JÁNOS KOLLÁR (éd.)

Flips and abundance for algebraic threefolds - A summer seminar at the University of Utah (Salt Lake City, 1991)

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FLIPS AND ABUNDANCE FOR ALGEBRAIC THREEFOLDS

János KOLLÁR

A summer seminar at the University of Utah
(Salt Lake City, 1991)

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PREFACE

These notes originated at the Second Algebraic Geometry Summer Seminar held at the University of Utah during August 1991. The seminar was the continuation of the First Summer Seminar held in 1987 whose notes appeared in [CKM88].

The aim of the First Summer Seminar was to give an introduction to three dimensional birational geometry, especially to Mori's Program (also called the Minimal Model Program). We are very happy to note that in the last few years this program has become much better known among algebraic geometers. This was reflected in the number of participants. In 1987 there were 16 participants for an introductory seminar; in 1991 there were 30 for a more advanced one.

Because of these changes, instead of starting at the beginning, the Second Summer Seminar concentrated on reviewing recent developments in higher dimensional birational geometry. We surveyed two of the most important recent directions.

The first topic was the existence of flips in dimension three, the final step in the three dimensional Minimal Model Program. In surface theory it is well known that repeated contraction of -1-curves yields a minimal surface. Similarly, starting with a threefold X, Mori's Program produces another threefold X', birational to X, which can reasonably be called minimal in analogy with the surface case. The required operations are however more complicated. One of them is called flip.

The existence of flips was first proved by [Mori88]. Recently a very different approach to a more general type of flipping problem (still in dimension three) was found by [Shokurov91]. We owe special thanks to Miles Reid who prepared an English translation of [Shokurov91] in a very short time. Shokurov's article contains many new ideas, but unfortunately it is very difficult to understand. Numerous parts required a truly joint effort of the participants and some details were understood only after several discussions with the author. Eventually we discovered an error in [ibid, 8.3]. Unfortunately, there was no opportunity to reconvene the seminar and study the new version [Shokurov92].

The first part of the notes (Chapters 4–8) presents a new proof of log flips using [Mori88]. The third part (Chapters 16–21) presents a reworked version of [Shokurov91, 1–7].

The second topic (Chapters 9–15) is the Abundance Conjecture proposed in [Reid83]. It is a natural continuation of Mori's Program. Starting with the threefold X' produced above, the conjecture states that a suitable multiple of the canonical class determines a base point free linear system (unless all such are empty). The proof of this result was completed in the series of articles [Kawamata84,85,91b; Miyaoka87a,b,88a,b]. Again we succeeded in simplifying several of the steps and generalizing many intermediate results.

A more detailed explanation of the results and an outline of the proofs is given in Chapter 1.

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PREREQUISITES

In writing these notes we tried to keep the prerequisites to the minimum. The reader is assumed to have a basic general knowledge of algebraic geometry. Some familiarity with higher dimensional techniques is necessary. We tried to rely only on Chapters 1–13 of [CKM88]. There are however two topics not adequately covered in [CKM88].

1. In [CKM88] the Cone Theorem and related results are proved only for the canonical divisor K_X instead of an arbitrary log terminal divisor $K_X + \Delta$. The proofs in the more general log terminal case are essentially the same as the proofs given in [CKM88]. A reader who understands Chapters 9–13 of [CKM88] should have no problem with the more general log versions. However we usually refer to [KMM87] where the precise results are stated and proved.

2. [CKM88, Chapter 6] collects the most important results on terminal and canonical singularities in dimension three, mostly without proofs. The reader who is happy to accept these results does not need to know more. For those who want proofs, the list of prerequisites gets longer. The survey article [Reid87] presents a very readable and elementary overview with proofs. Unfortunately even [Reid87] relies on detailed properties of elliptic Gorenstein surface singularities [Laufer77; Reid75] which are by no means basic. We could not offer any significant improvements; thus there was no reason to reproduce the results.

3. The first proof of the existence of log flips (Chapters 4–8) uses the very difficult results of [Mori88]. We need however only the statements and none of the techniques.

4. In Chapter 9 we discuss the abundance problem only for regular threefolds. The irregular case was solved earlier using the ideas of Iitaka's program which are not related to the methods discussed here.

No other result from higher dimensional birational geometry is used without proof.

We also need some other results which are not part of basic algebraic geometry.

Naturally we need Hironaka's resolution of singularities.

Simultaneous resolution of flat deformations of Du Val singularities (= rational double points) is an important result [Brieskorn71] which is not treated in textbooks.

In Chapters 9–10 we use several properties of stable vector bundles. Also in Chapter 9 we need some properties of foliations in positive characteristic. In all cases we state the results we use and give precise references.

Finally there are occasional uses of a few other topics: mixed Hodge structures, Lefschetz type theorems, relative duality and the existence of the Hilbert scheme.

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1. LOG FLIPS AND ABUNDANCE: AN OVERVIEW

JÁNOS KOLLÁR

The aim of these notes is to present two of the most important recent directions of three dimensional algebraic geometry. We generalize the following two theorems from surfaces to threefolds. (For the surface case, see for example, [BPV84,VI.1.1,V.12.1,VII.5.2].):

1.1 Theorem. *Let X be a smooth projective surface. Then there is a birational morphism $X \to X'$ to another smooth projective surface X', where X' satisfies exactly one of the following conditions:*

(1.1.1) $K_{X'}$ is nef, i.e. $C \cdot K_{X'} \geq 0$ for every curve $C \subset X'$;

(1.1.2) X' is $\mathbb{P}^1$-bundle over a smooth curve D;

(1.1.3) $X' \cong \mathbb{P}^2$.

1.2 Theorem. *Let Y be a smooth projective surface. Assume that K_Y is nef. Then $|mK_Y|$ is base point free for some $m > 0$.*

The approach to the higher dimensional version of (1.1) is called Mori's program or the Minimal Model Program, initiated in [Mori82]. (See [KMM87; Kollár90; Kollár91] for introductions.) Its general features have been well understood for a few years and they were presented in [CKM88,1–13] in a fairly elementary way. The major remaining open problem was to prove the existence of flips. This was finally done in [Mori88]. Recently a new proof (of a more general result) was given in [Shokurov91]; we present two proofs of this result. The first one (Chapters 4-8) is short, but relies on [Mori88]. The approach of [Shokurov91] is presented in Chapters 16–22.

The higher dimensional version of (1.2) is called the Abundance Conjecture [Reid83, 4.6]. In dimension three it is now a theorem; proved in the second part (Chapters 9–15).

Before giving a detailed outline of the three dimensional proofs, I give a very short sketch of the surface case and discuss the new features of the three dimensional case.

The proof of (1.1) is relatively easy (cf. [BPV84,III.4.1,VI.2.4], [CKM88,3]). One proves that if X does not satisfy any of the conditions (1.1.1–3) then it contains a smooth rational curve $C \subset X$ such that $C \cdot K_X = -1$. C can be contracted by a morphism $p : X \to X_1$ and X_1 is again smooth. We repeat this as many times as necessary. At every step the second Betti number drops by one, and therefore eventually the procedure must stop.

In dimension three, life is more complicated. The very first step $X \to X_1$ was analyzed in [Mori82]. He showed that in some cases X_1 is necessarily singular. At first sight this seems a major trouble; however, the techniques to handle the singularities that occur have been worked out. The big problem is that in subsequent steps we may arrive at a situation when the contraction forced upon us by the program is of the following type:

Small Extremal Contraction. $f : X \to Z$ is a proper birational morphism between threefolds such that the exceptional set of f is a curve $C \subset X$ and K_X is negative on C.

In this case Z has "very bad" singularities. This makes it necessary to find a new type of birational transformation, the flip.

Flips. Let $f : (C \subset X) \to (P \in Z)$ be a proper birational morphism such that $f : (X - C) \to (Z - P)$ is an isomorphism. Assume that K_X is negative on C. The flip of f is a proper birational morphism $f^+ : (C^+ \subset X^+) \to (P \in Z)$ such that $f^+ : X^+ - C^+ \to Z - P$ is an isomorphism, and K_{X^+} is positive on C^+. This gives the following diagram:

$$\begin{tikzcd} C \subset X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & C^+ \subset X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

(Frequently the birational map $\phi = (f^+)^{-1} \circ f : X \dashrightarrow X^+$ is also called the flip.)

Informally, we take C out of X and replace it with another curve C^+. The main point is that the canonical class becomes positive near C^+. Aside from the sign restriction on K, the flip might seem to be a symmetric operation; however, the negativity of $K_X \cdot C$ is crucial.

The existence of flips was the main open problem of three dimensional birational geometry for six years, until it was finally settled by [Mori88].

The first and third parts of these notes present a generalized version of flips. We look at perturbations of K_X of the form $K_X + \sum b_i B_i$ where the B_i are effective and $0 \le b_i \le 1$. There are further strong restrictions on the singularities of X and of the B_i which are not important for the general picture. Instead of requiring that $K_X \cdot C$ be negative, we require that $(K_X +$

$\sum b_i B_i) \cdot C$ be negative. This generalization gives us considerable flexibility in certain problems which is crucial in many applications.

The proofs of [Mori88] and of [Shokurov91] proceed along very different lines. The technical heart of [Mori88] is a method to understand the structure of X along C. Once we understand the structure sufficiently well, it is not too hard to construct the flip. This approach was further developed into a fairly complete structure theory of all possible pairs $C \subset X$ [Kollár-Mori92]. In particular, this method gives a very good description of X^+ in all cases.

[Shokurov91] has a more general situation where a complete description may very well be intractable. Thus he concentrates on trying to prove the existence of flips. His method is to have various results to the effect that if certain flips exist then some more general flips also exist. There are about five main types of reductions, each applied several times. This has the consequence that we know very little about X^+.

At the end of the program we obtain the following theorems. First we state the original version of [Mori88], then the generalized version of [Shokurov91].

1.3 Theorem. *(Existence of minimal models) Let X be a smooth projective threefold. Then there is a birational map $X \dashrightarrow X'$ to another projective threefold X' (with terminal singularities), where X' satisfies exactly one of the following conditions:*

(1.3.1) $K_{X'}$ is nef, i.e. $C \cdot K_{X'} \geq 0$ for every curve $C \subset X'$;

(1.3.2) There is a morphism $p : X' \to Z'$ onto a lower dimensional variety Z' such that $K_{X'}$ is negative on the fibers of f.

1.4 Theorem. *(Existence of log minimal models) Let X be a smooth projective threefold. Let $D = \sum d_i D_i$ where the D_i are different irreducible divisors,* Supp D *has only normal crossings and $0 \leq d_i \leq 1$.*

Then there is a birational map $\phi : X \dashrightarrow X'$ to another projective threefold X' such that $(X', D' = \phi_*(D))$ is log terminal (see (2.13)), and X' satisfies exactly one of the following conditions:

(1.4.1) $K_{X'} + D'$ is nef, i.e. $C \cdot (K_{X'} + D') \geq 0$ for every curve $C \subset X'$.

(1.4.2) There is a morphism $p : X' \to Z'$ onto a lower dimensional variety Z' such that $K_{X'} + D'$ is negative on the fibers of p.

A lot of work has been done on the structure of X' in the second case of (1.3) and (1.4), especially when $D = \emptyset$. Some of the most important contributions are [Sarkisov81,82; Miyaoka-Mori86; Iskovskikh87; Kawamata91a; Alexeev92; Corti92]. We do not say much about this direction, except for some very special examples in Chapter 23.

The second part of these notes concerns the following generalization of (1.2) conjectured in [Reid83,4.6] and proved in a series of articles [Kawamata84,85,91b; Miyaoka87a,b,88a,b].

1.5 Theorem. *Let Y be a projective threefold with terminal singularities such that K_Y is nef. Then $|mK_Y|$ is base point free for some $m > 0$.*

In order to see the difficulties of the proof, we recall the main steps of the two dimensional argument. By Riemann–Roch we have

$$h^0(\mathcal{O}_Y(mK_Y)) + h^2(\mathcal{O}_Y(mK_Y)) \geq \chi(\mathcal{O}_Y(mK_Y)) = \frac{m(m-1)}{2}K_Y^2 + \chi(\mathcal{O}_Y).$$

If $K_Y^2 > 0$ then $h^0(\mathcal{O}_Y(mK_Y)) \to \infty$, and therefore we have lots of sections. This corresponds to the case when $|mK_Y|$ gives a birational morphism for $m \gg 1$.

Thus assume that $K_Y^2 = 0$. Here we are in trouble since we only get $h^0(\mathcal{O}_Y(mK_Y)) \geq \chi(\mathcal{O}_Y) - 1$. In the elliptic surface case we have to prove that both $h^0(\mathcal{O}_Y(mK_Y))$ and $h^1(\mathcal{O}_Y(mK_Y))$ go to infinity, but they cancel each other out.

We have two different cases.

Irregular surfaces. We use the Albanese morphism $Y \to \mathrm{Alb}(Y)$ to get some information. Subvarieties of Abelian varieties are rather special, hence we can expect that analyzing the morphism gives us all necessary information. This part can be generalized rather successfully to higher dimensions, and it leads to several general conjectures of Iitaka, most of which were proved by Ueno, Fujita, Viehweg, Kawamata, Kollár and others. See [Mori87] for a survey.

Regular surfaces. In this case $\chi(\mathcal{O}_Y) \geq 1$,

$$h^0(\mathcal{O}(2K_Y)) + h^0(\mathcal{O}(-K_Y)) \geq 1.$$

Therefore we can find an effective divisor $D \in |2K_Y|$. If we expect that $2K_Y$ is trivial (i.e., $K3$ or Enriques surfaces) then $D = \emptyset$ and we are done. Otherwise we expect that Y is an elliptic surface and D is supported on fibers of the elliptic fibration. We need to show that (some multiple of) D moves in a pencil. There are two problems here. First, D can be very singular. Second, it is not at all obvious that D moves, even if it is smooth. This part is rather delicate even for surfaces.

The three dimensional version proceeds along the same main lines. The irregular case has been treated earlier by the methods of the Iitaka conjectures mentioned above [Viehweg80]. We do not deal with this part. Thus we are left with the regular case.

First we look at Riemann–Roch. Because of the singularities, the precise form is not easy to work out. It was done by Barlow–Fletcher–Reid

[Reid87,10.3]:

$$\begin{aligned} h^0(\mathcal{O}_Y(mK_Y)) + h^2(\mathcal{O}_Y(mK_Y)) \geq & \chi(\mathcal{O}_Y(mK_Y)) \\ = & \frac{m(m-1)(2m-1)}{6} K_Y^3 + \frac{m}{12} K_Y \cdot c_2(Y) \\ & + \chi(\mathcal{O}_Y) + l(Y,m), \end{aligned}$$

where $l(Y,m)$ is a periodic function of m, depending only on the singularities of Y.

If $K_Y^3 > 0$ then general methods of the Base Point Free Theorem give the result (see, e.g., [CKM88,9.3]). The next main step, due to [Miyaoka87a,b], is to show that $K_Y \cdot c_2(Y) \geq 0$. After further difficulties, we can at least show that if K_Y is nef then $|mK_Y| \neq \emptyset$ for some $m > 0$ [Miyaoka88a].

A further step was taken by [Miyaoka88b] who settled the problem completely in the case when we expect Y to be a pencil of K3–surfaces. The arguments are analogous to the elliptic surface case, but technically much more involved.

The elliptic threefold case was first studied by [Matsuki90], using the ideas of [Miyaoka88b]. He was able to achieve only partial results. Finally, this method was further developed in [Kawamata91b]. He improved Matsuki's argument at a decisive point. In general, one needs to deal with the possibility that $D \in |mK_Y|$ is badly singular. Kawamata considers a log minimal model for $K + \operatorname{red} D$. While we get more complicated threefold singularities, the resulting member of $|mK|$ becomes much better, which is crucial.

Before we give a detailed outline of the proofs, we need to discuss a little about the relevant singularities.

Singularities

Singularities enter into the program already at the first step [Mori82] and [Reid80,83], and understanding them is an indispensable initial part of three dimensional birational geometry. See [Reid87] for a general introduction.

The following observations lead to the correct classes of singularities.

(1.6.1). Our main interest is in studying the canonical class K_X and in being able to compute its intersection numbers with curves. Thus we need K_X to be Cartier or at least $\mathbb{Q}$-Cartier (i.e., a multiple of K_X is Cartier). Frequently we may even restrict ourselves to the case when every Weil divisor is $\mathbb{Q}$-Cartier.

(1.6.2). Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism. We can write

$$K_Y \equiv f^*K_X + \sum a(E,X)E,$$

where $E \subset Y$ are exceptional divisors, $a(E, X) \in \mathbb{Q}$ and $\equiv$ denotes numerical equivalence.

$a(E, X)$ is called the *discrepancy* of E with respect to X. $f(E) \subset X$ is called the *center* of E on X and is denoted by $\mathrm{Center}_X(E)$. A divisor E is called *exceptional* if $\dim \mathrm{Center}_X(E) \leq \dim X - 2$.

If $f' : Y' \to X$ is another proper birational morphism and $E' \subset Y'$ is the birational transform (2.4.1) of E on Y' then $a(E, X) = a(E', X)$ and $\mathrm{Center}_X(E) = \mathrm{Center}_X(E')$. In this sense $a(E, X)$ and $\mathrm{Center}_X(E)$ depend only on the divisor E but not on Y. This is the reason why Y is suppressed in the notation. A more invariant description is obtained by considering the rank one discrete valuation of the function field $\mathbb{C}(X)$ corresponding to a divisor. Thus we obtain a function

$$a(\ , X) : \{\text{divisors of } \mathbb{C}(X) \text{ with nonempty center on } X\} \to \mathbb{Q}.$$

(If X is proper then every divisor has a nonempty center.)

(1.6.3) It turns out to be very natural to measure the singularities of a variety X by the behavior of the discrepancy function. The most important measure is given by

$$\mathrm{discrep}(X) = \inf_E \{a(E, X) | E \text{ exceptional, } \mathrm{Center}_X(E) \neq \emptyset\} \in \mathbb{R} \cup \{-\infty\}.$$

The following is clear by considering the blow up of a codimension two subvariety:

1.7 Claim. *If X is smooth then* $\mathrm{discrep}(X) = 1$.

This property is close to characterizing smooth varieties. The precise statement is the following.

1.8 Conjecture. *Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. Then X is smooth iff*

$$a(E, X) \geq \dim X - \dim(\mathrm{Center}_X(E)) - 1 \quad \textit{for every } E.$$

This is true if $\dim X \leq 3$ (cf. (17.1.2)).

For arbitrary varieties the following result limits the possibilities:

1.9 Proposition. *[CKM88,6.3] Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. Then one of the following holds:*

(1.9.1) $\mathrm{discrep}(X) \in [-1, 1]$ *and the* inf *is a minimum;*

(1.9.2) $\mathrm{discrep}(X) = -\infty$.

For most singular varieties we have (1.9.2) and the first case should be considered very special. In general, the larger $\mathrm{discrep}(X)$, the milder the singularities of X.

There are four classes that deserve special attention:

1.10 Definition. Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. We say that

$$X \text{ has } \begin{cases} \text{terminal} \\ \text{canonical} \\ \text{log terminal} \\ \text{log canonical} \end{cases} \text{singularities if } \operatorname{discrep}(X) \begin{cases} > 0, \\ \geq 0, \\ > -1, \\ \geq -1. \end{cases}$$

In dimension two these classes correspond to well-known classes of singularities:

1.11 Theorem. *Let $0 \in X$ be a (germ of a) normal surface singularity over $\mathbb{C}$. Then X is*

$$\begin{aligned} \textit{terminal} &\Leftrightarrow \textit{smooth;} \\ \textit{canonical} &\Leftrightarrow \mathbb{C}^2/(\textit{finite subgroup of } SL(2,\mathbb{C})); \\ \textit{log terminal} &\Leftrightarrow \mathbb{C}^2/(\textit{finite subgroup of } GL(2,\mathbb{C})); \\ \textit{log canonical} &\Leftrightarrow \textit{simple elliptic, cusp, smooth or a quotient of these} \end{aligned}$$

All of these classes occupy an important place in the theory:

(1.12.1). Terminal singularities are the smallest class in which Mori's program can work, even if we start with smooth and projective varieties.

(1.12.2). Canonical singularities are precisely those that appear on the canonical models of smooth varieties of general type. [Reid80]

(1.12.3). Log terminal singularities are precisely those that appear on the canonical models of smooth varieties of **non**general type. [Kawamata85; Nakayama88]

Log canonical singularities appear naturally in a different context:

1.13 Conjecture. *Let X be a proper and normal variety such that K_X is $\mathbb{Q}$-Cartier.*

(1.13.1) If X has log canonical singularities then

$$H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X) \quad \textit{is surjective for every } i.$$

(1.13.2) Log canonical is the largest class where the above surjectivity holds.

(More precisely, there is a local version of the above surjectivity involving De Rham complexes [DuBois81; Steenbrink83], and this local version should characterize log canonical singularities.)

Both directions are true if X has isolated singularities [Ishii85].

Next we introduce the "perturbations" of K which are crucial in the sequel. Instead of concentrating on K_X we consider pairs (X, D), where X is a normal variety and $D = \sum d_i D_i$ is a divisor such that D_i distinct and $0 \leq d_i \leq 1$. Such a divisor is called a *boundary*. There are at least three reasons to consider these:

(1.14.1) Flexibility. By choosing D appropriately, we are able to analyze situations when K_X is small (e.g., $K_X \equiv 0$).

(1.14.2) Open varieties. Let X be a smooth variety and let $X \subset \bar{X}$ be a compactification such that $D = \bar{X} - X$ is a divisor with normal crossings. $H^j(\bar{X}, \Omega^i_{\bar{X}})$ are basic cohomological invariants of $\bar{X}$, but they depend on $\bar{X}$, not only on X. [Grothendieck66] discovered that the groups

$$H^j(\bar{X}, \Omega^i_{\bar{X}}(\log D))$$

depend only on X, not on the completion $\bar{X}$. The simplest one is

$$H^0(\bar{X}, \omega_{\bar{X}}(D)) \quad \text{or more generally} \quad H^0(\bar{X}, (\omega_{\bar{X}}(D))^{\otimes m}).$$

Thus if we want to study properties that reflect the choice of X, it is natural to consider the divisor $K_{\bar{X}} + D$.

(1.14.3) Fiber spaces. The simplest example is Kodaira's canonical bundle formula for elliptic surfaces [BPV84,V.12.1]. Let $f : S \to C$ be a minimal elliptic surface. Let $m_i F_i = f^*(c_i)$ be the multiple fibers. Then

$$\begin{aligned} K_S &= f^*K_C + f^*(f_*K_{S/C}) + \sum (m_i - 1)F_i \\ &\equiv f^*\left[K_C + (f_*K_{S/C}) + \sum \left(1 - \frac{1}{m_i}\right)[c_i]\right]. \end{aligned}$$

Thus the study of K_S can be reduced to the study of a divisor of the form $K_C + D$ where D has rational coefficients. The same happens in general for fiber spaces $f : X \to Y$ where the general fiber has trivial canonical class.

The notion of discrepancy is again the fundamental measure of the singularities of (X, D).

1.15 Definition. Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor (not necessarily effective) such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism. Then we can write

$$K_Y \equiv f^*(K_X + D) + \sum a(E, X, D)E$$

where $E \subset Y$ are distinct prime divisors and $a(E, X, D) \in \mathbb{Q}$. The right hand side is not unique because we allow nonexceptional divisors too. In order to make it unique we adopt the convention that a nonexceptional divisor F appears in the sum iff $F = D_i$ for some i, and then with the coefficient $a(F, X, D) = -d_i$.

We frequently write $a(E, D)$ if no confusion is likely.

As explained in (1.6.2), $a(E, X, D)$ depends only on the divisor E but not on Y. Thus we obtain a function

$$a(\ , X, D) : \{\text{divisors of } \mathbb{C}(X) \text{ with nonempty center on } X\} \to \mathbb{Q}.$$

$a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define as in (1.6.3)

$$\operatorname{discrep}(X, D) = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \operatorname{Center}_X(E) \neq \emptyset\}.$$

We also use the notation $\operatorname{logdiscrep}(X, D) = 1 + \operatorname{discrep}(X, D)$.

1.16 Definition. Let X be a normal variety. Let $D = \sum d_i D_i$ be an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that

$$(X, D) \text{ or } K_X + D \quad \text{is} \quad \begin{cases} \text{terminal} \\ \text{canonical} \\ \text{purely log terminal} \\ \text{log canonical} \end{cases} \quad \text{if } \operatorname{discrep}(X) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1, \\ \geq -1. \end{cases}$$

We say that (X, D) is Kawamata log terminal if (X, D) is purely log terminal and $d_i < 1$ for every i.

1.17 Remark. If $D = \emptyset$ then these definitions agree with (1.10). One should note that if $D \neq \emptyset$ then the terminal and canonical conditions on a log variety (X, D) are not preserved under extremal contractions in general.

The divisors $K + D$ that appear in the context of (1.14.3) are Kawamata log terminal, but the divisors appearing in (1.14.2) are not. Arbitrary log canonical singularities form a too large class; for instance, they need not be rational.

Kawamata log terminal seems to be the largest class where the proofs of [CKM88,9–13] go through with only minor modifications (see [KMM87]).

Thus the need arises for a suitable class between Kawamata log terminal and log canonical. There can be two different objectives in defining such a class.

(1.18.1) Minimalist. Take the smallest class that is necessary in order for the Minimal Model Program to work starting with a pair (X, D) where X is smooth and D is a boundary whose components are smooth and have normal crossings only.

(1.18.2) Maximalist. Take the largest class where all the relevant theorems still hold.

There are several proposed definitions (2.13). However, in my opinion none of them satisfies any of the above objectives fully. The lack of a good class leads to technical difficulties later.

Description of the Chapters

We start with two introductory chapters: Chapter 2 gives the precise definitions and basic properties of log terminal threefolds and their log canonical models. Many of the results are rather technical and are used only toward the end of the notes. The reader should skip (2.16–35) at the first reading and refer back only as necessary.

Chapter 3 gives the folklore classification of log canonical surface singularities (X, B) with reduced (possibly empty) boundary B. This was first written down in [Kawamata88]. Here we present an elementary proof, due to Alexeev, which works in any characteristic and generalizes well to fractional coefficients.

Log Flips I.

The aim of the first major part of the notes (Chapters 4–8) is to give our first proof of the existence and termination of log flips. This proof relies on [Mori88], but is otherwise fairly short.

Chapter 4 deals with flops and flips on threefolds with terminal singularities. First we prove the existence of flops due to [Reid83] and the termination of flops and flips. The arguments are taken from [Kawamata88, Kollár89, Matsuki91, Kawamata91c] with several improvements. The main result is the following:

1.19=4.15 Theorem. *(Termination of flips for canonical 3-folds) Let X be a normal three dimensional $\mathbb{Q}$-factorial variety and D an effective $\mathbb{Q}$-divisor such that (X, D) is canonical. Then any sequence of flips for (X, D) terminates, i.e., there is no infinite sequence*

$$\begin{array}{ccccccccc} (X_0, D_0) & & \dashrightarrow & & (X_1, D_1) & & \dashrightarrow & & (X_2, D_2) & \dashrightarrow \\ & \phi_0 \searrow & & \swarrow \phi_0^+ & & \phi_1 \searrow & & \swarrow \phi_1^+ & & \phi_2 \searrow & & \cdots \\ & & Z_0 & & & & Z_1 & & & & Z_2 \end{array}$$

where $X_{i+1} = (X_i)^+$ *is a* $K_{X_i}+D_i$*-flip of* X_i *and* D_i *is the birational transform of* $D_0 = D$.

Chapter 5 shows that log flips exist in the special case when (X, D) is terminal or canonical (4.9). This is the point where [Mori88] is used.

Chapter 6 presents the so called Backtracking Method of flipping (6.4) which is used several times to construct flips. The first two applications are the Crepant Descent Theorems (6.10–11). These are based on earlier cases worked out in [Kawamata88, Kollár89, Kawamata91c]. The main idea is the following. We want to flip $f : X \to Z$. Assume that we can find a birational morphism $h : X' \to X$ such that $K_{X'} = h^*K_X$. Then we are able to construct the flip of g by constructing various flips on X'. In many cases, X' exists and its singularities are simpler than the singularities of X. The main application is the following:

1.20=6.15 Theorem. *Assume that three dimensional terminal flips exist. Let* (X, B) *be a log terminal* $\mathbb{Q}$*-factorial threefold. Then log flips exist, and any sequence of them is finite.*

Chapter 7 discusses the question of termination of log flips in a special case. The arguments are taken from [Shokurov91] with several improvements.

Finally, in Chapter 8 we strengthen the previous results by proving that flips exist if (X, D) is log canonical (as opposed to log terminal). The techniques are independent of the previous chapters. At the end we extend the method to prove the following log canonical version of (1.5):

1.21=8.4 Theorem. *Let* X *be a proper threefold. Assume that* $K_X + D$ *is log canonical, nef and big. Then* $m(K_X + D)$ *is base point free for suitable* $m > 0$. *Thus*

$$\sum_{s=0}^{\infty} H^0(X, \mathcal{O}(s(K_X + D))) \quad \textit{is finitely generated.}$$

ABUNDANCE

While the general abundance problem can be formulated only for minimal models, some of its most difficult aspects were originally conjectured in a form not involving the notion of minimal models. This approach is based on the following:

1.22 Definition. A variety X^n is called *uniruled* if there exists a variety Y^{n-1} and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$. (Equivalently, there is a rational curve through every point of X.)

1.23 Theorem. *[Miyaoka-Mori86] Let X' be as in (1.3). Then*
(1.23.1) If $K_{X'}$ is nef (1.3.1) then X' is not uniruled.
(1.23.2) If X' is as in (1.3.2) then X' is uniruled.

The first important part of abundance is the following old question, which from the new point of view is a combination of (1.3) and (1.5):

1.24 Conjecture. *Let X be a smooth projective variety. Then X satisfies exactly one of the following conditions:*
(1.24.1) X is uniruled; or
(1.24.2) $h^0(X, \mathcal{O}(mK_X)) > 0$ for some $m > 0$.

It is in this form that the first substantial result was achieved:

1.25 Theorem. *[Viehweg80,Satz I] Let X be a smooth projective threefold over $\mathbb{C}$. Assume that $h^1(\mathcal{O}_X) > 0$. Then exactly one of the following holds:*
(1.25.1) X is uniruled.
(1.25.2) X is birational to a smooth variety X' such that $mK_{X'} \sim 0$ for some $m > 0$.
(1.25.3) $h^0(X, \mathcal{O}(mK_X)) \geq 2$ for some $m > 0$.

As already mentioned, the proof relies on the (by now) usual techniques of the Iitaka conjectures, and we do not present it. We, however, use this result to concentrate on regular threefolds only.

While (1.24) can be stated without minimal models, its proof in dimension three requires the theory of minimal models. There are two major steps. The first one is the generic semipositivity of Ω^1_X [Miyaoka87a,b,88a]. To be precise:

1.26=9.0.1 Theorem. *Let X^n be a smooth projective variety and assume that X is not uniruled. Let H be sufficiently ample on X and let C be the complete intersection of $(n-1)$ general members of $|H|$. Then $\Omega^1_X|C$ does not have any quotients of negative degree.*

The original proof of Miyaoka is very technical and complicated. In Chapter 9 we give a simpler proof due to Shepherd-Barron.

This result implies that various Chern numbers are nonnegative (in particular $-c_1(X)c_2(X) \geq 0$), which is exactly what we need in the Riemann–Roch formula. However, even if the linear term is positive, we are not done since there is no vanishing result for the $h^2(\mathcal{O}_X(mK_X))$ term. In the case when X is a pencil of surfaces with trivial canonical class, both $h^0(\mathcal{O}_X(mK_X))$ and $h^2(\mathcal{O}_X(mK_X))$ go to infinity. The way out is to observe that if $h^2(\mathcal{O}_X(mK_X)) \neq 0$ then we obtain a nontrivial extension

$$0 \to \mathcal{O}_X((1-m)K_X) \to E \to \mathcal{O}_X \to 0.$$

Analyzing the stability of E leads to the necessary result. This part relies on the results of [Donaldson85]. We finally achieve the first major step toward abundance.

1.27=9.0.6 Theorem. *(1.24) is true in dimension three.*

The next three chapters are preliminary in nature. Chapter 10 deals with the theory of Chern classes applied to Q-bundles. Q-bundles are locally the quotients of vector bundles by finite groups; one can expect that most of the relevant results go through. [Kawamata91b] sketches the analytic approach, we present an algebraic one. The Bogomolov inequality for stable Q-sheaves (10.11) and an improved Bogomolov–Miyaoka–Yau inequality for log surfaces (10.14) are due to Megyesi.

Chapter 11 proves abundance for log canonical surfaces. This was settled by [Kawamata79; Sakai83; Fujita84] (in fact their results are more general). We present only those results needed in subsequent chapters. Our proofs are adapted from three dimensional methods.

For later applications we also need to consider certain nonnormal surfaces with so called semi log canonical singularities. These are considered in Chapter 12. The key results (given in section 12.3) describe some special features of normal surfaces that were used in [Shokurov91, 6.9] for different purposes. The main ideas seem to apply in all dimensions. We also prove a version of (1.13) for semi log canonical surfaces.

With these preparations behind us, the threefold case is not that hard. First we divide the problem into four parts using the following notion.

1.28 Definition. Let L be a nef line bundle on a proper variety X. (I.e. $L \cdot C \geq 0$ for every curve $C \subset X$.) We define its *numerical Kodaira dimension* by

$$\nu(L) = \max\{k| \underbrace{L \cdots L}_{k-\text{factors}} \text{ is not zero in } H^{2k}(X, \mathbb{Q}).\}$$

Clearly $0 \leq \nu(L) \leq \dim X$.

Two of the cases are easy to dispense with:

1.29 Theorem. *Let X be a projective n-fold with terminal singularities. Assume that K_X is nef and let $D \in |mK_X|$.*

(1.29.1) If $\nu(K_X) = 0$ then $D = \emptyset$ hence $mK_X \sim 0$.

(1.29.2) If $\nu(K_X) = n$ then by [CKM88,9.3], $|rK_X|$ is base point free for some $r > 1$.

In dimension three we are left with two cases: $\nu = 1, 2$. The first case was treated by [Miyaoka88b], the second by [Kawamata91b], who also simplified the proof in the first case.

Let $D \in |mK_X|$. The argument of Kawamata starts by replacing (X, D) with another model (X', D') such that $K_{X'} + \operatorname{red} D'$ is log terminal. The origins of this procedure can be traced to the double projection method of G. Fano. By the results of Chapter 16, $\operatorname{red} D'$ is semi log canonical. In both cases we perform some further modifications to simplify the model.

Chapter 13 considers the case $\nu(K_X) = 1$. Here we find a model (X'', D'') such that every connected component of D'' is irreducible and $\operatorname{red} D''$ is semi log canonical (13.3.1–2). The crucial property of D'' is that (1.13.1) holds for semi log canonical surfaces. Once this is established, the argument of [Miyaoka88b] improved by [Kawamata91b] shows that D'' moves in a pencil.

Chapter 14 considers the case $\nu(K_X) = 2$. Our arguments are different from the one given in [Kawamata91b]. By choosing a suitable model (X'', D'') a crucial Todd class computation becomes rather easy (14.3). Furthermore, we can lift sections of $\mathcal{O}(nK_{X''})|D''$ to X'' directly. These results however only give a pencil in $|mK_X|$ while we expect a morphism onto a surface.

The remaining problem was settled earlier by [Kawamata85] in a general form. His argument relies on a very technical generalization of the Base Point Free Theorem. In Chapter 15 we present a shorter geometric argument, which is however probably restricted to dimension three.

Log Flips II.

In the third major part (Chapters 16–22) we return to Shokurov's proof of log flips. This approach does not use [Mori88], and our presentation is self-contained (assuming of course [CKM88]). This proof also uses (7.1). Furthermore at the present it does not yield termination of a sequence of log flips, so that (6.11) is also needed to complete this approach to prove (1.4).

Let $S \subset X$ be a Weil divisor. In Chapter 16 we define the different Diff of a divisor in a variety. $\operatorname{Diff}_S(0)$ essentially measures the failure of the adjunction formula $K_S = (K_X + S)|S$ in the presence of singularities. [Shokurov91] considers this under some restrictive assumptions; the general case was discovered and worked out by Corti. We also classify three dimensional log terminal singularities (X, B) where B is "large".

Then we want to use the different to relate properties of X to properties of S. This is done in Chapter 17. The main result for the present applications is the following, called "inversion of adjunction".

1.30=17.6 Theorem. *Let X be normal and $S \subset X$ an irreducible divisor. Let $B = \sum b_i B_i$ be an effective $\mathbb{Q}$-divisor such that $b_i < 1$ for every i, and assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then $K_X + S + B$ is purely log terminal in a neighborhood of S iff $K_S + \operatorname{Diff}(B)$ is Kawamata log terminal.*

In [Shokurov91] this was proved in dimension three by a rather elaborate argument. The proof in Chapter 17 works in all dimensions and is fairly short.

Chapter 18 contains the first two reduction steps. (1.30) allows us to simplify the proofs of Shokurov considerably while generalizing various parts to higher dimensions. The conclusion is the following result (still restricted to dimension three):

1.31=18.9 Theorem. *Assume that the flip exists for every small contraction $g : (U, K+S) \to V$ such that S is reduced, irreducible and has negative intersection with C (these are called special flips).*

Then the flip exists for any small contraction $f : (X, K+D) \to Z$ such that $K+D$ is Kawamata log terminal.

During the proof of (1.31) we encounter one of the major discoveries of [Shokurov88,91]. Let me describe a similar phenomenon where the complete result is known. (See [Alexeev89] for a more difficult example.)

Let $D_2 = \{\text{discrep}(X) | \ X \text{ is a log canonical surface}\}$.

1.32 Theorem. *(Shokurov, unpublished) Notation as above. Then*

(1.32.1) Any increasing subsequence of D_2 is finite;

(1.32.2) The accumulation points of D_2 are exactly

$$-1 \quad \textit{and} \quad -1+\frac{1}{2}, -1+\frac{1}{3}, -1+\frac{1}{4}, \dots$$

Shokurov's observation is that similar results hold in many different contexts. See (18.16) for the precise conjectures.

Chapter 19 considers complements on surfaces. The notion of a complement is another one of the major new inventions of [Shokurov91]. The main result (19.4) says that in many situations we can replace the boundary $\sum b_i B_i$ with another divisor $\sum b'_i B'_i$ such that

$$b'_i \in \frac{1}{n}\mathbb{N} \quad \text{for every } i, \text{ where} \quad n \in \{1, 2, 3, 4, 6\}.$$

Some other important technical results are also proved.

Unfortunately the flips required in (1.31) are still very hard to construct, and we need several preparatory results, presented in Chapter 20. We prove that the flip of $f : (X, K+B) \to Z$ exists if B has at least two reduced components intersecting the contracted curve C (20.7). This is used repeatedly in the next two chapters.

The special flips $g : (U, K+S) \to V$ of (1.31) are studied in Chapters 21–22. In this case we have $K_S = (K+S)|S$, and therefore $g|S : S \to S'$ is a K_S-negative contraction. Furthermore, by the results of Chapter 16, S

has only log terminal singularities. Thus we are in the situation of Chapter 19 and we can analyze g in terms of the properties of the surface S. By the results of Chapter 19 mentioned above we can find a reduced divisor B and an integer $n \in \{1, 2, 3, 4, 6\}$ such that

$$C \cdot \left(K + S + \frac{1}{n}B\right) = 0 \quad \text{and} \quad K + S + \frac{1}{n}B \quad \text{is log canonical.}$$

Different values of n present different levels of difficulty for flipping. Everything is easy if $n = 1$ (21.4). The cases where $n = 3, 4, 6$ are reduced to the $n = 2$ case in (21.10).

The really hard part is the $n = 2$ case. This is where [Shokurov91] contains an error ([ibid,8.3] is false). A new version ([Shokurov92]) was completed in February '92. In Chapter 22 we restrict ourselves to presenting the main line of the arguments. Hopefully this helps the reader to study the complete version.

Chapter 23 is independent of the rest of the notes. It reviews the proof of an old theorem of [Morin40] and [Predonzan49] saysing that complete intersections in $\mathbb{P}^n$ of very low degree are unirational. This was done independently by [Ramero90].

Further Developments

Several of the participants have continued to work on the problems discussed in these notes. Alexeev proved that $\mathcal{S}_2$(fano) and hence $\mathcal{S}_3$(local) and $\mathcal{S}_4^0$(local) satisfy the ascending chain condition (cf. Chapter 18). Fong, Keel, Matsuki and M^{c}Kernan proved several results about log abundance for threefolds. Szabó is doing some foundational work which should clarify the various different flavors of log terminal given in (2.13).

Flowcharts

The following diagrams exhibit the logical structure of the proofs of the principal results. The arrows indicate only the main lines of the arguments. There are many other occasional references to other parts.

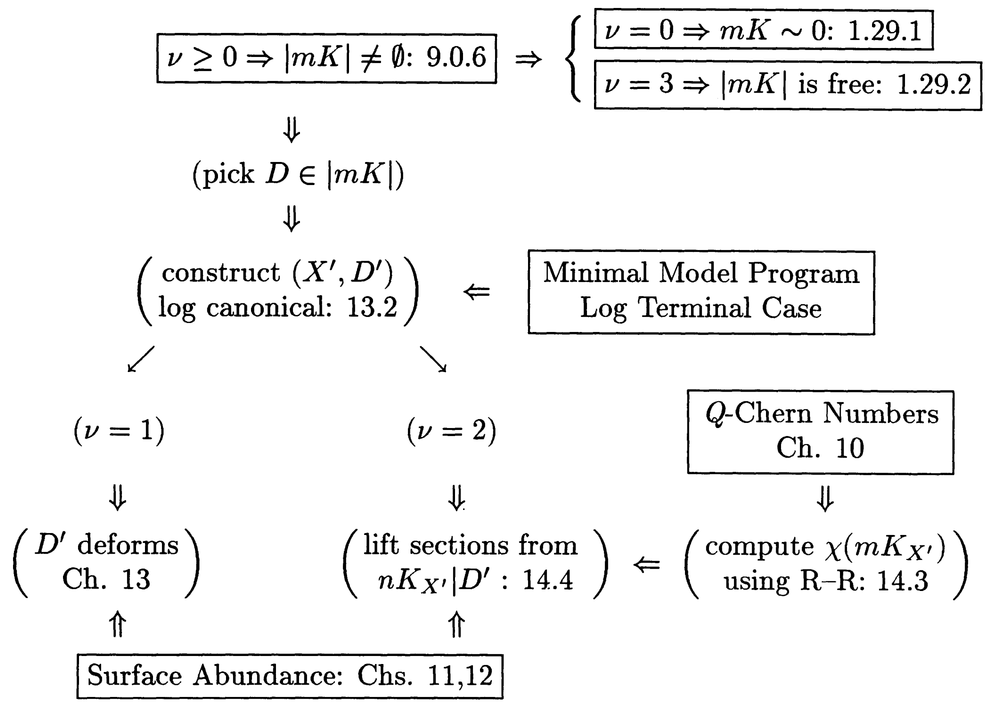
Flops and Easy Flips

$$
\begin{array}{ccc}
\boxed{\text{Existence of Terminal Flops: 4.8}} & & \\
\Downarrow & & \\
\begin{pmatrix}\text{Crepant Descent}\\ \text{6.10}\end{pmatrix} & \Leftarrow & \boxed{\text{Termination of Flops: 4.11}} \\
\Downarrow & & \\
\boxed{\text{Canonical Flops: 6.14}} & & \\
\Downarrow & & \\
\boxed{\text{Easy Flips: 20.7}} & \Leftarrow & \left\{\begin{array}{l}\boxed{\text{Covering Methods: Ch. 20}}\\ \boxed{\text{Complements on Surfaces: Ch. 19}}\end{array}\right.
\end{array}
$$

Log Minimal Model Program I

$$
\begin{array}{ccccc}
 & & \boxed{\text{Mori88}} & & \\
 & & \Downarrow & & \\
 & & \boxed{\text{Existence of Canonical Flips: 5.4}} & & \\
 & & \Downarrow & & \\
 & & \begin{pmatrix}\text{Crepant Descent}\\ \text{6.11}\end{pmatrix} & \Leftarrow & \left\{\begin{array}{l}\boxed{\text{Termination of Flips:4.10}}\\ \boxed{\text{Existence of Flops: 4.8}}\end{array}\right. \\
 & & \Downarrow & & \\
\boxed{\text{LC Flips: 8.1}} & \Leftarrow & \boxed{\text{Log Terminal Flips: 6.15}} & & \\
 & & \Downarrow & & \\
 & & \boxed{\begin{array}{c}\text{Minimal Models}\\ \text{Log Terminal Case}\end{array}} & \Leftarrow & \boxed{\text{Termination of Flips: 7.1}}
\end{array}
$$

ABUNDANCE FOR THREEFOLDS



m stands for a sufficiently large and divisible natural number.
If K is nef then $\nu = \nu(X)$ is defined in (1.28).
(For threefolds we have four cases: $\nu \in \{0, 1, 2, 3\}$.)

Log Flips II

Different and Adjunction: Ch. 16

$\Downarrow$

Inversion of Adjunction: Ch. 17

$\Downarrow$

Ascending Chain Condition: 18.25

$\Downarrow$

Reduction to Special Flips: 18.9 $\Leftarrow$ Termination of Flips: 7.1

$\Downarrow$

(Subdivison by index $n = 1, 2, 3, 4, 6$) $\Leftarrow$ Complements on Surfaces: Ch. 19

$\Downarrow$

(Backtracking Method 6.4-5; 21.6-9) $\Leftarrow$ Easy Flips: 20.7

$\swarrow$ $\downarrow$ $\searrow$

$n = 1$	$n = 3, 4, 6$	$n = 2$
21.4	21.10	Ch. 22

2. LOG CANONICAL MODELS

ANTONELLA GRASSI and JÁNOS KOLLÁR

In the following we consider normal algebraic schemes or normal complex analytic spaces. All the propositions are stated in terms of the algebraic case, although the proofs work for the analytic case with minor modifications.

BASIC DEFINITIONS

2.1 Definition.

(2.1.1) $f : Y \dashrightarrow X$ denotes a **map**; $f : Y \to X$ a **morphism**, that is, a map everywhere defined. We try to be very systematic about using dash arrows for maps and solid arrows for morphisms.

(2.1.2) A **modification** $f : Y \dashrightarrow X$ is a birational map.

(2.1.3) A proper morphism $f : Y \to X$ is a **contraction** if $f_*\mathcal{O}_Y = \mathcal{O}_X$.

(2.1.4) Let $f : Y \to X$ be a contraction with $\dim Y = \dim X$. f is a **birational contraction** (or **blow down**) if X is viewed as constructed from Y; **extraction** (or **blow up**) if Y is viewed as constructed from X.

(2.1.5) A **modification** of a proper morphism $f : X \to Z$ into a proper morphism $g : Y \to Z$ is a commutative diagram

$$\begin{array}{ccc} X & \stackrel{\phi}{\dashrightarrow} & Y \\ & f\searrow \quad \swarrow g & \\ & Z & \end{array}$$

where $\phi : X \dashrightarrow Y$ is a modification.

(2.1.6) A birational contraction is **small** if it is an isomorphism in codimension one. Equivalently, the exceptional set has codimension ≥ 2. (The literature is rather inconsistent about the definiton of small morphism. All definitions that we know of agree in dimension three but not in higher dimensions.)

2.2 Definition. In the following X is an n-dimensional normal variety:

(2.2.1) $D = \sum d_i D_i$ with D_i distinct prime Weil divisors, $d_i \in \mathbb{R}$ (or $\in \mathbb{Q}$) is called an **$\mathbb{R}$-divisor** (**$\mathbb{Q}$-divisor**).

(2.2.2) An $\mathbb{R}$ (or $\mathbb{Q}$)-**Cartier** divisor D is an $\mathbb{R}$ (or $\mathbb{Q}$)-linear combination of Cartier divisors which need not to be irreducible or reduced. The **index** of D is the smallest natural number m such that mD is Cartier. The **index** of X is the index of K_X (if it makes sense).

(2.2.3) Let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor as in (2.2.1). Set

$$\operatorname{Supp} D = \cup\{\operatorname{Supp} D_i \text{ such that } d_i \neq 0\}.$$

(2.2.4) An $\mathbb{R}$-divisor as in (2.2.1) is a **subboundary** if $d_i \leq 1$ for all i and a **boundary** if $0 \leq d_i \leq 1$ for all i.

(2.2.5) For $r \in \mathbb{R}$ let $\llcorner r \lrcorner = \max\{t \in \mathbb{Z} \text{ such that } t \leq r\}$ and $\ulcorner r \urcorner = -\llcorner -r \lrcorner$. (These are pronounced round down resp. round up.) Let $\{r\} = r - \llcorner r \lrcorner$ denote the fractional part of r.

(2.2.6) Assume that $D = \sum d_i D_i$ such that all the D_i's are distinct. Let $\llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i$ and $\{D\} = \sum \{d_i\} D_i$. If D is a boundary, $\llcorner D \lrcorner$ is the **reduced part** of D; $\{D\}$ is the **fractional part** of D.

Warning: If D is $\mathbb{Q}$-linearly equivalent to D', it does not follow that $\llcorner D \lrcorner$ is linearly equivalent to $\llcorner D' \lrcorner$.

2.3 Definition. Let (X, D_X) be a normal variety X together with a boundary D_X. (X, D_X) is a called a **log variety** with **log canonical divisor** $K_X + D_X$. If there is no danger of confusion we will denote this simply by (X, D).

We think of $K_X + D_X$ as a mixed object: K_X is a linear equivalence class, while D_X is a fixed Weil divisor.

2.4 Definition.

(2.4.1) Let $f : X \dashrightarrow Y$ be a map which is a morphism in codimension 1 and let D be a Weil divisor on X. We denote the image of D as Weil divisor by $f_*(D)$. This extends by linearity to the set of all $\mathbb{R}$-Weil divisors. If f is birational then $f_*(D)$ is called the **birational**, (or proper, or strict) **transform** of D. This notation will frequently be used when $f = g^{-1}$, in which case the notation $g_*^{-1}(D) = (g^{-1})_*(D)$ looks slightly unusual.

(2.4.2) A **log morphism** $f : (Y, D_Y) \to (X, D_X)$ is a morphism $f : Y \to X$ such that $f_*(D_Y) \subset D_X$.

2.5 Definition-Proposition. (cf. (1.15))

(2.5.1) Let $K_X + D_X$ be an $\mathbb{R}$-Cartier divisor on a normal variety X, and $f : Y \to X$ any extraction. Choose representatives of K_X and K_Y such that $f^*(K_X)$ and K_Y coincide on the smooth locus of Y. Then

$$K_Y + f_*^{-1}(D_X) = f^*(K_X + D_X) + \sum a(E_i, D_X) E_i,$$

for some real numbers $a(E_i, D_X)$; where the $\{E_i\}$ are the exceptional divisors.

(2.5.2) The number $a(E_i, D_X)$ does not depend on the choices made. It is called the **discrepancy** of E_i with respect to (X, D). When there is no danger of confusion we write $a(E_i)$ for $a(E_i, \emptyset)$.

(2.5.3) $1 + a(E_i, D_X)$ is the **log discrepancy** (denoted by $a_\ell(E_i, D_X)$).

(2.5.4) In general we define the discrepancy of any divisor F of the function field $\mathbb{C}(X)$ with center on X (see also (1.6)). If $c(F)$ is the coefficient of F in D_X, then we set by definition the discrepancy of F to be $a(F, D_X) = -c(F)$, while the log discrepancy is $a_\ell(F, D_X) = 1 + a(F, D_X) = 1 - c(F)$.

The log discrepancy behaves better in certain formulas (cf. e.g. (20.3)).

2.6 Remark. We will sometimes need the notion of discrepancy in cases where X is not normal. Instead of trying to work out the most general case, we restrict ourselves to the following special situation:

(2.6.1) X is reduced, equidimensional and if $P \in X$ is a codimension one point then P is either smooth or two smooth branches of X intersect transversally at P.

If X and Y both satisfy (2.6.1) then we say that $f : Y \to X$ is birational if

(2.6.2) f and f^{-1} are isomorphisms at the generic points of X and Y and also at codimension one singular points of X and Y.

In this situation one can define discrepancies exactly as in (2.5).

2.7 Definition. Let $f : X \dashrightarrow Y$ be any modification. Let $\{F_i\}$ be the exceptional divisors of f^{-1}.

If $K_X + D_X$ is $\mathbb{R}$-Cartier, let $\mathcal{F} = \{f(F_i)\}$ be a sequence of real numbers such that $1 \geq f(F_i) \geq \min\{1, -a(F_i, D_X)\}$. The **$\mathcal{F}$-birational transform** of D_X is defined as

$$(D_X)_{\mathcal{F},Y} = f_*(D_X) + \sum f(F_i)F_i.$$

We always assume that $K_Y + (D_X)_{\mathcal{F},Y}$ is $\mathbb{R}$-Cartier. Thus $a(F_i, (D_X)_{\mathcal{F},Y}) = -f(F_i)$. We will frequently write $D_{\mathcal{F},Y}$ instead of $(D_X)_{\mathcal{F},Y}$. If $f(F_i) = 1$ for every i or $K_X + D_X$ is not $\mathbb{R}$-Cartier then set

$$(D_X)_Y = f_*(D_X) + \sum F_i.$$

Note that $K_Y \neq (K_X)_Y$.

2.8 Remark. The most important case of the $\mathcal{F}$-birational transform is given by the special choice $f(F_i) = 1$. It turns out that in many cases the choice of $\mathcal{F}$ does not matter (cf. (2.22.1)). The freedom in our definition is sometimes convenient in intermediate steps of the proofs.

2.9 Definition. $f : Y \to X$ is a **log resolution** of the log variety (X, D) if Y is smooth and the irreducible components of $\operatorname{Supp}(D_Y)$ are non singular and cross normally.

It may be more natural to require only that $\operatorname{Supp}(D_Y)$ is locally analytically a normal crossing divisor (i.e. irreducible components are allowed to selfintersect). Our stronger requirement makes statements and proofs simpler.

2.10 Definition. Let D_X be a boundary on a normal variety X.

$K_X + D_X$ is **log canonical (lc)** (or (X, D) is log canonical) if $K_X + D_X$ is $\mathbb{R}$-Cartier and $a(E, D_X) \geq -1$ for all divisors E of $\mathbb{C}(X)$ with center on X (or equivalently $a_\ell(E, D_X) \geq 0$).

It is sufficient to check the above condition in (2.10) for one log resolution [CKM88, 6.5].

The following proposition allows us to consider only $\mathbb{Q}$-Cartier divisors:

2.11 Proposition. *Let $D = \sum d_i D_i$ be an $\mathbb{R}$-Cartier divisor on X. Then for every $\epsilon > 0$, there is a $\mathbb{Q}$-Cartier divisor $D' = \sum d'_i D_i$ such that*

(2.11.1) $|d_i - d'_i| < \epsilon$ for all i

(2.11.2) If C is a curve and $D \cdot C \in \mathbb{Q}$, then $D \cdot C = D' \cdot C$.

(2.11.3) Assume in addition that $K + D$ is $\mathbb{R}$-Cartier. Let F be a divisor of $\mathbb{C}(X)$ with center on X. If F has rational discrepancy, then $a(F, D) = a(F, D')$

Proof. By definition the D_i are Cartier divisors. (2.11.2) and (2.11.3) give a system of (possibly infinitely many) rational linear equations in $\sum \mathbb{R} D_i$, considered as real vector space with $\mathbb{Q}$ structure. We can replace (2.11.1) by a system of rational inequalities. These systems define a nonempty rational polyhedron, whose vertices have rational coordinates. Any vertex will do as $\{d'_i\}$. $\square$

2.12 Corollary. *A lc $\mathbb{R}$-divisor can be replaced with a lc $\mathbb{Q}$-divisor without changing rational intersection numbers and rational discrepancies.* $\square$

The following are variants of the notion of log terminal that have been introduced in the literature. Let (X, D) be a log variety. If every coefficient in D is < 1 then the natural notion is (2.13.5), which was already defined in (1.16). If we allow some coefficients to be 1, then the natural notion seems to be log canonical. This however seems too general for most theorems to hold. This leads to a slew of variants, four of which are introduced below. We feel that the only way to understand these is to see them used in proofs.

2.13 Definition. Let (X, D) be a log variety.

(2.13.1) (X, D) is **log terminal (lt)** if there *exists* a log resolution $f : Y \to X$ where all the f-exceptional divisors have positive log discrepancies $(a_\ell(E_i, D) > 0)$.

(2.13.2) (X, D) is **purely log terminal (plt)** if the log discrepancy of every exceptional divisor of $\mathbb{C}(X)$ with center on X is strictly positive.

(2.13.3) (X, D) is **divisorial log terminal (dlt)** if there *exists* a log resolution such that the exceptional locus consists of divisors with strictly positive log discrepancies.

(2.13.4) (X, D) is **weakly Kawamata log terminal (wklt)** if there exists a log resolution $f : Y \to X$ such that all the log discrepancies of the exceptional divisors with center on X are positive and there exists an f- anti ample divisor whose support coincides with that of the exceptional locus of f.

(2.13.5) (X, D) is **Kawamata log terminal (klt)** if every divisor of $\mathbb{C}(X)$ having center on X has positive log discrepancy. (Note that the singularities that we call klt are called "log terminal" in [KMM87, 0-2-10].)

2.14 Example. Let X be a smooth surface and D an irreducible curve with a node. The identity map is not a log resolution and (X, D) has log canonical but not log terminal singularities.

Let X be a smooth surface and D a divisor consisting of 2 reduced irreducible smooth curves intersecting transversely: then (X, D) is log terminal but not plt.

In both cases the exceptional divisor obtained by blowing up the singular point of D has log discrepancy 0.

The analogs of minimal models are the various versions of log minimal models (cf. (1.3–4)).

2.15 Definition.

(2.15.1) $g : (Y, D_Y) \to Z$ is a **relative log minimal model** or is g **log terminal** if $K_Y + D_Y$ is g-nef and log terminal. (Y, D_Y) is a **log minimal model** if $K_Y + D_Y$ is nef and log terminal.

(2.15.2) $g : (Y, D_Y) \to Z$ is a **relative log canonical model** if $K_Y + D_Y$ is g-ample and log canonical. (Y, D_Y) is a **log canonical model** if $K_Y + D_Y$ is ample and log canonical.

(2.15.3) $g : (Y, D_Y) \to Z$ is a **relative weak log canonical model** if $K_Y + D_Y$ is g-nef and log canonical.

Questions of uniqueness are discussed in (2.22).

BASIC TECHNICAL RESULTS

We advise the reader to skip this part at the first reading and to refer back to it only as necessary.

2.16 Proposition.

(2.16.1) By definition klt $\Longrightarrow$ plt $\Longrightarrow$ lt and wklt $\Longrightarrow$ dlt $\Longrightarrow$ lt.

(2.16.2) Let (X, D) be $\mathbb{Q}$-factorial and log terminal. Let $f : Y \to X$ be the log resolution whose existence is assumed in the definition. If f is projective then (X, D) is also wklt. (The assumption of projectivity might not be necessary.)

(2.16.3) (X, D_X) is plt iff it is lt and $\llcorner f_*^{-1}(D) \lrcorner$ is smooth.

(2.16.4) Wklt singularities are always rational.

Proof. (2.16.2) Let $f : Y \to X$ be a log resolution and H an f-ample divisor on Y. Then $H + E = f^*(f_*(H))$, for some effective divisor E whose support coincides with that of the exceptional locus of f. E is also f-anti ample.

(2.16.3) Consider $f : Y \to X$ and let $K_Y + f_*^{-1}(D) + \sum h_i H_i \equiv f^*(K_X + D)$ where $h_i = -a(H_i, D) < 1$ for every i since (X, D) is lt. Let ν be any divisor of $\mathbb{C}(X) = \mathbb{C}(Y)$. Apply (4.12.1.2) with

$$E = \llcorner f_*^{-1}(D) \lrcorner \quad \text{and} \quad H = \sum h_i H_i + \{f_*^{-1}(D)\}.$$

By assumption E is smooth, so $\mathrm{center}_Y(\nu)$ is contained in at most one component of E. Thus (4.12.1.2) implies that

$$a_\ell(\nu, X, D) = a_\ell(\nu, Y, f_*^{-1}(D) + \sum h_i H_i) > -1,$$

unless ν is one of the components of $\llcorner f_*^{-1}(D) \lrcorner$. Thus (X, D) is plt.

(2.16.4) is proved in [KMM87, 1-3-6]; we will not need it. □

The following proposition is a consequence of the definitions and of (2.11):

2.17 Proposition. *Let X be a variety.*

(2.17.1) The set of boundaries D for which $K + D$ is log canonical (nef, or numerically ample) is convex.

(2.17.2) The set of boundaries D with support in a finite union $\cup D_i$ for which $K + D$ is log canonical is a rational convex polyhedron in $\sum \mathbb{R} D_i$.

(2.17.3) If $D' \le D$ are such that $K + D$ is log canonical (respectively log terminal) and $K + D'$ is $\mathbb{R}$-Cartier, then $K + D'$ is also log canonical (respectively log terminal). Moreover, $a(E_i, D) \le a(E_i, D')$.

