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INTRODUCTION

Avec le soutien du C.N.R.S et de la D.R.E.D, l'année académique 1990/91 fût une année spéciale consacrée aux méthodes semiclassiques.

A l'origine les méthodes semi-classiques désignaient les techniques utilisées par les physiciens pour essayer de comprendre les relations subtiles existant entre la mécanique classique de Newton et la mécanique quantique de Heisenberg-Schrödinger (lorsque la constante de Planck $\hbar$ devient négligeable par rapport aux autres grandeurs physiques: masse, énergie, distances, ...). L'exemple fondamental est la méthode B.K.W (Brillouin, Kramers, Wentsel) qui consiste à construire des solutions asymptotiques, par rapport à la constante de Planck, de l'équation de Schrödinger. Cette méthode est restée longtemps formelle. La justification mathématique rigoureuse a nécessité l'élaboration de théories sophistiquées qui ont vu le jour dans les années 1970 (indice de Maslov, opérateurs intégraux de Fourier-Hörmander). A partir de ces travaux de base, de nombreux mathématiciens se sont attaqués avec succès à divers problèmes issus de la physique et se traduisant par l'étude spectrale d'opérateurs pseudo-différentiels, dépendant de paramètres. Citons quelques exemples parmi les plus connus:

• le comportement du spectre de l'opérateur de Schrödinger lorsque la constante de Planck tend vers zéro (régle de Bohr-Sommerfeld, effet tunnel)

• le comportement asymptotique des grandes valeurs propres (formules du type Weyl)

• la trace du noyau de la chaleur lorsque la température tend vers zéro et les invariants géométriques associés

• diffusion quantique ou acoustique: problèmes à plusieurs corps, problèmes inverses, résonances

• systèmes périodiques: analyse du spectre de bande, problèmes inverses

• description de certains systèmes quantiques désordonnés: potentiels quasi périodiques, équation de Harper, chaos quantique

• limite thermodynamique.

Durant ces quinze dernières années, les méthodes semi-classiques se sont beaucoup enrichies avec le développement de l'analyse microlocale des équations aux dérivées partielles et de leurs solutions. De nombreux mathématiciens (et physiciens!) ont participé à ce développement. Parmi les travaux que l'on peut considérer comme fondamentaux mentionnons en particulier ceux de S. Agmon, Y. Colin de Verdière, J. Chazarain, L. Hörmander, V. Ivrii, J. Leray,V. Maslov, R. Melrose, J. Sjöstrand, A. Voros (je cite ces noms car il me semble bien représenter le rapprochement fructueux qui s'est effectué durant cette période entre l'analyse des équations aux dérivées partielles et la physique-mathématique).

Deux volumes de la collection **Astérisque** regroupent les actes de l'Ecole d'Eté et du Colloque International organisés à Nantes, en Juin 1991. L'Ecole d'Eté était centrée sur quatre cours: V. Ivrii (Asymptotiques Spectrales); M.A Shubin (Théorie spectrale sur les variétés non compactes); A. Soffer (Problèmes à N-corps) et G. Ulhmann (Problèmes inverses). Le Colloque International comportait vingt conférences portant sur des thèmes variés, illustrant la puissance des méthodes semi-classiques appliquées aux équations de la mécanique quantique ou à l'équation des ondes acoustiques. Les sujets abordés concernent principalement l'équation de Schrödinger sous différents aspects: N-corps, champs magnétiques, limite thermodynamique, solitons, cristaux. Deux exposés sont consacrés à la diffusion acoustique par un obstacle et à la conjecture de Lax-Philips sur les résonances.

En conclusion, je voudrais remercier les institutions et les personnes qui ont permis le succès de cette année spéciale sur les méthodes semiclassiques, en premier lieu le C.N.R.S en la personne de J.P Ferrier et la D.R.E.D en la personne de J. Giraud. Je remercie également tous ceux qui ont participé à l'organisation des différents colloques qui se sont déroulés entre Novembre 1990 et Juin 1991, en particulier les collègues suivants: J. Bellissard, J.M.Bismut, A. Ben.Arous, J.M. Combes, C. Gérard , A. Grigis , J.C Guillot, B. Helffer, A. Martinez, J.F.Nourrigat, F. Pham, J. Sjöstrand, A.Unterberger, A. Voros.
Je remercie l'université de Nantes et le conseil général de Loire-Atlantique pour le soutien qu'ils nous ont apporté.
D. Macé-Ramette a assuré avec dévouement et compétence le secrétariat de cette année spéciale, je l'en remercie.

Nantes, le 21 Décembre 1992

D. Robert

RESUMES

1. IVRII Victor. *Semiclassical spectral asymptotics*

These lectures are devoted to semiclassical spectral asymptotics with accurate remainder estimates and their applications to spectral asymptotics of other types.In 0.*Introduction*, the brief description of the hyperbolic operator method is given. In I. *Why one should study local semiclassical spectral asymptotics?* we show how starting from rather classical theorems concerning LSSA (local semiclassical spectral asymptotics) one can weaken their conditions. In 2. *How local semiclassical spectral asymptotics yield standard spectral asymptotics?* we show how LSSA yield asymptotics of eigenvalues tending to $+\infty$ for operators on compact manifolds and for operators on $\mathbf{R}^d$ with potentials increasing at infinity and asymptotics of eigenvalues tending to -0 for operators in $\mathbf{R}^d$ with potentials decreasing at infinity. In 3 *How can one derive local semiclassical spectral asymptotics in the general case?*, we present basic ideas permitting us to use the hyperbolic operator method for general matrix operators and for operators on manifolds with boundary. In 4. *Propagation of singularities*, we apply the short-time propagation of singularities in order to justify the previous section construction; then in 5. *Tauberian theorem*, we derive LSSA. We treat the long-time propagation of singularities in order to improve the remainder estimate in LSSA in 6. *How to improve remainder estimate in the case of non periodic trajectories?* and in 7. *How to improve remainder estimate in the case of periodic trajectories?*, In the last case the final formula contains non-Weylian term. In 8. *Eigenvalue estimates and asymptotics for spectral problems with singularities*, we split LSSA and Lieb-Cwickel-Rozenbljum eigenvalue estimate and derive estimates above and below for the number of the eigenvalues for the Schödinger operator. Taking this operator depending on some parameters we obtain asymptotics with respect to this parameter. In 9. *Generalizations. Non-Weylian asymptotics*, more advanced development of the theory is presented.

2. SHUBIN Michael.*Spectral theory of elliptic operators on non-compact manifolds*

General aspects of spectral theory of elliptic operators on non-compact manifolds are studied. Methods of proving the coincidence of minimal and maximal operators are descibed and a review of the known results are given. Exponential weight estimates for the decay of the Green function on manifolds of bounded geometry are proved. Applications of these estimates to Schnol type theorems are given (these theorem give conditions of growth to be imposed on a non-trivial generazed eigenfunction to garantee that the corresponding eigenvalue is in the spectrum). This is done in particular on manifold of bounded geometry with the exponential growth of the volume of the balls. Estimates of growth of generalized eigenfunctions for almost all points in the spectrum (with respect to the spectral measure) are given.

3. SOFFER Avy. *On the many body problem in quantum mechanics*
I describe some of the main ideas and tools of the N-body scattering theory. Emphasis is given to the recent developments based on phase space and time dependent techniques. Also, the relationship to some problems in harmonic analysis, PDE and spectral theory, is noted.

4. ULHMANN Gunther. *Inverse boundary value problems and applications*
In these notes we give an overview of inverse boundary value problems. In these problems one attemps to discover internal properties of a body by making measurements at the boundary. We concentrate mainly in the problem of determining the conductivity of a body by making measurements of voltage potentials and corresponding fluxes at the boundary. This problem is often referred to as *Electric Impedance Tomography.* We give applications to inverse scattering as well as inverse spectral problems.

We consider first the *isotropic case.* In this case the conductivity does not depend on direction. We reduce the problem to an inverse boundary value problemfor the Schrödinger equation, at zero energy, for a compactly supported potential. More precisely the known information is encoded by the it Dirichlet to Neumann map. In this notes we describe how the construction of exponential growing solutions allows to prove that the potential is uniquely determined by knowledge of this map in dimension $n > 2$. We also discuss progress made in the two dimensional case. The method allow also to find reconstruction methods as well as to obtain estimates of the potential in terms of the given Dirichlet to Neumann map. The same techniques are used to prove similar results for the inverse scattering problem at a fixed energy. In the most general case in which the conductivity depends on direction, usually referred to as *anisotropic case*, there is a natural obstruction to uniqueness. We report on the progress made on this problem.

VICTOR IVRII

Semiclassical spectral asymptotics

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SEMICLASSICAL SPECTRAL ASYMPTOTICS

BY VICTOR IVRII

0. Introduction

The problem of the spectral asymptotics, in particular the problem of asymptotic distribution of eigenvalues is one of the central problems of the spectral theory of partial differential operators. It is also very important for the general theory of partial differential operators. Apart from applications in the quantum mechanics, radiophysics, continuum media mechanics (elasticity, hydrodynamics, theory of shells) etc, there are also applications to the mathematics itself and moreover there are deep though non-obvious links with differential geometry, dynamic systems theory and ergodic theory; even the term "spectral geometry" has arisen. All these circumstances make this topic very attractive for a mathematician.

This problem originated in 1911 when H.Weyl published a paper devoted to eigenvalue asymptotics for the Laplace operator in a bounded domain with a regular boundary. After this article there was published a huge number of papers devoted to the spectral asymptotics and numerous prominent mathematicians were among their authors. The theory was developed in two directions: first of all this theory was extended and there were considered more and more general operators and boundary conditions as well as geometrical domains on which these operators were given; on the other hand the theory was improved and more and more accurate remainder estimates were derived. Namely in the later way the links with differential geometry, dynamic systems theory and ergodic theory appeared. Even the theory of eigenvalue asymptotics for the Laplace (or Laplace-Beltrami) operator has a long, dramatic and yet non-finished history. At a certain moment apart of asymptotics with respect to the spectral parameter there appeared asymptotics with respect to other parameters; the most important among them are (in my opinion) semiclassical asymptotics, i.e. asymptotics with respect to the small parameter h (Planck constant in physics) tending to $+0$. For a long time these asymptotics were in the shadow: most attention was paid to the eigenvalue asymptotics for operators on compact manifolds (with or without a boundary); the results which had been obtained here then were proved again for operators in $\mathbb{R}^d$ such as the Schrödinger operator $-h^2\Delta + V(x)$ with fixed $h > 0$ and with $V(x) \to +\infty$ as $|x| \to \infty$; less attention was paid to semiclassical asymptotics (i.e. asymptotics of eigenvalues less than some fixed level λ as $h \to +0$); moreover the asymptotics of the small negative eigenvalues were considered in the case of fixed h and $V(x)$ decreasing at infinity as $|x|^{2m}$ with $m \in (-1, 0)$; under reasonable conditions in this case the discrete spectrum of an operator has an accumulation point -0 and the essential spectrum coincides with $[0, +\infty)$. The result of the development of the theory described above was that at a certain moment there existed four parallel (though not equally developed) theories and the statements in each of them had to be proved separately. However now this plurality has been finished (at least in my papers) because all the other results are easily derived from the *local semiclassical spectral asymptotics* (*LSSA* in what follows), which are the main object of these lectures All other results are obtained as their applications.

In his papers H.Weyl applied the *variational* method (Dirichlet-Neumann bracketing) invented by himself; later this method was improved in various directions by many mathematicians. Other methods also appeared later and I would like to mention only the *method of a hyperbolic operator* due to B.M.Levitan and Avvakumovič[1)]. All the asymptotics with the most accurate remainder estimates were obtained by this method. It is based on the fact that the fundamental solution to the Cauchy problem (or the initial-boundary value problem) $u(x, x, t)$ for the operator $D_t - A$ is the Schwartz' kernel of the operator $\exp itA$ (where $D_t = -i\partial_t$, etc) and it is connected with the eigenvalue counting function of an operator A by the formula

$$\int u(x, x, t)dx = \int e^{it\lambda}\, d_\lambda N(\lambda); \tag{0.1}$$

in the case of a matrix operator A $u(x, y, t)$ is a matrix-valued function and in the left-hand expression it should be replaced by its trace. Here and below $N(\lambda)$ is the number of eigenvalues of A less than λ (and in this place we consider only operators semi-bounded from below with purely discrete spectra). Then by means of the inverse Fourier transform we can recover $N(\lambda)$ provided we have constructed $u(x, y, t)$ by means of the methods of theory of partial differential operators. However, in fact we are never able (excluding some very special

[1)]This method is a special case of Tauberian methods due to T.Carleman; resolvent method, method of complex power and method of heat equation are other Tauberian methods. The method of the almost spectral projector due to M.Shubin and V.Tulovskii lies between variational and Tauberian methods.

and rare cases when all this machinery is not necessary) to construct $u(x, y, t)$ precisely and for all the values $t \in \mathbb{R}$. Usually (now we assume that A is an elliptic first-order pseudo-differential operator) the fundamental solution is constructed approximately (modulo smooth functions) for t belonging to some interval $[-T, T]$ with $T > 0$. As a consequence we obtain modulo $O(\lambda^{-K})$ with any arbitrarily chosen K an expression for

$$F_{t\to\tau}\chi_T(t)\int u(x,x,t)dx = \int \hat{\chi}_T(\tau-\lambda)d_\lambda N(\lambda) \tag{0.2}$$

where χ is a fixed smooth function supported in $[-1, 1]$, $\chi_T(t) = \chi(\frac{t}{T})$ and a hat as well as $F_{t\to\tau}$ mean the Fourier transform. Then if we know the left-hand expression, using the Tauberian theorem due to Hörmander we are able to recover approximately $N(\lambda)$ by the formula

$$N(\lambda) = \int_{-\infty}^{\lambda} (F_{t\to\tau}\chi_T(t)\sigma)(\tau))d\tau + O(\lambda^{d-1}) \tag{0.3}$$

where d is the dimension of the domain,

$$\sigma(t) = \int u(x,x,t)dx \tag{0.4}$$

and the explicit construction of $u(x, x, t)$ in this situation yields the formula

$$N(\lambda) = c_0\lambda^d + O(\lambda^{d-1}) \tag{0.5}$$

with the leading coefficient

$$c_0 = (2\pi)^{-d}\int_{a(x,\xi)<1} dxd\xi, \tag{0.6}$$

where $a(x, \xi)$ is a principal symbol of A.

We see that the crucial step in this approach is the construction of the fundamental solution. This construction by means of Fourier integral operators[2)] is standard and well-known now, provided we consider a scalar operator for an operator with constant multiplicities of the eigenvalues of the principal symbol and we construct $u(x, y, t)$ at the compact K contained in the interior of our domain X (and T depends on the distance between K and ∂X). If one of these assumptions is violated then the construction is more sophisticated and possible only under some very restrictive conditions. In the presence of a boundary (but only in the case of the constant multiplicities of the eigenvalues of the principal symbol) this construction was realized in certain papers due to R.Seeley, D.Vasil'ev, R.Melrose. However, it is possible to avoid all the troubles by means of another approach suggested by V.Ivrii[4] (see also L.Hörmander [3]) based on the investigation of the propagation of singularities

[2)] This construction due to L.Hörmander played a very important and stimulating role in the development of Fourier integral operators theory.

for $u(x, y, t)$ and construction of an "approximation" (in a rather exotic sense) for this distribution leading to an approximation in the reasonable sense for $\sigma(t)$ for $|t| \leq T$ with appropriate $T > 0$. For h-pseudo-differential operators which are the main subject of this article this approach is essentially more simple and transparent because there is a selected parameter h. We'll discuss this case below. We'll be able to prove in this way the asymptotics (0.3) for an arbitrary self-adjoint m-th order elliptic operator with $m > 0$ and the spectral parameter λ^m now on a compact manifold without or with a boundary (in the former case the boundary conditions are also supposed to be elliptic), scalar or matrix, semi-bounded from below or non-semi-bounded at all (in this case $N(\lambda)$ is replaced by $N^{\pm}(\lambda)$ which is a number of eigenvalues lying between 0 and $\pm\lambda^m$); the formula for c_0 should be changed if it is necessary.

At the same time the two-terms asymptotics

$$N(\lambda) = c_0\lambda^d + c_1\lambda^{d-1} + o(\lambda^{d-1}) \tag{0.7}$$

suggested by H.Weyl (who also gave a formula for c_1) fails to be true unless some additional condition is fulfilled. It is certainly wrong for $d = 1$ and for the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$ of any dimension (this is due to the high multiplicities of its eigenvalues). Moreover, this asymptotics remains wrong in the case when this Laplace-Beltrami operator is perturbed by a potential or even by a symmetric first-order operator with small coefficients; in this case all the eigenvalues of high multiplicities will generate narrow eigenvalue clusters separated by lacunae. On the other hand under some conditions of the global nature the asymptotics (0.7) is valid. For a scalar operator on a compact manifold without a boundary this condition is "*The measure of the* {*set of all the points of the cotangent bundle periodic with respect to the Hamiltonian flow generated by the principal symbol* } *equals to 0*"[3)]. This condition is more complicated for matrix operators. For a scalar second-order operator on a compact manifold with a boundary one needs to consider only trajectories transversal to the boundary and reflecting according to the geometrical optics law. Though there are some points of the cotangent bundle through which such infinitely long trajectory doesn't pass, but the measure of these dead-end points vanishes and we do not have to take them into account. For higher-order operators as well as for matrix operators the trajectories reflected from the boundary can branch and in this case it is necessary to follow every branch. This makes the situation much more complicated and the following additional condition (which isn't automatically fulfilled now) appears "*the measure of the* {*set of all the dead-end points*} *equals to* 0".

Let us clarify for the scalar first-order operator on a manifold without boundary a link between asymptotics (0.7) and periodic Hamiltonian trajec-

[3)]This condition appeared first in the papers of J.J.Duistermaat and V.Guillemin.

tories. It is well-known that singularities of the solutions of the hyperbolic equations propagate along Hamiltonian trajectories. This fact leads us to a conclusion that the singular support $\sigma(t)$ is contained in the set of all the periods of the Hamiltonian trajectories including $t = 0$; in particular $t = 0$ is an isolated point of this singular support (this fact remains true in very general situations). Hence if there is no periodic trajectory with the period not exceeding T, we know that on the interval $[-T, T]$ the distribution $\sigma(t)$ is singular only at $t = 0$ and hence we know $\sigma(t)$ on this interval modulo a smooth function. The Tauberian theorem permits us to obtain that the remainder in the asymptotics (0.7) doesn't exceed $\frac{C}{T}\lambda^{d-1} + O(\lambda^{d-2})$ with the constant C which doesn't depend on T; however "O" here isn't necessarily uniform with respect to T. Hence if $t = 0$ was the unique period (I am aware that it is impossible!) then we would choose T and obtain the remainder estimate $o(\lambda^{d-1})$. In the general (realistic) case one should consider the partition of unity given by two pseudo-differential operators Q_j for every chosen T such that the support of the first operator contains no periodic point with the period not exceeding T and the measure of the support of the symbol of the second operator is less than ϵ with arbitrary chosen $\epsilon > 0$ (due to our condition all periodic trajectories with the period not exceeding T form a closed nowhere dense set of measure 0). Applying the Tauberian theorem to every term

$$N_j(\lambda) = \int (Q_j e)(x, x, \lambda)dx$$

in $N(\lambda)$ we obtain the remainder estimate $\frac{C}{T}\lambda^{d-1} + O(\lambda^{d-2})$ for $j = 1$ and $C\epsilon\lambda^{d-1} + O(\lambda^{d-2})$ for $j = 2$ and these estimates imply (0.7) again. Here and in what follows $e(x, y, \lambda)$ is a Schwartz' kernel of the spectral projector. Moreover, under certain more restrictive conditions to the Hamiltonian flow one can improve the remainder estimate in (0.7) to $O(\lambda^{d-1}/\log\lambda)$ or even $O(\lambda^{d-1-\delta})$ with a small exponent $\delta > 0$[4)].

It has been discovered recently that even in the presence of the periodic trajectories and in the presence of eigenvalue clusters one can have the two-term asymptotics of the form

$$N(\lambda) = c_0\lambda^d + f(\lambda)\lambda^{d-1} + o(\lambda^{d-1}) \tag{0.8}$$

with the explicitly calculable function $f(\lambda)$ which is bounded and oscillating as $\lambda \to +\infty$ with the characteristic "period" of oscillations $\asymp 1$. In particular, this fact enables us to obtain an asymptotic distribution of eigenvalues inside of clusters[5)]. Moreover, under some assumptions including an assumption that

[4)] See papers of P.Bérard and B.Randoll and more recent papers of A.Volovoy and the author.

[5)] See papers of Yu.Safarov and more recent papers of the author.

all the trajectories are periodic one can obtain the asymptotics (0.8) with the remainder estimate $O(\lambda^{d-2})$!

It is well-known that in a large number of cases the Weylian formula fails to be applicable in its standard form. In these cases it is necessary either to remove from the domain of integration some part of the phase space or to divide variables x and ξ in two parts: $x = (x', x'')$, $\xi = (\xi', \xi'')$ and consider only the variables (x', ξ') as Weylian; that means that one should consider the operator in question as a (partial) differential operator with respect to x' with operator-valued coefficients and apply the Weylian procedure to this operator. In a more general case one should divide the phase space in a few parts. One of them should be removed from consideration and in the other parts the "Weylevization" (preceded by a certain transform) should be made only with respect to certain variables.

These remarks do not pretend to be a survey (even an incomplete one). Their goal is only to motivate this article in particular and all my works in general. I would like to recommend to the reader the books of M.S.Birman and M.Z.Solomyak [1] and G.Rozenblyum, M.Z.Solomyak and M.Shubin [11] as the best surveys. One can find accurate references in these books and additional references in the book of D.Robert [10] and in the author' preprints [7.1-7.9].

1. Why One Should Study Local Semiclassical Spectral Asymptotics

Local semiclassical spectral asymptotics (LSSA) are asymptotics of

$$\operatorname{Tr}\psi E(\lambda_1,\lambda_2) = \int \psi(x)\operatorname{tr} e(x,x,\lambda_1,\lambda_2)dx \tag{1.1}$$

as $h \to +0$ where $E(\lambda_1,\lambda_2)$ is a spectral projector of the operator $A = A_h$ depending on a small parameter h and corresponding to the interval $[\lambda_1,\lambda_2)$, $e(x,y,\lambda_1,\lambda_2)$ is its Schwartz kernel, ψ is a C_0^∞-function and Tr and tr mean operator and matrix traces respectively (for scalar operators tr in the right-hand expression is absent). If $\psi = 1$, we obtain $N(\lambda_1,\lambda_2)$. Therefore we hope that taking an appropriate partition of unity we can obtain asymptotics for $N(\lambda_1,\lambda_2)$ starting from LSSA.

Moreover, it is often better to start from microlocal semiclassical spectral asymptotics when ψ is an h-pseudo-differential operator with compactly supported symbol.

Now I would like to present a few well-known results (in a slightly stronger form[6)]) and show how they can "improve themselves". We consider only the

[6)] The improvement is that the remainder estimates are uniform and that no condition outside the ball $B(0,1)$ is assumed to be fulfilled. This enhancement adds no difficulties in the proofs but is very important for applications.

Schrödinger operator

$$A = -h^2\Delta + V(x) \tag{1.2}$$

in the domain $X \subset \mathbb{R}^d$, where we assume that X contains the unit ball $B(0,1)$ and V is uniformly smooth in this ball:

$$|D^\alpha V| \le c \qquad \forall \alpha : |\alpha| \le K \tag{1.3}$$

where $K = K(d)$ is large enough. We assume that A is self-adjoint in $L^2(X)$, $D(A) \supset C_0^2(B(0,1))$ and in $B(0,1)$ operator A is given by (1.2). One can take $\lambda_1 = -\infty$ now and without any loss of generality one can take $\lambda_2 = 0$.

Theorem 1.1. *(duetoJ.Chazarain). Let A be a self-adjoint operator of the form* (1.2) *and let condition* (1.3) *be fulfilled in $B(0,1) \subset X$. Then for $h \in (0,1]$*

(i) In the general case

$$|e(x,y,-\infty,0)| \le Ch^{-d} \qquad \forall x,y \in B(0,\frac{1}{2}) \tag{1.4}$$

where $C = C(d,c)$;

(ii) If $B(0,1)$ is classically forbidden, i.e. if

$$V \ge \epsilon \qquad \text{in } B(0,1); \tag{1.5}$$

for some $\epsilon > 0$ then

$$|e(x,y,-\infty,0)| \le C'h^s \qquad \forall x,y \in B(0,\frac{1}{2}); \tag{1.6}$$

where s is arbitrary and $C' = C'(d,c,s,\epsilon)$;

(iii) Finally, if 0 isn't a critical value of V, i.e.

$$|V| + |\nabla V| \ge \epsilon_0 \qquad \forall x \in B(0,1) \tag{1.7}$$

then

$$\Big|\int \psi(e(x,x,-\infty,0) - \varkappa(x)h^{-d})dx\Big| \le Ch^{1-d} \tag{1.8}$$

where

$$\varkappa(x) = (2\pi)^{-d}\int_{\{|\xi|^2+V(x)\le 0\}} d\xi = (2\pi)^{-d}\omega_d V_-^{\frac{d}{2}}, \tag{1.9}$$

$V_\pm = \max(\pm V, 0)$, ω_k is a volume of the unit ball in $\mathbb{R}^k$ and we assume that $\psi \in C_0^K(B(0,\frac{1}{2}))$ and

$$|D^\alpha \psi| \le c \qquad \forall \alpha : |\alpha| \le K. \tag{1.10}$$

We don't discuss here (1.8)-type asymptotics without spatial mollification (this asymptotics holds when 0 isn't value of V) and or asymptotics with more accurate remainder estimate (when some condition on the classical dynamic system should be assumed). In this theorem the boundary conditions are not important (because the boundary doesn't intersect $B(0,1)$) and, moreover, the nature of the operator and domain outside $B(0,1)$ isn't important (the self-adjointness is the only assumption of the global nature).

In order to improve this theorem let us reformulate it first for ball $B(\bar{x},\gamma)$ with arbitrary $\gamma>0$. By dilatation $x_{\mathrm{new}}=\frac{1}{\gamma}(x-\bar{x})$ and multiplication by ρ^{-2} this case can be reduced to the previous one; we consider γ,ρ as additional parameters. After reduction we obtain an operator of the form (1.2) again with $h_{\mathrm{new}}=\frac{h}{\rho\gamma}$ and with $V_{\mathrm{new}}=\rho^{-2}V(\bar{x}+\gamma x)$. Then one should assume that $h_{\mathrm{new}}\in(0,1]$ i.e. that

$$\rho\gamma\ge h>0 \tag{1.11}$$

instead of the previous condition $h\in(0,1]$ and in order to fulfill conditions (1.3),(1.10) after reduction one should assume that

$$|D^\alpha V|\le c\rho^2\gamma^{-|\alpha|}\qquad\forall\alpha:|\alpha|\le K \tag*{(1.3)$'$}$$

and

$$|D^\alpha\psi|\le c\gamma^{-|\alpha|}\qquad\forall\alpha:|\alpha|\le K \tag*{(1.10)$'$}$$

where now $\psi\in C_0^K(B(\bar{x},\frac{1}{2}\gamma))$. Taking in account that $e(x,x,-\infty,0)$ is a density i.e. that $e(x,x,-\infty,0)dx$ is invariant in this procedure we obtain

Theorem 1.1$'$. *Let A be a self-adjoint operator of the form* (1.2) *and let in $B(\bar{x},\gamma)\subset X$ condition* (1.3$'$) *be fulfilled. Then for $h\le\rho\gamma$*

(i) In the general case

$$|e(x,y,-\infty,0)|\le Ch^{-d}\rho^d\qquad\forall x,y\in B(\bar{x},\frac{1}{2}\gamma) \tag*{(1.4)$'$}$$

where $C=C(d,c)$;

(ii) If $B(\bar{x},\gamma)$ is classically forbidden i.e. if

$$V\ge\epsilon\rho^2\qquad\text{in } B(\bar{x},\gamma); \tag*{(1.5)$'$}$$

with some $\epsilon>0$ then

$$|e(x,y,-\infty,0)|\le C'h^s\rho^{-s}\gamma^{-d-s}\qquad\forall x,y\in B(\bar{x},\frac{1}{2}\gamma); \tag*{(1.6)$'$}$$

where s is arbitrary and $C'=C'(d,c,s,\epsilon)$;

(iii) Finally, if 0 isn't a critical value of V i.e.

$$|V|+|\nabla V|\gamma\ge\epsilon_0\rho^2\qquad\forall x\in B(\bar{x},\gamma) \tag*{(1.7)$'$}$$

then

$$|\int\psi(e(x,x,-\infty,0)-\varkappa(x)h^{-d})dx|\le Ch^{1-d}\rho^{1-d}\gamma^{1-d} \tag*{(1.8)$'$}$$

where $\varkappa$ is given by (1.9) *and $\psi\in C_0^K(B(\bar{x},\frac{1}{2}\gamma)$ satisfies* (1.10)$'$.

Let us treat the case $\rho = \gamma = 1$ without condition (1.7). Let us introduce the function

$$\gamma = \epsilon_1(|V| + |\nabla V|^2)^{\frac{1}{2}} + h^{\frac{1}{2}}. \tag{1.12}$$

Then it is easy to check that

$$|\nabla\gamma| \le \frac{1}{2} \tag{1.13}$$

for small enough constant $\epsilon_1 = \epsilon_1(d,c)$ and that for $\bar{x} \in B(0,\frac{3}{4})$ in $B(\bar{x},\gamma)$ conditions (1.11) and (1.3)′ are fulfilled with $\rho = \gamma = \gamma(\bar{x})$. Moreover, for $\gamma \ge 2h^{\frac{1}{2}}$ condition (1.7)′ is also fulfilled. Let us take a γ-admissible partition of unity $\{\psi_n\}$ in $B(0,\frac{3}{4})$; this means that ψ_n is supported in $B(x_n, \frac{1}{2}\gamma(x_n))$ and satisfies (1.10)′ with $\gamma = \gamma(x_n)$ and that the multiplicity of the covering of $B(0,\frac{3}{4})$ by balls $B(x_n,\gamma(x_n))$ doesn't exceed $C_0 = C_0(d)$. Condition (1.13) implies that this partition exists. Then the contribution of every ball with $\gamma(x_n) \ge h^{\frac{1}{2}}$ to the remainder estimate in (1.8) doesn't exceed

$$Ch^{1-d}\gamma(x_n)^{2d-2} = h^{1-d}\gamma^{d-2}\int_{B(x_n,\gamma(x_n))} dx$$

and therefore the total contribution of all the balls of this type doesn't exceed Ch^{-d} for $d \ge 2$ and $h^{-\frac{1}{2}}$ for $d = 1$. According to (i) $e(x,x,-\infty,0) \le Ch^{-\frac{d}{2}}$ if $\gamma(x) \asymp h^{\frac{1}{2}}$ and hence the total contribution of this zone to every term of asymptotics (1.8) can be estimated in the same way.

We have therefore proved

Theorem 1.2. *In frames of theorem 1.1 estimate* (1.8) *holds in the general case for* $d \ge 2$. *Moreover, for* $d = 1$ *the left-hand expression of* (1.8) *doesn't exceed* $Ch^{-\frac{1}{2}}$.

Let us treat case $d = 1$ more carefully.

Theorem 1.3. *Let* $d = 1$ *and conditions of theorem 1.1 be fulfilled. Then*

(i) If

$$|V| + |V'| + \cdots + |V^{(n)}| \ge \epsilon_0 \tag{1.14}$$

for some $n \ge 1$ *then the left-hand expression of* (1.8) *doesn't exceed* $C(|\log h| + 1)^{n-1}$ *where now* $K = K(n)$ *in conditions* (1.3), (1.10).

(ii) In the general case the left-hand expression of (1.8) *doesn't exceed* $Ch^{-\delta}$ *with arbitrary* $\delta > 0$ *where now* $K = K(\delta)$ *in conditions* (1.3), (1.10).

Proof. (i) For $n = 1$ this statement has been proved. Assume we have proved it for $n < \bar{n}$ and let us consider the case $n = \bar{n}$. Let us introduce the function

$$\gamma(x) = \epsilon_1 \left(\sum_{0 \le k \le n-1} |V^{(k)}|^{\frac{n}{n-k}} \right)^{\frac{1}{n}} + h^{\frac{2}{n+2}}$$

with small enough $\epsilon_1 = \epsilon_1(n, c, \epsilon_0)$. It is easy to check that (1.13) is fulfilled and that in $B(x, \gamma)$ conditions $(1.3)', (1.10)'$ are fulfilled with $\gamma = \gamma(x)$ and $\rho = \gamma^{\frac{n}{2}}$. Moreover, if $\gamma \ge 2h^{\frac{2}{n+2}}$ then after dilatation and multiplication condition $(1.14)_{n-2}$ is fulfilled; so contribution of this ball doesn't exceed $C(|\log h| + 1)^{n-1}$ (more refined estimate doesn't improve the final answer). Furthermore, it is easy to check that under condition $(1.14)_n$ $\gamma(x) \ge \epsilon_2 \min_l |x - x_{(l)}|$ for appropriate points $x_{(l)}$ with $l = 1, \dots, L \le L_0 = L_0(c, n)$, $\epsilon_1 = \epsilon_1(c, n, \epsilon_0) > 0$ and then the total contribution of these balls (intervals) to remainder estimate in (1.8) doesn't exceed $C(|\log h| + 1)^{n-1}$. Moreover, it is easy to check that the contribution of remaining L intervals doesn't exceed $C(|\log h| + 1)^{n-1}$ either and then the induction step is made. The statement (i) is proved.

(ii) Applying the same arguments as before and using (i) we obtain the remainder estimate $Ch^{-2/(n+2)}(|\log h| + 1)^{n-2}$ with arbitrary n because $(1.14)_n$ isn't assumed to be fulfilled. This yields (ii). □

Remark 1.4. This proof can be extended to a wide class of scalar operators in other dimensions. On the other hand, even in dimension $d = 1$ the propagation of singularities arguments improve the final answer.

2. How LSSA Yield Standard Spectral Asymptotics

Now I would like to discuss three examples which show how LSSA yield asymptotics with respect to the spectral parameter.

(i) Let X be a compact Riemannian manifold without a boundary, Δ the Laplace-Beltrami operator on X. Let us consider the Schrödinger operator $A_h = -h^2\Delta + V(x)$; we know that

$$N^-(A_h) = c_0 h^{-d} + O(h^{1-d})$$

with the Weylian constant c_0 provided either $d \ge 2$ or 0 isn't a critical value of $V(x)$ (otherwise asymptotics with a worse remainder estimate holds). I recall that $N^-(A_h)$ is the number of negative eigenvalues of A_h counting their multiplicities. The Birman-Schwinger principle implies that

$$N^-(A_h) - N^-(A_0) = N(h^{-2})$$

where $N(\lambda)$ is the number of eigenvalues $\mu \in (0, \lambda)$ of the spectral problem $(-\Delta + \mu V(x))u = 0$ and $N^-(A_0) = \lim_{h \to +0} N^-(A_h) < \infty$. These two equalities immediately yield the asymptotics

$$N(\lambda) = c_0 \lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-1}{2}}) \tag{2.1}$$

provided either $d \geq 2$ or 0 isn't a critical value of $V(x)$. Let somebody try to obtain this result directly in the case when $V(x)$ vanishes at some point! This is striking that nobody has yet observed this non-trivial new result (provided 0 isn't critical value of V) which trivially follows from two well-known facts!

(ii) Let us consider the Schrödinger operator $A = -\Delta + V(x)$ in $\mathbb{R}^d$ where Δ is the Laplacian and a real-valued potential $V(x)$ satisfies the following conditions:

$$|D^\alpha V| \leq c\langle x\rangle^{2m-\alpha} \qquad \forall\alpha : |\alpha| \leq K, \tag{2.2}$$

$$V \geq \epsilon_0 |x|^{2m} \qquad \text{for } x : |x| \geq c \tag{2.3}$$

where $\langle x\rangle = (|x|^2+1)^{\frac{1}{2}}$ here and in what follows. Let us take a γ-admissible partition of unity with $\gamma(x) = \frac{1}{4}\langle x\rangle$. If we make a dilatation transforming $B(\bar{x}, \gamma(\bar{x}))$ into $B(0,1)$ and multiply $A-\lambda$ by λ^{-1} we obtain for $|x| \leq C\lambda^{1/2m}$ in $B(0,1)$ the Schrödinger operator with standard restrictions on potential and with $h = \frac{1}{\gamma(\bar{x})\sqrt{\lambda}}$. Let us apply theorems 1.2,1.1 (then one should assume that either $d \geq 2$ or $V - \lambda$ is non-degenerating, i.e.

$$|\nabla V| \geq \epsilon_0 |x|^{2m-1} \qquad \text{for } |x| \geq c); \tag{2.4}$$

then we obtain for spatial means of $e(x,x,\lambda)$ in $B(\bar{x},\gamma(\bar{x}))$ the Weylian asymptotics with the remainder estimate $O(h^{1-d}) = O(\lambda^{\frac{d-1}{2}}\gamma(\bar{x})^{d-1})$; here is no additional factor because $e(x,x,\lambda)$ is a density but not a function. Summing with respect to the partition of unity we obtain remainder estimate $O(\lambda^{(d-1)l})$ with $l = \frac{1}{2m}(1+m)$. If we consider the ball $B(\bar{x},\gamma(\bar{x}))$ with $|x| \geq C\lambda^{\frac{1}{2m}}$ then after dilatation and multiplication of $A - \lambda$ by $\rho^{-2}(\bar{x})$ with $\rho(\bar{x}) = \langle x\rangle^m$ we obtain the Schrödinger operator with standard restrictions on the potential $v(x) \geq \epsilon_1$ and with $h = \frac{1}{\gamma(\bar{x})\rho(\bar{x})}$; then we derive an estimate $|e(x,x,\lambda)| \leq Ch^s\gamma(\bar{x})^{-d}$ with an arbitrary s and hence the contribution of the domain $\{|x| \geq C\lambda^{\frac{1}{2m}}\}$ in the remainder estimate is $O(\lambda^{-s})$ where we decrease s if it is necessary. We obtain also the asymptotics

$$N(\lambda) = \mathcal{N}(\lambda) + O(\lambda^{(d-1)l}) \tag{2.5}$$

with

$$\mathcal{N}(\lambda) = c_0 \int (\lambda - V)_+^{\frac{d}{2}} dx \tag{2.6}$$

as $\lambda \to +\infty$ and $\mathcal{N}(\lambda) \asymp \lambda^{dl}$.

(iii) Let us consider the Schrödinger operator in $\mathbb{R}^d$ with a potential $V(x)$ satisfying (2.2) with $m \in (-1,0)$ and let now $\lambda < 0$ be a small parameter. Then for the same ρ and γ as before the dilatation of $B(\bar{x},\gamma(\bar{x}))$ into $B(0,1)$ and multiplication of $A - \lambda$ by $\rho^{-2}(\bar{x})$ give the Schrödinger operator with the standard restrictions on the potential and with $h = \frac{1}{\gamma(\bar{x})\rho(\bar{x})}$ provided $|x| \leq |\lambda|^{\frac{1}{2m}}$; hence for the spatial mean of $e(x,x,\lambda)$ in $B(\bar{x},\gamma(\bar{x}))$ the Weylian formula holds with the remainder estimate $O(h^{1-d}) = O(\rho(\bar{x})^{d-1}\gamma(\bar{x})^{d-1})$ (provided that either $d \geq 2$ or condition (2.4) is fulfilled). On the other hand, for $|x| \geq C|\lambda|^{\frac{1}{2m}}$ dilatation and multiplication by $|\lambda|^{-1}$ give the Schrödinger operator with the standard restrictions to potential $v(x) \geq \epsilon_1$ and with $h = \frac{1}{\gamma(\bar{x})\sqrt{|\lambda|}}$. Hence the estimate $|e(x,x,\lambda)| \leq Ch^s\gamma(\bar{x})^{-d}$ holds again. Summing with respect to the partition of unity we obtain asymptotics (2.5)-(2.6) as $\lambda \to -0$ with the same l as in (i); moreover, $\mathcal{N}(\lambda) \asymp |\lambda|^{dl}$ provided

$$V \leq -\epsilon_1|x|^{2m} \qquad \text{for } x : |x| \geq c \tag{2.7}$$

in some non-empty open cone in $\mathbb{R}^d$.

In these three cases for $d = 1$ without the non-degeneracy condition the final remainder estimate is slightly worse.

3. How One Can Derive LSSA in the General Case

The only possible (or at least the best) way to derive spectral asymptotics with accurate remainder estimate is the hyperbolic operator method (I don't discuss here special cases when one can find eigenvalues explicitly). There are few implementation of this method for semiclassical spectral asymptotics; for example, for Schrödinger operator one can consider $U(t) = \exp ith^{-1}B$ with either $B = A$ or $B = A^{\frac{1}{m}}$ where $m = 2$ is the order of operator. The second definition leads to the wave equation $h^2D_t^2u = A_hu$ which is hyperbolic in the classical sense and has the useful finite speed of propagation property; however, difficulties arise in the case when A isn't semi-bounded from below and in the case of higher-order operator make this way rather poor. The best way is to consider $U(t) = \exp ith^{-1}A$; the corresponding non-stationary Schrödinger equation

$$hD_tu = A_hu \tag{3.1}$$

isn't hyperbolic in the classical sense but it has all the useful properties of hyperbolic equations (with reasonable modifications). In particular, there is a finite speed of propagation property in the compact domains of the phase space. Thus, as we have mentioned, one should apply methods of partial differential equations and construct $\operatorname{Tr} QU(t)$ at some interval $[-T,T] \ni t$ and then use the formula

$$F_{t\to h^{-1}\tau} \operatorname{Tr} \chi_T(t)QU(t) = T\int \hat{\chi}(\frac{(\tau-\lambda)T}{h})d_\lambda \operatorname{Tr} QE(\lambda) \tag{3.2}$$

and the Tauberian theorem in order to recover asymptotics of $\operatorname{Tr} QE(\tau_1,\tau_2)$; here and below Q is an h-pseudo-differential operator with a compactly supported symbol. For a scalar operator A in the interior of domain (or in similar cases) the construction of QU is well-known for $T = \text{const} > 0$ (or even on longer interval under very restrictive conditions); however, for matrix operators and near the boundary this explicit construction at this interval is either very complicated or impossible. The idea which was suggested eleven years ago by the author in order to avoid this difficulty is very transparent in the semi-classical case: to construct QU for a shorter interval $[-T', T']$ (with T' depending on h) and then to prove that $\operatorname{Tr} QU$ is negligible at $[-T, -T'] \cup [T', T]$.

In order to make the first step we apply the successive approximation method. Let us consider the equation (3.1) with the initial data

$$U|_{t=0} = \delta(x-y). \tag{3.3}$$

Hence

$$(hD_t - A)U^{\pm} = \mp ih\delta(x-y)\delta(t) \tag{3.4}$$

where $U^{\pm} = \theta(\pm t)U$ and θ is Heaviside function; therefore

$$U^{\pm} = \mp ihG^{\pm}\delta(x-y)\delta(t) \tag{3.5}$$

where $G^{\pm}$ is the parametrix of the problem

$$(hD_t - A)v = f, \qquad v|_{\pm t<0} = 0. \tag{3.6}$$

On the other hand, (3.4) yields that

$$(hD_t - \bar{A})U^{\pm} = \mp ih\delta(x-y)\delta(t) + RU^{\pm} \tag{3.7}$$

where $\bar{A} = A(y, hD_x, 0)$ is operator obtained from A by freezing coefficients at y and dropping lower-order terms (in the semi-classical sense) and $R = A - \bar{A}$. Therefore

$$U^{\pm} = \mp ih\bar{G}^{\pm}\delta(x-y)\delta(t) + \bar{G}^{\pm}RU^{\pm}$$

where $\bar{G}$ is the parametrix of problem (3.6) for operator $\bar{A}$. Iterating this equality and using (3.5) once we obtain that

$$U^{\pm} = \mp ih \sum_{n\le N-1} (\bar{G}^{\pm}R)^n\bar{G}^{\pm}\delta(x-y)\delta(t) \mp ih(\bar{G}^{\pm}R)^N G^{\pm}\delta(x-y)\delta(t) \tag{3.8}$$

with arbitrarily large N. Let us notice that

$$R = \bar{R} + R' = \sum_{1\le|\alpha|+k\le M-1} h^k(x-y)^{\alpha}B_{\alpha,k}(y, hD_x) + \sum_{|\alpha|+k=M} B_{\alpha,k}(x,y,hD_x). \tag{3.9}$$

Let us substitute (3.9) for (3.8) and let us move all the factors $(x_j - y_j)$ to the right. If one factor reaches $\delta(x-y)$ the corresponding term vanishes; so the term survives only if this factor is killed on his way. The factor can be killed by commutation either with some h-pseudo-differential operator or with some parametrix. In the first case a factor h arises. In the second case one can apply equality

$$[G^\pm, x_j - y_j] = G^\pm [A, x_j - y_j] G^\pm \tag{3.10}$$

and the similar equality for $\bar{A}$, $\bar{G}$; the first equality is due to identity

$$(hD_t - A)(x_j - y_j)v = -[A, x_j - y_j]v + (x_j - y_j)(hD_t - A)v$$

which yields that

$$(x_j - y_j)v = -G^\pm [A, x_j - y_j]v + G^\pm (x_j - y_j)(hD_t - A)v$$

provided $v|_{\pm t<0} = 0$; substituting $v = G^\pm f$ with $f|_{\pm t<0} = 0$ we obtain (3.10). Thus, in this commutation $(x_j - y_j)$ is replaced by an additional factor h and an additional parametrix appears.

The Duhamel formula yields that the operator norms of $G^\pm$ and $\bar{G}^\pm$ in $L^2([-T,T], \mathbb{R}^d)$ don't exceed $C\frac{T}{h}$. Let us note that in the original expansion every parametrix was accompanied either by h or by $(x_j - y_j)$ factors. This yields that under the condition

$$T \le h^{\frac{1}{2}+\delta} \tag{3.11}$$

with arbitrarily small $\delta > 0$ for $M = M(d, s, \delta)$ and $N = N(d, s, \delta)$ the remainder term (with $n = N$) in (3.8) is negligible (i.e., less than h^s) and the equality (3.8) remains true modulo negligible terms if one replaces R by $\bar{R}$[7)].

Now only operators with symbols not depending on x remain and the Fourier transform on x and Fourier-Laplace transform on t provide us with the final answer:

$$F_{t\to h^{-1}\tau} \operatorname{Tr} QU^\pm = \pm \int F(\tau, y, \xi, h)\, dy d\xi \tag{3.12}$$

for $\mp\tau > 0$ and

$$F_{t\to h^{-1}\tau} \operatorname{Tr} QU = \int \mathcal{F}(\tau, y, \xi, h)\, dy d\xi \tag{3.13}$$

for $\tau \in \mathbb{R}$. Here F is a sum of the terms of the type

$$\operatorname{tr}(\tau - a(y,\xi))^{-1} b_1(y,\xi)(\tau - a(y,\xi))^{-1} \dots \\ b_{r-1}(y,\xi)(\tau - a(y,\xi))^{-1} b_r(y,\xi) h^{-d+n}$$

(a is the principal symbol of A) and $\mathcal{F}(\tau, ., ., .) = F(\tau - i0, ., ., .) - F(\tau + i0, ., ., .)$. Thus at rather short time interval $\operatorname{Tr} QU$ is constructed modulo negligible term. Multiplying $\operatorname{Tr} QU$ by $\varphi(hD_t/L)\bar{\chi}_T(t)$ with $\bar{\chi} \in C_0^K([-1,1])$, $\bar{\chi} = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$,

[7)]In fact this deduction works only under some restrictions on A. It is sufficient to assume that its symbol is compactly supported; otherwise the appropriate cutoff should be done.

$\bar{\chi}_T(t) = \bar{\chi}(t/T)$, $\varphi \in C_0^K(\mathbb{R})$, $LT \ge h^{1-\delta}$ and setting $t = 0$ we obtain complete asymptotics of the spectral mean

$$(3.14) \qquad \int \phi(\frac{\tau}{L}) \operatorname{Tr} Q d_\tau E(\tau) \sim \sum_n h^{-d+n} \int \phi(\frac{\tau}{L}) \varkappa_k'(\tau) d\tau$$

provided $L \ge h^{\frac{1}{2}-\delta}$.

In order to derive asymptotics without mollifications additional arguments linked with propagation of singularities should be applied.

4. Propagation of Singularities

In fact, the results of this type (namely, locally finite speed of propagation) were used in order to justify the construction in the previous section. However here more refined results are necessary.

Let us assume first that A is a scalar operator. Let $\mathcal{V}$ be a small neighborhood of the point $(\bar{x}, \bar{\xi})$ such that $a(\bar{x}, \bar{\xi}) = 0$; if $|a(\bar{x}, \bar{\xi})| \ge \epsilon > 0$ then standard elliptic arguments yield that $F_{t \to h^{-1}\tau} Q_x U = O(h^s)$ for $|\tau| \le \epsilon_1 = \epsilon_1(d, c, \epsilon) > 0$ provided the symbol of Q is supported in $\mathcal{V}$. Let us assume first that

$$(4.1) \qquad |\nabla_\xi a(\bar{x}, \bar{\xi})| \ge c^{-1};$$

without loss of generality one can assume that $\partial_{\xi_1} a(\bar{x}, \bar{\xi}) \ge \epsilon_0$; otherwise one can reach it by change of co-ordinates. Then singularities in the neighborhood of $(\bar{x}, \bar{\xi})$ propagate with velocity disjoint from 0 in the x_1-direction (one can obtain this from the classical Hamiltonian system) and since at $t = 0$ all the singularities of U lie on $\{x = y, \xi = -\eta\}$ then for $0 \le \pm t \le T_0$ all the singularities of QU lie in $\{\mp(x_1 - y_1) \ge \pm\epsilon_0 t\}$; therefore there is no singularity of $QU|_{x=y}$ in $[-T_0, T_0] \setminus 0$ where $T_0 = T_0(d, c, \epsilon_0) > 0$ is small enough. Then $F_{t \to h^{-1}\tau} \chi_T' QU|_{x=y} = O(h^s)$ and $F_{t \to h^{-1}\tau} \chi_T' \operatorname{Tr} QU = O(h^s)$ for $\chi \in C_0^K([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $0 < T \le T_0$; this estimate is uniform for T disjoint from 0. However, this result can be improved by means of dilatation method and the derived estimates are uniform for $h^{1-\delta} \le T \le T_0$ with arbitrarily small exponent $\delta > 0$ (this restriction is due to uncertainty principle). Therefore under the above condition the construction of the previous section provides us with $F_{t \to h^{-1}\tau} \bar{\chi}_T \operatorname{Tr} QU$ for $T = T_0$ (because intervals $[-T_0, -h^{1-\delta}]$ (or $[h^{1-\delta}, T_0]$) and $[-h^{\frac{1}{2}+\delta}, h^{\frac{1}{2}+\delta}]$ overlap for small $\delta > 0$).

This and the arguments at the end of the previous section yield immediately the complete asymptotics for spectral means with mollification parameter $L \ge h^{1-\delta}$.

The asymptotics of $\operatorname{Tr} QE(\tau_1, \tau_2)$ without mollification and with remainder estimate $O(h^{1-d})$ follows from the construction of section 3 (extended to interval $[-T_0, T_0]$ now) via the Tauberian theorem (see below). Moreover, one can replace condition (4.1) by condition

$$|\nabla_{x,\xi} a(\bar{x},\bar{\xi})| \geq \epsilon_0. \tag{4.2}$$

Actually, if this condition is fulfilled one can always reach (4.1) by means of symplectic change of phase co-ordinates (and there is always implementation by unitary Fourier integral operator preserving trace but not restriction to the diagonal[8)]). Moreover, referring to above elliptic arguments one can replace (4.2) by

$$|a| + |\nabla_{x,\xi} a| \geq \epsilon_0 \qquad \text{in } \mathcal{V}. \tag{4.3}$$

Then we obtain the asymptotics of the above type for $|\tau_i| \leq \epsilon_1$ with a small enough constant $\epsilon_1 > 0$.[9)] Finally, the condition that $\mathcal{V}$ is small isn't necessary: one always can use an appropriate partition of unity.

This construction is done rigorously in [Ivrii 7.2]. There is also generalization to matrix operators and condition (4.3) is replaced by *microhyperbolicity* condition

$$\langle (\mathcal{T}a)(x,\xi)v, v\rangle \geq \epsilon_0 |v|^2 - c|a(x,\xi)v|^2 \qquad \forall v \tag{4.4}$$

for appropriate $\mathcal{T} \in T_{x,\xi}\mathcal{V}$ depending on (x,ξ) and such that $|\mathcal{T}| \leq 1$. Moreover this construction can be done near the boundary but with more sophisticated *microhyperbolicity* condition involving also the boundary operators [Ivrii 7.3]. One of the main statements obtained in [Ivrii 7.2] is the following

Theorem 4.1. *Let A be a self-adjoint h-pseudo-differential operator in $X = \mathbb{R}^d$ and let Q be a (fixed) h-pseudo-differential operator with the symbol supported in $\Omega \subset \{|x-\bar{x}| \leq c,\ |x - \bar{\xi}| \leq c\}$. Let at every point $(x,\xi) \in \Omega$ the microhyperbolicity condition (4.4) be fulfilled with $\mathcal{T} \in T_{(x,\xi)}T^*X$, $|\mathcal{T}| \leq 1$. Then the estimate*

$$|\operatorname{Tr} QE(\tau_1,\tau_2) - \varkappa_0 h^{-d}| \leq Ch^{1-d} \quad \forall \tau_1,\tau_2 \in [-\epsilon_1,\epsilon_1] \tag{4.5}$$

holds with

$$\varkappa_0 = (2\pi)^{-d} \int \operatorname{tr} q^0(x,\xi)\mathcal{E}(x,\xi,\tau_1,\tau_2)\,dx d\xi \tag{4.6}$$

where q^0 is the principal symbol of Q, $\mathcal{E}$ the spectral projector of $a(x,\xi)$ and $C = C(d,c,c')$, $\epsilon_1 = \epsilon_1(d,c,\epsilon_0)$; here c' is a constant in the routine smoothness conditions to symbol of Q.

