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D. YAFAEV

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RADIATION CONDITIONS AND SCATTERING THEORY FOR THREE-PARTICLE HAMILTONIANS

D.Yafaev

1. INTRODUCTION

One of the main problems of scattering theory is a description of asymptotic behaviour of N interacting quantum particles for large times. The complete classification of all possible asymptotics (channels of scattering) is called asymptotic completeness. The final result can easily be formulated in physics terms. Two particles can either form a bound state or are asymptotically free. In case $N \geq 3$ a system of N particles can also be decomposed asymptotically into its subsystems (clusters). Particles of the same cluster form a bound state and different clusters do not interact with each other.

There are two essentially different approaches to a proof of asymptotic completeness for multiparticle ($N \geq 3$) quantum systems. The first of them, started by L. D. Faddeev [1], relies on the detailed study of a set of equations derived by him for the resolvent of the corresponding Hamiltonian. This approach was developped in [1] for the case of three particles and was further elaborated in [2, 3]. The attempts [4, 5] towards a straightforward generalization of Faddeev's method to an arbitrary number of particles meet with numerous difficulties. However, the results of [6] for weak interactions are quite elementary.

Another approach relies on the commutator method [7] of T. Kato. In the theory of N-particle scattering it was introduced by R. Lavine [8, 9] for repulsive potentials. A proof of asymptotic completeness in the general case is much more complicated and is due to I. Sigal and A. Soffer [10]. In the recent paper [11] G. M. Graf gave an accurate proof of asymptotic completeness in the time-dependent framework. The distinguishing feature of [11] is that all intermediary results are also purely time-dependent and most of them have a direct classical interpretation. Papers [10, 11] were to a large extent inspired by V. Enss (see e.g. [12]) who was the first to apply a time-dependent technique for the proof of asymptotic completeness.

The aim of the present paper is to give an elementary proof of asymptotic completeness (for the precise statement, see section 2) for three-particle Hamiltonians with short-range potentials which fits into the theory of smooth perturbations [7, 13]. Our approach admits a straightforward generalization to an arbitrary number of particles. This will be discussed elsewhere. Our proof of asymptotic completeness relies on new estimates which establish some kind of radiation conditions for three-particle systems. Compared to the limiting absorption principle (see below) radiation conditions-estimates give us an additional information on the asymptotic behaviour of a quantum system for large distances or large times. Limiting absorption principle suffices for a proof of asymptotic completeness in case of two-particle Hamiltonians with short-range potentials. However, radiation conditions-estimates are crucial in scattering for long-range potentials (see e.g. [14]), in scattering by unbounded obstacles [15, 16] and in scattering for anisotropically decreasing potentials [17]. In the latter paper the role of radiation conditions was also advocated for three-particle Hamiltonians. Our proof of radiation conditions-estimates hinges on the commutator method rather than the integration-by-parts machinery used in the two-particle case (see e.g. [14]).

Our interpretation of radiation conditions is, of course, different from the two-particle case. Before discussing their precise form let us introduce the generalized three-particle Hamiltonians. We consider the self-adjoint Schrödinger operator $H = -\Delta + V(x)$ in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^d)$. Suppose that some finite number α_0 of subspaces X^α of $X := \mathbb{R}^d$ is given and let x^α, x_α be the orthogonal projections of $x \in X$ on X^α and $X_\alpha = X \ominus X^\alpha$, respectively. We assume that

$$V(x) = \sum_{\alpha=1}^{\alpha_0} V^\alpha(x^\alpha), \tag{1.1}$$

where V^α are decreasing real functions of variables x^α. We prove asymptotic completeness under the assumption that V^α are short-range functions of x^α but many intermediary results (in particular, radiation conditions-estimates) are as well true for long-range potentials. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any conical neighbourhood of X_α and $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case. Operators H considered here were introduced in [18] and are natural generalizations of N-particle Hamiltonians. We further assume that

$$X_\alpha \cap X_\beta = \{0\}, \quad \alpha \neq \beta, \tag{1.2}$$

so that regions where different V^α "live" have compact intersection (for potentials of compact support). For the Schrödinger operator this is true only for the case of three particles. Thus the assumption (1.2) distinguishes the three-particle problem.

Our proof of asymptotic completeness requires only the "angular part" of radiation conditions. Let $\langle \cdot, \cdot \rangle$ be the scalar product in the space $\mathbb{C}^d$ and let $\nabla^{(s)}$,

$$\nabla^{(s)} u(x) = \nabla u(x) - |x|^{-2} \langle \nabla u(x), x \rangle x, \tag{1.3}$$

be the projection of the gradient ∇ on the plane, orthogonal to x. Denote by χ_0 the characteristic function of any closed cone Γ_0 such that $\Gamma_0 \cap X_\alpha = \{0\}$ for all α. We prove that the operator

$$G_0 = \chi_0 (|x| + 1)^{-1/2} \nabla^{(s)} \tag{1.4}$$

is locally (away from thresholds and eigenvalues of H) H-smooth (in the sense of T. Kato – see e.g. [19]). In neighbourhoods of X_α we have only a weaker result. Namely, let ∇_{x_α} be the gradient in the variable x_α (i.e. $\nabla_{x_\alpha} u$ is the orthogonal projection of ∇u on X_α),

$$\nabla_{x_\alpha}^{(s)} u(x) = \nabla_{x_\alpha} u(x) - |x_\alpha|^{-2} \langle \nabla_{x_\alpha} u(x), x_\alpha \rangle x_\alpha \tag{1.5}$$

and let χ_α be the characteristic function of such a closed cone Γ_α that $\Gamma_\alpha \cap X_\beta = \{0\}$ for all $\beta \neq \alpha$. Then the operator

$$G_\alpha = \chi_\alpha (|x| + 1)^{-1/2} \, \nabla_{x_\alpha}^{(s)} \tag{1.6}$$

is locally H-smooth. A definition of H-smoothness of the operators G_0 and G_α can be given either in terms of the resolvent of the operator H or of its unitary group $U(t) = \exp(-iHt)$. In both versions results are formulated as certain estimates which we call radiation conditions-estimates.

Our proof in section 3 of H-smoothness of the operators G_0 and G_α is based on consideration of the commutator $[H, M] := HM - MH$, where M is a self-adjoint first-order differential operator with *bounded* coefficients. We find an operator M such that $i[H, M]$ is essentially bounded from below by $G_0^* G_0$ and $G_\alpha^* G_\alpha$. Here we take into account that certain terms, those vanishing as $O(|x|^{-\rho}), \rho > 1$, at infinity, are negligible. This is a consequence of local H-smoothness of the operator $(|x|+1)^{-r}, r > 1/2$, (limiting absorption principle) which, in turn, is ensured by the Mourre estimate [20, 21, 22]. We emphasize that all our considerations are localized in energy.

The H-smoothness of the operators G_0 and G_α suffices for the proof in section 4 of existence of suitable wave operators (both "direct" and "inverse") with non-trivial identifications which are first-order differential operators. The sum of these identifications equals M, which allows us to find the asymptotics of $MU(t)f$ for large t. Since the limit $M^{\pm}$ as $t \to \pm\infty$ of the observable $U^*(t)\,M\,U(t)$ also exists, this gives the asymptotics of the function $U(t)f$ for f from the range of the operator $M^{\pm}$. Using again the Mourre estimate, we prove (also in section 4) that actually this range coincides with the whole absolutely continuous subspace of the Hamiltonian H. Finally, in section 5 we conclude our proof of asymptotic completeness.

2. BASIC NOTIONS OF SCATTERING THEORY

Let us briefly recall some basic definitions of the scattering theory. For a self-adjoint operator H in a Hilbert space $\mathcal{H}$ we introduce the following standard notation: $\mathcal{D}(H)$ is its domain; $\sigma(H)$ is its spectrum; $E(\Omega; H)$ is the spectral projection of H corresponding to a Borel set $\Omega \subset \mathbb{R}$; $\mathcal{H}^{(ac)}(H)$ is the absolutely continuous subspace of H; $P^{(ac)}(H)$ is the orthogonal projection on $\mathcal{H}^{(ac)}(H)$; $\mathcal{H}^{(p)}(H)$ is the subspace spanned by all eigenvectors of the operator H; $\sigma^{(p)}(H)$ is the spectrum of the restriction of H on $\mathcal{H}^{(p)}(H)$, i.e. $\sigma^{(p)}(H)$ is the closure of the set of all eigenvalues of H. Norms of vectors and operators in different spaces are denoted by the same symbol $\|\cdot\|$; I is always the identity operator; $\mathcal{B}$ and $\mathcal{K}_\infty$ are the classes of bounded and compact operators (in different spaces) respectively; C and c are positive constants whose precise values are of no importance; "$s-\lim$" means the strong operator limit. Note that

$$s-\lim_{|t|\to\infty} K\exp(-iHt)P^{(ac)}(H) = 0, \quad \text{if} \quad K \in \mathcal{K}_\infty. \tag{2.1}$$

Let K be H-bounded operator, acting from $\mathcal{H}$ into, possibly, another Hilbert space $\mathcal{H}'$. It is called H-smooth (in the sense of T. Kato) on a Borel set $\Omega \subset \mathbb{R}$ if for every $f = E(\Omega; H)f \in \mathcal{D}(H)$

$$\int_{-\infty}^{\infty} \|K\exp(-iHt)f\|^2\,dt \le C\|f\|^2.$$

Obviously, BK is H-smooth on Ω if K has this property and $B \in \mathcal{B}$.

Let now H_j, $j = 1, 2$, be a couple of self-adjoint operators and let J be a bounded operator in a Hilbert space $\mathcal{H}$. The wave operator for the pair H_1, H_2 and the "identification" J is defined by the relation

$$W^{\pm}(H_2, H_1; J) = s-\lim_{t\to\pm\infty} \exp(iH_2t)J\exp(-iH_1t)P^{(ac)}(H_1) \tag{2.2}$$

under the assumption that this limit exists. We emphasize that all definitions and considerations for " + " and " − " are independent of each other. It suffices to verify existence of the limit (2.2) on some set dense in $\mathcal{H}$. If the wave operator (2.2) exists, then the intertwining property

$$E_2(\Omega)W^{\pm}(H_2, H_1; J) = W^{\pm}(H_2, H_1; J)E_1(\Omega) \tag{2.3}$$

($\Omega \subset \mathbb{R}$ is any Borel set and $E_i(\Omega) = E_i(\Omega; H_i)$) holds. It follows that the range $R(W^\pm(H_2, H_1; J))$ of the operator (2.2) is contained in $\mathcal{H}^{(ac)}(H_2)$ and its closure is an invariant subspace of H_2. Moreover, if the wave operator is isometric on some subspace $\mathcal{H}_1$, then the restrictions of H_1 and H_2 on the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2 = W^\pm(H_2, H_1; J)\mathcal{H}_1$, respectively, are unitarily equivalent. This equivalence is realized by the wave operator. Clearly, for every $f_2 = W^\pm(H_2, H_1; J)f_1$

$$\exp(-iH_2t)f_2 \sim J\exp(-iH_1t)f_1, \quad t \to \pm\infty,$$

where " $\sim$ " means that the difference between left and right sides tends to zero. In case $J = I$ we omit dependence of wave operators on J. The operator $W^\pm(H_2, H_1)$ is obviously isometric on $\mathcal{H}^{(ac)}(H_1)$. The operator $W^\pm(H_2, H_1)$ is called complete if $R(W^\pm(H_2, H_1)) = \mathcal{H}^{(ac)}(H_2)$. This is equivalent to existence of the wave operator $W^\pm(H_1, H_2)$.

