e^{-t}\frac{dt}{t}\,.$$

Moreover, one has the trivial identity

(7.38)
$$\Gamma'(1)=\int_0^1\left(e^{-t}-1\right)\frac{dt}{t}+\int_1^{+\infty}e^{-t}\frac{dt}{t}\,.$$

From (7.33), (7.35), (7.37), (7.38), we see that as $T_0\to+\infty$,

(7.39)
$$\int_1^{+\infty}\operatorname{Tr}_s\left[N\exp\left(-tD_{T_0}^2\right)P_{T_0}^{]0,1]}\right]\frac{dt}{2t}+\frac{1}{2}\operatorname{Tr}_s\left[N\operatorname{Log}\left(D_{T_0}^{2,]0,1]}\right)\right]$$
$$\to\frac{1}{2}\Gamma'(1)\left(\widetilde{\chi}'(F)-\chi'(F)\right).$$

Also by Theorem 7.8, we find that as $T_0\to+\infty$,

(7.40)
$$\int_1^{+\infty}\operatorname{Tr}_s\left[N\exp\left(-tD_{T_0}^2\right)P_{T_0}^{]1,+\infty[}\right]\frac{dt}{2t}\to 0\,.$$

Using (7.29), (7.30), (7.31), (7.39), (7.40), we get

(7.41)
$$I_1^1+\frac{1}{2}\operatorname{Tr}_s\left[N\operatorname{Log}\left(D_{T_0}^{2,]0,1]}\right)\right]\to I_1^2=-\frac{1}{2}\widetilde{\chi}'(F)\operatorname{Log}(\varepsilon)+\frac{1}{2}\Gamma'(1)\left(\widetilde{\chi}'(F)-\chi'(F)\right).$$

γ) $\underline{\varepsilon\to 0}$

Set

$$I_1^3 = \frac{1}{2}\Gamma'(1)\left(\widetilde{\chi}'(F) - \chi'(F)\right). \tag{7.42}$$

Clearly

$$I_1^2 + \frac{1}{2}\widetilde{\chi}'(F)\operatorname{Log}(\varepsilon) = I_1^3. \tag{7.43}$$

2) The term I_2^0. We have the obvious equality

$$I_2^0 = \int_0^{T_0} \operatorname{Tr_s}\left[f \exp\left(-AD_T^2\right)\right] dT. \tag{7.44}$$

α) $\underline{A \to +\infty}$

Clearly, as $A \to +\infty$,

$$I_2^0 \to I_2^1 = \int_0^{T_0} \operatorname{Tr_s}\left[f P_T\right] dT. \tag{7.45}$$

Proposition 7.16. *The following identity holds*

$$I_2^1 = -\frac{1}{2}\operatorname{Log}\left(\frac{\|\ \|^{RS}_{\det H^\bullet(M,F),T_0}}{\|\ \|^{RS}_{\det H^\bullet(M,F)}}\right)^2. \tag{7.46}$$

Proof. We proceed as in [BL2, Theorem 6.12]. By Hodge theory the map $s \in \mathbb{F}^{\{0\}} \to P_T\, s \in \mathbb{F}_T^{\{0\}}$ is the canonical isomorphism of $\mathbb{F}^{\{0\}}$ with $\mathbb{F}_T^{\{0\}}$ (these two finite dimensional $\mathbb{Z}$-graded vector spaces are identified with $H^\bullet(M,F)$). In particular, if $s \in \mathbb{F}^{\{0\}}, 0 \le T \le T'$, then

$$P_{T'}s = P_{T'}P_T s. \tag{7.47}$$

Using (7.47), we see that if $s \in \mathbb{F}^{\{0\}}, s' \in \mathbb{F}^{\{0\}}$, then

$$\begin{aligned} &\frac{\partial}{\partial T}\langle P_T s, P_T s'\rangle_{\mathbb{F},T} \\ &= \left\langle \frac{\partial P_T}{\partial T}P_T s, P_T s'\right\rangle_{\mathbb{F},T} + \left\langle P_T s, \frac{\partial P_T}{\partial T}P_T s'\right\rangle_{\mathbb{F},T} - 2\langle f P_T s, P_T s'\rangle_{\mathbb{F},T}. \end{aligned} \tag{7.48}$$

Since $P_T^2 = P_T$, then

$$\frac{\partial P_T}{\partial T}P_T + P_T\frac{\partial P_T}{\partial T} = \frac{\partial P_T}{\partial T}. \tag{7.49}$$

From (7.49), we deduce that $\frac{\partial P_T}{\partial T}$ maps $\mathbb{F}_T^{\{0\}}$ in its orthogonal with respect to the scalar product $\langle\ \ \rangle_{\mathbb{F},T}$. We then rewrite (7.48) in the form

$$\frac{\partial}{\partial T}\left\langle P_T s, P_T s'\right\rangle_{\mathbb{F},T} = -2\left\langle f P_T s, P_T s'\right\rangle_{\mathbb{F},T}. \tag{7.50}$$

Using (7.50), we obtain

$$\frac{\partial}{\partial T}\operatorname{Log}\left(|\ \ |^{RS}_{\det H^\bullet(M,F),T}\right)^2 = -2\operatorname{Tr_s}\left[f P_T\right]. \tag{7.51}$$

From (7.51), we get (7.46). □

$\beta)$ $\underline{T_0 \to +\infty}$

Tautologically

$$I_2^1 + \frac{1}{2}\operatorname{Log}\left(\frac{|\ \ |^{RS}_{\det H^\bullet(M,F),T_0}}{|\ \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 = 0. \tag{7.52}$$

$\gamma)$ $\underline{\varepsilon \to 0}$

Nothing is left.

3) The term I_3^0. Recall that $D = D_0$. We have the identity

$$I_3^0 = -\int_\varepsilon^A \operatorname{Tr_s}\left[N\exp\left(-tD^2\right)\right]\frac{dt}{2t}. \tag{7.53}$$

$\alpha)$ $\underline{A \to +\infty}$

Clearly, as $A \to +\infty$, then

$$\begin{aligned} I_3^0 + \frac{1}{2}\chi'(F)\operatorname{Log}(A) \to I_3^1 = &-\int_\varepsilon^1 \operatorname{Tr_s}\left[N\exp\left(-tD^2\right)\right]\frac{dt}{2t} \\ &-\int_1^{+\infty}\left(\operatorname{Tr_s}\left[N\exp\left(-tD^2\right)\right] - \chi'(F)\right)\frac{dt}{2t}. \end{aligned} \tag{7.54}$$

$\beta)$ $\underline{T_0 \to +\infty}$

As $T_0 \to +\infty$, I_3^1 remains constant and equal to I_3^2.

$\gamma)$ $\underline{\varepsilon \to 0}$

Set

$$a_{-1} = \operatorname{rk}(F)\int_M\int^B L\exp\left(-\frac{\dot{R}^{TM}}{2}\right), \qquad a_0 = \frac{n}{2}\chi(F). \tag{7.55}$$

Observe that

$$(7.56)\qquad \begin{aligned} a_{-1} &= 0 \quad \text{if} \quad n \text{ is even},\\ a_0 &= 0 \quad \text{if} \quad n \text{ is odd}. \end{aligned}$$

By Theorem 7.10, we know that as $t \to 0$,

$$(7.57)\qquad \begin{aligned} \operatorname{Tr_s}\left[N \exp\left(-tD^2\right)\right] &= a_0 + O(t) \quad \text{if } n \text{ is even},\\ &= \frac{a_{-1}}{\sqrt{t}} + O(\sqrt{t}) \text{ if } n \text{ is odd}. \end{aligned}$$

From (7.57), we see that as $t \to 0$, then

$$(7.58)\qquad \operatorname{Tr_s}\left[N \exp\left(-tD^2\right)\right] = \frac{a_{-1}}{\sqrt{t}} + a_0 + O(\sqrt{t}).$$

Using (7.58), we find that as $\varepsilon \to 0$, then

$$(7.59)\qquad \begin{aligned} I_3^2 + \operatorname{rk}(F) \int_M \int^B L \exp\left(-\frac{\dot{R}^{TM}}{2}\right) \frac{1}{\sqrt{\varepsilon}} - \frac{n}{4}\chi(F) \operatorname{Log}(\varepsilon)\\ \to I_3^3 = -\int_0^1 \left(\operatorname{Tr_s}\left[N \exp\left(-tD^2\right)\right] - \frac{a_{-1}}{\sqrt{t}} - a_0\right) \frac{dt}{2t} \end{aligned}$$

$$-\int_1^{+\infty} \left(\operatorname{Tr_s}\left[N \exp\left(-tD^2\right)\right] - \chi'(F)\right) \frac{dt}{2t} + \operatorname{rk}(F) \int_M \int^B L \exp\left(-\frac{\dot{R}^{TM}}{2}\right).$$

δ) Evaluation of I_3^3

Recall that the function $\theta^{\mathbb{F}}(s)$ was defined in Definition 2.1.

Theorem 7.17. *The following identity holds*

$$(7.60)\qquad I_3^3 = \frac{1}{2}\frac{\partial \theta^{\mathbb{F}}}{\partial s}(0) - \left(\frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F)\right)\Gamma'(1).$$

Proof. For $s \in \mathbb{C}, \operatorname{Re}(s) > \frac{\dim M}{2}$, using (7.58), we get

$$(7.61)\qquad \begin{aligned} \theta^{\mathbb{F}}(s) = -\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\operatorname{Tr_s}\left[N \exp\left(-tD^2\right)\right] - \frac{a_{-1}}{\sqrt{t}} - a_0\right) dt\\ -\frac{1}{\Gamma(s)} \int_1^{+\infty} t^{s-1} \left(\operatorname{Tr_s}\left[N \exp\left(-tD^2\right)\right] - \chi'(F)\right) dt\\ -\frac{a_{-1}}{\Gamma(s)\left(s-\frac{1}{2}\right)} - \frac{(a_0 - \chi'(F))}{\Gamma(s+1)}. \end{aligned}$$

From (7.59), (7.61), we get (7.60). □

4) The term I_4^0. Clearly

$$I_4^0 = -\int_0^{T_0} \mathrm{Tr_s}\left[f \exp\left(-\varepsilon D_T^2\right)\right] dT. \tag{7.62}$$

α) $\underline{A \to +\infty}$

The term I_4^0 remains constant and is equal to I_4^1.

β) $\underline{T_0 \to +\infty}$

By Theorem 7.11, we know that there exists $c > 0$ such that as $T \to +\infty$,

$$\mathrm{Tr_s}\left[f \exp\left(-\varepsilon D_T^2\right)\right] = \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f] + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T} + O\left(c^{-cT}\right). \tag{7.63}$$

Using (7.62), (7.63), we see as $T_0 \to +\infty$,

$$\begin{aligned} I_4^1 + \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f]T_0 + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\mathrm{Log}\,(T_0) \\ \to I_4^2 = -\int_0^1 \left(\mathrm{Tr_s}\left[f \exp\left(-\varepsilon D_T^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f]\right) dT \\ -\int_1^{+\infty} \left\{\mathrm{Tr_s}\left[f \exp\left(-\varepsilon D_T^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f] - \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T}\right\} dT. \end{aligned} \tag{7.64}$$

γ) $\underline{\varepsilon \to 0}$

As in Bismut-Lebeau [BL2, Section 6, eq. (6.57)], this step is quite difficult. Set

$$\varepsilon' = \sqrt{\varepsilon}. \tag{7.65}$$

Put

$$\begin{aligned} J_1^0 &= -\int_0^1 \frac{1}{\varepsilon'}\left(\mathrm{Tr_s}\left[f \exp\left(-\varepsilon'^2 D_{T/\varepsilon'}^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f]\right) dT, \\ J_2^0 &= -\int_1^{1/\varepsilon'} \frac{1}{\varepsilon'}\left(\mathrm{Tr_s}\left[f \exp\left(-\varepsilon'^2 D_{T/\varepsilon'}^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f]\right) dT, \\ J_3^0 &= -\int_1^{+\infty} \frac{1}{\varepsilon'^2}\left\{\mathrm{Tr_s}\left[f \exp\left(-\varepsilon'^2 D_{T/\varepsilon'^2}^2\right)\right] - \mathrm{rk}(F)\,\mathrm{Tr_s^B}[f]\right. \end{aligned} \tag{7.66}$$

$$- \varepsilon'^2 \left(\frac{n}{4} \chi(F) - \frac{1}{2} \widetilde{\chi}'(F) \right) \frac{1}{T} \Big\} dT.$$

Clearly

$$I_4^2 = J_1^0 + J_2^0 + J_3^0 - \left(\frac{n}{4} \chi(F) - \frac{1}{2} \widetilde{\chi}'(F) \right) \operatorname{Log}(\varepsilon). \tag{7.67}$$

By Theorem 7.12, there exists $C > 0$ such that for $\varepsilon \in]0,1], T \in [0,1]$,

$$\begin{aligned} &\Big| \operatorname{Tr_s} \left[f \exp \left(-\varepsilon'^2 D^2_{T/\varepsilon'} \right) \right] - \operatorname{rk}(F) \int_M f \int^B \exp(-B_{T^2}) \\ &+ \varepsilon' \int_M \frac{\theta}{2} \left(F, g^F \right) \int^B \widehat{df} \exp\left(-B_{T^2} \right) \Big| \leq C \varepsilon'^2. \end{aligned} \tag{7.68}$$

From (7.66), (7.68), we see that as $\varepsilon \to 0$,

$$\begin{aligned} &J_1^0 + \operatorname{rk}(F) \int_0^1 \left\{ \int_M f \left(\int^B \exp\left(-B_{T^2} \right) - \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \delta_x \right) \right\} \\ &dT \left(\frac{1}{\sqrt{\varepsilon}} \right) \to J_1^1 = \frac{1}{2} \int_0^1 \left\{ \int_M \theta \left(F, g^F \right) \int^B \widehat{df} \exp\left(-B_{T^2} \right) \right\} dT. \end{aligned} \tag{7.69}$$

Also

$$\begin{aligned} J_2^0 = &-\int_{\varepsilon'}^1 \frac{1}{\varepsilon'^2} \Big\{ \operatorname{Tr_s} \left[f \exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2} \right) \right] - \operatorname{rk}(F) \int_M f \int^B \exp\left(-B_{(T/\varepsilon')^2} \right) \\ &+ \varepsilon' \int_M \frac{\theta}{2} \left(F, g^F \right) \int^B \widehat{df} \exp\left(-B_{(T/\varepsilon')^2} \right) \Big\} dT \\ &- \frac{\operatorname{rk}(F)}{\varepsilon'} \int_1^{1/\varepsilon'} \left\{ \int_M f \left(\int^B \exp\left(-B_{T^2} \right) - \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \delta_x \right) \right\} dT + \\ &\frac{1}{2} \int_1^{1/\varepsilon'} \left\{ \int_M \theta \left(F, g^F \right) \int^B \widehat{df} \exp\left(-B_{T^2} \right) \right\} dT. \end{aligned} \tag{7.70}$$

By Propositions 5.4 and 5.5 and by Theorem 7.13, we know that for $T > 0$, as $\varepsilon' \to 0$,

$$\begin{aligned} &\frac{1}{\varepsilon'^2} \left(\operatorname{Tr_s} \left[f \exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2} \right) \right] - \operatorname{rk}(F) \operatorname{Tr_s^B}[f] \right) \\ &\to \left(\frac{n}{4} \chi(F) - \frac{1}{2} \widetilde{\chi}'(F) \right) \frac{\cosh(T)}{T \sinh(T)}. \end{aligned} \tag{7.71}$$

With the notation of (3.59), using (7.12), we find that if $x \in B$, then

$$\operatorname{Tr}\left[A_x^{-1}\right] = n - 2\operatorname{ind}(x). \tag{7.72}$$

By Theorem 3.20, we see that for $T > 0$, as $\varepsilon' \to 0$, then

(7.73)

$$\frac{1}{\varepsilon'^2}\left(\int_M f \int^B \exp\left(-B_{(T/\varepsilon')^2}\right) - \operatorname{Tr}_s^B[f]\right) \to \frac{1}{\operatorname{rk}(F)}\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T^2},$$

$$\frac{1}{\varepsilon'}\int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{(T/\varepsilon')^2}\right) \to 0.$$

Using (7.71), (7.73), we find that for $T > 0$,

$$\begin{aligned}\lim_{\varepsilon' \to 0}\frac{1}{\varepsilon'^2}\Big\{&\operatorname{Tr}_s\left[f\exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2}\right)\right] - \operatorname{rk}(F)\int_M f\int^B \exp\left(-B_{(T/\varepsilon')^2}\right)\\ &+\varepsilon'\int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{(T/\varepsilon')^2}\right)\Big\}\\ &= \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\left(\frac{\cosh(T)}{\sinh(T)} - \frac{1}{T}\right)\frac{1}{T}.\end{aligned} \tag{7.74}$$

On the other hand, by Propositions 5.4 and 5.5 and by Theorem 7.12, we know that there exists $C > 0$ such that for $0 < \varepsilon' \leq 1, \varepsilon' \leq T \leq 1$, then

$$\begin{aligned}\Big|\frac{1}{\varepsilon'^2}\Big\{&\operatorname{Tr}_s\left[f\exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2}\right)\right] - \operatorname{rk}(F)\int_M f\int^B \exp\left(-B_{(T/\varepsilon')^2}\right)\\ &+\varepsilon'\int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{(T/\varepsilon')^2}\right)\Big\}\Big| \leq C.\end{aligned} \tag{7.75}$$

Using (7.74), (7.75) and dominated convergence, we find that as $\varepsilon \to 0$,

$$\begin{aligned}-\int_{\varepsilon'}^1 \frac{1}{\varepsilon'^2}\Big\{&\operatorname{Tr}_s\left[f\exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2}\right)\right] - \operatorname{rk}(F)\int_M f\int^B \exp\left(-B_{(T/\varepsilon')^2}\right)\\ &+\varepsilon'\int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{(T/\varepsilon')^2}\right)\Big\}dT\\ &\to -\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\int_0^1\left(\frac{\cosh(T)}{\sinh(T)} - \frac{1}{T}\right)\frac{dT}{T}.\end{aligned} \tag{7.76}$$

Also by using in particular Theorem 3.20, we get

$$(7.77)\qquad \int_1^{1/\varepsilon'} \int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right) dT$$
$$= \int_1^{+\infty} \left\{ \int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right)\right\} dT$$
$$- \int_{1/\varepsilon'}^{+\infty} \left\{ \int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right)\right.$$
$$\left. - \frac{1}{T^2\,\mathrm{rk}(F)}\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\right\} dT - \frac{\varepsilon'}{\mathrm{rk}(F)}\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right).$$

By Theorem 3.20 and by (7.72), we find that

$$(7.78)\qquad \left| \frac{1}{\varepsilon'}\int_{1/\varepsilon'}^{+\infty} \left\{ \int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right)\right.\right.$$
$$\left.\left. - \frac{1}{T^2\,\mathrm{rk}(F)}\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\right\} dT \right| \ \le\ C\varepsilon'.$$

Using (7.77), (7.78), we see that as $\varepsilon \to 0$,

$$(7.79)\qquad -\frac{\mathrm{rk}(F)}{\varepsilon'}\int_1^{1/\varepsilon'} \left\{ \int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right)\right\} dT$$
$$= -\,\mathrm{rk}(F)\int_1^{+\infty} \left\{ \int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right)\right\} dT\left(\frac{1}{\sqrt{\varepsilon}}\right)$$
$$+\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right) + O\left(\sqrt{\varepsilon}\right).$$

Finally, by Theorem 3.18, we find that as $\varepsilon \to 0$,

$$(7.80)\qquad \frac{1}{2}\int_1^{1/\varepsilon'} \left\{ \int_M \theta\left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right)\right\} dT$$
$$\to \frac{1}{2}\int_1^{+\infty} \left\{ \int_M \theta\left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right)\right\} dT.$$

From (7.70), (7.76), (7.79), (7.80), we see that as $\varepsilon \to 0$,

$$(7.81)\quad J_2^0 + \operatorname{rk}(F)\int_1^{+\infty}\left\{\int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\operatorname{ind}(x)}\delta_x\right)\right\}$$

$$dT\left(\frac{1}{\sqrt{\varepsilon}}\right) \to J_2^1 = -\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\int_0^1\left(\frac{\cosh(T)}{\sinh(T)} - \frac{1}{T}\right)\frac{dT}{T}$$

$$+\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right) + \frac{1}{2}\int_1^{+\infty}\left\{\int_M \theta\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)\right\}dT.$$

By Theorem 7.13, we find that for $T > 0$,

$$(7.82)\quad \frac{1}{\varepsilon'^2}\Big(\operatorname{Tr_s}\left[f\exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2}\right)\right] - \operatorname{rk}(F)\operatorname{Tr}_s^B[f]$$

$$-\varepsilon'^2\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T}\Big)$$

$$\to \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\left(\frac{\cosh(T)}{\sinh(T)} - 1\right)\frac{1}{T}.$$

Moreover by Propositions 5.4 and 5.5 and by Theorem 7.14, there exist $c > 0, C > 0$ such that for $0 < \varepsilon' \le 1, T \ge 1$, then

$$(7.83)\quad \Big|\frac{1}{\varepsilon'^2}\Big(\operatorname{Tr_s}\left[f\exp\left(-\varepsilon'^2 D^2_{T/\varepsilon'^2}\right)\right] - \operatorname{rk}(F)\operatorname{Tr}_s^B[f]$$

$$-\varepsilon'^2\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T}\Big)\Big| \quad \le \quad c\exp(-CT).$$

From (7.66), (7.82), (7.83), we conclude that as $\varepsilon \to 0$,

$$(7.84)\quad J_3^0 \to J_3^1 = -\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\int_1^{+\infty}\left(\frac{\cosh(T)}{\sinh(T)} - 1\right)\frac{dT}{T}.$$

Using (7.67), (7.69), (7.81), (7.84), we see that as $\varepsilon \to 0$,

$$(7.85)\quad I_4^2 + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\operatorname{Log}(\varepsilon)$$

$$+\operatorname{rk}(F)\int_0^{+\infty}\left\{\int_M f\left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x\in B}(-1)^{\operatorname{ind}(x)}\delta_x\right)\right\}dT\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

$$\rightarrow I_4^3 = \frac{1}{2}\int_0^{+\infty}\left\{\int_M \theta\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)\right\}dT$$

$$-\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\left(\int_0^1\left(\frac{\cosh(T)}{\sinh(T)}-\frac{1}{T}\right)\frac{dT}{T}\right.$$

$$\left.+\int_1^{+\infty}\left(\frac{\cosh(T)}{\sinh(T)}-1\right)\frac{dT}{T}-1\right).$$

δ) Evaluation of I_4^3

Theorem 7.18. *The following identity holds*

$$I_4^3 = \frac{1}{2}\int_M \theta\left(F, g^F\right)(\nabla f)^*\psi\left(TM, \nabla^{TM}\right) \tag{7.86}$$
$$+\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\left(\operatorname{Log}(\pi)+\Gamma'(1)\right).$$

Proof. By (3.19), (3.30), it is clear that

$$\frac{1}{2}\int_0^{+\infty}\left\{\int_M \theta\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)\right\}dT \tag{7.87}$$
$$= \frac{1}{2}\int_M \theta\left(F, g^F\right)(\nabla f)^*\psi\left(TM, \nabla^{TM}\right).$$

Clearly

$$\frac{\cosh(T)}{\sinh(T)}-1 = \frac{2e^{-2T}}{1-e^{-2T}}. \tag{7.88}$$

Let $\zeta(s)$ be the Riemann zeta function. By (7.88), we easily deduce that for $s \in \mathbb{C}, \operatorname{Re}(s) > 1$, then

$$\frac{1}{\Gamma(s)}\int_0^{+\infty} T^{s-1}\left(\frac{\cosh(T)}{\sinh(T)}-1\right)dT = 2^{1-s}\zeta(s). \tag{7.89}$$

Also for $s \in \mathbb{C}, \operatorname{Re}(s) > 1$, we have the identity

$$\frac{1}{\Gamma(s)}\int_0^{+\infty} T^{s-1}\left(\frac{\cosh(T)}{\sinh(T)}-1\right)dT = \frac{1}{\Gamma(s)}\int_0^1 T^{s-1}\left(\frac{\cosh(T)}{\sinh(T)}-\frac{1}{T}\right)dT \tag{7.90}$$
$$+\frac{1}{\Gamma(s)}\int_1^{+\infty} T^{s-1}\left(\frac{\cosh(T)}{\sinh(T)}-1\right)dT - \frac{1}{\Gamma(s+1)}+\frac{1}{\Gamma(s)(s-1)}.$$

Both sides of (7.89) of extend into a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$. Using (7.89), (7.90), and taking derivatives at 0, we get

$$\text{(7.91)} \qquad \int_0^1 \left(\frac{\cosh(T)}{\sinh(T)} - \frac{1}{T}\right)\frac{dT}{T} + \int_1^{+\infty}\left(\frac{\cosh(T)}{\sinh(T)} - 1\right)\frac{dT}{T} \\ +\Gamma'(1) - 1 = -2\operatorname{Log}(2)\zeta(0) + 2\zeta'(0).$$

Classically,

$$\text{(7.92)} \qquad \begin{aligned} \zeta(0) &= -\frac{1}{2}, \\ \zeta'(0) &= -\frac{1}{2}\operatorname{Log}(2\pi). \end{aligned}$$

Using (7.91), (7.92), we find that

$$\text{(7.93)} \qquad \int_0^1 \left(\frac{\cosh(T)}{\sinh(T)} - \frac{1}{T}\right)\frac{dT}{T} + \int_1^{+\infty}\left(\frac{\cosh(T)}{\sinh(T)} - 1\right)\frac{dT}{T} - 1 \\ = -\operatorname{Log}(\pi) - \Gamma'(1).$$

From (7.85), (7.87), (7.93), we get (7.86) . □

e) Matching the divergences

Theorem 7.19. *The following identity holds*

(7.94)

$$I_1^3 + I_3^3 + I_4^3 - \frac{1}{2}\operatorname{Log}\left(\frac{\| \ \|^{R,K}_{\det H^\bullet(M,F)}}{| \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 - \left(n\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\operatorname{Log}(\pi) = 0.$$

Proof. Recall that by (7.27),

$$\text{(7.95)} \qquad \sum_{k=1}^{4} I_k^0 = 0.$$

As $A \to +\infty$, the following divergences which concern the terms I_1^0 and I_3^0 in (7.29) and (7.54) appear

$$\text{(7.96)} \qquad -\frac{1}{2}\chi'(F)\operatorname{Log}(A) + \frac{1}{2}\chi'(F)\operatorname{Log}(A) = 0.$$

Since these divergences cancel out, we get from (7.95)

$$\sum_{k=1}^{4} I_k^1 = 0. \tag{7.97}$$

By Theorem 7.6, we know that

$$\lim_{T_0 \to +\infty} \left\{ \frac{1}{2} \operatorname{Log} \left(\frac{| \ |^{RS}_{\det H^\bullet(M,F),T_0}}{| \ |^{RS}_{\det H^\bullet(M,F)}} \right)^2 + \frac{1}{2} \operatorname{Tr}_s \left[N \operatorname{Log} \left(D_{T_0}^{2,]0,1]} \right) \right] \right.$$
$$\left. + \operatorname{rk}(F) \operatorname{Tr}_s^B[f] T_0 + \left(\frac{n}{4} \chi(F) - \frac{1}{2} \widetilde{\chi}'(F) \right) \operatorname{Log} \left(\frac{T_0}{\pi} \right) \right\}$$
$$= \frac{1}{2} \operatorname{Log} \left(\frac{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}{| \ |^{RS}_{\det H^\bullet(M,F)}} \right)^2. \tag{7.98}$$

In view of (7.41), (7.52), (7.64), (7.97), (7.98), we find that for $0 < \varepsilon < 1$,
(7.99)

$$I_1^2 + I_3^2 + I_4^2 - \frac{1}{2} \operatorname{Log} \left(\frac{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}{| \ |^{RS}_{\det H^\bullet(M,F)}} \right)^2 - \left(\frac{n}{4} \chi(F) - \frac{1}{2} \widetilde{\chi}'(F) \right) \operatorname{Log}(\pi) = 0.$$

As $\varepsilon \to 0$, the following divergences appear, which concern the terms I_1^2, I_3^2, I_4^2 in (7.43), (7.59), (7.85),

$$\left(\frac{1}{2} \widetilde{\chi}'(F) - \frac{n}{4} \chi(F) + \frac{n}{4} \chi(F) - \frac{1}{2} \widetilde{\chi}'(F) \right) \operatorname{Log}(\varepsilon) \tag{7.100}$$
$$+ \operatorname{rk}(F) \left(\int_M \int^B L \exp \left(- \frac{\dot{R}^{TM}}{2} \right) \right.$$
$$\left. + \int_0^{+\infty} \left\{ \int_M f \left(\int^B \exp\left(-B_{T^2}\right) - \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \delta_x \right) \right\} dT \right) \frac{1}{\sqrt{\varepsilon}}.$$

Because of (7.99), the sum of these divergences should be 0. This is exactly the case for the coefficient of $\operatorname{Log}(\varepsilon)$. The coefficient of $\frac{1}{\sqrt{\varepsilon}}$ must also vanish. This is in fact a result which was proved in Theorem 3.19.

From (7.99), (7.100), we get (7.94). □

f) Proof of Theorem 7.1

By (7.42), (7.60), (7.86), (7.94), we get

$$
\begin{aligned}
(7.101)\quad & \left\{\frac{1}{2}\left(\widetilde{\chi}'(F)-\chi'(F)\right)-\frac{n}{4}\chi(F)+\frac{1}{2}\chi'(F)+\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right\}\Gamma'(1) \\
& +\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)-\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\right)\operatorname{Log}(\pi)+\frac{1}{2}\frac{\partial\theta^{\mathbb{F}}}{\partial s}(0) \\
& -\frac{1}{2}\operatorname{Log}\left(\frac{\|\ \|^{\mathcal{M},\nabla f}_{\det H^{\bullet}(M,F)}}{|\ |^{RS}_{\det H^{\bullet}(M,F)}}\right)^{2}+\frac{1}{2}\int_M \theta\left(F,g^F\right)(\nabla f)^{*}\psi\left(TM,\nabla^{TM}\right) = 0.
\end{aligned}
$$

The coefficients of $\Gamma'(1)$ and $\operatorname{Log}(\pi)$ in (7.101) vanish identically. Equation (7.101) is then equivalent to

$$
(7.102)\quad \operatorname{Log}\left(\frac{\|\ \|^{RS}_{\det H^{\bullet}(M,F)}}{\|\ \|^{\mathcal{M},\nabla f}_{\det H^{\bullet}(M,F)}}\right)^{2} = -\int_M \theta\left(F,g^F\right)(\nabla f)^{*}\psi\left(TM,\nabla^{TM}\right),
$$

which is exactly Theorem 7.1. □

g) Proof of Theorem 0.3

Let

$$
(7.103)\quad (F^{\bullet},v):0\to F^0\underset{v}{\to}F^1\underset{v}{\to}\dots\underset{v}{\to}F^m\to 0
$$

be a flat exact sequence of real flat vector bundles on M. Let σ be the canonical nonzero section of the line bundle $\det F^{\bullet}=\bigotimes_{j=0}^{m}(\det\ F^j)^{(-1)^j}$ constructed in [KMu], [BGS1, Section 1.a)].

Let $\tau\in\det H^{\bullet}(M,F^{\bullet})=\bigotimes_{j=0}^{m}(\det H^{\bullet}(M,F^j))^{(-1)^j}$ be the corresponding nonzero section constructed in [KMu], which is associated to the exact sequence (7.103).

Let $g^{F^0},\dots,g^{F^m}$ be Euclidean metrics on $F^0,\dots,F^m$. Let $\|\ \|_{\det F^{\bullet}}$ be the corresponding metric on the line bundle $\det F^{\bullet}$. Let g^{TM} be an Euclidean metric on TM.

Let $\| \ \|^{RS}_{\det H^{\bullet}(M,F^0)}, \ldots, \| \ \|^{RS}_{\det H^{\bullet}(M,F^m)}$ be the Ray-Singer metrics on the lines $\det H^{\bullet}(M,F^0), \ldots, \det H^{\bullet}(M,F^m)$ associated to the metrics $g^{TM}, g^{F^0}, \ldots, g^{F^m}$. Let $\| \ \|^{RS}_{\det H^{\bullet}(M,F^{\bullet})}$ denote the corresponding metric on the line $\det H^{\bullet}(M,F^{\bullet})$.

Now, we will prove Theorem 0.3, which we restate for convenience.

Theorem 7.20. *The following identity holds,*

$$\operatorname{Log}\left(\|\tau\|^{RS,2}_{\det H^{\bullet}(M,F^{\bullet})}\right) = \int_M \operatorname{Log}\left(\|\sigma\|^2_{\det F^{\bullet}}\right) e(TM, \nabla^{TM}). \tag{7.104}$$

Proof. We use the notation of Sections 7a)–b). Let $\| \ \|^{\mathcal{M},X}_{\det H^{\bullet}(M,F^0)}, \ldots, \| \ \|^{\mathcal{M},X}_{\det H^{\bullet}(M,F^m)}$ be the Milnor metrics on the lines $\det H^{\bullet}(M,F^0), \ldots, \det H^{\bullet}(M,F^m)$ attached to the metrics $\| \ \|_{\det F^0_x}, \ldots, \| \ \|_{\det F^m_x}$ $(x \in B)$. Let $\| \ \|^{\mathcal{M},X}_{\det H^{\bullet}(M,F^{\bullet})}$ denote the corresponding metric on the line $\det H^{\bullet}(M,F^{\bullet})$.

Clearly, we have the exact sequence of Thom-Smale complexes

$$0 \to (C^{\bullet}(W^u,F^0),\partial) \underset{v}{\to} (C^{\bullet}(W^u,F^1),\partial) \to \ldots \underset{v}{\to} (C^{\bullet}(W^u,F^m),\partial) \to 0. \tag{7.105}$$

Set

$$\det C^{\bullet}(W^u,F^{\bullet}) = \bigotimes_{j=0}^{m} (\det C^{\bullet}(W^u,F^j))^{(-1)^j}. \tag{7.106}$$

By (1.48), we have the canonical isomorphism

$$\det C^{\bullet}(W^u,F^{\bullet}) \simeq \det H^{\bullet}(M,F^{\bullet}). \tag{7.107}$$

Let τ' be the nonzero section of $\det C^{\bullet}(W^u,F^{\bullet})$ constructed in [KMu], [BGS1, Section 1.a)], which is attached to the acyclic complex (7.105). Then $\tau' \in \det C^{\bullet}(W^u,F^{\bullet})$ corresponds to $\tau \in \det H^{\bullet}(M,F^{\bullet})$ via the canonical isomorphism (7.107). It should now be clear that

$$\operatorname{Log}\left(\|\tau\|^{\mathcal{M},X,2}_{\det H^{\bullet}(M,F^{\bullet})}\right) = \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \operatorname{Log}\left(\|\sigma_x\|^2_{\det F^{\bullet}_x}\right). \tag{7.108}$$

Set

$$\theta(F^{\bullet}, g^{F^{\bullet}}) = \sum_{j=0}^{m} (-1)^j \theta(F^j, g^{F^j}). \tag{7.109}$$

Since σ is a nonzero flat section of $\det F^\bullet$, we see that

$$d\,\mathrm{Log}\left(\|\sigma\|^2_{\det F^\bullet}\right) = \theta(F^\bullet, g^{F^\bullet}). \tag{7.110}$$

By Theorem 0.2, we get

$$\begin{aligned}\mathrm{Log}\left(\|\tau\|^{RS,2}_{\det H^\bullet(M,F^\bullet)}\right) = \mathrm{Log}\left(\|\tau\|^{\mathcal{M},X,2}_{\det H^\bullet(M,F)}\right)\\ - \int_M \theta(F^\bullet, g^{F^\bullet}) X^* \psi(TM, \nabla^{TM}).\end{aligned} \tag{7.111}$$

Using (7.110) and proceeding as in (6.5), (6.6), we find that

$$\begin{aligned}- \int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}) = \int_M \mathrm{Log}\left(\|\sigma\|^2_{\det F^\bullet}\right)\\ e(TM, \nabla^{TM}) - \sum_{x \in B} (-1)^{\mathrm{ind}(x)} \mathrm{Log}\left(\|\sigma_x\|^2_{\det F^\bullet_x}\right)\end{aligned} \tag{7.112}$$

From (7.108)–(7.112), we get (7.104).

The proof of Theorem 7.20 is completed. □

Remark 7.21. Of course a direct analytic proof of Theorem 7.20 can be given, which is much simpler than the proof of Theorem 7.1.

VIII. The asymptotic structure of the matrix of the d^F operator on the Helffer-Sjöstrand orthogonal base

The purpose of this Section is to describe the construction by Helffer-Sjöstrand [HSj1–4] of an orthogonal base for the direct sum of the eigenspaces of the operator $\widetilde{D}_T^2$ associated to eigenvalues $\lambda \in [0,1]$, and to calculate the asymptotics of the corresponding matrix of d_T^F in terms of the corresponding Thom-Smale complex. The results of this Section will also be used in Section 9, where the asymptotics of the L_2 metric $| \ |^{RS}_{\det H^\bullet(M,F),T}$ on $\det H^\bullet(M,F)$ as $T \to +\infty$ is calculated, and where Theorem 7.6 is proved.

The results of this Section on the asymptotics of the matrix of d^F were already established in Helffer-Sjöstrand [HSj4, Theorem 3.1 and Proposition 3.3], in the case where F is the trivial Euclidean line bundle $\mathbb{R}$. Here the main difference with respect to the situation considered in [HSj4] is that F is a vector bundle, and more fundamentally that the metric g^F is not flat.