5. Tauberian Theorem

The following Tauberian theorem (a variant of Tauberian theorem due

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[8)]Therefore we cannot replace (4.1) by (4.2) in the case when we are interested in asymptotics without spatial mollification.

[9)]The condition (4.1) can be changed in a similar way.

to Hörmander) and its modification linked with Fourier transform plays the central role in the proof of our results:

Theorem 5.1. *Let $\nu(\tau)$ be a monotone non-decreasing function such that*

$$|\nu(\tau)| \le M'(|\tau|+1)^p \qquad \forall \tau \in \mathbb{R}. \tag{5.1}$$

Let $\chi \in C_0^K([-1,1])$ be a fixed function equal to 1 on $[-\frac{1}{2},\frac{1}{2}]$.

(i) Let us assume that

$$|\int \widehat{\chi_T}(\tau-\tau')d_{\tau'}\nu(\tau')| \le M'h^s \qquad \forall \tau \in [-\epsilon,\epsilon] \tag{5.2}$$

with $T \ge h^{1-\delta}$, $\delta > 0$. Then

$$|\nu(\tau)-\nu(0)| \le C'Mh^{s-q} \qquad \forall \tau \in [-\frac{\epsilon}{2},\frac{\epsilon}{2}] \tag{5.3}$$

where $q=q(\delta,p)$, $K=K(s,\delta,p)$, $C'=C'(s,\delta,p,\epsilon,\chi)$;

(ii) Let us assume that

$$\int \widehat{\chi_T}(\tau-\tau')d_{\tau'}\nu(\tau) = \vartheta(\tau) \qquad \forall \tau \in [-\epsilon,\epsilon] \tag{5.4}$$

with $h^{-p} \ge T \ge h^{1-\delta}$. Let $\phi \in C_0^K([-c,c])$ be a fixed function. Then

$$\begin{aligned} &|\int (\nu(\tau)-\nu(\tau')-\Theta(\tau,\tau'))\phi(\tau')d\tau'| \le \\ &\qquad CMT^{-1} + C'M'h^{s-q} \qquad \forall \tau \in [-\frac{\epsilon}{2},\frac{\epsilon}{2}] \end{aligned} \tag{5.5}$$

where

$$\Theta(\tau,\tau') = h^{-1}\int_{\tau'}^{\tau} \vartheta(\tau'')d\tau'', \tag{5.6}$$

$C=C(\phi,\chi,\epsilon)$, $M=\sup_{[-\epsilon,\epsilon]}|\vartheta(\tau)|$ and K, q, C' are the same exponents and constants as before. Moreover, if

$$|\vartheta(\tau)| \le M_0 + M|\tau| \qquad \forall \tau \in [-\epsilon,\epsilon] \tag{5.7}$$

then

$$\begin{aligned} &|\int (\nu(0)-\nu(\tau')-\Theta(0,\tau'))\phi(\tau')d\tau'| \le \\ &\qquad CM_0T^{-1} + CMhT^{-2} + C'M'h^{s-q} \qquad \forall \tau \in [-\frac{\epsilon}{2},\frac{\epsilon}{2}]. \end{aligned} \tag{5.8}$$

Thus, one can see easily that in order to obtain a good remainder estimate in spectral asymptotics one needs to construct $\operatorname{Tr} QU(t)$ for large T and estimate $F_{t\to h^{-1}\tau}\chi_T(\tau)\operatorname{Tr} QU(t)$ in an appropriate way (the condition that Q is non-negative definite operator provides that $\nu(\tau)=\operatorname{Tr} QE(0,\tau)$ is monotone non-decreasing function; one can easily extend the final remainder estimate to an arbitrary h-pseudo-differential operator Q). In the proof of theorem 4.1 the microhyperbolicity condition is used twice: in order to go from $T = h^{\frac{1}{2}+\delta}$ to $T = \mathrm{const} > 0$ and in order to estimate $F_{t\to h^{-1}\tau}\chi_T(\tau)\operatorname{Tr} QU(t)$ when it has been calculated by the method of successive approximations.

6. How to Improve Remainder Estimate in the Case of Non-Periodic Trajectories

The above Tauberian theorem yields that in order to improve the remainder estimate in the asymptotics one should increase T in our analysis. We treat only scalar operators in this and in the following subsections; certain generalizations can be found in [Ivrii 7.2,7.3]. In this section we consider the easiest case when all the singularities of $\operatorname{Tr} QU(t)$ in$[-T,T]$ lie in fact in $[-T',T']$ with $T' = h^{1-\delta}$ where T is either a large constant or even temperately large parameter; in view of section 4 one should prove that intervals $[-T,-T_0]$ and $[T_0,T]$ contain no singularity, where $T_0 > 0$ is an arbitrarily small constant (if T is a large parameter then one should prove that $\operatorname{Tr} QU(t)$ is uniformly negligible at these intervals). The properties of the trace yield that $\operatorname{Tr} QU(t)$ can be replaced by $\operatorname{Tr} QU(t)Q'$ in this analysis. Moreover, taking an adjoint operator we obtain $\operatorname{Tr} Q_1U(-t)Q_1'$ with $Q_1 = Q'^*$ and $Q_1' = Q^*$. Therefore it is enough to consider only one of the intervals $[-T,-T_0]$ and $[T_0,T]$, This is quite different from the analysis of resonances.

It is well-known that in the scalar case singularities propagate along classical Hamiltonian trajectories. This means that all the trajectories of the length $\le T$ starting from supp Q' in the positive (or negative) direction don't meet the boundary and don't leave the zone where the coefficients of the operator are regular. If for $T_0 \le t \le T$ they are disjoint from supp Q then interval $[T_0,T]$ (or $[-T_0,T]$ respectively) contains no singularity of $QU(t)Q'$ and both intervals contain no singularity of its trace; in the latter case Q and Q' are supported in the neighborhood of the same point. Here and below a support of h-pseudo-differential operator means a support of its symbol. If we refer to classical results then "disjoint" means that distance is greater than some positive constant. However, it is possible to obtain the same result for "disjoint" meaning that the distance is greater than $\bar\gamma = h^{\frac{1}{2}-\delta}$ with an arbitrarily small exponent $\delta > 0$. Moreover , we can treat the case when T is a large parameter; in this case one should assume that $T \le h^{-\sigma}$, $|J| \le h^{-\sigma}$ and $|D^\alpha a_\beta| \le h^{-\sigma}$ along trajectories where a_β are coefficients of operator, J means a Jacobi matrix of the Hamiltonian flow and $\sigma = \sigma(d,\delta,s) > 0$ is a small enough exponent.

The easiest proof uses the Heisenberg' representation. Let us introduce $Q_t = U(-t)QU(t)$. To prove that $QU(t)Q'$ is negligible it is sufficient to prove thatQ_tQ' is negligible. Moreover,

$$D_tQ_t = -h^{-1}[A,Q_t] \tag{6.1}$$

and

$$Q_0 = Q. \tag{6.2}$$

Let us assume for a moment that Q_t is a h-pseudo-differential operator. Then Cauchy problem (6.1)-(6.2) yields a sequence of Cauchy problems for different terms of the symbol. We can prove under our hypothesis that this sequence of problems has a solution belonging to the appropriate symbol class. Then the quantization of this symbol $\tilde{Q}_t$ satisfies Cauchy problem (6.1)-(6.2) modulo a negligible operator. It is easy to show that $Q_t - \tilde{Q}_t$ is also negligible. Here the conjecture that Q_t was a pseudo-differential operator was used only in order to pass from the operator to its symbol but we can write the Cauchy problem for a symbol formally, and justify this conjecture a posteriori when we prove that the solution is an admissible symbol and therefore operator $\tilde{Q}_t$ is an appropriate operator.

Moreover, even the case when the trajectory meets the boundary can be treated under certain hypothesis (see [Ivrii 7.3]). Let us assume $\operatorname{diam}\operatorname{supp} Q \leq \bar{\gamma} = h^{\frac{1}{2}-\delta}$ [10] Then the arguments of the previous sections yield the estimate

$$\begin{gathered} |\int \phi(\tau)(\operatorname{Tr} QE(\tau,0) - \Theta(\tau,0))d\tau| \leq \\ Ch^{1-d}T^{-1}\int_{\Sigma_0\cap\mathcal{V}} d\mu_0 + Ch^{1-d+\sigma'}\bar{\gamma}^{2d-1} + C'h^s \end{gathered} \tag{6.3}$$

where $\Theta(\tau',\tau)$ is given by (5.6) with $\vartheta(\tau) = F_{t\to h^{-1}\tau}\chi_T(t)\operatorname{Tr} QU(t)$, $\Sigma_\tau = \{(x,\xi) : a(x,\xi) = \tau\}$ is an energy surface in the phase space, $\mu_\tau = dxd\xi : da|_{\Sigma_\tau}$ is a natural density on Σ_τ, $\mathcal{V}$ is a $\bar{\gamma}$-neighborhood of supp Q, $\sigma' = \sigma'(d,\delta) > 0$ is a small enough exponent and under weak conditions C' depends on T but under more restrictive conditions C' doesn't depend on T.

Let us take partition of unity of diameter $\bar{\gamma}$ in the neighborhood of supp Q_0 where Q_0 is a h-pseudo-differential operator supported in a fixed ball in the phase space. Applying estimate (6.3) we obtain the estimate

$$\begin{gathered} |\int \phi(\tau)(\operatorname{Tr} QE(\tau,0) - h^{-d}\varkappa_0(\tau,0) - h^{1-d}\varkappa_1(\tau,0))d\tau| \leq \\ C\sum_{0\leq i\leq n} h^{1-d}T_i^{-1}\int_{\Lambda_i\cap\mathcal{V}} d\mu_0 + Ch^{1-d+\sigma'} + C'h^s \end{gathered} \tag{6.4}$$

where Λ_i $(i = 1,\dots,n)$ are closed subsets of Σ_0 on which appropriate conditions discussed above including the condition of non-periodicity

$$\operatorname{dist}((x,\xi),\Phi_t(x,\xi)) \geq \bar{\gamma} \tag{6.5}$$

[10] In our analysis x and ξ are of equal right and we should remember about uncertainty principle $\Delta x \cdot \Delta\xi \geq h^{1-\delta}$.

are fulfilled with T_i instead of T and $\Lambda_0 = \Sigma_0 \setminus (\Lambda_1 \cup \cdots \cup \Lambda_n)$, $T_0 = 1$. Of course, one can take Λ_i depending on h and under appropriate conditions the right-hand expression of (6.4) is $o(h^{1-d})$ or even better (up to $O(h^{1-d+\sigma'})$).

7. How to Improve the Remainder Estimate in the Case of Periodic Trajectories

Let us consider the case when all the trajectories are periodic (or at least there is a domain Ω in the phase space such that all the trajectories starting in Ω remain there and are periodic). As before we consider the scalar case and we assume that

$$|\nabla a| \geq \epsilon_0 \qquad \text{in } \Omega. \tag{7.1}$$

It is well-known that under these conditions generic period is a function of the energy level (exceptional subperiodic trajectories are possible):

$$T(x,\xi) = T(a(x,\xi)) \qquad \forall (x,\xi) \in \Omega.$$

Replacement $A \to A_1 = f(A)$ yields $T(x,\xi) \to T_1(x,\xi) = T(x,\xi)/f'(a(x,\xi))$ where prime means the derivative here; taking $f'(\tau) = T(\tau)$ we obtain $T_1 \equiv 1$ (at least for $(x,\xi) \in \Omega$). On the other hand, spectral projectors of A and $f(A)$ are linked obviously. Therefore without loss of generality one can assume that

$$\Phi_t(\Omega) = \Omega, \qquad \Phi_1(x,\xi) = (x,\xi) \qquad \forall (x,\xi) \in \Omega. \tag{7.2}$$

It is well-known [3,10] that in this case

$$e^{ih^{-1}A}Q \equiv e^{iB}Q$$

provided Q is compactly supported in Ω. Here B is an h-pseudo-differential operator with the principal symbol

$$b(x,\xi) = T(x,\xi)^{-1} \int_0^{T(x,\xi)} a^s(\Phi_t(x,\xi))dt + \alpha$$

(with $T \equiv 1$ here but this formula is invariant under the above replacement) where a^s is the subprincipal symbol and $\alpha = \alpha_1 h^{-1} + \alpha_2$ is the Maslov' constant: $\alpha_1 = \text{action}/T$ and $4\alpha_2/\pi$ is the Maslov' index of closed trajectory ($\alpha, \alpha_1, \alpha_2$ don't depend on trajectory in our case). Without loss of generality one can assume that $\alpha = 0$. In fact, $A \to A + \mu$ yields $B \to B + h^{-1}\mu$ for constant μ.

In order to understand the role of B let us assume first that $B = 0$. Moreover, let us assume that $e^{ih^{-1}A} = I$ (from the heuristic point of view these assumptions are almost equivalent); then

$$e^{ith^{-1}A} = I \tag{7.3}$$

for $t \in \mathbb{Z}$ and therefore $e^{ith^{-1}A} = e^{it'h^{-1}A}$ for $t \in \mathbb{R}$ where t' is the fractional part of t. Therefore in order to construct $\operatorname{Tr} QU(t)$ on $\mathbb{R}$ it is sufficient to construct it on the interval $[-\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ with arbitrarily small $\epsilon > 0$ and then apply the partition of unity $1=\sum_{n\in\mathbb{Z}}\chi(t-n)$ on $\mathbb{R}$ with appropriate $\chi\in C_0^K([-\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon])$. However , this doesn't lead to better remainder estimates in semiclassical spectral asymptotics because one can estimate $|F_{t\to h^{-1}\tau}\bar{\chi}_T \operatorname{Tr} QU|$ only by CTh^{1-d} for large T and increasing T we gain nothing (see the Tauberian theorem). It is reasonable: equality (7.3) yields that $\operatorname{Spec}(A)\subset\mathbb{Z}$ and therefore eigenvalues of A are highly degenerated (with multiplicities $\asymp h^{1-d}$). In this case we can obtain complete asymptotics *inside* spectral gaps.

Let us consider a more general case and let us assume that

$$e^{ih^{-1}A}Q\equiv e^{i\eta B}Q \tag{7.4}$$

for all h-pseudo-differential operators Q supported in Ω. Here B is an h-pseudo-differential operator and $\eta\in(h^n,h^{\delta-1})$ is an additional parameter. A small parameter η can appear because first terms of "original" B vanish and large parameter η can appear because one perturbs A by $\eta hA'$.

One can then easily prove that $e^{inh^{-1}A}Q\equiv e^{in\eta B}Q$ for $n\in\mathbb{Z}$, $|n|\le\epsilon/h\eta$ and Q compactly supported in Ω where $\epsilon>0$ depends on $\operatorname{dist}(\operatorname{supp}Q,\partial\Omega)$ and that

$$e^{ith^{-1}A}Q\equiv e^{it'h^{-1}A}e^{it''\eta h^{-1}B}Q\qquad \forall t:|t|\le\frac{\epsilon}{h\eta} \tag{7.5}$$

for same Q and ϵ as before, where $t''=[t]h\eta$ and $t'=\{t\}$. Hence one can study a long-time propagation of singularities in this case: singularities propagate along Hamiltonian trajectories of a drifting with velocity $h\eta$ along Hamiltonian trajectories of b. The uncertainty principle yields that one can notice this drift only for $|h\eta t|\ge h^{1-\delta'}$. Let us assume that

$$|\nabla_\Sigma b|\ge\epsilon_0\qquad \text{at } \Sigma_\tau \tag{7.6}$$

where ∇_Σ means differential along Σ_τ. Then the periodicity of trajectories is destroyed for $|n|\ge h^{-\delta'}\eta^{-1}$. If $\eta\ge h^{-\delta}$ then the periodicity of trajectories is destroyed after one turn and if we assume that

(7.7) There is no subperiodic trajectory of a

then the singularity of $\operatorname{Tr}QU(t)$ located in a neighborhood of 0 is the only singularity in the interval $[-T,T]$ with $T=\epsilon/h\eta$. The Tauberian theorem yields then the standard semiclassical spectral asymptotics with the remainder estimate $Ch^{1-d}\eta^{-1}$.

Let us assume that $\eta\le h^{-\delta}$. Then (under condition (7.7)) singularities of $\operatorname{Tr}QU(t)$ in $[-T,T]$ are located in $[-T',T']$ where $T=\epsilon/h\eta$ and $T'=1/h^\delta\eta$. Then

$$|F_{t\to h^{-1}\tau}\chi_T \operatorname{Tr} QU|\le CT'h^{1-d} \tag{7.8}$$

and the Tauberian theorem yields the semiclassical spectral asymptotics with the remainder estimate $O(h^{2-d-\delta})$ and with additional term $h^{1-d}F(\tau,\tau/h)$ due to singularities of $\operatorname{Tr} QU(t)$ located in $[-T',T']$ and different from 0. Namely,

$$F(t,z) = (2\pi)^{-d}\int_{\Sigma_\tau} \Upsilon(z-\eta b)\,d\mu_\tau \tag{7.9}$$

where $\Upsilon(z)$ is 2π-periodic function on $\mathbb{R}$ equal to $\pi - z$ at $[0,2\pi)$. Moreover, under conditions (7.6) and (7.7) more accurate calculations yield the estimate (7.8) with $T' = 1/\eta + 1$ and we obtain the remainder estimate $O(h^{2-d})$. Thus the estimate

$$|\operatorname{Tr} QE(\tau',\tau) - \varkappa_0(\tau',\tau)h^{-d}\varkappa_1(\tau',\tau)h^{1-d} - F(\tau,\tau/h)h^{1-d}| \le Ch^{2-d}(1+\eta) \tag{7.10}$$

holds.

In this asymptotics the non-Weylian term $F(\tau,\tau/h)h^{1-d}$ is of the same order as the second Weylian term $\varkappa_1 h^{1-d}$. However, at intervals of the length $\asymp h$ oscillations of the non-Weylian term are of the same order as the oscillations of the principal term (at least in the situation described below).

The detailed analysis and generalizations can be found in [Ivrii 1.2]. In particular, conditions (7.6) and (7.7) are weakened there. Moreover, there is proved that under conditions (7.1),(7.2),(7.4) $\operatorname{Tr} QE(\tau',\tau)$ is negligible when τ' and τ belong to the same gap in the semiclassical approximation to spectrum. These gaps are defined (for $\eta \le \epsilon$ only) by the condition

$$|b(x,\xi,h)\eta - ih^{-1}\tau - 2\pi m| \ge \epsilon\eta \qquad \forall m \in \mathbb{Z}. \tag{7.11}$$

In the case of τ and τ' belonging to different gaps complete asymptotics (with oscillating non-Weylian terms) is derived.

8. Eigenvalue Estimates and Asymptotics for Spectral Problems with Singularities

Singularity means the non-smoothness of the coefficients and (or) boundary, unboundedness (exit to infinity) of the domain X or of the classically allowed zone $\{x, V(x) \le \tau\}$ for the Schrödinger operator etc. For a sake of simplicity we consider only Schrödinger operator in dimension $d \ge 3$. It is well-known that in this case (under Dirichlet boundary condition)

$$N^-(A) \le c_0 h^{-d}\int V_-^{\frac{d}{2}}\,dx. \tag{8.1}$$

This is the Lieb-Cwickel-Rozenblyum estimate and many other estimates of this are known for the Schrödinger operator and more general operators. Under certain conditions it is possible to combine this estimate and local semiclassical spectral asymptotics. Namely, let us consider the Schrödinger operator in $\mathbb{R}^d$

or in a domain $X \subset \mathbb{R}^d$ (in this case we refer to LSSA near boundary), $d \geq 2$. Let us assume that in X functions γ and ρ are given such that

$$\gamma > 0,\ \rho > 0,\ |\nabla\gamma| \leq 1 \tag{8.2}$$

and in subdomain $X' = \{x \in X, \rho\gamma \geq h\}$ the following conditions are fulfilled:

$$y \in X', x \in B(y, \gamma(y)) \implies c^{-1} \leq \frac{\rho(y)}{\rho(x)} \leq c,\ |\nabla\rho| \leq c\rho\gamma^{-1}, \tag{8.3}$$

$$|D^\alpha V| \leq c\rho^2\gamma^{-|\alpha|} \qquad \forall\alpha : |\alpha| \leq K \tag{8.4}$$

and

$$\begin{aligned} X \cap B(y,\gamma(y)) = \{x_k = \phi_k(x_{\hat{k}}) \cap B(y,\gamma(y)),\\ |D^\alpha\phi_k| \leq c\gamma^{1-|\alpha|} \qquad \forall\alpha : 1 \leq |\alpha| \leq K\end{aligned} \tag{8.5}$$

for some $k = k(y)$ where $x_{\hat{k}} = (x_1, \ldots, x_{k-1}, \ldots, x_{k+1}, \ldots, x_d)$. Furthermore, let us assume that

(8.6) For $y \in X'$ on $\partial X \cap B(y,\gamma(y))$ either the Dirichlet or the Neumann condition is satisfied. Then

$$N^-(A) = \int \psi' e(x,x,-\infty,0)dx + \int \psi'' e(x,x,-\infty,0)dx$$

where $\psi' + \psi'' = 1$, ψ' and ψ'' are supported in (the closures of) $\{x \in X, \rho\gamma \geq \frac{5}{4}h\}$ and $\{x \in X, \rho\gamma \leq \frac{6}{4}h\}$ respectively and

$$|D^\alpha\psi'| \leq c\gamma^{-|\alpha|} \qquad \forall\alpha : |\alpha| \leq K.$$

Moreover, LSSA and dilatation-multiplication procedure yield estimates

$$\mathcal{N}^- - CR_1 \leq \int \psi' e(x,x,-\infty,0)dx \leq \mathcal{N}^- + CR_1 + C'R_2$$

where

$$\mathcal{N}^- = (2\pi)^{-d}\omega_d \int \psi' V_-^{\frac{d}{2}} dx \tag{8.7}$$

is the Weylian approximation and

$$R_1 = h^{1-d} \int_{X' \cap \{V \leq \epsilon\rho^2\}} \rho^{d-1}\gamma^{-1}dx, \tag{8.8}$$

$$R_2 = h^s \int_{X'} \rho^{-s}\gamma^{-d-s}dx, \tag{8.9}$$

$\epsilon > 0$ is arbitrary and C, C' depend on ϵ. Therefore

$$\mathcal{N}^- - CR_1 \leq N^- \leq \mathcal{N}^- + CR_1 + CR_2 + \int \psi'' e(x,x,-\infty,0)dx$$

and the lower estimate is derived. In order to derive an upper estimate one should derive an upper estimate for $\int \psi'' e(x,x,-\infty,0)dx$. Let us consider operator A'' in the domain $X'' = \{x \in X, \rho\gamma \le 2h\}$, coinciding in $\{\rho\gamma \le 7h/4\}$ with A (taking in account boundary conditions). Let us assume that A'' is a self-adjoint operator in $L^2(X'')$ and that the estimate

$$N^-(A'' - tJ) \le R''(t+1)^n \qquad \forall t \ge 0 \tag{8.10}$$

holds for some n and some function $J > 0$ coinciding with ρ^2 in $X' \cap X''$. Then it is possible to prove the following estimate:

$$\int \psi'' e(x,x,-\infty,0)dx \le C''R'' + C'R_2$$

where C'' depends on n and C', C'' doen't depend on the choice of J. Therefore the final estimates are

$$\mathcal{N}^- - CR_1 \le N^- \le \mathcal{N}^- + CR_1 + CR_2 + C''R''. \tag{8.11}$$

Moreover, if $d \ge 3$ and Dirichlet boundary condition is given on $\partial X \cap \{\rho\gamma \le 2h\}$ then one can take $A'' = A$ in X'' with the Diriclet boundary condition on $\partial X''$ and estimate (8.1) yields for appropriate J that

$$R'' = C\int_{X''} V_-^{\frac{d}{2}} dx \tag{8.12}$$

There are some useful modifications. Moreover, one can apply LSSA with more accurate remainder estimates (under restrictions to Hamiltonian trajectories). Finally, other improvements also can be done.

The same idea of splitting works even for operators non semi-bounded from below (for example, for the Dirac operator). In this case it is very useful to reduce the original problem to some modified problems via the Birman-Schwinger principle. This principle is very useful for semi-bounded operators too.

The upper and lower estimates for a number of negative eigenvalues or eigenvalues lying in an interval are very useful. Let us take $h = 1$ (however, in the deduction $h_{\text{eff}} \ll 1$ in some balls). Let us consider operator depending on other parameter(s). Then estimates derived here yield asymptotics with respect to new parameters. Some examples of this type were treated in as in section 2. A number of more sophisticated examples can be found in [Ivrii 5,6,7.7]

9. Generalizations. Non-Weylian Asymptotics

I list here some situations when the results of section 8 are not applicable and where non-Weylian spectral asymptotics arise.

1. Schrödinger and Dirac operators with the strong magnetic field. Let us start from local asymptotics. There are two parameters now: $h \ll 1$ and a coupling parameter $\mu \gg 1$. We use the same ideas as before: the hyperbolic operator method including the construction of $\operatorname{Tr} QU$ at short time interval by successive approximations and then extension to larger interval based on certain version of microhyperbolicity. Depending on the problem at hand, the microlocal canonical form of operators in question is used in both parts of analysis in the second one only.Thus the successive approximations method is applied either to the original operator or to the reduced one. For example, if the magnetic intensity is constant, $d = 2, 3$ and $|\nabla V| \geq \epsilon_0$ the first (second) approach works for $1 \leq \mu \leq h^{\delta-1}$ ($h^{-\delta} \leq \mu$ respectively). The asymptotics derived by this method are different and contain many terms which one can calculate only "in principle". However, a comparison of these two asymptotics in zone where both of them hold provides much simpler and more effective answer.

When local semiclassical spectral asymptotics are derived we generalize them by dilatation-multiplication method. Then in order to attack global problems we use partition of unity, treat the singular zones and derive eigenvalue estimates. Finally, we consider operators depending on parameter(s) and derive eigenvalue asymptotics with respect to these parameter(s).

Operators in domains with thick cusps. Spectral asymptotics (with accurate remainder estimates) for operators in domains with thin cusps are due to results of section 8. However, if the cusp is thick, the remainder estimate is not so good or we even may fail to derive asymptotics at all. In this case we change the co-ordinates and transform our cusp to the cylinder. Then we treat the reduced operator as d'-dimensional one with operator-valued coefficients where d' is dimension of the cusp (usually $d' = 1$). We can obtain local semiclassical spectral asymptotics for such operator by methods described above. Moreover, we can use this approach either only in order to extend the time interval (so successive approximations method is applied to original operator) or from the beginning of our analysis. The first (second) approach is useful in the part of cusp near to (far from respectively) origin and we can split these asymptotics.

The same ideas work for the Schrödinger operator in $\mathbb{R}^d$ when V fails to tend to $+\infty$ along some directions. If the "canyons" in the $d+1$-dimensional graph of function $V(x)$ are narrow then results of section 8 yield the desired answer. Otherwise the similar operators with respect to part of variables in the auxiliary Hilbert space should be treated etc.

The same ideas work also in the case when the operator degenerates on a symplectic manifold.

3. Riesz means. In the framework of section 1 the asymptotics of spectral Riesz means can be treated and the remainder estimate $O(h^{1+\vartheta-d})$ can be obtained in local semiclassical spectral asymptotics where ϑ is the order of the Riesz mean. Moreover, under appropriate condition to Hamiltonian flow this remainder estimate can be improved. However, if we apply the approach of section 8 in order to treat the case when V has singularities then we may or may not be able to recover the remainder estimate obtained in the smooth case. It depends on d, ϑ and order of singularity. For example, for Coulomb-like singularity and $\vartheta = 1$ (the most interesting case from the physical point of view) the remainder estimate is $O(h^{2-d})$ for $d \geq 5$ but we obtain $O(h^{-2}\log h)$ for $d = 4$ and $O(h^{-2})$ for $d = 2, 3$. However, under appropriate assumptions we can obtain asymptotics with remainder estimate $O(h^{2-d})$ or even better for $d = 2, 3, 4$. The main idea here is to treat the operator in question as a perturbation of the Schrödinger operator with homogeneous potential and estimate the difference between $\int e_\vartheta(x, x, -\infty, 0)\psi(x/r)dx$ for perturbed and unperturbed operators where subscript ν means that ϑ-th order Riesz means is calculated, $\psi \in C_0^K(\mathbb{R}^d)$ is a fixed function equal 1 near 0 and r is an appropriate parameter. This estimate is based on equality

$$\operatorname{Tr}(E_\vartheta(\tau; A_1) - E_\vartheta(\tau; A_0)) = -\vartheta \int_0^1 \operatorname{Tr} E_{\nu-1}(\tau; A_t) B dt$$

where $A_t = A_0 + Bt$ and $\vartheta \geq 1$; for $0 < \vartheta < 1$ some interpolation arguments are used. On the other hand, for

$$\int (\psi(x) - \psi(\frac{x}{r})) e_\vartheta(x, x, -\infty, 0) dx$$

we apply the local semiclassical spectral asymptotics approach (possibly with remainder estimates improved by very accurate treatment of propagation of singularities near origin). The details and generalizations can be found in [Ivrii 7.9, Ivrii& Sigal 8.1,8.2].

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SPECTRAL THEORY OF ELLIPTIC OPERATORS ON NON-COMPACT MANIFOLDS.

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SPECTRAL THEORY OF ELLIPTIC OPERATORS ON NON—COMPACT MANIFOLDS

M.A. SHUBIN

Introduction

This paper contains an enlarged and modified part of my five lectures given in June 1991 at Nantes during the Summer School on Semiclassical Methods. Of course the whole subject as given in the title is inexhaustible since even the "simplest" particular case of the Schrödinger operator on euclidean space can not be exhausted because it contains the whole Quantum Mechanics and hence its complete understanding would provide us with the complete understanding of a considerable part of the Universe. So I did not pretend to be complete in my lectures and I make even less pretensions in this paper. Actually this paper contains only a description of some qualitative results on the spectra of elliptic operators on non–compact manifolds. The lectures contained also a beginning of a quantitative theory, namely integrated density of states and applications of von Neumann algebra techniques to this topic. I hope that these things some day will be described in a second part of this paper but they seemed to me too voluminous and disorderly to include in this paper now.

This paper contains two chapters each having an Appendix. In Chapter 1 we discuss the first question which natually arises when you begin to study a differential operator: what is the natural domain, where this operator is defined? Actually, if the operator is to be considered in a Banach space, one can always take minimal and maximal domain arriving in this way to minimal and maximal operators in this Banach space. We concentrate on the question whether these operators coincide because then they provide a natural operator in the Banach space associated with the given differential operator. We describe several methods of proving the coincidence based on finite speed propagation for evolution equations, regularity results and estimates of the Green function. The necessary technique concerning manifolds of bounded geometry and behaviour of the Green function is described in Appendix 1 to this chapter. Note that a non–trivial difference between minimal and maximal operator would mean that boundary conditions should be imposed but this certainly goes out of the scope of this paper. The only thing we do about it here is that we explain how to write the unique solution of the hyperbolic Cauchy problem in operator terms in case when the corresponding generating second order operator is symmetric but not essentially self–adjoint due to the behaviour of lower–order terms at infinity (Theorem 3.4).

In Chapter 2 we discuss some general topics concerning elliptic operators on manifolds of bounded geometry. Namely first we apply the general abstract eigenfunction expansion theorem, described in Appendix 2, to provide weighted Sobolev spaces which contain complete orthonormal system of generalized eigenfunctions for any self–adjoint operator. We use the ellipticity to narrow these spaces by use of regularity theorems. Next we discuss Schnol–type theorems giving sufficient conditions for the given complex number λ to belong to the spectrum if a non–trivial and non–square –integrable eigenfunction with an appropriate behaviour at infinity is given.

Some parts of this paper are based on methods and technique that were described in [44] and [45], and I felt free to borrow from these papers which were only published in a volume of the PDE seminar in École Polytechnique. But many of the results of [44] are essentially improved here and also some clarifications are added.

We are very grateful to the organizers of the Summer School on Semiclassical Methods at Nantes (and especially to Professor D. Robert) for providing the opportunity to lecture there and so to see the topics discussed here from a renewed point of view. We are also very grateful to the Sloan Foundation and M.I.T. for their support during the time when this text was being written, and to

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Numerational convention. We numbered all formulas and also Definitions, Theorems etc. separately in every Chapter or Appendix. Inside a Chapter or an Appendix we refer to a formula, Definition, Theorem etc. from the same Chapter or Appendix without any indication of the division where it belongs.

Chapter 1. Minimal and maximal operators.

1.1. Abstract preliminaries

Let $\mathcal{H}$ be a complex Hilbert space, A a densely defined linear operator in $\mathcal{H}$ (the domain of A will be denoted $D(A)$). Suppose that A has a closure $\overline{A}$ or, equivalently, that the adjoint operator A^* is densely defined (see e.g. [32]). We shall denote by G_A the graph of A i.e. the set of pairs $\{u, Au\}$, $u \in D(A)$. Then $G_{\overline{A}} = \overline{G}_A$, i.e. the graph of $\overline{A}$ is the closure of the graph of A. Moreover $\overline{A} = A^{**} = (A^*)^*$.

Now let A^+ be another densely defined linear operator in $\mathcal{H}$.

DEFINITION 1.1. A^+ is called *formally adjoint* to A if

$$(1.1) \qquad (Au, v) = (u, A^+v), \ u \in D(A), \ v \in D(A^+),$$

where $(\cdot,\cdot)$ is the scalar product in $\mathcal{H}$.

If $A = A^+$ then A is called *symmetric* or *formally self-adjoint*.

Note that since A, A^+ are densely defined, both A and A^+ have closures.

DEFINITION 1.2. Let A, A^+ be as in Definition 1.1. Then the *minimal* and the *maximal* operator for A are defined as follows:

$$A_{\min} = \overline{A} = A^{**}, \ A_{\max} = (A^+)^*.$$

Note that both $A_{\min}$ and $A_{\max}$ are closed and $A_{\min} \subset A_{\max}$ i.e. $D(A_{\min}) \subset D(A_{\max})$ and $A_{\max}$ is an extension of $A_{\min}$. The important question that arises in analytic situations and will be discussed later is whether $A_{\min} = A_{\max}$ or not. In an important particular case $A = A^+$ the coincidence $A_{\min} = A_{\max}$ means that A is *essentially self-adjoint* i.e. A is a self-adjoint operator in $\mathcal{H}$.

Now let us consider a more general abstract context. Let $\mathcal{B}, \mathcal{B}'$ be complex Banach spaces and a continuous non–degenerated pairing $\mathcal{B} \times \mathcal{B}' \to \mathbb{C}$ be given which will be denoted $\langle \cdot, \cdot \rangle$. Here continuity may be understood as separate continuity i.e. continuity with respect to each variable. Non–degeneracy means first that if $u \in \mathcal{B}$ and $\langle u, v \rangle = 0$ for all $v \in \mathcal{B}'$ then $u = 0$, and second that if $v \in \mathcal{B}'$ and $\langle u, v \rangle = 0$ for all $u \in \mathcal{B}$ then $v = 0$. Also this pairing may supposed to be bilinear as well as hermitean i.e. linear with respect to the first variable and antilinear with respect to the second variable (in the latter case we shall denote it by $(\cdot, \cdot)$. Now let two pairs $\mathcal{B}_i, \mathcal{B}_i'$, $i = 1, 2$, be given with continuous non–degenerated pairings described as before. Suppose that $A : \mathcal{B}_1 \to \mathcal{B}_2$ and $A^t : \mathcal{B}_2' \to \mathcal{B}_1'$ are two densely defined linear operators. Then A^t is called a formally transposed operator to A if

$$(1.2) \qquad \langle Au, v \rangle = \langle u, A^t v \rangle, \ u \in D(A), \ v \in D(A^t).$$

If we have hermitean pairings between $\mathcal{B}_i$ and $\mathcal{B}_i'$ and (1.1) is satisfied for two densely defined linear operators $A : \mathcal{B}_1 \to \mathcal{B}_2$ and $A^+ : \mathcal{B}_2' \to \mathcal{B}_1'$ then A^+ is called formally adjoint to A. In both situations the following definition is applicable

DEFINITION 1.2$'$. $A_{\min} = \overline{A}$, $A_{\max} = (A^t)^*$ or $A_{\max} = (A^+)^*$

Here $\overline{A}$ is the operator whose graph is the closure of the graph of A in $\mathcal{B}_1 \times \mathcal{B}_2$ and $(A^t)^*$ and $(A^+)^*$ are naturally defined as the maximal operators such that the following natural identities hold:

$$(1.3) \qquad \langle (A^t)^* u, v \rangle = \langle u, A^t v \rangle, \ u \in D((A^t)^*), \ v \in D(A^t),$$

$$(1.3') \qquad ((A^+)^* u, v) = (u, A^+ v), \ u \in D((A^+)^*, \ v \in D(A^+).$$

It is easy to see that $A_{\min}$ is well defined as for the case of Hilbert space and $A_{\min} \subset A_{\max}$. Now it is natural to ask about the conditions of coincidence $A_{\min}$ and $A_{\max}$.

Sometimes it is useful to pass from a couple A, A^t (or A, A^+) to the matrix

$$(1.4) \qquad \mathfrak{a} = \begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix} \text{ (or } \begin{bmatrix} 0 & A \\ A^+ & 0 \end{bmatrix}\text{)} : \mathcal{B}_2' \oplus \mathcal{B}_1 \to \mathcal{B}_2 \oplus \mathcal{B}_1'$$

Then we naturally have $\mathfrak{a}^t = \mathfrak{a}$ (or $\mathfrak{a}^+ = \mathfrak{a}$).

Proposition 1.3. *Equality $\mathfrak{a}_{\min} = \mathfrak{a}_{\max}$ is equivalent to the simultaneous fulfilment of two equalities*

(1.5)
$$A_{\min} = A_{\max} \textit{ and } (A^t)_{\min} = (A^t)_{\max} \textit{ (or } (A^+)_{\min} = (A^+)_{\max}).$$

(So the trick of passing to the matrix operator $\mathfrak{a}$ allows to reduce the proof of the equalities (1.5) to a similar equality for a "symmetric" operator $\mathfrak{a}$.)

Proof. It is easy to check that

$$\overline{\mathfrak{a}} = \begin{bmatrix} 0 & \overline{A} \\ \overline{A}^t & 0 \end{bmatrix} \text{ and } \mathfrak{a}^* = \begin{bmatrix} 0 & (A^t)^* \\ A^* & 0 \end{bmatrix}$$

(and similar equalities for hermitean case are valid too). The Proposition immediately follows. □

Now it it well known that for a symmetric densely defined operator A in a Hilbert space essential self–adjointness is equivalent to two equalities

$$(1.6) \qquad \operatorname{Ker}(A^* - iI) = 0, \ \operatorname{Ker}(A^* + iI) = 0$$

It easily follows that actually they are equivalent to inclusions

$$(1.7) \qquad \operatorname{Ker}(A^* - iI) \subset D(\overline{A}), \ \operatorname{Ker}(A^* + iI) \subset D(\overline{A})$$

(see e.g. [41]). Also the following proposition is sometimes useful.

Proposition 1.4 ([42]). *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces,*

$$A: \mathcal{H}_1 \to \mathcal{H}_2, \ A^+: \mathcal{H}_2 \to \mathcal{H}_1$$

a pair of densely defined linear operators and (1.1) is fulfilled. Suppose that the operator A^+A is densely defined and essentially self–adjoint. Then $A_{\min} = A_{\max}$ and $(A^+)_{\min} = (A^+)_{\max}$.

This statement actually means that A and A^+ are "essentially adjoint" to each other i.e.

$$\overline{A} = (A^+)^* \text{ and } \overline{A^+} = A^*.$$

So Proposition 1.4 in a sense gives an inverse statement to the well–known fact (first established by von Neumann) that if A is a closed densely defined linear operator in a Hilbert space then the operator A^*A is self–adjoint.

Now we shall recall some facts concerning a connection between self–adjointness and evolution equations (see e.g. [4]). First let us consider the following Cauchy problem for functions of a real variable t with values in a Hilbert space $\mathcal{H}$ where a densely defined symmetric operator A is given:

$$\ddot{u} = -A^*u, \ u(0) = u_0, \ \dot{u}(0) = u_1. \tag{1.8}$$

Here $\dot{u} = \frac{du}{dt}$, $\ddot{u} = \frac{d^2u}{dt^2}$ and the derivatives are understood as the limits in the norm–topology of $\mathcal{H}$ and they may be supposed continuous in this topology. Also the solutions u may supposed to be defined for all real values of t. Actually we shall only speak about the uniqueness of the solutions of (1.8) and in the context given all the uniqueness statements are equivalent. So the uniqueness of the solution of (1.8) can be formulated as follows: if $u: \mathbb{R} \to \mathcal{H}$, u is continuous, $\dot{u}$, $\ddot{u}$ exists in the norm sense and are continuous, $u(t) \in D(A^*)$ for every $t \in \mathbb{R}$ and (1.8) are satisfied for all t with $u_0 = u_1 = 0$ then $u \equiv 0$.

Proposition 1.5 ([4]). *Suppose that A is semi–bounded from below i.e.*

$$(1.9) \qquad (Au, u) \geq -C(u, u), \ u \in D(A)$$

with a real constant C. Suppose that we have the uniqueness of solutions for the Cauchy problem (1.8). Then A is essentially self–adjoint.

The idea of the proof is as follows: if A is not essentially self–adjoint then it has at least two different semi–bounded from below self–adjoint extensions. But for any such an extension $\tilde{A}$ we can write the solution of (1.8) in the form

$$(1.10) \qquad u(t) = (\cos\sqrt{\tilde{A}}t)u_0 + \frac{\sin\sqrt{\tilde{A}}t}{\sqrt{\tilde{A}}}u_1$$

(the choice of the branch of the square roots does not matter because both functions

$$\mu \mapsto \cos\mu t, \ \mu \mapsto \frac{\sin\mu t}{\mu}$$

are even). So using two different semi–bounded from below extensions $\tilde{A}_1$ and $\tilde{A}_2$ in (1.10) and taking the difference $u = u^{(1)} - u^{(2)}$ of two solutions u_1 and u_2 obtained in this way with the same initial values $u_0, u_1 \in D(A)$ we shall come to a non–zero function satisfying (1.8) with vanishing initial values.

Observe that if, vice versa, A is essentially self–adjoint (and semi–bounded from below) than even the uniqueness of the weak solution of (1.8) can be easily proved by the use of the Holmgren principle.

There is a possiblity to use a first–order evolution problem (of heat equation type)

$$(1.11) \qquad \dot{u} = -A^*u, \ u(0) = u_0.$$

Then the statement of Proposition 1.5 is still true if we change (1.8) to (1.11) in this statement (and the proof does not change). But there is also a possibility to avoid the semiboundedness requirement (1.9) by considering a Schrödinger–type evolution equation

$$\dot{u} = iA^*u, \; u(0) = u_0. \tag{1.12}$$

Let us introduce "deficiency indices"

$$\kappa_\pm = \dim \operatorname{Ker}(A^* \pm iI) \tag{1.13}$$

(which may be non–negative integers or $+\infty$)

Proposition 1.6 ([4]). *Suppose that $\kappa_+ = \kappa_-$ and there is the uniqueness of solutions for the Schrödinger type Cauchy problem (1.12). Then A is essentially self–adjoint.*

Here the uniqueness should be understood in the sense which is similar to that described before Proposition 1.5 for the problem (1.8) (of course only continuity of u and $\dot{u}$ is required). The idea of the proof is also similar to the one of the Proposition 1.5 (the condition $\kappa_+ = \kappa_-$ is necessary and sufficient for self–adjoint extensions to exist and $\kappa_+ = \kappa_- > 0$ implies that there are at least two such extensions).

1.2. Minimal and maximal operators, essential self–adjointness for differential operators (basic definitions and notations).

Let us consider a linear differential operator

$$A : \; C^\infty(X, E_1) \to C^\infty(X, E_2), \tag{2.1}$$

where X is a C^∞–manifold, E_1, E_2 are complex C^∞–vector bundles over X, $C^\infty(X, E_i)$ is the space of all C^∞–sections of E_i over X.

When we want to study such a differential operator, especially spectral properties of this operator, the first thing to be done is to supply it with an appropriate domain so as to make it a reasonable operator in a Hilbert space or, more generally, in a Banach space. So we begin by describing different possibilities to do this.

Let $\Omega = \Omega(X)$ be the vector bundle of (complex) densities (or 1–densities) on X. Integration of densities gives a linear map

$$\int : \ C_0^\infty(X,\Omega) \to \mathbb{C}, \ \omega \longmapsto \int_X \omega,$$

where $C_0^\infty(X,E)$ for any vector bundle E over X denotes the space of all compactly supported C^∞–sections of E over X. Now for any vector bundle E over X we define (following [3]) the dual bundle $E^* = \mathrm{Hom}_{\mathbb{C}}(E,\Omega)$. Hence we have a natural bilinear pairing of bundles $E \times E^* \to \Omega$, hence applying integration we obtain natural bilinear pairings in sections

(2.2)
$$C_0^\infty(X,E) \times C^\infty(X,E^*) \to \mathbb{C}, \ C^\infty(X,E) \times C_0^\infty(X,E^*) \to \mathbb{C},$$

which we will denote $\langle \cdot , \cdot \rangle$. Now the transposed operator to A is a differential operator

$$A^t : \ C^\infty(X,E_2^*) \to C^\infty(X,E_1^*),$$

defined by the identity

$$\langle Au, v\rangle = \langle u, A^t v\rangle, \ u \in C_0^\infty(X,E), \ v \in C_0^\infty(X,F^*). \tag{2.3}$$

Now let $\mathcal{D}'(X,E)$ denote the space of all distributional sections of E over X which is the dual space to $C_0^\infty(X,E^*)$, i.e. the space of all linear forms on $C_0^\infty(X,E^*)$ which are continuous in the usual sense (see e.g. [22], Ch. 2). Then we have a natural inclusion $C^\infty(X,E) \subset \mathcal{D}'(X,E)$ and the identity (2.3) allows then to extend A to a linear operator

$$(2.4) \qquad A: \mathcal{D}'(X, E_1) \to \mathcal{D}'(X, E_2)$$

which we denote A again because it does not lead to a confusion.

DEFINITION 2.1. Suppose that we are given Banach spaces $\mathcal{B}_1, \mathcal{B}_2$ such that $C_0^\infty(X, E_i) \subset \mathcal{B}_i \subset \mathcal{D}'(X, E_i)$, $i = 1, 2$, and the inclusions $\mathcal{B}_i \subset \mathcal{D}'(X, E_i)$ are continuous in the weak topology of $\mathcal{D}'(X, E_i)$ (which means that if $\lim_{k\to\infty} u_k = u$ in the norm of $\mathcal{B}_i$ then $\lim_{k\to\infty} \langle u_k, \psi\rangle = \langle u, \psi\rangle$ for every $\psi \in C_0^\infty(X, E_i^*)$).

The *minimal operator* $A_{\min} : \mathcal{B}_1 \to \mathcal{B}_2$ is the closure of $A : C_0^\infty(X, E_1) \to C_0^\infty(X, E_2)$ i.e. a linear operator from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that its graph in $\mathcal{B}_1 \times \mathcal{B}_2$ is the closure of the set of pairs $\{u, Au\}$ with $u \in C_0^\infty(X, E_1)$. The *maximal operator* is a linear operator $A_{\max} : \mathcal{B}_1 \to \mathcal{B}_2$ such that its domain $D(A_{\max}) = \{u | u \in \mathcal{B}_1, \; Au \in \mathcal{B}_2\}$, where A is applied in the sense of distributions (i.e. as in (2.4)) and $A_{\max}$ is a restriction of the operator (2.4) (i.e. $A_{\max} u = Au$ if $u \in D(A_{\max})$).

It is easy to see that the minimal operator is well defined and $A_{\min} \subset A_{\max}$ i.e. $D(A_{\min}) \subset D(A_{\max})$ and $A_{\max}$ is an extension of $A_{\min}$. The important question we will discuss below is whether $A_{\min}$ and $A_{\max}$ coincide or not.

An example of the spaces $\mathcal{B}_i$ appears if we have an hermitean metric on each bundle E_i, $i = 1, 2$, and also a positive C^∞–density $d\mu$ on X. Then we can define a space $L^p(X, E_i)$, $1 \le p < \infty$ which is the completion of $C_0^\infty(X, E_i)$ with respect to the norm

$$\|u\|_p = [\int_X |u(x)|^p d\mu(x)]^{1/p},$$

where $|u(x)|$ denotes the norm of $u(x)$ induced by the hermitian metric in the fiber. So we can take $\mathcal{B}_i = L^{p_i}(X, E_i)$, $i = 1, 2$, and speak about the coincidence of $A_{\min}$ and $A_{\max}$ from L^{p_1} to L^{p_2}. In case of $p_1 = p_2 = p$ we will just speak about the coincidence of $A_{\min}$ and $A_{\max}$ in L^p.

The case when $\mathcal{B}_i = L^{p_i}(X, E_i)$ can be also viewed as a particular case of the setting described in Sect. 1.1 if we take $\mathcal{B}_i' = L^{p_i'}(X, E_i^*)$ with $1/p_i' + 1/p_i = 1$.

Now instead of the usual space $L^\infty(X,E)$ it is often more convenient to use the Banach space $\tilde{C}(X,E)$ of all continuous sections of E vanishing at infinity. We shall also denote this space by $\tilde{L}^\infty(X,E)$. It also has a natural non–degenerated duality with $L^1(X,E)$ but is more convenient than $L^\infty(X,E)$ because $C_0^\infty(X,E)$ is dense in $\tilde{L}^\infty(X,E)$ (but not in $L^\infty(X,E)$). We shall also define $\tilde{L}^p(X,E) = L^p(X,E)$, $1 \le p < \infty$, to be able to use the whole scale $\tilde{L}^p(X,E)$, $1 \le p \le \infty$.

Now instead of linear duality between E and E^* we can also consider an hermitean duality. We will actually use only the case when $E = E^*$ so E is supplied with a fiberwise positive hermitean map $E \times E \to \Omega(X)$. Then we get a Hilbert space $L^2(X,E)$. Suppose that we have $E_1 = E_2 = E$ in (2.1) and A is symmetric. Then the coincidence $A_{\min} = A_{\max}$ means that A is essentially self–adjoint.

1.3. Finite speed propagation and essential self–adjointness.

Here we describe how the finite speed propagation for hyperbolic equations and systems allows to make use of abstract Propositions 1.5 and 1.6 in order to prove essential self–adjointness of some differential operators. The idea to apply uniqueness for evolution equations to prove essential self–adjointness is due to A. Ja. Povzner ([31]), it was formulated in an abstract form by Ju. M. Berezanskii ([4]) and later rediscovered and applied in geometric situations by P. Chernoff ([9]).

Let X be a Riemannian manifold, Δ is the scalar Laplacian on X. This means that $\Delta = -\delta d$ where $d: C^\infty(X) \to \Lambda^1(X)$ is the standard differential ($\Lambda^1(X)$ is the space of all smooth 1–forms on X), δ.: $\Lambda^1(X) \to C^\infty(X)$ is the formally adjoint operator to d. The simplest example of the application of Proposition 1.5 is given by the following

Theorem 3.1. *Let X be a complete Riemannian manifold i.e. all geodesics can be extended indefinitely. Let $A: C^\infty(X) \to C^\infty(X)$ be a linear differential operator of the form*

$$(3.1) \qquad A = -\Delta + B, \text{ ord } B \leq 1.$$

Suppose that A is formally self–adjoint and semibounded from below on $C_0^\infty(X)$. Then A is essentially self–adjoint.

Proof. Consider the Cauchy problem

$$(3.2) \qquad \frac{\partial^2 u}{\partial t^2} = -Au, \ u|_{t=0} = u_0, \ \frac{\partial u}{\partial t}|_{t=0} = u_1.$$

The equation in (3.2) is strictly hyperbolic, the bicharacteristic flow is essentially the geodesic flow on X. Hence due to the finite speed propagation we can always find a solution $u \in C^\infty(\mathbb{R}, C_0^\infty(X))$ provided $u_0, u_1 \in C_0^\infty(X)$. (Here $C^\infty(\mathbb{R}, C_0^\infty(X))$ denotes the space of functions $u : \mathbb{R} \times X \to \mathbb{C}$, such that $t \mapsto u(t, \cdot)$ is a C^∞–function of t with values in $C_0^\infty(X)$; this implies in particular $\operatorname{supp} u \cap ([-T, T] \times X)$ is a compact for every $T > 0$). Hence the standard application of the Holmgren principle gives the uniqueness of the Cauchy problem required to apply Proposition 1.5. □

Theorem 3.1 was formulated in a slightly weaker form by P. Chernoff ([9]) (for the case when ord $B = 0$ i.e. when A is the Schrödinger operator) though the reasoning given in [9] works for the operator (3.1) too. The arguments in [9] directly use the evolution equations like (3.2) considering invariance properties of domains of operators i.e. they do not appeal to abstract statements like Propositions 1.5, 1.6. Therefore they allow to prove the self–adjointness for all powers of A as well as for self–adjoint geometric matrix differential operators e.g. Laplacians or signature operator $d+\delta$ on differential forms on complete Riemannian manifolds. Remark that the proof of essential self–adjointness of $d+\delta$ can be done by use of Proposition 1.6 if we use the Friedrichs theory of symmetric hyperbolic systems ([13]). Besides any zero order terms (which do not change formal self–adjointness) can be added to $d+\delta$ without changing the essential self–adjointness. As we will see below this is in a sharp contrast with the behaviour of the second order operators where lower order terms may be of crucial importance.

Observe that the essential self–adjointness of pure Laplacian Δ (without lower order terms) on differential forms on a complete Riemannian manifold was first stated and proved by M.P. Gaffney [14–16] with the help of cut–off functions and Friedrichs mollifiers, and independently by W. Roelcke [34]. H.O. Cordes [10] used a beautiful inequality technique to prove essential self–adjointness of the powers of the scalar Laplacian and some Schrödinger operators. The essential self–adjointness of generalized Dirac operators on complete Riemannian manifolds was proved by M. Gromov and H.B. Lawson ([21]).

There exist a lot of results about essential self–adjointness of elliptic operators in $\mathbb{R}^n$ or in open subsets of $\mathbb{R}^n$. We shall mention only a very small part of them which is most closely connected with the results on manifolds that we have discussed here.

