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Equivariant Euler-Poincaré Characteristics and Tameness

Ted CHINBURG† and Boas EREZ‡

Introduction

In this paper we give a reasonably self-contained discussion of the Euler-Poincaré characteristics defined by Chinburg in the first part of [Ch1]. We show how these arise naturally when studying actions of a finite group G on coherent sheaves. More precisely, suppose $f : X \to Y$ is a tame G-covering of schemes which are proper and of finite type over a noetherian ring A. Let T be a coherent sheaf on X which has an action of G compatible with the action of G on O_X. (The construction we will give applies to bounded complexes of sheaves having coherent terms, but for simplicity we will assume in this introduction that T is a single sheaf.) One then has a naive coherent Euler-Poincaré characteristic

$$\chi(G,T) = \sum_i (-1)^i \cdot [H^i(X,T)]$$

in the Grothendieck group $G_0(AG)$ of all finitely generated AG-modules. We will show here how to lift $\chi(G,T)$ in a canonical way to a more refined Euler-Poincaré characteristic $\chi R\Gamma^+(f_*(T))$ in the Grothendieck group $CT(AG)$ of all finitely generated cohomologically trivial AG-modules. The natural forgetful homomorphism $CT(AG) \to G_0(AG)$ is in general neither surjective nor injective. Thus the existence of a canonical $\chi R\Gamma^+(f_*(T))$ in $CT(AG)$ mapping to $\chi(G,T)$ restricts the possibilities for $\chi(G,T)$ and also provides a more subtle invariant of T.

The motivation for defining $\chi R\Gamma^+(f_*(T))$ is to combine the insight into Euler-Poincaré characteristics arising from classical algebraic geometry and

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from the theory of the Galois module structure of rings of algebraic integers in tame extensions. For example, many results in algebraic geometry have to do with computing the multiplicities of irreducible representations of G in various cohomology groups. The connection of these multiplicities to character functions, such as Gauss sums, becomes clearer when one uses 'Hom-descriptions' of Grothendieck groups, as suggested by work on algebraic integers. Various results about rings of integers have close geometric counterparts, which in turn may suggest new approaches to studying rings of integers. As one example, Taylor's Theorem connects the stable isomorphism class of the ring of integers in a tame finite Galois extension of number fields to the root numbers appearing in the functional equations of L-series. In [Ch1, Ch2], some conjectural generalizations of Taylor's Theorem to tame G-coverings of schemes are discussed, and results in this direction are proved in the case of smooth projective varieties over a finite field. Over finite fields, one has an alternate approach using l-adic cohomology to proving the Galois Gauss sum congruences which are the deepest arithmetic part of the proof of Taylor's Theorem. (See [Ch1, Sect. 8].) This suggests looking for a new, geometric proof of Taylor's Theorem for rings of integers; at this time we know of no such proof.

In this paper we will focus on how to define $\chi R\Gamma^+(f_*(T))$. The generality of the definition makes it possible to consider examples of a widely varied nature. At the same time, we would like to stress that the definition provides a way of calculating $\chi R\Gamma^+(f_*(T))$.

We now give a quick survey of this paper. Sections 1 - 3 are mainly a summary of known results, definitions and examples which prepare the stage for the new results presented in Sections 4 and 5. In Section 1 we recall the two examples which have motivated essentially all research on Galois module theory. Section 2 is an exposition of some well-known applications of Euler-Poincaré characteristics. In Section 3 we recall the notions of G-coverings and Galois G-coverings of schemes and of quasicoherent G-sheaves. In Section 4 we introduce what we call numerically tame G-coverings. These coverings are more general than the ones considered in [Ch1] and [Ch2], and it is for them that we may define $\chi R\Gamma^+(f_*(T))$. The construction of $\chi R\Gamma^+(f_*(T))$ is carried out in Section 5. The Appendix contains a proof –based on Abhyankar's Lemma– that G-coverings which are tame in codimension 1 are numerically tame.

1. Two basic examples

The following examples are included to give the reader an idea of the kind of information on the Galois module structure of G-coverings one should expect to extract from a description of the classes defined in Section 5.