In [HSj4, Sections 2 and 3], in the case where $F = \mathbb{R}$, the solutions of the WKB equations for the eigenvectors of $\widetilde{D}_T^2$ associated to eigenvalues $\lambda \in [0,1]$, were calculated, by solving in particular transport equations near $W^u(x)$ and $W^s(x)(x \in B)$. If the metric g^F on F is flat, then the calculations of [HSj4] can be used without change. If not, the operator $\widetilde{D}_T^2$ which we consider here is more complicated than in [HSj4]. In fact the analogues of [HSj4, Proposition 2.3 and 2.4], where Helffer-Sjöstrand calculate the leading term of the WKB equation for $\widetilde{D}_T^2$ along $W^s(x)$ and $W^u(x)$ are Propositions 8.24 and 8.25. On $W^s(x)$, parallel transport with respect to the connection ∇^F is used to solve the transport equation, while on $W^u(x)$, it is the dual connection ∇^{F*} (which itself depends on the metric g^F) which is needed. This reflects in fact Poincaré duality for flat vector bundles which are not orthogonally flat.

Because the situation we deal with is different from the one in [HSj4], we have felt necessary to give a detailed exposition of some of the results and techniques of Helffer-Sjöstrand [HSj1–4], referring when necessary to the original work. Our own contribution in this Section is in fact to simply apply the general techniques of [HSj1–3] to a situation which is slightly more complicated than in [HSj4].

This Section is organized as follows. In a), we introduce the Agmon metric $|\nabla f|^2 g^{TM}$. In b), we recall simple results of Witten [W] on the harmonic oscillator one can attach to each $x \in B$. In c), we describe the results of [HSj1–3] concerning eigenvectors of the operators $\widetilde{D}_T^2$ with certain Dirichlet boundary conditions. In d), we construct a corresponding orthonormal base of eigenvectors.

In e), following [HSj1–3], we construct an orthonormal base $\{\widetilde{e}_{T,x,k}\}_{\substack{x\in B \\ 1\leq k\leq \mathrm{rk}(F)}}$ of the eigenspaces of $\widetilde{D}_T^2$ associated to eigenvalues $\lambda \in [0,1]$.

In f), we describe the WKB equation for $\widetilde{D}_T^2$. In g) and h), we solve the corresponding transport equation over $W^s(x)$ and $W^u(x) (x \in B)$. Finally in i), we establish in Theorem 8.30 the main result of this Section, which is the asymptotic structure of the action of the operator d_T^F on the considered eigenspaces of $\widetilde{D}_T^2$. This generalizes a corresponding result of Helffer-Sjöstrand [HSj4, Proposition 3.3].

In this Section, we use the notation of Sections 1, 2, 4 and 7. Also the simplifying assumptions of Section 7b) will be in force in the whole section.

a) The Agmon metric $|\nabla f|^2 g^{TM}$

If $z \in M, \varepsilon > 0$, let $B^M(z,\varepsilon)$ be the open ball of center z and radius ε with respect to the Riemannian distance associated to the metric g^{TM}, and let $B^{T_zM}(0,\varepsilon)$ be the open ball of center 0 and radius ε in (T_zM, g^{T_zM}).

In the sequel, we assume that $\varepsilon > 0$ is small enough so that the balls $B^M(x,2\varepsilon)$ ($x \in B$) do not intersect each other, that (7.12) is verified on the balls $B^M(x,\varepsilon)$ ($x \in B$), and also the metric g^F is flat on the balls $B^M(x,\varepsilon)$ ($x \in B$).

Definition 8.1. Let g_A^{TM} be the Agmon metric on TM associated to the potential $|\nabla f|^2$, i.e.

$$g_A^{TM} = |\nabla f|^2 g^{TM}. \tag{8.1}$$

Then g_A^{TM} is a degenerate metric on TM, which degenerates on $B \subset M$. Let $d_A^M(\cdot,\cdot)$ be the Agmon distance associated to the metric g_A^{TM}. By [HSj1, Section 6], we know that if $x, x' \in M$, there exists a minimizing geodesic γ for the distance d_A^M, which is smooth on $\gamma \backslash B$.

Take $x \in B$. For $z \in M$, set

$$\varphi_x(z) = d_A^M(x,z). \tag{8.2}$$

Then, φ_x is a Lipschitz function.

b) The harmonic oscillator of Witten

Recall that by (7.12), if $x \in B$, there exists a coordinate system $y = (y^1, \cdots, y^n) \in \mathbb{R}$ on $B^M(x,\varepsilon)$ such that 0 represents x, and moreover,

$$\begin{aligned} g^{TM} &= \sum_1^n \left|dy^i\right|^2, \\ f(y) &= f(x) + \frac{1}{2}\left(-\sum_1^{\operatorname{ind}(x)} \left|y^i\right|^2 + \sum_{\operatorname{ind}(x)+1}^{n} \left|y^i\right|^2\right). \end{aligned} \tag{8.3}$$

One verifies easily that if $|y| \leq \varepsilon$, then

$$\varphi(y) = \frac{1}{2}|y|^2. \tag{8.4}$$

Recall that for $x \in B$, the metric g^F is flat on $B^M(x,\varepsilon)$. On $B^M(x,\varepsilon)$, we trivialize F by using the connection $\nabla^F = \nabla^{F,e}$. The fibres of F on $B^M(x,\varepsilon)$ are identified to F_x.

Then $\mathbb{R}^n$ splits canonically into

$$\mathbb{R}^n = \mathbb{R}^{\operatorname{ind}(x)} \oplus \mathbb{R}^{n-\operatorname{ind}(x)}. \tag{8.5}$$

Recall that we have identified an open neighborhood of $x \in B$ in M to an open neighborhood of 0 in $\mathbb{R}^n$. At $x \in B$, the splitting (8.5) coincides with the obvious splitting

$$T_x M = T_x W^u(x) \oplus T_x W^s(x). \tag{8.6}$$

Since $T_x W^u(x)$ is oriented, we find that in (8.5), $\mathbb{R}^{\mathrm{ind}(x)}$ inherits the corresponding orientation. Let ρ_x be the volume form of the Euclidean oriented vector space $\mathbb{R}^{\mathrm{ind}(x)}$. Of course, one can assume that the coordinates $y^1, \cdots, y^{\mathrm{ind}(x)}$ are such that

$$\rho_x = dy^1 \wedge \cdots \wedge dy^{\mathrm{ind}(x)}. \tag{8.7}$$

From (8.5), we deduce that near x,

$$\Lambda(T^*M) = \Lambda\left(\mathbb{R}^{\mathrm{ind}(x)*}\right) \widehat{\otimes} \Lambda\left(\mathbb{R}^{(n-\mathrm{ind}(x))*}\right). \tag{8.8}$$

Of course at x, (8.8) corresponds to

$$\Lambda(T^*M) = \Lambda\left(T_x^* W^u(x)\right) \widehat{\otimes} \Lambda\left(T_x^* W^s(x)\right). \tag{8.9}$$

Let N^-, N^+ be the number operators acting in $\Lambda(\mathbb{R}^{\mathrm{ind}(x)*}), \Lambda(\mathbb{R}^{(n-\mathrm{ind}(x)*)})$, so that near x, $N = N^+ + N^-$. Let $\Delta^{\mathbb{R}^n}$ be the usual Laplacian on $\mathbb{R}^n$. We now give a simple formula of Witten [W].

Proposition 8.2. *Near $x \in B$, for any $T \geq 0$, the following identity holds,*

$$\widetilde{D}_T^2 = -\Delta^{\mathbb{R}^n} + T^2|y|^2 - Tn + 2T\left(N^+ + \mathrm{ind}(x) - N^-\right). \tag{8.10}$$

Proof. Equation (8.10) follows easily from (4.29) and (5.13). □

Let $\widetilde{D}_{T,x}^{2,\mathbb{R}^n}$ be the obvious action of the operator (8.10) on the vector space of smooth sections of $\Lambda(\mathbb{R}^{n*}) \otimes F_x$ over $\mathbb{R}^n$. Another simple result of Witten [W] is as follows.

Proposition 8.3. *The operator $\widetilde{D}_{T,x}^{2,\mathbb{R}^n}$ has discrete spectrum and compact resolvent. Its spectrum is exactly $2T\mathbb{N}$. The kernel $\widetilde{D}_{T,x}^{2,\mathbb{R}^n}$ is of dimension $\mathrm{rk}(F)$. More precisely*

$$\mathrm{Ker}\, \widetilde{D}_{T,x}^{2,\mathbb{R}^n} = \left\{ \left(\frac{T}{\pi}\right)^{n/4} e^{-\frac{T|y|^2}{2}} \rho_x \right\} \otimes F_x. \tag{8.11}$$

Proof. Let G_T be map $f(y) \to f(\frac{y}{\sqrt{T}})$. Then

$$G_T \widetilde{D}_{T,x}^{2,\mathbb{R}^n} G_T^{-1} = T\left(-\Delta^{\mathbb{R}^n} + |y|^2 - n\right) + 2T\left(N^+ + \operatorname{ind}(x) - N^-\right). \tag{8.12}$$

The operator $-\Delta^{\mathbb{R}^n} + |y|^2 - n$ is twice the harmonic oscillator. It has compact resolvent and its spectrum is exactly 2N. The operator $2(N^+ + \operatorname{ind}(x) - N^-)$ is nonnegative and its spectrum is included in 2N. Also the kernel of $-\Delta^{\mathbb{R}^n} + |y|^2 - n$ acting on smooth real functions is one dimensional and spanned by the functions $e^{-|y|^2/2}$. Finally if $\alpha \in \Lambda(\mathbb{R}^{n*}) \otimes F_x$, then $(N^+ + \operatorname{ind}(x) - N^-)\alpha = 0$ if and only if $\alpha \in \Lambda^{\operatorname{ind}(x)}(\mathbb{R}^{\operatorname{ind}(x)*}) \otimes F_x$. Equation (8.11) follows. □

c) The estimates of Helffer and Sjöstrand for the eigenforms of $\widetilde{D}_T^2$ with Dirichlet boundary conditions

For $\eta > 0, x \in B$, set

$$M_x = M \backslash \bigcup_{\substack{y \in B\backslash\{x\} \\ \operatorname{ind}(y) = \operatorname{ind}(x)}} B^M(y, \eta). \tag{8.13}$$

For $\eta > 0$ small enough, M_x is a smooth manifold with boundary.

Let $\mathbb{F}_x = \bigoplus_{i=0}^n \mathbb{F}_x^i$ be the vector space of smooth sections of $\Lambda(T^*M) \otimes F = \bigoplus_{i=0}^n \Lambda^i(T^*M) \otimes F$ over M_x. We equip $\mathbb{F}_x$ with the scalar product $\langle\ ,\ \rangle_{\mathbb{F}_x}$ given by

$$\alpha, \alpha' \in \mathbb{F}_x \to \langle \alpha, \alpha' \rangle_{\mathbb{F}_x} = \int_{M_x} \langle \alpha, \alpha' \rangle_{\Lambda(T^*M)\otimes F}\, dv_M. \tag{8.14}$$

Let $\widetilde{D}_{T,x}^2$ be the obvious action of $\widetilde{D}_T^2$ on $\mathbb{F}_x$ with Dirichlet boundary conditions on ∂M_x.

Definition 8.4. For $0 \le i \le n, T \ge 0$, let $\widetilde{D}_{T,x}^{2,i}$ be the restriction of $\widetilde{D}_{T,x}^2$ to $\mathbb{F}_x^i$. For $T \ge 0$, let $K_{T,x}^{[0,1]} = \bigoplus_{i=0}^n K_{T,x}^{[0,1],i}$ be the direct sum of the eigenspaces of $\widetilde{D}_{T,x}^2$ associated to eigenvalues $\lambda \in [0,1]$. Let $Q_{T,x}^{[0,1]}$ be the orthogonal projection operator from $\mathbb{F}_x$ on $K_{T,x}^{[0,1]}$.

Take $c > 0$. Following [HSj3, Lemma 1.5], we will write that as $T \to +\infty$,

$$A(T) = \widetilde{O}\left(e^{-Tc}\right) \tag{8.15}$$

if for any $\gamma > 0$, there exist $\eta(\gamma) > 0$ such that if $0 < \eta < \eta(\gamma)$, as $T \to +\infty$

$$A(T) = O\left(e^{-T(c-\gamma)}\right). \tag{8.16}$$

If in (8.16), $A(T)$ and c depend themselves on an extra parameter, it is understood that (8.16) is uniform in this parameter.

For $0 \leq i \leq n$, set

$$\begin{aligned} B^i &= \{x \in B; \mathrm{ind}(x) = i\}, \\ M^i &= \mathrm{card}(B^i). \end{aligned} \tag{8.17}$$

We first state a result of Helffer-Sjöstrand [HSj4, Theorem 1.4 and Lemma 1.6].

Theorem 8.5. *For $T > 0$ large enough, then*

$$\mathrm{rk}\left(K_{T,x}^{[0,1],i}\right) = \begin{matrix} \mathrm{rk}(F) & \textit{if} & i = \mathrm{ind}(x), \\ 0 & \textit{if} & i \neq \mathrm{ind}(x). \end{matrix} \tag{8.18}$$

If $\varphi \in K_{T,x}^{[0,1],\mathrm{ind}(x)}$ is of norm 1, as $T \to +\infty$,

$$\varphi(x') = \widetilde{O}\left(e^{-d_A^M(x,x')T}\right). \tag{8.19}$$

Set

$$c_x = 2 \inf_{y \in B^{\mathrm{ind}(x)-1} \cup B^{\mathrm{ind}(x)+1} \cup B^{\mathrm{ind}(x)} \setminus \{x\}} d_A^M(x,y). \tag{8.20}$$

If λ is an eigenvalue of $\widetilde{D}_{T,x}^2$ in $[0,1]$, then

$$\lambda = \widetilde{O}\left(e^{-c_x T}\right). \tag{8.21}$$

Proof.. The main difference with [HSj4] is that here, the kernel of the operator $\widetilde{D}_{T,x}^{2,\mathbb{R}^n}$ considered in Proposition 8.3 is of dimension $\mathrm{rk}(F)$ and not necessarily of dimension 1. However all the arguments of [HSj1, Section 4] on which [HSj4] is based can still be used in this case. $\square$

d) An orthonormal base for Dirichlet eigenspaces associated to small eigenvalues

Definition 8.6. For $x \in B, T > 0$, let $r_{T,x}$ be the map

$$s \in \mathbb{F}_x^{\mathrm{ind}(x)} \to r_{T,x}s = \left(\frac{\pi}{T}\right)^{n/4} s_x \in \left(\Lambda^{\mathrm{ind}(x)}(T^*M) \otimes F\right)_x. \tag{8.22}$$

Let γ be a smooth function defined on $\mathbb{R}$ with values in $\mathbb{R}^+$, such that

$$\begin{aligned} \gamma(a) &= 1 \quad \text{for} \quad a \leq \tfrac{\varepsilon}{2}, \\ &= 0 \quad \text{for} \quad a > \varepsilon. \end{aligned} \tag{8.23}$$

If $y \in \mathbb{R}^n$, set

$$\mu(y) = \gamma(|y|). \tag{8.24}$$

We can consider μ as a smooth function defined on M with values in $\mathbb{R}^+$, which vanishes on $M \backslash \bigcup_{x \in B} B^M(x, \varepsilon)$.

Set

$$\alpha_T = \int_{\mathbb{R}^n} \mu^2(y) \exp\left(-T|y|^2\right) dy. \tag{8.25}$$

Clearly, there exists $c > 0$ such that

$$\alpha_T = \frac{\pi^{n/2}}{T^{n/2}} + O\left(e^{-cT}\right). \tag{8.26}$$

Recall that if $x \in B$, on $B^M(x,\varepsilon)$, the fibres of F have been identified to F_x.

Definition 8.7. For $x \in B, T > 0$, let $J_{T,x}$ be the linear map from F_x in $\mathbb{F}_x^{\mathrm{ind}(x)}$

$$f \in F_x \to J_{T,x}f(y) = \frac{1}{(\alpha_T)^{1/2}} \mu(y) \exp\left(-\frac{T|y|^2}{2}\right) \rho_x \otimes f \in \mathbb{F}_x^{\mathrm{ind}(x)}. \tag{8.27}$$

Clearly $J_{T,x}$ is an isometry from F_x into $\mathbb{F}_x^{\mathrm{ind}(x)}$. Also

$$r_{T,x} J_{T,x} f = \frac{\left(\frac{\pi}{T}\right)^{n/4}}{(\alpha_T)^{1/2}} \rho_x \otimes f, \tag{8.28}$$

so that by (8.26), as $T \to +\infty$,

$$r_{T,x} J_{T,x} f = \rho_x \otimes f + O\left(e^{-cT}\right) \|f\| . \tag{8.29}$$

Theorem 8.8. *Take $\eta > 0$ small enough. There exists $c > 0$ such that for any $x \in B, f \in F_x$, then as $T \to +\infty$,*

$$Q^{[0,1]}_{T,x} J_{T,x} f - J_{T,x} f = O\left(e^{-cT}\right) \|f\|_{F_x} \quad \text{uniformly on } M_x. \tag{8.30}$$

In particular, if $f \in F_x$, as $T \to +\infty$,

$$\left| r_{T,x} Q^{[0,1]}_{T,x} J_{T,x} f - \rho_x \otimes f \right| = O\left(e^{-cT}\right) |f|_{F_x} . \tag{8.31}$$

Proof. We proceed as in [BL2, Section 10]. Let δ be the oriented circle of center 0 and radius $1/2$ in $\mathbb{C}$. By (8.21), we know that for $T \geq 0$ large enough,

$$Q^{[0,1]}_{T,x} = \frac{1}{2\pi i} \int_\delta \left(\lambda - \widetilde{D}^2_{T,x}\right)^{-1} d\lambda. \tag{8.32}$$

Moreover, if $\lambda \in \mathbb{C}^*$, then

$$\left(\lambda - \widetilde{D}^2_{T,x}\right) \frac{J_{T,x} f}{\lambda} - J_{T,x} f = -\frac{\widetilde{D}^2_T J_{T,x} f}{\lambda}, \tag{8.33}$$

and so

$$\frac{J_{T,x} f}{\lambda} - \left(\lambda - \widetilde{D}^2_{T,x}\right)^{-1} J_{T,x} f = -\left(\lambda - \widetilde{D}^2_{T,x}\right)^{-1} \frac{\widetilde{D}^2_{T,x} J_{T,x} f}{\lambda}. \tag{8.34}$$

For $p \geq 1$, let $\mathbb{F}_{x,p}$ be the p-th Sobolev space of sections of $\Lambda(T^*M) \otimes F$ over M_x. Since $\mu(y) = 1$ for $|y| \leq \varepsilon/2$, we deduce from Proposition 8.3 that for any $p \geq 1$, there is $c > 0$ such that

$$\left\| \widetilde{D}^2_T J_{T,x} f \right\|_{\mathbb{F}_{x,p}} = O\left(e^{-cT}\right). \tag{8.35}$$

Let $\mathbb{F}^0_x$ be the vector space of sections $s \in \mathbb{F}_x$ such that $s_{|\partial M_x} = 0$. Take $q \in \mathrm{N}^*$. By [Tay, p. 108], there exists $C > 0$ such that if $s \in \mathbb{F}^0_x$, then

$$\|s\|_{\mathbb{F}_{x,2q}} \leq C \left(\left\| D^2 s \right\|_{\mathbb{F}_{x,2q-2}} + \|s\|_{x,0} \right). \tag{8.36}$$

Also using (5.16), (5.17), we see that there exists $C' > 0$ such that for $\lambda \in \delta, T \geq 1, s \in \mathbb{F}_x$,

$$\left\| \left(\lambda - \widetilde{D}^2_T + D^2\right) s \right\|_{\mathbb{F}_{x,2q-2}} \leq C' T^2 \|s\|_{\mathbb{F}_{x,2q-2}} . \tag{8.37}$$

By (8.36), (8.37), we find that there exists $C'' > 0$ such that for $\lambda \in \delta, T \geq 1, s \in \mathbb{F}_x^0$, then

$$\|s\|_{\mathbb{F}_{x,2q}} \leq C'' \left(\left\| \left(\lambda - \widetilde{D}_T^2\right) s \right\|_{\mathbb{F}_{x,2q-2}} + T^2 \|s\|_{\mathbb{F}_{x,2q-2}} \right). \tag{8.38}$$

Using (8.38), we see that there exists $C > 0$ such that for $\lambda \in \delta, T \geq 1, s \in \mathbb{F}_x^0$, then

$$\|s\|_{\mathbb{F}_{x,2q}} \leq CT^{2q} \left(\left\| \left(\lambda - \widetilde{D}_T^2\right) s \right\|_{\mathbb{F}_{x,2q-2}} + \|s\|_{\mathbb{F}_{x,0}} \right). \tag{8.39}$$

By Theorem 8.5, we know that for $T \geq 1$ large enough, if $\lambda \in \delta$, then $\lambda \notin \mathrm{Sp}(\widetilde{D}_{T,x}^2)$. More precisely, there exists $C' > 0$ such that for $T \geq 1$ large enough, $s \in \mathbb{F}_x$, then

$$\left\| \left(\lambda - \widetilde{D}_{T,x}^2\right)^{-1} s \right\|_{\mathbb{F}_{x,0}} \leq C' \|s\|_{\mathbb{F}_{x,0}}. \tag{8.40}$$

Moreover for $\lambda \in \delta, T \geq 1$ large enough, if $s \in \mathbb{F}_x$, then $(\lambda - \widetilde{D}_{T,x}^2)^{-1} s \in \mathbb{F}_x^0$.

Using (8.39), (8.40), we see that there exists $C'' > 0$ such that if $\lambda \in \delta, T \geq 1$, $s \in \mathbb{F}_x$, then

$$\left\| \left(\lambda - \widetilde{D}_{T,x}^2\right)^{-1} s \right\|_{\mathbb{F}_{x,2q}} \leq C'' T^{2q} \|s\|_{\mathbb{F}_{x,2q-2}}. \tag{8.41}$$

From (8.35), (8.41), we deduce that there is $c > 0$ such that for $T \geq 1$ large enough,

$$\left\| \left(\lambda - \widetilde{D}_{T,x}^2\right)^{-1} \widetilde{D}_{T,x}^2 J_{T,x} f \right\|_{\mathbb{F}_{x,2q}} = O\left(e^{-cT}\right) \|f\|_{F_x} \quad \text{uniformly in } \lambda \in \delta. \tag{8.42}$$

Using (8.42) and Sobolev's inequalities, we see that there exists $c > 0$ such that for $T \geq 1$, for any $f \in F_x$,

$$\left| \left(\lambda - \widetilde{D}_{T,x}^2\right)^{-1} \widetilde{D}_{T,x}^2 J_{T,x} f \right| \leq O\left(e^{-cT}\right) \|f\|_{F_x} \quad \text{uniformly on } M. \tag{8.43}$$

From (8.32), (8.34), (8.43), we obtain (8.30). Equation (8.31) is an obvious consequence of (8.29) and (8.30). □

Let $(Q_{T,x}^{[0,1]} J_{T,x})^*$ be the adjoint of $Q_{T,x}^{[0,1]} J_{T,x}$. Then $(Q_{T,x}^{[0,1]} J_{T,x})^*$ maps $K_{T,x}^{[0,1]}$ into F_x.

Definition 8.9. For $x \in B$, set

$$H_{T,x} = \left(Q_{T,x}^{[0,1]} J_{T,x}\right)^* Q_{T,x}^{[0,1]} J_{T,x}. \tag{8.44}$$

Then $H_{T,x}$ is self-adjoint in $\operatorname{End}(F_x)$.

Theorem 8.10. *For $T \geq 0$ large enough, for any $x \in B$, the linear map*

$$f \in F_x \to Q_{T,x}^{[0,1]} J_{T,x} f \in K_{T,x}^{[0,1],\mathrm{ind}(x)} \tag{8.45}$$

is one to one. Also there is $c > 0$ such that as $T \to +\infty$, for any $x \in B$, then

$$H_{T,x} = 1 + O\left(e^{-cT}\right). \tag{8.46}$$

Proof. Recall that $J_{T,x}$ is an isometry from F_x into $\mathbb{F}_x$. From (8.30), it follows that for T large enough, the linear map (8.45) is injective. By Theorem 8.5, for T large enough, F_x and $K_{T,x}^{[0,1],\mathrm{ind}(x)}$ have the same rank, and so the linear map (8.45) is one-to-one. Since $J_{T,x}$ is an isometry, (8.46) follows from (8.30) and from the previous considerations. □

For every $x \in B$, let $f_{x,1}, \cdots, f_{x,\mathrm{rk}(F)}$ be an orthonormal base of F_x with respect to the metric g^{F_x}. This base is fixed once and for all. By (8.46), for $T \geq 0$ large enough, $H_{T,x}$ is invertible.

Definition 8.11. For $T \geq 0$ large enough, $1 \leq j \leq \mathrm{rk}(F)$, set

$$\varphi_{T,x,j} = Q_{T,x}^{[0,1]} J_{T,x} H_{T,x}^{-1/2} f_{x,j}. \tag{8.47}$$

Proposition 8.12. *For $T \geq 0$ large enough, $\varphi_{T,x,1}, \cdots, \varphi_{T,x,\mathrm{rk}(F)}$ is an orthonormal base of the vector space $K_{T,x}^{[0,1],\mathrm{ind}(x)}$.*

Proof. This is a trivial consequence of Theorem 8.10. □

e) The orthonormal base of Helffer-Sjöstrand of the eigenspaces of the operator $\widetilde{D}_T^2$ asssociated to small eigenvalues

For $\eta > 0, y \in B$, let θ_y be a smooth function defined on M with values in $[0,1]$ such that $\theta_y = 1$ on $B^M(y,\eta)$, and $\theta_y = 0$ on $M \backslash B^M(y, 2\eta)$.

If $x \in B$, set

$$\chi_x = 1 - \sum_{\substack{y \in B\backslash\{x\} \\ \mathrm{ind}(y)=\mathrm{ind}(x)}} \theta_y. \tag{8.48}$$

For $\eta > 0$ small enough, χ_x vanishes on $\bigcup_{\substack{y \in B\backslash\{x\} \\ \mathrm{ind}(y)=\mathrm{ind}(x)}} B^M(y,\eta)$.

Definition 8.13. For $T \geq 0$ large enough, set

$$\psi_{T,x,j} = \chi_x \varphi_{T,x,j} \quad , \quad 1 \leq j \leq \mathrm{rk}(F). \tag{8.49}$$

For $T \geq 0$ large enough, and $0 \leq i \leq n$, let $\widetilde{\mathbb{G}}_T^{[0,1],i}$ be the vector subspace of $\mathbb{F}^i$ spanned by the $\psi_{T,x,j}$'s with $\mathrm{ind}(x) = i, 1 \leq j \leq \mathrm{rk}(F)$. Set

$$\widetilde{\mathbb{G}}_T^{[0,1]} = \bigoplus_{i=0}^{n} \widetilde{\mathbb{G}}_T^{[0,1],i}. \tag{8.50}$$

Definition 8.14. For $0 \leq i \leq n, T \geq 0$, let $\widetilde{D}_T^{2,i}$ be the restriction of $\widetilde{D}_T^2$ to $\mathbb{F}^i$. For $0 \leq i \leq n, T \geq 0$, let $\widetilde{\mathbb{F}}_T^{[0,1]} = \bigoplus_{i=0}^n \widetilde{\mathbb{F}}_T^{[0,1],i}$ be the direct sum of the eigenspaces of $\widetilde{D}_T^2$ associated to eigenvalues $\lambda \in [0,1]$. Let $\widetilde{P}_T^{[0,1]}$ be the orthogonal projection operator from $\mathbb{F}$ on $\widetilde{\mathbb{F}}_T^{[0,1]}$ with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F}}$ on $\mathbb{F}$.

If H_1, H_2 are closed vector subspaces of a Hilbert space H, if p^{H_1}, p^{H_2} are the orthogonal projection operators from H on H_1, H_2, set

$$\overrightarrow{d}(H_1,H_2) = \left\|p^{H_1} - p^{H_2}p^{H_1}\right\| = \left\|p^{H_1} - p^{H_1}p^{H_2}\right\|. \tag{8.51}$$

For $0 \leq i \leq n$, set

$$S^i = \inf_{\substack{x,y \in B^i \\ x \neq y}} d_A^M(x,y). \tag{8.52}$$

The following result is proved in [HSj3, Theorem 1.2], [HSj4, Proposition 1.7].

Theorem 8.15. *For $T \geq 0$ large enough, for any $i, 0 \leq i \leq n$, the eigenvalues of the operator $\widetilde{D}_T^{2,i}$ contained in $[0,1]$ can be put in one-to-one correspondence with the union of the eigenvalues of the operators $\widetilde{D}_{T,x}^{2,i}$ $(x \in B^i)$ contained in $[0,1]$, so that the difference of the corresponding eigenvalues is $\widetilde{O}(e^{-S^i T})$.*

For $T \geq 0$ large enough, for any $i, 0 \leq i \leq n$, the vector spaces $\widetilde{\mathbb{F}}_T^{[0,1],i}$ and $\widetilde{\mathbb{G}}_T^{[0,1],i}$ have the same dimension $\mathrm{rk}(F)M^i$, and moreover

$$\overrightarrow{d}\left(\widetilde{\mathbb{F}}_T^{[0,1],i}, \widetilde{\mathbb{G}}_T^{[0,1],i}\right) = \overrightarrow{d}\left(\widetilde{\mathbb{G}}_T^{[0,1],i}, \widetilde{\mathbb{F}}_T^{[0,1],i}\right) = \widetilde{O}\left(e^{-TS^i}\right). \tag{8.53}$$

Remark 8.16. As pointed out in Helffer-Sjöstrand [HSj4, Corollary 1.8], Morse inequalities for $H^\bullet(M,F)$ immediately follow from the fact that for T large enough, $\dim \widetilde{\mathbb{F}}_T^{[0,1]} = \mathrm{rk}(F)M^i$.

For $x \in B$, set

$$v_{T,x,j} = \widetilde{P}_T^{[0,1]} \psi_{T,x,j} \qquad 1 \leq j \leq \mathrm{rk}(F). \tag{8.54}$$

If $x \in B, x' \in M$, set

$$\delta_x(x') = \inf_{y \in B^{\mathrm{ind}(x)} \setminus \{x\}} \left(d_A^M(x,y) + d_A^M(y,x')\right). \tag{8.55}$$

By [HSj2, eq.(2.1.17)], [HSj4, eq. (1.38)], we know that

(8.56)

$$\left(v_{T,x,j} - \psi_{T,x,j}\right)(x') = \widetilde{O}\left(e^{-\delta_x(x')T}\right) \text{ uniformly together with its derivatives.}$$

From (8.19), (8.56), we deduce that

$$v_{T,x,j}(x') = \widetilde{O}\left(e^{-Td_A^M(x,x')}\right) \text{ uniformly together with its derivatives.} \tag{8.57}$$

Definition 8.17. For $0 \leq i \leq n$, and for $T \geq 0$ large enough, let V_T^i be the $(\mathrm{rk}(F)M^i, \mathrm{rk}(F)M^i)$ self-adjoint matrix

$$V_T^i = \langle v_{T,x,j}, v_{T,y,j'} \rangle_{\mathbb{F}}, \quad x, y \in B^i, \qquad 1 \leq j, j' \leq \mathrm{rk}(F). \tag{8.58}$$

As in [HSj2, Section 2.1], we observe that for $0 \leq i \leq n$, if $x, y \in B^i, 1 \leq j, j' \leq \mathrm{rk}(F)$ then

(8.59)

$$\langle v_{T,x,j}, v_{T,y,j'} \rangle_{\mathbb{F}} = \langle \psi_{T,x,j}, \psi_{T,y,j'} \rangle_{\mathbb{F}} - \langle v_{T,x,j} - \psi_{T,x,j}, v_{T,y,j'} - \psi_{T,y,j'} \rangle_{\mathbb{F}}.$$

From(8.59), Helffer and Sjöstrand [HSj2, Section 2.1], [HSj4, eq. (1.43)] deduce important estimates on the matrices V_T^i. A trivial consequence of (8.56), (8.57) is that for $0 \leq i \leq n$, there exists $c_i > 0$ such that as $T \to +\infty$,

$$V_T^i = 1 + O\left(e^{-c_i T}\right). \tag{8.60}$$

In the sequel, for $0 \leq i \leq n$, we consider $(v_{T,x,j})_{\substack{x \in B^i \\ 1 \leq j \leq \mathrm{rk}(F)}}$ as a linear map from $\mathbb{R}^{\mathrm{rk}(F) M^i}$ into $\widetilde{\mathbb{F}}_T^{[0,1],i}$, which we note v_T^i.

Definition 8.18. For $T \geq 0$ large enough, $0 \leq i \leq n$, set

$$\widetilde{e}_T^i = v_T^i \left(V_T^i\right)^{-1/2}. \tag{8.61}$$

The linear map $\widetilde{e}_T^i$ defines vectors $(\widetilde{e}_{T,x,k})_{\substack{x \in B^i \\ 1 \leq k \leq \mathrm{rk}(F)}}$ in $\widetilde{\mathbb{F}}_T^{[0,1],i}$.

Proposition 8.19. *For $T \geq 0$ large enough, for $0 \leq i \leq n$, $\{\widetilde{e}_{T,x,j}\}_{\substack{x \in B^i \\ 1 \leq j \leq \mathrm{rk}(F)}}$ is an orthonormal base of $\widetilde{\mathbb{F}}_T^{[0,1],i}$. Also as $T \to +\infty$, for $x \in B, 1 \leq k \leq \mathrm{rk}(F)$,*

$$\widetilde{e}_{T,x,k}(x') = \widetilde{O}\left(e^{-T d_A^M(x,x')}\right) \text{ uniformly together with its derivatives.} \tag{8.62}$$

Proof. The first part of the Proposition follows from Theorem 8.16 and from (8.60). Equation (8.62) follows from (8.57) and from the estimates on the matrices $V_T^i (0 \leq i \leq n)$ proved in [HSj2, Section 2.1], [HSj4, eq. (1.43) and (3.12)]. □

f) The WKB equation for $\widetilde{D}_T^2$

Let U be a non empty open set in M. Let $\mathbb{F}_U = \bigoplus_{i=0}^n \mathbb{F}_U^i$ be the vector space of smooth sections of $\Lambda(T^*M) \otimes F = \bigoplus_{i=0}^n \Lambda^i(T^*M) \otimes F$ over U. We equip $\mathbb{F}_U$ with the scalar product $\langle\ ,\ \rangle_{\mathbb{F}_U}$ which is the obvious analogue of the scalar product $\langle\ ,\ \rangle_{\mathbb{F}}$ on $\mathbb{F}$.

If Y is a smooth vector field on U, let L_Y be the Lie derivative operator associated to Y. Then L_Y acts on $\mathbb{F}_U$. Let L_Y^* be the formal adjoint of L_Y with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F}_U}$.

Let $e_1, \cdots, e_n$ be an orthonormal base of TM.

Definition 8.20. If $h : U \to \mathbb{R}$ is a smooth function, let $\tau(h)$ be the first order differential operator acting on $\mathbb{F}_U$

$$\tau(h) = L_{\nabla f} + L^*_{\nabla f} + L_{\nabla h} - L^*_{\nabla h}. \tag{8.63}$$

Proposition 8.21. *For any smooth function $h : U \to \mathbb{R}$, the following identity holds*
(8.64)

$$\tau(h) = 2\nabla_{\nabla h} + \sum_{1\leq i,j\leq n} \left\langle \nabla^{T^*M}_{e_i} df, e_j \right\rangle c(e_i)\widehat{c}(e_j) + \Delta h + \omega\left(F, g^F\right)\left(\nabla(h-f)\right).$$

Proof. We have the trivial formula

$$L_{\nabla f} = \nabla_{\nabla f} + \sum_{1\leq i,j\leq n} \left\langle \nabla^{TM}_{e_i} \nabla f, e_j \right\rangle e^i \wedge i_{e_j}. \tag{8.65}$$

From (8.65), we deduce that

$$L^*_{\nabla f} = -\nabla_{\nabla f} - \Delta f + \sum_{1\leq i,j\leq n} \left\langle \nabla^{T^*M}_{e_i} df, e_j \right\rangle e^i \wedge i_{e_j} - \omega\left(F, g^F\right)(\nabla f). \tag{8.66}$$

Similar identities hold for $L_{\nabla h}, L^*_{\nabla h}$. Equation (8.64) follows. □

We now reprove a formula of [HSj4, Lemma 2.1].

Proposition 8.22. *Let $h : U \to \mathbb{R}$ be a smooth function. Then*

$$e^{Th}\widetilde{D}^2_T e^{-Th} = D^2 + T\tau(h) + T^2\left(|df|^2 - |dh|^2\right). \tag{8.67}$$

Proof. Using (5.12), we get

$$\begin{aligned} e^{Th} d^F_T e^{-Th} &= d^F + Td(f-h)\wedge, \\ e^{Th} \delta^F_T e^{-Th} &= d^{F*} + Ti_{\nabla(f+h)}. \end{aligned} \tag{8.68}$$

From (5.10), (8.68), we obtain
(8.69)

$$e^{Th}\widetilde{D}^2_T e^{-Th} = D^2 + T\left(L_{\nabla f} + L^*_{\nabla f} + L_{\nabla h} - L^*_{\nabla h}\right) + T^2\left(|df|^2 - |dh|^2\right).$$

Equation (8.67) follows. □

Take now $x \in B$. Recall that φ_x is the function $\varphi_x(x') = d^M_A(x, x')$. If $x' \in W^u(x)$, there exists an integral curve γ of the vector field $-\nabla f$, with $\gamma_{-\infty} = x, \gamma_a = x'(-\infty \leq a < +\infty)$. This integral curve is obviously unique. In particular it avoids the points in $B\backslash\{x\}$. By proceeding as in [HSj4, Appendix 2], we see that γ is the unique geodesic connecting x and x' with respect to the Agmon metric g_A^{TM}. It easily follows that the function φ_x is smooth on an open neighborhood of $\gamma([-\infty, a[)$. Therefore φ_x is smooth on an open neighborhood of $W^u(x)$. Similarly φ_x is smooth on an open neighborhood of $W^s(x)$.