The essential self–adjointness of semi–bounded elliptic second–order symmetric operator in $\mathbb{R}^n$ was first proved by E. Wienholtz ([49]; see also a very simple exposition for the Schrödinger operator in the Glazman's book [18]).

Now let us mention the following Titchmarsh–Sears theorem (see [48], [39] and an exposition in [5]).

Theorem 3.2. *Let $A = -\Delta + V(x)$ be a Schrödinger operator on $\mathbb{R}^n$ and $V(x) \geq -Q(|x|)$, where Q is a positive non–decreasing function on $[0, \infty)$ such that*

$$\int_0^\infty Q(r)^{-1/2} dr = \infty \tag{3.3}$$

Then A is essentially self–adjoint.

Observe that condition (3.3) is satisfied for $Q(r) = (1+r)^\alpha$ if and only if $\alpha \leq 2$. On the other hand the Schrödinger operator with the potential $V(x) = -(1+|x|^2)^{\alpha/2}$ is essentially self–adjoint if and only if $\alpha \leq 2$ (see [5]) which shows that the condition (3.3) is relatively precise. Note that if we consider the classical Hamiltonian on $\mathbb{R}^{2n}$

$$H(p,q) = |p|^2 - (1 + |q|^2)^{\alpha/2}$$

corresponding to the quantum Hamiltonian $A = -\Delta - (1 + |x|^2)^{\alpha/2}$ then the condition $\alpha \leq 2$ is equivalent to the completeness of the classical dynamics for H (i.e. the existence of solutions for the corresponding Hamiltonian system for all values of t–variable). Hence in this example the properties to be well–defined for the corresponding classical and quantum systems are equivalent though no direct connection has been established. Note that the completeness condition for the manifold in Theorem 3.1 (and in other similar more general results mentioned before) are also in fact conditions of completeness of the corresponding classical systems. We refer the reader to P. Chernoff [9] for a beautiful speculation why lower order terms do not matter for the first–order operators from this point of view: first–order operators correspond to relativistic systems and no conditions are needed to infinity because the particle never gets there.

Theorem 3.2 was improved and generalized in many directions. T. Ikebe and T. Kato ([23]) extended it to Schrödinger operators with magnetic field so as to include quantum Hamiltonians of Stark and Zeeman effects. Many improvements and generalizations (e.g. for the cases where no spherically symmetric minorante is required) were made by F.S. Rofe–Beketov and his collaborators (see e.g. [37], review papers [35], [36] and references there).

T. Kato ([24]) used the evolution equation approach by P. Chernoff to prove that if $A = -\Delta + V$ is a Schrödinger operator in $\mathbb{R}^n$ with a real valued $V \in C^\infty(\mathbb{R}^n)$ and $A \geq -a - b|x|^2$ on $C_0^\infty(\mathbb{R}^n)$ with some constants a and b then A is essentially self-adjoint. This means that we can use a minorante like $-a - b|x|^2$ not only for the potential V but also for the operator A itself.

Many other results and references about the essential self–adjointness of Schrödinger operators in $\mathbb{R}^n$ can be found in [32], vol. II.

Recently Igor Oleinik ([30]) proved the following generalization of Theorem 3.2 to manifolds.

Theorem 3.3. *Let X be a Riemannian manifold, and aasume that there exists a point $x_0 \in X$ such that the exponential map $\exp_{x_0} : T_{x_0}X \to X$ is a diffeomorphism. Consider the Schrödinger operator $A = -\Delta + V(x)$ on X and suppose that $V(x) \geq -Q(r)$, where $r = dist(x, x_0)$ and Q is a positive non–decreasing function on $[0, \infty)$ satisfying (3.3). Then A is essentially self–adjoint.*

The condition on the exponential map is probably not necessary but let us mention that it is satisfied for all rotationally symmetric manifolds (e.g. for the hyperbolic space).

The proof of Theorem 3.3 may be given along the same lines as for the euclidean case $X = \mathbb{R}^n$ with the standard metric (see e.g. [5]) but with the use of refined Green's formulas and cut–off functions.

Now we turn to the situation when a formally self–adjoint elliptic second–order operator is not essentially self–adjoint due to the lower order terms. What happens with the solution of the corresponding hyperbolic Cauchy problem like (1.8)? Can it be expressed in operator terms by a formula like (1.10)? We shall give now a more precise statement of the problem and the answer in a simplest case.

Let X be a complete Riemannian manifold and $A = \Delta + V$ be the Schrödinger operator with a real–valued potential $V \in C^\infty(X)$. Hence A is formally self–adjoint but not necessarily semibounded. We can consider the Cauchy problem (3.2) which will be a strictly hyperbolic problem, hence well posed in spaces like $C_0^\infty(X)$, $C^\infty(X)$, $L^2_{comp}(X)$, $L^2_{loc}(X)$ etc. due to the finite speed of propagation.

Now suppose that $u_0, u_1 \in C_0^\infty(X)$. Then we can find a unique $u \in C^\infty(\mathbb{R}, C_0^\infty(X))$ which is a solution of (3.2). Obviously $u(t, \cdot) \in D(A) = C_0^\infty(X)$ for all $t \in \mathbb{R}$, in particular $u(t, \cdot) \in D(A_{\min})$ for all $t \in \mathbb{R}^n$. How can this solution be expressed in operator terms?

Note that A is a real operator hence it has equal deficiency indices (complex conjugation interchanges $\mathrm{Ker}(A^* - iI)$ and

$\mathrm{Ker}(A^* + iI))$. Therefore there exists a self-adjoint extension of A which we shall denote $\tilde{A}$ (it may not be unique,namely when the deficiency indices do not vanish).

We shall need cut-offs $\tilde{A}_N$ for the operator $\tilde{A}$ which are defined as $E((-N,\infty);\tilde{A})\tilde{A}$, where $E(I;\tilde{A})$ means the spectral projection of $\tilde{A}$ corresponding to the interval I i.e. $E(I;\tilde{A}) = \chi_I(\tilde{A})$ where $\chi_I : \mathbb{R} \to \{0,1\}$, $\chi_I(\lambda) = 1$ if $\lambda \in I$, $\chi_I(\lambda) = 0$ if $\lambda \notin I$. Hence $\tilde{A}_N \geq -NI$. Now for every $u_0, u_1 \in C_0^\infty(X)$ we can consider

$$\tilde{u}_N(t) = (\cos t\sqrt{\tilde{A}_N})u_0 + \frac{\sin t\sqrt{\tilde{A}_N}}{\sqrt{\tilde{A}_N}}u_1 \tag{3.4}$$

(The choice of the branch for the square root is not important because the functions $\lambda \mapsto \cos t\sqrt{\lambda}$ and $\lambda \mapsto (\sin t\sqrt{\lambda})/\sqrt{\lambda}$ are even; the fraction in the right hand side of (3.4) should be understood as the result of substitution of $\tilde{A}_N$ into the second function.) Now we can state the result.

Theorem 3.4. *Let A be a Schrödinger operator on a complete Riemannian manifold X with the real potential $V \in C^\infty(X)$. Let u be the solution of (3.2) with initial values $u_0, u_1 \in C_0^\infty(X)$, $\tilde{A}$ a self-adjoint extension of A, $\tilde{u}_N$ are defined by (3.4). Then*

$$\tilde{u}_N \in C^\infty(\mathbb{R}, L^2(X)) \cap C^\infty(\mathbb{R}\times X) \tag{3.5}$$

and

$$\lim_{N\to\infty} \tilde{u}_N = u \text{ in } C^\infty(\mathbb{R}\times X). \tag{3.6}$$

In particular the limit in (3.6) does not depend on the choice of the self-adjoint extension $\tilde{A}$.

Proof. The inclusion $\tilde{u}_N \in C^\infty(\mathbb{R}, L^2(X))$ is obvious since u_0, u_1 belong to the domain $D(\tilde{A}^k)$ hence to $D(\tilde{A}_N^k)$ for every $k \in \mathbb{Z}_+$.

The operator inclusion $\tilde{A} \subset A^*$ and the ellipticity of A imply now that $\tilde{u}_N \in C^\infty(\mathbb{R} \times X)$.

Let us decompose $u_j, j = 0, 1$, as follows

$$u_j = u'_{j,N} + u''_{j,N},$$

$$u'_{j,N} = E((-N, \infty); \tilde{A})u_j, \; u''_{j,N} = E((-\infty, -N]; \tilde{A})u_j.$$

Then (3.4) can be rewritten as

$$\tilde{u}_N(t) = \tilde{u}'_N(t) + \tilde{u}''_N(t),$$

where

$$\tilde{u}'_N(t) = \cos t\sqrt{\tilde{A}_N} u'_{0,N} + \frac{\sin t\sqrt{\tilde{A}_N}}{\sqrt{\tilde{A}_N}} u'_{1,N}; \; \tilde{u}''_N(t) = u''_{0,N} + tu''_{1,N}.$$

Now note that $\tilde{u}'_N$ is the solution of the Cauchy problem (3.2) with u_0, u_1 replaced by $u'_{0,N}, u'_{1,N}$ because $\tilde{A}^k_N u'_{j,N} = \tilde{A}^k u'_{j,N}$ for every $k \in \mathbb{Z}_+$, $j = 0, 1$. Since $\lim\limits_{N\to\infty} \tilde{A}^k u''_{j,N} = 0$ in $L^2(X)$ for every $k \in \mathbb{Z}$, it follows due to the ellipticity of A that $\lim\limits_{N\to\infty} u''_{j,N} = 0$ in $C^\infty(X)$, $j = 0, 1$, hence $\lim\limits_{N\to\infty} \tilde{u}''_N = 0$ in $C^\infty(\mathbb{R} \times X)$ and $\lim\limits_{N\to\infty} u'_{j,N} = u_j$ in $C^\infty(X)$, $j = 0, 1$. It remains to notice that then $\lim\limits_{N\to\infty} \tilde{u}'_N = u$ in $C^\infty(\mathbb{R} \times X)$ due to the well known local energy estimates for the Cauchy problem (3.2). □

1.4. Minimal and maximal operators on manifolds of bounded geometry.

We shall use definitions, notations and facts about manifolds of bounded geometry, which are collected in Appendix 1 to this Chapter.

Let X be a manifold of bounded geometry, E, F are vector bundles of bounded geometry on X and

$$(4.1) \qquad A:\ C_0^\infty(X,E) \to C_0^\infty(X,F)$$

is a C^∞–bounded uniformly elliptic differential operator of order m. Recall that A can be extended to a bounded linear operator

$$(4.2) \qquad A:\ W_p^m(X,E) \to L^p(X,F),\ 1 \leq p \leq \infty$$

Lemma 1.4 from Appendix 1 easily implies that $A_{\min} = A_{\max}$ in $L^p(X,E)$ if $1 < p < \infty$. More exactly

Proposition 4.1. *If $1 < p < \infty$ and A is a uniformly elliptic operator (4.1) then $A_{\min} = A_{\max}$ in $L^p(X,E)$ and*

$$(4.3) \qquad D(A_{\min}) = D(A_{\max}) = W_p^m(X,E).$$

Proof. Clearly due to the continuity of A in (4.2)

$$W_p^m(X,E) \subset D(A_{\min}) \subset D(A_{\max})$$

But Lemma 1.3 from Appendix 1 implies that $D(A_{\max}) \subset W_p^m(X,E)$, hence $D(A_{\min}) = D(A_{\max}) = W_p^m(X,E)$. $\square$

Corollary 4.2. *Let A be as in Proposition 4.1 with $E = F$ and let E have a hermitean C^∞–bounded scalar product on fibers, $(\cdot,\cdot)$ is the scalar product on $L^2(X,E)$ induced by the scalar product on fibers and the Riemannian density on X. Let A be formally self–adjoint with respect to this scalar product. Then A is essentially self–adjoint in $L^2(X,E)$.*

Proposition 4.1 does not cover exceptional values $p = 1$ and ∞ but actually $A_{\min} = A_{\max}$ also for the case $p = 1$. As to the case $p = \infty$, a natural modification is necessary: we have to consider $\tilde{L}^\infty = \tilde{C}$ instead of L^∞ (see notations in Sect. 1.2). So we have

Theorem 4.3 ([45]). *Let A be a C^∞–bounded uniformly elliptic operator acting as in (4.1). Then $A_{\min} = A_{\max}$ in $\tilde{L}^p(X, E)$ for all $p \in [1, \infty]$.*

Following [45] we shall give a proof that uses the theory of operators with a parameter. A much more complicated parabolic operation approach was suggested by Yu. A. Kordyukov [27],[28] who proved the same statement in the case where $E = F$ and A has a positive–hermitian principal symbol. Many authors have obtained the equality $A_{\min} = A_{\max}$ (in L^1 or $\tilde{C}$) or results which imply this in various special cases. E.B. Davies [12] obtains such results for second order operators on homogeneous spaces, Lie groups and on some more general manifolds. The work of R.S. Strichartz [47] also treats the second order case on manifolds. T. Kato [25] studies the Schrödinger operator on $\mathbb{R}^n$ with non smooth potential. H.B. Stewart [46] studies strongly elliptic operators in the Euclidean case and obtains resolvent estimates in the case $p = 1, \infty$. He also refers to some unpublished seminar notes of Masuda.

First we shall suppose that the following Agmon–Agranovich–Vishik condition is satisfied:

(H) $E = F$ and there exist constants $\rho \in \mathbb{C}$ and $C > 0$ with $|\rho| = 1$ such that $\|(a_m(\nu) - \rho\lambda)^{-1}\| \le C$ for all $\nu \in T^*X$ with $|\nu| = 1, \lambda > 0$.

Here a_m is the principal symbol of A, $|\nu|$ means the norm of the cotangent vector ν with respect to the given Riemannian metric and $\|\cdot\|$ is the operator norm in fibers of E which is taken in local trivializations of E making it a vector bundle of bounded geometry (see Appendix 1).

The following Lemma summarizes the necessary part of the Agmon–Agranovich–Vishik theory of the operators satisfying (H) (see e.g. [1], [2], [7], [40], or [41]):

Lemma 4.4. *There exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the operator $A - \lambda\rho I : W_2^{m+\ell}(X, E) \to W_2^\ell(X, E)$ is bijective for* *every $\ell \in \mathbb{R}$ with a bounded inverse $(A-\lambda\rho)^{-1}: W_2^\ell(X,E) \to W_2^{m+\ell}(X,E)$ satisfying the estimate*

$$\begin{aligned} &\|(A-\lambda\rho)^{-1}u\|_{m+\ell} + \lambda^{1/m}\|(A-\lambda\rho)^{-1}u\|_{m+\ell-1} + \dots \\ &+ \lambda\|(A-\lambda\rho)^{-1}u\|_\ell \le C\|u\|_\ell \end{aligned} \tag{4.4}$$

for every $u \in W_2^\ell(X,E)$. Here $\|\cdot\|_s$ denotes the norm in $W_2^s(X,E)$ and $C>0$ is a constant which is independent of u and of λ.

Proof. We first notice that it is enough to prove the result with A replaced by $\rho^{-1}A$, which satisfies $\|(\rho^{-1}a_m(x,\xi)-\lambda)^{-1}\| \le C$, $x \in X$, $|\xi| = 1$. This is the usual uniform Agmon condition so we can apply the Seeley construction of a local parametrix of $(\rho^{-1}A-\lambda)$ which will satisfy uniform estimates. (See [40].) We then get a global parametrix by using the uniform partition of unity of Lemma 1.3 in Appendix 1. (Making use of the fact that A is a differential operator, one can give simpler proofs, see for instance [41].) □

Later on we shall abbreviate $W_2^s(X,E)$ to W_2^s, $L^p(X,E)$ to L^p etc.

Let $f \in C^\infty(X)$ have the property that $\nu(x,\partial_x)f$ is a C^∞–bounded function for every C^∞–bounded vector field ν. Then:

$$e^f \circ A \circ e^{-f} = A + B_f, \tag{4.5}$$

where B_f is a C^∞–bounded differential operator of order $m-1$. We then have:

$$e^f \circ (A-\lambda\rho) \circ e^{-f} = (A-\lambda\rho) + B_f,$$

and if we choose $\lambda > \lambda_0$, where λ_0 is given in Lemma 4.1, then in the sense of bounded operators from $W_2^{m+\ell}$ to W_2^ℓ, we can write

$$e^f \circ (A-\lambda\rho) \circ e^{-f} = (1 + B_f(A-\lambda\rho)^{-1}) \circ (A-\lambda\rho). \tag{4.6}$$

If $\lambda > 0$ is large enough (depending only on the bounds of $\partial^\alpha f$ for $1 \leq |\alpha| \leq m$ in canonical coordinates), the norm of $B_f(A-\lambda\rho)^{-1}$: $L^2 \to L^2$ is smaller than $\frac{1}{2}$. We conclude that the right hand side of (4.6), viewed as an operator $W_2^m \to L^2$, is bijective with a uniformly bounded inverse when $\lambda > \lambda_1$, and $\lambda_1 > 0$ is large enough. The identity (4.6) is of course to be understood in the sense of distributions, but we have:

Lemma 4.5. *Let f be as above. Then there exists a constant $\lambda_1 > 0$ depending only on the bounds of $\partial^\alpha f$ for $1 \leq |\alpha| \leq m$ (in canonical coordinates) such that for $\lambda > \lambda_1$ the uniformly bounded inverse, G_λ of the operator $A - \lambda\rho :\ W_2^m \to L^2$ (which exists according to Lemma 4.4) has the following property: The operator $e^f \circ G_\lambda \circ e^{-f}$ (which a priori maps $L^2 \cap \mathcal{E}'$ into $W_{2,\mathrm{loc}}^m$) has a bounded extension $L^2 \to W_2^m$, and the norm can be bounded by a constant which is independent of λ and of f.*

Proof. If f is a bounded function, then multiplication by $e^{\pm f}$ is a bounded operator on all the spaces W_2^s, and we see that $e^f \circ G_\lambda \circ e^{-f}$ is the inverse of the operator (4.6), and the proposition follows in that case. If f is not a bounded function, we let $\psi(s)$ be a smooth increasing real valued function with $\psi(s) = s$ for $-1 \leq s \leq 1$, $\psi(s) = 2$ for $s \geq 3$, $\psi(s) = -2$ for $s \leq -3$ and put $\psi_\epsilon(s) = \epsilon^{-1}\psi(\epsilon s)$, for $0 < \epsilon \leq 1$. Notice that $|\partial_s^k \psi_\epsilon(s)| \leq C_k$ for $k = 1, 2, \ldots$, where C_k are independent of s and of ϵ, so that the functions $f_\epsilon = \psi_\epsilon \circ f$ satisfy $|\partial^\alpha f_\epsilon(x)| \leq \tilde{C}_\alpha$ for $1 \leq |\alpha| \leq m$, with $\tilde{C}_\alpha$ independent of ϵ. We can then apply Lemma 4.5 with f replaced by f_ϵ. We conclude that $e^{f_\epsilon} \circ G_\lambda \circ e^{-f_\epsilon}$ is bounded as an operator $L^2 \to W_2^m$, uniformly with respect to λ and ϵ. If $u \in L^2 \cap \mathcal{E}'$, then for $\epsilon > 0$ small enough, we have $f_\epsilon = f$ on the support of u, and if K is an arbitrary compact subset in X, then for $\epsilon > 0$ small enough, we have $e^f G_\lambda e^{-f} u = e^{f_\epsilon} G_\lambda e^{-f_\epsilon} u$ on K, hence $\|e^f G_\lambda e^{-f} u\|_{m,K} \leq C\|u\|_0$, with a constant $C > 0$ which is independent of u and K. Here $\|\cdot\|_{m,K}$ denotes the W_2^m-norm over K. Since K is arbitrary, we conclude that $e^f G_\lambda e^{-f} u$ belongs to W_2^m and $\|e^f G_\lambda e^{-f} u\|_m \leq C\|u\|_0$. It is then clear that

$e^f \circ G_\lambda \circ e^{-f}$ extends to a bounded operator $L^2 \to W_2^m$. $\square$

Notice that the distribution kernel of $e^f \circ G_\lambda \circ e^{-f}$ is of the form $e^{f(x)-f(y)} K_{G_\lambda}(x,y)$, if we denote the distribution kernel of G_λ by $K_{G_\lambda}(x,y)$. Also notice that K_{G_λ} is C^∞ outside the diagonal. We shall apply the above result with $f = f_x(y) = (t+1)\tilde{d}(x,y)$, where $\tilde{d}$ is the function constructed by Kordyukov (see Lemma 2.1 in Appendix 1). Here x may be an arbitrary point of X, and $t > 0$ may be arbitrary but fixed. Then the hypotheses of Lemma 4.5 are satisfied uniformly when x varies in X and as in Theorem 2.2 of Appendix 1 we obtain:

Lemma 4.6. *Let $t > 0$. Then there exists $\lambda(t) > 0$ such that for $\lambda \geq \lambda(t)$ we have the following: For every $\delta > 0$ and all multiindices α, β there exists $C_{\alpha,\beta,\delta} > 0$ such that*

$$|\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)| \leq C_{\alpha,\beta,\delta} e^{-td(x,y)} \text{ for all } x, y \in X \text{ with } d(x,y) > \delta. \tag{4.7}$$

The study of K_{G_λ} in the region $d(x,y) < \delta$ goes through exactly as in section 3 of Appendix 1, and we obtain the following analogue of Theorem 3.7 of Appendix 1:

Theorem 4.7. *Let $t > 0$. Then there exists $\lambda(t) > 0$ such that for $\lambda \geq \lambda(t)$ we have the following: For all multiindices α, β there exists a constant $C_{\alpha,\beta} > 0$ such that when $m < n$ and $x \neq y$:*

$$|\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)| \leq C_{\alpha,\beta} d(x,y)^{m-n-|\alpha|-|\beta|} e^{-td(x,y)}, \tag{4.8}$$

and when $m \geq n$ and $x \neq y$:

$$\begin{aligned} &|\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)| \\ &\leq C_{\alpha,\beta}(1 + d(x,y)^{m-n-|\alpha|-|\beta|} |\log(d(x,y))|) e^{-td(x,y)}. \end{aligned} \tag{4.9}$$

We here also notice that it is well known that the kernel is locally integrable in y for every fixed x and in x for every fixed y.

We have the following result where the only assumption is that X is of bounded geometry:

Lemma 4.8. *Let $B(x,r) = \{y \in X; d(y,x) < r\}$. There exists a constant $C = C(X)$ such that for all $x \in X$ and $r \geq 0$:*

$$\text{(4.10)} \qquad \textit{Vol}(B(x,r)) \leq e^{Cr}.$$

Proof. We supply a simple proof for the sake of completeness. A more general result due to Bishop, can be found in the book of M. Gromov [20]. We shall use reasoning as in the proof of Lemma 1.2 of Appendix 1. Let us take a maximal system of points $\{x_j | j = 1, 2, \dots, N\} \subset B(x,r)$ such that the balls $B(x_i, \varepsilon)$ and $B(x_j, \varepsilon)$ do not intersect if $i \neq j$. Then $B(x,r)$ will be covered by the balls $B(x_i, 2\varepsilon)$, $i = 1, 2, \dots, N$. Now evidently

$$N \leq C_1(\varepsilon)\mathrm{Vol}B(x,r), \text{ where } C_1(\varepsilon) = [\inf_{x\in X} \mathrm{Vol}B(x,\varepsilon)].$$

Since the ball $B(x, r+\varepsilon)$ is covered by the balls $B(x_i, 3\varepsilon)$, $i = 1, \dots, N$, we have

$$\mathrm{Vol}\ V(x, r+\varepsilon) \leq C(\varepsilon)\ \mathrm{Vol}\ V(x,r)$$

where $C(\varepsilon) = C_1(\varepsilon) \sup_{x\in X} \mathrm{Vol}\ B(x, 3\varepsilon)$. Now (4.10) evidently follows. $\square$

Using the lemma one obtains the following corollary of Theorem 4.7.

Corollary 4.9. *There exists $\lambda_0 > 0$, such that if $\lambda > \lambda_0$, then:*

(4.11)
$$\sup_{x\in X} \int |K_{G_\lambda}(x,y)| dy < +\infty, \ \sup_{y\in X} \int |K_{G_\lambda}(x,y)| dx < +\infty.$$

Proof. Using (4.8), (4.9), it is easy to see that

$$\sup_{x\in X,|x-y|\leq\delta}\int |K_{G_\lambda}(x,y)|dy < +\infty,$$

$$\sup_{y\in X|x-y|\leq\delta}\int |K_{G_\lambda}(x,y)|dx < +\infty,$$

so we only have to estimate the corresponding integrals over the domain $|x-y| > \delta$, and here we may use (4.7): We get for $\lambda \geq \lambda(t)$

$$\int_{|x-y|>\delta} |K_{G_\lambda}(x,y)|dy \leq C_0 \int_0^{+\infty} e^{-td(x,y)}dy = C_0 \int_0^{+\infty} e^{-tr}dV(r),$$

where $V(r) = \mathrm{Vol}(B(x,r))$. We choose t strictly larger than the constant "C" which appears in Lemma 4.8. Then the last integral is convergent and an integration by parts gives:

$$\int_0^{\infty} e^{-tr}dV(r) = \int_0^{\infty} te^{-tr}V(r)dr \leq \int_0^{\infty} te^{(C-t)r}dr = t/(t-C).$$

The same estimate is valid for the x–integrals and the corollary follows. $\square$

From now on we take $\lambda > 0$ sufficiently large so that Corollary 4.9 applies. By Schur's lemma (see e.g. Lemma 18.1.12 in [22], vol. 3) we then know that the restriction of G_λ to C_0^∞ has a unique bounded extension $\tilde{L}^p \to \tilde{L}^p$, when $1 \leq p < \infty$. It is also easy to see (using also (4.11)), that G_λ has a unique bounded extension, $\tilde{L}^\infty \to \tilde{L}^\infty$. Working with some fixed p, we denote this extension $\tilde{G}_\lambda$. For $u \in C_0^\infty$ we have $(A-\lambda\rho)G_\lambda u = u$, and using the continuity of $\tilde{G}_\lambda$ in $\tilde{L}^p$ and the continuity of $A-\lambda\rho$ for the weak topology of distributions, we get:

$$(A-\lambda\rho)\tilde{G}_\lambda = I \text{ on } \tilde{L}^p \tag{4.12}$$

Let $u \in D(A_{\max})$ so that u and Au belong to $\tilde{L}^p$. Then if $\varphi \in C_0^\infty$, we get formally:

(4.13)
$$(\tilde{G}_\lambda(A - \lambda\rho)u, \varphi) = ((A - \lambda\rho)u, G_\lambda^*\varphi) = (u, (A - \lambda\rho)^* G_\lambda^*\varphi),$$

where the scalar products are taken either in L^2 and $*$ indicates that we take the formal complex adjoint in the sense of distributions. To justify these manipulations we may use the cut–off functions constructed as follows:

$$\chi_N(x) = \sum_{1 \le i \le N} \varphi_i(x), \tag{4.14}$$

where $\{\varphi_i | i = 1, 2, \dots\}$ is the partition of unity described in Lemma 1.3 of Appendix 1. Clearly $\chi_N \in C_0^\infty(X)$, $0 \le \chi_N \le 1$ and for every compact $K \subset X$ there exists N such that $\chi_N = 1$ in a neighbourhood of K. Moreover $|\partial^\alpha \chi_N| \le C_\alpha$ in canonical coordinates uniformly with respect to N. Now we can begin with the obvious equality

$$(\tilde{G}_\lambda \chi_N (A - \lambda\rho)u, \varphi) = (u, (A - \lambda\rho)^* \chi_N G_\lambda^* \varphi)$$

and then take limit as $N \to \infty$. Using the boundedness $\tilde{G}_\lambda : \tilde{L}^p \to \tilde{L}^p$ in the left–hand side and the estimates (4.8), (4.9) in the right–hand side we shall conclude that the limits exist and (4.13) is fulfilled. Now $(A - \lambda\rho)^* G_\lambda^* \varphi = \varphi$ as can be seen by replacing u by a C_0^∞–section ψ in (4.13) and using that $\tilde{G}_\lambda(A - \lambda\rho)\psi = G_\lambda(A - \lambda\rho)\psi = \psi$. Thus (4.13) reduces to:

$$(\tilde{G}_\lambda(A - \lambda\rho)u, \varphi) = (u, \varphi). \tag{4.15}$$

and varying φ we conclude that:

$$\tilde{G}_\lambda(A - \lambda\rho) = I \text{ on } D(A_{\max}). \tag{4.16}$$

Thus we have proved that for λ sufficiently large, $(A - \lambda\rho)$ is bijective from $D(A_{\max})$ onto $\tilde{L}^p$ and that the inverse is $\tilde{G}_\lambda$.

We can now end Proof of Theorem 4.3. First suppose that (H) is satisfied. Let $u \in D(A_{\max})$ and $v = Au$. Let w_j, $j = 1, 2, \ldots$ be a sequence of C_0^∞–sections converging to $v - \lambda\rho u$ in $\tilde{L}^p$, and put $u_j = \tilde{G}_\lambda w_j \in \tilde{L}^p \cap C^\infty$. Then $u_j \to u$ in $\tilde{L}^p$ and $Au_j = w_j + \lambda\rho u_j \to u$ in $\tilde{L}^p$. It only remains to prove that u_j belongs to $D(A_{\min})$. We note that if $\Omega_j = \operatorname{supp}(w_j)$ then

$\sup_x \int_{\Omega_j} (1 - \chi_N(x))|K_{G_\lambda}(x, y)|dy$ and

$\sup_{y \in \Omega_j} \int (1 - \chi_N(x))|K_{G_\lambda}(x, y)|dx$ tend to zero when N tends to infinity, and similarly when $(1 - \chi_N(x))K_{G_\lambda}$ is replaced by some x–derivative of the same function. (Indeed, this is proved in the same way as Corollary 4.9.) Hence (still with j fixed) $\chi_N u_j \to u_j$ and $A(\chi_N u_j) \to Au_j$ in $\tilde{L}^p$ when $N \to \infty$, and the proof is complete provided (H) is satisfied.

Now consider the general case. Here we just have to apply Proposition 1.3. We may assume that E and F are uniformly C^∞–bounded hermitean vector bundles. Let A^+ denote the formal complex adjoint of A, and consider the uniformly elliptic C^∞–bounded formally self adjoint operator: $\mathfrak{a} : C^\infty(M; F \oplus E) \to C^\infty(M; F \oplus E)$ given by the matrix

$$\mathfrak{a} = \begin{pmatrix} 0 & A \\ A^+ & 0 \end{pmatrix}$$

We notice that $\mathfrak{a}$ satisfies (H) with $\rho = \sqrt{-1}$, so we know that $\mathfrak{a}_{\max} = \mathfrak{a}_{\min}$. It follows due to Proposition 1.3 that $A_{\max} = A_{\min}$ q.e.d. $\square$

Appendix 1. Analysis on manifolds of bounded geometry.

In this Appendix we mostly follow [44].

A1.1. Preliminaries. Let X be a Riemannian manifold $n = \dim X$. In what follows we shall always suppose for the sake of simplicity that X is connected. Then the Riemannian distance

$d: X \times X \to [0, +\infty)$ is well defined; namely $d(x, y)$ is the infinium of Riemannian lengths of all arcs connecting x and y.

Denote by T_xX the tangent space of X at a point $x \in X$ and let $\exp_x : T_xX \to X$ be the usual exponential geoddesic map: $\exp_x v = \gamma(1)$, where $\gamma(t)$ is the geodesic (with a canonical parameter which is proportional to the arc length) starting at x with the initial speed $v \in T_xX$, i.e. $\gamma(0) = x$, $\dot\gamma(0) = v$. We shall always suppose that X is complete or equivalently that $\exp_x$ is defined everywhere i.e. for every $x \in X$ and $v \in T_xX$ the corresponding geodesic $\gamma(t)$ can be defined for all $t \in \mathbb{R}$. The exponential map $\exp_x : T_xX \to X$ is a diffeomorphism of a ball $B_x(0, r) \subset T_xX$ of radius $r > 0$ with the center 0 on a neighborhood $U_{x,r}$ of x in X. (Actually for a fixed x this neighborhood $U_{x,r}$ will be the ball $B(x, r)$ of the radius r centered at x on the manifold X with respect to the distance d induced by the given Riemannian metric, provided r is sufficiently small). Denoting by r_x the supremum of possible radii of such balls we can define the injectivity radius of X as $r_{inj} = \inf_{x \in X} r_x$. If $r_{inj} > 0$ then taking $r \in (0, r_{inj})$ we see that $\exp_x : B_x(0, r) \to U_{x,r}$ will be a diffeomorphism for every $x \in X$. Euclidean coordinates in T_xX (associated with an orthonormal frame in T_xX) define coordinates on $U_{x,r}$ (by means of $\exp_x$) which are called canonical.

DEFINITION 1.1 (see e.g. [8] or [33]) X is called a *manifold of bounded geometry* if the following two conditions are satisfied:

a) $r_{inj} > 0$

b) $|\nabla^k R| \le C_k$, $k = 0, 1, 2, \ldots$ (i.e. every covariant derivative of the Riemann curvature tensor is bounded).

Note that a) implies that X is complete i.e. all geodesics can be extended indefinitely. It follows that X is complete as a metric space with the metric given by the Riemannian distance d, and every ball $\{x | d(x, x_0) \le r\}$ is compact whatever $x_0 \in X$, $r > 0$.

The property b) can be replaced by the following equivalent property which will be more convenient for the use here

b′) let us fix any $r \in (0, r_{inj})$ and let $U_{x,r}, U_{x',r}$ be two domains of canonical coordinates $y: U_{x,r} \to \mathbb{R}^n, y' : U_{x',r} \to \mathbb{R}^n$ such that $U_{x,r} \cap U_{x',r} \ne \emptyset$: consider the vector function $y' \circ y^{-1}$:

$y(U_{x,r} \cap U_{x',r}) \to \mathbb{R}^n$; then

$$|\partial_y^\alpha (y' \circ y^{-1})| \leq C_{\alpha,r}$$

for every multiindex α.

Examples of manifolds of bounded geometry are Lie groups or more general homogeneous manifolds (with invariant metrics), covering manifolds of compact manifolds (with a Riemannian metric which is lifted from the base manifold), leaves of a foliation on a compact manifold (with a Riemannian metric which is induced by a Riemannian metric of the compact manifold).

Below we shall always use only canonical coordinates with a fixed $r \in (0, r_{inj})$. Then all the change of coordinate functions have bounded derivatives of all orders. This property allows to formulate a correct notion of C^k–boundedness ($k = 0, 1, 2, \cdots$) or C^∞–boundedness for functions, vector fields, exterior forms and other tensor fields on X. Namely a function $f : X \to \mathbb{C}$ is called C^k–bounded if $f \in C^k(X)$ and $|\partial_y^\alpha f(y)| \leq C_\alpha$ for every multiindex α with $|\alpha| \leq k$ and for any choice of canonical coordinates. A function $f : X \to \mathbb{C}$ is called C^∞–bounded if $f \in C^\infty(X)$ and f is C^k–bounded for every $k = 0, 1, 2, \cdots$. Let $C_b^k(X)$ be the space of all C^k–bounded complex–valued functions on X (here $k = 0, 1, 2, \cdots$ or $k = \infty$). Of course C^k–boundedness of a function $f \in C^k(X)$ is equivalent to the estimate $|\nabla^k f(x)| \leq C$ but the formulation in local coordinates is sometimes more convenient.

Similarly a vector field, an exterior form on any general tensor field on X is called C^k–bounded ($k = 0, 1, 2, \cdots$ or $k = \infty$) if all components of the field in any canonical coordinate system are C^k–bounded as C^k–functions of corresponding coordinates (with bounds depending only on the order of the differentiation but not on the chosen coordinate neighbourhood).

Let $A : C^\infty(X) \to C^\infty(X)$ be a differential operator of order m with C^∞–coefficients. We shall call it *C^∞–bounded* if in any canonical coordinate system A is written in the form

$$A = \sum_{|\alpha| \le m} a_\alpha(y) \partial_y^\alpha \tag{1.1}$$

where the coefficients a_α are (complex–valued) functions satisfying the estimates $|\partial_y^\beta a_\alpha(y)| \le C_\beta$ for any multiindex β (with a constant C_β which does not depend on the chosen canonical neighbourhood). A C^∞–bounded vector field defines a C^∞–bounded differential operator of order 1.

Let E be a complex vector bundle on X. We shall say that E is a **bundle of bounded geometry** if it is supplied by an additional structure: trivializations of E on every canonical coordinate neighbourhood U such that the corresponding matrix transition functions $g_{UU'}$ on all intersections $U \cap U'$ of such neighbourhoods are C^∞–bounded i.e. all their derivatives $\partial_y^\alpha g_{UU'}(y)$ with respect to canonical coordinates are bounded with bounds C_α which do not depend on the chosen pair U, U'. Examples of vector bundles of bounded geometry are: trivial bundle $X \times \mathbb{C}$, complexified tangent and cotangent bundles $TX \otimes \mathbb{C}$ and $T^*X \otimes \mathbb{C}$, complexified exterior powers $\Lambda^\ell T^*X \otimes \mathbb{C}$ of the cotangent bundle (C^∞–sections of $\Lambda^\ell T^*X \otimes \mathbb{C}$ are exterior complex–valued ℓ–forms on X), complexified tensor bundles etc. The definition of C^∞–bounded differential operator is easily generalized to the case of operators

$$A : \ C^\infty(X, E) \to C^\infty(X, F) \tag{1.2}$$

acting between spaces of C^∞–sections of vector bundles of bounded geometry E, F (the definition is the same as for scalar operators but with the use of the representation (1.1) in canonical coordinates and chosen trivializations. Examples of C^∞–bounded differential operators in this more general context are the exterior differentiation de Rham operator $d : \ \Lambda^\ell(X) \to \Lambda^{\ell+1}(X)$ where $\Lambda^\ell(X) = C^\infty(X, \Lambda^\ell T^*X \otimes \mathbb{C})$, operators of covariant differentiation of tensors, Laplace–Beltrami operators on functions or forms etc.

If E is a vector bundle of bounded geometry on X then the notion of C^ℓ–boundedness and the corresponding spaces $C_b^\ell(X,E)$ of C^ℓ–bounded sections are also defined for $\ell = 0,1,2,\cdots$ or $\ell = \infty$. Also the space $L^p(X,E)$ of the sections with the integrable p–th power of a fiber norm $(1 \le p < \infty)$ is naturally defined as well as the spaces $\tilde{L}^p(X,E)$, $1 \le p \le \infty$.

The following Lemma is essentially due to M. Gromov [19].

Lemma 1.2. *There exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0,\varepsilon_0)$ then there exists a countable covering of X by balls of the radius ε : $X = \cup B(x_i,\varepsilon)$ such that the covering of X by the balls $B(x_i,2\varepsilon)$ with the double radius and the same centers has a finite multiplicity.*

Here the multiplicity (or index in the terminology of [19]) of the covering by balls is the maximal number of the balls with non–empty intersection in this covering.

Proof. Let us choose $\varepsilon_0 > 0$ so that $3\varepsilon_0 < r_{inj}$, hence the canonical coordinates are defined on the ball $B(x,3\varepsilon)$ for every $x \in X$ and the transition functions from one set of canonical coordinates to another have bounded derivatives of every order (see Definition 1.1). Also the components g_{ij} and g^{ij} of the Riemannian metric have bounded derivatives of every order in chosen canonical coordinates. It follows in particular that there exists $C > 0$ such that

$$C^{-1} \le \frac{V(x,r)}{V(y,r)} \le C, \;\; x,y \in X, \;\; r \in (0,3\varepsilon_0),$$

where $V(x,r) = \mathrm{Vol}\, B(x,r)$ (here Vol means volume with respect to the standard Riemannian density).

Let us choose a maximal set of disjoint balls $B(x_1,\varepsilon/2)$, $B(x_2,\varepsilon/2),\dots$ (such a set exists due to Zorn Lemma and is obviously countable). For every $x \in X$ there exists i such that $d(x,x_i) < \varepsilon$ (otherwise we could add $B(x,\varepsilon/2)$ to the chosen balls). Hence $X = \cup B(x_i,\varepsilon)$.

Now if $y \in B(x_i,2\varepsilon)$ then $B(x_i,\varepsilon/2) \subset B(y,3\varepsilon)$. Hence if y is covered by each of different balls $B(x_{i_k},2\varepsilon)$, $k = 1,\dots,N$, then

$\sum_{1\leq k\leq N} V(x_{i_k}, \varepsilon/2) \leq V(y, 3\varepsilon)$ and we get the required estimate of multiplicity

$$N \leq (\sup_{y\in X} V(y,3\varepsilon))(\inf_{x\in X} V(x,\varepsilon/2)). \quad \square$$

Lemma 1.1 implies the existence of "uniform" partition of unity which is subordinate to a covering by balls from Lemma 1.1. Let us choose $\varepsilon < r/2$ where $r \in (0, r_{inj})$ is fixed as before.

Lemma 1.3. *For every $\varepsilon > 0$ there exists a partition of unity $1 = \Sigma_{i=1}^{\infty}\varphi_i$ on X such that*

1) $\varphi_i \geq 0$, $\varphi_i \in C_0^\infty(X)$, supp $\varphi_i \subset B(x_i, 2\varepsilon)$, where $\{x_i\}$ is the sequence of points from Lemma 1.2;

2) $|\partial_y^\alpha \varphi_i(y)| \leq C_\alpha$ for every multiindex α in canonical coordinates uniformly with respect to i (i.e. with the constant C_α which does not depend on i).

This Lemma is a useful tool to construct global objects on X from their local prerequisites. One of the important examples is the uniform Sobolev or Besov spaces $W_p^s(X)$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$ (see e.g. [33] in case $p = 2$). First introduce the Sobolev norm $\|.\|_{s,p}$ on $C_0^\infty(X)$ by the formula

$$\|u\|_{s,p}^p = \Sigma_{i=1}^{\infty} \|\varphi_i u\|_{s,p;B(x_i,2\varepsilon)}^p, \tag{1.3}$$

where $\|.\|_{s,p;B(x_i,2\varepsilon)}$ means the usual Sobolev (Besov or Bessel potential) norm of order s in canonical coordinates on $B(x_i, 2\varepsilon)$. Actually we only need the case $s \in \mathbb{Z}_+$; then the local Sobolev norm can be written for every open set $\Omega \subset \mathbb{R}^n$ as

$$\|v\|_{s,p;\Omega} = (\Sigma_{|\alpha|\leq s} \int_\Omega |\partial^\alpha v(y)|^p dy)^{1/p}, \ 1 \leq p < \infty,$$

$$\|v\|_{s,\infty;\Omega} = \Sigma_{|\alpha|\leq s} \operatorname{ess\,sup}_\Omega |\partial^\alpha v(y)|$$

Also if we choose a system $Y_1, \cdots, Y_N$ of C^∞–bounded vector fields on X such that $Y_1(x), \cdots, Y_N(x)$ generate T_xX for every $x \in X$ then we can introduce the following norm which is equivalent to (1.3)

(1.3′)

$$\|u\|_{s,p}^p = \sum_{k=0}^{s} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} \int_X |Y_{i_1} \cdots Y_{i_k} u(x)|^p dx, \ 1 \le p < \infty,$$

where dx is the standard Riemannian density on X,

$$\|u\|_{s,\infty} = \sum_{k=0}^{s} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} \operatorname*{ess\,sup}_X |Y_{i_1} \cdots Y_{i_k} u(x)|.$$

Another equivalent norm for $s \in \mathbb{Z}_+$ is given by

$$\|u\|_{s,p}^p = \Sigma_{k=0}^{s} \int_X |\nabla^k u(x)|^p dx, \ 1 \le p < \infty,$$

$$\|u\|_{s,\infty} = \sum_{k=0}^{s} \operatorname*{ess\,sup}_X |\nabla^k u(x)|$$

(here $|\cdot|$ is understood as the norm induced by the Riemannian metric on tensors).

Now we can introduce the uniform Sobolev space $W_p^s(X)$ as the completion of $C_0^\infty(X)$ with respect to the norm (1.3). The spaces $W_p^s(X)$ have the same properties as the corresponding spaces in the case $X = \mathbb{R}^n$. All of them are naturally included in the space of distributions $\mathcal{D}'(X)$. The space $W_2^s(X)$ has a natural Hilbert structure and will be also denoted $H^s(X)$. The usual embedding theorems are true, e.g. $W_p^0(X) = L^p(X)$ if $1 \le p < \infty$, $W_p^s(X) \subset C_b^k(X)$ if $s > k + n/p$. If E is a vector bundle of bounded geometry then the Sobolev norms of sections and the corresponding Sobolev spaces of sections $W_p^s(X,E)$ are defined in the same way.

Denote $W_p^{-\infty}(X) = \cup_{s\in\mathbb{R}} W_p^s(X), W_p^\infty(X) = \cap_{s\in\mathbb{R}} W_p^s(X)$ and the similar meaning have the notations $W_p^{-\infty}(X,E)$, $W_p^\infty(X,E)$.

Let A be a differential operator of order m acting as in (1.2) between spaces of sections of vector bundles of bounded geometry. The principal symbol of A gives a family of linear maps

$$a_m(x,\xi): \ E_x \to F_x$$

where $x \in X$, $(x,\xi) \in T_x^*X$ is a cotangent vector based at x, E_x and F_x are fibers of bundles E and F over x. Let us choose admissible trivializations of E and F over a neighbourhood of x. Then $a_m(x,\xi)$ becomes a (complex) matrix. The operator A is called elliptic if this matrix is invertible for every (x,ξ) with $\xi \neq 0$. It is called **uniformly elliptic** if there exists $C > 0$ such that

$$(1.4) \qquad |a_m^{-1}(x,\xi)| \leq C|\xi|^{-m}, \ (x,\xi) \in T^*X, \xi \neq 0.$$

Here $|\xi|$ is the length of (x,ξ) with respect to the given Riemannian metric, $|a_m^{-1}(x,\xi)|$ is the operator norm of the matrix $a_m^{-1}(x,\xi)$ in the above mentioned trivializations.

Let A be a C^∞–bounded differential operator of order m on M. Then A defines a bounded linear operator $A: \ W_p^s(X) \to W_p^{s-m}(X)$ for every $s \in \mathbb{R}$, $1 \leq p \leq \infty$ (if A acts as in (1.2) then it defines a bounded linear operator $A: \ W_p^s(X,E) \to W_p^{s-m}(X,F)$). Now we shall formulate regularity properties and a priori estimates which follow from uniform ellipticity.

Lemma 1.4. *Let A be a C^∞–bounded uniformly elliptic differential operator acting as in (1.2) between spaces of sections of vector bundles of bounded geometry. Then for every $s,t \in \mathbb{R}$, $p \in (1,+\infty)$ there exists $C > 0$ such that*

$$(1.5) \qquad \|u\|_{s,p} \leq C(\|Au\|_{s-m,p} + \|u\|_{t,p}), \ u \in C_0^\infty(X,E).$$

Moreover if $u \in W_p^{-\infty}(X,E)$ *and* $Au \in W_p^{s-m}(X,F)$ *then* $u \in W_p^s(X,E)$.

Proof. Let us choose the points $x_1, x_2, \dots$ and $\varepsilon > 0$ as in Lemma 1.1. We have the usual local a priori estimate

$$(1.6)\quad \|u\|^p_{s,p;B(x_i,\varepsilon)} \le C_1(\|Au\|^p_{s-m,p;B(x_i,2\varepsilon)} + \|u\|^p_{t,p;B(x_i,2\varepsilon)})$$

with a constant C_1 which does not depend on i. Summing over all i we evidently obtain an estimate which is equivalent to (1.5). The last statement also follows from the corresponding local regularity result and the estimate (1.6). □

A1.2. Weight estimates and decay of the Green function.

We begin with a construction which gives a substitute with natural smoothness properties for the distance $d = d(x,y)$ on a connected Riemannian manifold X of bounded geometry. Such a substitute will be a function which we shall denote by $\tilde{d} = \tilde{d}(x,y)$. For the case of Lie groups it can be constructed as a convolution of $d(x,.)$ with a C_0^∞–function ([29]). The general case requires a more complicated procedure which we shall give now ([27],[28]).

Lemma 2.1. *(Yu.A. Kordyukov). There exists a function* $\tilde{d}$: $X \times X \to [0,+\infty)$ *satisfying the following conditions:*

(i) *there exists* $\rho > 0$ *such that*

$$|\tilde{d}(x,y) - d(x,y)| < \rho$$

for every $x, y \in X$;

(ii) *for every multiindex* α *with* $|\alpha| > 0$ *there exists a constant* $C_\alpha > 0$ *such that*

$$|\partial_y^\alpha \tilde{d}(x,y)| \le C_\alpha,\ x,y \in X,$$

where the derivative ∂_y^α *is taken with respect to canonical coordinates.*

Moreover for every $\varepsilon > 0$ the exists a function $\tilde{d} : X \times X \to [0, \infty)$ satisfying (i) with $\rho < \varepsilon$.

Proof. Let us choose a covering $X = \cup B(x_i, 2\varepsilon)$ and a partition of unity $1 = \Sigma\varphi_i$ described in Lemmas 1.2 and 1.3. We shall suppose that an orthonormal frame is chosen in every tangent space $T_{x_i}X$, $i = 1, 2, \cdots$, so $T_{x_i}X$ is identified with $\mathbb{R}^n$ and the exponential maps at the points x_i can be considered as the maps $\exp_{x_i} : \mathbb{R}^n \to X$.

Let us choose a function $\theta_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\theta_1 \geq 0$, supp $\theta_1 \subset \{x \| x| < 1\}$, $\int_{\mathbb{R}^n} \theta_1(x)\mathrm{dx} = 1$ and define $\theta_\delta(x) = \delta^{-n}\theta_1(x/\delta)$ for any $\delta > 0$. Now choosing δ sufficiently small we can define

$$(2.1)\quad \tilde{d}(x,y) = \Sigma_{i=1}^{\infty}\varphi_i(y) \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x, \exp_{x_i}(z)) dz.$$

Subtracting the evident identity

$$d(x,y) = \Sigma_{i=1}^{\infty}\varphi_i(y) \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x,y) dz$$

from (2.1) and using the triangle inequality we obtain the estimate

$$|\tilde{d}(x,y) - d(x,y)| \leq \Sigma_{i=1}^{\infty}\varphi_i(y) \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(\exp_{x_i}(z), y) dz.$$

It follows from the bounded geometry conditions that there exists $C > 0$ such that $d(\exp_{x_i}(z), y) < C\delta$ if $y \in$ supp φ_i and $|\exp_{x_i}^{-1}(y) - z| < \delta$, so we obtain

$$|\tilde{d}(x,y) - d(x,y)| < C\delta$$

which proves (i) with small ρ provided δ is chosen sufficiently small.

To prove (ii) let us consider first the case $|\alpha| = 1$.

Using the notation $\partial_j = \partial/\partial y_j$ in some canonical coordinates we obtain

(2.2)

$$\partial_j \tilde{d}(x,y) = \Sigma_{i=1}^{\infty}[\partial_j \varphi_i(y)] \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x, \exp_{x_i}(z)) dz +$$

$$\Sigma_{i=1}^{\infty} \varphi_i(y) \Sigma_{k=1}^{n} \int_{\mathbb{R}^n} b_{ijk}(y) [\frac{\partial}{\partial z_k} \theta_\delta(\exp_{x_i}^{-1}(y) - z)] d(x, \exp_{x_i}(z)) dz$$

where b_{ijk} are some functions (in the chosen canonical coordinates) which are C^∞–bounded uniformly with respect to i, j, k and the chosen coordinates. The same arguments as we used in proving (i) show that the first term in the right hand side of (2.2) is estimated by a constant. To estimate the second term we can subtract from it a similar term which is obtained by changing $d(x, \exp_{x_i}(z))$ to $d(x,y)$ (this modified term evidently vanishes). Following then the reasoning used for the proof of (i) we obtain that the second term is estimated by a constant.

Further inductive reasoning shows that (ii) is true for every α q.e.d. $\square$

Now we can introduce exponential weights $f_{\varepsilon,y} \in C^\infty(X)$ by

$$f_{\varepsilon,y}(x) = \exp(\varepsilon \tilde{d}(y,x)), \ x, y \in X, \tag{2.3}$$

where $\varepsilon \in \mathbb{R}$ (usually ε will be sufficiently small).

Let us introduce a weight Sobolev space

$$W_{p,\varepsilon}^s(X) = \{u | u \in \mathcal{D}'(X), f_{\varepsilon,y} u \in W_p^s(X)\}$$

where $s \in \mathbb{R}$, $p \in [1, \infty]$ and y is any fixed point in X. It is easy to check that

$$f_{\varepsilon,y_1}^{-1} f_{\varepsilon,y_2} \in C_b^\infty(X)$$

for any fixed points $y_1, y_2 \in X$. It follows that the space $W_{p,\varepsilon}^s(X)$ does not depend on the chosen point y. The space $W_{p,\varepsilon}^s(X)$ is a Banach space with the norm

$$\|u\|_{s,p;\varepsilon,y} = \|f_{\varepsilon,y}u\|_{s,p}. \tag{2.4}$$

These norms obtained by use of different points y are equivalent but the dependence on y is sometimes essential.

Now we shall consider a C^∞–bounded uniformly elliptic operator $A: C^\infty(X,E) \to C^\infty(X,E)$ where E is a vector bundle of bounded geometry. Then $A_{\min} = A_{\max}$ in $L^p(X,E)$, $1 < p < \infty$ (see Sect. 1.4 in Ch. 1) and we denote $\sigma_p(A)$ the spectrum of $A_{\min}$ (or $A_{\max}$) in $L^p(X,A)$. Let us suppose that $\lambda \in \mathbb{C}\backslash\sigma_p(A)$ for $p \in (1,+\infty)$. Then there is a bounded everywhere defined inverse operator

$$(A-\lambda I)^{-1}: \ L^p(X,E) \to L^p(X,E).$$

The L. Schwartz kernel of this inverse operator will be denoted $G = G(x,y)$ and will be called **the Green function** (p and λ are fixed). We are ready to prove estimates of decay of the Green function off the diagonal $\Delta = \{(x,x)|x \in X\} \subset X \times X$. Note that G is a distributional section of the bundle $E \otimes E^*$ on $X \times X$ (the fiber of $E \otimes E^*$ over a point $(x,y) \in X \times X$ is $E_x \otimes E_y^*$, where E_y^* is the dual linear space to E_y). We identify the density bundle over M with a trivial bundle by use of the standard Riemannian density.