1.1. Galois structure of differentials and generalizations. One of the first uses of character theory outside group theory was Hecke's determination of the Galois module structure of the complex vector space V of cusp forms of weight 2 and level a prime number p. Hecke's goal was to simplify the problem of studying such cusp forms by decomposing V into isotopic components under the natural action of $PSL(2,p)$ (see [He, 1-2]). As is well known V can be identified with the space $H^0(X,\Omega^1)$ of holomorphic differentials on the Riemann surface X which is the compactification of the orbit space of the action by the congruence subgroup $\Gamma(p)$ on the upper half plane (see [L], [Sh, 2.17]): so Hecke was considering the covering $X \to Y = \mathbf{P}^1$ with the group $PSL(2,p)$ acting on X and Ω^1. In [C-W] [W2] Chevalley and Weil generalize part of Hecke's work and deal with G-coverings $X \to Y$ of compact Riemann surfaces with an arbitrary finite group G as group of automorphisms. They give a formula for the multiplicity m_χ of any irreducible character χ of G in $H^0 = H^0(X,\Omega^1)$ in terms of the genus and ramification data. We observe that this can interpreted as follows. The space H^0 is a finite dimensional module over the semisimple algebra $\mathbf{C}G$ and hence determines a class $[H^0]$ in the Grothendieck group $K_0(\mathbf{C}G) = R(G)$ of (projective) $\mathbf{C}G$-modules – this is of course nothing but the group of virtual characters of G. Since $R(G)$ is the free abelian group on the irreducible characters of G, there is a natural isomorphism from $R(G)$ to $\mathrm{Hom}(R(G),\mathbf{Z})$; this isomorphism sends $[H^0]$ to the homomorphism which on an irreducible character χ of G takes the value m_χ.

These results have been considerably generalized in two directions. The first generalization concerns equivariant Euler-Poincaré characteristics of coherent sheaves other than the sheaf of differentials, mainly for covers of varieties which are smooth and proper over an algebraically closed field (see for example [G-G-H], [E-L], [N,1-3], [V-M]). The second generalization concerns Euler-Poincaré characteristics of sheaves for the étale topology. The formulas that have been obtained in this case are generalizations of Weil's interpretation of the fact that in characteristic 0 the determination of the G-action on $H^0(X,\Omega^1)$ determines the action on the space of harmonic forms: this space is dual to the first homology space of X and so we should study the Tate module of the Jacobian of the curve, which plays the role of the first homology group for the étale topology. This leads to a formula for the Artin conductor attached to a G-covering in positive characteristic (see [W1, p. 79],[S1] [Mi],[R]).

1.2. Galois structure of rings of integers. Let N/K be a finite Galois extension of number fields with group G. The ring of integers O_N is a $\mathbf{Z}G$-module and one can show that it is projective if and only if the extension is tamely ramified. Suppose N/K is tamely ramified, so there is a class

(O_N) in the *reduced* Grothendieck group $Cl(\mathbf{Z}G)$ of projective $\mathbf{Z}G$-modules (of rank 0). By results of Fröhlich one can describe $Cl(\mathbf{Z}G)$ in terms of homomorphisms from the group of virtual characters $R(G)$ into the group of idèles of an algebraic closure $(\mathbf{Q}^c)$ of $\mathbf{Q}$. The class (O_N) is then shown to be determined by a homomorphism defined *via* the Galois-Gauss sums of the characters of G (see [F, Ch. 1]).

We observe that most of the results in (1.1) deal with the determination of the actual isomorphism class of the modules involved whereas in (1.2) the emphasis is shifted to the determination of the *stable* isomorphism class. In both cases, however, what one really does when computing the classes is to determine a homomorphism on the virtual characters.

2. Applications of Euler-Poincaré characteristics

In this section we recall some elementary applications of Euler-Poincaré characteristics and of their equivariant generalizations.

2.1. The Riemann Problem. Let X be a projective variety over an algebraically closed field k. The Riemann problem has do to with determining the dimension of the k-vector space $H^0(X,T)$ of global sections of a coherent sheaf T on X. The classical approach to this problem has two steps: (a) Find an expression for the *Euler Poincaré characteristic*

$$\chi(T) = \sum_{i=0}^{\infty}(-1)^i \cdot dim_k H^i(X,T)$$

by means of Generalized Riemann-Roch Theorems, and (b) Prove a Vanishing Theorem which asserts that under suitable hypotheses, $H^i(X,T) = 0$ for $i > 0$. For such T, (a) gives an expression for $\chi(T) = dim_k H^0(X,T)$.

If one has compatible actions of a group G on X and on T, then one can refine the Riemann problem by asking for the kG-module structure of $H^0(X,T)$, as in Sect. (1.1). In step (a), it is then natural to consider the Euler-Poincaré characteristic

$$\chi(G,T) = \sum_{i=0}^{\infty}(-1)^i \cdot [H^i(X,T)]$$

in $G_0(kG)$. (Recall that $G_0(k) = K_0(k)$ can be identified to the ring of integers via the dimension map.) If one cannot accomplish step (b), one may be able nonetheless to restrict the possibilities for the class of $H^i(X,T)$ in $G_0(kG)$ for $i > 0$. For example, one might be able to show that various irreducible representations cannot occur in $H^i(X,T)$ for $i > 0$. In this way one may still deduce from $\chi(G,T)$ information about the class of $H^0(X,T)$ in $G_0(kG)$.