We note also the multiplication theorem

$$W^\pm(H_3, H_1; \tilde{J}J) = W^\pm(H_3, H_2; \tilde{J})W^\pm(H_2, H_1; J). \tag{2.4}$$

More precisely, if both wave operators in the right side exist, then the wave operator in the left side also exists and the equality (2.4) holds.

We need the following sufficient condition of existence of wave operators.

Proposition 2.1 *Let an operator $\mathcal{J}$ be H_1-bounded and let its adjoint $\mathcal{J}^*$ be H_2-bounded. Suppose that for some $N < \infty$*

$$H_2\mathcal{J} - \mathcal{J}H_1 = \sum_{n=1}^{N} K_{2,n}^* K_{1,n}$$

(in the precise sense this should be understood as an equality of sesquilinear forms on $\mathcal{D}(H_1) \times \mathcal{D}(H_2)$), where the operators $K_{j,n}$ are H_j-bounded and are H_j-smooth on some bounded interval Λ. Then the wave operators

$$W^\pm(H_2, H_1; \mathcal{J}E_1(\Lambda)), \quad W^\pm(H_1, H_2; \mathcal{J}^*E_2(\Lambda))$$

exist.

Proof for the case $\mathcal{J} = I$ can be found e.g. in [19]. For arbitrary $\mathcal{J}$ the proof is practically the same [23]. Unboundedness of $\mathcal{J}$ is inessential because real identifications $\mathcal{J}E_1(\Lambda)$ and $\mathcal{J}^*E_2(\Lambda)$ are bounded operators. We use Proposition 2.1 only in the case $\mathcal{D}(H_1) = \mathcal{D}(H_2)$ and $\mathcal{J} = \mathcal{J}^*$.

We consider an operator $H = T + V$ in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^d)$ where $T = -\Delta$ and V is multiplication by a function $V(x)$ defined by (1.1). We do not usually distinguish in notation a function and the operator of multiplication by this function. Assume that real functions V^α are sums of short-range V_s^α and long-range V_l^α terms:

$$V^\alpha = V_s^\alpha + V_l^\alpha. \tag{2.5}$$

We say that a potential V^α is short-range if $V_l^\alpha = 0$. It is convenient to split all conditions on V^α into two parts. To formulate them we need to introduce the operator $T^\alpha = -\Delta_{x^\alpha}$ in the space $\mathcal{H}^\alpha = L_2(X^\alpha)$.

Assumption 2.2 *Operators*

$$V^\alpha(T^\alpha + I)^{-1}, \quad (|x^\alpha| + 1)V_s^\alpha(T^\alpha + I)^{-1}, \quad (|x^\alpha| + 1)|\nabla V_l^\alpha|(T^\alpha + I)^{-1}$$

are compact in the space $\mathcal{H}^\alpha$.

Assumption 2.3 *For some* $\rho > 1$ *operators*

$$(|x^\alpha| + 1)^\rho V_s^\alpha(T^\alpha + I)^{-1}, \quad (|x^\alpha| + 1)^\rho|\nabla V_l^\alpha|(T^\alpha + I)^{-1}$$

are bounded in the space $\mathcal{H}^\alpha$.

Compactness of $V^\alpha(T^\alpha + I)^{-1}$ ensures that the operator H is self-adjoint on the domain $\mathcal{D}(H) = \mathcal{D}(T) =: \mathcal{D}$ and H is semi-bounded from below. Set

$$U(t) = \exp(-iHt), \quad E(\cdot) = E(\cdot; H).$$

The condition (1.2) is always assumed. Dimensions d^α of the subspaces X^α are arbitrary. In particular, we do not exclude that one of the subspaces X^α, say X^{α_0}, coincides with the whole space $X = \mathbb{R}^d$. Thus the (three-particle) potential $V^{\alpha_0}(x)$ tends to zero in all directions.

Assumption 2.2 has a preliminary nature. It is required for the Mourre estimate. Practically we use only that for $2r = \rho$ the operators

$$((x^\alpha)^2 + 1)^{r/2}|V_s^\alpha|^{1/2}(T^\alpha + I)^{-1/2} \quad \text{and} \quad ((x^\alpha)^2 + 1)^{r/2}|\nabla V_l^\alpha|^{1/2}(T^\alpha + I)^{-1/2}$$

are bounded in the space $\mathcal{H}^\alpha$. This is a consequence of Assumption 2.3 in virtue of the Heinz inequality. It follows that considered in the space $\mathcal{H}$ the operators $|V_s^\alpha|^{1/2}$ and $|\nabla V_l^\alpha|^{1/2}$ admit the representations

$$|V_s^\alpha|^{1/2} = B_s^\alpha(T + 1)^{1/2}((x^\alpha)^2 + 1)^{-r/2}, \quad B_s^\alpha \in \mathcal{B}, \tag{2.6}$$

$$|\nabla V_l^\alpha|^{1/2} = B_l^\alpha(T + 1)^{1/2}((x^\alpha)^2 + 1)^{-r/2}, \quad B_l^\alpha \in \mathcal{B}. \tag{2.7}$$

Let us introduce operators $H^\alpha = T^\alpha + V^\alpha$, $1 \leq \alpha \leq \alpha_1 := \alpha_0 - 1$, in the spaces $\mathcal{H}^\alpha$ playing the role of "two-particle" Hamiltonians. The point spectrum of H^α consists of eigenvalues accumulating, possibly, at the point $\lambda = 0$ only. The set of thresholds Υ_0 for H is defined as the union

$$\Upsilon_0 = \bigcup_{1 \le \alpha \le \alpha_1} \sigma^{(p)}(H^\alpha) \cup \{0\}.$$

We need the following basic result (see [20, 21, 22]) of spectral theory of multiparticle Hamiltonians. It is formulated in terms of the auxiliary operator

$$A = \sum_{j=1}^{d} (x_j D_j + D_j x_j), \quad D_j = -i\partial_j, \quad \partial_j = \partial/\partial x_j.$$

Proposition 2.4 *Let Assumption 2.2 hold. Then eigenvalues of H may accumulate only at Υ_0 so that the "exceptional" set $\Upsilon = \Upsilon_0 \cup \sigma^{(p)}(H)$ is closed and countable. Furthermore, for every $\lambda \in \mathbb{R} \setminus \Upsilon$ there exists a small interval $\Lambda_\lambda \ni \lambda$ such that the estimate (the Mourre estimate) for the commutator holds:*

$$i([H, A]u, u) \ge c\|u\|^2, \quad c = c_\lambda > 0, \quad u \in E(\Lambda_\lambda)\mathcal{H}. \tag{2.8}$$

Remark. The quadratic form in the left side of (2.8) defined originally for $u \in \mathcal{D}(H) \cap \mathcal{D}(A)$ extends by continuity to all $u \in \mathcal{D}(H)$. Thus it is well-defined for $u \in E(\Lambda_\lambda)\mathcal{H}$.

Let Q be multiplication by $(x^2+1)^{1/2}$. *Below Λ is always an arbitrary bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$*, where $\bar{\Lambda}$ is the closure of Λ. One of the main consequences of (2.8) is the following

Proposition 2.5 *Let Assumptions 2.2 and 2.3 hold. Then for any $r > 1/2$ the operator Q^{-r} is H-smooth on Λ.*

The proof of this assertion under our assumptions can be found in [17].

Corollary 2.6 *The operator H is absolutely continuous on $E(\Lambda)\mathcal{H}$. In particular, it does not have any singular continuous spectrum, i.e.*

$$\mathcal{H} = \mathcal{H}^{(p)}(H) \oplus \mathcal{H}^{(ac)}(H).$$

Note that Propositions 2.4 and 2.5 hold true also for the two-particle case. Thus the operator $(|x^\alpha| + 1)^{-r}, r > 1/2$, is H^α-smooth on any bounded positive interval separated from the point 0. According to Proposition 2.1 this implies that for short-range V^α the wave operators $W^\pm(H^\alpha, T^\alpha)$ exist and are complete.

Let us give the precise formulation of the scattering problem for three-particle Hamiltonians. We introduce auxiliary Hamiltonians $H_\alpha = T + V^\alpha$, $1 \le \alpha \le \alpha_1$, in the space $\mathcal{H}$ with only one pair potential each. Since $X = X_\alpha \oplus X^\alpha, \mathcal{H}$ splits into a tensor product

$$L_2(X) = L_2(X_\alpha) \otimes L_2(X^\alpha). \tag{2.9}$$

Let us introduce also the "free" operator $T_\alpha = -\Delta_{x_\alpha}$ in the space $\mathcal{H}_\alpha = L_2(X_\alpha)$. In the representation (2.9)

$$H_\alpha = T_\alpha \otimes I + I \otimes H^\alpha. \tag{2.10}$$

Denote by P^α the orthogonal projection in $\mathcal{H}^\alpha$ on the subspace $\mathcal{H}^{(p)}(H^\alpha)$ and set $P_\alpha = I \otimes P^\alpha$. Clearly, the orthogonal projection P_α commutes with H_α and its functions. Set also $V^0 = 0, H_0 = T, P_0 = I$. *Below indice a (and b) takes all values* $0, 1, ..., \alpha_1$. We use notation

$$U_a(t) = \exp(-iH_a t), \quad E_a(\cdot) = E(\cdot; H_a).$$

The basic result of the scattering theory for three-particle Hamiltonians is the following

Theorem 2.7 *Suppose that functions V^α satisfy Assumptions 2.2 and 2.3 and are short-range, i.e. $V^\alpha = V_s^\alpha$. Then the wave operators*

$$W_a^\pm = W^\pm(H, H_a; P_a) \tag{2.11}$$

exist and are isometric on $P_a\mathcal{H}$. The ranges $R(W_a^\pm)$ of $W_a^\pm$ are mutually orthogonal and the asymptotic completeness holds:

$$\sum_a \oplus R(W_a^\pm) = \mathcal{H}^{(ac)}(H). \tag{2.12}$$

Our assumptions on V^α are somewhat larger than those of I. M. Sigal and A. Soffer [10] or G .M. Graf [11] since we do not require anything about derivatives of V^α.

Scattering theory for the operator H_α containing only one pair potential reduces to that for the two-particle case. Indeed, comparing formula (2.10) and

$$H_0 = T_\alpha \otimes I + I \otimes T^\alpha,$$

we find that

$$U_\alpha(t)U_0(t) = I \otimes \exp(iH^\alpha t)\exp(-iT^\alpha t).$$

So wave operators $W^\pm(H_\alpha, H_0)$ and $W^\pm(H^\alpha, T^\alpha)$ exist at the same time and

$$W^\pm(H_\alpha, H_0) = I \otimes W^\pm(H^\alpha, T^\alpha).$$

Since wave operators $W^\pm(H^\alpha, T^\alpha)$ exist and are complete we have the following

Proposition 2.8 *In conditions of Theorem 2.7 the wave operators $W^\pm(H_\alpha, H_0)$ exist and*

$$R(W^\pm(H_\alpha, H_0)) = (I - P_\alpha)\mathcal{H}.$$

In particular, for every $f \in \mathcal{H}$ and $f_0^\pm = (W^\pm(H_\alpha, H_0))^* f$

$$U_\alpha(t)f \sim U_0(t)f_0^\pm + U_\alpha(t)P_\alpha f, \quad t \to \pm\infty. \tag{2.13}$$

We conclude this section with some standard technicalities.