(2.17.4) Let $K_X + D = K_X + \sum d_i D_i$ be a log terminal divisor. Then there exists a positive number ϵ such that $K + D'$ is log terminal for all $\mathbb{R}$-Cartier

divisors $K + D' = K + \sum d_i' D_i$ *such that* $d_i' \le \min\{1, d_i + \epsilon\}$. *In addition if* $K + D$ *is plt, then* $K + D'$ *is plt.*

(2.17.5) If $K_X + D$ *is plt and* $K_X + D + D'$ *is lc, then* $K + D + tD'$ *is plt for all* $t \in [0, 1]$. $\square$

2.18 Proposition. *Let* $g : Y \to Z$ *be birational. Set* $D_Z = g_*(D_Y)$ *and assume that* $K_Z + D_Z$ *is* $\mathbb{R}$*-Cartier. If* $g : (Y, D_Y) \to Z$ *is a relative weak log canonical model, then*

$$K_Y + D_Y \equiv g^*(K_Z + D_Z) - \sum c(E_i)E_i \quad \text{with} \quad c(E_i) \ge 0, \ \forall i.$$

If $c(E_i) = 0$ *for every* i *then* (K_Z, D_Z) *is lc. Conversely, if* (K_Z, D_Z) *is lc and* $\llcorner D_Y \lrcorner$ *contains the exceptional divisor of* g *then* $c(E_i) = 0$ *for every* i.

Proof. This follows from (2.19), which is sometimes called "Kodaira Lemma". (Others attribute it to Zariski.) $\square$

2.19 Lemma. *Let* $f : Y \to X$ *be a proper birational morphism. Assume that* Y *is normal. Let* $F_i \subset Y$ *be the* f*-exceptional divisors. Let* L *be a line bundle on* X*; let* M *be an* f*-nef line bundle on* Y*, and let* $G \subset Y$ *be an effective divisor such that none of the* F_i *is a component of* G*. Assume that*

$$f^*(L) \equiv M + G + \sum f_i F_i.$$

Then

(2.19.1) $f_i \ge 0$ *for every* i.

(2.19.2) $f_i > 0$ *if* M *is not numerically* f*-trivial on some* F_j *such that* $f(F_i) = f(F_j)$.

Proof. The proof is taken from [Kollár91, 5.2.5.3] with some changes. See also [Shokurov91, 1.1].

If f is not projective, by the Chow Lemma there is a birational projective morphism $f' : Y' \to Y \to X$. If (2.19) holds for f' then it also holds for f. Thus assume that f is projective.

Fix an F_i. By cutting with $\dim f(F_i)$ general hypersurfaces in X we may assume that $f(F_i)$ is zero dimensional. Let $S \subset Y$ be the intersection of $\dim Y - 2$ general hypersurfaces containing a general point of F_i. Let $E_j = S \cap F_j$; this is either an irreducible curve or empty. By assumption E_i is nonempty. $M' = (M + G)|S$ is f-nef, thus

$$0 = f^*L| \cup E_j \equiv (M' + \sum f_i F_i)| \cup E_j \equiv (M' + \sum f_j E_j)| \cup E_j,$$

where the second sum runs only over those E_j which are nonempty. By assumption M is nef on $\cup E_j$, thus everything is implied by the following abstract linear algebra result (cf. [Artin62]).

2.19.3 Lemma. *Let $Q(\ ,\ )$ be an inner product on $\mathbb{R}^n$ with basis $\{E_i\}$. Assume that for every $i \neq j$ we have $Q(E_i, E_i) < 0$, $Q(E_i, E_j) \geq 0$ and Q is negative definite. Then*

(2.19.3.1) Let $F = \sum \alpha_i E_i$ be such that $Q(F, E_i) \geq 0$. Then $\alpha_i \leq 0$ for every i and strict inequality holds unless $F = 0$.

(2.19.3.2) The matrix $(Q(E_i, E_j))^{-1}$ has only negative entries.

Proof. Let $F = F^+ - F^-$ where

$$F^+ = \sum_{i:\alpha_i > 0} \alpha_i E_i \quad \text{and} \quad F^- = \sum_{i:\alpha_i \leq 0} -\alpha_i E_i.$$

Assume that $F^+ \neq 0$. Then for some j, $\alpha_j > 0$ and $Q(E_j, F^+) < 0$ since Q is negative definite. Thus $Q(E_j, F) = Q(E_j, F^+) - Q(E_j, F^-) < 0$.

Each column of the inverse satisfies the assumptions of the first part, thus they have only negative entries. □

2.20 Proposition. *Let $g : Y \to Z$ be a morphism:*

(2.20.1) The set of boundaries D for which g is a relative log canonical model forms a convex subset in the set of all boundaries.

(2.20.2) The set of rational boundaries is dense in the set of all boundaries D for which g is a relative log canonical model.

(2.20.3) If $g : Y \to Z$ is a relative log canonical model, then g is projective.

Proof. This follows from (2.11) and (2.17). □

2.21 Definition. Let $g : Y \to Z$ be a modification of the proper morphism $f : X \to Z$. Choose $\mathcal{F}$ as in (2.7). We obtain a diagram

$$\begin{tikzcd}
(X, D_X) \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & (Y, D_{\mathcal{F},Y}) \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

(2.21.1) $g : (Y, D_{\mathcal{F},Y}) \to Z$ is a **weak log canonical model** (with respect to $\mathcal{F}$) of $f : X \to Z$ if g is a relative weak log canonical model and $a(G_i, D_{\mathcal{F},Y}) \geq a(G_i, D_X)$ for all divisors $G_i \subset X$ that are ϕ-exceptional. Note that by (2.7) if F_i is a ϕ^{-1}-exceptional divisor then $a(F_i, D_{\mathcal{F},Y}) \leq \max\{-1, a(F_i, D_X)\}$, thus the inequality is reversed.

(2.21.2) $g : (Y, D_{\mathcal{F},Y}) \to Z$ is called a **log terminal model** of $f : X \to Z$ (with respect to $\mathcal{F}$) if g is also a relative log minimal model (2.15.1).

(2.21.3) $g : (Y, D_{\mathcal{F},Y}) \to Z$ is called a **log canonical model** of $f : X \to Z$ (with respect to $\mathcal{F}$) if g is also a relative log canonical model (2.15.2).

(2.21.4) If $f(F_i) = 1$ for every i then we drop $\mathcal{F}$ from the notation and call $g : (Y, D_Y) \to Z$ a weak log canonical model etc.

2.21.5 Remark. It follows from (2.23.3) that weak log canonical models of f can be described in the following more invariant way. $\phi : (X, D_X) \dashrightarrow (Y, D_Y)$ is a weak log canonical model of f iff

(2.21.5.1) $(Y, D_Y) \to Z$ is a relative weak log canonical model (2.15),

(2.21.5.2) $a(E, D_Y) \geq a(E, D_X)$ for every divisor E of $\mathbb{C}(X)$, and

(2.21.5.3) $a(E, D_Y) = \max\{-1, a(E, D_X)\}$ for every exceptional divisor of $g : Y \to Z$. (If f is not birational, we consider every divisor $E \subset Y$ to be exceptional.)

One of our eventual main aims is to show that log terminal or log canonical models exist under various assumptions. Here we do not address the question of existence; rather, we consider basic properties of log models assuming that they exist.

2.22 Theorem.

(2.22.1) A log canonical model for $f : X \to Z$ is unique; in particular it does not depend on the choice of $\mathcal{F}$.

(2.22.2) If $g : Y \to Z$ is a weak log canonical model and $g^c : Y^c \to Z$ a log canonical model then there is a unique morphism $\rho : Y \to Y^c$ such that $g = \rho \circ g^c$.

(2.22.3) Let $g : Y \to Z$ be a weak log canonical model. Then a log canonical model $g^c : Y^c \to Z$ exists iff some multiple of $K_Y + D_Y$ is g-free, and then Y^c/Z is given as the image of Y/Z under $m(K_Y + D_Y)$ for suitable $m > 0$.

2.22.4 Remark. (2.19) implies that if $g : Y \to Z$ is the log canonical model of (X, D) and $E \subset Y$ is a g-exceptional divisor then $a(E, D) < -1$. For such divisors the coefficient in $\mathcal{F}$ is -1, which explains why Y is independent of $\mathcal{F}$.

The proof relies on the following variant of [Shokurov91, 1.5.5–6].

2.23 Theorem. *Let $g : Y \to Z$ be a weak log canonical model of $f : X \to Z$ as in (2.21). Let W be a normal scheme, proper and birational over both X and Y such that the following diagram is commutative:*

$$\begin{tikzcd} & W \arrow[dl, "\tilde{f}"'] \arrow[dr, "\tilde{g}"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Let $\{E_i, F_i, G_i\} \subset W$ *be all the exceptional divisors such that* $\{E_i\}$ *are both* $\tilde{g}$ *and* $\tilde{f}$*-exceptional,* $\{F_i\}$ *are* $\tilde{f}$ *exceptional but not* $\tilde{g}$*-exceptional and* $\{G_i\}$ *are* $\tilde{g}$ *exceptional but not* $\tilde{f}$ *exceptional, for every* i*. Set:*

$$\tilde{f}_*^{-1}(D_X) = \sum d_i D_i + \sum g_i G_i$$
$$\tilde{g}_*^{-1}(D_{\mathcal{F},Y}) = \sum d_i D_i + \sum f_i F_i.$$

Note that the f_i*'s are the coefficients defining* $D_{\mathcal{F},Y}$.

(2.23.1) There exists a Zariski decomposition:

$$\begin{aligned} &K_W + \tilde{f}_*^{-1}(D_X) + \sum f_i F_i + \sum E_i \\ &\equiv \tilde{g}^*(K_Y + D_{\mathcal{F},Y}) + \sum [a(E_i, D_{\mathcal{F},Y}) + 1] E_i + \sum [a(G_i, D_{\mathcal{F},Y}) + g_i] G_i, \end{aligned}$$

where $a(E_i, D_{\mathcal{F},Y}) + 1 \geq 0$ *and* $a(G_i, D_{\mathcal{F},Y}) + g_i \geq 0$.

(2.23.2) If $K_X + D_X$ *is log canonical, then there exists a (Zariski-type) decomposition:*

$$\begin{aligned} &K_W + \tilde{f}_*^{-1}(D_X) + \sum f_i F_i + \sum E_i \\ &\equiv \tilde{f}^*(K_X + D_X) + \sum [a(F_i, D_X) + f_i] F_i + \sum [a(E_i, D_X) + 1] E_i \end{aligned}$$

where $a(F_i, D_X) + f_i \geq 0$ *and* $a(E_i, D_X) + 1 \geq 0$.

(2.23.3) Let B *be a divisor of* $\mathbb{C}(X)$*. Then*

$$a(B, D_{\mathcal{F},Y}) \geq a(B, D_X).$$

Furthermore if $K_Y + D_{\mathcal{F},Y}$ *is* g*-ample (i.e.* g *is a log canonical model) and* ϕ *is not a morphism at the generic point of* $\mathrm{Center}_X(B)$ *then*

$$a(B, D_{\mathcal{F},Y}) > a(B, D_X).$$

Proof. The displayed formulas in (2.23.1-2) are formal equalities. The inequalities $a(E_i, D_{\mathcal{F},Y}) + 1 \geq 0$ and $a(E_i, D_X) + 1 \geq 0$ follow from the definition of lc. $a(G_i, D_{\mathcal{F},Y}) + g_i \geq 0$ follows from the definition (2.21) and $a(F_i, D_X) + f_i \geq 0$ from the definition (2.7).

(2.23.3) We may assume that B is a divisor on W. From (2.23.1-2) we obtain

$$\begin{aligned} &\tilde{f}^*(K_X + D_X) \\ &\equiv \tilde{g}^*(K_Y + D_{\mathcal{F},Y}) + \sum [a(G_i, D_{\mathcal{F},Y}) + g_i] G_i \\ &+ \sum [-f_i - a(F_i, D_X)] F_i + \sum [a(E_i, D_{\mathcal{F},Y}) - a(E_i, D_X)) E_i. \end{aligned} \tag{2.23.4}$$

Here $\sum[a(G_i, D_{\mathcal{F},Y}) + g_i]G_i$ is effective and $\tilde{g}^*(K_Y + D_{\mathcal{F},Y})$ is $\tilde{f}$-nef. The first part follows from (2.19).

Assume that ϕ is not a morphism at the generic point of $\operatorname{Center}_X(B)$. Then

$$\dim \tilde{g}(\tilde{f}^{-1}(\tilde{f}(B))) > 0,$$

thus $\tilde{g}^*(K_Y + D_{\mathcal{F},Y})$ is not numerically trivial on $\tilde{f}^{-1}(\tilde{f}(B))$. Thus again (2.19) applies. $\square$

2.24 Corollary. *Let $g : Y \to Z$ be a log model of the proper morphism $f : X \to Z$. Then:*

(2.24.1) If $K_X + D_X$ is log canonical, and $g : Y \to Z$ is the log canonical model of f, then ϕ^{-1} does not contract any divisor.

(2.24.2) If $g_i : (Y_i, D_i) \to Z \quad (i = 1, 2)$ are weak log canonical models of f then g_2 is a weak log canonical model of g_1.

(2.24.3) If $f : X \to Z$ is a weak log canonical model, then the modification ϕ to the log canonical model is a morphism.

(2.24.4) Assume that $K_X + D_X$ is log canonical, $f : X \to Z$ is birational and f is small or $-(K_X + D_X)$ is f-nef. Then $g : Y \to Z$ is a small contraction.

Proof. Let F_i be an exceptional divisor of ϕ^{-1}. If $K_X + D_X$ is log canonical, then by (2.23.2) $-f_i - a(F_i, D_X) \leq 0$, while $g : Y \to Z$ relative log canonical model implies $-f_i - a(F_i, D_X) > 0$. This proves (2.24.1) and also (2.24.4) for f small.

If $-(K_X + D_X)$ is f-nef, then let $L \subset Y$ be a g-exceptional divisor. By the above, $\phi_*^{-1}(L)$ is a divisor. Restrict both sides of (2.23.4) to $\tilde{g}_*^{-1}(L)$. The left hand side is negative, the right hand side is big + effective. Again a contradiction.

(2.24.2) Let $\psi : Y_1 \dashrightarrow Y_2$ be the induced map. If $E_1 \subset Y_1$ is ψ-exceptional then by (2.21)

$$a(E_1, D_1) = \max\{-1, a(E_1, D_X)\} \leq \max\{-1, a(E_1, D_2)\}.$$

Similarly, if $E_2 \subset Y_2$ is ψ^{-1}-exceptional then by (2.21)

$$a(E_2, D_2) = \max\{-1, a(E_2, D_X)\} \leq \max\{-1, a(E_2, D_1)\}.$$

By assumption (Y_i, D_i) are lc thus all discrepancies are at least -1. Therefore

$$a(E_1, D_1) \leq a(E_1, D_2), \quad \text{and} \quad a(E_2, D_2) \leq a(E_2, D_1).$$

These together imply that g_2 is a weak log canonical model of g_1.

(2.24.3) Take a common resolution as in (2.23); we have:

$$\begin{aligned}\tilde{g}^*(K_Y + D_{\mathcal{F},Y}) \equiv & \tilde{f}^*(K_X + D_X) - \sum[(a(G_i, D_{\mathcal{F},Y}) + g_i]G_i \\ & - \sum[(a(E_i, D_{\mathcal{F},Y}) - a(E_i, D_X)]E_i.\end{aligned}$$

Using (2.19) and (2.23.3) we obtain that $a(E_i, D_{\mathcal{F},Y}) - a(D_X, E_i) = 0$ and $a(G_i, D_{\mathcal{F},Y}) + g_i = 0$ for every index i and thus $\tilde{g}^*(K_Y + D_{\mathcal{F},Y}) = \tilde{f}^*(K_X + D_X)$. If ϕ is not a morphism, then there exists a curve $C \subset W$ which is $\tilde{f}$-exceptional but not $\tilde{g}$-exceptional. Then $K_Y + D_{\mathcal{F},Y}$ g-ample implies

$$0 = (K_X + D_X) \cdot C = (K_Y + D_{\mathcal{F},Y}) \cdot C > 0$$

which is a contradiction. □

Proof of (2.22). Assume that we have two log canonical models Y_1 and Y_2. By (2.24.2-3) there are morphisms $Y_1 \to Y_2$ and $Y_2 \to Y_1$; these must be inverses of each other. (2.22.2) is the same as (2.24.3). Finally (2.22.3) clearly follows from (2.22.2). □

2.25 Corollary. *(2.25.1) Let $X_1 \to Z$ be a modification of $f : X \to Z$ as in the diagram:*

$$\begin{array}{ccc} X & \overset{\tau}{\dashrightarrow} & X_1 \\ f \searrow & & \swarrow f_1 \\ & Z & \end{array}$$

If no divisorial component $G_i \subset X$ is contracted by τ, then a log model for f_1 is also a log model for $f : X \to Z$.

(2.25.2) Notation as in (2.23). If $K_X + D_X$ is log canonical, $r(K_X + D_X)$ is Cartier for some $r \in \mathbb{N}$ and $g : Y \to Z$ is the log canonical model, then

$$Y = \operatorname{Proj} \oplus_{n \geq 0} f_* \mathcal{O}_X(nr(K_X + D_X)).$$

Proof. (2.25.1) follows from the definition.

(2.25.2) Without loss of generality we can assume that $r(K_Y + D_Y)$ is also Cartier. Let W be as in (2.23). $K_Y + D_Y$ is g-ample, thus

$$\begin{aligned} Y &= \operatorname{Proj} \oplus_{n \geq 0} g_* \mathcal{O}_Y(nr(K_Y + D_Y)) \\ &= \operatorname{Proj} \oplus_{n \geq 0} (g\tilde{g})_* \mathcal{O}_W(\tilde{g}^*(nr(K_Y + D_Y))) \\ &= \operatorname{Proj} \oplus_{n \geq 0} (f\tilde{f})_* \mathcal{O}_W(\tilde{g}^*(nr(K_Y + D_Y))) \\ &= \operatorname{Proj} \oplus_{n \geq 0} (f\tilde{f})_* \mathcal{O}_W(\tilde{f}^*(nr(K_X + D_X))). \end{aligned}$$

$K_X + D_X$ is log canonical and thus, by (2.24.1) $\{F_i\} = \emptyset$. The result follows from (2.23.1-2). □

We will frequently need various versions of the Minimal Model Program. Next we describe a general variant whose steps do not exist in complete generality but which provides the right framework in all cases that we use later.

2.26 Minimal Model Program. Let (X, E) be a scheme X (over a base scheme S which we suppress in the notation) together with an $\mathbb{R}$-Cartier divisor E (not necessarily effective). By the E-**Minimal Model Program** (E-**MMP** for short) we mean a sequence

$$(X_0, E_0) \overset{g_0}{\dashrightarrow} (X_1, E_1) \overset{g_1}{\dashrightarrow} (X_2, E_2) \overset{g_2}{\dashrightarrow} \cdots$$

constructed as follows.

(2.26.1) $(X_0, E_0) = (X, E)$;

(2.26.2) Assume that (X_i, E_i) is already constructed. If E_i is nef, we stop.

(2.26.3) If E_i is not nef then assume that there is a contraction $f_i : X_i \to Z_i$ such that $-E_i$ is f_i-ample and $\rho(X_i/Z_i) = 1$. If $f_i(E_i)$ is $\mathbb{R}$-Cartier (this happens usually when the exceptional set of f_i is an irreducible $\mathbb{Q}$-Cartier divisor) then set $g_i = f_i$ and $(X_{i+1}, E_{i+1}) = (Z_i, f_i(E_i))$. (We are stuck if f_i does not exist.)

(2.26.4) If $f_i(E_i)$ is not $\mathbb{R}$-Cartier, then we try to find a diagram

$$\begin{tikzcd} (X_i,E_i) \arrow[rr, dashed, "g_i"] \arrow[dr, "f_i"'] & & (X_{i+1}, E_{i+1}) \arrow[dl, "f_i^+"] \\ & Z_i & \end{tikzcd}$$

with the following properties

(2.26.4.1) f_i^+ is a small morphism,

(2.26.4.2) E_{i+1} is f_i^+-ample,

(2.26.4.3) $E_{i+1} = (g_i)_*(E_i)$.

Such a diagram is called the **generalized opposite** or **generalized flip** of f_i with respect to E_i. If f_i itself is small then the diagram is called the **opposite** or **flip** of f_i with respect to E_i, or an E_i-**flip**, or an E-**flip**. (We are stuck again if the flip does not exist.)

(2.26.5) Further terminology:

(2.26.5.1) The modification described in (2.26.4) has collected various labels since it was first introduced. The name "flip" has been traditionally used to describe the above situation when $E = K_X$ while "**log flip**" is reserved for the case of a log divisor $E = K_X + B_X$. If K_{X_i} is f_i-trivial, then the flip of f_i with respect to the divisor E_i is called the E-**flop** or E_i-**flop**.

(2.26.5.2) X^+, f^+ and ϕ are also called the "flip of f ".

(2.26.5.3) The birational transform of E_i is often denoted by E_i^+.

2.27 Proposition. *Let $f : X \to Z$ be a small birational contraction such that $-(K_X + D_X)$ is f-ample; then the log canonical model of f is the flip with respect to $K_X + D_X$ and conversely.*

A flip or log canonical model of f is also a log canonical model of Z for $K_Z + f(D)$. Therefore the discrepancies do not decrease under flips.

Proof. This follows from (2.24.2), (2.24.4) and (2.23). □

The inequality between the discrepancies is also implied by the following more general result which will be useful in many situations:

2.28 Proposition. *Let $f : X \to Z$ and $g : Y \to Z$ be proper birational morphisms between varieties. Let $D \subset Z$ be a divisor and let E_X (resp. E_Y) be f (resp. g)-exceptional divisors. (Not necessarily effective.) Assume that*

(2.28.1.1) $-(K_X + f_*^{-1}(D) + E_X)$ is $\mathbb{R}$-Cartier and f-nef;

(2.28.1.2) $K_Y + g_*^{-1}(D) + E_Y$ is $\mathbb{R}$-Cartier and g-nef;

Let B be any divisor of $\mathbb{C}(Z)$ and let $b \in \mathrm{Center}_Z(B)$ be the generic point. Then

(2.28.2.1) $a(B, Y, g_*^{-1}(D) + E_Y) \geq a(B, X, f_*^{-1}(D) + E_X)$; and

(2.28.2.2) equality holds iff $K_X + f_*^{-1}(D) + E_X$ is numerically trivial on $f^{-1}(b)$ and $K_Y + g_*^{-1}(D) + E_Y$ is numerically trivial on $g^{-1}(b)$

Proof. Let W be a normal variety such that there are proper birational morphisms $\bar{f} : W \to X$ and $\bar{g} : W \to Y$. Then

$$M = \bar{g}^*(K_Y + g_*^{-1}(D) + E_Y) + \bar{f}^*(-(K_X + f_*^{-1}(D) + E_X))$$

is nef on W/Z. Furthermore it is supported on the exceptional locus. Thus by (2.19) $M \equiv -F$ where F is an effective divisor supported on the exceptional locus of $W \to Z$. Therefore

$$\begin{aligned} a(B, Y, g_*^{-1}(D) + E_Y) &= a(B, W, \bar{g}^*(K_Y + g_*^{-1}(D) + E_Y) - K_W) \\ &\geq a(B, W, \bar{g}^*(K_Y + g_*^{-1}(D) + E_Y) - K_W + F) \\ &= a(B, W, \bar{f}^*(K_X + f_*^{-1}(D) + E_X) - K_W) \\ &= a(B, X, f_*^{-1}(D) + E_X), \end{aligned}$$

and strict inequality holds iff $\mathrm{Center}_W(B) \subset \mathrm{Supp}\, F$. Thus (2.28.2.1) is clear and (2.28.2.2) follows from (2.19.2). □

2.28.3 Remark. (2.28.3.1) We will frequently use the above result in the special case when f or g is an isomorphism. If $f : X \to Z = Y$ is an extremal divisorial contraction then the result says that discrepancies increase for divisors whose center is contained in the exceptional divisor of f.

(2.28.3.2) It is easy to see that (2.28) also holds if X, Y, Z satisfy (2.6.1) and f and g are birational in the sense of (2.6.2).

2.29 Proposition.

(2.29.1) Let (X, D_X) be klt and $f : X \to Z$ a small birational contraction. Assume that $g : Y \to Z$ is a weak log canonical model, as in (2.23.3). Then the flip of f exists.

(2.29.2) Let $g : Y \to Z$ be a weak log canonical model of $f : X \to Z$. Assume that $\rho(X/Z) = 1$ and either X is $\mathbb{Q}$-factorial or g is projective. If g is small then g is the flip of $f : X \to Z$.

Proof. (29.1) If (X, D_X) is klt then g is small by (2.24.4) and the Base Point Free Theorem [KMM87, 3-3-1] applies: thus the flip exists.

(2.29.2) Up to a constant multiple, $K_Y + D_Y$ is the only relative divisor on Y. Thus $K_Y + D_Y$ is ample and the flip exists. □

Shokurov introduces a systematic method of decreasing the coefficients of D while preserving the intersection numbers with the exceptional curves of f and preserving rationality under an extra condition.

2.30 Definition. Let $f : X \to S$ be a contraction and $K + D$ a log divisor on X. We say that D is an LSEPD (=Locally (over S) the Support of an Effective Principal Divisor) divisor if the following holds: for every $s \in S$ there is an open neighborhood $s \in U_s \subset S$ and a regular function $h_s \in \mathcal{O}(U_s)$ such that

$$f^{-1}(U_s) \cap \llcorner D \lrcorner \subset \operatorname{Supp}(f^* h_s = 0) \subset f^{-1}(U_s) \cap \ulcorner D \urcorner = f^{-1}(U_s) \cap \operatorname{Supp} D.$$

I.e., $\operatorname{Supp}(f^* h_s = 0)$ contains every component of D which has coefficient 1 and $\operatorname{Supp}(f^* h_s = 0)$ is contained in the support of D.

2.31 Remark.

(2.31.1) Let $f : Y \to S$ be a small contraction such that $\rho(Y/S) = 1$, $R^1 f_* \mathcal{O}_Y = 0$ and Y is $\mathbb{Q}$-factorial. A reduced boundary D is LSEPD if and only if

either all the components of D are numerically zero with respect to f,

or at least one component is f-positive and one f-negative.

(2.31.2) Let $X \xrightarrow{h} Z \to S$ be proper morphism. Let D_X (resp. D_Z) be divisors on X (resp. Z). Then

(2.31.2.1) D_X LSEPD $\Rightarrow h_*(D_X)$ LSEPD;

(2.31.2.2) D_Z LSEPD $\Rightarrow h^*(D_Z)$ LSEPD;

(2.31.2.3) Assume that $X^+ \to Z$ is the opposite of $X \to Z$. Then D_X LSEPD $\Rightarrow D_X^+$ LSEPD.

Next we prove some results which allow us to change D without changing the log flip.

2.32 Proposition. *Let $f : X \to Z$ be a small morphism.*

(2.32.1) If $\rho(X/Z) = 1$ and $R^1 f_* \mathcal{O}_X = 0$, then the opposite of f with respect to E does not depend on the choice of $E = \sum e_i E_i$. In particular we are free to increase or decrease the coefficients of E as long as $-E$ remains f-ample..

(2.32.2) Let $f : (X, D) \to Z$ be log terminal, with D LSEPD. Then there exists a divisor D' such that $K_X + D'$ is klt and D' is f- equivalent to D.

Proof. (2.32.1) If E and $\alpha E'$ ($\alpha > 0$) are numerically equivalent over Z then the opposite with respect to E is the same as the opposite with respect to E'.

(2.32.2) If D is LSEPD and $K + D$ lt then there exists a positive number ϵ such that $D - \epsilon(f \circ h = 0)$ is effective, $K_X + D - \epsilon(f \circ h = 0)$ is lt and $\llcorner D - \epsilon(f \circ h = 0) \lrcorner = \emptyset$. □

2.33 Proposition. *Let $f : X \to Z$ be a small morphism with Z affine. Assume that $K_X + dD + D'$ is lc (resp. plt) where D is a Weil divisor. Let $n \in \mathbb{N}$. Then there is a reduced divisor $\tilde{D}$ such that*

(2.33.1) $\tilde{D} \sim nD$ (hence $K_X + dD + D' \equiv K_X + \frac{d}{n}\tilde{D} + D'$);

(2.33.2) $K_X + \frac{d}{n}\tilde{D} + D'$ is also lc (resp. plt).

Proof. Let $\bar{D}$ be a general element of the linear system $|nf(D)|$ on Z. Since Z is affine, $\bar{D}$ is reduced. Let $\tilde{D}$ be the birational transform of $\bar{D}$. $\tilde{D} \sim nD$ since f is small. Let $g : Y \to X$ be any log resolution with exceptional divisors E_i. Then

$$g_*^{-1}(\tilde{D}) \sim g_*^{-1}(nD) + \sum e_i E_i,$$

where $e_i \geq 0$. Thus

$$a(E_i, \frac{d}{n}\tilde{D} + D') = a(E_i, dD + D') + \frac{d}{n} e_i. \quad \square$$

We will use the following two special cases:

2.34 Corollary. *Let $f : X \to Z$ be a small morphism where Z is affine. Assume that $K_X + D$ is lc (resp. plt). Then*

(2.34.1) There is a divisor $\bar{D}$ such that $K_X + D \equiv K_X + \bar{D}$, $\llcorner \bar{D} \lrcorner = \emptyset$ and $K_X + \bar{D}$ is lc (resp. plt).

(2.34.2) Assume that D is a Weil divisor. There is a Weil divisor $\tilde{D}$ such that $K_X + D \equiv K_X + \frac{1}{2}\tilde{D}$ and $K_X + \frac{1}{2}\tilde{D}$ is lc (resp. plt). □

The following result will be needed in Chapters 5 and 18.

2.35 Proposition. *Let (X, B) be lc and let $f : X \to Y$ be proper and birational. Then there are only finitely many f-extremal rays if one of the following conditions are satisfied:*

(2.35.1) (X, B) is plt and $\llcorner B \lrcorner$ does not contain any exceptional divisors;

(2.35.2) (X, B) is lt outside $\llcorner B \lrcorner$ and B is LSEPD with respect to f.

Proof. (see [KMM87,4-2-4]) Assume (2.35.2). The problem is local on Y so by shrinking Y we may assume that there is an effective principal divisor $M \subset Y$ such that

$$\operatorname{Supp}\llcorner B \lrcorner \subset \operatorname{Supp} f^*M \subset \operatorname{Supp} B.$$

Thus $(X, B - \epsilon f^*M)$ is klt for $0 < \epsilon \ll 1$ and has the same extremal rays as (X, B). Therefore (2.35.1) implies (2.35.2).

Let $\mathcal{O}_X(1)$ be f-ample. Choose $H \in |\mathcal{O}_X(-1)|$ such that $\operatorname{Supp} H$ and $\operatorname{Supp}\llcorner B \lrcorner$ do not have common irreducible components. Thus $(X, B + \epsilon H)$ is still plt for $0 < \epsilon \ll 1$. By the cone theorem [KMM87,4-2-1] if M is f-ample then there are only finitely many $(K + B + \epsilon H)$-extremal rays R such that $R \cdot (B + \epsilon H + M) \leq 0$. Choose $M = \epsilon(-H)$ to conclude. $\square$

3. CLASSIFICATION OF LOG CANONICAL SURFACE SINGULARITIES: ARITHMETICAL PROOF

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(3.0.0). Notation. Let (X, P) be a germ of a normal surface singularity and $B = \sum b_i B_i$ a formal sum of irreducible Weil divisors, passing through P, with rational coefficients $0 \leq b_i \leq 1$. Since X is normal, we can assume that P is the only singularity of X. Also, we have a well defined linear equivalence class of canonical Weil divisors K_X.

We use the usual definitions for *log canonical, log terminal and purely log terminal* (2.13).

(3.0.1). If $B = \emptyset$ and the characteristic of the base field is 0, log terminal singularities of surfaces are the same as quotient singularities [Kawamata84] and were classified by [Brieskorn68]. [Iliev86] contains an arithmetical proof.

In the case B is *reduced*, i.e. all the $b_i = 1$, [Kawamata88] classified all log canonical and log terminal singularities (the latter turn out to be also purely log terminal with one trivial exception: when X is nonsingular and B consists of two normally crossing nonsingular curves). This classification is given in Fig.3. The notation is explained in (3.1).

The proof of [Kawamata88] is slightly tricky and uses the log canonical cover of (X, P). Arithmetical proofs were given in [Sakai87] for the case $b_i = 0$ and by S. Nakamura in an appendix to [Kobayashi90].

(3.0.2). Here we suggest a purely arithmetical and quite elementary approach for the classification. The idea is the following: let $f : Y \to X$ be the minimal resolution of the singularity (X, P) (a priori not a good resolution of (X, P)).

Let $f_*^{-1}C \subset Y$ denote the birational transform of a curve $C \subset X$. Write

$$K_Y + \sum f_*^{-1} B_i + \sum E_j = f^*(K_X + \sum B_i) + \sum a_j E_j.$$

Then for any $j = 1, \dots, n$, by the adjunction formula, we have

$$\begin{aligned}2p_a(E_j) &= E_j(K_Y + E_j) + 2 = \\ &= E_j(f^*(K_X + B) + \sum_k a_k E_k - \sum f_*^{-1} B_i - \sum_{k \neq j} E_k) + 2 = \\ &= E_j(\sum a_k E_k - \sum f_*^{-1} B_i - \sum_{k \neq j} E_k) + 2\end{aligned}$$

Therefore we get the following system of n linear equations in n variables

$$(*) \qquad \sum_{k=1}^{n} a_k E_k \cdot E_j = -c_j,$$

where $c_j = 2 - 2p_a(E_j) - (\sum f_*^{-1} B_i + \sum_{k \neq j} E_k) E_j$. Equivalently,

$$(**) \qquad \sum_{k=1}^{n} (a_k - 1) E_k \cdot E_j = -d_j,$$

where $d_j = 2 - 2p_a(E_j) + E_j^2 - \sum f_*^{-1} B_i \cdot E_j$.

(3.0.3). Now our strategy is very simple: solve the system $(*)$, find the a_k and check the conditions $a_k \geq 0$.

(3.0.4). Some of the formulas for the coefficients a_k are contained in [Alexeev89, 4.7,4.8]. Note also that in the log terminal case with $B = \emptyset$, our treatment has some intersections with [Iliev86]. However, our proof is more explicit and direct.

J. Kollár points out that the present proof works in any characteristic. This follows from the fact that the system $(*)$ has a unique solution independent of the characteristic of the base field.

3.1. Solution of $(*)$.

(3.1.0). First, note that $(*)$ does have a unique solution since by [Mumford61] the matrix $(E_k \cdot E_j)$ is negative definite.

(3.1.1). The *weighted dual graph* Γ of the resolution $f : Y \to X$ is the following: each curve E_j corresponds to a vertex v_j. Two vertices v_{j_1} and v_{j_2} are connected by an edge of weight m if the corresponding curves intersect: $E_{j_1} \cdot E_{j_2} = m$. Each vertex v_j has a positive weight $n_j = -E_j^2$.

Since the resolution f is minimal, we have $d_j = 2 - 2p_a(E_j) + E_j^2 - \sum f_*^{-1}B_iE_j \leq 0$ for all j.

(3.1.2). By (2.19.3) every coefficient of the inverse matrix of $(E_k \cdot E_j)$ is strictly negative. Therefore, $(**)$ implies that either all $d_j = 0$, and then for all k, $a_k - 1 = 0$ or at least one $d_j < 0$, and for all k, $a_k - 1 < 0$. The former happens only if all $p_a(E_j) = 0$, $E_j^2 = -2$ and $E_j \cdot \sum f_*^{-1}B_i = 0$. Such singularities (and the corresponding graphs) are called *Du Val singularities* (resp. *Du Val graphs*).

The following result is easy.

(3.1.3). Lemma. *(cf. [Alexeev89,3.2(ii-iii)]) Let Γ be a weighted graph corresponding to a minimal resolution, in particular such that all $d_j \leq 0$. Let $\Gamma' \subset \Gamma$, $\Gamma' \neq \Gamma$ be a subgraph in the sense that all the vertices of Γ' are at the same time vertices of Γ with the same weight n_j, the weights of edges of Γ' and p_a of vertices in Γ' do not exceed the corresponding weights and p_a in Γ, and $E_j \cdot \sum f_*^{-1}B_i$ in Γ' do not exceed the corresponding $E_j \cdot \sum f_*^{-1}B_i$ in Γ.*

Then the corresponding coefficients satisfy $a_k \leq a'_k$ and if Γ is not a Du Val graph, then $a_k < a'_k$.

Proof. Compare the corresponding systems $(**)$ of linear equations and use (3.1.2). □

(3.1.4). Suppose that $\Gamma' = \{v_1\}$ and $p_a(E_1) = 1$. Then in $(*)$ $c_1 = 2 - 2p_a(E_1) - 0 = 0$ and $a'_1 = 0$. If E_1 is a smooth elliptic curve, this is Case 4 of Fig.3. If E_1 is a rational curve with a node then after a single blow up we are in Case 5 of Fig.3. If E_1 is a rational curve with a cusp it is easy to show that after two blow ups one gets a log discrepancy $a_3 = -1$, so this is not a log canonical singularity.

(3.1.5). Suppose that $\Gamma' = \{v_1, v_2, \dots, v_l\}$ is a circle of smooth rational curves. Then in $(*)$ $c_j = 2 - 0 - 2 = 0$ and all $a'_j = 0$. This is Case 5 of Figure 3. Note that all the curves E_j should intersect normally: if a circle contains two or three vertices and two corresponding curves have a common tangent, or three curves intersect at one point, then two or one blow ups give a log discrepancy $a'_3 = -1$.

(3.1.6). Now (3.1.2-5) imply that:

(3.1.6.1). The graph of a log canonical singularity does not contain a vertex v_j with $p_a(E_j) > 1$ or an edge of weight > 2.

(3.1.6.2). If $\Gamma \neq \Gamma'$ as in (3.1.4) or (3.1.5), then Γ contains only vertices that correspond to smooth rational curves, all edges are simple, i.e. of weight 1, and Γ is a tree.

From now on we always assume that we are in this final case.

(3.1.7). For any subgraph $\Gamma' \subset \Gamma$, we define $\Delta' = \Delta(\Gamma')$ as the absolute value of the determinant of the submatrix $(E_k \cdot E_j)$, made up by the columns and rows corresponding to the vertices of Γ'.

Note that if Γ' is a disjoint union of graphs Γ_1 and Γ_2, then $\Delta' = \Delta_1 \cdot \Delta_2$. We set $\Delta(\emptyset) = 1$ by definition.

The following lemmas are easy exercises.

3.1.8 Lemma. *Let Γ be a graph with simple edges, v a vertex of Γ of weight n, and $v_1, \ldots, v_s$ the vertices adjacent to v. Then*

$$\Delta(\Gamma) = n \cdot \Delta(\Gamma - v) - \sum_i \Delta(\Gamma - v - v_i).$$

3.1.9 Lemma. *Let Γ be a tree with simple edges, v_{j_1}, v_{j_2} two vertices of Γ. Then the (j_1, j_2) cofactor of the matrix $(E_k \cdot E_j)$ is*

$$A_{j_1 j_2} = (-1)^{j_1+j_2} M_{j_1 j_2} = -(-1)^n \Delta\big(\Gamma - (\text{path from } v_{j_1} \text{ to } v_{j_2})\big)$$

Note that since Γ is a tree there is a unique (shortest) path joining v_{j_1} and v_{j_2}.

(3.1.10). The previous lemma gives the solution of $(*)$:

$$(***) \qquad \begin{aligned} a_j &= \frac{1}{\Delta(\Gamma)} \sum_{k=1}^{n} \Delta\big(\Gamma - (\text{path from } v_j \text{ to } v_k)\big) \cdot c_k, \\ c_k &= 2 - \big(\sum f_*^{-1} B_i + \sum_{l \neq k} E_l\big) E_k. \end{aligned}$$

Here $(\sum f_*^{-1} B_i + \sum_{l\neq k} E_l) E_k$ is the number of connections of the vertex v_k with adjacent vertices (among $\sum f_*^{-1} B_i$ and the other E_l). Therefore, $c_k = 0$ if and only if v_k has exactly 2 neighbours, $c_k = 1$ if it has 1 neighbour and $c_k < 0$ if if has ≥ 3 neighbours. By $(***)$, a_j is a sum of c_k with positive coefficients. We are interested in the cases when $a_j \geq 0$, therefore we call vertices with $c_k = 1$ (resp. $c_k < 0$) *bonus* (resp. *penalty*) vertices.

Now our aim is to simplify the use of the formulas $(***)$.

(3.1.11). We need the following well known description of weighted chains. Every weighted chain with positive integer weights (from the left to right) $n_1, \ldots, n_s \geq 2$ corresponds in unique way to the pair (Δ, q), where $\Delta = \Delta(\Gamma)$ and $1 \leq q < \Delta$ is an integer coprime to Δ defined by:

$$\frac{\Delta}{q} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\cdots \cfrac{1}{n_s}}}$$

Let us show how to get this description. Let v be the end vertex of the chain Γ. Then by (3.1.8), $\Delta = \Delta(\Gamma)$ can be expressed in terms of $q = \Delta(\Gamma - v)$ and $\Delta(\Gamma - v - v_1)$, then $\Delta(\Gamma - v)$ can be expressed in terms of $\Delta(\Gamma - v - v_1)$ and $\Delta(\Gamma - v - v_1 - v_2)$ and so on, the last determinant will be $\Delta(\emptyset) = 1$. One can easily see that this procedure is nothing other than the Euclidean algorithm for finding the greatest common divisor, so $(\Delta, q) = 1$, and one gets the given formula.

3.1.12 Lemma. *Suppose that a graph Γ contains a subgraph Γ' such that Γ' is a chain with weights $n_j \geq 2$ and the interior vertices of this chain have no other neighbors in Γ or $\sum B_j$. Let v_{j_1} be one of the middle vertices, a_{j_1} the corresponding log discrepancy of Γ. Then the graph of the function a_j at the vertex v_{j_1} is concave up if $a_{j_1} \geq 0$ and is concave down if $a_{j_1} \leq 0$.*

Proof. Note that from $(*)$

$$a_{j_1-1} - n_{j_1} a_{j_1} + a_{j_1+1} = 0,$$

so that

$$|a_{j_1}| = \left|\frac{a_{j_1-1} + a_{j_1+1}}{n_{j_1}}\right| \leq \left|\frac{a_{j_1-1} + a_{j_1+1}}{2}\right|.$$

The rest is obvious. □

3.1.13 Lemma. *Let Γ be a tree with simple edges and all weights $n_j \geq 2$ (all these conditions hold in our situation). Then all the log discrepancies of Γ are nonnegative (resp. positive) if and only if the same holds for all vertices with at least 3 neighbours and for all vertices neighbouring $\sum f_*^{-1} B_i$.*

Proof. Indeed, if $\Gamma' \subset \Gamma$ is a subchain such that each middle vertex has exactly 2 neighbours and one of this middle vertices has $a_{j_1} \leq 0$ (resp. $a_j < 0$), then by (3.1.12) the same holds for the ends of Γ'.

Moreover, we can exclude the vertices with exactly 1 neighbour, because from $(*)$ we have

$$a_{j_1+1} - n_{j_1} a_{j_1} = -1$$

and $a_{j_1} \leq 0$ implies $a_{j_1+1} < a_{j_1}$. $\square$

(3.1.14). We explain the notation of Fig.3. We consider a minimal resolution $f : Y \to X$ (with the exception of Case 5). $\circ$ denotes an exceptional curve of f, $\bullet$ denotes (local branches of) B_i. Long empty ovals denote any chain (Δ, q), attached at an end.

3.2. The case $B = \emptyset$. We first consider several simple possibilities for the graph Γ

(3.2.1). Let Γ be a chain. Then by (3.1.13) Γ corresponds to a log terminal singularity, because none of the vertices has ≥ 3 neighbours.

(3.1.10) gives the formula for the log discrepancies. Let v_j be a vertex of Γ, so that $\Gamma - v_j = \Gamma_1 - \Gamma_2$ is a disjoint union of two chains (Γ_1 or Γ_2 could be empty), let Δ_1, Δ_2 be the corresponding (absolute values of) the determinants ($\Delta(\emptyset) = 1$ by definition). In our situation we have only 2 bonus vertices, namely the ends of the chain Γ. Therefore

$$a_j = \frac{1}{\Delta}(\Delta_1 + \Delta_2) = \frac{\Delta_1 \Delta_2}{\Delta}(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}).$$

This is Case 1 of Fig.3.

(3.2.2). Let Γ be a graph having a single fork at a vertex v_j and suppose that $\Gamma - v_j = \Gamma_1 + \Gamma_2 + \Gamma_3$, and $\Delta_i = \Delta(\Gamma_i)$ for $i = 1, 2, 3$. In order for Γ to correspond to a log terminal (resp. log canonical) singularity one should have $a_j > 0$ (resp. $a_j \geq 0$). In this situation we have 3 bonus vertices, namely the simple ends of Γ_1, Γ_2, Γ_3 and 1 penalty vertex which is v_j itself. Therefore, by (3.1.10) one has

$$\begin{aligned} a_j &= \frac{1}{\Delta}(\Delta_1\Delta_2 + \Delta_2\Delta_3 + \Delta_3\Delta_1 - \Delta_1\Delta_2\Delta_3) = \\ &= \frac{\Delta_1\Delta_2\Delta_3}{\Delta}(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1}{\Delta_3} - 1). \end{aligned}$$

So this is a log terminal singularity in the cases

(3.2.2.1). $\qquad (\Delta_1, \Delta_2, \Delta_3) = (2, 2, n), \quad n \geq 2$

(3.2.2.2). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 3)$

(3.2.2.3). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 4)$

(3.2.2.4). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 5)$

and a log canonical (but not log terminal) singularity in the cases

(3.2.2.5). $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 6)$

(3.2.2.6). $(\Delta_1, \Delta_2, \Delta_3) = (2, 4, 4)$

(3.2.2.7). $(\Delta_1, \Delta_2, \Delta_3) = (3, 3, 3)$

This gives Cases 2 and 6 of Fig.3.

(3.2.3). Now let Γ be a graph with a single fork at the vertex v_j and suppose that $\Gamma - v_j = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, $\Delta_i = \Delta(\Gamma_i)$ for $i = 1, \ldots, 4$.

Then

$$a_j = \frac{\Delta_1\Delta_2\Delta_3\Delta_4}{\Delta}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1}{\Delta_3} + \frac{1}{\Delta_4} - 2\right)$$

and gives a log canonical singularity only if

(3.2.3.1) $(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = (2, 2, 2, 2)$

This is Case 8 of Fig.3.

(3.2.4). In the case of graph Γ with a single fork at a vertex v_j, breaking up Γ into $N \geq 5$ subgraphs we get a non-log canonical singularity, because

$$a_j = \frac{\prod \Delta_i}{\Delta}\left(\sum_{i=1}^{N} \frac{1}{\Delta_i} - (N-2)\right) < 0$$

for $\Delta_i \geq 2$ and $N \geq 5$.

(3.2.5). Now suppose that we are in the situation of Fig.1 of a graph Γ with at least 2 forks, one of them at the vertex v_j. Suppose that $\Gamma - v_j = \Gamma_1 + \Gamma_2 + \Gamma_3$, and let Δ_1, Δ_2, Δ_3, Δ_A, Δ_B be the corresponding determinants. Then by (3.1.10),

$$a_j = \frac{\Delta_1\Delta_2\Delta_3}{\Delta}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \frac{1 - (\Delta_A - 1)(\Delta_B - 1)}{\Delta_3} - 1\right).$$

This is nonnegative (actually, equal to zero) only in the case

$$\Delta_1 = \Delta_2 = \Delta_A = \Delta_B = 2.$$

By (3.1.10), this is also the sufficient condition for Γ to give a log canonical singularity. This is Case 7.

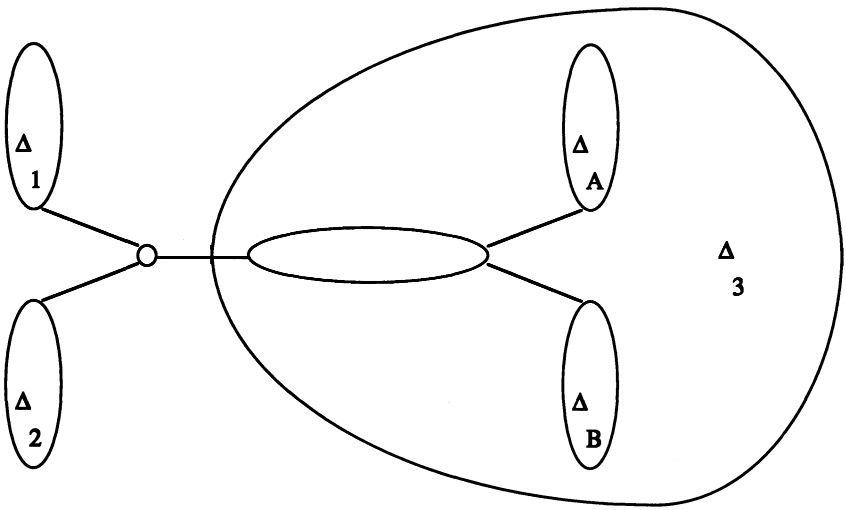


Fig.1

(3.2.6). Using (3.1.10) one can easily show that in the graphs of Fig.2 the marked vertices have negative log discrepancies, hence these graphs define non-log canonical singularities.

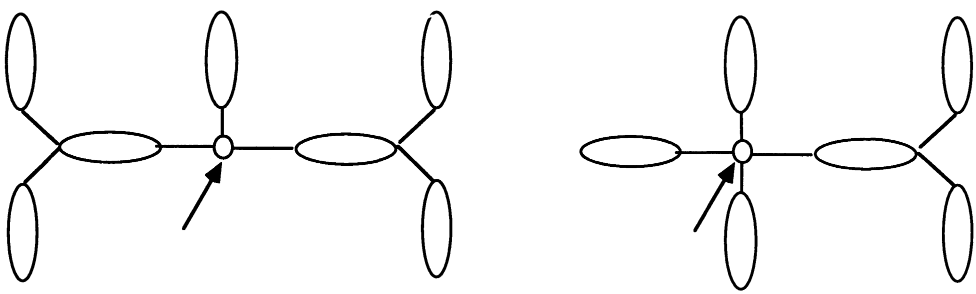


Fig.2

3.2.7 Lemma. *If Γ corresponds to a log terminal (log canonical) singularity then Γ is one of the graphs listed in (3.2.1–2.5).*

1st proof. (3.2.5) gives the general rule for what happens to a log discrepancy when we add an additional fork: the term, denote it by T, that corresponds to the part of the graph after the new fork is changed to a number

$$T \cdot (\Delta_A - \Delta_B(\Delta_A - 1))$$

with the corresponding $\Delta_A, \Delta_B \geq 1$. The other terms don't change.

Therefore, starting from (3.2.3), (3.2.4) or (3.2.6), adding a fork always gives a negative log discrepancy.

2nd proof. By (3.1.3) the subgraph $\Gamma' \subset \Gamma$ also defines a log canonical singularity. Therefore Γ cannot have subgraphs as in (3.2.4) or (3.2.6). □

(3.2.8). Note that Case 8 is essentially a subcase of 7.

3.3. The case $B \neq \emptyset$.

(3.3.1). In addition to the restrictions of (3.2) we have to consider additional penalties for the connections with $f_*^{-1}B$. Now it is an easy excercise to get the remaining Cases of Fig.3.

(3.3.2). From Fig.3 one can see that the minimal resolution is a good resolution for $K+B$. Note that in Case 9 with a chain containing a single vertex v_1, the curves corresponding to the black vertices do not intersect E_1. Otherwise, a single blow up gives a log discrepancy $a_2 = -1$.

(3.3.3). Note that in the Case 9 of Fig.3 all the discrepancies are zero because we have neither bonuses nor penalties.

(3.3.4). The index of a rational singularity, i.e. the least natural number N such that NK_X is a Cartier divisor, is at the same time the least common denominator of all the log discrepancies a_j. One can easily see that in the Cases 6–8 indices are 2,3,4 or 6.

3.4. Final remarks.

(3.4.1). Note that the only restriction on the unmarked weights on Fig.3 is that the quadratic form of the whole graph Γ should be negative definite. This is essential only in Cases 6–8 (where at least one weight should be > 2), and also in Case 5 (where either all weights are at least two and at least one at least three; or there are two vertices, one of them has weight one and the other has weight at least five).

An easy case by case check shows that in Cases 1–3 and 6-10 any (contractible) graph defines a rational singularity, so by [Artin66] a configuration can be contracted to a normal singular point. In cases 4–5 if the quadratic form is negative definite, then a configuration can be contracted in the analytic situation. In the algebraic situation this is a necessary condition (but not sufficient).

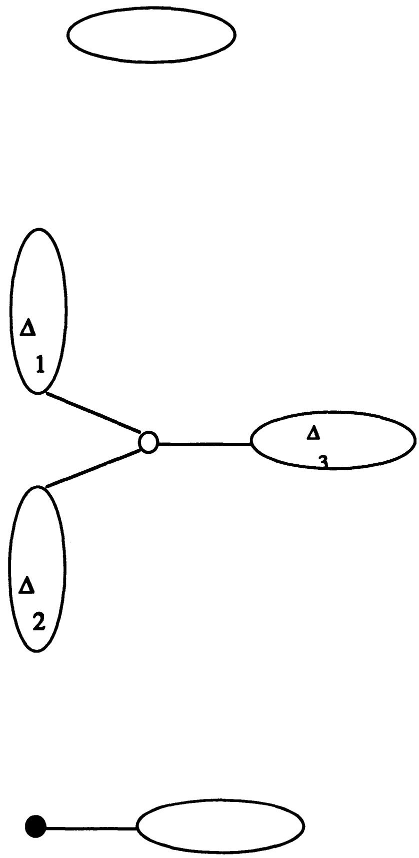
(3.4.2). Our method allows one in principle to classify log terminal or log canonical surface singularities $(X, K+B)$ when B may have fractional coefficients with denominators $2, 3, \dots$, if this should turn out to be necessary. There will be a large number of new cases.

K+B is log terminal, B is reduced

(1)

(2)

(3)



$(\Delta_1, \Delta_2, \Delta_3) =$

(2,2,n)

(2,3,3)

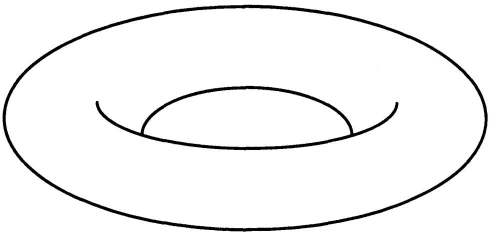
(2,3,4)

(2,3,5)

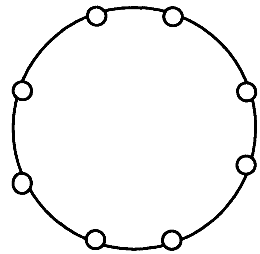
Fig.3, beginning

K+B is log canonical but not log terminal, B is reduced

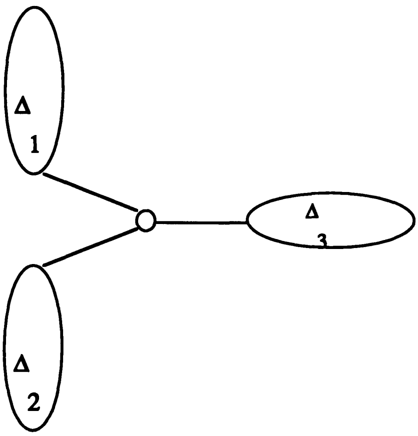
(4)



(5)



(6)



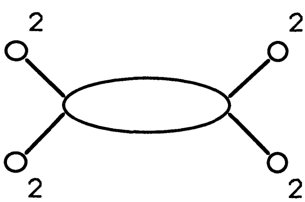
$(\Delta_1, \Delta_2, \Delta_3) =$

(3,3,3)

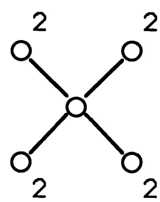
(2,4,4)

(2,3,6)

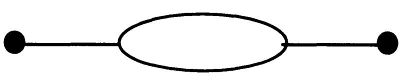
(7)



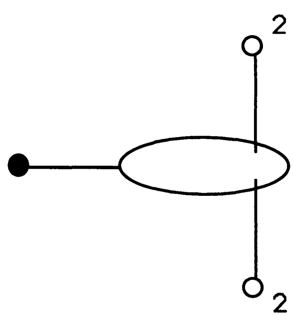
(8)



(9)



(10)



4. TERMINATION OF CANONICAL FLIPS

JÁNOS KOLLÁR and KENJI MATSUKI

The aim of this chapter is to study flops and flips for terminal and canonical threefolds. First we prove the basic finite generation theorem of [Reid83]. The second main result is termination of flips (and flops) for canonical pairs (X, D) (4.10). We start with some general results that hold for arbitrary schemes.

4.1 Definition. Let X be a normal scheme. A *small modification* of X is a proper birational morphism $f : Y \to X$ such that Y is normal and the exceptional set of f has codimension ≥ 2. We usually exclude the trivial case $Y \cong X$.

The following proposition relates projective small modifications to the divisor class group Weil(X) (cf. (16.3.1)).

4.2 Proposition. *[Kawamata88,3.1] Let X be a normal scheme and let D be a Weil divisor on X (not necessarily effective). The following two statements are equivalent:*

(4.2.1) $\sum_{m=0}^{\infty} \mathcal{O}_X(mD)$ is a finitely generated $\mathcal{O}_X$-algebra.

(4.2.2) There is a small modification $f : Y \to X$ such that D', the birational transform of D on Y, is $\mathbb{Q}$-Cartier and f-ample.

Furthermore f is nontrivial iff no positive multiple of D is Cartier.

Proof. Assume that $f : Y \to X$ exists. Let $C \subset Y$ be the exceptional set. First we claim that

$$f_*\mathcal{O}_Y(mD') = \mathcal{O}_X(mD) \quad \text{for } m \geq 0. \tag{4.2.3}$$

It is always true that $f_*\mathcal{O}_Y(mD') \subset \mathcal{O}_X(mD)$. Let $C \subset Y$ be the exceptional set of f. Let $s : \mathcal{O}_X \to \mathcal{O}_X(mD)$ be a section. We can pull it back to a section

$$s : \mathcal{O}_{Y-C} \to \mathcal{O}_{Y-C}(mD').$$

Since C has codimension ≥ 2, this extends to a section $s : \mathcal{O}_Y \to \mathcal{O}_Y(mD')$. This proves (4.2.3). Then (4.2.1) follows since D' is f-ample, and hence

$$\sum_{m=0}^{\infty} f_* \mathcal{O}_Y(mD')$$

is finitely generated.

Replacing D by rD for some $r > 0$ we may assume that $\mathcal{O}_X(D)$ generates $\sum \mathcal{O}_X(mD)$. Let

$$Y = \operatorname{Proj}_X \sum_{m=0}^{\infty} \mathcal{O}_X(mD),$$

and let D' be the birational transform of D on Y (hence $\mathcal{O}_Y(D') = \mathcal{O}_Y(1)$). Let $C \subset Y$ be the exceptional set and assume that it contains a divisor E. For $m \gg 1$ we have an exact sequence

$$0 \to f_* \mathcal{O}_Y(mD') \to f_* \mathcal{O}_Y(mD' + E) \to f_* \left(\mathcal{O}_Y(mD') \otimes (\mathcal{O}_Y(E)/\mathcal{O}_Y) \right) \to 0,$$

since $R^1 f_* \mathcal{O}_Y(mD') = 0$. Therefore for $m \gg 1$

$$\mathcal{O}_X(mD) = f_* \mathcal{O}_Y(mD') \subsetneq f_* \mathcal{O}_Y(mD' + E).$$

This is impossible since $\mathcal{O}_X(mD)$ is reflexive and

$$\mathcal{O}_X(mD)|X - f(C) = f_* \mathcal{O}_Y(mD' + E)|X - f(C).$$

Finally, assume that mD is Cartier. Then mD' and $f^*(mD)$ are two $\mathbb{Q}$-Cartier divisors on Y which agree outside a set of codimension two. Thus $mD' = f^*(mD)$. Since D is f-ample and $f_* \mathcal{O}_Y = \mathcal{O}_X$, this is possible only if $Y \cong X$. $\square$

4.3 Remark. (4.3.1) If X is affine, then one can always find an ideal sheaf $I \subset \mathcal{O}_X$ which is isomorphic to $\mathcal{O}_X(D)$ (as a sheaf), and then the m^{th} symbolic power of I is by definition $I^{(m)} \cong \mathcal{O}_X(mD)$. For this reason the algebra $\sum_{m=0}^{\infty} \mathcal{O}_X(mD)$ is called the symbolic power algebra of D.

(4.3.2) The equivalent statements of (4.2) are both false in general. However it is not easy to come up with nice examples (see e.g. [Cutkosky88]).

4.4 Definition. Let D be a Weil divisor on X. We say that *finite generation holds* for D on X if the equivalent conditions of (4.2) are satisfied.