2.2. Group actions on varieties. Euler-Poincaré characteristics are useful in studying the possible actions of a finite group G on a projective variety X. Suppose for example that G acts freely on X, i.e. that there is a quotient morphism of schemes $f : X \to Y = X/G$ which is an (étale) Galois covering with group G, in the sense of Definition 3.3 below. Then for any coherent sheaf T on X which admists a G-action compatible with the one on X, we have the identity

$$\chi(T) = |G| \cdot \chi((f_*(T))^G)$$

where $(f_*(T))^G$ denotes the sheaf on Y of invariants in $f_*(T)$ (see [Mum, Sect. 12, Theorems 1B and 2]). In particular $|G|$ must divide $\chi(T)$. Thus, for example, if the arithmetic genus $\chi(O_X)$ equals 1, then no nontrivial finite group can act *freely* on X. Some interesting X with arithmetic genus 1 are given in [Ha1, Ex. II.8.4(f)].

By [E-L, Theorem 2.4] and [B-K, Prop. 1.2], one can refine the above formula to the equality

$$\chi(G,T) = \chi((f_*(T))^G) \cdot [kG]$$

in $G_0(kG)$ provided that G acts freely on X. Thus $\chi(G,T)$ gives a further obstruction to the freeness of an action of G on X. This is because there may exist non-free actions for which $\chi(G,T)$ is not the class of a free module in $G_0(kG)$ even though $|G|$ divides $\chi(T)$.

Suppose now only that $f : X \to Y$ is a *tame* G-covering of proper schemes of finite type over k. Nakajima shows in [Na2] that while the cohomology groups $H^i(X,T)$ of a coherent G-sheaf T on X need not be projective kG-modules in general, one can always express $\chi(G,T)$ in terms of projective kG-modules. This result will be a consequence of the more general construction in Section 5. For other uses of Euler-Poincaré characteristics, see [B-K] and [N1, Sect. 3].

3. G-covers and G-sheaves ([SGA 1, Ch. I and V], [Mum, Sect. 5 and 12])

In this section we describe the set up for the rest of this paper. Let $S = Spec(A)$ be the spectrum of a noetherian ring A. All the schemes we consider are of finite type over S. Let (X, O_X) be an S-scheme and let G be a finite group. In what follows we will assume that the group G acts admissibly on X by S-automorphisms (on the right)([SGA 1, Ch.I and V]). This means that there is a morphism $f : X \to Y$ of S-schemes such that $O_Y \xrightarrow{\sim} f_*(O_X)^G$. Then $Y = X/G$ is the quotient of X by G. If f is also finite, we will say that f is a *G-covering*.

Definition 3.1. A sheaf T of O_X-modules is a *G-sheaf* on X if G acts on T in a way compatible with its action on O_X and with the action of O_X on T (i.e. $O_X \times T \to T$ is a G-morphism).
Examples: (a) The direct image $f_*(T)$ of a G-sheaf T on X is a G-sheaf on Y. (b) If G acts freely on X then every G-sheaf T on X is the inverse image $f^{-1}(F)$ of some sheaf F on Y (see [Mum, Sect. 12, Theorem 1B]). (c) The structure sheaf O_X and the sheaf $\Omega^1_{X/S}$ of relative differentials are G-sheaves. (d) The sheaf associated to any ambiguous ideal in a Galois extension N/K of number fields with group $G = Gal(N/K)$ is a G-sheaf on $Spec(O_N)$. (e) The invertible sheaf $L(D)$ associated to a G-invariant divisor on X is a G-sheaf. (f) If G acts trivially on X, then the identity morphism $X \to X$ is a G-covering according to our definitions. In this case a G-sheaf on X is simply a sheaf of modules for $O_X - G$, on which the actions of G and O_X commute.
Remark: If each of the cohomology groups $H^i(X,T)$ are finitely generated AG-modules and almost all of these groups vanish, then one may define the Euler-Poincaré characteristic as the alternating sum

$$\chi(G,T) = \sum_i (-1)^i \cdot [H^i(X,T)] \tag{3.2}$$

of the classes $[H^i(X,T)]$ of the AG-modules $H^i(X,T)$ in the Grothendieck group $G_0(AG)$ of all finitely generated AG-modules. This will be the case, for example, if X is proper and of finite type over S and if T is a coherent G-sheaf.

For any scheme Z we denote by G_Z the *constant Z-group scheme*

$$G_Z = \coprod_{g \in G} Z_g$$

where for all $g \in G$, Z_g is isomorphic to Z. The group scheme structure of G_Z is induced by the identity maps $Z_g \times_Z Z_h \to Z_{gh}$ for $g, h \in G$. This shows that an S-action of the group G is the same as an action of the group scheme G_S.
Definition 3.3. A G-covering $f : X \to Y$ is *Galois* if X is a G-torsor over Y, in the sense that X is faithfully flat over Y and the map $(x,g) \mapsto (x,xg)$ defines an isomorphism

$$X \times_Y G_Y \simeq X \times_Y X.$$

A G-covering is Galois if and only if its inertia groups are trivial [SGA1, V.2.6]; in particular, Galois G-coverings are étale.