Let V be an open neighborhood of $W^u(x) \cup W^s(x)$ such that φ_x is smooth on V. Then φ_x verifies the Hamilton-Jacobi equation

$$|\nabla \varphi_x|^2 = |\nabla f|^2 \quad \text{on } V. \tag{8.70}$$

Now, we proceed as in [HSj4, Section 2]. Set

$$\begin{aligned} f_x^+ &= \frac{1}{2}\left(\varphi_x + f - f(x)\right), \\ f_x^- &= \frac{1}{2}\left(\varphi_x - f + f(x)\right). \end{aligned} \tag{8.71}$$

With the notation of Helffer and Sjöstrand in [HSj4, eq. (2.6)], then

$$f_x^+ = \frac{1}{2} g_- \quad , \qquad f_x^- = \frac{1}{2} g_+. \tag{8.72}$$

Clearly

$$\begin{aligned} f &= f(x) + f_x^+ - f_x^-, \\ \varphi_x &= f_x^+ + f_x^-. \end{aligned} \tag{8.73}$$

The functions f_x^+ and f_x^- are positive Lipschitz functions, which are smooth on V.

Using (8.70), (8.73), it is clear that

$$\left\langle \nabla f_x^+, \nabla f_x^- \right\rangle = 0. \tag{8.74}$$

Also by proceeding as in [HSj4, Lemma A.2.2], we see that

$$\begin{aligned} \varphi_x &= f - f(x) \quad \text{on} \quad W^s(x), \\ &= -f + f(x) \quad \text{on} \quad W^u(x). \end{aligned} \tag{8.75}$$

Since over $W^u(x) \cup W^s(x)$, the minimizing geodesics for the Agmon distance are integral curves of the vector field $-\nabla f$, we find easily that

$$\begin{aligned} \nabla \varphi_x &= \nabla f \quad \text{on} \quad W^s(x), \\ &= -\nabla f \quad \text{on} \quad W^u(x). \end{aligned} \tag{8.76}$$

From (8.76), we deduce that f_x^+ vanishes to order 2 on $W^u(x)$, and f_x^- vanishes to order 2 on $W^s(x)$.

Let

$$\alpha_T = \sum_{k=0}^{+\infty} \frac{\alpha_k}{T^k} \tag{8.77}$$

be a formal power series with values in smooth sections of $\Lambda(T^*M) \otimes F$ over V.

We now look for a solution of an equation of WKB type

$$\frac{1}{T^2} e^{T\varphi_x} \widetilde{D}_T^2 e^{-T\varphi_x} \alpha_T = O\left(\frac{1}{T^\infty}\right) \alpha_T \quad \text{on } V. \tag{8.78}$$

Using Proposition 8.22 and (8.70), we see that equation (8.78) is equivalent to

$$\left(\frac{1}{T^2} D^2 + \frac{1}{T} \tau\left(\varphi_x\right)\right) \alpha_T = O\left(\frac{1}{T^\infty}\right) \alpha_T \quad \text{on } V. \tag{8.79}$$

By cancelling the coefficient of $\frac{1}{T}$ in the left-hand side of (8.79), we get

$$\tau\left(\varphi_x\right) \alpha_0 = 0 \tag{8.80}$$

Equivalently, by using Proposition 8.21, we find that

$$\begin{aligned} \Big(2\nabla_{\nabla \varphi_x} + \sum_{1 \leq i,j \leq n} \left\langle \nabla_{e_i}^{T^*M} df, e_j \right\rangle c\left(e_i\right) \widehat{c}\left(e_j\right) & \\ + \Delta \varphi_x + \omega\left(F, g^F\right) \left(\nabla(\varphi_x - f)\right) \Big) \alpha_0 &= 0. \end{aligned} \tag{8.81}$$

Equation (8.81) holds in particular at x, where $\nabla f = 0, \nabla \varphi_x = 0$. Therefore

$$\left(\sum_{1 \leq i,j \leq n} \langle \nabla_{e_i}^{TM} \nabla f(x), e_j \rangle c\left(e_i\right) \widehat{c}\left(e_j\right) + \Delta \varphi(x) \right) \alpha_0 = 0. \tag{8.82}$$

Now we use the notation of Proposition 8.2. By (8.3), equation (8.82) is equivalent to

$$2\left(N^+ + \operatorname{ind}(x) - N^-\right) \alpha_0(x) = 0 \tag{8.83}$$

The same argument as in the proof of Proposition 8.3 shows that (8.83) holds if and only if there is $g \in F_x$ such that

$$\alpha_0(x) = \rho_x \otimes g. \tag{8.84}$$

Then once $\alpha_0(x)$ taken as in (8.84) is fixed, since the operator $N_+ + \operatorname{ind}(x) - N_-$ is nonnegative and self-adjoint, one sees easily that equation (8.81) has a unique solution.

Recall that near x, (8.3) holds. We trivialize F on $B^M(x,\varepsilon)$ using the flat connection ∇^F. Moreover since the metric g^F is flat on $B^M(x,\varepsilon)$, $\omega(F,g^F)$ vanishes on $B^M(x,\varepsilon)$. As in Proposition 8.3, we extend $\rho_x \otimes g$ into a "constant" section of $\Lambda(T^*M)\otimes F$ on $B^M(x,\varepsilon)$. Then

(8.85)

$$\nabla_{\nabla\varphi_x}(\rho_x\otimes g) = 0 \quad \text{on } B^M(x,\varepsilon),$$

$$\Big(\sum_{1\leq i,j\leq n} \langle \nabla^{TM}_{e_i}\nabla f, e_j\rangle c(e_i)\widehat{c}(e_j) + \Delta\varphi_x + \omega\left(F,g^F\right)\left(\nabla(\varphi_x - f)\right)\Big)(\rho_x\otimes g)$$

$$= 0 \quad \text{on } B^M(x,\varepsilon).$$

Therefore, on $B^M(x,\varepsilon)$, the constant $\alpha_0 = \rho_x\otimes g$ is exactly the solution of equation (8.81). Also, on $B^M(x,\varepsilon)$, $D^2 = -\Delta^{\mathbb{R}^n}$, and so we see that

(8.86) $$D^2(\rho_x\otimes g) = 0 \quad \text{on } B^M(x,\varepsilon).$$

So by Proposition 8.21 and by (8.85), (8.86), we find that

(8.87) $$\left(\frac{1}{T^2}D^2 + \frac{1}{T}\tau(\varphi_x)\right)(\rho_x\otimes g) = 0 \quad \text{on } B^M(x,\varepsilon).$$

By Proposition 8.22 and by (8.70), (8.87) is equivalent to

(8.88) $$e^{T\varphi_x}\widetilde{D}^2_T e^{-T\varphi_x}(\rho_x\otimes g) = 0 \quad \text{on } B^M(x,\varepsilon).$$

The fact that (8.88) holds permits us to assume that in (8.77),

(8.89) $$\text{for any} \quad j\geq 1, \alpha_j = 0 \quad \text{on } B^M(x,\varepsilon).$$

If V is small enough, the equivalent equations (8.78) and (8.79) can then be solved by a trivial recursion procedure.

As in Helffer-Sjöstrand [HSj4, Section 2], it will now be crucial to solve the transport equation (8.80) along $W^s(x)$ and $W^u(x)$. In fact ∇f is tangent to $W^s(x)$ and $W^u(x)$. By (8.76), $\nabla\varphi_x$ is tangent to $W^s(x)$ and $W^u(x)$ and so the same is true for $\nabla f^\pm_x$.

g) The transport equation on $W^s(x)$

By (8.3) and (8.4), it is clear that near x,

$$f_x^+(y) = \frac{1}{2} \sum_{\mathrm{ind}(x)+1}^{n} |y^i|^2,$$

(8.90)

$$f_x^-(y) = \frac{1}{2} \sum_{1}^{\mathrm{ind}(x)} |y^i|^2.$$

Using (8.90), we see that near x, f_x^+ vanishes exactly to order 2 on $W^u(x)$. Moreover by (8.71), (8.76), $\nabla f_x^+ = \nabla f$ on $W^s(x)$, and so on $W^s(x)$, ∇f_x^+ only vanishes at x.

Let V be an open neighborhood of $W^s(x)$. From the previous considerations, we see that if V is small enough, the restriction of ∇f_x^+ to V vanishes only on $W^u(x)$.

Let $(y^1, \cdots y^n)$ be the system of coordinates near x considered in (8.3). Then $(y^1, \cdots y^{\mathrm{ind}(x)})$ is a system of coordinates on $W^u(x)$ near x.

As in [HSj4, eq. (2.21)], we consider the transport equation

(8.91)

$$L_{\nabla f_x^+} \bar{y}_j = 0 \qquad 1 \leq j \leq \mathrm{ind}(x),$$

$$\bar{y}_{j|W^u(x)} = y_{j|W^u(x)}.$$

Equation (8.91) means exactly that $(\bar{y}^1, \cdots, \bar{y}^{\mathrm{ind}(x)})$ is constant along the trajectories of the gradient vector field ∇f_x^+. The considerations we made before guarantee that $(\bar{y}^1, \cdots, \bar{y}^{\mathrm{ind}(x)})$ defines a system of coordinates transverse to $W^s(x)$, which vanishes on $W^s(x)$. Note that near x, $(\bar{y}^1, \cdots, \bar{y}^{\mathrm{ind}(x)})$ coincides with $(y^1, \cdots, y^{\mathrm{ind}(x)})$.

Over $W^s(x)$, we define the section $\bar{\rho}_x$ of $\Lambda^{\mathrm{ind}(x)}(T^*M)$ by the formula

(8.92)
$$\bar{\rho}_x = d\bar{y}^1 \wedge \cdots \wedge d\bar{y}^{\mathrm{ind}(x)}.$$

Of course, near x, $\bar{\rho}_x$ restricts to the section ρ_x of $\Lambda^{\mathrm{ind}(x)}(T^*M)$ considered in (8.7). Similarly, if $g \in F_x$, we extend g to a smooth section $\bar{g}_x$ of $F_{|W^s(x)}$ by parallel transport with respect to the connection ∇^F.

Near x, $\bar{\rho}_x \otimes \bar{g}_x$ coincides with the restriction to $W^s(x)$ of the section $\rho_x \otimes g$ which was considered in (8.84). We now prove the analogue of [HSj4, Proposition 2.3].

Proposition 8.24. *Over $W^s(x)$, if $g \in F_x$, then the following identity holds*

$$\tau(\varphi_x)\left(\bar{\rho}_x \otimes \bar{g}_x\right) = 0. \tag{8.93}$$

Proof. By (8.63), (8.70), it is clear that

$$\tau(\varphi_x) = 2L_{\nabla f_x^+} - 2L^*_{\nabla f_x^-}. \tag{8.94}$$

Since $\bar{g}_x$ is a flat section of $F_{|W^s(x)}$, from (8.91), we get

$$L_{\nabla f_x^+}\left(\bar{\rho}_x \otimes \bar{g}_x\right) = 0. \tag{8.95}$$

Using (8.66), we know that

$$L^*_{\nabla f_x^-} = -\nabla_{\nabla f_x^-} - \Delta f_x^- + \sum_{1 \le i,j \le n} \left\langle \nabla^{T^*M}_{e_i} df_x^-, e_j \right\rangle e^i \wedge i_{e_j} - \omega\left(F, g^F\right)\left(\nabla f_x^-\right). \tag{8.96}$$

As we saw after (8.76), f_x^- vanishes to order 2 on $W^s(x)$. Then, one verifies easily that

$$\left(-\Delta f_x^- + \sum_{1 \le i,j \le n} \left\langle \nabla^{T^*M}_{e_i} df_x^-, e_j \right\rangle e^i \wedge i_{e_j}\right)\left(\bar{\rho}_x \otimes \bar{g}_x\right) = 0 \quad \text{on} \quad W^s(x). \tag{8.97}$$

Also $\nabla f_x^- = 0$ on $W^s(x)$. Using (8.96), (8.97), we get

$$L^*_{\nabla f_x^-}\left(\bar{\rho}_x \otimes \bar{g}_x\right) = 0. \tag{8.98}$$

Equation (8.95) follows from (8.94), (8.95), (8.98). □

h) The transport equation on $W^u(x)$

The coordinate system $y = (y^1, \cdots, y^n)$ near $x \in B$ is still taken as in (8.3). Then $(y^{\mathrm{ind}(x)+1}, \cdots, y^n)$ is a system of coordinates on $W^s(x)$ near x.

As in [HSj4, eq. (2.30)], instead of (8.91), we consider the transport equation on $W^u(x)$

$$(8.99)\qquad \begin{aligned} &L_{\nabla f_x^-}\bar{y}^j = 0 \quad \operatorname{ind}(x)+1 \le j \le n, \\ &\bar{y}^j_{|W^s(x)} = y^j_{|W^s(x)}. \end{aligned}$$

The same considerations as the ones we made after (8.90) guarantee that equation (8.99) has a unique solution near $W^u(x)$. Then $(\bar{y}^{\operatorname{ind}(x)+1}, \cdots, \bar{y}^n)$ is a system of coordinates transverse to $W^u(x)$, which vanishes on $W^u(x)$. Also near $x, (\bar{y}^{\operatorname{ind}(x)+1}, \cdots, \bar{y}^n)$ coincides with $(y^{\operatorname{ind}(x)+1}, \cdots, y^n)$. Since $TW^u(x)$ is oriented, $d\bar{y}^{\operatorname{ind}(x)+1} \wedge \cdots \wedge d\bar{y}^n$ is a section of $\Lambda^{n-\operatorname{ind}(x)}(T^*M) \otimes o(TM)$.

Recall that $*$ is the Hodge operator for the metric g^{TM}. Set

$$(8.100)\qquad \bar{\rho}_x^* = (-1)^{\operatorname{ind}(x)(n-\operatorname{ind}(x))} * (d\bar{y}^{\operatorname{ind}(x)+1} \wedge \cdots \wedge d\bar{y}^n).$$

Then, $\bar{\rho}_x^*$ is a section of $\Lambda^{\operatorname{ind}(x)}(T^*M)$. Also near $x, \bar{\rho}_x^*$ coincides with ρ_x.

Take $g \in F_x$. Let $\bar{g}_x^*$ be the flat section of $F_{|W^u(x)}$ with respect to the flat connection ∇^{F*}, defined in (3.2), which extends g to $W^u(x)$. Since the metric g^F is flat near $x, \bar{g}_x^*$ coincides with g near x.

Near x, $\bar{\rho}_x^* \otimes \bar{g}_x^*$ coincides with the restriction to $W^u(x)$ of the section $\rho_x \otimes g_x$ considered in (8.84).

We now prove the following important extension of [HSj4, Proposition 2.4].

Proposition 8.25. *Over $W^u(x)$, the following identity holds*

$$(8.101)\qquad \tau(\varphi_x)\,(\bar{\rho}_x^* \otimes \bar{g}_x^*) = 0.$$

Proof. Recall that $i : F \to F^*$ is the canonical identification of F and F^* associated to the metric g^F. Let $L^{F^*}_{\nabla f_x^\pm}$ be the analogue of the operator $L_{\nabla f_x^\pm}$ acting on smooth sections of $\Lambda(T^*M) \otimes F^*$. Clearly

$$(8.102)\qquad L^*_{\nabla f_x^\pm} = -(* \otimes i)^{-1} L^{F^*}_{\nabla f_x^\pm}(* \otimes i).$$

Using (8.94), (8.102), we see that

$$(8.103)\qquad (* \otimes i)\tau(\varphi_x)(* \otimes i)^{-1} = 2L^{F^*}_{\nabla f_x^-} - 2L^{F^*,*}_{\nabla f_x^+}.$$

Comparing with (8.94), we find that the operator (8.103) is still an operator of the type $\tau(\varphi_x)$, with F replaced by F^*, and f by $-f$. We can then use Proposition 8.24 and obtain (8.101). □

Remark 8.26. The proof of Proposition 8.25 reflects Poincaré duality in a rather subtle way.

We now describe the solutions of the WKB equation (8.78) on $W^s(x) \cup W^u(x)$. Recall that $r_{T,x}$ was defined in Definition 8.6.

Theorem 8.27. *Let* $\alpha(g) = (\frac{T}{\pi})^{n/4} \sum_0^{+\infty} \frac{\alpha_j(g)}{T^j}$ *be the* WKB *solution of*

$$\begin{aligned} &\frac{1}{T^2} e^{T\varphi_x} \widetilde{D}_T^2 e^{-T\varphi_x} \alpha(g) = O\left(\frac{1}{T^\infty}\right) \alpha(g), \\ &r_{T,x}\alpha(g) = \rho_x \otimes g. \end{aligned} \tag{8.104}$$

Then

$$\begin{aligned} \alpha_0(g) &= \bar{\rho}_x \otimes \bar{g}_x \quad \text{on } W^s(x), \\ &= \bar{\rho}_x^* \otimes \bar{g}_x^* \quad \text{on } W^u(x). \end{aligned} \tag{8.105}$$

Proof. This follows trivially from Propositions 8.24 and 8.25. □

i) The matrix of d_T^F in the base $\widetilde{e}_{T,x,k}$

By [HSj4, Lemma A.2.1], we know that if $x \in B, y \in M$,

$$d_A^M(x,y) \geq f(x) - f(y). \tag{8.106}$$

Proposition 8.28. *Let* $x \in B,\ y \in M$. *Then*

$$d_A^M(x,y) = f(x) - f(y) \tag{8.107}$$

if and only if $y \in \overline{W^u}(x)$. *Moreover if* $y \in B, y \neq x$, *and if (8.107) holds, then*

$$\operatorname{ind}(x) \geq \operatorname{ind}(y) + 1. \tag{8.108}$$

Proof. If $x \in B, y \in W^u(x)$, then (8.107) holds. Therefore (8.107) also holds on $\overline{W^u(x)}$.

Conversely assume that (8.107) holds. For $a \in [-\infty, +\infty]$, let $[-\infty, +\infty] \cup \cdots \cup [-\infty, a]$ be a finite union of intervals $[-\infty, +\infty]$ and of the interval $[-\infty, a]$. We denote by $-\dot{\infty}$ the first of the $-\infty$. Let $t \in [-\dot{\infty}, +\infty] \cup \cdots \cup [-\infty, a] \to \gamma_t \in M$ be a minimizing geodesic with respect to the Agmon distance d_A^M, such that $\gamma_{-\dot{\infty}} = x$, $\gamma_a = y$. By [HSj4, Lemmas A 2.1 and A 2.2], we find that γ is a generalized integral curve of the vector field $-\nabla f$, and f is decreasing along γ. If γ is parametrized by $[-\infty, a]$, it is obvious that $y \in \overline{W^u(x)}$. If γ is parametrized by $[-\dot{\infty}, +\infty] \cup [-\infty, a]$, set $x_2 = \gamma_{+\infty}$. Then $x_2 \in B \cap \overline{W^u(x)}$, $x_2 \neq x$. As before, $y \in \overline{W^u(x_2)}$. Now by [Ro, Lemma 1], or by Proposition 2 in the Appendix, since ∇f verifies the Smale transversality conditions, then $\overline{W^u(x_2)} \subset \overline{W^u(x)}$, and so $y \in \overline{W^u(x)}$. A trivial recursion argument shows that in full generality, $y \in \overline{W^u(x)}$.

Suppose that $y \in B, y \neq x$ and that (8.107) holds. Let $x_2 \in B$ be the first critical point of f distinct from x visited by γ. Then

$$W^u(x) \cap W^s(x_2) \neq \emptyset. \tag{8.109}$$

Since the vector field ∇f verifies the Smale transversality conditions, we find that

$$\mathrm{ind}(x) \geq \mathrm{ind}(x_2) + 1. \tag{8.110}$$

By iterating (8.110), we get (8.108). □

Remark 8.29. Proposition 8.28 is very important, since it guarantees that assumption H1 of Helffer-Sjöstrand [HSj4] is verified.

Assumption H2 of [HSj4] is verified because ∇f satisfies the Smale transversality conditions.

If $x \in B$, recall that $[W^u(x)]^*$ is the line dual to the line $[W^u(x)]$. Let $W^u(x)^* \in [W^u(x)]^*$ be dual to $W^u(x) \in [W^u(x)]$, so that $\langle W^u(x)^*, W^u(x)\rangle = 1$. Then $C^\bullet(W^u, F)$ is spanned by the $W^u(x)^* \otimes f$'s $(x \in B, f \in F_x)$.

The metric g^F induces metrics g^{F_x} on F_x $(x \in B)$. The lines $[W^u(x)]^*$ $(x \in B)$ can be equipped with the obvious metrics which give the norm 1 to $W^u(x)^*$ $(x \in B)$. Therefore if $x \in B$, $[W^u(x)]^* \otimes F_x$ is naturally equipped with a scalar product. We equip $C^\bullet(W^u, F) = \bigoplus_{x \in B} [W^u(x)]^* \otimes F_x$ with the scalar product $\langle\ \rangle_{C^\bullet(W^u,F)}$, which is the direct sum of the previous scalar products.

We now establish an extension of a fundamental result of Helffer-Sjöstrand [HSj4].

Theorem 8.30. *For* $0 \leq i \leq n$, $x \in B^{i+1}, x' \in B^i$, *for* $1 \leq k, k' \leq \mathrm{rk}(F)$, *as* $T \to +\infty$,

$$\left\langle d_T^F \widetilde{e}_{T,x',k'}, \widetilde{e}_{T,x,k} \right\rangle_{\mathbb{F}} = \left(\frac{T}{\pi}\right)^{1/2} e^{-T(f(x)-f(x'))} \tag{8.111}$$

$$\left(\left\langle \widetilde{\partial}\left(W^u(x')^* \otimes f_{x',k'}\right), W^u(x)^* \otimes f_{x,k}\right\rangle_{C^\bullet(W^u,F)} + O\left(\frac{1}{T^{1/2}}\right)\right).$$

Proof. We essentially follow Helffer-Sjöstrand [HSj4, Section 3]. Still we have to modify their argument and computations, because of the presence of the flat vector bundle F.

Take η, with $0 < \eta < \frac{1}{2} d_A^M(x,x')$. Let $\chi_{x,x'}$ be a smooth function from M into $[0,1]$ such that

$$(8.112) \qquad \begin{aligned} \chi_{x,x'} &= 1 \quad \text{in} \quad B_A^M\left(x, \frac{1}{2} d_A^M(x,x') - \eta\right), \\ &= 0 \quad \text{in} \quad B_A^M\left(x', \frac{1}{2} d_A^M(x,x') - \eta\right). \end{aligned}$$

Recall that for T large enough, the $\psi_{T,x,j}$'s $(x \in B, 1 \leq j \leq \operatorname{rk}(F))$ were defined in Definition 8.13, and depend also on $\eta > 0$.

By proceeding as in [HSj4, Theorem 3.1], and using Proposition 8.28, we find that there exists $\alpha > 0$ such that as $T \to +\infty$,

(8.113)

$$\left\langle d_T^F \widetilde{e}_{T,x',k'}, \widetilde{e}_{T,x,k}\right\rangle_{\mathbb{F}} = -\left\langle \psi_{T,x,k}, d\chi_{x,x'} \wedge \psi_{T,x',k'}\right\rangle_{\mathbb{F}} + \widetilde{O}\left(e^{(-\alpha - d_A^M(x,x'))T}\right).$$

Using (8.19), (8.49), it is clear that

$$(8.114) \qquad \left\langle \psi_{T,x,k}, d\chi_{x,x'} \wedge \psi_{T,x',k'}\right\rangle_{\mathbb{F}} = \widetilde{O}\left(e^{-d_A^M(x,x')T}\right).$$

By (8.106), we know that $f(x) - f(x') \leq d_A^M(x,x')$. If $f(x) - f(x') < d_A^M(x,x')$, from (8.113), (8.114), we deduce that there exists $\alpha' > 0$ such that

$$(8.115) \qquad \left\langle d_T^F \widetilde{e}_{T,x',k'}, \widetilde{e}_{T,x,k}\right\rangle_{\mathbb{F}} = e^{-T(f(x)-f(x'))} \widetilde{O}\left(e^{-\alpha' T}\right).$$

Moreover if there was an integral curve $\gamma : [-\infty, +\infty]$ of $-\nabla f$ with $\gamma_{-\infty} = x, \gamma_{+\infty} = x'$ it would follow that $f(x) - f(x') = d_A^M(x,x')$. So if $f(x) - f(x') < d_A^M(x,x')$, then $W^u(x) \cap W^s(x') = 0$. From (8.115), we find that (8.111) holds.

So we now consider the case where $f(x) - f(x') = d_A^M(x,x')$. By Proposition 8.28, we know that $x' \in \overline{W^u(x)}$. Since $\operatorname{ind}(x') = \operatorname{ind}(x) - 1, W^u(x) \cap W^s(x')$ consists of a finite set $\Gamma(x,x')$ of minimizing geodesics γ for the Agmon distance, with $\gamma_{-\infty} = x, \ \gamma_{+\infty} = x'$.

Take $\gamma \in \Gamma(x,x')$. Let V_γ be an open neighborhood of γ in M. Using (8.19), (8.49), it is clear that there exists $\alpha'' > 0$ such that

$$(8.116) \qquad - \langle \psi_{T,x,k}, d\chi_{x,x'} \wedge \psi_{T,x',k'} \rangle_{\mathbb{F}}$$
$$= - \sum_{\gamma \in \Gamma(x,x')} \int_{V_\gamma} \langle d\chi_{x,x'} \wedge \psi_{T,x',k'} \wedge * \psi_{T,x,k} \rangle_F + \widetilde{O}\left(e^{-(d_A^M(x,x') + \alpha'')T} \right).$$

Recall that $\varphi_{T,x,k} (1 \leq k \leq \operatorname{rk}(F))$ was defined in Definition 8.11. By (8.30), (8.46), (8.47), there exists $c > 0$ such that as $T \to +\infty$, then

$$(8.117) \qquad \varphi_{T,x,k} = J_{T,x} f_{x,k} + O\left(e^{-cT}\right) \quad \text{uniformly on } M.$$

Take $\varepsilon > 0$ as in Section 8a). Let $\mathbb{F}_{B^M(x,\varepsilon),0}$ be the Hilbert space of the L_2 sections of $\Lambda(T^*M) \otimes F$ over $B^M(x,\varepsilon)$. By [HSj1, eq. (5.9)] and by (8.89), if $\eta > 0$ is small enough, there exists a $(\operatorname{rk} F, \operatorname{rk} F)$ orthogonal matrix $c_{T,x}$ such that
(8.118)

$$\varphi_{T,x,k} = \left(\frac{T}{\pi}\right)^{n/4} e^{-T\varphi_x} \left[\rho_x \otimes \sum_{1}^{\operatorname{rk}(F)} c_{T,x,k}^{k'} f_{x,k'} \right] + O\left(\frac{1}{T^\infty}\right) \quad \text{in } \mathbb{F}_{B^M(x,\varepsilon),0}.$$

Comparing with (8.117), we obtain

$$(8.119) \qquad \varphi_{T,x,k} = \left(\frac{T}{\pi}\right)^{n/4} e^{-T\varphi_x} \rho_x \otimes f_{x,k} + O\left(\frac{1}{T^\infty}\right) \quad \text{in } \mathbb{F}_{B^M(x,\varepsilon),0}.$$

We use the notation of Theorem 8.27. Let $\mathcal{W}$ be an open neighborhood of $\gamma \backslash B^M(x,\eta)$. By [HSj1, Theorem 5.8] and by (8.119), we see that if $\eta > 0$ and $\mathcal{W}$ are small enough, for any $j \in \mathbb{N}$, as $T \to +\infty$,

$$(8.120) \qquad \left\| e^{T\varphi_x} \varphi_{T,x,k} - \left(\frac{T}{\pi}\right)^{n/4} \sum_{0}^{j} \frac{\alpha_i (f_{x,k})}{T^i} \right\|_{\mathbb{F}_{\mathcal{W},0}} = O\left(\frac{1}{T^{j+1-\frac{n}{4}}}\right).$$

From (8.49) and (8.120), we deduce that if $\eta > 0$ and $\mathcal{W}$ are small enough, then

$$(8.121) \qquad \left\| e^{T\varphi_x} \psi_{T,x,k} - \left(\frac{T}{\pi}\right)^{n/4} \sum_{0}^{j} \frac{\alpha_i (f_{x,k})}{T^i} \right\|_{\mathbb{F}_{\mathcal{W},0}} = O\left(\frac{1}{T^{j+1-\frac{n}{4}}}\right).$$

Let $\mathcal{W}'$ be an open neighborhood of $\gamma\backslash B^M(x',\eta)$. Then if $\eta>0$ and $\mathcal{W}'$ are small enough, the analogue of (8.121) is

$$\left\| e^{T\varphi_{x'}}\psi_{T,x',k'} - \left(\frac{T}{\pi}\right)^{n/4}\sum_0^j \frac{\alpha_i(f_{x',k'})}{T^i}\right\|_{\mathbb{F}_{\mathcal{W}',0}} = O\left(\frac{1}{T^{j+1-\frac{n}{4}}}\right). \tag{8.122}$$

By (8.71), we know that

$$\varphi_x(t)+\varphi_{x'}(t) = f(x)-f(x')+2\left(f_x^+(t)+f_{x'}^-(t)\right), \tag{8.123}$$

and so

$$\varphi_x(t)+\varphi_{x'}(t)\geq f(x)-f(x'). \tag{8.124}$$

Let $(\overline{y}^1,\cdots,\overline{y}^i)$ be the system of coordinates transverse to $W^s(x')$ taken as in (8.91). Similarly, let $(\overline{z}^1,\cdots,\overline{z}^{n-i-1})$ be the system of coordinates transverse to $W^u(x)$ considered in (8.99) (under the name of $\overline{y}^{i+1},\cdots,\overline{y}^n$). As in [HSj4, proof of Proposition 3.3], we observe that since $W^u(x)$ and $W^s(x')$ are transversal, the forms $d\overline{y}^1,\cdots,d\overline{y}^i$, $d\overline{z}^1\cdots d\overline{z}^{n-i-1}$ are linearly independent near γ.

Equation (8.73) is equivalent to

$$L_{\nabla f_x^-} f_x^+ = 0. \tag{8.125}$$

Using (8.90), (8.99), (8.125) we find that

$$f_x^+ = \frac{1}{2}\sum_1^{n-i-1}\left|\overline{z}^j\right|^2 \quad \text{near } W^u(x). \tag{8.126}$$

Similarly

$$f_x^- = \frac{1}{2}\sum_1^{i}\left|\overline{y}^j\right|^2 \quad \text{near } W^s(x'). \tag{8.127}$$

From (8.121), (8.122), (8.124), we deduce that if $\eta>0$ and V_γ are small enough, then for j large enough,

$$\begin{aligned} & -\int_{V_\gamma}\langle d\chi_{x,x'}\wedge\psi_{T,x',k'}\wedge *\psi_{T,x,k}\rangle_F \\ &= -\left(\frac{T}{\pi}\right)^{n/2}\int_{V_\gamma} d\chi_{x,x'}\wedge\sum_0^j\frac{\alpha_i(f_{x,k})}{T^i}\wedge *\sum_0^j\frac{\alpha_i(f_{x',k'})}{T^i}e^{-T(\varphi_x+\varphi_{x'})} \\ &\qquad + e^{-T(f(x)-f(x'))}O(1). \end{aligned} \tag{8.128}$$

Let $N_{W^u(x)/M}$, $N_{W^s(x')/M}$ be the normal bundles to $W^u(x)$, $W^s(x')$. Using Theorem 8.27 and (8.123), (8.126), (8.127), we find that

$$(8.129)\quad -\left(\frac{T}{\pi}\right)^{n/2}\int_{V_\gamma}\langle d\chi_{x,x'}\wedge\alpha_0(f_{x',k'})\wedge *\alpha_0(f_{x,k})\rangle_F e^{-T(\varphi_x+\varphi_{x'})}$$
$$= -e^{-T(f(x)-f(x'))}\Bigg(\left(\frac{T}{\pi}\right)^{1/2}\frac{1}{\pi^{(n-1)/2}}\int_\gamma\left\langle\bar f_{x',k'},\bar f^*_{x,k}\right\rangle_F d\chi_{x,x'}$$
$$\int_{N_{W^s(x')/M}|_\gamma} e^{-|\bar y|^2}d\bar y^1\wedge\cdots\wedge d\bar y^i\int_{N_{W^u(x)/M}|_\gamma} e^{-|\bar z|^2}d\bar z^1\wedge\cdots\wedge d\bar z^{n-i-1}+O(1)\Bigg).$$

We orient γ positively by the standard orientation of $[-\infty,+\infty]$, i.e. from x to x', and we denote by $\overrightarrow{\gamma}$ the corresponding oriented geodesic. One sees easily that, if $n_\gamma(x,x')$ is defined as in (1.28), then

$$(8.130)\qquad -\frac{1}{\pi^{(n-1)/2}}\int_\gamma\left\langle\bar f_{x',k'},\bar f^*_{x,k}\right\rangle_F d\chi_{x,x'}$$
$$\int_{N_{W^s(x')/M}|_\gamma} e^{-|\bar y|^2}d\bar y^1\wedge\cdots\wedge d\bar y^i\int_{N_{W^u(x)/M}|_\gamma} e^{-|\bar z|^2}d\bar z^1\wedge\cdots\wedge d\bar z^{n-i-1}$$
$$= -\int_{\overrightarrow{\gamma}}\left\langle\bar f_{x',k'},\bar f^*_{x,k}\right\rangle_F d\chi_{x,x'}n_\gamma(x,x').$$

Now recall that $f_{x',k'}$ is parallel along γ with respect to the connection ∇^F, and that $f^*_{x,k}$ is parallel along γ with respect to the connection ∇^{F*}. It follows that $\langle\bar f_{x',k'},\bar f^*_{x,k}\rangle_F$ is constant along γ. Also $-\int_{\overrightarrow{\gamma}}d\chi_{x,x'}=1$. Therefore

$$(8.131)\qquad -\int_{\overrightarrow{\gamma}}\left\langle\bar f_{x',k'},\bar f^*_{x,k}\right\rangle d\chi_{x,x'}=\langle f_{x',k'}(x),f_{x,k}\rangle_{F'_x}.$$

Also it is clear that

$$(8.132)\qquad \sum_{\gamma\in\Gamma(x,x')}\langle f_{x',k'}(x),f_{x,k}\rangle_{F_x} n_\gamma(x,x')$$
$$=\left\langle\widetilde{\partial}\left(W^u(x')^*\otimes f_{x',k'}\right),W^u(x)^*\otimes f_{x,k}\right\rangle_{C^\bullet(W^u,F)}.$$

The same argument as in (8.129) can be used to handle the other terms in (8.128). Using (8.112), (8.116), (8.128)–(8.132), we find that

$$\begin{aligned}(8.133)\qquad \left\langle d^F_T \widetilde{e}_{T,x',k'}, \widetilde{e}_{T,x,k}\right\rangle_{\mathbb{F}} &= \left(\frac{T}{\pi}\right)^{1/2} e^{-T(f(x)-f(x'))} \\ &\left(\left\langle \widetilde{\partial}\left(W^u(x')^* \otimes f_{x',k'}\right), W^u(x)^* \otimes f_{x,k}\right\rangle_{C^\bullet(W^u,F)} + O\left(\frac{1}{T^{1/2}}\right)\right),\end{aligned}$$

i.e. we still get (8.111).

The proof of Theorem 8.30 is completed. □

IX. Proof of Theorem 7.6

The purpose of this Section is to prove Theorem 7.6, i.e. to calculate the asymptotics of $T \to +\infty$ of

$$\operatorname{Tr}_s \left[N \operatorname{Log} \left(D_T^{2,]0,1]} \right) \right] + \operatorname{Log} \left(\frac{\| \ \|^{RS}_{\det H^\bullet(M,F),T}}{\| \ \|^{RS}_{\det H^\bullet(M,F)}} \right)^2 .$$

A key input is provided by Theorem 8.30, which allows us to calculate the asymptotics of the matrix of d^F on $\mathbb{F}_T^{[0,1]}$. This asymptotics contains exponentially small terms. A first step is then to modify the scalar product on $\mathbb{F}_T^{[0,1]}$ so that these exponentially small terms disappear.

Once this is done, a second key and essentially new step in the proof of Theorem 7.6 is Theorem 9.15, where the asymptotics of the scalar product on the cohomology of $(\mathbb{F}_T^{[0,1]}, d^F)$ with respect to the new scalar product on $\mathbb{F}_T^{[0,1]}$ is calculated in terms of the corresponding scalar product on the cohomology of $(C^\bullet(W^u, F), \widetilde{\partial})$. This uses again the WKB approximation of the eigenvectors of $\widetilde{D}_T^2$ associated to eigenvalues $\lambda \in [0,1]$, which was given in Section 8. The de Rham map $P_\infty : (\mathbb{F}, d^F) \to (C^\bullet(W^u, F), \widetilde{\partial})$, which identifies $H^\bullet(\mathbb{F}, d^F)$ and $H^\bullet(C^\bullet(W^u, F), \widetilde{\partial})$, appears explicitly from the analysis.

By putting together these two arguments, we establish Theorem 7.6.

This Section is organized as follows. In a), we define a new scalar product on $\mathbb{F}_T^{[0,1]}$. In b), we construct the corresponding harmonic elements in $(\mathbb{F}_T^{[0,1]}, d^F)$. In c), we establish the key Theorem 9.15, in which we calculate the asymptotics as $T \to +\infty$ of the modified scalar product on $H^\bullet(M, F)$. In d), we obtain the asymptotics of the corresponding metric on $\det H^\bullet(M, F)$. Finally, in e), we prove Theorem 7.6.

In this Section, we use the notation of Sections 1, 4, 7, 8. Again, the simplifying assumptions of Section 7 b) will be in force in the whole Section.

a) A modified scalar product on $\mathbb{F}_T^{[0,1]}$

Recall that for $T \geq 0$, the scalar product $\langle\ \rangle_{\mathbb{F},T}$ on $\mathbb{F}$ was defined in (5.2). Also the finite dimensional $\mathbb{Z}$-graded vector space $\mathbb{F}_T^{[0,1]}$ was defined in Definition 7.4. In the sequel, we will often write $\mathbb{F}_T^{[0,1],\bullet}$ instead of $\mathbb{F}_T^{[0,1]}$, to emphasize the $\mathbb{Z}$-grading.

