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A SHORT PROOF OF THE ALBERT-BRAUER-HASSE-NOETHER THEOREM

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We present a short proof of the Albert-Brauer-Hasse-Noether theorem on the Brauer group of a global field. The connection between Galois cohomology and algebraic tori theory is emphasized. Let K/k be a finite Galois extension of arbitrary fields with group G, then the relative Brauer group is $Br(K/k) \cong H^2(G,K^*) \cong H^1(G,T_1(K))$, where T_1 is the algebraic k-torus associated to the augmentation ideal I_G of G. When k is a global field, we use fundamental facts from algebraic tori theory, Tate-Nakayama duality and modern versions of Grunwald-Wang's lemma to deduce the short exact sequence

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_v Br(k_v) \longrightarrow Q/Z \longrightarrow 0,$$

where k_v runs over the completions of k at all places v of k.

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1. Preliminaries.

Since the seminal work by Manin(1970), the Brauer group plays an increasingly important role in Number Theory and especially in questions related to the existence of rational points on algebraic varieties (see Lang(1991), chap. X). A recent introduction to the Brauer group over a field is Kersten(1990).

Besides the original proofs of the classical theorem of Albert-Brauer-Hasse-Noether, one finds proofs fitting the context of Diophantine Geometry in Shatz(1972) and Artin/Tate(1968). Thus the topic discussed in this note is well-known, only the conceptual presentation may be new. As pointed out by Serre(1962) for C_1-fields, the Brauer group of fields may be computed via algebraic tori theory. This follows also from our Theorem 1 in the special case n=2. Indeed the relative Brauer group of a Galois extension with finite group G is

$$Br(K/k) \cong H^2(G,K^*) \cong H^1(G,T_1(K)),$$

where T_1 is the algebraic k-torus associated to the augmentation ideal I_G of G.

Let K/k be a finite Galois extension of arbitrary fields with group G. The category of finitely generated ZG-modules which are free as abelian groups is denoted by L_G. The Z-dual of a ZG-module M is the ZG-module M^o=Hom(M,Z). One knows that L_G is in duality with the category T(K/k) of algebraic tori defined over k and split by K (see Borel(1969) for example). A k-torus T ε T(K/k) corresponds in this duality to the dual $X(T)^o$ of the character module $X(T)=Hom(T,G_m)$. Under the K-rational points of T, one understands the group $T(K)=Hom(X(T),K^*)$. Two algebraic k-tori T_1 and T_2 are called ***k-stably birationally equivalent*** if the k-varieties $T_1 x_k A^r(k)$ and $T_2 x_k A^s(k)$ are k-birationally equivalent for appropriate choices of the integers r and s. For an arbitrary ZG-module M, one denotes by $H^n(G,M)$ the Tate cohomology groups defined for all integers n. A ZG-module M is called a ***flasque module*** if $H^{-1}(H,M)=0$ for all subgroups $H \subset G$. It is shown in Colliot-Thélène/Sansuc(1977), Prop. 5 and 6, that there exists an ***invariant*** ρ, defined on T(K/k) with values in a "semigroup of similarity classes of flasque ZG-modules", and which characterizes the equivalence classes of k-stably birational equivalent tori. For a precise construction, the interested reader is referred to the mentioned work.

Let us link now Galois cohomology with algebraic tori theory. From the standard free resolution of Z, one obtains the short exact sequences

$$(1.1)\quad \begin{array}{l} 0 \longrightarrow X(T_1):=I_G \longrightarrow ZG \longrightarrow X(T_0):=Z \longrightarrow 0, \\ 0 \longrightarrow X(T_{n+1}):=Ker(\varepsilon_n) \longrightarrow \bigoplus_{\mathbf{r}\varepsilon G^n} ZG\cdot u(\mathbf{r}) \xrightarrow{\varepsilon_n} X(T_n) \longrightarrow 0, \quad n \geq 1, \end{array}$$