4.5 Corollary. *Let X be a normal scheme and assume that* $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X - \operatorname{Sing} X)/\operatorname{Pic}(X) = 1$. *Then X has at most two small projective modifications.*

Proof. Let $f_i : Y_i \to X$ be a small modification and let D_i' be an f_i-ample divisor. Let $D_i = f_*(D_i')$. Then by (4.2)

$$Y_i = \operatorname{Proj}_X \sum_{j=0}^{\infty} \mathcal{O}_X(jD_i).$$

If $nD_1 \sim mD_2$ for some $n, m > 0$ then since Proj is unchanged on truncating a graded ring

$$\begin{aligned}\operatorname{Proj}_X \sum_{j=0}^{\infty} \mathcal{O}_X(jD_1) &\cong \operatorname{Proj}_X \sum_{j=0}^{\infty} \mathcal{O}_X(jnD_1)\\ &\cong \operatorname{Proj}_X \sum_{j=0}^{\infty} \mathcal{O}_X(jmD_2) \cong \operatorname{Proj}_X \sum_{j=0}^{\infty} \mathcal{O}_X(jD_2).\end{aligned}$$

Therefore the two possible modifications correspond to the positive and negative parts of $\mathbb{Z}$. □

4.6 Proposition. *[Kawamata88,3.2] Let X and Z be normal, irreducible schemes and $g : Z \to X$ a finite and surjective morphism. Let E be a Weil divisor on X and $E_Z = g^*E$. Then finite generation holds for E iff it holds for E_Z.*

Proof. (Assume for simplicity that g is separable.) Suppose that finite generation holds for E. Let $f : Y \to X$ be a small modification such that the birational transform E' of E is f-ample. Let $p : Y_Z \to Y$ be the normalization of $Y \times_X Z$. Then $h : Y_Z \to Z$ is a small modification and $p^*(E')$ is h-ample. Also, $p^*(E')$ is the birational transform of E_Z. Thus finite generation holds for E_Z.

Assume finite generation for E_Z. Let $q : U \to Z \to X$ be the Galois closure of Z over X and G the Galois group of U/X. Set $E_U = q^*E_Z$. By the previous case finite generation holds for E_U; thus there is a small modification $f_U : Y_U \to U$ such that the birational transform E_U' of E_U is f_U-ample. Clearly G acts on Y_U. Take $Y = Y_U/G$. E_U' descends to a divisor E_U'/G on Y which is the birational transform of E. Thus finite generation holds for E. □

4.7 Theorem. *[Reid83] Let X be a threefold with terminal singularities. Let $D \subset X$ be a Weil divisor. Then*

$$\sum_{m=0}^{\infty} \mathcal{O}_X(mD)$$

is a finitely generated $\mathcal{O}_X$-algebra.

Proof. The problem is local, thus we may assume that $\mathcal{O}_X(mK_X) \cong \mathcal{O}_X$ for some $m > 0$.

By (4.6) it is sufficient to prove (4.7) for the index one cover of X. Thus we may assume that X is terminal with index one. By [Reid83] X is a cDV point; thus it can be viewed as a one parameter family $g : X \to \Delta(t)$ of surfaces with DuVal singularities. By [Brieskorn71] (see also [Artin74]) there is a base change $t = s^m$ such that the resulting threefold $X' = X \times_{\Delta(t)} \Delta(s)$ admits a small resolution. That is, there is a small modification $h : Y' \to X'$ such that Y' is smooth. By (4.6) it is sufficient to prove finite generation on X'. Let D be a Weil divisor on X' and let H be its birational transform on Y'.

We apply the $(K + \epsilon H)$-MMP on Y'/X' with some $0 < \epsilon \ll 1$ (see (2.26)). The existence of flops is given by (4.8) while termination is proved in (4.11). Finally we obtain $h^+ : Y^+ \to X'$ such that H^+ is h^+-nef. By Base Point Freeness [KMM87,3-1-2], there is a morphism

$$h^+ : Y^+ \xrightarrow{p} \bar{Y} \xrightarrow{q} X'$$

such that $p(H^+)$ is $\mathbb{Q}$-Cartier and q-ample. Thus $q : \bar{Y} \to X'$ shows finite generation for D. $\square$

4.8 Theorem. *[Reid83] Let $f : Y \to X$ be a small modification between threefolds. Assume that Y has isolated cDV points only and K_Y is numerically f-trivial. Let H be a divisor on Y such that H is negative on Y/X. Then the flop $f^+ : Y^+ \to X$ of f with respect to H exists and has isolated cDV points only.*

Proof. A very simple proof, due to Mori, is given in [CKM88,16.8-9]. $\square$

4.9 Definition. [Kawamata91c] Let (X, D) be a log variety. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier (K_X need not be $\mathbb{Q}$-Cartier). We say that (X, D) is *terminal* (resp. *canonical*) if $a(E, D) > 0$(resp. ≥ 0) for every exceptional divisor of $\mathbb{C}(X)$ with center on X (cf. (1.6)). If $D = \emptyset$, this coincides with the usual definition of terminal (resp. canonical).

4.9.1 Exercise. Let $(0 \in S, \sum b_i B_i)$ be the germ of a normal surface. Then
(4.9.1.1) $(S, \sum b_i B_i)$ is terminal iff

$$S \text{ is smooth and } \quad \sum b_i \operatorname{mult}_0 B_i < 1.$$

Therefore if (X, D) is terminal (any dimension) then $\llcorner D \lrcorner = \emptyset$ and X is smooth in codimension two.

(4.9.1.2) $(S, \sum b_i B_i)$ is canonical iff either

$$S \text{ is smooth and } \quad \sum b_i \operatorname{mult}_0 B_i \leq 1; \quad \text{or}$$
$$S \text{ is Du Val and } \quad \sum b_i B_i = 0.$$

From (2.28) we see that terminal is preserved under flops and flips. It is however not preserved under extremal contractions.

In the rest of this section, flips are assumed to exist whenever they are mentioned.

Next we prove the termination of a sequence of flips for terminal 3-folds (X, D). The flop version was first proved by [Kawamata88] for a special case, then in general by [Kollár89]. Finally [Kawamata91c] noticed that the right context is the more general form (4.10).

Here we emphasise the analogy between [Kollár89] and [Shokurov91, 4.1] whose proof is presented in Chapter 7. Roughly speaking, the proofs consist of two major steps (the $D = \emptyset$ case can be treated as a special case of (I)):

(I) Show that there is a *finite* set of special discrete valuations associated to the flipped curves such that the cardinality of the set (or some other invariant) drops if a flipped curve is contained in the boundary. This step shows that, after finitely many flips, no flipped curve is contained in the boundary.

(II) Now use the finiteness of the Picard number of the irreducible components of the boundary to conclude that, after finitely many flips, no flipping curve can be contained in the boundary.

4.10 Theorem. *(Termination of flips for canonical 3-folds) Let X be a normal three dimensional $\mathbb{Q}$-factorial scheme of finite type over a field of characteristic zero and D an effective $\mathbb{Q}$-divisor. Assume that (X, D) is canonical and $\llcorner D \lrcorner = \emptyset$. Then any sequence of flips for (X, D) terminates, i.e., there is no infinite sequence*

$$\begin{tikzcd}[column sep=small]
(X_0, D_0) \arrow[rr, dashed] \arrow[dr, "\phi_0"'] & & (X_1, D_1) \arrow[rr, dashed] \arrow[dl, "\phi_0^+"] \arrow[dr, "\phi_1"'] & & (X_2, D_2) \arrow[r, dashed] \arrow[dl, "\phi_1^+"] \arrow[dr, "\phi_2"'] & \cdots \\
& Z_0 & & Z_1 & & Z_2
\end{tikzcd}$$

where $X_{i+1} = (X_i)^+$ is a $(K_{X_i} + D_i)$-flip of X_i for each i and D_i is the birational transform of $D_0 = D$.

4.11 Corollary. *(Termination of flops for terminal 3-folds) Let X be a normal $\mathbb{Q}$-factorial 3-fold with only terminal singularities and D an effective $\mathbb{Q}$-Cartier divisor. Then any sequence of D-flops terminates.*

Proof. For $0 < \epsilon \ll 1$ the pair $(X, \epsilon D)$ is terminal and any D-flop is a $(K_X + \epsilon D)$-flip. □

The proof of (4.10) is done in several steps.

4.12 Discrepancy Lemmas.

4.12.1 Lemma. *Let Y be a smooth variety with a (not necessarily effective) $\mathbb{Q}$-divisor $B = \sum b_i B_i$ such tat $\sum B_i$ has simple normal crossings.*

(4.12.1.1) If ν is a divisor of $\mathbb{C}(Y)$ then there are $k, n_i \in \mathbb{N}$ such that

$$a_\ell(\nu, Y, B) = k + \sum n_i(1 - b_i) = k + \sum n_i a_\ell(B_i, Y, B).$$

$n_i = 0$ unless $\mathrm{Center}_Y(\nu) \subset B_i$ and $k + \sum n_i \geq \mathrm{codim}(\mathrm{Center}_Y(\nu), Y)$.

(4.12.1.2) Let $B = E + H = \sum e_j E_j + \sum h_k H_k$ such that $e_j \leq 1$. Assume that $1 - h_k \geq c$ for every k, where c is some fixed constant with $1 \geq c \geq 0$. Let ν be a divisor of $\mathbb{C}(Y)$ such that

$$\#\{j | \mathrm{Center}_Y(\nu) \subset E_j\} < \mathrm{codim}(\mathrm{Center}_Y(\nu), Y).$$

Then $a_\ell(\nu, Y, E + H) \geq c$.

(4.12.1.3) Assume that $(1-b_k)+(1-b_l) \geq 2$ whenever B_k and B_l intersect. If ν is a discrete valuation with small center on Y such that $a_\ell(\nu, B) < 2$ then ν is obtained by blowing up the generic point of a subvariety $W \subset Y$ such that $\mathrm{codim}_Y W = 2$, only one of the B_k (say B_{k_0}) contains W and $b_{k_0} > 0$.

Proof. Let ν be any discrete valuation of Y. Let $Z_1 \subset Y_1 = Y$ be the center of ν on Y. Let Y_2 be the blow up of Y_1 along Z_1. Let Z_2 be the center of ν on Y_2. Then Y_2 is smooth at the generic point of Z_2 and we can continue the blowing up procedure. After finitely many steps the center of ν on Y_k becomes a divisor. (This is a basic result of Zariski. See [Artin86, 5.2] for a simple self-contained proof.) Thus if we understand the behavior of log discrepancies under a single (smooth) blow up, then we understand them for all discrete valuations.

With this in mind, (4.12.1.1-3) are easy computations. See [Kollár89,3.2] for details. □

4.12.2 Lemma. *Let (X, D) be a log variety, where $D = \sum d_j D_j$ is an effective $\mathbb{Q}$-divisor on X. Assume that (X, D) is klt.*

(4.12.2.1) There is a finite set of valuations $\{\nu_i\}$ such that if

$$a_\ell(\nu, D) < \min\{2, 1 + \mathrm{logdiscrep}(X, D)\} \quad \text{and} \quad \nu \notin \{\nu_i\}$$

then ν is obtained from blowing up the generic point of a subvariety $W \subset D \subset X$ such that D and X are generically smooth along W (and thus only one of the D_j contains W) and $\dim W = \dim X - 2$.

(4.12.2.2) There are only finitely many exceptional divisors ν such that

$$a_\ell(\nu, D) < \min\{1 + \operatorname{logdiscrep}(X, D), 2 - \max_j\{d_j\}\}.$$

Proof. The second claim is a consequence of the first. To see the first, take a good resolution $f : Y \to X$ such that $F = f_*^{-1}(D)$ is smooth and let $K_Y + \sum h_k H_k \equiv f^*(K_X + D)$. (Thus F is a summand of $\sum h_k H_k$.) Then $a_\ell(\nu, D) = a_\ell(\nu, \sum h_k H_k)$ for every ν.

We want to change Y so that (4.12.1.3) is satisfied. Assume that it fails for a pair (k, l). Blow up $H_k \cap H_l$. Let H' be the new exceptional divisor. Then

$$a_\ell(H', D) = a_\ell(H_k, D) + a_\ell(H_l, D).$$

Let $c = \min\{1 - d_j, \operatorname{discrep}(X, D) + 1\}$. Then

$$a_\ell(H', D) + a_\ell(H_k, D) \geq a_\ell(H_k, D) + a_\ell(H_l, D) + c.$$

Repeating this procedure a finite number of times, we can finally achieve that the assumption of (4.12.1.3) is satisfied. By a slight abuse of notation we assume that $f : Y \to X$ itself satisfies (4.12.1.3).

Thus we obtain (4.12.2.1) except that (4.12.1.3) gives information about the centers on Y and not on X.

Assume that ν is a discrete valuation such that

$$a_\ell(\nu, D) = a_\ell(\nu, \sum h_k H_k) < 2.$$

By (4.12.1.3) all but finitely many of these are obtained by blowing up a smooth codimension one point on $\sum H_k$. If the center of ν is contained in H_j and H_j is f-exceptional then

$$a_\ell(\nu, D) = 1 + a_\ell(H_j, D) \geq 1 + \operatorname{logdiscrep}(X, D).$$

Therefore the center of ν is contained in F. Among these ν, there are only finitely many ν whose center on X does not satisfy (4.12.2.1.1). (The exceptions come from the exceptional divisors of $F \to D$, the singular locus of D and the singular locus of X.) $\square$

4.12.3 Definition. Let $(X, D = \sum d_j D_j)$ be a canonical pair. Assume that $\llcorner D \lrcorner = \emptyset$. Fix an integer $N \in \mathbb{N}$ such that ND is a Weil divisor (i.e. $Nd_j \in \mathbb{N}$ for every j). Let $d = \max\{d_j\}$. Let

$$d_N(X, D) = \sum_{i=Nd}^{\infty} \# \left\{ \begin{array}{c} \text{discrete valuations } \nu \text{ with small center on } X \\ \text{such that } a_\ell(\nu, D) < 2 - i/N \end{array} \right\}.$$

This is a weighted version of the "difficulty" introduced by [Shokurov85] (see also [Kollár89]). $d_N(X, D) < \infty$ by (4.12.2.2).

Shokurov pointed out that even if (X, D) is not canonical, $d_N(X, D) < \infty$ if $d \geq 1 - \operatorname{logdiscrep}(X, D)$.

4.12.4 Lemma. *Let (X, D) be a canonical pair. Assume that $\llcorner D \lrcorner = \emptyset$. Then $d_N(X, D)$ is finite and nonincreasing under flips.*

Proof. Let ν be a discrete valuation with center of codimension ≥ 2 on Y such that $a_\ell(\nu, D) < 2$. If ν is obtained by blowing up a smooth codimension one point of F_j then $a_\ell(\nu, D) = 2 - d_j \geq 2 - d$. Thus finiteness follows from (4.12.2).

(2.28) implies the second part. □

(4.13) Proof of (4.10).

Let (X, D) be as in (4.10). Let $D = \sum d_j F_j$, so $D_i = \sum d_j F_j^i$. Consider a sequence of $(K_X + D)$-flips. We prove termination by descending induction on the coefficients d_j of D, combined with the strategy explained at the beginning of the chapter. As before set $d = \max\{d_j\}$ ($d = 0$ if $D = \emptyset$) and let $G := \sum_{d_l = d} F_l$ be the divisor consisting of the F_j with the biggest coefficient ($G = X$ if $D = \emptyset$). We prove the following two statements:

$(I)_G$ After some flips no flipped curve is contained in (the birational transform of) G.

$(II)_G$ After some flips no flipping curve is contained in (the birational transform of) G.

(Here by a *flipping curve* we mean any component of a fiber of ϕ_i and by a *flipped curve* any component of a fiber of ϕ_i^+.)

4.13.1 Subclaim. *Suppose a flipped curve C is contained in G_{i+1} (the birational transform of G on $X_{i+1} = X_i^+$). Let E_C be the divisor obtained from blowing up the generic point of C. Then there is a $k(C) \in \mathbb{N}$ such that*

$$a_\ell(E_C, D_i) < a_\ell(E_C, D_{i+1}) = 2 - \frac{k(C)}{N} \leq 2 - d.$$

Proof. By (4.9.1) the generic point of C lies in the smooth locus of X_{i+1}. By explicit computation

$$a_\ell(E_C, D_{i+1}) = 2 - \sum m_j d_j,$$

where m_j is the multiplicity of F_j^{i+1} along the generic point of C. Set $k(C) = N \sum m_j d_j$. If $C \subset G$ then $k(C) \geq Nd$. By (2.23.3)

$$a_\ell(E_C, D_i) < a_\ell(E_C, D_{i+1}). \quad \square$$

4.13.2 Claim. *$(I)_G$ is true.*

Proof. If $\psi_i : X_i \dashrightarrow X_{i+1}$ is a flip and a flipped curve is contained in G then by (4.13.1) $d_N(X_{i+1}, D_{i+1}) < d_N(X_i, D_i)$. Since $d_N(\ , \ )$ is nonnegative, this can happen only finitely many times. $\square$

4.13.3 Claim. *$(II)_G$ is true.*

Proof. By virtue of $(I)_G$, we may assume that no flipped curve is contained in G_i. This implies that the induced birational map $\psi_i : \{G_i\}^\nu \to \{G_{i+1}\}^\nu$ is actually a morphism, and moreover contracts a curve whenever a flipping curve is contained in G_i. ($\{ \ \}^\nu$ denotes the normalization.) This cannot be repeated infinitely many times, and thus we have the claim $(II)_G$. $\square$

If $D = \emptyset$ then (4.13.2) completes the proof. Otherwise after finitely many flips neither the flipping nor the flipped curve is contained in the birational transform of G. In the $\mathbb{Q}$-factorial case this implies that the birational transform of G is disjoint from the flipping curves. Indeed, assume that C intersects G but is not contained in it. Then the $\mathbb{Q}$-factoriality of X implies that there exists a component G_0 of G such that $C \cdot G_0 > 0$. This in turn implies $C^+ \cdot G_0^+ < 0$ and hence $C^+ \subset G_0^+ \subset G^+$.

Thus we may replace (X, D) by $(X \setminus G, \sum_{d_j < d} d_j F_j)$ and use induction on the number of irreducible components of D. $\square$

4.14 Remark. Szabó observed that it is not too difficult to modify the above proof in case X is not $\mathbb{Q}$-factorial. We cannot guarantee that G becomes disjoint from the flipping curves. We need to modify the definition (4.12.3) by counting only those discrete valuations ν which are not obtained by blowing up the generic point of a curve in G. Once neither the flipping nor the flipped curves are contained in G, this definition is independent of further flips.

We also need a slight strengthening of (4.10):

4.15 Theorem. *Let X be a normal three dimensional $\mathbb{Q}$-factorial scheme of finite type over a field of characteristic zero and D an effective $\mathbb{Q}$-divisor. Assume that (X, D) is canonical. Then any sequence of flips for (X, D) terminates.*

Proof. Let $g : (X, D) \dashrightarrow (X^+, D^+)$ be a flip and let C^+ be a flipped curve. Assume that $C^+ \subset \llcorner D^+ \lrcorner$. Then by (4.9.1) X^+ is generically smooth along C^+. Let E be the exceptional divisor obtained by blowing up the generic point of C^+. Then $0 = a(E, D^+) > a(E, D) \geq 0$ gives a contradiction. Thus $C^+ \not\subset \llcorner D^+ \lrcorner$.

As in (4.13.3) we see that after finitely many steps no flipping curve can be contained in $\llcorner D \lrcorner$. Thus after finitely many flips we can replace X by $X \setminus \llcorner D \lrcorner$ and termination is reduced to (4.10). $\square$

4.15.1 Remark. Shokurov pointed out that the above proof of (4.15) works in positive characteristic as well.

5. EXISTENCE OF CANONICAL FLIPS

Alessio Corti and János Kollár

The aim of this chapter is to prove that if (X, D) is canonical and three dimensional then flips exist. Unfortunately, the proof assumes the existence of flips in the $D = \emptyset$ case, which is a very difficult result of [Mori88]. For technical reasons we need to consider pairs (X, D) which are slightly more general than terminal.

5.1 Definition. We say that the pair (X, D) satisfies condition $(*)$ if the following assumptions hold:

(5.1.1) X is a normal $\mathbb{Q}$-factorial threefold and $D = \sum d_i D_i$ is a $\mathbb{Q}$-Cartier divisor; and

(5.1.2) $a(E, D) \geq 0$ for every exceptional divisor E with equality holding only if E is obtained by blowing up (the generic point of) a curve contained in $\llcorner D \lrcorner$.

5.2 Proposition. *Assume that (X, D) satisfies $(*)$. If (X', D') is obtained from (X, D) by a sequence of D-flips or extremal contractions which do not contract any components of D, then (X', D') also satisfies $(*)$.*

Proof. It is sufficient to consider one flip or contraction $g : X \dashrightarrow X'$. Let $C' \subset X'$ be the exceptional set of g^{-1}. If E is an exceptional divisor over X' such that $\mathrm{Center}_{X'}(E) \not\subset C'$ then $a(E, D') = a(E, D)$. Thus assume that $\mathrm{Center}_{X'}(E) \subset C'$. In this case, $a(E, D') > a(E, D)$ by (2.23.3) and (2.28.3).

The only case that needs attention is when g is a divisorial contraction and E the exceptional divisor of g (since E is not exceptional over X). If E is not a component of D then $a(E, D) = 0$. Otherwise $a(E, D) < 0$, hence $E \subset \mathrm{Supp}\, D$, which was excluded. $\square$

5.3 Lemma. *Assume (X, D) satisfies $(*)$. Then*

(5.3.1) X has terminal singularities;

(5.3.2) If $x \in \llcorner D \lrcorner$ then X and D are smooth at x.

Proof. The first part is clear.

The second part can be done by hand, but it is easier to use adjunction. Assume that $x \in \llcorner D \lrcorner$. Pick a component $S \subset \llcorner D \lrcorner$ containing x. By (17.2) $(S, \mathrm{Diff}(D-S))$ is terminal, and thus by (4.9.1) S is smooth. Let $p : (x', X') \to (x, X)$ be the index one cover in a neighborhood of x. The covering is étale outside x and has degree equal to index$(x \in X)$. S is smooth, thus $p^{-1}(S)$ is a union of index$(x \in X)$ irreducible components intersecting at x'. $p^{-1}(S)$ is a $\mathbb{Q}$-Cartier divisor on the cDV variety X', thus Cartier by (6.7.2). Therefore $p^{-1}(S)$ is S_2. Let $i : U = p^{-1}(S) - \{x'\} \to p^{-1}(S)$ be the injection. Since $p^{-1}(S)$ is S_2, $i_* \mathcal{O}_U \cong \mathcal{O}_{p^{-1}(S)}$. This implies that $p^{-1}(S)$ is irreducible. Therefore index$(x \in X) = 1$, X is a cDV point and S is Cartier. Hence X is also smooth. □

5.4 Theorem. *Assume that (X, D) is canonical. Let $f : X \to Z$ be a small extremal contraction such that $-(K_X + D)$ is f-ample and $\rho(X/Z) = 1$. Then the flip of f exists.*

Proof. The proof is in two steps. First we establish the result in the case when D is reduced and satisfies $(*)$. Then we prove the general case by induction on the number of irreducible components of D.

5.4.1 Step 1. (5.4) holds if $D = \sum D_i$ is reduced and satisfies $(*)$.

Let $C \subset X$ be the exceptional curve. By shrinking Z, we may assume that C is connected. If $C \cdot D_i \geq 0$ then we can discard D_i. If we discard all the D_j then $C \cdot K_X < 0$. Then the flip exists by [Mori88] (the flips with respect to $K_X + D$ and with respect to K_X coincide, cf. (2.32.1)).

If we assume that $C \cdot D_1 < 0$, then $C \subset D_1$. Thus no other component of D intersects C by (5.3.2) and $S = D_1$ is smooth along C. Consider the contraction $f : S \to f(S)$. $K_S = K + S|S$. Thus $-K_S$ is $(f|S)$-ample. Therefore $f|S$ is the contraction of a single -1-curve C and $(K+S) \cdot C = -1$.

Suppose that $S \cdot C = -m$, so that $K \cdot C = m - 1$. Furthermore,

$$\mathcal{O}_X(K+S)^{\otimes m} \cong \mathcal{O}_X(D),$$

at least in a neighborhood of C. Using the natural section of $\mathcal{O}_X(S)$ we can construct a degree m cyclic cover $p : X^m \to X$ ramified along S. Let Z^m be the normalization of Z in X^m and $f^m : X^m \to Z^m$ the induced contraction of $C^m = p^{-1}(C)$. By the ramification formula

$$C^m \cdot K_{X^m} = C^m \cdot p^* \left(K_X + \frac{m-1}{m} S \right) = C \cdot K_X + \frac{m-1}{m} C \cdot S = 0.$$

Therefore f^m is a flopping contraction and the opposite $(X^m)^+ \to Z^m$ exists by (4.8). Thus $X^+ = (X^m)^+ / \mathbb{Z}_m$ is the flip of f. □

By (2.35) for fixed η there are only finitely many $(K+B-(\epsilon+\eta)B_j)$-extremal rays. Thus we may assume that if R generates a $(K+B-(\epsilon+\eta)B_j)$-extremal ray then

$$R \cdot (K + B - \epsilon B_j) = 0.$$

Therefore, if C_i is a flipping curve, then

$$(5.4.2.4) \qquad (K + B - (\epsilon + \eta)B_j) \cdot C_i = -\eta B_j \cdot C_i < 0.$$

Set

$$B' = \sum_{i<j} d_i h_*^{-1}(D_i) + \sum_{i \geq j+1} h_*^{-1}(D_i).$$

From (5.4.2.4) we conclude that

$$B_j \cdot C_i > 0 \quad \text{and} \quad (K + B') \cdot C_i < 0.$$

Thus the flip required is also a $(K + B')$-flip, which exists by induction since B' has one fewer irreducible components. After some flips and contractions we can increase the value of ϵ to $\epsilon' \geq \epsilon+\eta$. Next apply the $(K+B-(\epsilon'+\eta')B_j))$-MMP, and so on.

We claim that after finitely many steps we reach $\epsilon = 1 - d_j$. As usual, the only question is the termination of flips. As was remarked above, every flip is a $(K + B')$-flip, and so termination follows from (4.15). In the end we obtain

$$h^{j+1} : (Y^{j+1}, B^{j+1}) \to Z.$$

(5.4.2.5) If D has k components then iterating (5.4.2.3) we obtain

$$h^{k+1} : (Y^{k+1}, B^{k+1}) \to Z \quad \text{such that} \quad B^{k+1} = (h^{k+1})_*^{-1}(D).$$

Thus we can take $\bar{X} = X^{k+1}$. □

5.5 Remark. One can consider the $(K + D)$-MMP for terminal or canonical pairs. In general it can occur that an extremal contraction creates a pair (X', D') which is not canonical. This can happen when we contract an irreducible component of D. There are some geometric conditions which ensure that this does not occur.

The simplest case is when we do the relative MMP with respect to a morphism $f : X \to Y$ such that f is generically finite on every irreducible component of D. Another example is when D is reduced and every irreducible component has nonnegative Kodaira dimension.

Assume that we avoid the above problem and the $(K+D)$-MMP terminates with a pair (X^m, D^m) such that $K + D^m$ is nef and satisfies $(*)$. In general

X^m is not unique in codimension one since we can always blow up a smooth curve inside the smooth locus of $\llcorner D^m \lrcorner$ to obtain another minimal model. In order to remedy the situation we introduce the following notion:

5.6 Definition. We say that a pair (X', D') is a $(K+(1-0)D')$-minimal model if the following conditions are satisfied:

(5.6.1) (X', D') is canonical;

(5.6.2) (X', D') is terminal outside $\ulcorner D' \urcorner$ (equivalently, X' has terminal singularities);

(5.6.3) $K + (1-\epsilon)D'$ is nef for every $0 \leq \epsilon \ll 1$.

5.7 Construction. Assume that (X, D) satisfies $(*)$. The construction of $(K+(1-0)D)$-minimal models proceeds along the lines of the MMP. First we apply the $(K+D)$-MMP. Thus eventually we obtain (X^m, D^m), unless we run into a forbidden contraction as in (5.5).

If $K + (1-\epsilon)D^m$ is nef for some $0 < \epsilon$ then we can take $X' = X^m$. Otherwise, we choose ϵ such that every $(K + (1-\epsilon)D^m)$-extremal ray R has zero intersection with $K + D^m$ and apply the $(K + (1-\epsilon)D^m)$-MMP.

Assume that we need to flip a curve $C \subset X^m$. Then

$$(K + (1-\epsilon)D^m) \cdot C = -\epsilon D^m \cdot C < 0,$$

and therefore $(1-\epsilon)D^m \cdot C > 0$ and $K \cdot C < 0$. Thus every such log flip is a K-flip. Hence they exist by [Mori88] and any sequence terminates. Of course again there is the possibility that we contract a component of D.

These models have the same uniqueness property as ordinary minimal models:

5.8 Proposition. *Assume that (X, D) satifies $(*)$ and let (X^i, D^i) $(i = 1, 2)$ be two $(K + (1-0)D)$-minimal models. Then the natural birational map $X^1 \dashrightarrow X^2$ is an isomorphism in codimension one.*

Proof. Choose ϵ so that $K + (1-\epsilon)D^i$ is nef for $i = 1, 2$. Then $(X^i, (1-\epsilon)D^i)$ are terminal. The rest of the proof is essentially the same as in [Kollár89,4.3]. We do not use this result in the rest of the notes. □

5.9 Remark. It is interesting to note that the above notions can be used to unify flops, flips and inverses of flips. Consider pairs (X, D) with D reduced which are canonical and terminal outside D. The flops in this category are precisely the following:

terminal flops (D is a member of $|\mathcal{O}_X| = |-K_X|$);

terminal flips (D is a member of $|-K_X|$, [Kollár-Mori92, 1.7]); and

inverses of terminal flips (D is a member of $|-K_X|$, [Kollár-Mori92, Ch.3]).

6. CREPANT DESCENT

JÁNOS KOLLÁR

The aim of this chapter is to develop a reduction method for log flips. The main results say that if $f : Y \to X$ is a birational morphism such that $K_Y \equiv f^*K_X$ (i.e. f is *crepant*) then flipping on X can be reduced to flipping on Y. This method first appeared in [Kawamata88] and was further developed in [Kollár89] and [Kawamata91c]. First we outline the general method of doing this, called the Backtracking Method. Then we prove the two main applications in (6.10–11).

We start with three auxiliary lemmas.

6.1 Lemma. *Let $h : U \to Z$ be a projective morphism such that $h_*\mathcal{O}_U = \mathcal{O}_Z$. Assume that $\rho(U/Z) = 2$. Then there are at most two normal and projective schemes $V_j \to Z$ $(j = 1, 2)$ giving nontrivial factorizations*

$$U \to V_j \to Z$$

such that $U \to V_j$ has connected fibers.

Proof. Let H_j be ample on V_j/Z and let M_j be the pull-back of H_j to U. Then M_j is nef and trivial on the curves that are contained in the fibers of $U \to V_j$.

$$\{[D] | D \cdot M_j = 0\} \subset \overline{NE}(U/Z) \subset \mathbb{R}^2$$

is an extremal face which determines V_j. A convex cone in $\mathbb{R}^2$ has only two edges, thus there can be at most two contraction morphisms $U \to V_j$. $\square$

6.2 Lemma. *Let Y be a normal $\mathbb{Q}$-factorial variety. Let $q : Y \to X$ be a projective birational morphism such that $\rho(Y/X) = 1$. Let $q' : Y' \to X$ be another projective birational morphism with a unique exceptional divisor $E' \subset Y'$. Assume that the composite birational map $q^{-1} \circ q' : Y' \dashrightarrow Y$ is an isomorphism at the generic point of E'. Then $q^{-1} \circ q'$ is an isomorphism.*

Proof. Let H' be an effective, irreducible q'-ample divisor. Its birational transform H on Y is an irreducible and effective divisor which does not contain the exceptional set of q. Since $\rho(Y/X) = 1$, this implies that H is q-ample. Thus $q^{-1} \circ q'$ is an isomorphism in codimension one and transforms the q'-ample divisor H' into the q-ample divisor H. This easily implies that $q^{-1} \circ q'$ is an isomorphism. $\square$

6.3 Lemma. *Let $g : Y \to Z$ and $g' : Y' \to Z$ be proper birational morphisms. Let $\phi : Y \dashrightarrow Y'$ be a Z-map, isomorphic in codimension one. Let H be a divisor on Y and let $H' = \phi(H)$. Assume that both H and H' are $\mathbb{Q}$-Cartier. If $-H$ is g-ample and H' is g'-nef then g and g' are both small.*

Proof. Let $F' \subset Y'$ be the closed subset where ϕ^{-1} is not an ismorphism. For $m \gg 1$, $|-mH|$ is g-very ample, hence base point free. Thus $\phi_*|-mH|$ is base point free outside F'. If $g' : Y' - F' \to Z$ is not an immersion then there is a proper curve $C' \subset Y'$ such that $g'(C') =$ point, C' intersects F' but is not contained in it. Thus $C' \cdot (-H') > 0$, a contradiction. $\square$

6.4 Backtracking Method.

Let $f : X \to Z$ be a small contraction with $\rho(X/Z) = 1$ and let H be a $\mathbb{Q}$-Cartier divisor on X such that $-H$ is f-ample. The aim of the method is to construct the opposite of f with respect to H.

Set $X = X_0$. As a first step we construct a birational projective morphism $q_1 : Y_1 \to X_0$ such that $\rho(Y_1/X) = 1$ and $\rho(Y_1/Z) = 2$. (The latter is automatic if X is $\mathbb{Q}$-factorial.) If X_0 is $\mathbb{Q}$-factorial, this implies that the exceptional set of q_1 is an irreducible divisor.

Assume that $q_i : Y_i \to X_{i-1} \to Z$ is already constructed. By (6.1) there are at most two nontrivial factorizations

$$Y_i \to V_j \to Z.$$

X_{i-1} is one of them. The corresponding extremal ray is denoted by Q_i. Let R_i be the other extremal ray and let $r_i : Y_i \to X_i$ be the corresponding contraction (provided it exists). If r_i is a divisorial contraction, we stop. Our hope is that X_i is the opposite of $X \to Z$. If r_i is a small contraction then let $q_{i+1} : Y_{i+1} \to X_i$ be the opposite (if it exists).

We have to be a little more careful if $Y_1 \to Z$ is small. (This never happens in most applications.) In this case we stop the method when the birational transform of $-q_1^*H$ becomes nef on Y_i. If it becomes ample then $Y_i \to Z$ is the flip of $X_0 \to Z$. Otherwise $-q_1^*H$ should descend to X_i, thus $X_i \to Z$ is the required flip.

In working with the method we always use the above notation. Also, if D is a divisor on X or on Y_1, its birational transform on Y_i is denoted by D_i. We usually write simply K instead of K_{Y_i} or K_X if no confusion is likely.

Thus starting with $q_1 : Y_1 \to X_0$ we define a unique chain of projective Z-schemes and morphisms:

$$\begin{array}{ccccccc} & (Y_1, D_1) & & \dashrightarrow & (Y_2, D_2) & & \dashrightarrow \\ \swarrow & q_1 \quad r_1 & \searrow & \swarrow & q_2 \quad r_2 & \searrow & \qquad \cdots \\ X_0 & & & X_1 & & & X_2 \end{array}$$

The necessary steps for the success of this approach are the following:

(6.4.1) The construction of $q_1 : Y_1 \to X_0$ (mostly easy).

(6.4.2) Proof that the contractions r_i exist (easy).

(6.4.3) Proof that the opposites $q_{i+1} : Y_{i+1} \to X_i$ exist (this is the hardest).

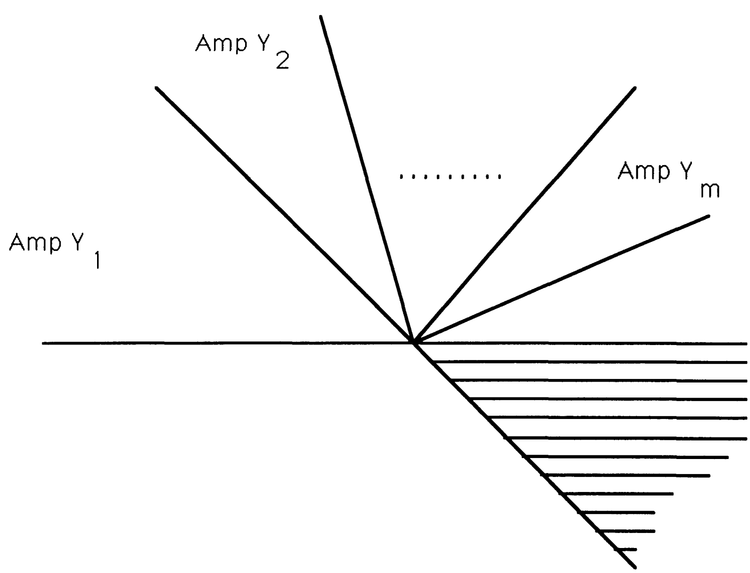
(6.4.4) Proof that eventually we get a divisorial contraction $r_j : Y_j \to X_j$ (easy using Chapters 4 and 7).

(6.4.5) Proof that $X_j \to Z$ is indeed the opposite of $X \to Z$ (easy).

It is convenient to imagine the bactracking method by drawing a picture of the ample cones. By assumption there are natural isomorphism

$$N^1(Y_1) \cong N^1(Y_2) \cong \ldots \stackrel{\text{def}}{=\!=} N^1 \cong \mathbb{R}^2.$$

For each i, let $\operatorname{Amp} Y_i \subset N^1$ be the closed cone generated by the relatively ample divisors of Y_i/Z. The two edges of the cone $\operatorname{Amp} Y_i$ correspond to the two contractions q_i and r_i: they are given by pull backs of ample divisors from X_{i-1} and from X_i. In particular, the cones $\operatorname{Amp} Y_i$ and $\operatorname{Amp} Y_{i+1}$ share a common edge corresponding to the pull back of ample divisors from X_i. Thus we obtain a subdivision of N^1 into a collection of cones.



6.4.6 Lemma. *Notation as above. All the cones* $\operatorname{Amp} Y_i$ *are in one of the half planes determined by the line* $\mathbb{R}[q_1^* H]$. *In particular, all the cones* $\operatorname{Amp} Y_i$ *are different, hence* Y_i *and* Y_j *are not isomorphic over* Z *if* $i \neq j$.

Proof. The shaded area represents those divisors F for which $-F$ is ample on Y_1. Thus if $Y_1 \to Z$ is not small then by (6.3) the cones $\operatorname{Amp} Y_m$ are disjoint from the shaded area. If $Y_1 \to Z$ is small then we stop the method when $q_1^* H$ becomes nef. Thus in both cases we stay in the halfplane containing $\operatorname{Amp} Y_1$. □

6.5 General Properties of the Backtracking Method.

6.5.1. When applying the Backtracking Method, the choice of $q_1 : Y_1 \to X_0$ is our only freedom. In some cases it is easy, in some other cases it is fairly hard to prove that a choice with very good properties exists.

6.5.2 Claim. *Notation as above. Assume that* $-D_1$ *is relatively ample on* Y_1/Z. *Then the steps of the Backtracking Method are steps of the* D_1*-MMP applied to* (Y_1, D_1).

Proof. We assumed the $i = 1$ case. By induction assume next that this holds for i. Thus $D_i \cdot R_i < 0$. Then $D_{i+1} \cdot Q_{i+1} > 0$. If $Y_1 \to Z$ is not small, then by (6.3) D_{i+1} is not nef on Y_{i+1}, thus $D_{i+1} \cdot R_{i+1} < 0$.

If $Y_1 \to Z$ is small, then it can happen that D_{i+1} is nef on Y_{i+1}. This however was declared to be the last step of the method. □

6.5.2.1 Complement. Notation as above. Assume that X_0 is $\mathbb{Q}$-factorial and $H = K + \Delta$ where (X_0, Δ) is klt. Let $E_1 \subset Y_1$ be the exceptional divisor. Then $D_1 = K + \Delta_1 \stackrel{\text{def}}{=} q_1^*(K + \Delta) + \epsilon E_1$ is klt and negative on Y_1/Z for $0 < \epsilon \ll 1$. Therefore the steps of the backtracking method become the steps of the $(K + \Delta_1)$-MMP. In particular the contractions r_i exist.

6.5.3. In the general framework I cannot say anything about the existence of the opposites. In the applications the crucial point is to show that the singularities of Y_i are "simpler" than the singularities of X_0. Thus we prove existence of flips by reduction to "simpler" singularities. Unfortunately the notions of "simplicity" used seem rather artificial and it is not clear how to generalize them to higher dimensions.

6.5.4. Termination of flips is again a problem. In the applications the results of Chapters 4 and 7 imply that eventually we get a divisorial contraction $r_m : Y_m \to X_m$.

6.5.5 Proposition. *Notation and assumptions as above. Assume furthermore that* X *and* Y_1 *are* $\mathbb{Q}$*-factorial. Assume that eventually we get a divisorial contraction* $r_m : Y_m \to X_m$. *Then* $X_m \to Z$ *is the opposite of* $X \to Z$.

Proof. By (4.5), if X is $\mathbb{Q}$-factorial then Z has at most two small modifications, X and its opposite. Therefore it is sufficient to show that

$$\psi : X \to Z \leftarrow X_m$$

is not an isomorphism. (Warning! It can easily happen that X and X_m are isomorphic as varieties, but they are not isomorphic over Z.)

Assume the contrary. Then

$$\phi : Y_1 \xrightarrow{q_1} X \cong X_m \xleftarrow{r_m} Y_m$$

is also an isomorphism by (6.2). This is however impossible by (6.4.6). □

Next we formulate the crepant descent theorems. First we collect properties of flops and terminal flips of threefolds that are needed during the proof of the descent theorems. The lists are complete in the sense that if (6.7) (resp. (6.9)) holds in dimension n then (6.10) (resp. (6.11)) also holds in dimension n. (Unfortunately, as Matsuki pointed out to me, (6.7.2) has no analog in dimension ≥ 4.)

6.6 Definition. Let (X, B) be a klt threefold. By (4.12.1) there are only finitely many exceptional divisors (i.e. valuations) with log discrepancy ≤ 1. The number of these divisors is denoted by $e(X, B)$. If $B = \emptyset$, then we write $e(X)$. Thus (X, B) is terminal (4.9) iff $e(X, B) = 0$.

6.7 Proposition. *Let Y be a threefold with terminal singularities.*

(6.7.1) Flops exist and terminate with respect to any effective Cartier divisor.

(6.7.2) Let E be a $\mathbb{Q}$-Cartier Weil divisor on Y. Then $\operatorname{index}(Y)E$ *is Cartier.*

(6.7.3) The index is unchanged under flops.

(6.7.4) [Reid80,83] Let X be a threefold with canonical singularities. Then there is a threefold with $\mathbb{Q}$-factorial terminal singularities Y and a projective morphism $f : Y \to X$ such that $K_Y \equiv f^*K_X$.

Proof. (6.7.1) was proved in Chapter 4.

(6.7.2) follows from the following local result (in the analytic topology): if D is $\mathbb{Q}$-Cartier then $\operatorname{index}(0 \in Y)D$ is Cartier. To prove this let $p : Y' \to Y$ be the index one cover. Then $p_*p^*D = \operatorname{index}(0 \in Y)D$, thus it is sufficient to consider the index one case. An index one terminal singularity is a hypersurface in $\mathbb{C}^4$. Therefore $Y' - \{0\}$ is simply connected (see e.g. [Milnor68]), and thus $H^2(Y' - \{0\}, \mathbb{Z})$ is torsion free. Therefore any $\mathbb{Q}$-Cartier divisor is Cartier.

Let $X \xrightarrow{f} Z \xleftarrow{f^+} X^+$ be a flop. Then by the Base Point Free Theorem [KMM87,3-1-1] $\operatorname{index}(X) = \operatorname{index}(Z) = \operatorname{index}(X^+)$ which proves (6.7.3).

(6.7.4) is the most difficult. First, by [Reid80] (cf. [CKM88,6.19-25]) there is a morphism $g' : Y' \to X$ such that $K_{Y'} \equiv g'^* K_X$ and Y' has only terminal singularities. Thus it is sufficient to prove the exsitence of a small morphism $Y \to Y'$ such that Y is $\mathbb{Q}$-factorial and terminal. (We could just resolve Y' and run the MMP. However, it is desirable to give a proof which uses less.)

In order to prove this we first note the easy result that $\mathrm{Weil}(Y')/\operatorname{Pic}(Y')$ is finitely generated since Y' has rational singularities (see (16.3.1) for the definition of Weil). Let $D \in \mathrm{Weil}(Y')/\operatorname{Pic}(Y')$ be a nontorsion element. By (4.7) there is a small morphism $Y_1' \to Y'$ such that the birational transform D_1 of D is torsion of order m_1 in $\mathrm{Weil}(Y_1')/\operatorname{Pic}(Y_1')$. Since

$$\mathrm{Weil}(Y_1') = \mathrm{Weil}(Y') \quad \text{and} \quad \operatorname{Pic}(Y_1') \supset \langle \operatorname{Pic}(Y'), m_1 D_1 \rangle,$$

we see that

$$\operatorname{rank}_{\mathbb{Z}} \mathrm{Weil}(Y_1')/\operatorname{Pic}(Y_1') \leq \operatorname{rank}_{\mathbb{Z}} \mathrm{Weil}(Y')/\operatorname{Pic}(Y') - 1.$$

Therefore after finitely many steps we obtain a small projective morphism $Y = Y_m' \to Y'$ such that Y is $\mathbb{Q}$-factorial. □

6.8 Definition. Let (X, B) be an lc threefold. Let H be a Cartier divisor on X. Let $f : X \to Z$ be a small contraction such that $K_X + B$ is numerically f-trivial and $-H$ is f-ample. The opposite of f with respect to H is called an H-flop with respect to $K + B$ or simply an H-flop. If (X, B) is klt then $(X, B + \epsilon H)$ is klt for $0 < \epsilon \ll 1$ and an H-flop is a $(K + B + \epsilon H)$-log flip.

6.9 Proposition. *Let (Y, D) be a terminal threefold.*

(6.9.1.1) H-flops exist and terminate with respect to any effective Cartier divisor H.

(6.9.1.2) Terminal flips exist and terminate.

(6.9.2) [Kawamata91c] Set $r(Y, D) = (4\ulcorner \mathrm{discrep}(Y, D)^{-1} \urcorner)!$. Let E be a $\mathbb{Q}$-Cartier Weil divisor on Y. (Assume for simplicity that K_Y is $\mathbb{Q}$-Cartier.) Then $r(Y, D)E$ is Cartier.

(6.9.3) Let (X, B) be a klt threefold. discrep(X, B) *is nondecreasing under flops and flips.*

(6.9.4) [Kawamata91c] Let (X, B) be a klt threefold. Then there is a terminal threefold (Y, D) with $\mathbb{Q}$-factorial singularities and a projective morphism $f : Y \to X$ such that $f_* D = B$ and $K_Y + D \equiv f^*(K_X + B)$.

Proof. (6.9.1.1) was proved in Chapter 4.

(6.9.1.2) was proved in Chapters 4 and 5.

(6.9.2) can be proved as follows. Let $y \in Y$ be a singular point. Then $y \in Y$ is terminal. Let r be its index. By (6.7.2) rE is Cartier at y. We see in (6.9.7) that there is an exceptional divisor E_y dominating y such that

$$\frac{4}{r} \geq a(E_y, \emptyset) \geq a(E_y, D) \geq \operatorname{discrep}(Y, D).$$

Thus r divides $r(Y, D)$, and hence $r(Y, D)E$ is Cartier at y.

(6.9.3) is a special case of (2.28) and holds in all dimensions.

Finally consider (6.9.4). Let $h_0 : V_0 \to X$ be a log resolution such that:

(6.9.5.1) If ν_j is a discrete rank one valuation with log discrepancy at most one, then $E_j = \operatorname{Center}_{V_0}(\nu_j)$ is a divisor;

(6.9.5.2) $\cup E_j \cup$(birational transform of B) has smooth support (i.e. different components are disjoint).

We write

$$K_{V_0} \equiv h_0^*(K_X + B) + E^+ - E^-,$$

where E^+, E^- are effective $\mathbb{Q}$-divisors without common components. By (6.9.5.2) $\operatorname{Supp} E^-$ is smooth. Therefore $K_{V_0} + E^-$ is terminal.

Apply the $(K_{V_0} + E^-)$-minimal model program to V_0/X. Assume that we have already constructed $V_0 \overset{r_i}{\dashrightarrow} V_i \xrightarrow{h_i} X$ and the birational transform $E_i^- = (r_i)_* E^-$ such that

(6.9.6.1) r_i does not contract any irreducible components of E^-; and

(6.9.6.2) $K_{V_i} + E_i^-$ is terminal.

Let $p_i : V_i \to Z_i$ be the contraction of a $(K_{V_i} + E_i^-)$-extremal ray. If p_i is small, the flip exists by (6.9.1.2). Assume that p_i is divisorial with exceptional divisor F_i.

$$K_{V_i} + E_i^- = h_i^*(K_X + B) + (r_i)_* E^+,$$

and $F_i \subset \operatorname{Supp}(r_i)_* E^+$. Hence $r_{i+1} = p_i \circ r_i$ satisfies (6.9.6.1) and this implies (6.9.6.2) for $i+1$. Thus eventually we obtain $h_m : V_m \to X$ such that

$$K_{V_m} + E_m^- \equiv h_m^*(K_X + B) + (r_m)_* E^+ \quad \text{is } h_m\text{-nef}.$$

Therefore $(r_m)_* E^+ = \emptyset$. Set $(Y, D) = (V_m, E_m^-)$. □

It is quite likely that one can prove (6.9.4) by explicit blow ups as is the case for (6.7.4).

6.9.7 Lemma. *(Kawamata in appendix to [Shokurov91]) Let* $(0 \in X)$ *be a three dimensional terminal singularity. Then* $4/\operatorname{index}(X) \geq \operatorname{discrep}(X)$.

Proof. Kawamata shows that in fact $\operatorname{discrep}(X) = 1/\operatorname{index}(X)$. However we need only this weaker version.

The claim is clear if $\operatorname{index}(X) \leq 4$. If $\operatorname{index}(X) \geq 5$ then X is of the form

$$(xy + f(z, w^r) = 0)/\mathbb{Z}_r(a, -a, 0, 1) \subset \mathbb{C}^4/\mathbb{Z}_r(a, -a, 0, 1).$$

Let $k = \operatorname{ord} f(s,t)$ and consider the weighted blow up $W' \to \mathbb{C}^4/\mathbb{Z}_r(a,-a,0,1)$ given by weights

$$wt(x,y,z,w) = (a+ir, kr-ir-a, r, 1).$$

Let $X' \subset W'$ be the birational transform of X. Explicit computation yields that the unique exceptional divisor has discrepancy $1/r$. □

6.9.8 Definition. (6.9.8.1) A morphism $f : Y \to X$ is called *crepant* if $K_Y = f^*K_X$.

(6.9.8.2) A log morphism $f : (Y, D_Y) \to (X, D_X)$ is called *log crepant* if $K_Y + D_Y = f^*(K_X + D_X)$.

(6.9.8.3) Let (X, B) be a klt threefold. By (6.9.4) there is a terminal threefold (Y, D) with $\mathbb{Q}$-factorial singularities and a projective morphism $f : Y \to X$ such that $f_*D = B$ and $K_Y + D \equiv f^*(K_X + B)$. Set $r(X,B) = r(Y,D)$. By construction discrep(Y,D) is the minimum of the *positive* discrepancies of exceptional divisors over (X,B). Thus $r(X,B)$ is well defined.

A special case of (6.10) was proved in [Kawamata88], the general form is in [Kollár89]. (6.11) is a strengthening of [Kawamata91c] using the method of [Kollár89].

6.10 Theorem. *(Crepant Descent of Flops) Let X be a threefold with canonical singularities. Then*

(6.10.1) There is a small projective morphism $f : \bar{X} \to X$ such that $\bar{X}$ is $\mathbb{Q}$-factorial.

(6.10.2) If $e(X) > 0$ and X is $\mathbb{Q}$-factorial then there is a morphism $q : X' \to X$ such that $\rho(X'/X) = 1$ and $K_{X'} \equiv q^*K_X$. In particular, $e(X') = e(X) - 1$.

(6.10.3) If X is $\mathbb{Q}$-factorial then H-flops exist for any effective divisor H.

(6.10.4) If X is $\mathbb{Q}$-factorial then H-flops terminate for any effective divisor H.

(6.10.5) Let D be a $\mathbb{Q}$-Cartier Weil divisor on X. Then mD is Cartier for some

$$1 \le m \le \operatorname{index}(X)^{2^{e(X)}} 3^{2^{e(X)}-1}.$$

6.11 Theorem. *(Crepant Descent of Flips) Let (X,B) be a threefold with klt singularities. Then*

(6.11.1) There is a small projective morphism $f : (\bar{X}, \bar{B}) \to (X, B)$ such that $\bar{X}$ is $\mathbb{Q}$-factorial.

(6.11.2) If $e(X,B) > 0$ and X is $\mathbb{Q}$-factorial then there is a morphism $q : (X', B') \to (X, B)$ such that (X', B') is $\mathbb{Q}$-factorial and klt, $\rho(X'/X) = 1$ and $K_{X'} + B' \equiv q^*(K_X + B)$. In particular, $e(X', B') = e(X, B) - 1$.

(6.11.3.1) If X is $\mathbb{Q}$-factorial then H-flops exist for any effective divisor H.

$E \subset X'$ be the exceptional divisor. Let $(1-\alpha)$ be the coefficient of E in B'. (Thus $\alpha \geq \text{logdiscrep}(X, B)$.) Let $p : E' \to E$ be the normalization. Then by (16.5)

$$\begin{aligned} K_{E'} &\equiv p^*(K_{X'} + E) - \text{Diff}(0) \\ &\equiv p^*(K_{X'} + B') + p^*(\alpha E) - p^*(B' - (1-\alpha)E) - \text{Diff}(0) \\ &\equiv p^*(\alpha E) - (\text{effective } \mathbb{Q}\text{-divisor}). \end{aligned}$$

Pick a general $x \in q(E)$ and let $x \in U \subset X$ be an affine neighborhood. Let $H \subset U$ be very ample and let $H' \subset q^{-1}(U)$ be q-very ample. Then intersecting $\dim q(E)$ general members of $|H|$ containing x and $\dim E - \dim q(E) - 1$ general members of $|H'|$ we obtain a surface $B \subset X'$ such that $A = B \cap E$ is a curve contracted by q. Thus A has negative selfintersection in B. Hence

$$K_{E'} \cdot p^{-1}A \leq \alpha E \cdot A = \alpha A \cdot_B A < 0.$$

In our case $\dim E' = 2$ and from surface classification we know that the minimal resolution of E' is a ruled surface. Thus E' is covered by rational curves C'_λ such that $0 > C'_\lambda \cdot K_{E'} \geq -3$. (In higher dimensions one can use [Miyaoka-Mori86].)

Thus there are rational curves $C_\lambda = p(C'_\lambda) \subset X'$ such that $q(C_\lambda)$ is a point and $0 > C_\lambda \cdot E \geq -3\,\text{logdiscrep}(X, B)^{-1}$. Let D' be the birational transform of D on Y. By 5_{e-1} we can find m_1 and m_2 such that $m_1 E$ and $m_2 D'$ are Cartier. Thus

$$(m_1 E \cdot C_\lambda) m_2 D' - (m_2 D' \cdot C_\lambda) m_1 E$$

is Cartier and is numerically q-trivial. Therefore by the base point free theorem [KMM87,3.1.2] it descends to a Cartier divisor on X. Thus $(m_1 E \cdot C_\lambda) m_2 D$ is Cartier. $0 < -(m_1 E \cdot C_\lambda) m_2 \leq 3\,\text{logdiscrep}(X, B)^{-1} m_1 m_2$, which proves 5_e.

The proof of 3_e and 4_e relies on the Backtracking Method.

Let $f : X \to Z$ be a small contraction which we want to flop or flip. The flop or flip of f can be obtained as a sequence of flops or flips where the relative Picard number is one. Thus we only need to deal with the case $\rho(X/Z) = 1$.

Set $(X^0, B^0) = (X, \eta H)$ for some $0 < \eta \ll 1$ in case (6.10) and $(X^0, B^0) = (X, B)$ in case (6.11). 2_e gives $q_1^0 : (Y_1^0, B_1^0) \to (X^0, B^0)$ such that $K + B_1^0 \equiv (q_1^0)^*(K + B^0)$. Let $D_1^0 = K + B_1^0$. $K + B_1^0$ is klt. Hence by (6.5.2) and [KMM87,3-2-1] the contractions r_i exist and the existence of the opposites follows from 3_{e-1}.

The sequence of flips terminates by 4_{e-1}. Thus eventually we get a divisorial contraction $r_{m_0}^0 : Y_{m_0}^0 \to X_{m_0}^0$. By (6.5.5) $X_{m_0}^0 = X^1$ is the flop (resp. flip) of f.

In order to see 4_e consider a sequence of flops (resp. flips)

$$X^0 \dashrightarrow X^1 \dashrightarrow \cdots . \tag{6.13.1}$$

Our method of flipping starts with a $Y_1^0 \to X^0$ and produces a sequence of flips

$$Y_1^0 \dashrightarrow Y_2^0 \dashrightarrow \cdots \dashrightarrow Y_{m_0}^0,$$

ending finally with a contraction $r_{m_0}^0 : Y_{m_0}^0 \to X^1$. We can take

$$r_{m_0}^0 = q_1^1 : Y_{m_0}^0 = Y_1^1 \to X^1$$

as the starting point of the sequence of flips constructing $X^1 \dashrightarrow X^2$. In this way the sequence of flips (6.13.1) gives another sequence

$$Y_1^0 \dashrightarrow \cdots \dashrightarrow Y_{m_0}^0 = Y_1^1 \dashrightarrow \cdots \dashrightarrow Y_{m_1}^1 = Y_1^2 \dashrightarrow \cdots . \tag{6.13.2}$$

(6.13.2) is a sequence of flips but when we go from $Y_{m_i}^i$ to Y_1^{i+1} the relevant divisor may change. Indeed, $D_{m_i}^i$ is seminegative on $Y_{m_i}^i \to X^{i+1}$ while D_1^{i+1} is numerically trivial on $Y_{m_i}^i = Y_1^{i+1} \to X^{i+1}$. Therefore

$$B_1^{i+1} = B_{m_i}^i - c_i E_i \quad \text{for some } c_i \geq 0, \tag{6.13.3}$$

where E_i is the exceptional divisor of q_1^i. Let $c(E_i)$ be the coefficient of E_i in B_1^i. Then by (6.13.3)

$$c(E_j) = c(E_0) - \sum_{i=0}^{j-1} c_i. \tag{6.13.4}$$

Choose N such that that NB^0 is a Weil divisor on X^0. Then so are the birational transforms NB^i on X^i for every i. By 5_e there is a universal $M(e)$ such that $M(e)(NK + NB^i)$ is Cartier for every i. Thus

$$M(e)N(K + B_1^i) = (q_1^i)^* M(e)(NK + NB^i)$$

is a Cartier, hence a Weil divisor. Thus $M(e)NB_{m_i}^i$ is also a Weil divisor. Comparing this with (6.13.3) we conclude that $M(e)Nc_i$ is an integer.

If $c_i = 0$ for $i \geq N$ then the sequence of flips starting with X^N lifts to an infinite sequence of flips starting with Y_1^N, which is impossible. Otherwise $c_i \geq 1/(M(e)N)$ for infinitely many values of i, hence $c(E_j) < 0$ for some j.

In case (6.10) this is impossible since $c(E_j)$ is the coefficient of E_j in the effective divisor $B_1^j = \eta(q_1^j)^* H_j$.

In case (6.11) this means that the discrepancy of E_j in $Y_1^j \to X^j$ is greater than 0. Thus $e(X^j, B^j) < e(X, B)$ and again we are done by induction. □

One of the main applications of (6.10) is the following generalization of (4.7):

6.14 Theorem. *[Kawamata88] Let X be a threefold with log terminal singularities. Let D be a Weil divisor on X. Then*

$$\sum_{m=0}^{\infty} \mathcal{O}_X(mD)$$

is a finitely generated $\mathcal{O}_X$-algebra.

Proof. By taking the index one cover as in (4.7) it is sufficient to consider the case when K_X is Cartier and hence X has canonical singularities.

Let $p : \bar{X} \to X$ be given by (6.10.1). Let $\bar{D}$ be the birational transform of D on $\bar{X}$.

We apply the $(K + \epsilon\bar{D})$-MMP on $\bar{X}/X$ for some $0 < \epsilon \ll 1$. The existence and termination of flops is given by (6.10.3-4). Finally we obtain $p^+ : \bar{X}^+ \to X$ such that $\bar{D}^+$ is p^+-nef. By base point freeness [KMM87,3-1-2] there is a morphism

$$p^+ : \bar{X}^+ \xrightarrow{s} Y \xrightarrow{q} X$$

such that $s(\bar{D}^+)$ is $\mathbb{Q}$-Cartier and q-ample. Thus by (4.2) the exsitence of $q : Y \to X$ proves finite generation for D. $\square$

The following strengthening of (6.11) is needed in Chapter 8.

6.15 Proposition. *Let (X, B) be a log terminal $\mathbb{Q}$-factorial threefold. Then log flips exist and any sequence of them is finite.*

Proof. Let $g : (X, B) \to Z$ be a small contraction such that $-(K + B)$ is g-ample. Then $-(K+(1-\epsilon)B)$ is g-ample and $(X, (1-\epsilon)B)$ is klt for $0 < \epsilon \ll 1$. Thus the flip exists by (6.11) (cf. (2.32.1)).