The operator d^F acts on $\mathbb{F}_T^{[0,1],\bullet}$. Then $(\mathbb{F}_T^{[0,1],\bullet}, d^F)$ is a complex, and morever

$$H^\bullet\left(\mathbb{F}_T^{[0,1],\bullet}, d^F\right) \simeq H^\bullet(M,F). \tag{9.1}$$

Let $\langle\ ,\ \rangle_{\mathbb{F}_T^{[0,1]},T}$ be the scalar product on $\mathbb{F}_T^{[0,1]}$ induced by $\langle\ ,\ \rangle_{\mathbb{F},T}$. The operator $D_T^{2,[0,1]}$ is exactly the associated Laplacian acting on $\mathbb{F}_T^{[0,1]}$.

From (1.4) and (9.1), we deduce that

$$\det H^\bullet(M,F) \simeq \det \mathbb{F}_T^{[0,1],\bullet}. \tag{9.2}$$

The $\mathbb{Z}$-graded vector space $\widetilde{\mathbb{F}}_T^{[0,1]}$ was defined in Definition 8.14. Recall that for $T \geq 0$ large enough, for $0 \leq i \leq n$, $\{\widetilde{e}_{T,x,k}\}_{\substack{x\in B^i \\ 1\leq k\leq \mathrm{rk}(F)}}$ is the orthonormal base of $\widetilde{\mathbb{F}}_T^{[0,1],i}$ with respect to the scalar product induced by $\langle\ ,\ \rangle_{\mathbb{F}}$, which was defined in Definition 8.18.

Definition 9.1. For $T \geq 0$ large enough, $x \in B$, set

$$e_{T,x,k} = e^{Tf}\widetilde{e}_{T,x,k} \quad 1 \leq k \leq \mathrm{rk}(F). \tag{9.3}$$

By Propositions 5.3 and 5.4, for $0 \leq i \leq n$, $(e_{T,x,k})_{\substack{x\in B^i \\ 1\leq k\leq \mathrm{rk}(F)}}$ is an orthonormal base of $\mathbb{F}_T^{[0,1],i}$ with respect to the scalar product induced by $\langle\ ,\ \rangle_{\mathbb{F},T}$.

Definition 9.2. For $T \geq 0$ large enough, for $0 \leq i \leq n, x \in B^i$, let $\mathbb{F}_{T,x}^{[0,1]}$ be the vector subspace of $\mathbb{F}_T^{[0,1],i}$ spanned by $e_{T,x,1}, \cdots, e_{T,x,\mathrm{rk}(F)}$.

For $0 \leq i \leq n$, $\mathbb{F}_T^{[0,1],i}$ splits orthogonally into

$$\mathbb{F}_T^{[0,1],i} = \bigoplus_{x \in B^i} \mathbb{F}_{T,x}^{[0,1]}. \tag{9.4}$$

Definition 9.3. For $T \geq 0$ large enough, let $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$ be the scalar product on $\mathbb{F}_T^{[0,1]}$, which is such that

— The various $\mathbb{F}_{T,x}^{[0,1]}$'s are mutually orthogonal in $\mathbb{F}_T^{[0,1]}$ with respect to $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$.

— If $x \in B$, and if $\alpha, \beta \in \mathbb{F}_{T,x}^{[0,1]}$, then

$$\langle \alpha, \beta \rangle'_{\mathbb{F}_T^{[0,1]},T} = \left(\frac{\pi}{T}\right)^{\operatorname{ind}(x)-n/2} e^{2Tf(x)} \langle \alpha, \beta \rangle_{\mathbb{F},T}. \tag{9.5}$$

Definition 9.4. For $T \geq 0$ large enough, $x \in B, 1 \leq k \leq \operatorname{rk}(F)$, set

$$e'_{T,x,k} = \left(\frac{T}{\pi}\right)^{\frac{\operatorname{ind}(x)}{2}-n/4} e^{-Tf(x)} e_{T,x,k}. \tag{9.6}$$

For $x \in B$, $e'_{T,x,1}, \cdots, e'_{T,x,\operatorname{rk}(F)}$ is an orthonormal base of $\mathbb{F}_{T,x}^{[0,1]}$ with respect to the scalar product $\langle \cdot, \cdot \rangle'_{\mathbb{F}_T^{[0,1]},T}$.

Theorem 9.5. *For $0 \leq i \leq n$, if $x \in B^{i+1}, x' \in B^i$, for $1 \leq k, k' \leq \operatorname{rk}(F)$, then as $T \to +\infty$*

$$\begin{aligned} &\left\langle d^F e'_{T,x',k'}, e'_{T,x,k} \right\rangle'_{\mathbb{F}_T^{[0,1]},T} \\ &= \left\langle \widetilde{\partial}\left(W^u(x')^* \otimes f_{x',k'}\right), W^u(x)^* \otimes f_{x,k} \right\rangle_{C^\bullet(W^u,F)} + O\left(\frac{1}{T^{1/2}}\right). \end{aligned} \tag{9.7}$$

Proof. By Proposition 5.3 and by (9.5), (9.6), it is clear that

$$\begin{aligned} &\left\langle d^F e_{T,x',k'}, e_{T,x,k} \right\rangle_{\mathbb{F},T} = \left\langle d_T^F \widetilde{e}_{T,x',k'}, \widetilde{e}_{T,x,k} \right\rangle_{\mathbb{F}}, \\ &\left\langle d^F e'_{T,x',k'}, e'_{T,x,k} \right\rangle'_{\mathbb{F}_T^{[0,1]},T} = e^{T(f(x)-f(x'))} \left(\frac{\pi}{T}\right)^{1/2} \left\langle d^F e_{T,x',k'}, e_{T,x,k} \right\rangle_{\mathbb{F},T}. \end{aligned} \tag{9.8}$$

Using Theorem 8.30 and (9.8), we get (9.7). □

Definition 9.6. For $T \geq 0$ large enough, let $\mathcal{F}$ be the operator acting on $\mathbb{F}_T^{[0,1]}$ by multiplication by $f(x)$ on $\mathbb{F}_{T,x}^{[0,1]}$.

The operator $\mathcal{F}$ is self-adjoint with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F}_T^{[0,1]},T}$. Moreover, if $\alpha, \beta \in \mathbb{F}_T^{[0,1],i}$, then

$$\langle \alpha, \beta \rangle'_{\mathbb{F}_T^{[0,1]},T} = \left(\frac{\pi}{T}\right)^{i-n/2} \langle e^{T\mathcal{F}}\alpha, e^{T\mathcal{F}}\beta \rangle_{\mathbb{F}_T^{[0,1]},T}. \tag{9.9}$$

Recall that d^F and d_T^{F*} act on $\mathbb{F}_T^{[0,1]}$.

Definition 9.7. Let $d_T^{F*'}$ be the adjoint of the restriction of d^F to $\mathbb{F}_T^{[0,1]}$ with respect to the scalar product $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$.

Proposition 9.8. *The following identity of operators acting on* $\mathbb{F}_T^{[0,1]}$ *holds*

$$d_T^{F*'} = \frac{\pi}{T} e^{-2T\mathcal{F}} d_T^{F*} e^{2T\mathcal{F}}. \tag{9.10}$$

Proof. The operator $e^{T\mathcal{F}}$ is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{F}_T^{[0,1]},T}$. Using (9.9), (9.10) follows. □

Definition 9.9. For $T \geq 0$ large enough, set

$$\mathbb{F}_T'^{\{0\}} = \left\{ s \in \mathbb{F}_T^{[0,1]};\, d^F s = 0, d_T^{F*'} s = 0 \right\}. \tag{9.11}$$

Let Π_T be the orthogonal projection operator from $\mathbb{F}_T^{[0,1]}$ on $\mathbb{F}_T'^{\{0\}}$ with respect to the scalar product $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$.

In the sequel, we write often $\mathbb{F}_T'^{\{0\},\bullet}$ instead of $\mathbb{F}_T'^{\{0\}}$, to emphasize the $\mathbb{Z}$-grading.

b) The harmonic elements in $\mathbb{F}_T^{[0,1]}$ for the new scalar product

Recall that $(\mathbb{F}_T^{[0,1],\bullet}, d^F)$ is a complex. Then $\mathbb{F}_T'^{\{0\}}$ is the vector space of harmonic elements in $\mathbb{F}_T^{[0,1]}$ with respect to the scalar product $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$. By (9.1), it is clear that there is a canonical identification of $\mathbb{Z}$-graded vector spaces

$$\mathbb{F}_T'^{\{0\},\bullet} \simeq H^\bullet(M, F). \tag{9.12}$$

Recall that $P_T^{[0,1]}$ is the orthogonal projection operator from $\mathbb{F}$ on $\mathbb{F}_T^{[0,1]}$ with respect to the scalar product $\langle\ ,\ \rangle_{\mathbb{F},T}$.

Take $[\omega] \in H^\bullet(M,F)$. Let ω be any closed current on M representing $[\omega]$. Then since $P_T^{[0,1]}$ has a smooth kernel, $P_T^{[0,1]}\omega$ is well-defined and lies in $\mathbb{F}_T^{[0,1]}$.

Theorem 9.10. *For $T \geq 0$ large enough, if $[\omega] \in H^\bullet(M,F)$, if ω is a closed current on M representing $[\omega]$, $\Pi_T P_T^{[0,1]}\omega$ only depends on $[\omega]$. The map*

$$[\omega] \in H^\bullet(M,F) \to \Pi_T P_T^{[0,1]}\omega \in \mathbb{F}_T'^{\{0\}} \tag{9.13}$$

is in fact the canonical isomorphism $H^\bullet(M,F) \simeq \mathbb{F}_T'^{\{0\}}$.

Proof. Let $\mathcal{D}'(M,F)$ be the vector space of currents on M with values in F. The map $P_T^{[0,1]} : (\mathcal{D}'(M,F), d^F) \to (\mathbb{F}_T^{[0,1]}, d^F)$ is a quasi-isomorphism of complexes. Our Theorem is now obvious. □

If $[\omega] \in H^\bullet(M,F)$ is taken as in Theorem 9.10, we will write $\Pi_T P_T^{[0,1]}[\omega]$ instead of $\Pi_T P_T^{[0,1]}\omega$.

Recall that the scalar product $\langle\ ,\ \rangle_{C^\bullet(W^u,F)}$ on $C^\bullet(W^u,F)$ was defined in Section 8i).

Definition 9.11. Let $\widetilde{\partial}^*$ be the adjoint of $\widetilde{\partial}$ with respect to the scalar product $\langle\ ,\ \rangle_{C^\bullet(W^u,F)}$ on $C^\bullet(W^u,F)$. Set

$$C^{\{0\},\bullet}(W^u,F) = \left\{h \in C^\bullet(W^u,F), \widetilde{\partial}h = 0, \widetilde{\partial}^*h = 0\right\}. \tag{9.14}$$

By Hodge theory, we have a canonical identification of $\mathbb{Z}$-graded vector spaces

$$C^{\{0\},\bullet}(W^u,F) \simeq H^\bullet\left(C^\bullet(W^u,F), \widetilde{\partial}\right). \tag{9.15}$$

Definition 9.12. Let Π_∞ be the orthogonal projection operator from $C^\bullet(W^u,F)$ on $C^{\{0\},\bullet}(W^u,F)$ with respect to the scalar product $\langle\ ,\ \rangle_{C^\bullet(W^u,F)}$.

Recall that if $\alpha \in \mathbb{F}, P_\infty\alpha \in C^\bullet(W^u,F)$ was defined in Definition 2.8 by

$$P_\infty\alpha = \sum_{x\in B} W^u(x)^* \otimes \int_{W^u(x)} \alpha. \tag{9.16}$$

Theorem 9.13. *If* $[\omega] \in H^\bullet(M,F)$ *and if* $\omega \in \mathbb{F}$ *is a smooth closed form representing* $[\omega]$, $\Pi_\infty P_\infty \omega$ *only depends on* $[\omega]$. *The map*

$$[\omega] \in H^\bullet(M,F) \to \Pi_\infty P_\infty \omega \in C^{\{0\},\bullet}(W^u,F) \tag{9.17}$$

provides the canonical isomorphism $H^\bullet(M,F) \simeq C^{\{0\},\bullet}(W^u,F)$.

Proof. By Theorem 2.9, the map $\alpha \in (\mathbb{F}, d^F) \to P_\infty \alpha \in (C^\bullet(W^u,F), \widetilde{\partial})$ is a quasi-isomorphism. Our Theorem is now obvious. □

If $\omega, [\omega]$ are taken as in Theorem 9.13, we will write $\Pi_\infty P_\infty[\omega]$ instead of $\Pi_\infty P_\infty \omega$.

Remark 9.14. The class of closed currents ω to which Theorem 9.13 applies is larger than the smooth ones.

c) The asymptotics as $T \to +\infty$ of the modified scalar product on $H^\bullet(M,F)$.

The following result is one of the essential results of this Section.

Theorem 9.15. *For any* $[\omega]$, $[\omega'] \in H^\bullet(M,F)$, *then*

(9.18)

$$\lim_{T\to+\infty} \left\langle \Pi_T P_T^{[0,1]}[\omega], \Pi_T P_T^{[0,1]}[\omega'] \right\rangle'_{\mathbb{F}_T^{[0,1]},T} = \langle \Pi_\infty P_\infty[\omega], \Pi_\infty P_\infty[\omega'] \rangle_{C^\bullet(W^u,F)} .$$

Proof. Take $i, 0 \le i \le n$, and assume that $\deg[\omega] = \deg[\omega'] = i$. Let $\omega, \omega' \in \mathbb{F}^i$ be smooth closed representatives of $[\omega], [\omega']$. Clearly, for $T \ge 0$ large enough,

$$P_T^{[0,1]}[\omega] = \sum_{\substack{x \in B^i \\ 1 \le k \le \mathrm{rk}(F)}} \left(\int_M \langle \omega \wedge * e_{T,x,k} \rangle_F \, e^{-2Tf} \right) e_{T,x,k}. \tag{9.19}$$

Using (9.3), (9.6), (9.19), we see that

(9.20)

$$P_T^{[0,1]}[\omega] = \sum_{\substack{x \in B^i \\ 1 \le k \le \mathrm{rk}(F)}} \left(\frac{T}{\pi} \right)^{n/4 - i/2} \left(\int_M \langle \omega \wedge * \widetilde{e}_{T,x,k} \rangle_F \, e^{-T(f - f(x))} \right) e'_{T,x,k},$$

and so,
(9.21)
$$\Pi_T P_T^{[0,1]}[\omega] = \sum_{\substack{x\in B^i \\ 1\le k\le \mathrm{rk}(F)}} \left(\frac{T}{\pi}\right)^{n/4-i/2} \left(\int_M \langle \omega \wedge *\widetilde{e}_{T,x,k}\rangle_F \, e^{-T(f-f(x))}\right) \Pi_T e'_{T,x,k}.$$

Let $\overline{W^{u,i-1}}$ be the union of the cells $\overline{W^u(x)}, x \in B, \mathrm{ind}(x) \le i-1$. Then, the class $[\omega]$ can be represented by a smooth closed form on M which vanishes on an open neighborhood V of $\overline{W^{u,i-1}}$. In effect by Proposition 7 by Laudenbach in the Appendix, $[\omega]$ can be represented by a current γ which is a linear combination of the $g\delta_{\overline{W^s}(x)}$ (where $x \in B^i$ and g is a flat section of $F_{|\overline{W^s}(x)}$). By de Rham regularization [Rh2, Chapter XV], we obtain a closed form $\omega \in \mathbb{F}^i$ which has the required property. Another simple proof of this fact is as follows. Assume temporarily that f is a nice function. Then with the notation of Remark 1.8, $H^i(V_{i-1}, F) = 0$. So any closed form in $\mathbb{F}^i$ is exact on V_{i-1}. This implies that $[\omega]$ can be represented by $\omega \in \mathbb{F}^i$ having the required property. In the sequel we assume that ω is chosen in this way.

Recall that by (8.62), if $x \in B$,

$$\widetilde{e}_{T,x,k} = \widetilde{O}\left(e^{-\varphi_x T}\right) \quad , \quad 1 \le k \le \mathrm{rk}(F). \tag{9.22}$$

Also by [HSj4, Lemma A.2.1], if $t \in M$,

$$\varphi_x(t) + f(t) - f(x) \ge 0. \tag{9.23}$$

By Proposition 8.28, if there is equality in (9.23), then $t \in \overline{W^u(x)}$.

Let $\mathcal{W}_x$ be an open neigborhood of $\overline{W^u(x)}$ in M. From (9.22), (9.23), we deduce that there exists $c > 0$ such that for $x \in B^i$,

$$\begin{aligned} &\left(\frac{T}{\pi}\right)^{n/4-i/2} \int_M \langle \omega \wedge *\widetilde{e}_{T,x,k}\rangle \, e^{-T(f-f(x))} \\ &= \left(\frac{T}{\pi}\right)^{n/4-i/2} \int_{\mathcal{W}_x} \langle \omega \wedge *\widetilde{e}_{T,x,k}\rangle \, e^{-T(f-f(x))} + \widetilde{O}\left(e^{-cT}\right). \end{aligned} \tag{9.24}$$

Recall that δ_x was defined in (8.55). By [HSj2, Section 2.1] and [HSj4, eq. (3.12)], we know that

$$\widetilde{e}_{T,x,k} - v_{T,x,k} = \widetilde{O}\left(e^{-\delta_x T}\right). \tag{9.25}$$

Using (8.56) and (9.25), we get

$$\widetilde{e}_{T,x,k} - \psi_{T,x,k} = \widetilde{O}\left(e^{-\delta_x T}\right). \tag{9.26}$$

By [Ro, Lemma 1] or by Proposition 2 in the Appendix, we know that $\overline{W^u(x)}$ is obtained from $W^u(x)$ by adding certain $\overline{W^u(x')} \subset \overline{W^{u,i-1}}$. So we find that $\overline{W^u(x)} \backslash V \subset W^u(x)$. Moreover $\overline{W^u(x)}\backslash V$ is compact. Therefore there exists $\alpha > 0$ such that

$$\delta_x \geq \varphi_x + \alpha \quad \text{on} \quad \overline{W^u(x)}\backslash V. \tag{9.27}$$

So if $\mathcal{W}_x$ is small enough,

$$\delta_x \geq \varphi_x + \alpha/2 \quad \text{on} \quad \mathcal{W}_x \backslash V. \tag{9.28}$$

By using (9.26), (9.28) and [HSj1, Theorem 5.8] as in (8.120), we find that if $\eta > 0$ and $\mathcal{W}_x$ are small enough, then

(9.29)
$$\left\| e^{T\varphi_x}\left(\widetilde{e}_{T,x,k} - \left(\frac{T}{\pi}\right)^{n/4} \sum_0^j \left(\frac{\alpha_i\left(f_{x,k}\right)}{T^i}\right)\right)\right\|_{\mathbb{F}_{\mathcal{W}_x \backslash V,0}} = O\left(\frac{1}{T^{j+1-n/4}}\right).$$

Recall that ω vanishes on V. Using (8.71), (9.29), we get for j large enough,

$$\begin{aligned} &\left(\frac{T}{\pi}\right)^{n/4-i/2} \int_{\mathcal{W}_x} \langle \omega \wedge * \widetilde{e}_{T,x,k}\rangle_F\, e^{-T(f-f(x))} \\ &= \left(\frac{T}{\pi}\right)^{n/2-i/2}\left[\int_{\mathcal{W}_x}\left\langle \omega \wedge * \sum_0^j \left(\frac{\alpha_i\left(f_{x,k}\right)}{T^i}\right)\right\rangle_F e^{-2Tf_x^+}\right] + O\left(\frac{1}{\sqrt{T}}\right). \end{aligned} \tag{9.30}$$

We use now the coordinates $(\overline{y}^{\operatorname{ind}(x)+1}, \cdots, \overline{y}^n)$ transverse to $W^u(x)$ which were constructed in Section 8h). By using Theorem 8.27 and by (8.126) we find that as $T \to +\infty$,

$$\left(\frac{T}{\pi}\right)^{n/2-i/2} \int_{\mathcal{W}_x} \langle \omega \wedge * \alpha_0(f_{x,k})\rangle\, e^{-2Tf_x^+} \to \int_{W^u(x)} \left\langle \omega, \overline{f}^*_{x,k}\right\rangle_F. \tag{9.31}$$

Over $W^u(x)$, $\overline{f}^*_{x,k}$ is parallel with respect to the connection ∇^{F*}. Then, we see that

$$\int_{W^u(x)} \left\langle \omega, \overline{f}^*_{x,k}\right\rangle_F = \left\langle \int_{W^u(x)} \omega, f_{x,k}\right\rangle_{F_x}. \tag{9.32}$$

The other terms in the sum appearing in the right-hand side of (9.30) can be dealt with in the same way as in (9.31). Using (9.24), (9.30)–(9.32), we find that as $T \to +\infty$,

$$\left(\frac{T}{\pi}\right)^{n/4-i/2} \int_M \langle \omega \wedge *\widetilde{e}_{T,x,k}\rangle_F \, e^{-T(f-f(x))} \to \left\langle \int_{W^u(x)} \omega, f_{x,k} \right\rangle_{F_x}. \tag{9.33}$$

Let $\underline{d}^F$ be the matrix of d^F with respect to the base $(e'_{T,x,k})_{\substack{x\in B\\ 1\le k\le \mathrm{rk}(F)}}$ of $\mathbb{F}_T^{[0,1]}$, and let $\underline{\widetilde{\partial}}$ be the matrix of $\widetilde{\partial}$ with respect to the base $(W^u(x)^* \otimes f_{x,k})_{\substack{x\in B\\ 1\le k\le \mathrm{rk}(F)}}$ of $C^\bullet(W^u,F)$. Then by Theorem 9.5, as $T \to +\infty$,

$$\underline{d}^F = \underline{\widetilde{\partial}} + O\left(\frac{1}{T^{1/2}}\right). \tag{9.34}$$

Moreover, and this is *essential*, by Theorem 1.16 and by (9.1), the complexes $(\mathbb{F}_T^{[0,1]}, d^F)$ and $(C^\bullet(W^u,F), \widetilde{\partial})$ have the same Betti numbers. Let $\underline{\Pi}_T$ be the matrix of Π_T with respect to the base $(e'_{T,x,k})_{\substack{x\in B\\ 1\le k\le \mathrm{rk}(F)}}$, and let $\underline{\Pi}_\infty$ be the matrix of Π_∞ with respect to the base $(W^u(x)^* \otimes f_{x,k})_{\substack{x\in B\\ 1\le k\le \mathrm{rk}(F)}}$. It follows from (9.34) that as $T \to +\infty$,

$$\underline{\Pi}_T \to \underline{\Pi}_\infty. \tag{9.35}$$

Let ω' be a smooth closed form of degree i representing $[\omega']$ and verifying the same support conditions as ω. The obvious analogue of (9.33) still holds. Using (9.21), (9.33), (9.35), we find that

$$\begin{aligned} &\lim_{T\to+\infty} \langle \Pi_T P_T[\omega], \Pi_T P_T[\omega']\rangle'_{\mathbb{F}_T^{[0,1]},T} \\ &= \Bigg\langle \sum_{\substack{x\in B^i\\ 1\le k\le \mathrm{rk}(F)}} \left\langle \int_{W^u(x)} \omega, f_{x,k} \right\rangle_{F_x} \Pi_\infty \left(W^u(x)^* \otimes f_{x,k}\right), \\ &\sum_{\substack{x'\in B^i\\ 1\le k'\le \mathrm{rk}(F)}} \left\langle \int_{W^u(x')} \omega', f_{x',k'} \right\rangle_{F_x} \Pi_\infty \left(W^u(x')^* \otimes f_{x',k'}\right) \Bigg\rangle_{C^\bullet(W^u,F)}, \end{aligned} \tag{9.36}$$

which is equivalent to (9.18). □

d) The asymptotics of the modified metric on $\det H^\bullet(M,F)$

Definition 9.16. Let $\| \ \|_{\det \mathbb{F}_T^{[0,1],\bullet},T}$ be the metric on the line $\det \mathbb{F}_T^{[0,1],\bullet}$ associated to the scalar product $\langle\ ,\ \rangle_{\mathbb{F}_T^{[0,1]},T}$ on $\mathbb{F}_T^{[0,1]}$. For $T \geq 0$ large enough, let $\| \ \|'_{\det \mathbb{F}_T^{[0,1],\bullet},T}$ be the metric on the line $\det \mathbb{F}_T^{[0,1],\bullet}$ associated to the scalar product $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$ on $\mathbb{F}_T^{[0,1],\bullet}$. Let $\| \ \|^{\sim}_{\det H^\bullet(M,F),T}$, $\| \ \|^{\sim,\prime}_{\det H^\bullet(M,F),T}$ be the metrics on the line $\det H^\bullet(M,F)$ corresponding to the metrics $\| \ \|_{\det \mathbb{F}_T^{[0,1],\bullet},T}$, $\| \ \|'_{\det \mathbb{F}_T^{[0,1],\bullet},T}$ via the canonical isomorphism $\det H^\bullet(M,F) \simeq \det \mathbb{F}_T^{[0,1],\bullet}$.

Proposition 9.17. *For any $T \geq 0$, the following identity holds*

(9.37)
$$\operatorname{Log}\left(\frac{|\ |^{RS}_{\det H^\bullet(M,F),T}}{|\ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 + \operatorname{Tr_s}\left[N \operatorname{Log}\left(D_T^{2,]0,1]}\right)\right] = \operatorname{Log}\left(\frac{\|\ \|^{\sim}_{\det H^\bullet(M,F),T}}{|\ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 .$$

Proof. Using [BGS1, Proposition 1.5], (9.37) follows. □

Proposition 9.18. *For $T \geq 0$ large enough, the following identity holds,*

(9.38)
$$\operatorname{Log}\left(\frac{\|\ \|^{\sim,\prime}_{\det H^\bullet(M,F),T}}{\|\ \|^{\sim}_{\det H^\bullet(M,F),T}}\right)^2 = 2\operatorname{rk}(F)\operatorname{Tr}_s^B[f]T + \left(\frac{n}{2}\chi(F) - \widetilde{\chi}'(F)\right)\operatorname{Log}\left(\frac{T}{\pi}\right).$$

Proof. This follows trivially from (9.9). □

The following result is now crucial.

Theorem 9.19. *The following identity holds*

$$\lim_{T\to+\infty} \operatorname{Log}\left(\frac{\|\ \|^{\sim,\prime}_{\det H^\bullet(M,F),T}}{|\ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 = \operatorname{Log}\left(\frac{\|\ \|^{\mathcal{M},\nabla f}}{|\ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 . \tag{9.39}$$

Proof. Recall that the vector space $\mathbb{F}_T^{\prime\{0\}}$ was defined in (9.11). By (9.12), we get

$$\det \mathbb{F}_T^{\prime\{0\}} \simeq \det H^\bullet(M,F). \tag{9.40}$$

Let $|\ \ |_{\det \mathbb{F}_T^{\prime\{0\}},T}$ be the metric on the line $\det \mathbb{F}_T^{\prime\{0\}}$ induced by the scalar product $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$ restricted to $\mathbb{F}_T^{\prime\{0\}}$. Let $|\ \ |^{\sim,\prime}_{\det H^\bullet(M,F),T}$ be the corresponding metric on the line $\det H^\bullet(M,F)$ via the canonical isomorphism (9.40).

Let D'_T be the operator acting on $\mathbb{F}_T^{[0,1]}$,

$$D'_T = d_T^F + d_T^{F*\prime}. \tag{9.41}$$

Then D'_T is self-adjoint with respect to the metric $\langle\ ,\ \rangle'_{\mathbb{F}_T^{[0,1]},T}$. Also (9.11) says that

$$\mathbb{F}_T^{\prime\{0\}} = \operatorname{Ker} D'_T. \tag{9.42}$$

Let $D_T^{\prime 2,>0}$ be the restriction of $D_T^{\prime 2}$ to the nonzero eigenspaces of $D_T^{\prime 2}$. By [BGS1, Proposition 1.5], we know that

(9.43)

$$\operatorname{Log}\left(\frac{\|\ \ \|^{\sim,\prime}_{\det H^\bullet(M,F),T}}{|\ \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 = \operatorname{Log}\left(\frac{|\ \ |^{\sim,\prime}_{\det H^\bullet(M,F),T}}{|\ \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 + \operatorname{Tr_s}\left[N \operatorname{Log}\left(D_T^{\prime 2,>0}\right)\right].$$

Recall that $\mathbb{F}^{\{0\}}$ was defined in (2.4). Clearly $\mathbb{F}^{\{0\}} = \mathbb{F}_0^{\{0\}}$. By Theorem 9.10, for $T \geq 0$ large enough, the linear map

$$\omega \in \mathbb{F}^{\{0\}} \to \Pi_T P_T^{[0,1]} \omega \in \mathbb{F}_T^{\prime\{0\}} \tag{9.44}$$

is one to one and provides the canonical isomorphism of $\mathbb{F}^{\{0\}}$ with $\mathbb{F}_T^{\prime\{0\}}$. By Theorem 2.9, the linear map

$$\omega \in \mathbb{F}^{\{0\}} \to \Pi_\infty P_\infty \omega \in C^{\{0\}}(W^u, F) \tag{9.45}$$

is one to one and provides the canonical isomorphism of $\mathbb{F}^{\{0\}}$ with $C^{\{0\}}(W^u, F)$.

Let $|\ \ |_{\det C^{\{0\},\bullet}(W^u,F)}$ be the metric on the line $\det C^{\{0\},\bullet}(W^u,F)$ induced by the scalar product $\langle\ ,\ \rangle_{C^\bullet(W^u,F)}$. Let $|\ \ |^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}$ be the corresponding metric on the line $\det H^\bullet(M,F)$. Using Theorem 9.15, it is clear that

$$\lim_{T\to+\infty} \operatorname{Log}\left(\frac{|\ \ |^{\sim,\prime}_{\det H^\bullet(M,F),T}}{|\ \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 = \operatorname{Log}\left(\frac{|\ \ |^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}{|\ \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2. \tag{9.46}$$

Let $\underline{D}_T^{\prime 2}$ be the matrix of $D_T^{\prime 2}$ with respect to the orthonormal base $\{e'_{T,x,k}\}_{\substack{x\in B\\ 1\leq k\leq \mathrm{rk}(F)}}$ of $\mathbb{F}_T^{[0,1]}$. Set

$$D' = \widetilde{\partial} + \widetilde{\partial}^*. \tag{9.47}$$

Then,

$$C^{\{0\}}(W^u, F) = \operatorname{Ker} D'. \tag{9.48}$$

Let $D'^{2,>0}$ be the restriction of D'^2 to the eigenspaces of D'^2 associated to positive eigenvalues. By [BGS1, Proposition 1.5], we know that

$$\operatorname{Log}\left(\frac{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}{| \ |^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}\right)^2 = \operatorname{Tr_s}\left[N \operatorname{Log}\left(D'^{2,>0}\right)\right]. \tag{9.49}$$

Let $\underline{D}'^2$ be the matrix of D'^2 with respect to the orthonormal base $\{W^u(x)^* \otimes f_{x,k}\}_{\substack{x\in B\\ 1\leq k\leq \mathrm{rk}(F)}}$ of $C^\bullet(W^u,F)$. By Theorem 9.5, it is clear that as $T\to+\infty$,

$$\underline{D}_T^{\prime 2} \to \underline{D}'^2. \tag{9.50}$$

Also for $T>0$ large enough, the $\mathbb{Z}$-graded kernels of the matrices $\underline{D}_T^{\prime 2}$ and $\underline{D}'^2$ have the same dimension. From (9.50), we deduce that as $T\to+\infty$,

$$\operatorname{Tr_s}\left[N \operatorname{Log}\left(D_T^{\prime 2,>0}\right)\right] \to \operatorname{Tr_s}\left[N \operatorname{Log}\left(D'^{2,>0}\right)\right]. \tag{9.51}$$

Using (9.43), (9.46), (9.49), (9.51), we get (9.39). □

e) Proof of Theorem 7.6

We now prove Theorem 7.6, which we restate for convenience.

Theorem 9.20. *As* $T\to+\infty$,

$$\lim_{T\to+\infty}\left\{\operatorname{Tr_s}\left[N\operatorname{Log}\left(D_T^{2,]0,1]}\right)\right] + \operatorname{Log}\left(\frac{| \ |^{RS}_{\det H^\bullet(M,F),T}}{| \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2 \right. \tag{9.52}$$

$$\left. +2\operatorname{rk}(F)\operatorname{Tr_s^B}[f]T + \left(\frac{n}{2}\chi(F) - \widetilde{\chi}'(F)\right)\operatorname{Log}\left(\frac{T}{\pi}\right)\right\} = \operatorname{Log}\left(\frac{\| \ \|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F)}}{| \ |^{RS}_{\det H^\bullet(M,F)}}\right)^2.$$

Proof. This follows from Propositions 9.17 and 9.18 and from Theorem 9.19. □

X. The asymptotics as $T \to +\infty$ of certain traces associated to the operator D_T^2

The purpose of this Section is to establish Theorems 7.7, 7.8 and 7.9. These results concern the asymptotics as $T \to +\infty$ or $t \to +\infty$ of supertraces involving the operator $\exp(-tD_T^2)$ and also the asymptotics of the eigenvalues $\lambda \in [0,1]$ of D_T^2.

To establish these results, we use the techniques of [BL2, Sections 8 and 9], where a much more difficult problem was considered.

This Section is organized as follows. In a), we describe the operator $\widetilde{D}_T$ near B. In b), following [BL2], we prove Theorem 7.7, in c), we establish Theorem 7.8, and in d), we prove Theorem 7.9.

a) The operator $\widetilde{D}_T$ near B

By (5.12), we know that

$$\widetilde{D}_T = D + T\widehat{c}(\nabla f), \tag{10.1}$$

and so,

$$\widetilde{D}_T^2 = D^2 + T\,[D, \widehat{c}(\nabla f)] + T^2 |df|^2. \tag{10.2}$$

Observe that by (5.17), $[D, \widehat{c}(\nabla f)]$ is a matrix valued operator, i.e. an operator of order 0.

Also, $|df|^2$ is positive on $M \backslash B$. Therefore the situation is formally identical to the one described by Bismut and Lebeau in [BL2], with Y replaced by B and V^2 by $|df|^2$. We will pursue this analogy further.

Take $i, 0 \leq i \leq n$. We equip $\mathbb{R}^n$ with its canonical scalar product, and we identify $\mathbb{R}^n$ and $\mathbb{R}^{n*}$ by the scalar product. We split $\mathbb{R}^n$ orthogonally into

$$\mathbb{R}^n = \mathbb{R}^i \oplus \mathbb{R}^{n-i}. \tag{10.3}$$

Then

$$\Lambda(\mathbb{R}^{n*}) = \Lambda(\mathbb{R}^{i*}) \widehat{\otimes} \Lambda\left(\mathbb{R}^{(n-i)*}\right). \tag{10.4}$$

Let N, N^-, N^+ be the number operators on $\Lambda(\mathbb{R}^n)$, $\Lambda(\mathbb{R}^{i*})$, $\Lambda(\mathbb{R}^{(n-i)*})$, so that

$$N = N^+ + N^-. \tag{10.5}$$

If $y \in \mathbb{R}^n$, we write y in the form

$$y = y^- + y^+ \ ; \quad y^- \in \mathbb{R}^i \ , \quad y^+ \in \mathbb{R}^{n-i}. \tag{10.6}$$

Let $\mathbf{F}$ be the vector space of smooth sections of $\Lambda(\mathbb{R}^n) \otimes \mathbb{R}^k$ over $\mathbb{R}^n$. Let $\mathbf{F}_0$ be the space of square-integrable sections of $\Lambda(\mathbb{R}^n) \otimes \mathbb{R}^k$ over $\mathbb{R}^n$. We equip $\mathbf{F}_0$ with the scalar product

$$\alpha, \beta \in \mathbf{F}_0 \to \langle \alpha, \beta \rangle_{\mathbf{F}_0} = \int_{\mathbb{R}^n} \langle \alpha \wedge * \beta \rangle_{\mathbb{R}^k}. \tag{10.7}$$

The operator $d + (y^+ - y^-)\wedge$ acts on $\mathbf{F}$. Its formal adjoint with respect to the scalar product (10.7) is the operator $d^* + i_{(y^+ - y^-)}$. Set

$$\widetilde{D}^{\mathbb{R}^n} = d + \left(y^+ - y^-\right) \wedge + d^* + i_{(y^+ - y^-)}. \tag{10.8}$$

Let $\Delta^{\mathbb{R}^n}$ be the flat Laplacian on $\mathbb{R}^n$. By Proposition 8.2, we know that

$$\left(\widetilde{D}^{\mathbb{R}^n}\right)^2 = -\Delta^{\mathbb{R}^n} + |y|^2 - n + 2\left(N^+ + i - N^-\right). \tag{10.9}$$

Let ρ be the volume form of $\mathbb{R}^i$ with respect to the Euclidean scalar product of $\mathbb{R}^i$ equipped with its canonical orientation.