Examples: (a) $G_Y \to Y$ is called the *trivial* Galois G-covering of Y. (b) Let K be a field. A Galois G-covering X of $Y = Spec(K)$ is necessarily the spectrum of a separable finite dimensional K-algebra N. These algebras are known as *Galois G-algebras*. The trivial G-covering gives the standard Galois G-algebra $\mathrm{Map}(G, K)$ of all set theoretic maps from G to K. If N is a field, then the isomorphism in Definition 3.3 is nothing but the basic isomorphism from Galois theory

$$Map(G, N) \simeq N \otimes_K N.$$

(c) If O_N and O_K are the rings of integers in a G-Galois extension N/K of number fields, then the natural map $Spec(O_N) \to Spec(O_K)$ is a Galois G-covering if and only if O_N is unramified over O_K in the usual sense (observe that $Map(G, O_N)$ is the maximal O_K–order in $Map(G, N)$).

4. Tame G-coverings

We now investigate under which conditions we can ensure that the Euler-Poincaré characteristic defined in (3.2) can be lifted to the group of finitely generated cohomologically trivial AG-modules. Our guiding example will be Example (1.2).

4.1. Cohomologically trivial modules. Recall that a G-module M is *cohomologically trivial* if for every subgroup H of G, the (reduced) Tate cohomology groups $\tilde{H}^i(H, M)$ are zero for all i. This is equivalent to the condition that the non-reduced cohomology group $H^i(G, M)$ vanishes for two consecutive values of $i > 0$. If M is an A-module, then $H^i(H, M)$ and $\tilde{H}^i(H, M)$ are A-modules. If B is a flat A-algebra, then

$$H^i(H, B \otimes_A M) = B \otimes_A H^i(H, M)$$

for all $i \geq 0$. Thus if B is faithfully flat over A, then M is cohomologically trivial if and only if $B \otimes_A M$ is. We also see that M is cohomologically trivial if and only if each of the localizations of M at prime ideals of A are cohomologically trivial.

We now recall a well-known sufficient condition for an AG-module to be cohomologically trivial.

Lemma 4.2. *Suppose H is a subgroup of G of order prime to the residue characteristics of A , i.e. the order of H is a unit in every localization of A at a prime ideal. Suppose that M is isomorphic to the induced AG-module $Ind_H^G(M')$ associated to some AH-module M'. Then M is cohomologically trivial for G.*

Proof. By Shapiro's Lemma and Mackey's formula, it will suffice to show that M' is cohomologically trivial for H. Let H' be a subgroup of H and let A_P

be a localization of A at a prime ideal P. Then the order of H' is a unit of A_P which annihilates

$$H^i(H', A_P \otimes_A M') = A_P \otimes_A H^i(H', M')$$

for all $i > 0$. It follows that M' is cohomologically trivial for H', as required.

4.3. Čech cohomology. Recall now that we can compute the cohomology of a *quasicoherent* sheaf F on a separated scheme Y by means of the Čech complex. Choose a finite open affine covering $\mathcal{U} = \{U_i\}_i$ of Y and consider the family of abelian groups

$$C^p(\mathcal{U}, F) = \prod_{i_0 < \cdots < i_p} F\left(U_{i_0} \cap \cdots \cap U_{i_p}\right) .$$

These form a complex the i-th homology group of which is the group $H^i(Y, F)$ (see [Ha1, III 4]). Suppose now that G acts trivially on (Y, O_Y) and that F is a G-sheaf on Y. Since Y is separated, the intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ are affine. Hence to show that the $C^p(\mathcal{U}, F)$ are cohomologically trivial AG-modules, it is sufficient to show – say – that for any affine open subset $U \subset Y$ the module $F(U)$ is cohomologically trivial. By the preceding remarks this is equivalent to showing that the stalks of F at every point of Y are cohomologically trivial (see also Prop. 4.7 below).

Consider the case of a G-covering $f : X \to Y$ and a sheaf F which is the direct image $f_*(T)$ of a (quasicoherent) G-sheaf T on X. Then the study of the stalk of F at any point y of Y can be done by base change. More precisely, we will use the following. Let $\alpha : Y' \to Y$ be a *flat* base change containing y in its image and let $f' : X' \to Y'$ and T' be the base change of f and T respectively. Then f' is again a G-covering and for $y' \in Y'$ such that $\alpha(y') = y$ we have

$$(f'_*T')_{y'} = O_{Y',y'} \otimes_{O_{Y,y}} (f_*T)_y .$$

In particular—since $O_{Y',y'}$ is faithfully flat over $O_{Y,y}$ —$(f_*T)_y$ is cohomologically trivial if and only if $(f'_*T')_{y'}$ is.