Theorem 2.2. *Let $A: \ C^\infty(X,E) \to C^\infty(X,E)$ be a C^∞-bounded uniformly elliptic differential operator. Let $p \in (1,+\infty)$ and $\lambda \in \mathbb{C}\backslash\sigma_p(A)$ be fixed, $G = G(x,y)$ the Green function. Then $G \in C^\infty(X \times X\backslash\Delta)$ and there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every multiindices α,β there exists $C_{\alpha\beta\delta} > 0$ such that*

$$|\partial_x^\alpha \partial_y^\beta G(x,y)| \le C_{\alpha\beta\delta}\exp(-\varepsilon d(x,y)) \ \textit{if}\ d(x,y) \ge \delta. \tag{2.5}$$

Here the derivatives ∂_x^α and ∂_y^β are taken with respect to canonical coordinates and absolute value in the left hand side is taken in the corresponding fibers.

Proof. Without loss of generality we can suppose that $\lambda = 0$. For the sake of simplicity of notations we shall only considei the scalar case i.e. the case of trivial $E = X \times \mathbb{C}$. Let us for every $\varepsilon \in \mathbb{R}$, $y \in X$ consider a differential operator $A_{\varepsilon,y} = F_{\varepsilon,y} A F_{\varepsilon,y}^{-1}$ where $F_{\varepsilon,y}$ is the multiplication operator $(F_{\varepsilon,y}u)(x) = f_{\varepsilon,y}(x)u(x)$ with $f_{\varepsilon,y}$ defined by (2.3). Choosing any $s \in R$ we obtain a commutative diagram

$$\begin{array}{ccc} W_p^s(X) & \xrightarrow{A_{\varepsilon,y}} & W_p^{s-m}(X) \\ \uparrow{\scriptstyle F_{\varepsilon,y}} & & \uparrow{\scriptstyle F_{\varepsilon,y}} \\ W_{p,\varepsilon}^s(X) & \xrightarrow{A} & W_{p,\varepsilon}^{s-m}(X) \end{array} \tag{2.6}$$

where the vertical arrows are linear topological isomorphisms and even isometries if we use the norm (2.4) in $W_{p,\varepsilon}^s(X)$ and the corresponding norm in $W_{p,\varepsilon}^{s-m}(X)$. It follows from the properties of $\tilde{d}$ described in Lemma 2.1 that

$$A_{\varepsilon,y} = A + \varepsilon B_{\varepsilon,y}, \tag{2.7}$$

where $\{B_{\varepsilon,y} | y \in X, |\varepsilon| < 1\}$ is a family of uniformly C^∞–bounded differential operators of order $m-1$. It follows that the operator norm

$$\|A_{\varepsilon,y} - A : \ W_p^s(X) \to W_p^{s-m}(W)\|$$

tends to 0 as $\varepsilon \to 0$. The required invertibility of A implies now that A defines a linear topological isomorphism of Banach spaces

$$A : \ W_p^s(X) \to W_p^{s-m}(X),$$

so $A_{\varepsilon,y}$ in the diagram (2.6) also defines a linear topological isomorphism if $|\varepsilon| < \varepsilon_0$ where $\varepsilon_0 > 0$ is sufficiently small. Besides all norm estimates are uniform with respect to $y \in X$. Hence A in the diagram is also uniformly topologically invertible if $|\varepsilon| < \varepsilon_0$.

Now notice that

$$G(x,y) = [A^{-1}\delta_y(\cdot)](x), \tag{2.8}$$

where δ_y is the standard Dirac δ–measure on X supported at $y \in X$. The Sobolev embedding theorem implies that if $s < -n/p$ then $\delta_y \in \cap_{\varepsilon \in \mathbb{R}} W^s_{p,\varepsilon}(X)$ and $\|\delta_y\|_{s,p;\varepsilon y} \le C_{s,p}$ uniformly over $y \in X$ and ε with $|\varepsilon| < 1$. It follows from (2.8) that

$$\|G(\cdot,y)\|_{s+m,p;\varepsilon,y} \le C_{s,p} \tag{2.9}$$

if $|\varepsilon| < \varepsilon_0$.

Now note that

$$A_x G(x,y) = 0 \text{ if } x \ne y.$$

It follows from (2.9) and the uniform local a priori estimate like (1.6) that for every $\delta > 0$, $s \in \mathbb{R}$, $p \in (1,+\infty)$, $y \in X$ and $x \in X$ with $d(x,y) > \delta$

$$\|G(\cdot,y)\|_{s,p,B(x,\delta/2)} \le C_{s,p,\delta} \exp(-\varepsilon d(x,y)).$$

The Sobolev embedding theorem implies now that the required estimate (2.5) is satisfied if $\beta = 0$. Now the same reasoning can be applied with respect to y because we can use the uniformly elliptic equation

$$A^t_y G(x,y) = 0 \text{ if } x \ne y$$

where A^t is the formally transposed operator to A defined by the equality

$$\langle Au, v\rangle = \langle u, A^t v\rangle, \ u,v \in C_0^\infty(X),$$

where

$$\langle f, g \rangle = \int_X f(x)g(x)dx,$$

dx is the Riemannian density on X. This immediately leads to the estimates (2.5). $\square$

Actually estimates (2.5) prove to be adequate only in case of subexponential growth of the volume of the balls on X. For the case of exponential growth stronger estimates in terms of L^p–norms are available.

Theorem 2.3. *Let $p \in (1, +\infty)$ and $\lambda \in \mathbb{C}\backslash\sigma_p(A)$ be fixed, $G = G(x,y)$ the Green function. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every multiindices α, β there exists $C_{\alpha\beta\delta} > 0$ such that*

$$\int_{x:d(x,y)\geq\delta} |\partial_x^\alpha \partial_y^\beta G(x,y)|^p \exp(\varepsilon d(x,y))dx \leq C_{\alpha\beta\delta} \tag{2.10}$$

$$\int_{y:d(x,y)\geq\delta} |\partial_x^\alpha \partial_y^\beta G(x,y)|^{p'} \exp(\varepsilon d(x,y))dy \leq C_{\alpha\beta\delta}, \tag{2.11}$$

where $1/p' + 1/p = 1$, the derivatives and absolute values are understood as in Theorem 2.2.

Proof. We should just return to (2.9) but use it a little differently. Namely, using the same reasoning as in the proof of Theorem 2.2 we can evidently conclude from (2.9) that for every $s \in \mathbb{R}$

$$\sum_{j=1}^{\infty} \|G(\cdot, y)\|^p_{s,p,B(x_j,\delta/2)} \exp(\varepsilon d(x_j, y)) < \infty$$

where x_j are chosen as in Lemma 1.2 (with ε replaced by δ there). Then (2.10) obviously follows. To prove (2.11) we should apply (2.10) to the transposed operator A^t. $\square$

We need also uniform local estimates of the Green function near the diagonal but the simplest way to obtain them is in a use of pseudo–differential operators. This will be done in the next Section.

A1.3. Uniform properly supported pseudo–differential operators and structure of inverse operators.

We shall introduce here classes of uniform properly supported pseudo–differential operators on a manifold X of bounded geometry which coincide locally with well–known Hörmander classes Ψ^m and Ψ^m_{phg} ([22], vol. 3). Such classes were inroduced first on Lie groups in [29] and later in the general case in [28]

DEFINITION 3.1. $U\Psi^{-\infty}(X)$ is a class of all operators R with a L. Schwartz kernel $K_R \in C^\infty(X \times X)$ satisfying the following conditions

(i) there exists $C_R > 0$ such that $K_R(x,y) = 0$ if $d(x,y) > C_R$;
(ii) $|\partial_x^\alpha \partial_y^\beta K_R(x,y)| \le C_{\alpha\beta}$, $x, y \in X$, where the derivatives are taken in canonical coordinates.

The class $U\Psi^{-\infty}(X)$ will serve as a class of negligible operators in our context. Notice that an operator $R \in U\Psi^{-\infty}(X)$ is not necessarily compact e.g. in $L^2(X)$.

In the next definition we fix $r \in (0, r_{inj})$ as was already done before.

DEFINITION 3.2. $U\Psi^m(X)$ is a class of all operators $A : C_0^\infty(X) \to C_0^\infty(X)$ satisfying the following conditions:

(i) there exists $C_A > 0$ such that $K_A(x,y) = 0$ if $d(x,y) > C_A$ (here K_A is the L. Schwartz kernel of A);
(ii) let $B(x_0, r)$ be a ball on X, then in canonical coordinates on $B(x_0, r)$ the operator

$$A_{x_0} = A|_{C_0^\infty(B(x_0,r))} : \ C_0^\infty(B(x_0,r)) \to C^\infty(B(x_0,r)),$$
$$u \mapsto Au|_{B(x_0,r)}$$

can be written as

$$A_{x_0} = a_{x_0}(x, D_x) + R_{x_0} \tag{3.1}$$

where $a_{x_0} \in S^m$ uniformly with respect to x_0, i.e.

$$|\partial_\xi^\alpha \partial_x^\beta a_{x_0}(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$$

with $C_{\alpha\beta}$ which do not depend on x_0, and R_{x_0} is an operator with a L. Schwartz kernel $K_{R_{x_0}} \in C^\infty(B(x_0,r) \times B(x_0,r))$ satisfying the following estimates

$$|\partial_x^\alpha \partial_y^\beta K_{R_{x_0}}(x,y)| \leq C'_{\alpha\beta}$$

with constants $C'_{\alpha\beta}$ which do not depend on x_0.

DEFINITION 3.3. $U\Psi^m_{phg}(X)$ is a class of operators $A \in U\Psi^m(X)$ which have polyhomogeneous local symbols $a_{x_0}(x,\xi)$ with uniform estimates of homogeneous terms in local representations (3.1). More exactly it is required that there exist $a_{x_0,j} = a_{x_0,j}(x,\xi)$, $j = 0,1,2,\dots$, such that the following conditions are satisfied:

(i) $a_{x_0,j}(x,\xi)$ is defined when $x \in B(x_0,r), \xi \neq 0$ and is homogeneous of degree $m-j$ with respect to ξ, i.e.

$$a_{x_0,j}(x,t\xi) = t^{m-j} a_{x_0,j}(x,\xi), \;\; x \in B(x_0,r), \;\; \xi \in \mathbb{R}^n \backslash 0, t > 0;$$

(ii) $a_{x_0,j} \in C^\infty$ when $\xi \neq 0$ and $|\partial_\xi^\alpha \partial_x^\alpha a_{x_0,j}(x,\xi)| \leq C_{\alpha\beta j}$ when $x \in B(x_0,r)$ and $|\xi| = 1$ with the constants $C_{\alpha\beta j}$ which do not depend on x_0;

(iii) let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$ when ξ is close to 0, and χ is fixed, then for every N, α, β, x_0

$$|\partial_\xi^\alpha \partial_x^\beta [a_{x_0}(x,\xi) - \Sigma_{j=0}^{N-1}(1-\chi(\xi)) a_{x_0,j}(x,\xi)]| \leq C_{\alpha\beta N}(1+|\xi|)^{m-N}$$

with $C_{\alpha\beta N}$ which do not depend on x_0.

So the classes $U\Psi^m, U\Psi^m_{phg}$ are just usual Hörmander classes of properly supported pseudo–differential operators but with appropriate uniformity conditions.

The classes $U\Psi^m, U\Psi^m_{phg}$ are defined for all $m \in \mathbb{R}$. The class $U\Psi^m_{phg}(X)$ can be defined also for $m \in \mathbb{C}$ as a class of operators $A \in U\Psi^{\mathrm{Re}\ m}(X)$ such that the conditions (i), (ii), (iii) of Definition 3.3 are satisfied if we replace m by Re m in (iii).

The usual algebraic and continuity properties are satisfied for the classes $U\Psi^m(X)$, $U\Psi^m_{phg}(X)$.

In particular the following statements are easily checked:

(a) if $A_j \in U\Psi^{m_j}(X)$, $j = 1, 2$, then $A_1 A_2 \in U\Psi^{m_1+m_2}(X)$; the same is true for the classes $U\Psi^m_{phg}(X)$;

(b) if $A \in U\Psi^m(X)$ (or $U\Psi^m_{phg}(X)$) then $A^* \in U\Psi^m(X)$ (resp. $U\Psi^{\bar{m}}_{pgh}(X)$ where $\bar{m}$ is complex conjugate to m).

(c) if $A \in U\Psi^m(X)$ then A defines for every $s \in \mathbb{R}$, $p \in (1, +\infty)$ a continuous linear operator

$$A: \ W^s_p(X) \to W^{s-m}_p(X)$$

Proposition 3.4. *Let A be a C^∞-bounded uniformly elliptic differential operator of order m on X. Then there exists $B \in U\Psi^{-m}_{phg}(X)$ such that $I - AB, I - BA \in U\Psi^{-\infty}(X)$.*

Proof. The operator B with required properties is easily constructed by use of inform local parametrices B_i for A in the balls $B(x_i, \varepsilon)$ from Lemma 1.2 and then patching them up by the formula

$$B = \Sigma_i \Psi_i B_i \Phi_i,$$

where Φ_i, Ψ_i are multiplication operators $\Phi_i u(x) = \varphi_i(x) u(x)$, $\Psi_i u(x) = \psi_i(x) u(x), \varphi_i$ is taken from the partition of unity of Lemma 1.3, $\psi_i \in C_0^\infty(B(x_i, 2\varepsilon))$ are chosen to be uniformly C^∞–bounded and such that $\psi_i(x) = 1$ in a neighbourhood of supp φ_i. □

REMARK 3.5. Choosing $\varepsilon > 0$ sufficiently small we can obtain the parametrix B with a L. Schwartz kernel K_B with

$$\operatorname{supp} K_B \subset \{(x, y) | d(x, y) < \varepsilon_1\}$$

where $\varepsilon_1 = \varepsilon_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we can describe the structure of the operator $(A - \lambda I)^{-1}$ in case $\lambda \notin \sigma_p(A)$ more precisely.

First note that all the definitions and statements of this Section can be easily generalized to operators acting in spaces of sections of vector bundles of bounded geometry on X. The corresponding classes of operators $A: C_0^\infty(X, E) \to C_0^\infty(X, F)$ will be denoted $U\Psi^{-\infty}(X; E, F)$, $U\Psi^m(X; E, F)$, $U\Psi^m_{phg}(X; E, F)$ or $U\Psi^{-\infty}(X, E)$ etc. in case $E = F$.

Theorem 3.6. *Let $A: C_0^\infty(X, E) \to C_0^\infty(X, F)$ be a uniformly elliptic C^∞–bounded differential operator of order m. Let the closure of A in $L^p(X, E)$ have an everywhere defined bounded inverse A^{-1}. Then there exists $\varepsilon > 0$ and a representation:*

$$A^{-1} = B + T, \tag{3.2}$$

where $B \in U\Psi^{-m}_{phg}(X; F, E)$, T has a L. Schwartz kernel $K_T \in C^\infty$ satisfying the following estimates

$$|\partial_x^\alpha \partial_y^\beta K_T(x, y)| \leq C_{\alpha\beta} \exp(-\varepsilon d(x, y)). \tag{3.3}$$

Also

$$\int_X |\partial_x^\alpha \partial_y^\beta K_T(x, y)|^p \exp(\varepsilon d(x, y)) dx \leq C_{\alpha\beta} \tag{3.3'}$$

$$\int_X |\partial_x^\alpha \partial_y^\beta K_T(x, y)|^{p'} \exp(\varepsilon d(x, y)) dy \leq C_{\alpha\beta} \tag{3.3''}$$

where $1/p' + 1/p = 1$. Here the derivatives and the norm in the left–hand side are taken with respect to the canonical coordinates and canonical trivializations of E and F.

Proof. For the sake of simplicity of notations we shall consider the case of trivial $E = F = X \times \mathbb{R}$. It follows from Proposition 3.4 that there exists $B \in U\Psi^{-m}_{phg}(X)$ such that

$$AB = I - R,$$

where $R \in U\Psi^{-\infty}(X)$. Multiplying by A^{-1} from the left we obtain (3.2) with $T = A^{-1}R$. Now it is clear that

$$K_T(x,y) = [A^{-1}K_R(\cdot,y)](x). \tag{3.4}$$

Notice that $K_R(\cdot,y) \in C_0^\infty(X)$ and supp $K_R(\cdot,y) \subset B(y,r_0)$ for some $r_0 > 0$ which does not depend on y. Hence it follows from (3.4) and Theorem 2.1 that the estimates (3.3) are fulfilled if $d(x,y) \geq r_0$ with $r_0 > 0$ arbitrarily small so the estimates (3.3) are proved outside δ–neighbourhood of the diagonal for every $\delta > 0$.

Now we have to prove (3.3) in the set

$$\{(x,y)|d(x,y) < \delta\}$$

where $\delta > 0$ can be chosen arbitrarily small. But then (3.3) reduces to the boundedness of all derivatives which follows from the Sobolev embedding theorem and the boundedness of the operator

$$A^{-1}: \ W_p^s(X) \to W_p^{s+m}(X)$$

for every $s \in \mathbb{R}$ which is due to the regularity properties (Lemma 1.4) and the closed graph theorem.

Now to prove (3.3$'$) we use (3.4) again but apply the boundedness of $A_{\varepsilon,y}^{-1}$ instead of Theorem 2.2 itself. The estimate (3.3$''$) is proved by applying the same arguments to A^t instead of A. $\square$

Now we can prove estimates of the Green function near the diagonal.

Theorem 3.7. *Let A, p, λ satisfy the conditions of Theorem 2.2, G be the Green function (the L. Schwartz kernel of $(A - \lambda I)^{-1}$). Then there exists $\varepsilon > 0$ such that*

$$|\partial_x^\alpha \partial_y^\beta G(x,y)| \leq C_{\alpha\beta} d(x,y)^{m-n-|\alpha|-|\beta|} \exp(-\varepsilon d(x,y)) \tag{3.5}$$

provided $m < n$;

$$|\partial_x^\alpha \partial_y^\beta G(x,y)| \leq \tag{3.6}$$
$$C_{\alpha\beta}[1 + d(x,y)^{m-n-|\alpha|-|\beta|}|\log d(x,y)|]\exp(-\varepsilon d(x,y))$$

provided $m \geq n$.

Proof. As usual we shall consider the scalar case. Due to Theorem 2.2 it is sufficient to prove (3.5) and (3.6) for $x, y \in X$ such that $d(x,y) \leq \delta$ with some fixed $\delta > 0$. Let us consider the representation (3.2). Clearly the L. Schwartz kernel K_T satisfies the required estimates due to (3.3). Now we have to consider K_B and to do this let us present B locally in $B(x_0, r)$ in the form (3.1)

$$B_{x_0} = b_{x_0}(x, D_x) + R_{x_0}$$

where the L. Schwartz kernel of R_{x_0} satisfies the required estimates and $b_{x_0} = b_{x_0}(x,\xi)$ is a polyhomogeneous symbol with uniform estimates. The L. Schwartz kernel of $b_{x_0}(x, D_x)$ in local canonical coordinates near x_0 is equal to

$$K_{x_0}(x,y) = F_{\xi\to x-y} b_{x_0}(x,\xi) = (2\pi)^{-n} \int b_{x_0}(x,\xi) e^{i\langle x-y,\xi\rangle} d\xi$$

so to prove the necessary estimates it is sufficient to use the well known properties of the Fourier transform of homogeneous functions or their appropriate distributional regularizations (see e.g. [22], vol. 1). $\square$

REMARK 3.8. Most part of the results described here can be generalized to pseudo–differential operators. Namely, Theorem 3.6 is true for uniformly elliptic pseudo–differential operators $A \in U\Psi^m_{phg}(X; E, F)$ if $m > 0$. Also if $A \in U\Psi^m(X; E, F)$ is uniformly elliptic in appropriate sense (see [29] for the case of Lie groups) then the statement of Theorem 3.6 is true with $B \in U\Psi^{-m}(X;F,E)$. So Theorem 3.7 is also true in the case $A \in U\Psi^m_{phg}(X;E,F)$ if $m > 0$ (the estimate (3.5) will be true when $m < n$ or $m - n \notin \mathbb{Z}$).

In fact it is not necessary to consider only pseudo–differential operators which are properly supported. Everything is true e.g. for the operators like the right–hand side in (3.2) i.e. for the operators of the form $A = A_0 + T$, where $A_0 \in U\Psi^m_{phg}(X;E,F)$ and T satisfies some decay conditions as in the formulation of Theorem 3.6. Moreover the requirement of exponential decay of the kernel off the diagonal can also be relaxed if the volume of balls on X grows even more slowly. The corresponding machinery was developed in [29] for Lie groups and is perfectly suitable for general manifolds of bounded geometry so we omit the details.

Chapter 2. Eigenfunctions and spectra.

2.1. Generalized eigenfunctions.

Let X be a manifold of bounded geometry which we shall suppose to be connected for the sake of simplicity, $\dim X = n$, and E a complex vector bundle of bounded geometry on X. We shall always suppose that E is provided with an hermitian scalar product of bounded geometry on fibers. In particular the Hilbert space of sections $L^2(X,E)$ is well defined. We shall construct a special Hilbert–Schmidt rigging of this space, hence its negative space will contain a complete orthonormal system of generalized eigenfunctions of any self–adjoint operator (see Appendix 2 after this Chapter). In the elliptic case additional regularity properties of these generalized eigenfunctions will be proved.

Denote $V_x(r) = \mathrm{Vol}B(x,r)$, $V(r) = \sup_{x\in X} V_x(r)$. Lemma 4.4 from Chapter 1 immediately implies that there exists $a > 0$ such that

$$V(r) \leq e^{ar}. \tag{1.1}$$

Also both $V_x(r), V(r)$ are increasing functions on $[0,\infty)$ with values in $[0,\infty)$, positive on $(0,\infty)$. The reasoning in the proof of Lemma 4.4, Ch.1 shows that there exists $C > 0$ such that

$$V_x(r+1) \le CV_x(r), \; r \ge 1, \; x \in X. \tag{1.2}$$

Taking supremum over $x \in X$ on both sides we obtain

$$V(r+1) \le CV(r), \; r \ge 1. \tag{1.2'}$$

with the same constant C. Hence (again with the same constant $C > 0$) we obtain

$$C^{-1}V_x(r) \le V_x(\rho) \le CV_x(r) \text{ if } \rho \in [r-1, r+1], \tag{1.3}$$

$$C^{-1}V(r) \le V(\rho) \le CV(r) \text{ if } \rho \in [r-1, r+1]. \tag{1.3'}$$

REMARK 1.1. It is not always possible to estimate $V_x(r)$ from below by $C^{-1}V(r)$ whatever $C > 0$. For example, if we take a manifold X which is diffeomorphic to $\mathbb{R}^n$ with coordinates $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ with the hyperbolic metric $x_n^{-2}(dx'^2 + dx_n^2)$ in $\{x | x_n \ge 1\}$ and the euclidean metric $dx'^2 + dx_n^2$ in $\{x | x_n \le -1\}$ with a smooth transition in $\{x | -1 \le x_n \le 1\}$ making X a manifold of bounded geometry then $V_x(r)$ for a fixed r varies at least between volumes of the euclidean and the hyperbolic ball of radius r (the first one being $0(r^n)$ and the second growing exponentially as $r \to +\infty$).

Lemma 1.2. *There exist increasing C^∞ functions $\tilde{V} : [0, \infty) \to (0, \infty)$, $\tilde{V}_x : [0, \infty) \to (0, \infty)$ such that*

$$C^{-1}V_x(r) \le \tilde{V}_x(r) \le CV_x(r), \; r \ge 1, \tag{1.4}$$

$$(1.4') \qquad C^{-1}V(r) \le \tilde{V}(r) \le CV(r), \ r \ge 1,$$

with the same constant C as in (1.2), (1.2′), (1.3), (1.3′). Besides

$$(1.5) \qquad |\partial_r^k \tilde{V}_x(r)| \le C_k \tilde{V}_x(r), \ |\partial_r^k \tilde{V}(r)| \le C_k \tilde{V}(r)$$

for every $k = 0, 1, 2, \ldots$.

Proof. Let us extend $V_x(r), V(r)$ by 0 on $(-\infty, 0)$ and then take

$$\tilde{V}_x(r) = \int V_x(r+s)\varphi(s)ds, \ \tilde{V}(r) = \int V(r+s)\varphi(s)ds,$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \ge 0$, $\int \varphi(s)ds = 1$ and $\operatorname{supp}\varphi \subset [-1/4, 1/4]$. The estimates (1.4), (1.4′), (1.5) now obviously follow from (1.3), (1.3′). Also $\tilde{V}_x, \tilde{V}$ are increasing due to the same property of V_x, V. □

Now let us define positive weight C^∞ functions

$$(1.6) \qquad f_{x_0}(x) = \tilde{V}_{x_0}(\tilde{d}(x_0, x)), \ f(x) = \tilde{V}(\tilde{d}(x_0, x)),$$

where $\tilde{d}$ is the smoothed distance–function constructed in Lemma 2.1 of Appendix 1.

Lemma 1.3. *In canonical coordinates*

$$(1.7) \qquad |\partial^\alpha f_{x_0}(x)| \le C_\alpha f_{x_0}(x), \ |\partial^\alpha f(x)| \le C_\alpha f(x), \ x \in X$$

with constants C_α which do not depend on x.

Proof. The estimates (1.7) obviously follow from (1.5), the “derivative of composition formula”, e.g.

$$(1.8)\quad \partial^\alpha f(x) = \sum_{\substack{\alpha_1+\ldots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1,\ldots,\alpha_k}(\partial_r^k \tilde{V})(\tilde{d}(x_0,x))\partial_x^{\alpha_1}\tilde{d}(x_0,x)\ldots\partial_x^{\alpha_k}\tilde{d}(x_0,x),$$

and boundedness of the derivatives $\partial^\alpha \tilde{d}(x_0,x)$ for $|\alpha|>0$ (see Lemma 2.1 in Appendix 1). □

Now change f_{x_0}, f to real powers of these functions.

Lemma 1.4. *For any $t\in\mathbb{R}$ in canonical coordinates*

$$(1.9)\qquad |\partial^\alpha f_{x_0}^t(x)| \le C_{\alpha,t} f_{x_0}^t(x),\ |\partial^\alpha f^t(x)| \le C_{\alpha,t} f^t(x).$$

Proof. Using "derivative of composition formula" like (1.8) we obtain e.g.

$$(1.10)\quad \partial^\alpha f^t(x) = \sum_{\substack{\alpha_1+\ldots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1,\ldots,\alpha_k} f^{t-k}(x)\partial^{\alpha_1}f(x)\ldots\partial^{\alpha_k}f(x)$$

and (1.9) follows from Lemma 1.3. □

Lemma 1.5. *If $t>1/2$ then $f_{x_0}^{-t}, f^{-t}\in L^2(X)$. Also*

$$f_{x_0}^{-1/2}(\log f_{x_0})^{-1/2-\varepsilon}, f^{-1/2}(\log f)^{-1/2-\varepsilon} \in L^2(X)$$

for every $\varepsilon>0$.

Proof. Let us fix $t>1/2$. We clearly have due to Lemma 1.2

$$\int_{x:d(x,x_0)\ge 1} f_{x_0}^{-2t}(x)dx \le C_1\int_1^\infty V_{x_0}^{-2t}(r)dV_{x_0}(r) =$$
$$C_1\int_{V_{x_0}(1)}^\infty \lambda^{-2s}d\lambda < \infty.$$

with a consant $C_1 > 0$. Hence $f_{x_0}^{-t} \in L^2$. Now $V_x(r) \le V(r)$, therefore $V^{-t}(r) \le V_{x_0}^{-t}(r)$ and $f^{-t}(x) \le C_2 f_{x_0}^{-t}(x)$. Hence $f^{-t} \in L^2(X)$. Other inclusions are checked similarly. $\square$

Now let $g: X \to (0,\infty)$ be a positive C^∞–function such that

$$|\partial^\alpha g(x)| \le C_\alpha g(x), \ x \in X. \tag{1.10}$$

Examples of such functions are $f_{x_0}^t, f^t$ due to Lemma 1.4. We could also take $f_{x_0}^t(\log f_{x_0})^{t_1}$ or $f^t(\log f)^{t_1}$ with $t, t_1 \in \mathbb{R}$.

Now let us define the weighted Sobolev space $H_g^s = H_g^s(X,E)$ with $s \in \mathbb{R}$ as follows

$$H_g^s(X,E) = \{u | u \in \mathcal{D}'(X,E), \ gu \in H^s(X,E)\},$$

where

$$H^s(X,E) = W_2^s(X,E)$$

is the uniform Sobolev space defined in Sect. A1 of Appendix 1.

Clearly $H_g^s(X,E) \supset C_0^\infty(X,E)$, hence $H_g^s(X,E)$ continuously included and dense in $L^2(X,E)$ provided $s \ge 0$ and $g(x) \ge g_0 > 0$. Therefore in this case we can use H_g^s as a positive space to construct a rigging of $L^2(X,E)$.

Lemma 1.6. *If we use $H_g^s(X,E)$ with $s \ge 0$ and $g(x) \ge g_0 > 0$ as a positive space to construct a rigging of $L^2(X,E)$ then the corresponding negative (dual) space will be equal to $H_{g^{-1}}^{-s}(X,E)$*

Proof. Denote $\mathcal{H}_+ = H_g^s(X,E)$. Then in the notations of Appendix 2 we obviously have:

$$\mathcal{H}_- = \{u | u \in \mathcal{D}'(X,E), \ g^{-1}u \in H^{-s}(X,E)\} = H_{g^{-1}}^{-s}(X,E)$$

due to the standard duality by $H^s(X,E)$ and $H^{-s}(X,E)$. $\square$

Proposition 1.7. *Suppose that $s > n/2$, $g \in C^\infty(X)$ satisfies (1.10), $g(x) \geq g_0 > 0$ and $g^{-1} \in L^2(X)$. Then the rigging of $\mathcal{H} = L^2(X,E)$ with the positive space $\mathcal{H}_+ = H^s_g(X,E)$ is a Hilbert–Schmidt rigging.*

Proof. Choosing an elliptic pseudo–differential operator $B \in U\Psi^{s/2}_{phg}(X,E)$ we may take $A = I + B^*B \in U\Psi^s_{phg}(X,E)$ which will be elliptic invertible self–adjoint operator of order s. Hence $u \in H^s(X,E)$ if and only if $u \in L^2(X,E)$ and $Au \in L^2(X,E)$. Now obviously $H^s(X,E) = \mathrm{Im}(\bar{A}^{-1})$, where $\bar{A}$ is the self–adjoint operator defined by A on $L^2(X,E)$ with the domain $D(\bar{A}) = H^s(X,E)$. Hence

$$H^s_g(X,E) = \{g^{-1}\bar{A}^{-1}u | u \in L^2(X,E)\} = g^{-1}\bar{A}^{-1}L^2(X,E).$$

Therefore it is sufficient to establish that $g^{-1}\bar{A}^{-1}$ is a Hilbert–Schmidt operator. But his Schwartz kernel is given by

$$K(x,y) = g^{-1}(x)G(x,y)$$

where $G(\cdot,\cdot)$ is the Schwartz kernel of $\bar{A}^{-1}$ (or the Green function of A). Now we can use Theorems 2.3 and 3.7 from Appendix 1 to conclude that

$$\int_X |G(x,y)|^2 dy \leq C < \infty$$

It follows that

$$\int_{X\times X} |K(x,y)|^2 dxdy \leq C \int_X g^{-2}(x)dx < \infty$$

hence $g^{-1}\bar{A}^{-1}$ is a Hilbert–Schmidt operator. $\square$

Now applying Theorem 2.3 from Appendix 2 we immediately obtain

Theorem 1.8. *Suppose that $s > n/2$ and g satisfies the conditions in Proposition 1.7. Then for any self-adjoint operator A in $L^2(X,E)$ the space $H^{-s}_{g^{-1}}(X,E)$ contains complete orthonormal system of generalized eigenfunctions of A in the sense of Definition 2.2 of Appendix 2.*

Corollary 1.9. *For any $\varepsilon > 0$, $\delta > 0$ both spaces $H^{-n/2-\varepsilon}_{f^{-1/2-\delta}}(X,E)$, $H^{-n/2-\varepsilon}_{f^{-1/2}(\log f)^{-1/2-\delta}}$ contain complete orthonormal system of generalized eigenfunctions of any self-adjoint operator A in $L^2(X,E)$.*

REMARK 1.10. Using the composition formula for pseudo-differential operators of classes $U\Psi^m$ we can describe the space $H^s_g(X,E)$ also in a dual way as the space of all $u \in \mathcal{D}'(X,E)$ such that $gBu \in L^2(X,E)$ for every $B \in U\Psi^s(X,E)$. If $s \in \mathbb{Z}_+$ then we can equivalently write $g\partial^\alpha u \in L^2(X,E)$ for every multiindex α with $|\alpha| \le s$ (here $\partial^\alpha u$ can be taken in canonical coordinates for any piecewise constant choice of such coordinates induced by coverings described in Lemma 1.2 of Appendix 1). Using this description we can skip the requirement of smoothness of g and estimates (1.10) defining e.g. $H^s_{f^t}$ for $s \in \mathbb{Z}_+$ as the space of sections $u \in L^2(X,E)$ such that $[1 + V(d(\cdot,x_0))]^t\partial^\alpha u \in L^2(X,E)$ for every α with $|\alpha| \le s$. Hence the dual space $H^{-s}_{f^{-t}}$ consists of distributions which have the form

$$v = \sum_{\substack{k\le s\\ X_1\ldots X_k}} X_1\ldots X_k[(1+V(d(\cdot,x_0)))^t v_\alpha],\ v_\alpha \in L^2(X,E),$$

where $X_1,\ldots,X_s$ are first-order uniformily C^∞-bounded differential operators in $C^\infty(X,E)$, the sum is taken over a finite set of such tuples $X_1,\ldots,X_k$ with $k \le s$. Similarly for general $s > 0$ the space $H^{-s}_{f^{-t}}$ consists of sections $u \in \mathcal{D}'(X,E)$ of the form

$$u = \sum_{j=1}^{N} B_j[(1+V(d(\cdot,x_0))^t v_j],\ v_j \in L^2(X,E),\ B_j \in U\Psi^s(X,E)$$

where N and the set $B_1, \dots, B_N$ depend on u.

EXAMPLE 1.11. If $X = \mathbb{R}^n$ with the standard euclidean metric then $V(d(x,0)) = c_n|x|^n$ and for any $\varepsilon > 0$, $\delta > 0$ we can take the space $H^{-n/2-\varepsilon}_{(1+|x|^2)^{-n/4-\delta}}(\mathbb{R}^n)$ as the negative space containing a complete orthonormal space of generalized eigenfunctions of any self–adjoint operator in $L^2(\mathbb{R}^n)$.

EXAMPLE 1.12. If $X = \mathbb{H}^n$ is the hyperbolic space with the curvature -1 then $V_x(r) = V(r) \sim c_n e^{(n-1)r}$ as $r \to \infty$. Let us denote $|x| = d(x,0)$, where 0 is a fixed point in $\mathbb{H}^n$, and choose a positive C^∞–function $x \mapsto \langle x\rangle$ coinciding with $|x|$ if $|x| \geq 1$. Then for any $\varepsilon > 0$, $\delta > 0$ we can take one of the spaces $H^{-n/2-\varepsilon}_{\exp(-(n-1+\delta)\langle x\rangle/2)}(\mathbb{H}^n)$ or $H^{-n/2-\varepsilon}_{\exp(-(n-1)\langle x\rangle/2)\langle x\rangle^{-1/2-\delta}}(\mathbb{H}^n)$ as the desired negative space for any self–adjoint operator in $L^2(\mathbb{H}^n)$.

Now suppose that we consider not a general self–adjoint operator but a uniformly elliptic C^∞–bounded self–adjoint differential operator $A : C^\infty(X,E) \to C^\infty(X,E)$. Then we can use local a priori estimates to increase $-n/2 - \varepsilon$ up to any s. Actually any generalized eigenfunction will be a solution of a uniformly elliptic equation, hence it should be a C^∞–function (or rather C^∞–section). Hence we arrive to the following

Theorem 1.13. *Let $A : C^\infty(X,E) \to C^\infty(X,E)$ be a C^∞–bounded uniformly elliptic self–adjoint operator. Let g be a positive C^∞–function on X, satisfying (1.10), such that $g^{-1} \in L^2(X)$. Then there exist a complete orthonormal system of eigenfunctions for A, such that any eigenfunction ψ in this system satisfies the following estimates*

$$\int_X |\partial^\alpha \psi(x)|^2 g^{-2}(x) < \infty, \; x \in X, \tag{1.11}$$

for any multiindex α.

Now using locally (on balls of a fixed radius) the Sobolev imbedding theorem we obtain

Corollary 1.14. *Under the conditions of Theorem 1.13 there exists a complete orthonormal system of eigenfunctions such that any eigenfunction ψ in this system satisfies estimates*

$$|\partial^\alpha \psi(x)| \leq C_\alpha g(x). \tag{1.12}$$

REMARK 1.15. Clearly g here can be replaced by a positive function g_1 such that

$$C^{-1} g(x) \leq g_1(x) \leq C g(x)$$

with a constant $C > 0$. In particular both Theorem 1.13 and Corollary 1.14 remain true if we replace g by one of the following functions:

$$[1 + V(d(\cdot, x_0))]^{1/2+\varepsilon}, \ [1 + V(d(\cdot, x_0))]^{1/2} \ \log[2 + V(d(\cdot, x_0))]^{1+\varepsilon},$$

where $\varepsilon > 0$.

2.2. Schnol–type theorems.

In the previous section we gave a sufficient condition for a space to contain a complete orthonormal system of generalized eigenfunctions for a self–adjoint operator. The corresponding eigenvalues then will be in the spectrum of this operator (at least almost everywhere) and actually the closure of the set of these eigenvalues constitutes the spectrum in L^2. In this section we will consider an opposite question: assume that for some $\lambda \in \mathbb{C}$ we know a solution ψ of the equation $A\psi = \lambda\psi$ satisfying some estimates at infinity; when can we conclude that λ is in the spectrum $\sigma(A)$ of the operator A in L^2?

An example of the sort is the well known Schnol theorem ([38], [11]) which (with some simplifying restrictions) states that if $A = -\Delta + q(x)$ is a Schrödinger operator in $L^2(\mathbb{R}^n)$ with the potential $q \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ such that $q(x) \geq -C$ for all $x \in \mathbb{R}^n$ and there exists a non–trivial solution ψ of the equation $A\psi = \lambda\psi$ such that for every $\varepsilon > 0$

$$\psi(x) = O(\exp(\varepsilon|x|))$$

then $\lambda \in \sigma(A)$. Another Schnol theorem ([38]) also concerning the Schrödinger operator states that if the negative part $q_-(x) = \min(0, q(x))$ satisfies the estimate

$$q_-(x) = o(|x|^2)$$

then the existence of a non–trivial polynomially bounded solution (i.e. a solution ψ such that $\psi(x) = 0((1+|x|)^N)$ with some $N > 0$) for the equation $A\psi = \lambda\psi$ implies that $\lambda \in \sigma(A)$.

T. Kobayashi, K. Ono and T. Sunada ([26]) introduced.

DEFINITION 2.1. An operator A satisfies the **weak Bloch property** (WBP) if the following implication is true:

$$\{\text{there exists a bounded } \psi \not\equiv 0 \text{ such that } A\psi = \lambda\psi\} \Longrightarrow \lambda \in \sigma(A)$$

So each of the mentioned Schnol theorems implies that the Schrödinger operator on $\mathbb{R}^n$ with a locally bounded and semi–bounded below potential satisfies WBP.

On the other hand the Laplacian Δ of the standard Riemannian metric on the hyperbolic space $\mathbb{H}^n$ does not satisfy WBP because $\Delta 1 = 0$ but $0 \notin \sigma(\Delta)$.

It is natural to investigate the following WBP–problem: **describe classes of manifolds and operators which satisfy WBP.**

It is easy to notice that the WBP–problem is closely connected with the problem of coincidence of spectra of an operator in spaces $L^p(X)$ for different p: if all these spectra for $1 \leq p \leq \infty$ coincide then WBP evidently holds because if $\sigma_p(A)$ means the spectrum of A in $L^p(X)$ then the existence of a non–trivial bounded solution ψ of $A\psi = \lambda\psi$ implies that $\lambda \in \sigma_\infty(A)$ so $\lambda \in \sigma_2(A) = \sigma(A)$. The problem of the coincidence of spectra was considered on discrete metric spaces in [43] where it was pointed out that the coincidence follows from the exponential decay of the Green function off the diagonal provided the space has a subexponential growth of the number of points lying in a ball of the radius r as $r \to +\infty$. The exponential decay of the Green function off the diagonal was proved in [43] for some operators which were called pseudodifference operators, e.g. difference operators with a finite radius of action and bounded coefficients on discrete groups etc.

The same reasoning works also for continuous objects when the appropriate estimates of the Green function hold. Such estimates were obtained in [29] for uniformly elliptic operators on unimodular Lie groups and in [27],[28] on general manifolds of bounded geometry. It follows (though it was not noticed in [29] or [27],[28]) that the spectra of corresponding operators in $L^p(X)$ coincide for all $p \in (1, +\infty)$ provided the volumes of balls of radius r grow subexponentially as $r \to +\infty$, and also that WBP is satisfied in this situation. The main ideas of this approach will be explained here in detail. The important point here is a use of some weighted Sobolev spaces with exponential weights. In [26] the authors used an entirely different method which is quite close to the original Schnol method (see also [11]). The WBP was proved in [26] for the Schrödinger operators with periodic potentials on Riemannian manifolds X with a subexponential growth of volumes of balls and with a discrete group of isometries Γ such that the orbit space X/Γ is compact.

Now let X be a complete connected Riemannian manifold, $d(x, y)$ be the Riemannian distance between x and y, $x, y \in X$. Let A be a differential operator on X. Denote by $\sigma(A)$ its spectrum in $L^2(X)$.

DEFINITION 2.2.

i) The operator A satisfies the **weak Schnol property** (WSP) if the existence of a non–trivial solution ψ of the equation $A\psi = \lambda\psi$ satisfying an estimate of the form

$$|\psi(x)| = O(1 + d(x, x_0)^N)$$

(with some $N > 0$ and a fixed x_0) implies that $\lambda \in \sigma(A)$.

ii) The operator A satisfies the **strong Schnol property** (SSP) if the following implication is true: if there exists a non–trivial solution ψ of the equation $A\psi = \lambda\psi$ such that for every $\varepsilon > 0$

$$|\psi(x)| = O(\exp(\varepsilon d(x, x_0)))$$

(with a fixed x_0) then $\lambda \in \sigma(A)$.

Clearly SSP implies WSP, and WSP implies WBP. We shall prove that if X is a manifold of bounded geometry with a subexponential growth of volumes of balls and A is a uniformly elliptic differential operator with C^∞–bounded coefficients on X then A satisfies (SSP) and even stronger property: if for every $\varepsilon > 0$ there exists a non–trivial solution ψ_ε of $A\psi_\varepsilon = \lambda\psi_\varepsilon$ with

$$|\psi_\varepsilon(x)| = O(\exp(\varepsilon d(x, x_0)) \tag{2.1}$$

(with a fixed x_0) then $\lambda \in \sigma(A)$. We even prove the following Theorem which does not require any subexponential growth conditions

Theorem 2.3. *Let X be a manifold of bounded geometry, E a vector bundle of bounded geometry on X,*

$$A : \ C^\infty(X, E) \to C^\infty(X, E)$$

a uniformly elliptic C^∞–bounded differential operator. Let $p \in (1, \infty)$, $\lambda \in \mathbb{C}$ and for every $\varepsilon > 0$ there exists $\psi_\varepsilon \in C^\infty(X, E)$ such that $A\psi_\varepsilon = \lambda\psi_\varepsilon$, $\psi_\varepsilon \not\equiv 0$ and

$$\psi_\varepsilon \exp(-\varepsilon d(\cdot, x_0)) \in L^p(X, E). \tag{2.2}$$

Then $\lambda \in \sigma_p(A)$.

Here $\sigma_p(A)$ means the spectrum of $A_{\min} = A_{\max}$ in $\tilde{L}^p(X, E)$ (see Sect. 1.4 in Ch. 1), $1 \le p \le \infty$.

Before proving Theorem 2.3 we will give its corollaries and particular cases.

DEFINITION 2.4. Let X be a manifold of bounded geometry. We shall say that X has a *subexponential growth* (or is a *manifold of subexponential growth*) if for every $\varepsilon > 0$

$$V(r) = O(e^{\varepsilon r}),\ r \to \infty \tag{2.3}$$

where $V(\cdot)$ is introduced in Sect. 1.

Corollary 2.5. *Suppose that X is a manifold of subexponential growth, A is a uniformly elliptic C^∞–bounded differential operator on X and $\lambda \in \mathbb{C}$. Suppose that for every $\varepsilon > 0$ there exists $\psi_\varepsilon \not\equiv 0$ satisfying $A\psi_\varepsilon = \lambda\psi_\varepsilon$ and the estimate (2.1). Then $\lambda \in \sigma_p(A)$, $1 < p < \infty$. In particular (WBP), (WSP) and (SSP) are satisfied for A in this case.*

Proof. To apply Theorem 2.3 we have to check that $\exp(-\varepsilon d(\cdot, x_0)) \in L^p(X)$ for any $\varepsilon > 0$, $1 \le p \le \infty$. This can be proved if we notice that (2.3) implies for any $\varepsilon > 0$, $\delta > 0$

$$\exp(-\varepsilon d(\cdot, x_0)) \le V^{-1-\delta}(d(\cdot, x_0)) \le V_{x_0}^{-1-\delta}(d(\cdot, x_0))$$

and then use the same reasoning as in the proof of Lemma 1.5. □

Corollary 2.5 gives the same sufficient condition for $\lambda \in \sigma_p(A)$ to be true whatever $p \in (1, \infty)$. So we may expect that $\sigma_p(A)$ does not depend on p in the case of subexponential growth. We shall prove this and even give some information about extremal cases $p = 1$ and $p = \infty$.

Proposition 2.6. *Let X, A be as in Corollary 2.5 (in particular X has a subexponential growth). Then the spectrum $\sigma_p(A)$ does not depend on $p \in (1, \infty)$. Moreover denoting this spectrum by $\sigma(A)$ we have*

$$\sigma_1(A) \subset \sigma(A),\ \sigma_\infty(A) \subset \sigma(A). \tag{2.4}$$

Proof. For the sake of simplicity of notations let us consider the case of trivial bundle E with the fiber $\mathbb{C}$. We have to prove that if $\lambda \in \mathbb{C} - \sigma_{p_0}(A)$ for some $p_0 \in (1,\infty)$ then $\lambda \notin \sigma_p(A)$ for all $p \in [1,\infty]$. Now we may also suppose that $\lambda = 0$.

Due to Theorem 3.7 of Appendix 1 we obtain for the Green function $G(\cdot,\cdot)$ (the L. Schwartz kernel of A^{-1}) that

$$\sup_y \int |G(x,y)|dx < \infty, \ \sup_x \int |G(x,y)|dy < \infty.$$

Hence due to the well known Schur lemma (see e.g. Lemma 18.1.12 in [22], vol. 3) we obtain that the integral operator G with the Schwartz kernel $G(\cdot,\cdot)$ can be extended to a linear bounded operator

$$G: \ L^p(M) \to L^p(M)$$

for every $p \in [1,\infty]$. Let us introduce for any $\varepsilon > 0$ a space W_ε which contains functions $\varphi \in C^\infty(X)$ such that

$$|\partial^\alpha \varphi(x)| = O(\exp(-\varepsilon d(x, x_0)))$$

for every multiindex α (with the derivative ∂^α in canonical coordinates) and a chosen fixed $x_0 \in X$ (the condition does not depend on x_0). The subexponentiality condition clearly implies that $W_\varepsilon \subset L^p(X)$ for all $\varepsilon > 0$, $p \in [1,\infty]$ and moreover

$$W_\varepsilon \subset \bigcap_{p\in[1,\infty]} \bigcap_{s\in\mathbb{R}} W_p^s(X), \ \varepsilon > 0. \tag{2.5}$$

Now it follows from Theorem 3.6 of Appendix 1 that G maps $C_0^\infty(X)$ into W_ε with some $\varepsilon > 0$. Evidently $AG = GA = I$ on $C_0^\infty(X)$. Note that the first equality implies that $A_x G(x,y) = \delta_y(x)$ and the second implies that $A^t G^t = I$ on $C_0^\infty(X)$, hence $A_y^t G(x,y) = \delta_x(y)$. Another important algebraic corollary is that $G^t A^t = I$ on $C_0^\infty(X)$.

Now it is easy to check that $AG = I$ on $L^p(X)$ for every $p \in [1,\infty]$ if A is applied in the sense of distributions. In fact if $u \in L^p(X)$, $v \in C_0^\infty(X)$ then

$$\langle AGu, v\rangle = \langle Gu, A^t v\rangle = \langle u, G^t A^t v\rangle = \langle u, v\rangle,$$

hence $AGu = u$. It follows that $Gu \in D_p(A)$ where $D_p(A)$ is the domain of A in $\tilde{L}^p(X)$. Hence $A:\ D_p(A) \to L^p(X)$ is surjective.

Let us prove that $GA = I$ on $D_p(A)$, $p \in [1,\infty]$. If $u \in D_p(A)$, $v \in C_0^\infty(X)$ then

$$\langle GAu, v\rangle = \langle Au, G^t v\rangle$$

due to the Fubini theorem. Note that $G^t v \in W_\varepsilon$ for some $\varepsilon > 0$. So it is enough to prove that

$$\langle Au, \varphi\rangle = \langle u, A^t\varphi\rangle,\ u \in D_p(A),\ \varphi \in W_\varepsilon \tag{2.6}$$

Let us define a cut–off function

$$\chi_N(x) = \Sigma_{i=1}^N \varphi_i(x)$$

where φ_i are the functions from the partition of unity of Lemma 1.3 in Appendix 1. It is clear that $\chi_N \in C_0^\infty(X)$, $0 \le \chi_N \le 1$ and for every compact $K \subset X$ there exists N such that $\chi_N = 1$ in a neighbourhood of K. Moreover $|\partial^\alpha \chi_N| \le C_\alpha$ in canonical coordinates uniformly with respect to N.

Now we can begin with the equality

$$\langle Au, \chi_N\varphi\rangle = \langle u, A(\chi_N\varphi)\rangle,\ u \in D_p(A),\ \varphi \in W_\varepsilon, \tag{2.7}$$

and try to take limit as $N \to \infty$ to obtain (2.6). Note that $(Au)\varphi \in L^1(X)$ due to (2.5), therefore $\lim\limits_{N\to\infty} \langle Au, \chi_N\varphi\rangle = \langle Au, \varphi\rangle$ due to the dominated convergence theorem. The same reasoning can be applied to the right–hand side of (2.7) due to the estimates of derivatives of χ_N, so we obtain (2.6).

We have proved that the operators $A: \; D_p(A) \to L^p(X)$ and $G: \; L^p(X) \to D_p(A)$ are mutually inverse as required. $\square$

Proposition 2.6 immediately implies that WBP holds under its conditions, i.e. if X has a subexponential growth, A is uniformly elliptic C^∞–bounded operator on X, $\lambda \in \mathbb{C}$ and there exists $u \in L^\infty$, $u \not\equiv 0$ such that $Au = \lambda u$, then $\lambda \in \sigma_p(A)$, $1 < p < \infty$, because $\sigma_\infty(A) \subset \sigma_p(A)$. But Theorem 2.3 will give us a stronger result as mentioned in Corollary 2.5.

Corollary 2.5 and Proposition 2.6 were proved in the paper [44] which was inspired by the beautiful paper [26], though the paper [44] relied heavily on ideas contained in [43], [29] and [28]. Theorem 2.3 improves the results of [44] extending it to general manifolds of bounded geometry.

Now we are ready for the proof of the main theorem.

Proof of Theorem 2.3. Let us consider the scalar case and suppose that $\lambda = 0$. We should repeat arguments given in the proof of Proposition 2.6. Let us suppose that $0 \notin \sigma_p(A)$. Then we can construct the Green operator $G = A^{-1}$ which has a Schwartz kernel $G(\cdot,\cdot)$ satisfying estimates (2.10), (2.11) in Theorem 2.3 of Appendix 1.

Using the local a priori estimates it is easy to prove that (2.2) implies the same inclusion for derivatives of ψ_ε:

$$|\partial^\alpha \psi_\varepsilon(\cdot)| \exp(-\varepsilon d(\cdot, x_0)) \in L^p(X) \tag{2.8}$$

for every multiindex α (with the derivatives taken in local coordinates). But (2.8) and the estimate (2.11) in Appendix 1 imply now that $GA\psi_\varepsilon$ makes sense due to the Hölder inequality if $\varepsilon > 0$ is sufficiently small. Moreover $GA\psi_\varepsilon = \psi_\varepsilon$. Indeed for every $v \in C_0^\infty(X)$ we obtain using the Fubini theorem and estimates (2.10), (2.11) from Appendix 1:

$$\langle GA\psi_\varepsilon, v\rangle = \langle A\psi_\varepsilon, G^t v\rangle = \langle \psi_\varepsilon, A^t G^t v\rangle = \langle \psi_\varepsilon, v\rangle$$

(the middle equality is obtained by a limit procedure with the same use of the cut–off functions χ_N as in the proof of Proposition

2.6. On the other hand $A\psi_\varepsilon = 0$ implies $GA\psi_\varepsilon = 0$, hence $\psi_\varepsilon = 0$, so we get a contradiction which proves the theorem. □

REMARK 2.7. Suppose that X has a free isometric action of a discrete group Γ such that X/Γ is compact. Let Δ be the scalar Laplacian on X. Then R. Brooks [6] proved that $0 \in \sigma(\Delta)$ if and only if Γ is amenable. Note that we always have $\Delta 1 = 0$, hence $0 \in \sigma_\infty(\Delta)$ and WBP does not hold on X if Γ is not amenable. However it is not clear whether something like this is true for more general operators (e.g. Schrödinger operator with a Γ–invariant potential, which is the case where WBP was proved in [26] for the case of subexponential growth).