Proposition 10.1. *The kernel of the operator* $(\widetilde{D}^{\mathbb{R}^n})^2$ *is of dimension* k. *If* $f_1, \cdots, f_k$ *is an orthonormal base of* $\mathbb{R}^k$, *then* $\operatorname{Ker}(\widetilde{D}^{\mathbb{R}^n})^2$ *is spanned by* $\frac{1}{\pi^{n/4}} e^{-\frac{|y|^2}{2}} \rho \otimes$

$f_1, \cdots, \frac{1}{\pi^{n/4}} e^{-\frac{|y|^2}{2}} \rho \otimes f_k$. *Moreover if* $f \in \mathbb{R}^k$, *then*

$$(10.10) \qquad \begin{aligned} &\left(d + \left(y^+ - y^-\right) \wedge\right) \left(\frac{e^{-\frac{|y|^2}{2}}}{\pi^{n/4}} \rho \otimes f \right) = 0. \\ &\left(d^* + i_{(y^+ - y^-)}\right) \left(\frac{e^{-\frac{|y|^2}{2}}}{\pi^{n/4}} \rho \otimes f \right) = 0. \end{aligned}$$

Proof. The first part of our Proposition was already established in Proposition 8.3. Moreover (10.10) follows from an easy direct computation. □

b) Proof of Theorem 7.7

By Proposition 5.4,

$$(10.11) \qquad \operatorname{Tr_s}\left[N \exp\left(-tD_T^2\right)\right] = \operatorname{Tr_s}\left[N \exp\left(-t\widetilde{D}_T^2\right)\right].$$

In view of (10.2) and of Proposition 10.1, we see that the situation is formally similar to the corresponding situation in Bismut-Lebeau [BL2, Theorems 6.4 and 8.3]. Of course it is much simpler here, since the set $B = \{y, |df|^2(y) = 0\}$ is finite, while its analogue Y in [BL2] is a union of submanifolds. Also by Proposition 8.2, if $x \in B$, the operator $\widetilde{D}_T^2$ is exactly an harmonic oscillator on a whole neighborhood of x, while in [BL2], only the corresponding infinitesimal analogue is true. Since B consists of isolated points, the analogue of the operator D^Y in [BL2] is the zero operator acting on $\bigoplus_{x \in B} F_x$.

So by proceeding as in [BL2, Section 9], we find that for ε, A with $0 < \varepsilon < A < +\infty$, there exist $c > 0, C > 0$ such that if $\varepsilon \le t \le A, T \ge 1$, then

$$(10.12) \qquad \left| \operatorname{Tr_s}\left[N \exp\left(-t\widetilde{D}_T^2\right)\right] - \operatorname{rk}(F) \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \operatorname{ind}(x) \right| \le \frac{C}{\sqrt{T}}.$$

Using (10.11), (10.12), we get

$$(10.13) \qquad \left| \operatorname{Tr_s}\left[N \exp\left(-tD_T^2\right)\right] - \widetilde{\chi}'(F) \right| \le \frac{C}{\sqrt{T}},$$

which is exactly Theorem 7.7.

c) Proof of Theorem 7.8

Recall that $\widetilde{P}_T^{[0,1]}$ was defined in Definition 8.14. By Proposition 5.4, we get

$$\mathrm{Tr}_s\left[N\exp\left(-tD_T^2\right)P_T^{]1,+\infty[}\right]=\mathrm{Tr}_s\left[N\exp\left(-t\widetilde{D}_T^2\right)\widetilde{P}_T^{]1,+\infty[}\right]. \tag{10.14}$$

Let $\Delta=\Delta_+\cup\Delta_-$ be the oriented contour in $\mathbb{C}$

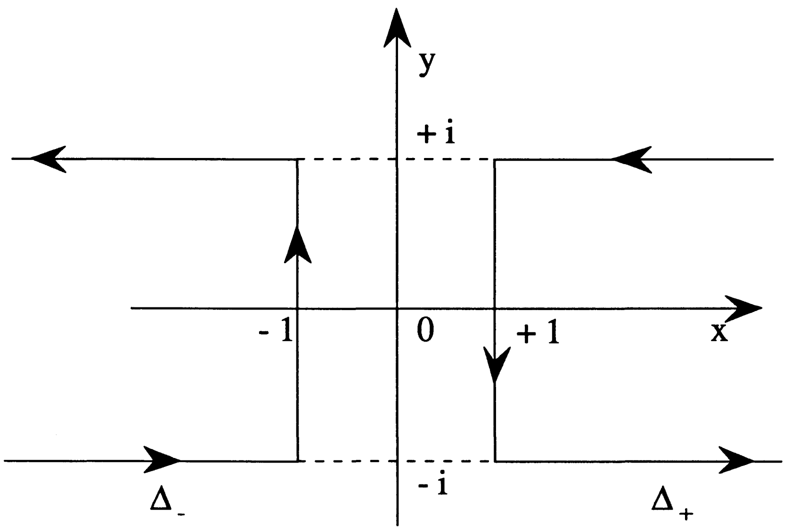

Figure 2

The analogue of the operator D^Y in [BL2] is the zero operator acting on $\bigoplus_{x\in B}F_x$. By the analogue of [BL2, Theorem 9.25], we find that for $T\geq 0$ large enough,

$$\mathrm{Sp}\left(\widetilde{D}_T\right)\cap\Delta=\emptyset. \tag{10.15}$$

Take $p\in\mathbb{N}, p\geq n+2$. Let f_p be the unique holomorphic function defined on $\mathbb{C}\backslash\sqrt{-1}\,\mathbb{R}$ with values in $\mathbb{C}$, which has the following properties :

— As $\lambda\to\pm\infty, f_p(\lambda)\to 0$.

— The following identity holds

$$\frac{f_p^{(p-1)}(\lambda)}{(p-1)!}=\exp\left(-\lambda^2\right). \tag{10.16}$$

Using (10.15), we see that for $T \geq 0$ large enough,

$$\exp\left(-t\widetilde{D}_T^2\right)\widetilde{P}_T^{]1,+\infty[} = \frac{1}{2\pi i}\int_\Delta \exp\left(-t\lambda^2\right)\left(\lambda - \widetilde{D}_T\right)^{-1} d\lambda. \tag{10.17}$$

Equivalently

$$\exp\left(-t\widetilde{D}_T^2\right)\widetilde{P}_T^{]1,+\infty[} = \frac{1}{2\pi i}\int_\Delta \frac{f_p\left(\sqrt{t}\lambda\right)}{\left(\sqrt{t}\right)^{p-1}}\left(\lambda - \widetilde{D}_T\right)^{-p} d\lambda. \tag{10.18}$$

Also

$$\int_\Delta \frac{f_p\left(\sqrt{t}\lambda\right)}{\left(\sqrt{t}\right)^{p-1}}\lambda^{-p} d\lambda = 0 \tag{10.19}$$

Using (10.18), (10.19) and by proceeding as in [BL2, Section 9g)], we find that (7.18) holds. Also by proceeding as in [BL2, Section 9h)], we get (7.19). The proof of Theorem 7.9 is completed. □

d) Proof of Theorem 7.9

Let $D_T^{2,i}$ be the restriction of D_T^2 to $\mathbb{F}^i$. Recall that $M^i = \mathrm{card}(B^i)$. By using Proposition 10.1 and by proceeding as in [BL2, Section 9], we see that for any $t > 0$,

$$\lim_{T\to+\infty} \mathrm{Tr}\left[\exp\left(-tD_T^{2,i}\right)\right] = \mathrm{rk}(F)M^i. \tag{10.20}$$

From (10.20), and from elementary properties of the Laplace transform, (7.20) and (7.21) follow. The proof of Theorem 7.9 is completed. □

Remark 10.12. To prove Theorems 7.8 and 7.9, one can also proceed as in [BL2, proof of Theorem 9.25], by using in particular the analogue of [BL2, eq. (9.154), (9.155)]. However the conclusions of [BL2, Theorem 9.25] are not valid any more. In [BL2, Theorem 9.25], one shows that for $T \geq 0$ large enough, if $\lambda \in \mathbb{R}$ is an eigenvalue of the analogue of $\widetilde{D}_T^2$ which is such that $|\lambda| \leq 1$, then $\lambda = 0$. This follows from a purely algebraic argument, which has no equivalent here. In general, Morse inequalities are indeed inequalities and not equalities.

Theorem 7.9 can also be proved by using the much stronger Theorems 8.5 and 8.15.

XI. The asymptotics of $\mathrm{Tr_s}[N\exp(-tD^2)]$ as $t \to 0$

The purpose of this Section is to prove Theorem 7.10, i.e. to calculate the asymptotics as $t \to 0$ of $\mathrm{Tr_s}[N\exp(-tD^2)]$. This asymptotics has already been obtained by Dai and Melrose [D] in the case where the metric g^F is flat.

We will obtain Theorem 7.10 as a trivial consequence of Theorem 4.20.

Here we make the same assumptions as in Section 2, i.e. we may work with an arbitrary metric g^{TM} on TM.

We use the notation of Section 4. Let $e_1, \cdots, e_n$ be an orthonormal base of TM. Then one has the trivial

$$N = \frac{1}{2}\sum_1^n c(e_i)\widehat{c}(e_i) + \frac{n}{2}. \tag{11.1}$$

By proceeding as in the proof of Theorem 4.20 (and more specifically as in (4.55)–(4.63)), we find easily that if n is odd

(11.2)

$$\lim_{t\to 0} \sqrt{t}\,\mathrm{Tr_s}\left[\left(\frac{1}{2}\sum_1^n c(e_i)\widehat{c}(e_i)\right)\exp\left(-tD^2\right)\right] = \mathrm{rk}(F)\int_M \int^B L\exp\left(-\frac{\dot{R}^{TM}}{2}\right).$$

If n is odd, using standard results on asymptotic expansion of traces of heat kernels, we get the second identity in (7.22).

We now assume that n is even. In view of Theorem 4.14, of Proposition 4.15 and of equation (4.74) in the proof of Theorem 4.20, it is clear that

$$\lim_{t\to 0} \mathrm{Tr_s}\left[\left(\frac{1}{2}\sum_1^n c(e_i)\widehat{c}(e_i)\right)\exp\left(-tD^2\right)\right] \tag{11.3}$$

$$= -\int_M \left\{ \int^B \nabla^{TM} \left(\frac{L}{2}\right) \exp\left(-\frac{\dot{R}^{TM}}{2}\right) \wedge \varphi\theta\left(F, g^F\right) \right\}.$$

By Proposition 3.15, we get

$$\nabla^{TM} L = 0 \tag{11.4}$$

From (11.3), (11.4), we deduce that

$$\lim_{t\to 0} \operatorname{Tr_s}\left[\left(\frac{1}{2}\sum_1^n c(e_i)\widehat{c}(e_i)\right) \exp\left(-tD^2\right)\right] = 0. \tag{11.5}$$

Incidently note here that (11.5) also follows directly from Proposition 4.15 and from Theorem 4.20.

By standard properties of traces of heat kernels, we find from (11.5) that as $t \to 0$,

$$\operatorname{Tr_s}\left[\left(\frac{1}{2}\sum_1^n c(e_i)\widehat{c}(e_i)\right) \exp\left(-tD^2\right)\right] = O(t). \tag{11.6}$$

Moreover by the McKean-Singer formula [McKS], we get

$$\operatorname{Tr_s}\left[\frac{n}{2} \exp\left(-tD^2\right)\right] = \frac{n}{2}\chi(F). \tag{11.7}$$

From (11.1), (11.6), (11.7), we obtain the first identity in (7.22).

The proof of Theorem 7.10 is completed.

XII. An asymptotic expansion for $\mathrm{Tr_s}[f\exp(-tD_T^2)]$ as $T \to +\infty$

The purpose of this Section is to prove Theorem 7.11, i.e. to caculate, for a fixed $t > 0$, the asymptotic expansion for $\mathrm{Tr_s}[f\exp(-tD_T^2)]$ as $T \to +\infty$.

This Section is organized as follows. In a), we give an estimate for the kernel of $\exp(-t\widetilde{D}_T^2)$ away from B. In b), using the fact that the metrics g^{TM} and g^F are flat near B, we show that near B, the kernel for $\exp(-t\widetilde{D}_T^2)$ is well approximated by the kernel of a corresponding harmonic oscillator. Finally in c), we prove Theorem 7.11.

Let us point out that in our proof of our mains results in Theorem 7.1, we only need to establish Theorem 7.11 for $t = \varepsilon$ small enough. This simplifies the arguments of Section 12 b), where part of the difficulty comes from the fact that we establish certain estimates for arbitrary (i.e. not necessarily small) $t > 0$.

As already explained, we suppose the simplifying assumptions of Section 7 b) (which concern the form of g^{TM}, f and g^F near B) to be in force.

a) An estimate of the kernel of $\exp(-t\widetilde{D}_T^2)$ on $M\setminus\bigcup_{x\in B} B^M(x,\varepsilon)$

Definition 12.1. For $t > 0, T > 0$, let $P_{t,T}(z,z')(z,z' \in M)$ be the smooth kernel of the operator $\exp(-t\widetilde{D}_T^2)$ with respect to the volume element dv_M over M.

Then if $s \in \mathbb{F}$, for any $z \in M$

$$\exp\left(-t\widetilde{D}_T^2\right) s(z) = \int_M P_{t,T}(z,z')s(z')dv_M(z'). \tag{12.1}$$

Proposition 12.2. *For any $t > 0, \alpha > 0$, there exist $c > 0, C > 0$ for which if $z \in M$ is such that $d(z, B) \geq \alpha$, for $T \geq 0$,*

$$|P_{t,T}(z,z)| \leq c \exp(-CT). \tag{12.2}$$

Proof. Using (10.2) and the fact that $[D^X, \widehat{c}(\nabla f)]$ is an operator of order 0, (12.2) can be proved by the same methods as the stronger [BL2, Proposition 13.1]. □

Remark 12.3. The proof of [BL2, Proposition 13.1] uses the nonnegativity of the operator $\widetilde{D}_T^2$, and also probabilistic estimates for $P_{\frac{t}{T},T}(z,z)$. Still using the nonegativity of $\widetilde{D}_T^2$ and an argument using finite propagation speed, one can also give another proof of (12.2).

b) A harmonic oscillator approximation for the kernel of $\exp(-t\widetilde{D}_T^2)$ near B

Let $r > 0$ be the injectivity radius of (M, g^{TM}).

Take $\varepsilon \in]0, r/2]$ small enough so that for any $x \in B$, the balls $B^M(x, 2\varepsilon)(x \in B)$ do not intersect each other, that (7.12) holds on $B^M(x,\varepsilon)$, and moreover the metric g^F is flat on $B^M(x,\varepsilon)$.

Take $x \in B$. We use the notation of Section 8 b) or of Section 10, with $T_xM = T_xW^u(x) \oplus T_xW^s(x)$ replacing $\mathbb{R}^n = \mathbb{R}^i \oplus \mathbb{R}^{n-i}$. In particular, if $y \in T_xM, y^+$ and y^- denote the orthogonal projection of y on $T_xW^s(x)$ and $T_xW^u(x)$. Also recall that TM and T^*M are identified by the metric.

Let $\mathbf{F}_x$ be the vector space of smooth sections of $(\Lambda(T^*M) \otimes F)_x$ on T_xM. Let dv_{T_xM} be the volume element of T_xM with respect to the metric g^{T_xM}. We equip $\mathbf{F}_x$ with the scalar product

$$\alpha, \alpha' \in \mathbf{F}_x \to \langle \alpha, \alpha' \rangle_{\mathbf{F}_x} = \int_{T_xM} \langle \alpha, \alpha' \rangle_{(\Lambda(T^*M)\otimes F)_x} (y) dv_{T_xM}(y). \tag{12.3}$$

The operators $d^F + T(y^+ - y^-)\wedge$ and $d^{F*} + Ti_{y^+ - y^-}$ act on $\mathbf{F}_x$.

Definition 12.4. Set

$$\begin{aligned} \widetilde{D}_T^{T_xM} &= d^F + T(y^+ - y^-) \wedge + d^{F*} + Ti_{y^+ - y^-}, \\ \widetilde{D}^{T_xM} &= d^F + (y^+ - y^-) \wedge + d^{F*} + i_{y^+ - y^-}. \end{aligned} \tag{12.4}$$

Let G_T be the map

$$s(y) \in \mathbf{F}_x \to s\left(\frac{y}{\sqrt{T}}\right) \in \mathbf{F}_x. \tag{12.5}$$

Then

$$G_T \widetilde{D}_T^{T_xM} G_T^{-1} = \sqrt{T}\, \widetilde{D}^{T_xM}. \tag{12.6}$$

Let Δ^{T_xM} be the standard Laplacian on (T_xM, g^{T_xM}). By Proposition 8.2, we know that

$$\left(\widetilde{D}_T^{T_xM}\right)^2 = -\Delta^{T_xM} + T^2|y|^2 - Tn + 2T(N^+ + \mathrm{ind}(x) - N^-). \tag{12.7}$$

Let $\mathcal{L}$ be the harmonic oscillator

$$\mathcal{L} = \frac{1}{2}\left(-\Delta^{T_xM} + |y|^2 - n\right). \tag{12.8}$$

Then

$$\left(\widetilde{D}_T^{T_xM}\right)^2 = 2TG_T^{-1}\left(\mathcal{L} + N^+ + \mathrm{ind}(x) - N^-\right) G_T. \tag{12.9}$$

Definition 12.5. For $t > 0, T \geq 0$, let $Q^x_{t,T}(y,y')(y,y' \in T_xM)$ be the smooth kernel associated to the operator $\exp(-t(\widetilde{D}_T^{T_xM})^2)$ with respect to the volume element dv_{T_xM}.

We then use the coordinates $y = (y^1, \cdots, y^n)$ considered in (7.12) near x. In particular if $z \in M, d^M(x,z) < \varepsilon, Q^x_{t,T}(z,z)$ is well defined.

Theorem 12.6. *For any $t > 0$, there exist $c > 0, C > 0$ such that if $x \in B$, $z \in B^M(x,\varepsilon), T \geq 0$, then*

$$\left\| \left(P_{t,T} - Q^x_{t,T}\right)(z,z) \right\| \leq c\exp(-CT). \tag{12.10}$$

Proof. Let $P^D_{t,T}(z,z')(z,z' \in B^M(x,\varepsilon))$ be the smooth kernel associated to the operator $\exp(-t\widetilde{D}_T^2)$ and Dirichlet boundary conditions on $\partial B^M(x,\varepsilon)$. We claim that there exist $t_0 > 0,\ C > 0$ for which, given $t \in]0, t_0]$, there is $c > 0$, such that if $z \in B^M(x,\varepsilon), z' \in B^M(x,\varepsilon), T \geq 0$, then

$$\left\| \left(P_{t,T} - P^D_{t,T}\right)(z,z') \right\| \leq c\exp(-CT). \tag{12.11}$$

To establish (12.11), we will use a simple probabilistic method.

In fact by Theorem 4.13 and by (5.16), we know that there exists smooth sections A_0, A_1 of $\mathrm{End}(\Lambda(T^*M)\otimes F)$ such that for any $T\geq 0$

$$\widetilde{D}_T^2 = -\Delta^e + A_0 + TA_1 + T^2|df|^2. \tag{12.12}$$

For $z\in M, z'\in M$, let $R^t_{z,z'}$ be the probability law on $\mathcal{C}([0,1];M)$ of the Brownian bridge $s\in[0,1]\to x\in M$ associated to the metric $\frac{g^{TM}}{2t}$, starting at z and ending at z'. Tautologically, $R^t_{z,z'}(z_0=z)=R^t_{z,z'}(z_1=z')=1$. Under $R^t_{z,z'}$, $z_.$ is exactly the Brownian motion associated to the metric $\frac{g^{TM}}{2t}$, starting at z at 0 and conditioned to be z' at 1. For the definition of the Brownian bridge, we refer to [B2, Chapter 2]. Let $E^{R^t_{z,z'}}$ be the expectation operator associated to $R^t_{z,z'}$.

For $0\leq s\leq 1$, let τ^0_s be the parallel transport operator along the curve z from $(\Lambda(T^*M)\otimes F)_z$ into $(\Lambda(T^*M)\otimes F)_{z_s}$. Set $\tau^s_0=(\tau^0_s)^{-1}$. Observe that by [B2, Chapter 2], these operators are well-defined for any $s\in[0,1]$, $R^t_{z,z'}$ $a.s.$.

Under $R^t_{z,z'}$, consider the differential equation

$$\begin{aligned}\frac{dV_s^{t,T}}{ds} &= -V_s^{t,T}\tau_0^s\left(tA_0(z_s)+tTA_1(z_s)\right)\tau_s^0,\\ V_0^{t,T} &= 1_{(\Lambda(T^*M)\otimes F)_z}.\end{aligned} \tag{12.13}$$

In (12.13), $V_s^{t,T}$ lies in $\mathrm{End}_z(\Lambda(T^*M)\otimes F)$.

Let S be the stopping time

$$S=\inf\left\{s\geq 0; z_s\in\partial B^M(x,\varepsilon)\right\}. \tag{12.14}$$

Let Δ^{TM} be the Laplace-Beltrami operator on M, and let $p_t(z,z')(t>0,z,z'\in M)$ be the corresponding heat kernel associated to the semi group $e^{t\Delta^{TM}}$. A standard application of Ito's formula shows that if $z,z'\in B^M(x,\varepsilon)$, then

$$\begin{aligned}&\left(P_{t,T}-P^D_{t,T}\right)(z,z')\\ &= p_t(z,z')E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_0^1|df(z_s)|^2ds\right\}V_1^{t,T}\tau_0^1 1_{S\leq 1}\right].\end{aligned} \tag{12.15}$$

Clearly, there exists $\gamma>0$ such that for any $t>0, T\geq 0$,

$$\left|V_1^{t,T}\right|\leq\exp(\gamma t(1+T)). \tag{12.16}$$

From (12.15), (12.16), we deduce

$$\left|\left(P_{t,T}-P^D_{t,T}\right)(z,z')\right|\leq\exp(\gamma t(1+T))p_t(z,z') \tag{12.17}$$

$$E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_0^1 |df(z_s)|^2 ds\right\}1_{S\le 1}\right].$$

Estimating the right-hand side of (12.17) is now a scalar problem. We fix $t>0$. In the sequel, the constants $c'>0, c''>0\cdots$ may depend on $t>0$ but not on $T>0$. Clearly

$$\begin{aligned}(12.18)\qquad & E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_0^1 |df(z_s)|^2 ds\right\}1_{S\le 1}\right]\\ &\le E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_0^1 |df(z_s)|^2 ds\right\}1_{S\le 1/2}\right]\\ &+E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_0^1 |df(z_s)|^2 ds\right\}1_{1/2\le S\le 1}\right].\end{aligned}$$

By using time reversal, the two quantities in the right-hand side (12.16) are deduced from each other by exchanging z and z'. So we only need to estimate the first one.

Set

$$(12.19)\qquad S'=\inf\left\{s\ge S, z_s\in\bigcup_{y\in B}\partial B^M\left(y,\frac{\varepsilon}{2}\right)\right\}.$$

Then for $0\le a\le 1/4$, we have the obvious

$$\begin{aligned}(12.20)\qquad & E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_0^1 |df(z_s)|^2 ds\right\}1_{S\le 1/2}\right]\\ &\le R^t_{z,z'}\left[S\le 1/2, S'-S\le a\right]\\ &+E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_S^{S+a} |df(z_s)|^2 ds\right\}1_{S\le 1/2, S'-S\ge a}\right].\end{aligned}$$

Now there exists $\beta>0$ such that

$$(12.21)\qquad |df|^2\ge\beta \quad\text{on}\quad M\backslash\bigcup_{y\in B}B^M\left(y,\frac{\varepsilon}{2}\right).$$

Therefore

(12.22)

$$E^{R^t_{z,z'}}\left[\exp\left\{-tT^2\int_S^{S+a} |df(z_s)|^2 ds\right\}1_{S\le 1/2, S'-S\ge a}\right]\le\exp\left(-\beta a t T^2\right).$$

Let R_z^t be the probability law on $\mathcal{C}([0,1];M)$ of the standard Brownian motion z on M associated to the metric $\frac{g^{TM}}{2t}$, with $R_z^t(z_0=z)=1$.

Recall that $t>0$ is fixed. By [B2, Definition 2.4], on the σ-field $\mathcal{B}(z_s|s\leq 3/4)$, $R_{z,z'}^t$ has a bounded density with respect to R_z^t. Using the estimates of Varadhan [V, Proof of Theorem 5.1] on R_z^t, one finds easily that there exists $c'>0$ such that for $z,z'\in B^M(x,\varepsilon)$, $0<a\leq 1/4$,

$$R_{z,z'}^t\left[S\leq 1/2, S'-S\leq a\right]\leq c'\exp\left(-\frac{\varepsilon^2}{32at}\right). \tag{12.23}$$

From (12.17)–(12.23), we find there exists $c''>0$ such that for $T\geq 0$, $0<a\leq 1/4$,

(12.24)

$$\left|\left(P_{t,T}-P_{t,T}^D\right)(z,z')\right|\leq c''\exp(\gamma t(1+T))\left(c'\exp\left(-\frac{\varepsilon^2}{32at}\right)+\exp\left(-\beta atT^2\right)\right).$$

Take

$$a=\frac{\varepsilon}{\sqrt{32\beta}\,tT}. \tag{12.25}$$

It is clear that for $T\geq 0$ large enough, then $0<a\leq\frac{1}{4}$. Also

$$\frac{\varepsilon^2}{32at}=\beta atT^2=\varepsilon\sqrt{\frac{\beta}{32}}\,T. \tag{12.26}$$

Set

$$t_0=\frac{\varepsilon\sqrt{\beta}}{8\gamma}. \tag{12.27}$$

Then, if $t\leq t_0$

$$\varepsilon\sqrt{\frac{\beta}{32}}-\gamma t>0. \tag{12.28}$$

Using (12.24), (12.28), we get (12.11).

By a strictly similar proof, we see that for $0<t\leq t_0$, there exists $c>0$ such that if $x,x'\in B$, $x\neq x'$ and if $z\in B^M(x,\varepsilon), z'\in B^M(x',\varepsilon)$, if $T\geq 0$, then

$$|P_{t,T}(z,z')|\leq c\exp(-CT). \tag{12.29}$$

Also an application of Ito's formula shows that

$$P_{t,T}(z,z) = p_t(z,z)E^{R^t_{z,z}}\left[\exp\left\{-tT^2\int_0^1 |df(z_s)|^2\,ds\right\}V_1^{t,T}\right]. \tag{12.30}$$

Take $A > 0$. By (12.16), (12.30), there exists $c > 0$, such that for $t \in]0,A], T \in [0,\frac{1}{t}], z \in M$,

$$|P_{t,T}(z,z)| \leq \frac{c}{t^{n/2}}\cdot \tag{12.31}$$

Since the operator $(\widetilde{D}_T)^2$ is nonnegative, for any $z \in M$, the function $t \in \mathbb{R}_+^* \to \operatorname{Tr}[P_{t,T}(z,z)]$ is decreasing. Moreover $P_{t,T}(z,z) \in \operatorname{End}(\Lambda(T^*M)\otimes F)$ being self-adjoint and nonnegative, we find that if $|\ \ |$ denote the norm of trace, $t \to |P_{t,T}(z)|$ is decreasing. In particular, for any $t > 0$, for $T \geq \frac{1}{t}, z \in M$

$$|P_{t,T}(z,z)| \leq \left|P_{\frac{1}{T},T}(z,z)\right|. \tag{12.32}$$

From (12.16), (12.31), (12.32), we find for $t \in]0,A], T \geq \frac{1}{t}$,

$$|P_{t,T}(z,z)| \leq cT^{n/2}. \tag{12.33}$$

From (12.31), (12.33), we find that given $A > 0$, there exists $c > 0$ such that for $0 \leq t \leq A, z \in M$,

$$\begin{aligned} |P_{t,T}(z,z)| &\leq \frac{c}{t^{n/2}} \quad \text{if } 0 \leq T \leq \frac{1}{t}, \\ &\leq cT^{n/2} \quad \text{if } 0 \leq T \leq \frac{1}{t}. \end{aligned} \tag{12.34}$$

Since $\exp(-t\widetilde{D}_T^2)$ is a self-adjoint positive operator, if $z, z' \in M$,

$$|P_{t,T}(z,z')| \leq |P_{t,T}(z,z)|^{\frac{1}{2}}\,|P_{t,T}(z',z')|^{\frac{1}{2}}. \tag{12.35}$$

Take $t > 0$ which we fix once and for all. For $m \in \mathbb{N}$ large enough, $\frac{t}{m} \in]0,t_0]$. If $x \in B$, and if $z \in B^M(x,\varepsilon)$, then

$$\begin{aligned} P_{t,T}(z,z) = \int_{M^{m-1}} & P_{\frac{t}{m},T}(z,x_1)P_{\frac{t}{m},T}(x_1,x_2)\cdots \\ & \cdots P_{\frac{t}{m},T}(x_{m-1},z)dv_M(x_1)\cdots dv_M(x_{m-1}). \end{aligned} \tag{12.36}$$

Using (12.2), (12.29), (12.34)–(12.36), it is clear that given $t > 0$, there exist $c' > 0, C' > 0$ such that if $x \in B, z \in B^M(x,\varepsilon)$, $T \geq 1$, then

(12.37) $$\left|P_{t,T}(z,z) - \int_{(B^M(x,\varepsilon))^{m-1}} P_{\frac{t}{m},T}(z,x_1)\cdots P_{\frac{t}{m},T}(x_{m-1},z)dv_M(x_1) \cdots dv_M(x_{m-1})\right| \leq c'\exp(-C'T).$$

Also the same argument as in (12.30)–(12.34) shows that given $A > 0$, there is $c > 0$ such that if $t \in]0,A], T \geq 0$, then if $z \in B^M(x,\varepsilon)$,

(12.38) $$\begin{aligned} \left|P^D_{t,T}(z,z)\right| &\leq \frac{c}{t^{n/2}} \quad \text{if } 0 \leq T \leq \frac{1}{t}, \\ &\leq cT^{n/2} \quad \text{if } T \geq \frac{1}{t}. \end{aligned}$$

So, by proceeding as in (12.35), we get for $z, z' \in B^M(x,\varepsilon)$,

(12.39) $$\left|P^D_{t,T}(z,z')\right| \leq \left|P^D_{t,T}(z,z)\right|^{1/2} \left|P^D_{t,T}(z',z')\right|^{1/2}.$$

From (12.11), (12.34), (12.37)–(12.39), we find that given $t > 0$, there exist $c'' > 0, C'' > 0$ such that for $T \geq 1$,

(12.40) $$\left|P_{t,T}(z,z) - \int_{(B(x,\varepsilon))^{m-1}} P^D_{\frac{t}{m},T}(z,x_1)\cdots P^D_{\frac{t}{m},T}(x_{m-1},z)dv_M(x_1) \cdots dv_M(x_{m-1})\right| \leq c''\exp(-C''T).$$

Moreover

(12.41)
$$P^D_{t,T}(z,z) = \int_{(B(x,\varepsilon))^{m-1}} P^D_{\frac{t}{m},T}(z,x_1)\cdots P^D_{\frac{t}{m},T}(x_{m-1},z)dv_M(x_1)\cdots dv_M(x_{m-1}).$$

From (12.40), (12.41), we deduce that given any $t > 0$, there exist $c'' > 0, C'' > 0$ such that if $z \in B^M(x,\varepsilon)$, $T \geq 1$,

(12.42) $$\left|\left(P_{t,T} - P^D_{t,T}\right)(z,z)\right| \leq c''\exp(-C''T).$$

Let $Q^{x,D}_{t,T}(z,z')(z,z' \in B^{T_xM}(0,\varepsilon))$ be the smooth heat kernel associated with the operator $\exp(-t(\widetilde{D}^{T_xM}_T)^2)$ and Dirichlet boundary conditions on $\partial B^{T_xM}(0,\varepsilon)$.

One can prove as in (12.11) that there exist $t_0 > 0, C > 0$ such that if $0 < t \leq t_0$, there is $c > 0$ such that if $z \in B^{T_x M}(0, \varepsilon), T \geq 0$, then

$$\left| \left(Q^x_{t,T} - Q^{x,D}_{t,T} \right) (z, z) \right| \leq c \exp(-CT). \tag{12.43}$$

The obvious analogue of (12.34) holds. Moreover the kernel $Q^x_{t,T}(z, z')$ is explicitly known by Mehler's formula [GlJ, Theorem 1.5.10]. One can then easily obtain estimates at infinity for $Q^x_{t,T}(z, z')$, and show that the analogue of (12.37) holds. We deduce that given $t > 0$, there exist $c'' > 0, C'' > 0$ such that if $z \in B^{T_x M}(0, \varepsilon), T \geq 0$, then

$$\left| \left(Q^x_{t,T} - Q^{x,D}_{t,T} \right) (z, z) \right| \leq c \exp(-CT). \tag{12.44}$$

Finally, if $z \in B^M(x, \varepsilon)$, one has the obvious

$$P^D_{t,T}(z, z) = Q^{x,D}_{t,T}(z, z). \tag{12.45}$$

Equation (12.10) now follows from (12.42), (12.44), (12.45). □

c) Proof of Theorem 7.11

Here $t > 0$ is fixed. By Proposition 5.4, we get

$$\mathrm{Tr_s}\left[f \exp\left(-tD^2_T\right)\right] = \mathrm{Tr_s}\left[f \exp\left(-t\widetilde{D}^2_T\right)\right]. \tag{12.46}$$

Moreover,

$$\mathrm{Tr_s}\left[f \exp\left(-t\widetilde{D}^2_T\right)\right] = \int_M \mathrm{Tr_s}\left[f(z) P_{t,T}(z, z)\right] dv_M(z). \tag{12.47}$$

By Proposition 12.2, we know that there exist $c > 0, C > 0$, such that

$$\left| \int_{M \setminus \bigcup_{x \in B} B^M(x, \varepsilon)} \mathrm{Tr_s}\left[f(z) P_{t,T}(z, z)\right] dv_M(z) \right| \leq c \exp(-CT). \tag{12.48}$$

Also by Theorem 12.6, there exist $c' > 0, C' > 0$ such that if $x \in B$,

$$\left| \int_{B^M(x, \varepsilon)} \mathrm{Tr_s}\left[f(z) \left(P_{t,T} - Q^x_{t,T}\right)(z, z)\right] dv_M(z) \right| \leq c' \exp(-C'T). \tag{12.49}$$

Using (12.9) and Mehler's formula [GlJ, Theorem 1.5.10], we get for $y \in T_xM$,

$$Q^x_{t,T}(y,y) = \left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2} \tag{12.50}$$

$$\exp\left\{-T\tanh(tT)|y|^2\right\}\exp\left\{-2tT\left(N^+ + \mathrm{ind}(x) - N^-\right)\right\}.$$

Moreover by (7.12), if $|y| \leq \varepsilon$,

$$f(y) = f(x) + \frac{1}{2}\left(\left|y^+\right|^2 - \left|y^-\right|^2\right). \tag{12.51}$$

Then

$$\int_{B^M(x,\varepsilon)} \mathrm{Tr_s}\left[f(z)Q^x_{t,T}(z,z)\right] dv_M(z) \tag{12.52}$$

$$= \left\{\mathrm{rk}(F)f(x)\int_{|y|\leq\varepsilon}\left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2}\exp\left\{-T\tanh(tT)|y|^2\right\}dy\right.$$

$$\left.+\,\mathrm{rk}(F)\int_{|y|\leq\varepsilon}\frac{1}{2}\left(\left|y^+\right|^2 - \left|y^-\right|^2\right)\left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2}\exp\left\{-T\tanh(tT)|y|^2\right\}dy\right\}$$

$$\mathrm{Tr_s}^{\Lambda(T^*_xM)}\left[e^{-2tT\left(N^+ + \mathrm{ind}(x) - N^-\right)}\right].$$

Also

$$\int_{|y|\leq\varepsilon}\left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2}\exp\left(-T\tanh(tT)|y|^2\right)dy \tag{12.53}$$

$$= \left(\frac{1}{1-e^{-2tT}}\right)^n \int_{|y|\leq[T\tanh(tT)]^{1/2}\varepsilon} e^{-|y|^2}\frac{dy}{(\pi)^{n/2}},$$

and so there exists $c > 0$ such that as $T \to +\infty$,

$$\int_{|y|\leq\varepsilon}\left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2}\exp\left(-T\tanh(tT)|y|^2\right)dy = 1 + O\left(e^{-cT}\right). \tag{12.54}$$

Moreover

$$\int_{|y|\leq\varepsilon}\frac{1}{2}\left(\left|y^+\right|^2 - \left|y^-\right|^2\right)\left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2}\exp\left(-T\tanh(tT)|y|^2\right)dy \tag{12.55}$$

$$= \left(\frac{1}{1-e^{-2tT}}\right)^n \frac{1}{2T\tanh(tT)}\int_{|y|\leq[T\tanh(tT)]^{1/2}\varepsilon}\left(\left|y^+\right|^2 - \left|y^-\right|^2\right)e^{-|y|^2}\frac{dy}{\pi^{n/2}}.$$

From (3.80), (12.55), we deduce that there is $c > 0$ such that as $T \to +\infty$,

(12.56)

$$\int_{|y|\leq\varepsilon} \frac{1}{2}\left(|y^+|^2 - |y^-|^2\right)\left(\frac{Te^{2tT}}{2\pi\sinh(2tT)}\right)^{n/2} \exp\left(-T\tanh(tT)|y|^2\right) dy$$

$$= \frac{1}{4T}(n - 2\,\mathrm{ind}(x)) + O\left(e^{-cT}\right).$$

Also, there is $c' > 0$ such that as $T \to +\infty$,

$$\mathrm{Tr}_s^{\Lambda(T_x^*M)}\left[e^{-2tT\left(N^+ + \mathrm{ind}(x) - N^-\right)}\right] = (-1)^{\mathrm{ind}(x)} + O\left(e^{-cT}\right). \tag{12.57}$$

Using (12.46)–(12.57), we get (7.23). The proof of Theorem 7.11 is completed.

□

XIII. An estimate for $\mathrm{Tr_s}[f\exp(-(tD+T\widehat{c}(\nabla f))^2)]$ in the range $0 < t \leq 1\,,\ \ 0 \leq T \leq \frac{d}{t}$

The purpose of this Section is to prove Theorem 7.12, i.e. to establish an estimate involving $\mathrm{Tr_s}[f\exp(-(tD+T\widehat{c}(\nabla f))^2)]$ in the range $t\in]0,1], T\in [0,\frac{d}{t}]$. The results of this Section are essential in explaining the appearance of the term $-\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM})$ in Theorem 7.1.