Lemma 2.9 *For any $r \in [0,1]$ the operator $[H, Q^r](T+I)^{-1/2} \in \mathcal{B}$.*

Proof. – Clearly,

$$[H, Q^r] = [T, Q^r] = -2\nabla q_r \nabla - \Delta q_r, \quad q_r(x) = (x^2+1)^{r/2}.$$

Since $r \leq 1$, functions ∇q_r and Δq_r are bounded. □

Lemma 2.10 *Let $\psi \in C_0^\infty(\mathbb{R})$ and $r \in [0,1]$. Then $[\psi(H), Q^r] \in \mathcal{B}$.*

Proof. – Note that

$$[U(t), Q^r] = -i \int_0^t U(s)[H, Q^r]U(t-s)ds.$$

Thus in virtue of Lemma 2.9

$$\|[U(t), Q^r](|H|+I)^{-1/2}\| \leq C|t|. \tag{2.14}$$

For an arbitrary ψ we have that

$$[\psi(H), Q^r] = \int_{-\infty}^{\infty}[U(t), Q^r]\hat{\psi}(t)dt, \quad 2\pi\hat{\psi}(t) = \int_{-\infty}^{\infty} \exp(i\lambda t)\psi(\lambda)d\lambda.$$

By (2.14), it follows that

$$[\psi(H), Q^r](|H|+I)^{-1/2} \in \mathcal{B}, \quad \text{if} \quad \int_{-\infty}^{\infty} |t\hat{\psi}(t)|dt < \infty. \tag{2.15}$$

Finally, let $\psi_1 \in C_0^\infty(\mathbb{R})$ and $\psi_1(\lambda) = 1$ on support of ψ so that $\psi = \psi\psi_1$. Then

$$[\psi(H), Q^r] = \psi(H)[\psi_1(H), Q^r] + [\psi(H), Q^r]\psi_1(H)$$

and both terms in the right side are bounded in virtue of (2.15). □

Lemma 2.11 *For $r \in [0,1]$ and arbitrary $z \notin \sigma(H)$ the operator $Q^{-r}(T+I)(H-z)^{-1}Q^r$ is bounded.*

Proof. – Clearly,

$$(H-z)^{-1}Q^r = Q^r(H-z)^{-1} - (H-z)^{-1}[H, Q^r](H-z)^{-1}$$

and, by Lemma 2.9, $[H, Q^r](H-z)^{-1} \in \mathcal{B}$. Thus it remains to check that

$$Q^{-r}(T+I)Q^r(T+I)^{-1} \in \mathcal{B}.$$

To that end we commute T with Q^r and remark that the gradient and Laplacian of $q_r(x) = (x^2+1)^{r/2}$ are bounded. □

Quite similarly we obtain the following result.

Lemma 2.12 *Suppose that a function v obeys the estimate*

$$|v(x)| + |(\nabla v)(x)| + |(\Delta v)(x)| \leq C(|x| + 1)^{-r}, \quad r \in [0,1]. \tag{2.16}$$

Then

$$(T+I)v(T+I)^{-1}Q^r \in \mathcal{B}, \quad (T+I)^{1/2}vD_j(T+I)^{-1}Q^r \in \mathcal{B}, \quad j = 1, \dots, d.$$

Combining Lemma 2.11 with Proposition 2.5 we immediately obtain

Proposition 2.13 *For every $r > 1/2$ the operator $Q^{-r}(T+I)$ is H-smooth on Λ.*

Proof. – For any $z \notin \sigma(H)$

$$Q^{-r}(T+I)U(t)f = (Q^{-r}(T+I)(H-z)^{-1}Q^r)\, Q^{-r}U(t)(H-z)f.$$

Since the first factor in the right side is bounded it suffices to apply the definition of H-smoothness to the element $(H-z)f \in E(\Lambda)\mathcal{H}$. □

In virtue of Lemma 2.12, Proposition 2.13 is more general than Proposition 2.5. Therefore we usually give references below only to Proposition 2.13. Similarly, by Lemma 2.12, Proposition 2.13 ensures H-smoothness of the operators $Q^{-r}D_j$ where $r > 1/2$ and $j = 1, \dots, d$.

Of course, all results formulated for the operator H are as well true for H_0 and H_α.

3. POSITIVE COMMUTATORS AND RADIATION CONDITIONS

Our approach relies on consideration of the commutator of H with a first-order differential operator

$$M = \sum_{j=1}^{d}(m_j D_j + D_j m_j), \quad m_j = \partial m / \partial x_j, \tag{3.1}$$

where m is suitably chosen real function. To give an idea of this choice we note that for $m(x) = |x|$ there is the identity

$$i[H_0, M] = 4\nabla^{(s)}|x|^{-1}\nabla^{(s)}, \quad H_0 = T = -\Delta, \tag{3.2}$$

which can be deduced e.g. from the formulas (3.3) and (3.13) below. The arguments of [7] (reproduced in the proof of Theorem 3.5) show that the identity (3.2) ensures H_0-smoothness of the operator $Q^{-1/2}\nabla^{(s)}$. Furthermore, since $[V^{\alpha_0}, M] = O(|x|^{-\rho}), \rho > 1, |x| \to \infty$, using Proposition 2.5, we can prove smoothness of $Q^{-1/2}\nabla^{(s)}$ with respect to the "two-particle" Hamiltonian $H_{\alpha_0} = H_0 + V^{\alpha_0}$. However, the functions $[V^\alpha, M], 1 \leq \alpha \leq \alpha_1$, decrease only as $|x|^{-1}$ at infinity. Actually, one can not expect that the operator $Q^{-1/2}\nabla^{(s)}$ is H-smooth. To prove a weaker result about H-smoothness of the operators (1.4) and (1.6) the function $m(x)$ should be modified in such a way that $[V^\alpha, M] = O(|x|^{-\rho}), \rho > 1$ for all α. The last relation holds if $m(x)$ depends only on the variable x_α in some cone where $V^\alpha(x^\alpha)$ is concentrated. This is similar to the idea of G. M. Graf applied in [11] in the time-dependent context.

Suppose for a moment that m is an arbitrary smooth function. We start with the standard calculation of the commutator $[H_0, M]$.

Lemma 3.1 *Let an operator M be defined by (3.1). Then*

$$i[H_0, M] = 4\sum_{j,k} D_j m_{jk} D_k - (\Delta^2 m), \quad m_{jk} = \partial^2 m/\partial x_j \partial x_k. \tag{3.3}$$

Proof. – Let us consider

$$[\partial_j^2, m_k \partial_k] = \partial_j^2 m_k \partial_k - m_k \partial_k \partial_j^2. \tag{3.4}$$

Commuting ∂_j with m_k we find that the first term in the right side equals

$$\partial_j^2 m_k \partial_k = \partial_j(m_{jk} + m_k \partial_j)\partial_k.$$

Similarly, the second term

$$\begin{aligned} m_k\partial_k\partial_j^2 &= (m_k\partial_j)(\partial_k\partial_j) = (-m_{jk} + \partial_j m_k)(\partial_k\partial_j) = \\ &= -(\partial_j m_{jk} - m_{jjk})\partial_k + \partial_j m_k \partial_k \partial_j, \quad m_{jjk} = \partial^3 m/\partial x_j^2 \partial x_k. \end{aligned}$$

Inserting these expressions into (3.4) we obtain that

$$[\partial_j^2, m_k\partial_k] = 2\partial_j m_{jk}\partial_k - m_{jjk}\partial_k.$$

It follows that

$$\begin{aligned} [\partial_j^2, m_k\partial_k + \partial_k m_k] &= [\partial_j^2, m_k\partial_k] + [\partial_j^2, m_k\partial_k]^* = \\ &= 2(\partial_j m_{jk}\partial_k + \partial_k m_{jk}\partial_j) - m_{jjk}\partial_k + \partial_k m_{jjk} = \\ &= 2(\partial_j m_{jk}\partial_k + \partial_k m_{jk}\partial_j) + m_{jjkk}, \quad m_{jjkk} = \partial^4 m/\partial x_j^2 \partial x_k^2. \end{aligned}$$

Summing up these relations in j and k we arrive at (3.3). □

We choose $m(x)$ as a homogeneous function of degree 1. Such functions have singularities at $x = 0$. In virtue of Proposition 2.5 values of $m(x)$ in a bounded domain are inessential. Therefore we can get rid of singularity of $m(x)$ replacing it in a neighbourhood of $x = 0$ by an arbitrary smooth function. In such a way we obtain C^∞-function which satisfies the relation $m(sx) = sm(x)$ if $|x| \geq c > 0$ and $s \geq 1$. We say that m is homogeneous for $|x| \geq c$. A function m is constructed differently in neighbourhoods of subspaces X_α and in a "free" region, which is separated from all X_α. In order to describe necessary properties of m it is convenient to define a conical neighbourhood

$$\Gamma_\alpha(\varepsilon) = \{|x_\alpha| > (1-\varepsilon)|x|\}, \quad \varepsilon \in (0,1), \quad 1 \leq \alpha \leq \alpha_1,$$

of $X_\alpha \setminus \{0\}$. For sufficiently small ϵ and $\varepsilon \leq \epsilon$ these neighbourhoods are separated from each other, i.e. $\overline{\Gamma_\alpha(\varepsilon)} \cap \overline{\Gamma_\beta(\varepsilon)} = \{0\}$ for $\alpha \neq \beta$. This is a consequence of the assumption (1.2). Set also

$$\Gamma_0(\varepsilon) = \{(1-\varepsilon)|x| > |x_\alpha|, \ 1 \leq \alpha \leq \alpha_1\}.$$

We always assume that $\varepsilon \in (0, \epsilon)$ so that cones $\Gamma_0(\varepsilon)$ are not empty. Clearly, $\Gamma_0(\varepsilon)$ gets larger if ε decreases but never intersects with X_α. More precisely, $\Gamma_0(\varepsilon) \cap \Gamma_\alpha(\varepsilon) = \emptyset$ and

$$\overline{\Gamma_0(\varepsilon)} \cup \bigcup_{\alpha=1}^{\alpha_1} \Gamma_\alpha(\varepsilon) = X. \tag{3.5}$$

Let us subtract from $\Gamma_a(\varepsilon)$ the unit ball, that is we set

$$\overset{o}{\Gamma}_a(\varepsilon) = \Gamma_a(\varepsilon) \cap \{|x| > 1\}.$$

We submit $m(x)$ to the following requirements:

1^0 $m(x)$ is a real nonnegative C^∞-function, which is homogeneous of degree 1 for $|x| \geq 1$ and $m(x) = 0$ for $|x| \leq 1/2$.

2^0 $m(x) > 0$ if $|x| = 1$.