where the $u(\mathbf{r})=u(r_1,...,r_n)$, r_i ε G, are symbols such that the s.u(**r**), s ε G, form a Z-basis of $\oplus$ ZG·u(**r**). The action of G on u(**r**) is trivial and ε_n is the ZG-homomorphism defined by $\varepsilon_n(u(\mathbf{r}))=(d_{n-1}u)(\mathbf{r})$, where d_{n-1} is the usual coboundary in the cohomology theory of groups. To the ZG-module $X(T_n)$ corresponds by duality the algebraic k-tori T_n split by K. For n negative one defines T_n by specifying its character module $X(T_n)=X(T_{-n})^o$. Then one applies the functor $Hom(\cdot,K^*)$ to the sequence (1.1). Since $Hom(ZG,K^*)=ZG\otimes K^*$ is G-induced, one has $H^n(G,Hom(ZG,K^*))=0$ for all n. From the associated long exact sequences of cohomology one derives the following result (mentioned by Opolka in Schappacher/Scholz(1992), p. 18).

THEOREM 1. *For all finite Galois extensions of arbitrary fields K/k with group G and for all positive integers i, n such that* $1 \le i \le n$, *one has* $H^n(G,K^*) \cong H^i(G,T_{n-i}(K))$, *and a similar result holds for negative integers.*

Recall now some standard facts from algebraic number theory. Assume in the following that K and k are global fields. By A_K, J_K and $C_K=J_K/K^*$, one denotes respectively the ***adele ring*** of K, the ***idele group*** of K and the ***idele classgroup*** of K. For $T \varepsilon T(K/k)$ one defines $T(A_K)=Hom(X(T),J_K)$ and $T(C_K)=T(A_K)/T(K)=Hom(X(T),C_K)$. The set of all places v of k is denoted by P.

The following result is the generalization of one of the main theorems in classfield theory, which states that the finite Galois extensions K/k build up a ***classformation*** with respect to the idele classgroup C_K.

THEOREM 2. (**Tate-Nakayama duality**) *Let K/k be a finite Galois extension of global fields with group G. For every integer n and for any* $T \varepsilon T(K/k)$, *there is a (non-canonical) isomorphism* $H^n(G,T(C_K)) \cong H^{2-n}(G,X(T))$.

Proof. See the work of Tate and Nakayama or Ono(1963), 2.2.1.

As a ***special case*** when $T=G_m$, one gets $H^n(G,C_K)\cong H^{2-n}(G,Z)$, which is a main theorem in classfield theory.

By application of the functor $Hom(\cdot,J_K)$ to the sequences (1.1) and by passing to the long exact sequences of cohomology using that $H^n(G,Hom(ZG,J_K))=0$ for all n (since $Hom(ZG,J_K)=ZG\otimes J_K$ is G-induced!), one gets the analogue of Theorem 1.

THEOREM 3. *For all finite Galois extensions of global fields K/k with group G and for all positive integers i, n such that* $1 \le i \le n$, *one has* $H^n(G,J_K)\cong H^i(G,T_{n-i}(A_K))$.

Finally for a torus $T \varepsilon T(K/k)$ one denotes by $Ш^n(T)$ the kernel of the natural map $H^n(G,T(K)) \longrightarrow H^n(G,T(A_K))$. The group $Ш^1(T)$, simply written $Ш(T)$, is the so-called Shafarevich-Tate group of T. The cokernel of the map $H^1(G,T(K)) \longrightarrow H^1(G,T(A_K))$ is denoted by Ч(T).

2. Computation.

Let us evaluate the ***Brauer group*** Br(k) of a global field k. Let K/k be a finite Galois extension of global fields with group G. By Theorem 1 one has $H^2(G,K^*)=H^1(G,T_1(K))$ and by Theorem 3 one has $H^2(G,J_K)=H^1(G,T_1(A_K))$. By definition of the groups Ш(·) and Ч(·) one gets the exact sequence

(2.1) $$0 \longrightarrow Ш(T_1) \longrightarrow H^2(G,K^*) \longrightarrow H^2(G,J_K) \longrightarrow Ч(T_1) \longrightarrow 0.$$

The evaluation of the ρ-invariant of T_1 yields $\rho(T_1)=\rho(I_G)=0$ (see Colliot-Thélène/Sansuc(1977), prop. 5 and 6). From the same work, prop. 18, it follows that $Ш(T_1)=0$. On the other hand from the long exact sequence associated to the sequence

$$1 \longrightarrow T_1(K) \longrightarrow T_1(A_K) \longrightarrow T_1(C_K) \longrightarrow 1,$$

one derives the exact sequence

(2.2) $$0 \longrightarrow Ч(T_1) \longrightarrow H^1(G,T_1(C_K)) \longrightarrow Ш^2(T_1) \longrightarrow 0.$$