The proof of termination works in the more general case when (X, B) is lc and is klt outside $\llcorner B \lrcorner$. Let $(X_0, B_0) = (X, B)$ and consider a sequence of log flips

$$(X_i, B_i) \xrightarrow{g_i} Z_i \xleftarrow{g_i^+} (X_i^+, B_i^+) = (X_{i+1}, B_{i+1}).$$

Let $C_i \subset X_i$ be the flipping curve. By (7.1) $C_i \cap \llcorner B_i \lrcorner = \emptyset$ for all but finitely many values of i. Thus by shifting the index i we may assume that $C_i \cap \llcorner B_i \lrcorner = \emptyset$ for every i. We may as well replace X_i by $X_i \setminus \llcorner B_i \lrcorner$; hence we may assume that $\llcorner B_i \lrcorner = \emptyset$, which implies (cf. (2.13)) that (X_i, B_i) is klt for every i. Termination follows from (6.11). $\square$

Finally we prove a result about partial resolutions of singularities of threefolds.

7. TERMINATION OF 3–FOLD LOG FLIPS NEAR THE REDUCED BOUNDARY

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In this chapter we prove termination of a sequence of 3-fold log flips near the reduced part of the boundary. The role of this result is two fold. First, it completes the results about existence and termination of log flips proved in Chapters 4–6. Second, it is an essential part of the second proof of log flips. (7.1) is slightly more general than the original theorem in [Shokurov91,4.1]. Kawamata kindly informed us that Shokurov himself announced the theorem in this generalized form in a letter.

As in Chapter 4, the proof consists of two major steps:

(I) By considering a *finite* set of special discrete valuations associated to the flipped curves, we show that after finitely many flips no flipped curve is contained in (the birational transform of) the reduced part of the boundary.

(II) Then, using the finiteness of the Picard number of the reduced part of the boundary, we show that after finitely many flips no flipping curve is contained in it.

7.1 Theorem. *Let X be a normal 3-fold and B an effective $\mathbb{Q}$-divisor such that (X, B) is log canonical. Assume that X is $\mathbb{Q}$-factorial. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:*

$$\begin{array}{ccccccccc} (X_0,B_0) & & \dashrightarrow & & (X_1,B_1) & & \dashrightarrow & & (X_2,B_2) & \dashrightarrow \\ & \phi_0 \searrow & & \swarrow \phi_0^+ & & \phi_1 \searrow & & \swarrow \phi_1^+ & & \phi_2 \searrow & & \cdots \\ & & Z_0 & & & & Z_1 & & & & Z_2 \end{array}$$

where $\phi_i : X_i \to Z_i$ is a contraction of an extremal ray R_i with $(K_{X_i} + B_i) \cdot R_i < 0$, and $\phi_i^+ : X_i^+ (= X_{i+1}) \to Z_i$ is the log flip. Then after finitely many flips, all the flipping curves (and thus all the flipped curves) are disjoint from $\llcorner B_i \lrcorner$.

Proof. The proof is given in several steps.

7.2 Definition. The notions of semi log category are explained in Chapter 16. Let (X, B) be semi log canonical (16.9). We say that a (not necessarily closed) point $p \in X$ is a *maximally log canonical* point of (X, B) if there is a divisor E dominating p such that $a(E, X, B) = -1$.
For example, the maximally log canonical points of an slc surface (S, D) are the following: double curves of S; irreducible components of $\llcorner D \lrcorner$; closed points where (S, D) is not slt; and singular points of $\llcorner D \lrcorner$.

7.2.1 Proposition. *If X is a variety (any dimension) then the number of maximally log canonical points of (X, B) is finite.*

Proof. Let $f : Y \to X$ be a log resolution of (X, B). (4.12.1) implies that the maximally log canonical points are the (general points of) $f(E_{i_1} \cap \cdots \cap E_{i_k})$ where $E_{i_j} \subset Y$ are divisors with $a(E_{i_j}, X, B) = -1$. □

7.2.2 Notation. Let $S_i = \llcorner B_i \lrcorner$ and $D_i = \operatorname{Diff}_{S_i}(B_i - \llcorner B_i \lrcorner)$. Note that by (16.9) (S_i, D_i) is semi log canonical. Furthermore let $\pi_i : S_i^\nu \to S_i$ be the normalization of S_i and let $\pi_i^*(K + D_i) = K + D_i^\nu$. Thus $D_i^\nu = \pi_i^*(D_i) + \Theta_i$ where $\Theta_i \subset S_i^\nu$ is the *divisor of double curves.* Set $E_i = \llcorner D_i^\nu \lrcorner$.

7.3 First Reduction Step. *After finitely many flips, no flipping curve contains a maximally log canonical point of X or of (S_i, D_i).*

Proof. If a maximally log canonical point of X is contained in a flipping curve then after a flip the number of maximally log canonical points decreases by (2.28). Essentially by (16.9), a maximally log canonical point of (S_i, D_i) is also a maximally log canonical point of X. □

7.4 Second Reduction Step. *After finitely many flips no flipping curve intersects $\pi_i(E_i)$.*

Proof. This is achieved by analyzing the sequence of pairs (S_i^ν, D_i^ν). Let $\psi_i : S_i^\nu \dashrightarrow S_{i+1}^\nu$ be the induced map. By (7.3) we may assume that no flipping curve contains a maximally log canonical point of (S_i, D_i). In particular, E_i is smooth at the indeterminacies of ψ_i, thus it induces an isomorphism $\psi_i : E_i \cong E_{i+1}$. Set $E = E_1$ and let $\sigma_i : E \cong E_i$ be the induced isomorphism.

7.4.1 Claim. *Under the above isomorphism*

$$\operatorname{Diff}_{E_{i+1}}(D_{i+1}^\nu - E_{i+1}) = \psi_i \left(\operatorname{Diff}_{E_i}(D_i^\nu - E_i) - H_i\right),$$

where H_i is an effective $\mathbb{Q}$-divisor and

$$\operatorname{Supp} H_i = \pi_i^{-1}\left(\pi_i(E_i) \cap \textit{flipping curve}\right).$$

Proof. Let T_i be the normalization of $\phi_i(S_i^\nu)$. By construction we have morphisms

$$S_i^\nu \xrightarrow{\rho_i} T_i \xleftarrow{\rho_i^+} (S_i^\nu)^+ = S_{i+1}^\nu, \tag{7.4.1.1}$$

$-(K_{S_i^\nu} + D_i^\nu)$ is ρ_i-ample and $K_{S_{i+1}^\nu} + D_{i+1}^\nu$ is ρ_i^+-ample. If F is any divisor then by (2.28)

$$a(F, S_i^\nu, D_i^\nu) \le a(F, S_{i+1}^\nu, D_{i+1}^\nu), \tag{7.4.1.2}$$

and strict inequality holds if ψ_i is not an isomorphism at $\mathrm{Center}_{S_i^\nu}(F)$.

The coefficient of the different can be related to discrepancies as follows.

Let W_i be a common good resolution of (S_i^ν, D_i^ν) and $(S_{i+1}^\nu, D_{i+1}^\nu)$. Let $p \in E_i$ be a point and let $p' \in E_i' \subset W_i$ be the corresponding point of the birational transform. Since W is a good resolution, there is at most one exceptional curve $F \subset W_i$ intersecting E_i' at p' (by further blowing up we may assume that F is exceptional over both S_i^ν and S_{i+1}^ν). By (17.2.3) the coefficient of p in $\mathrm{Diff}_{E_i}(D_i^\nu - E_i)$ is exactly $-a(F, S_i^\nu, D_i^\nu)$. Thus (7.4.1.2) implies (7.4.1). □

7.4.2 Corollary. *Notation as above. If a flipping curve intersects $\pi_i(E_i)$ then it intersects it at a point of*

$$\pi_i\left(\mathrm{Supp}(\mathrm{Diff}_{E_i}(D_i^\nu - E_i))\right). \quad \square$$

In order to use (7.4.1) we need two further results:

7.4.3 Lemma. *[Shokurov91, 4.2] Let $0 < b_i \le 1$, $n_j, l \in \mathbb{N}^+$ and $k_{ij}, l_j \in \mathbb{N}$. Assume that*

$$d_j = \frac{n_j - 1}{n_j} + \sum_i \frac{k_{ij} b_i}{n_j} \le 1, \quad \textit{and} \tag{7.4.3.1}$$

$$p = \frac{l-1}{l} + \sum_j \frac{l_j d_j}{l} < 1. \tag{7.4.3.2}$$

Then there are $m, m_i \in \mathbb{N}$ such that

$$p = \frac{m-1}{m} + \sum_i \frac{m_i b_i}{m}.$$

Proof. If $n_j = 1$ for all j with $l_j \ge 1$, then this is obvious. Otherwise, there exists a unique j_0 such that $n_{j_0} \ge 2$ and $l_{j_0} \ge 1$, for if there were 2 or more, then

$$p \ge \frac{l-1}{l} + \frac{1}{l}\left(\frac{1}{2} + \frac{1}{2}\right) = 1.$$

Similarly we obtain $l_{j_0} = 1$. Hence

$$p = \frac{l-1}{l} + \frac{1}{l}\left(\frac{n_{j_0}-1}{n_{j_0}} + \sum_i \frac{k_{j_0 i}}{n_{j_0}} b_i\right) + \sum_{j \neq j_0} \frac{l_j}{l}\left(\sum_i k_{ji} b_i\right)$$
$$= \frac{n_{j_0} l - 1}{n_{j_0} l} + \sum_i \frac{k_{j_0 i} + \sum_{j \neq j_0} n_{j_0} l_j k_{ji}}{n_{j_0} l} b_i. \quad \square$$

7.4.4 Lemma. *Fix a sequence of numbers $0 < b_i \leq 1$ and $c > 0$. Then there are only finitely many possible values $m, m_i \in \mathbb{N}$ such that*

$$\frac{m-1}{m} + \sum_i \frac{m_i b_i}{m} \leq 1 - c.$$

Proof. It is easy to see that $m \leq c^{-1}$ and $m_i \leq c^{-1} b_i^{-1}$. $\square$

7.4.5 Proof of (7.4). Let $B = \sum b_j B_j$ and let $D_i^\nu = \sum d_j^i D_j^i$. By (16.6.4), we can write d_j^i in the form (7.4.3.1). Let p be any of the coefficients occurring in $\operatorname{Diff}_{E_i}(D_i^\nu - E_i)$. Then by (16.6.4) p is of the form (7.4.3.2). Thus by (7.4.4) there are only finitely many possible values for p. By (7.4.1) $\sigma_i^* \operatorname{Diff}_{E_i}(D_i^\nu - E_i)$ is a decreasing sequence of effective divisors on E which is strictly decreasing whenever the flipping curve intersects $\pi_i(E_i)$. Since there are only finitely many possibilities for the coefficients, the sequence must stabilize. $\square$

7.5 Third Reduction Step. *After finitely many flips no flipped curve is contained in S_i^+.*

Proof. By (7.3.2) we may assume that no flipping curve intersects $\pi_i(E_i)$. We introduce another version of difficulty (cf. (4.12.3)):

7.5.1 Definition. Fix a finite set of positive numbers $\mathbf{b} = \{b_j\}$. Let (S, D) be an slc surface. Assume first that S does not contain any maximally log canonical points (i.e. it is sklt). Let

$$d_{\mathbf{b}}(S, D) = \sum_{m \in \mathbb{N}^+, r_j \in \mathbb{N}} \#\left\{E \middle| a(E, S, D) < -\left(1 - \frac{1}{m} + \sum \frac{r_j b_j}{m}\right)\right\}.$$

In general, if $Z \subset S$ is the set of maximally log canonical points then let

$$d_{\mathbf{b}}(S, D) \stackrel{\text{def}}{=} d_{\mathbf{b}}(S - \bar{Z}, D).$$

7.5.2 Lemma. *Let (S, D) be an slc surface. Then $d_{\mathbf{b}}(S, D) < \infty$.*

Proof. We may assume that S has no maximally log canonical points. Each of the summands in (7.5.1) is finite by (4.12.2). By (7.4.4) we have only finitely many nonzero summands. □

7.5.3 Lemma. *Let*

$$X_i \xrightarrow{\phi_i} Z_i \xleftarrow{\phi_i^+} X_{i+1}$$

be a flip. Assume that the flipping curve does not intersect $\pi_i(E_i)$. Let $\mathbf{b} = \{b_j\}$ be the set of coefficients of the irreducible components B_i. Then

$$d_{\mathbf{b}}(S_i, D_i) \geq d_{\mathbf{b}}(S_i^+, D_i^+) = d_{\mathbf{b}}(S_{i+1}, D_{i+1}).$$

Furthermore, the inequality is strict if S_i^+ contains a flipped curve.

Proof. Let $T_i = \phi_i(S_i)$. By construction we have morphisms $S_i \to T_i \leftarrow S_i^+ = S_{i+1}$. Furthermore, $-(K_{S_i} + D_i)$ is (S_i/T_i)-ample and $K_{S_i^+} + D_i^+$ is (S_i^+/T_i)-ample. If E is any divisor then by (2.28) $a(E, S_i, D_i) \leq a(E, S_i^+, D_i^+)$ which shows the first claim.

Assume that ϕ_i^+ is not an isomorphism. Let C^+ be an exceptional curve of ϕ_i^+. Then by (2.28) and (16.6.7)

$$a(C^+, S_i, D_i) < a(C^+, S_i^+, D_i^+) = -\left(1 - \frac{1}{m} + \sum \frac{r_j b_j}{m}\right)$$

for some $m, r_j \in \mathbb{N}$. Thus $d_{\mathbf{b}}(S_i, D_i) > d_{\mathbf{b}}(S_{i+1}, D_{i+1})$. □

Clearly (7.5.2) and (7.5.3) imply (7.5). □

7.6 Fourth Reduction Step. *Assume that no flipped curve is contained in $S_i^+ = \llcorner B_i^+ \lrcorner$. Then after finitely many flips no flipping curve is contained in $S_i = \llcorner B_i \lrcorner$.*

Proof. Using the notation of (7.5.3) we obtain that $T_i \cong S_i^+$ and $S_i \to T_i$ contracts a curve. Thus the Picard number of S_i decreases after a flip. This cannot be repeated infinitely many times. □

7.7 Proof of (7.1). By (7.5) and (7.6) after finitely many steps neither a flipping nor a flipped curve can be contained in the reduced part of the boundary. As in (4.13.3) this implies that the flipping curves are disjoint from the reduced part of the boundary. This completes the proof. □

8. LOG CANONICAL FLIPS

SEAN KEEL and JÁNOS KOLLÁR

The aim of this chapter is to prove the existence of log flips in the log canonical case. In (8.4) we extend this to a general base point freeness result for log canonical threefolds.

8.1 Theorem. *Let (Y, Δ) be log canonical and let $g : Y \to Z$ be a small contraction such that $K_Y + \Delta$ is g-negative. Then the flip of g exists.*

Proof. The problem is local on Z. Thus we may assume that Z is a neighborhood of a point $0 \in Z$ which we shrink if necessary without further comments. As in (2.34) we may assume that $\llcorner \Delta \lrcorner = \emptyset$. This somewhat simplifies the argument.

Let $Y' \to Y$ be a log resolution of Y. Let $\Delta' = \Delta_{Y'}$ (cf. (2.7)). Apply the log MMP to $(Y', \Delta') \to Z$. During the program every occurring pair (Y_i', Δ_i') is log terminal and $\mathbb{Q}$-factorial. Log flips exist and terminate by (6.15). Thus eventually the program stops with $f : (X, \Delta_X) \to Z$ such that $K_X + \Delta_X$ is f-nef, X is $\mathbb{Q}$-factorial and (X, Δ_X) is log terminal.

In general f is not an isomorphism over $Z - 0$. If $L \subset Z - 0$ is a curve along which (Z, Δ_Z) is not log terminal then f is not an isomorphism over L but gives a log terminal model. Therefore

$$(8.1.1) \qquad m_0(K_X + \Delta_X | X - f^{-1}(0)) \cong f^*(m_0(K_Z + \Delta_Z)|Z - 0)$$

for suitable $m_0 > 0$.

Let $h : V \to X$ be any resolution and let

$$K_V = h^*(K_X + \Delta_X) + \sum a_i F_i.$$

Since Y is lc, $a_i \geq -1$ for every i. Furthermore, by (2.23.3) X has the following property:

$$(8.1.2) \qquad \text{if } f \circ h(F_i) = 0 \in Z \text{ then } a_i > -1.$$

By (2.22.3) we need to show that $\mathcal{O}_X(n(K_X+\Delta_X))$ is generated by global sections for some $n>0$. The usual base point freeness theorem ([KMM87,3-1-2]) does not apply, since (X,Δ_X) is not klt.

Let $\Theta = \llcorner \Delta_X \lrcorner$. We want to modify our model X to achieve that $K_X+\Delta_X-\epsilon\Theta$ is f-nef for $1 \gg \epsilon > 0$. Let $p \in f(\Theta)$ be a generic point. Then $\operatorname{Spec}\mathcal{O}_{p,Z}$ is a log canonical surface singularity and $f: X \to Z$ is a log terminal model of $\operatorname{Spec}\mathcal{O}_{p,Z}$. From the list of Chapter 3 we see that Θ is negative semidefinite on general fibers of $\Theta \to Z$. This implies that a $(K_X+\Delta_X-\epsilon\Theta)$-extremal contraction never contracts a component of Θ.

Let $C = f^{-1}(0)$ (with reduced structure). Choose $1 \gg \epsilon > 0$. $K_X + \Delta_X - \epsilon\Theta$ is klt and if $B \subset C$ is an irreducible component such that $(K_X + \Delta_X - \epsilon\Theta)\cdot B < 0$ then $(K_X+\Delta_X)\cdot B = 0$. If $B \subset C$ and B generates a $(K_X+\Delta_X-\epsilon\Theta)$-extremal ray then the flip of B is a $(K_X+\Delta_X)$-flop. Therefore condition (8.1.2) still holds after such a flip and any sequence of such flips is finite by (6.11).

Thus (up to renaming) we may assume that $K_X+\Delta_X$ is lc, $K_X+\Delta_X-\epsilon\Theta$ is klt and f-nef for $1 \gg \epsilon > 0$. By [KMM87,3-1-2] there is an $m_1 > 0$ such that $m_1(K_X+\Delta_X-\epsilon\Theta)$ is f-base point free. Thus

$$m_1(K_X+\Delta_X) = m_1(K_X+\Delta_X-\epsilon\Theta) + m_1\epsilon\Theta$$

is base point free outside $\operatorname{Supp}\Theta$. Therefore it remains to prove base point freeness on Θ itself.

To this end consider the exact sequence

$$\begin{aligned}0 \to \mathcal{O}_X(m_1(K_X+\Delta_X)-\Theta) &\to \mathcal{O}_X(m_1(K_X+\Delta_X)) \\ &\to \mathcal{O}_\Theta(m_1(K_X+\Delta_X)|\Theta) \to 0.\end{aligned} \tag{8.1.3}$$

Observe that

$$m_1(K_X+\Delta_X)-\Theta \equiv K_X + (\Delta_X-\Theta) + (m_1-1)(K_X+\Delta_X),$$

and $K_X+(\Delta_X-\Theta)$ is klt by our assumptions. Thus $R^1f_*\mathcal{O}_X(m_1(K_X+\Delta_X)-\Theta)=0$ by [KMM87,1-2-6]. Therefore

$$f_*\mathcal{O}_X(m_1(K_X+\Delta_X)) \to f_*\mathcal{O}_\Theta(m_1(K_X+\Delta_X)|\Theta) \tag{8.1.4}$$

is surjective. Thus it is sufficient to prove that $\mathcal{O}_\Theta(m_1(K_X+\Delta_X)|\Theta)$ is generated by global sections for suitable $m_1>0$.

Let Θ_i be the irreducible components of Θ. By (8.1.2) we see that $\operatorname{Sing}\Theta_i$ and $\Theta_i \cap \Theta_j$ $(i \neq j)$ are finite over Z. (Otherwise we would get a divisor with discrepancy ≤ -1 lying over $\operatorname{Sing}\Theta_i$ or $\Theta_i\cap\Theta_j$.) By (8.1.1) $m_0(K_X+\Delta_X)|\Theta$ is linearly equivalent to a (not necessarily effective) divisor D supported on the fiber over $0 \in Z$. It is also nef, thus by (8.1.5) $m_2m_0(K_X+\Delta_X)|\Theta \sim 0$ for some $m_2>0$. By (8.1.4) the constant section of $\mathcal{O}_\Theta(m_0m_1m_2(K_X+\Delta_X)|\Theta)$ lifts to a section of $\mathcal{O}_X(m_0m_1m_2(K_X+\Delta_X))$ which is nowhere zero along Θ. $\square$

8.1.5 Claim. *Let $f : \Theta \to C$ be a proper morphism with connected fibers from a surface to a smooth affine curve. Assume that Θ is normal at all generic points of $f^{-1}(0)$. Let D be a (not necessarily effective) $\mathbb{Q}$-Cartier divisor supported on $f^{-1}(0)$. Assume that D is nef. Then $mD \sim 0$ for some $m > 0$.*

Proof. Let $\operatorname{Supp} f^{-1}(0) = \cup C_i$ and let $D = \sum d_i C_i$. By adding a suitable multiple of $f^{-1}(0)$ to D we may assume that

$$D + \frac{a}{b}[f^{-1}(0)] = \sum d_i' C_i \quad \text{where} \quad d_i' \leq 0,$$

with equality holding for at least one index i. Since D is nef, this implies that $d_i' = 0$ for every i. Thus $bD \sim -a[f^{-1}(0)]$. □

We are now ready to put the termination of flips in the following final form, due to Matsuki and Mori.

8.2 Theorem. *Let (X, B) be a log canonical threefold. Then any sequence of $(K + B)$-log flips is finite.*

Proof. The case when X is $\mathbb{Q}$-factorial and log terminal was done in (6.15). Next assume that X is lc and let

$$(8.2.1) \qquad \begin{array}{ccccccccc} (X_0, B_0) & & \dashrightarrow & & (X_1, B_1) & & \dashrightarrow & & (X_2, B_2) & \dashrightarrow \\ & \phi_0 \searrow & & \swarrow \phi_0^+ & & \phi_1 \searrow & & \swarrow \phi_1^+ & & \phi_2 \searrow & \cdots \\ & & Z_0 & & & & Z_1 & & & & Z_2 \end{array}$$

be a sequence of flips.

Let $q_0 : (Y_0, D_0) \to (X_0, B_0)$ be a $\mathbb{Q}$-factorial log terminal model as in (8.2.2). $K + D_0 = q_0^*(K + B_0)$, thus $K + D_0$ is log terminal and not nef on Y_0/Z_0. There is a sequence of divisorial contractions and flips (whose existence and termination is guaranteed by (6.15)) such that at the end we obtain $(Y_1, D_1) \to Z_0$ such that $K + D_1$ is log terminal and relatively nef. By definition, $(Y_1, D_1) \to Z_0$ is a weak log canonical model (2.21) of $Y_0 \to Z_0$. Thus by (2.22.3) there is a morphism $q_1 : Y_1 \to X_1$ such that $K + D_1 = q_1^*(K + B_1)$. We can continue as before using $Y_1 \to X_1 \to Z_1$. This way a sequence of flips on X lifts to a sequence of flips and divisorial contractions on Y_0. By (6.15) the sequence terminates on Y_0, hence the sequence of flips (8.2.1) is also finite. □

8.2.2 Lemma. *Let (X, B) be an lc threefold. Then there is a projective morphism $q : (Y, D) \to (X, B)$ such that (Y, D) is $\mathbb{Q}$-factorial, log terminal and $K + D = q^*(K + B)$.*

Proof. Let $f : X' \to X$ be a log resolution of (X, B) with reduced exceptional divisor E. Apply the $(K + f_*^{-1}(B) + E)$-MMP on X'/X. By (6.15) all the steps exist and the program terminates with $q : (Y, D) \to (X, B)$ such that $K + D$ is q-nef. Thus $q : Y \to X$ is a weak log canonical model of (X, B). Since (X, B) is lc, it is its own log canonical model, hence by (2.22.3) $K + D = q^*(K + B)$. $\square$

The method of (8.1) can be generalized to yield the finite generation of log canonical rings for threefolds (X, Δ) of log general type. This is the $\kappa = 3$ part of the Abundance Conjecture for lc threefolds. Most of the proof involves analysis of semi log canonical surfaces, therefore it should be read after Chapter 12.

If (X, Δ) is klt then the result is a special case of base point freeness. [Kawamata91d] settled the lc case under some technical assumptions.

8.3 Definition. Let X be a proper and irreducible variety over a field. Let L be a line bundle on X. We say that L is *big* if there is an $\epsilon > 0$ such that

$$h^0(X, L^m) > \epsilon m^{\dim X} \quad \text{for } m \gg 1.$$

(8.3.2) Let $f : X \to Z$ be a proper morphism; X irreducible. Let L be a line bundle on X. We say that L is f-*big* if L is big on the fiber of f over the generic point of $f(X)$.

Thus if f is generically finite then every line bundle is f-big.

(8.3.3) Let (X, B) be proper, irreducible and lc. We say that it is of *log general type* if $K + B$ is big.

8.4 Theorem. *Let X be an irreducible threefold and let Δ be an effective $\mathbb{Q}$-divisor on X. Assume that $K_X + \Delta$ is log canonical. Let $f : X \to Z$ be a proper morphism and assume that $K_X + \Delta$ is f-nef and f-big. Then $m(K_X + \Delta)$ is f-base point free for suitable $m > 0$. Thus*

$$\sum_{s=0}^{\infty} f_* \mathcal{O}_X(s(K_X + \Delta)) \quad \textit{is a finitely generated } \mathcal{O}_Z\textit{-algebra.}$$

Proof. The proof is similar to the proof of (8.1). As a first step we reduce the problem to abundance on $\llcorner \Delta \lrcorner$. This was already done in [Kawamata91d]. Here we present another proof in the spirit of (8.1) which however uses more.

First we take a log terminal model $h : X' \to X$ to obtain (X', Δ'). As in (8.1), after some contractions and flips we obtain $f'' : (X'', \Delta'') \to Z$ such that

(8.4.1) (X'', Δ'') is lc; and

(8.4.2) $K_{X''} + \Delta'' - \epsilon \llcorner \Delta'' \lrcorner$ is klt, f''-nef and f''-big for $1 \gg \epsilon > 0$.

(Here we can not exclude the possibility that we contract a component of $\llcorner\Delta'\lrcorner$.) From now on we drop the $''$ from our notation. Let $\Theta = \llcorner\Delta\lrcorner$. As in the proof of (8.1) we obtain that

(8.4.3) $m_1(K+\Delta)$ is f-base point free outside $\operatorname{Supp}\Theta$ for suitable $m_1 > 0$, and

(8.4.4) $f_*\mathcal{O}_X(m_1(K+\Delta)) \to f_*\mathcal{O}_\Theta(m_1(K+\Delta)|\Theta)$ is surjective.

Therefore (8.4) is implied by the following (just set $S = \Theta$ and $\Delta = \operatorname{Diff}_S(\Delta - \Theta)$):

8.5 Theorem. *Let S be a reduced surface and let Δ be a $\mathbb{Q}$-Weil divisor on S. Let $f : S \to Z$ be a proper morphism; Z affine. Assume that $K_S + \Delta$ is $\mathbb{Q}$-Cartier, f-nef and semi log canonical.*

Then the linear system $|m(K_S + \Delta)|$ is base point free for suitable $m > 0$.

Proof. Most of the work is done in Chapter 12 where this is established under the additional assumption that Z =point and $(K_S + \Delta)^2 = 0$. We use the notation and terminology of Chapter 12. As in (12.4) we may assume that S is semismooth.

Let $D \subset S$ be the union of those double curves which are

(i) either contained in at least one irreducible componet of S on which $K_S + \Delta$ is f-big;

(ii) or proper over Z and contained in a nonproper component of S.

Let $p : \bar{S} \to S$ be the surface obtained by blowing up D. The connected components of $\bar{S}$ are as follows:

(8.5.1) One (not necessarily connected) proper, smooth and semi log canonical surface (X, Θ) such that $K + \Theta$ is f-big on every component;

(8.5.2) One (not necessarily connected) proper semi log canonical surface (Y_1, Ξ_1) such that $(K + \Xi_1)^2 = 0$.

(8.5.2) One (not necessarily connected) surface Y_2 whose irreducible components are not proper and the restricion of $K + \Delta$ is not f-big on any component. Clearly, Y_2 satisfies the assumptions of (12.4.7.1), where B is the normalisation of $f(Y_2)$.

Let $Y = Y_1 \cup Y_2$. Let $D_X = p^{-1}(D)| \cup X_i$ and $D_Y = p^{-1}(D)|Y$. We can decompose $D = D^1 \cup D^2 \cup D^3$ where D^1 is the union of those curves whose preimages under p are both in X, D^3 is the union of those curves whose preimages under p are both in Y, and D^2 are the rest. Together with the morphism p these fit in the following diagram:

$$\begin{array}{ccccc} D_X & = & D^1_X \cup D^2_X & & D^2_Y \\ & 2:1 \downarrow & \text{bir} \searrow & & \swarrow \text{bir} \\ & D^1 & & D^2 & \end{array}$$

where the arrows marked bir are birational. Finally let $C_X = \cup \llcorner \Theta_i \lrcorner$.

By (12.1.1) and (12.4.7.1) abundance holds for (Y_i, Ξ_i). For the other components we use the following:

8.6 Lemma. *Let (X,Θ) be an irreducible and log canonical surface and $f : X \to Z$ a proper morphism; Z affine. Assume that $K+\Theta$ is f-nef and f-big. Let $C = \llcorner \Theta \lrcorner$. Let $m > 0$ be such that $m(K+\Theta)$ is Cartier and let*

$$s_i \in H^0(C, \mathcal{O}(m(K_X+\Theta)|C))$$

be sections without common zeros. Let $x \in X$ be an arbitrary point.

Then there is an $r > 0$ and a section $s \in H^0(X, \mathcal{O}(rm(K_X+\Theta)))$ such that

(8.6.1) $s(x) \neq 0$;

(8.6.2) the image of s under the restriction map

$$\operatorname{res} : H^0(X, \mathcal{O}(rm(K_X+\Theta))) \to H^0(C, \mathcal{O}(rm(K_X+\Theta)|C))$$

is one of the sections s_i^r.

Proof. Let us prove first that $k(K+\Theta)$ is base point free for some $k > 0$. As before, we may assume that $K+\Theta-\epsilon C$ is klt and nef for $1 \gg \epsilon > 0$. Thus $k(K+\Theta)$ is base point free outside C and we are reduced to establishing base point freeness for $(C, (K+\Theta)|C)$. $(K+\Theta)|C = K_C + \operatorname{Diff}(\Theta - C)$, hence base point freeness holds by (12.2.11). Thus we obtain base point freeness for $k(K+\Theta)$. This gives a factorisation

$$f : (X,\Theta) \xrightarrow{h} (X',\Theta') \xrightarrow{f'} Z,$$

such that $k(K+\Theta')$ is an f'-ample Cartier divisor and $k(K+\Theta) = h^*(k(K+\Theta'))$.

Assume first that $h(x) = h(c)$ for some $c \in C$. Choose s_i such that $s_i(c) \neq 0$. As in (8.1.4) res is surjective, thus there is $s \in H^0(X, \mathcal{O}(m(K_X+\Theta)))$ such that $\operatorname{res}(s) = s_i$. Since s pulls back from X', we conclude that $s(x) \neq 0$.

If $h(x)$ and $h(C)$ are disjoint, choose r large enough so that in the following diagram the horizontal arrows are surjective ($\mathbb{C}(x)$ is the residue field of $x \in X$):

$$\begin{array}{ccc}
f'_*\mathcal{O}(rm(K+\Theta')) & \longrightarrow & \mathbb{C}(x') + f'_*\mathcal{O}(rm(K+\Theta')|h(C)) \\
\downarrow \cong & & \downarrow \cong \\
f_*\mathcal{O}(rm(K+\Theta)) & \longrightarrow & \mathbb{C}(x) + f_*\mathcal{O}(rm(K+\Theta)|C).
\end{array}$$

Thus any of the s_i^{rk} can be lifted to a suitable s. $\square$

(8.7) Proof of (8.5). By (12.1.1) and (12.4.7.1) $\mathcal{O}(m(K_Y+\Xi))$ is generated by normalised sections σ_j for suitable $m>0$. These restrict to normalized sections

$$\sigma_j|D_Y^2 \in H^0(D_Y^2, \mathcal{O}(mp^*(K_S+\Delta)|D_Y^2)) = H^0(D_Y^2, \mathcal{O}(m(K_Y+\Xi)|D_Y^2)).$$

Thus $\sigma_j|D_Y^2$ induces a normalised section

$$\rho_j \in H^0(D_X^2, \mathcal{O}(mp^*(K_S+\Delta)|D_X^2)).$$

On D_X^1 we can choose normalised sections

$$\tau_k \in H^0(D_X^1, \mathcal{O}(m(K_S+\Delta)|D_X^1))$$

which have no common zeros (if necessary we may replace m by $12m$). By (12.2.11) ρ_j and $p^*\tau_k$ extend to a normalised section

$$s_{jk} \in H^0(C_X, \mathcal{O}(mp^*(K_S+\Delta)|C_X)),$$

and we may assume that the s_{jk} have no common zeros (this may require several extensions for each pair (j,k) but we ignore this in the notation).

Finally by (8.6) we can extend these to sections

$$s_{jk}^r \in H^0(X, \mathcal{O}(mrp^*(K_S+\Delta)))$$

such that the s_{jk}^r have no common zeros (we may assume that r does not depend on j,k). By construction s_{jk}^r and σ_j^r glue together to sections of $\mathcal{O}(mr(K_S+\Delta))$ without common zeros. $\square$

9. MIYAOKA'S THEOREMS ON THE GENERIC SEMINEGATIVITY OF T_X AND ON THE KODAIRA DIMENSION OF MINIMAL REGULAR THREEFOLDS.

N.I. SHEPHERD-BARRON

9.0 Introduction

In this chapter the aim is to prove the following results of [Miyaoka87a,88a], concerning normal complex projective varieties X.

9.0.1 Theorem. *If X is not uniruled, then Ω^1_X is generically semipositive (equivalently, T_X is generically seminegative).*

In recalling what this means, we use the following notation, which will be fixed throughout this chapter:

X : a normal projective n-fold.

$H_1, \dots, H_{n-1}, H$: ample divisors on X.

$\{C_t\}_{t\in S}$: the complete family of curves of the form $D_1 \cap \dots \cap D_{n-1}$, where $D_i \in |m_i H_i|$ and $m_i \gg 0$.

C : a geometric generic member of $\{C_t\}$.

Then Ω^1_X is *generically semipositive* if every torsion free quotient of $\Omega^1_X|_C$ has nonnegative degree.

This result follows immediately from the next result.

9.0.2 Theorem. *Assume that there is a subsheaf $\mathcal{E} \subset T_X$ such that $c_1(\mathcal{E}) \cdot C > 0$. Then there is a saturated $\mathcal{F} \subset T_X$ such that $c_1(\mathcal{F}) \cdot C > 0$ and there is a rational curve M through a generic point x of X such that*

(i) M is smooth at x,

(ii) $T_M(x) \hookrightarrow \mathcal{F}(x)$ and

(iii) $(H \cdot M) \leq 2n(H \cdot C)/(c_1(\mathcal{F}) \cdot C)$.

This result can be extended. Define a variety to be *rationally chain connected* if two general points on it can be joined by a chain of rational curves.

9.0.3 Theorem. *Assume the hypotheses and notation of (9.0.2). Then the sheaf $\mathcal{F}$ defines a foliation on X whose leaves are compact and are rationally chain connected varieties.*

Remark. By abuse of language, we confuse the notions of foliation and integrable distribution, and we say that a foliation with singularities has compact leaves if the closure of each leaf is a projective variety that contains the leaf as a Zariski open subset.

One of the main consequences of (9.0.1) concerns the second Chern class. Recall that X is minimal if it has only terminal singularities and K_X is nef.

9.0.4 Theorem. *Suppose that X is minimal and that $\rho : Y \to X$ is a resolution. Then $c_2(Y) \cdot \rho^* H_1 \cdot \dots \cdot \rho^* H_{n-2} \geq 0$.*

Apart from (9.0.1) to prove (9.0.4) we need a consequence of Bogomolov's theorem on unstable vector bundles, which is proved in Chapter 10.

9.0.5 Theorem. *Suppose that $\mathcal{E}$ is a reflexive sheaf on Y. Put $\mathcal{F} = (\rho_* \mathcal{E})^{\vee\vee}$, and assume that $\mathcal{F}$ is generically semipositive and that $c_1(\mathcal{F})(= \rho_* c_1(\mathcal{E}))$ is nef. Then $c_2(\mathcal{E}) \cdot \rho^* H_1 \cdot \dots \cdot \rho^* H_{n-2} \geq 0$.*

Then we deduce

9.0.6 Theorem. *If X is a minimal regular threefold, then $\kappa(X) \geq 0$.*

In the course of proving this, we shall assume the corresponding result for irregular threefolds. For this, we refer to [Ueno82] and [Viehweg80].

As we said above, all of these theorems are due to Miyaoka, and indeed our proofs of (9.0.4) and (9.0.6) follow his very closely (except for a slightly slicker use of Donaldson's theorem [Donaldson85] on stable bundles with trivial Chern classes, which was suggested by conversations with Kollár and Kotschick). However, Miyaoka's proof of (9.0.2) uses his theory of deformations of morphisms along foliations [Miyaoka87a], whereas our proof seems to be considerably simpler.

9.1 **Foliations**

In this section we prove

9.0.2(bis) Theorem. *Assume that $\mathcal{F} \hookrightarrow T_X$ is a subsheaf such that $\mathcal{F}$ is a piece of the Harder-Narasimhan filtration of T_X and $\mu_{min}(\mathcal{F}|_C) > 0$. (This notation is explained in (9.1.1).)*

Then through a geometric generic point x of X, there is a rational curve M such that

(i) M is smooth at x,

(ii) $T_M(x) \subseteq \mathcal{F}(x)$ and

(iii) $(H \cdot M) \leq 2n(H \cdot C)/(c_1(\mathcal{F}) \cdot C)$.

To prove this we carry out the following steps:
(1) Show that $\mathcal{F}$ is closed under Lie bracket. (2) Reduce X modulo p. (3) Show that $\mathcal{F}$ is closed under Lie bracket and taking pth powers, for $p \gg 0$. (4) Divide X by $\mathcal{F}$, giving a purely inseparable morphism $\rho : X \to Y = X/\mathcal{F}$. (5) Note that $(\rho^* c_1(Y) \cdot C) > 0$ for $p \gg 0$. (6) Find rational curves on Y. (7) Pull them back to rational curves on X. (8) Lift back to characteristic zero, and check the conclusions of (9.0.2(bis)).

9.1.1 **Some facts about vector bundles**

We collect here, without proofs, some well known definitions and theorems about vector bundles. (See e.g. [Seshadri82, Part 1] for an introduction over curves and [Siu87, Chapter 1] for the higher dimensional properties.)

Let g denote the genus of the curve C above.

9.1.1.1 Suppose that $\mathcal{E}$ is a vector bundle on C. Write $\mu(\mathcal{E}) = \deg(\mathcal{E})/rk(\mathcal{E})$. Then there is a unique filtration (the *Harder–Narasimhan filtration* or H.-N. filtration) of $\mathcal{E}$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that if $\mathcal{G}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, then $\mathcal{G}_i$ is a semistable vector bundle and

$$\mu(\mathcal{G}_1) > \ldots > \mu(\mathcal{G}_r).$$

Write $\mu(\mathcal{G}_1) = \mu_{max}(\mathcal{E})$ and $\mu(\mathcal{G}_r) = \mu_{min}(\mathcal{E})$.

9.1.1.2 If $\mathcal{A}$ and $\mathcal{B}$ are vector bundles on C and $\mu_{min}(\mathcal{A}) > \mu_{max}(\mathcal{B})$, then $\operatorname{Hom}(\mathcal{A}, \mathcal{B}) = 0$.

9.1.1.3 $\mu(\mathcal{A} \otimes \mathcal{B}) = \mu(\mathcal{A}) + \mu(\mathcal{B})$ and $\mu(\bigwedge^2 \mathcal{A}) = 2\mu(\mathcal{A})$.

9.1.1.4 (char $= 0$) If $\mathcal{A}$ and $\mathcal{B}$ are semistable, then so are $\bigwedge^2 \mathcal{A}$ and $\mathcal{A} \otimes \mathcal{B}$.

N.B. In characteristic $p > 0$, tensor bundles of semistable bundles can be unstable.

9.1.1.5 For any vector bundle $\mathcal{E}$ on C, the tensor product $\mathcal{E} \otimes \mathcal{O}(A)$ is generated by its sections if $A \in \operatorname{Pic} C$ with $\deg A > 2g - 1 - \mu_{min}(\mathcal{E})$.

9.1.1.6 Given a general point x of $\mathbb{P}(\mathcal{E})$, there is a section C' of $\mathbb{P}(\mathcal{E})$ passing through x, where C' corresponds to a surjection $\mathcal{E} \to \mathcal{L}$ with $\mathcal{L} \in \operatorname{Pic} C$ and

$$\deg \mathcal{L} = \deg \mathcal{E} \ + \ (\operatorname{rk} \mathcal{E} \ - 1) \cdot \ulcorner (2g - \mu_{min}(\mathcal{E})) \urcorner.$$

This follows immediately from (9.1.1.5), by considering the tautological linear system on a suitable $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}(A)$.

9.1.1.7 [Mehta-Ramanathan82] If $\mathcal{E}$ is a torsion-free sheaf on X, then there is a unique filtration (the *Harder–Narasimhan filtration*)

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

whose graded pieces are torsion-free and which restricts to the Harder–Narasimhan filtration of $\mathcal{E}|_C$ (cf. [ibid, Introduction and 6.1]).

9.1.1.8 If $X \to S$ is a family of varieties, where S is a scheme of finite type over a field or excellent Dedekind domain, and $\mathcal{E}$ is a torsion-free sheaf on X, flat over S, then the points $s \in S$ such that $\mathcal{E}_s$ is a stable (resp. semistable) torsion-free sheaf on X_s form an open subset of S.

9.1.2 **Foliations in positive characteristic**

All we need is what follows. For most of the proofs, we refer to [Ekedahl87], see esp. [ibid,2.4,3.4,4.2].

9.1.2.1 Proposition. *Given a normal variety X in char. $p > 0$, there is a one-to-one correspondence between*

(A) factorizations $X \xrightarrow{\rho} Y \xrightarrow{\sigma} X^{(1)}$ of the geometric Frobenius morphism $F_X : X \to X^{(1)}$, where $\deg \rho = p^r$, and

(B) saturated coherent subsheaves $\mathcal{F} \hookrightarrow T_X$ such that $\operatorname{rk} \mathcal{F} = r$, $\mathcal{F}$ is closed under Lie bracket and $\mathcal{F}$ is closed under pth powers. □

Such an $\mathcal{F}$ is a *1-foliation.* Write $Y = X/\mathcal{F}$. (Given $\mathcal{F}$, we have $\mathcal{O}_Y = \mathcal{O}_X^{\mathcal{F}}$, the algebra of functions annihilated by $\mathcal{F}$. Given ρ, we get $\mathcal{F} = \ker d\rho$.) If X is smooth, then Y is smooth if and only if $\mathcal{F}$ is a subbundle of T_X.

9.1.2.2 Proposition. *Suppose that $\mathcal{F} \hookrightarrow T_X$ is a saturated subsheaf. Then $\mathcal{F}$ is a 1-foliation if*

(i) $\operatorname{Hom}_{\mathcal{O}_U}(\bigwedge^2 \mathcal{F}, T_X/\mathcal{F}) = 0$ and

(ii) $\operatorname{Hom}_{\mathcal{O}_U}(F^*\mathcal{F}, T_X/\mathcal{F}) = 0$,

where F is the absolute Frobenius and U is the locus where both X is smooth and $\mathcal{F}$ is a subbundle. □

9.1.2.3 Proposition. *Suppose that $\mathcal{F} \hookrightarrow T_X$ is a 1-foliation. Put $\mathcal{G} = T_X/\mathcal{F}$, and let $\rho : X \to Y = X/\mathcal{F}$ be the quotient by $\mathcal{F}$. Then $\rho^* c_1(Y) = p \cdot c_1(\mathcal{F}) + c_1(\mathcal{G})$.*

Proof. We have a factorization of F_X as in (9.1.2.1(A)) and exact sequences

$$0 \to \mathcal{F} \to T_X \to \mathcal{G} \to 0,$$

$$0 \to \mathcal{A} \to T_Y \to \mathcal{B} \to 0,$$

say, where $\mathcal{G} \cong \operatorname{im}(T_X \to \rho^* T_Y) = \rho^* \mathcal{A}$ and $\mathcal{B} \cong \operatorname{im}(T_Y \to \sigma^* T_{X^{(1)}}) = \sigma^* \mathcal{F}^{(1)}$. ($X$ and $X^{(1)}$ are conjugate k-varieties; if $\mathcal{Z}$ is a sheaf on X, then $\mathcal{Z}^{(1)}$ is its conjugate on $X^{(1)}$.) So

$$\rho^* c_1(Y) = \rho^*(c_1(\mathcal{A}) + c_1(\mathcal{B})) = \rho^*(c_1(\mathcal{A}) + \sigma^* c_1(\mathcal{F}^{(1)})) = c_1(\mathcal{G}) + p \cdot c_1(\mathcal{F}),$$

since $c_1(F_X^* \mathcal{Z}^{(1)}) = p \cdot c_1(\mathcal{Z})$ for any sheaf $\mathcal{Z}$ on X. □

9.1.3 **Proof of (9.0.2(bis)) (first step)**
We suppose given

$$0 \to \mathcal{F} \to T_X \to \mathcal{G} \to 0$$

such that $\mathcal{F}$ is a term in the H.-N. filtration of T_X, and $\mu_{min}(\mathcal{F}|_C) > 0$ (so that in particular, $(c_1(\mathcal{F}) \cdot C) > 0$).

9.1.3.1 Lemma. *(char. $= 0$) $\mathcal{F}$ is closed under Lie bracket.*

Proof. Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are the composition factors in the H.-N. filtration of $\mathcal{F}|_C$, with $\mu(\mathcal{A}_1) > \ldots > \mu(\mathcal{A}_m)$. Then the composition factors in the H.-N. filtration of $\bigwedge^2 \mathcal{F}|_C$ are all of the form $\mathcal{A}_i \otimes \mathcal{A}_j$ or $\bigwedge^2 \mathcal{A}_i$, so that $\mu_{min}(\bigwedge^2 \mathcal{F}_C) = 2\mu_{min}(\mathcal{F}|_C)$. Hence $\mu_{min}(\bigwedge^2 \mathcal{F}_C) > \mu_{max}(\mathcal{G}|_C)$, so that $\mathrm{Hom}(\bigwedge^2 \mathcal{F}|_C, \mathcal{G}|_C) = 0$ and then $\mathrm{Hom}(\bigwedge^2 \mathcal{F}, \mathcal{G}) = 0$. □

Now reduce modulo p. (This will hold until the end of (9.1.3).)

9.1.3.2 Lemma. *(char $= p > 0$) $\mathcal{F}$ is closed under Lie bracket.*

Proof. Immediate from (9.1.3.1), by specialization. □

To prove that $\mathcal{F}$ is closed under pth powers, it would be enough to know that $F^*\mathcal{F}|_C$ is semistable. Unfortunately, this need not be true. However, the following result will suffice.

9.1.3.3 Proposition. *(char $= p > 0$) Suppose that $\mathcal{E}$ is a semistable vector bundle of rank r over a curve C of genus g, such that $F^*\mathcal{E} = \tilde{\mathcal{E}}$, say, is unstable. Then*

$$\mu_{max}(\tilde{\mathcal{E}}) - \mu_{min}(\tilde{\mathcal{E}}) \leq (r^r - 1)(2g + 1)r/(r - 1).$$

([Lange-Stuhler77] have already found such a bound when $r = 2$.)

Proof. Recall first that $F^*\mathcal{E} = F_C^*\mathcal{E}^{(1)}$. Suppose that

$$0 \to \mathcal{A} \to \tilde{\mathcal{E}} \to \mathcal{B} \to 0$$

fits into the H.-N. filtration of $\tilde{\mathcal{E}}$. So $\mu_{min}(\mathcal{A}) > \mu_{max}(\mathcal{B})$. Put $\mathbb{P}_1 = \mathbb{P}(\mathcal{B})$, $\tilde{\mathbb{P}} = \mathbb{P}(\tilde{\mathcal{E}})$, $\mathbb{P} = \mathbb{P}(\mathcal{E}^{(1)})$. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{P}_1 & \xrightarrow{\iota} & \tilde{\mathbb{P}} & \xrightarrow{\rho} & \mathbb{P} \\ & & \tilde{\pi}\downarrow & & \downarrow\pi \\ & & C & \xrightarrow{F_C} & C^{(1)} \end{array}$$

where the square is Cartesian and ι is the natural inclusion.

Let σ denote the composite $\mathbb{P}_1 \to C$.

By (9.1.2.2), there is a line subbundle $\mathcal{H}$ of $T_{\tilde{\mathbb{P}}}$ which is a foliation, such that $\mathbb{P} = \tilde{\mathbb{P}}/\mathcal{H}$. Since the square is Cartesian, we see that $\mathcal{H} \to \tilde{\pi}^* T_C$ is an isomorphism.

Now if $\mathcal{H}|_{\mathbb{P}_1} \hookrightarrow T_{\mathbb{P}_1}$, then $\mathcal{H}|_{\mathbb{P}_1}$ is a nonsingular foliation on $\mathbb{P}_1$, and $\mathbb{P}_1$ maps p-to-1 to its image in $\mathbb{P}$, giving a subscroll of $\mathbb{P}$ that destabilizes $\mathcal{E}^{(1)}$. Hence $\mathcal{H}|_{\mathbb{P}_1} \hookrightarrow \mathcal{N}_{\mathbb{P}_1/\tilde{\mathbb{P}}}$, so that

$$(\circledast) \qquad \mathcal{H}|_{\mathbb{P}_1} \hookrightarrow \mathcal{O}_{\mathbb{P}_1}(1) \otimes \sigma^* \mathcal{A}^{\vee}.$$

By (9.1.1.6), there is a section $C' \hookrightarrow \mathbb{P}_1$ in general position such that

$$\deg(\mathcal{O}_{\mathbb{P}_1}(1)|_{C'}) = \deg \mathcal{B} + (\operatorname{rk} \mathcal{B} - 1).\ulcorner 2g - \mu_{min}(\mathcal{B}) \urcorner.$$

Restricting $\circledast$ to C', we get $T_{C'} \hookrightarrow (\mathcal{O}_{\mathbb{P}_1}(1)|_{C'}) \otimes \mathcal{A}^{\vee}$. Hence

$$2 - 2g \leq \deg(\mathcal{B}) + (\operatorname{rk}(\mathcal{B}) - 1) \cdot \ulcorner 2g - \mu_{min}(\mathcal{B}) \urcorner + \mu_{max}(\mathcal{A}^{\vee}).$$

Therefore

$$\begin{aligned} 2 - 2g &\leq \deg(\mathcal{B}) + (\operatorname{rk}(\mathcal{B}) - 1)(2g + 1 - \mu_{min}(\mathcal{B})) - \mu_{min}(\mathcal{A}), \\ 0 &\leq \operatorname{rk}(\mathcal{B}) \cdot (\mu(\mathcal{B}) - \mu_{min}(\mathcal{B}) + 2g + 1) + \mu_{min}(\mathcal{B}) - \mu_{min}(\mathcal{A}), \text{ and} \\ \mu_{min}(\mathcal{A}) - \mu_{min}(\mathcal{B}) &\leq r \cdot (\mu_{max}(\mathcal{B}) - \mu_{min}(\mathcal{B}) + 2g + 1). \end{aligned}$$

Let

$$\mu_{max}(\tilde{\mathcal{E}}) = \mu_1 > \mu_2 > \ldots > \mu_m = \mu_{min}(\tilde{\mathcal{E}})$$

be the slopes of the composition factors in the Harder-Narasimhan filtration of $\tilde{\mathcal{E}}$, so that also $\mu_{min}(\mathcal{B}) = \mu_m$.

Put $M_i = \mu_i - \mu_m$. Then we get

$$M_i \leq r \cdot (M_{i+1} + 2g + 1),$$

which leads by descending induction on i to

$$M_1 \leq (r^r - 1) r (2g + 1)/(r - 1),$$

as stated. $\square$

9.1.3.4 Remark. This bound is clearly crude. Its virtue, however, is that it is independent of p.

9.1.3.5 Proposition. *$\mathcal{F}$ is p-closed if $p \gg 0$.*

Proof. $\mu(F^*\mathcal{F}|_C) = p \cdot \mu(\mathcal{F}|_C) \geq p$. Then if $p \gg 0$, we have, by (9.1.3.3), $\mu_{min}(F^*\mathcal{F}|_C) > \mu_{max}(\mathcal{G}|_C)$. Hence $\mathrm{Hom}_{\mathcal{O}_X}(F^*\mathcal{F}, \mathcal{G}) = 0$ for $p \gg 0$. □

Now let $\rho : X \to Y = X/\mathcal{F}$ be the quotient by $\mathcal{F}$ ($p \gg 0$). There is $G \in \operatorname{Pic} Y$ such that $\rho^*G = pH$, and G is ample. By (9.1.2.3),

$$C \cdot \rho^* c_1(Y) = p \cdot (C \cdot c_1(\mathcal{F})) + (C \cdot c_1(\mathcal{G}))$$

Put $(C \cdot c_1(\mathcal{F})) = \gamma$. For all β with $0 < \beta < \gamma$, we have $(C \cdot \rho^* c_1(Y)) \geq \beta \cdot p$ for $p \gg 0$. Let $f : C \to Y$ be the composite. Then

$$\dim_{[f]} \operatorname{Mor}(C, Y) \geq \beta \cdot p + n(1-g),$$

so that for every $b \in \mathbb{N}$ with $\beta \cdot p + n(1-g) - bn > 0$ and for every subscheme $B \subset C$ of length b, we can deform f nontrivially, keeping B fixed.

Then by [Miyaoka-Mori86, Theorem 4], through a general point of $f(C)$ there is a rational curve L such that

$$G \cdot L \leq 2 \deg(f^*G)/(\beta p - g).$$

(N.B. [Miyaoka-Mori86, Theorem 4] is stated for morphisms $f : C \to X$ where X is projective and smooth. However, the proof given there carries over verbatim to the case where X is allowed to be singular, provided that $f(C)$ lies in the smooth locus of X.)

Hence for any α with $0 < \alpha < \beta$, we have

$$L \cdot G \leq 2n \cdot \deg(f^*G)/\alpha p = 2n(C \cdot H),$$

independently of p (provided that $p \gg 0$).

Since ρ is purely inseparable, L pulls back to give a rational curve M through a general point x of X. □

9.1.3.6 Lemma. *$M \to L$ is purely inseparable.*

Proof. If not, then $M \to L$ is birational. Then

$$p(M \cdot H) = M \cdot \rho^*G = L \cdot G \leq 2n(C \cdot H)/\alpha.$$

This is absurd for $p \gg 0$. □

9.1.3.7 Lemma. $M \cdot H \leq 2n(H \cdot C)/\alpha$.

Proof. By (9.1.3.6), $p(M \cdot H) = M \cdot \rho^* G = p(L \cdot G)$. Then $M \cdot H \leq 2n(C \cdot H)/\alpha$. □

9.1.4 **Conclusion of proof of (9.0.2(bis))**

By (9.1.3.7) and the properties of the Hilbert scheme, through a general point x of X (in characteristic zero) there is a rational curve M with $M \cdot H \leq 2n(C \cdot H)/\alpha$.

This holds for all α with $0 < \alpha < C \cdot c_1(\mathcal{F})$, and so

$$M \cdot H \leq 2n(C \cdot H)/(C \cdot c_1(\mathcal{F})).$$

The final thing to check is that $T_M(x) \hookrightarrow \mathcal{F}(x)$ for general $x \in M$. This can be checked after reduction modulo p, for all $p \gg 0$, and now it is equivalent to (9.1.3.6) □

9.1.5 **Proof of (9.0.2).**

Given $\mathcal{E} \hookrightarrow T_X$ with $c_1(\mathcal{E}) \cdot C > 0$, we certainly have $\mu_{max}(T_X|_C) > 0$. Hence we can take as $\mathcal{F}$ any term in the Harder- Narasimhan filtration of T_X such that $\mu_{min}(\mathcal{F}|_C) > 0$. □

9.1.6 **Compactifying the leaves of $\mathcal{F}$.**

The classical theorem of Frobenius et al. shows that, given $\mathcal{F} \hookrightarrow T_X$ closed under Lie bracket, the leaves of $\mathcal{F}$ exist locally *analytically* away from the singularities of X and of $\mathcal{F}$. That is, locally analytically there is a morphism $\rho : X \to Y$ with $\mathcal{F} = \ker d\rho$. In general the leaves are not compact; however, we now show (9.0.3) which says that if $\mathcal{F}$ is positive in the above sense of Miyaoka, then the rational curves that have been constructed tangent to $\mathcal{F}$ can be bundled together to give compact leaves of $\mathcal{F}$.

Proof of (9.0.3). Consider the family $\{M_t\}$ of rational curves tangent to $\mathcal{F}$ constructed above. Pick a geometric generic point ξ of X. Define inductively an ascending chain of subvarieties V_i of X, as follows:

$V_0 = \{\xi\}$, and for $i > 0$ V_i is an irreducible component of the scheme swept out by those curves M_t passing through a general point of V_{i-1}.

Let m denote the least value of i such that $V_i = V_{i+1}$, and put $V_m = V$.

9.1.6.1 Lemma. *V is tangent at its generic point to $\mathcal{F}$.*

Proof. V is covered by curves M_t, so that if η is a geometric generic point of V the generic curve M_t through η is smooth there, and the tangent lines $T_{M_t}(\eta)$ sweep out a Zariski open subset of the tangent space $T_V(\eta)$. Since $T_{M_t}(\eta) \subset \mathcal{F}(\eta)$ for all t, it follows that $T_V(\eta) \subset \mathcal{F}(\eta)$ also, as stated. □

9.1.6.2 Lemma. *There is a rational map $\sigma : X \dashrightarrow Z$ such that* $\dim Z < \dim X$ *and* $\ker d\sigma(\xi) \subset \mathcal{F}(\xi)$.

Proof. By construction, there is a unique subvariety V as described above passing through ξ. Since the Hilbert scheme of X has only countably many components and the field $\mathbb{C}$ is uncountable, there is an irreducible algebraic family of subvarieties $\{V_z\}_{z \in Z}$ in X that covers X, with the property that there is a unique member V through ξ. Hence there is a rational map $\sigma : X \dashrightarrow Z$ sending each point to the subvariety through it. By (9.1.5.1) we have $\ker d\sigma(\xi) \subset \mathcal{F}(\xi)$, as required. $\square$

Consider the map $\sigma : X \dashrightarrow Z$, and say that $\dim X - \dim Z = r$. If $\ker d\sigma = \mathcal{F}$ at ξ, then there is nothing to do. If not, then there is an exact sequence

$$0 \to \mathcal{A} \to \sigma^* T_Z \to \mathcal{G} \to 0,$$

where $\mathcal{A} = \mathcal{F}/\ker d\sigma$ and $\mathcal{G} = T_X/\mathcal{F}$.

Define $W = D_1 \cap \ldots \cap D_m$, where $D_i \in \mid m_i H_i \mid$ is general, so that W is generically finite over Z and $\mu_{min}(\mathcal{A} \mid_W) > \max\{\mu_{max}(\mathcal{G} \mid_W),\ 0\}$. Let Q be the Galois closure of $W \to Z$ and let $\mathcal{L}$ (resp. $\mathcal{M}$) be the pull-back of $\mathcal{A} \mid_W$ (resp. T_Z) to Q. It is clear from consideration of the slopes of these sheaves (restricted to the inverse image of C) that $\mathcal{L}$ is a Galois invariant subsheaf of $\mathcal{M}$. Hence $\mathcal{A}$ descends to a subsheaf $\mathcal{H}$ of T_Z and the curves $\{C_t\}$ form a covering family of curves on Z whose general member misses any given codimension two subset of Z such that, letting $f : C \to Z$ be the composite of $C \hookrightarrow X \dashrightarrow Z$, we have $\mu_{min}(f^*\mathcal{H}) > \max\{\mu_{max}(f^*(T_Z/\mathcal{H})),\ 0\}$.

We can now follow (9.1.3) and (9.1.4) to find rational curves on Z that are tangent to $\mathcal{H}$, so that a trivial inductive argument completes the proof of (9.0.3). $\square$

9.1.7 (9.1.3.3) allowed us to avoid the following issue. Suppose that X is a normal (or smooth) n-dimensional projective variety in char. $p > 0$ and that $\mathcal{E}$ is a reflexive (or locally free) sheaf on X of rank $r \leq n$. Then it seems likely that for $\mathcal{E}$ to be semistable while $F^*\mathcal{E}$ is unstable should impose strong conditions on X; e.g., maybe X should be uniruled. The exact meaning of stability here is deliberately unclear, but when $r = n = 2$ and instability is taken in Bogomolov's sense, then results along these lines have been established and used in [Shepherd-Barron91]. However, when $n \geq r \geq 2$ this is not known.