3^0 $m(x)$ is a (locally) convex function for $|x| \geq 1$, i.e.

$$\sum_{j,k} m_{jk}(x)\xi_j\bar{\xi}_k \geq 0, \quad \forall \xi \in \mathbb{C}^d, \ |x| \geq 1. \tag{3.6}$$

4^0 For every $\alpha = 1, \ldots, \alpha_1$ there exist $\epsilon_\alpha \in (0, \epsilon)$ and $\mu_\alpha > 0$ such that $m(x) = \mu_\alpha |x_\alpha|$ if $x \in \overset{o}{\Gamma}_\alpha(\epsilon_\alpha)$. Furthermore, there exist $\epsilon_0 > \max\{\epsilon_\alpha\}$ and $\mu_0 > 0$ such that $m(x) = \mu_0|x|$ if $x \in \overset{o}{\Gamma}_0(\epsilon_0)$.

The final property is, strictly speaking, related to the family of functions satisfying $1^0 - 4^0$.

5^0 By a choice of $m(x) = m^{(\epsilon_0)}(x)$ a number ϵ_0 can be made arbitrary small (i.e. for arbitrary small neighbourhoods of X_α one can construct $m(x)$ in such a way that $m(x) = \mu_0|x|$ for $|x| \geq 1$ outside of these neighbourhoods).

Below ϵ, ϵ_0 and ϵ_α are always numbers specified here; in particular, $\epsilon > \epsilon_0 > \epsilon_\alpha$, $\alpha = 1, \dots, \alpha_1$.

Let us give an example of a function $m(x)$ obeying all conditions $1^0 - 5^0$. First, we introduce a family of functions $m_\varepsilon(x)$ satisfying all the properties except smoothness and then average $m_\varepsilon(x)$ over ε. Set

$$m_\varepsilon(x) = \max\{|x_1|, ..., |x_{\alpha_1}|, (1-\varepsilon)|x|\}, \quad 0 < \varepsilon < \epsilon.$$

By definition, $m_\varepsilon(x)$ is a homogeneous function of degree 1. Being maximum of convex functions, $m_\varepsilon(x)$ is convex, i.e.

$$m_\varepsilon(r_1 x_1 + r_2 x_2) \leq r_1 m_\varepsilon(x_1) + r_2 m_\varepsilon(x_2), \quad r_j \in [0,1], \ r_1 + r_2 = 1.$$

Clearly, $m_\varepsilon(x) = |x_\alpha|$ if $x \in \Gamma_\alpha(\varepsilon)$, and $m_\varepsilon(x) = (1-\varepsilon)|x|$ if $x \in \Gamma_0(\varepsilon)$. In other words,

$$m_\varepsilon(x) = \sum_{\alpha=1}^{\alpha_1} |x_\alpha| \theta(|x_\alpha| - (1-\varepsilon)|x|) + (1-\varepsilon)|x|(1 - \sum_{\alpha=1}^{\alpha_1} \theta(|x_\alpha| - (1-\varepsilon)|x|)), \quad (3.7)$$

where $\theta(s) = 1$ for $s \geq 0$ and $\theta(s) = 0$ for $s < 0$.

Let $\varphi(\varepsilon)$ be some smooth nonnegative function supported in a closed interval $[\epsilon_1, \epsilon_0]$, $0 < \epsilon_1 < \epsilon_0 < \epsilon$. Define

$$m(x) = \int_0^\epsilon m_\varepsilon(x) \varphi(\varepsilon) d\varepsilon. \quad (3.8)$$

Obviously, $m(x)$ is again homogeneous function of degree 1. It satisfies the property 2^0 because

$$m_\varepsilon(x) \geq 1 - \varepsilon \geq 1 - \epsilon_0 > 0, \quad |x| = 1.$$

Being an integral of convex functions, $m(x)$ is convex. Comparing (3.7) with (3.8) and denoting

$$\Phi(s) = \int_s^\epsilon \varphi(\varepsilon) d\varepsilon, \quad \tilde{\Phi}(s) = \int_s^\epsilon (1-\varepsilon) \varphi(\varepsilon) d\varepsilon,$$

we find that

$$m(x) = \sum_{\alpha=1}^{\alpha_1} |x_\alpha| \Phi(1 - |x|^{-1}|x_\alpha|) + |x|(\tilde{\Phi}(0) - \sum_{\alpha=1}^{\alpha_1} \tilde{\Phi}(1 - |x|^{-1}|x_\alpha|)). \quad (3.9)$$

Functions $\Phi(s)$ and $\tilde{\Phi}(s)$ are smooth, they equal zero if $s \geq \epsilon_0$ and they equal constants $\Phi(0)$ and $\tilde{\Phi}(0)$, respectively, if $s \leq \epsilon_1$. Therefore the function (3.9) belongs to $C^\infty(\mathbb{R}^d \setminus \{0\}), m(x) = \Phi(0)|x_\alpha|$ if $x \in \Gamma_\alpha(\epsilon_1)$ and $m(x) = \tilde{\Phi}(0)|x|$ if $x \in \Gamma_0(\epsilon_0)$. Thus the property 4^0 (with $\mu_\alpha = \Phi(0), \epsilon_\alpha = \epsilon_1$ and $\mu_0 = \tilde{\Phi}(0)$) holds. Since ϵ_0 is an arbitrary small number, the property 5^0 is also fulfilled.

Finally, one can get rid of local singularity of $m(x)$ at $x = 0$ replacing it by $\tau(x)m(x)$ where $\tau \in C^\infty(\mathbb{R}^d)$, $\tau(x) = 0$ for $|x| \leq 1/2$ and $\tau(x) = 1$ for $|x| \geq 1$.

Actually, the concrete construction of the function m is of no importance for us and we always use only its properties $1^0 - 5^0$ listed above. By the property 1^0 derivatives m_j of m are homogeneous functions of degree 0, m_{jk} are homogeneous of degree -1 and m_{jjkk} are homoneneous of degree -3. Therefore

$$(\Delta^2 m)(x) = O(|x|^{-3}), \quad |x| \to \infty, \tag{3.10}$$

and the main contribution to the commutator (3.3) is determined by the operator

$$L = L(m) = \sum_{j,k} D_j m_{jk} D_k. \tag{3.11}$$

To estimate it we first compute the matrix

$$\mathbf{M}(x) = \{m_{jk}(x)\} = \text{Hess } m(x)$$

in the region where $m(x) = \mu_0|x|$:

$$m_j(x) = \mu_0|x|^{-1}x_j, \quad m_{jk}(x) = \mu_0(|x|^{-1}\delta_{jk} - |x|^{-3}x_jx_k). \tag{3.12}$$

Here $\delta_{jj} = 1$ and $\delta_{jk} = 0$ if $j \neq k$. By the definition (1.3), the angular part of the gradient ∇u obeys the identity

$$|\nabla^{(s)}u|^2 = |\nabla u|^2 - |x|^{-2}|\langle \nabla u, x\rangle|^2 = \sum_j |u_j|^2 - |x|^{-2}|\sum_j u_jx_j|^2 =$$
$$= \sum_j (1 - |x|^{-2}x_j^2)|u_j|^2 - |x|^{-2}\sum_{j\neq k} x_jx_ku_j\bar{u_k}, \quad u_j = \partial u/\partial x_j.$$

According to (3.12) it follows that

$$\sum_{j,k} m_{jk}u_j\bar{u_k} = \mu_0|x|^{-1}|\nabla^{(s)}u|^2. \tag{3.13}$$

In the region where $m(x) = \mu_\alpha|x_\alpha|$ all calculations hold true if x is replaced by x_α. Thus we obtain the following

Lemma 3.2 *Let $\nabla^{(s)}u$ and $\nabla^{(s)}_{x_\alpha}u$ be defined by (1.3), (1.5) respectively and let $\overset{o}{\Gamma}_0\ (\epsilon_0), \overset{o}{\Gamma}_\alpha\ (\epsilon_\alpha)$ be the truncated cones introduced in the condition 4^0 on $m(x)$. For $x \in \overset{o}{\Gamma}_0\ (\epsilon_0)$ the identity (3.13) holds and for $x \in \overset{o}{\Gamma}_\alpha\ (\epsilon_\alpha)$*

$$\sum_{j,k} m_{jk}u_j\bar{u_k} = \mu_\alpha|x_\alpha|^{-1}|\nabla^{(s)}_{x_\alpha}u|^2, \quad \alpha = 1, ..., \alpha_1. \tag{3.14}$$

Note that in case $\dim X_\alpha = 1$ both sides of (3.14) equal zero.

By the condition (3.6) on $m(x)$

$$(Lu,u)=\sum_{j,k}\int m_{jk}u_j\bar{u}_k dx \geq \sum_{j,k}\int_\Gamma m_{jk}u_j\bar{u}_k dx - c\int_{|x|<1}|\nabla u|^2 dx,$$

where Γ is any region lying outside of the unit ball. Combining this inequality with Lemma 3.2 we obtain

Proposition 3.3 *In notation of Lemma 3.2 for every $u\in\mathcal{D}$*

$$(Lu,u)\geq \mu_0\int_{\overset{o}{\Gamma}_0(\epsilon_0)}|x|^{-1}|\nabla^{(s)}u|^2dx - c\int_{|x|<1}|\nabla u|^2dx$$

and

$$(Lu,u)\geq \mu_\alpha\int_{\overset{o}{\Gamma}_\alpha(\epsilon_\alpha)}|x_\alpha|^{-1}|\nabla^{(s)}_{x_\alpha}u|^2dx - c\int_{|x|<1}|\nabla u|^2dx,\quad \alpha=1,...,\alpha_1.$$

It turns out that due to the property 4^0 the commutator $[V,M]$ is in some sense small. The precise formulation is given in the following

Proposition 3.4 *Suppose that V^α is defined by (2.5) where V_s^α and V_l^α satisfy Assumptions 2.2 and 2.3. Let m obey the property 1^0 and $m(x)=m(x_\alpha)$ if $x\in\overset{o}{\Gamma}_\alpha(\varepsilon_\alpha)$ for some $\varepsilon_\alpha>0$. Then*

$$|([V^\alpha,M]u,u)|\leq C\|Q^{-r}(T+I)u\|^2,\quad u\in\mathcal{D},\quad 2r=\rho. \tag{3.15}$$

Proof. – Suppose first that $1\leq\alpha\leq\alpha_1$. Let us introduce a smooth homogeneous (for $|x|\geq 2$) function ζ_α of degree zero such that $0\leq\zeta_\alpha(x)\leq 1$, $\zeta_\alpha(x)=1$ if $x\notin\overset{o}{\Gamma}_\alpha(\varepsilon_\alpha)$ and $\zeta_\alpha(x)=0$ if $x\in\Gamma_\alpha(\varepsilon)$ for some $\varepsilon\in(0,\varepsilon_\alpha)$ and $|x|\geq 2$. The long-range part of V^α is differentiable so that

$$i[V_l^\alpha,M]=2[V_l^\alpha,\sum_{j=1}^d m_j\partial_j]=-2\sum_{j=1}^d m_j\partial V_l^\alpha/\partial x_j=-2\langle\nabla m(x),\nabla V_l^\alpha(x^\alpha)\rangle.$$

This scalar product equals zero for $x\in\overset{o}{\Gamma}_\alpha(\varepsilon_\alpha)$ because m depends only on x_α in this region and, consequently, $\nabla m(x)\in X_\alpha$ whereas $\nabla V_l^\alpha\in X^\alpha$. Since $|\nabla m(x)|$ is bounded, it follows that

$$|\langle\nabla m(x),\nabla V_l^\alpha(x^\alpha)\rangle|\leq C\zeta_\alpha(x)|\nabla V_l^\alpha(x^\alpha)|\zeta_\alpha(x). \tag{3.16}$$

Using the representation (2.7) we find that

$$|([V_l^\alpha,M]u,u)|\leq C\|(T+I)^{1/2}w_\alpha u\|^2,\quad w_\alpha(x)=((x^\alpha)^2+1)^{-r/2}\zeta_\alpha(x).$$

The function $w_\alpha(x)$ obeys the condition (2.16) because $\zeta_\alpha(x)=0$ if $x\in\Gamma_\alpha(\varepsilon)$ and $|x|\geq 2$. Therefore, taking into account Lemma 2.12, we obtain the bound (3.15) for V_l^α.