The proofs rely on the Berezin integral formalism of Section 3, and also on the local index techniques we developed in Section 4.

This Section is organized as follows. In a), we show that the problem considered in Theorem 7.12 is local on M. In b), we prove certain estimates on the kernel of the operator $\exp(-(tD+T\widehat{c}(\nabla f))^2)$ in the range $t\in]0,1], 0\leq T\leq T_0$. In c), we extend these estimates to the range $t\in]0,1], 0\leq T\leq \frac{d}{t}$ on compact sets of $M\backslash B$. Finally in d), we prove Theorem 7.12.

In the whole Section, the simplifying assumptions of Section 7 b) will be in force. Also we use the notation of Sections 3 and 4.

a) Localization of the problem

Let $r>0$ be the injectivity radius of (M,g^{TM}). Take $b\in]0,r/2]$.

Definition 13.1. For $t>0, T\geq 0$, let $S_{t,T}(z,z')$ $(z,z'\in M)$ be the smooth kernel associated to the operator $\exp(-(tD+T\widehat{c}(\nabla f))^2)$ with respect to the volume element dv_M.

Comparing with Definition 12.1, we get

$$S_{t,T}(z,z') = P_{t^2,\frac{T}{t}}(z,z'). \tag{13.1}$$

Definition 13.2. Given $z_0 \in M$, let $S_{t,T}^{D,z_0}(z,z')(z,z' \in B^M(z_0,b))$ be the smooth kernel associated to the operator $\exp(-(tD+T\widehat{c}(\nabla f))^2)$ and Dirichlet boundary conditions on $\partial B^M(z_0,b)$.

Proposition 13.3. *For any $d>0$ there exist $c>0, C>0$ such that if $z_0 \in M, t \in]0,1], T \in [0,d/t], z \in B^M(z_0,b/2)$, then*

$$\left|\left(S_{t,T}-S_{t,T}^{D,z_0}\right)(z,z)\right| \leq c\exp\left(-C/t^2\right). \tag{13.2}$$

Proof. In view of (10.2), and of the fact that $[D,\widehat{c}(\nabla f)]$ is of order 0, the proof of Proposition 13.3 is the same as the proof of [BL2, Proposition 11.10]. □

b) An estimate for the kernel of $\exp(-(tD+T\widehat{c}(\nabla f))^2)$ in the range $t \in]0,1], T \in [0,T_0]$.

In the sequel, dv_M is considered as a section of $\Lambda^n(T^*M)\otimes o(TM)$.

Theorem 13.4. *For any $T_0 \geq 0$, there exists $c>0$ such that if $z \in M, t \in]0,1], 0 \leq T \leq T_0$, then*

(13.3)

$$\left|\mathrm{Tr_s}\left[S_{t,T}(z,z)\right]dv_M - \mathrm{rk}(F)\int^B \exp\left(-B_{T^2}\right) - td\int^B \frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp(-B_{T^2})\right| \leq Ct^2.$$

Proof. Let $e_1,\cdots,e_n$ be an orthonormal base of TM. By Theorem 4.13 and Proposition 5.5, we know that

$$\begin{aligned}(tD+T\widehat{c}(\nabla f))^2 = {} & -t^2\Delta^e + \frac{t^2K}{4} + \frac{t^2}{8}\sum_{1\leq i,j,k,\ell\leq n}\left\langle e_k, R^{TM}(e_i,e_j)e_\ell\right\rangle \\ & c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) + \frac{t^2}{4}\sum_{1\leq i\leq n}\left(\omega\left(F,g^F\right)(e_i)\right)^2 \\ & -\frac{t^2}{8}\sum_{1\leq i,j\leq n}\left(c(e_i)c(e_j)-\widehat{c}(e_i)\widehat{c}(e_j)\right)\left(\omega\left(F,g^F\right)\right)^2(e_i,e_j) \\ & -\frac{t^2}{4}\sum_{1\leq i,j\leq n}c(e_i)\widehat{c}(e_j)\left(\nabla^F_{e_i}\omega\left(F,g^F\right)(e_j)+\nabla^F_{e_j}\omega\left(F,g^F\right)(e_i)\right) \\ & -tT\omega\left(F,g^F\right)(\nabla f) + tT\sum_{1\leq i,j\leq n}\left\langle\nabla^{T^*M}_{e_i}df, e_j\right\rangle c(e_i)\widehat{c}(e_j) + T^2|df|^2.\end{aligned} \tag{13.4}$$

Take $z \in M$. We identify $B^{T_zM}(0,b)$ with $B^M(z,b)$ using geodesic coordinates centered at z. Also if $y \in T_zM, |y| \leq b$, we identify T_yM with T_zM (resp. F_y with F_z) by parallel transport with respect to the connection ∇^{TM}(resp.$\nabla^{F,e}$) along the geodesic $s \in [0,1] \to sy \in M$. Therefore if $y \in B^M(x,b), (\Lambda(T^*M) \otimes F)_y$ is identified with $(\Lambda(T^*M) \otimes F)_z$.

Let γ be a smooth function $\mathbb{R} \to \mathbb{R}_+$ such that

$$\begin{aligned} \gamma(s) &= 1 \text{ if } s \leq 1/2, \\ &= 0 \text{ if } s \geq 1. \end{aligned} \tag{13.5}$$

If $y \in T_zM$, set

$$\rho(y) = \gamma\left(\frac{|y|}{b}\right). \tag{13.6}$$

Then

$$\begin{aligned} \rho(y) &= 1 \text{ if } |y| \leq b/2, \\ &= 0 \text{ if } |y| \geq b. \end{aligned} \tag{13.7}$$

Let $\mathbf{F}_z$(resp.$\mathbf{F}_{z,0}$) be the vector space of smooth (resp. square integrable) sections of $(\Lambda(T^*M) \otimes F)_z$ over T_zM. Let Δ^{T_zM} be the Euclidean Laplacian on T_zM.

Let $J^{1,z}_{t,T}$ be the operator acting on $\mathbf{F}_z$

$$J^{1,z}_{t,T} = \left(1 - \rho^2(y)\right)\left(-t^2\Delta^{T_zM} + T^2\right) + \rho^2(y)\left(tD + T\widehat{c}(\nabla f)\right)^2. \tag{13.8}$$

Let $S^{1,z}_{t,T}(y,y')(y,y' \in T_zM)$ be the smooth kernel associated to the operator $\exp(-J^{1,z}_{t,T})$ with respect to the volume element dv_{T_zM}. By Proposition 13.3, there exist $c > 0, C > 0$ such that if $t \in]0,1], T \in [0,d/t]$, then

$$\left|S_{t,T}(z,z) - S^{1,z}_{t,T}(0,0)\right| \leq c\exp\left(-\frac{C}{t^2}\right). \tag{13.9}$$

Let H_t be the linear map

$$s(y) \in \mathbf{F}_z \to s\left(\frac{y}{t}\right) \in \mathbf{F}_z. \tag{13.10}$$

Set

$$J^{2,z}_{t,T} = H_t^{-1} J^{1,z}_{t,T} H_t. \tag{13.11}$$

Let $e_1, \cdots, e_n$ be an orthonormal base of T_zM, and let $e^1, \cdots, e^n$ be the corresponding dual base of T_z^*M. For $1 \le i \le n$, set

$$
\begin{aligned}
c'_t(e_i) &= \frac{e^i}{\sqrt{t}} - \sqrt{t}\, i_{e_i}, \\
\widehat{c}'_t(e_i) &= \frac{\widehat{e}^i}{\sqrt{t}} + \sqrt{t}\, i_{\widehat{e_i}}.
\end{aligned} \tag{13.12}
$$

Let $J^{3,z}_{t,T}$ be the operator obtained from $J^{2,z}_{t,T}$ by replacing the operators $c(e_i), \widehat{c}(e_i)$ by $c'_t(e_i), \widehat{c}'_t(e_i)$ $(1 \le i \le n)$. Let $S^{3,z}_{t,T}(y,y')(y,y' \in T_zM)$ be the smooth kernel associated to the operator $\exp(-J^{3,z}_{t,T})$. Then $S^{3,z}_{t,T}(0,0)$ can be expanded in the form

(13.13)

$$
S^{3,z}_{t,T}(0,0) = \sum_{\substack{1 \le i_1 < i_2 \cdots < i_p \le n \\ 1 \le i'_1 < i'_2 \cdots < i'_{p'} \le n \\ 1 \le j_1 < j_2 \cdots < j_q \le n \\ 1 \le j'_1 < j'_2 \cdots < j'_{q'} \le n}} e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \widehat{e^{i'_1}} \wedge \cdots \wedge \widehat{e^{i'_{p'}}} \wedge i_{e_{j_1}} \cdots i_{e_{j_q}}
$$

$$
i_{\widehat{e_{j'_1}}} \cdots i_{\widehat{e_{j'_{q'}}}} \otimes Q^{j_1,\cdots j_q, j'_1, \cdots j'_{q'}}_{i_1,\cdots i_p, i'_1, \cdots i'_{p'}} \quad ; \quad Q^{j_1,\cdots j_q, j'_1, \cdots j'_{q'}}_{i_1,\cdots i_p, i'_1, \cdots i'_{p'}} \in \operatorname{End}(F_z).
$$

Set

$$
\left[S^{3,z}_{t,T}(0,0)\right]^{\max} = Q_{1,\cdots n,1,\cdots,n} \in \operatorname{End}(F_z). \tag{13.14}
$$

By Proposition 4.11, it is clear that

$$
\operatorname{Tr_s}\left[S^{1,z}_{t,T}(0,0)\right] = 2^n(-1)^{\frac{n(n+1)}{2}} \operatorname{Tr}\left[S^{3,z}_{t,T}(0,0)\right]^{\max}. \tag{13.15}
$$

Let $\Gamma^{TM}, \Gamma^{F,e}$ be the connection forms for $\nabla^{TM}, \nabla^{F,e}$ with respect to the considered trivializations of TM, F near z. By [ABoP, Proposition 3.7], we know that

$$
\begin{aligned}
\Gamma^{TM}_y &= \frac{1}{2} R^{TM}_z(y, \cdot) + O\left(|y|^2\right), \\
\Gamma^{F,e}_y &= O(|y|).
\end{aligned} \tag{13.16}
$$

In the sequel for $m \in \mathbb{Z}$, $O(|y|^m)$ denotes any matrix valued operator depending smoothly on y, which may also depend on $t > 0$, and is such that for any $k \in \mathbf{N}$, there is $C_k > 0$ such that

$$|\partial^k O(|y|^m)| \leq C_k |y|^{m-k}. \tag{13.17}$$

The geodesic coordinate system $y = (y^1, \cdots, y^n)$ defines a canonical trivialization of TM near x (which is distinct from the one considered before). It is well-known that in this trivialization, the Christoffel symbols of the connection ∇^{TM} still vanish at $y = 0$. If $e \in T_z M, y \in T_z M, |y| \leq \varepsilon$, let $\tau(e)(y)$ be the parallel transport of e along the geodesic $s \in [0,1] \to sy \in M$ with respect to this trivialization. It follows that

$$\tau e(y) = e + O\left(|y|^2\right). \tag{13.18}$$

Then by using (4.28), (4.31), (13.4), (13.8) and proceeding as in the proof of Theorem 4.20, we find that

$$\begin{aligned}
J^{3,z}_{t,T} &= \left(1-\rho^2(ty)\right)\left(-\Delta^{T_z M}+T^2\right) \\
&+\rho^2(ty)\Bigg\{-\Bigg(\nabla_{e_i + t^2 O(|y|^2)} + \frac{t}{4}\sum_{1\leq k,\ell\leq n}\left\langle\left(R^{TM}_z(y,e_i)+tO\left(|y|^2\right)\right)e_k, e_\ell\right\rangle \\
&\left(\left(e^k\wedge -ti_{e_k}\right)\left(e^\ell\wedge -ti_{e_\ell}\right)-\left(\widehat{e}^k\wedge +ti_{\widehat{e_k}}\right)\left(\widehat{e}^\ell\wedge +ti_{\widehat{e_\ell}}\right)\right)+t^2 O\left(|y|\right)\Bigg)^2 \\
&+\frac{1}{8}\sum_{1\leq i,j,k,\ell\leq n}\left(\left\langle e_k, R^{TM}_z(e_i,e_j)e_\ell\right\rangle + tO\left(|y|\right)\right) \\
&\left(e^i\wedge -ti_{e_i}\right)\left(e^j - ti_{e_j}\right)\left(\widehat{e}^k + ti_{\widehat{e_k}}\right)\left(\widehat{e}^\ell + ti_{\widehat{e_\ell}}\right) \\
&+T\sum_{1\leq i,j\leq n}\left(\left\langle \nabla^{T^*M}_{e_i} df(z), e_j\right\rangle + tO\left(|y|\right)\right) \\
&\left(e^i - ti_{e_i}\right)\left(\widehat{e}^j + ti_{\widehat{e_j}}\right) + T^2\left(|df(z)|^2 + tO(|y|)\right) \\
&-t\Bigg[\frac{1}{8}\sum_{1\leq i,j\leq n}\Bigg(\left(e^i\wedge -ti_{e_i}\right)\left(e^j\wedge -ti_{e_j}\right) \\
&-\left(\widehat{e}^i\wedge +ti_{\widehat{e_i}}\right)\left(\widehat{e}^j\wedge +ti_{\widehat{e_j}}\right)\Bigg)\left(\left(\omega_z\left(F,g^F\right)\right)^2(e_i,e_j)+tO(|y|)\right)
\end{aligned} \tag{13.19}$$

$$+\frac{1}{4}\sum_{1\leq i,j\leq n}\left(e^i\wedge -ti_{e_i}\right)\left(\widehat{e^j}\wedge +ti_{\widehat{e_j}}\right)\left(\nabla^F_{e_i}\omega_z\left(F,g^F\right)(e_j)\right.$$

$$\left.\left.\left.+\nabla^F_{e_j}\omega_z\left(F,g^F\right)(e_i)+tO(|y|)\right)+T\left(\omega_z\left(F,g^F\right)(\nabla f)+tO(|y|)\right)\right]+t^2O(1)\right\}.$$

Now we use the notation of Section 3 f). Set

$$J^{3,z}_{0,T}=-\Delta^{T_zM}+B_{T^2},\tag{13.20}$$

$$K^{3,z}_{0,T}=-\frac{1}{2}\sum_{1\leq i,k,\ell\leq n}\left\langle R^{TM}_z(y,e_i)e_k,e_\ell\right\rangle$$

$$\left(e_k\wedge e_\ell-\widehat{e}^k\wedge\widehat{e}_\ell\right)\nabla_{e_i}+\frac{1}{8}\sum_{1\leq i,j,k,\ell\leq n}\left\langle e_k,R^{TM}_z(e_i,e_j)e_\ell\right\rangle$$

$$\left(e^i\wedge e^j\wedge\left(i_{\widehat{e_k}}\widehat{e}_\ell\wedge+\widehat{e}_k\wedge i_{\widehat{e_\ell}}\right)-\left(i_{e_i}e_j\wedge+e^i\wedge i_{e_j}\right)\widehat{e}^k\wedge\widehat{e}^\ell\right)$$

$$+T\sum_{1\leq i,j\leq n}\left\langle\nabla^{T^*M}_{e_i}df(z),e_j\right\rangle\left(e^i\wedge\widehat{i_{e_j}}-i_{e_i}\widehat{e}^j\wedge\right)$$

$$-\left[\frac{1}{8}\sum_{1\leq i,j\leq n}\left(e^i\wedge e^j-\widehat{e}^i\wedge\widehat{e}^j\right)\left(\omega_z\left(F,g^F\right)\right)^2(e_i,e_j)\right.$$

$$\left.+\frac{1}{4}\sum_{1\leq i,j\leq n}e^i\wedge\widehat{e^j}\left(\nabla^F_{e_i}\omega_z\left(F,g^F\right)(e_j)+\nabla^F_{e_j}\omega_z\left(F,g^F\right)(e_i)\right)+T\omega_z\left(F,g^F\right)(\nabla f)\right].$$

In the sequel, $O_T(t^2)$ denotes a second order differential operator acting on $\mathbf{F}_z$, whose coefficients are $O(t^2)$ as $t\to 0$. From (13.19), we see that there is an explicitly computable matrix valued operator $L_T(y)$, depending linearly on $y\in T_zM$ such that as $t\to 0$,

$$J^{3,z}_{t,T}=J^{3,0}_{0,T}+t\left(K^{3,z}_{0,T}+L^z_T(y)\right)+O_T(t^2).\tag{13.21}$$

Let $S^{3,z}_{0,T}(y,y')(y,y'\in T_zM)$ be the smooth kernel associated to the operator $\exp(-J^{3,z}_{0,T})$. Clearly,

$$S^{3,z}_{0,T}(0,0)=\frac{1}{2^n\pi^{n/2}}\exp\left(-B_{T^2,z}\right).\tag{13.22}$$

We define $[S^{3,z}_{0,T}(0,0)]^{\max}$ as in (13.14). From (13.22), we deduce that

$$(13.23) \qquad 2^n(-1)^{\frac{n(n+1)}{2}} \operatorname{Tr}\left[S^{3,z}_{0,T}(0,0)\right]^{\max} dv_M = \operatorname{rk}(F)\int^B \exp\left(-B_{T^2,z}\right).$$

For $t\in[0,1], s>0$, let $S^{3,z}_{t,T,s}(y,y')(y,y'\in T_zM)$ be the smooth kernel associated to the operator $\exp(-sJ^3_{t,T})$. In particular,

$$(13.24) \qquad S^{3,z}_{t,T,1} = S^{3,z}_{t,T}.$$

If $p_s(y,y')$ denotes the standard scalar heat kernel associated with the operator $\exp(s\Delta^{T_zM})$, then

$$(13.25) \qquad S^{3,z}_{0,T,s}(y,y') = p_s(y,y')\exp\left(-sB_{T^2}\right).$$

By Duhamel's formula, we know that

$$(13.26) \qquad S^{3,z}_{t,T,s} - S^{3,z}_{0,T,s} = \int_{0\leq s_1\leq s} S^{3,z}_{t,T,s_1}\left(J^{3,z}_{0,T} - J^{3,z}_{t,T}\right) S^{3,z}_{0,T,s-s_1}ds_1.$$

From (13.24), (13.26) we get
(13.27)

$$\left(S^{3,z}_{t,T} - S^{3,z}_{0,T}\right)(0,0) = \int_{0\leq s_1\leq 1}\left(S^{3,z}_{0,T,s_1}\left(J^{3,z}_{0,T} - J^{3,0}_{t,T}\right)S^{3,z}_{0,T,1-s_1}\right)(0,0)ds_1$$

$$+\int_{0\leq s_1\leq s_2\leq 1}\left(S^{3,z}_{t,T,s_1}\left(J^{3,z}_{0,T} - J^{3,z}_{t,T}\right)S^{3,z}_{0,T,s_2-s_1}\left(J^{3,z}_{0,T} - J^{3,z}_{t,T}\right)S^{z}_{0,T,1-s_2}\right)(0,0)ds_1ds_2.$$

Take $T_0\geq 0, s_0\in]0,1]$. By proceeding as in [BL2, Theorem 11.31], for any $s_0\geq 0$, one easily obtains uniform bounds in $s\in[s_0,1], t\in[0,1]\, 0\leq T\leq T_0$, on $S^{3,z}_{t,T,s}(y,y')$ together with its derivatives over compact sets of $T_zM\times T_zM$, and also uniform bounds in $s\in[0,1], t\in[0,1], 0\leq T\leq T_0$, on $S^{3,z}_{t,T,s}$ as an operator acting on $\mathbf{F}_{z,0}$. Incidently note that one here does not need the complicate system of L_2 norms with weights depending on the grading which is used in [BL2], this essentially because in (13.19), $\langle R^{TM}_z(y,e_i),e_k,e_\ell\rangle$ appears with the coefficient t, while in [BL2], a similar term appeared with the coefficient 1. The standard L_2 norm over $\mathbf{F}_{z,0}$ is here quite enough.

Similarly, using the techniques of [BL2, Theorem 11.30], or finite propagation speed methods, one can obtain adequate uniform control in $s\in[0,1], t\in[0,1], 0\leq T\leq T_0$, of the kernels $S^{3,z}_{t,T,s_1}(y,y')$ as $|y|$ or $|y'|\to+\infty$.

From (13.21), (13.27), we find that as $t \to 0$,

$$\left(S^{3,z}_{t,T} - S^{3,z}_{0,T}\right)(0,0) \tag{13.28}$$

$$= -t \int_{0 \le s_1 \le 1} \left(S^{3,z}_{0,T,s_1}\left(K^{3,z}_{0,T} + L^z_T(y)\right) S^{3,z}_{0,T,1-s_1}\right)(0,0)ds_1 + O_T(t^2),$$

and in (13.28), $O_T(t^2)$ is such that there exists $C > 0$ for which if $t \in [0,1], 0 \le T \le T_0$, then,

$$\left|O_T(t^2)\right| \le Ct^2. \tag{13.29}$$

We now use (13.25). Since $L_T(y)$ depends linearly on y, it is clear that for $0 \le s_1 \le 1$,

$$\left(S^{3,z}_{0,T,s_1} L_T(y) S^{3,z}_{0,T,1-s_1}\right)(0,0) = 0 \tag{13.30}$$

Also by Proposition 3.10, B_{T^2} is a sum of forms of type (p,p), and so for $0 \le s_1 \le 1$,

$$\begin{aligned}
&\left[\exp\left(-s_1 B_{T^2}\right) e^i \wedge e^j \exp\left(-(1-s_1)B_{T^2}\right)\right]^{\max} = 0,\\
&\left[\exp\left(-s_1 B_{T^2}\right) \widehat{e^i} \wedge \widehat{e^j} \exp\left(-(1-s_1)B_{T^2}\right)\right]^{\max} = 0,\\
&\left[\exp\left(-s_1 B_{T^2}\right) e^i \wedge e^j \left(i_{\widehat{e_k}}\widehat{e_\ell} + \widehat{e_k} i_{\widehat{e_\ell}}\right) \exp\left(-(1-s_1)B_{T^2}\right)\right]^{\max} = 0,\\
&\left[\exp\left(-s_1 B_{T^2}\right) \left(i_{e_i} e_j \wedge + e^i \wedge i_{e_j}\right) \widehat{e^k} \wedge \widehat{e^\ell} \exp\left(-(1-s_1)B_{T^2}\right)\right]^{\max} = 0,\\
&\left[\exp\left(-s_1 B_{T^2}\right) e^i \wedge \widehat{i_{e_j}} \exp\left(-(1-s_1)B_{T^2}\right)\right]^{\max} = 0,\\
&\left[\exp\left(-s_1 B_{T^2}\right) i_{e_i} \wedge \widehat{e^j} \wedge \exp\left(-(1-s_1)B_{T^2}\right)\right]^{\max} = 0.
\end{aligned} \tag{13.31}$$

So from (13.20), (13.25), (13.30), (13.31), we get

$$-2^n(-1)^{\frac{n(n+1)}{2}} \left[\int_{0 \le s_1 \le 1} \left(S^{3,z}_{0,T,s_1}\left(K^{3,z}_{0,T} + L^z_T(y)\right) S^z_{0,T,1-s_1}\right)(0,0)ds_1\right]^{\max} \tag{13.32}$$

$$= \int^B \exp\left(-B_{T^2}\right)\left(\frac{1}{4}\sum_{1 \le i,j \le n} e^i \wedge \widehat{e^j}\left(\nabla^F_{e_i}\omega\left(F,g^F\right)(e_j) + \nabla^F_{e_j}\omega\left(F,g^F\right)(e_i)\right)\right.$$

$$\left. + T\omega\left(F,g^F\right)(\nabla f)\right).$$

Using (4.73), (13.32), we obtain

(13.33)

$$-2^n(-1)^{\frac{n(n+1)}{2}}\operatorname{Tr}\left[\int_{0\le s_1\le 1}\left(S^{3,z}_{0,T,s_1}\left(K^{3,z}_{0,T}+L^z_T(y)\right)S^{3,z}_{0,T,1-s_1}\right)(0,0)ds_1\right]^{\max} dv_M$$

$$=\int^B\left(\frac{1}{2}\nabla\widehat{\theta}\left(F,g^F\right)+i_{T\widehat{\nabla f}}\widehat{\theta}\left(F,g^F\right)\right)\exp\left(-B_{T^2}\right).$$

Now by Theorem 3.2, we see that

$$d\int^B\frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp\left(-B_{T^2}\right) \tag{13.34}$$

$$=\int^B\left(\frac{1}{2}\nabla\widehat{\theta}\left(F,g^F\right)+i_{T\widehat{\nabla f}}\widehat{\theta}\left(F,g^F\right)\right)\exp\left(-B_{T^2}\right).$$

From (13.15), (13.23), (13.28), (13.29), (13.33), (13.34), we get (13.3). The proof of Theorem 13.4 is completed. □

c) An estimate for the kernel of $\exp(-(tD+T\widehat{c}(\nabla f))^2)$ in the range $t\in]0,1],T\in[0,\frac{d}{t}]$

Theorem 13.5. *Take $\alpha>0,\ d>0$. There exists $C>0$ such that for any $z\in M$ with $d^M(z,B)\ge\alpha$, for any $t\in]0,1],T\in[0,d/t]$, then*

$$\Big|\operatorname{Tr_s}\left[S_{t,T}(z,z)\right]dv_M-\operatorname{rk}(F)\int^B\exp\left(-B_{T^2}\right) \tag{13.35}$$

$$-td\Big(\int^B\frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp\left(-B_{T^2}\right)\Big)\Big|\ \le\ Ct^2.$$

Proof. For uniformly bounded T, (13.35) was proved in Theorem 13.4. To establish (13.35), we will take advantage of the fact $d^M(z,B)\ge\alpha$.

We may and we will assume that in Proposition 13.3, $b\le\frac{\alpha}{2}$. By (13.2), it is clear that to establish (13.35), we only need to work 'locally' near $z\in M$. This exactly means that all the constructions in the proof of Theorem 13.4 remain valid.

Set

$$\beta=\inf_{d(z,B)\ge\alpha/2}|df|^2(z)\wedge 1. \tag{13.36}$$

We will use Duhamel's formula as in (13.26), (13.27). The main point is that since $T \leq \frac{d}{t}$, the norm of pointwise estimates on the kernels $S^{3,z}_{t,T,s}$ can be improved by a factor $\exp(-s\beta T^2)$. This can be proved by using the Feynman-Kac formula.

Alternatively, by proceeding as in [BL2, Section 11], one can show that for any $k \in \mathbb{N}$, for $t \in]0,1], 0 \leq T \leq \frac{d}{t}$, the estimates we established for the kernel $S^{3,z}_{t,T}(y,y')$ in Theorem 13.4 remain valid here for the kernel $T^k S^{3,z}_{t,T}(y,y')$.

Now $J^3_{0,T} - J^3_{t,T}$ is quadratic in T. By proceeding as in (13.28), (13.29), it easily follows that (13.28), (13.29) hold uniformly in $T \in [0, d/t]$.

As in the proof of Theorem 13.4, we get (13.35). □

d) Proof of Theorem 7.12

In the sequel, the constants $c > 0, C > 0$ may vary from line to line.

Take $\varepsilon \in]0, \frac{r}{2}]$ small enough so that the metric g^F is flat on $\bigcup_{x\in B} B^M(x,\varepsilon)$, and (7.12) holds on $\bigcup_{x\in B} B^M(x,\varepsilon)$. Clearly

$$\text{(13.37)} \qquad \mathrm{Tr_s}\left[f \exp\left(-\left(tD + T\widehat{c}(\nabla f)\right)^2\right)\right] = \int_M f\, \mathrm{Tr_s}\left[S_{t,T}(z,z)\right] dv_M .$$

Then

$$\text{(13.38)} \qquad \int_M f\, \mathrm{Tr_s}\left[S_{t,T}(z,z)\right] dv_M$$

$$= \int_{\left\{z;d(z,B)>\frac{\varepsilon}{2}\right\}} f\, \mathrm{Tr_s}\left[S_{t,T}(z,z)\right] dv_M + \int_{\left\{z,d(z,B)\leq\frac{\varepsilon}{2}\right\}} f\, \mathrm{Tr_s}\left[S_{t,T}(z,z)\right] dv_M .$$

Now by Theorem 13.5, for $t \in]0,1], 0 \leq T \leq d/t$,

(13.39)

$$\left| \int_{\left\{z,d(z,B)>\frac{\varepsilon}{2}\right\}} f\, \mathrm{Tr_s}\left[S_{t,T}(z,z)\right] dv_M - \int_{\left\{z,d(z,B)>\frac{\varepsilon}{2}\right\}} f\left(\mathrm{rk}(F)\int^B \exp\left(-B_{T^2}\right)\right.\right.$$

$$\left.\left. -td\int^B \frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp\left(-B_{T^2}\right)\right)\right| \quad \leq \quad Ct^2 .$$

Now, we use the notation of Section 12 a). If $x \in B$, let $A^x_{t,T}$ be the operator acting on $\mathbf{F}_x$,

$$\text{(13.40)} \qquad A^x_{t,T} = -t^2\Delta^{T_xM} + T^2|y|^2 - ntT + tT\left(N^+ + \mathrm{ind}(x) - N^-\right).$$

With the notation of (12.7), $A^x_{t,T} = t^2(\widetilde{D}^{T_xM}_{T/t})^2$.

Definition 13.6. Let $U^x_{t,T}(y,y')(y,y' \in T_xM)$ be the smooth kernel associated to the operator $\exp(-A^x_{t,T})$. Let $U^{D,x}_{t,T}(y,y')(y,y' \in T_xM, |y|,|y'| \leq \varepsilon)$ be the smooth kernel associated to the operator $\exp(-A^x_{t,T})$, with Dirichlet conditions on $\partial B^M(x,\varepsilon)$.

By the same arguments as in the proof of [BL2, Proposition 11.10], which were already used in the proof of Proposition 13.3, we find that if $t \in]0,1], T \in [0,\frac{d}{t}], y \in B^M(x,\varepsilon/2)$, then

$$\left|\left(U^x_{t,T} - U^{D,x}_{t,T}\right)(y,y)\right| \leq c\exp\left(-\frac{C}{t^2}\right). \tag{13.41}$$

In Definition 13.1, we take $b=\varepsilon$. Then

$$S^{D,x}_{t,T}(y,y) = U^{D,x}_{t,T}(y,y)\ ,\ y \in B^M(x,\varepsilon). \tag{13.42}$$

By (13.2), (13.41), (13.42), we see that if $t \in]0,1], T \in [0,\frac{d}{t}], y \in B^M(x,\frac{\varepsilon}{2})$,

$$\left|\left(S_{t,T} - U^x_{t,T}\right)(y,y)\right| \leq c\exp\left(-\frac{C}{t^2}\right). \tag{13.43}$$

So from (13.43), we see that if $t \in]0,1], T \in [0,\frac{d}{t}]$, then

$$\left|\int_{|y|\leq\varepsilon/2} f\left(\mathrm{Tr_s}\left[S_{t,T}(y,y)\right] - \mathrm{Tr_s}\left[U^x_{t,T}(y,y)\right]\right) dv_M\right| \leq c\exp\left(-\frac{C}{t^2}\right). \tag{13.44}$$

Using Mehler's formula [GlJ, Theorem 1.5.10], as in (12.50), with $U^x_{t,T} = Q^x_{t^2,T/t}$, we get

$$\begin{aligned}U^x_{t,T}(y,y) = \left(\frac{Te^{2tT}}{2\pi t\sinh(2tT)}\right)^{n/2} \exp\left(-\frac{T}{t}\tanh(tT)|y|^2\right)\\ \exp\left(-2tT\left(N^+ + \mathrm{ind}(x) - N^-\right)\right).\end{aligned} \tag{13.45}$$

Now

$$\begin{aligned}&\mathrm{Tr_s}\left[\exp\left(-2tT\left(N^+ + \mathrm{ind}(x) - N^-\right)\right)\right]\\ &= \mathrm{rk}(F)\left(1-e^{-2tT}\right)^{n-\mathrm{ind}(x)} e^{-2tT\,\mathrm{ind}(x)}\left(1-e^{2tT}\right)^{\mathrm{ind}(x)}.\end{aligned} \tag{13.46}$$

Equivalently,

(13.47)

$$\operatorname{Tr}_s\left[\exp\left(-2tT\left(N^+ + \operatorname{ind}(x) - N^-\right)\right)\right] = \operatorname{rk}(F)(-1)^{\operatorname{ind}(x)}\left(1-e^{-2tT}\right)^n .$$

So by (13.45),(13.47), we get

(13.48)

$$\operatorname{Tr}_s\left[U^x_{t,T}(y,y)\right] = (-1)^{\operatorname{ind}(x)}\operatorname{rk}(F)\left(\frac{T}{\pi t}\tanh(tT)\right)^{n/2}\exp\left(-\frac{T}{t}\tanh(tT)|y|^2\right).$$

In particular, we deduce from (13.48) that for any $T \geq 0$, as $t \to 0$,

(13.49)

$$\operatorname{Tr}_s\left[fU^x_{t,T}(y,y)\right] = (-1)^{\operatorname{ind}(x)}\operatorname{rk}(F)f\left(\frac{T^2}{\pi}\right)^{n/2}\exp\left(-T^2|y|^2\right) + O(t^2),$$

which fits with (13.3) and (13.43).

Now using (13.48), we find that

$$(13.50)\quad \int_{|y|\leq\varepsilon/2} f\operatorname{Tr}_s\left[U^x_{t,T}(y,y)\right]dy - (-1)^{\operatorname{ind}(x)}\operatorname{rk}(F)\int_{|y|\leq\varepsilon/2} f\left(\frac{T^2}{\pi}\right)^{n/2}\exp\left(-T^2|y|^2\right)dy$$
$$= (-1)^{\operatorname{ind}(x)}\operatorname{rk}(F)\left\{\int_{|y|\leq\varepsilon/2\left(\frac{T}{t}\tanh(tT)\right)^{1/2}} f\left(\left(\frac{t}{T\tanh(tT)}\right)^{1/2}y\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} - \int_{|y|\leq\varepsilon/2T} f\left(\frac{y}{T}\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\right\}.$$

Recall that y^+, y^- are the projections of $y \in T_xM$ on $T_xW^s(x), T_xW^u(x)$. Then by (7.12),

$$(13.51)\qquad f(y) = f(x) + \frac{1}{2}\left(\left|y^+\right|^2 - \left|y^-\right|^2\right), |y| \leq \varepsilon.$$

Set

$$(13.52)\qquad T' = tT$$

Then

$$(13.53)\qquad (T'\tanh(T')) \leq T'.$$

Moreover

$$\int_{|y|\leq \frac{\varepsilon T'}{2t}} \exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} - \int_{|y|\leq \frac{\varepsilon}{2t}(T'\tanh T')^{1/2}} \exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} \tag{13.54}$$
$$= \int_{\frac{\varepsilon}{2t}(T'\tanh T')^{1/2}\leq |y|\leq \frac{\varepsilon T'}{2t}} \exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}.$$

Now if $0 < a < b < +\infty$,

(13.55)

$$\int_{|y|\in[a,b]} \exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} = C_n \int_a^b \exp\left(-r^2\right) r^{n-1}dr. \leq C\exp\left(-a^2\right) b^{n-1}(b-a).$$

From (13.55), we deduce that

$$\int_{\frac{\varepsilon}{2t}(T'\tanh T')^{1/2}\leq |y|\leq \frac{\varepsilon T'}{2t}} \exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} \tag{13.56}$$
$$\leq C\exp\left(-\frac{\varepsilon^2}{4t^2}T'\tanh(T')\right)\left(\frac{\varepsilon T'}{2t}\right)^{n-1}\frac{\varepsilon}{2t}\left(T' - (T'\tanh(T'))^{1/2}\right).$$

Take now $d > 0$. Then there exist $c > 0, c' > 0$, such that for $T' \in [0, d]$,

$$\begin{gathered}\left|T' - (T'\tanh(T'))^{1/2}\right| \leq cT'^3,\\ T'\tanh(T') \geq c'T'^2.\end{gathered} \tag{13.57}$$

By (13.56), (13.57), we deduce that for $T' \in [0, d]$,

$$\frac{1}{t^2}\int_{\frac{\varepsilon}{2t}(T'\tanh T')^{1/2}\leq |y|\leq \frac{\varepsilon T'}{2t}} \exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} \tag{13.58}$$
$$\leq C\left(\frac{T'}{t}\right)^{n+2}\exp\left(-\frac{\varepsilon^2 c'}{4}\frac{T'^2}{t^2}\right) \leq C'.$$

Similarly,

$$\frac{1}{t^2}\left|\frac{t^2}{T'^2}\int_{|y|\leq \frac{\varepsilon T'}{2t}} \left(\left|y^+\right|^2 - \left|y^-\right|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\right. \tag{13.59}$$
$$\left. - \frac{t^2}{T'\tanh(T')}\int_{|y|\leq \frac{\varepsilon}{2t}(T'\tanh(T'))^{1/2}} \left(\left|y^+\right|^2 - \left|y^-\right|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\right|$$
$$\leq C\left|\frac{1}{T'\tanh(T')} - \frac{1}{T'^2}\right|$$

$$+\frac{1}{T'^2}\left|\int_{\frac{\varepsilon}{2t}(T'\tanh(T'))^{1/2}\le|y|\le\frac{\varepsilon T'}{2t}}\left(\left|y^+\right|^2-\left|y^-\right|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\right|.$$

Also there is $C'>0$ such that if $T'\in[0,d]$,

$$\left|\frac{1}{T'\tanh(T')}-\frac{1}{T'^2}\right|\le C. \tag{13.60}$$

Moreover by using (13.58), we get for $T'\in[0,d]$,

$$\begin{aligned}\frac{1}{T'^2}&\left|\int_{\frac{\varepsilon}{2t}(T'\tanh(T'))^{1/2}\le|y|\le\frac{\varepsilon T'}{2t}}\left(\left|y^+\right|^2-\left|y^-\right|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\right|\\ &\le\frac{\varepsilon^2}{4t^2}\int_{\frac{\varepsilon}{2t}(T'\tanh(T'))^{1/2}\le|y|\le\frac{\varepsilon T'}{2t}}\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\le C'.\end{aligned} \tag{13.61}$$

By (13.50), (13.51), (13.54), (13.58), (13.59)–(13.61), we find that there exists $C>0$ such that if $t\in]0,1]$, $0\le T\le\frac{d}{t}$, then

$$\begin{aligned}\Bigg|\int_{|y|\le\frac{\varepsilon}{2}}f\left[\mathrm{Tr_s}\left[U^x_{t,T}(y,y)\right]\right]-(-1)^{\mathrm{ind}(x)}\,\mathrm{rk}(F)&\\ \int_{|y|\le\frac{\varepsilon}{2}}f\left(\frac{T^2}{\pi}\right)^{n/2}\exp\left(-T^2|y|^2\right)dy\Bigg|&\le Ct^2.\end{aligned} \tag{13.62}$$

From (13.39), (13.40), (13.62), we see that there exists $C>0$ such that if $t\in]0,1],0\le T\le\frac{d}{t}$, then

$$\begin{aligned}&\Bigg|\mathrm{Tr_s}\left[f\exp\left(-\left(tD+T\widehat{c}(\nabla f)\right)^2\right)\right]\\ &-\mathrm{rk}(F)\int_M f\int^B\exp\left(-B_{T^2}\right)-t\int_M fd\int^B\frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp\left(-B_{T^2}\right)\Bigg|\le Ct^2.\end{aligned} \tag{13.63}$$

Also

$$\begin{aligned}\int_M fd\int^B\frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp\left(-B_{T^2}\right)&\\ =-\int_M\int^B df\frac{1}{2}\widehat{\theta}\left(F,g^F\right)\exp\left(-B_{T^2}\right)&\\ =\int_M\int^B\frac{1}{2}\widehat{\theta}\left(F,g^F\right)df\exp\left(-B_{T^2}\right).&\end{aligned} \tag{13.64}$$

By Theorem 3.13, we find that

(13.65)
$$\int_M \int^B \frac{1}{2}\widehat{\theta}\left(F, g^F\right) df \exp\left(-B_{T^2}\right) = -\int_M \frac{1}{2}\theta\left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right).$$

From (13.63), (13.65), we get (7.24). The proof of Theorem 7.12 is completed. □

XIV. The asymptotics as $t \to 0$ of $\mathrm{Tr_s}[f\exp(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2)]$

The purpose of this Section is to prove Theorem 7.13, i.e. to calculate the asymptotics as $t \to 0$ of $\mathrm{Tr_s}[f\exp(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2)]$. In this Section, we assume that the simplifying assumptions of Section 7 b) are in force. Also we use the notation of Section 13.