Now the amenability of Γ is equivalent to the amenability of X which means the existence of compacts $K_j \subset X$, $j = 1, 2, \ldots$, such that

$$\lim_{j\to\infty} \frac{\mathrm{Vol}[(K_j)_1 - K_j]}{\mathrm{Vol}\, K_j} = 0$$

where $(K_j)_1 = \{x \mid \mathrm{dist}(x, K_j) \le 1\}$. This makes sense for general manifolds of bounded geometry. So it is natural to ask whether WBP is true for general C^∞–bounded uniformly elliptic operators on amenable manifolds. The positive answer for the Schrödinger operator in the Γ–periodic case was conjectured in [26].

Similar questions may be asked for WSP and SSP (for SSP the natural question is whether the subexponential growth condition can be weakened or not).

REMARK 2.8. There is an essential gap between Theorems 1.8 (or 1.13) and 2.3. Namely Theorems 1.8 and 1.13 do not allow to exclude C^∞–bounded functions from negative spaces where we are trying to find a complete orthogonal system of generalized eigenfunctions. On the other hand the condition (2.2) in Theorem 2.3 (in case $p = 2$) is not satisfied for $\psi_\varepsilon \equiv 1$ unless X has a subexponential growth. The gap disappears for the manifolds of subexponential growth but it is natural to try to fill it in case of manifolds of exponential growth (like $\mathbb{H}^n$). No considerable improvement can be expected in Theorem 2.3 because its growth conditions come close to those which exist in examples like $\mathbb{H}^n$. On the other hand the abstract Theorem 2.3 from Appendix 2 can not be improved in the sense that the condition on the rigging to be a Hilbert–Schmidt rigging is necessary if we want the negative space to contain a complete orthonormal system of generalized eigenvectors for any self–adjoint operator. So possible improvement can be made here only if we switch from abstract operators e.g. to C^∞–bounded uniformly elliptic ones.

REMARK 2.9. It is sufficient to have only a sequence $\varepsilon_j \to 0$, and it is not necessary to keep λ fixed when we change ε. For instance in Theorem 2.3 we can only require that there exist a sequence $\varepsilon_j > 0$, $\varepsilon_j \to 0$ as $j \to +\infty$, and sections $\psi_j \not\equiv 0$, such that $\psi_j \exp(-\varepsilon_j d(\cdot, x_0)) \in L^p(X, E)$, $A\psi_j = \lambda_j \psi_j$ with $\lambda_j \to \lambda$ as $j \to \infty$. Then we can easily prove by the same reasoning that $\lambda \in \sigma_p(A)$. (Here $1 < p < \infty$.)

In case of subexponential growth it is easy to prove, using Theorem 1.8, that for the self-adjoint operators satisfying the conditions of Theorem 2.3 this condition is also necessary (hence necessary and sufficient) for the inclusion $\lambda \in \sigma(A)$, as well as the existence of sequences $\lambda_j \to \lambda$, $\psi_j \in C^\infty(X, E)$, $\psi_j \not\equiv 0$, such that $A\psi_j = \lambda_j \psi_j$ and $|\psi_j(x)| = 0(\exp \varepsilon_j d(x, x_0))$ where $\varepsilon_j \to 0$ as $j \to 0$.

Appendix 2. Rigged spaces and generalized eigenvectors of self–adjoint operators.

In this Appendix we shall briefly describe some well–known results about rigged spaces and generalized eigenvectors of abstract self–adjoint operators. We will mainly follow [5] referring the reader to the book for proofs and more details. An alternative approach can be found in [4].

A2.1. Rigged Hilbert spaces.

Usually Hilbert spaces arise in Analysis as spaces of square–integrable functions, sections of a vector bundle etc. But in this case usually additional restrictions of smoothness or (and) decay may be imposed to form a smaller Hilbert space. Also then the dual to this smaller space can be defined as a Hilbert space which naturally includes the basic Hilbert space. This situation is described in the following

DEFINITION 1.1. A *rigged Hilbert space* is a triple

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \tag{1.1}$$

where $\mathcal{H}, \mathcal{H}_+, \mathcal{H}_-$ are (complex) Hilbert spaces (with the scalar products and norms denoted by $(\cdot,\cdot)$, $(\cdot,\cdot)_+$, $(\cdot,\cdot)_-$, $\|\cdot\|$, $\|\cdot\|_+$, $\|\cdot\|_-$ respectively) and the following conditions are satisfied:

i) Both inclusions $\mathcal{H}_+ \subset \mathcal{H}$ and $\mathcal{H} \subset \mathcal{H}_-$ are linear continuous operators with dense image.

ii) The scalar product $(\cdot,\cdot)$ in $\mathcal{H}$ can be extended to a continuous hermitian form $(\cdot,\cdot) : \mathcal{H}_+ \times \mathcal{H}_- \to \mathbb{C}$ which is non–degenerate in the following strong sense: every linear continuous functional $\ell : \mathcal{H}_+ \to \mathbb{C}$ can be uniquely represented in the form $\ell(\cdot) = (\cdot, f)$ where $f \in \mathcal{H}_-$ and $\|f\|_- = \|\ell\|$ where $\|\ell\|$ is the usual (operator) norm of ℓ; similarly, every anti–linear continuous functional $\ell' : \mathcal{H}_- \to \mathbb{C}$ can be uniquely represented in the form $\ell'(\cdot) = (g, \cdot)$ with $g \in \mathcal{H}_+$ and $\|g\|_+ = \|\ell'\|$.

The triple (1.1) is called then a rigging for the Hilbert space $\mathcal{H}$. Spaces $\mathcal{H}_+, \mathcal{H}_-$ (and norms $\|\cdot\|_+, \|\cdot\|_-$) are usually called positive and negative spaces (and norms) respectively. Actually the negative space $\mathcal{H}_-$ can be obviously reconstructed if only the couple $\mathcal{H}_+ \subset \mathcal{H}$ is given with the continuous imbedding operator having dense image.

A convenient general procedure of constructing a rigging for a given Hilbert space $\mathcal{H}$ is to use a continuous linear operator $K : \mathcal{H} \to \mathcal{H}$ such that Ker $K = 0$ and Ker $K^* = 0$ (hence with a dense image $K\mathcal{H}$). Having such an operator we can put

$$\mathcal{H}_+ = K\mathcal{H}, \ (Ku, Kv)_+ = (u, v), \ u, v \in \mathcal{H}. \tag{1.2}$$

Then $\mathcal{H}_-$ can be reconstructed as the dual space to $\mathcal{H}_+$ or as the completion of $\mathcal{H}$ with respect to the norm $\|h\|_- = \|K^*h\|$.

Actually without loss of generality K can be chosen self–adjoint because replacing K by $|K| = \sqrt{K^*K}$ does not change the space $\mathcal{H}_+$ (and its norm).

DEFINITION 1.2. A *Hilbert–Schmidt rigging* is a rigging constructed with the help of a Hilbert–Schmidt operator K in (1.2).

Hilbert–Schmidt riggings play a special role in spectral theory as we shall see in the next section.

Supposing that $K^* = K$ we may consider $A = K^{-1}$ as a self–adjoint operator in $\mathcal{H}$; besides Ker $A = 0$. If such an operator is given then we can construct the rigging by putting $\mathcal{H}_+ = D(A)$ and $(u, v)_+ = (Au, Av)$. This will be a Hilbert–Schmidt rigging if and only if A has a discrete spectrum and its eigenvalues $\{\lambda_j | j = 1, 2, \dots\}$ satisfy

$$\sum_{j=1}^{\infty} \lambda_j^{-2} < \infty \tag{1.3}$$

Note that only separable Hilbert space $\mathcal{H}$ may have a Hilbert–Schmidt rigging in the sense described here. But this is the only case which we need in applications.

A2.2. Generalized eigenvectors.

First recall a general formulation of the spectral theorem for self–adjoint operators (see e.g. [5] or [32]).

Theorem 2.1. *Let A be a self–adjoint operator in a Hilbert space $\mathcal{H}$. Then there exists a measure space (M, μ), a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$ and a real–valued measurable function a on M which is defined and finite almost everywhere such that*

(i) $\psi \in D(A)$ if and only if $a(\cdot)(U\psi)(\cdot) \in L^2(M, d\mu)$

(ii) If $\varphi \in U(D(A))$ then $(UAU^{-1}\varphi)(m) = a(m)\varphi(m)$.

In other words A can be represented as a multiplication operator M_a given by $(M_a\varphi)(m) = a(m)\varphi(m)$ in $L^2(M, d\mu)$ with a real–valued measurable and almost everywhere finite function a. More exactly $A = U^{-1}M_aU$ with a unitary U. Let us recall that under the given conditions the operator M_a with the natural domain

$$D(M_a) = \{\varphi | \varphi \in L^2(M, d\mu),\ a\varphi \in L^2(M, d\mu)\}$$

is self–adjoint.

Now let us consider a rigging (1.1) of $\mathcal{H}$. Let A be a self–adjoint operator in $\mathcal{H}$. Suppose further that we are given a measure space (M, μ) and a vector–valued function $\Phi : M \to \mathcal{H}_-$ (which may be actually defined almost everywhere) with values in the negative space of the rigging.

DEFINITION 2.2. A vector–valued function $\Phi : M \to \mathcal{H}_-$ is called a *complete orthonormal system of generalized eigenvectors* (or *eigenfunctions*) of the operator A if the following conditions are fulfilled:

(i) for any $h_+ \in \mathcal{H}_+$ the function $m \mapsto (h_+, \Phi(m))$ on M belongs to $L^2(M, d\mu)$;

(ii) the map $h_+ \mapsto (h_+, \Phi(\cdot))$ can be extended to a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$ which gives a spectral representation of A as in Theorem 2.1.

The reader can find motivations and explanations of this definition in [5]. Let us remark only that $\Phi(m)$ is really a generalized eigenfunction of A with an eigenvalue $a(m)$ in a reasonable sense. For example if we take any complex–valued Borel function $f : \mathbb{R} \to \mathbb{C}$ then

$$(\Phi(m), \bar{f}(A)g) = f(a(m))(\Phi(m), g) \tag{2.1}$$

for any $g \in \mathcal{H}_+ \cap \bar{f}(A)^{-1}\mathcal{H}_+$ (i.e. $g \in \mathcal{H}_+ \cap D(\bar{f}(A))$ and $\bar{f}(A)f \in \mathcal{H}_+$) and for almost every $m \in M$. In particular

$$(\Phi(m), Ag) = a(m)(\Phi(m), g)$$

for any g such that $g \in \mathcal{H}_+$ and $Ag \in \mathcal{H}_+$, and for almost every $m \in M$.

Actually the set $M_0 \subset M$ where all relations (2.1) are true (and such that $\mu(M - M_0) = 0$) may be choosen independent of g provided $\mathcal{H}_+$ is separable (see [5], Proposition 2.7 in Supplement 1).

Now we shall remind the main result about riggings and generalized eigenfunctions. It is due to Ju. M. Berezanskii but in a weaker form it was proved earlier by I.M. Gelfand and A.G. Kostyuchenko [17].

Theorem 2.3. *Given a Hilbert–Schmidt rigging (1.1) of $\mathcal{H}$ and a self–adjoint operator A in $\mathcal{H}$, there exists in (1.1) a complete orthonormal system of generalized eigenvectors for the operator A.*

A simple proof can be found in [5]. Remark that the condition on the rigging (1.1) to be Hilbert–Schmidt is necessary in a natural sense (see [4]).

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On the Many Body Problem in Quantum Mechanics

Avy Soffer*

Section 1. Introduction

The aim of these lectures is to describe some of the modern mathematical techniques of N–body Scattering and with particular mention of their relations to other fields of analysis.

Consider a system of N quantum particles moving in $\mathbb{R}^n$, interacting with each other via the pair potentials V_α; the Hamiltonian (with center of mass removed) for such a system is given by

$$H = -\Delta + \sum_{i<j} V_{ij}(x_i - x_j) \quad \text{on} \quad L^2(\mathbb{R}^{n\,N-n}) \ .$$

Here $1 \leq i,\, j \leq N$, $x_i \in \mathbb{R}^n$. $-\Delta$ is the Laplacian on $L^2(\mathbb{R}^{n\,N-n})$ with metric

$$x \cdot y = \sum_{i=1}^{N} m_i x_i \cdot y_i \qquad ; \quad m_i > 0 \ .$$

The m_i are the masses of the particles. The main problem of scattering theory is to describe the spectral properties of H and find the asymptotic behavior of $e^{-iHt}\varphi$ for $\varphi \in L^2$, as $t \to \pm\infty$.

There are two reasons for that: one, the behavior is much simpler as $t \to \pm\infty$. Secondly it determines the full properties of the system. Since the

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sum $\sum\limits_{i<j} V_{ij}$ does not vanish as $|x| \to \infty$ in certain directions, the perturbation of $-\Delta$ is not negligible at infinity. The spectral properties and asymptotic behavior of H are therefore radically different than that of $-\Delta$.

This is the generic multichannel problem. There are many different asymptotic behaviors possible, depending on the choice of φ. Thus the main theorem can be phrased as: given $\varphi \in L^2(\mathbb{R}^n)$, find hamiltonians H_a and functions $\varphi_a^\pm$, s.t.

$$e^{-iHt}\varphi - \sum_a e^{-iH_a t}\varphi_a^\pm \approx 0 \qquad \text{as } t \to \pm\infty \; .$$

Accepting the physicist's dogma that every state of the system is described asymptotically in terms of *particles* (or bound clusters of particles) we conclude that the only possible H_a are the subhamiltonians of the system:

$$\begin{aligned} H_a &= H - I_a \\ I_a &\equiv \sum_{(i,j)\subset a} V_{ij}(x_i - x_j) \end{aligned}$$

and a stands for arbitrary disjoint cluster decomposition of $\{1, 2, \ldots, N\}$.

I_a is called the intercluster interaction. The Hamiltonian that describes the bound clusters of a decomposition a, is denoted by H^a. Not much is known for Multichannel Non Linear Scattering; see however [Sof-We and cited ref.].

The approach to studying $e^{-iHt}\psi$ for large $|t|$ is by first reducing the problem via channel decoupling (or other methods) to the study of the localization in the phase space of $e^{-iHt}\psi$. Then, we develop a theory of propagation in the phase space for H. The channel decoupling is achieved by constructing a partition of unity of the space, with two main properties: one, on the support of each member of the partition the motion $e^{-iHt}\psi$ is simple (= one channel) and can be described by one fixed hamiltonian. The second property is that the boundary of the partitions is localized in regions where we can prove that no propagation of $e^{-iHt}\psi$ is possible there for large times; in this way we conclude that no switching back and forth between channels is possible as $|t| \to \infty$ which implies the desired results.

The first part, based on the construction of partitions of unity relies mainly on geometric analysis combined with the kinematics of (freely) moving particles. Different techniques are now known, each with its own importance, and I will describe some of the main constructions. The second part of the proof is analytic; it provides an approach to finding the asymptotic behavior of $e^{-iHt}\psi$ as $|t| \to \infty$, which is *complementary to that of stationary phase*. As I will describe below it replaces the (central) notion of *oscillation* by that of *microlocal monotonicity*. The distinctive feature of this approach allows the study of general pseudo differential operators H on equal footing with constant coefficient operators.

The first proof of Asymptotic Completeness (AC) for N-body systems along these lines was given in [Sig-Sof1]. Since then, different proofs were developed, with new useful implications [Der1, Kit, Gr, Ta] (see also [En2, Ger2-3]). Further developments concentrated on the long range problem. The three body case was first solved by Enss [En2]. (See also [Sig-Sof3].) Local decay and minimal and maximal velocity bounds were proved for N-body hamiltonian, including ones with time dependent potentials in [Sig-Sof2]. This approach is further utilized in [Sk, FrL, Ger2 , Ger-Sig, H-Sk]. A method of dealing with the problem of AC for long range many body scattering is developed in [Sig-Sof4,5]; the case of $N = 4$ is solved there.

A final comment; the phase space approach to N-body scattering originated with the fundamental works of Enss [En1,2]. A comprehensive description of the Enss method can be found in [Pe], including applications to many problems in spectral theory. References of many of by now classical results, including the works of Mourre, until about 1983 can be found in [CFKS]. We refer the reader to this book also as the basic reference used here on spectral and scattering theory.

Section 2. Microlocal Propagation Theory

Let H be a self adjoint operator on $L^2(\mathbb{R}^n)$ arising from the quantization of a classical Hamiltonian h. By solving the Hamilton-Jacoby equations for h it makes sense to talk about the classical trajectories (or bi-characteristics) of h (or H). As $t \to \pm\infty$ the (unbounded) trajectories concentrate, in general, in a certain set of the phase space.

DEFINITION 2.1. A bounded p.d.o. j with symbol homogeneous of degree 0 in x is said to be supported away from the *propagation set* (at energy E) of H if the following estimate holds

$$\int_{\pm 1}^{\pm\infty} \| \frac{1}{\langle x\rangle^{1/2}} j\, e^{-iHt}\psi\|^2 dt \leq c\|\psi\|^2 \quad \text{for all} \quad \psi = E_\Omega(H)\psi \ .$$

Here $\langle x\rangle^2 \equiv 1 + x^2$, $E_\Omega(H)$ is the spectral projection of H, with Ω any sufficiently small interval containing E.

Our aim is to identify the (conical) set PS_E of the phase space, with the property that any j is supported away from the propagation set in the sense of the above definition if and only if it is supported away from PS_E. We can therefore think of PS_E as the *propagation set of* H *at energy* E.

The main tool to proving that a given conical set $\tilde{K}$ is away from the propagation set PS_E will be to prove (microlocal) monotonicity of the flow generated by H in $\tilde{K}$.

The claim is that the *classical* flow generated by H is moving out of any such $\tilde{K}$ monotonically in t, for large t. By finding a lower bound for this monotone flow in $\tilde{K}$ we can then absorb the effects of quantization and other potential perturbations of H.

I chose to describe the above approach first when applied to $H = -\Delta$, and along the way prove some known and new smoothing estimates for $-\Delta$. The proofs are easy but allow the introduction of some of the other fundamental notions and arguments repeatedly used later.

DEFINITION 2.2. The Heisenberg derivative of an operator family $F(t)$, $DF(t)$, w.r.t. to H is defined by

$$DF(t) \equiv i[H, F] + \frac{\partial F}{\partial t} \ ;$$

DEFINITION 2.3. A bounded family of linear operators $F(t)$ on $L^2(\mathbb{R}^n)$ is called a *propagation observable* for H if its Heisenberg derivative is positive –

lower order terms. For some $\theta > 0$:

$$DF(t) \geq \theta B^*B - O((B^*B)^{1-\varepsilon}) - O(L^1(dt))$$

we then say that $DF(t)$ majorates B^*B ($DF(t) \geq \theta B^*B$).

Basic Lemma 2.4 *Let $F(t)$ be a propagation observable which majorates B^*B. Then*

$$\int_{\pm 1}^{\pm\infty} \|Be^{-iHt}\psi\|^2 dt \leq c\|\psi\|^2 \ .$$

The proof follows by the fundamental theorem of calculus and Heisenberg equations of motion:

$$\frac{d}{dt}(e^{iHt}\psi, F\, e^{-iHt}\psi) = (e^{iHt}\psi, DF\, e^{-iHt}\psi) \ . \qquad \blacksquare$$

The Basic Lemma reduces the proof that a given j is supported away from PS to finding a propagation observable majorating $j^*\langle x\rangle^{-1}j$. When F is chosen to be a p.d.o., one can often use Görding's inequality to check majoration, which reduces the problem to finding a lower bound for the Poisson bracket $\{h, f\}$.

Theorem 2.5 (Microlocal Smoothing Estimate) *Let j be a bounded homogeneous of degree* 0 *(in x) symbol, with support away from*

$$PS \equiv \{(x,\xi) \in T^*X \mid x \| \xi\} \ .$$

Then

a) $$\int_0^T \left\| \frac{1}{\langle x\rangle^{1/2}} J\langle p\rangle^{1/2} e^{+i\Delta t}\psi \right\|^2 dt \leq C_T\|\psi\|^2 \quad \psi \in L^2(\mathbb{R}^n)$$

b) $$\int_0^T \| \frac{1}{\langle x\rangle^{1/2+\varepsilon}} \langle p\rangle^{1/2} e^{+i\Delta t}\psi\|^2 dt \leq c_T\|\psi\|^2 \ .$$

Furthermore, in case the dimension $n \geq 3$, C_T can be chosen independent of T. The same is true in any dimension if $\hat{\psi}(\xi)$ is supported away from 0.

REMARK. Part b) of the theorem is known as local smoothing estimate. If was proved in [Co-S, Sj, V] (see also [Be-K, G-V2, Ka-Ya]) and found since then many important applications in both linear and nonlinear PDE see e.g. [JSS], [KPV].

PROOF. The proof for a general H replacing Δ is given in [So2]. Here I sketch the main steps: By the Basic Lemma we have to find operators F_1, F_2 bounded and s.t.

$$DF_1 \geq \langle x\rangle^{-1/2} J\langle p\rangle J^* \langle x\rangle^{-1/2} + O(1)$$

and

$$DF_2 \geq \langle x\rangle^{-1/2-\varepsilon/2}\langle p\rangle\langle x\rangle^{-1/2-\varepsilon/2} + O(1)$$

where $O(1)$ stands for an operator of order zero (in ξ). Using p.d. calculus it is easy to check that

$$F_i \equiv \hat{\gamma}_i \equiv (\hat{x}_i \cdot \hat{p} + \hat{p} \cdot \hat{x}_i)\frac{1}{2} \qquad i = 1, 2$$

satisfy both of the above;

$$\hat{x}_i \equiv x/(1 + x^2 + g_i(x))^{1/2} \qquad \hat{p} \equiv p/\langle p\rangle \quad g_1(x) \equiv 0, g_2(x) \equiv |x|^{2-\varepsilon} .$$

REMARK 1.. The original proofs of b) uses stationary phase analysis, and therefore does not extend to cases where the kernel of e^{-itH} is not explicitly constructible, e.g. $H = -\Delta + V$, V singular. The above argument trivially extends to such general H.

REMARK 2. The above theorem shows that the notion of propagation set is relevant also for finite time behavior of $e^{-iHt}\psi$

The operator $\hat{\gamma}$ comes from regularizing the operator $\gamma = \frac{1}{2}(\hat{x} \cdot p + p \cdot \hat{x})$. Different versions of γ appeared in scattering theory [L, M1-2, M-R-S].

Its centrality for the N–body problem was first realized in [Sig-Sof1]. To see its importance, let us compute the Poisson bracket of ξ^2 with the symbol of γ

$$\{\xi^2, \hat{x}\cdot\xi\}_{PB} = 2\xi \cdot \nabla_i(\hat{x}\cdot\xi) = \frac{2}{\langle x\rangle}(\xi^2 - (\hat{x}\cdot\xi)^2)$$

and it is clear that the above bracket is positive $(0(\frac{1}{\langle x\rangle}\xi^2))$ iff (x,ξ) is localized away from $\{\hat{x}\|\xi\}$. We can therefore identify the PS of $-\Delta$ with $\{\hat{x}\|\xi\}$, which is not surprising since $x(t) = x_0 + 2\xi t$ are the classical trajectories of $-\Delta$, and they concentrate where $\frac{x}{t} = 2\xi$.

Section 3. Hamiltonians and Kinematics

Consider an N–body system in the physical space R^ν. The configuration space in the center–of–mass frame is

$$X = \{x \in R^{\nu N} \mid \sum_{i=1}^{N} m_i x_i = 0\} \tag{3.1}$$

where $x = (x_1, \ldots, x_N)$ with $x_i \in \mathbb{R}^\nu$, with the inner product

$$\langle x, y\rangle = 2\sum_{i=1}^{N} m_i x_i \cdot y_i \ . \tag{3.2}$$

Here $m_i > 0$ are masses of the particles in question. The *Schrödinger operator* of such a system is

$$H = -\Delta + V(x) \quad \text{on} \quad L^2(X) \ .$$

Here Δ is the Laplacian on X and

$$V(x) = \Sigma V_{ij}(x_i - x_j) \ ,$$

where (ij) runs through all the pairs satisfying $i < j$.

We assume that the potentials V_{ij} are real and obey: $V_{ij}(y)$ are Δ_y–compact. It is shown in [Com] (see also [CFKS]) that under this condition Kato theorem applies and H is self–adjoint on $D(H) = D(\Delta)$. Moreover, by a simple application of Hölder and Young inequalities and by a standard approximation argument (see [CFKS]) one shows that if V_{ij} are Kato potentials, i.e.

$$V_{ij} \in L^r(R^\nu) + (L^\infty(R^\nu))_\varepsilon \text{ ,where } r > \frac{\nu}{2} \text{ if } \nu > 4 \text{ and } r = 2 \text{ if } \nu \leq 3 \text{ ,}$$

and the subscript ε indicate that the L^∞–component can be taken arbitrarily small, then V_{ij} is Laplacian compact.

Now we describe the decomposed system. Denote by $a, b, \ldots,$ *partitions* of the set $\{1, \ldots, N\}$ into non–empty disjoint subsets, called clusters. The relation $b < a$ means that b is a refinement of a and $b \neq a$. Then $a_{\min}$ is the partition into N clusters $(1), \ldots, (N)$. Usually, we assume that partitions have at least two clusters. $\#(a) \equiv |a|$ denotes the number of clusters in a. We also identify pairs $\ell = (ij)$ with partitions having $N - 1$ clusters: $(ij) \leftrightarrow \{(ij)(1)\ldots(i)\ldots(j)\ldots(N)\}$. We emphasize that the relation $\ell \subsetneq a$ (resp. $\ell \subseteq a$) with $\ell = (ij)$ is equivalent to saying that i and j belong to different clusters (resp. to same cluster) of a.

We define the *intercluster interaction* for a partition a as $I_a =$ sum of all potentials linking different clusters in a, i.e.

$$I_a = \sum_{\ell \not\subseteq a} V_\ell \ . \tag{3.3}$$

For each a we introduce the truncated Hamiltonian:

$$H_a = H = I_a \ . \tag{3.4}$$

These operators are clearly self–adjoint. They describe the motion of the original system broken into non–interacting clusters of particles.

For each cluster decomposition a, define the configuration space of relative motion of the clusters in a:

$$X_a = \{x \in X \mid x_i = x_j \text{ if } i \text{ and } j \text{ belong to same cluster of } a\}$$

and the configuration space of the internal motion within those clusters:

$$X^a = \{x \in X \mid \sum_{j \in C_i} m_j x_j = 0 \text{ for all } C_i \in a\} \ .$$

Clearly X_a and X^a are orthogonal (in our inner product) and they span X:

$$X = X^a \oplus X_a .$$

Given generic vector $x \in X$, its projections on X^a and X_a will be denoted by x^a and x_a, respectively.

If i and j belong to some cluster of a, then $x_i - x_j = (\pi^a x)_i - (\pi^a x)_j$, where π^a is the orthogonal projection in X on X^a. This elementary fact and the fact that $-\Delta = \langle p, p\rangle$ with $p = -i\nabla_x$ (see equation (3.6) and the sentence after it) yield the following decomposition:

$$H_a = H^a \oplus 1 + 1 \oplus T_a \text{ on } L^2(X) = L^2(X^a) \oplus L^2(X_a) . \tag{3.5}$$

Here H^a is the Hamiltonian of the non–interacting a–clusters with their centers–of–mass fixed at the originating on $L^2(X^a)$, and $T_a =$ $-$(Laplacian on X_a), the kinetic energy of the center–of–mass motion of those clusters.

The eigenvalues of H^a, whenever they exist, will be denoted by ε^α, where $\alpha = (a, m)$ with m, the number of the eigenvalue in question *counting the multiplicity*. For $a = a_{\min}$, we set $\varepsilon^\alpha = 0$. The set $\{\varepsilon^\alpha,\ \text{all } \alpha\}$ is called the *threshold set* of H and ε^α are called the *thresholds* of H. For $\alpha = (a, m)$ we denote $|\alpha| = |\alpha|$ and $a(\alpha) = a$.

Our method is based on localization of operators in the *phase–space* $T^*X = X \times X'$. Hence and henceforth, the prime stands for taking dual of the space in question. The dual (momentum) space X' is identified with

(3.6)
$$X' = \{k \in R^{\nu N} \mid \Sigma k_i = 0\} \text{ with the inner product } \langle k, u\rangle = \Sigma \frac{1}{2m_i} k_i \cdot u_i .$$

Thus $|k|^2$ is the symbol of $-\Delta$ and $-\Delta = |p|^2$. We use extensively the natural bilinear form on $X \times X' : \langle x, k\rangle = \Sigma x_i \cdot k_i$. Given generic vector $k \in X'$, its projections on X'_a and $(X^a)'$ will be denoted by k_a and k^a, correspondingly. Accordingly, the momenta canonically conjugate to x_a and x^a and corresponding to k_a and k^a will be denoted by p_a and p^a, respecitvely. Thus $T_a = |p_a|^2$. Using the bilinear form above we define the generator of dilations as

$$A = \frac{1}{2}(\langle p, x\rangle + \langle x, p\rangle)$$

and the self–adjoint operator γ,

$$\gamma = \frac{1}{2}(\langle p, \hat{x}\rangle + \langle \hat{x}, p\rangle) \ ,$$

associated with the angle between the velocity and coordinate. Again, for decomposed systems A splits into the operator

$$A^a = \frac{1}{2}(\langle p^a, x^a\rangle + \langle x^a, p^a\rangle)$$

corresponding to the internal motion of the clusters, and the operator

$$A_a = \frac{1}{2}(\langle p_a, x_a\rangle + \langle x_a, p_a\rangle)$$

corresponding to the motion of the centers–of–mass of the clusters.

Finally, we mention some notation. We denote $E_\Delta = f(H \in \Delta)$ for an interval $\Delta \subset R$ and set $H_\Omega = HE_\Omega$. P will stand for the orthogonal projection the pure point spectral subspace of H.

Section 4. Partitions of Units

The configuration space of N particles moving in ν dimensions with the center of mass removed is

$$X = \{x \in \mathbb{R}^{\nu N} \mid \sum_{i=1}^{N} m_i x_i = 0\} \ .$$

Here the m_i's are the masses of the particles and $x_i \in \mathbb{R}^\nu$ their position. Let a be any disjoint cluster decomposition of $\{1, 2, \dots, N\}$. Denote by X_a the subspace of X given by

$$X_a = \{x \mid x_i - x_j = 0 \ \text{ if } (ij) \text{ belong to the same cluster in } a\} \ .$$

Define $|x|_a = \min_{(ij)\not\subset a} |x_i - x_j|$. We can now prove the existence of a *two cluster* partition of unity $\{j_a\}$.

Proposition 4.1 *There exists a partition of unity of* X, $\{j_a\}_{\#(a)=2}$ *s.t.*

i) $\sum\limits_a j_a^2(x) = 1$ *on* X.

ii) Each $0 \le j_a(x) \le 1$ *is smooth and homogeneous of degree* 0 *outside the unit ball of* X.

iii) $j_a(x) = 1$ *for some neighborhood of* $X_a/\{|x| \le 1\}$

iv) $j_a(x) = 0$ *for* $|x|_a \le \varepsilon_a |x|$ *for some positive* ε_a.

The proof follows by finding a covering of the unit sphere of X by neighborhoods of X_a, $\#(a) = 2$. For each member of the covering we then associate a smooth characteristic function which we then extend by homogeneity to $|x| > 1$ and in a smooth but otherwise arbitrary way to $|x| < 1$. We then normalize these functions so that $\sum_a j_a^2 = 1$.

The partition constructed in the proposition above is called a two–cluster partition of unity and it appeared already in the Haag-Ruelle theory [GJ]. By generalizing the construction above to neighborhoods of X_a, a any two or more cluster decomosition we can construct a k-cluster decomposition $\{j_a\}_{\#(a) \le k}$ to obtain

Proposition 4.2 *There exists a partition of unity of* X, $\{j_a\}$ *s.t.*

i) $\sum_{\#(a) \le k} j_a^2(x) = 1$

ii) Each $0 \le j_a(x) \le 1$ *is smooth and homogeneous of degree zero outside* $\{|x| \le 1\}$

iii) $j_a(x) = 1$ *for some neighborhood of* $X_a/\{|x| \le 1\}$

iv) supp $j_a(x) \subset \{|x|_a \ge \delta |x|\}$ *for some* $\delta > 0$.

This kind of partitions will allow us to use induction on number of cluster decompositions. Such partitions were used extensively in [Ag, Sig-Sof 1]. Partitions of unity are the basic tool to decouple channels of propagation from each other. In spectral geometry they are used to decouple different neighborhoods of infinity from each other [FHP1-2], see also [CFKS chapter 11].

Theorem 4.3 (Hunziker Van Winter Zislin) (HVZ)

$$\sigma_{\mathrm{ess}}(H) = \bigcup_a \sigma_{\mathrm{ess}}(H_a) \qquad \#(a) = 2\ .$$

PROOF. [Sig 3] Using trail functions it is easy to prove that

$$\sigma_{\text{ess}}(H) \supseteq \bigcup_a \sigma_{\text{ess}}(H_a) \ .$$

To prove that $\sigma_{\text{ess}}(H) \subseteq \bigcup_a \sigma_{\text{ess}}(H_a)$ we use the two cluster partition of unity $\{j_a\}_{\#(a)=2}$:

$$\begin{aligned} H &= \frac{1}{2} \sum_a (j_a^2 H + H j_a^2) = \sum_a j_a H j_a + \sum_a j_a [j_a, H] \\ &= \sum_a j_a (H_a + I_a) j_a + \frac{1}{2} \sum_a [j_a, [j_a, H]] \ . \end{aligned}$$

Since $H = -\Delta + V(x)$ it follows that

$$[j_a, [j_a, H]] = [j_a, [j_a, -\Delta]] = O(|x|^{-2})$$

by property (ii) of the partitions j_a. Furthermore, by property (iv) of j_a we conclude that

$$j_a I_a = O(|x|^{-\mu}) \ .$$

Hence

$$H = \sum_a j_a H_a j_a + O(|x|^{-\mu}) + O(|x|^{-1}) \ .$$

Since $O(|x|^{-\mu})$ is $-\Delta$ (and hence H) compact it follows by Weyl's Lemma that

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(\sum_a j_a H_a j_a)$$

from which the result follows by an elementary argument. ∎

In the study of the asymptotic behavior of multichannel systems the decoupling by $j_a(x)$ is not sufficient (see however [D-S,] [Sig3]). This is due to the fact that the flow under H can move through the boundary of one partition into the other and back. Therefore, to achieve a true decoupling between channels a new approach is needed.

This has been done in [Sig-Sof1] by introducing a phase space partition of unity with the boundary of the partitions localized away from the PS_E of H. In this way the decoupling is achieved between channels, modulo quantum oscillations capable of tunneling through the classically forbidden regions. The contribution of such oscillations is then controlled by the microlocal propagation estimates as explained in the previous section.

Proposition 4.4 (phase space partition of unity) *There exists a partition of* $X \times X'(\equiv T^*X), j_{a,E}(x,\xi_a)$ *s.t.*

i) $\sum_{\#(a)\geq 2} j^2_{a,E}(x,\xi_a) = 1$

ii) Each $0 \leq j_a \leq 1$ *is homogeneous of degree zero in* x

iii) supp $j_a(x,\xi_a) \subset \{|x|_a > \delta|x|\}$ *and* $j_a = 1$ *in some neighborhood of* X_a.

iv) supp $\nabla_x j_a(x,\xi_a)$ *is away from* PS_E.

The construction of such partition can be found in [Sig-Sof1]. The main building blocks of such a partition are $\chi_{x_0}(x)$, and $\tilde{\chi}_{\xi_0}(\xi)$; $\chi_{x_0}(x)$ is a (smooth) characteristic function of a cone in x near the direction x_0 and $\tilde{\chi}_{\xi_0}(\xi)$ in ξ near ξ_0; the support of the $\tilde{\chi}_{\xi_0}$ is taken to be either strictly inside that of χ_{x_0} or strictly outside. It is then easy to see that $\nabla_x(\chi_{x_0}(x)\tilde{\chi}_{\xi_0}(\xi))$ is supported where $x \mid\mid \xi$.

The main application of the above partition is

Theorem 4.5 (Channel Decoupling) *Let* E *be a given non threshold energy of* H. *Then AC follows from the propagation theorem on* PS_E.

PROOF.

$$\begin{aligned} e^{-iHt}\psi &= \sum_a j_a^2 e^{-iHt}\psi \\ &= \sum_a e^{-iH_a t}(e^{iH_a t} j_a^2 e^{-iHt}\psi) \ . \end{aligned}$$

It is therefore left to show that

$$e^{iH_a t} j_a^2 e^{-iHt}\psi \to \psi_a^{\pm} \quad \text{as} \quad t \to \pm\infty \ .$$

By Cook's method this is reduced to proving that

$$\int \|(H_a j_a^2 - j_a^2 H)e^{-iHt}\psi\|_2 dt \leq c\|\psi\| \ .$$

But

$$H_a j_a^2 - j_a^2 H = H_a j_a^2 - j_a^2 H_a - j_a^2 I_a$$

$j_a^2 I_a = 0(|x|^{-\mu})$ and for $\mu > 1$ the above estimate holds by local decay since E is away from the thresholds (see section 7).

$$H_a j_a^2 - j_a^2 H_a = \mathcal{O}(|x|^{-1})\tilde{j}_a(x, \xi_a)$$

where $\tilde{j}_a(x, \xi_a)$ lives away from the PS_E by property iv). Applying the propagation theorem to this term the result follows. ■

A very interesting partition of unity of X was constructed in [Gr]. (A simpler construction is given in [Der3].) It is an N–cluster partition of unity with further property on the boundary which implies monotonicity of the flow there in a certain sense:

Proposition 4.6 (Monotonic Partition of Unity) *There exists an N–cluster partition of unity $\{q_a\}$; furthermore, the derivative of q_a along x_a on the boundary is nonnegative:*

$$\sum_a x_a \otimes \nabla q_a(x) \geq 0 \ .$$

The idea behind the construction of q_a is the observation that $\nabla F(|x| \leq c) = -\nabla F(|x| \geq c)$ and q_a is a product of such F's with $x \to x_b^a$ and $c \to c_b^a$. One can then cancel the negative terms in the sum $\sum_a x_a \otimes \nabla q_a(x)$ by the corresponding positive ones, using that

$$x_a \otimes \nabla F(|x_a^b| \geq c_a^b) + x_b \otimes \nabla F(|x_a^b| \leq c_a^b) \geq 0 \ .$$

Using the above partition, one can construct new propagation observables with monotone Heisenberg derivative by "clustering" (see [FHP1] for the first such procedure) the corresponding two body analog: In one channel nonlinear scattering one uses the propagation observable $(p - \frac{x}{t})^2 + V(\varphi)$ ($V(\varphi)$ stands for the nonlinearity), to derive the pseudo-conformal identities for the NLS equation [G-V1]. For N-body systems, using q_a one replaces $\frac{x}{t}$ by $v(x,t)$ where

$$v(x,t) \equiv \sum_a \frac{x_a}{t} w_a$$

where w_a are appropriately scaled (in time) q_a. One can then show that

$$-K \equiv (p - v(x,t))^2 + V(x)$$

is a propagation observable. Other observables can also be constructed, e.g. $\sum\limits_a w_a \frac{A_a}{t} w_a$.

In the study of Long Range Scattering one is led to study the asymptotics of N–body systems at *threshold energies*. This requires zooming on zero velocities (coming from the critical points of the symbols of $E_a + H_a$) which we do by scaling in the time variable (see section 8). A natural partition of unity used in such an analysis is multiscaled [Sig-Sof5]:

Proposition 4.7 (Multiscale partition of unity) *There exists a k–cluster $\{j_a\}$ partition of X, depending on time, s.t. on support $j_a(x,t)$ $|x|_a \geq \delta_1 t^{\alpha(a)}$ and $|x^a| < \delta_2 t^{\beta(a)}$ (where of course $\alpha(a) > \beta(a)$). The partition is multiscaled since $\alpha(a) > \alpha(a')$ if $a \subset a'$.*

Just like with the monotonic partition it is possible to cluster operators using the multiscale partition leading to new propagation observables.

Section 5. The Channel Expansion

Recall that in the two body case the dilation generator A has positive commutator with H $(\equiv -\Delta + V(x))$ for sufficiently regular $V(x)$ and when the commuator is localized away from the thresholds of H. The question arises whether we can "cluster" A to prove similar bounds for the N–body case. (See [Hu1] for the case of classical mechanics.) This was first shown by Mourre in the case $N = 3$ and later generalized for all N in [PSS, FH1, BG2]. In [Sig-Sof1] it is shown that the commutator of H with certain global observables, including A and γ, can be approximated to arbitrary accuracy by a sum of contributions from the *open channels* of the system, at the given energy. This became a central technical result in the study of spectral and scattering of N-body systems. It implies, for example, that the Mourre estimate holds at non threshold energies with *precise lower and upper bounds* on the commutator $i[H,A]$. This section is devoted to proving the theorem using an important simplifying idea of Hunziker [Hu2].

The channel expansion theorem state that certain commutators with H, as well as the identity can be approximated, arbitrarily close, by a finite sum of contributions of open channels only. The approximation gets better as we add more and more *open* channels to the sum and shrink the interval Δ around the energy E. We first need a few definitions.

Let $\{\varepsilon_j(a)\}_{j=1}^{\infty}$ be the eigenvalues of H^a, with corresponding projections $P_j(a)$. Here the $P_j(a)$ are all chosen to be finite dimensional. Denote by $P(a) = \sum_j P_j(a) = 1 - \overline{P}(a)$ the projection on $\mathcal{H}_{p.p}(H^a)$ and $\overline{P}(a)$ is the projection on the continuous spectral subspace of H^a.

$$P^N(a) = \sum_{j=1}^{N} P_j(a) \quad \text{and} \quad \overline{P}^N(a) = 1 - P^N(a)\ .$$

We drop the index a when $H^a \equiv H$ $(\#a = 1)$. For the smooth spectral projection F_Δ of H^a, we let

$$F_\Delta^N = F_\Delta \overline{P}^N(a)\ .$$

A cluster decomposition a and a choice of eigenvalue for H^a determines a channel α. So, we let P_α be the projection on the channel bound state, p_α be the channel momentum ($p_\alpha = p_{a(\alpha)}$), T_α its kinetic energy ; $T_\alpha = T_{a(\alpha)} = |p_{a(\alpha)}|^2 = |p_\alpha|^2$, $m(\alpha) -$ order number of the eigenvalue $\varepsilon_j(\alpha)$.

Theorem 5.1 (Channel Expansion) *Given $E \in \mathbb{R}$, $\delta, \varepsilon > 0$ there exists integers $\{N_i\}_{i=1}^{N}$ and a* **finite** *set of channels C_ε:*

$$\alpha \in C_\varepsilon \quad \textit{iff} \quad m(a) \le N_{\#(\alpha)}\ .$$

For each $\alpha \in C_\varepsilon$ there exists a smooth characteristic function of the origin χ_α, with sharpness less than δ, and an interval $\Delta \supset E$ s.t.

i)

$$F^{N_1}_{\Delta_1} i[H,A] F^{N_1}_{\Delta_1} \stackrel{\varepsilon}{=} F^{N_1}_{\Delta_1}\Big(\sum_{\alpha\in C_\varepsilon}\ \sum_{S=(a_2,\dots,a_k,a(\alpha))} j_S 2p_\alpha^2 \chi_\alpha(p_\alpha^2+\varepsilon(\alpha)-E\Big)j_S^*\Big)F^{N_1}_{\Delta_1}$$

$$(F^{N_1}_{\Delta_1})^2 \stackrel{\varepsilon}{=} F^{N_1}_{\Delta_1}\Big(\sum_{\alpha\in C_\varepsilon}\ \sum_{S=(a_2,\dots,a_k,a(\alpha))} j_S \chi_\alpha(p_\alpha^2+\varepsilon(\alpha)-E)j_S^*\Big)F^{N_1}_{\Delta_1}$$

where $\Delta_1 \subset \Delta$, and for $S=(a_2,\dots,a_k,a(\alpha))$

$$j_S = \varphi^{a_1}_{a_2}\overline{P}^{N_2}_{a_2}\varphi^{a_2}_{a_3}\overline{P}_{a_3}\dots\varphi^{a_{k-1}}_{a_k}P\alpha\ .$$

PROOF. We sketch the proof before giving the details. The idea is to try to mimic the proof of the HVZ theorem, by reducing the problem to H_a using the two cluster partition of unity j_a. We then get

$$i[H,A] = \sum_{\substack{a\\ \#(a)=2}} j_a i[H,A] j_a + K$$

where K stands for a relatively compact operator (w.r.t. H). Next, we want to replace $i[H,A]$ by $i[H_a,A_a]$, using that

$$H = H_a + I_a \quad \text{and} \quad J_a I_a = K$$

and

$$A = A^a + A_a\ .$$

Doing that, we get

$$i[H,A] = \sum_{a_2} j_a i[H_a,A_a]j_a + \sum_{a_2} j_a i[H_a,A^a]j_a + K\ .$$

The first sum on the r.h.s. involves a "one body" commutator and produces the $2p_a^2$ term. It is left to consider $i[H_a,A^a]$, which we rewrite now as

$$i[H^a\otimes 1 + p_a^2\otimes 1, A^a] = i[H^a,A^a]\ .$$

This last commutator is exactly the same as $i[H, A]$ but for a subhamiltonian coming from some two cluster decomposition a. We can therefore *assume* it satisfies the theorem and proceed by induction to conclude the proof.

The difficulty lies at this stage: the original commutator is localized with $H \sim E$, but the new one, $i[H^a, A^a]$ has H^a localized near $E - 2p_a^2$ which varies over a large interval, in general, and can hit bound states of H^a for example, where the induction hypothesis is useless. To proceed, we use a resolution of the identity

$$1 = \sum_{j=1}^{N} P_j(a) + \overline{P}^N(a)$$

and study

(H1) $P_j(a)i[H^a, A^a]P_j(a)$

(H2) $\overline{P}^N(a)i[H^a, A^a]\overline{P}^N(a)$

(H3) $\overline{P}^N(a)i[H^a, A^a]P_j(a)$ (and its adjoint).

Case (H1) is shown to contribute zero by applying the virial theorem and localization using F_Δ. Case (H2) is treated by the induction hypothesis. Case (H3) is shown to be small in norm by compactness. The main simplification in the proof below compared to [Sig-Sof1] is that the induction hypothesis is formulated and used for H_a, rather than H^a. In this we follow Hunziker [Hu2].

The induction on clusters begins with $n = N$ and descends to 1. For $n = N$, H is reduced to $-\Delta$ where the proof is straightforward. For the sake of notation we only do the last step of the induction: proving it for H assuming it for all H_a, $\#(a) \geq 2$. Using the IMS localization formula:

$$B_{\Delta_1}^{N_1} \equiv F_{\Delta_1}^{N_1} i[H, A] F_{\Delta_1}^{N_1} \stackrel{\varepsilon}{=} \sum_{\#(a)=2} F_{\Delta_1}^{N_1} j_a i[H_a, A] j_a F_{\Delta_1}^{N_1}$$

for Δ_1 sufficiently small, and N_1 sufficiently large, depending on ε.

$$B_{\Delta_1}^{N_1} \stackrel{\varepsilon}{=} \sum_{a_2} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}(H_a) i[H_a, A] F_{\Delta_2}(H_a) j_a F_{\Delta_1}^{N_1}$$

for all $\Delta_2 \supset \Delta_1$, s.t. $F_{\Delta_2} F_{\Delta_1} = F_{\Delta_1}$.

To prove the last inequality we used that for all $\varepsilon' > 0$

$$\|F_{\Delta_1}^{N_1} j_a F_{\Delta_2}(H_a) - F_{\Delta_1}^{N_1} j_a\| < \varepsilon'$$

for $|\Delta_1| < \delta(\varepsilon')$ and $N_1 > N_1'(\varepsilon)$.

To prove it, observe that since $j_a J_a = K$ and $H = H_a + I_a$ then

$$j_a F_{\Delta_2}(H_a) = j_a F_{\Delta_2}(H) + K = F_{\Delta_2}(H) j_a + K \ .$$

Next, we use

$$i[H_a, A] = i[H^a, A^a] + 2p_a^2 \ .$$

Hence, using the resolution of identity $\sum_1^N P_j + F_\Delta^N = 1$ we get

$$\begin{aligned} B_{\Delta_1}^{N_1} &\overset{\varepsilon}{=} \sum_{a_2} F_{\Delta_1}^{N_1} j_{a_2} F_{\Delta_2}^{N_2} i[H_a, A] F_{\Delta_2}^{N_2} j_{a_2} F_{\Delta_1}^{N_1} \\ &+ \sum_{\substack{\alpha \\ \#(a)=2 \\ m(\alpha)\le N_1}} F_{\Delta_1}^{N_1} j_a P_\alpha 2p_\alpha^2 F_{\Delta_1}^2 (p_\alpha^2 + \varepsilon_\alpha) P_\alpha j_a F_{\Delta_1}^{N_1} \\ &+ \sum_{\substack{a_2 \\ m(\alpha)\le N_2 \\ m(\alpha')\le N_2 \\ \alpha\ne\alpha'}} F_{\Delta_1}^{N_1} j_{a_2} F_{\Delta_2} P_\alpha i[H_{a_2}, A] P_\alpha F_{\Delta_2} j_{a_2} F_{\Delta_1}^{N_1} \\ &+ \sum_{\substack{a_2 \\ m(\alpha)\le N_2}} F_{\Delta_1}^{N_1} j_{a_2} F_{\Delta_2}^{N_2} i[H_{a_2}, A] P_\alpha F_{\Delta_2} j_{a_2} F_{\Delta_1}^{N_1} + h.c. \end{aligned}$$

h.c. stands for hermitian conjugate.

The second term on the r.h.s. is derived by using the virial theorem:

$$P_\alpha i[H^a, A] P_\alpha = 0 \ .$$

The third term is zero, since P_α localizes H^a near ε_α (in F_{Δ_2}) and $P_{\alpha'}$ localizes H^a near $\varepsilon_{\alpha'} \ne \varepsilon_\alpha$ (in F_{Δ_2}). Therefore, since $\varepsilon_{\alpha'} \ne \varepsilon_\alpha$, choosing the sharpness of F_{Δ_2} sufficiently small either $F_{\Delta_2} P_\alpha = 0$ or $F_{\Delta_2} P_{\alpha'} = 0$ (or both).

The fourth term and its hermitian conjugate are made smaller than any $\varepsilon > 0$, by observing that

$$F_{\Delta_2}^{N_2} i[H_{a_2}, A] P_\alpha$$

is a (relatively) compact operator, since P_α is compact on $L^2(X^a)$. Hence, letting $|\Delta_2| \downarrow 0$, $N_2 \to \infty$, we see that $F_{\Delta_2}^{N_2} \xrightarrow{s} 0$ and hence

$$\|F_{\Delta_2}^{N_2} i[H_{a_2}, A]P_\alpha\| \to 0 \ .$$

One technical tool used here and in the above compactness arguments is reduction to the subspace $L^2(X^a)$ using the fibre representation for H_a is fibres of p_a:

$$F_\Delta(H_a) = \int_\oplus F_\Delta(H^a + \xi_a^2)\, d\xi_a \ .$$

Since H is semibounded from below the sum over ξ_a extends over a compact set. This allows us to use compactness arguments in $L^2(X^a)$ for each fibre ξ_a.