9.2 The nonnegativity of the Kodaira dimension for regular minimal threefolds

Throughout this section, X will denote a minimal threefold of index r and irregularity $q(X) = 0$. X has isolated singularties. We shall fix a resolution

$\rho : Y \to X$ such that ρ^{-1} is an isomorphism over the smooth locus X^0 of X. Our aim is to prove (9.0.6), so that we may assume that $p_g(X) = 0$. Hence $\chi(\mathcal{O}_X) \geq 1$.

Theorem 9.3. *(i) $c_2(Y) \cdot \rho^* H \geq 0$. (That is, $c_2(Y)$ is pseudo-effective.)*
(ii) $c_2(Y) \cdot \rho^* D \geq 0$ for all nef $\mathbb{Q}$-divisors D on X.

Proof. (i) Put $\mathcal{F} = (\rho_* \Omega^1_Y)^{\vee\vee}$. Since X is not uniruled, by the main result of [Miyaoka-Mori86], (9.0.1) shows that $\mathcal{F}$ is generically semipositive, while $c_1(\mathcal{F})$ is nef by definition. It is shown in (10.12) that now $c_2(Y) \cdot \rho^* H \geq 0$, as required.
(ii) D is a limit of ample divisors, so that (ii) follows from (i). $\square$

Recall the Riemann–Roch formula, where $n \equiv 0(\mod r)$:

$$\begin{aligned}
&\chi(Y,\ \rho^* \mathcal{O}(nK_X)) \\
&= \frac{2n^3 - 3n^2}{12}(\rho^* K_X)^3 + \frac{n}{12}(\rho^* K_X) \cdot (K_Y^2 + c_2(Y)) + \chi(\mathcal{O}_X) \\
&= \frac{2n^3 - 3n^2}{12} K_X^3 + \frac{n}{12} K_X \cdot (K_X^2 + \rho_* c_2(Y)) + \chi(\mathcal{O}_X) \\
&\geq \chi(\mathcal{O}_X) \geq 1.
\end{aligned}$$

Proof of (9.0.6). We shall consider various cases separately:
(1) $K_X^2 \not\equiv 0$.
(2) $K_X \not\equiv 0$, $K_X^2 \equiv 0$ and $\pi_1^{alg}(X^0)$ is finite (X^0 being the smooth locus of X).
(3) $K_X \not\equiv 0$, $K_X^2 \equiv 0$ and $\pi_1^{alg}(X^0)$ is infinite.
(4) $K_X \equiv 0$.

Case (1): Fix a smooth ample divisor H on X. Taking cohomology of

$$0 \to \mathcal{O}(nK_X) \to \mathcal{O}(nK_X + H) \to \mathcal{O}_H(nK_X + H) \to 0$$

gives an exact sequence

$$\begin{aligned}
H^1(X, \mathcal{O}(nK_X + H)) &\to H^1(H, \mathcal{O}_H(nK_X + H)) \\
&\to H^2(X, \mathcal{O}(nK_X)) \to H^2(X, \mathcal{O}(nK_X + H)).
\end{aligned}$$

If H is sufficiently ample, then the first and last terms vanish, giving

$$H^1(H,\ \mathcal{O}_H(nK_X + H)) \cong H^2(X,\ \mathcal{O}(nK_X)).$$

Assume that these groups are nonzero; then Serre duality on H gives

$$H^1(H,\ \mathcal{O}_H(-(n-1)K_X)) \neq 0.$$

However, $\mathcal{O}_H(K_X)$ is nef and big, so that by the Kodaira–Ramanujam vanishing theorem $H^1(H, \mathcal{O}_H(-(n-1)K_X)) = 0$.

Hence $H^2(X, \mathcal{O}(nK_X)) = 0$. Since $R^i\rho_*\mathcal{O}_Y = 0$ for $i > 0$, we get that $H^2(Y, \rho^*\mathcal{O}(nK_X)) = 0$, and R–R. gives $P_n(Y) = P_n(X) \geq 1$.

Case (2): Let $\sigma : \tilde{X} \to X$ be the finite cover inducing the universal algebraic cover of X^0, with $\tilde{X}$ normal. $K_{\tilde{X}} = \sigma^* K_X$, thus $\tilde{X}$ is minimal, and it is enough to show that $\kappa(X) \geq 0$. Hence we may assume that $\pi_1^{alg}(X^0) = 1$.

Again let H be a smooth ample divisor on X. Then $\pi_1^{alg}(H) = 1$, by [Grothendieck68]. As in case (1), we can assume that $H^2(X, \mathcal{O}(nK_X)) \neq 0$. So by Serre duality, there is a nonsplit extension

$$0 \to \mathcal{O}(K_X) \to \mathcal{E} \to \mathcal{O}(nK_X) \to 0.$$

Assume that $\mathcal{E}$ is H-stable. Then if deg $H \gg 0$, $\mathcal{E}|_H = \mathcal{F}$, say, is H-stable. Consider $\mathcal{F} \otimes \mathcal{F}^\vee$ = End $\mathcal{F} = \mathcal{G}$, say. $\mathcal{G}$ is polystable (i.e., a direct sum of stable bundles of the same slope). We have $c_1(\mathcal{G}) = 0$ and $c_2(\mathcal{G}) = 4c_2(\mathcal{F}) - c_1(\mathcal{F})^2 = 0$, since $K_X^2 \cdot H = 0$. Then by a theorem of [Donaldson85], $\mathcal{G}$ is induced from a representation of $\pi_1(H)$. Since $\pi_1^{alg}(H) = 1$ and finitely generated subgroups of complex linear groups are residually finite, this representation is trivial. That is, $\mathcal{G}$ is trivial. Hence $H^0(\text{End } \mathcal{F}) = 4$, so that, by the Cayley–Hamilton theorem, $\mathcal{F}$ has a nonzero nilpotent endomorphism. This, however, contradicts the H-stability of $\mathcal{F}$.

Hence $\mathcal{E}$ is not H-stable. So there is an exact sequence

$$0 \to \mathcal{A} \to \mathcal{E} \to \mathcal{B} \to 0$$

which destabilizes $\mathcal{E}$; then the composite arrows $\mathcal{A} \to \mathcal{O}(nK)$ and $\mathcal{O}(K) \to \mathcal{B}$ are nonzero.

We have $\mathcal{A}^{\vee\vee} = \mathcal{O}(A)$ and $\mathcal{B}^{\vee\vee} = \mathcal{O}(B)$ with A, B Weil divisors on X. Put $A|_H = a$, $B|_H = b$, $H|_H = h$ and $K_X|_H = k$. We obtain that $a + b = (n+1)k$, $h \cdot (a - b) \geq 0$ and $a \cdot b \leq n \cdot k^2 = 0$.

Suppose that $(a-b)^2 > 0$. Then $(a - b) \in C_{++}(H)$, the positive cone of H, so that $h^0(\mathcal{O}(m(a-b))) = O(m^2)$ for $m \gg 0$. Since $b - k$ and $nk - a$ are effective, we get

$$h^0(\mathcal{O}(m(n-1)k)) = O(m^2) \quad \text{for} \quad m \gg 0.$$

However, k is nef and $k^2 = 0$, so this is impossible. Hence $(a-b)^2 = 0$.

Since $\mathcal{O}(a) \hookrightarrow \mathcal{O}(nk)$ and $\mathcal{O}(k) \hookrightarrow \mathcal{O}(b)$, we get $\mathcal{O}(a-b) \hookrightarrow \mathcal{O}((n-1)k)$. Since $k^2 = 0$ and k is nef, we find $k \cdot (a-b) = 0$. Since k and $a - b$ both lie in the closure of $C_{++}(H)$, the index theorem gives $\mathbb{Q} \cdot k = \mathbb{Q} \cdot (a-b)$

in $\mathrm{NS}(H) \otimes \mathbb{Q}$. Since $\pi_1^{alg}(H) = 1$ and $\mathrm{Weil}(X) \hookrightarrow \mathrm{Pic}(H)$, it follows that $\mathbb{Q} \cdot K_X = \mathbb{Q} \cdot (A - B)$ in $\mathrm{Weil}(X) \otimes \mathbb{Q}$.

There is a primitive element $D \in \mathrm{Weil}(X)$ such that $A \sim \alpha D$, $B \sim \beta D$ and $K_X \sim \kappa D$ for some $\alpha, \beta, \kappa \in \mathbb{Z}$. Since $\mathcal{O}(A) \hookrightarrow \mathcal{O}(nK_X)$ we obtain that $h^0(\mathcal{O}(n\kappa - \alpha)D) > 0$, hence $P_{n\kappa-\alpha}(X) > 0$.

Case (3): By the proof of (6.7.2) $\pi_1^{alg,loc}(X,P)$ is finite. (This is actually true for any isolated 3-fold canonical singularity.) Hence finite étale Galois covers $\tilde{X}^0$ of X^0 of sufficiently high degree extend to varieties $\tilde{X}$ that factorize as

$$\tilde{X} \xrightarrow{\beta} X_1 \xrightarrow{\alpha} X$$

where α is of bounded degree and is étale over X^0, X_1 is minimal and β is étale.

Since it is enough to show that $\kappa(X_1) \geq 0$, we may assume that $X = X_1$, i.e., that $\pi_1^{alg}(X)$ is infinite. Also, we may assume that all these finite covers are regular, since irregular minimal 3-folds are known to have $\kappa \geq 0$.

Replacing X by $\tilde{X}$, we can assume that $\chi(\mathcal{O}_X) \geq 4$. Fixing a resolution $\rho : Y \to X$, we get $h^0(\Omega_Y^2) \geq 3$. Choosing three linearly independent sections in $H^0(\Omega_Y^2)$, we get a homomorphism $\gamma : \mathcal{O}_Y^3 \to \Omega_Y^2$. Say rank $\gamma = r$, and let $\mathcal{E}$ denote im γ. Since $\Omega_Y^2 \otimes \mathcal{O}(-K_Y) \cong T_Y$, Theorem 1 gives

$$(c_1(\mathcal{E}) - rK_Y) \cdot \rho^* H \cdot \rho^* L \leq 0$$

for all ample $H, L \in Pic(X) \otimes \mathbb{Q}$. Since $K_Y \cdot \rho^* H \cdot \rho^* L = K_X \cdot H \cdot L$, we see, letting $L \to K_X$, that

$$c_1(\mathcal{E}).\rho^* H \cdot \rho^* K_X \leq rH \cdot K_X^2 = 0,$$

since $K_X^2 \equiv 0$. But $c_1(\mathcal{E})$ is effective, and so

$$c_1(\mathcal{E}) \cdot \rho^* K_X \cdot \rho^* H = 0.$$

Since $h^0(\mathcal{O}(c_1(\mathcal{E}))) \geq 3$, we can write $|c_1(\mathcal{E})| = B + |M|$, where $|M|$ has no fixed component and $\dim |M| \geq 2$. We get $M \cdot \rho^* K_X \cdot \rho^* H \geq 0$.

Suppose that H is sufficiently ample and that S is a general member of $|\rho^* H|$. Then by the Hodge index theorem on S, we get $\mathbb{Q} \cdot M|_S = \mathbb{Q} \cdot (\rho^* K_X)$ in $\mathrm{NS}(S) \otimes \mathbb{Q}$. Since $q(X) = 0$, it follows that $m \cdot \rho_* M \sim n \cdot K_X$ for some $m, n \in \mathbb{N}$, and so $\kappa(X) \geq 0$.

Case (4): Since $q(X) = 0$, we have $|\mathrm{Tors}\,\mathrm{Pic}(X)| \cdot rK_X \sim 0$, and so $\kappa(X) \geq 0$. $\square$

10. CHERN CLASSES OF Q–SHEAVES

GÁBOR MEGYESI

In this chapter we introduce the notions of Q-varieties, Q-sheaves, Chern classes for Q-sheaves, and we extend some results, such as the condition for semistability and the Bogomolov–Miyaoka–Yau inequality $c_1^2 \le 3c_2$, from smooth varieties to Q-varieties. One of our main aims is to calculate the Chern classes of the Q-sheaves of log differentials. Kawamata's original approach was more analytic, using Chern forms; we take a different, algebraic approach. This also enables us to define Chern classes for Q-sheaves in general, not just Q-vector bundles.

We work over an algebraically closed field of characteristic 0 throughout.

10.1 Definition. [Mumford83, §2.] A *Q-variety* is an irreducible, normal, quasiprojective algebraic variety X with only quotient singularities, together with a finite atlas of charts

$$\begin{tikzcd} X_\alpha \arrow[dd, "p_\alpha"'] \arrow[dr] & \\ & X_\alpha/G_\alpha \arrow[dl, "p'_\alpha"] \\ U_\alpha & \end{tikzcd}$$

where U_α is a Zariski open subset of X, $X = \cup_\alpha U_\alpha$, p'_α is étale, quasifinite, Galois, surjective, and finite in a neighbourhood of any singular point, X_α is smooth and quasiprojective, G_α is a finite group acting faithfully on X_α, freely in codimension one, so that $X_\alpha \to X_\alpha/G_\alpha$ is finite, Galois and étale in codimension 1. We also require the compatibility condition that the natural projections from the normalisation $X_{\alpha\beta}$ of $X_\alpha \times_X X_\beta$ to X_α and X_β should be étale.

X can also be constructed globally as the quotient of a quasiprojective variety $\tilde{X}$ by a finite group. Take a Galois extension of the function field $k(X)$ containing all the function fields $k(X_\alpha)$, and let $\tilde{X}$ be the normalisation of X in this field. Then $G = \mathrm{Gal}(k(\tilde{X})/k(X))$ acts faithfully on $\tilde{X}$, and $X = \tilde{X}/G$.

$\mathcal{F}_\alpha = \left(\mathcal{I}_{C|_{U_\alpha}} \cdot \mathcal{O}_{X_\alpha}\right)^{\vee\vee}$. There is also a short Q-exact sequence of Q-sheaves $0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0$.

(iii) If (X, B) is a log canonical surface, where X has only quotient singularities, then $\hat{\Omega}^1_X(\log B)$ exists as a Q-vector bundle. Let $C_\alpha = p_\alpha^{-1}(B\,|_{U_\alpha})$. By the classification of Chapter 3, there are three possibilities in the neighbourhood of a point of C_α.

(*a*) (X_α, C_α) is analytically isomorphic to $(\mathbb{A}^2, x = 0)$, $G_\alpha \cong \mathbb{Z}_n$ acting by $(x, y) \to (\zeta x, \zeta^a y)$, where ζ is a primitive n-th root of unity, $(a, n) = 1$,
(*b*) $(X_\alpha, C_\alpha) \cong (\mathbb{A}^2, xy = 0)$, $G_\alpha \cong \mathbb{Z}_n$ acting by $(x, y) \to (\zeta x, \zeta^a y)$, or
(*c*) $(X_\alpha, C_\alpha) \cong (\mathbb{A}^2, xy = 0)$, G_α is the binary dihedral group of order $4n$ acting by $(x, y) \to (\zeta x, \zeta^a y)$ and $(x, y) \to (-y, x)$.

In each case C_α has normal crossings, therefore $\mathcal{F}_\alpha = \Omega^1_X(\log C_\alpha)$ is a locally free sheaf, so $\hat{\Omega}^1_X(\log B)$ is Q-locally free.

Considering the normalization C_α^ν of C_α, we see that the G_α action extends naturally to $\mathcal{O}_{C_\alpha^\nu}$. Therefore we can define the Q-sheaf $\mathcal{O}_{B^\nu}$, the Q-normalisation of B, by the collection of sheaves $\mathcal{O}_{C_\alpha^\nu}$ on the X_α. If $B_1, \dots, B_s$ are the components of B, then $\mathcal{O}_{B^\nu} = \bigoplus_{i=1}^{s} \mathcal{O}_{B_i^\nu}$, and we have a Q-exact sequence

$$0 \to \hat{\Omega}^1_X \to \hat{\Omega}^1_X(\log B) \to \bigoplus_{i=1}^{s} \mathcal{O}_{B_i^\nu} \to 0,$$

whose Q-exactness follows from the exactness of

$$0 \to \Omega^1_{X_\alpha} \to \Omega^1_{X_\alpha}(\log C_\alpha) \to \mathcal{O}_{C_\alpha^\nu} \to 0.$$

For any quasiprojective variety Z we can define the Chow ring $A_*(Z) = \oplus_{k=0}^{\dim Z} A_k(Z)$, where A_k is the group of k dimensional cycles on Z modulo rational equivalence, and for Y smooth, we can also define $A^*(Y) = \oplus_{k=0}^{\dim Y} A^k(Y)$, where A^k is the group of k codimensional cycles on Y modulo rational equivalence. A morphism $h : Z \to Y$ induces a cap product $A^k(Y) \times A_l(Z) \xrightarrow{\cap} A_{l-k}(Z)$, [Fulton75, §2].

10.4 Definition. For V a possibly singular quasiprojective variety, we define

$$A^*(V) = \operatorname{Im}\Big\{ \varinjlim_{f:V\to Y} A^*(Y) \to \prod_{g:Z\to V} \operatorname{End}(A_*(Z)) \Big\}$$

where Y, Z are quasiprojective, Y is smooth, and the map is induced by the cap product (cf. [Fulton75, §2.] or the definition of *op*$A^\cdot$ in [Mumford83, §1.]).

This definition agrees with the original one for V smooth. Moreover, A^* is a contravariant functor, $A^*(V)$ inherits a natural ring structure, cap products can be defined, and most importantly for our purposes, for any coherent sheaf $\mathcal{F}$ on V with finite locally free resolution, we can define Chern classes $c_k(\mathcal{F}) \in A^k(V)$ [Fulton75, §3.2].

In some of the following we need that $\tilde{X}$ is Cohen–Macaulay, therefore we assume it from now on. As remarked above, this assumption is satisfied for surfaces. The following lemma explains its significance.

10.5 Lemma. [Mumford83, Proposition 2.1.] If X is quasi projective and $\tilde{X}$ is Cohen–Macaulay, then any coherent sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ arising from a Q-sheaf $\mathcal{F}$ on X has a finite locally free resolution.

Proof. Let $n = \dim X$. Let $0 \to \tilde{\mathcal{E}}_n \to \tilde{\mathcal{E}}_{n-1} \to \ldots \to \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_0 \to \tilde{\mathcal{F}}$ be a resolution of $\tilde{\mathcal{F}}$, with $\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_{n-1}$ locally free $\mathcal{O}_{\tilde{X}}$-modules. As X_α is smooth, $\mathcal{F}_\alpha$ has a locally free resolution of length at most n. The morphism $\tilde{X}_\alpha \to X_\alpha$ is flat, since $\tilde{X}_\alpha$ is Cohen–Macaulay and X_α is smooth, therefore the resolution of $\mathcal{F}_\alpha$ pulls back to a locally free resolution of $\tilde{\mathcal{F}}|_{\tilde{X}_\alpha}$ of length at most n. By Schanuel's lemma, if $\tilde{\mathcal{F}}|_{\tilde{X}_\alpha}$ has a locally free resolution of length at most n, then $\tilde{\mathcal{E}}_0|_{\tilde{X}_\alpha}, \tilde{\mathcal{E}}_1|_{\tilde{X}_\alpha}, \ldots, \tilde{\mathcal{E}}_{n-1}|_{\tilde{X}_\alpha}$ locally free implies that $\tilde{\mathcal{E}}_n|_{\tilde{X}_\alpha}$ is also locally free. Hence $\tilde{\mathcal{E}}_n$ is locally free and so $\tilde{\mathcal{F}}$ has a finite locally free resolution. $\square$

Hence for any coherent sheaf on $\tilde{X}$ we can define Chern classes in $A^*(\tilde{X})$, and using this we can define Chern classes for Q-sheaves on X.

10.6 Definition. The Chern classes $\hat{c}_k$ of the Q-sheaf $\mathcal{F}$ on X are given by $\hat{c}_k(\mathcal{F}) = \dfrac{1}{|G|} c_k(\tilde{\mathcal{F}}) \in A^k(\tilde{X}) \otimes \mathbb{Q}$.

By [Mumford83, Theorem 3.1] there exist canonical isomorphisms $\gamma : A_{n-k}(X) \otimes \mathbb{Q} \to A^k(\tilde{X})^G \otimes \mathbb{Q}$ for $0 \leq k \leq n$, where $n = \dim X$. Identifying the Chow groups via γ, $A_*(X) \otimes \mathbb{Q}$ obtains a ring structure and we can define Chern classes in it. There exists a degree map $\deg : A^n(\tilde{X})^G \otimes \mathbb{Q} \to \mathbb{Q}$; to get the correct intersection numbers on X we have to take into account that $p : \tilde{X} \to X$ has degree $|G|$, so we define $\deg : A_0(X) \otimes \mathbb{Q} \to \mathbb{Q}$ by $\deg Z = \deg \gamma(Z)/|G|$ for $Z \in A_0(X) \otimes \mathbb{Q}$. We can define the total Chern class by $\hat{c}(\mathcal{E}) = \sum_{k=0}^{n} \hat{c}_k(\mathcal{E})$. As a Q-exact sequence of Q-sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ on X pulls back to a short exact sequence of sheaves $0 \to \tilde{\mathcal{E}} \to \tilde{\mathcal{F}} \to \tilde{\mathcal{G}} \to 0$ on $\tilde{X}$, we have $\hat{c}(\mathcal{F}) = \hat{c}(\mathcal{E})\hat{c}(\mathcal{G})$.

For a Q-sheaf on a subvariety of X we can not in general define Chern classes in this way. We need this only in one case, for Q-sheaves on a curve B on a surface X with only quotient singularities such that (X, B) is log canonical; then the cover $C_\alpha \subset X_\alpha$ is a curve with at most simple nodes as singularities and we can define $\hat{c}_1$ for a Q-sheaf on B.

By considering the sheaves in codimension 1 only, we see that $\hat{c}_1(\hat{\Omega}^1_X) = K_X$, and if (X, B) is log canonical, then $\hat{c}_1(\hat{\Omega}^1_X(\log B)) = K_X + B$. Calculating $\hat{c}_2(\hat{\Omega}^1_X)$ and $\hat{c}_2(\hat{\Omega}^1_X(\log B))$ is one of the main aims of this chapter. For this, we need the notion of the orbifold Euler number.

10.7 Definition. Let X be a quasiprojective variety with only isolated quotient singularities and let Y be an open or closed subset of X. The orbifold Euler number of Y is defined as

$$e_{orb}(Y) = e_{top}(Y) - \sum_{P \in Y \cap \operatorname{Sing} X} \left(1 - \frac{1}{r(P)}\right),$$

where e_{top} is the usual topological Euler number and $r(P)$ is the order of the local fundamental group. It should be noted that if Y is closed then $e_{orb}(Y)$ depends not only on Y but also on the embedding $Y \subset X$. In our case, this does not lead to any confusion.

10.8 Theorem. *Let X be a normal projective surface with only quotient singularities, B a reduced Weil divisor on X such that (X, B) is log canonical. Then*

$$\hat{c}_2(\hat{\Omega}^1_X(\log B)) = e_{orb}(X \setminus B).$$

Proof. First we consider the case $B = \emptyset$ to prove that $\hat{c}_2(\hat{\Omega}^1_X) = e_{orb}(X)$.

Fix a projective embedding of X. A generic pencil of hyperplane sections has reduced elements only, and its base locus is reduced and disjoint from $\operatorname{Sing} X$ and B. Blowing up this base locus we obtain a morphism $f : \hat{X} \to \mathbb{P}^1$ with reduced fibres. Since both sides of the required equality increase by 1 under blowing up a smooth point, we may assume that in fact we have a morphism from X, $f : X \to \mathbb{P}^1$ with reduced fibres. Let g be the genus of the general fiber.

There exists a Q-exact sequence

$$(10.8.1) \qquad 0 \to f^*\Omega^1_{\mathbb{P}^1} \to \hat{\Omega}^1_X \to \hat{\omega}_{X/\mathbb{P}^1} \to \mathcal{O}_Z \to 0,$$

where Z is a 0-dimensional scheme supported on $\operatorname{Sing} X$ together with the nonsingular points where $df(x) = 0$.

Let $P \in Z$, $P \in U_\alpha$. Assume that $f(P) = 0$. $f_\alpha = f \circ p_\alpha$ is a G_α-invariant function on X_α. P has $\deg p_\alpha / r(P)$ inverse images in X_α. For $0 < |t| \ll 1$, $f_\alpha^{-1}(t)$ has the homotopy type of a wedge of μ_P circles in the neighbourhood of each point $Q \in p_\alpha^{-1}(P)$, hence its Euler number is $1 - \mu_P$. Therefore if we fix a small neighbourhood of P, the intersection of $f^{-1}(t)$ with this neighbourhood has orbifold Euler number $\dfrac{1-\mu_P}{r(P)}$ for $0 < |t| \ll 1$. Thus

$$e_{top}(X) = 2(2-2g) + \sum_{P \in Z} \Big(\frac{\mu_P - 1}{r(P)} + 1 \Big)$$

and

$$e_{orb}(X) = 2(2-2g) + \sum_{P \in Z} \frac{\mu_P}{r(P)}. \tag{10.8.2}$$

μ_P can also be calculated as $\operatorname{length}(\mathcal{O}_{X_\alpha,Q}/(\partial f_\alpha/\partial x, \partial f_\alpha/\partial y))$ by Milnor's Theorem [Milnor68, §7]. Define a 0-dimensional subscheme Z_α of X_α with ideal $(\partial f_\alpha/\partial x, \partial f_\alpha/\partial y))$ at each $Q \in p_\alpha^{-1}(P)$. The $\mathcal{O}_{Z_\alpha}$ define the Q-sheaf structure of $\mathcal{O}_Z$.

Z_α is a local complete intersection as X_α is smooth, so we can define $\tilde{Z}$ by $\tilde{Z}\,|_{\tilde{X}_\alpha} = q_\alpha^*(Z_\alpha)$, where q_α^* is the scheme theoretic inverse image. $\tilde{Z}$ is also a local complete intersection. We have the following lemma.

10.9 Lemma. *If $\tilde{Z}$ is a zero dimensional local complete intersection subscheme of $\tilde{X}$, then $c_2(\mathcal{O}_{\tilde{Z}}) = -\deg \tilde{Z}$.*

Proof. Both sides are clearly additive over subschemes with disjoint supports.

If $\tilde{Z}$ is a (reduced) smooth point P, then there exist smooth hyperplane sections H_1, H_2 such that $P \in H_1 \cap H_2$, every point of intersection of H_1 and H_2 is smoothin X and H_1, H_2 meet transversally there. Let $Y = H_1 \cap H_2$. From the exact sequence

$$0 \to \mathcal{O}(-H_1 - H_2) \to \mathcal{O}(-H_1) \oplus \mathcal{O}(-H_2) \to \mathcal{I}_Y \to 0$$

we can calculate $c_2(\mathcal{I}_Y) = H_1 \cdot H_2$, hence $c_2(\mathcal{O}_Y) = -H_1 \cdot H_2$. c_2 is invariant in an algebraic family, all points of Y are algebraically equivalent on H_1, therefore $c_2(\mathcal{O}_P) = -1$.

In the general case, since $\tilde{Z}$ is a local complete intersection, there exists a sufficiently ample divisor H such that $\mathcal{O}_{\tilde{Z}}(H)$ is generated by global sections and there exist $H_1, H_2 \in |H|$ whose local equations generate the ideal of $\tilde{Z}$ in $\mathcal{O}_{\tilde{X},P}$ for each point $P \in \operatorname{Supp} \tilde{Z}$, and all their other intersections are transversal and lie at smooth points of $\tilde{X}$. Let Y be the scheme theoretic intersection of H_1 and H_2; then from the exact sequence we have $c_2(\mathcal{O}_Y) = -H_1 \cdot H_2$ as before, and each point of $\operatorname{Supp} Y \setminus \operatorname{Supp} \tilde{Z}$ contributes -1. $c_2(\mathcal{O}_{\tilde{Z}}) = -\deg \tilde{Z}$ in the general case. $\square$

In our case

$$\deg \tilde{Z} = \sum_{P\in Z} \frac{\mu_P \deg q_\alpha \deg p_\alpha}{r(P)} = |G| \sum_{P\in Z} \frac{\mu_P}{r(P)},$$

hence

$$\hat{c}_2(\mathcal{O}_Z) = \frac{1}{|G|} c_2(\mathcal{O}_{\tilde{Z}}) = -\frac{1}{|G|} \deg \tilde{Z} = -\sum_{P\in Z} \frac{\mu_P}{r(P)}. \tag{10.8.3}$$

From (10.8.1), (10.8.2) and (10.8.3) we obtain

$$\begin{aligned} \hat{c}_2(\hat{\Omega}^1_X) &= \hat{c}_1(f^*\Omega^1_{\mathbb{P}^1})\hat{c}_1(\hat{\Omega}_{X/\mathbb{P}^1}) - \hat{c}_2(\mathcal{O}_Z) \\ &= 2(2-2g) + \sum_{P\in Z} \frac{\mu_P}{r(P)} = e_{orb}(X). \end{aligned} \tag{10.8.4}$$

Let $B_1, B_2, \ldots, B_s$ be the components of B. There exist Q-exact sequences

$$0 \to \mathcal{O}(-B_i) \to \mathcal{O}_X \to \mathcal{O}_{B_i} \to 0, \tag{10.8.5}$$

$$0 \to \mathcal{O}_{B_i} \to \mathcal{O}_{B_i^\nu} \to \mathcal{O}_{W_i} \to 0 \tag{10.8.6}$$

and

$$0 \to \hat{\Omega}^1_X \to \hat{\Omega}^1_X(\log B) \to \bigoplus_{i=1}^{s} \mathcal{O}_{B_i^\nu} \to 0, \tag{10.8.7}$$

where W_i is a 0-dimensional subscheme of X supported at those points of B_i which are either nodes of B_i or singular points of X of type (c) in Examples 10.3. (iii) on B_i. The Q-sheaf structure of $\mathcal{O}_{W_i}$ is given by $\mathcal{O}_{p_\alpha^{-1}(W_i\cap U_\alpha)}$ on X_α, where p_α^{-1} denotes the set theoretic inverse image.

From (10.8.5) we see that $\hat{c}_1(\mathcal{O}_{B_i}) = B_i$ and $\hat{c}_2(\mathcal{O}_{B_i}) = B_i^2$, while from (10.8.6) and (10.9), $\hat{c}_2(\mathcal{O}_{B_i^\nu}) = \hat{c}_2(\mathcal{O}_{B_i}) + \hat{c}_2(\mathcal{O}_{W_i}) = B_i^2 - \sum_{P\in W_i} \frac{1}{r(P)}$. Thus from (10.8.7) we obtain

$$\hat{c}_2(\hat{\Omega}^1_X(\log B)) = \hat{c}_2(\hat{\Omega}^1_X) + K_X\cdot B + \sum_{1\le i\le j\le s} B_i\cdot B_j - \sum_{i=1}^{s}\sum_{P\in W_i} \frac{1}{r(P)}. \tag{10.8.8}$$

We have Q-exact sequences

$$0 \to \hat{\mathcal{N}}^\vee_{B_i/X} \to \hat{\Omega}^1_X|_{B_i} \to \hat{\Omega}^1_{B_i} \to 0,$$

where $\hat{\mathcal{N}}^\vee_{B_i/X}$ is the conormal Q-sheaf, obtained by taking the G_α-invariants of $\mathcal{N}^\vee_{C_\alpha/X_\alpha}$, where $C_\alpha = p_\alpha^{-1}(B_i\,|_{U_\alpha})$. Now $\hat{c}_1(\hat{\Omega}^1_X\,|_{B_i}) = K_X \cdot B_i$, while $\hat{c}_1(\hat{\mathcal{N}}^\vee_{B_i/X}) = -B_i^2 + \sum_{P\in W_i} \frac{1}{r(P)}$, since each simple node of C_α contributes $+1$ on X_α. Hence

$$\hat{c}_1(\hat{\Omega}^1_{B_i}) = K_X \cdot B_i + B_i^2 - \sum_{P\in W_i} \frac{1}{r(P)}. \tag{10.8.9}$$

10.10 Lemma. $\hat{c}_1(\hat{\Omega}^1_{B_i}) = -e_{orb}(B_i)$.

Proof. By an argument similar to the above, we can find a morphism $f : B_i \to \mathbb{P}^1$ such that f has only ordinary ramification points and these are all smooth points of X and not nodes of B_i. Let d be the degree of this map, a the number of ramification points, b the number of nodes of B_i. Then $e_{top}(B_i) = 2d-a-b$, and hence $e_{orb}(B_i) = 2d-a-b-\sum_{P\in B_i\cap \operatorname{Sing} X}\left(1-\frac{1}{r(P)}\right)$.

We determine $\hat{c}_1(\hat{\Omega}^1_{B_i})$ from the Q-exact sequence

$$0 \to f^*\Omega^1_{\mathbb{P}^1} \to \hat{\Omega}^1_{B_i} \to \hat{\Omega}^1_{B_i/\mathbb{P}^1} \to 0;$$

the argument is similar to Hurwitz's formula.

Note first that $\hat{c}_1(f^*\Omega^1_{\mathbb{P}^1}) = -2d$. Each ramification point of f and each node of B_i which is a smooth point of X contributes 1 to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$. If a node $P \in B_i$ is a singular point of X, then it is type (c) in Examples 10.3. (iii). Let $P \in U_\alpha$, $f_\alpha = f \circ p_\alpha$. On X_α, $C_\alpha = p_\alpha^{-1}(B_i\,|_{U_\alpha})$ has a simple node at each $Q \in p_\alpha^{-1}(P)$, and Q is a ramification point of index $r(P)$ of f_α on each branch, therefore the contribution to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$ at P is $\frac{2(r(P)-1)+1}{r(P)} = \left(2-\frac{1}{r(P)}\right)$. If $P \in B_i$ is a singular point of X which is not a node of B_i then it is of type (a) or (c) in Examples 10.3. (iii). Let $P \in U_\alpha$. If P is of type (a), then on X_α each $Q \in p_\alpha^{-1}(P)$ is a ramification point of f_α of index $r(P)$, so the contribution to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$ is $1-\frac{1}{r(P)}$. If P is of type (c), then $r(P) = 4l$, C_α has a node at each $Q \in p_\alpha^{-1}(P)$, and Q is a ramification point of index $2l$ on both branches of C_α, so the contribution to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$ at P is $\frac{2(2l-1)+1}{4l} = \left(1-\frac{1}{r(P)}\right)$ in this case too.

Hence

$$\begin{aligned}\hat{c}_1(\hat{\Omega}^1_{B_i}) &= \hat{c}_1(f^*\Omega_{\mathbb{P}^1}) + \hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})\\ &= -2d+a+b+\sum_{P\in B_i\cap \operatorname{Sing} X}\left(1-\frac{1}{r(P)}\right) = -e_{orb}(B_i). \quad \square\end{aligned}$$

since H^* is nef and not numerically trivial.

Then $\pi_*\mathcal{E}^* \hookrightarrow \pi_*\mathcal{F}^* = \mathcal{F}$ and $c_1(\pi_*\mathcal{E}^*)\cdot\tilde{H} = c_1(\tilde{\mathcal{E}})\cdot\tilde{H}$, since $\tilde{H}$ is ample, so some multiple of it can be moved away from the singular locus of $\tilde{X}$, and the sheaves $\pi_*\mathcal{E}^*, \tilde{\mathcal{E}}$ agree on the smooth locus of $\tilde{X}$. Having obtained the instability of $\tilde{\mathcal{F}}$ on $\tilde{X}$, we can now choose a G-invariant destabilizing subsheaf, namely the first step $\mathcal{E}_1$ in the Harder–Narasimhan filtration of $\tilde{\mathcal{F}}$ for $\tilde{H}$ [Miyaoka87b, Theorem 2.1], which is unique, therefore G-invariant. Taking G-invariants, we obtain the required destabilizing Q-subsheaf $\mathcal{E} = \tilde{\mathcal{E}}_1^G$ of $\mathcal{F}$, then (10.11.1) follows from

$$\frac{c_1(\mathcal{E}_1)\cdot\tilde{H}}{\operatorname{rk}\mathcal{E}_1} > \frac{c_1(\tilde{\mathcal{F}})\cdot\tilde{H}}{r}. \quad \square$$

10.12 Proposition. *Let $\mathcal{E}$ be a Q-locally free sheaf on a normal projective surface X with only quotient singularities such that $\hat{c}_1(\mathcal{E})$ is nef and $\mathcal{E}$ is generically semipositive, i.e., for any nef divisor D on X and for any torsion free quotient Q-sheaf $\mathcal{F}$, $\hat{c}_1(\mathcal{F})\cdot D \geq 0$. Then $\hat{c}_2(\mathcal{E}) \geq 0$.*

Proof. Let H be an ample divisor on X, t a positive rational number, then $H_t = \hat{c}_1(\mathcal{E}) + tH$ is an ample $\mathbb{Q}$-divisor. Let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = \mathcal{E}$ be the Harder–Narasimhan filtration for $\mathcal{E}$ with respect to H_t, which is obtained by taking the G-invariants of the Harder–Narasimhan filtration for $\tilde{\mathcal{E}}$ with respect to p^*H_t. Let $\mathcal{G}_i = (\mathcal{E}_i/\mathcal{E}_{i-1})^{\vee\vee}$, $r_i = \operatorname{rk}\mathcal{G}_i$. $\mathcal{E}_i/\mathcal{E}_{i-1} \subset \mathcal{G}_i$ with skyscraper cokernel, therefore $\hat{c}_2(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq \hat{c}_2(\mathcal{G}_i)$ by (10.9), while $\hat{c}_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = \hat{c}_1(\mathcal{G}_i)$, since they agree in codimension 1. $\hat{c}(\mathcal{E}) = \prod_{i=1}^{s}\hat{c}(\mathcal{E}_i/\mathcal{E}_{i-1})$, where $\hat{c}$ is the total Chern class, therefore

$$\begin{aligned}
\hat{c}_2(\mathcal{E}) &\geq \prod_{1\leq i<j\leq s} \hat{c}_1(\mathcal{G}_i)\hat{c}_1(\mathcal{G}_j) + \sum_{i=1}^{s}\hat{c}_2(\mathcal{G}_i)\\
&= \frac{1}{2}(\hat{c}(\mathcal{E}))^2 + \sum_{i=1}^{s}\hat{c}_2(\mathcal{G}_i) - \frac{1}{2}\sum_{i=1}^{s}(\hat{c}_1(\mathcal{G}_i))^2\\
&\geq \frac{1}{2}(\hat{c}_1(\mathcal{E}))^2 - \sum_{i=1}^{s}\frac{1}{2r_i}(\hat{c}_1(\mathcal{G}_i))^2,
\end{aligned}$$

where in the last step we used the semistability of the $\mathcal{G}_i$ and Lemma 10.11.

Let $\alpha_i = \dfrac{\hat{c}_1(\mathcal{G}_i)\cdot H_t}{r_i H_t^2}$; then $\alpha_1 > \alpha_2 > \ldots \alpha_s \geq 0$ by definition of the Harder–Narasimhan filtration and the generic semipositivity of $\mathcal{E}$. By the

Hodge Index Theorem, $(\hat{c}_1(\mathcal{G}_i))^2 \leq r_i^2\alpha_i^2H_t^2$. Hence

$$\begin{aligned}
\hat{c}_2(\mathcal{E}) &\geq \frac{1}{2}(\hat{c}_1(\mathcal{E}))^2 - \sum_{i=1}^{s}\frac{1}{2r_i}(\hat{c}_1(\mathcal{G}_i))^2 \\
&\geq \frac{1}{2}\left((\hat{c}_1(\mathcal{E}))^2 - \sum_{i=1}^{s} r_i\alpha_i^2H_t^2\right) \\
&\geq \frac{1}{2}\left(((\hat{c}_1(\mathcal{E}))^2 - H_t^2) + \left(1 - \sum_{i=1}^{s} r_i\alpha_i^2\right)H_t^2\right) \\
&\geq \frac{1}{2}\left(((\hat{c}_1(\mathcal{E}))^2 - H_t^2) + \left(1 - \alpha_1\sum_{i=1}^{s} r_i\alpha_i\right)H_t^2\right) \\
&= \frac{1}{2}\left(((\hat{c}_1(\mathcal{E}))^2 - H_t^2) + (1-\alpha_1)H_t^2\right).
\end{aligned}$$

Now $\alpha_1 = \dfrac{\hat{c}_1(\mathcal{G}_1)\cdot H_t}{r_1H_t^2} \leq \dfrac{\hat{c}_1(\mathcal{E})\cdot H_t}{r_1H_t^2} < 1$, whereas $(\hat{c}_1(\mathcal{E}))^2 - H_t^2 \to 0$ as $t \to 0$, so that $\hat{c}_2(\mathcal{E}) \geq 0$. □

10.13 Theorem. *Let X be a normal projective threefold, B a reduced Weil divisor on X, such that (X,B) is log canonical, $(X,\emptyset)$ is log terminal, K_X+B is nef and X is not uniruled. Let S be a general hyperplane section of X; then $\hat{c}_2(\hat{\Omega}_X^1(\log B)\,|_S) \geq 0$.*

Proof. X has quotient singularities in codimension 2, so $\hat{\Omega}_X^1(\log B)$ can be defined as a Q-vector bundle except at finitely many points. S has only quotient singularities, $(S, B\,|_S)$ is log canonical, so $\hat{\Omega}_X^1(\log B)\,|_S$ is a Q-vector bundle. $\hat{\Omega}_X^1\,|_S$ is generically semipositive by (9.0.1), therefore so is $\hat{\Omega}_X^1(\log B)\,|_S$. $\hat{c}_1(\hat{\Omega}_X^1(\log B)\,|_S) = (K_X+B)\,|_S$ is nef by assumption, therefore we can apply (10.12) to deduce the result. □

We prove a generalization of the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 3c_2$. This inequality was proved for smooth surfaces of general type in [Miyaoka77, Theorem 4] and for smooth surfaces with c_1 negative in [Yau77, Theorem 4.]. It was generalised to $c_1^2(\Omega_X^1(\log B)) \leq 3c_2(\Omega_X^1(\log B))$ in [Sakai80, Theorem 7.6] for the case when X is a smooth surface and $B \subset X$ is a semistable curve, which implies that K_X+B is nef and (X,B) is log canonical. [Miyaoka84, Theorem 1.1] deals with the log case on surfaces with quotient singularities when the curve B does not pass through the singular points of the surface. A version of this inequality for log canonical surfaces with fractional boundary divisor with K_X+B ample is proved in [KNS89, Theorem 12]. We give a new method of proof for the case when X has only quotient singularities, (X, B) is log canonical, $K_X + B$ is nef. Our result is more general than [Miyaoka84] in that we also allow the curve B to pass through the singular points.

10.14 Theorem. *Let X be a normal projective surface with only quotient singularities, $B \subset X$ a curve such that (X, B) is log canonical and $K_X + B$ is nef. Then*

$$\hat{c}_1^2(\hat{\Omega}^1_X(\log B)) \leq 3\hat{c}_2(\hat{\Omega}^1_X(\log B)).$$

Proof. We prove this theorem by reducing it to the smooth case. Let $\mathcal{F} = \hat{\Omega}^1_X(\log B)$. Let $\pi : X^* \to \tilde{X}$ be an embedded resolution of $(\tilde{X}, p^{-1}(B)_{red})$, let $B^* = ((\pi \circ p)^{-1}(B))_{red}$, $\mathcal{F}^* = \pi^*\tilde{\mathcal{F}}$. Since $c_i(\mathcal{F}^*) = \pi^* c_i(\tilde{\mathcal{F}})$, it is sufficient to prove that $c_1^2(\mathcal{F}^*) \leq 3c_2(\mathcal{F}^*)$.

$\mathcal{F}$ is locally free of rank 2, therefore so is $\mathcal{F}^*$. $\mathcal{F}|_{\tilde{X}_\alpha} = q_\alpha^* \Omega^1_{X_\alpha}(\log C_\alpha)$, where $C_\alpha = p_\alpha^{-1}(B|_{U_\alpha})$, hence $\mathcal{F}^*|_{\pi^{-1}\tilde{X}_\alpha} = \pi^* q_\alpha^* \Omega^1_{X_\alpha}(\log C_\alpha) \subset \Omega^1_{X^*}(\log B^*)|_{\pi^{-1}X_\alpha}$, therefore $\mathcal{F}^* \subset \Omega^1_{X^*}(\log B^*)$. If $B = \emptyset$, $\mathcal{F}^* \subset \Omega^1_{X^*}$.

If $\omega \in H^0(X^*, \Omega^1_{X^*}(\log B^*))$, then ω is d-closed by [Deligne71]. (See also [Griffiths–Schmid73, 6.5] for a simpler proof.) Thus we can prove that if $\mathcal{L} \hookrightarrow \Omega^1_{X^*}(\log B^*)$ is an invertible sheaf, then $h^0(X, \mathcal{L}^{\otimes n}) \leq cn$ for some constant c [Sakai80, Lemma 7.5]. Using this, and the fact that $c_1(\mathcal{F}) = \pi^* p^*(K_X + B)$ is nef, we can follow Miyaoka's original proof for the non-log case [Miyaoka77, Theorem 4] to obtain $c_1^2(\mathcal{F}^*) \leq 3c_2(\mathcal{F}^*)$. □

10.15 Corollary. [Miyaoka84, Proposition 2.1.1] Let $\hat{X}$ be a minimal surface of nonnegative Kodaira dimension. Then the number of disjoint smooth rational curves on $\hat{X}$ is at most $\dfrac{2}{9}(3c_2(\hat{X}) - c_1^2(\hat{X}))$.

Proof. $K_{\hat{X}}$ is nef as $\hat{X}$ is minimal, so $C^2 \leq -2$ for any smooth rational curve on $\hat{X}$ by the adjunction formula. Let X be the surface obtained by contracting some disjoint smooth rational curves to singular points. Contracting a smooth rational curve with selfintersection $-n$ increases $\hat{c}_1^2$ by $\dfrac{(n-2)^2}{n}$, decreases $\hat{c}_2$ by $2 - \dfrac{1}{n}$, so $3\hat{c}_2 - \hat{c}_1^2$ decreases by at least $\dfrac{9}{2}$. K_X is still nef, so by the previous theorem $3\hat{c}_2(X) - \hat{c}_1^2(X) \geq 0$, which gives the bound on the number of contracted curves. □

11. LOG ABUNDANCE FOR SURFACES

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11.1 INTRODUCTION

Chapters 11–14 present Kawamata and Miyaoka's proof of the abundance theorem for threefolds.

11.1.1 Abundance Theorem. *A three dimensional minimal model X has a free pluricanonical system, that is, there exists a positive integer m such that $|mK_X|$ has no base points.*

(1.22–29) contains a general introduction to Abundance, and to the contents of Chapters 11–14. The division of labour indicated by the authors listed for each chapter is somewhat arbitrary; every author has made a significant contribution to each chapter. We would like to thank Kawamata for answering questions regarding his original version of [Kawamata91b]. We would also like to thank Shepherd-Barron, and Corti among others for helpful discussions and comments.

The purpose of this chapter is to gather together and prove some facts concerning log abundance for surfaces. These facts will be needed in Chapters 12–14 to prove the abundance conjecture for threefolds. We collect together some standard definitions and notation.

11.1.2 Notation

$\kappa(X,D)$ denotes the Iitaka dimension of the pair (X,D). By definition $\kappa(X,D) = -\infty$ iff $h^0(\mathcal{O}_X(nD)) = 0$ for every $n > 0$, and $\kappa(X,D) = k > -\infty$ iff

$$0 < \limsup \frac{h^0(\mathcal{O}_X(nD))}{n^k} < \infty.$$

One can see that $\kappa(X,D) \in \{-\infty, 0, 1, \dots, \dim X\}$.

$\kappa(X) \doteq \kappa(X, K_X)$ is the Kodaira dimension of X.

In case the divisors are nef, we can define the numerical counterparts (cf. (1.28)):

$\nu(X,D) = \max\{\, n \in \mathbb{N} \cup 0 \mid (D^n) \text{ not numerically } 0 \,\}$.

$\nu(X) = \nu(X, K_X)$.

The log abundance theorem for a normal surface X asserts the following:

11.1.3 Theorem. *Let (X, Δ) be a normal surface with boundary Δ (see (2.2.4) for a definition). If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, nef and log canonical then $|m(K_X + \Delta)|$ is basepoint free for some m (and in particular $\nu(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$).*

We need (11.1.3) in the cases $\nu(K_X + \Delta) = 0$ and 1, and content ourselves with proving these cases only. Readers interested in seeing the other case should consult [Fujita84]. The proof presented here is different from that in [Fujita84] at various points, and is adapted from Miyaoka's proof of the abundance theorem in the threefold case, as will be evident to the readers. Following Miyaoka's idea, we extend (11.1.3) to the semi log canonical case in Chapter 12.

The idea of the proof is as follows: we first show that the linear system $|m(K_X+\Delta)|$ contains a divisor D (11.2.1). Then we replace D with $B = D_{\text{red}}$, and apply the log minimal model program to $(X, \Delta + B)$, so that $K_X + \Delta + B$ becomes nef (11.3.2). Then we use a further series of log extremal contractions to make each connected component of B irreducible (11.3.4). Next we make a cyclic cover of a neighborhood of a connected component of B, to improve how it sits inside X (11.3.6). Finally using some simple cohomological arguments, one can show that this component moves to any infinitesimal order (11.3.7).

11.2 Existence of an effective member

We start with the following lemma.

11.2.1 Lemma. *Let (X, Δ) be a smooth surface with boundary Δ. If $K_X+\Delta$ is nef then $\kappa(X, K_X + \Delta) \geq 0$. In other words, there exists a member $D \in |m(K_X + \Delta)|$ for some $m > 0$.*

11.2.2 Remark An analog of this result for threefolds is proved in Chapter 9.

Proof. (cf. [Fujita84, §2]) If $\kappa(X, K_X) \geq 0$ then the conclusion is clear. Thus we may assume that X is ruled. There are two cases to consider, X is rational or irrational.

First consider the case when X is rational. Let $G = K_X+\Delta$. G is nef by assumption. Since X is rational, $h^1(\mathcal{O}_X) = 0$. Therefore if G is numerically trivial, then $mG \sim 0$ for some m. Otherwise $h^2(mG) = h^0(-(m-1)G - \Delta) = 0$ for $m \geq 2$ and sufficiently divisible. Now $\chi(\mathcal{O}_X) = 1$, and so Riemann–Roch reads

$$h^0(X, mG) = h^1(X, mG) + \frac{1}{2}mG \cdot (mG - K_X) + 1.$$

Note that $mG - K_X = (m-1)G + \Delta$ and Δ is a sum of effective divisors. Since G is nef, we have $G \cdot (mG - K_X) \geq 0$ and therefore $h^0(X, mG) > 0$. This proves the lemma for rational surfaces.

Next consider the case when X is irrational. We write $\Delta = \Delta_1 + \Delta_2$, where Δ_1 and Δ_2 are boundaries, in such a way that Δ_1 has no vertical components, and furthermore $(K_X + \Delta_1) \cdot F = 0$. (11.2.1) follows if we show that $|m(K_X + \Delta_1)| \neq \emptyset$. Thus we may as well assume that $(K_X + \Delta) \cdot F = 0$ to start with, i.e., we prove the stronger statement:

11.2.3 lemma. *Let (X, Δ) be an irrational ruled surface with boundary Δ. Ssuppose that Δ has no vertical components and $(K_X + \Delta) \cdot F = 0$. Then $\kappa(X, K_X + \Delta) \geq 0$.*

The proof is by induction on the Picard number $\rho(X)$. Consider the case when X is a $\mathbb{P}^1$-bundle.

$\rho(X) = 2$, and the cone $\overline{\mathrm{NE}}(X)$ has two edges. One is the class generated by F, a fibre of the ruling $\pi : X \to C$ with C of genus $g > 0$. Suppose that the other edge is generated by H. Since $F^2 = 0$ and $H^2 \leq 0$ (see [CKM88, 4.4]), we must have $H \cdot F > 0$. We normalize H by taking $H \cdot F = 1$.

Let $\Delta = \sum k_i \Delta_i$, where the Δ_i are the prime components of Δ. We have $\Delta_i \equiv a_i H + F_i$, where $a_i \in \mathbb{Z}$ and $F_i = \pi^*(D_i)$ for some divisor D_i on C. Let $b_i = \deg(D_i)$. Since Δ_i is not a vertical component, $a_i > 0$. We also know that $K_X \equiv -2H + F_0$, with $F_0 = \pi^*(D_0)$, $\deg(D_0) = H^2 + 2g - 2$. Hence

$$K_X + \Delta \equiv \left(-2 + \sum k_i a_i\right) H + \sum F_i. \tag{11.2.3.1}$$

By assumption $(K_X + \Delta) \cdot F = 0$, and so $\sum k_i a_i = 2$. Now $\sum F_i = \pi^*(\sum D_i)$ and $\deg(\sum D_i) = H^2 + 2g - 2 + \sum k_i b_i$.

Look at $H \cdot \Delta_i$. If $H \cdot \Delta_i \geq 0$, then $b_i \geq -a_i H^2 \geq 0$. Otherwise $H \cdot \Delta_i < 0$, but since H is an edge of $\overline{\mathrm{NE}}(X)$, this implies that $\Delta_i^2 < 0$. Hence Δ_i is a section of the ruling of X with negative selfintersection. Moreover according to [CKM88, 4.5], the class of Δ_i is an edge, and so Δ_i is proportional to H. By the normalization $H \cdot F = 1$, H is the class generated by Δ_i, and we can replace numerical equivalence in (11.2.3.1) by linear equivalence. In particular, $a_i = 1$ and $D_i = 0$.

Now we have the following two cases:

Case (i). If $H \cdot \Delta_i \geq 0$ for all i, then $\sum k_i b_i \geq -\left(\sum a_i k_i\right) H^2 = -2H^2$, and so $H^2 + 2g - 2 + \sum k_i b_i \geq -H^2 + 2g - 2$.

Case (ii). The other possibility is that $H \cdot \Delta_1 < 0$, in which case H is generated by Δ_1 and $H \cdot \Delta_i \geq 0$ for all $i \neq 1$. Then $\sum k_i b_i \geq -2H^2 + k_1 H^2$, and so $H^2 + 2g - 2 + \sum k_i b_i \geq -(1 - k_1)H^2 + 2g - 2$. Note that since Δ is a boundary, $k_1 \leq 1$.

When $g > 1$, in either case, deg $(\sum D_i) > 0$, which implies (11.2.3).

We are left with the case $g = 1$ and deg $(\sum D_i) = 0$. This implies, in case (ii), that $\Delta_1 = H$ is an elliptic curve, $H^2 < 0$, $k_1 = 1$ and $\Delta_i \cdot H = 0$ for $i \geq 2$. Thus H is disjoint from Δ_i, for $i \geq 2$ and so $(K_X + \Delta)|_H = K_H \sim 0$. Therefore $K_X + \Delta \sim 0$.

In case (i), this implies $H^2 = 0$ and $b_i = 0$ for all i. Then H is the class of a section with selfintersection 0 and we denote the section by H. We also replace the numerical equivalence by linear equivalence. Then $K_X \sim -2H$, and $\Delta_i \cdot \Delta_j = 0$ for any i and j. Applying adjunction to Δ_i, we see that each Δ_i is a smooth elliptic curve. Now π restricts to an étale map from Δ_i to C of degree a_i.

As the Δ_i are disjoint, we can find an étale cover $p : \tilde{C} \to C$, so that on the fibre product $\tilde{\pi} : \tilde{X} \to \tilde{C}$, the pull back by $\tilde{p}$ of Δ is a disjoint union of $n = \sum a_i$ sections of $\tilde{\pi}$. Since $\tilde{p}$ is étale, $\tilde{p}^*(K_X) = K_{\tilde{X}}$. Now if $n \geq 3$, then $\tilde{X}$ is actually $\tilde{C} \times \mathbb{P}^1$, and $\tilde{p}^*(K_X + \Delta)$ is trivial. If $n < 3$, as $\sum k_i a_i = 2$, n must be 2, and on $\tilde{X}$, $\tilde{p}^*(\Delta) = \tilde{\Delta}_1 + \tilde{\Delta}_2$. It is then clear that both $K_{\tilde{X}}$ and $\mathcal{O}_{\tilde{X}}(-\tilde{\Delta}_1 - \tilde{\Delta}_2)$ are the relative dualizing sheaf for $\tilde{\pi}$. Thus $\tilde{p}^*(K_X + \Delta)$ is still trivial. But $\tilde{p}^*(K_X + \Delta) = \tilde{\pi}^* p^*(\sum D_i)$. Therefore $p^*(\sum D_i) \sim 0$, and $r(\sum D_i) = p_* p^*(\sum D_i) \sim 0$, where r is the degree of p, that is $\sum D_i$ is a torsion class on C. This finishes the proof of (11.2.3) when X is a $\mathbb{P}^1$-bundle.

Now suppose that π has a singular fibre and E is a component of the singular fibre. If E is not a -1-curve, then $E \cdot K_X \geq 0$. Since Δ contains no vertical component, $(K_X + \Delta) \cdot E \geq 0$. By assumption, $(K_X + \Delta) \cdot F = 0$, hence $(K_X + \Delta) \cdot E \leq 0$ for some exceptional curve E of the singular fibre. We may blow down E to get $p : X \to X'$. Set $\Delta' = p_* \Delta$. We have $K_X + \Delta = p^*(K_{X'} + \Delta') + \alpha E$, with $\alpha = -E \cdot (K_X + \Delta) \geq 0$. Clearly $K_{X'} + \Delta'$ satisfies the inductive assumption, hence $\kappa(X', K_{X'} + \Delta') \geq 0$. It follows at once that $\kappa(X, K_X + \Delta) \geq 0$. □

The log abundance theorem for the case $\nu(X, K_X + \Delta) = 0$ is a direct consequence of (11.2.1).

11.2.4 Lemma. *Let X be a proper surface and assume that (X, Δ) is log canonical. If $K_X + \Delta$ is nef then $\kappa(X, K_X + \Delta) \geq 0$.*

Proof. We want to find a member in $|m(K_X + \Delta)|$. For this let $\phi : X' \to X$ be the minimal resolution, and write $K_{X'} + \Delta_{X'} = \phi^*(K_X + \Delta)$. Since $K_X + \Delta$ is log canonical and ϕ is minimal, $\Delta_{X'}$ is a boundary. As $K_{X'} + \Delta_{X'}$ is nef, (11.2.1) implies that $|m(K_{X'} + \Delta_{X'})| \neq \emptyset$. But $\mathrm{H}^0(m(K_{X'} + \Delta_{X'})) = \mathrm{H}^0(m(K_X + \Delta))$, and so we can find $D \in |m(K_X + \Delta)|$. □

11.3 The case $\nu(K_X + \Delta) = 1$

This section is devoted to a proof of the following result.

11.3.1 Theorem. *Let (X, Δ) be a normal surface with boundary Δ. If $K_X + \Delta$ is nef, $\mathbb{Q}$-Cartier, log canonical and $\nu(X, K_X + \Delta) = 1$, then $|m(K_X + \Delta)|$ is free for some m.*

We first observe that to prove (11.3.1), it is enough to show that $\kappa(X, K_X + \Delta) = 1$. In fact suppose $M + B \in |m(K_X + \Delta)|$, where M moves in a pencil, and B is the fixed part. Now $M \cdot B \geq 0$, as $|M|$ has no one dimensional base locus, and since $(M + B)^2 = 0$, this implies $M(M + B) = M^2 = 0$. Thus $|M|$ is free, and so it defines a map of S to a smooth curve C. As $M \cdot B = 0$ and the numerical class of M is equivalent to a multiple of a fibre, the divisor B is linearly equivalent to the pullback of a divisor from C. But then some multiple of B is base point free.

Here is the first step of (11.3.1).

11.3.2 Lemma. *There exists a surface $\hat{X}$ birational to X, and divisors $\hat{\Delta}$, $\hat{B}$ and $\hat{D}$ such that:*

(1) *$(\hat{X}, \hat{\Delta} + \hat{B})$ is $\mathbb{Q}$-factorial and log canonical and $\hat{D} \in |m(K_{\hat{X}} + \hat{\Delta} + \hat{B})|$. Moreover $\hat{B} = \hat{D}_{\text{red}}$.*
(2) *$K_{\hat{X}} + \hat{\Delta} + \hat{B}$ is nef.*
(3) *$\nu(X, K_X + \Delta) = \nu(X, K_{\hat{X}} + \hat{\Delta} + \hat{B})$ and $\kappa(X, K_X + \Delta) = \kappa(\hat{X}, K_{\hat{X}} + \hat{\Delta} + \hat{B})$.*

Proof. By (11.2.4), we may find $D \in |m(K_X + \Delta)|$. Pick a minimal good resolution $\mu : X_0 \longrightarrow X$ of the pair $(X, D + \Delta)$, and write $K_{X_0} + \tilde{\Delta} = \mu^*(K_X + \Delta)$. As (X, Δ) is log canonical, $\tilde{\Delta}$ is effective. Set $B_0 = (\mu^* D)_{\text{red}}$ and replace $\tilde{\Delta}$ with Δ_0, where we only include those components of $\tilde{\Delta}$ which are not components of B_0. With this choice of Δ_0, $\Delta_0 + B_0$ is a boundary, and there is a divisor $D_0 \in |m(K_{X_0} + \Delta_0 + B_0)|$.