To consider $[V_s^\alpha, M]$ we use again that the function $\zeta_\alpha(x)$ differs from 1 only if $x \in \overset{o}{\Gamma}_\alpha (\varepsilon_\alpha)$. In this region the function m does not depend on x^α. It follows that the operator

$$i\eta_\alpha M = 2\eta_\alpha(\nabla_{x_\alpha} m)\nabla_{x_\alpha} + \eta_\alpha(\Delta_{x_\alpha} m), \quad \eta_\alpha(x) = 1 - \zeta_\alpha^2(x),$$

commutes with V_s^α and hence $[V_s^\alpha, M] = [V_s^\alpha, \zeta_\alpha^2 M]$. Simple computations show that

$$[V_s^\alpha, \zeta_\alpha^2 M] = 2\sum_{j=1}^{d}(V_s^\alpha \xi_{\alpha,j} D_j - D_j V_s^\alpha \xi_{\alpha,j} - iV_s^\alpha \partial\xi_{\alpha,j}/\partial x_j), \quad \xi_{\alpha,j} = \zeta_\alpha^2 m_j. \quad (3.17)$$

Note that the functions m_j are bounded together with their derivatives and $\xi_{\alpha,j} = 0$ if $x \in \Gamma_\alpha(\varepsilon)$ and $|x| \geq 2$. In virtue of the representation (2.6) for $|V_s^\alpha|^{1/2}$ the last term in (3.17) is estimated exactly as the right side of (3.16). Similarly,

$$|(V_s^\alpha \xi_{\alpha,j} D_j u, u)| \leq C\|(T+I)^{1/2} w_\alpha D_j u\| \, \|(T+I)^{1/2} w_\alpha u\|,$$

which is estimated by the right side of (3.15) according to Lemma 2.12. In the case $\alpha = \alpha_0$ the estimates are the same but the cut-off by ζ_α is no longer necessary. □

Given Propositions 3.3 and 3.4 the proof of the main result of this section is quite standard. We formulate it only for the operator H since H_0 and H_α are its special cases.

Theorem 3.5 *Suppose that V^α are defined by (2.5) where V_s^α and V_l^α satisfy Assumptions 2.2 and 2.3. Let $\chi_a(\varepsilon;\cdot)$, $a = 0, 1, \ldots, \alpha_1$, be the characteristic function of a cone $\Gamma_a(\varepsilon)$, where $\varepsilon \in (0, \epsilon)$ is arbitrary. Then the operators*

$$G_0(\varepsilon) = \chi_0(\varepsilon) Q^{-1/2}\nabla^{(s)}, \quad G_\alpha(\varepsilon) = \chi_\alpha(\varepsilon) Q^{-1/2}\nabla^{(s)}_{x_\alpha},$$

acting from the space $L_2(\mathbb{R}^d)$ into the vector-spaces $L_2(\mathbb{R}^d)\otimes\mathbb{C}^d$ and $L_2(\mathbb{R}^d)\otimes \mathbb{C}^{d_\alpha}$, $d_\alpha = \dim X_\alpha$, respectively, are H-smooth on arbitrary bounded interval $\Lambda, \bar\Lambda \cap \Upsilon = \emptyset$.

Proof. – Let us consider

$$d(MU(t)f, U(t)f)/dt = i([H, M]f_t, f_t), \quad (3.18)$$

where $f_t = U(t)f$, $f \in \mathcal{D}$. By (3.3), (3.11)

$$i([H, M]f_t, f_t) = 4(Lf_t, f_t) - ((\Delta^2 m)f_t, f_t) + i([V, M]f_t, f_t).$$

Taking into account (3.10) and applying Propositions 3.3, 3.4 to elements $u = f_t$ we find that (under the assumption $\rho \leq 3$)

$$i([H, M]f_t, f_t) \geq c_1\|G_a(\epsilon_a)f_t\|^2 - c_2\|Q^{-r}(T+I)f_t\|^2, \quad 2r = \rho, \quad (3.19)$$

for any $a = 0, 1, ..., \alpha_1$. Here we have omitted $\|Q^{-r} f_t\|^2$ and the integral of $|\nabla f_t|^2$ over the unit ball because they are estimated by the last term in the right side of (3.19). Integrating (3.18), (3.19) over $t \in (t_1, t_2)$ we obtain that

$$\int_{t_1}^{t_2} \|G_a(\epsilon_a) f_t\|^2 dt \leq C(|(M f_t, f_t)\Big|_{t_1}^{t_2} | + \int_{t_1}^{t_2} \|Q^{-r}(T + I) f_t\|^2 dt). \tag{3.20}$$

Suppose now that $f = E(\Lambda) f$. Then the first term in the right side of (3.20) is bounded by $C\|f\|^2$ because $ME(\Lambda) \in \mathcal{B}$ for bounded Λ. The second term admits the same estimate according to Proposition 2.13. It follows that the integral in the left side of (3.20) is bounded by $C\|f\|^2$ so that each of the operators $G_a(\epsilon_a)$ is H-smooth on Λ. By the property 5^0 of the function $m(x)$ a number ϵ_0 can be arbitrary small. This concludes the proof of H-smoothness of $G_0(\varepsilon)$ for arbitrary $\varepsilon > 0$. Since

$$|\nabla_{x_\alpha}^{(s)} u| \leq |\nabla^{(s)} u|, \tag{3.21}$$

H-smoothness of $G_\alpha(\varepsilon)$ for arbitrary $\varepsilon \in (0, \epsilon)$ is now a consequence of that fact for some $\varepsilon > 0$. $\square$

Remark. Let us give for completeness a proof of (3.21). We can assume that u is real. By definitions (1.3), (1.5) the estimate (3.21) is equivalent to the bound

$$|\xi_\alpha|^2 + |x|^{-2} |\langle \xi, x \rangle|^2 \leq |\xi|^2 + |x_\alpha|^{-2} |\langle \xi_\alpha, x_\alpha \rangle|^2, \tag{3.22}$$

where ξ ($\xi = \nabla u$) is an arbitrary vector of X and ξ_α ($\xi_\alpha = \nabla_{x_\alpha} u$) is the orthogonal projection of ξ on X_α. It suffices to prove (3.22) with $|\langle \xi, x \rangle|$ replaced by $|\langle \xi_\alpha, x_\alpha \rangle| + |\xi^\alpha||x^\alpha|$. By identical transformations such an estimate can be reduced to the obvious inequality

$$2|x_\alpha|^2 |\langle \xi_\alpha, x_\alpha \rangle| |\xi^\alpha| |x^\alpha| \leq |\langle \xi_\alpha, x_\alpha \rangle|^2 |x^\alpha|^2 + |x_\alpha|^4 |\xi^\alpha|^2.$$

Remark. By (3.21), Theorem 3.5 gives us more information about $U(t) f$ in the "free" region Γ_0 compared to that in the regions Γ_α where potentials V^α are concentrated.

Remark. The notion of H-smoothness can be equivalently reformulated in terms of the resolvent of H. Thus radiation conditions-estimates given by Theorem 3.5 also admit a stationary formulation.

Remark. In the two-particle case (where $H = T + V^{\alpha_0}$) the result of Theorem 3.5 reduces to H-smoothness of the operator $Q^{-1/2} \nabla^{(s)}$ on any bounded positive interval separated from the point 0. This is different from the usual form of the radiation condition (see e.g. [14]). First, we consider only the angular part of $\nabla U(t) f$. Second, the estimate of [14] implies that

$$\int_{-\infty}^{\infty} \|Q^{-r} \nabla^{(s)} U(t) f\|^2 dt < \infty. \tag{3.23}$$

Here r is some number smaller than $1/2$ whereas we require that $r = 1/2$ which is less informative. On the other hand, in (3.23) f should belong to some dense (in $\mathcal{H}$) set whereas our estimate is uniform for all $f \in \mathcal{H}$.

Note, finally, that in [24] a radiation condition for N-particle case was derived in the free region Γ_0. From the viewpoint of the previous remark the result of [24] is similar to the two-particle radiation condition and thus it differs from Theorem 3.5. Results of [24] can probably be used (see the discussion at the beginning of the next section) for a proof of asymptotic completeness in the three-particle case. However, an information about $U(t)f$ in a free region only is not sufficient for the case of $N > 3$ particles.

4. MODIFIED WAVE OPERATORS

In order to explain an idea of the subsequent proof of asymptotic completeness let us recall that, as remarked by P. Deift and B. Simon [25], it is equivalent to existence of wave operators $W^{\pm}(H_a, H; J^{(a)})$, $a = 0, 1, \ldots, \alpha_1$. Here identifications $J^{(a)}$ are multiplications by smooth homogeneous functions $\eta^{(a)}$ of zero order such that $\sum \eta^{(a)}(x) = 1$. Furthermore, $\eta^{(\alpha)}(x) = 1$ in a neighbourhood Γ_α of the subspace X_α and $\eta^{(0)}(x) = 1$ if x is sufficiently far from all of them. The main contribution to the "perturbation" $HJ^{(a)} - J^{(a)}H_a$ is given by the term $\nabla\eta^{(a)}\nabla$, which equals $\nabla\eta^{(a)}\nabla^{(s)}$ because $\langle\nabla\eta^{(a)}(x), x\rangle = 0$. Remark also that $\nabla\eta^{(a)}(x)$ decays as $|x|^{-1}$ at infinity and differs from zero in a free region Γ_0 only. Therefore convergence of the integral (cf. with the last remark in section 3)

$$\int_{-\infty}^{\infty} \|\chi_0 Q^{-r}\nabla^{(s)}U(t)f\|^2 dt < \infty$$

for some $r < 1/2$ and for elements f from some set dense in $\mathcal{H}$ would have been sufficient (see [17] for more details about such a plan of the proof) for existence of the wave operators $W^{\pm}(H_a, H; J^{(a)})$.

The result of Theorem 3.5 allows us to accomodate the terms $G_a^* G_a$ which are similar to $\nabla\eta^{(a)}\nabla^{(s)}$ but are second-order differential operators. Thus we are compelled to change the identifications $J^{(a)}$. We choose new identifications as first-order differential operators $M^{(a)}$ constructed by means of functions $\eta^{(a)}m$. Coefficients of $M^{(a)}$ equal zero outside of a region Γ_a and $\sum M^{(a)} = M$. We emphasize that our proof of existence of the wave operators $W^{\pm}(H_a, H; M^{(a)}E(\Lambda))$ requires H-smoothness of all operators G_a (not only of G_0). To remove the identifications $M^{(a)}$ we introduce also the auxiliary wave operator $W^{\pm}(H, H; ME(\Lambda))$. At the end of this section we show that this operator is invertible on the subspace $E(\Lambda)\mathcal{H}$. As was explained in section 1, this is an essential step in our proof of asymptotic completeness.