Similarly, we prove the channel expansion for the identity:

$$\begin{aligned}(F_{\Delta_1}^{N_1})^2 &= \sum_{a_2} F_{\Delta_1}^{N_1} j_{a_2}^2 F_{\Delta_1}^{N_1} \\ &\stackrel{\varepsilon}{=} \sum_{\substack{a \\ \#(a)=2}} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^2(H_a) j_a F_{\Delta_1}^{N_1} \ .\end{aligned}$$

Since $\|F_{\Delta_1}^{N_1} K F_{\Delta_1}^{N_1}\| \leq \varepsilon$ for all sufficiently small $|\Delta_1|$ and large N_1. Writing

$$F_{\Delta_2}^2(H_a) = F_{\Delta_2}^{N_2}(H_a)^2 + (F_{\Delta_2} P^{N_2}(a))^2$$

we set

$$\begin{aligned}(F_{\Delta_1}^{N_1})^2 = &\sum_{\substack{a \\ \#(a)=2}} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^{N_2}(H_a)^2 j_a F_{\Delta_1}^{N_1} \\ &+ \sum_{\substack{\#(a)=2 \\ m(\alpha)\leq N_2}} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^2(\varepsilon_\alpha + p_\alpha^2) j_a F_{\Delta_1}^{N_1}\end{aligned}$$

and we redefine

$$F_{\Delta_2}^2(\varepsilon_2 + p_\alpha^2) \equiv \chi_\alpha(p_\alpha^2 + \varepsilon_\alpha - E) \ .$$

Using the induction hypothesis and using local compactness to prove

$$F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^{N_2} \stackrel{\varepsilon}{=} F_{\Delta_1}^{N_1} j_a \overline{P}_a^{N_2}$$

the result follows. ■

Next we prove the *Localization Lemma*:

Lemma 5.2 *Let F_i, $i = 1, 2$ be smooth functions s.t. supp $F_1 \subset (-\delta/2, \delta/2)$ and supp $F_2 \subset [\frac{3}{4}\delta, \infty)$. Then*

$$F_1(|p| - \kappa)F_2(\pm\gamma - \kappa) = O(|x|^{-1}) .$$

PROOF. Pick F_3 so that supp $F_3 \subset (-(2/3)\delta, (2/3)\delta)$ and $F_3 = 1$ on supp F_1. Denote $g = F_3(|p| - \kappa)$, $F_1(p) = F_1(|p| - \kappa)$ and $F_2(\gamma) = F_2(\pm\gamma - \kappa)$. The operator $\gamma_g \equiv g\gamma g$ is symmetric and bounded:

$$|\langle \hat{x} \cdot pgu, gu\rangle \leq \|pgu\| \; \|\hat{x}gu\| \leq (\kappa + \frac{2}{3}\delta) \; \|gu\|^2 ,$$

where we have used that

$$\|\hat{x}f\| \leq \|f\| .$$

This shows that

$$\pm g\gamma g \leq \kappa + \frac{2}{3}\delta .$$

Now observe

$$F_1(p)F_2(\gamma) = F_1(p)(F_2(\gamma) - F_2(\gamma_g)) .$$

Using the Fourier representation

$$F(\gamma) = \int \hat{F}_2(s)e^{i\gamma s} \, ds$$

and using a continuity argument in order to extend the following result from $\zeta(\mathbb{R})$-functions to smooth bounded functions F_2 with C_0^∞ derivatives, we obtain

$$\begin{aligned} F_2(\gamma) - F_2(\gamma_g) &= \int_{-\infty}^{\infty} \hat{F}_2(s)(e^{i\gamma s} - e^{i\gamma_g s}) \, ds \\ &= -i \int_{-\infty}^{+\infty} ds \hat{F}_2(s)e^{i\gamma s} \int_0^s du \; e^{-i\gamma u}(\gamma - g\gamma g)e^{i\gamma_g u} . \end{aligned}$$

This implies

$$F_1(p)F_2(\gamma) = -i\int_{-\infty}^{\infty} ds\hat{F}_2(s)\int_0^s duF_1(p)e^{i\gamma(s-u)}(\gamma - g\gamma g)e^{i\gamma_g u} .$$

We commute $F_1(p)$ on the r.h.s. of this expression through $e^{i\gamma(s-u)}$. The result is

$$F_1(p)F_2(\gamma) = B_1 + B_2 ,$$

where

$$B_1 = -i\int_{-\infty}^{\infty} ds\hat{F}_2(s)\int_0^s du[F_1(p), e^{i\gamma(s-u)}](\gamma - g\gamma g)e^{i\gamma_g u}$$

and

$$B_2 = i\int_{-\infty}^{\infty} ds\hat{F}_2(s)\int_0^s due^{i\gamma(s-u)}F_1(p)\gamma(1-g)e^{i\gamma_g u} .$$

Next we show that

$$(\alpha) \qquad [F_1(p), \gamma] = O(|x|^{-1}) .$$

Using that

$$[F_1(p), \gamma] = \Sigma[p_i F_1(p), \frac{x_i}{\langle x\rangle}] + O(|x|^{-1})$$

and using the commutator formulas [Section 6] we obtain (α). Now the relation

$$[F_1(p), e^{i\gamma t}] = -i\int_0^t e^{i\gamma(t-s)}[F_1(p), \gamma]e^{i\gamma s}\, ds ,$$

Equation (α) and commuator formulas imply that

$$[F_1(p), e^{i\gamma t}] = O(|x|^{-1}\, t^2) .$$

Equations (α) and the above equation together with the relations

$$F_1(p)(1-g) = 0$$

and

$$\int_{-\infty}^{\infty} |\hat{F}_2(s)|\ |s|^n\, ds < \infty \quad \text{for} \quad n = 1, 2, 3 ,$$

imply

$$B_i = O(|x|^{-1}) \ , \quad i = 1, 2 \ .$$

■

Using the Localization Lemma, we can then sharpen the statement of the channel expansion theorem, by replacing $\chi_\alpha(p_\alpha^2 + \varepsilon(\alpha) - E)$ by $F_\alpha \chi_\alpha(p_\alpha^2 + \varepsilon(\alpha) - E) F_\alpha$, with

$$F_\alpha \equiv \chi(\gamma_{a(\alpha)} = \sqrt{E - \varepsilon(\alpha)}) \ .$$

Section 6. Some Operator Calculus

In this section we derive various estimates on functions of self-adjoint operators following [Sig-Sof2]. We begin with a few remarks.

Let A be a self-adjoint operator on $L^2(\mathbb{R}^n)$. If f is a measurable function with integrable Fourier transform then we define

$$f(A) = \int \hat{f}(s) e^{iAs} ds \tag{6.1}$$

where $\hat{f}(s)$ is the Fourier transform of $f(\lambda)$ and the limit defining the integral is taken in the strong sense. With some care this formula can be extended to a broader class of functions. For positive powers of positive operators we use the representation on $D(A^{[\alpha]+1})$

$$A^\alpha = \frac{\sin \pi(\alpha - [\alpha])}{\pi} \int_0^\infty \frac{w^{\alpha - [\alpha] - 1}}{A + w} \, dw A^{[\alpha]+1} \ , \tag{6.2}$$

where $[\alpha]$ is the integer part of α.

Expansion of Commutators

Let H and A be self-adjoint operators on the same Hilbert space $\mathcal{H}$. We assume that $D(A) \cap D(H)$ is dense in $\mathcal{H}$, and for some $n \geq 1$

(6.3) $\quad ad_A^k(H)$ extends to a bounded operator for all $1 \leq k \leq n$.

Here $ad_A^k(H) \equiv [\cdots [H, A], A], \cdots A] k$-times are defined initially as forms on $D(A) \cap D(H)$.

Property (F):

We consider a class of smooth functions f whose Fourier transforms $\hat{f}$ obey

$$\|\hat{f}^{(n)}\|_1 \equiv \int |\hat{f}(s)|\,|s|^n ds < \infty \,. \tag{6.4}$$

Here, $f^{(k)}$ stands for the k^{th} derivative of f. We derive Taylor-type expansions for the commutator $[H, f(A)]$.

Lemma 6.1 (Leibnitz Rule)

Let for some $n > 0$ H obey (6.3)*and f obey condition (F). Let $[H, f(A)]$ be defined as a form on $D(A^n)$. Then,*

$$[H, f(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H) + R_n(f) \tag{6.5}$$

in the form sense with the remainder $R_n(f)$ satisfying

$$\|R_n(f)\| \le C \|\hat{f}^{(n)}\|_1 \|ad_A^n(H)\| \tag{6.6}$$

Consequently, $[H, f(A)]$ defines an operator on $D(A^{n-1})$.

PROOF. We begin with $f \epsilon C_0^\infty$ functions for which representation (6.1) is well-defined and then extend the expressions obtained to the class of interest. Thus on $D(H) \times D(H)$

$$[H, f(A)] = \int ds \hat{f}(s)[H, e^{iAs}] \,. \tag{6.7}$$

We have

$$[H, e^{iAs}] = e^{iAs}(e^{-iAs} H e^{iAs} - H) \,.$$

Using that

$$\frac{d}{ds}(e^{-iAs} H e^{iAs}) = i e^{-iAs} ad_A(H) e^{iAs}$$

and the Fundamental Formula of calculus we compute

$$e^{-iAs} H e^{iAs} - H = i \int_0^s du e^{-iAu} ad_A(H) e^{iAu} \,. \tag{6.8}$$

The above two formulas are first derived with H replaced by $H_\epsilon \equiv \frac{H}{1+i\varepsilon H}$, $\varepsilon > 0$. Then, we let $\varepsilon \downarrow 0$ and use the boundedness of $ad_A(H)$ to prove that this limit exists.

Subtracting from and adding to the integrand $ad_A(H)$ gives

$$e^{-iAs}He^{iAs} - H = is\, ad_A(H) + i \int_0^s du(e^{-iAu} ad_A(H) e^{iAu} - ad_A(H)) \ .$$

Iterating this relation $n-1$ times we obtain

$$e^{-iAs}He^{iAs} - H = \sum_{k=1}^{n-1} \frac{(is)^k}{k!} ad_A^k(H) + R_n(s)$$

where

(6.9) $$R_n(s) = \int_0^s du_1 \cdots \int_0^{u_{n-1}} du_n e^{-iAu_n} ad_A^n(H) e^{iAu_n} \ .$$

This together with (6.7) and the relation

$$\int_{-\infty}^{\infty} \hat{f}(s)(is)^k e^{iAs} ds = f^{(k)}(A)$$

yields

(6.10) $$[H, f(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H) + R_n(f)$$

where

(6.11) $$R_n(f) = \int_{-\infty}^{\infty} \hat{f}(s) e^{iAs} R_n(s) ds \ .$$

Since $ad_A^n(H)$ is bounded we have that

$$\|R_n(s)\| \leq \text{const. } |s|^n \|ad_A^n(H)\|$$

which yields

$$\|R_n(f)\| \leq \text{const.} \int_{-\infty}^{\infty} |\hat{f}(s)|\,|s|^n ds \|ad_A^n(H)\| \ . \tag{6.12}$$

Finally we extend the above analysis to arbitrary function f satisfying condition F.

First assume that H is a bounded operator. Then the form

$$A(f) \equiv i[H, f(A)] - \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H)$$

is well defined on $D(A^n)$.

Now, let $f_j \epsilon C_0^\infty(\mathbb{R})$ $j = 1, \ldots \infty$ with $f_j \to f$ in the F topology

$$\|f\|_F \equiv \|\hat{f}^{(n)}\|_1 \ .$$

It readily follows that $A(f_j) \to A(f)$ in the form sense since $f_j \xrightarrow{F} f$ implies

$$\|\langle\lambda\rangle^{-n}(f_j(\lambda) - f(\lambda)\|_\infty \to 0 \ .$$

By the estimate (6.12)

$$R_n(f_j) \to R_n(f)$$

with $Rn(f)$ bounded.

Equality (6.10) then implies

$$A(f) - R_n(f) = 0$$

in the form sense on $D(A^n)$.

Since $R_n(f)$ is bounded and $\sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H)$ is an operator defined on $D(A^{n-1})$ the above equality extends to $D(A^{n-1})$. The result for unbounded H now follows by approximating H by $\frac{H}{1+i\varepsilon H}$ and a simple continuity argument. ■

REMARK. For similar expansions, based on resolvents see [B-G1 and cited ref.].

Lemma 6.2 *Let $A(t)$ be a commutative family of self-adjoint operators with common domain $\mathcal{D}$. We assume that $A(t)$ is norm differentiable in t : $A(t+$*

$\delta) - A(t)$ are bounded for δ small and the following norm limit exist:

$$\lim_{\delta \to 0} \frac{1}{\delta} \left(A(t+\delta) - A(t)\right) \equiv \frac{dA(t)}{dt} \ .$$

Then, we have

(a) For all bounded smooth functions f, with $\hat{f}^{(1)} \epsilon L^1$, $f\left(A(t)\right)$ is norm differentiable.

(b) Assume that $A(t) > 0$ for all t. Let $\alpha \geq 0$. Then $A(t)^\alpha f\left(A(t)\right)$ is differentiable in the strong resolvent sense and the chain rule applies for $g(\lambda) \equiv \lambda^\alpha f(\lambda)$:

$$\frac{d}{dt} g\left(A(t)\right) = g'\left(A(t)\right) \frac{dA(t)}{dt} \ .$$

PROOF.

(a) As in the proof of Lemma 6.1 we derive the formula

$$\frac{d}{dt} e^{i\lambda A(t)} = i\lambda \frac{dA(t)}{dt} e^{i\lambda A(t)} \ .$$

Using this formula and equation (6.1) we get

$$\frac{d}{dt} f\left(A(t)\right) = \frac{dA(t)}{dt} f'\left(A(t)\right) \tag{6.13}$$

where the limits defining $\frac{d}{dt}$ are taken in the norm sense.

(b) Since the $A(t)$ have a common domain $\mathcal{D}$, $A(t)/\left(A(s) + i\right)^{-1}$ are bounded for all s and t by the closed graph theorem. We can then compute directly that

$$\frac{d}{dt} A(t)^\alpha = \alpha A(t)^{\alpha - 1} \frac{dA(t)}{dt}$$

for all positive integers α. Thus by the Chain rule it suffices to consider the case $0 < \alpha < 1$.

To this end we compute

$$\frac{1}{\delta} \left[A(t+\delta)^\alpha - A(t)^\alpha\right] == \frac{1}{\delta} A(t)^\alpha \left[\left(1 + \delta\beta(t)\right)^\alpha - 1\right]$$

where

$$\beta(t) \equiv \frac{\left(A(t+\delta) - A(t)\right)}{\delta} A(t)^{-1} \ .$$

Note that since $A(t) > 0$ and norm differentiable and, $A(t)$ are commuting for different values of $t, \beta(t)$ is a bounded self-adjoint operator for each t. Using that by the spectral theorem

$$(1+\delta\beta)^{\alpha} \; - 1 \; = \; \alpha\delta\beta \; + \; 0(\delta)$$

the result follows.

Some Domain Estimates

Lemma 6.3 *Let H and $A \geq 0$ be self adjoint operators on a Hilbert space $\mathcal{H}$. Assume that $D(H) \cap D(A)$ is dense in $\mathcal{H}$. Then, for $\alpha \geq 0$ s.t.*

(6.15) $\qquad ad_H^k(A)$ *are bounded operators for* $1 \leq k \leq [\alpha]+1$

we have

$$g(H)\colon \; D(A^{\alpha}) \to D(A^{\alpha})$$

for all $g(\lambda)\epsilon C_0^{\infty}(\mathbb{R})$.

PROOF. By interpolation it is sufficient to prove the lemma for integer α.

We first show that $ad_A^k(g(H))$ are bounded for $1 \leq k \leq [\alpha]+1$; using eq. (6.8) we derive, in the sense for forms on $D(H) \cap D(A) \equiv D$,the following equality

$$[A, e^{isH}] = e^{isH}(e^{-isH}Ae^{isH} - A =)$$

$$= ie^{isH} \int_0^s d\mu e^{-i\mu H} ad_H(A) \quad e^{i\mu H}, \tag{6.16}$$

hence

$$\sup_{\substack{\varphi,\psi\epsilon D \\ \|\phi\|=\|\psi\|=1}} \; |<\phi, [A, e^{isH}]\psi>| = \|[A, e^{isH}]\| \leq s\|ad_H(A)\| \leq cs \tag{6.17}$$

due to (6.15) (with $k=1$).

Iterating the formula (6.16) and using the estimate (6.17) we get

$$\|ad_A^n(e^{isH})\| \leq cs^n \qquad\qquad n \leq [a]+1 \tag{6.18}$$

By (6.7) and (6.18)

$$(6.19) \quad \|ad_A^n(g(H))\| = \| \int_{-\infty}^{\infty} ds\, \hat{g}(s) ad_A^n\, (e^{isH}) \| \leq C \int_{-\infty}^{\infty} |s^n\, \hat{g}(s)| ds < \infty\, .$$

Since $\|ad_A^n(g(H))\| < \infty$ for $n = [\alpha] + 1$, taking g-real valued we apply Lemma 6.1 (with $f(\lambda) \equiv \lambda^\alpha$, α integer) to get:

$$(6.20) \qquad [g(h), A^\alpha] = \sum_{k=1}^{[\alpha]} \frac{C_k}{k!} ad_A^k(g(H)) A^{\alpha-k} + R_{[\alpha]+1}$$

and $R_{[\alpha]+1}$ is a bounded operator due to (6.6). (Note that since α is an integer, one can derive eq. (6.20) directly, without reference to Lemma 6.1.)

Hence, if we let $u \epsilon D(A^\alpha)$, then using eq. (6.20) we obtain

$$\begin{aligned} A^\alpha g(H) u &= g(H) A^\alpha u + \; [A^\alpha, g(H)] u \\ &= g(H) A^\alpha u + \sum_{i=1}^{[\alpha]} B_i A^{\alpha-i} u + R_{[\alpha]+1} u \epsilon \mathcal{H} \end{aligned}$$

since $B_i \equiv \frac{C_i}{i!} ad_A i(g(H))$ are all bounded by (6.19). ■

Section 7. Local Decay, Velocity Bounds and Spectral Theory

Recall the notion of threshold energy: $\mathcal{K}_\alpha = 0$ for some cluster decomposition $a(\alpha)$ and channel α. Here $\mathcal{K}_\alpha = E - \varepsilon_\alpha$ where ε_α is an eigenvalue of H^a. Thus thresholds E are eigenvalues of subhamiltonians H^a. The set of all thresholds is denoted $\mathcal{T}$. It is known that $\mathcal{T}$ is discrete and bounded. Furthermore points of $\mathcal{T}$ can accumulate (at $\mathcal{K} \in \mathcal{T}$) only from below. These properties of H follow from the Mourre estimate [see e.g. CFKS] which we now turn to:

For $E \notin \mathcal{T}$, the channel expansion for H gives (the Mourre estimate) [M1-2, PSS, FH1, BG2]. For more general Hamiltonians see [Der2, Ger1, FHP1].

$$E_\Delta(H) i[H, A] E_\Delta(H) \geq \theta E_\Delta^2(H) + K\ , \quad \theta > 0$$

with K compact ($E \subset \Delta$).

Let $\tilde{\Delta} \subset \Delta$ be s.t. H has no eigenvalues in $\tilde{\Delta}$. Then $E_{\tilde{\Delta}}(H) \xrightarrow{s} 0$ as $|\tilde{\Delta}| \to 0$. Since K is compact, we can choose $\tilde{\Delta}$ sufficiently small s.t.

$$E_{\tilde{\Delta}} i[H, A] E_{\tilde{\Delta}} \geq \theta E^2_{\tilde{\Delta}}(H) - \varepsilon E^2_{\tilde{\Delta}} > (\theta - \varepsilon) E^2_{\tilde{\Delta}}(H) \ .$$

A general spectral theory has been developed with the Mourre estimate as the main tool. This theory can be thought of infinitesimal and microlocal version of scaling theory in PDE (see also [L]). The Mourre estimate determines the way an infinitesimal scaling affects the operator H. Let us describe few notable consequences of the Mourre estimate. (A comprehensive analysis of the continuous spectral part of H is done in [ABG], [BG1].) See [Iw] for applications to systems of equations and [We] to nonhomogeneous media.

Theorem 7.1 (Mourre) *Assume H satisfies the Mourre estimate for an interval Δ. Then H has only finitely many eigenvalues in Δ; assume moreover that the commutator $i[[H, A], A]$ is H bounded. Then H has no singular spectrum in Δ.*

Theorem 7.2 (Local Decay) *Assume H satisfies the Mourre estimate for an interval Δ and $ad^2_A(H)$ is H bounded. Then local decay holds:*

$$\int_{-\infty}^{\infty} \|\langle A\rangle^{-\frac{1}{2}-\varepsilon} e^{-iHt}\psi\|_2^2 dt \leq c\|\psi\|_2^2 \quad \text{for all } \psi = E_\Delta \psi \ .$$

In case A is the dilation generator it is easy to show that $\langle A\rangle$ can be replaced by $\langle x\rangle$, in the above local decay estimate.

Theorem 7.3 (Minimal and Maximal Velocity Bounds) *[Sig-Sof2] Assume as before that the Mourre estimate holds for some energy E. Let θ_m and θ_M be the lower and upper bounds:*

$$\theta_m E^2_\Delta \leq E_\Delta i[H, A] E_\Delta \leq \theta_M E^2_\Delta \ .$$

Assume furthermore that $ad_A^2(H)$ and ad_A^3 are H bounded. Then

$$\int \|F(\frac{A}{t} \leq \theta_m - \varepsilon)e^{-iHt}\psi\|^2 \frac{dt}{t^\alpha} \leq c\|\langle x\rangle\beta(\alpha)\psi\|_2^2 \qquad \beta(\alpha) = \frac{1-\alpha}{2} \ ,$$

and

$$\int \|F(\frac{A}{t} > \theta_M + \varepsilon)e^{-iHt}\psi\|\frac{dt}{t^\alpha} \leq c\|\langle x\rangle^{\beta(\alpha)}\psi\|_2^2 \ .$$

Here $\psi = E_\Delta\psi$.

REMARK. The upper bound inequality for $i[H, A]$ is called the reverse Mourre estimate. Sharp values of θ_m and θ_M can be found for a general N–body hamiltonian using the Channel Expansion Theorem.

A corollary of Theorems 1 and 2 is a proof of asymptotic completeness for the two body case. Further results can also be established by the analysis leading to the above theorems, e.g. propagation estimates for the region of phase space where $A < 0$ and analytic properties in certain weighted spaces of the resolvent of H. But not less important and impressive are the results about *eigenfunctions* of H and its resonances.

Theorem 7.4 (Froese-Herbst) *Exponential decay of eigenfunctions. Let $H = -\Delta + V$ where V satisfies:*

i) V is $-\Delta$ bounded with bound less than 1

ii) $x \cdot \nabla V$ is bounded from $\mathcal{H}^1$ to $\mathcal{H}^{-2}$.

Suppose that $H\psi = E\psi$. Then $e^{\lambda\langle x\rangle}\psi \in L^2$ for all $\lambda^2 \leq M^\sim(H) - E$. Here $\eta \in M(H)$ iff the Mourre estimate holds at $\eta = E$ (see CFKS, ch.4).

The results on absence of embedded eigenvalues uses:

Theorem 7.5 (Froese-Herbst) *Absence of embedded eigenvalues. Let $H = -\Delta + V$ where V satisfies conditions (i), (ii) of the above theorem and furthermore, $x \cdot \nabla V$ is Δ bounded with bound less than 2. Then, if $e^{\lambda\langle x\rangle}\psi \in L^2$ for all λ real, then $\psi = 0$ (see CFKS, ch. 4).*

There are also interesting results about the existence and characterization of resonances using the Mourre estimate in [Or] and to Nonlinear instability of periodic solutions [Sig2]. See also [FL, J, Na].

PROOF. The proof of the first part of Theorem 7.1 is very simple: Let $\{\psi_n\}_{n=1}^{\infty}$ be eigenfunctions of H with eigenvalues in Δ. Then

$$0 = \langle \psi_n, [H, iA]\psi_n \rangle = \langle \psi_n, E_\Delta [H, iA] E_\Delta \psi_n \rangle \geq \theta \|\psi_n\|^2 + \langle \psi_n, K\psi_n \rangle \ .$$

Now, let $n \to \infty$; then $\langle \psi_n, K\psi_n \rangle \to 0$ since K is compact. We used also the virial theorem to prove the first equality.

The absence of singular continuous spectrum in Δ follows from Local Decay (Theorem 7.2).

The original proof of Theorem 7.2 given by Mourre was based on proving differential inequality for the complex distorted resolvent of H, the distortion is generated by the group $e^{i\lambda A}$.

Later a new proof, more general, was given by the methods of microlocal propagation estimates [Sig-Sof2]. This proof implied also the minimal and maximal velocity bounds as well as pointwise decay estimates (in time) in certain regions of the phase–space [Sig-Sof2, Sk, Ger2, H-Sk, Ger-Sig, Her]. This approach could also be extended to time dependent hamiltonians of the type

$$H(t) = H + W(x, t)$$

that arise in long range scattering theory.

IDEA OF PROOF. (Theorems 7.2,3) We construct a sequence of *negative propagation observables* of singular operators. Each one, when used, implies a propagation estimate which is then used to control the remainder terms of the next, more singular observable. Denote for a moment by $\lambda(t)$ a monotone increasing function of t, and let $\alpha, \beta \geq 0$. Then

$$\phi_{\alpha,\beta}(\lambda, t) \equiv -\Big(\frac{-\lambda(t)}{\tau^\beta}\Big)^\alpha F\Big(\frac{\lambda(t)}{t} < -\delta\Big)$$

satisfies

i) $\phi_{\alpha,\beta} \leq 0$

ii) $\frac{d}{dt}\phi_{\alpha,\beta}(t) \geq 0$ for any $\delta > 0$.

The Mourre estimate then suggest the use of

$$\lambda(t) \sim e^{iHt} A e^{-iHt} \ .$$

To make the analysis go, we take $\phi_{\alpha,\beta}(A,t) \equiv F_\alpha$ as the propagation observables and work inductively in α.

Using the Leibnitz Rule for operators and the Mourre estimate we can then bound $E_\Delta D F_\alpha E_\Delta$ from below, to prove the theorems 2, 3 by invoking the Basic Lemma.

This approach allows very precise localization of the orbit $e^{-itH}\psi$ in the phase space. For general two body hamiltonians of the PDO type one can prove asymptotic completeness by proving sharp localization of the solution near the classical trajectories [Sig1]. The previous approach to this problem required intricate stationary phase analysis and resolvent estimates due to Agmon; see [Hö IV, last chapter], see also [Kit-Ku], [Comb] for another approach.

The above method of proving theorems 2, 3 suggests a way of getting *finite propagation speed* behavior to Schrödinger type equations and may be useful in the study of propagation of singularities. For some progress in this direction see [Ger-Sig].

Section 8. The N–body Long Range Scattering

The results of this section are based on [Sig-Sof4,5].

Using the minimal velocity bounds we infer that, for large times, $|x| \gtrsim ct$, when the total energy of the state ψ_+ is localized away from the thresholds of H.

In the Long Range case, the two body potentials (at least some of them) vanish like $|x|^{-\mu}$ with $\mu \leq 1$. Therefore $|x_{ij}|^{-\mu} \sim t^{-\mu}$ for large $|t|$ which is not integrable. In particular the proof of AC fails, since now

$$I_a J_a = O(|x|^{-\mu}) \notin L^1(dt) \ .$$

(by this we mean $(\psi(t), |x|^{-\mu}\psi(t)) \neq L^1(dt)$).
It can be shown that the asymptotic motions of subsystems cannot be free. The modification needed makes the asymptotic hamiltonians *time dependent.*

For each cluster decomposition a we define the hamiltonian

$$H_a(t) \equiv H_a + I_a(x^a, x_a)\mid_{x_a = v_a t} \ .$$

Using $H_a(t)$ instead of H_a in the proof of existence of the Deift-Simon Wave Operators, the $J_a I_a$ term is replaced by

$$J_a(I_a(x^a, x_a) - I_a(x^a, v_a t)) = O(|x|^{-1-\mu})|x_a - v_a t|J_a$$

We therefore need to prove a *sharp propagation estimate*

$$\|J_a|x_a - v_a t|\psi(t)\| = O(t^{\mu-\varepsilon}) \quad \text{for some } \varepsilon > 0$$

to conclude the proof in the Long Range Case.

In practice we modify the time dependent part of $H_a(t)$ further, to include the known minimal and maximal velocity bounds:

$$H_a(t) = H_a + W_a(x,t)$$

where

$$W_a(x,t) = F_{a,E}(x,t) I_a(x^a, v_a t) \ .$$

Here $F_{a,E}(x,t)$ localizes $m \leq |x|/t \leq M$ with m, M depending on a and E. We therefore have

$$|\partial_x^\alpha \partial_t^\beta W| \leq C_{\alpha,\beta}(1 + |x| + |t|)^{-\mu-|\alpha|-|\beta|} \ .$$

Let

$$\phi_a \equiv F(m \leq \frac{|x|}{t} \leq M) j_a(x, p_a)$$

where j_a is a phase–space partition of unity of T^*X. Then it is easy to verify that

Theorem 8.1

a) $\sum_a \phi_a + F(\frac{|x|}{t} \leq m) + F(\frac{|x|}{t} > M) = 1 + O(t^{-1}) \quad |t| > 1$

b) ϕ_a *are supported in* $\{|x|_a > \delta|x|\} \cap \{mt \leq |x| < Mt\} \times \{|k_a| < R\} \equiv \Omega_a$

c) $\langle t\rangle^{|\alpha|+\beta} \partial_x^\alpha \partial_t^\beta \phi_a$ *are supported in* $\Omega_a \backslash PS_E^t$.

Here $PS_E^t \equiv PS_E \cap \{m < \frac{|x|}{t} < M\}$.

We then have (Sig-Sof4)

Theorem 8.2 (Sharp Propagation Estimate) *Assume* $\mu > 0$. *Let* E *be away from the thresholds and eigenvalues of* H. *Then, there exists an interval* Δ *around* E *s.t.*

$$\int_1^\infty \left\| |\frac{x_a}{t} - v_a|^{1/2} \phi_a \psi_t \right\|^2 \frac{dt}{t} \leq c\|\psi\|^2 \ .$$

The proof follows by studying the following propagation observables. Let

$$\Lambda_a \equiv |\frac{x_a}{t} - v_a|^2 + t^{-2\beta-2} \quad t \geq 1$$

and define the propagation observables

$$F_a \equiv \phi_a \Lambda_a \phi_a \ .$$

The Heisenberg derivative of F_a consists of two (kinds of) terms:

$$\phi_a (D\Lambda_a) \phi_a \leq 0$$

and

$$(D\phi_a)\Lambda_a\phi_a + \phi_a\Lambda_a D\phi_a \ .$$

This second term lives away from the PS_E^t by the properties of ϕ_a. Hence the original propagation theorem for PS_E and the minimal/maximal velocity bounds show that this term is $L^1(dt)$ which completes the proof. Alternatively, one can use Graf's argument and consider

$$\sum_a \tilde{\phi}_a \Lambda_a \tilde{\phi}_a$$

and try to arrange that, by choosing different $\tilde{\phi}_a$

$$\sum_a (D\tilde{\phi}_a)\Lambda_a\tilde{\phi}_a + \sum_a \tilde{\phi}_a \lambda_a D\tilde{\phi}_a \leq 0 + O(L^1(dt)) \ .$$

This indeed can be done using the monotonic partitions of unity [Gr, Der3]. So far we studied the asymptotic behavior of $e^{-iHt}\psi$ for t large and E away from the thresholds and eigenvalues of H. This establishes *Asymptotic Clustering*

Theorem 8.3 (Asymptotic Clustering) *Let $\mu = 1$ and E be away from the thresholds and eigenvalues of H. Then Asymptotic Clustering holds for any number of particles:*

$$\lim_{t\to\pm\infty} \|e^{-iHt}\psi - \sum_a U_a(t)\psi_a^\pm\| \to 0 \ .$$

The proof follows from the Sharp propagation estimate and construction of the Deift–Simon wave operators as in the short range case [Sig-Sof4]. Recently, by using an intermediate asymptotic dynamics [Ge-Der] improved the above theorem to include all cases of $\frac{1}{2} < \mu$.

We now turn to the problem of Asymptotic Completeness in the Long Range Case. The new feature is the need to analyze the asymptotic behavior of an N–body system with time dependent perturbation $W(x,t)$ added. In this case we have to redevelop all the local decay, velocity bounds, etc. for such hamiltonians. The first problem we are faced with is that the energy is not conserved by time dependent hamiltonians: $E_\Delta(H_a)U_a(t) \neq U_a(t)E_\Delta(H_a)$.

Asymptotic Energy Operators

To treat the lack of energy conservation, we need the method of asymptotic microlocalization. We show that asymptotically the energy distribution is constant and we will microlocalize using the asymptotic observables build from the energy projections [Sig-Sof2-5] see also [Sol]. Let Ω be any bounded interval. Then

$$E_\Omega^\pm(H) \equiv s - \lim_{t\to\pm\infty} U(t)^* E_\Omega(H) U(t)$$

exists, for any $\mu > 0$.

Here $U(t)$ is generated by $H + W(x,t)$ and $W(x,t)$ satisfies the conditions of this section. Moreover

$$\|U(t)E_\Omega^\pm - E_\Omega U(t)\| \leq C|\Omega|^{-1}\langle t\rangle^{-\mu} \ .$$

The proof is simple; it uses Cook's argument to $U(t)^* E_\Delta U(t)$ and the decay properties of $W(x,t)$. From now on, we refer to $U_a(t)$ as $U(t)$ (with $H_a \to H$). By considering $E \notin \mathcal{T}$ (the threshold set of H) and letting

$$\psi^\pm = E_\Delta^\pm(H)\psi \quad \Delta \supset E, |\Delta| \text{ small enough}$$

we can prove now asymptotic clustering for $U(t)E_\Delta^\pm$. This is because the local decay and minimal and maximal velocity bounds can be proved for $H(t)$ by the methods of [Sig-Sof2] as for the independent case. The same is true for the sharp propagation estimates. The main difference now is that we have to estimate $D_{H(t)}F$ instead of $D_H F$ for the propagation observables F. Furthermore, we need to localize using $E_\Delta^\pm$ instead of E_Δ. Both of these can be achieved using the decay properties of W. It is left to consider states in the range of the *singular asymptotic projections*: $E_0^\pm$:

Let $E \in \mathcal{T}$ and $\Omega \supset E$. Then, as before $E_\Omega^\pm$ exists. Since $\mathcal{T}$ is discrete, by density argument we can reduce the problem to an arbitrary small interval around E. We then are left to consider

$$\mathcal{H}^\pm_{\text{thres.}}(H) = \bigcup_{E\in\mathcal{T}} \left\{ \overline{\lim_{\substack{|\Omega|\downarrow 0 \\ \Omega\supset E}}}\, E_\Omega^\pm(H)\psi \mid \psi \in L^2 \right\} .$$

The scattering theory for initial states in $\mathcal{H}^\pm_{\text{thres.}}(H)$ is fundamentally different than that of states in the orthogonal complement. Such states, if they exist, can only diffuse in certain channels (open) rather than scatter, because they are localized on threshold energies. Consequently the propagation theory is very different. To begin with, the Mourre estimate does not hold and therefore local decay, velocity bounds fail.

Asymptotic Microlocalization and Propagation

We defined the space of (asymptotic) thresholds $\mathcal{H}_{\text{thres.}}(H)$ in terms of the singular projections $\overline{\lim}_{|\Delta|\downarrow 0} E_\Delta^\pm(H)$ $\Delta \supset E$, $E \in \mathcal{T}$. Since we cannot expect $|x| \sim ct$ for such states we need another way of getting some local decay. The first step is then the following time dependent decomposition of the space:

(A) $\quad |x| < ct^\alpha$

(B) $\quad |x| \geq ct^\alpha$

for some $\alpha < 1$.
In region (A) scattering is not possible. However we treat states in this region using the following wave–operator argument:

$$W(x,t) = W(x,t) - W(0,t) + W(0,t) = O(\frac{|x|}{t^2}) + W(0,t) = 0(t^{-2+\alpha}) + W(0,t)$$

in the region (A). Hence we expect the following wave operator to exist:

$$U_D(t)^* \Big(\frac{|x|}{t^\alpha} \leq 1) U(t) \xrightarrow{s} \Omega_D^\pm .$$

By Cook's argument and the observation above, it is reduced to proving that $DF(\frac{|x|}{t^\alpha} \leq 1) \in L^1(dt)$. Since DF lives in the region $\frac{|x|}{t^\alpha} \sim 1$, the problem is reduced to the region $\frac{|x|}{t^\alpha} \geq 1$, where scattering is expected.

In region (B) $|x| \geq t^\alpha$, but the momentum can be arbitrarily close to zero. This suggests another sharp decomposition:

(I) $\quad |p| > ct^{-\beta}$

(II) $\quad |p| \leq ct^{-\beta}$.

Our aim is now to get positive commutators in the region (I), (II) by using that for free flow $x_a \sim v_a t$. Let us consider a simple example to illustrate this approach. Assume that $U(t)$ is generated by a three particle Hamiltonian plus $W(x,t)$. (This is the hard case for $N = 4$). Let $E \in \mathcal{T}$ be negative. In this case, there is no propagation on the three cluster decomposition, by energy conservation.

Consider the following propagation observable, for any two cluster decomposition a:

$$F_a = F_1\Big(\frac{|x^a|}{|x|} \leq \varepsilon^a\Big) F_2\Big(\frac{|x_a|}{t^\alpha} \geq 1\Big) F_3(|p_a| < \alpha t^{\alpha-1}) .$$

We estimate $D_0 F_a \equiv D_{-\Delta} F_a$. Clearly

$$D_0 F_3 \leq 0 \quad \text{since} \quad [p_a, \Delta] = 0 .$$

$$D_0 F_2 = F_2' t^{-\alpha}\Big(\gamma_a - \frac{\alpha|x_a|}{t}\Big) + O(t^{-2\alpha}) \ .$$

for $\alpha > \frac{1}{2}$, $O(t^{-2\alpha}) \in L^1(dt)$. F_2' localizes $\frac{|x_a|}{t^\alpha} \sim 1$, hence

$$-\frac{\alpha|x_a|}{t} \sim -\alpha t^{\alpha-1} \ .$$

Finally,

$$|\gamma_a| F_3(|p_a| < \alpha t^{\alpha-1}) < \alpha t^{\alpha-1} + O(t^{-1})$$

by the Localization Lemma. Combining all the above we conclude that

$$D_0 F_2 \leq -\delta t^{-1} + O(t^{-2\alpha}) \quad \text{for some positive } \delta \ ,$$

on support of F_3.

Finally, observe that DF_1 lives where $|x^a| \sim \varepsilon^a |x|$ and therefore on the free channel (recall $\#(a) = 2$ and the total number of particles is 3). There is no propagation on support DF_1, since $E < 0$. We therefore conclude that

$$D_0 F_a \leq \frac{-\delta}{t} F_1 F_2' F_3 + O(L^1(dt)) \ .$$

To show that the potential parts do not change the monotonicity estimate, observe that F_a lives in the region $\{|x|_a \geq t^\alpha, \frac{|x^a|}{|x|} \leq \varepsilon^a\}$ which is a two cluster decomposition a with $|x| \geq ct^\alpha$. Furthermore F_a, as a phase space operator is independent of p^a. Hence the commutator with V decays like $|t|^{-(1+\mu)\alpha} \in L^1(dt)$. The commutator with $W(x,t)$ is estimated by the formulas for the commutator of functions of operators to give a contribution of order

$$O(t^{-1-\mu} t^{1-\alpha}) = O(t^{-\mu-\alpha}) \in L^1(dt) \ .$$

We therefore conclude that $-F_a$ is a propagation observable for $H(t)$, when $H(t) = H + W(x,t)$ and H is a three particle hamiltonian.

The resulting propagation estimate and similar analysis for $|p_a| \geq \alpha t^{\alpha-1}$ and a time dependent two cluster partition of unity, shows that there is no propagation in the region $\frac{|x|}{t^\alpha} = 1$. Other types of estimates are needed to complete the proof for $N = 4$.

Let us consider another case. Let $E = 0$. Then $E \in \mathcal{T}$ and the system can propagate on 3 cluster decompositions, so the previous observable does not work, since now we do not know if $DF_1 \in L^1(dt)$. However, in this case, we compute

$$DF_1 = \frac{1}{\langle x\rangle} F_1'(\gamma^a - \frac{|x^a|}{|x|}\gamma) + O(|x|^{-2}) .$$

We use F_2 to conclude that

$$O(|x|^{-2})F_2 = O(t^{-2\alpha}) \in L^1(dt) .$$

Furthermore, F_1' localizes on three cluster decompositions and $\frac{|x^a|}{|x|} \sim \varepsilon^a$. On three cluster decompositions $\sum\limits_{i<j} V = 0(|x|^{-\mu})$.

Hence, if we pull a sharp energy localization projection of the type $F(|H| < t^{-\beta})$ from $\psi \in \mathcal{H}_{\text{thres.}}$, (Recall that now $E = 0$) we get

$$F_1'F(|H| < t^{-\beta})F_2 = F_1'F(|p^2| < t^{-\beta})F_2 + O(t^{-\mu\alpha+\beta}) .$$

Using now that $|p^2| < t^{-\beta}$ we use the localization Lemma to conclude that

$$\langle x\rangle^{-1}\gamma^a \text{ and } \langle x\rangle^{-1}\gamma \text{ are both of order } t^{-\alpha}t^{-\beta/2} \in L^1(dt)$$

if we choose $\alpha + \beta/2 > 1$. A complete solution of the 4 body problem along these lines is given in [Sig-Sof5].

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Note Added: After completion of this survey the following results have been obtained.

1) Asymptotic Completeness for N-body Long Range Scattering is proved by extending the methods described, here by I. M. Sigal, A. Soffer, preprint by J. Derezinski, in preparation, and by L. Zielinski, in preparation.

2) Asymptotic Completeness for 3-Particle Short Range Systems was proved in D. Yafaev, "Radiation Condition and Scattering Theory for Three-Particle Hamiltonians," preprint.

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Inverse boundary value problems and applications

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Inverse boundary value problems and applications

Gunther Uhlmann*

0. Introduction

The main purpose of these lecture notes, which are a revised and expanded version of the survey paper [S-U V], is to give an overview of the mathematical developments in the last few years in inverse boundary value problems. In these problems one attempts to discover internal properties of a body by making measurements at the boundary. We concentrate mainly in the problem of determining the conductivity of a body from measurements of voltage potentials and corresponding current fluxes at the boundary. This problem which is often referred to as *Electrical Impedance Tomography* arose in geophysics from attempts to determine the composition of the earth. More recently it has been proposed as a potentially valuable diagnostic tool for the medical sciences. The methods developed to study this problem have lead to new results in inverse scattering and inverse spectral problems. We also give an account of some of these developments in these notes.

1. Electrical impedance tomography; the isotropic case.

In this section we formulate the inverse conductivity problem and a similar problem for the Schrödinger equation at zero energy.

Let $\Omega \subseteq \mathbf{R}^n$ $n \geq 2$, be a smooth bounded domain. If the conductivity of Ω is independent of direction (isotropic case) it is represented by a positive function, which we assume in $C^{1,1}(\overline{\Omega})$, with a positive lower bound. If we assume that there are no sources or sinks of current in Ω, the conductivity equation for the potential u in Ω is

$$L_\gamma u = \operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega. \tag{1.1}$$

If f represents the induced potential on the boundary (assume $f \in H^{\frac{1}{2}}(\partial\Omega)$), $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} L_\gamma u &= 0 \qquad \text{in } \Omega \\ u|_{\partial\Omega} &= f. \end{aligned} \tag{1.2}$$

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The Dirichlet to Neumann map is then defined by

$$\Lambda_\gamma(f) = \gamma\frac{\partial u}{\partial \nu} \tag{1.3}$$

where u is the solution of (1.2) and ν is the unit outer normal to the boundary. The map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is selfadjoint and is often called the voltage to current map because $\gamma\frac{\partial u}{\partial \nu}$ measures the current flux at the boundary.

The inverse conductivity problem consists of the study of various properties of the map

$$\gamma \xrightarrow{\Phi} \Lambda_\gamma. \tag{1.4}$$

These properties include the injectivity, range, and continuity of the map and its inverse (when an inverse exists). From the point of view of applications, an even more important problem is to give a method to reconstruct γ(or at least to deduce as much information as possible about γ) from Λ_γ.

A closely related problem is to consider instead of the conductivity equation, the Schrödinger equation at zero energy

$$L_q = \Delta - q \tag{1.5}$$

where $q \in L^\infty(\Omega)$.

If 0 is not an eigenvalue of L_q, we can solve the Dirichlet problem

$$\begin{aligned} L_q u &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= f \end{aligned} \tag{1.6}$$

and define the Dirichlet to Neumann map by

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu} \tag{1.7}$$

where u is the solution of (1.6). We want to study the map

$$q \xrightarrow{\widetilde{\Phi}} \Lambda_q. \tag{1.8}$$

Λ_γ and Λ_q are related in the following way: If u is a solution of (1.1) then

$$w = \gamma^{\frac{1}{2}} u$$

is a solution of $L_q w = 0$ with $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. It is a straightforward computation to see that

$$\Lambda_q = \gamma^{-\frac{1}{2}}\Lambda_\gamma\gamma^{-\frac{1}{2}} + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}. \tag{1.9}$$

Thus if we know $\Lambda_\gamma, \gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$ we can determine Λ_q. In the next section we shall see that Λ_γ determines $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$, so that knowledge of Λ_γ determines Λ_q.

2. Results at the boundary

Kohn and Vogelius ([K-V, I]) proved that if $\gamma \in C^\infty(\overline{\Omega})$ one can determine $\frac{\partial^j\gamma}{\partial\nu^j}|_{\partial\Omega} \quad \forall j$.

Theorem 2.1. *Let $\gamma_i (i = 1, 2)$ be in $L^\infty(\Omega)$ with a positive lower bound. Let $x_0 \in \partial\Omega$ and let B be a neighborhood of x_0 relative to $\overline{\Omega}$. Suppose that*

$$\gamma_i \in C^\infty(B), \quad i = 1, 2$$

and

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega) \qquad \text{with}$$

supp $f \subset B \cap \partial\Omega$, then

$$\left(\frac{\partial}{\partial x}\right)^\alpha \gamma_1(x_0) = \left(\frac{\partial}{\partial x}\right)^\alpha \gamma_2(x_0)$$

where

$$\left(\frac{\partial}{\partial x}\right)^\alpha \text{ denotes } \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Sketch of proof.

Kohn and Vogelius proved this result by cleverly choosing boundary data. We outline here a different approach taken in [S-U, I] which makes use of the fact that Λ_γ is a pseudodifferential operator of order 1. This means that, in local coordinates near $x_0 \in \partial\Omega$ which we denote by x', and for f supported near x_0,

$$\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'}\lambda_\gamma(x', \xi')\widehat{f}(\xi')d\xi'. \tag{2.2}$$

$\lambda_\gamma(x', \xi')$ is the full symbol of Λ_γ and has an asymptotic expansion for large $|\xi'|$

$$\lambda_\gamma(x', \xi') \sim \sum_{j\leq 1} \lambda_\gamma^{(j)}(x', \xi') \tag{2.3}$$

with $\lambda_\gamma^{(j)}$ homogeneous of degree j in ξ'. We have $\lambda_\gamma^{(1)}(x',\xi') = \gamma|_{\partial\Omega}(x')|\xi'|$ and it was proven in [S-U, I] that $\lambda_\gamma^{(j)}(x',\xi')$ determines inductively $\frac{\partial^{j-1}}{\partial\nu^{j-1}}\gamma|_{\partial\Omega}$ (For a simpler proof of this see the paper [L-U] and also the sketch in section 9 of this paper.) □

The previous result implies the injectivity of Φ at real-analytic conductivities. Kohn and Vogelius extended this result further to cover piecewise real-analytic conductivities ([K-V, II]).

Sylvester and Uhlmann ([S-U I]) used the proof of Theorem 2.1 outlined above to give continuous dependence estimates at the boundary.

Theorem 2.4. *Let γ_i, $i = 1,2$ be in $L^\infty(\Omega)$ with a positive lower bound. Then*

(a) $$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}.$$

If γ_1, γ_2 are continuous, then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C_1\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}.$$

(b) If γ_1, γ_2 are Lipschitz continuous then

$$B_i = \Lambda_{\gamma_i} - \gamma_i\Lambda_1 \quad \textit{satisfy}$$

$\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} \leq C_2\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\Omega)}$

and $\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\partial\Omega)} + \|\frac{\partial}{\partial\nu}(\gamma_1 - \gamma_2)\|_{L^\infty(\partial\Omega)}$

$$\leq C_3(\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})$$

On the operators we use the operator norm. C_1 depends only on Ω and the lower bound of the γ_i's. C_2, C_3 depends only on Ω and the γ_i's are normalized to have Lipschitz norm less than or equal to one.

3. Linearization at constants; Calderón's approach

Calderón formulated the inverse conductivity problem in a different way. He considered the Dirichlet integral associated to the solution of (1.2)

$$Q_\gamma(f) = \int_\Omega \gamma|\nabla u|^2, \tag{3.1}$$

$Q_\gamma(f)$ measures the power necessary to maintain the potential f on the boundary.

Polarizing the quadratic form Q_γ we obtain the bilinear form

$$Q_\gamma(f,g) = \int_\Omega \gamma \nabla u \cdot \nabla v \tag{3.2}$$

where u is a solution of (1.2) and v solves

$$\begin{aligned} L_\gamma v &= 0 \quad \text{in } \Omega \\ v|_{\partial\Omega} &= g\,. \end{aligned} \tag{3.3}$$

The divergence theorem gives

$$Q_\gamma(f,g) = \int_{\partial\Omega} g \Lambda_\gamma f \tag{3.4}$$

In other words $\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is the unique selfadjoint operator associated to the quadratic form Q_γ with domain $H^{\frac{1}{2}}(\partial\Omega)$. The inverse conductivity problem can then be reformulated as the study of the map

$$\gamma \xrightarrow{Q} Q_\gamma. \tag{3.5}$$

For the Schrödinger equation L_q we look at the Dirichlet form

$$Q_q(f,g) = \int_\Omega \nabla u \cdot \nabla v + quv \tag{3.6}$$

where u, v solve

$$\begin{aligned} L_q u = L_q v &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} = f; \quad v|_{\partial\Omega} &= g \end{aligned} \tag{3.7}$$

and we can consider the map

$$q \xrightarrow{\widetilde{Q}} Q_q\,. \tag{3.8}$$

Calderón computed the formal linearization of Q near γ. He obtained

$$\lim_{\varepsilon\to 0} \frac{(Q_{\gamma+\varepsilon\varphi} - Q_\gamma)}{\varepsilon}(f,g) = \int_\Omega \varphi \nabla u \cdot \nabla v \tag{3.9}$$

with u, v as in (1.2) and (3.3).

An analogous computation shows that

$$\lim_{\varepsilon \to 0} \frac{(\widetilde{Q}_{q+\varepsilon\varphi} - \widetilde{Q}_q)}{\varepsilon}(f,g) = \int_\Omega \varphi u v \tag{3.10}$$

with u, v as in (3.7).

Formulas (3.9) and (3.10) imply that the formal linearization of Q (resp. $\widetilde{Q}$) at γ (resp. q) is injective iff the linear span of the inner products of gradients of solutions to (3.3)(resp. products of solutions to (3.7)) is dense in $L^2(\Omega)$; or equivalently that any function orthogonal to all such inner products (resp. products) is identically zero.

Calderón exploited this by proving:

Theorem 3.11. *The linear span of the inner products of gradients of solutions of harmonic functions (or the product of harmonic functions) is dense in $L^2(\Omega)$.*

Proof. Calderón chose the complex exponential harmonic functions

$$\begin{aligned} u &= e^{x\cdot\rho} \\ v &= e^{-x\cdot\overline{\rho}} \end{aligned} \tag{3.12}$$

where $\rho \in \mathbf{C}^n$. These functions are harmonic iff

$$\rho \cdot \rho = 0\,. \tag{3.13}$$

For $\rho = \eta + ik$, with $\eta, k \in \mathbf{R}^n$, (3.13) is satisfied iff $\eta \cdot k = 0$, $|\eta| = |k|$. Inserting (3.12) into (3.9)(resp. (3.10)) yields

$$\int_\Omega \varphi \nabla u \cdot \nabla v = -2|k|^2 \int_\Omega e^{2ix\cdot k}\varphi(x)dx$$

and

$$\int_\Omega \varphi u v = \int_\Omega e^{2ix\cdot k}\varphi(x)dx\,.$$

In both cases we conclude by the Fourier inversion formula that $\varphi = 0$ in Ω. $\square$

4. Special solutions

Motivated by Calderón's approach, Sylvester and Uhlmann constructed an analog for the elliptic equations (3.3) (or (3.7)) of the geometrical optics solutions for hyperbolic equations. These solutions behave like the complex exponentials $e^{x\cdot\rho}$, $\rho \cdot \rho = 0$ for large complex frequencies ρ.

Theorem 4.1. *Let $q \in L^\infty(\Omega)$ so that $q = 0$ in Ω^c.*

Let $\rho \in \mathbf{C}^n$, $n \geq 2$ be such that

$$\rho \cdot \rho = 0 \tag{4.2}$$

and

$$|\rho| > \|(1 + |x|^2)^{\frac{1}{2}} q\|_{L^\infty}, \tag{4.3}$$

then there exists a unique solution to

$$L_q u = 0 \quad \text{in } \mathbf{R}^n$$

of the form

$$u(x, \rho) = e^{x\cdot\rho}(1 + \psi_q(x, \rho)) \tag{4.4}$$

where $\psi_q(\cdot, \rho) \in L^2_\delta(\mathbf{R}^n)$, $-1 < \delta < 0$.

Furthermore

$$\|\psi_q\|_{H^m_\delta} \leq \frac{C}{|\rho|}\|q\|_{H^m_{\delta+1}}, \quad m \geq 0. \tag{4.5}$$

$L^2_\delta(\mathbf{R}^n)$ is the weighted L^2-space

$$L^2_\delta(\mathbf{R}^n) = \{f; \int |f|^2(1 + |x|^2)^\delta dx < \infty\},$$

$H^m_\delta(\mathbf{R}^n)$ is the corresponding Sobolev space.

An analogous statement is valid for the conductivity equation. Extend $\gamma \in C^{1,1}(\Omega)$ to $\gamma \in C^{1,1}(\mathbf{R}^n)$ with $\gamma = 1$ outside a ball. Then the solution (4.4) is replaced by

$$u(x, \rho) = e^{x\cdot\rho}\gamma^{-\frac{1}{2}}(1 + \psi_\gamma(x, \rho)) \tag{4.6}$$

ψ_q (resp. ψ_γ) in (4.4) (resp. (4.6)) satisfy the "transport" equation

$$\Delta\psi_q + 2\rho \cdot \nabla\psi_q - q\psi_q = q \tag{4.7}$$

(resp. $\Delta\psi_\gamma + 2\rho \cdot \nabla\psi_\gamma - q_\gamma\psi_q = q_\gamma$ with $q_\gamma = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$). The solution of the singular perturbation problem (4.7) with growth condition at infinity is easily seen to be a regular perturbation of the following proposition (see [S-U, II], Prop. 2.1)

Proposition 4.8. *Suppose*

$$\rho \cdot \rho = 0, |\rho| > B > 0, -1 < \delta < 0$$

and $f \in L^2_{\delta+1}$. *Then there exists a unique* $\phi \in L^2_\delta$ *solving*

$$\Delta\phi + \rho \cdot \nabla\phi = f.$$

Moreover,

$$\|\phi\|_{H^m_\delta} \leq \frac{C(B,\delta)}{|\rho|}\|f\|_{H^m_{\delta+1}}, \quad m \geq 0.$$

Theorem 4.1 has been extended to more singular potentials (see for instance [Ch]). Isakov [Is I] has given a different construction of special solutions which also applies to other equations with constant coefficient principal part. However, he doesn't obtain weighted estimates for the solutions.

5. Uniqueness and continuous dependence, $n \geq 3$

Sylvester and Uhlmann [S-U, II] proved that the map Φ (resp. $\widetilde{\Phi}$) is injective for smooth conductivities (potentials). The smoothness assumptions were relaxed to $\gamma \in C^{1,1}(\overline{\Omega})$ ($q \in L^\infty(\Omega)$) in [N-S-U].