4.4. Tame extensions of D.V.R.'s. Let R be a discrete valuation ring with field of fractions K and residue field k. A finite Galois extension L/K with group G is said to be *tamely ramified* (or simply *tame*) if for any maximal ideal of the integral closure W of R in L the inertia group has order prime to the characteristic of k and the residue field extension is separable ([SGA 1, XIII 2], [G-M]). If R is *strictly henselian*, i.e. if all of its connected étale extensions are trivial, then the situation is very simple: L/K is tame if and only if the degree $[L : K]$ is prime to the residue characteristic. Moreover since in this case the decomposition group equals the inertia group, we see that L/K is cyclic. It is shown in [SGA 1,XIII 2.0.2] and [G-M, 2.2.8] that tameness for any extension W/R of D.V.R.'s can be detected by going over to the extension $\hat{W}/\hat{R}$ of the strict henselizations. Recall that $W \to \hat{W}$ is faithfully flat ([EGA IV] [Ra, Chap. VIII.4]).

Proposition 4.5. ([Ra, X Lemma 1]) *Let R be any strictly henselian ring and let G be a finite group. Let W be a (finite) R-algebra on which G acts in such a way that $R = W^G$. Choose a maximal ideal q of W. Then*

$$W \simeq Map_H(G, C)$$

where C is the localization of W at q, and where the algebra $Map_H(G, C)$ is the algebra of elements in the standard algebra $Map(G, C)$ invariant under the inertia group $H = I_q$.

Thus for example a tame G-extensions of a strictly henselian discrete valuation ring is induced from a subgroup of order prime to the residue characteristic (compare with the structure of the semilocalization of a ring of integers in tame G-extensions of number fields). Observe that in the situation of the proposition any WG-module M is isomorphic as an RG-module to a module $\mathrm{Ind}_H^G(M')$ induced from some RH-module M' (see the proof of Prop. 4.7).

We now make a geometric definition motivated by the above discussion.

Definition 4.6. Let $f : X \to Y$ be a G-covering. We say that f is *numerically tame* if for every point y of Y there is a flat morphism $Y' \to Y$ having y in its image and a Y'-scheme Z for which the following conditions are satisfied.

(a) The structure morphism $Z \to Y'$ is an H-covering for some group H of order prime to the residue characterics of Y', and

(b) There is a homomorphism from H to G for which there is a G-equivariant isomorphism of Y'-schemes

$$X \times_Y Y' \simeq (Z \times_{Y'} G_{Y'})/H.$$

Remarks: (a) We do not require the group H to be abelian, we are only imposing a condition on its order (this is a "numerical" condition). (b) A surjective Galois covering $f : X \to Y$ is clearly numerically tame (take $Y' = X = Z$ and $H = 1$). (c) By (4.4) any tame finite Galois extensions of number fields defines a numerically tame $Gal(N/K)$-covering $Spec(O_N) \to Spec(O_K)$. (d) If G has order prime to all of the residue characteristics of Y, then $f : X \to Y$ is numerically tame, since we can let $Y' = Y$,$Z = X$ and $H = G$. (e) In the Appendix we relate numerical tameness to a more ramification theoretic notion of tameness, thereby giving many examples of numerically tame G-coverings.

The following result was proved in [Ch1] under the more restrictive hypothesis that $f : X \to Y$ is a tame G-covering in the sense of [G-M, 2.2.2] (see also A.1 of the Appendix).

Proposition 4.7. *Let $f : X \to Y$ be a numerically tame G-covering of S-schemes where $S = Spec(A)$ and let $F = f_*(T)$ be the direct image of a quasicoherent G-sheaf T on X. Then the following two equivalent conditions are satisfied*
(a) all the stalks of F are cohomologically trivial AG-modules
(b) for every open affine subset U of Y, $F(U)$ is a cohomologically trivial AG-module.

Proof. If $U = Spec(B)$ in (b), then $F(U)$ is a BG-module which is cohomologically trivial if and only if for all $y \in U$, the localization $F(U)_y$ of $F(U)$ at y is cohomologically trivial. Hence the equivalence of (a) and (b) follows from the fact that $F(U)_y$ is the stalk F_y of F at y.

By the remarks in (4.3), to show that F_y is cohomologically trivial, we are free to make a base change by a flat morphism $Y' \to Y$ whose image contains y. Thus in view of Definition 4.6, we can assume that $X = (Z \times_Y G_Y)/H$ for some finite group H mapping homomorphically to G and for some H-cover $Z \to Y$. By [G-M, Lemma 1.5.2(iii)] we can replace H by its image in G and Z by its image in X so as to be able to assume that H is a subgroup of G. Choose a set $\mathcal{S}$ of representatives for the cosets $H \backslash G$ which contains the identity element e of G. Over Y we have an isomorphism of schemes

$$(Z \times_Y G_Y)/H = \coprod_{s \in \mathcal{S}} Z \cdot s. \tag{4.8}$$

This is a G-isomorphism when we let G act on the right hand side according to our choice of coset representatives $\mathcal{S}$ and according to the (right) action of H on Z.