We now apply the log minimal model program to $(X_0, \Delta_0 + B_0)$. We inductively construct a sequence X_i, Δ_i, B_i and D_i satisfying (1). If $K_{X_i} + \Delta_i + B_i$ is not nef, then there is a divisorial contraction ϕ_i associated to some log extremal ray of $K_{X_i} + \Delta_i + B_i$ (clearly ϕ_i is not of fibre type), and we put $B_{i+1} = \phi_{i*}(B_i)$, $\Delta_{i+1} = \phi_{i*}(\Delta_i)$, and $D_{i+1} = \phi_{i*}(D_i)$. (By [KMM87, 5-1-6] $(X_{i+1}, \Delta_{i+1} + B_{i+1})$ is $\mathbb{Q}$-factorial and log terminal.)

Since at each step the Picard number drops by one, this process must terminate at some i, and we set $\hat{X} = X_i$, $\hat{B} = B_i$, $\hat{\Delta} = \Delta_i$ and $\hat{D} = D_i$.

Conditions (1) and (2) are automatic from the construction. (3) follows from the (11.3.3) applied to the pullbacks of the divisors $m(K_X + \Delta)$ and D_i to X_0 (cf. (13.2.4)). $\square$

Note that in fact the pair $(\hat{X}, \hat{\Delta} + \hat{B})$ is log terminal; we do not need this.

11.3.3 Lemma. *Let X be a proper variety of dimension n and G_1, G_2 two effective nef divisors with the same support. Then $\nu(X, G_1) = \nu(X, G_2)$ and $\kappa(X, G_1) = \kappa(X, G_2)$.*

Proof. Let $\nu(X, G_i) = \nu_i$ and $\kappa(X, G_i) = \kappa_i$. Choose a_1 so that $a_1 G_1 - G_2$ is effective.

(1) Let H be any ample divisor. Then

$$\begin{aligned}\left((a_1G_1)^{\nu_2} \cdot H^{n-\nu_2}\right) &\geq \left((a_1G_1)^{\nu_2-1} \cdot G_2 \cdot H^{n-\nu_2}\right)\\ &\vdots\\ &\geq \left(G_2^{\nu_2} \cdot H^{n-\nu_2}\right) > 0,\end{aligned}$$

and therefore $\nu_1 \geq \nu_2$.

(2) $\mathrm{H}^0(mG_2) \hookrightarrow \mathrm{H}^0(ma_1G_1)$, therefore $\kappa_1 \geq \kappa_2$.

Now reverse the roles of G_1 and G_2. $\square$

Now Riemann–Roch for $n\hat{D}$ reads:

$$\begin{aligned}\chi(n\hat{D}) &= \frac{n\hat{D} \cdot (n\hat{D} - K_{\hat{X}})}{2} + \chi(\mathcal{O}_X)\\ &= \frac{n(n-1/m)}{2}(\hat{D}^2) + \frac{n}{2}\hat{D} \cdot (\hat{\Delta} + \hat{B}) + \chi(\mathcal{O}_X).\end{aligned}$$

We know already that $\hat{D}^2 = 0$ and so from now on we can assume $\hat{D} \cdot \hat{\Delta} = 0$, since otherwise (11.3.1) follows immediately (because $h^2(n\hat{D}) = h^2(K_{\hat{X}} - n\hat{D}) = 0$ for large n, as G is not numerically trivial, and we only need to show $\kappa(X, K_X + \Delta) = 1$). Since we have chosen $\hat{B}$ so that $\hat{\Delta}$ and $\hat{B}$ have no components in common, this implies that $\hat{\Delta}$ and $\hat{B}$ do not intersect.

Choose an integer m so that $\hat{L} = \mathcal{O}_{\hat{X}}(m(K_{\hat{X}} + \hat{\Delta} + \hat{B})) \in \operatorname{Pic}(\hat{X})$ and $|\hat{L}|$ is non-empty. Note that $\hat{L}$ is nef.

11.3.4 Lemma. *There are X', Δ', B' and D' satisfying (11.3.2.1–3) and in addition*

(4) *Every connected component of B' is irreducible.*

Proof. Pick an irreducible component S of $\hat{B}$. Suppose S meets another component $\hat{S}$ of $\hat{B}$. Now $\nu(X, \hat{L}) = 1$, so that $\hat{L}^2 = 0$. But $\hat{L}^2 = \hat{L} \cdot (\hat{D} - \hat{S}) + \hat{L} \cdot \hat{S}$, and both terms are non-negative as $\hat{L}$ is nef. It follows that $\hat{L} \cdot \hat{S} = 0$ and moreover that $(K_{\hat{X}} + \hat{\Delta} + \hat{B} - S) \cdot \hat{S} < 0$. But then there is a log extremal ray of $(K_{\hat{X}} + \hat{\Delta} + \hat{B} - S)$ associated to $\hat{S}$, and so a log extremal contraction, which must be divisorial. Such a contraction decreases the Picard number of $\hat{X}$, and so eventually we may isolate every component of $\hat{B}$. □

Pick any prime component S of B', and let U be an open subset of X' which retracts to S [BPV84, page 27]. Let L' be the line bundle $\mathcal{O}_U(m(K_U+S)) = \mathcal{O}_U(m(K_U+\Delta'+S))$.

11.3.5 Lemma. *$L'|_S$ is a torsion element of* $\operatorname{Pic}(S)$ *(i.e some multiple of $L'|_S$ is isomorphic to $\mathcal{O}_S$).*

Proof. If we apply adjunction to S in U, we get

$$(K_U+S)|_S = K_S + P$$

where $P = \operatorname{Diff}$ is effective. If $P=0$, then S is elliptic or nodal rational and so $K_S+P=0$. If $P\neq 0$ then S is a smooth $\mathbb{P}^1$. □

Now we make a cover of U to improve S and how it sits inside U (compare [Miyaoka88b], where this argument first appears).

11.3.6 Lemma. *Let U be a normal analytic space, and S a compact subspace. If the inclusion $i : S \to U$ induces isomorphisms*

$$i^* : H^j(U,\mathbb{Z}) \cong H^j(S,\mathbb{Z}) \quad \textit{for } j=1,2,$$

then

(1) *the kernel of the restriction map*

$$\operatorname{Pic}(U) \longrightarrow \operatorname{Pic}(S)$$

is a $\mathbb{C}$-vector space. In particular it is divisible, and torsion free.

Moreover if G is a $\mathbb{Q}$-Cartier integral divisor on U such that $G|_S$ is torsion, then

(2) *there is a finite Galois cover $\pi : \tilde{U} \longrightarrow U$, étale in codimension one, such that π^*G is a Cartier divisor, which restricts to a divisor linearly equivalent to zero on π^*S.*

Proof. Compare the cohomology exact sequences of the exponential sequences on S and U:

$$\begin{tikzcd}
H^1(S,\mathbb{Z}) \arrow[r, "\alpha_S"] & H^1(S,\mathcal{O}_S) \arrow[r] & H^1(S,\mathcal{O}_S^*) \arrow[r] & H^2(S,\mathbb{Z}) \\
H^1(U,\mathbb{Z}) \arrow[r, "\alpha_U"] \arrow[u, "\beta_1"] & H^1(U,\mathcal{O}_U) \arrow[r] \arrow[u, "\beta_2"] & H^1(U,\mathcal{O}_U^*) \arrow[r] \arrow[u, "\beta_3"] & H^2(U,\mathbb{Z}). \arrow[u, "\beta_4"]
\end{tikzcd}$$

Now Pic $(U) = H^1(U, \mathcal{O}_U^*)$, Pic $(S) = H^1(S, \mathcal{O}_S^*)$, and β_3 is just the restriction map. By assumption, β_1 and β_4 are isomorphisms, and so the kernel of β_3 is isomorphic to the kernel of β_2, which in turn is a subvector space of the $\mathbb{C}$-vector space $H^1(U, \mathcal{O}_U)$. Hence (1) holds.

For (2), let r be the smallest integer such that rG is Cartier and $rG|_S \sim 0$. The class of $rG|_S$ in $H^2(S, \mathbb{Z})$ is zero, and as β_4 is an isomorphism, the class of rG is zero in $H^2(U, \mathbb{Z})$. As $H^1(U, \mathcal{O}_U)$ is divisible, there is a line bundle M on U such that:

$$\mathcal{O}_U(rG) \otimes M^r = \mathcal{O}_U \quad \square$$

We are going to apply (11.3.6.2) to ensure that both the pullback of K_X and the class of S are multiples of the same Cartier divisor $\tilde{G}$, which will itself restrict to a divisor linearly equivalent to zero on the pullback of S.

As S is irreducible, there is a divisor $D \in |m(K_U + S)|$ such that $D = eS$ for some positive integer e. But then $dS \sim mK_U$, where $d = e - m$. Note that either d and m are nonnegative or d is negative, but $-d < m$. Let c be the highest common factor of m and d. We may find integers m', d', b_1 and b_2 such that:

$$m = m'c, \qquad d = d'c, \qquad c = b_1 m + b_2 d.$$

Let G be the Weil divisor $b_1 S + b_2 K_U$. We have

$$c(S - m'G) = (b_1 m + b_2 d)S - m(b_1 S + b_2 K_U) \sim 0 \sim c(K_U - d'G),$$

and so

$$c(K_U + S - (m' + d')G) \sim 0.$$

Thus the three divisors

$$(S - m'G)|_S, \qquad (K_U - d'G)|_S \quad \text{and} \quad G|_S$$

are all torsion (the third by (11.3.5)). Now we apply (11.3.6.2) three times to these divisors. Thus there is a finite Galois cover $\pi : \tilde{U} \longrightarrow U$, étale in codimension one, such that, if we put $\tilde{S} = \pi^* S$ and $\tilde{G} = \pi^* G$,

$$\tilde{G}|_{\tilde{S}} \sim 0 \qquad \tilde{S} \sim m'\tilde{G}, \qquad K_{\tilde{U}} \sim d'\tilde{G},$$

and so

$$\omega_{\tilde{S}} = \mathcal{O}_{\tilde{U}}(\tilde{S})|_{\tilde{S}} = \mathcal{O}_{\tilde{S}}.$$

The next lemma shows that $\tilde{S}$ moves in $\tilde{U}$ infinitesimally (cf. [Miyaoka88b, 4.2]). First some notation; let V be an analytic space, and S a Cartier divisor on V. Denote by S_n the analytic subspace of V defined by the sheaf of ideals $\mathcal{O}_U(-nS)$ and set $A_n = \text{Spec } \mathbb{C}[\epsilon]/(\epsilon)^n$.

11.3.7 Lemma. *Let V be a Cohen–Macaulay complex space, and S a divisor on V. Assume that K_V and S are both multiples d' and m' of the same Cartier divisor G, and that the following three conditions hold*

(1) $d' + (n-1)m' \neq 0$ *for any* $n \geq 2$,

(2) $\omega_S \simeq \mathcal{O}_S$,

(3) *the restriction* $H^p(S_n, \mathcal{O}_{S_n}) \longrightarrow H^p(S, \mathcal{O}_S)$ *is surjective for every* p

Then S moves infinitesimally in V, to any order.

Proof. We prove the following statements by induction on n.

$(\mathrm{i})_n$ There are proper flat morphisms $\xi_i : S_i \longrightarrow A_i$ $(i \leq n)$ such that the following diagram is commutative

$$\begin{array}{ccc} S_{i-1} & \longrightarrow & S_i \\ {\scriptstyle \xi_{i-1}}\downarrow & & {\scriptstyle \xi_i}\downarrow \\ A_{i-1} & \longrightarrow & A_i \end{array}$$

$(\mathrm{ii})_n$ The sheaves $R^p\xi_{n*}\mathcal{O}_{S_n}$ are locally free.

$(\mathrm{iii})_n$ $\omega_{S_n} \simeq \mathcal{O}_{S_n}$.

Note that if $(\mathrm{i})_n$ holds for every n, then S moves infinitesimally to any order, by definition.

For $n = 1$, we take ξ_1 to be the structure map. Then $(\mathrm{ii})_1$ is automatic, and $(\mathrm{iii})_1$ is just (2).

Otherwise suppose that all three statements are true for all integers less than n. As $K_V + (n-1)S$ is Cartier and S_{n-1} is Cohen Macaulay (it is a Cartier divisor in a Cohen Macaulay scheme), we may apply adjunction to S_{n-1}:

$$\begin{aligned} \omega_{S_{n-1}} &= \omega_V((n-1)S) \otimes \mathcal{O}_{S_{n-1}} \\ &= \mathcal{O}_{S_{n-1}}((d' + (n-1)m')G). \end{aligned}$$

On the other hand $(\mathrm{iii})_{n-1}$ implies that $\omega_{S_{n-1}}$ is linearly equivalent to zero. We may apply (11.3.6) (1) to S and S_{n-1} to deduce that G is linearly equivalent to zero on S_{n-1}. In particular $\mathcal{O}_{S_{n-1}} \otimes \mathcal{O}_V(-S) \simeq \mathcal{O}_{S_{n-1}}$.

Consider the exact sequence of sheaves on V,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{S_n} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

where $\mathcal{K}$ is defined by exactness. It is clear that the support of $\mathcal{K}$ is S_{n-1}. In fact

$$\mathcal{O}_S = \mathcal{O}_V/\mathcal{O}_V(-S) \qquad \mathcal{O}_{S_n} = \mathcal{O}_V/\mathcal{O}_V(-(n+1)S)$$

and so $\mathcal{K} \cong \mathcal{O}_{S_{n-1}} \otimes \mathcal{O}_V(-S)$ as a sheaf of $\mathcal{O}_V$-modules. Now we have shown that this is the trivial line bundle on S_{n-1}.

Let e be the image of the global section 1 of the sheaf $\mathcal{K}$ in the vector space $H^0(S_n, \mathcal{O}_{S_n})$. Define a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[\epsilon]/(\epsilon)^n$ to $H^0(S_n, \mathcal{O}_{S_n})$, by sending ϵ to e. This gives $H^0(S_n, \mathcal{O}_{S_n})$ a flat $\mathbb{C}[\epsilon]/(\epsilon)^n$-module structure, which since A_n is affine, is equivalent to a proper flat morphism $\xi_n : S_n \longrightarrow A_n$. It is not hard, from the definition of ξ_n, to check that the diagram

$$\begin{array}{ccc} S_{n-1} & \longrightarrow & S_n \\ {\scriptstyle \xi_{n-1}}\big\downarrow & & {\scriptstyle \xi_n}\big\downarrow \\ A_{n-1} & \longrightarrow & A_n \end{array}$$

commutes. This proves $(\mathrm{i})_n$.

Condition (3) now implies $(\mathrm{ii})_n$ (see for example [Hartshorne77, III 12.11]). It follows by duality, that $R^p\xi_{n*}\omega_{S_n}$ are also locally free, for every p. (Unfortunately this seems to require relative duality theory, see e.g. [Hartshorne66].) As ω_S is isomorphic to the trivial line bundle, $\xi_{n*}\omega_{S_n}$ has a global non vanishing section, which we may pullback to ω_{S_n}. Thus ω_{S_n} is trivial also, which is $(\mathrm{iii})_n$. □

11.3.8 Example. There is an interesting example which indicates the necessity for the somewhat strange assumptions of (11.3.7). Take X to be a $\mathbb{P}^1$-bundle over an elliptic curve, given by the unique rank two vector bundle of degree zero which does not split. X has a unique section S of selfintersection zero, which does not move. However it does move to first order. Of course there is no divisor G such that both the class of the curve and its dualizing sheaf are multiples of G.

11.3.9. Now we check that the conditions of (11.3.7) apply to $\tilde{S}$ in $\tilde{U}$. In fact (1) follows as m' is always positive, and if d' is negative, $-d' < m'$, (2) has already been verified, and so we are left with (3). But as $\tilde{S}$ is a curve, certainly

$$H^1(\tilde{S}_n, \mathcal{O}_{\tilde{S}_n}) \longrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}),$$

is surjective, as the obstruction is the second cohomology of the kernel of the natural map $\mathcal{O}_{\tilde{S}_n} \longrightarrow \mathcal{O}_{\tilde{S}}$, which always vanishes. This leaves

$$H^0(\tilde{S}_n, \mathcal{O}_{\tilde{S}_n}) \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) \simeq \mathbb{C},$$

which is again certainly surjective.

Now we are in a position to finish the proof of (11.3.1). Let G be the Galois group of the cover $\tilde{U} \longrightarrow U$ of degree r. The Cartier divisor $(rS)_n$ pulls back, under π, to the Cartier divisor $\tilde{S}_{nr}$. Thus $\tilde{S}_{nr}$ descends to $(rS)_n$, and moreover G acts naturally on $H^0(\tilde{S}_{nr}, \mathcal{O}_{\tilde{S}_{nr}})$. But this may be identified, via ξ_{nr}, with $\mathbb{C}[\epsilon]/(\epsilon)^{nr}$. It follows that $(rS)_n$ maps to $A_{ns} = \operatorname{Spec}(\mathbb{C}[\epsilon]/(\epsilon)^{nr})^G$, for some s dividing r. Since the Hilbert scheme is of finite type we are done.

12. SEMI LOG CANONICAL SURFACES

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12.1 INTRODUCTION

In this chapter we collect together some results concerning semi log canonical surfaces (see (12.2) for the definitions and basic properties). The first of these is log abundance for semi log canonical surfaces in the cases $\nu = 0$ or $\nu = 1$.

12.1.1 Theorem. *Let S be a reduced projective surface and let Δ be a $\mathbb{Q}$-Weil divisor on S. Assume that $K_S + \Delta$ is $\mathbb{Q}$-Cartier, nef and semi log canonical and $\nu(S, K_S + \Delta) = 0$ or 1.*

Then the linear system $|m(K_S + \Delta)|$ is base point free for suitable $m > 0$ (and in particular $\nu(S, K_S + \Delta) = \kappa(S, K_S + \Delta)$).

The idea is to show that we can descend sections to S from the normalization of S (here we use (11.1.3)). In both cases the arguments are a little delicate; we have to analyze carefully the patching data.

The second result is a version of (1.13) (which is proved in (12.5)).

12.1.2 Theorem. *Let S be a reduced projective surface with semi log canonical singularities. Then the natural map induced by $\mathbb{C}_S \subset \mathcal{O}_S$*

$$i_p : H^p(S, \mathbb{C}_S) \longrightarrow H^p(S, \mathcal{O}_S) \qquad \textit{is surjective for every } p.$$

When S is smooth (12.1.2) is a standard result. Therefore we just need to analyze how the cohomology of S differs from the cohomology of a resolution. We split this analysis into two steps; in one step we consider how to resolve the bad singularities at isolated points of S, and in the other step we remove the one dimensional singular locus via a finite map. However we introduce a new twist; rather than first normalizing S for the second step, which loses too much information about the singularities of S, we make S as nice as possible by altering S at a finite set of points, and then normalize.

12.2 Basic results

We collect together here some of the properties of semi log canonical surface singularities.

Let X be a scheme with at worst double normal crossings in codimension one. The next set of definitions introduces the appropriate notion of log canonical (these definitions were given in [KSB88] for surfaces).

12.2.1 Definition.

(1) An n-dimensional singularity $(x \in X)$ is called a *double normal crossing point*, resp. a *pinch point* if it is analytically (or formally) isomorphic to

$$(0 \in (x_0x_1 = 0)) \subset (0 \in \mathbb{C}^{n+1}) \quad \text{resp.} \quad (0 \in (x_0^2 = x_1x_2^2)) \subset (0 \in \mathbb{C}^{n+1}).$$

(2) An n-fold X is *semismooth* if every closed point $(x \in X)$ is either smooth or double normal crossing point or pinch point. The singular locus of X is then a smooth $(n-1)$-fold D_X. The normalization $\nu : X^\nu \to X$ is smooth and $D_\nu = \nu^{-1}(D_X) \to D_X$ is a double cover ramified along the pinch locus.

(3) A morphism $f : Y \to X$ is called a *semiresolution* if f is proper, Y is semismooth, no component of D_Y is f-exceptional, and there is a codimension two closed subset $S \subset X$ such that $f|f^{-1}(X \setminus S) : f^{-1}(X \setminus S) \to X \setminus S$ is an isomorphism.

(4) Let X be a reduced scheme, $\Delta \subset X$ a $\mathbb{Q}$-Weil divisor (cf. (16.2)). Let $f : Y \to X$ be a semiresolution with exceptional divisors E and exceptional set $Ex(f) \subset Y$.

f is a *good* semiresolution (resp. a *good divisorial* semiresolution) of $\Delta \subset X$ if the union $E \cup D_Y \cup f_*^{-1}(\Delta)$ (resp. $Ex(f) \cup D_Y \cup f_*^{-1}(\Delta)$) is a divisor with global normal crossings on Y.

(5) Let S be a reduced surface. A semiresolution $f : T \to S$ is *minimal* if ω_T is f-nef. (In the nonnormal case, minimal resolutions are not unique.)

(6) Let X be a reduced S_2 scheme, $\Delta \subset X$ a boundary (i.e., a $\mathbb{Q}$-Weil divisor $\Delta = \sum d_i\Delta_i$ with $0 \le d_i \le 1$). We say that $K_X + \Delta$ is *semi log terminal* (resp. *divisorial semi log terminal*, resp. *semi log canonical*) (frequently abbreviated as **slt** resp. **dslt** resp. **slc**) if it is $\mathbb{Q}$-Cartier and there is a good semiresolution (resp. a good divisorial semiresolution, resp. a good semiresolution) $f : Y \to X$ of $\Delta \subset X$ such that:

$$K_Y + f_*^{-1}(\Delta) = f^*(K_X + \Delta) + \sum a_iE_i,$$

where the E_i are the f-exceptional divisors and all $a_i > -1$ (resp. $a_i > -1$ resp. $a_i \geq -1$). We leave it to the reader to formulate the analogous definition of the various flavors of semi log terminal.

(7) Let $f : Y \to X$ be a semiresolution. We say X has semirational singularities, if $f_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$. As in the normal case, this is independent of the semiresolution chosen.

(8) A scheme X (over an algebraically closed field) is called seminormal if the following condition holds:

Every finite and surjective morphism $X' \to X$ which is one-to-one on closed points is an isomorphism.

12.2.2 Notation. Let (X, Δ) be slc. Let $\mu : X^\mu \longrightarrow X$ be the normalization. Let $D \subset X$ (resp. $D_\mu \subset X^\mu$) be the double intersection locus. Thus $\mu|D_\mu : D_\mu \longrightarrow D$ is a double cover. Let $\Theta = \mu^{-1}\Delta + D_\mu$. Thus

$$K_{X^\mu} + \Theta = \mu^*(K_X + \Delta).$$

The irreducible components of X^μ are frequently denoted by X_i and then Θ_i denotes the restriction of Θ to X_i.

12.2.3 Proposition. *[vanStraten87] Let S be a surface which is semismooth in codimension one. Then S has a minimal semiresolution. If $\Delta \subset S$ is a Weil divisor then (S, Δ) has a good semiresolution.*

Proof. Let S be a surface, with normal crossings in codimension one, and choose a good resolution (T_0, D_0) of the pair (S^μ, D_μ). If $\Delta = \emptyset$ then D_μ is reduced and we may assume in addition that $K_{T_0} + D_0$ is nef on T_0/S^μ. The map $D_\mu \longrightarrow D_S$ is two-to-one, and defines an involution τ on D_0. It is easy to see (cf. [Artin70] for the general theory) that one can find an analytic (or algebraic) space T, which is obtained from T_0 by gluing together points of D_0 that are conjugate under the involution τ. Moreover it is not hard to see that T is semismooth; pinch points correspond to fixed points of the involution τ. There is a morphism $f : T \longrightarrow S$ with fibres which are either points or curves. Thus f is projective, hence T is also projective and so f is a semiresolution. □

The following is clear from the definitions (cf. (2.6)):

12.2.4 Proposition. *Notation as above. Then*

$$\mathrm{discrep}(X, \Delta) = \mathrm{discrep}(X^\mu, \Theta). \quad \square$$

It might seem from (12.2.4) that one could define the semi log versions of lt, lc etc. by requiring the corresponding notion to hold for the normalization. However, $K_X + \Delta$ is usually not $\mathbb{Q}$-Cartier even when $\Delta = \emptyset$ and (X^μ, Θ) is log canonical. In dimension two one can give the following necessary (and sufficient) condition.

12.2.5 Proposition. *Let (S, Δ) be a slc surface. Let $D_1 \subset S$ be a double curve such that $\mu^{-1}(D_1) = D_1' \cup D_1''$ has two components. Then (see (16.6) for the definition of* Diff*)*

$$\mathrm{Diff}_{D_1'}(\Theta - D_1') = \mathrm{Diff}_{D_1''}(\Theta - D_1'').$$

Proof. Let S_1, S_2 be analytic neighborhoods of D_1', D_1'' respectively. We abuse notation, and identify S_1 and S_2 with their images under μ. Now we may compute the different at any point of D_1' or D_1'', on the surface S, by first restricting to S_1 or S_2. In either case this is equivalent to restricting $K_S + \Delta$ to the double curve D_1. □

12.2.6 Corollary. *Let (S, Δ) be a germ of a slc surface. Assume that S^μ has two irreducible components S_1^μ, S_2^μ. Then*

$$(S_1^\mu, \Theta_1) \cong (\mathbb{C}^2, \mathbb{C}) \quad \Leftrightarrow \quad (S_2^\mu, \Theta_2) \cong (\mathbb{C}^2, \mathbb{C}).$$

Proof. Note that by (16.6) (S_i, Θ_i) is isomorphic to $(\mathbb{C}^2, \mathbb{C})$ iff Θ_i is irreducible and the different is zero. □

12.2.7 Corollary. *Let (S, B) be a germ of a slt surface. Then S has one or two irreducible components.*

Proof. Assume that S has at least three irreducible components. Then there is a component S_1 which intersects at least two other components along curves. Thus $\Theta_i = \Theta|S_i^\mu$ contains at least two reduced curves. By Chapter 3, this implies that (S_i^μ, Θ_i) is not lt. □

12.2.8 Proposition – Definition. *Let (S, Δ) be a germ of an slc surface. Let $f : T \to S$ be a minimal semiresolution (of S). Let $E_i \subset T$ be the exceptional divisors. Then*

(1)
$$K_T + f_*^{-1}(\Delta) = f^*(K_S + \Delta) + \sum a_i E_i,$$

where $0 \geq a_i \geq -1$. Let $E = \sum_{a_i=-1} E_i$.

(2) $R^1 f_* \mathcal{O}_T(-E) = 0$.

(3) *If (S, Δ) is not semirational then $\Delta = \emptyset$ and S is either simple elliptic, a cusp or a degenerate cusp; where we define S to be*

(i) simple elliptic, *if $E = Ex(f)$ is a smooth elliptic curve, and S is normal,*

(ii) a cusp *(resp.* degenerate cusp*), if S is normal (resp. not normal, but T has no pinch points, locally about E), if $E = Ex(f)$ is a cycle of $\mathbb{P}^1$ or a nodal $\mathbb{P}^1$.*

Proof. Let $\nu : T^\nu \longrightarrow T$ be the normalization of T. We get a commutative diagram

$$\begin{tikzcd} T^\nu \arrow[r, "g"] \arrow[d, "\nu"'] & S^\mu \arrow[d, "\mu"] \\ T \arrow[r, "f"] & S. \end{tikzcd}$$

Now $g : T^\nu \longrightarrow S^\mu$ is a resolution of S^μ. Thus

$$K_{T^\nu} + g_*^{-1}(\Theta) = g^*(K_{S^\mu} + \Theta) + \sum a_i F_i,$$

where F_i are the exceptional divisors of g and $0 \geq a_i$ follows from (2.19).

Let $F = \sum_{a_i=-1} F_i$. Then $f^*E = F$, and so to show that $R^1 f_* \mathcal{O}_T(-E) = 0$, it is enough to show that $R^1 g_* \mathcal{O}_{T^\nu}(-F) = 0$, as the morphisms ν and μ are finite. But as

$$-F = K_{T^\nu} + \left(g_*^{-1}(\Theta) + \sum_{a_i > -1} -a_i F_i \right) - g^*(K_{S^\mu} + \Theta),$$

this follows by Kawamata–Viehweg vanishing [KMM87, 1-2-3].

Now if S is not semirational, $h^1(\mathcal{O}_E) \geq 1$, by (2). Applying adjunction to E, we have:

$$K_E = (K_T + E)|E = \sum_{0 \geq a_i > -1} (a_i E_i - f_*^{-1}(\Delta))|E,$$

which is negative unless $E = Ex(f)$ and $\Delta = \emptyset$. Thus $H^1(\mathcal{O}_E) = 0$ unless $E = Ex(f)$ and $\Delta = \emptyset$. In the latter case E has arithmetic genus one, and so it is an elliptic curve, a cycle of $\mathbb{P}^1$ or a nodal $\mathbb{P}^1$. Therefore if S is not normal then D^μ has two components on every component of S^μ and every (S^μ, D_μ) falls to case (9) of Figure 3 in the classification of Chapter 3. Thus S is a degenerate cusp. This proves (3). $\square$

12.2.9 Definition. Let (C, Δ) be a semi log canonical curve and Δ a $\mathbb{Q}$-divisor.

Let $n : \bar{C} = \cup C_i \to C$ be the normalization and define Δ_i by

$$n^*(K_C + \Delta)|C_i = K_{C_i} + \Delta_i.$$

Assume that $m(K_{C_i} + \Delta_i)$ is an integral divisor. For every $P \in \llcorner \Delta_i \lrcorner$ let z_P be a local parameter at P. A section $s_i \in \Gamma(C_i, \mathcal{O}(m(K_{C_i} + \Delta_i)))$ is *normalized* if $s_i - (dz_P/z_P)^m$ vanishes at P. This is easily seen to be independent of the choice of z_P.

A section $s \in \Gamma(C, \mathcal{O}(m(K_C + \Delta)))$ is *normalized* if $n^*(s)|C_i$ is normalized for every i.

On the nodal curve $(xy = 0) \subset \mathbb{C}^2$ consider the 1-form $\sigma = dx/x = -dy/y$. Even powers of σ are normalized and there are no normalized sections if m is odd.

All normalized sections form an *affine* subspace in the space of sections. This will be denoted by

$$\Gamma^n(C, \mathcal{O}(m(K_C + \Delta))).$$

12.2.9.1 Complement. If C_i is such that $\llcorner \Delta_i \lrcorner = 0$ then C_i is a smooth connected component of C and the above definition imposes no restrictions on sections of $\mathcal{O}(m(K + \Delta_i))$. For our purposes it will be convenient to make the following convention. Assume that C_i is an elliptic curve such that $\Delta_i = 0$. $\mathrm{Aut}(C)$ acts trivially on $H^0(C, \mathcal{O}_C(12K_C))$. We fix a nonzero section for every elliptic curve and call it (and its powers in $H^0(C, \mathcal{O}_C(12mK_C))$) normalized.

12.2.10 Definition. Let (X, Δ) be an slc surface. As in (12.2.2) let $n : (X^\mu, \Theta) \to (X, \Delta)$ be the normalization. As section $s \in \Gamma(X, \mathcal{O}(m(K_X + \Delta)))$ is *normalized* if

$$n^* s|\llcorner\Theta\lrcorner \in \Gamma(\llcorner\Theta\lrcorner, \mathcal{O}(m(K_{\llcorner\Theta\lrcorner} + \mathrm{Diff}(\Theta - \llcorner\Theta\lrcorner))))$$

is normalized.

All normalized sections form an affine subspace $\Gamma^n(X, \mathcal{O}(m(K_X + \Delta)))$ in the space of all sections.

12.2.11 Proposition. *Let (C, Δ) be an slc curve and let m be a natural number such that $m\Delta$ is integral. Then*

(12.2.11.1) $\Gamma^n(C, \mathcal{O}(2m(K_C + \Delta))) = \prod_i \Gamma^n(C_i, \mathcal{O}(2m(K_{C_i} + \Delta_i)))$;

(12.2.11.2) If $K_C + \Delta$ is nef then $\Gamma^n(C, \mathcal{O}(12m(K_C + \Delta)))$ generates $\mathcal{O}(12m(K_C + \Delta))$.

Proof. The first part is clear. Using the first part, it is sufficient to prove the second for C irreducible and smooth.

We distinguish two cases:

(12.2.11.3) $\deg(K_C + \Delta) = 0$. Then either $g(C) = 1$ and $\Delta = 0$ or $g(C) = 0$ and $\llcorner\Delta\lrcorner$ is at most two points of C. $\mathcal{O}(12m(K_C + \Delta))$ has one section (up to scalars) and a suitable multiple is normalized if $\llcorner\Delta\lrcorner$ is at most one point. If $\llcorner\Delta\lrcorner = \{0, \infty\}$ then $(dz/z)^{12m}$ is normalized.

(12.2.11.4) $\deg(K_C + \Delta) > 0$. Let P be any point different from $\llcorner\Delta\lrcorner$. Consider the exact sequence

$$0 \to \mathcal{O}(12m(K_C + \Delta) - \llcorner\Delta\lrcorner - P) \to \mathcal{O}(12m(K_C + \Delta)) \to \mathbb{C}(P) + \mathbb{C}(\llcorner\Delta\lrcorner) \to 0.$$

Since

$$\begin{aligned}&\deg(12m(K_C+\Delta)-\llcorner\Delta\lrcorner-P)\\&=\deg(K_C+11m(K_C+\Delta))+\deg((m-1)(K_C+\Delta)+\{\Delta\})-1\\&\geq\deg K_C+11-1=\deg K_C+10,\end{aligned}$$

we conclude that

$$H^0(C,\mathcal{O}(12m(K_C+\Delta)))\to H^0(C,\mathbb{C}(P)+\mathbb{C}(\llcorner\Delta\lrcorner))$$

is surjective. □

12.3 The reduced boundary of lc surfaces

Let (S,Θ) be an lc surface. Our aim is to analyze $\llcorner\Theta\lrcorner$ in the cases when $\nu(S,\Theta)\in\{0,1\}$.

12.3.1 Proposition. *[Shokurov91, 6.9] Let (S,Θ) be a proper lc surface. Assume that $K+\Theta\equiv 0$. Then (S,Θ) satisfies one of the following conditions:*

(1) *$\llcorner\Theta\lrcorner$ is connected and for every $C\in\llcorner\Theta\lrcorner$ the pair $(C,\mathrm{Diff}(\Theta-C))$ is* **not** *klt, (i.e.,* $\mathrm{Diff}(\Theta-C)$ *contains a point with multiplicity 1.)*
(2) *$\llcorner\Theta\lrcorner$ is irreducible and for $C=\llcorner\Theta\lrcorner$ the pair $(C,\mathrm{Diff}(\Theta-C))$ is klt.*
(3) *$\llcorner\Theta\lrcorner$ has two connected components, for every $C\subset\llcorner\Theta\lrcorner$ the pair $(C,\mathrm{Diff}(\Theta-C))$ is klt and there is a morphism onto a curve $g:S\to B$ such that $\llcorner\Theta\lrcorner$ consists of two sections of g. (B is either rational or elliptic.)*

Proof. Let $h:S'\to S$ be an lt modification of S and let $K+\Theta'=h^*(K+\Theta)$. Then (S',Θ') is lt and it is sufficient to prove that the result holds for (S',Θ'). In this case $(C,\mathrm{Diff}(\Theta'-C))$ is not klt iff C intersects another irreducible component of $\llcorner\Theta'\lrcorner$.

We prove a stronger relative version:

12.3.2 Proposition. *Let (S,Θ) be a log terminal surface. Let $f:S\to R$ be a proper morphism with connected fibers. Assume that $K+\Theta$ is numerically f-trivial. Let $r\in R$ be arbitrary. Then one of the following holds:*

(1) *$\llcorner\Theta\lrcorner$ is connected in a neighborhood of $f^{-1}(r)$;*
(2) *$\llcorner\Theta\lrcorner$ has two connected components in a neighborhood of $f^{-1}(r)$, both components are smooth and there is a morphism onto a curve $g:S/R\to B/R$ such that $\llcorner\Theta\lrcorner$ consists of two sections of g.*

Proof. If f is birational then (17.4) implies that we have (1). Thus we may assume that f has positive dimensional fibers and that $\llcorner\Theta\lrcorner \neq \emptyset$.

We apply the $(K+\Theta-\epsilon\llcorner\Theta\lrcorner)$-MMP on S/R for $0<\epsilon\ll 1$. The end result is a proper birational morphism $p : S/R \to Z/R$ such that $K_Z + p(\Theta)$ is lc and $K_Z + p(\Theta) - \epsilon\llcorner p(\Theta)\lrcorner$ is lt. We claim that

$$p(\llcorner\Theta\lrcorner) = \llcorner p(\Theta)\lrcorner.$$

Indeed, since $K+\Theta$ is numerically f-trivial, $K = p^*(K_Z + p(\Theta)) - \Theta$. If $z \in p(\llcorner\Theta\lrcorner) - \llcorner p(\Theta)\lrcorner$ then

$$K = p^*(K_Z + p(\Theta)) - \Theta = p^*(K_Z + p(\Theta) - \epsilon\llcorner p(\Theta)\lrcorner) - \Theta$$

in a neighborhood of $p^{-1}(z)$, which shows that $K_Z + p(\Theta) - \epsilon\llcorner p(\Theta)\lrcorner$ is not lt at z, a contradiction. In particular $\llcorner p(\Theta)\lrcorner \neq \emptyset$. By (17.4) the fibers of $\llcorner\Theta\lrcorner \to \llcorner p(\Theta)\lrcorner$ are connected, hence $\llcorner p(\Theta)\lrcorner$ is connected iff $\llcorner\Theta\lrcorner$ is connected.

Now we distinguish several cases.

(i) $K_Z + p(\Theta) - \epsilon\llcorner p(\Theta)\lrcorner$ is numerically trivial over R. This can only happen if the fibers of $Z \to R$ are one dimensional and $\llcorner p(\Theta)\lrcorner$ is the union of some fibers, thus $\llcorner p(\Theta)\lrcorner$ is connected near any fiber. Otherwise there is a $(K_Z + p(\Theta) - \epsilon\llcorner p(\Theta)\lrcorner)$-extremal contraction $u : Z/R \to V/R$. Here there are two subcases:

(ii) u contracts Z to a point. Then $\rho(Z) = 1$, hence any two curves in Z intersect. Thus $\llcorner p(\Theta)\lrcorner$ is connected.

(iii) u contracts Z to a curve and the generic fiber is $\mathbb{P}^1$. Therefore $\llcorner p(\Theta)\lrcorner$ intersects the generic fiber in at most two points. For any $v \in V$, the fiber $u^{-1}(v) \subset Z$ is an irreducible curve. Thus if $\llcorner p(\Theta)\lrcorner$ is not connected in the neighborhood of a fiber of $Z \to S$ then $\llcorner p(\Theta)\lrcorner$ is the union of two sections of u near that fiber. Thus $\llcorner\Theta\lrcorner$ also has two connected components.

In order to prove (2), consider the morphism $u\circ p : S \to V$. In a neighborhood of $(u\circ p)^{-1}(v)$, $\llcorner\Theta\lrcorner$ consists of two sections and possibly some other curves $C = \cup C_i \subset (u\circ p)^{-1}(v)$ which are p-exceptional. If C is not empty then $(u\circ p)^{-1}(v) - C$ is contractible, and the resulting contraction contradicts (17.4). Thus C is empty and (2) holds. □

As a straightforward corollary we obtain:

12.3.3 Theorem. *Let (S,Δ) be a proper, connected slc surface such that $K+\Delta\equiv 0$. Let (S_i,Θ_i) be the irreducible components of the normalization. Then one of the following conditions is satisfied:*

(1) *$\llcorner\Theta_i\lrcorner$ is connected for every i and for every irreducible curve $C \subset \llcorner\Theta_i\lrcorner$ the different $(C, \mathrm{Diff}(\Theta_i - C))$ is **not** klt.*

(2) *For every i and for every irreducible curve $C \subset \llcorner\Theta_i\lrcorner$ the different $(C, \mathrm{Diff}(\Theta_i - C))$ is klt.* $\square$

The combinatorial description of the intersections of the irreducible components of S is very subtle in case (1). (See [Friedman–Morrison83] for an overview of the special case of semistable degenerations of surfaces.) In the second case the combinatorics is easy but we need further information about the relationship between the two components of $\llcorner\Theta_i\lrcorner$.

12.3.4 Theorem. *Let (S, Θ) be an lc surface. Let $f : S \to B$ be a proper morphism onto a curve, with connected fibers. Assume that $K + \Theta$ is numerically f-trivial and $\llcorner\Theta\lrcorner \supset C_1 \cup C_2$ where the C_i are sections of f. Let $f_i = f|C_i$. Then*

(1) $(f_1)_* \mathrm{Diff}_{C_1}(\Theta - C_1) = (f_2)_* \mathrm{Diff}_{C_2}(\Theta - C_2)$; *let us call this $\mathbb{Q}$-divisor P.*

(2) *For some $m > 0$ we have an isomorphism $\psi : f^*\mathcal{O}_B(mK + mP) \cong \mathcal{O}_S(mK + m\Theta)$.*

(3) *Let ψ_i denote the composite isomorphism*

$$\begin{aligned}\psi_i : \mathcal{O}_B(mK + mP) &\cong f_*(f^*\mathcal{O}_B(mK + mP)) \\ &\overset{\psi}{\cong} f_*\mathcal{O}_S(mK + m\Theta) \\ &\cong f_*(\mathcal{O}_S(mK + m\Theta)|C_i) \\ &\cong (f_i)_*\mathcal{O}_{C_i}(mK + m\,\mathrm{Diff}(\Theta - C_i)).\end{aligned}$$

Then

$$\psi_2 \circ \psi_1^{-1} : (f_1^{-1} \circ f_2)^*\mathcal{O}_{C_1}(mK + m\,\mathrm{Diff}(\Theta - C_1)) \to \mathcal{O}_{C_2}(mK + m\,\mathrm{Diff}(\Theta - C_2))$$

and the natural isomorphism

$$(f_1^{-1} \circ f_2)_* : (f_1^{-1} \circ f_2)^*\mathcal{O}_{C_1}(mK + m\,\mathrm{Diff}(\Theta - C_1)) \to \mathcal{O}_{C_2}(mK + m\,\mathrm{Diff}(\Theta - C_2))$$

differ by the sheaf multiplication (-1).

Proof. Let $h : (S', \Theta') \to (S, \Theta)$ be a proper morphism such that $K + \Theta' \equiv h^*(K + \Theta)$. Then the theorem holds for (S, Θ) iff it holds for (S', Θ'). Thus as in (11.2.4) we may reduce to the case when S is smooth, and then by contracting (-1)-curves in the fibers we may assume that $f : S \to B$ is a $\mathbb{P}^1$-bundle. Thus Θ consists of two sections and some fibers (with coefficients), which clearly implies (1). (2) and (3) are not affected by the vertical components of Θ, thus we may even assume that $\Theta = C_1 \cup C_2$. By further elementary transformations we may also assume that C_1 and C_2 are disjoint. It is now clear that

$$\psi : \mathcal{O}_S(K + C_1 + C_2) \cong f^*\mathcal{O}_B(K).$$

In order to see (3) we may restrict our attention to a local chart on B. Thus S is of the form $\mathbb{P}^1 \times B$. Let $(s : t)$ be coordinates on $\mathbb{P}^1$ and let $C_1 = (s = 0)$ and $C_2 = (t = 0)$. Let z be a parameter on B and let $g(z)dz$ be a 1-form. Under the isomorphism ψ we obtain

$$\psi^*(g(z)dz) = \lambda \frac{ds}{s} \wedge g(z)dz,$$

where λ is an unknown constant. Thus ψ_1 is given by

$$\psi_1(g(z)dz) = \lambda g(f_1^*(z))d(f_1^*(z)).$$

Changing from s to t we obtain

$$\psi^*(g(z)dz) = -\lambda \frac{dt}{t} \wedge g(z)dz,$$

hence

$$\psi_2(g(z)dz) = -\lambda g(f_2^*(z))d(f_2^*(z)).$$

This proves (3). □

12.4 Abundance

In this section we present a proof of (12.1.1).

Let $f : T \to S$ be a minimal semiresolution. By (12.2.8.1) there is a boundary Δ_T on T such that (T, Δ_T) is log canonical and $K + \Delta_T = f^*(K + \Delta)$. Thus abundance for (S, Δ) is equivalent to abundance for (T, Δ_T). In several instances it will be convenient to consider only the case when our surface S is already semismooth.

12.4.1 Claim. *(12.1.1) is true if $\nu = 0$ and we are in case (1) of (12.3.3).*

Proof. We may assume S is semismooth. Choose m such that $m(K + \Theta_i)$ is a linearly trivial Cartier divisor for every i. We claim that $12m(K + \Delta) \sim 0$.

In order to see this we have to choose sections $\sigma_i \in \mathcal{O}_{S_i}(12m(K + \Theta_i))$ such that they patch together along the double curves. By assumption $\llcorner \Theta_i \lrcorner$ is connected and $K + \Theta_i$ is numerically trivial; thus

$$H^0(\llcorner \Theta_i \lrcorner, \mathcal{O}_{\llcorner \Theta_i \lrcorner}(12m(K + \operatorname{Diff}(\Theta_i - \llcorner \Theta_i \lrcorner))))$$

is one dimensional, and it contains a unique normalized section ρ_i. Choose σ_i such that it restricts to ρ_i. If $C \subset \llcorner \Theta_i \lrcorner$ is a proper subcurve then $\rho_i | C$ is the unique normalized section of $\mathcal{O}_C(12m(K + \operatorname{Diff}(\Theta_i - \llcorner \Theta_i \lrcorner))|C)$. Thus the σ_i automatically patch together to a global section $\sigma \in H^0(S, \mathcal{O}(12m(K_S + \Delta)))$. □

12.4.2 Claim. *(12.1.1) is true in the following cases:*

(1) $\nu = 0$ *in case (2) of (12.3.3); and*

(2) $\nu = 1$ *provided* $\nu(S_i, \Theta_i) = 1$ *for every irreducible component* S_i *of* S^ν *and* $S_i \cap S_j$ *has no vertical components for* $i \neq j$.

Proof. Let $\mu : S^\mu \to S$ be the normalization and let $D_i \subset S_i$ be the inverse images of the double curves. By assumption D_i has one or two irreducible components. Moreover, except when D_i is irreducible, it makes sense to talk about horizontal and vertical components of Θ_i. If $\nu = 0$ then (12.3.1.3) provides a morphism onto a curve, in the second case the morphism is given by abundance for (S_i, Θ_i).

By suitable indexing of the components S_i $(1 \leq i \leq n)$ of S^μ we may assume the following conditions

$$\llcorner \Theta_i \lrcorner = D_i^- \cup D_i^+ \cup (\text{vertical parts}) \qquad (D_1^- \text{ or } D_n^+ \text{may be empty}); \text{ and}$$
$$D_i^+ \cong \mu(D_i^+) = \mu(D_{i+1}^-) \cong D_{i+1}^- \quad \text{for } 1 \leq i \leq n-1.$$

We distinguish two cases according to the behaviour of μ on the curves D_1^- and D_n^+.

(chain) $D_1^- \to \mu(D_1^-)$ and $D_n^+ \to \mu(D_n^+)$ are isomorphisms and $\mu(D_1^-) \neq \mu(D_n^+)$. If $D_1^- \to \mu(D_1^-)$ or $D_n^+ \to \mu(D_n^+)$ is two-to-one, let τ_1 (resp. τ_n) denote the corresponding involution of D_1^- (resp. D_n^+). Otherwise let τ_1 and τ_n be the identity.

(cycle) $D_n^+ \cong \mu(D_n^+) = \mu(D_1^-) \cong D_1^-$.

The following obvious proposition describes $H^0(S, \mathcal{O}(mK + m\Delta))$ in terms of S^μ:

12.4.3 Proposition. *Suppose that m is sufficiently divisible. Set*

$$\begin{aligned} H(i) &= H^0(S_i, \mathcal{O}(mK + m\Theta_i)) \\ H(i^-) &= H^0(D_i^-, \mathcal{O}(mK + m\,\mathrm{Diff}(\Theta_i - D_i^-)) \\ H(i^+) &= H^0(D_i^+, \mathcal{O}(mK + m\,\mathrm{Diff}(\Theta_i - D_i^+)), \end{aligned} \tag{12.4.3.1}$$

and let

$$\begin{aligned} \psi_i^- &: H(i) \to H(i^-) \\ \psi_i^+ &: H(i) \to H(i^+) \\ \phi_i &: H(i^+) \to H((i+1)^-) \\ \phi_n &: H(n^+) \to H(0^-) \quad \textit{(for cycle only)} \end{aligned} \tag{12.4.3.2}$$

be the natural isomorphisms.

Then the sections of $H^0(S, \mathcal{O}(mK + m\Delta))$ *are exactly those sequences* $\{\eta_i \in H(i)\}$ *which satisfy the following assumptions:*

(chain) $\psi_{i+1}^-(\eta_{i+1}) = \phi_i(\psi_i^+(\eta_i))$, $\phi_1^-(\eta_1)$ *is* τ_1*-invariant and* $\phi_n^+(\eta_n)$ *is* τ_n*-invariant.*

(cycle) $\psi_{i+1}^-(\eta_{i+1}) = \phi_i(\psi_i^+(\eta_i))$ *and* $\psi_1^-(\eta_1) = \phi_n(\psi_n^+(\eta_n))$. □

The choice of η_1 and the compatibility conditions $\psi_{i+1}^-(\eta_{i+1}) = \phi_i(\psi_i^+(\eta_i))$ automatically determine the other η_i uniquely. Let η denote any set $\{\eta_i\}$ which satisfy these compatibility conditions.

We also need the following:

12.4.4 Lemma. *The image G of* $\mathrm{Aut}(D_1^-, \mathrm{Diff}(\Theta_1 - D_1^-))$ *in* $H(1^-)$ *is finite.*

Proof. This is clear unless $D_1^- \cong \mathbb{P}^1$. If this holds then $\mathrm{Diff}(\Theta_1 - D_1^-)$ is klt in case $\nu = 0$ and has degree > 2 in case $\nu = 1$. Thus $\mathrm{Supp}\,\mathrm{Diff}(\Theta_1 - D_1^-)$ consists of ≥ 3 points, hence $\mathrm{Aut}(D_1^-, \mathrm{Diff}(\Theta_1 - D_1^-))$ is itself finite. □

12.4.5 Corollary. *Notation as above. Let* $G = \{g_1, \dots, g_k\}$. *Then*

$$(g_1^*(\eta) \otimes g_2^*(\eta) \otimes \cdots \otimes g_k^*(\eta))^{\otimes 2}$$

descends to a section of

$$\mathcal{O}_S(2kmK + 2km\Delta).$$

Proof. Note first that by (12.3.4) all the pairs $(D_i^-, \mathrm{Diff}(\Theta_i - D_i^-))$ and $(D_i^+, \mathrm{Diff}(\Theta_i - D_i^+))$ are isomorphic, and thus all the corresponding groups are the same. Furthermore, any isomorphism obtained by a combination of the isomorphisms in (12.4.3.2) is, up to a sign, induced by an isomorphism of the underlying pairs. Therefore, the second set of compatibility conditions are satisfied for η up to an element of G and up to a sign.

Therefore, in the cycle case, there is an element $g \in G$ such that

$$\psi_1^-(\eta_1) = \pm g^*(\phi_n(\psi_n^+(\eta_n))),$$

and similarly for chains. By taking the product over all $g_i \in G$ and taking the square we get rid of the ambiguities. □

12.4.6 Claim. *(12.1.1) is true if* $\nu = 1$.

Proof. Let (S, Δ) be slc with $\nu = 1$. As we remarked earlier, it is sufficient to consider the case when S is semismooth, and hence D is smooth. Let $D = D_0 \cup D_1$, where D_0 is the union of those irreducible components D^j such that $\nu(D^j, K_S + \Delta) = 0$ and at least one of the irreducible components S_i containing D^j has $\nu = 1$.

Let $\pi : S' \to S$ be the morphism obtained by normalizing in a neighborhood of D_0. The connected components of (S', Θ') have either $\nu = 0$ or $\nu = 1$ and they satisfy the assumptions of (12.4.2.2). Thus abundance holds for (S', Θ'). We need to analyze the patching of sections along $\pi^{-1}(D_0)$.

12.4.7 Lemma. *Assume that (S, Δ) satisfies the assumptions of (12.4.2.2). Let $p : S \to B$ be the morphism given by a large multiple of $K + \Delta$. Let Δ' be the vertical part of Δ. Then $\llcorner \Delta' \lrcorner$ is the union of fibers of p. In particular for every irreducible $C \subset \llcorner \Delta' \lrcorner$ the restriction $(C, \operatorname{Diff}_C(\Delta - C))$ is either not klt or C is a smooth elliptic curve and $\operatorname{Diff}_C(\Delta - C) = 0$. Furthermore, there are sections*

$$\tau \in H^0(S, \mathcal{O}_S(2mK + 2m\Delta))$$

whose restriction to $\llcorner \Delta' \lrcorner$ is the unique normalized section of

$$\mathcal{O}_{\llcorner \Delta' \lrcorner}(2m(K + \Delta)|\llcorner \Delta' \lrcorner).$$

These sections have no common zeros.

Proof. The first claim follows from (12.3.2) applied to the normalization of S. Let $b_i \in B$ be the points corresponding to $\llcorner \Delta \lrcorner$. For some $m > 1$ we have

$$\mathcal{O}_S(mK + m\Delta) = p^*(\mathcal{O}_B(mK + m\sum[b_i] + mP))$$

for some $\mathbb{Q}$-divisor P. Since $K_B + \sum[b_i] + P$ is ample, for $m \gg 1$, it follows that there are sections of $\mathcal{O}_B(mK + m\sum[b_i] + mP)$ taking any preassigned value at the points b_i. Furthermore these sections will not have any common zeros. $\square$

12.4.7.1 Complement. It is easy to see that (12.4.7) also holds if (S, Δ) is a semi-smooth surface, B is an affine curve and $p : S \to B$ is a proper and flat morphism such that $K + \Delta$ is p-trivial and every double curve of S is horizontal.

Now we can finish the proof of (12.4.6). By (12.4.7) and (12.4.1) we can choose sections of $\mathcal{O}_{S'}(2mK + 2m\Theta')$ which induce the unique normalized section of

$$\mathcal{O}(2mK_{S'} + 2m\Theta'|\pi^{-1}(D_0)).$$

These sections will descend to S and they have no common zeros. $\square$

12.5 Hodge Theory

In this section we prove (12.1.2). The following lemma is useful in comparing the cohomology of S, with that of a partial resolution of S.

12.5.1 Lemma. *Consider the following commutative diagram of Abelian groups*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\alpha"'] & W' \arrow[r] \arrow[d, "\omega"'] & B' \arrow[r, "d'"] \arrow[d, "\beta"'] & C' \arrow[d, "\gamma"'] \\ A \arrow[r] & W \arrow[r] & B \arrow[r, "d"] & C. \end{tikzcd}$$

If the rows are exact, α and β are surjective, and

$$d'(\ker\beta) = \operatorname{im} d' \cap \ker\gamma, \tag{12.5.1.1}$$

then ω is surjective. (The last condition holds for example if there are compatible splittings β' and γ' of the maps β and γ, or if γ is an isomorphism.)

Proof. An easy diagram chase, left to the reader. □

We first prove (12.1.2) assuming that S is semismooth.

12.5.2 Lemma. *If S is semismooth then the natural map*

$$i_p : H^p(S, \mathbb{C}) \longrightarrow H^p(S, \mathcal{O}_S) \qquad \text{is surjective for every } p.$$

Proof. Let $g : S^\mu \longrightarrow S$ be the normalization of S; S^μ is smooth. We compare the cohomology of S and S^μ. There are two relevant exact sequences:

$$0 \longrightarrow \mathbb{C}_S \longrightarrow g_*\mathbb{C}_{S^\mu} \longrightarrow \mathcal{G} \longrightarrow 0. \tag{12.5.3}$$

$$0 \longrightarrow \mathcal{O}_S \longrightarrow g_*\mathcal{O}_{S^\mu} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{12.5.4}$$

We identify the sheaves $\mathcal{F}$ and $\mathcal{G}$, which are defined at the moment as cokernels in (12.5.3–4).

D_μ is smooth and maps two-to-one to $D = D_S$. Let τ be the natural involution on D_μ. The involution τ acts naturally on the sheaves $g_*(\mathcal{O}_{D_\mu})$ and $g_*(\mathbb{C}_{D_\mu})$. Under this action, these sheaves decompose into invariant and anti-invariant parts; the sheaves $\mathcal{F}$ and $\mathcal{G}$ are then the anti-invariant parts. Let P be the union of all the pinch points and let $L^2 \cong \mathcal{O}(P)$ the line bundle defining the double cover. It is an easy computation to check that $\mathcal{F} = L^{-1}$.

Now we compare the two long exact sequences of (12.5.3) and (12.5.4):
(12.5.5)

$$\begin{tikzcd} H^{p-1}(D,\mathcal{G}) \arrow[r] \arrow[d, "k_{p-1}"'] & H^p(S,\mathbb{C}) \arrow[r] \arrow[d, "i_p"'] & H^p(S^\mu,\mathbb{C}) \arrow[r, "c_p"] \arrow[d, "j_p"'] & H^p(D,\mathcal{G}) \arrow[d, "k_p"'] \\ H^{p-1}(D,\mathcal{F}) \arrow[r] & H^p(S,\mathcal{O}_S) \arrow[r] & H^p(S^\mu,\mathcal{O}_{S^\mu}) \arrow[r, "d_p"] & H^p(D,\mathcal{F}). \end{tikzcd}$$

Here the diagram commutes, and the horizontal sequences are exact. As previously observed, since S^μ is smooth the map j_p is surjective.

Now we have to find compatible splittings of the maps j_p and k_p; these are given by Hodge theory. In fact the cohomology groups $H^p(D_\mu,\mathbb{C})$ decompose into invariant and anti-invariant subspaces under the action of τ and $H^p(D,\mathcal{G})$ is just the anti-invariant part. As such $H^p(D,\mathcal{G})$ inherits a filtration from the natural Hodge filtration on $H^p(D_\mu,\mathbb{C})$. Now consider the commutative square

(12.5.6)
$$\begin{tikzcd} H^p(S^\mu,\mathbb{C}) \arrow[r, "e_p"] \arrow[d, "j_p"'] & H^p(D_\mu,\mathbb{C}) \arrow[d] \\ H^p(S^\mu,\mathcal{O}_{S^\mu}) \arrow[r, "f_p"] & H^p(D_\mu,\mathcal{O}_{D_\mu}). \end{tikzcd}$$

Clearly the maps e_p and f_p preserve the Hodge filtrations. But the horizontal maps c_p and d_p of (12.5.5) factor through the horizontal maps e_p and f_p of (12.5.6). Thus there is a natural splitting of the map k_p, compatible with the splitting of j_p. Now apply (12.5.1) to deduce i_p is surjective. $\square$

We are now in a position to prove (12.1.2).

Proof. Let $f : T \longrightarrow S$ be a semiresolution of S. By (12.5.2), the natural maps

$$j_p : H^p(T,\mathbb{C}) \longrightarrow H^p(T,\mathcal{O}_T)$$

are surjective.

We wish to compare the cohomology of T and S. There are two relevant spectral sequences; the Leray–Serre spectral sequences associated to the map f and the sheaves $\mathbb{C}_T$, $\mathcal{O}_T$. The respective E_2 terms of the two spectral sequences are $H^p(S,R^qf_*\mathbb{C}_T)$ and $H^p(S,R^qf_*\mathcal{O}_T)$. Both spectral sequences degenerate at the E_3 level, and converge to $H^*(T,\mathbb{C})$ and $H^*(T,\mathcal{O}_T)$ respectively.

Let F be the exceptional locus of the map f. As F is one dimensional, the only interesting cohomology groups to identify at the E_2 level are

$$H^0(S,R^1f_*\mathbb{C}_T) = H^1(F,\mathbb{C}_F) \quad \text{and} \quad H^0(S,R^1f_*\mathcal{O}_T) = H^1(F,\mathcal{O}_F).$$

The first identification is easy; given any open neighbourhood of F, we can always find a smaller one which retracts to F. For the second we use (12.2.8). In fact if we push down the short exact sequence

$$0 \longrightarrow \mathcal{O}_T(-F) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_F \longrightarrow 0$$

by f, we obtain a sequence

$$0 \longrightarrow R^1 f_* \mathcal{O}_T(-F) \longrightarrow R^1 f_* \mathcal{O}_T \longrightarrow R^1 f_* \mathcal{O}_F = H^1(F, \mathcal{O}_F) \longrightarrow 0.$$

The two spectral sequences give rise to the following commutative diagram of cohomology groups, with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H^1(S, \mathbb{C}_S) \arrow[r] \arrow[d, "i_1"] & H^1(T, \mathbb{C}_T) \arrow[r] \arrow[d, "j_1"] & H^1(F, \mathbb{C}_F) \arrow[d, "k_1"] \\ 0 \arrow[r] & H^1(S, \mathcal{O}_S) \arrow[r] & H^1(T, \mathcal{O}_T) \arrow[r] & H^1(F, \mathcal{O}_F). \end{tikzcd} \tag{12.5.7}$$

We apply (12.5.1). We need to find a compatible splitting for k_1. Let F_j be the connected compnents of F. By (12.2.8) these come in three types. If F_j is a tree of rational curves then $H^1(F_j, \mathbb{C}) = 0$. If F_j is a cycle of rational curves then $H^1(F_j, \mathbb{C}) \to H^1(F, \mathcal{O}_F)$ is an isomorphism. Finally if F_j is a smooth elliptic curve then $H^1(T, \mathbb{C}_T) \to H^1(F, \mathbb{C}_F)$ factors through $H^1(S^\mu, \mathbb{C})$ hence the splitting of $k_1|F_j$ provided by Hodge decomposition works.

i_0 is automatically surjective, and there is a similar commutative diagram

$$\begin{tikzcd} H^1(F, \mathbb{C}_S) \arrow[r] \arrow[d, "k_1"] & H^2(S, \mathbb{C}_T) \arrow[r] \arrow[d, "i_2"] & H^2(T, \mathbb{C}_F) \arrow[r] \arrow[d, "j_2"] & 0 \\ H^1(F, \mathcal{O}_S) \arrow[r] & H^2(S, \mathcal{O}_T) \arrow[r] & H^2(T, \mathcal{O}_F) \arrow[r] & 0 \end{tikzcd} \tag{12.5.8}$$

(12.5.1.1) is vacuously satisfied, hence i_2 is surjective. □

12.5.8 Remark. One can see that the kernel of i_p in (12.1.2) is precisely $F^1 H^p(S, \mathbb{C}_S)$ (given by the natural mixed Hodge structure, cf. [Griffiths-Schmid73]). The proof given above could have been shortened by using more difficult Hodge theoretic methods.