The real number $T > 0$ is fixed in the whole Section.

Proposition 14.1. *Take $\alpha > 0$. There exist $c > 0, C > 0$ such that for $z \in M$, with $d^M(z,B) \geq \alpha$, and any $t \in]0,1]$, then*

$$\left|S_{t,\frac{T}{t}}(z,z)\right| \leq c\exp\left(-\frac{C}{t^2}\right). \tag{14.1}$$

Proof. In view of (10.2), the proof of (14.1) is identical to the proof of [BL2, Proposition 12.1]. □

Clearly

$$\mathrm{Tr_s}\left[f\exp\left(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2\right)\right] = \int_M f\,\mathrm{Tr_s}\left[S_{t,\frac{T}{t}}(z,z)\right]dv_M. \tag{14.2}$$

It easily follows from (13.44), (14.1), (14.2) that there exist $c > 0, C > 0$ such that if $t \in]0,1]$, then,

$$\left|\mathrm{Tr_s}\left[f\exp\left(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2\right)\right] - \sum_{x\in B}\int_{|y|\leq\frac{\varepsilon}{2}} f(y)\,\mathrm{Tr_s}\left[U^x_{t,\frac{T}{t}}(y,y)\right]dy\right| \leq c\exp\left(-\frac{C}{t^2}\right). \tag{14.3}$$

Take $x \in B$. By (13.48), we know that
(14.4)

$$\operatorname{Tr_s}\left[U^x_{t,\frac{T}{t}}(y,y)\right] = \operatorname{rk}(F)(-1)^{\operatorname{ind}(x)}\left(\frac{T}{\pi t^2}\tanh(T)\right)^{n/2}\exp\left(-\frac{T}{t^2}\tanh(T)|y|^2\right).$$

Using (13.51) and (14.4), we see that

$$\begin{aligned}(14.5)\quad & \frac{1}{t^2}\left\{\int_{|y|\le\frac{\varepsilon}{2}} f(y)\operatorname{Tr_s}\left[U_{t,\frac{T}{t}}(y,y)\right]dy - \operatorname{rk}(F)(-1)^{\operatorname{ind}(x)}f(x)\right\} \\ &= \operatorname{rk}(F)(-1)^{\operatorname{ind}(x)}\left\{\frac{1}{t^2}f(x)\left(\int_{|y|\le\frac{\varepsilon}{2}\frac{(T\tanh(T))^{1/2}}{t}}\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}-1\right)\right. \\ &\quad \left.+\frac{1}{T\tanh(T)}\int_{|y|\le\frac{\varepsilon}{2}\frac{(T\tanh(T))^{1/2}}{t}}\frac{1}{2}\left(|y^+|^2-|y^-|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\right\}.\end{aligned}$$

Clearly there are $c>0, C>0$ such that for $t\in]0,1]$,

$$(14.6)\quad \left|\int_{|y|\le\frac{\varepsilon}{2}\frac{(T\tanh(T))^{1/2}}{t}}\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}-1\right| \le c\exp\left(-\frac{CT\tanh(T)}{t^2}\right).$$

Moreover by (3.80),

$$\begin{aligned}(14.7)\quad \lim_{t\to 0}\frac{1}{T\tanh(T)}&\int_{|y|\le\frac{\varepsilon}{2}\frac{(T\tanh(T))^{1/2}}{t}}\frac{1}{2}\left(|y^+|^2-|y^-|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} \\ &= \frac{1}{T\tanh(T)}\int_{T_xM}\frac{1}{2}\left(|y^+|^2-|y^-|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}} \\ &= \frac{1}{T\tanh(T)}\left(\frac{n}{4}-\frac{\operatorname{ind}(x)}{2}\right).\end{aligned}$$

In view of (14.3), (14.5)–(14.7), we see that

$$\begin{aligned}(14.8)\quad \lim_{t\to 0}\frac{1}{t^2}&\left(\operatorname{Tr_s}\left[f\exp\left(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2\right)\right]-\operatorname{rk}(F)\operatorname{Tr}_s^B[f]\right) \\ &= \left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T\tanh(T)}.\end{aligned}$$

This is exactly Theorem 7.13. □

XV. The asymptotics of $\mathrm{Tr_s}[f\exp(-(tD+\frac{T}{t}\hat{c}(\nabla f))^2)]$ for $0 < t \leq 1,\ T \geq 1$

The purpose of this Section is to prove Theorem 7.14, i.e. to obtain an estimate involving $\mathrm{Tr_s}[f\exp(-(tD+\frac{T}{t}\hat{c}(\nabla f))^2)]$ in the range $0<t\leq 1, T\geq 1$.

As in Sections 13 and 14, we denote by $S_{t,\frac{T}{t}}(z,z')(z,z'\in M)$ the kernel of the operator $\exp(-(tD+\frac{T}{t}\hat{c}(\nabla f))^2)$.

This Section is organized as follows. In a) we give an estimate for $S_{t,\frac{T}{t}}(z,z)$ on the compact sets of $M\backslash B$. In b), we show that near $x\in B, S_{t,\frac{T}{t}}(z,z)$ is well approximated by the kernel $U^x_{t,\frac{T}{t}}(z,z)$ defined in Definition 13.6. Finally in c), we establish Theorem 7.14.

The organization of Section 15 b) is closely related to the organization of Section 12 b), although we work here in a different range of parameters. Also, in our proof of our main result, given in Theorem 7.1, we only need to establish Theorem 7.14 for $t=\varepsilon$ small enough. This simplifies the arguments of Section 15 b), where part of the difficulty is to extend the estimates in the range $t\in]0,t_0]$ (with $t_0\in]0,1]$) to the range $t\in]0,1]$.

In the whole Section, the simplifying assumptions of Section 7 b) will be in force. Also we use the notation of Section 13. In particular $\varepsilon>0$ is chosen as in Section 13 d).

a) An estimate for $S_{t,\frac{T}{t}}(z,z)$ on compact sets of $M\backslash B$

Proposition 15.1. *Take $\alpha > 0$. There exist $c > 0, C > 0$ such that for any $z \in M$ with $d^M(z,B) \geq \alpha$, and any $t \in]0,1], T \geq 1$, then*

$$\left|S_{t,\frac{T}{t}}(z,z)\right| \leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.1}$$

Proof. We proceed as in [BL2, Proposition 13.1]. Let $|S_{t,\frac{T}{t}}(z,z)|$ be the norm of the matrix $S_{t,\frac{T}{t}}(z,z)$ with respect to the trace. Since the operator $(tD+\frac{T}{t}\widehat{c}(\nabla f))^2$ is self-adjoint and nonnegative, we find that for any $\beta \in]0,1]$,

$$\left|S_{t,\frac{T}{t}}(z,z)\right| \leq \left|S_{t\beta,\frac{T\beta}{t}}(z,z)\right|. \tag{15.2}$$

Assume that $t \in]0,1], T \geq 1$. By taking $\beta = \frac{1}{\sqrt{T}}$ in (15.2), we get

$$\left|S_{t,\frac{T}{t}}(z,z)\right| \leq \left|S_{\frac{t}{\sqrt{T}},\frac{\sqrt{T}}{t}}(z,z)\right|. \tag{15.3}$$

Now $\frac{t}{\sqrt{T}} \in]0,1]$. By Proposition 14.1, we obtain,

$$\left|S_{\frac{t}{\sqrt{T}},\frac{\sqrt{T}}{t}}(z,z)\right| \leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.4}$$

From (15.3), (15.4), (15.1) follows. □

b) The kernel $S_{t,\frac{T}{t}}(z,z)$ near B and the harmonic oscillator

Theorem 15.2. *There exist $c > 0, C > 0$ such that if $t \in]0,1], T \geq 1$, if $x \in B, z \in B^M(x,\varepsilon)$, then*

$$\left|\left(S_{t,\frac{T}{t}} - U^x_{t,\frac{T}{t}}\right)(z,z)\right| \leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.5}$$

Proof. Let $S^{D,x}_{t,\frac{T}{t}}(z,z'))(z,z' \in B^M(x,\varepsilon))$ be the smooth kernel associated to the operator $\exp(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2)$, with Dirichlet boundary conditions on $\partial B^M(x,\varepsilon)$.

We claim that there exist $c>0, C>0$ such that if $t\in]0,1], T\geq 1, x\in B, z\in B^M(x,\varepsilon)$, then

$$\left|\left(S_{t,\frac{T}{t}}-S^{D,x}_{t,\frac{T}{t}}\right)(z,z)\right|\leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.6}$$

To establish (15.6), we use the notation and the methods in the proof of Theorem 12.6. Recall that $S_{t,\frac{T}{t}}=P_{t^2,\frac{T}{t^2}}, S^{D,x}_{t,\frac{T}{t}}=P^{D,x}_{t^2,\frac{T}{t^2}}$. By (12.15), we get for $z,z'\in B^M(x,\varepsilon)$,

(15.7)

$$\left(S_{t,\frac{T}{t}}-S^{D,x}_{t,\frac{T}{t}}\right)(z,z)=p_{t^2}(z,z)E^{R^{t^2}_{z,z}}\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1|df(z_s)|^2\,ds\right\}V_1^{t^2,\frac{T}{t^2}}\tau_0^1 1_{S\leq 1}\right].$$

By (12.16), there exists $\gamma>0$ such that if $t\in]0,1], T\geq 1$,

$$\left|V_1^{t^2,\frac{T}{t^2}}\right|\leq\exp(\gamma T). \tag{15.8}$$

From (15.7), (15.8), we get

$$\begin{aligned}\left|\left(S_{t,\frac{T}{t}}-S^{D,x}_{t,\frac{T}{t}}\right)(z,z')\right|\leq{}&\exp(\gamma T)p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\\&\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1|df(z_s)|^2\,ds\right\}1_{S\leq 1}\right].\end{aligned} \tag{15.9}$$

As in (12.18), we have

$$\begin{aligned}&p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1|df(z_s)|^2\,ds\right\}1_{S\leq 1}\right]\\ \leq{}&p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1|df(z_s)|^2\,ds\right\}1_{S\leq \frac{1}{2}}\right]\\&+p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1|df(z_s)|^2\,ds\right\}1_{\frac{1}{2}\leq S\leq 1}\right].\end{aligned} \tag{15.10}$$

By using time reversal, we find that the two quantities in the right-hand side of (15.10) are deduced from each other by interchanging z and z'. So we only need to estimate the first one.

We still define the stopping time S' as in (12.19). By the analogue of (12.20)–(12.22), we obtain for $0\leq h\leq 1/4$,

$$p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1|df(z_s)|^2\,ds\right\}1_{S\leq \frac{1}{2}}\right] \tag{15.11}$$

$$\leq \quad p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[S \leq 1/2, S'-S \leq h\right] + p_{t^2}(z,z')\exp\left\{-\frac{T^2}{t^2}\beta h\right\}.$$

Let $R^{t^2}_z$ be the probability law of the Brownian motion z associated to the metric $\frac{g^{TM}}{2t^2}$, with $z_0 = z$. By [B2, Definition 2.4], we know that since $h \leq 1/4$,

(15.12)

$$p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[S \leq 1/2, S'-S \leq h\right] = E^{R^{t^2}_z}\left[1_{S\leq 1/2, S'-S\leq h}\, p_{\frac{t^2}{4}}\left(z_{3/4}, z'\right)\right].$$

For any $s > 0$, the operator $\exp(s\Delta^M)$ is positive. Therefore if $\bar{z}, \bar{z}' \in M$,

$$p_s(\bar{z},\bar{z}') \leq p_s^{1/2}(\bar{z},\bar{z})p_s^{1/2}(\bar{z}',\bar{z}'). \tag{15.13}$$

From (15.13), we deduce that there exists $C > 0$ such that for $s \in]0,1], \bar{z}, \bar{z}' \in M$,

$$p_s(\bar{z},\bar{z}') \leq \frac{C}{s^{n/2}}. \tag{15.14}$$

Moreover, by [V, proof of Theorem 5.1], we see that there exists $c > 0$ such that for any $z \in B^M(x,\varepsilon)$,

$$R^{t^2}_z\left[S \leq 1/2, S'-S \leq h\right] \leq c\exp\left(-\frac{\varepsilon^2}{32ht^2}\right). \tag{15.15}$$

So from (15.12)–(15.15), we obtain

$$\begin{aligned} p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}&\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1 |df(z_s)|^2\, ds\right\}1_{S\leq 1/2}\right] \\ &\leq \frac{C}{t^n}\left[\exp\left(-\frac{\varepsilon^2}{32ht^2}\right) + \exp\left(-\frac{T^2}{t^2}\beta h\right)\right]. \end{aligned} \tag{15.16}$$

In (15.16), we take

$$h = \inf\left\{\frac{\varepsilon}{\sqrt{32\beta}T}, \frac{1}{4}\right\}. \tag{15.17}$$

Then we find that there exist $c > 0, C > 0$ such that if $t \in]0,1], T \geq 0, x \in B, z, z' \in B^M(x,\varepsilon)$,

(15.18)

$$p_{t^2}(z,z')E^{R^{t^2}_{z,z'}}\left[\exp\left\{-\frac{T^2}{t^2}\int_0^1 |df(z_s)|^2\, ds\right\}1_{S\leq 1/2}\right] \leq \frac{c}{t^n}\exp\left(-\frac{CT}{t^2}\right).$$

From (15.9), (15.10), (15.18), we deduce that there exist $c>0, C>0$ such that for $t\in]0,1], T\geq 0, x\in B, z,z'\in B^M(x,\varepsilon)$,

$$\left|\left(S_{t,\frac{T}{t}}-S^{D,x}_{t,\frac{T}{t}}\right)(z,z')\right|\leq \frac{c}{t^n}\exp\left(-\left(C-\gamma t^2\right)\frac{T}{t^2}\right). \tag{15.19}$$

Using (15.19), we find that there exist $t_0\in]0,1]$ and $c>0, C>0$ such that for $t\in]0,t_0], T\geq 0, x\in B, z,z'\in B^M(x,\varepsilon)$, then

$$\left|\left(S_{t,\frac{T}{t}}-S^{D,x}_{t,\frac{T}{t}}\right)(z,z')\right|\leq \frac{c}{t^n}\exp\left(-\frac{CT}{t^2}\right). \tag{15.20}$$

So (15.6) is proved for $t\in]0,t_0]$.

By the same arguments as before, we see that if $t\in]0,t_0], T\geq 0, x,x'\in B, x\neq x'$, if $z\in B^M(x,\varepsilon), z'\in B^M(x',\varepsilon)$, then

$$\left|S_{t,\frac{T}{t}}(z,z')\right|\leq \frac{c}{t^n}\exp\left(-\frac{CT}{t^2}\right). \tag{15.21}$$

Also by (12.34), for any $\tau>0$, there exists $C'>0$ such that for $t\in]0,1], T\geq \tau, z\in M$, then

$$\left|S_{t,\frac{T}{t}}(\bar{z},\bar{z})\right|\leq C'\left(\frac{T}{t^2}\right)^{n/2}. \tag{15.22}$$

Since $\exp(-(tD+\frac{T}{t}\hat{c}(\nabla f))^2)$ is a positive operator, then if $\bar{z},\bar{z}'\in M$,

$$\left|S_{t,\frac{T}{t}}(\bar{z},\bar{z}')\right|\leq \left|S_{t,\frac{T}{t}}(\bar{z},\bar{z})\right|^{1/2}\left|S_{t,\frac{T}{t}}(\bar{z}',\bar{z}')\right|^{1/2}. \tag{15.23}$$

Clearly there exists $m\in \mathrm{N}$ such that if $t\in [t_0,1]$, then $\frac{t}{\sqrt{m}}\in]0,t_0]$. Moreover, if $z\in B^M(x,\varepsilon)$,

$$\begin{aligned} S_{t,\frac{T}{t}}(z,z)=\int_{M^{m-1}} & S_{\frac{t}{\sqrt{m}},\frac{T}{t\sqrt{m}}}(z,x_1)\cdots \\ & \cdots S_{\frac{t}{\sqrt{m}},\frac{T}{t\sqrt{m}}}(x_{m-1},z)\,dv_M(x_1)\cdots dv_M(x_{m-1}). \end{aligned} \tag{15.24}$$

Using (15.1), (15.20)–(15.24), we see that there exist $c>0, C>0$ such that for $t\in [t_0,1], T\geq 1, x\in B, z\in B^M(x,\varepsilon)$, then

$$\Bigg|S_{t,\frac{T}{t}}(z,z)-\int_{(B^M(x,\varepsilon))^{m-1}} S_{\frac{t}{\sqrt{m}},\frac{T}{t\sqrt{m}}}(z,x_1)\cdots \tag{15.25}$$

$$\cdots S_{\frac{t}{\sqrt{m}},\frac{T}{t\sqrt{m}}}\left(x_{m-1},z\right)dv_M\left(x_1\right)\cdots dv_M\left(x_{m-1}\right)\Big| \leq c\exp\left(-\frac{CT}{t^2}\right).$$

By (12.38), we find that for any $\tau > 0$, there exists $c > 0$ such that for $t \in]0,1], T \geq \tau, \ z \in B^M(x,\varepsilon)$,

$$\left|S^{D,x}_{t,\frac{T}{t}}(z,z)\right| \leq c\left(\frac{T}{t^2}\right)^{n/2}. \tag{15.26}$$

Also as in (15.23), if $z, z' \in B^M(x,\varepsilon)$, then

$$\left|S^{D,x}_{t,\frac{T}{t}}(z,z')\right| \leq \left|S^{D,x}_{t,\frac{T}{t}}(z,z)\right|^{1/2}\left|S^{D,x}_{t,\frac{T}{t}}(z',z')\right|^{1/2} \tag{15.27}$$

Using (15.20), (15.21), the fact that if $t \in [t_0,1]$, then $\frac{t}{\sqrt{m}} \in]0,t_0]$, and also (15.25), (15.27), we find that there exist $c' > 0, C' > 0$ such that if $t \in [t_0,1], T \geq 1, x \in B, z \in B^M(x,\varepsilon)$, then

$$\left|\left(S_{t,\frac{T}{t}} - S^{D,x}_{t,\frac{T}{t}}\right)(z,z)\right| \leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.28}$$

Equation (15.6) follows from (15.20) and (15.28).

Let $U^{D,x}_{t,T}(y,y')$ $(y,y' \in B^{T_xM}(0,\varepsilon))$ be the smooth kernel associated to the operator $\exp(-A^x_{t,T})$ with Dirichlet boundary conditions on $\partial B^{T_xM}(0,\varepsilon)$. By proceeding as in (15.7)–(15.20), one finds that there exist $t_0 \in]0,1], c > 0, C > 0$ such that if $t \in]0,t_0], T \geq 1, y, y' \in B^{T_xM}(0,\varepsilon)$, then

$$\left|\left(U_{t,\frac{T}{t}} - U^{D,x}_{t,\frac{T}{t}}\right)(y,y')\right| \leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.29}$$

Moreover the kernel $U^x_{t,\frac{T}{t}}(y,y')$ is explicitly known by Mehler's formula [GlJ, Theorem 1.5.10]. One can then easily obtain estimates at infinity for $U_{t,\frac{T}{t}}(y,y')$ and show that the obvious analogue of (15.25)–(15.28) holds. As in (15.6), we deduce that there exist $c' > 0, C' > 0$ such that for any $t \in]0,1], T \geq 1, y \in B^{T_xM}(0,\varepsilon)$,

$$\left|\left(U^x_{t,\frac{T}{t}} - U^{D,x}_{t,\frac{T}{t}}\right)(y,y)\right| \leq c\exp\left(-\frac{C'T}{t^2}\right). \tag{15.30}$$

Finally, if $z \in B^M(x,\varepsilon)$, one has the obvious

$$S^{D,x}_{t,\frac{T}{t}}(z,z) = U^{D,x}_{t,\frac{T}{t}}(z,z). \tag{15.31}$$

Using (15.16), (15.30), (15.31), we get (15.5). The proof of Theorem 15.2 is completed. $\square$

c) Proof of Theorem 7.14.

Clearly,

(15.32)
$$\mathrm{Tr_s}\left[f\exp\left(-(tD+\frac{T}{t}\hat{c}(\nabla f))^2\right)\right]=\int_M f(z)\,\mathrm{Tr_s}\left[S_{t,\frac{T}{t}}(z,z)\right]dv_M(z).$$

Now by Proposition 15.1, we know that

$$\left|\int_{\{z,d(z,B)>\varepsilon\}} f(z)\,\mathrm{Tr_s}\left[S_{t,\frac{T}{t}}(z,z)\right]dv_M(z)\right|\leq c\exp\left(-\frac{CT}{t^2}\right). \tag{15.33}$$

Moreover if $x\in B$, by Theorem 15.2, we get

$$\left|\int_{|y|\leq\varepsilon} f(y)\,\mathrm{Tr_s}\left[\left(S_{t,\frac{T}{t}}-U^x_{t,\frac{T}{t}}\right)(y,y)\right]dy\right|\leq c'\exp\left(-\frac{C'T}{t^2}\right). \tag{15.34}$$

Also by (14.4), we have

$$\int_{|y|\leq\varepsilon} f(y)\,\mathrm{Tr_s}\left[U^x_{t,\frac{T}{t}}(y,y)\right]dy \tag{15.35}$$
$$=(-1)^{\mathrm{ind}(x)}\,\mathrm{rk}(F)\int_{|y|\leq\frac{\varepsilon}{t}(T\tanh(T))^{1/2}} f\left(\frac{t}{(T\tanh(T))^{1/2}}y\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}.$$

Equivalently, using (13.51) and (15.35), we find that

$$\int_{|y|\leq\varepsilon} f(y)\,\mathrm{Tr_s}\left[U^x_{t,\frac{T}{t}}(y,y)\right]dy \tag{15.36}$$
$$=\mathrm{rk}(F)(-1)^{\mathrm{ind}(x)}\Big\{f(x)\int_{|y|\leq\frac{\varepsilon}{t}(T\tanh(T))^{1/2}}\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}$$
$$+\frac{t^2}{T\tanh(T)}\int_{|y|\leq\frac{\varepsilon}{t}(T\tanh(T))^{1/2}}\frac{1}{2}\left(\left|y^+\right|^2-\left|y^-\right|^2\right)\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}\Big\}.$$

Clearly,

(15.37)
$$1-\int_{|y|\leq\frac{\varepsilon}{t}(T\tanh(T))^{1/2}}\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}=\int_{|y|>\frac{\varepsilon}{t}(T\tanh(T))^{1/2}}\exp\left(-|y|^2\right)\frac{dy}{\pi^{n/2}}$$

So there exist $c > 0, C > 0$ such that if $t \in]0,1], T \geq 1$

(15.38)
$$\left| \frac{1}{t^2} \left(1 - \int_{|y| \leq \frac{\varepsilon}{t}(T\tanh(T))^{1/2}} \exp\left(-|y|^2\right) \frac{dy}{\pi^{n/2}} \right) \right| \leq c \exp\left(-\frac{CT}{t^2}\right)$$

Also by (3.80),

$$\int_{T_xM} \frac{1}{2} \left(\left|y^+\right|^2 - \left|y^-\right|^2 \right) \exp\left(-|y|^2\right) \frac{dy}{\pi^{n/2}} = \left(\frac{1}{4}n - \frac{1}{2}\operatorname{ind}(x) \right). \tag{15.39}$$

From (15.39), we deduce that

$$\begin{aligned} &\frac{1}{t^2} \left[\frac{t^2}{T\tanh(T)} \int_{|y| \leq \frac{\varepsilon}{t}(T\tanh(T))^{1/2}} \frac{1}{2} \left(\left|y^+\right|^2 - \left|y^-\right|^2 \right) \right. \\ &\qquad \left. \exp\left(-|y|^2\right) \frac{dy}{\pi^{n/2}} - \frac{t^2}{T} \left(\frac{1}{4}n - \frac{1}{2}\operatorname{ind}(x) \right) \right] \\ &= -\frac{1}{T\tanh(T)} \int_{|y| > \frac{\varepsilon}{t}(T\tanh(T))^{1/2}} \frac{1}{2} \left(\left|y^+\right|^2 - \left|y^-\right|^2 \right) \exp\left(-|y|^2\right) \frac{dy}{\pi^{n/2}} \\ &\qquad + \frac{1}{T} \left(\frac{1}{\tanh(T)} - 1 \right) \left(\frac{1}{4}n - \frac{1}{2}\operatorname{ind}(x) \right). \end{aligned} \tag{15.40}$$

Clearly, there exist $c > 0, C > 0$ such that for $t \in]0,1], T \geq 1$,

$$\begin{aligned} &\left| \frac{1}{T\tanh(T)} \int_{|y| > \frac{\varepsilon}{t}(T\tanh(T))^{1/2}} \frac{1}{2} \left(\left|y^+\right|^2 - \left|y^-\right|^2 \right) \exp\left(-|y|^2\right) \frac{dy}{\pi^{n/2}} \right| \\ &\qquad \leq c \exp\left(-\frac{CT}{t^2}\right). \end{aligned} \tag{15.41}$$

Moreover as $T \to +\infty$,

$$\frac{1}{T} \left(\frac{1}{\tanh(T)} - 1 \right) = \frac{1}{T} O\left(e^{-2T}\right). \tag{15.42}$$

Using (15.36), (15.38), (15.40)–(15.42), we find that there exist $c > 0, C > 0$ such that for any $x \in B, t \in]0,1], T \geq 1$,

$$\begin{aligned} &\frac{1}{t^2} \left| \int_{|y| \leq \varepsilon} f(y) \operatorname{Tr_s}\left[U^x_{t,\frac{T}{t}}(y,y) \right] dy \right. \\ &\qquad \left. - \operatorname{rk}(F)(-1)^{\operatorname{ind}(x)} \left(f(x) + \frac{t^2}{T} \left(\frac{1}{4}n - \frac{1}{2}\operatorname{ind}(x) \right) \right) \right| \leq c \exp(-CT). \end{aligned} \tag{15.43}$$

From (15.33), (15.34), (15.43), we see that there exist $c>0, C>0$ such that if $t\in]0,1], T\geq 1$, then

$$
\begin{aligned}
(15.44) \qquad & \left| \frac{1}{t^2}\left\{ \mathrm{Tr}_s\left[f\exp\left(-(tD+\frac{T}{t}\widehat{c}(\nabla f))^2\right)\right] \right.\right. \\
& \left.\left. -\operatorname{rk}(F)\,\mathrm{Tr}_s^B[f] - \frac{t^2}{T}\left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\right\}\right| \leq c\exp(-CT).
\end{aligned}
$$

The proof of Theorem 7.14 is completed. □

XVI. A direct proof of a formula comparing two Milnor metrics

Let M be a compact manifold. Let F be a flat vector bundle on M, and let g^F be a smooth metric on F.

Let $f, g : M \to \mathbb{R}$ be two Morse functions. Let $g_0^{TM}, g_0^{\prime TM}$ be two smooth metrics on TM, and let X, X', be the gradient vector fields of f, g with respect to the metric $g_0^{TM}, g_0^{\prime TM}$.

We assume that X and X' verify the Smale transversality conditions.

Let B and B' be the zero sets of X and X'. As in Section 7 a), let $\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}$ and $\| \ \|^{\mathcal{M},X'}_{\det H^\bullet(M,F)}$ be the Milnor metrics on the line $\det H^\bullet(M,F)$ determined by the $g_x^F (x \in B)$ and the $g_{x'}^F (x' \in B')$.

Let g^{TM} be a smooth metric on TM, and let ∇^{TM} be the Levi-Civita connection on (TM, g^{TM}).

Theorem 16.1. *For any smooth metric g^{TM} on TM, the following identity holds*

$$(16.1) \qquad \operatorname{Log}\left(\frac{\| \ \|^{\mathcal{M},X'}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M},X}_{\det H^\bullet(M,F)}}\right)^2 = \int_M \theta\left(F, g^F\right) X'^{*}\psi\left(TM, \nabla^{TM}\right) - \int_M \theta\left(F, g^F\right) X^{*}\psi\left(TM, \nabla^{TM}\right).$$

Proof. Clearly (16.1) is a trivial consequence of Theorem 7.1. Here, we will give a direct proof of (16.1).

By Proposition 6.1 and Theorem 6.3, we see that the right-hand side of (16.1) does not depend on the metric g^{TM}.

Assume first that $f = g$. Then X and X' are gradient vector fields of f. Observe that one can modify f so that X and X' are still gradient vector fields for f, and f takes distinct values on B. By Proposition 6.1,

$$(16.2)\quad \int_M \theta\left(F, g^F\right) X'^* \psi\left(TM, \nabla^{TM}\right) - \int_M \theta\left(F, g^F\right) X^* \psi\left(TM, \nabla^{TM}\right) = 0.$$

In the Appendix, Laudenbach constructs a smooth path $t \in [0,1] \to X_t$ of gradient vector fields for f, which verify the Thom-Smale transversality conditions except at a finite set $\{t_1, \cdots, t_q\} \subset [0,1]$, with $0 < t_1 < \cdots < t_q < 1$. For $t \notin \{t_1, \cdots, t_q\}$, let $(C^\bullet(W,F), \partial_t)$ be the Thom-Smale complex associated to X_t. As the notation indicates, the $\mathbb{Z}$-graded vector space $C^\bullet(W,F)$ does not depend on t, only the chain map ∂_t depends on t.

Clearly ∂_t is constant on the intervals $[0, t_1[,]t_1, t_2[, \cdots,]t_q, 1[$. For $1 \le i \le q$, let $(C^\bullet(W,F), \partial_{t_i}^-)$ and $(C^\bullet(W,F), \partial_{t_i}^+)$ be the Thom-Smale complexes on the left of t_i and on the right of t_i. By a result of Laudenbach given in Propositions 9 and 11 of the Appendix, there is an invertible linear map A, acting on the $\mathbb{Z}$-graded vector space $C^\bullet(W,F)$, which is a chain homomorphism from $(C^\bullet(W,F), \partial_{t_i}^-)$ into $(C^\bullet(W,F), \partial_{t_i}^+)$ and which identifies canonically the corresponding cohomology groups. By the Appendix, it is clear that for $1 \le j \le q$, the determinant of the action of A on each $C^j(W,F) (0 \le j \le n)$ is equal to 1. It then follows from the previous considerations that for $1 \le i \le q$,

$$(16.3)\quad \| \ \|^{\mathcal{M}, X_{t_i^-}}_{\det H^\bullet(M,F)} = \| \ \|^{\mathcal{M}, X_{t_i^+}}_{\det H^\bullet(M,F)}.$$

We deduce from (16.3) that

$$(16.4)\quad \| \ \|^{\mathcal{M}, X}_{\det H^\bullet(M,F)} = \| \ \|^{\mathcal{M}, X'}_{\det H^\bullet(M,F)}.$$

Using (16.2), (16.4), we see that if X and X' are the gradient vector fields of a common Morse function f, both sides of (16.1) are equal to 0.

Since the Milnor metric $\| \ \|^{\mathcal{M}, X}_{\det H^\bullet(M,F)}$ depends only on f, we will write $\| \ \|^{\mathcal{M}, f}_{\det H^\bullet(M,F)}$ instead of $\| \ \|^{\mathcal{M}, X}_{\det H^\bullet(M,F)}$.

Let now f and g be arbitrary Morse functions. Let $t \in [0,1] \to f_t$ be a smooth Cerf path [Ce] of smooth functions mapping M into $\mathbb{R}$, such that

$f_0 = f, f_1 = g$, which are Morse, except at a finite set of parameters $t_1, \cdots, t_q$ such that $0 < t_1 \cdots < t_q < 1$, where two critical points y'_t and y''_t of index j and $j+1$ $(0 \le j \le n-1)$ appear or disappear at a birth or death point $y \in M$. The form of $f_t(x)$ near (t_i, y) is given by Laudenbach in the Appendix, equation (8).

We claim that the continuous function $t \in [0,1] \backslash \{t_1, \cdots, t_q\} \to \int_M \theta(F, g^F)$ $(\nabla f_t)^* \ \psi(TM, \nabla^{TM}) \in \mathbb{R}$ extends to a continuous function $t \in [0,1] \to \mathbb{R}$. In fact we only need to consider the case where $t = t_i (1 \le i \le q)$. If $\theta(F, g^F)$ vanishes near the birth or death point $y \in M$, it is clear that t_i is also a point of continuity. More generally, there is a closed form $\theta'(F, g^F)$, which vanishes near $y \in M$, which is cohomologous to $\theta(F, g^F)$, i.e. there exists a smooth function $V : M \to \mathbb{R}$ such that

$$\theta'\left(F, g^F\right) - \theta\left(F, g^F\right) = dV. \tag{16.5}$$

By using the equation of currents (3.33), we see that if $t \in [0,1] \backslash \{t_1, \cdots, t_q\}$ and if B_t is the set of critical points of f_t, then

$$\begin{aligned}\int_M \theta\left(F, g^F\right)\left(\nabla f_t\right)^* \psi\left(TM, \nabla^{TM}\right) &= \int_M \theta'\left(F, g^F\right)\left(\nabla f_t\right)^* \psi\left(TM, \nabla^{TM}\right) \\ &\quad + \int_M V e\left(TM, \nabla^{TM}\right) - \sum_{x \in B_t} (-1)^{\mathrm{ind}(x)} V(x).\end{aligned} \tag{16.6}$$

Now the first two terms in the right-hand side of (16.6) are clearly continuous at $t = t_i$. Assume that when t increases, y is a birth point of two critical points, of index j and $j+1$. Then

$$\sum_{x \in B_{t_j^+}} (-1)^{\mathrm{ind}(x)} V(x) = \sum_{x \in B_{t_j^-}} (-1)^{\mathrm{ind}(x)} V(x) + V(x) - V(x). \tag{16.7}$$

Equivalently, the function $\sum_{x \in B_t} (-1)^{\mathrm{ind}(x)} V(x)$ extends to a continuous function near t_i. Of course this is still true if y is a death point. We have thus proved that $\int_M \theta(F, g^F)(\nabla f_t)^* \ \psi(TM, \nabla^{TM})$ extends to a continuous function on $[0,1]$.