Let T_Z be the restriction of the G-sheaf T on $X = (Z \times_Y G_Y)/H$ to the component $Z \cdot e$ on the right hand side of (4.8). Let $\alpha : Z \cdot e \to Y$ be the restriction of $f : X \to Y$. Then we find that the stalk $f_*(T)_y = F_y$ at y is isomorphic as a left $O_{Y,y}[G]$-module to the induced module $Ind_H^G \alpha_*(T_Z)_y$, where $\alpha_*(T_Z)_y$ is a left $O_{Y,y}[H]$-module. Because of condition (a) of Definition 4.6, we can now conclude from Lemma 4.2 that F_y is cohomologically trivial, which completes the proof.

5. Equivariant Euler-Poincaré characteristics for numerically tame coverings

In this section we carry out the construction of equivariant Euler-Poincaré characteristics for numerically tame coverings. In particular, we deal with the problem of producing bounded complexes of finitely generated cohomologically trivial AG-modules which compute sheaf cohomology. As will be seen, we consider a slightly more general situation than the one we have considered so far, in that we replace the G-sheaf T by a bounded complex of sheaves. The additional freedom this allows is useful in applications, where one considers complexes which arise from truncating the de Rham complex $\Omega^\bullet_{X/A}$ (see [Ch1, Ch2]).

As in Section 3, let $f : X \to Y = X/G$ be a G-covering of schemes of finite type over a noetherian ring A, where G is a finite group. Let $K^+(Y, A, G)$ be the category whose objects are complexes $F^\bullet$ of sheaves of AG-modules on Y which are bounded below. Morphisms between consecutive terms of $F^\bullet$ are assumed to respect the AG-module structure, where the actions of A and G commute. Note that we do not assume that O_Y acts on the terms of $F^\bullet$. Morphisms in $K^+(Y, A, G)$ are homotopy classes of morphisms of complexes.

Let $K^+(A, G)$ be the (homotopy) category of complexes of AG-modules which are bounded below. A morphism between two such complexes is a quasi-isomorphism if it induces an isomorphism in cohomology. The derived category $D^+(A, G)$ of $K^+(A, G)$ is the localization of $K^+(A, G)$ with respect to quasi-isomorphisms (see [Ha2, p.37] or [K-S, Chap. 1 and 2]). Since there are enough injectives in the category of sheaves of AG-modules, the construction given in [Ha1, Prop. III.2.2] shows that there are enough injectives in the category of sheaves of AG-modules on Y. Hence by e.g. [Ha2, Cor. I.5.3] the global section functor has a right derived functor $R\Gamma^+ : K^+(Y, A, G) \to D^+(A, G)$. The basic fact that we will need to know about $R\Gamma^+$ is that under suitable assumptions on Y and $F^\bullet$, one can compute $R\Gamma^+(F^\bullet)$ by a Čech hypercohomology complex $\mathbf{H}(\mathcal{U}, F^\bullet)$. In the case of a complex reduced to the term F in degree 0, this is the usual Čech complex. More precisely we will need the

Facts 5.1. [EGA III, 0.12.4.7] Let $\mathcal{U} = \{U_\alpha\}$ be a finite open affine cover of Y and let

$$C^i(\mathcal{U}, F^j) = \prod_{k_0 < \cdots < k_i} F^j (U_{k_0} \cap \cdots \cap U_{k_i}) \ .$$

One can define differentials on the $C^i(\mathcal{U}, F^j)$ so as to get a double complex $C^\bullet(\mathcal{U}, F^\bullet)$. Let $\mathbf{H}(\mathcal{U}, F^\bullet)$ be the total complex of this double complex. Suppose now that Y is separated over A. Suppose further that the terms of $F^\bullet$ are quasicoherent O_Y-modules, and that the A-module structure given on each of these terms is compatible with the structure morphism $Y \to \mathrm{Spec}(A)$. Then $\mathbf{H}(\mathcal{U}, F^\bullet)$ is isomorphic in $D^+(A, G)$ to $R\Gamma^+(F^\bullet)$.

The following result was proved in [Ch1] under the more restrictive hypothesis that $f : X \to Y$ is a tame G-covering in the sense of [G-M, 2.2.2] and A.1 of the Appendix.