Theorem 5.1. *(a) Let* $n \geq 3$, $\gamma_1, \gamma_2 \in C^{1,1}(\overline{\Omega})$ *with a positive lower bound and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2\,.$$

(b) Let $n \geq 3$, $q_1, q_2 \in L^\infty(\Omega)$ *and*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Proof. We first prove (b). An easy application of Green's theorem gives

$$\int_\Omega (q_1 - q_2)u_1 u_2 = \int_{\partial\Omega} f_1 \Lambda_{q_1} f_2 - f_2 \Lambda_{q_2} f_1 \tag{5.2}$$

where u_i is solution of $L_{q_i} u_i = 0$ and $f_i = u_i|_{\partial\Omega}$, $i = 1, 2$. Since Λ_q is a self adjoint map we obtain the identity proven by Alessandrini ([A])

$$\int_\Omega (q_1 - q_2)u_1 u_2 = \int_{\partial\Omega} f_1(\Lambda_{q_1} - \Lambda_{q_2}) f_2\,. \tag{5.3}$$

If $\Lambda_{q_1} = \Lambda_{q_2}$ we have

$$\int_\Omega (q_1 - q_2)u_1 u_2 = 0 \tag{5.4}$$

for all u_i which solve $L_{q_i} u_i = 0\,, \quad i = 1, 2$. We let

$$u_i = e^{x\cdot\rho_i}(1 + \psi_{q_i}(x, \rho_i)) \tag{5.5}$$

with ρ_i as in Theorem 4.1 and choose (in order to guarantee (4.2))

$$\begin{aligned} \rho_1 &= \frac{\eta}{2} + \frac{i(r\omega + k)}{2} \\ \rho_2 &= -\frac{\eta}{2} + \frac{i(-r\omega + k)}{2} \end{aligned} \tag{5.6}$$

where $\eta, \omega, k \in \mathbf{R}^n$, $|\omega| = 1$, $r \in \mathbf{R}$ with

$$\eta \cdot k = \eta \cdot \omega = \omega \cdot k = 0$$

and

$$|\eta|^2 = r^2 + k^2\,.$$

Substituting (5.5) into (5.4) gives

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2) = -\int_\Omega e^{ix\cdot k}(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})(q_1 - q_2)\,. \tag{5.7}$$

However, the estimate (4.5) implies that $\psi_{q_i} \to 0$ in $\overline{\Omega}$ as $r \to \infty$. Therefore

$$\widehat{q_1}(k) = \widehat{q_2}(k)$$

and thus

$$q_1 = q_2\,.$$

A proof of part (a) follows from the fact that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\Lambda_{q_{\gamma_1}} = \Lambda_{q_{\gamma_2}}$ with $q_{\gamma_i} = \frac{\Delta\sqrt{\gamma_i}}{\sqrt{\gamma_i}}$ because of (1.9) and Theorem (2.1). Now it is easy to check (see for example [S-U II]) that $q_{\gamma_1} = q_{\gamma_2}$ implies $\gamma_1 = \gamma_2$. $\square$

A very interesting problem is to extend the uniqueness result above to the case of piecewise continuous conductivities. Isakov [Is II] has proven such a result for conductivities with jump type singularities across the boundary of an open bounded subset of Ω.

Alessandrini ([Al]) used the identity (5.3), the special solutions of Theorem (4.1) and the continuous dependence estimate at the boundary (Theorem 2.4) to prove a stability estimate (i.e. a logarithmic continuous dependence result, which depends on an a-priori bound in a high Sobolev norm) for the conductivity.

Theorem 5.8. *Let $s > \frac{n}{2}, n \geq 3$, $\gamma_i \in H^{s+2}(\Omega)$ with*

$$0 < \alpha \leq \gamma_i(x) \quad x \in \overline{\Omega}$$

and

$$\|\gamma_i\|_{H^{s+2}(\Omega)} \leq \frac{1}{\alpha}, \quad i = 1, 2.$$

Then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C_\alpha w(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})$$

where

$$w(t) = (\frac{1}{-\log t})^\delta, \quad 0 < t < 1$$

and δ, $0 < \delta < 1$ depends only on n and s.

It is not known whether this is the best possible continuous dependence result. However, for conductivities having special features better continuous dependence results are known. Friedman and Vogelius ([F-V]) have shown that if one seeks to find spheres of zero or infinite conductivity inside a medium with ambient constant conductivity, then the radii and diameters are Lipschitz continuous functions of the measurements in two dimensions. It would be useful to understand the mechanism of ill-posedness in the general problem in order to better study special problems where the dependence could be better.

6. Complex frequency Born approximation, $n \geq 3$

In this section we discuss briefly the relationship between the Dirichlet to Neumann map Λ_q and the function T defined in the $\overline{\partial}$-approach to multidimensional inverse scattering theory by Ablowitz and Nachman ([N-A]) and Beals and Coifman ([B-C I]). In one dimension it had been developed earlier in [B-C II]. For more details the reader should look at those papers and the more recent ones like [N-H], [N] and [No] and the references indicated there.

Let us assume $q \in L^\infty(\Omega)$ with $q = 0$ outside Ω. The scattering amplitude can then be written in terms of the outgoing eigenfunction (see for example [Ag])

$$a(\lambda, \theta, \omega) = c_n \int e^{-i\lambda x\cdot\theta} q(x)\psi_+(\lambda, x, \omega)dx, \tag{6.1}$$

where $\lambda \in \mathbf{R}, \theta, \omega \in S^{n-1}$ and $\psi_+(\lambda, x, \omega)$ is the outgoing eigenfunction of $-\Delta + q$ i.e. ψ_+ is the solution of the Lippmann-Schwinger equation

$$\psi_+(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} - \int G_\lambda^+(x-y)q(y)\psi_+(\lambda, y, \omega)dy \tag{6.2}$$

where G_+^λ is the outgoing Green's kernel

$$G_\lambda^+(x) = (2\pi)^{-n}\int \frac{e^{ix\cdot k}}{k^2 - \lambda^2 - i0}dk\,. \tag{6.3}$$

The outgoing eigenfunction ψ_+ has the asymptotic expression for large $|x|$ (see [Ag])

$$\psi_+(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} + \frac{a(\lambda, \theta, \omega)}{|x|^{\frac{n-1}{2}}}e^{i\lambda|x|} + O(|x|^{\frac{-(n+1)}{2}}), \tag{6.3$'$}$$

where $\theta = \frac{x}{|x|}$.
Moreover the following estimate holds (see [Ag])

$$\|\psi_+ - e^{i\lambda x\cdot\omega}\|_{L^2_\delta} \le \frac{C}{\sqrt{\lambda}}\|q\|_{L^2_\delta}, \quad \delta < -\frac{1}{2}. \tag{6.4}$$

¿From (6.4) and (6.1) it is easy to derive the Born approximation for the scattering amplitude.

Faddeev [F] proposed to construct exponentially growing eigenfunctions of

$$(-\Delta + q)u(x, \zeta) = \zeta^2 u(x, \zeta) \tag{6.5}$$

where $\zeta \in \mathbf{C}^n$ is arbitrary but non-real, by solving the integral equation

$$u(x, \zeta) = e^{x\cdot\zeta} - \int G_\zeta(x-y)q(y)u(y, \zeta)dy \tag{6.6}$$

where $G_\zeta(x)$ is a new Green's kernel for $\Delta - \zeta^2$:

$$G_\zeta(x) = \frac{1}{(2\pi)^n}e^{x\cdot\zeta}\int \frac{e^{ix\cdot k}}{-|k|^2 + 2i\zeta\cdot k}\,dk\,. \tag{6.7}$$

Notice that G_ζ satisfies formally

$$(\Delta - \zeta^2)G_\zeta = \delta(x)\,. \tag{6.8}$$

Faddeev proposed using these generalized eigenfunctions for complex parameters ζ with imaginary part tending to zero as a generalization to 3 dimensions of the Gelfand-Levitan approach to inverse scattering in one dimension.

Notice that $h_\zeta(x) = e^{-x\cdot\zeta}G_\zeta(x)$ is the solution of

$$(\Delta + 2\zeta\cdot\nabla)h_\zeta = \delta(x)\,. \tag{6.9}$$

Proposition (4.8) implies, for $\zeta\cdot\zeta = 0$ and $|\zeta|$ large, the integral equation (6.6) has a unique solution. These generalized eigenfunctions were also considered by Ablowitz and Nachman ([N-A]) and Beals and Coifman [B-C I,II] in their $\overline{\partial}$-approach to the study of the scattering amplitude. In particular, in analogy with (6.1) they considered the function

$$T_q(k,\zeta) = \int e^{-ix\cdot k}q(x)u(x,\zeta)dx \tag{6.10}$$

with u solution of (6.6). The point is that the compatibility conditions for the $\overline{\partial}$-equation leads to compatibility conditions for the range of the map

$$q \xrightarrow{T} T_q. \tag{6.11}$$

Henkin and Novikov ([N-H]) gave a characterization of T for sufficiently smooth potentials (the derivation in [N-A] is formal and Beals and Coifman [B-C] gave proofs for small potentials and $\zeta\cdot\zeta = 0$). The relationship between $T(k,\zeta)$ and the physical scattering amplitude has been studied ([L-N]) and [N-H] but there is still not complete understanding of this. We want to point out here the relation between $T(k,\zeta)$ (or rather a closely related function; see below) and Λ_q. For this we shall give yet another proof of Theorem 4.1 which appeared in [N-S-U]. We define

$$t(k,\rho) = \int e^{-ix\cdot k}e^{-x\cdot\rho}q(x)u(x,\rho)dx \tag{6.12}$$

where $u(x,\rho)$ is the solution of $L_q u = 0$ in Theorem 4.1 and we require, for $k \in \mathbf{R}^n$, that $\rho \in \mathbf{C}^n$ satisfy:

$$\rho\cdot\rho = 0, \quad (ik+\rho)\cdot(ik+\rho) = 0, \quad |\rho| > \|(1+|x|^2)^{\frac{1}{2}}q(x)\|_{L^\infty}\,. \tag{6.13}$$

We then have

$$\Delta e^{ix\cdot k + x\cdot\rho} = 0, \quad \Delta u = qu. \tag{6.14}$$

Using (6.14) and Green's theorem we see that

$$t(k,\rho) = \int_{\partial\Omega} e^{-ix\cdot k} e^{-x\cdot\rho} [\Lambda_q u|_{\partial\Omega} + (ik+\rho)\cdot \nu u|_{\partial\Omega}]\, dS \tag{6.15}$$

with dS euclidean surface measure on $\partial\Omega$.

Hence we can compute $t(k,\rho)$ for (k,ρ) satisfying (6.13)) if we know Λ_q and the boundary values of the special solution $u(x,\rho)$. moreover, we prove next (see [S-U, I]) that $u|_{\partial\Omega}$ is actually determined uniquely by Λ_q.

Proposition 6.16. *Let $q_1, q_2 \in L^\infty(\Omega)$ such that*

$$\Lambda_{q_1} = \Lambda_{q_2}.$$

Let u_1, u_2 be solution of

$$L_{q_i} u_i = 0, \ i = 1,2$$

as in Theorem 4.1. Then

$$u_1|_{\Omega^c} = u_2|_{\Omega^c}$$

Proof. Let us consider the solution of

$$L_{q_1} w = 0 \tag{6.17}$$

$$w|_{\partial\Omega} = u_2.$$

Let us define

$$z = \begin{cases} w & \text{in } \Omega \\ u_2 & \text{in } \Omega^c. \end{cases} \tag{6.18}$$

Now, z obviously satisfies (6.17) in $\mathbf{R}^n \backslash \partial\Omega$; in addition,

$$\frac{\partial z}{\partial \nu}|_{\partial\Omega} = \Lambda_{q_1} z|_{\partial\Omega} = \Lambda_{q_1}(u_2|_{\partial\Omega}) = \Lambda_{q_2}(u_2|_{\partial\Omega}) = \frac{\partial u_2}{\partial \nu}|_{\partial\Omega}. \tag{6.19}$$

Hence $z \in C^{1,1}(\overline{\Omega})$ and solves (6.17) in all of $\mathbf{R}^n$. Because z satisfies the required growth conditions at ∞, the uniqueness part of Theorem 4.1 implies that $w = u_1$, concluding the proof. □

Proposition (6.16) implies that, if $\Lambda_{q_1} = \Lambda_{q_2}$, then $t_1(k,\rho) = t_2(k,\rho)$ with (k,ρ) as in (6.13).

Now for these (k, ρ)

$$\lim_{|\rho|\to\infty} t(k,\rho) = \int e^{-ix\cdot k} q(x)dx = \widehat{q}(k)\,. \tag{6.20}$$

Proposition (6.16) and (6.20) provide another proof of Theorem 5.1. Equation (6.20) may be thought of as an analog of the Born-approximation for complex-frequencies. Nachman ([N]) observed that $u(\cdot,\rho)|_{\partial\Omega}$ as in (4.4) satisfies a Fredholm integral equation on the boundary. Because $q = 0$ in Ω^c, $u(x,\rho)$ must satisfy

$$\Delta u = 0 \text{ in } \Omega^c \tag{6.21}$$

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega} = \Lambda_q u|_{\partial\Omega}.$$

Because u has the same asymptotics as G_ρ in (6.7), it must be a combination of the single and double layer potentials

$$S_\rho f(x) = \int_{\partial\Omega} G_\rho(x-y) f(y) dS_y \tag{6.22}$$

$$B_\rho f(x) = \int_{\partial\Omega} \frac{\partial G_\rho}{\partial \nu}(x-y) f(y) dS_y. \tag{6.23}$$

Nachman showed that $u(x,\rho)|_{\partial\Omega}$ was the unique solution to

$$f(x,\rho) = e^{x\cdot\rho} - (S_\rho\Lambda_q - B_\rho - \frac{1}{2}) f(x,\rho) \tag{6.24}$$

for every $x \in \partial\Omega$.

The point is that equation (6.24) does not depend on q and therefore provides a direct method for finding $u(x,\rho)|_{\partial\Omega}$ without a priori knowledge of q. Novikov [No] studied similar integral equations.

7. The two dimensional case

The Schwartz kernel of the Dirichlet to Neumann map is a distribution of $(n-1)+(n-1) = 2n-2$ variables, while the conductivity itself is a function of n variables. Hence the inverse conductivity problem is formally overdetermined in dimension $n \geq 3$ and formally determined in dimension 2. This is reflected in the lack of freedom to choose enough exponential solutions as in the proof of Theorem 4.1. The first result in this case was proven by Sylvester and Uhlmann [S-U III] for conductivities (resp. potentials) close to constant (resp. zero).

Theorem 7.1.

(a) Let $\gamma_i \in W^{3,\infty}(\Omega)$, $i = 1, 2$ with positive lower bound. There exists $\varepsilon > 0$ such that if

$$\|\gamma_i - 1\|_{W^{3,\infty}(\Omega)} < \varepsilon, \quad i = 1, 2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

(b) Let $q_i \in W^{1,\infty}(\Omega)$ such that L_{q_i}, $i = 1, 2$ does not have zero as an eigenvalue. There exists $\varepsilon > 0$ such that if

$$\|q_i\|_{W^{1,\infty}(\Omega)} < \varepsilon, \quad i = 1, 2$$

and

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Brief Sketch of proof. Again we only indicate how to prove (b). As in the proof of Theorem 5.1, we substitute the special solutions (5.5) into the identity (5.4). However, in two dimensions we may not choose ρ_i as in (5.6), but must be content with

$$\rho_1 = \frac{l + ik}{2} \qquad (7.2)$$

$$\rho_2 = \frac{-l + ik}{2}$$

where $l \cdot k = 0$ and $|l|^2 = |k|^2 = \frac{1}{2}|\rho|^2$ is sufficiently large. This yields estimates for the Fourier transform of $q_1 - q_2$ for all sufficiently large frequencies.

We may estimate the Fourier transform of $q_1 - q_2$ at sufficiently low frequencies by inserting into (5.4) solutions of $L_{q_i}\widetilde{u}_i = 0$ of the form

$$\widetilde{u}_1 = e^{x\cdot\rho} + \delta\widetilde{u}_1, \quad \delta\widetilde{u}_1|_{\partial\Omega} = 0, \qquad (7.3)$$

$$\widetilde{u}_2 = e^{-x\cdot\widetilde{\rho}} + \delta\widetilde{u}_2, \quad \delta\widetilde{u}_2|_{\partial\Omega} = 0.$$

If q_1 q_2 are small enough, both estimates combine to produce an inequality which can be satisfied only when $q_1 - q_2$ is identically zero. □

The uniqueness question for the inverse conductivity problem for smooth conductivity remains open. We report in this section on the progress obtained. The "transport" equation (4.7) has special features in two dimensions.

Let $\rho \in \mathbf{C}^2$ be such that

$$\rho \cdot \rho = 0, |\rho| > \|(1+|x|^2)^{\frac{1}{2}} q\|_{L^\infty}. \tag{7.4}$$

We write such a ρ in the form

$$\rho = \frac{\eta + ik}{2}, \eta \cdot k = 0, |\eta| = |k|; \eta, k \in \mathbf{R}^2, \quad k = (k_1, k_2).$$

Then the equation for ψ in two dimensions can be written in the form

$$\overline{\partial}\partial\psi + (k_2 + ik_1)\partial\psi - q\psi = q \tag{7.5}$$

where

$$\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}), \quad \partial = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}). \tag{7.6}$$

In [S-U, III] it was proven that ψ can be written in the form

$$\psi(x,k) = \frac{a(x)}{k_2 + ik_1} + \frac{b(x,k)}{(k_2 + ik_1)^2} \tag{7.7}$$

with

$$\|a\|_{H^1_\delta}, \|a\|_{L^\infty(\Omega)}, \|b\|_{H^1_\delta} \leq C\|q\|_{W^{1,\infty}(\Omega)}. \tag{7.8}$$

Moreover a solves

$$\overline{\partial} a = q \tag{7.9}$$

¿From Proposition (6.16) and the expansion (7.7) we conclude

Proposition 7.10. *Suppose $q_i \in L^\infty(\Omega)$, $i = 1, 2$, $q_i = 0$ in Ω^c and L_{q_i} has not zero as eigenvalue. Suppose*

$$\Lambda_{q_1} = \Lambda_{q_2}$$

then

$$a_1 = a_2 \text{ in } \Omega^c$$

where a_i are as in (7.7).

We can write a_i in terms of q_i using the Cauchy integral representation

(7.11) $$a_i(x) = \frac{1}{2\pi i} \int \frac{q_i(w)}{x-w} dw \wedge d\overline{w} \, .$$

For $|x|$ sufficiently large, we can write

(7.12) $$a_i(x) = \frac{1}{2\pi i x} \sum_{n=0}^{\infty} \int q_i(w) \frac{w^n}{x^n} dw \wedge d\overline{w}$$

Therefore we conclude from Proposition 7.10 the following result proven in [S-U, IV] and [Su, I]

Theorem 7.13. *(a) Let γ_i, $i = 1,2$ be in $W^{3,\infty}(\Omega)$ with a positive lower bound. Assume Ω simply connected and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\int_\Omega (q_{\gamma_1} - q_{\gamma_2})h = 0$$

for all h harmonic in $\overline{\Omega}$.

(b) Let q_i be in $W^{1,\infty}(\Omega)$ so that L_{q_i} has no eigenvalue 0, $i = 1,2$. Assume Ω simply connected and

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$\int_\Omega (q_1 - q_2)h = 0$$

for all h harmonic in $\overline{\Omega}$.

In particular one can prove the global uniqueness result

Corollary 7.14. *Let $\gamma_1 \in W^{3,\infty}(\Omega)$ with a positive lower bound. Suppose $\gamma_2 =$ constant > 0 and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2 = \text{ constant.}$$

Sun ([Su, II]) has observed that Theorem 7.13 gives the following global uniqueness result for conductivities:

Theorem 7.15. *Let $\gamma_i \in W^{3,\infty}(\Omega)$, $i = 1,2$ with positive lower bound. Assume γ_2^α is harmonic for some $\alpha \in \mathbf{R}$ or $\log \gamma_2$ is harmonic. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

Sun also gave a logarithmic continuous dependence result for conductivities γ_1, γ_2 as in the hypothesis of Theorem 7.15 under an priori $C^4(\overline{\Omega})$ bound on γ_i's, $i = 1,2$. For local uniqueness Sun ([Su, II]) improved on the local result, Theorem 7.1, to prove

Theorem 7.16. *Let $\gamma_i \in W^{3,\infty}(\Omega)$, $i = 1,2$ with positive lower bound. Let $\gamma_0 \in C^3(\overline{\Omega})$ be such that either (a) γ_0^α is harmonic for some $\alpha \in \mathbf{R}$ or (b) $\gamma_0 = e^{\operatorname{Re}\phi}$ where ϕ is an injective conformal map in $\overline{\Omega}$. Then there is $\varepsilon > 0$ such that if*

$$\|\gamma_i - \gamma_0\|_{W^{3,\infty}(\Omega)} < \varepsilon\,, \quad i = 1,2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

All the results assume some a priori restriction on the conductivities or potentials besides smoothness. Recently Sun and Uhlmann [Su-U I] proved that for almost all conductivities or potentials injectivity and local injectivity for the map Φ and $\widetilde{\Phi}$ holds. More precisely:

Theorem 7.17. *(a) There exists an open and dense set $\mathcal{O}$ in $W^{3,\infty}_{pos}(\Omega)$*. If $\gamma \in \mathcal{O}$ there exists an $\varepsilon > 0$ such that if*

$$\|\gamma_i - \gamma\|_{W^{3,\infty}(\Omega)} < \varepsilon\,, \quad i = 1,2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

* *$W^{3,\infty}_{pos}(\Omega)$ denotes the set of positive functions in $W^{3,\infty}(\Omega)$.*

(b) There exists an open and dense set $\mathcal{O}$ in $W^{1,\infty}(\Omega)$. If $q \in \mathcal{O}$ there exists $\varepsilon > 0$ such that if

$$\|q_i - q\|_{W^{1,\infty}(\Omega)} < \varepsilon\,, i = 1, 2$$

and

$$\Lambda_{q_1} = \Lambda_{q_2}$$

then

$$q_1 = q_2.$$

For global uniqueness it was proven in [Su-U I]:

Theorem 7.18.

(a) There exists an open dense set $\mathcal{O}$ in $W^{3,\infty}_{pos}(\Omega) \times W^{3,\infty}_{pos}(\Omega)$, such that if $(\gamma_1, \gamma_2) \in \mathcal{O}$ and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then

$$\gamma_1 = \gamma_2\,.$$

(b) There exists an open dense set $\mathcal{O}$ in $W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ such that if $(q_1, q_2) \in \mathcal{O}$ and

$$\Lambda_{q_1} = \Lambda_{q_2}$$

then

$$q_1 = q_2.$$

Sketch of proof.

We indicate how to prove part (b) of Theorems 7.17 and 7.18. Part (a) follows in a similar way to the proof of part (a) of Theorem 5.1 from part (b).

The proof of Theorem (7.17) is reduced to show that

Lemma 7.19. *Let $q \in L^\infty(\Omega)$. Then*

$$D_q = \{uv; u, v \text{ are solutions of } L_q u = L_q v = 0 \text{ in } \Omega\}$$

is complete in $L^2(\Omega)$ for $q \in \mathcal{O}$ where $\mathcal{O}$ is an open and dense set in $W^{1,\infty}(\Omega)$.

Sketch of proof of Lemma 7.19

Consider the q's in $W^{1,\infty}(\Omega), q = 0$ in Ω^c with $\|q\|_{W^{1,\infty}(\Omega)} < R$. By Theorem 4.1 there exists $L_R > 0$ and solutions u, v of $L_q u = L_q v = 0$ in $\mathbf{R}^2$ of the form

$$\begin{aligned} u_1(x,k) &= e^{x\cdot\rho}(1 + \psi(x,k)) \text{ for } |k| > L_R \\ u_2(x,k) &= e^{-x\cdot\overline{\rho}}(1 + \widetilde{\psi}(x,k)) \text{ for } |k| > L_R\,. \end{aligned} \tag{7.20}$$

We require further that L_q does not have zero as an eigenvalue (this set of q's is easily seen to be open and dense in $W^{1,\infty}(\Omega)$) and denote by $\widetilde{u}_i$ the solutions of the Dirichlet problem (7.3).

Next we define the operator

$$A_q f(k) = \begin{cases} \int f e^{ix\cdot k} + \int_\Omega e^{ix\cdot k} f(\psi + \widetilde{\psi} + \psi \cdot \widetilde{\psi}) & \text{for } |k| > L_R \\ \int f e^{ixk} + \int_\Omega f(\widetilde{u}_1 \widetilde{u}_2 - e^{ix\cdot k}) & \text{for } |k| \leq L_R \end{cases} \tag{7.22}$$

The operator M_q is defined by

$$A_q f = \widehat{f} + M_q f \tag{7.23}$$

and K_q is defined by taking the inverse Fourier transform

$$(A_q f)^\vee = f + K_q f. \tag{7.24}$$

It is easy to see that D_q is complete in $L^2(\Omega)$ if A_q is injective in $L^2(\mathbf{R}^2)$.

The next two propositions are the main technical points of the proof. Because of the decay in $|k|$ of the lower order terms ψ and $\widetilde{\psi}$ and the representation (7.7) one can prove:

Proposition (7.25). *$K_q : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is compact .*

Moreover the explicit construction of ψ as in (7.7) allows to prove

Proposition (7.26). *K_q depends analytically on q, that is, $K_{q_0+\lambda q_1} f$ has a convergent power series in $L^2(\mathbf{R}^2)$ for those λ's so that $\|q_0+\lambda q_1\|_{W^{1,\infty}(\Omega)} < R$ and $L_{q_0+\lambda q_1}$ does not have zero as an eigenvalue.*

Then for $\lambda \in \mathbf{C}$

$$(A_{\lambda q} f)^\vee = (Id + K_{\lambda q}) f\,,$$

$K_{\lambda q}$ is an analytic function of λ for λ's so that $|\lambda| \|q\|_{W^{1,\infty}(\Omega)} < R$ and $L_{\lambda q}$ does not have zero as eigenvalue. By the analytic Fredholm theorem then $(A_{\lambda q} f)^\vee$ is an isomorphism except for a discrete set of λ's. This sketches the proof of Theorem 7.17. For more details see [Su-U]. □

The proof of the global result Theorem 7.18 proceeds along similar lines. If $\Lambda_{q_1} = \Lambda_{q_2}$ relation (5.4) motivates the definition of a similar operator to K_q above. Let $q_1, q_2 \in W^{1,\infty}(\Omega)$ so that $\|q_i\|_{W^{1,\infty}(\Omega)} < R$ and L_{q_i}, $i = 1, 2$ does not have zero as eigenvalue. We define

$$(A_{q_1,q_2} f)^\vee = f + K_{q_1,q_2} f \tag{7.27}$$

where

$$(K_{q_1,q_2}f)^\wedge = \begin{cases} \int_\Omega e^{ix\cdot k} f(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2}), |k| > L_R \\ \int_\Omega f(\widetilde{u}_{q_1}\widetilde{u}_{q_2} - e^{ix\cdot k}), |k| \leq L_R \end{cases},$$

$$u_i = e^{x\cdot\rho}(1 + \psi_{q_i}), \quad |k| > L_R \tag{7.28}$$

is a solution of $L_{q_i}u_i = 0$ $i = 1,2$ as in theorem 4.1, and $\widetilde{u}_{q_i}$ solves

$$L_{q_i}\widetilde{u}_{q_i} = 0; \quad \widetilde{u}_{q_1}|_{\partial\Omega} = e^{x\cdot\rho}|_{\partial\Omega}; \quad \widetilde{u}_{q_2}|_{\partial\Omega} = e^{-x\cdot\overline{\rho}}|_{\partial\Omega}. \tag{7.29}$$

Again, $K_{q_1,q_2} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is compact and $K_{\lambda q_1,\lambda q_2}$ depends analytically on λ for λ such that $\|\lambda q_i\|_{W^{1,\infty}} < R$, $i = 1,2$. Then by the analytic Fredholm theorem $A_{\lambda q_1,\lambda q_2}$ is an isomorphism except for a discrete set of λ's.

Now if $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 - q_2$ is in the kernel of A_{q_1,q_2} (see (5.4)). Then for an open dense set $\mathcal{O}$ in $W^{1,\infty} \times W^{1,\infty}$ if $(q_1, q_2) \in \mathcal{O}$ and $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$. This finishes the sketch of proof of Theorem 7.17. □

8. Determining Lamé parameters by boundary measurements

Another inverse boundary value problem which arises in applications is to determine the elastic properties of a material by measuring the stress energy to maintain it in a prescribed shape. We formulate below more precisely the mathematical problem.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary which will be considered in this paper as a linear, inhomogeneous, isotropic, elastic medium. The elastic properties of Ω are determined by the pair of Lamé parameters $\gamma = (\lambda, \mu) \in L^\infty(\Omega)$. Moreover we assume the strong convexity assumption

$$\mu > 0, n\lambda + 2\mu > 0 \text{ on } \Omega. \tag{8.1}$$

Under the assumption (8.1) we can solve uniquely, with $\vec{u} \in H^1(\Omega)$, the displacement boundary value problem:

$$\begin{cases} (L_\gamma \overrightarrow{u})_i = \sum_{j,k,\ell=1}^n \partial_{x_j}(c_{ijk\ell}\partial_{x_k}u_\ell) = 0 \quad \text{in } \Omega, i = 1,\dots,n, \\ \overrightarrow{u}|_{\partial\Omega} = \overrightarrow{\phi} \in H^{\frac{1}{2}}(\partial\Omega), \end{cases} \tag{8.2}$$

where the elasticity tensor is given by

$$c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \tag{8.3}$$

the displacement vector is denoted by $\overrightarrow{u} = (u_1, \dots, u_n)$, and δ_{ij} denotes the Kronecker delta.

Associated to the displacement vector $\overrightarrow{u}$, there are two tensor fields

$$\epsilon(\overrightarrow{u}) = \frac{1}{2}(\nabla \overrightarrow{u} +^t \nabla \overrightarrow{u}) \tag{8.4}$$

and

$$\tau(\overrightarrow{u}) = \lambda(\text{trace } \epsilon(\overrightarrow{u}))I + 2\mu\epsilon(\overrightarrow{u}) \tag{8.5}$$

which are called the strain tensor and stress tensor respectively. Here $\overrightarrow{u} = (u_1, \dots, u_n)$, $\nabla \overrightarrow{u} = (\partial_{x_j} u_i; \begin{smallmatrix} i \downarrow 1, \dots, n \\ j \rightarrow 1, \dots, n \end{smallmatrix})$, trace $\epsilon(\overrightarrow{u}) = \Sigma_{j=1}^n \partial_{x_j} u_j$. The equation in (8.2) simply means that the stress tensor is divergence free (i.e. there no source or sinks of stress):

$$L_\gamma \overrightarrow{u} = \nabla \cdot \tau(\overrightarrow{u}) = 0 \text{ in } \Omega. \tag{8.6}$$

The energy associated to a solution $\overrightarrow{u}$ of (8.6) is given by

$$\begin{aligned} Q_\gamma(\overrightarrow{\phi}) &= \inf_{\overrightarrow{v} \in H^1(\Omega)} \int_\Omega \text{trace}(\tau(\overrightarrow{u})\overline{\epsilon(\overrightarrow{v})})dx \\ &= \int_\Omega \sum_{i,j,k,\ell} c_{ijk\ell} \partial_{x_j} u_i \overline{\partial_{x_\ell} u_k} dx \\ &= \int_\Omega \{\lambda |\text{ div } \overrightarrow{u}|^2 + 2\mu|\epsilon(\overrightarrow{u})|^2\} dx. \end{aligned} \tag{8.7}$$

The stress energy form obtained by polarization of (8.7) is given by

$$Q_\gamma(\overrightarrow{\phi}, \overrightarrow{\psi}) = \int_\Omega (\lambda \text{ div } \overrightarrow{u} \cdot \overline{\text{div } \overrightarrow{u}} + 2\mu\epsilon(\overrightarrow{u}) \cdot \overline{\epsilon(\overrightarrow{v})}dx \tag{8.8}$$

where $\overrightarrow{u}$, $\overrightarrow{v}$ are solutions of

$$L_\gamma \overrightarrow{u} = L_\gamma \overrightarrow{v} = 0 \quad \text{in } \Omega, \overrightarrow{u}|_{\partial\Omega} = \overrightarrow{\phi}, \overrightarrow{v}|_{\partial\Omega} = \overrightarrow{\psi}. \tag{8.9}$$

By using Green's theorem one can easily prove that

$$Q_\gamma(\overrightarrow{\phi}, \overrightarrow{\psi}) = \int_{\partial\Omega} \tau(\overrightarrow{u})\nu \cdot \overline{\overrightarrow{v}} dS = \int_{\partial\Omega} \Lambda_\gamma \overrightarrow{\phi} \cdot \overline{\overrightarrow{\psi}} dS \tag{8.10}$$

where ν denotes the unit outer normal to $\partial\Omega$ and dS denotes surface measure on $\partial\Omega$. The Dirichlet to Neumann map is defined by

$$(8.11) \qquad (\Lambda_\gamma \overrightarrow{\phi})_i = (\tau(\overrightarrow{u})|_{\partial\Omega} \cdot \nu)_i = \sum_{j,k,\ell=1}^{n} \nu_j c_{ijk\ell} \partial_{x_k} u_\ell|_{\partial\Omega}.$$

Physically, $\Lambda_\gamma \overrightarrow{f} = T\overrightarrow{u}$ where T measures traction on the boundary.

The inverse problem is whether knowledge of $\Lambda_\gamma \overrightarrow{\phi}$ for any $\overrightarrow{\phi} \in H^{\frac{1}{2}}(\partial\Omega)$, which involves only boundary measurements, determines the Lamé parameters λ and μ in $\overline{\Omega}$. That is we want to determine the injectivity of the map

$$L^\infty(\Omega) \times L^\infty(\Omega) \ni \gamma = (\lambda, \mu) \xrightarrow{\Lambda} \Lambda_\gamma.$$

Because of (8.10) knowledge of the selfadjoint map

$$\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is equivalent to knowledge of $Q_\gamma(\overrightarrow{\phi}, \overrightarrow{\psi})$ for any $\overrightarrow{\phi}, \overrightarrow{\psi} \in H^{\frac{1}{2}}(\partial\Omega)$.

In [N-U] it was proven, in two dimensions, local injectivity of Λ in a $W^{31,\infty}(\Omega)$ neighborhood of constant λ, μ.

Let $\gamma_* = (\lambda_*, \mu_*)$ denote a pair of constant Lamé parameters in Ω satisfying (8.1). Then we have

(8.12) Theorem. *Let $n = 2$. There exists $\epsilon > 0$ such that if $\gamma_j = (\lambda_j, \mu_j)$ satisfy* (8.1),

$$\|\lambda_j - \lambda_*\|_{W^{31,\infty}(\Omega)} + \|\mu_j - \mu_*\|_{W^{31,\infty}(\Omega)} < \epsilon$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then $\gamma_1 = \gamma_2$ on $\overline{\Omega}$.

There are several new difficulties in extending the method used in [S-U II] for the conductivity problem to this case. First of all L_γ is an elliptic system and second we have to determine two functions λ, μ of $\gamma = (\lambda, \mu)$. To underscore these difficulties let us look at the linearized problem. The Fréchet derivative of Λ at a constant pair $\gamma_* = (\lambda_*, \mu_*)$ in a direction $h = (h_1, h_2)$ is given by

$$(8.13) \qquad \begin{aligned} &(d\Lambda_{\gamma_*}(h)(\overrightarrow{u}_*|_{\partial\Omega}), (\overrightarrow{v}_*|_{\partial\Omega})) = \\ &\int_\Omega \{h_1 \operatorname{div} \overrightarrow{u}_* \cdot \overline{\operatorname{div} \overrightarrow{v}_*} + 2h_2(\epsilon(\overrightarrow{u}_*) \cdot \overline{\epsilon(\overrightarrow{v}_*)})\} dx \end{aligned}$$

where $\overrightarrow{u}_*, \overrightarrow{v}_*$ are solutions of

$$L_{\gamma_*}(\overrightarrow{u}_*) = L_{\gamma_*}(\overrightarrow{v}_*) = 0 \quad \text{in } \mathbf{R}^2. \tag{8.14}$$

We first construct analogous solutions of (8.14) to the ones considered by Calderón for the linearized problem for the inverse conductivity problem. Namely we take

$$\overrightarrow{u}_* = \nabla e^{x\cdot\zeta}, \overrightarrow{v}_* = \nabla e^{-x\cdot\overline{\zeta}} \text{ with } \zeta \in \mathbf{C}^2, \zeta\cdot\zeta = 0. \tag{8.15}$$

Notice that $\overrightarrow{u}_*, \overrightarrow{v}_*$ are vector-valued harmonic functions. Substituting (8.15) in (8.13) we find that

$$(d\Lambda_*(h)(\overrightarrow{u}_*|_{\partial\Omega}))\overline{(\overrightarrow{v}_*|_{\partial\Omega})}) = |k|^2 \int_\Omega 2h_2 e^{ix\cdot k}dx, \tag{8.16}$$

where

$$\Lambda_* = \Lambda_\gamma \text{ with } \gamma = \gamma_*$$

and

$$\zeta = \frac{1}{2}(Jk + ik), J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = (k_1, k_2) \in \mathbf{R}^2. \tag{8.17}$$

If $d\Lambda_*(h) = 0$, then we get by the Fourier inversion formula $h_2 = 0$ in Ω.

So we need different solutions of (8.14) to get information about h_1. Ikehata [I] used a different set of solutions of (8.14) other than (8.15) that allowed him to prove injectivity of the linearized map (8.16) at the constant pair γ_*.

Ikehata found these by constructing new solutions of the biharmonic operator Then he used the so called Boussineq-Somigliana- Garlekin method to construct solutions of the elasticity system at a constant pair. Namely if g solves

$$\Delta^2 g = 0 \text{ in } \Omega \tag{8.18}$$

then

$$u = (\lambda_* + 2\mu_*)\Delta g - (\lambda_* + \mu_*)\nabla(\nabla\cdot g) = F(g), \tag{8.19}$$

solves

$$L_{\gamma_*}(u) = 0 \quad \text{in } \Omega. \tag{8.20}$$

Ikehata considered

$$g_1 = -\frac{1}{2}|\zeta|^{-2}(x \cdot \overline{\zeta})e^{-x\cdot\zeta} \tag{8.21}$$

$$g_2 = -\frac{1}{2}|\overline{\zeta}|^{-2}(x \cdot -\zeta)e^{x\cdot\overline{\zeta}},$$

with ζ as in (8.17), as solutions of (8.18) and $u_* = F(g_1), v_* = F(g_2)$ as solutions of (8.14). Plugging these in (8.13) we find that
(8.22)

$$d\Lambda_{\gamma_*}(h)(u_*|_{\partial\Omega}, v_*|_{\partial\Omega}) = \mu_*^2\frac{|k|^2}{4}\int_\Omega e^{ix\cdot k}h_1(x)dx + \mu_*^2\frac{|k|^2}{4}\int_\Omega e^{ix\cdot k}h_2(x)dx.$$

$$+(\lambda_* + \mu_*)^2\frac{|k|^4}{8}\int_\Omega |x|^2 h_2(x)e^{ix\cdot k}dx$$

We already know that $h_2 = 0$ if $d\Lambda_{\gamma*}(h) = 0$, therefore we conclude that $h_1 = 0$ concluding the proof that the linearized problem is injective at constant Lamé parameters.

The main difficulty in the non-linear case is to construct for high frequencies the analog of the solutions (8.21). This was done in [N-U]. We outline some of the ideas.

Akamatsu, Nakamura and Steinberg [A-N-S] proved the analog of the Kohn-Vogelius result in this case. We have

(8.23) Theorem. *(Akamatsu, Nakamura, Steinberg) Let $\gamma_j \in C^2(\overline{\Omega})(j = 1,2)$ satisfying (8.1). Assume*

$$\Lambda_1 = \Lambda_2 \text{ where } \Lambda_j = \Lambda_\gamma \text{ with } \Lambda = \Lambda_j (j = 1,2).$$

Then

$$\partial^\alpha \gamma_1|_{\partial\Omega} = \partial^\alpha \gamma_2|_{\partial\Omega} \quad (|\alpha| \le 2).$$

Hence we may assume $\gamma_1 - \gamma_2 \in C_0^2(\overline{\Omega}), \gamma_j - \gamma_* \in C_0^2(B(0,r_0))(j = 1,2)$, where $B(0,r_0) = \{x \in \mathbf{R}^2; |x| < r_0\} \supset \overline{\Omega}$.

Another important fact is that in two dimensions one can diagonalize the elasticity system to a system whose principal part is the biharmonic operator Δ^2.

(8.24) Proposition. *Let $n = 2$ and $\gamma = (\lambda, \mu) \in C^2(\overline{\Omega})$ satisfying (8.1). Moreover let*

$$T(D) = \begin{bmatrix} D_1 & D_2 \\ D_2 & -D_1 \end{bmatrix}$$

where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. *Then*

(8.25)

$$T(D)L_\gamma(x,D)T(D) = \begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}(-\Delta^2)I + M_\gamma^3(x,D) + M_\gamma^2(x,D),$$

where

(8.26)

$$M_\gamma^3(x,\xi) = 2i|\xi|^2 \begin{bmatrix} \nabla(\lambda+2\mu)\cdot\xi & -(\xi\wedge\nabla)\mu \\ (\xi\wedge\nabla)\mu & \nabla\mu\cdot\xi] \end{bmatrix}, \xi\wedge\nabla = \xi_1\frac{\partial}{\partial x_2} - \xi_2\frac{\partial}{\partial x_1}$$

and $M_\gamma^2(x,D)$ *is a system of second order differential operators whose coefficients consist of second order derivatives of* γ.

Let

$$M_\gamma(x,D) = -\Delta^2 I + \begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}^{-1}(M_\gamma^3(x,D)) + M_\gamma^2(x,D)). \tag{8.27}$$

Factorizing $-\Delta$ in (1.25) we get

$$M_\gamma(x,D) = (-\Delta)\{\Delta I + \widetilde{M}_\gamma^1(x,D) + \widetilde{M}_\gamma^0(x,D)\} \tag{8.28}$$

where

$$\widetilde{M}_\gamma^1(x,\xi) = \begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}^{-1} M_\gamma^3(x,\xi)|\xi|^{-2} \tag{8.29}$$

and $\widetilde{M}_\gamma^0(x,D)$ is a pseudodifferential operator of order 0 such that

$$M_\gamma(x,D) = (-\Delta)\widetilde{M}_\gamma^0(x,D) \tag{8.30}$$

is a system of second order differential operator whose coefficients are p-th $(2 \le p \le 4)$ order derivatives of γ.

For each compact set high frequency solutions of

$$(\Delta I + \widetilde{M}_\gamma^1(x,D) + \widetilde{M}_\gamma^0(x,D)\overrightarrow{w} = 0 \text{ or a constant vector in } \mathbf{R}^2 \tag{8.31}$$

are constructed of the form

$$\overrightarrow{w} = e^{x\cdot\zeta}(A_0(x,\zeta) + A_{-1}(x,\zeta)), \zeta\in\mathbf{C}^2, \zeta\cdot\zeta = 0, \tag{8.32}$$

where $A_0(x,\zeta), |\zeta|A_{-1}(x,\zeta)$ are uniformly bounded. Here we remark that although $\overrightarrow{w}$ is constructed on a compact set, $\Delta\overrightarrow{w}$ has a natural extension to $\mathbf{R}^2$ so that it satisfies (8.31).

One difference with the conductivity equation is that in that case $A_0(x,\zeta)$ is independent of ζ (in fact $A_0(x,\zeta) = \gamma^{-\frac{1}{2}}$ where γ is the conductivity). Moreover one does not solve the transport equation for A_{-1} in a unique fashion in an appropriate weighted class. However, it is solved in every compact set. Everything works out since one can check that

$$\overline{D^1}\{L\gamma(x,D)T(D)\overrightarrow{w}\} \in L^p_\delta(\mathbf{R}^2) = L^p(\mathbf{R}^2;(1+|x|^2)^{\frac{p\delta}{2}}dx);$$
$$1 < p < \infty, -\frac{2}{p} < \delta < 1 - \frac{2}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where $\overline{D^k} = (\partial^\alpha/\partial x^\alpha; |\alpha| \leq k)$ for $k \in N$. Since $T(D)^2 = -\Delta I$ one gets by standard estimates that in fact

$$L_\gamma(x,D)T(D)\overrightarrow{w} = 0 \quad \text{in } \mathbf{R}^2.$$

One must also match the two types of low frequency solutions that are constructed (as in [I], but slightly different) with the high frequency solutions. Full details are in the paper [N-U].

9. Electrical impedence tomography; the anisotropic case

If the conductivity of Ω depends on direction then it is represented by a positive definite symmetric matrix $\gamma = (\gamma^{ij})$ in $\overline{\Omega}$ which we assume to be smooth. Kohn and Vogelius ([K-V III]]) suggested a constructive approach to the isotropic case based on an algorithm developed by Wexler et al ([W-F-N]). This consists of minimizing an appropriate functional. The functional is not quasi-convex and, therefore, a minimizing sequence will not in general converge to a solution, but will in general have limit points which are solutions to the "relaxed problem". Kohn and Vogelius computed the relaxation of one such problem which turned out to be a variational problem for an anisotropic conductivity. Numerical performance of this method has been recently studied by Kohn and McKenney ([K-M]). Thus the anisotropic problem occurs naturally even when considering isotropic conductivities. We now formulate more precisely the inverse conductivity problem in the anisotropic case.

The conductivity equation is

$$L_\gamma u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\gamma^{ij}\frac{\partial}{\partial x_j}u) = 0 \quad \text{in } \Omega. \tag{9.1}$$

The Dirichlet to Neumann map is defined by

$$\Lambda_\gamma f = \sum_{i,j=1}^{n} \nu^i\gamma^{ij}\frac{\partial u}{\partial x_j}|_{\partial\Omega}dS \tag{9.2}$$

where ν^i is the i-th component of the unit euclidean conormal, dS represents the $(n-1)$ dimensional euclidean surface measure on $\partial\Omega$ and u is the solution of the Dirichlet problem

$$(9.3) \qquad L_\gamma u = 0 \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = f,$$

It is convenient to define the Dirichlet to Neumann map as a $(n-1)$ form since in actual measurements one integrates the current flux rather than measure it pointwise. Moreover it helps to understand the invariance properties of Λ_γ under the action of diffeomorphisms. We have, again, using the divergence theorem

$$(9.4) \qquad \int_{\partial\Omega} g\Lambda_\gamma(f) = \int_\Omega \sum_{i,j=1}^n \gamma^{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dV$$

where dV is the euclidean volume element in Ω, u as in (8.3) and v solves

$$(9.5) \qquad L_\gamma v = 0 \quad \text{in } \Omega$$

$$v|_{\partial\Omega} = g.$$

Again, instead of considering the map

$$(9.6) \qquad \gamma \xrightarrow{\Phi} \Lambda_\gamma$$

we can consider the map

$$(9.7) \qquad \gamma \xrightarrow{Q} Q_\gamma$$

where Q_γ is the quadratic form

$$(9.8) \qquad Q_\gamma(f) = \int_\Omega \sum_{i,j=1}^n \gamma^{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dV$$

with u solution of (9.3).

Unfortunately injectivity of Φ (or Q) is not valid in the anisotropic case. The following observation can be found in [K-V IV]: Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism so that $\Psi|_{\partial\Omega} = Id$. Let

$$(9.9) \qquad \widetilde{\gamma} = \frac{(D\Psi)^T \cdot \gamma \cdot D\Psi) \circ \Psi^{-1}}{|detD\Psi|}$$

where $D\Psi$ denotes the differential of Ψ and $(D\Psi)^T$ its transpose. The relevant point is that

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \ (Q_{\widetilde{\gamma}} = Q_\gamma). \tag{9.10}$$

This is a consequence of the following observation:

Proposition 9.11. *Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism so that $\Psi|_{\partial\Omega} = Id$. Then if u solves*

$$L_\gamma u = 0 \quad \text{in } \Omega$$
$$u|_{\partial\Omega} = f,$$

then $u \circ \psi^{-1} = \widetilde{u}$ solves

$$L_{\widetilde{\gamma}} \widetilde{u} = 0 \quad \text{in } \Omega$$
$$\widetilde{u}|_{\partial\Omega} = f$$

with $\widetilde{\gamma}$ as in (9.9).

More generally, let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism so that $\Psi|_{\partial\Omega} = \psi$. Then

$$Q_\gamma(f) = Q_{\widetilde{\gamma}}(f \circ \psi^{-1}) \tag{9.12}$$

with $\widetilde{\gamma}$ as in (9.9).

We disgress a little to discuss the corresponding relation for Λ_γ given by (9.4). It is convenient to give an invariant interpretation of (9.9), (9.10) and (9.12). For more details see the discussion in the introduction in [S]. Ohm's law (or rather its differential version) in a wire is given by

$$i(x) = \gamma(x) du(x)$$

where $u(x)$ is the voltage potential, $i(x)$ the current flowing through x and $\gamma(x) = 1/\rho(x)$ where $\rho(x)$ is the resistivity.

In higher dimensions the current i is represented by an $(n-1)$ form. Then it is natural to interpret the conductivity as a map from 1 forms $(du(x))$ to $(n-1)$ forms $(i(x))$. The conductivity is then a map

$$\gamma : \Lambda^1(\overline{\Omega}) \to \Lambda^{n-1}(\overline{\Omega}) \tag{9.13}$$

which is symmetric and positive definite as explained below.

In standard Euclidean coordinates $x^1, \dots, x^n$ and ω_k the $(n-1)$ forms

$$\omega_k = (-1)^{k-1} dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n \tag{9.14}$$

then the components γ^{ij} of γ are given by

$$\gamma(dx^i) = \sum_{i=1}^{n} \gamma^{ij} \omega_j \,,$$

with γ^{ij} a symmetric, positive definite matrix in $\overline{\Omega}$.

If $\Psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism then the push forward of γ as in (9.13) is given by

$$(\Psi_* \gamma)\alpha = \Psi_*(\gamma(\Psi^* \alpha)) \tag{9.15}$$

where $\Psi^*\alpha$ denotes the pull back of the 1-form α and $\Psi_* = (\Psi^{-1})^*$ denotes the pull back by Ψ^{-1} acting on the $(n-1)$ form $\gamma(\Psi^*\alpha)$. In coordinates (9.15) reads

$$(\Psi_* \gamma(y))^{\ell m} = \frac{\frac{\partial \Psi^\ell}{\partial x^i} \gamma^{ij} \frac{\partial \Psi^m}{\partial x^j}}{|\det(\frac{\partial \Psi}{\partial x})|} \tag{9.16}$$

which is exactly the relation (9.9). Thus we may rewrite the relation (9.9) in an invariant way as

$$\widetilde{\gamma} = \Psi_* \gamma \tag{9.17}$$

Now we define the Dirichlet to Neumann map by

$$\Lambda_\gamma f = \gamma du|_{\partial\Omega}$$

which, in coordinates, is just (9.2).

If $\Psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism with $\Psi|_{\partial\Omega} = \psi$ we can define the push forward $\Psi_* \Lambda_\gamma$ by

$$(\psi_* \Lambda_\gamma) f = \psi_*(\Lambda_\gamma(\psi^* f))$$

where $\psi^* f = f \circ \psi^{-1}$. Then the relation (9.12) can be rewritten as

$$\Lambda_{\Psi_* \gamma} = \psi_* \Lambda_\gamma \tag{9.18}$$

Of course, if $\Psi|_{\partial\Omega} = Id$ we obtain

$$\Lambda_{\Psi_* \gamma} = \Lambda_\gamma \tag{9.19}$$

which is (9.10).

The natural conjecture is that (9.19) is the only obstruction to uniqueness.

Conjecture 9.20. Let γ_1, γ_2 be smooth anisotropic symmetric and positive definite conductivities in $\overline{\Omega}$. Suppose

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}.$$

Then there exists a diffeomorphism $\Psi : \overline{\Omega} \to \overline{\Omega}$ diffeomorphism such that $\Psi|_{\partial\Omega} = Id$ so that

$$\Psi_* \gamma_1 = \gamma_2 .$$

Progress has been made in proving the conjecture even though the general case remains unsolved. In the two dimensional case Sylvester ([S]) proved

Theorem 9.21. *($n = 2$) Let γ_i be C^3 anisotropic conductivities with*

$$\|\gamma_i\|_{C^3(\overline{\Omega})} \leq M, \quad i = 1, 2.$$

Then there exists $\varepsilon(\Omega, M)$ such that if

$$\|\log(\det \gamma_i)\|_{C^3(\overline{\Omega})} < \varepsilon \quad i = 1, 2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then there exists a C^3 diffeomorphism Ψ with

$$\gamma_1 = \Psi_* \gamma_2, \quad \Psi|_{\partial\Omega} = Id.$$

Sketch of Proof of Theorem 9.21. The first step in the proof to use the existence of isothermal coordinates (see for instance [A]) to reduce the proof to a new isotropic problem.

Proposition (9.22). *(Isothermal coordinates) Given a C^3 anisotropic conductivity γ with*

$$\|\gamma\|_{C^3(\overline{\Omega})} \leq M,$$

we can find a constant $k = k(M)$, and a C^3 diffeomorphism Ψ such that

$$\Psi : \overline{\Omega} \to \overline{D} = \{x \in \mathbf{R}^2; |x| \leq 1\},$$

$$\|\Psi\|_{C^3} \leq k$$

and

$$\Psi_* \gamma \quad \text{is isotropic.} \tag{9.23}$$

Let γ_i belong to $C^3(\overline{\Omega})$. Then there exists a diffeomorphism $\Psi_i : \overline{\Omega} \to \overline{D}$ so that

$$(\Psi_i)_* \gamma_i = \alpha_i e \quad i = 1, 2 \tag{9.24}$$

where $\alpha_i \in C^3(\overline{\Omega})$ has a positive lower bound, $i = 1, 2$ and e is the euclidean conductivity.