13. ABUNDANCE FOR THREEFOLDS, CASE $\nu(X) = 1$

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This chapter treats the proof of the following result, proved in [Miyaoka88b] (see (11.1.2) for definitions):

13.1 Theorem. *Let X be a minimal threefold. If the numerical dimension $\nu(X)$ is one, then $|mK_X|$ is base point free for some $m > 0$.*

The main steps in the proof are almost identical to those of Chapter 11, but of course some steps are harder. Here is a generalization of (11.3.2) to dimension three.

13.2 Lemma. *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair (2.13.5), $\dim X = 3$. Suppose there is a nef divisor $D \in |m(K_X + \Delta)|$ such that $X \setminus D$ has terminal singularities. Let $B = D_{\mathrm{red}}$. Then there is a threefold $\hat{X}$, with boundary $\hat{\Delta} + \hat{B}$, where $\hat{B}$ is reduced, such that:*

1. *The pair $(\hat{X}, \hat{\Delta} + \hat{B})$ has $\mathbb{Q}$-factorial log canonical singularities, $\hat{X} \setminus \hat{B}$ is isomorphic to $X \setminus B$ and there is a divisor $\hat{D} \in |m(K_{\hat{X}} + \hat{\Delta} + \hat{B})|$. Moreover $\hat{D}_{\mathrm{red}} = \hat{B}$.*
2. *$K_{\hat{X}} + \hat{\Delta} + \hat{B}$ is nef.*
3. *$\nu(X, K_X + \Delta) = \nu(\hat{X}, K_{\hat{X}} + \hat{\Delta} + \hat{B})$ and $\kappa(X, K_X + \Delta) = \kappa(\hat{X}, K_{\hat{X}} + \hat{\Delta} + \hat{B})$.*

Proof. By (6.16.1) or (20.9) there is a projective partial resolution of singularities $\mu : X_0 \to X$ such that

1. the divisor $B_0 = (\mu^* B)_{\mathrm{red}}$ is a normal crossing divisor,
2. $\mu : (X_0 \setminus B_0) \to (X \setminus B)$ is an isomorphism.

As (X, Δ) has klt singularities, $m(K_{X_0} + \mu_*^{-1}\Delta + E) = \mu^* D + \Gamma = \tilde{D}$, where E is the union of the μ-exceptional divisors and Γ is effective and supported on the exceptional locus. In particular, $\operatorname{Supp} \tilde{D} = \operatorname{Supp} \mu^* D$. Now we replace

$\mu_*^{-1}\Delta$ with Δ_0 where we only include those components of $\mu_*^{-1}\Delta$ which are not components of B_0. With this choice of Δ_0, $\Delta_0 + B_0$ is a boundary, and there is a divisor $D_0 \in |m(K_{X_0} + \Delta_0 + B_0)|$

We now apply the log minimal model program to $(X_0, \Delta_0 + B_0)$. We construct X_i, Δ_i, B_i and D_i satisfying (1) inductively. If $K_{X_i} + \Delta_i + B_i$ is not nef, there exists an elementary contraction $\phi_i : X_i \to Z_i$ [KMM87, 4-2-1 and 3-2-1] associated to a log extremal ray with respect to $K_{X_i} + \Delta_i + B_i$. Clearly the map ϕ_i is birational.

If ϕ_i is a divisorial contraction, we set $X_{i+1} = Z_i$, $\Delta_{i+1} = \phi_{i*}(\Delta_i)$, $B_{i+1} = \phi_{i*}B_i$ and $D_i = \phi_{i*}D_i$. (By [KMM87, 5-1-6], the image $(X_{i+1}, \Delta_{i+1} + B_{i+1})$ is $\mathbb{Q}$-factorial log terminal, and (13.2.4) implies that B_{i+1} and D_{i+1} are divisors.)

Otherwise there is a log flip, i.e., a small birational morphism $\phi_i^+ : X_{i+1} \to Z_i$. We take Δ_{i+1}, B_{i+1} and D_{i+1} to be the birational transforms of B_i and D_i under ϕ_i. (By [KMM87, 5-1-11] the log flip $(X_{i+1}, \Delta_{i+1} + B_{i+1})$ is log canonical and $\mathbb{Q}$-factorial in a neighborhood of B_{i+1}. The pluricanonical class pushes across the flip, because it may be defined using differential forms on a complement of any codimension 2 locus.)

$K_{X_i} + \Delta_i + B_i$ is negative relative to the morphism ϕ_i. Since $D_i \in |m(K_{X_i} + \Delta_i + B_i)|$ is supported on B_i, the exceptional locus of ϕ_i is contained in B_i. By (7.1) the process we have just described must terminate at some i, and we set $\hat{X} = X_i$, $\hat{\Delta} = \Delta_i$, $\hat{B} = B_i$ and $\hat{D} = D_i$.

Conditions (1) and (2) are automatic from the construction. (3) follows from (11.3.3) and (13.2.4) applied to the pullbacks of the divisors $m(K_X + \Delta)$ and D_i to a common resolution. □

13.2.4 Lemma. *The set theoretic image of an effective nef divisor under a birational morphism is divisorial.*

Proof. Let $f : X \to Y$ be a birational morphism, and let L be an effective nef $\mathbb{Q}$-Cartier divisor on X. We may assume that L is Cartier. Let $M = f_*L$ be the cycle theoretic push forward, and let $M_0 = f(\operatorname{Supp} L)$ be the set theoretic image. Write $M_0 - \operatorname{Supp} M = C_0 \cup \ldots \cup C_i$ where C_i are distinct irreducible components. By taking generic hyperplane sections of Y, we may assume that $\min\{\dim C_i\} = 0$. Using generic hyperplane sections on X we may assume $\dim X = 2$. Choosing a resolution of singularities for X and pulling back L, we may assume that X is smooth. But by the Hodge index theorem the intersection matrix of divisors supported on the exceptional locus of f over C_i is negative definite, and L is supported on this locus near C_i, a contradiction. □

13.3 Conclusion of Proof of (13.1).

Let X be a minimal threefold, i.e. a threefold with $\mathbb{Q}$-factorial terminal singularities such that K_X is nef. Suppose that $D \in |mK_X|$ (9.0.6) and let

$B = D_{\text{red}}$.

13.3.1 Lemma. *There is a threefold $\hat{X}$, birational to X, with reduced boundary $\hat{B}$ such that:*

(1) *The pair $(\hat{X}, \hat{B})$ is $\mathbb{Q}$-factorial and log canonical, $\hat{X} \setminus \hat{B}$ has terminal singularities, and there is a divisor $\hat{D} \in |mK_{\hat{X}}|$. Moreover $\hat{D}_{\text{red}} = \hat{B}$.*

(2) *$K_{\hat{X}} + \hat{B}$ is nef.*

(3) *$\nu(X) = \nu(\hat{X}, K_{\hat{X}} + \hat{B})$ and $\kappa(X) = \kappa(\hat{X}, K_{\hat{X}} + \hat{B})$.*

Proof. This is an immediate consequence of (13.2.1). □

13.3.2 Lemma. *There is a threefold X', birational to X, with a reduced boundary B' satisfying conditions (1–3) of (13.2.1) and*

(4) *every connected component of B' is irreducible.*

Proof. It remains to modify $\hat{X}$ further to achieve (4). Suppose S is a prime component of $\hat{B}$ which is not isolated in $\hat{B}$. We will apply the log minimal model program to $K_{\hat{X}} + \hat{B} - S$.

Suppose we have constructed a sequence of pairs (X_j, B_j) $((X_0, B_0) = (\hat{X}, \hat{B}))$ and birational morphisms $\phi_j : X_j \dashrightarrow X_{j+1}$, with respect to $K_{X_j} + B_j - S_j$ for $j \leq i-1$, where B_{j+1} and S_{j+1} are respectively either ${\phi_j}_*(B_j)$ and ${\phi_j}_*(S_j)$, if ϕ_j is a divisorial contraction, or the birational transforms of B_j and S_j under ϕ_j, if ϕ_j is a log flip. As in (13.2.1), (X_j, B_j) satisfies properties (1–3).

Suppose S_i is still not isolated in B_i. Then there is another component S' of B_i which meets S_i in a curve C (recall X_i is $\mathbb{Q}$-factorial). Let H be an ample divisor and set $C' = H \cap S'$. Let L_i be the line bundle $\mathcal{O}_{X_i}(m(K_{X_i} + B_i))$. It is automatic that $\nu(S, L_i|_S) = 0$ and so deg $L_i|_{C'} = 0$, as the curve C' lies in S'. On the other hand, as H is ample, $S \cdot C' = H \cdot C > 0$, and so

$$(K_{X_i} + B_i - S_i) \cdot C' < 0.$$

As L_i is nef, the Theorem on the Cone [KMM87, 4.2.1] implies there is a log extremal ray R such that

$$(K_{X_i} + B_i - S_i) \cdot R < 0.$$

with $L \cdot R = 0$. Now $S \cdot R > 0$ and the support of the base locus of K_{X_i} is a subset of supp (B_i) and so $R \subset$ supp $(B_i - S)$. By (8.1), there is a log flip of R with respect to L and by (7.1) this sequence of log flips terminates. Thus at some stage S_i is isolated in B_i.

However if T is another prime component of B' and T is isolated in B', then T_i (the component of B_i corresponding to T) is isolated in B_i (as each

ϕ_j only modifies points of S_j). In this way we isolate every component of B', one by one. $\square$

Proof of (13.1.1). Pick a component S of B', and put $\Delta = \mathrm{Diff}_S(0)$. By (16.9.1), the pair (S, Δ) is semi log canonical and so (12.1.1) implies $K_S + \Delta$ is torsion. Just as before, by (11.3.6), we may find a finite Galois cover $\pi : \tilde{U} \longrightarrow U_1$, étale in codimension one, such that

$$\tilde{S} \sim m'\tilde{G}, \qquad K_{\tilde{U}} + \tilde{S} \sim d'\tilde{G}, \qquad \text{and} \qquad \omega_{\tilde{S}} = \mathcal{O}_{\tilde{U}}(\tilde{S})|_{\tilde{S}} = \mathcal{O}_{\tilde{S}}.$$

where $\tilde{S} = \pi^* S$. Now if we can apply (11.3.7), then we may conclude just as in Chapter 11. Conditions (1) and (2) of (11.3.7) are automatic.

Consider the commutative square

$$\begin{array}{ccc} H^p(\tilde{S}_n, \mathbb{C}) & \longrightarrow & H^p(\tilde{S}_n, \mathcal{O}_{\tilde{S}_n}) \\ \downarrow & & \rho\downarrow \\ H^p(\tilde{S}, \mathbb{C}) & \xrightarrow{i_p} & H^p(\tilde{S}, \mathcal{O}_{\tilde{S}}). \end{array}$$

where $\tilde{S}_n$ is defined as in Chapter 11. As the first vertical map is an isomorphism (the support of $\tilde{S}$ and $\tilde{S}_n$ are the same), and the map i_p is surjective (this is (12.1.2)), the map ρ is surjective as well, which is condition (3) of (11.3.7). $\square$

14. ABUNDANCE FOR THREEFOLDS,

$\nu(X) = 2$ IMPLIES $\kappa(X) \geq 1$

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We continue our treatment of Miyaoka's and Kawamata's proof of the abundance conjecture for threefolds. In this chapter, we look at the case $\nu(X) = 2$. The method we use in sections (14.3-4) is due to Kollár.

14.1 A special case

Let us first consider the following very special case, which gives some idea about the line of proof in the general case.

Assume X is a *smooth* minimal model with $\nu(X) = 2$, and assume the existence of a smooth member $D \in |mK_X|$. As $\nu(D, K_D) = 1$, by abundance for surfaces (11.3.1), $\kappa(D) = 1$ and D is an elliptic surface over some curve.

Let H be a hyperplane section of X. Kodaira vanishing on H gives

$$H^i(X, mK_X + lH) \simeq H^i(X, mK_X + (l+1)H)$$

for $i \geq 2$, $m > 1$ and $l \geq 0$. But since H is ample, this group vanishes for large l. Therefore we get

$$H^i(X, mK_X) = 0 \quad \text{for} \quad i \geq 2.$$

We now use Riemann–Roch. Since $K_X^3 = 0$, the coefficient of the leading (linear) term in $\chi(nK_X)$ is $K \cdot c_2(X)$, which is proportional to $c_2(D)$, which is nonnegative [BPV84, p. 188]. Hence $\chi(nK_X) \geq C$ for some constant C, and $h^0(X, nK_X) \geq C + h^1(X, nK_X)$. From the exact sequence

$$0 \to \mathcal{O}_X((n-m)K_X) \to \mathcal{O}_X(nK_X) \to \mathcal{O}_D(nK_X|_D) \to 0,$$

we get $h^2(D, nK_X|_D) = 0$ and

$$H^1(X, nK_X) \to H^1(D, nK_X|_D) \to 0.$$

Riemann–Roch on D implies that $\chi(D, nK_X|_D)$ is constant. Since $\nu(K_X|_D) = 1$, the abundance theorem on surfaces implies that both $h^0(D, nK_X|_D)$ and $h^1(D, nK_X|_D)$ grow with n. Hence $h^1(X, nK_X)$ grows with n. This proves that $\kappa(X) > 0$.

We begin with a construction similar to that of (13.2).

14.2 Lemma. *There is a normal threefold X', birational to X, with reduced boundary B', and such that*

(1) *(X', B') has $\mathbb{Q}$-factorial log canonical singularities and $(X', 0)$ has only log terminal singularities. There exists a divisor $D' \in |mK_{X'}|$. Moreover $B' = D'_{\mathrm{red}}$.*
(2) *$K_{X'} + B'$ is nef.*
(3) *$\nu(X) = \nu(X', K_{X'} + B')$ and $\kappa(X) = \kappa(X', K_{X'} + B')$.*
(4) *$L' = m(K_{X'} + B')$ is Cartier.*
(5) *If C is a curve in X' with $C \cdot (K_{X'} + B') = 0$, then $C \cdot K_{X'} \geq 0$. If D is a curve in X' with $D \cdot (K_{X'} + B') > 0$, then (X', B') is log terminal along the generic point of D.*

Proof. As in (13.2), we can apply the log minimal model program to construct a threefold $\tilde{X}$, with boundary $\tilde{B}$ satisfying (1), (2) and (3).

Replacing m by a multiple, we can assume that $L = m(K_{\tilde{X}} + \tilde{B})$ is a Cartier divisor.

We construct X' inductively. Take $X'_0 = \tilde{X}$, $B'_0 = \tilde{B}$ and $D'_0 = \tilde{D}$. If there exists a curve C in X'_i such that $C \cdot (K_{X'_i} + B'_i) = 0$ and $C \cdot K_{X'_i} < 0$, then there is some $K_{X'_i}$ extremal ray R_i lying on the hyperplane $\{\Gamma \mid \Gamma \cdot (K_{X'_i} + B'_i) = 0\}$. We have a divisorial contraction or a log flip $\phi_i : X'_i \dashrightarrow X'_{i+1}$ associated with R_i. Put $B'_{i+1} = \phi_{i*}(B'_i)$ and $D'_{i+1} = \phi_{i*}(D'_i)$. Then (1–4) are clearly satisfied. Since $(\tilde{X}, 0)$ has log terminal singularities, this process will stop and gives X'.

It remains to check inductively that if we contract a divisor by ϕ_i, we still have log terminal singularities generically along curves having positive intersection with $K_{X'_{i+1}} + B'_{i+1}$. Since $\nu(B'_i, L'_i|_{B'_i}) = 1$, (12.1.1) implies that $|m'L'|_{B'}|$ defines a morphism f from B'_i to some curve. Let S be a component of B'_i on which L'_i is not numerically trivial, and let $\lambda : S^\lambda \to S$ be the normalization. Consider the different Θ defined by $\lambda^*(K_{X'_i} + B'_i) = K_{S^\lambda} + \Theta$ (cf. (16.6)). Θ lies over the nonnormal locus of B'_i and the singular locus of X'_i. Let Θ_h be the horizontal part of Θ. If the generic fibre F of $f \circ \lambda$ is a smooth elliptic curve, then $\Theta_h = 0$. Otherwise $F \cong \mathbb{P}^1$, and $\Theta_h \cdot F = 2$. Decompose Θ_h as $\sum c_k \Gamma_k + \sum d_l \Delta_l$ in a neighborhood of F, where the Γ_k map under λ to the singular locus of X'_i and the Δ_l to the nonnormal locus of B'_i. Then

$$\sum c_k + \sum d_l = 2. \tag{14.2.1}$$

By the inductive assumption, (X'_i, B'_i) has log terminal singularities along $\lambda(\Gamma_k)$ and $\lambda(\Delta_l)$. In particular, X'_i is smooth along $\lambda(\Delta_l)$ and $d_l = 1$, while along $\lambda(\Gamma_k)$, X'_i has index m_k quotient singularities ($m_k \geq 2$) and $c_k = 1 - \frac{1}{m_k}$

(See (16.6)). The set of solutions $(d_1, \cdots; m_1, m_2, \cdots)$ of (14.2.1) can be easily enumerated:

Case 1 (1,1;),

Case 2 (1;2,2), and

Case 3 (;2,4,4), (;3,3,3), (;2,3,6), and (;2,2,2,2).

Suppose that ϕ_i contracts a $K_{X_i'}$ extremal curve C on X_i'. $C \cdot D_i' < 0$ implies that $C \subset B_i'$. $C \cdot (K_{X_i'} + B_i') = 0$, and hence C is contained in a fibre of f. $C \cdot B_i' > 0$ implies that C has to intersect a component of B_i' positively. We see at once that ϕ_i can never contract components that are in Case 3. In Case 1, S intersects two other components S_1 and S_2 of B_i'. X_i' is smooth in a neighborhood of $\lambda(F)$, hence X_{i+1}' is generically smooth along the intersection of $\phi_i(S_1)$ and $\phi_i(S_2)$. In Case 2, X_i' has only two curves of A_1-singularities in a neighborhood of $\lambda(F)$, hence it is canonical. Therefore X_{i+1}' has terminal singularities in a neighborhood of $\phi_i(\lambda(F))$ (2.28.3), thus X_{i+1}' is generically smooth along $\phi_i(S)$. (In fact one can see that in this case $\lambda(F) \cdot K \geq 0$, and therefore we never have to contract $\lambda(F)$.) □

14.3 Computing the second Todd class.

We now proceed with the proof of the abundance conjecture and establish an inequality involving the second Todd class on a resolution of X'. This is used in the final step when we apply Riemann–Roch.

14.3.1 Lemma. *X', B' and L' as in section 14.2. Let $\mu : V \to X'$ be a resolution of singularities. Then we have*

$$\mu^* L' \cdot \left(K_V^2 + c_2(V)\right) \geq L' \cdot \left(K_{X'}^2 + \hat{c}_2(\hat{\Omega}^1_{X'})\right).$$

Proof. Let

$$\mu_* \left(K_V^2 + c_2(V)\right) - \left(K_{X'}^2 + \hat{c}_2(\hat{\Omega}^1_{X'})\right) = \sum a_i C_i.$$

Then all the 1-cycles C_i are supported on the singular locus of X', and in particular they lie in B'. (By (13.2.4) X' has isolated singularities outside B'.) Because we are interested in the intersection of $\sum a_i C_i$ with $L' = m(K_{X'}+B')$, we only need to consider 1-cycles on components S of B' on which $\nu(L') \neq 0$, and focus on cycles C_i 'horizontal' to the map f defined in (14.2). They are contained in the Θ_h considered in (14.2) and we have a complete list of possible singularities there.

We can compute the numbers a_i by taking a transversal slice Γ_i at a general point P_i on C_i, and reduce the computation to the surface case. Let $\mu : \tilde{\Gamma}_i = \mu^{-1}(\Gamma_i) \to \Gamma_i$ be the resolution induced by μ. Notice that the number $c_1^2 + c_2$

does not change on blowing up a smooth point of a surface. We may assume that μ_i is the minimal resolution. If P_i is an index m_i point, then (10.8) (with $B = 0$) tells us that

(14.3.1.2)

$$\begin{aligned} a_i &= \mu_* \left(K_{\tilde{\Gamma}_i}^2 + c_2(\tilde{\Gamma}_i)\right) - \left(K_{\Gamma_i}^2 + \hat{c}_2(\Gamma_i)\right) \\ &= \left(K_{\tilde{\Gamma}_i} - \mu^* K_{\Gamma_i}\right)^2 + e_{\mathrm{top}}(\mu^{-1}(P_i)) - \frac{1}{m_i}. \end{aligned}$$

If the singularity is a Du Val singularity, then we can work out by explicit computation that a_i is $\frac{3}{2}$, $\frac{8}{3}$, $\frac{15}{4}$ and $\frac{35}{6}$, when m_i is 2, 3, 4 and 6 repectively. Otherwise, a_i is $\frac{4}{3}$, $\frac{3}{4}$ and $-\frac{5}{6}$, when m_i is 3, 4 and 6 respectively. Now an index 6 point is always accompanied by an index 2 point and an index 3 point, hence the sum of the corresponding a_i is at least 2. This completes the proof of the lemma. □

14.3.2 Lemma. $L' \cdot \hat{c}_2(\hat{\Omega}^1_{X'}) \geq L' \cdot \hat{c}_2\left(\hat{\Omega}^1_{X'}(\log B')\right) - L' \cdot (K_{X'} + B') \cdot B'$.

Proof. By (10.8.8), the difference $\hat{c}_2(\hat{\Omega}^1_{X'}) - \hat{c}_2\left(\hat{\Omega}^1_{X'}(\log B')\right) - (K_{X'} + B') \cdot B'$ is an effective 1–cycle supported on the singular locus of X'. □

14.3.3 Lemma. *Let $\mu : V \to X'$ be a resolution of singularities. Then we have*

$$\mu^* L' \cdot (K_V^2 + c_2(V)) \geq 0.$$

Proof. By (14.3.1) and (14.3.2), we have

$$\mu^* L' \cdot (K_V^2 + c_2(V)) \geq L' \cdot K_X^2 + L' \cdot \hat{c}_2\left(\hat{\Omega}^1_{X'}(\log B')\right) - L' \cdot (K_{X'} + B') \cdot B'.$$

It follows from (10.13) that $L' \cdot \hat{c}_2\left(\hat{\Omega}^1_{X'}(\log B')\right) \geq 0$. $L' = m(K_{X'} + B')$ and $\nu(B', L'|_{B'}) = 1$, so that $L' \cdot (K_{X'} + B') \cdot B' = 0$. Write $K_{X'}$ as $\sum b_i S_i$, where $b_i \geq 0$ and S_i are components of B'. Moreover $S_i \cdot L'$ is equivalent to an effective sum of curves having zero intersection with $(K_{X'} + B')$. By condition (5) of (14.2), this implies $S_i \cdot L' \cdot K_{X'} \geq 0$. Hence $L' \cdot K_{X'}^2 \geq 0$. This completes the proof of the lemma. □

14.3.4 Remarks.

(i) From the proof we see that the inequality in (14.3.3) is strict, unless the map f_0 has smooth elliptic fibres on all the components of B where $\nu(L') = 1$.

(ii) The above proof works in any dimension.

14.4 Proving that $\kappa(X) > 0$. We now can prove the main theorem along the lines of the smooth case as in (14.1).

14.4.1 Theorem. [Kawamata91b] *Let X be a minimal 3-fold over $\mathbb{C}$. Suppose that $\nu(X) = 2$. Then $\kappa(X) > 0$.*

Proof. Construct X' and L' as in (14.2). Let $\mu : V \to X'$ be a desingularization of X'. Since X' is log terminal, X' has only rational singularities. Therefore

(14.4.1.1)

$$\begin{aligned}\chi(X', nL') =&\chi(V, n\mu^* L') \\ =&\frac{n}{12}\left(K_V^2 + c_2(V)\right) \cdot \mu^* L' + \chi(\mathcal{O}_V).\end{aligned}$$

(14.3.3) shows that the linear term in (14.4.1.1) is nonnegative. Therefore

(14.4.1.2) $$\chi(X', nL') \geq C \qquad \text{for some constant } C.$$

Now look at the exact sequence:

(14.4.1.3) $$0 \to \mathcal{O}_{X'}(nL'(-B')) \to \mathcal{O}_{X'}(nL') \to \mathcal{O}_{B'}(nL'|_{B'}) \to 0.$$

Recall that $L' = m(K_{X'} + B')$, thus $nL'(-B') \equiv K_{X'} + (nm - 1)M'$ where $M' = K_{X'} + B'$. Take a general ample hyperplane section H' of X'. Using the restriction exact sequence and the Kawamata–Viehweg vanishing theorem [KMM87, 1-2-5] we see that

$$H^i\left(X', nL'(-B') + lH'\right) \simeq H^i\left(X', nL'(-B') + (l+1)H'\right)$$

for $i \geq 2$ and $l \geq 0$. The last group vanishes when l is large, thus $H^2\left(X', nL'(-B')\right) = 0$. Moreover, since B' is Cohen-Macaulay,

$$h^2\left(B', nL'|_{B'}\right) = h^0\left(B', \omega_{B'}(-nL'|_{B'})\right) = 0$$

for n large. Therefore we have $H^2(X', nL') = 0$ for large n. Combined with (14.4.1.2), this shows that

$$h^0(X', nL') \geq h^1(X', nL') + C.$$

Thus it is sufficient to prove that $h^1(X', nL')$ grows linearly with n. Note that $\chi(X', K_{X'} + (n-1)L') = -\chi(X', (1-n)L')$ has the same linear term as in (14.4.1.1). Hence it follows from (14.4.1.3) that $\chi(B', nL'|_{B'})$ is actually a constant. Then (12.1.1) together with the vanishing of $H^2(B', nL'|_{B'})$ implies that both $h^0(B', nL'|_{B'})$ and $h^1(B', nL'|_{B'})$ grow with n. We have

$$H^1(X', nL') \to H^1(B', nL'|_{B'}) \to H^2(X', nL'(-B')) = 0.$$

This shows that $h^1(X, nL')$ grows with n and completes the proof of the theorem. $\square$

15. LOG ELLIPTIC FIBER SPACES

JÁNOS KOLLÁR

The aim of this chapter is to complete the proof of abundance for threefolds. Instead of the cohomological approach of [Kawamata85] we present a rather geometric one. Many of the arguments work for an arbitrary nef divisor B such that $\nu(B) = 2$ and $\kappa(B) \geq 1$. The underlying variety can have arbitrary dimension or even positive characteristic. We however formulate everything for a klt divisor $K_X + \Delta_X$ in characteristic zero, where the necessary flips are known to exist.

15.1 Definition. (15.1.1) A *log elliptic fiber space* is a proper morphism $g : (V, \Delta_V) \to W$ such that $g_* \mathcal{O}_V = \mathcal{O}_W$, the generic fiber E_g is an irreducible curve and $(K_V + \Delta_V) \cdot E_g = 0$.

(15.1.2) Let (X, Δ_X) be a log variety. A *log elliptic structure* on X is a diagram

$$\begin{array}{ccc} (X, \Delta_X) & \xleftarrow{\ h\ } & (V, \Delta_V) \\ & & \big\downarrow{\scriptstyle g} \\ & & W \end{array}$$

where h is a birational morphism, $g : (V, \Delta_V) \to W$ is a log elliptic fiber space and $K_V + \Delta_V = h^*(K_X + \Delta_X) + F$, where F is effective and $\operatorname{Supp} F$ contains every h-exceptional divisor.

15.1.3 Comments. The second definition is motivated by two examples. First, assume that (V, Δ_V) is a log elliptic fiber space and assume that (X, Δ_X) is obtained from (V, Δ_V) by $(K_V + \Delta_V)$-extremal contractions and flips. Then (X, Δ_X) has a log elliptic structure (we may have to blow up V a little).

Second, if X has terminal singularities, $\Delta_X = 0$ and X is birational to an elliptic fiber space $(V, 0)$ then X has an elliptic structure.

15.2 Proposition. *Let (X, Δ_X) be a proper klt variety. Assume that $K_X + \Delta_X$ is nef and that X has a log elliptic structure. Then there is an open set $U \subset X$ and a proper morphism $f_U : U \to Z$ which is a log elliptic fiber space.*

Proof. Let $E_g \subset V$ be the generic fiber of g. Then

$$0 = E_g \cdot (K_V + \Delta_V) = E_g \cdot h^*(K_X + \Delta_X) + E_g \cdot F \geq E_g \cdot F.$$

Thus F is disjoint from E_g and h is an isomorphism in a neighborhood of E_g. □

For higher dimensional fibers the situation is more complicated. The following result (which is not used in the sequel) generalizes [Grassi91, 1.8].

15.3 Theorem. *Let X be a proper variety with $\mathbb{Q}$-factorial terminal singularities. Assume that $mK_X = 0$ for some $m > 0$ and $\rho(X) = 1$. Let $p : X \dashrightarrow Z$ be a dominant rational map with connected fibers. Then $p^{-1}(z)$ is of general type for every general $z \in Z$.*

Proof. Let $g : Y \to X$ be a proper birational morphism such that $f = p \circ g : Y \to Z$ is a morphism. Let $E \subset Y$ be the exceptional divisor of g. We may assume that Y is smooth. Let $H \subset Z$ be a divisor. Then $g(f^*(H))$ is an effective divisor on X, hence ample.

$$g^*(g(f^*(H))) = f^*(H) + F_1 \quad \text{where } \operatorname{Supp} F_1 \subset E.$$

Let $z \in Z - H$ be a point such that $f^{-1}(z)$ is smooth and $g|f^{-1}(z)$ is birational. Then

$$g^*(g(f^*(H)))|f^{-1}(z) = F_1|f^{-1}(z)$$

is the pull back of an ample divisor by the birational morphism $g|f^{-1}(z)$. In particular it is big. On the other hand, $K_Y = g^*K_X + F_2$ where $\operatorname{Supp} F_2 = E$. Thus

$$mK_{f^{-1}(z)} = mK_Y|f^{-1}(z) = mF_2|f^{-1}(z).$$

Since $\operatorname{Supp} F_1 \subset \operatorname{Supp} F_2$ this implies that $K_{f^{-1}(z)}$ is big. □

15.4 Theorem. *Let (X, Δ_X) be a projective $\mathbb{Q}$-factorial threefold such that $K_X + \Delta_X$ is klt. Assume that*

(15.4.1) $K_X + \Delta_X$ is nef;

(15.4.2) $\dim |m(K_X + \Delta_X)| \geq 1$ for some $m > 0$;

(15.4.3) there is an open set $U \subset X$ and a proper morphism $f_U : U \to Z$ which is a log elliptic fiber space.

Then $K_X + \Delta_X$ is eventually free.

15.4.4 Remark. If $p : X \to Y$ is a morphism such that $K_X + \Delta_X = p^*(K_Y + \Delta_Y)$ then $K_X + \Delta_X$ is eventually free iff $K_Y + \Delta_Y$ is. Similarly, if $p : X \dashrightarrow Y$

is a $(K_X + \Delta)$-flop then $K_X + \Delta_X$ is eventually free iff $K_Y + p_*(\Delta_X)$ is. We use these observations to change X.

15.5 End of the proof of (11.1.1). Let X be a minimal threefold. The only case still open is when $\nu(X) = 2$. We woud like to check the conditions of (15.4) in case $\Delta_X = 0$. (15.4.1) is assumed and (14.4.1) shows (15.4.2). (15.4.3) requires a little work.

Lt (X', B') be as in (14.2). B' is a semi log canonical surface with $\nu(B') = 1$. Thus by (11.3.1) it has an irreducible component which is birational to either a ruled or to an elliptic surface. We already know that $\dim |m(K_X + \Delta_X)| \geq 1$. Assume that we can construct (X', B') such that B' moves in a pencil. We obtain that X' contains a pencil of ruled or elliptic surfaces. X' is not uniruled, thus it has a pencil of elliptic surfaces. Therefore X is birational to an elliptic threefold, hence (15.2) implies (15.4.3).

Let us go back to the construction in (14.2) which was started in (13.2). (We use the notation employed there.) If $D \in |m(K_X + \Delta_X)|$ moves in a pencil then we can choose $\mu : X_0 \to X$ such that $\tilde{D}$ still moves in a pencil. This pencil survives in all the contractions and flips. At the end we obtain (X', B') as in (14.2) such that B' moves in a pencil $\{B'_t\}$ and (X', B'_t) is log canonical for general t. At least one of the moving components of B'_t has $\nu(B'_t) = 1$. Thus the above argument applies and (15.4) completes the proof of the abundance theorem for threefolds. □

15.6 Definition. We say that an effective divisor $D \subset X$ is $(K_X + \Delta_X)$-*trivially connected* if for any two points $x_1, x_2 \in D$ there is a connected curve $x_1, x_2 \in C \subset D$ such that $K_X + \Delta_X$ is numerically trivial on every irreducible component of C.

15.7 Lemma. *Assume (15.4.1 and 2). Let $D \subset X$ be $(K_X + \Delta_X)$-trivially connected. Then one of the following holds:*

(15.7.1) $K_X + \Delta_X$ is eventually free and is composed of a pencil,

(15.7.2) there is an effective divisor D' and natural numbers d, m such that $dD + D' \in |m(K_X + \Delta_X)|$, $\operatorname{Supp} D \not\subset \operatorname{Supp} D'$ and $D \cap D' \neq \emptyset$.

Proof. Let $|m(K_X + \Delta_X)| = F + |M|$ where F is the fixed part. Assume first that $|M|$ is composed of a free pencil. Let $p : X \to C$ be the corresponding morphism with connected fibers. Assume that we can not find $dD + D'$ as required. Then $\operatorname{Supp} D$ is a fiber of p, hence F is contained in a union of fibers. Since F is nef, F is the sum of rational multiples of fibers, hence some multiple of $K_X + \Delta_X$ is the pull-back of an ample divisor from C.

Otherwise there is a pencil $F' + |N_t| \subset |m(K_X + \Delta_X)|$ such that every N_t is connected and $|N_t|$ has a base point $b \in X$. $D \subset X$ is $(K_X + \Delta_X)$-trivially connected, thus if $B \in |m(K_X + \Delta_X)|$ intersects D then D is an irreducible component of B. If $b \in D$ then $D \subset F$ and any general N_t intersects D but is different from it. If $b \notin D$ then there is a t_0 such that N_{t_0} intersects D. N_{t_0} is connected and also contains b, thus we are again done. □

15.8 Lemma. *Assumptions as in (15.7) and assume that (15.7.2) holds. Then $K_X + \Delta_X + \epsilon D$ is not nef for $\epsilon > 0$.*

Proof. By assumption there is an irreducible curve $C \subset D$ such that $C \cdot (K_X + \Delta_X) = 0$ satisfying $C \cap D' \neq \emptyset$ and $C \not\subset D'$. Thus

$$0 = C \cdot (dD + D') = dC \cdot D + C \cdot D', \quad \text{hence} \quad C \cdot D < 0.$$

Therefore $C \cdot ((K_X + \Delta_X) + \epsilon D) = \epsilon C \cdot D < 0$. □

15.9 Corollary. *Assumptions as in (15.4). Then one of the following holds:*

(15.9.1) $|n(K_X + \Delta_X)|$ is composed of a free pencil for some $n > 0$; or

(15.9.2) there is a log variety $(X', \Delta_{X'})$ which is log birational to (X, Δ_X) and satisfies all the assumptions of (15.4) and such that X' does not contain any $(K_{X'} + \Delta_{X'})$-trivially connected divisors.

Proof. Assume that X contains a $(K_X + \Delta_X)$-trivially connected divisor D. Then either (15.7.1) holds or $K_X + \Delta_X + \epsilon D$ is not nef. After a sequence of D-flops (with respect to $K_X + \Delta_X$) the birational transform of D becomes contractible. For this it is sufficient to observe that the birational transform of D under a sequence of flops stays $(K + \Delta)$-trivially connected. The general fiber of the elliptic fibration is disjoint from D, thus (15.4.3) is preserved under flops and $(K + \Delta)$-trivial contractions. Repeating this procedure, we eventually stop at X'. □

15.10 Theorem. *Assumptions as in (15.4). Assume furthermore that X does not contain any $(K_X + \Delta_X)$-trivially connected divisors. Then f_U extends to a morphism $f : X \to \bar{Z}$ with 1-dimensional fibers.*

Proof. By shrinking Z we may assume that f_U is flat. Thus we get a morphism $Z \to \text{Chow}(X)$. (See [Hodge-Pedoe52, X.6-8] for basic results about Chow varieties.) Let $\bar{Z}$ be the normalization of the closure of the image and let $g : \bar{U} \to \bar{Z}$ be the universal family. Let $u : \bar{U} \to X$ be the natural morphism. We prove that u is an isomorphism.

u is an isomorphism over $g^{-1}(Z)$. Assume that $F \subset \bar{U}$ is a divisor contracted by u. Then $g(F)$ is at most one dimensional. Since g has one dimensional fibers, $g(F)$ is one dimensional. Let $E = g^{-1}(g(F))$. $\dim u(E) = 2$ since a 1-dimensional subvariety of X supports only countably many different cycles in $\text{Chow}(X)$. (This is the point where we need Chow instead of Hilb.) Thus there are divisors $E_1, E_2 \subset E$ such that

$$\dim u(E_1) = 2; \quad \dim u(E_2) \leq 1 \quad \text{and} \quad E_1 \cap E_2 \text{ dominates } g(F).$$

We claim that $u(E_1)$ is $(K_X+\Delta_X)$-trivially connected. Indeed, $u(E_1\cap E_2)\subset u(E_2)$ and every curve in $u(E_2)$ has zero intersection with $K_X+\Delta_X$. Any two points of $u(E_1)$ can be connected by images of fibers of $E_1\to g(F)$ and by $u(E_1\cap E_2)$.

This contradiction shows that u does not contract any divisors. Since X is $\mathbb{Q}$-factorial, u can not contract curves, and thus u is an isomorphism. □

15.11 Lemma. *Let X be a variety with log terminal singularities. Let $f: X\to Z$ be a proper morphism onto a normal variety Z such that every fiber has dimension k for some fixed k. If $\dim Z>2$ then assume that K_Z is $\mathbb{Q}$-Cartier. Then Z has only log terminal singularities.*

Proof. Choose a projective embedding of X. Fix $z\in Z$. Let $H\subset X$ be a complete intersection of k general hyperplanes. $H\to Z$ is dominant and we may assume that $H\to Z$ is finite over z. H has a log terminal singularities (cf. [Reid80, 1.13]) thus by (20.3.1) Z has a log terminal singularity at z. □

15.11.1 Remark. Shokurov pointed out that under the assumptions of (15.11) if X is $\mathbb{Q}$-factorial then so is Z.

15.12 Proposition. *Let $f:(X,\Delta_X)\to Z$ be a log elliptic fiber space with 1-dimensional fibers. Assume that (X,Δ_X) is lc and nef. Then there is a line bundle L on Z such that*

$$n(K_X+\Delta_X)\sim f^*L \quad \text{for some } n>0.$$

Proof. A general fiber E_g of f is either an elliptic curve (which is disjoint from Δ_X) or is a rational curve. In either case a multiple of $K_X+\Delta_X$ is linearly equivalent to zero on the generic fiber. Thus there is a (not necessarily effective) divisor D which is disjoint from E_g and is linearly equivalent to $n_0(K_X+\Delta_X)$ for some $n_0>0$. Let $C_i\subset Z$ be the irreducible components of $f(\operatorname{Supp} D)$. We can write $D=\sum D_i$ where the D_i are those components that map onto C_i. Let z_i be a general point of C_i. Then D_i is nef on $f^{-1}(z_i)$, thus D_i is a rational multiple of $f^*(C_i)$. Hence $n_iD_i=f^*(m_iC_i)$ for some $n_i>0$ (possibly $m_i<0$). Choose M such that

$$M\sum\frac{m_i}{n_i}C_i$$

is Cartier. Then

$$Mn_0(K_X+\Delta_X)\sim f^*\mathcal{O}_Z\left(M\sum\frac{m_i}{n_i}C_i\right). \quad \square$$

(15.13) Proof of (15.4). If $|n(K_X + \Delta_X)|$ is composed of a base point free pencil then we are done. Otherwise $\nu(K_X + \Delta_X) \geq 2$.

By (15.9) there is a series of flops and $(K + \Delta)$-trivial contractions $X \dashrightarrow X'$ such that X' does not contain $(K + \Delta)$-trivially connected surfaces. By (15.4.4) it is sufficient to show that $K_{X'} + \Delta_{X'}$ is eventually free. (15.10) gives a proper morphism $f : X' \to \bar{Z}$ and by (15.12) there is a line bundle L on $\bar{Z}$ such that $n(K_{X'} + \Delta_{X'}) \sim f^*L$.

I claim that L is ample. This is proved using the Nakai–Moishezon criterion. Let H be ample on X and let E_g be a general fiber of f. Then

$$(E_g \cdot H)(L \cdot L) = H \cdot f^*L \cdot f^*L = n^2 H \cdot (K_{X'} + \Delta_{X'}) \cdot (K_{X'} + \Delta_{X'}) > 0.$$

If $C \subset Z$ is an irreducible curve such that $C \cdot L = 0$ then $K_{X'} + \Delta_{X'}$ is numerically trivial on $f^{-1}(C)$, a contradiction. Thus L is ample, and hence a suitable multiple of L is generated by global sections. $\square$

16. ADJUNCTION OF LOG DIVISORS

ALESSIO CORTI

In this chapter we discuss several matters connected with the adjunction formula for a Weil divisor $S \subset X$ inside a normal space X. The first goal is to define a *different* Diff which is a $\mathbb{Q}$-divisor on S so that the following adjunction formula holds:

$$K_S + \mathrm{Diff} = K_X + S \mid_S .$$

[Shokurov91,Ch.3] defines the different as a divisor on the normalization S^ν of S, and uses the notion to establish some elementary properties of log terminal singularities. However, it is desirable to deal with the reduced part of the boundary of a log divisor without normalizing it. For this reason we define the different directly on S.

Once the different is defined, we use it to relate properties of (X, S) to (S, Diff).

We begin with some preliminaries on Weil divisors on nonnormal varieties. In the following, X is a pure dimensional reduced scheme. After (16.7) we always assume that X is defined over an algebraically closed field of characteristic zero. X may be reducible and not necessarily S_2. $K(X)$ denotes the sheaf of total quotient rings (see e.g. [Hartshorne77, II.6]).

16.1 Definition.

(16.1.1) A *Weil divisorial subsheaf* is a coherent $\mathcal{O}_X$-module $\mathcal{L}$, which is principal in codimension one and saturated, *together with the choice of an embedding* $\mathcal{L} \subset K(X)$. The condition that $\mathcal{L}$ is free in codimension one implies $\mathcal{L} \cong \mathcal{L}^{**}$, provided X is S_2. The embedding $\mathcal{L} \subset K(X)$ is very important, although, following common useage in the literature, I will occasionally be sloppy about it (see 16.3.3).

(16.1.2) Define the product $\mathcal{L} \cdot \mathcal{L}' \subset K(X)$ in the natural way (i.e. $\mathcal{L} \cdot \mathcal{L}'$ is the saturation of the product of sheaves $\mathcal{L}\mathcal{L}' \subset K(X)$). Note that in general the natural homomorphism $\mathcal{L} \otimes \mathcal{L}' \to \mathcal{L} \cdot \mathcal{L}'$ is neither injective nor surjective (it is, however, an isomorphism, whenever $\mathcal{L}$ or $\mathcal{L}'$ is locally $\mathcal{O}_X$-free). We also write $\mathcal{L}^{[n]}$ for the product of $\mathcal{L}$ with itself n-times. With these laws, the set of Weil divisorial subsheaves is a group which we denote by $\mathrm{WSh}(X)$. In a natural way $\mathcal{L}^* = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X) = \mathcal{L}^{-1} = \{x \in K(X) \mid x \cdot \mathcal{L} \subset \mathcal{O}_X\} \subset K(X)$.

Equivalently, let $\mathrm{CDiv}(U)$ be the group of Cartier divisors on a scheme U. Then

$$\mathrm{WSh}(X) = \operatorname{proj}\lim \mathrm{CDiv}(X \setminus S)$$

where the limit is over all closed subschemes $S \subset X$ such that $\operatorname{codim}_X S \geq 2$.

If X is normal then this is the usual definition. However for nonnormal schemes unexpected things can happen. Let for instance

$$X = \operatorname{Spec} \mathbb{C}[x, y, z, z^{-1}]/(x^2 - zy^2).$$

The ideals (x) and (y) define different Weil divisorial subsheaves such that $(x)^{[2]} = (y)^{[2]}$.

(16.1.3) The group of $\mathbb{Q}$-Weil divisorial sheaves is defined as $\mathrm{WSh}(X)_{\mathbb{Q}} = \mathrm{WSh}(X) \otimes \mathbb{Q}$.

(16.1.4) To each unit $x \in K(X)^*$ there is a naturally associated Weil divisorial subsheaf $(x) = x \cdot \mathcal{O}_X \subset K(X)$. We say that two Weil divisorial subsheaves $\mathcal{L}$ and $\mathcal{L}'$ are *linearly equivalent* and write $\mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L}^{-1} \cdot \mathcal{L}' = (x)$ for some $x \in K(X)^*$.

(16.1.5) If $\mathcal{L}$ is a Weil divisorial subsheaf, we define the *support* of $\mathcal{L}$ to be the Zariski closed subset $\mathrm{Supp}(\mathcal{L}) \subset X$ of points where $\mathcal{L} \neq \mathcal{O}_X$.

(16.1.6) $\mathcal{L} \subset K(X)$ is effective if $\mathcal{O}_X \subset \mathcal{L} \subset K(X)$.

16.2 Definition.

(16.2.1) A Weil divisor on X is a formal linear combination:

$$D = \sum n_\Gamma \Gamma,$$

where the sum extends over all points of codimension one $\Gamma \subset X$ such that $\mathcal{O}_{X,\Gamma}$ is a DVR, and n_Γ are integers, only finitely many of which are nonzero. The group of all Weil divisors is denoted by $\mathrm{WDiv}(X)$. As in (16.1.3), $\mathrm{WDiv}(X)_{\mathbb{Q}} = \mathrm{WDiv}(X) \otimes \mathbb{Q}$

(16.2.2) There is a natural injective group homomorphism $\mathrm{WDiv}(X) \ni D \mapsto \mathcal{O}(D) \in \mathrm{WSh}(X)$. Let $\Gamma \subset X$ be a codimension one prime of X, then $\mathcal{O}(D)$ is uniquely determined by $\mathcal{O}(D)_\Gamma = \mathcal{O}_{X,\Gamma}$ if X is not regular at Γ, and $\mathcal{O}(D)_\Gamma = t^{n_\Gamma} \cdot \mathcal{O}_{X,\Gamma}$ if $\mathcal{O}_{X,\Gamma}$ is a DVR.

If $\mathcal{L}$ is a Weil divisorial subsheaf, $\mathcal{L}(D)$ as usual denotes $\mathcal{L} \cdot \mathcal{O}(D)$.

We say that D and D' are linearly equivalent if the corresponding sheaves are.

Also, perhaps inappropriately, we say that a Weil divisorial subsheaf $\mathcal{L} \subset K(X)$ is a Weil divisor if $\mathcal{L} = \mathcal{O}(D)$ for some Weil divisor D. Of course, this is equivalent to saying that no codimension one component of the support of $\mathcal{L}$ is contained in the singular locus of X.

16.3 Remarks and more definitions.

(16.3.1) The inclusion $\mathrm{WDiv}(X) \subset \mathrm{WSh}(X)$ induces an isomorphism

$$\mathrm{WDiv}(X)/\sim\, \cong \mathrm{WSh}(X)/\sim,$$

and we denote any of these two groups by $\mathrm{Weil}(X)$.

(16.3.2) $\mathcal{O}_X \subset K(X)$ is a Weil divisorial subsheaf precisely when X is S_2.

(16.3.3) The dualizing sheaf ω_X (as in [Hartshorne77, III.7], that is, $\omega_X = H^{-d}(\omega^{\cdot}_X)$ if $\omega^{\cdot}_X$ is the normalized dualizing complex) is torsion free of rank one, and admits therefore an embedding $\omega_X \subset K(X)$. Since we also know that ω_X is saturated (see e.g. [Reid80, App. to §1]), ω_X is a Weil divisorial subsheaf precisely when X is Gorenstein in codimension one. This is why later (16.5) we shall assume this condition (which is satisfied for example if X has normal crossings in codimension one). If this is the case then with an appropriate choice of embedding $\omega_X \subset K(X)$, ω_X is actually a Weil divisor, whose linear equivalence class is denoted by K_X.

(16.3.4) Weil divisors and sheaves are codimension one constructions. This means that X may always be replaced with any open subset $U \subset X$ such that $\operatorname{codim}_X(X \setminus U) \geq 2$. This principle is used in many natural constructions like pullbacks and restrictions, as well as in many proofs (sometimes without explicit mention).

(16.3.5) Let $p : X' \to X$ be a finite dominant morphism. There is a natural pullback

$$p^w : \mathrm{WSh}(X) \to \mathrm{WSh}(X').$$

This is defined on $\mathcal{L}$ by taking $U \subset X$ open with $\operatorname{codim}_X(X \setminus U) \geq 2$, and such that $\mathcal{L}$ is locally free on U. Then on $V = p^{-1}(U)$, $p^w(\mathcal{L}) = p^*(\mathcal{L})$ is a locally free subsheaf of $K(V)$, and defines a Weil divisorial subsheaf on X' (16.3.4).

(16.3.6) Similarly, let $i : S \hookrightarrow X$ be a subscheme of pure codimension one. Denote by $\mathrm{WSh}_S(X)$ the subgroup of sheaves $\mathcal{L}$ which are $\mathbb{Q}$-Cartier at all points $P \subset S$ of codimension one, and such that S and $\mathrm{Supp}(\mathcal{L})$ have no common irreducible components (if these conditions are satisfied we say that $\mathcal{L}$ has good support on S). Then we have a natural restriction homomorphism:

$$i^w : \mathrm{WSh}_S(X) \to \mathrm{WSh}(S)_{\mathbb{Q}}.$$

This is defined as follows. If $\mathcal{L}$ is Cartier at points $P \subset S$ of codimension one, let $U \subset X$ be an open subset such that $\operatorname{codim}_X(X\setminus U) \geq 2$, $\operatorname{codim}_S(S\setminus U) \geq 2$ and $\mathcal{L}$ is Cartier on U. Then on $V = S \cap U$, $i^w\mathcal{L}$ is the usual restriction of a Cartier divisor (and, because $\mathcal{L}$ has good support on S, $\mathcal{L} \subset K(X)$ induces $i^w\mathcal{L} \subset K(S)$). This determines $i^w\mathcal{L}$ on S. If $\mathcal{L} \in \mathrm{WSh}_S(X)$, then $\mathcal{L}^{[n]}$ is Cartier at points $P \subset S$ of codimension one for some $n > 0$. $i^w\mathcal{L}$ is defined to be $\frac{1}{n}i^w\mathcal{L}^{[n]}$. This is independent of the choice of n. We also write $\mathcal{L}|S$ instead of $i^w(\mathcal{L})$. The whole point of this construction is of course that we want to define i^w in such a way that it is functorial and a group homomorphism.

Next we state the adjunction formula for a divisor $i : S \hookrightarrow X$. If $\mathcal{F}$ is a sheaf on X, we write $i^*\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_S$ and

$$i^{[*]}\mathcal{F} \overset{\text{def}}{=\!=} \text{saturation of } (i^*\mathcal{F}/\mathrm{Torsion}_{\mathcal{O}_S}(i^*\mathcal{F})).$$

16.4 Proposition. *Let X be a normal scheme (actually it is enough that X is S_2), and $i : S \hookrightarrow X$ a reduced subscheme of pure codimension one. Then there is a canonical isomorphism:*

$$\omega_S = i^{[*]}\omega_X(S).$$

In particular:

(16.4.1) If X is S_3 and S is a Cartier divisor, then $\omega_S = \omega_X \otimes \mathcal{O}_X(S) \otimes \mathcal{O}_S$.

(16.4.2) If $\omega_X(S)$ is locally free and S is S_2, then $\omega_S = \omega_X(S) \otimes \mathcal{O}_S$. In particular ω_S is locally free and S is Gorenstein if it is CM (=Cohen-Macaulay).

(16.4.3) If $\omega_X(S)$ is Cartier at every codimension one point $P \in S$, then $\omega_S = i^w\omega_X(S)$. In particular then S is Gorenstein in codimension one, and choosing suitable embeddings we may write the above isomorphism in the form $K_S = K_X + S|S$.

Proof. By assumption X is CM outside a set Z of codimension three; by considering $X \setminus Z$ we may assume that X is CM.

Along the lines of [Hartshorne77, III.7] it is easy to check that $\omega_S = \mathcal{E}xt^1(\mathcal{O}_S, \omega_X)$ is a dualizing sheaf for S. Applying $\mathcal{H}om_{\mathcal{O}_X}(\,\cdot\,, \omega_X)$ to the exact sequence:

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

(since S is a Weil divisor, $\mathcal{I}_S = \mathcal{O}_X(-S) \subset \mathcal{O}_X$, with the notation of (16.2.2)), we obtain an exact sequence:

$$0 \to \omega_X \to \omega_X(S) \to \omega_S \to 0,$$

which fits into a commutative diagram (with exact rows):

$$\begin{array}{ccccccccc}
 & & \omega_X(S) \otimes \mathcal{O}_X(-S) & \longrightarrow & \omega_X(S) & \longrightarrow & \omega_X(S) \otimes \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \omega_X & \longrightarrow & \omega_X(S) & \longrightarrow & \omega_S & \longrightarrow & 0.
\end{array}$$

This shows that $\alpha : \omega_X(S) \otimes \mathcal{O}_S \to \omega_S$ is *surjective*. S is a Weil divisor, and hence X is smooth at every generic point of S. Therefore α is an isomorphism at generic points of S, so $\omega_X(S) \otimes \mathcal{O}_S/\mathrm{Torsion}_{\mathcal{O}_S}(\omega_X(S) \otimes \mathcal{O}_S) \cong \omega_S$, which is what we want. □

16.4.4 Example. Let $A \subset \mathbb{P}^{n-1}$ be a smooth, projectively normal, Abelian surface and let $X \subset \mathbb{P}^n$ be the cone over A with vertex $x \in X$. Then X is normal, lc and $\omega_X \cong \mathcal{O}_X(-1)$. However X is not S_3. Let $x \in H \subset X$ be a hyperplane section, smooth outside x. H is not normal; let $p : \bar{H} \to H$ be the normalisation. Then

$$\omega_H = p_*(\mathcal{O}_{\bar{H}}) \neq \mathcal{O}_H = \mathcal{E}xt^1(\mathcal{O}_H, \omega_X).$$

The aim is to generalize the adjunction formula (16.4.3) to the case where $\omega_X(S)$ is only $\mathbb{Q}$-Cartier at codimension one points $P \subset S$. This is accomplished in the following:

16.5 Proposition - Definition. *Let X be a normal scheme, $i : S \hookrightarrow X$ a reduced subscheme of pure codimension one. Assume that S is Gorenstein in codimension one and that $\omega_X(S) \in \mathrm{WSh}_S(X)$. Then there is a naturally defined effective different $\mathcal{D}iff(0) \in \mathrm{WSh}(S)_{\mathbb{Q}}$ so that:*

$$\omega_S \cdot \mathcal{D}iff(0) = i^w \omega_X(S).$$

If $B \in \mathrm{WSh}_S(X)_{\mathbb{Q}}$, we also define the different of B by $\mathcal{D}iff(B) = \mathcal{D}iff(0) \cdot i^w B$.

Proof. We systematically remove codimension 2 subsets $Z \subset S$, whenever needed, without warning.

From the adjunction formula (16.4) we know that $\omega_S = \omega_X(S)_S$. Suppose that $\omega_X(S)^{[n]}$ is Cartier at every codimension one point $P \in S$. Consider the sequence of maps

$$\left(\omega_X(S) \otimes \mathcal{O}_S\right)^{\otimes n} \cong \omega_X(S)^{\otimes n} \otimes \mathcal{O}_S^{\otimes n} \to \omega_X(S)^{\otimes n} \otimes \mathcal{O}_S \to \omega_X(S)^{[n]} \otimes \mathcal{O}_S.$$

Taking the quotient by the torsion submodules we obtain

$$\omega_S^{[n]} \xrightarrow{\delta} \omega_X(S)^{[n]} \otimes \mathcal{O}_S$$

which is an isomorphism at the generic points of S because X is normal. δ defines a Weil divisorial subsheaf $\mathcal{D}$ on S so that $\omega_S^{[n]} \cdot \mathcal{D} = i^w \omega_X(S)^{[n]}$. Since the isomorphism of the adjunction formula is natural, $\mathcal{D}$ is well defined (i.e., it does not depend on the embedding $\omega_X(S) \subset K(X)$). Set $\mathcal{D}iff(0) = \frac{1}{n}\mathcal{D}$. □

We now apply the different to study log canonical and log terminal singularities. The nice fact is that if $K_X + S$ is log canonical in codimension two, the different is actually a Weil divisor (i.e., no codimension one component of the support of $\mathcal{D}iff$ is contained in the singular locus of S). Also, under the same assumptions, we compute the different.

16.6 Proposition. *Let X be a normal space, $S \subset X$ a reduced subscheme of pure codimension one and B a $\mathbb{Q}$-Weil divisor. Assume that K_X+S+B is log canonical in codimension two. Then S has normal crossings in codimension one, so the assumptions of (16.5) are satisfied. Moreover the different $\mathcal{D}iff(B)$ is a $\mathbb{Q}$-Weil divisor (that is, no codimension one component of the support of $\mathcal{D}iff(B)$ is contained in the singular locus of S), which is denoted by* $\operatorname{Diff}(B)$.

Let $P \subset S$ be a codimension one point of S. The following computes the coefficient p of the different $\operatorname{Diff}(0)$ *at P:*

(16.6.1) If S has two branches at P then $P \not\subset \operatorname{Supp} B$ and $p = 0$.
This follows from the more precise result that one of the following holds:

(16.6.1.1) $K + S$ is lt at P, X is smooth at P, and S is a normal crossing divisor at P.

(16.6.1.2) $K + S$ is lc but not lt at P. Then $K + S$ is Cartier at P. More precisely, locally analytically at P, $S \subset X$ is isomorphic to $(C \subset T) \times \mathbb{C}^{d-2}$, where $(C \subset T) \cong \big((xy = 0) \subset \mathbb{C}^2/\mathbb{Z}_m\big)$ and $\mathbb{Z}_m$ acts with weights $(1, q)$ with $(q, m) = 1$.

(16.6.2) If S has one branch at P, and $K + S$ is lc but not lt at P, then $p = 1$.
More precisely $K + S$ has index two at P. Let $\pi : X' \to X$ be the index one cover, and $S' = \pi^{-1}(S)$. Then $S' \subset X'$ is as in (16.6.1.2).

(16.6.3) If S has one branch at P and $K + S$ is lt at P, then, locally analytically at P, $S \subset X$ is isomorphic to $(C \subset T) \times \mathbb{C}^{d-2}$, where $(C \subset T) \cong \big((x = 0) \subset \mathbb{C}^2/\mathbb{Z}_m\big)$ and $\mathbb{Z}_m$ acts with weights $(1, q)$ with $(q, m) = 1$. Also, the local class group $\operatorname{Weil}(\mathcal{O}_{X,P}) \cong \mathbb{Z}_m$, and X is smooth at P iff $m = 1$. In particular:

$$p = \frac{m-1}{m},$$

where m is characterized by any of the following properties:

(16.6.3.1) m is the index of $K + S$ at P;

(16.6.3.2) m is the index of S at P;

(16.6.3.3) m is the order of the cyclic group $\operatorname{Weil}(\mathcal{O}_{X,P})$.