Theorem 5.2. *Let $f : X \to Y$ be a numerically tame G-covering of schemes which are proper and of finite type over A. Suppose $T^\bullet$ is a bounded complex of sheaves of abelian groups on X which has the following properties. Each term of $T^\bullet$ is assumed to be a coherent O_X-module which has an action of G compatible with the action of G on O_X. The structure morphism $X \to Spec(A)$ then makes each term of $T^\bullet$ an AG-module. The morphism between consecutive terms of $T^\bullet$ are assumed to respect the AG-module structure (but not necessarily the O_X-structure). Under these assumptions, $f_*(T^\bullet)$ is an object in $K^+(Y, A, G)$. Furthermore, there is a bounded complex $M^\bullet = (M^i)$ of finitely generated cohomologically trivial AG-modules wich is isomorphic to $R\Gamma^+(f_*(T^\bullet))$ in $D^+(A, G)$. Let $CT(AG)$ be the Grothendieck group of all finitely generated AG modules which are cohomologically trivial as G-modules. The (equivariant) Euler-Poincaré characteristic*

$$\chi R\Gamma^+(f_*(T^\bullet)) = \sum_{i=-\infty}^{\infty} (-1)^i \cdot [M^i]$$

in $CT(AG)$ depends only on $T^\bullet$ and not on the choice of $M^\bullet$.

Proof. Since $f : X \to Y$ is finite, the complex $F^\bullet = f_*(T^\bullet)$ is a bounded complex of coherent O_Y-modules. The A-module and G-module structure of the terms of $T^\bullet$ make $F^\bullet$ into an object in $K^+(Y, A, G)$. The action of O_Y on each term of $F^\bullet$ commutes with the action of G, and is compatible with the A-module structure of this term via the structure morphism $Y \to \mathrm{Spec}(A)$. In light of the Facts 5.1, as a first approximation to the complex $M^\bullet$ we may take the Čech hypercohomology complex $M_0^\bullet = \mathbf{H}(\mathcal{U}, F^\bullet)$. By our assumptions on $T^\bullet$ and $\mathcal{U}$, $M_0^\bullet$ is bounded. Since the G-covering $f : X \to Y$ is numerically tame and the intersections $U_{k_0} \cap \cdots \cap U_{k_i}$ are affine, we know by Proposition 4.7 that all the $F^j(U_{k_0} \cap \cdots \cap U_{k_i})$ are cohomologically trivial AG-modules. So the terms of $M_0^\bullet$ are cohomologically trivial. We now use the assumption that the terms of $T^\bullet$ are coherent to get a complex with the same properties as $M_0^\bullet$, but with finitely generated terms. There is a standard way of doing this (see [Ha, III 12.3] [EGA III, 0.11.9.1] [Mum, Sect. 5 Lemma 1] [SGA 6, I 1.4]). The idea is as follows. Suppose that $L^\bullet = (L^i)$ and $N^\bullet = (N^i)$ are complexes with terms in some nice category and let $u : L^\bullet \to N^\bullet$ be a morphism of complexes inducing an isomorphism in cohomology for $i > n$ and an epimorphism for $i = n$. Suppose also that the n-th cohomology of the mapping cone complex of u satisfies some reasonable finiteness condition. Then one can modify u in degree n so as to get an isomorphism in cohomology for $i = n$ and an epimorphism for $i = n - 1$ (i.e. we move one step to the left). Moreover the modification simply consists of adding to L^n an object of the category satisfying the same finiteness condition as the cohomology of the cone. We will apply this with $L^\bullet = M_0^\bullet$, $N^\bullet = R\Gamma^+(F^\bullet)$ and proceed by descending induction to get a complex $M_1^\bullet$ with finitely generated cohomologically trivial terms and with the same cohomology as $R\Gamma^+(F^\bullet)$: the assumption on the mapping cone is satisfied since we are considering complexes with coherent cohomology sheaves. The complex $M_1^\bullet$ is not necessarily bounded below. We get the complex $M^\bullet$ of the statement by truncating $M_1^\bullet$ in degree 0 so as not to change its cohomology, i.e. we let $M^i = M_1^i$ for $i \geq 1$ and $M^0 = M_1^0/\mathrm{im}(M_1^{-1})$. It follows from the fact that we have isomorphisms in cohomology that M^0 is also cohomologically trivial. Suppose now that $M'^\bullet$ is another bounded complex of finitely generated AG-modules which are cohomologically trivial for G, and that there is an isomorphism between $M^\bullet$ and $M'^\bullet$ in the derived category. This isomorphism is represented by a pair of quasi-isomorphisms of complexes $N^\bullet \to M^\bullet$ and $N^\bullet \to M'^\bullet$, where $N^\bullet$ is bounded below. Applying the construction above, we can replace $N^\bullet$ by a complex of finitely generated AG-modules which are cohomologically trivial as G-modules. Thus to prove $\chi(M) = \sum_{i=0}^{\infty}(-1)^i \cdot [M^i]$ equals $\chi(M')$ in $CT(AG)$, we can reduce to the case in which there is a quasi-isomorphism of complexes $\tau : M^\bullet \to M'^\bullet$. The mapping cone complex $L^\bullet$ of τ is now acylic and consists of finitely generated cohomologically trivial AG-modules. Hence $0 = \chi(L^\bullet) = \chi(M^\bullet) - \chi(M'^\bullet)$, as claimed. This concludes the proof of the theorem.