Let $\Psi_i|_{\partial\Omega} = \psi_i$ for $i = 1, 2$. Then using (9.24) and (9.18) we obtain

$$\Lambda_{\alpha_1 e} = \Lambda_{(\Psi_1)_* \gamma_1} = (\psi_1)_* \Lambda_{\gamma_1} \, . \tag{9.25}$$

Using the hypothesis $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$,(9.18), and (9.24) we have

$$(\psi_1)_* \Lambda_{\gamma_1} = (\psi_1)_* (\psi_2^{-1})_* (\psi_2)_* \Lambda_{\gamma_2} = (\phi)_* \Lambda_{(\Psi_2)_* \gamma_2} = \phi_* \Lambda_{\alpha_2 e}$$

where

(9.26). $$\phi = \psi_1 \psi_2^{-1}$$

We conclude that

$$(\phi)_* \Lambda_{\alpha_2 e} = \Lambda_{\alpha_1 e} \tag{9.27}$$

which is a relation between two isotropic conductivities. The main technical result in [S] is

Lemma (9.28). *Let α_i be C^3 isotropic conductivity, $i = 1, 2$ such that (9.27) is satisfied. Then there exists a C^3 conformal map $\Phi : \overline{D} \to \overline{D}$, such that*

$$\Phi|_{\partial D} = \phi. \tag{9.29}$$

Assuming the lemma for a moment we complete the proof of Theorem 9.21. Let Φ be as in (9.29). Then by (9.18) and (9.27)

$$\Lambda_{\alpha_1 e} = (\phi)_* \Lambda_{\alpha_2 e} = \Lambda_{\Phi_*(\alpha_2 e)}. \tag{9.30}$$

Since α_2 is isotropic and Φ is conformal, then $(\Phi)_*(\alpha_2 e)$ is also isotropic. The smallness hypothesis in Theorem 9.21 and the bounds for Ψ_i imply that α_i is close to 1, $i = 1, 2$. Using now the local result of [SU-II] (see Theorem (7.1)) for the isotropic case in dimension 2 we conclude

$$\alpha_1 = \Phi_*(\alpha_2 e)$$

and by (9.9) since Φ is conformal

$$\alpha_1 = \alpha_2 \circ \Phi.$$

Unwinding the definitions then yields

$$\gamma_1 = (\Psi_1^{-1} \Phi \Psi_2)_* \gamma_2.$$

It follows from (9.26) that, on the boundary,

$$\psi_1^{-1} \phi \psi_2 = Id$$

which proves the theorem. □

The proof of the lemma begins by constructing a C^2-diffeomorphism

$$\Phi : \overline{D^c} \to \overline{D^c}, \Phi = Id \text{ for } |x| > R\,, \Phi|_{\partial D} = \phi$$

such that the (anisotropic) conductivity given by

$$\gamma_{12} = \begin{cases} \alpha_2 & \text{for } |x| < 1 \\ \Phi_* \alpha_1 & \text{for } |x| > 1 \end{cases} \tag{9.31}$$

is in $C^{1,1}(\mathbf{R}^2)$ where α_2 has been extended as a $C^{1,1}$ function to $\mathbf{R}^2$. To see that such a Φ exists involves the formal solution to a Beltrami equation as well as the computation of the two first two terms in the expansion of the full symbol of $\phi_* \Lambda_{\alpha_1}$ and Λ_{α_2} (see [S], Prop. 3.1).

A more precise version of the existence of isothermal coordinates allows the construction of a unique C^2-diffeomorphism $F^{12} : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$(F^{12})_* \gamma_{12} = (\det \gamma_{12} \circ (F^{1,2})^{-1})^{\frac{1}{2}} e \tag{9.32}$$

where e is the euclidean conductivity. If we consider F^{12} as a complex valued function, it is the unique solution to the Beltrami equation

$$\overline{\partial} F^{12} = \mu_{12} F^{12}. \tag{9.33}$$

which is asymptotic to z at infinity (see [S], Prop 2.1) for a more precise description). In (9.33), μ_{12} is a rational function in the coefficients of γ which is called the complex dilitation. In particular,

$$\mu_{12} = 0 \iff \gamma_{12} \text{ is isotropic }. \tag{9.34}$$

Therefore,

$$F^{12} = Id \iff \gamma_{12} \quad \text{is isotropic }. \tag{9.35}$$

This version of isothermal coordinates can be used to prove that the special solutions of Theorem (4.1) exist in the anisotropic case; that is, there exist unique solutions $u(z,k)$ which are asymptotic to e^{kz} at infinity (it is convenient to use complex notation, $k, z \in \mathbf{C}$) which solve

$$L_{\gamma_{12}} u = 0 \quad \text{in } \mathbf{R}^2 . \tag{9.36}$$

Moreover, one can show that

$$\lim_{|k|\to\infty} \frac{\log u(z,k)}{k} = F^{12} \tag{9.37}$$

uniformly on compact sets.

Arguments similar to those used in the proof of Proposition 6.16 can be used to show that

$$u = \begin{cases} v & \text{for } |x| < 1 \\ u_1 \circ \Phi^{-1} & \text{for } |x| \geq 1 \end{cases} \tag{9.38}$$

where Φ is as in (9.31), v solves the Dirichlet problem

$$L_{\alpha_2} v = 0 \text{ in } D \tag{9.39}$$

$$v|_{\partial\Omega} = u_1|_{\partial\Omega} \circ \phi^{-1} = u_1 \circ \Phi^{-1}|_{\partial\Omega} \quad ,$$

and $u_1(z,k)$ is the special solution of

$$L_{\alpha_1} u_1 = 0 \text{ in } \mathbf{R}^2 \tag{9.40}$$

which is asymptotic to e^{kz} at infinity. Now since γ_{12} is isotropic in D we have

$$\overline{\partial} F^{12} = 0 \text{ in } D. \tag{9.41}$$

For points on the boundary of D, (9.37) implies that (recall that u is smooth across ∂D)

$$F^{12} = \lim_{|k|\to\infty} \frac{\log u}{k} = \lim_{|k|\to\infty} \log \frac{u_1(z,k) \circ \Phi^{-1}}{k} = F_1 \circ \phi^{-1} . \tag{9.42}$$

where F_1 is the solution to the Beltrami equation associated to the conductivity α_1. Since α_1 is isotropic (9.35) implies that $F_1(z) = z$ and hence that

$$F^{12}|_{\partial D} = \phi^{-1}. \tag{9.43}$$

¿From (9.41) and (9.43) we conclude that F^{12} is the conformal map with boundary value ϕ^{-1}. Therefore $(F^{12})^{-1}$ is the desired conformal map. □

The proof above relied heavily on the construction of isothermal coordinates This is not available in dimension $n \geq 3$.

J. Lee and G. Uhlmann ([L-U]) have proved conjecture 9.20 in dimension $n \geq 3$ in the real-analytic category under certain restrictions.

First we note that in dimensions $n \geq 3$ we can identify Riemannian metrics and anisotropic conductivities.

Let g be a smooth Riemannian metric in $\overline{\Omega}$. We denote by Δ_g the Laplace-Beltrami operator associated to g. In local coordinates

$$\Delta_g u = \sum_{i,j=1}^{n} (\det g_{k\ell})^{-1/2} \frac{\partial}{\partial x_i}((\det g_{k\ell})^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x_j}) \tag{9.45}$$

where g^{ij} is the inverse of the metric g_{ij}.

We can solve the Dirichlet problem

$$\Delta_g u = 0 \quad \text{in } \Omega \tag{9.46}$$

$$u|_{\partial\Omega} = f$$

and define the Dirichlet to Neumann map as map from functions on the boundary to $(n-1)$ forms in the boundary, by

$$\Lambda_g f = \nabla_g u \lrcorner\, dV_g|_{\partial\Omega}, \tag{9.47}$$

where ∇_g denotes the gradient with respect to the metric g, dV_g is the Riemannian volume element and $\lrcorner$ denotes interior differentiation. (We recommend the book by Spivak [Sp] for the reader unfamiliar with the differential geometric terms used.)

Let γ be an anisotropic conductivity given in local coordinates by γ^{ij}. Then if $n \geq 3$

$$g_{ij} = (\det \gamma^{k\ell})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \tag{9.48}$$

is a Riemannian metric with

$$\Lambda_g = \Lambda_\gamma. \tag{9.49}$$

Conversely, if g is Riemannian metric given in local coordinates by g_{ij}, then

$$\gamma^{ij} = (\det g_{k\ell})^{1/2}(g_{ij})^{-1} \tag{9.50}$$

is an anisotropic conductivity satisfying (9.49). We shall identify in the rest of this section conductivities and Riemannian metrics.

We first compute the full symbol of Λ_g if g is a smooth Riemannian metric. For this it is convenient to use boundary normal coordinates. For each $x_0 \in \partial\Omega$, let γ_{x_0} be the unit speed geodesic starting at x_0 and normal to $\partial\Omega$. If $\{x^1, \ldots, x^{n-1}\}$ are local coordinates for $\partial\Omega$ near $p \in \partial\Omega$, we can extend them smoothly to functions in a neighborhood of p in $\overline{\Omega}$ by letting them be constant along each normal geodesic γ_{x_0}. If we then define x^n to be the parameter along each γ_{x_0}, it follows that $\{x^1, \ldots, x^n\}$ are coordinates in $\overline{\Omega}$, which we call *boundary normal coordinates* determined by $\{x^1, \ldots, x^{n-1}\}$. In these coordinates $x^n > 0$ in Ω and $\partial\Omega$ is locally characterized by $x^n = 0$. The metric g takes the form

$$g = \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}(x)dx^\alpha dx^\beta + (dx^n)^2, \tag{9.51}$$

and the Laplace-Beltrami operator is given by

$$-\Delta_g = D_{x^n}^2 + iE(x)D_{x^n} + Q(x, D_{x'}) \tag{9.52}$$

where

$$E(x) = -\frac{1}{2}\sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x)\partial_{x^n} g_{\alpha\beta}(x),$$

$$Q(x, D_{x'}) = \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x)D_{x^\alpha}D_{x^\beta} - i\sum_{\alpha,\beta=1}^{n-1}\left(\frac{1}{2}g^{\alpha\beta}(x)\partial_{x^\alpha}\log r(x) + \partial_{x^\alpha}g^{\alpha\beta}(x)\right)D_{x^\beta}$$

and $x = (x', x^n)$. Moreover

$$r(x) = \det(g_{ij}).$$

We use (9.52) to factorize Δ_g and give an easy way to compute the full symbol of the Dirichlet to Neumann map (see [L-U] Proposition 1.1).

Proposition 9.53. *There exists a pseudodifferential operator $A(x, D_{x'})$ of order one in x' depending smoothly on x^n such that*

$$-\Delta_g = (D_{x^n} + iE(x) - iA(x, D_{x'}))(D_{x^n} + iA(x, D_{x'})) \tag{9.54}$$

modulo a smoothing operator.

We can actually write the full symbol, $a(x,\xi')$, of $A(x,D_{x'})$

$$a(x,\xi') \sim \sum_{j\leq 1} a_j(x,\xi'), \quad \xi' \in \mathbf{R}^{n-1}, \tag{9.55}$$

with a_j homogeneous of degree j in ξ' and

$$a_1(x,\xi') = -\sqrt{q_2} \tag{9.56}$$

with q_2 the principal symbol of Q as in (9.52)

$$a_{m-1}(x',\xi') = \frac{1}{2\sqrt{q_2}} \sum_{\substack{j,k,K \\ m\leq j,k\leq 1 \\ |K|=j+k-m}} \frac{1}{K!}(\partial_{\xi'}^K(a_j)D_{x'}^K(a_k) + \partial_{x^n}a_m - Ea_m)$$

The main point is that

Proposition 9.57. $\Lambda_g f = (-1)^{n-1} r^{1/2} A f dx^1 \wedge \ldots \wedge dx^{n-1}|_{\partial\Omega}$ *modulo a smoothing operator.*

Sketch of Proof. This follows from the factorization (9.54). Let u satiusfies (9.46). Then using the factorization (9.54) we get that

$$(D_{x^n} + iA)u = v \tag{9.58}$$

$$u|_{x^n=0} = f$$

with

$$(D_{x^n} + iE - iA)v = h \in C^\infty([0,T]\times \mathbf{R}^{n-1}) \text{ for } T > 0. \tag{9.59}$$

It follows, since (9.59) can be viewed as a backwards generalized heat equation (make the substitution $t = T - x_n$), that v is also smooth (see [T]). Therefore from (9.58) and elliptic regularity we conclude $(D_{x^n}u) = -iAu$ modulo a smooth function and $\Lambda_g f = \frac{1}{i} D_{x^n} u|_{x^n=0}$ in boundary coordinates. $\square$

The computation (9.56), together with Proposition (9.57) shows (see [L-U], Prop. 1.3) that one can determine from a_j the full Taylor series of g in boundary normal coordinates. This is the analog in the anisotropic case of the Kohn-Vogelius result theorem in the isotropic case.

Theorem 9.60. *Let $n \geq 3$. Let $\{x^1,\ldots,x^{n-1}\}$ be any local coordinates for an open set $U \subset \partial M$ and let $\{a_j, j \leq 1\}$ denote the full symbol of A in* *these coordinates. For any $p \in U$, the full Taylor series of g at p in boundary normal coordinates is given by an explicit formula in terms of the functions $\{r^{1/2}a_j\}$ and their tangential derivatives at p.*

Now in case that $\partial\Omega$, g_1 and g_2 are real-analytic and $\Lambda_{g_1} = \Lambda_{g_2}$ we can use the last result to easily find a collar neighborhood of $\partial\Omega$ and a real-analytic diffeomorphism $\Psi_0 : U \to \overline{\Omega}$, $\Psi_0|_{\partial\Omega} = Id$ so that (see [L-U], Lemma 2.1)

$$\Psi_0^* g_1 = g_2.$$

One needs to extend the diffeomorphism Ψ_0 to Ω. In [L-U] this was done by analytic continuation along geodesics. We mention one of the results obtained (For a more general statement see Prop. 2.2 in [L-U]).

Theorem 9.61. *Let $g_i, i = 1, 2$ be real-analytic Riemannian metrics so that $\Lambda_{g_1} = \Lambda_{g_2}$. Assume Ω is simply connected and $\overline{\Omega}$ is strongly convex with respect to the metrics g_1, g_2. Then $\exists \Psi : \overline{\Omega} \to \overline{\Omega}$ real-analytic diffeomorphism so that*

$$\Psi^* g_1 = g_2, \Psi|_{\partial\Omega} = Id.$$

Theorems (9.21) and (9.61) use special features. In two dimensions isothermal coordinates are used to break the diffeomorphism invariance. In dimension $n \geq 3$, in the real-analytic case, geodesic flow is used to break the diffeomorphism invariance.

Jack Lee has suggested the use of harmonic maps to break this invariance. We discuss this idea in more detail. The material that follows is taken from [S-U VI].

For a general reference on harmonic maps see [Ha]. We shall only consider the case where the domain and range of a map is $\overline{\Omega}$, with Ω a smooth bounded domain in $\mathbf{R}^n$.

Let $f : (\overline{\Omega}, g) \to (\overline{\Omega}, h)$ be a smooth map where g and h are Riemannian metrics in $\overline{\Omega}$. The *energy* associated to the map f is given in local coordinates by

$$E(f) = \sum_{\alpha,\beta,i,j=1}^{n} \int_{\Omega} g^{ij}(x) h_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j} \sqrt{\det g} dx. \tag{9.62}$$

The Euler-Lagrange equation associated to the quadratic form (9.62) is given by the non-linear elliptic system

$$\frac{-2}{\sqrt{\det g}} \sum_{\alpha,i,j=1}^{n} \frac{\partial}{\partial x_j} \quad ((\sqrt{\det g}) g^{ij} h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x_i}) + \sum_{\alpha,\gamma,i,j=1}^{n} g^{ij} \frac{\partial h_{\alpha\gamma}}{\partial f_\beta} \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\gamma}{\partial x_j} = 0 \quad \forall \beta. \tag{9.63}$$

Definition 9.64. A C^∞ map $f : (\overline{\Omega}, g) \to (\overline{\Omega}, h)$ is called *harmonic* if it is a critical point of (9.62) (i.e., it is a solution of (9.63)).

Note that if h is the Euclidean metric, then (1.13) simply states that the components of f are harmonic functions with respect to the metric g.

We are going to reduce conjecture 9.20 to the proof of a uniqueness theorem by means of the following Proposition, which follows readily from the definition of a harmonic map.

(9.65) Proposition. *Let $(\overline{\Omega}, g)$ and $(\overline{\Omega}, h)$ be two smooth bounded domains with Riemannian metrics g and h. Suppose there is a harmonic map*

(9.66)
$$\psi : (\overline{\Omega}, g) \to (\overline{\Omega}, h) \text{ such that } \psi|_{\partial\Omega} = \text{Identity and } \psi \text{ a diffeomorphism.}$$

Then the Identity: $(\overline{\Omega}, g) \to (\overline{\Omega}, \psi^* h)$ is harmonic.

We shall show that conjecture 9.20 is reduced to prove

Conjecture 9.67. Suppose g and h are Riemannian metrics on $\overline{\Omega}$ and that Identity: $(\overline{\Omega}, g) \to (\overline{\Omega}, h)$ is harmonic and $\Lambda_g = \Lambda_h$.

Then $g = h$.

(9.68) Proposition. *Conjecture 9.67 $\Longrightarrow$ 9.20 if there exists harmonic ψ satisfying (9.66).*

Proof. If $\Lambda_g = \Lambda_h$, and there is a ψ with $\psi|_{\partial\Omega} =$ Identity and $\Lambda_g = \Lambda_{\psi^* h} = \Lambda_h$. Then using Proposition 9.65 and Conjecture 9.67 we conclude that $h = \psi^* g$.

The solvability of the harmonic Dirichlet problem (9.66) is known if h has nonpositive sectional curvature ([H]) or if g and h are sufficiently close in the C^3 topology to the euclidean metric ([L-M-S-U]).

Thus, we have reduced the proof of Conjecture (9.20) to the uniqueness statement in Conjecture (9.67), under the additional assumption of the existence of an harmonic diffeomorphism which is the identity on the boundary.

In [S-U VI] it was proven that the linearization at the identity of conjecture (9.67) holds. We sketch the proof. In analogy with (9.8) the quadratic form associated to Λ_g is given by

$$Q_g(f, g) = \sum_{i,j=1}^{n} \int_\Omega g^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \sqrt{\det g}\, dx \tag{9.69}$$

with u, v solution of $\Delta_g u = \Delta_g v = 0$ in Ω; $u|_{\partial\Omega} = g$. We consider the linearization of Q at the euclidean metric in the direction of the quadratic form $m \in C_0^\infty(\Omega)$

$$dQ_m(f, g) = \lim_{\epsilon \to 0} \frac{Q_{e+\epsilon m}(f, g) - Q_e(f, g)}{\epsilon}. \tag{9.70}$$

A computation yields:

$$dQ_m(f,g) = \sum_{i,j=1}^{n} \int_\Omega (m_{ij} - \frac{1}{2}trm) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \tag{9.71}$$

where $\Delta u = \Delta v = 0$ in Ω; $u|_{\partial\Omega} = f, v|_{\partial\Omega} = g$ and $trm = \sum_{i=1}^n m_{ii}$.

We assume that $dQ_m = 0$. As Calderón did for the isotropic case, we take

$$u = e^{x\cdot\xi}, v = e^{-x\cdot\overline{\xi}} \tag{9.72}$$

where $\xi \in \mathbf{C}^n$, $\xi = \eta + ik$ with $\eta, k \in \mathbf{R}^n$ and $\langle \eta, k\rangle = 0$, $|\eta| = |k|$. Substituting (9.72) in (9.71) we obtain

$$\sum_{i,j=1}^{n} \int_\Omega (m_{ij} - \frac{1}{2}trm) e^{i\langle x,k\rangle} (\eta_i\eta_j + k_ik_j) = 0.$$

We rewrite (9.72) in the form

$$k^t(\widehat{m} - \frac{1}{2}tr\widehat{m})k + \eta^t(\widehat{m} - \frac{1}{2}tr\widehat{m})\eta = 0 \tag{9.73}$$

where t denotes transpose and $\widehat{}$ the Fourier transform.

Now the fact that the identity is a harmonic map implies the following system of n first order linear partial differential equations for $m = g - h$ (g is the euclidean metric in this computation):

$$-2\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(m_{j\beta}) + \frac{\partial}{\partial x_\beta} trm = 0 \text{ in } \Omega, \quad \beta = 1, \cdots, n. \tag{9.74}$$

Taking the Fourier transform of (9.74) we obtain

$$-2\sum_{j=1}^{n} k_j \widehat{m}_{j\beta}(k) + k_\beta tr\widehat{m}(k) = 0, \quad \beta = 1, \cdots, n. \tag{9.75}$$

Let us take $k = (1, 0, \ldots, 0)$, $\eta \in k^\perp$ with $|\eta| = |k| = 1$. Using (9.75) we get

$$\begin{aligned} \widehat{m}_{1\beta}(k) &= 0, \quad \beta = 2, \cdots, n \\ \widehat{m}_{11}(k) &= \frac{1}{2}tr\widehat{m}(k). \end{aligned} \tag{9.76}$$

Using (9.73) we obtain

(9.77) $\widehat{m}_{\beta\gamma} - tr\widehat{m}(k) = -(\widehat{m}_{11} - tr\widehat{m})(k), \quad \beta = 2,\dots,n, \quad \gamma = 2,\dots,n.$

Combining (9.76) and (9.77) we conclude

$$tr\widehat{m}(k) = 0$$

Using (9.76) again we see that $\widehat{m}_{ij}(k) = 0 \quad i,j = 1,\dots,n$. Rotating coordinates shows that $\widehat{m}(k) = 0 \; \forall \; k$ and therefore $m = 0$.

10. The Borg-Levinson theorem. We consider in this section an application of the methods developed for the inverse conductivity problem to study an inverse spectral problem. This involves, in an essential way, the study of the Dirichlet to Neumann map for the equation $\Delta - q + \lambda$.

We consider the equation

(10.1) $$L_{q-\lambda} = \Delta - q + \lambda$$

with $q \in L^\infty(\Omega)$ and $\lambda \in \mathbf{C}$.

The following theorem appears in [N-S-U] :

Theorem 10.2. *Let $n \geq 2$ and $q_i \in L^\infty(\Omega)$, $\quad i = 1,2$. Suppose that, as meromorphic operator valued functions of λ,*

(10.3) $$\Lambda_{q_1-\lambda} = \Lambda_{q_2-\lambda} \quad \forall \lambda \in \mathbf{R}.$$

Then

$$q_1 = q_2 .$$

Remark 10.4. For $n \leq 3$ it is enough as a consequence of Theorem (5.1) to assume $\Lambda_{q_1-\lambda_0} = \Lambda_{q_2-\lambda_0}$ for λ_0 not an eigenvalue of L_{q_1} or L_{q_2}.

Sketch of proof of Theorem 10.2.

Because we know the Dirichlet to Neumann map $\Lambda_{q-\lambda}$ for all λ (except for a discrete set) we may use the scattering solutions (6.2) instead of the exponentially growing solutions from theorem (4.1). Let us take

(10.5) $$\psi_+^i = e^{ix\cdot\xi_i} + \Omega_i \quad i = 1,2$$

where $\xi_i \in \mathbf{R}^n$ and (assume $\lambda > 0$)

(10.6) $$\xi_i \cdot \xi_i = \lambda$$

and

$$\|\Omega_i\|_{L^2_\delta} \leq \frac{C}{\sqrt{\lambda}}\|q_i\|_{L^2_\delta}, \quad -\frac{1}{2} < \delta. \tag{10.7}$$

Using the hypothesis of the theorem we conclude, as in the proof of theorem (5.1),

$$\int_\Omega (q_1 - q_2)u_1 u_2 = 0 \tag{10.8}$$

for all u_i solution of $L_{q_i - \lambda} u_i = 0$, $i = 1, 2$.

We fix $k \in \mathbf{R}^n$ and choose

$$\begin{aligned} \xi_1 &= \frac{1}{2}(k + \ell), \quad k \cdot \ell = 0, |k|^2 + |\ell|^2 = \lambda \\ \xi_2 &= \frac{1}{2}(k - \ell). \end{aligned} \tag{10.9}$$

Now replacing (10.5), with ξ as in (10.9), in (10.8) and letting l and $\lambda \to \infty$ we conclude

$$\widehat{q_1}(k) = \widehat{q_2}(k)$$

which proves the theorem. □

The Dirichlet to Neumann map $\Lambda_{q-\lambda}$ can be related to the eigenvalues and eigenfunctions of the Schrödinger operator $\Delta - q$. We give only a formal argument here. The reader is referred to [N-S-U] for complete proofs.

Let $q \in L^\infty(\Omega)$ be real-valued and let $\{\lambda_i\}_{i=1}^\infty$ denote the Dirichlet eigenvalues of L_q. Let $G(\lambda, x, y), \lambda \neq \lambda_i$, be the Green's kernel for the Dirichlet problem

$$(\Delta - q + \lambda)G = \delta(x - y), \quad G(\lambda, \cdot, y)|_{\partial\Omega} = 0, \quad \forall y \in \Omega.$$

The solution of

$$\begin{aligned} L_{q-\lambda} u &= 0 \\ u|_{\partial\Omega} &= f \end{aligned} \tag{10.10}$$

has the representation

$$u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(\lambda, x, y) f(y) dS_y \tag{10.11}$$

where dS_y is the euclidean surface measure on $\partial\Omega$.

G can be written in term of the λ_i's and the corresponding set of orthonormal eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$

$$G(\lambda, x, y) = \sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(y)}{\lambda - \lambda_i}. \tag{10.12}$$

Inserting (10.12) into (10.11) we obtain

$$u(x) = \int_{\partial\Omega} \sum_{i=1}^{\infty} \varphi_i(x) \frac{\partial \varphi_i}{\partial \nu}(y) f(y) dS_y \tag{10.13}$$

and therefore

$$\Lambda_{q-\lambda} f = \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} \sum_{i=1}^{\infty} \frac{\frac{\partial \varphi_i}{\partial \nu}(x) \frac{\partial \varphi_i}{\partial \nu}(y) f(y) dS_y}{\lambda - \lambda_i}. \tag{10.14}$$

Formula (10.14) and Theorem 10.2 lead directly to the following result ([N-S-U]; Novikov [No] proved this result independently) which states that the Dirichlet eigenvalues and normal derivatives at the boundary of an orthonormal set of eigenfunctions uniquely determine the potential.

Theorem 10.15. *Let $q_i \in L^{\infty}(\Omega)$ $i = 1, 2$ be real-valued. Let $\lambda_j(q_i)$, $j = 1, 2, \ldots$ denote the Dirichlet eigenvalues of L_{q_i}, $i = 1, 2$ with $\lambda_j \geq \lambda_{j+1}$ and eigenvalues repeated according to their multiplicity. Assume*

$$\lambda_j(q_i) = \lambda_j(q_2) \quad \forall j. \tag{10.16}$$

For q_i, $i = 1, 2$ we choose orthonormal sets of eigenfunctions $\{\varphi_j(\cdot, q_i)\}_{i=1}^{\infty}$ with

$$\frac{\partial \varphi_j}{\partial \nu}(x, q_1) = \frac{\partial \varphi_j}{\partial \nu}(x, q_2). \tag{10.17}$$

Then

$$q_1 = q_2.$$

Remark 10.18. Theorem (10.15) can be thought of as an n-dimensional analog of the one-dimensional Borg-Levinson theorem, which states that the Dirichlet eigenvalues and the norming constants determine the potential uniquely. Alessandrini and Sylvester ([A-S]) have given stability estimates for the result Theorem 10.2. Roughly speaking, they showed that if q is a-priori bounded in some Sobolev norm, then, in some lower Sobolev norm, q

depends continuously on its Dirichlet eigenvalues and the normal derivatives of an orthonormal set of Dirichlet eigenfunctions.

11. The hyperbolic Dirichlet to Neumann map

We consider the mixed problem for the wave equation associated to the Schrödinger equation

$$((\frac{\partial^2}{\partial t^2} - \Delta) + q)u = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{11.1}$$

$$u|_{t=0} = \varphi, \frac{\partial u}{\partial t}|_{t=0} = \psi\,,$$

$$u|_{\partial\Omega \times (0,T)} = f$$

where $q \in L^\infty(\Omega)$.

The (hyperbolic) Dirichlet to Neumann map is then defined by

$$\Lambda_q^h(f) = \frac{\partial u}{\partial \nu}. \tag{11.2}$$

with u solution of (11.1). Notice that φ, ψ are fixed throughout. As shown in [Ra-S] the choice of φ, ψ is inmaterial. Rakesh and Symes ([Ra-S]) proved

Theorem 11.3. $(n \geq 2)$ *Let* $q_1, q_2 \in L^\infty(\Omega)$. *Assume* $\Lambda_{q_1}^h = \Lambda_{q_2}^h$ *for* $t \in [0,T]$ *with* $T >$ diam(Ω). *Then*

$$q_1 = q_2.$$

Remark 11.4

If one knows $\Lambda_q^h(f)$ for all t, then taking Fourier transform in the time variable, one obtains the Dirichlet to Neumann map $\Lambda_{q-\lambda^2}$ considered in Theorem 10.2. In Theorem 11.3, we require only knowledge of Λ_q^h in the interval $[0,T]$.

Sketch of proof.

Rakesh and Symes use geometrical optics solutions concentrated near lines with direction $\omega \in S^{n-1}$ and an identity similar to (5.4) to prove that one can recover the X-ray transform of q knowing Λ_q^h.

We indicate here another way of obtaining this information from the hyperbolic Dirichlet to Neumann map. We consider for simplicity the case $q \in C_0^\infty(\Omega)$. Let $q_1, q_2 \in C_0^\infty(\Omega)$ such that

$$\Lambda_{q_1}^h = \Lambda_{q_2}^h, \quad 0 \leq t \leq T, \text{ with } T > \text{diam}\,(\Omega)\,. \tag{11.5}$$

Let u_i, $i = 1, 2$ be the solution of

$$((\frac{\partial^2}{\partial t^2} - \Delta) + q_i)u_i = 0 \tag{11.6}$$

$$u_i = \delta(t - x \cdot \omega), t \ll 0$$

where $\omega \in S^{n-1}$ is the direction of the plane wave $\delta(t - x \cdot \omega)$. We proceed now to show that the information (11.5) implies $u_1 = u_2$ in $\Omega^c \times [0, T]$. We proceed as in (6.18). Let

$$z = \begin{cases} w & \text{in } \Omega \times [0, T] \\ u_2 & \text{in } \Omega^c \times [0, T] \end{cases} \tag{11.7}$$

where w solves the initial boundary value problem

$$(\frac{\partial^2}{\partial t^2} - \Delta + q_1)w = 0 \tag{11.8}$$

$$w = \delta(t - x \cdot \omega), t \ll 0$$

$$w|_{\partial\Omega \times (0,T)} = u_2|_{\partial\Omega \times (0,T)}.$$

Now

$$\frac{\partial w}{\partial \nu} = \Lambda^h_{q_1}(w|_{\partial\Omega \times [0,T]}) = \Lambda^h_{q_2}(u_2|_{\partial\Omega \times [0,T]}) = \frac{\partial u_2}{\partial \nu}.$$

Therefore z solves

$$(\frac{\partial^2}{\partial t^2} - \Delta + q_1)z = 0 \quad \text{in} \quad \mathbf{R}^n \times \mathbf{R} \tag{11.9}$$

$$z = \delta(t - x \cdot \omega), \quad t \ll 0\,.$$

By the uniqueness of the solution of (11.9) we obtain

$$z = u_1$$

proving that

$$u_1 = u_2 \text{ in } \Omega^c \times [0, T]\,. \tag{11.10}$$

Now one can use the progressive wave expansion of Courant-Lax ([C-L]) to conclude for u_i as in (11.6)

$$u_i = \delta(t - x \cdot \omega) + a_i(x, \omega)H(t - x \cdot \omega) + b_i(t, x, \omega) \tag{11.12}$$

where $b_i \in C^0(\mathbf{R} \times \mathbf{R}^n \times S^{n-1})$ $i = 1, 2$, $H(x)$ is the Heaviside function and

$$\nabla a_i \cdot w = \frac{-q_i(x)}{2}, \quad i = 1, 2,$$

$$a_i = 0 \text{ for } x \cdot \omega \ll 0.$$

Since $u_1 = u_2$ in $\Omega^c \times [0, T]$ we conclude

$$a_1 = a_2 \text{ in } \Omega^c.$$

But a_i, $i = 1, 2$, can be obtained as integration of the potential q_i in the direction ω, therefore implying that the X-ray transform of q_1 and q_2 coincide

$$\int_{-\infty}^{\infty} q_i(x + t\omega)dt = \int_{-\infty}^{\infty} q_2(x + t\omega)dt \quad \forall \omega, x.$$

Now by the inversion of the X-ray transform ([H]) we conclude

$$q_1 = q_2.$$

Stefanov [St] and Ramm and Sjöstrand [R-Sj] have extended Theorem 11.3 result to the case of potentials depending on time. Isakov [Is III] has considered the case of wave equation plus first order perturbations. In all these works geometrical optics solutions and the relationship between the hyperbolic Dirichlet to Neumann and the X-ray transform play a crucial role.

We now consider the hyperbolic Dirichlet to Neumann map in the anisotropic case. In particular we would like to describe the relationship of this map and the inverse kinematic problem in seismology. The material that follows is taken from [S-U VI] and is part of work in progress of the author with Jack Lee, Gerardo Mendoza and John Sylvester [L-M-S-U].

Let Ω be a smooth bounded domain in $\mathbf{R}^n$ and g a smooth Riemannian metric on $\overline{\Omega}$. We consider the initial boundary value problem

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_g\right) u = 0 \text{ in } \Omega \times (0, T), \quad T > 0 \tag{11.13}$$

$$u|_{t=0} = \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \text{ in } \Omega$$

$$u|_{\Omega \times (0,T)} = f.$$

We define the (hyperbolic) Dirichlet to Neumann map by

$$\Lambda_g^h(f) = \sum_{i,j=1}^{n} \left(\nu_i g^{ij} \frac{\partial u}{\partial x_j}\right)\Big|_{\partial\Omega} \tag{11.14}$$

where u is a solution of (11.13).

As in the elliptic case, it is easy to see that the map

$$g \xrightarrow{\Lambda^h} \Lambda_g^h \tag{11.15}$$

is not injective since $\Lambda^h_{\psi^* g} = \Lambda^h_g$ for any diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ such that $\psi|_{\partial\Omega} =$ Identity. One can show,as in the elliptic case, that knowledge of Λ^h_g determines the Taylor series of g at $\partial\Omega$ in boundary normal coordinates.

If $\Lambda_{g_0} = \Lambda_{g_1}$ one can extend $g_0 = g_1$ to Ω^c such that both are smooth and both are euclidean outside a ball. Using similar arguments to the ones used in the proof of Proposition 6.16 we have

(11.16) Proposition. *Let g_0, g_1 the smooth Riemannian metrics on $\overline{\Omega}$. Assume $\Lambda^h_{g_0} = \Lambda^h_{g_1}$. Let $(u_0, u_1) \in \mathcal{E}'(\mathbf{R}^n) \times \mathcal{E}'(\mathbf{R}^n)$, supp $u_k \subseteq \Omega^c$, $k = 0, 1$. The solution v_k of the initial value problem*

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{g_k}\right) v_k = 0 \text{ in } \mathbf{R}^n \times (0,T)$$

$$v_k|_{t=0} = u_0$$

$$\frac{\partial v_k}{\partial t}\Big|_{t=0} = u_1$$

satisfies $v_0 = v_1$ in $\Omega^c \times (0,T)$.

One can use the proposition above and the geometrical optics construction (2.7) to solve the wave equation with data supported outside Ω^c (say $u_0 = \delta_y, y \in \Omega^c, u_1 = 0$) to conclude that the geodesic distance function for points $y, x \in \Omega^c$ is the same. We are going to use an alternative method which is the Hadamard parametrix construction (see Hörmander [Hö], section 12.4).

Let $F_k(t,x,y)$ be the solution of

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{g_k}\right) F_k = 0, \quad k = 0, 1$$

$$F_k(0,x,y) = \delta(x-y), y \in \Omega^c$$

$$\frac{\partial F_k}{\partial t}(0,x,y) = 0.$$

Then, assuming that the exponential map for each of the metrics g_k is a global diffeomorphism near $\overline{\Omega}$ (i.e., no caustics in a neighborhood of Ω), we may write

$$F_k(t,x,y) = \sum_{j=0}^{N} A_j^k(x,y)\left(t^2 - (s_k(x,y))^2\right)_+^{-j+\frac{1}{2}(n-1)} + F_N^k \tag{11.17}$$

where $F_N^k \in C^{N+1-[\frac{1}{2}(n-1)]}(\mathbf{R}_t \times \mathbf{R}_x^n \times \mathbf{R}_y^n)$ and $A_j^k \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, $k = 0, 1$. Here $s_k(x,y)$ denotes the geodesic distance between x and y in the metric g_k, $k = 0, 1$. The distributions

$$\left(t^2 - (s(x,y)^2\right)_+^\lambda = \begin{cases} \frac{\left(t^2-(s(x,y))^2\right)^{-\lambda}}{\Gamma(1-\lambda)} & \text{for } t^2 > \left(s(x,y)\right) \\ 0 & t^2 < \left(s(x,y)\right) \end{cases}$$

are defined for Re $\lambda \ll 0$ and have an analytic continuation to $\lambda \in \mathbf{C}$.

Now from proposition (11.16) we know that if $\Lambda_{g_0}^h = \Lambda_{g_1}^h$, then $F_0(t,x,y) = F_1(t,x,y)$ in Ω^c, for $t > 0$. Therefore, comparing the most singular terms in (2.20) we conclude that

$$\left(t^2 - (s_0(x,y))^2\right)_+^{\frac{1}{2}(n-1)} = \left(t^2 - (s_1(x,y))^2\right)_+^{\frac{1}{2}(n-1)}.$$

Thus we have proved

Theorem 11.18. *Let g_0 and g_1 be Riemannian metrics with $\Lambda_{g_0}^h = \Lambda_{g_1}^h$. Then if the exponential map is a global diffeomorphism in $\overline{\Omega}$ for g_k, $k = 0, 1$ and $s_k(x,y)$ denotes the geodesic distance from x to y in the metric g_k, we have*

$$s_0(x,y) = s_1(x,y) \quad \forall x, y \in \partial\Omega.$$

The inverse kinematic problem in seismology is to recover g from $s_g(x,y), x, y \in \partial\Omega$. Again this is not possible since if $\psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism such that $\psi|_{\partial\Omega} =$ Identity, then $s_{\psi^* g} = s_g$. As in conjecture 1, the question is whether this is the only obstruction to uniqueness. It is proven in [S-U V] that the linearized version at the euclidean metric of this conjecture is valid using again the harmonic map equation.

Let g_ϵ be a family of Riemannian metrics in Ω, $g_\epsilon = e + \epsilon h$, where e is the euclidean metric. We also assume that $g_\epsilon = e$ in Ω^c and

$$s_{g_\epsilon}(x,y) = s_e(x,y) \ \forall \epsilon. \tag{11.19}$$

An easy computation shows that

$$\int_{\gamma(x,t,v)} (h_{ij})(v,v)dt = 0 \tag{11.20}$$

where $\gamma(x,t,v)$ denotes a straight line through x with direction v at time t. Formula (11.20) means that the X-ray transform of the quadratic form h_{ij} vanishes in the direction v.

We recall that the linearization at the identity of the harmonic map equation (in the direction h) is

$$-2\sum_{i=1}^n \frac{\partial}{\partial x_i}(h_{i\beta}) + \frac{\partial}{\partial x_\beta} trh = 0, \quad \beta = 1, \ldots, n. \tag{11.21}$$

Integrating (11.21) along the lines with direction v yields

$$\int_{\gamma(x,t,v)} v_j(h_{ij})w_\beta = 0 \ \forall w \in \mathbf{R}^n \text{ with } \langle w, v\rangle = 0.$$

Arguments similar to those at the end of section 9 show that

$$\int_{\gamma(x,t,v)} (h_{ij}(w,w) = 0 \quad \forall w \in \mathbf{R}^n$$

proving that the X-ray transform of $h_{ij}(w,w)$ is zero for all w and therefore that $h = 0$.

12. The scattering amplitude at fixed energy

In the previous section 10 we related the Dirichlet to Neumann map $\Lambda_{q-\lambda}$ to spectral information about q. One can also relate this to scattering information, now fixing the frequency λ (this is more or less implicit in the hyperbolic Dirichlet to Neumann map and in the analog to formula (6.15) for the scattering amplitude).

In the same way that we obtained (6.15), it is possible to show that the scattering amplitude satisfies

$$a(\lambda,\theta,\omega) = \int_{\partial\Omega} e^{i\lambda x\cdot\omega}(\Lambda_{q-\lambda^2}\psi_+ + i\lambda\nu\cdot\omega\psi_+)dS \tag{12.1}$$

where ψ_+ is the outgoing eigenfunction.

(In [N] and [No] an integral equation was derived for $\psi_+|_{\partial\Omega}$ in terms of $\Lambda_{q-\lambda^2}$ similar to (6.24). See also the nice exposition of Colton and Kress [C-K] on integral equation methods in scattering theory).

Arguments analogous to those in proof of Proposition 6.16 show that if

$$\Lambda_{q_1-\lambda_0^2} = \Lambda_{q_2-\lambda_0^2} \tag{12.2}$$

for $q_1, q_2 \in L^\infty(\Omega)$ then

$$\psi_+^{(1)} = \psi_+^{(2)} \quad \text{in } \Omega^c. \tag{12.3}$$

with $\psi_+^{(i)}, i = 1,2$ the outgoing eigenfunction associated to q_i. Then using (12.1) we conclude that if (12.3) is satisfied,

$$a_1(\lambda_0,\theta,\omega) = a_2(\lambda_0,\theta,\omega) \quad \forall\theta,\omega \in S^{n-1}$$

with a_i $i = 1,2$ the scattering amplitude associated to q_i. One can prove the converse

Theorem 12.4. $(n \geq 3)$. *Let* $q_i \in L^\infty(\mathbf{R}^n)$, supp $q_i \subseteq \Omega = \{x; |x| < R\}$, $i = 1, 2$ *such that*

$$a_1(\lambda_0, \theta, \omega) = a_2(\lambda_0, \theta, \omega) \tag{12.5}$$

for some $\lambda_0 \neq 0$, $\forall \theta, \omega \in S^{n-1}$. *Then if* λ_0^2 *is not an eigenvalue of* L_{q_1} or L_{q_2} *(in* Ω *with Dirichlet boundary conditions),*

$$\Lambda_{q_1 - \lambda_0^2} = \Lambda_{q_2 - \lambda_0^2}$$

and therefore

$$q_1 = q_2 \,.$$

Sketch of proof

Let $G_q(x, y, \lambda_0)$ be the outgoing Green's kernel for $-\Delta + q - \lambda_0^2$. The single-layer operator, which is an invertible operator from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega)$, is defined by

$$\mathcal{S}_{\lambda_0} f(x) = \int_{\partial B(0,R)} G_q(x, y, \lambda_0) f(y) dS \tag{12.6}$$

where dS denotes surface measure.

It was proven in [N] (see Theorem 1.6; the proof is also valid in two dimensions) that

$$\Lambda_{q - \lambda_0^2} \to \mathcal{S}_{\lambda_0} \tag{12.7}$$

is injective. More precisely (see (1.40) in [N])

$$\Lambda_{q - \lambda_0^2} = \Lambda_{-\lambda_0^2} + \mathcal{S}_{\lambda_0}^{-1} - (\mathcal{S}_{\lambda_0}^{+})^{-1} \tag{12.8}$$

where $\mathcal{S}_{\lambda_0}^{+}$ is as in (12.6) with $q = 0$. Next we sketch how to prove that the map

$$\mathcal{S}_{\lambda_0} \to \mathcal{A}_{\lambda_0} \tag{12.9}$$

is injective, where $\mathcal{A}_{\lambda_0}(q) = a(\lambda_0, \theta, \omega)$.

This is an old result of Berezanskii ([B]) who showed how to go from the far field ($\mathcal{A}_{\lambda_0}$) to the near field ($\mathcal{S}_{\lambda_0}$) in a quite explicit fashion. One can see the injectivity of (12.9) using the asymptotic expansion of the outgoing Green's kernel, namely

$$G_q(x, y, \lambda_0) = \frac{e^{i\lambda_0|x|}}{|x|^{\frac{n-1}{2}}} \psi(\lambda_0, y, \theta) + 0(|x|^{-\frac{(n-1)}{2}-1}) \tag{12.10}$$

with $\theta = -\frac{x}{|x|}$ and ψ_+ the outgoing eigenfunction. Now if $\mathcal{A}_{\lambda_0}(q_1) = \mathcal{A}_{\lambda_0}(q_2)$, by (12.10) and (6.3') we get

$$G_{q_1}(x,y,\lambda_0) - G_{q_2}(x,y,\lambda_0) = 0(|x|^{-\frac{(n-1)}{2}-1}|y|^{-\frac{(n-1)}{2}-1}). \tag{12.11}$$

Now

$$\varphi(x,y) = G_{q_1}(x,y,\lambda_0) - G_{q_2}(x,y,\lambda_0)$$

solves

$$(-\Delta_x - \lambda_0^2)\varphi = 0 \quad \text{for } |x| \geq R, |y| \geq R.$$

Therefore by Rellich's lemma we obtain that

$$G_{q_1}(x,y,\lambda_0) = G_{q_2}(x,y,\lambda_0) \quad \text{for } |x|, |y| \geq R$$

proving the injectivity of the map (12.9). □

In two dimensions Novikov [N II] proved injectivity of the map

$$q \to \mathcal{A}_{\lambda_0}(q) \tag{12.12}$$

for q close to 0. This result can be also proven using the method outlined in the proof of Theorem 12.4 and the local result in [S-U II] stated in section 7. Sun and Uhlmann [S-U II] used the generic results in [Su-U I] to prove generic injectivity of the map (12.2). More recently in [Su-U III] it was proven that in two dimensions for a singular potential having jump type discontinuities across a subdomain, knowledge of the map (12.2) determines both the location of the singularity and the jump at the singularity. This result follows from a corresponding one for the Dirichlet to Neumann map.

Remark 12.13. Ramm stated Theorem 12.4 in several papers. However some of his proofs, as indicated by Novikov ([No]), are incorrect (for instance [R I]). A corrected proof appears in [R II].The proof sketched above was communicated to us by A. Nachman. Stefanov [St II] has used similar ideas to obtain continuous dependence results for the map (12.12). Henkin and Novikov ([N-H]) had proved Theorem 12.4 earlier in the case of small potentials. Novikov ([No]) sketched a proof of Theorem 12.4 without the smallness assumption using the results in [N-H].

13. An analogous discrete problem

A discrete version of the inverse conductivity problem described in section 1 is to consider a network of resistors. The problem is to determine the resistances in the network by making voltage and current measurements at the boundary of the network. Of course the geometry of the network is important for uniquely determining the resistors. For instance it is easy to see that two resistances wired in series cannot be determined by making voltage and current measurements at the boundary.

We consider rectangular network Ω of resistors in the plane. We follow here the approach of [Cu-M I]. The nodes of Ω are the lattice points $p = (i, j)$ for which $a \leq i \leq b$ and $c \leq j \leq d$ with the four corner points (a, c), (b, c), (a, d) and (b, d) excluded. The set of nodes is denoted by Ω_0. The interior int Ω_0 consists of those nodes in Ω_0 all of whose four adjoint points are in Ω_0. The *edges* of Ω, Ω_1 are the horizontal and vertical line segments which connect each pair of adjacent points in Ω_0. The *conductivity* is a function

$$\gamma : \Omega_1 \to \mathbf{R}^+$$

where $\mathbf{R}^+$ is the set of positive numbers and $\frac{1}{\gamma(\sigma)}$ is the resistance of the edge.

The conductivity equation is easily obtained used Kirkhoff's law: The sum of all currents at an interior node is zero

$$L_\gamma u(p) = \sum_{q \sim p} \gamma(p, q)(u(q) - u(p)) = 0, \quad p \in \text{int } \Omega_0 \tag{13.1}$$

where $q \sim p$ means that q and p are nodes connected by a resistance; $\gamma(p, q)$ represents the conductivity associated to the edge joining p and q.

The discrete Dirichlet to Neumann is then defined by

$$\Lambda_\gamma^d f(p) = \gamma(p, q)(u(q) - u(p)), p \in \partial\Omega_0 \tag{13.2}$$

where q is the unique node in Ω_0 connected to p by an edge and u is the solution to the Dirichlet problem

$$\begin{aligned} L_\gamma u &= 0 \text{ in } \text{ int } \Omega_0 \\ u|_{\partial\Omega_0} &= f. \end{aligned} \tag{13.3}$$

Again $\Lambda_\gamma^d f(p)$ is the induced current at p by the potential u induced by the voltage f.

In analogy with the continuous case it is easy to see that if we consider the total power to maintain the potential f on the boundary, with u solution of (13.3)

$$Q_\gamma^d(f) = \sum_{q \sim p} \gamma(p, q)(u(q) - u(p))^2, \tag{13.4}$$

then

$$Q_\gamma^d(f) = \sum_{p \in \partial\Omega_0} f(p)\Lambda_\gamma^d f(p). \tag{13.5}$$

The inverse conductivity problem for the network of resistances can then be reduced to study the map

$$\gamma \xrightarrow{\Phi} \Lambda_\gamma^d \tag{13.6}$$

with Λ_γ^d as in (13.2) or equivalently the map

$$\gamma \xrightarrow{Q} Q_\gamma^d \tag{13.7}$$

with Q_γ^d as in (13.4).

Lawler and Sylvester ([L-S]) proved the injectivity of the map Φ (or Q) for conductivities which are a small deviation of constant conductivities. They used the analog of the growing exponential solutions of Calderón (section 2). In [Cu-M I] completely different solutions of (13.1) are constructed which don't have an analog in the continuous case. This allows to prove not only injectivity for Φ (or Q) and to give a reconstruction method to get γ from Λ_γ^d but also to give a characterization of all possible Λ_γ^d which arise ([Cu-M II]). We first state

Theorem 13.8. *Let Ω_0 be a network of resistors in the plane with edges Ω_1. Let γ_i, $i = 1,2$ be two conductivities $\gamma_i : \Omega_1 \to \mathbf{R}^+$. Assume*

$$\Lambda_{\gamma_1}^d = \Lambda_{\gamma_2}^d,$$

then

$$\gamma_1 = \gamma_2 .$$

Sketch of proof.

Similar to the approach taken in the continuous case, we look at Q_γ^d. Polarizing the quadratic form (13.4) we obtain the bilinear form

$$Q_\gamma^d(f,g) = \sum_{q\sim p} \gamma(p,q)(u(q) - u(p))(v(q) - v(p)) \tag{13.9}$$

where u is a solution of (13.1) and v solves the Dirichlet problem.

$$L_\gamma v = 0 \text{ in } \operatorname{int} \Omega_0 \tag{13.10}$$

$$v|_{\partial\Omega_0} = g.$$

One can easily prove the corresponding identity to (5.4) in the case of a network of resistors. Namely if $\gamma_i, i = 1,2$, are conductivities so that $\Lambda^d_{\gamma_1} = \Lambda^d_{\gamma_2}$, then

$$\sum_{q\sim p}(\gamma_1(p,q) - \gamma_2(p,q))(u_1(p) - u_1(q))(u_2(p) - u_2(q)) = 0 \tag{13.11}$$

where u_i is solution of

$$L_{\gamma_i}u_i = 0, \quad i = 1,2\,.$$

The main technique used in [Cu-M I] is "harmonic continuation". More precisely given a conductivity γ one can show that there exist solutions of

$$L_\gamma u = 0 \quad \text{in } \Omega_0$$

so that $u = 0$ below any line of slope plus or minus one (of course there is no analog of these solutions in the continuous case). By choosing u_1 in (13.11) to be zero below the appropiate line of slope one, and u_2 to be zero below a line of slope minus one, Curtis and Morrow proved

Proposition 13.12. *Given an edge joining p_0 and q_0 and two conductivities γ_1, γ_2 in a network, one can construct solutions*

$$L_{\gamma_i}u_i = 0 \text{ in } \operatorname{int}\Omega_0$$

so that for $q \sim p$

$$(u_1(q) - u_1(p))(u_2(q) - u_2(p)) = \delta_{q_0p_0} \tag{13.13}$$

where

$$\delta_{q_0p_0} = \begin{cases} 1 & q = q_0, p = p_0 \\ 0 & \textit{otherwise.} \end{cases}$$

The theorem follows immediately from Proposition (13.12) since we may insert u_1 and u_2 as in (13.13) into (13.11) to get

$$\gamma_1(p_0,q_0) = \gamma_2(p_0,q_0),$$

which proves the theorem. □

This method of proof allowed Curtis and Morrow to give a reconstructive procedure to get γ from Λ^d_γ and moreover to formulate necessary conditions for a matrix Λ_{ij} to be the Dirichlet to Neumann map associated to a conductivity. They have recently proved that these conditions are also sufficient ([Cu-M, II]).

Let Ω_0 be a square network of side $n\times n$ and $\gamma : \Omega_1 \to \mathbf{R}^+$ a conductivity. The Dirichlet to Neumann map Λ^d_γ is represented by the matrix $\Lambda_{i,j}$ (if we number the boundary nodes clockwise, then the functions which are one at the j'th node and zero elsewhere form a basis for functions on the boundary). Curtis and Morrow proved

Theorem 13.14. *Let $\Lambda_{i,j}$ be a $4n$ by $4n$ matrix representing the linear map Λ. Then there is a unique conductivity function γ on Ω_1, such that $\Lambda = \Lambda^d_\gamma$ iff $\Lambda_{i,j}$ satisfies the four properties listed below.*

(R1) Let k be an integer with $1 \leq k \leq n$, and take $m = 4n - k + 1$. Then there is a unique set of numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ such that for each i with $k < i < m$,

$$\Lambda_{i,m} + \sum_{i=1}^{k} \Lambda_{ij}\alpha_j = 0$$

A similar relation holds for any node in any face, and columns from faces either clockwise or anti-clockwise from that node.

(R2) Λ_γ is symmetric: $\Lambda_{i,j} = \Lambda_{j,i}$.Thus, there are relations similar to (R1) involving the rows of Λ_γ.

(R3) For each $i = 1, 2, \dots, 4n$,

$$\sum_{j=1}^{4n} \Lambda_{i,j} = 0$$

(DP) Each of the six $n \times n$ blocks which lie entirely above the diagonal, and each of their transposes has the Determinant Property –A matrix has the determinant property if any k by k submatrix M satisfies: det $M < 0$ if $k \equiv 1$ or 2 mod 4; det $M > 0$ if $k \equiv 3$ or 4 mod 4 .

An interesting open question is to analyze the relationship between the discrete and continuous Dirichlet to Neumann map.

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