Let us proceed to the formal exposition. We start with the following elementary observation.

Lemma 4.1 *Suppose that $m(x)$ is an arbitrary smooth homogeneous (for $|x| \geq 1$) function of degree 1. Let $\lambda_n(x)$ and $p_n(x)$ be eigenvalues and eigenvectors of the symmetric matrix $\mathbf{M}(x) = \{m_{jk}(x)\}$. Then vectors $p_n(x), |x| \geq 1$, corresponding to $\lambda_n(x) \neq 0$, are orthogonal to x.*

Proof. – Since $\mathbf{M}(x)$ is symmetric, it suffices to show that x is its eigenvector corresponding to the zero eigenvalue. Differentiating the identity $m(sx) = sm(x)$ in s and setting $s = 1$ we find that

$$\sum m_j(x)x_j = m(x)$$

(Euler's formula). Differentiation of this relation in x_k shows that

$$\sum_j m_{kj}(x)x_j = 0, \quad k = 1, \ldots, d.$$

Thus $\mathbf{M}(x)x = 0$. □

Let some function m satisfying conditions $1^0 - 4^0$ be given and let $\varepsilon_0 = \min \epsilon_\alpha$, $\alpha = 1, \ldots, \alpha_1$. We introduce homogeneous functions $\eta^{(\alpha)} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ of degree 0, $\alpha = 1, \ldots, \alpha_1$, such that supp $\eta^{(\alpha)} \subset \overline{\Gamma_\alpha(\epsilon)}$ (and hence supports of $\eta^{(\alpha)}$ for different α intersect only at zero) and $\eta^{(\alpha)}(x) = 1$ if $x \in \Gamma_\alpha(\epsilon_\alpha)$. The function

$$\eta^{(0)}(x) = 1 - \sum_{\alpha=1}^{\alpha_1} \eta^{(\alpha)}(x) \tag{4.1}$$

equals zero if $x \notin \Gamma_0(\varepsilon_0)$ and $\eta^{(0)}(x) = 1$ if $x \in \Gamma_0(\epsilon)$. Set $m^{(a)}(x) = \eta^{(a)}(x)m(x)$, $a = 0, 1, \ldots, \alpha_1$, and

$$M^{(a)} = \sum_{j=1}^{d}(m_j^{(a)}D_j + D_j m_j^{(a)}), \quad m_j^{(a)} = \partial m^{(a)}/\partial x_j. \tag{4.2}$$

Clearly, $m^{(a)}(x)$ satisfies the properties 1^0 and 4^0 (with $\mu_a^{(a)} = \mu_a$ and $\mu_b^{(a)} = 0$ for $b \neq a$) but the properties 2^0 and 3^0 are violated.

Theorem 4.2 *Suppose that functions V^α satisfy the assumptions of Theorem 2.7 and Λ is any bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$. Then the wave operators*

$$W^\pm(H, H_a; M^{(a)}E_a(\Lambda)), \quad W^\pm(H_a, H; M^{(a)}E(\Lambda)), \tag{4.3}$$

exist for all $a = 0, 1, \ldots, \alpha_1$.

Proof. – We shall show that the triple $H_a, H, M^{(a)}$ satisfies on Λ the conditions of Proposition 2.1. Let us consider

$$HM^{(a)} - M^{(a)}H_a = [T, M^{(a)}] + [V^a, M^{(a)}] + \sum_{\beta \neq a} V^\beta M^{(a)}, \quad \beta = 1, \ldots, \alpha_0, \tag{4.4}$$

$V^0 = 0$. We shall verify that each term in the right side can be factored into a product of H- and H_a-smooth operators. We start with the last two terms which can be estimated with the help of Proposition 2.13 only. The commutator $[V^\alpha, M^{(\alpha)}]$ was actually already considered in Proposition 3.4. Its assumptions are fulfilled because the function $m^{(\alpha)}$ satisfies the property 1^0 and $m^{(\alpha)}(x) = \mu_\alpha |x_\alpha|$ if $x \in \overset{o}{\Gamma}_\alpha\ (\epsilon_\alpha)$. The estimate (3.15) is equivalent to the representation

$$[V^\alpha, M^{(\alpha)}] = (T + I)Q^{-r}B^{(\alpha)}Q^{-r}(T + I), \quad 2r = \rho, \quad B^{(\alpha)} \in \mathcal{B},$$

where $Q^{-r}(T + I)$ is H- and H_α-smooth on Λ in virtue of Proposition 2.13.

We need short-range assumption on potentials only to treat $V^\beta M^{(a)}, \beta \neq a$. Suppose first that $\beta \neq \alpha_0$. Recall that $m^{(a)}(x) = 0$ if $x \in \Gamma_\beta(\epsilon_\beta)$. Therefore $m_j^{(a)}(x) = m_j^{(a)}(x)\zeta_\beta^2(x)$ and $m_{jj}^{(a)}(x) = m_{jj}^{(a)}(x)\zeta_\beta^2(x)$ for suitable $\zeta_\beta \in C^\infty(\mathbb{R}^d)$, homogeneous (for $|x| \geq 1$) of degree 0, such that $\zeta_\beta(x) = 0$ if $x \in \overset{o}{\Gamma}_\beta\ (\varepsilon)$ for some $\varepsilon \in (0, \epsilon_\beta)$. By (2.6), (4.2) the operator $V^\beta M^{(a)}$ consists of terms

$$V^\beta m_j^{(a)} D_j = w_\beta (T + I)^{1/2} B_j^{(a,\beta)} (T + I)^{1/2} w_\beta D_j$$

and

$$V^\beta m_{jj}^{(a)} = w_\beta (T + I)^{1/2} B_{jj}^{(a,\beta)} (T + I)^{1/2} w_\beta,$$

where $j = 1, \ldots, d$,

$$w_\beta(x) = ((x^\beta)^2 + 1)^{-r/2}\zeta_\beta(x), \quad 2r = \rho, \quad B_j^{(a,\beta)} \in \mathcal{B}, \quad B_{jj}^{(a,\beta)} \in \mathcal{B}.$$

The function $w_\beta(x)$ obeys the condition (2.16). Therefore, by Lemma 2.12, each of these terms equals $(T+I)Q^{-r}BQ^{-r}(T+I)$ with some bounded operator B. This proves the required factorization of $V^\beta M^{(a)}$ into a product of smooth operators. In case $\beta = \alpha_0$ the estimates are the same but the cut-off by ζ_β is no longer necessary.

Let us consider the first term in the right side of (4.4). According to Lemma 3.1 the commutator $[T, M^{(a)}]$ is defined by (3.3) with m replaced by $m^{(a)}$. Since $m^{(a)}$ is a homogeneous function of degree 1 the term $(\Delta^2 m^{(a)})(x) = O(|x|^{-3})$ as $|x| \to \infty$. Hence $\Delta^2 m^{(a)} = Q^{-3/2}B^{(a)}Q^{-3/2}$ where $B^{(a)}$ is multiplication by a bounded function and $Q^{-3/2}$ is H- and H_a-smooth by Proposition 2.5.

To estimate the operator $L^{(a)} = L(m^{(a)})$ defined by (3.11) we need Theorem 3.5. Its application relies on Lemma 4.1. Let $\lambda_n^{(a)}(x)$ and $p_n^{(a)}(x)$, $n = 1, \ldots, d$, be eigenvalues and normalized eigenvectors of the symmetric matrix $\mathbf{M}^{(a)}(x) = \{m_{jk}^{(a)}(x)\}$. Clearly, $\lambda_n^{(a)}(x)$ are homogeneous (for $|x| \geq 1$) functions of order -1 and $p_n^{(a)}(x)$ - of order 0. Diagonalizing the matrix $\mathbf{M}^{(a)}$ we find that

$$(L^{(a)}u, v) = \int_X \sum_{j,k} m_{jk}^{(a)}(x) D_k u(x) \overline{D_j v(x)} dx =$$

$$= \int_X \sum_n \lambda_n^{(a)}(x) \langle \nabla u(x), p_n^{(a)}(x) \rangle \langle p_n^{(a)}(x), \nabla v(x) \rangle dx = (K_1^{(a)}u, K_2^{(a)}v)_{\mathbf{H}},$$

where

$$(K_j^{(a)}u)(x) = \sum_n \nu_{n,j}^{(a)}(x) \langle \nabla u(x), p_n^{(a)}(x) \rangle p_n^{(a)}(x), \quad j = 1, 2, \tag{4.5}$$

$$\nu_{n,1}^{(a)}(x) = |\lambda_n^{(a)}(x)|^{1/2}, \quad \nu_{n,1}^{(a)}(x) \nu_{n,2}^{(a)}(x) = \lambda_n^{(a)}(x)$$

and $\mathbf{H} = L_2(\mathbb{R}^d) \otimes \mathbb{C}^d$. Let χ be the characteristic function of the ball $|x| \leq 1$ and $\overset{o}{\chi} = 1 - \chi$. Since

$$|(K_j^{(a)}u)(x)| \leq C |\nabla u(x)|,$$

H- and H_a-smoothness of the operators $\chi K_j^{(a)}$ is ensured by Proposition 2.13.

To treat the operators $\overset{o}{\chi}\, K_j^{(a)}$ we notice that, by the definition (1.3) and Lemma 4.1,

$$\langle \nabla u(x), p_n^{(a)}(x) \rangle = \langle \nabla^{(s)} u(x), p_n^{(a)}(x) \rangle, \quad |x| \geq 1,$$

if $\lambda_n^{(a)}(x) \neq 0$. It follows that

$$|(K_j^{(a)}u)(x)| \leq C |x|^{-1/2} |\nabla^{(s)} u(x)|, \quad |x| \geq 1, \quad C = \sup_{|x|=1} \sum_n \nu_{n,1}^{(a)}(x). \tag{4.6}$$

Set $\overset{o}{\chi}_a (\varepsilon) = \overset{o}{\chi}\, \chi_a(\varepsilon)$ where $\chi_a(\varepsilon)$ is the characteristic function of the cone $\Gamma_a(\varepsilon)$. By (4.6),

$$|(\overset{o}{\chi}_0 (\varepsilon) K_j^{(a)} u)(x)| \leq C |(G_0(\varepsilon) u)(x)|$$

so that the local H- and H_a-smoothness of the operators $\overset{o}{\chi}_0 (\varepsilon) K_j^{(a)}$ for arbitrary $\varepsilon > 0$ is a consequence of Theorem 3.5. Since $\mathbf{M}^{(0)}(x) = 0$ if $x \notin \Gamma_0(\varepsilon_0)$ we have that $K_j^{(0)} = \chi_0(\varepsilon_0) K_j^{(0)}$. Thus the operators $\overset{o}{\chi}\, K_j^{(0)}$ are H- and H_0-smooth. In case $a = \alpha$ we have that $\mathbf{M}^{(\alpha)}(x) = 0$ if $x \in \Gamma_\beta(\epsilon), \beta \neq \alpha$, and hence, by (3.5) ,

$$K_j^{(\alpha)} = \chi_0(\varepsilon) K_j^{(\alpha)} + \chi_\alpha(\varepsilon) K_j^{(\alpha)}, \quad \forall \varepsilon \in (0, \epsilon).$$