By Proposition 6.4, we know that

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_M \theta\left(F, g^F\right)\left(\nabla f_t\right)^* \psi\left(TM, \nabla^{TM}\right) \\ - \int_M \theta\left(F, g^F\right)\left(\nabla f_0\right)^* \psi\left(TM, \nabla^{TM}\right)\Big)\end{aligned} \tag{16.8}$$

$$= \sum_{x \in B_t} (-1)^{\mathrm{ind}(x)} \theta\left(F, g^F\right) \left(\frac{\partial x}{\partial t}\right) \quad \text{on } [0,1] \backslash \{t_1, \cdots, t_q\}.$$

On the other hand, it is clear from the equation of $f_t(x)$ near (t_i, y_i) given in the Appendix, equation (8), that the right-hand side of (16.8) is an integrable function on $[0,1]$. Since the function $t \in [0,1] \to \int_M \theta(F, g^F)(\nabla f_t)^* \psi(TM, \nabla^{TM}) - \int_M \theta(F, g^F)(\nabla f_0)^* \psi(TM, \nabla^{TM})$ is continuous, we have the equality of distributions on $[0,1]$,

$$\text{(16.9)} \qquad \frac{\partial}{\partial t}\Big(\int_M \theta\left(F, g^F\right) \left(\nabla f_t\right)^* \psi\left(TM, \nabla^{TM}\right) - \int_M \theta\left(F, g^F\right) \left(\nabla f_0\right)^* \psi\left(TM, \nabla^{TM}\right) \Big) = \sum_{x \in B_t} (-1)^{\mathrm{ind}(x)} \theta\left(F, g^F\right) \left(\frac{\partial x}{\partial t}\right).$$

Take $t \in [0,1] \backslash \{t_1, \cdots, t_q\}$, and let g^{TM} be a smooth metric on TM, such that the corresponding gradient vector field ∇f_t verifies the Smale transversality conditions. Then for $t' \in [0,1]$ close enough to $t, \nabla f_{t'}$ still verifies the Smale transversality conditions, and the Thom complex $(C^\bullet(W_{t'}, F), \partial)$, for $\nabla f_{t'}$ can be identified to the complex $(C^\bullet(W_t, F), \partial)$ for ∇f_t, but of course, the identification is in general not isometric. In fact one has the easy identity

$$\text{(16.10)} \qquad \frac{\partial}{\partial t} \mathrm{Log} \left(\frac{\| \ \|^{\mathcal{M}, f_t}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M}, f_0}_{\det H^\bullet(M,F)}} \right)^2 = \sum_{x \in B_t} (-1)^{\mathrm{ind}(x)} \theta\left(F, g^F\right) \left(\frac{\partial x}{\partial t}\right) \quad \text{on } [0,1] \backslash \{t_1, \cdots, t_q\}.$$

We claim that the function $t \in [0,1] \backslash \{t_1, \cdots, t_q\} \to \mathrm{Log}\big(\frac{\| \ \|^{\mathcal{M}, f_t}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathcal{M}, f_0}_{\det H^\bullet(M,F)}}\big)^2 \in \mathbb{R}$ extends to a continuous function from $[0,1]$ into $\mathbb{R}$. Take $i, 1 \le i \le q$ and let g^{TM} be a smooth metric on TM taken as in the Appendix with respect to t_i. Then for $t \neq t_i$ and t near t_i, the Thom-Smale complex $(C^\bullet(W_t, F), \partial)$ is constant on the left and the right of t_i. Assume again that $y \in M$ is a birth point of two critical points y'_t, y''_t of index j an $j+1$. In particular, for $t > t_i$ close enough to t_i, we may identify $F_{y'_t}$ and $F_{y''_t}$ to F_y by using a flat trivialization of F near y.

Let $(C^\bullet_y(F), \partial')$ be the complex concentrated in degree i and $i+1$

$$0 \to F_{y'_t} \underset{\partial'}{\to} F_{y''_t} \to 0. \tag{16.11}$$

In (16.11), ∂' denotes the canonical identification of $F_{y'_t}$ and $F_{y''_t}$. Of course $(C^\bullet_y(W,F), \partial')$ is acyclic.

Then by Propositions 8 and 11 of the Appendix, there exists a linear automorphism A of the $\mathbb{Z}$-graded vector space $C^\bullet(W_{t_i^-}, F) \oplus C^\bullet_y(F)$, which has determinant 1 in every degree, such that

$$\left(C^\bullet\left(W_{t_i}^+, F\right), \partial\right) = \left(C^\bullet\left(W_{t_i^-}, F\right) \oplus C^\bullet_y(F), A^{-1}(\partial \oplus \partial')A\right), \tag{16.12}$$

which induces the canonical identification of the cohomology groups. Also the identification (16.12) identifies the metrics. Since A has determinant 1, it preserves the obvious metric on $\det(C^\bullet(W_{t_i}^-, F) \oplus C^\bullet_y(F))$. Clearly

$$\det\left(C^\bullet\left(W_{t_i}^-, F\right) \oplus C^\bullet_y(F)\right) = \det C^\bullet\left(W_{t_i}^-, F\right) \otimes \det C^\bullet_y(F). \tag{16.13}$$

Using (16.12), (16.13), we see that

$$\det C^\bullet\left(W_{t_i}^+, F\right) = \det C^\bullet\left(W_{t_i}^-, F\right) \otimes \det C^\bullet_y(F). \tag{16.14}$$

Now $g^F_{y'_t}$ and $g^F_{y''_t}$ can be considered as metrics on F_y. Also $\det C^\bullet_y(F)$ has a canonical section $(\det \partial')^{-1}$, and moreover

$$\|(\det \partial')^{-1}\|^2_{\det C^\bullet_y(F)} = \left(\det\left(\frac{g^F_{y'_t}}{g^F_{y''_t}}\right)\right)^{(-1)^i}. \tag{16.15}$$

In particular

$$\lim_{\substack{t>t_i \\ t\to t_i}} \left\|\left(\det \partial'\right)^{-1}\right\|_{\det C^\bullet_y(F)} = 1. \tag{16.16}$$

Using (16.12)–(16.16), we find that

$$\operatorname{Log}\left(\frac{\| \quad \|^{\mathcal{M}, f_{t_i^+}}_{\det H^\bullet(M,F)}}{\| \quad \|^{\mathcal{M}, f_{t_i^-}}_{\det H^\bullet(M,F)}}\right)^2 = 0. \tag{16.17}$$

We have thus proved that $\operatorname{Log}\left(\frac{\|\ \|^{\mathcal{M},f_t}_{\det H^\bullet(M,F)}}{\|\ \|^{\mathcal{M},f_0}_{\det H^\bullet(M,F)}}\right)^2$ extends to a continuous function of $t \in [0,1]$. As in (16.9), we deduce from (16.10) that we have the equality of distributions on [0, 1],

$$\frac{\partial}{\partial t}\operatorname{Log}\left(\frac{\|\ \|^{\mathcal{M},f_t}_{\det H^\bullet(M,F)}}{\|\ \|^{\mathcal{M},f_0}_{\det H^\bullet(M,F)}}\right)^2 = \sum_{x\in B_t}(-1)^{\operatorname{ind}(x)}\theta\left(F,g^F\right)\left(\frac{\partial x}{\partial t}\right). \tag{16.18}$$

From (16.9), (16.18), it is now clear that for $t \in [0,1]$,

$$\operatorname{Log}\left(\frac{\|\ \|^{\mathcal{M},f_t}_{\det H^\bullet(M,F)}}{\|\ \|^{\mathcal{M},f_0}_{\det H^\bullet(M,F)}}\right)^2 \tag{16.19}$$

$$= \int_M \theta\left(F,g^F\right)(\nabla f_t)^*\,\psi\left(TM,\nabla^{TM}\right) - \int_M \theta\left(F,g^F\right)(\nabla f_0)^*\,\psi\left(TM,\nabla^{TM}\right).$$

By taking $t=1$ in (16.19), we get (16.1). □

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Appendix. On the Thom-Smale complex

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Appendix. On the Thom-Smale complex by François LAUDENBACH

Morse theory has been much studied and still it is the source of very interesting papers (Witten [W], Floer [F1], [F2] ; see also the review and comments by Bott [B]). Therefore, it seems very hard to write down any new ideas on the subject. Nevertheless, the generic structure of the gradient field of a Morse function is always hidden, though it should be very simple. The aim of this paper is to uncover this simplicity, at least partially. Then some applications to de Rham currents are given. The bifurcation theory in 1-parameter families of gradient fields is also considered.

From now on, M is a C^∞ closed manifold (i.e. compact, without boundary), $f : M \to \mathbb{R}$ is a Morse function and X is the gradient field of $-f$ with respect to a metric on TM. If x is a critical point, $W^u(x)$ (resp. $W^s(x)$) will denote the unstable (resp. stable) manifold of x for the vector field X. We recall that $W^u(x)$ is a submanifold (non closed), diffeomorphic to an open ball whose dimension is the index $i(x)$ of f at x. In the sequel, we make the assumption (T), which is generically satisfied in the space of gradient vector fields [S] :

(T) *For any pair* x, y *of critical points, the manifolds* $W^u(x)$ *and* $W^s(y)$ *are transversal.*

A gradient vector field X satisfying (T) will be said to be Morse-Smale. Then it is known [R] that the closure $\overline{W}^u(x)$ of $W^u(x)$ is obtained by adding a union of unstable manifolds of smaller index. This will be proved again in a special case. For an arbitrary Morse-Smale vector field, this closure may be very complicated ; but when the vector field is gradient and is of *special Morse* type near the singularities (see the condition (SM) below), the structure of $\overline{W}^u(x)$ is very simple and we will describe it.

a) Submanifolds with conical singularities

We define submanifolds with conical singularities (abridged : *smcs*) of dimension k in a smooth manifold N^p of dimension p by recursion on the dimension k. For $k = 0$, it is a discrete set of points. A stratified set $\Sigma = (\Sigma_0, \Sigma_1, \ldots, \Sigma_k)$ in a manifold N^p is a *smcs* of dimension k if the following conditions are satisfied.

(1) For any $i \leq k$, $\Sigma_i - \Sigma_{i+1}$ is a smooth submanifold of dimension $k - i$.

(2) For any point $x \in \Sigma_i - \Sigma_{i+1}$, there exist a neigbourhood V diffeomorphic to a product of discs $D^{k-i} \times D^{p-k+i}$ and a *smcs* $T = (T_0, \ldots, T_i)$ of dimension i in D^{p-k+i} such that :

$$V \cap (\Sigma_0, \Sigma_1, \ldots, \Sigma_k) = D^{k-i} \times (T_0, \ldots, T_i, \emptyset, \ldots, \emptyset).$$

(3) If $x \in \Sigma_k$, there is a C^1 p-ball B centered at x such that :

$\Sigma' = \Sigma \cap \partial B$ is a *smcs* of dimension $(k-1)$ in the $(p-1)$ -sphere ,

and

$$(B, B \cap \Sigma_0, \ldots, B \cap \Sigma_{k-1}) = (B, c\Sigma'_0, \ldots, c\Sigma'_{k-1}),$$

where $c\Sigma'_i$ denotes the cone on Σ'_i with respect to the linear structure of the C^1-parametrized ball B.

Of course, a submanifold with boundary is a *smcs*. Also the singular locus of Σ lies in Σ_1, but some strata of Σ_1 may consist of regular points of Σ. When one does not need to label each stratum, one denotes a *smcs* by Σ or by (Σ_0, Σ_1).

The following facts may be easily proved by recursion on the dimension :

(4) There exists a neighbourhood V of Σ_1 in N and a deformation retract of $(V, V \cap \Sigma_0)$ onto Σ_1.

A submanifold S is said to be *transversal* to a *smcs* Σ if S is transversal to each stratum.

Lemma 1. *1) If a submanifold S of codimension q in N^p is transversal to $\Sigma = (\Sigma_0, \ldots, \Sigma_k)$, then $(S \cap \Sigma_0, \ldots, S \cap \Sigma_{k-q})$ is a smcs of dimension $k - q$ in S.*

2) Suppose that S has a product neighbourhood $S \times D^q$ in N^p, with $S = S \times \{0\}$. Then there exists a germ of diffeomorphisms H of $S \times D^q$ along $S \times \{0\}$ commuting with the projection on D^q, such that $H(\Sigma) \subset (\Sigma \cap S) \times D^q$.

Proof. 1) The first part is local. For instance, take $x \in S \cap \Sigma_{k-q}$. By (2), there is a chart near x such that $\Sigma = D^q \times (T_0, \dots, T_{k-q})$, where $T = (T_0, \dots, T_{k-q})$ is a *smcs* in D^{p-q}. The projection $p : D^q \times D^{p-q} \to D^{p-q}$ induces a local diffeomorphism $\varphi : S \to D^{p-q}$. In the corresponding chart on S, $S \cap \Sigma = T$, and so $S \cap \Sigma$ is a *smcs*.

2) One has a local stratified projection $\varphi^{-1}p : D^q \times D^{p-q} \to S$; by stratified projection we mean a C^1-map which is the identity on S and preserves the stratification $T_i \to S \cap T_i$.

It is easy to construct a stratified projection π' defined on a small tube U around S glueing together local stratified projections by means of partition of unity.

On the other hand, one has the projection $\pi'' : U \to D^q$ given by the trivialization of the normal bundle of S. Then (π', π'') is a diffeomorphism near S which is the wanted H. □

b) The main result

If x is a critical point of index k of the Morse function f, the Morse lemma states there exist coordinates $x_1, \dots, x_n$ near x such that

$$f(x_1, \dots, x_n) = f(x) - x_1^2 \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2. \tag{5}$$

The gradient vector field X is said to be *Special Morse* (SM) if, near every critical point, there exists a coordinate system $(x_1, \dots, x_n)$ such that f can be written as in (5), and that X is the gradient of $-f$ with respect to the canonical Euclidean metric associated to the coordinates $x_1, \dots, x_n$.

Proposition 2. *Assume that X verifies (T) and (SM).*

a) If x is a critical point of index k, then $(\overline{W}^u(x), \overline{W}^u(x) - W^u(x))$ is a smcs of dimension k.

b) $\overline{W}^u(x) - W^u(x)$ is stratified by unstable manifolds of critical points of index strictly less than k.

Remark 3. This proposition says that the unstable manifolds give rise to a structure of CW-complex on M, with one cell for each critical point, the attaching maps of the cells being given by the retractions of (4). In **[T]**, René Thom anticipated such a decomposition.

This result can probably be extended to the case where X verifies only (T) . To do this, one needs to change the definition of a *smcs* by delinearizing the cone construction.

Proof of Proposition 2. Let x be a critical point of f. For $a \in \mathbb{R}$, set $S_a(x) = \overline{W}^u(x) \cap \{f = a\}$. Then if $a < f(x)$ is close enough to $f(x)$, $S_a(x)$ is a sphere. As a decreases, this picture remains stable, as long as a does not coincide with the value of f at a critical point x', which, by (T), is such that $i(x') < i(x)$. The set $\overline{W}^u(x) \cap f^{-1}(f(x') - \epsilon)$ is no longer a smooth manifold. However the next lemma states it is a *smcs* and that its structure remains of the same type as we pass the other critical values of f. The singular strata of this set will be also described.

Let $W \subset \mathbb{R}^n$ be the canonical Morse model : it is a cobordism from a level set $V_{-1} \cong S^{i-1} \times D^{n-i}$ to $V_{+1} \cong D^i \times S^{n-i-1}$. It is equipped with the canonical Morse function $q = -x_1^2 \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$. The gradient field X of $-q$ is calculated with respect to the canonical Euclidean metric. Of course $V_{\pm 1} = \{q = \pm 1\} \cap W$.

Put $S = S^{i-1} \times \{0\}$ in V_{-1} and $S' = \{0\} \times S^{n-i-1}$ in V_{+1}.

Lemma 4. *Let (Σ', Σ'_1) be a smcs of dimension k in V_{+1}, transversal to S' with non empty intersection. Let Σ (resp. Σ_1) be the closure in V_{-1} of the set of points which lie on a gradient line descending from Σ' (resp. Σ'_1). Then Σ contains S and ($\Sigma, \Sigma_1 \bigcup S$) is a smcs of dimension k.*

Proof. In V_{-1} (resp. V_{+1}), we use polar coordinates $(\phi, \psi, r) \in S^{i-1} \times S^{n-i-1} \times [0,1]$. With these coordinates and when $r > 0$, the map ($V_{+1} - S'$) $\to$ ($V_{-1} - S$) is the identity. Set $K = \Sigma' \cap S'$, which is a *smcs* by the transversality condition.

First, suppose that Σ' is $D^i \times K \subset D^i \times S^{n-i-1}$, that is :

$$\Sigma' - K = \{(\phi, \psi, r) | \phi \in S^{i-1}, \psi \in K, r > 0\}.$$

In V_{-1}, $\Sigma - S$ is given by the same formula and therefore one has : $\Sigma = S^{i-1} \times cK$, which is a cone fibration, whose vertices lie in S. More generally, by Lemma 1, there is a diffeomorphism H of the form $H(\varphi, \psi, r) = (\varphi, \overline{\psi}(\varphi, \psi, r), r)$ with

$\overline{\psi}(\varphi, \psi, 0) = \psi$, such that $H(D^i \times K) = \Sigma$ near $\{0\} \times K$. Then Σ can be expressed locally as the image of $S^{i-1} \times cK$ by the map $\widetilde{H}$, which is the map H considered as a map from $S^{i-1} \times D^{n-i}$ into itself. Because the radial derivatives of $\widetilde{H}$ exist and are continuous, one verifies easily that $\widetilde{H}$ is C^1-diffeomorphism. Therefore Σ' is a *smcs*. □

Remark 5. 1) Σ is not transversal to S, both sides of the cobordism don't play the same role.

2) The proof of the lemma shows that each stratum of $\overline{W}^u(x)$ is C^∞. However the way in which strata adhere to each other may only be C^1.

Now, we prove Proposition 2. By condition (T), $\overline{W}^u(x) \cap f^{-1}(f(x') + \epsilon)$ is transversal to the sphere S' of the Morse model of x'. Then $\overline{W}^u(x) \cap f^{-1}(f(x') - \epsilon)$ is a *smcs* with a new singular stratum. One then proceed by recursion. The proof of Proposition 2 is completed.

□

c) The Thom-Smale complex

In this section, we make the same assumptions as in Proposition 2. An orientation is chosen on each $W^u(x)$.

For critical points x and y of f, with $i(y) = i(x) - 1$, we define the integer $n(x, y)$ as follows: $n(x, y) = 0$ when $W^u(y)$ does not lie in the closure of $W^u(x)$; otherwise, near $W^u(y)$, $W^u(x)$ consists of $n_+ + n_-$ connected components, $W^u(y)$ being the oriented boundary of n_+ of these. Then $n(x, y) = n_+ - n_-$.

Here is an alternative definition for $n(x, y)$. As $W^s(y)$ is co-oriented (i.e. transversally oriented), to each gradient line in $W^u(x) \cap W^s(y)$ (which is the union of a finite number of gradient lines), one can attach a sign and $n(x, y)$ is the sum of these signs.

Let C_k denote the free abelian group generated by the critical points of index k. The boundary operator $\partial : C_k \to C_{k-1}$ is defined by

$$\partial x = \Sigma n(x, y) y, \tag{6}$$

the sum being over all critical points of index $i(x) - 1$. On the other hand, as the geometry of $W^u(x)$ is "finite" near its boundary, we can consider the oriented $\overline{W}^u(x)'s$ as currents, and we have the following Stokes formula.

Proposition 6. *For any smooth differential form* ω *of degree* $k-1$ *on* M, *one has :*

$$\int_{\overline{W}^u(x)} d\omega = \sum_y n(x,y) \int_{\overline{W}^u(y)} \omega . \tag{7}$$

Proof. Let U be a neighborhood of $\overline{W}^u(x) - W^u(x)$ which has property (4) in Section a). We apply Stokes theorem to ω on $\overline{W}^u(x) - U$. As we let U shrink, the Stokes formula is seen to converge to the right-hand side of (7), because the singular locus of $\overline{W}^u(x) - W^u(x)$ is negligible with respect to the $(k-1)$-dimensional Lebesgue measure. □

Corollary. $\partial \circ \partial = 0$.

Proof. For any critical point y of index $k-2$, there exists a $(k-2)$-form whose integral over $W^u(y)$ is nonzero and which vanishes over the other $(k-2)$-unstable manifolds. The result then follows from (6), (7) and from the fact that $d \circ d = 0$. □

Let $I_* : C_* \to R_*$ be the map, with values in the complex R_* of de Rham currents, which associates to each critical point x the current of integration over the oriented manifold $\overline{W}^u(x)$. By (7), I_* is a morphism of complexes. Of course, as the $W^u(x)$'s are the cells of a CW-complex, it is known that the homology of C_* is canonically isomorphic to the singular homology of M [M1, Appendix A]. But, in our context, the weaker result with real coefficients may be stated as follows.

Proposition 7. $I_* : C_* \otimes \mathbb{R} \to R_*$ *induces a homology isomorphism.*

Proof. The stable manifolds are naturally co-oriented and give rise to a complex $(\bar{C}_*, \bar{\partial})$, graded by the co-index of critical points : $\bar{i}(x) = n - i(x)$. The pairing $\langle x,x\rangle = 1, \langle x,y\rangle = 0$ when $x \neq y$, satisfies $\langle \bar{\partial} x, y\rangle = \pm\langle x, \partial y\rangle$ and creates a duality between $\bar{C}_{n-*}$ and C_*. Then $H_{n-k}(\overline{C}_*;\mathbb{R}) \cong \operatorname{Hom}(H_k(C_*);\mathbb{R})$.

Like the unstable manifolds, a co-oriented stable manifold of dimension $n-k$ defines a current, which can be paired with smooth $n-k$ forms twisted by the orientation bundle of TM. The de Rham regularization operator [Rh,§15] transforms such currents into smooth differential forms of degree k, and maps $\bar{\partial}$ to d.

Let $\sigma \in C_k$ be a cycle. If σ is not homologous to 0 in C_* , there exists a cycle $\bar{\sigma}$ in $\bar{C}_{n-k}$ such that $\langle \bar{\sigma}, \sigma \rangle \neq 0$; then the de Rham regularization operator transforms $\bar{\sigma}$ into a closed $k-$ form ω such that $\langle \bar{\sigma}, \sigma \rangle = \int_\sigma \omega$. Therefore σ is not homologous to 0 as a current, and so, I_* is injective in homology.

By duality, to show that I_* is surjective in homology, we only need to prove that if ω is a closed k-form on M such that $\int_\sigma \omega = 0$ for any $\sigma \in C_k$ with $\partial\sigma = 0$, then ω is exact. In fact, there exists $\xi \in \overline{C}_{n-k} \otimes \mathbb{R}$ such that for any critical point x, $\langle \xi, x \rangle = \int_{\overline{W^u(x)}} \omega$. Since $\langle \xi, \sigma \rangle = 0$ for any cycle σ, one has $\xi = \partial\eta, \eta \in \overline{C}_{n-k+1}$. By de Rham regularization, ξ is smoothed into a form ω', which is the differential of the de Rham regularized of η. Then, $\int_{\overline{W^u(x)}}(\omega - \omega') = 0$ for any x. The form $\omega - \omega'$ is shown to be exact by climbing the skeleton, and applying the Poincaré lemma to each cell ; this is detailed in [ST ; 6.2, Lemma 3]. In fact the structure of the closure of the unstable manifolds allows us to proceed in the same way as with the simplices of a triangulation. □

d) The Thom-Smale complex with local coefficients

Let F be a real flat vector bundle on M. Let $C_k(F)$ be the vector space generated by the $x \otimes f$, where x is a critical point of index k, and $f \in F_x$. Then if x, y are critical points of f such that $i(y) = i(x) - 1$, $W^u(x) \cap W^s(y)$ consists of a finite number of gradient lines. To each of these gradient lines, one can attach a sign ϵ and an identification $\alpha : F_x \to F_y$. Set $\partial = \Sigma\epsilon\alpha$. Then the obvious analogues of the results of $c)$ still hold.

e) Bifurcation of the Thom-Smale complex in a $1-$parameter family

Now we consider a smooth path of pairs (f_t, X_t), $t \in [0,1]$, where X_t is the gradient of $-f_t$ with respect to a metric μ_t. We assume that f_0 and f_1 are Morse functions, and that X_0 and X_1 verify (T) and (SM). One may ask how the Thom-Smale complexes of X_0 and X_1 are related to each other. Observe that the given path can be modified into any other path having the same ends. We allow ourselves modifications which are based on classical tranversality arguments, as well as on a by-product of the universal unfolding of the x^3 singularity. So we assume that the following assumptions are verified:

a) Except on a finite set $\{t_1, \ldots, t_k\}$ with $0 < t_1 < \ldots < t_m < 1, f_t$ is a Morse function.

b) Near t_k, the path f_t is an "elementary" path of birth or death of a pair of critical points. The word "elementary" means the path is described as in [C, p. 244 - 246] : near the degenerate critical point the path of functions is given by,

$$f_t(x) = \frac{1}{3}x_1^3 - (t - t_k)x_1 \pm x_2^2 \ldots \pm x_n^2 + \text{const.} \tag{8}$$

for $t \in [t_k - \epsilon, t_k + \epsilon]$, when the birth happens for increasing t.

c) For $t \in [t_k - \epsilon, t_k + \epsilon]$, the metric μ_t is constant. In the chart where (8) holds, μ_t is a small C^0-perturbation of the canonical Euclidean metric, so that (SM) holds at the two new critical points $(\pm\sqrt{\epsilon}, 0, \ldots, 0)$ of $f_{t_k+\epsilon}$.

d) The stable and unstable manifolds of X_{t_k} are transversal ; at the cubical singularities, they are manifolds with boundary.

e) For any t and any critical point x of f_t, distinct from the critical points which appear in the birth/death process when $t \in]t_k - \epsilon, t_k + \epsilon[$, the condition (SM) is satisfied at x with respect to the metric μ_t.

f) At the end points $t = t_k \pm \epsilon$, assumption (T) is verified.

To describe the modification of the Thom-Smale complex along such a path, we consider in succession the following two problems: how does the complex change when one passes a birth-death point, and how does it vary along a path of Morse gradient fields, at the points where (T) is not satisfied.

f) Modification of the Thom-Smale complex near a birth-death point

Change the orientation of the t-axis if necessary and assume that t_k is the birth point of a pair of critical points of index $i, i + 1$.

Set $g_- = f_{t_k-\epsilon}$, $g_0 = f_{t_k}$, $g_+ = f_{t_k+\epsilon}$. Let x be the cubic singularity of g_0 ; let x' (resp. x'') be the index i (resp. $i + 1$)-critical point of g_+ just created from x. The point x is a degenerate critical point with index i. Its local unstable manifold is a half-disc of dimension $i+1$ and its local unstable manifold is a half-disc of dimension $n-i$; they meet only at x which lies in their boundaries. The kernel of the Hessian at x is the unique direction tangent to the stable and unstable manifolds. The singularities of g_+, all quadratic, are those of g_- plus x' and x''.

If y (resp. z) is a critical point of index $i+1$ (resp. i) of g_0, the integer $n(y,x)$ (resp. $n(x,z)$) is well defined because the transversality condition is assumed for $g_0 = f_{t_k}$: it is the algebraic number of gradient lines descending from y to x (resp. from x to z).

The formulae which calculate the complex (C_+, ∂_+) associated to $(g_+, \mathrm{grad} g_+)$ from the complex (C_-, ∂_-) associated to $(g_-, \mathrm{grad} g_-)$ are the following :

$$\partial_+ p = \partial_- p \text{ for any critical point } p \text{ of } g_+ \text{ with } i(p) \neq i+1, i+2 \text{ and } p \neq x', \tag{9}$$

$$\partial_+ x'' = x' + \sum_{i(z)=i} n(x,z) z, \tag{10}$$

$$\partial_+ x' = - \sum_{i(z)=i} n(x,z) \partial_- z, \tag{11}$$

$$\partial_+ y = \partial_- y + \; n(y,x)[x' + \sum_{i(z)=i} n(x,z) z], \tag{12}$$

for any critical point y of g_+, $i(y) = i+1$ and $y \neq x''$;

$$\partial_+ y = \partial_- y - \; n(\partial_- y, x) x'' \tag{13}$$

for any critical point y of $g_+, i(y) = i+2$.

In (13), $n(\alpha_1 y_1 + \cdots + \alpha_k y_k, x) = \alpha_1 n(y_1, x_1) + \cdots + \alpha_k n(y_k, x)$, where the α_j 's are integers and the y_j 's are critical points of index $i+1$. These formulae are complicated, but, except when i is 0, $n-1$ or $n-2$, one can easily make all the $n(x,z)$ and $n(y,x)$ zero, in which case they become trivial. This is the case when the box where the new pair of critical points of index $i, i+1$ is far from the unstable manifolds of points of index $i+1$ and from the stable manifolds of points of index i.

All these formulae are consequences of the following geometrical fact : if L is a level set of g_+ just below x', then $L \cap W^u(x')$ is the boundary of $L \cap W^u(x'')$ which is a small deformation of $L \cap W^u(x)$; if L is a level set just above x'', then $L \cap W^s(x'')$ is the boundary of $L \cap W^s(x')$ which is a small deformation of $L \cap W^s(x)$.

Now we put these formulae in a more concentrated form. For this, we introduce the split extension (C_-^e, ∂_-^e) of (C_-, ∂_-) by the acyclic complex $0 \to \mathbb{Z}x'' \xrightarrow{\times 1} \mathbb{Z}x' \to 0$.

Consider the following automorphism A of C_-^e : in degree distinct from $i,\ i+1$, it is the identity. For $i(y) = i+1, y \neq x''$, put $A(y) = y + n(y,x)x''$ and $A(x') = x' + \Sigma_{i(z)=i}\, n(x,z)z$. This automorphism is "elementary" in the sense of algebraic K-theory. We get

Proposition 8. *(C_+, ∂_+) is obtained from (C_-, ∂_-) by setting $C_+ = C_-^e$ and $\partial_+ = A^{-1} \circ \partial_-^e \circ A$.*

g) The Thom-Smale complex near points where (T) is not satisfied

After the above discussion, we are reduced to consider a path of Morse functions $f_t,\ t \in [0,1]$, where both ends $f_i,\ i = 0,1$, are equipped with gradient vector fields X_i satisfying (T) and (SM). The Morse lemma holds with parameters and the space of Morse charts of a given Morse function, near one fixed critical point, is connected, up to the Euclidean symmetries of the model (Alexander trick). Then it is easy to construct a path of metrics μ_t such that $X_t = -\mathrm{grad}_{\mu_t} f_t$ satisfies (SM) for every $t \in [0,1]$ and coincides with the given vector fields for $t = 0,1$.

Now, by approximation, we can suppose that X_t satisfies the transversality condition (T) except for $0 < t'_1 < \ldots < t'_p < 1$; moreover, the $f_{t'_k}$'s have distinct critical values. The lack of transversality in a $1-$ parameter family can be described generically as follows : let L be a regular level of $f = f_{t'_k}$, $L = f^{-1}(a)$, just above a critical point x of index i. In $f^{-1}([a, +\infty[)$ and in $f^{-1}(]-\infty, a])$, the transversality condition (T) is valid for the stable and unstable manifolds of each cobordism considered alone. The unstable manifolds of critical points of f, with critical values $> a$, induce on L some stratification St with conical singularities. Let $S \subset L$ be the trace of the stable manifold of x: S is non transversal to exactly one stratum Σ of St; there is a unique point p where Σ and S meet non transversally and the tangency at p is a "codimension 1" singularity.

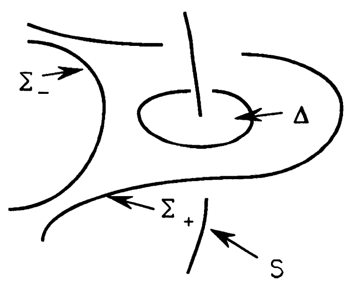
The stratification of the space of embeddings $S \to L$ induce by Σ is described in [C, p.123]. When going from $f_{t'_k-\epsilon}$ to $f_{t'_k+\epsilon}$, the picture of the stable-unstable manifolds is itself stable above L and below L. But the glueing of both pictures in L is not stable; it crosses a codimension 1 stratum in the space of embeddings mentionned above.

In the following, we only consider failures of transversality which a priori generate modifications of the associated algebraic complex. They are of two types:

First type. $\dim\Sigma + \dim S = \dim L = n-1$. In this case $\Sigma = W^u(y) \cap L$, where y is a critical point of index $i+1$; during the transition, some pair of gradient lines descending from y to x is created or cancelled. But the integer $n(x,y)$ is preserved and the algebraic complex does not change.

Second type. $\dim\Sigma + \dim S = \dim L - 1$.

In this case y is a critical point of index i. The transition is pictured in L : we have a small disc Δ cutting S in one point and one moves from Σ_- to Σ_+ through Δ.



As unstable manifolds are oriented, Σ is oriented and S is transversally oriented; therefore, the above operation comes with a sign ϵ. The boundary morphism changes from ∂_- to ∂_+ according to the following formulae :

(14) $$\partial_+(z) = \partial_-(z) - \epsilon\, n(z,y)x \text{ if } \operatorname{ind}(z) = i+1 \ ,$$

(15) $$\partial_+(z) = \partial_-(z) \text{ if } \operatorname{ind}(z) = i \text{ and } z \neq y,$$

$$\partial_+(y) = \partial_-(y) + \epsilon\partial_-(x), \tag{16}$$

and, for the other critical points, $\partial_+ = \partial_-$.

Here is a sketch of proof for (16). Let L' be a level set of f just below $f(x)$; as Δ is a small meridian disc of S, the gradient lines descending from $\partial\Delta$ intersect L' along a sphere parallel to $L' \cap W^u(x)$. If Σ'_+ (resp. Σ'_-) denotes the trace in L' of the gradient lines descending from Σ_+ (resp. Σ_-) then Σ'_+ is the connected sum of Σ'_- with a sphere parallel to $L' \cap W^u(x)$. Formula (16) follows.

If A is the "elementary" automorphism of the module C_* defined by $A(p) = p$ for any generator $p \neq y$ and by $A(y) = y + \epsilon x$, then we get :

Proposition 9. *(C_+, ∂_+) is obtained from (C_-, ∂_-) by setting $C_+ = C_-$ and $\partial_+ = A^{-1} \circ \partial_- \circ A$.*

The formulas from (9) to (16) still make sense with local coefficients. Then, if for some adhoc system of coefficients the complex becomes acyclic, its torsion (Franz-Reidemeister or Whitehead) does not depend on the pair - function, gradient vector field - chosen at the beginning. Of course, this fact is well known (compare Milnor [M2, §9]).

h) Final comments and complements

The only new fact proved in this appendix is that the pair (f, X) of a function and a gradient vector field (with some conditions) produces an embedding I_* of the Thom-Smale complex C_* into the complex R_* of de Rham currents, because the unstable manifolds of critical points are currents. Then, by Proposition 7, we have a canonical isomorphism between the Thom-Smale homology (homology of the Thom-Smale complex) and the de Rham homology. In this Section, we will verify directly that the identifications of complexes of Proposition 8 and 9 induce the corresponding canonical identifications of their homology groups.

When we need to specify the pair (f, X) which is used, $C_*(f, X)$ and $I_*(f, X)$ will denote the Thom-Smale complex and the corresponding embedding into the de Rham complex.

First, let us consider a one-parameter family (f_t, X_t), $t \in [0, 1]$, of Morse functions and gradient vector fields satisfying both conditions (T) and (SM) on the whole interval. In this case, $C_*(f_0, X_0)$ and $C_*(f_1, X_1)$ are the same as the critical points of both functions are in canonical correspondance and we have two embeddings of the same Thom-Smale complex. We claim that $I_*(f_0, X_0)$ and $I_*(f_1, X_1)$ are homotopic; this means that there exists a morphism K of degree $+1$ from C_* to R_* such that

$$I_*(f_1, X_1) - I_*(f_0, X_0) = \partial \circ K + K \circ \partial$$

This equation is satisfied if for each generator x of $C_k(f_0, X_0)$, we set $K(x) = \bigcup_t W^u(x_t)$. Here x_t is the critical point of f_t corresponding to x and $K(x)$ is of course a $(k+1)$-dimensional current; it is the direct image by the projection $M \times [0,1]$ to M of the obvious current $\bigcup_t W^u(x_t) \times \{t\}$ in $M \times [0,1]$. As a consequence, one has the following result.

Proposition 10. $I_*(f_0, X_0)$ *and* $I_*(f_1, X_1)$ *induce the same isomorphism in homology.*

The crossing of an "accident" along the path (f_t, X_t) - failure of transversality or birth-death point - involves a little bit more technicality. But with the notation of Propositions 8 and 9, and using homotopies like above, one can prove the following.

Proposition 11. *1) Near a generic no-transversality point, the morphisms* $I_*(f_+, X_+)$ *and* $I_*(f_-, X_-) \circ A$ *induce the same isomorphism in homology.*

2) Near a birth point, $I_*(f_+, X_+)$ *and* $I_*(f_-, X_-) \circ p \circ A$ *induce the same isomorphism in homology, where* p *is the natural projection of* C_-^e *onto* C_-.

To conclude this Appendix, we give a Fubini formula which only makes sense by our use of currents. Here (f, X) is a pair satifying the (T) and (SM) conditions, ω is a closed k-form, Ω is a closed orientation-twisted $(n-k)$-form; the k-dimensional unstable manifolds are oriented and the $(n-k)$-dimensional stable manifolds are co-oriented.

Proposition 12.

$$\int_M \omega \wedge \Omega = \sum_x \int_{\overline{W^u}(x)} \omega \int_{\overline{W^s}(x)} \Omega, \tag{17}$$

where the sum is taken over all the critical points of index k.

Proof. The transpose of I_k maps the cocycle ω to a cycle of $\overline{C}_{n-k}$, given by

$$\sum_x \Big(\int_{\overline{W}^u(x)} \omega\Big) x,$$

which itself gives rise to the twisted closed current

$$\sigma = \sum_x \left(\int_{\overline{W}^u(x)} \omega \right) \int_{\overline{W}^s(x)} .$$

Thus the homology class of σ only depends on the cohomology class of ω. Therefore, the right-hand side of (17) only depends on the cohomology classes of ω and Ω. The same is obviously true for the left-hand side. So we are reduced to the case where ω vanishes near the $(k-1)$-skeleton of the stratification by unstable manifolds and Ω vanishes near the $(n-k-1)$-skeleton of the stratification by stable manifolds. Then $\omega \wedge \Omega$ vanishes everywhere except on blocks $D^k \times D^{n-k}$, usually called handlebodies. On each handlebody the formula reduces more or less to Fubini. $\square$

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Résumé

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Résumé

Dans cet article, on étend le théorème de Cheeger et Müller, relatif à l'égalité des métriques de Reidemeister et de Ray-Singer sur le déterminant de la cohomologie d'un fibré plat muni d'une métrique plate ou d'une métrique unimodulaire, à des fibrés plats munis de métriques arbitraires. Le rapport de ces deux métriques s'exprime à l'aide de l'intégrale d'un courant de Chern-Simons. On montre également des formules d'anomalie pour les métriques de Ray-Singer.