By Tate-Nakayama duality, that is Theorem 2, one has

$$H^1(G,T_1(C_K)) = H^1(G,X(T_1)) = H^1(G,I_G) = H^0(G,Z) = Z/nZ,$$

where n is the ***order*** of the group G. Moreover applying Theorems 1 and 3 one gets

$$\begin{aligned} Ш^2(T_1) &= \mathrm{Ker}(\ H^2(G,T_1(K)) \longrightarrow H^2(G,T_1(A_K))\ ) \\ &= \mathrm{Ker}(\ H^3(G,K^*) \longrightarrow H^3(G,J_K)\ ) \\ &= H^3(G,K^*), \end{aligned}$$

the last equality following since $H^3(G,J_K)=0$ by classfield theory (see for example Neukirch(1969), III, (3.5) in the case of number fields, or Iyanaga(1975), chap. V, Theorem 1.4, in the general case). Furthermore the third cohomological group $H^3(G,K^*)$ is cyclic of order n/m where $m=m_{K/k}$ (depending on the extension K/k) is the least common multiple of the local degrees (see for example Cassels/Fröhlich(1967), p. 199). Introduced in (2.2) these two results provide the equality

(2.3) $$Ч(T_1) = Z/m_{K/k}Z.$$

Introduced in (2.1) together with $Ш(T_1)=0$ this generates a whole class of exact sequences whose members are

(2.4) $$0 \longrightarrow H^2(G,K^*) \longrightarrow H^2(G,J_K) \longrightarrow Z/m_{K/k}Z \longrightarrow 0.$$

This class defines a ***directed system*** of exact sequences of abelian groups with respect to inflation on cohomology groups and the divisibility property of the integer symbols $m_{K/k}$. For L/K, L/k and K/k all Galois extensions the ***partial ordering*** of this system, formally written $K/k \trianglelefteq L/k$, is provided by the existence of inflation homomorphisms $\mathrm{Inf}: H^2(G_{K/k},K^*) \longrightarrow H^2(G_{L/k},L^*)$ and $\mathrm{Inf}: H^2(G_{K/k},J_K) \longrightarrow H^2(G_{L/k},J_L)$ and the fact that $m_{K/k}$ divides $m_{L/k}$. Moreover since $H^1(G_{L/k},L^*)=H^1(G_{L/k},J_L)=0$ the inflation maps are actually injective. Therefore the partial ordering of this system can be viewed as ***inclusion*** of exact sequences. The system is ***directed***. Indeed given Galois extensions K/k, L/k such that $K/k \trianglelefteq L/k$, there always exists a Galois extension M/k such that $M \supset K \supset k$, $M \supset L \supset k$ are all Galois extensions, hence also $K/k \trianglelefteq M/k$ and $L/k \trianglelefteq M/k$. Now the integer symbols $m_{K/k}$ take on all values of $\mathbb{N}$. In fact this is already true for cyclic extensions (for example Neukirch(1969), III, (3.7) in the case of number fields and the generalized Grunwald-Wang theorems of Saltman(1982) in the general case). The functor ***direct limit*** yields from (2.4) the exact sequence

$$(2.5)\quad 0 \longrightarrow \lim_{K/k} H^2(G_{K/k},K^*) \longrightarrow \lim_{K/k} H^2(G_{K/k},J_K) \longrightarrow \lim_{K/k} \mathbb{Z}/m_{K/k}\mathbb{Z} \longrightarrow 0.$$

In view of the inclusion interpretation of the above inflation maps and completing with the necessary details (for example Neukirch(1969), III, p. 244), one obtains the desired famous result.

THEOREM 4. (**Albert-Brauer-Hasse-Noether**) *Let k be a global field. The following sequence*

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_{v \in P} Br(k_v) \overset{inv}{\longrightarrow} Q/Z \longrightarrow 0$$

is exact, where $inv=\sum inv_v$ *is the sum of the local invariant maps.*

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