Consequently, in order to obtain H-and H_α-smoothness of $\overset{o}{\chi}\ K_j^{(\alpha)}$ we must additionally consider only $\overset{o}{\chi}_\alpha\ (\varepsilon)K_j^{(\alpha)}$ for any $\varepsilon > 0$. Note that $m^{(\alpha)}(x) = m(x) = \mu_\alpha |x_\alpha|$ if $x \in \overset{o}{\Gamma}_\alpha\ (\epsilon_\alpha)$. In virtue of (3.14) for such x

$$(K_j^{(\alpha)} u)(x) = \mu_\alpha^{1/2} |x_\alpha|^{-1/2} (\nabla_{x_\alpha}^{(s)} u)(x)$$

(in this case all eigenvalues of $\mathbf{M}^{(\alpha)}(x)$, except zero, equal $\mu_\alpha |x_\alpha|^{-1}$) so that

$$|(\overset{o}{\chi}_\alpha\ (\varepsilon) K_j^{(\alpha)} u)(x)| \leq C |(G_\alpha(\varepsilon) u)(x)|, \quad \varepsilon \leq \epsilon_\alpha.$$

Therefore the H- and H_α-smoothness of the operators $\overset{o}{\chi}_\alpha\ (\varepsilon) K_j^{(\alpha)}$ is ensured again by Theorem 3.5. This concludes the proof of the required factorization of the right side of (4.4) into a product of H- and H_a-smooth operators.□

Let us now introduce the observable

$$M^\pm = M^\pm(\Lambda) := W^\pm(H, H; ME(\Lambda)). \tag{4.7}$$

Existence of these wave operators can be verified similarly to Theorem 4.2. Actually, let us consider

$$HM - MH = [T, M] + \sum_\alpha [V^\alpha, M].$$

The main contribution to $[T, M]$ is determined by the operator $L = K_2^* K_1$, where K_j are constructed by the formulas (4.5) in terms of eigenvalues $\lambda_n(x)$ and eigenvectors $p_n(x)$ of the matrix $\mathbf{M}(x)$. For any $\varepsilon > 0$ H-smoothness of the operators $\overset{o}{\chi}_0\ (\varepsilon) K_j$ is ensured by H-smoothness of the operator $G_0(\varepsilon)$. Similarly, H-smoothness of $\overset{o}{\chi}_\alpha\ (\epsilon_\alpha) K_j$ is ensured by H-smoothness of $G_\alpha(\epsilon_\alpha)$. Remaining terms in $[T, M]$ are estimated by Proposition 2.13. Finally, we apply Proposition 3.4 to the commutators $[V^\alpha, M]$. Note that potentials V^α may contain long-range parts since the short-range assumption was used in Theorem 4.2 only for the estimate of the term $V^\beta M^{(a)}$, $\beta \neq a$, which is absent now. Thus we have

Proposition 4.3 *Let M be the same operator as in section 3. Suppose that functions (2.5) satisfy Assumptions 2.2 and 2.3. Then the wave operators (4.7) exist.*

The operator $M^\pm(\Lambda)$ is, clearly, self-adjoint, bounded and commutes with H. Our goal is to show that it is invertible on the subspace $E(\Lambda)\mathcal{H}$. In fact, we shall see that $\pm M^\pm(\Lambda)$ is positively definite.

Let us give a classical interpretation of this assertion for a particle (of mass 1/2) in an external field. In this case the observable $U^*(t) M U(t)$ corresponds, in the Heisenberg picture of motion, to the projection $\mathcal{M}(t) =$

$|x(t)|^{-1}\langle\xi(t), x(t)\rangle$ of the momentum $\xi(t)$ of a particle on a vector $x(t)$ of its position. For positive energies λ and large t we have that $\xi(t) \sim \xi_\pm, \xi_\pm^2 = \lambda$, and $x(t) \sim 2\xi_\pm t + x_\pm$. Therefore $\mathcal{M}(t)$ tends to $\pm\lambda^{1/2}$ as $t \to \pm\infty$.

We shall consider $U(t)$ on elements $f = \varphi(H)g$ where $\varphi \in C_0^\infty(\Lambda)$ and $g \in \mathcal{D}(Q)$. Clearly, for different φ and g such elements are dense in $E(\Lambda)\mathcal{H}$. By Lemma 2.10 applied to $\psi(\lambda) = \exp(-i\lambda t)\varphi(\lambda)$, we have that $U(t)f \in \mathcal{D}(Q)$. Thus $mU(t)f$ are well defined.

Let $f_t = U(t)f$ and $h_t = U(t)h$ where $h \in \mathcal{H}$ is arbitrary. Integrating the identity

$$d(mf_t, h_t)/dt = i([H,m]f_t, h_t) = i([T,m]f_t, h_t) = (Mf_t, h_t),$$

we find that

$$(mf_t, h_t) = (mf, h) + \int_0^t (Mf_s, h_s)ds. \tag{4.8}$$

According to Proposition 4.3

$$|(Mf_s, h_s) - (M^\pm f, h)| \le \varepsilon(s)\|h\|, \tag{4.9}$$

where $\varepsilon(s)$ does not depend on h and tends to zero as $s \to \pm\infty$. Comparing (4.8) and (4.9) we obtain

Lemma 4.4 *Let $f = \varphi(H)g$ where $\varphi \in C_0^\infty(\Lambda)$ and $g \in \mathcal{D}(Q)$. Then*

$$U^*(t)mU(t)f = t\, M^\pm(\Lambda)f + o(|t|), \quad t \to \pm\infty.$$

Since $m \ge 0$, Lemma 4.4 implies that

$$\pm(M^\pm f, f) = \lim_{t\to\pm\infty} |t|^{-1}(mf_t, f_t) \ge 0.$$

The inequality $\pm(M^\pm f, f) \ge 0$ established on the dense set extends by continuity to the whole space $E(\Lambda)\mathcal{H}$. Thus we have

Corollary 4.5 *The operator $M^\pm(\Lambda) \ge 0$.*

To prove that $\pm M^\pm$ is positively definite we use Proposition 2.4. In virtue of the identity $i[H, Q^2] = 2A$, it follows from (2.8) that

$$2^{-1}d^2(Q^2 f_t, f_t)/dt^2 = d(Af_t, f_t)/dt = (i[H,A]f_t, f_t) \ge c\,\|f\|^2,$$
$$f = \varphi(H)g, \quad \varphi \in C_0^\infty(\Lambda_\lambda), \quad g \in \mathcal{D}(Q).$$

Integrating twice this inequality we find that for sufficiently large $|t|$

$$\|Qf_t\| \ge c\,|t|\,\|f\|. \tag{4.10}$$

On the other hand, according to Lemma 4.4

$$\|mf_t\| = \|M^\pm f\|\,|t| + o(|t|). \tag{4.11}$$

By property 2^0, $m(x) \geq m_0|x|$, $m_0 > 0$, $|x| \geq 1$, so that

$$\|Qf_t\|^2 \leq 2\|f_t\|^2 + m_0^{-2}\|mf_t\|^2.$$

Thus comparing (4.10) with (4.11) we obtain the inequality

$$\|M^{\pm}f\| \geq c\,\|f\|, \tag{4.12}$$

where $f = \varphi(H)g$, $g \in \mathcal{D}(Q)$, $\varphi \in C_0^\infty(\Lambda_\lambda)$ and $c = c_\lambda$. This inequality is, of course, true for all $f \in E(\Lambda_\lambda)\mathcal{H}$. The compact set $\bar{\Lambda}$ is covered by finite number of intervals Λ_λ. Since $M^\pm$ commutes with $E(\cdot)$, it follows that (4.12) extends to all $f \in E(\Lambda)\mathcal{H}$. Considering now Corollary 4.5 we obtain

Proposition 4.6 *Under the assumptions of Proposition 4.3 for every* $f \in E(\Lambda)\mathcal{H}$

$$\pm(M^\pm(\Lambda)f, f) \geq c\,\|f\|^2, \quad c = c(\Lambda) > 0.$$

Corollary 4.7 *In the space* $E(\Lambda)\mathcal{H}$ *the kernel of* $M^\pm(\Lambda)$ *is trivial and its range*

$$R(M^\pm(\Lambda)) = E(\Lambda)\mathcal{H}.$$

5. EXISTENCE AND COMPLETENESS OF WAVE OPERATORS

In this section we give the proof of Theorem 2.7. Its difficult part is, of course, asymptotic completeness. Actually, the relation (2.12) can be reformulated in basically equivalent form without wave operators (2.11). We start with the proof of this form of asymptotic completeness called asymptotic clustering in [10]. Let, as always, Λ be a bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$ and let M and $M^{(a)}$ be defined by (3.1) and (4.2), respectively. According to (4.1)

$$\sum_a M^{(a)} = M, \quad 0 \leq a \leq \alpha_1. \tag{5.1}$$

Theorem 5.1 *Under the assumptions of Theorem 2.7 for every* $f = E(\Lambda)f$ *there exist elements* $f_a^\pm$ *such that*

$$U(t)f \sim \sum_a U_a(t)f_a^\pm, \quad t \to \pm\infty. \tag{5.2}$$

Proof. – By Corollary 4.7, every $f \in E(\Lambda)\mathcal{H}$ admits the representation $f = M^\pm(\Lambda)f^\pm$, $f^\pm \in E(\Lambda)\mathcal{H}$, so that the asymptotic relation

$$U(t)f \sim MU(t)f^\pm, \quad t \to \pm\infty, \tag{5.3}$$

holds. On the other hand, Theorem 4.2 ensures that for every $a = 0, 1, \ldots, \alpha_1$

$$M^{(a)}U(t)f^{\pm} \sim U_a(t)f_a^{\pm}, \quad t \to \pm\infty, \tag{5.4}$$

where

$$f_a^{\pm} = W^{\pm}(H_a, H; M^{(a)}E(\Lambda))f^{\pm}.$$

Summing up the relations (5.4) and taking into account (5.1) we find that

$$MU(t)f^{\pm} \sim \sum_a U_a(t)f_a^{\pm}, \quad t \to \pm\infty.$$

Comparing it with (5.3) we arrive at (5.2). □

To complete the proof of Theorem 2.7 we need to establish existence of wave operators (2.11). Note that in the proof of Theorem 5.1 we have used only existence of the second set of wave operators (4.3). Now we rely on existence of $W^{\pm}(H, H_a; M^{(a)}E_a(\Lambda))$. Since elements $f = E_a(\Lambda)f$ are dense in the space $\mathcal{H} = \mathcal{H}^{(ac)}(H_a)$, this is equivalent to existence of the wave operators

$$W^{\pm}(H, H_a; M^{(a)}(H_a + i)^{-1}).$$

Here $-i$ can, of course, be replaced by an arbitrary regular point of H_a. Some minor technical complications below are related to unboundedness of the operators $M^{(a)}$. We start with some simple auxiliary assertions.