Appendix: A variant of Abhyankar's Lemma

We define G-coverings of normal schemes which are tame in codimension 1 and show that they are numerically tame. The statement comes form [G-M]. The proof is based on Abhyankar's observation that under suitable regularity assumptions one can "eliminate ramification from a tame covering by a tame base change which is completely determined by the covering".

Definition A.1. [G-M, 2.2.2] Let $f : X \to Y$ be a G-covering and suppose X and Y are *normal.* Let D be a divisor with normal crossings on Y which is of codimension at least 1. Write $U = Y \setminus D$. The covering f is said to be *tamely ramified in codimension 1* with respect to D if

a) $X \times_Y U$ is Galois over U with group G (see (3.3));
b) every irreducible component of X dominates an irreducible component of Y;
c) f is tamely ramified at every $y \in D$ of codimension 1 in the following sense. The local ring $O_{Y,y}$ is a discrete valuation ring by our assumptions on Y and y. Let $K = Q(O_{Y,y})$ be its quotient field. Then we want every field component of the function ring of the fiber $f^{-1}(y)$ to be tamely ramified over K (as in (4.4)).

The assumption on D implies that the local rings of all the points in the support of D are regular and that at any point we have nice local equations for the components of D after making an étale base change. The functions defining these equations together with the orders of the relevant inertia groups are the crucial elements needed for Abhyankar's method to work.

Theorem A.2. *A G-covering $f : X \to Y$ of normal schemes which is tame in codimension 1 (with respect to some divisor with normal crossings D) is numerically tame.*

Proof. We just string a number of propositions from [G-M] and [SGA 1] (note that a proof of the theorem is not given in [G-M] nor in [SGA 1, XIII]). For every $y \in Y$ it suffices to construct an étale neighborhood $g : Y' \to Y$ of y and a covering Z of Y' satisfying the conditions of Definition 4.6. We choose Y' to be the spectrum of the strict henselization of the local ring at y. The base change $f' : X' = X \times_Y Y' \to Y'$ of f is again a G-covering ([SGA 1, V 1.9]). Applying Proposition 4.5 to the covering f' we can replace X' by one of its connected components and G by the inertia group of the closed point of this component, so as to be able to assume that X' is the spectrum of a strictly henselian local ring. The G-covering f' is tame with respect to a divisor D' on Y' having normal crossings ([G-M, p. 27 and 2.2.8]).

Next we consider the Y'-scheme Y''

$$Y'' = \operatorname{Spec}\left(O_{Y'}[(T_i)]_{i\in I}\right) / \left((T_i^{n_i} - a_i)_{i\in I}\right) .$$

Here $\underline{a} = \{a_i\}$ is the finite family of non-unit global sections of $O_{Y'}$ defining the local equations of the divisor D' at the closed point y' of Y' and $\underline{n} = \{n_i\}$ is a family of integers determined by the orders of the inertia groups attached to f' (see [SGA 1, XIII 5.2]). The covering $Y'' \to Y'$ comes with an action of $\mu_{\underline{n}} = \prod_i \mu_i$ and is a Kummer covering in the sense of [G-M, 1.2]. Furthermore Y'' is the spectrum of a strictly henselian local ring (see the proof of [G-M, Lemma 1.8.6]). The main idea of the proof – which goes back to Abhyankar (see e.g. [S2]) – is to show that $X'' = X' \times_{Y'} Y''$ is étale over Y''; then since Y'' is strictly henselian, it will follow that $X'' \to Y''$ has a section. Composing this section with the natural projection $X'' \to X'$, exhibits $X' \to Y'$ as a quotient of the covering $Y'' \to Y'$. (To check this it is useful to apply Proposition 4.5 to the coverings $X'' \to X'$ and $X'' \to Y'$.)

To show that $X'' \to Y''$ is étale one reduces to the classical Abhyankar Lemma as stated in [SGA 1, X 3.6] by using the Theorem on the Purity of the Branch Locus [SGA 2, X 3.4] (see [SGA 1, XIII 5.3.0] for the reduction). The rest is all downhill. Since f' is a quotient of a Kummer covering it is a generalized Kummer covering (X', G). Then by [G-M, Prop. 1.6.2] there is a generalized Kummer covering (Z, H) such that $(X', G) \simeq ((Z \times G)/H, G)$. Here Z is the connected component of X' and H is the stabilizer of Z in G; compare Proposition 4.5. This concludes the proof.

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