Lemma 5.2 *Let $V^{\alpha}(T^{\alpha} + I)^{-1}$ be compact in $\mathcal{H}^{\alpha}$. Then*

$$s - \lim_{|t| \to \infty} V^{\alpha}U_0(t)(H_0 + i)^{-1} = 0.$$

Proof. – In terms of the tensor product (2.9)

$$V^{\alpha}U_0(t)(T^{\alpha} + I)^{-1} = \exp(-iT_{\alpha}t) \otimes V^{\alpha}(T^{\alpha} + I)^{-1}\exp(-iT^{\alpha}t).$$

According to (2.1), the second factor in the right side converges strongly to zero. Therefore the tensor product also tends strongly to zero. It remains to remark that $(T_{\alpha} + I)(H_0 + i)^{-1}$ is bounded. □

Lemma 5.3 *Let $\alpha = 1, \ldots, \alpha_1$. Suppose that ζ_{α} is a bounded function such that $\zeta_{\alpha}(x) = 0$ if $x \in \Gamma_{\alpha}(\varepsilon)$ for some $\varepsilon > 0$. Then*

$$s - \lim_{|t| \to \infty} \zeta_{\alpha}U_{\alpha}(t)P_{\alpha} = 0, \tag{5.5}$$

$$s - \lim_{|t| \to \infty} \zeta_{\alpha}\nabla U_{\alpha}(t)P_{\alpha}(H_{\alpha} + i)^{-1} = 0. \tag{5.6}$$

Proof. – It suffices to check (5.5) on elements $f = g \otimes \psi^\alpha$, where ψ^α is an eigenvector of the operator $H^\alpha, H^\alpha\psi^\alpha = \lambda^\alpha\psi^\alpha$, g is an arbitrary element of $\mathcal{H}_\alpha$ and the tensor product is defined by (2.9). Linear combinations of such elements f are dense in the space $P_\alpha\mathcal{H}$. According to (2.10)

$$U_\alpha(t)f = \exp(-i(T_\alpha + \lambda^\alpha))g \otimes \psi^\alpha \tag{5.7}$$

so that

$$\|\zeta_\alpha U_\alpha(t)f\| = \|\Psi_\alpha \exp(-iT_\alpha t)g\|_{\mathcal{H}_\alpha}, \tag{5.8}$$

where

$$\Psi_\alpha^2(x_\alpha) = \int_{X^\alpha} |\zeta_\alpha(x_\alpha, x^\alpha)|^2 |\psi^\alpha(x^\alpha)|^2 dx^\alpha \leq C \int_{|x^\alpha|\geq c|x_\alpha|} |\psi^\alpha(x^\alpha)|^2 dx^\alpha,$$

$c = c(\varepsilon) > 0$, by our assumptions on ζ_α. It follows that $\Psi_\alpha(x_\alpha) \to 0$ as $|x_\alpha| \to \infty$ and hence the operator $\Psi_\alpha(T_\alpha + I)^{-1}$ is compact in the space $\mathcal{H}_\alpha$. Therefore (5.8) tends to zero in virtue of (2.1).

Let us split the vector equality (5.6) into two parts corresponding to ∇_{x_α} and ∇_{x^α} (instead of ∇). The operator ∇_{x_α} commutes with $U_\alpha(t)P_\alpha$ and $\nabla_{x_\alpha}(H_\alpha + i)^{-1} \in \mathcal{B}$. So the part of (5.6) for ∇_{x_α} is a consequence of (5.5). To verify the same for ∇_{x^α} we remark that

$$\|\nabla_{x^\alpha} U_\alpha(t)P_\alpha(H_\alpha + i)^{-1}\| \leq \|\nabla_{x^\alpha}(H_\alpha + i)^{-1}\| < \infty$$

because $(H_\alpha + i)^{-1}$ commutes with $U_\alpha(t)P_\alpha$ and $\nabla_{x^\alpha}(H_\alpha + i)^{-1} \in \mathcal{B}$. Hence it suffices again to consider this limit on elements $f = g \otimes \psi^\alpha$. In this case $(H_\alpha + i)^{-1}f = \tilde{g} \otimes \psi^\alpha$, where $\tilde{g} = (T_\alpha + \lambda^\alpha + i)^{-1}g \in \mathcal{H}_\alpha$. Thus, by (5.7),

$$\zeta_\alpha \nabla_{x^\alpha} U_\alpha(t)P_\alpha(H_\alpha + i)^{-1}f = \zeta_\alpha \exp(-i(T_\alpha + \lambda^\alpha))\tilde{g} \otimes \nabla_{x^\alpha}\psi^\alpha.$$

Since $\nabla_{x^\alpha}\psi^\alpha \in \mathcal{H}^\alpha \otimes \mathbb{C}^{d^\alpha}$, this term can be estimated quite similarly to (5.8).□

Corollary 5.4 *For every $\alpha = 1, \ldots, \alpha_1$ and $b \neq \alpha$*

$$s - \lim_{|t|\to\infty} M^{(b)} U_\alpha(t)P_\alpha(H_\alpha + i)^{-1} = 0. \tag{5.9}$$

Proof. – According to (4.2)

$$iM^{(b)} = 2(\nabla m^{(b)})\nabla + \Delta m^{(b)}, \tag{5.10}$$

where, by the construction of $m^{(b)}$, the zero-degree homogeneous function $\nabla m^{(b)}$ vanishes in the cone $\Gamma_\alpha(\epsilon_\alpha)$. The contribution to (5.9) of the first term in the right side of (5.10) tends to zero in virtue of (5.6). The term $(\Delta m^{(b)})U_\alpha(t)$ converges strongly to zero because $(\Delta m^{(b)})(x) \to 0$ as $|x| \to \infty$. □

Now we are able to prove

Lemma 5.5 *The wave operators*

$$W^{\pm}(H, H_a; M^{(b)}(H_a + i)^{-1}) \tag{5.11}$$

exist for all $a, b = 0, 1, \ldots, \alpha_1$.

Proof. – According to Theorem 4.2 it suffices to consider the case $a \neq b$ only. Let first $a = 0$ and $b = \beta \neq 0$. By the multiplication theorem (2.4), the wave operator

$$W^{\pm}(H, H_0; M^{(\beta)}(H_\beta + i)^{-1}) = W^{\pm}(H, H_\beta; M^{(\beta)}(H_\beta + i)^{-1})W^{\pm}(H_\beta, H_0)$$

exists. Here we have taken into account that the wave operators in the right side exist in virtue of Theorem 4.2 and Proposition 2.8. Therefore in order to establish existence of $W^{\pm}(H, H_0; M^{(\beta)}(H_0 + i)^{-1})$ it remains to verify that

$$s - \lim_{|t| \to \infty} M^{(\beta)}((H_\beta + i)^{-1} - (H_0 + i)^{-1})U_0(t) = 0. \tag{5.12}$$

In virtue of the resolvent identity this is a direct consequence of Lemma 5.2.

In case $a = \alpha \neq 0$ and $b \neq \alpha$ we proceed from the relation (2.13). Let us apply to it the bounded operator $M^{(\beta)}(H_\alpha + i)^{-1}$. In virtue of Corollary 5.4 it follows that

$$M^{(b)}(H_\alpha + i)^{-1}U_\alpha(t)f \sim M^{(b)}(H_\alpha + i)^{-1}U_0(t)f_0^{\pm}, \quad t \to \pm\infty.$$

Furthermore, according to Lemma 5.2, we can replace (cf. with (5.12)) the operator $(H_\alpha + i)^{-1}$ in the right side by $(H_0 + i)^{-1}$. Therefore the existence of the wave operators (5.11) for $a \neq 0$ is ensured by their existence for $a = 0$.□

Corollary 5.6 *The wave operators* $W^{\pm}(H, H_a; M(H_a + i)^{-1})$ *exist for all* $a = 0, 1, \ldots, \alpha_1$.

Proof. – It suffices to "sum up" the wave operators (5.11) over all $b = 0, 1, \ldots, \alpha_1$ and to take into account the relation (5.1). □

Now we can get rid of the identification M.

Proposition 5.7 *The wave operators* $W^{\pm}(H, H_a)$ *exist for all* $a = 0, 1, \ldots, \alpha_1$.

Proof. – By Proposition 4.3 there exists

$$M_a^{\pm}(\Lambda) = W^{\pm}(H_a, H_a; ME_a(\Lambda))$$

(here it is sufficient to assume that an interval Λ is bounded, $0 \notin \bar{\Lambda}$ and $\sigma^{(p)}(H^\alpha) \cap \bar{\Lambda} = \emptyset$ if $a = \alpha$). By Corollary 4.7, $E_a(\Lambda)\mathcal{H} = R(M_a^{\pm}(\Lambda))$ so that for every $f \in E_a(\Lambda)\mathcal{H}$

$$U_a(t)f \sim MU_a(t)f_a^{\pm}, \quad t \to \pm\infty, \quad f = M_a^{\pm}(\Lambda)f_a^{\pm}, \quad f_a^{\pm} \in E_a(\Lambda)\mathcal{H}.$$

Thus Corollary 5.6 ensures existence of $W^{\pm}(H, H_a; E_a(\Lambda))$ and hence of $W^{\pm}(H, H_a)$. □

Since P_a commutes with $U_a(t)$, we have

Corollary 5.8 *The wave operators (2.11) exist and are isometric on $P_a\mathcal{H}$.*

In order to check that the ranges of the operators (2.11) are orthogonal, we shall show that

$$\lim_{t\to\pm\infty}(U_a(t)P_af_a, U_b(t)P_bf_b) = 0, \quad a \neq b. \tag{5.13}$$

If $b = 0$ (so that $P_b = I$) and $a = \alpha$, then, by Proposition 2.8, the limit (5.13) exists and equals

$$(P_\alpha f_\alpha, W^\pm(H_\alpha, H_0)f_0) = 0.$$

Let now $b = \beta$ and $a = \alpha \neq \beta$. The relation (5.5) implies that

$$U_\alpha(t)P_\alpha f_\alpha \sim \chi_\alpha(\varepsilon)U_\alpha(t)P_\alpha f_\alpha, \quad |t| \to \infty, \quad \alpha = 1, \dots, \alpha_1,$$

where $\chi_\alpha(\varepsilon)$ is the characteristic function of the cone $\Gamma_\alpha(\varepsilon)$ and $\varepsilon \in (0,1)$ is arbitrary. So it remains to recall that $\chi_\alpha(\varepsilon)\chi_\beta(\varepsilon) = 0$ if $\alpha \neq \beta$ and $\varepsilon < \epsilon$.

Let us finally verify the relation (2.12). According to (5.2) and (2.13) for every $f \in E(\Lambda)\mathcal{H}$ and some elements $\tilde{f}_0^\pm, f_\alpha^\pm$ the representation

$$U(t)f \sim U_0(t)\tilde{f}_0^\pm + \sum_{\alpha=1}^{\alpha_1} U_\alpha(t)P_\alpha f_\alpha^\pm, \quad t \to \pm\infty,$$

holds. Since the wave operators (2.11) exist, it follows that

$$f = W_0^\pm \tilde{f}_0^\pm + \sum_{\alpha=1}^{\alpha_1} W_\alpha^\pm f_\alpha^\pm$$

and hence f belongs to the left side of (2.12). Considering that linear combinations of elements $f = E(\Lambda)f$ for all admissible Λ are dense in $\mathcal{H}^{(ac)}(H)$, we conclude the proof of Theorem 2.7.

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D.Yafaev
Université de Nantes
Permanent address:
Math. Inst., Fontanka 27,
St. Petersburg, 191011 Russia