Proof. I may assume that X is a surface. All the statements then follow from the classification of log canonical surface singularities in Chapter 3. That $\mathcal{D}iff(B)$ is a Weil divisor also follows from the classification, more specifically from (16.6.1) above. In (16.6.2), it is easy to check that $K_{S'} = (\pi|S')^w(K_S + P)$. □

16.7 Corollary. *Assumptions as in (16.6). Let $B = \sum b_i B_i$. The coefficient of $[P]$ in $\mathrm{Diff}(B)$ is*

$$\begin{array}{ll} 0 & \textit{in case (16.6.1);} \\ 1 & \textit{in case (16.6.2);} \\ 1 - \dfrac{1}{m} + \sum \dfrac{r_i b_i}{m} & \textit{in case (16.6.3), for suitable } r_i \in \mathbb{N}. \end{array}$$

Proof. In the first two cases $P \notin \operatorname{Supp} B$, so (16.6) applies directly. In the last case the local class group has order m. Thus mB_i is Cartier at P hence $i^w(\mathcal{O}_X(B_i)) = (r_i/m)\mathcal{O}_S(P)$ for some $r_i \geq 0$. $\square$

16.8 Remark. The different is used in the following situation. Let X be a normal variety, and $K_X + S + B$ a log divisor with S reduced and $\llcorner B \lrcorner = 0$. Then if $K_X + S + B$ is lt, it should be true that $K_S + \mathrm{Diff}(B)$ is lt (and conversely) in some suitable sense. Now in general S is a variety with double normal crossings in codimension one and we need to use the appropriate notions of semi log terminal etc. introduced in (12.2).

Unfortunately we encounter the following technical problem:

The birational transform of $S \subset X$ in a log resolution of $(X, S + B)$ is in general not a semi resolution of S since different components may get separated. Also, the exceptional role of higher normal crossing points complicates the formulation of the result (cf. (16.9.2)). (Recent results of Szabó seem to have settled this problem.)

In dimension three one can overcome some of these problems. The results become somewhat cumbersome, mostly due to our choice of definition of log terminal.

16.9 Proposition. *Let X be a normal threefold, $K + S + B$ a log divisor with S reduced. Then:*

(16.9.1) If $K + S + B$ is lc then $K_S + \mathrm{Diff}(B)$ is slc.

(16.9.2) Let $K + S + B$ be dlt, and $f : Y \to X$ a good divisorial resolution. Assume that $\llcorner B \lrcorner = \emptyset$. Then, outside a number of triple normal crossing points at which f is an isomorphism, $K_S + \mathrm{Diff}(B)$ is semi lt. Moreover, S has a semiresolution without pinch points.

Proof. Let us prove (16.9.2) first. Let $S' = f_*^{-1}(S)$. Since $K + S + B$ is lc in codimension 2, S is semismooth outside a finite set. We have by definition:

$$K_Y + S' = f^*(K_X + S + B) + \sum a_i E_i \tag{16.9.3}$$

with all $a_i > -1$ ($\llcorner B \lrcorner = 0$). In particular, f is generically an isomorphism above the normal crossing locus of S. Also, because X is divisorial lt, no component of the double curve of S' is mapped to a point. All this says that $S' \to S$ is a good semiresolution outside the triple points. By our definitions, S' has no pinch points. Note that since f is divisorial, it sends a neighbourhood of the triple normal crossing locus of S' isomorphically to a neighbourhood of the triple normal crossing locus of S. Now from (16.9.3) and (16.5) we get that

$$K_{S'} = (f|S')^*(K_S + \mathrm{Diff}(B)) + \sum a_i E_i|S'. \tag{16.9.4}$$

We see later in (17.5) that S is S_2 and seminormal. This however is not important for the rest of the chapter.

(16.9.1) is similar but easier: it is not true that S' is a semiresolution of S, but this does not affect the slc property (cf. [KSB88, 4.30]). □

16.10 Corollary. *Let $(x \in X)$ be a three dimensional germ, $S \subset X$ a reduced boundary. If $K_X + S$ is divisorial log terminal and S has at least three components at x, $(x \in S \subset X)$ is analytically isomorphic to $\big(0 \in (xyz = 0) \subset \mathbb{C}^3\big)$.*

Proof. By (12.2.7) an slt point cannot have three or more components. □

16.11 Example. The assumption dlt is necessary in (16.9.2) and (16.10). Indeed, let $S \subset X$ be $(xw = 0) \subset ((xy + zw = 0) \subset \mathbb{C}^4)$. Then $K_X + S$ is lt, as can be seen on any of the two standard small resolutions. $K_X + S$ however is not dlt. Here $K_S = K_S + \mathrm{Diff}(0)$ and S has a log canonical quadruple point at the origin.

The rest of the chapter is devoted to the classification of log terminal singularities (X, D) in dimension three where $\llcorner D \lrcorner$ is "large". These results will not be used later. It gives however a good flavour of how to work with log terminal singularities and with the different.

The presence of a reduced boundary imposes strong restrictions on log terminal singularities; an example is (16.10). A key tool in classifying terminal and log terminal singularities are standard coverings of various kinds (cf. [CKM88,6.7]):

16.12 Lemma. *Let $0 \in X$ be a germ of a normal variety, $D \subset X$ a $\mathbb{Q}$-Cartier integral Weil divisor. There is a cyclic covering $p : X' \to X$, which is uniquely determined by the following properties:*

(16.12.1) $p^*D = D' \subset X'$ is a Cartier divisor.

(16.12.2) p is étale in codimension one and is (totally) ramified precisely along the locus where D is not Cartier.

X' can also be characterized as the smallest covering of X such that D' is Cartier. X' is called the index one cover relative to D. □

To a log divisor $K_X + B$ as above one can associate two index one covers $X' \to X$, relatively to $K_X + B$ or B. It is useful to be able to relate the log terminal property of X and X'.

16.13 Lemma. *Let X be a normal variety, $p : X' \to X$ any finite morphism which is étale in codimension one. Then:*

(16.13.1) If X has canonical (terminal) singularities, so does X'.

(16.13.2) Let $B \subset X$ be a boundary (possibly empty) and let $B' = p^*B$.

(16.13.2.1) $K_X + B$ is lc iff $K_{X'} + B'$ is lc.

(16.13.2.2) $K_X + B$ is plt iff $K_{X'} + B'$ is plt

(16.13.2.3) If p is a cyclic cover, X is a threefold and $K_X + B$ is dlt (resp. lt), then so is $K_{X'} + B'$. Furthermore,

$$(B' \subset X') \cong ((xyz = 0) \subset \mathbb{C}^3) \quad \Leftrightarrow \quad (B \subset X) \cong ((xyz = 0) \subset \mathbb{C}^3).$$

Proof. (16.13.1) is [CKM88, 6.7.(ii)]. (16.13.2.1-2) is proved in (20.3). We only prove (16.13.2.3) for dlt here, the lt case is the same. This also illustrates pretty well the difficulties involved in working with the notion of log terminal.

Let $f : Y \to X$ be a good divisorial resolution such that $K_Y + f_*^{-1}B + E = f^*(K_X + B) + \sum a_i E_i$ with all $a_i > 0$ where $E = \sum E_i$ is the f-exceptional divisor. Let $Y' = (Y \times_X X')^\nu$ be the normalized pull back, so that we have a diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & Y \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{p} & X \end{array}$$

Let E' be the f'-exceptional set. The crux of the argument is to be able to construct a good divisorial resolution $\varphi : \tilde{Y} \to Y'$, with the property that the image of the φ-exceptional locus is entirely contained in E'. The point is that since p is étale in codimension one, p' can only be ramified along E, and since E is a normal crossing divisor, Y' has toroidal singularities. Set $B' = p^*(B)$.

Pick a point $q \in f_*^{-1}B$. Choose local coordinates (x, y, z) near $q \in Y$ such that the components of $E \cup f_*^{-1}B$ are the coordinate planes. Locally, the covering is the normalization of

$$(t^d = x^a y^b z^c) \subset \mathbb{C}^1 \times \mathbb{C}^3.$$

The local equation of $f_*^{-1}B$ is one of the following: $(xyz = 0)$, $(xy = 0)$, $(x = 0)$ or $(1 = 0)$. In the first case p' is unramified along the coordinate planes, thus $a = b = c = 0$ and p' is étale above q. In the second case p' is unramified along two of the the coordinate planes, thus $a = b = 0$ and $(f')_*^{-1}B' \subset Y'$ is a (double) normal crossing point. In the third case p' is unramified along one of the the coordinate planes, thus $a = 0$. Let T be the normalization of the surface singularity $(t^d = y^b z^c)$. Then

$$[(f')_*^{-1}B' \subset Y'] \cong [T \times \{0\} \subset T \times \mathbb{C}].$$

Therefore Y' is smooth along $(f')_*^{-1}B'$, except possibly for some curves $C_i \subset Y'$ of cyclic quotient singularities that meet $(f')_*^{-1}B'$ transversally. We begin constructing a resolution by resolving Y' along C_i. (We care only about a neighborhood of $(f')_*^{-1}B'$ in this step.) This gives $\varphi' : Y'' \to Y'$. Y'' is smooth in a neighborhood of $(\varphi')_*^{-1}(f')_*^{-1}B'$, and $(\varphi')_*^{-1}(f')_*^{-1}B' + E''$ is a global normal crossing divisor in a neighborhood of $(\varphi')_*^{-1}(f')_*^{-1}B'$. It is clear that a good divisorial resolution can now be achieved by blowing up centers contained in E'' only (and not intersecting $(\varphi')_*^{-1}(f')_*^{-1}B'$).

The rest is an easy consequence of the log ramification formula (20.2). The situation now is the following:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{p}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{p} & X. \end{array}$$

Here $\tilde{f} : \tilde{Y} \to X'$ is a good divisorial resolution, $\tilde{p}$ is generically finite, and F_j being any $\tilde{f}$-exceptional component, $\tilde{p}(F_j) \subset E_i$ for some f-exceptional component E_i. Write

$$K_{\tilde{Y}} + \tilde{f}_*^{-1}B' = \tilde{f}^*(K_{X'} + B') + \sum b_j F_j.$$

Then if $\tilde{p}^* E_i = \sum_j e_{ij} F_j$, we have $b_j = \sum_i e_{ij} a_i + r_j$ with $r_j \geq 0$, by the log ramification formula. Since $\tilde{p}(F_j) \subset E_i$ for some i, we see that $b_j > 0$.

Finally, if $f : Y \to X$ is not the identity then by our construction $\tilde{f} : \tilde{Y} \to X'$ is not the identity, thus $(B' \subset X')$ is different from $((xyz = 0) \subset \mathbb{C}^3)$. □

16.14 Remark. The converse to (16.13.2.3) is probably also true. Here, however, the problem is to find a suitable resolution of X, without blowing up the double locus of B. This does not follow directly from Hironaka. (Recently Szabó settled this question.)

From now on X is a threefold, and $B \subset X$ a $\mathbb{Q}$-Cartier reduced boundary such that $K_X + B$ is divisorial log terminal. We begin by classifying these singularities.

16.15 Theorem. *Let $x \in B \subset X$ be a three dimensional germ, assume $K + B$ dlt and B $\mathbb{Q}$-Cartier. Then:*

(16.15.1) If B has three components, then $x \in B \subset X$ is analytically isomorphic to

$$0 \in (xyz = 0) \subset \mathbb{C}^3.$$

(16.15.2) If B has two components, both of which are $\mathbb{Q}$-Cartier, then $x \in B \subset X$ is analytically isomorphic to

$$0 \in (xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_m(q_1, q_2, 1) \quad \text{where} \quad (q_1, q_2, m) = 1.$$

(16.15.3) If B has two components, neither of which is $\mathbb{Q}$-Cartier, then $x \in B \subset X$ is analytically isomorphic to

$$0 \in (z = 0) \subset (xy + zf(z,t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_m(q_1, -q_2, 1, a)$$
$$\text{where} \quad (q_i, a, m) = (q_1, q_2, m) = 1.$$

Proof. (16.15.1) is a special case of (16.10), so let's prove (16.15.2–3).

Let $p : X' \to X$ be the index one cover relative to $K_X + B$, and set $B' = p^*B$. By (16.13.2.3), $K_{X'} + B'$ is dlt. Note that $K_B + \operatorname{Diff}(0) = K_B + \sum \frac{m_i - 1}{m_i} P_i$, where $P_i \subset B \subset X$ are codimension two singular points on X as in (16.6.3). Also by (16.6.3), B' is smooth at $p^{-1}(P_i)$, and $p|B'$ is ramified in codimension one precisely at $\sum m_i P_i$. It follows then from (16.13.2.3) that B' has two components. Also then $K_{B'} = (p|B')^*(K_B + \operatorname{Diff}(0))$ is semi log terminal of index one, by (16.9). Then by [KSB88,4.21] $B' = B_1' + B_2'$, where B_1' and B_2' are smooth and cross normally.

In case (16.15.2), each component of B' is $\mathbb{Q}$-Cartier and Cartier in codimension two. It is easy then to show that X' must be smooth along B'. Indeed let $p' : X'' \to X'$ be the index one cover relative to B_1'. Then, since B_1' is Cartier in codimension two, $p|B_1'' : B_1'' \to B_1'$ is unramified in codimension one. It follows that B_1'' is regular in codimension one. But X'' has rational singularities (it is log terminal), hence CM, so B_1'' is also CM, and normal by the Serre criterion. But then $p|B_1'' : B_1'' \to B_1'$ is a split cover, since it is unramified in codimension one and B_1' is smooth. This means that $p' = \mathrm{id}$, and since B_1' is smooth and Cartier, X' is smooth. Now (16.15.2) follows at once: $B' \subset X' \cong (xy = 0) \subset \mathbb{C}^3$, and $x \in B \subset X$ is analytically isomorphic to $0 \in (xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_m(q_1, q_2, q_3)$. We may assume $q_3 = 1$, because p is unramified along $B_1 \cap B_2$, and $(q_1, q_2, m) = 1$ because p is unramified in codimension one.

In case (16.15.3), let $p' : X'' \to X'$ be the index one cover relative to B'. Then it is clear that, as for B', $B'' = B_1'' + B_2''$, where B_1'' and B_2'' are smooth and cross normally. Then, $p'|B'' : B'' \to B'$ is a split covering, since it is unramified in codimension one. It follows that $p' = \mathrm{id}$ and B' is also Cartier. Then X' has cDV singularities and the result follows at once. $\square$

16.16 Remark. It should be possible to check directly (although I did not do it) that the singularities in (16.15.2–3) are dlt.

If B has only one component, it is not possible to give a compact description as above. Even if B is Cartier, we know from inversion of adjunction (16.9) that any $\mathbb{Q}$-Gorenstein deformation (in particular the trivial deformation) of a surface quotient singularity is log terminal. However, under further restrictions, it is possible to come up with a short list:

16.17 Proposition. *[KSB88] Let $x \in B \subset X$ be a three dimensional germ, assume $K + B$ is dlt and B is Cartier. Also assume that X is cDV outside B. Then $x \in B \subset X$ is analytically isomorphic to one of the following:*

(16.17.1) $0 \in (xyz = 0) \subset \mathbb{C}^3$;

(16.17.2) $0 \in (t = 0) \subset (x^2 + f(y,z,t) = 0) \subset \mathbb{C}^4$ *where* $(x^2 + f(y,z,0) = 0)$ *defines a Du Val singularity;*

(16.17.3) $0 \in (t = 0) \subset (xy + f(z^r, t) = 0) \subset \mathbb{C}^3/\mathbb{Z}_r(a, -a, 1, 0)$. $\square$

17. ADJUNCTION AND DISCREPANCIES

JÁNOS KOLLÁR

The aim of this chapter is to investigate the problem posed in Chapter 16 of comparing the discrepancies of $(X, S+B)$ and $(S, \mathrm{Diff}(B))$. Before formulating the first result, we need to define some other variants of discrep(X).

17.1 Definition. Let X be a normal scheme, $D = \sum d_i D_i$ a boundary and let $Z \subset S \subset X$ be closed subschemes. (More generally, we may allow X to be nonnormal as long as the conditions of (2.6) are satisfied.) We use the following refinements of (1.6):

$$
\begin{aligned}
&\mathrm{discrep}(X, D)\\
&\qquad = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \emptyset \neq \mathrm{Center}_X(E)\};\\
&\mathrm{discrep}(\mathrm{Center} \subset Z, X, D)\\
&\qquad = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \emptyset \neq \mathrm{Center}_X(E) \subset Z\};\\
&\mathrm{discrep}(S \cap \mathrm{Center} \subset Z, X, D)\\
&\qquad = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \emptyset \neq S \cap \mathrm{Center}_X(E) \subset Z\};
\end{aligned}
$$

One can also define versions where we allow E to be nonexceptional as well. These are denoted by totaldiscrep. Of course, totaldiscrep = discrep if Z has codimension at least two. We write discrep$(S \cap \mathrm{Center} \neq \emptyset, X, D)$ instead of discrep$(S \cap \mathrm{Center} \subset S, X, D)$ which is misleading in appearance.

17.1.1 Proposition. *(17.1.1.1) Any of the discrepancies defined above is either $-\infty$ or ≥ -1 and the infimum is a minimum.*

(17.1.1.2) For any $Z \subset S \subset X$

$$
\begin{aligned}
\mathrm{discrep}(\mathrm{Center} \subset Z, X, D) &\geq \mathrm{discrep}(S \cap \mathrm{Center} \subset Z, X, D)\\
&\geq \mathrm{totaldiscrep}(X, D);
\end{aligned}
$$

(17.1.1.3) If discrep(Center $\subset Z, X, D) \geq -1$ *then there is an open neighborhood* $Z \subset U \subset X$ *such that* totaldiscrep$(U, D) \geq -1$.

Proof. (17.1.1.2) is clear from the definition.

In order to see the other two claims, take a log resolution $f : Y \to (X, D)$. If $a(E, X, D) \geq -1$ for every divisor $E \subset Y$ then

$$\operatorname{totaldiscrep}(X, D) = \min_E \{a(E, X, D) | E \subset Y\}$$

by (4.12.1.2). Similarly, (4.12.1.1) implies (17.1.1.1) for the other versions.

Assume now that there is a divisor $E \subset X$ such that $a(E, X, D) = -1 - c$ for some $c > 0$. Let $p \in E$ be any point. Choose a general codimension one subvariety $p \in W \subset E$. Let $g_1 : Y_1 \to Y$ be the blow up of W and let $E_1 \subset Y_1$ be the exceptional divisor. If $g_i : Y_i \to Y$ and $E_i \subset Y_i$ are already defined then let $g_{i+1} : Y_{i+1} \to Y_i \to Y$ be the blow up of $E_i \cap (g_i)_*^{-1}(E)$ and let E_{i+1} be the exceptional divisor of $Y_{i+1} \to Y_i$. By an easy computation $a(E_j, X, D) = -jc$. Let $p_j \in E_j$ be a point such that $g_j(p_j) = p$ and let F_j be the divisor obtained by blowing up p_j. Then

$$a(F_j, X, D) \leq -jc + \text{const.} \quad \text{hence} \quad \operatorname{discrep}(\text{Center} \subset f(p), X, D) = -\infty.$$

Choosing p such that $f(p) \in Z$ completes the proof. □

An upper bound is harder to find:

17.1.2 Conjecture. *[Shokurov88] Let $0 \in (X, D)$ be an n-dimensional normal singularity. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then*

$$\operatorname{discrep}(\text{Center} \subset 0, X, D) \leq \dim X - 1,$$

and equality holds only if X is smooth and $0 \notin D$. (cf. (1.8)).

17.1.3 Remark. Assume that the conjecture fails for $0 \in X$. Then (X, D) is terminal. Thus if a list of terminal singularities is known, the conjecture can be verified. Therefore (17.1.2) is trivial if $\dim X \leq 2$. For $\dim X = 3$ it was checked by Markushevich (unpublished).

The following is the easy direction in comparing discrepancies:

17.2 Theorem. *Let X be a variety and let $S + B$ be a Weil divisor. Assume that S is reduced and $K + S + B$ is lc in codimension two. Assume furthermore that $K + S + B$ is $\mathbb{Q}$-Cartier. Let $Z \subset S$ be a closed subscheme. Then*

(17.2.1)

$$\begin{aligned} \operatorname{totaldiscrep}(\text{Center} \subset Z, S, \operatorname{Diff}(B)) &\geq \operatorname{discrep}(\text{Center} \subset Z, X, S + B) \\ &\geq \operatorname{discrep}(S \cap \text{Center} \subset Z, X, S + B). \end{aligned}$$

In particular,

$$\text{totaldiscrep}(S, \text{Diff}(B)) \geq \text{discrep}(X, S + B). \tag{17.2.2}$$

Proof. Set $Z = S$ in (17.2.1) to obtain (17.2.2). Also, the second inequality of (17.2.1) is obvious. For the rest we need a simple lemma which we state in a general setup:

17.2.3 Lemma. *Let $f : Y \to X$ be a proper birational morphism with exceptional divisors E_j. Assume that Y is normal. Let $S + B$ be a $\mathbb{Q}$-divisor on X and let S' be the birational transform of S on Y. Assume that (X, S) and (Y, S') are lc in codimension two. Let $D \subset S$ be the union of all codimension one points of S above which $S' \to S$ is not an isomorphism and let $D' \subset S'$ be the preimage of D. Finally let*

$$K_Y + f_*^{-1}(S + B) \equiv f^*(K_X + S + B) + \sum a(E_j, S + B)E_j.$$

Then

(17.2.4)

$$(f|S')_* \operatorname{Diff}_{S'}\left(f_*^{-1}B - \sum a(E_j, S + B)E_j\right) = \operatorname{Diff}_S(B) + 2[D]; \quad \textit{and}$$

$$K_{S'} + \operatorname{Diff}_{S'}\left(f_*^{-1}B - \sum a(E_j, S + B)E_j\right) \equiv (f|S')^*\left(K_S + \operatorname{Diff}_S(B)\right).$$

Proof. The left hand side of the second eqality is $f^*(K + S + B)|S'$ and the right hand side is $f^*(K + S + B|S)$. Thus the second equality is clear.

The first is a codimension one question on S, so that by shrinking X, we may assume that S is semismooth and $f : S' \to S$ is finite. Assume that $m(K_X + S + B)$ is Cartier. Then

$$\begin{aligned}
&mK_{S'} + m(f|S')_*^{-1}(\operatorname{Diff}(B)) + mD' \\
&\quad = (f|S')^*\left(m(K_S + \operatorname{Diff}(B))\right) \\
&\quad = f^*\left(m(K_X + S + B)|S'\right) \\
&\quad = mK_{S'} + m \operatorname{Diff}_{S'}\left(f_*^{-1}B - \sum a(E_j, S + B)E_j\right),
\end{aligned}$$

where all the equalities are equalities of divisors. Pushing this down to S gives the first equality. □

In order to see (17.2) let $f : Y \to X$ be a log resolution of $(X, S + B)$ with exceptional divisors E_j. Let $E_j \cap S' = \sum C_{jk} + \sum D_{jk}$ where the C_{jk} are the $(f|S')$-exceptional components of the intersection and $f|D_{jk}$ is birational. For simplicity assume that S' is disjoint from $f_*^{-1}(B)$.

Restricting (17.2.4) to S' we obtain:

$$\begin{aligned} K_{S'} &+ (f|S')_*^{-1}(\mathrm{Diff}(B)) + D' \\ &\equiv f^*(K_S + \mathrm{Diff}(B)) + \sum_{j,k} a(E_j, S+B) C_{jk}. \end{aligned}$$

Therefore

$$\text{(17.2.5)} \qquad \begin{aligned} a\left(C_{jk}, S, \mathrm{Diff}(B)\right) &= a(E_j, X, S+B), \quad \text{and} \\ a\left(D_{jk}, S, \mathrm{Diff}(B)\right) &= a(E_j, X, S+B). \end{aligned}$$

Every exceptional divisor over S appears as an irreducible component of $E_j \cap S'$ for a suitable choice of f. The only problem is that $f(C_{jk}) \subset Z$ does not imply $f(E_j) \subset Z$. However if we blow up C_{jk} then we obtain a new exceptional divisor E_{jk} such that

$$f(E_{jk}) = f(C_{jk}) \subset Z \quad \text{and} \quad a(E_{jk}, X, S+B) = a(E_j, X, S+B).$$

This proves (17.2.1). $\square$

The following conjecture asserts that the inequalities in (17.2) are equalities. Special cases were discussed earlier in [KSB88,Chapter 6; Stevens88; Shokurov91,3.3]. The conjecture (or similar results and conjectures) will be frequently referred to as *adjunction* (if we assume something about X and obtain conclusions about S) or *inversion of adjunction* (if we assume something about S and obtain conclusions about X).

17.3 Conjecture. *Notation as in (17.2). Then*

(17.3.1)

$$\begin{aligned} \mathrm{totaldiscrep}(\mathrm{Center} \subset Z, S, \mathrm{Diff}(B)) &= \mathrm{discrep}(\mathrm{Center} \subset Z, X, S+B) \\ &= \mathrm{discrep}(S \cap \mathrm{Center} \subset Z, X, S+B). \end{aligned}$$

In particular,

$$\text{(17.3.2)} \qquad \mathrm{totaldiscrep}\left(S, \mathrm{Diff}\left(B\right)\right) = \mathrm{discrep}\left(\mathrm{Center} \cap S \neq \emptyset, X, S+B\right).$$

Unfortunately, I do not know how to prove these in full generality. The rest of the chapter is devoted to proving some important special cases.

The following technical result is crucial in (17.6–7). It was proved by [Shokurov91,5.7] for surfaces.

17.4 Theorem. *Let X, Z be normal varieties (or analytic spaces) and let $h : X \to Z$ be a proper morphism with connected fibers. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X. Assume that*

(17.4.1) if $d_i < 0$ then $h(D_i)$ has codimension at least two in Z; and

(17.4.2) $-(K_X + D)$ is h-nef and h-big. (If h is birational then h-big is automatic.)

Let

$$f : Y \xrightarrow{g} X \xrightarrow{h} Z$$

be a resolution of singularities such that $\operatorname{Supp} g^{-1}(D)$ *is a divisor with normal crossings. Let*

$$K_Y = g^*(K_X + D) + \sum e_i E_i.$$

Further let

$$A = \sum_{i:e_i > -1} e_i E_i \quad \text{and} \quad F = -\sum_{i:e_i \le -1} e_i E_i.$$

Then $\operatorname{Supp} F = \operatorname{Supp} \llcorner F \lrcorner$ *is connected in a neighborhood of any fiber of f.*

Proof. By definition

$$\ulcorner A \urcorner - \llcorner F \lrcorner = K_Y + (-g^*(K_X + D)) + \{-A\} + \{F\},$$

and therefore by [KMM87,1-2-3]

$$R^1 f_* \mathcal{O}_Y(\ulcorner A \urcorner - \llcorner F \lrcorner) = 0.$$

Applying f_* to the exact sequence

$$0 \to \mathcal{O}_Y(\ulcorner A \urcorner - \llcorner F \lrcorner) \to \mathcal{O}_Y(\ulcorner A \urcorner) \to \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner) \to 0$$

we obtain that

(17.4.3) $$f_* \mathcal{O}_Y(\ulcorner A \urcorner) \to f_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)$$

is surjective. Let E_i be an irreducible component of $\ulcorner A \urcorner$. Then either E_i is g-exceptional or E_i is the birational transform of some D_i and $d_i = -e_i < 0$.

Thus $g_*(\ulcorner A \urcorner)$ is h-exceptional and

$$f_* \mathcal{O}_Y(\ulcorner A \urcorner) = h_*(\mathcal{O}_X(g_*(\ulcorner A \urcorner))) = \mathcal{O}_Z.$$

Assume that $\llcorner F \lrcorner$ has at least two connected components $\llcorner F \lrcorner = F_1 \cup F_2$ in a neighborhood of $f^{-1}(z)$ for some $z \in Z$. Then

$$f_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)_{(z)} \cong f_* \mathcal{O}_{F_1}(\ulcorner A \urcorner)_{(z)} + f_* \mathcal{O}_{F_2}(\ulcorner A \urcorner)_{(z)},$$

and neither of these summands is zero. Thus $f_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)_{(z)}$ cannot be the quotient of the cyclic module $\mathcal{O}_{z,Z} \cong f_* \mathcal{O}_Y(\ulcorner A \urcorner)_{(z)}$. $\square$

17.5 Corollary. *If (X, D) is lt then $\llcorner D \lrcorner$ is seminormal and it has a semiresolution with normal crossing points only. If (X, D) is dlt then $\llcorner D \lrcorner$ is seminormal and S_2. If (X, D) is dlt and every irreducible component of $\llcorner D \lrcorner$ is $\mathbb{Q}$-Cartier then every irreducible component of $\llcorner D \lrcorner$ is normal.*

Proof. We apply (17.4) to $h : X \cong Z$. Let $g : Y \to X$ be a log resolution. Then $F = \llcorner F \lrcorner$ is the birational transform of $\llcorner D \lrcorner$. By assumption F has only normal crossing points. In particular, F is seminormal and S_2. We can successively blow up the normal crossing points of multiplicity at least 3 starting with the highest multiplicity locus to obtain a semiresolution of $\llcorner D \lrcorner$ with normal crossing points only.

By (17.4.3) the composite

$$g_*\mathcal{O}_Y(\ulcorner A \urcorner) \cong \mathcal{O}_X \to \mathcal{O}_{\llcorner D \lrcorner} \hookrightarrow g_*\mathcal{O}_F \hookrightarrow g_*\mathcal{O}_F(\ulcorner A \urcorner)$$

is surjective, and hence

$$\mathcal{O}_{\llcorner D \lrcorner} \cong g_*\mathcal{O}_{\llcorner F \lrcorner}. \tag{17.5.1}$$

Let $n : B \to \llcorner D \lrcorner$ be the seminormalizsation of $\llcorner D \lrcorner$. Then $B \times_n F \to F$ is a homeomorphism, thus an isomorphism. Therefore $F \to \llcorner D \lrcorner$ factors through n. Thus by (17.5.1) $n_*\mathcal{O}_B = \mathcal{O}_{\llcorner D \lrcorner}$, hence n is an isomorphism.

Assume now that (X, D) is dlt. Let $Z \subset \llcorner D \lrcorner$ be a closed subset of codimension ≥ 2. I claim that $Z' = \operatorname{Sing} F \cap g^{-1}(Z)$ has codimension ≥ 2 in F. Assume the contrary. Then there is an irreducible component $Z'' \subset Z'$ such that $Z'' \subset Y$ has codimension two and it is contained in the exceptional set of g. Therefore Z'' is contained in an exceptional divisor E of g. Since $\operatorname{Supp} g^{-1}(D)$ is a normal crossing divisor, there is at most one irreducible component of F containing Z''. This contradicts $Z'' \subset \operatorname{Sing} F$.

Let $n' : B' \to \llcorner D \lrcorner$ be the S_2-ization of $\llcorner D \lrcorner$ [EGA, IV.5.10.16-17]. Then $B' \times_{n'} F \to F$ is finite and birational on every irreducible component. Furthermore, by the above considerations, it is a homeomorphism in codimension one. Since F is seminormal and S_2, this implies that it is an isomorphism. Therefore $F \to \llcorner D \lrcorner$ factors through n'. Thus by (17.5.1) $n'_*\mathcal{O}_{B'} = \mathcal{O}_{\llcorner D \lrcorner}$, hence n' is an isomorphism.

Assume that every irreducible component of $\llcorner D \lrcorner$ is $\mathbb{Q}$-Cartier and let $D_1 \subset \llcorner D \lrcorner$ be an irreducible component. We can replace D by $D' = D - (1/2)(\llcorner D \lrcorner - D_1)$. Then (X, D') is dlt and $\llcorner D' \lrcorner = D_1$. Thus D_1 is seminormal and S_2. By the classification of Chapter 3, it is also smooth in codimension one, hence normal. $\square$

17.5.2 Example. (cf. (16.11)) Let $X = (xy - uv = 0) \subset \mathbb{C}^4$ and

$$D = (x = u = 0) + (y = v = 0) + \frac{1}{2}\sum_{i=1}^{4}(x + 2^i u = y + 2^{-i} v = 0).$$

Then (X, D) is lt and $\llcorner D \lrcorner$ is two planes intersecting at a single point. Thus it is not S_2.

The most important application of the above connectedness result is to the problem of inversion of adjunction. The following theorem shows that in the notation of (17.3)

$$\operatorname{totaldiscrep}(S, \operatorname{Diff}(B)) > -1 \quad \Leftrightarrow \quad \operatorname{discrep}(\operatorname{Center} \cap S \neq \emptyset, X, S+B) > -1.$$

17.6 Theorem. *Let X be normal and let $S \subset X$ be an irreducible divisor. Let B be an effective $\mathbb{Q}$-divisor such that $\llcorner B \lrcorner = \emptyset$ and assume that K_X+S+B is $\mathbb{Q}$-Cartier. Then K_X+S+B is plt in a neighborhood of S iff $K_S+\operatorname{Diff}(B)$ is klt.*

Proof. Let $g : Y \to X$ be a resolution of singularities and as in (17.4) let

$$K_Y = g^*(K_X + S + B) + A - F.$$

Let $S' \subset Y$ be the birational transform of S and let $F = S' \cup F'$. By adjunction

$$K_{S'} = g^*(K_S + \operatorname{Diff}(B)) + (A - F')|S'.$$

$K_X + S + B$ is plt iff $F' = \emptyset$ and $K_S + \operatorname{Diff} B$ is plt iff $F' \cap S' = \emptyset$. Let $h : X \to X$ be the identity. By (17.4) $S' \cup F'$ is connected, hence $F' = \emptyset$ iff $F' \cap S' = \emptyset$. $\square$

17.7 Theorem. *Let X be normal and let $S \subset X$ be an irreducible divisor. Let B and B' be effective $\mathbb{Q}$-divisors such that $\llcorner B \lrcorner = \emptyset$. Assume furthermore that*

(17.7.1) B' is $\mathbb{Q}$-Cartier, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and

(17.7.2) $K_X + S + B$ is plt.

Then $K_X + S + B + B'$ is lc in a neighborhood of S iff $K_S + \operatorname{Diff}(B + B')$ is lc.

Proof. By (2.17.5) $K_X + S + B + B'$ (resp. $K_S + \operatorname{Diff}(B + B')$) is lc iff $K_X+S+B+tB'$ (resp. $K_S+\operatorname{Diff}(B+tB')$) is plt for every $0 \le t < 1$. Thus (17.6) implies (17.7). $\square$

The following corollary is very important in Chapter 18. (See (18.3) for the definition of maximally lc.)

17.8 Corollary. *Let X be normal, $\mathbb{Q}$-factorial and let $S \subset X$ be an irreducible divisor. Let $\sum d_i D_i$ be an effective $\mathbb{Q}$-divisor. Assume that $K_X + S$ is plt. Set*

$$\Delta = \operatorname{Diff}_S(0) \quad \text{and} \quad B_i = i^w \mathcal{O}_X(D_i),$$

where $i : S \to X$ is the natural injection and i^w is defined in (16.3.6).

Then $K_X + S + \sum d_i D_i$ is maximally lc near a point $x \in S$ iff $K_S + \Delta + \sum d_i B_i$ is maximally lc near $x \in S$. □

The rest of the chapter is devoted to showing that if the minimal model program works in dimension n then (17.3) holds for small discrepancies for $\dim X = n$. The precise assumptions are the following.

17.9 Assumption. For the rest of the chapter we use the following special case of the Log Minimal Model Program:

$(*_n)$. Let $f : Y \to X$ be a proper birational morphism. Assume that Y is normal, $\mathbb{Q}$-factorial and $\dim Y \leq n$. Let D be a $\mathbb{Q}$-Weil divisor on Y such that (Y, D) is log terminal. Then the steps of the $(K_Y + D)$-MMP (as described in (2.26)) all exist and the process terminates with a relative minimal model $\bar{f} : (\bar{Y}, \bar{D}) \to X$.

We know that $(*_2)$ and $(*_3)$ hold.

We start with the following result which is of considerable interest in itself. It is a generalisation of (6.9.4).

17.10 Theorem. *Assume $(*_n)$. Let (X, B) be a log canonical pair, $\dim X \leq n$. Let $f : Y \to X$ be a log resolution. Let $\mathcal{E}$ be a subset of the exceptional divisors $\{E_i\}$ such that*

(17.10.1.1) If $a(E_i, B) = -1$ then $E_i \subset \mathcal{E}$;

(17.10.1.2) If $E_j \subset \mathcal{E}$ then $a(E_j, B) \leq 0$.

Then there is a factorization

$$f : Y \overset{h}{\dashrightarrow} X(\mathcal{E}) \xrightarrow{g} X$$

with the following properties:

(17.10.2.1) h is a local isomorphism at every generic point of $\mathcal{E}$;

(17.10.2.2) h contracts every exceptional divisor not in $\mathcal{E}$;

$$\begin{aligned} & h_*\left(K_Y + f_*^{-1}(B) + \sum -a(E_i, B)E_i\right) \\ \text{(17.10.2.3)} \qquad & = K_{X(\mathcal{E})} + g_*^{-1}(B) + \sum_{E_i \subset \mathcal{E}} -a(E_i, B)h_*(E_i) \\ & \equiv g^*(K_X + B) \quad \textit{is log terminal.} \end{aligned}$$

Proof. For a small ϵ let

$$\text{(17.10.3)} \qquad d(E_i) = \begin{cases} -a(E_i, B) & \text{if } \ E_i \subset \mathcal{E}; \\ \max\{-a(E_i, B) + \epsilon, 0\} & \text{if } \ E_i \not\subset \mathcal{E}. \end{cases}$$

Then

$$K_Y + f_*^{-1}(B) + \sum d(E_i)E_i \equiv f^*(K_X + B) + \sum_{E_j \notin \mathcal{E}} (d_j + a(E_j, B))E_j.$$

Apply the $(K_Y + f_*^{-1}(B) + \sum d(E_i)E_i)$-MMP to Y/X. Every extremal ray is supported in (the birational transform of) $h_*(\mathcal{E})$. Also, an effective exceptional divisor is never nef. Thus the MMP stops with a factorization

$$f : Y \overset{h}{\dashrightarrow} X(\mathcal{E}) \xrightarrow{g} X$$

such that $h_*(\mathcal{E}) = \emptyset$ and h is an isomorphism at every generic point of $\mathcal{E}$. □

17.11 Corollary. *Assume* $(*_n)$. *Let* $(X, S + B)$ *be as in (17.2) such that* $\dim X \leq n$ *and* X *is* $\mathbb{Q}$*-factorial. Assume furthermore that either,*

(17.11.1) $(X, S + B)$ *is plt and* $d = \operatorname{discrep}(S \cap \operatorname{Center} \subset Z, X, S + B) \leq 0$*;*
or

(17.11.2) $(X, S + B)$ *is lc and* $d = -1$.

Then the equalities (17.3.1) hold.

Proof. Let $f : Y \to X$ be a log resolution of $(X, S + B)$ such that $f^{-1}(Z)$ is a divisor with normal crossings. Let $S'' \subset Y$ be the birational transform of S.

Let $\mathcal{E}$ be the set of exceptional divisors with discrepancy d such that $S \cap \operatorname{Center}_X(E) \subset Z$. By assumption $\mathcal{E} \neq \emptyset$. We apply the

$$\left(K_Y + f_*^{-1}(S + B) + \sum d(E_i)E_i\right)\text{-MMP on } f : Y \to X.$$

At the end we obtain $h : Y \dashrightarrow X(\mathcal{E})$ and $g : X(\mathcal{E}) \to X$ such that

$$h_* \left(K_Y + f_*^{-1}(S + B) + \sum d(E_i)E_i\right) = g^*(K + S + B).$$

Let $S' \subset X(\mathcal{E})$ be the birational transform of S. Since X is $\mathbb{Q}$-factorial, the exceptional set of g is exactly $h_*(\mathcal{E})$, hence S' intersects the exceptional divisor $h_*(\mathcal{E})$. $f(S') \cap f(h_*(\mathcal{E})) \subset Z$, hence every irreducible component $C \subset S' \cap h_*(\mathcal{E})$ lies above Z.

By (16.7) the coefficient $p(C)$ of $[C]$ in $\operatorname{Diff}(g_*^{-1}B - dh_*(\mathcal{E}))$ is

$$p(C) = 1 - \frac{1}{m} + \sum \frac{r_i b_i}{m} + \frac{r_0(-d)}{m} \geq 1 - \frac{1+d}{m} \geq -d,$$

and by (17.2.3) $a(C, S, \operatorname{Diff}(B)) = -p(C)$. Combining with (17.2) we are done. □

17.12 Corollary. *Assume* $(*_n)$. *Let* $(X, S+B)$ *be as in (17.2) such that* $\dim X \leq n$ *and* X *is* $\mathbb{Q}$*-factorial. Then*

$$\operatorname{totaldiscrep}(S, \operatorname{Diff}(B)) = \operatorname{discrep}(\operatorname{Center} \cap S \neq \emptyset, X, S+B).$$

Proof. Let $d = \operatorname{discrep}(\operatorname{Center} \cap S \neq \emptyset, X, S+B)$. By blowing up a codimension one smooth point of S we see that $d \leq 0$. If $d > -1$ then $(X, S+B)$ is plt, thus (17.11.1) implies the required equality.

If $d = -1$ then we can apply (17.11.2).

Finally assume that $d = -\infty$. We need to show that $(S, \operatorname{Diff}(B))$ cannot be lc. Let $f : (Y, f_*^{-1}(S+B)+E) \to X$ be a log terminal model of $(X, S+B)$ where E is the reduced exceptional divisor. Write

$$K_Y + f_*^{-1}(S+B) + E \equiv f^*(K_X + S + B) - F,$$

where by (2.19) F is effective and either $F = 0$ or $\operatorname{Supp} F = \operatorname{Supp} E$. In the former case $(X, S+B)$ is lc. In the latter case let $S' \subset Y$ denote the birational transform of S. Then S' and E intersect nontrivially and

$$K_{S'} + \operatorname{Diff}_{S'}(f_*^{-1}(B) + E + F) = f^*(K_S + \operatorname{Diff}_S(B))$$

contains a component with coefficient greater than 1 by (16.7).

Thus $(S, \operatorname{Diff}(B))$ is not lc. $\square$

18. REDUCTION TO SPECIAL FLIPS

ANTONELLA GRASSI and JÁNOS KOLLÁR

18.1 Conventions. In this chapter $f : (X, K+S+B) \to Z$ denotes a small contraction such that $-(K+S+B)$ is f-ample. We always assume that $K+S+B$ is log canonical, S is reduced and $B = \sum_1^n b_i B_i$ with $0 < b_i \leq 1$ where the B_i are distinct, irreducible and reduced. (In general B is allowed to have a reduced part.) The assumptions imply that S and B have no irreducible components in common. One should keep in mind that the sum $S+B$ does not determine S and B uniquely. Irreducible components with coefficient 1 can be either in S or in B.

Let $0 \in Z$ be a distinguished point and set $C = f^{-1}(0)$. In dimension three C is the whole exceptional set (after possibly shrinking Z) but not necessarily so in higher dimensions. Any irreducible curve in C is called a *flipping* curve.

We always assume that every irreducible component of $S+B$ intersects C.

18.2 Definition.

(18.2.1) The *type* of $S+B$ is the sequence $(b_1, \cdots, b_n)$. It is denoted by type$(S+B)$. We usually do not think of B with a specified ordering of the components in mind, so strictly speaking f has several types.

(18.2.2) We introduce an ordering on sequences of numbers as follows: $(b_1^s, \cdots, b_m^s) < (b_1^t, \cdots, b_n^t)$ if either $n < m$ or $n = m$ and $b_i^s \leq b_i^t$ $\forall i$, with strict inequality holding for at least one index i.

18.3 Definition. Let $K+\Delta+\sum d_i D_i$ be a log canonical divisor on X. Assume that D_i are $\mathbb{Q}$-Cartier Weil-divisors. We say that $K+\Delta+\sum d_i D_i$ is *maximally log canonical* near $Z \subset X$ if $(X, K+\Delta+\sum d_i' D_i)$ is not log canonical in any neighborhood of Z where $d_i' \geq d_i$ with inequality holding for at least one index i.

Warning: It is important to note that this definition depends on the Δ and the D_i, not just on $\Delta + \sum d_i D_i$.

The following is clear:

18.4 Lemma. *Let $K + \Delta + \sum d_i D_i$ be as above.*

(18.4.1) Let $f : Y \to X$ be a log resolution. $K + \Delta + \sum d_i D_i$ is maximally log canonical in a neighborhood of Z iff for every D_i there is a divisor $E_i \subset Y$ with log discrepancy zero such that $f(E_i) \subset \operatorname{Supp} D_i$ and $Z \cap f(E_i) \neq \emptyset$.

(18.4.2) There is a (nonunique) sequence $d'_i \geq d_i$ such that $(X, K + \Delta + \sum d'_i D_i)$ is maximally log canonical in a neighborhood of Z.

(18.4.3) Assume that $K + \Delta + \sum d_i D_i$ is log terminal, $\llcorner d_1 D_1 \lrcorner = 0$, and D_1 does not have any irreducible components in common with Δ or with $\sum_{k \neq 1} D_k$. Then we may assume that $d'_1 > d_1$. □

18.5 Definition. $f : (X, K + S + B) \to Z$ is a *limiting contraction* if

(18.5.1) X is $\mathbb{Q}$-factorial and f is small;

(18.5.2) S is irreducible and f-negative;

(18.5.3) every irreducible component of B is f-negative;

(18.5.4) $K + S + B$ is maximally log canonical in a neighborhood of C;

(18.5.5) $K + S$ is purely log terminal.

18.6 Definition. $f : (X, K + D) \to Z$ is a *pre limiting contraction* if

(18.6.1) X is $\mathbb{Q}$-factorial and f is small;

(18.6.2) there exists $S \subset \llcorner D \lrcorner$ such that S is f-negative;

(18.6.5) $K + D$ is log terminal.

18.7 Lemma. *Let $f : (X, K + S + B) \to Z$ be a pre limiting contraction. Assume that $\rho(X/Z) = 1$. Then there is a suitable B' such that*

(18.7.1) $K + S + B'$ is limiting.

(18.7.2) The flip of $K + S + B'$ is isomorphic to the flip of $K + S + B$ (assuming they exist).

(18.7.3) $\operatorname{type}(S + B') \geq \operatorname{type}(S + B)$ *and if $\{B\} \neq \emptyset$ then* $\operatorname{type}(S + B') > \operatorname{type}(S + B)$.

Proof. Since $\rho(X/Z) = 1$, the flip of f is independent of the choice of $S + B$ (2.32.1). We can throw away the components of B which are f-semipositive. This gives $K + S + B_1$. By (18.4) we can increase the coefficients of B_1 until we get B' which is maximally log canonical near C.

The type increased or remained unchanged in both steps. It is unchanged only if $B = B_1$ and $\{B_1\} = \emptyset$. □

18.8 Definition. $f : (X, K+S+B) \to Z$ is a *special contraction* if it is limiting, $K + S + B$ is lt and B is reduced (possibly empty).

Our aim is to show that if flips of special contractions exist then all flips exist.

18.9 Theorem. *(In dimension three only.) If the flip of any special contraction exists then the flip exists for any small contraction $f : (X, K + D) \to Z$ such that $K + D$ is klt.*

Proof. This follows from (18.11) and (18.26).

We start with an explanation of the basic idea behind the proof.

18.10 Reduction Strategy. Assume for simplicity that X is $\mathbb{Q}$-factorial. First we increase the coefficients of D until $K+D$ becomes maximally log canonical. Then take a log resolution $h : Y \to X \to Z$ and apply the MMP to $K_Y + D_Y$. Since $K_Y + D_Y$ is log terminal, during the program we stay in the category of log terminal singularities. Therefore the program never leads back to the original $f : X \to Z$. Moreover, each time we need to flip, we can increase the coefficients further as in the first step. Thus we can use descending induction on the coefficients of D. If all technical details work out then ultimately we are reduced to flips of contractions $g : (X', D') \to Z'$ where D' is reduced.

This simple picture has several technical and conceptual drawbacks.

(18.10.1) We need to know termination of flips in order to apply the procedure. Currently we know this in special cases only (cf. Chapter 7).

(18.10.2) The MMP stops when the birational transform of $K+D$ becomes nef. This is in general not the flip, only a log terminal model. The current base point freeness theorems are not strong enough to conclude the existence of the flip unless $\llcorner D \lrcorner = 0$. (See, however, Chapter 8.)

(18.10.3) The main problem is that we are left with too many cases. Assume that we need to flip $g : (X', D') \to Z'$ and D' is reduced. If $D' \cdot C' < 0$ then D' contains the flipping curve C', thus g is a special contraction. In this case the restriction $g|D' : D' \to g(D')$ captures many of the properties of $g : X' \to Z'$ and allows us to use results about (not necessarily small) contractions in dimension $\dim X - 1$.

However if $D' \cdot C' \geq 0$ then we might as well throw away D', and we have no boundary at all. These cases include all terminal flips, which are already very difficult to handle.

Our aim is to have a reduction procedure where we always end up in the first case $D' \cdot C' < 0$ of (18.10.3). This makes the reduction more complicated, but much more useful.

A large part of the proof applies in all dimensions. There are only two places where we use three dimensional results. The first result we need is that limiting flips terminate. The second result concerns log canonical singularities and is discussed in detail later (18.15–26).

18.11 Proposition. *Assume that flips of pre limiting contractions exist, and that any sequence of them terminates. Then*

(18.11.1) For every small contraction $f : (X, D) \to Z$ there exists a $\mathbb{Q}$-factorial log terminal model.

(18.11.2) If $K + D$ is klt then the flip of f exists.

The following improved version of (18.10) is based on [Shokurov91,6.4–5]. Our choice of H' is slightly different. The advantage is that we do not need to use semi stable flips later on.

18.12 Log Flipping Procedure.

Start with $(X, K + D)$ arbitrary and let $f : X \to Z$ be a small contraction. Let $T \subset Z$ be the exceptional set of f^{-1}.

(18.12.1) Let H' be a Cartier divisor on Z such that

(18.12.1.1) $H = f^*H'$ contains the exceptional locus of f.

(18.12.1.2) H' contains the singular locus of Z and the singular locus of the support of $f(D)$.

(18.12.1.3) Fix a resolution $\pi : Z' \to Z$. Let $F_j \subset Z'$ be divisors which generate $N^1(Z'/Z)$. We assume that H' contains $\pi(F_j)$ for every j. (This usually implies that H' is reducible.)

The main consequence of the last assumption is the following:

(18.12.1.4) Let $h : Y \to Z$ be any proper birational morphism such that Y is $\mathbb{Q}$-factorial. Then the irreducible components of the birational transform of H' and the exceptional divisors generate $N^1(Y/Z)$.

(18.12.2) We claim that there is a log resolution $h : Y \to X \to Z$ for $K + D + H$ which is an isomorphism over $Z \setminus H'$. Indeed, first we can resolve the singularities of Z; for this we need to blow up only inside the singular set. Then we resolve the singularities of the inverse image of $H' \cup D$; for this again we need to blow up only inside the singular set which is contained in the preimage of H'.

Then $K_Y + (D + H)_Y$ is $\mathbb{Q}$-factorial and log terminal. Observe that $h^*(H')$ contains $h^{-1}(T)$, $h^*(H')$ is LSEPD with respect to h and $h^*(H')$ contains all exceptional divisors.

(18.12.3) Apply the Y/Z-Minimal Model Program to $K_Y + (D + H)_Y$ over a neighborhood of T. We successively construct the objects $(h_i : Y_i \to Z, K_{Y_i} + (D + H)_{Y_i})$. $\llcorner(D + H)_{Y_i}\lrcorner$ contains the support of h_i^*H, and every flipping curve is contained in supp h_i^*H which is LSEPD. Termination of flips needs to be established. If we can perform the flips then we end up with a $\mathbb{Q}$-factorial log terminal model $\bar{h} : (\bar{Y}, K_{\bar{Y}} + (D + H)_{\bar{Y}}) \to Z$.

(18.12.4) Our next goal is to remove the birational transform $\bar{H}'$ of H' from $(D + H)_{\bar{Y}}$.

By definition $K_{\bar{Y}} + (D + H)_{\bar{Y}}$ is $\bar{h}$-nef. Consider the largest ϵ in the range $0 \le \epsilon \le 1$ such that $K_{\bar{Y}} + (D + H)_{\bar{Y}} - \epsilon\bar{H}'$ is $\bar{h}$-nef. If $\epsilon = 1$ then

$$K_{\bar{Y}} + D_{\bar{Y}} = K_{\bar{Y}} + (D + H)_{\bar{Y}} - \bar{H}'$$

is $\bar{h}$-nef, hence $\bar{h}:\bar{Y}\to Z$ is a log terminal model.

Otherwise we try to increase ϵ as follows. Take $0<\eta\ll\epsilon$. Then

$$K_{\bar{Y}}+(D+H)_{\bar{Y}}-(\epsilon+\eta)\bar{H}'$$

is not nef. We can apply the relative Minimal Model Program. We successively construct the objects

$$(\bar{h}_i:\bar{Y}_i\to Z, K_{\bar{Y}_i}+(D+H)_{\bar{Y}_i}-(\epsilon+\eta)\bar{H}'_i).$$

By construction $(D+H)_{\bar{Y}}-(\epsilon+\eta)\bar{H}'$ is LSEPD, thus by (2.35) there are only finitely many $(K_{\bar{Y}}+(D+H)_{\bar{Y}}-(\epsilon+\eta)\bar{H}')$-extremal rays. Therefore we may assume that if C_i is a flipping curve, then

$$(K_{\bar{Y}_i}+(D+H)_{\bar{Y}_i}-(\epsilon+\eta)\bar{H}'_i)\cdot C_i=-\eta\bar{H}'_i\cdot C_i<0, \tag{18.12.4.1}$$

hence $\bar{H}'_i\cdot C_i>0$. Also $0=\bar{h}_i^*H'\cdot C_i=\bar{H}'_i\cdot C_i+\sum\alpha_kE_k\cdot C_i$, where all the α_k are nonnegative integers and the E_k are $\bar{h}_i$-exceptional. Then $E_k\cdot C_i<0$ for some index k and $C_i\subset E_k\subset \llcorner(D+H)_{\bar{Y}_i}\lrcorner$.

If these flips exist and terminate then we obtain

$$(\bar{h}_k:\bar{Y}_k\to Z, K_{\bar{Y}_k}+(D+H)_{\bar{Y}_k}-(\epsilon+\eta)\bar{H}'_k)$$

such that $K_{\bar{Y}_k}+(D+H)_{\bar{Y}_k}-(\epsilon+\eta)\bar{H}'_k$ is $\bar{h}_k$-nef. Thus we can increase the value of ϵ to $\epsilon'\geq\epsilon+\eta$. Next apply the

$$K_{\bar{Y}_k}+(D+H)_{\bar{Y}_k}-(\epsilon'+\eta')\bar{H}'_k$$

Minimal Model Program as before, and so on.

We claim that after finitely many steps we reach $\epsilon=1$. The only question is the termination of flips. This is however slightly more delicate than usual since we have to account for the possibility that we have an infinite sequence of $(K+(D+H)_Y-\epsilon\bar{H})$-flips during which the choice of ϵ changes. However from (18.12.4.1) it follows that for every such flip

$$(K_{\bar{Y}_i}+(D+H)_{\bar{Y}_i}-\bar{H}'_i)\cdot C_i<0,$$

thus our sequence of $(K+(D+H)_Y-\epsilon\bar{H})$-flips is also a sequence of $(K+(D+H)_Y-\bar{H})$-flips. Hence we face only the usual termination problem which is settled in chapter 7.

(18.12.5) If all the above flips exist and terminate then at the end we obtain

$$\tilde{h}:(\tilde{Y},K_{\tilde{Y}}+D_{\tilde{Y}})\to Z$$

such that $\tilde{Y}$ is $\mathbb{Q}$-factorial, $K_{\tilde{Y}} + D_{\tilde{Y}}$ is log terminal and $\tilde{h}$-nef.

18.13 Proof of (18.11). Let $f : (X, K+D) \to Z$ be a pre limiting contraction. Apply the log flipping procedure (18.12).

We claim that during the procedure only pre limiting flips are used. If C_i is a flipping curve in (18.12.3) then $C_i \subset h_i^* H'$ and $C_i \cdot h_i^* H' = 0$. By (18.12.1.4) there is an irreducible component $F_i \subset h_i^* H'$ such that $C_i \cdot F_i \neq 0$. Thus a suitable irreducible component of $h_i^* H'$ intersects C_i negatively. We throw away those components of $(D+H)_{Y_i}$ which intersect C_i nonnegatively. Write the remaining components as $S + B$ where the components of S have coefficient one and the components of B have coefficient < 1.

In step (18.12.4) we proved that $C_i \cdot E_k < 0$, therefore we obtain a pre limiting contraction.

By assumption every step of the log flipping procedure exists and we assume termination. At the end we obtain a $\mathbb{Q}$-factorial log terminal model $\tilde{h} : \tilde{Y} \to Z$.

If $K + D$ is klt, then $K_{\tilde{Y}} + \tilde{D}_Y$ is also klt, hence the flip of f exists by (2.29). $\square$

The following refinement of (18.11) is crucial in the next step.

18.14 Proposition. *Let $f : (X, K+S+B) \to Z$ be a limiting contraction. Assume that $\rho(X/Z) = 1$. Assume moreover that*

(18.14.1) The flip of every limiting contraction of greater type exists.

(18.14.2) The flip of every special contraction exists.

(18.14.3) Pre limiting flips terminate.

Then the flip of f also exists.

Proof. As before let $T \subset Z$ be the exceptional set of f^{-1}.

As a first step we construct a log terminal model of $(X, K+S+B)$. To do this we take a log resolution $p : X' \to X$ and apply the $(K_{X'} + (S+B)_{X'})$-MMP relative to a neighborhood of $S \subset X$. In the course of the program we have to make certain flips with flipping curve C. All flipping curves are contained in

$$\operatorname{Supp}(p^* S) \subset \llcorner (S+B)_{X'} \lrcorner.$$

By the proof of (2.16.2) the exceptional divisor of p supports a divisor E such that $-E$ is p-ample. Thus $C \cdot E < 0$ and the contraction of C is pre limiting.

Let $B = \sum_1^k b_i B_i$. Then $\operatorname{type}(S+B) = (b_1, \dots, b_k)$. On X' the only divisors in $(S+B)_{X'}$ with coefficient < 1 are the birational transforms B'_i. In order to make a contraction limiting, first we throw away those B'_j which have nonnegative intersection with C. Then we can increase the coefficients as in (18.7). Thus the corresponding limiting contraction of C is either special or it has type strictly greater than $\operatorname{type}(S+B)$.

Thus the existence of the flip of C follows from the existence of limiting flips of greater type and of special flips.

At the end we obtain $g : Y \to X$ which is a $\mathbb{Q}$-factorial log terminal model. In particular,

$$K_Y + (S+B)_Y \equiv g^*(K_X + S + B).$$

Next let H' be a sufficiently general and sufficiently f-ample divisor on X. Let $H = g^*H'$. For some $0 < \epsilon < 1$, $K + S + B + \epsilon H'$ is numerically f-trivial and $K_Y + (S+B)_Y + \epsilon H$ is log terminal and numerically $f \circ g$-trivial. For some $0 < \eta \ll \epsilon$ apply the MMP for

$$K_Y + (S+B)_Y + (\epsilon - \eta)H$$

to Y/Z. During the course of the program the birational transform of $K_Y + (S+B)_Y + \epsilon H$ remains numerically trival over Z. Thus if C_i is a flipping curve in the i^{th}-step of the program then $C_i \cdot H_i > 0$. Therefore the contraction corresponding to C_i is a $(K_{Y_i} + (S+B)_{Y_i})$-extremal contraction. We claim that it is pre limiting and of type at least the type of $S + B$. The statement about the type can be proved as before.

Let $S_1 \subset S$ be such that $C \cdot S_1 < 0$. Since $H' \cdot C > 0$, there is an $\alpha > 0$ such that $S_1 + \alpha H'$ is numerically f-trivial. Thus $g^*(S_1 + \alpha H)$ is numerically $f \circ g$-trivial, and it contains $(f \circ g)^{-1}(T)$. The same properties continue to hold for its birational transform on Y_i for every i. By assumption $C_i \cdot H > 0$, hence $(g^*S_1)_i \cdot C_i < 0$. Therefore there is an irreducible component of

$$\operatorname{Supp}(g^*S_1)_i \subset \llcorner (S+B)_{Y_i} \lrcorner$$

which intersects C_i negatively.

At the end we obtain $\bar{g} : \bar{Y} \to Z$ such that

$$K_{\bar{Y}} + (S+B)_{\bar{Y}} + (\epsilon - \eta)\bar{H}$$

is $\bar{g}$-nef. $S + B + (\epsilon - \eta)H'$ is LSEPD with respect to f, and thus the flip of f exists by (2.32.2) and (2.29.1). $\square$

(18.14) is very useful if there is no infinite increasing sequence of limiting contractions. At first sight there is no reason why such a sequence should not exist. [Shokurov88,91] discovered that there are many situations where a similar ordering of the coefficients makes sense, and, at least conjecturally, there are no infinite increasing sequences. Below we define some of these sets of sequences. Later we prove some relationships between them and finally we show the nonexistence of infinite increasing sequences in low dimensional cases.

18.15 Definition.

(18.15.1) $\mathcal{S}_n$(fano) is the set of sequences $(b_1, \ldots, b_m)$ such that there is an a smooth and proper Fano variety X of dimension at most n and a divisor $\sum b_i B_i$ such that $\rho(X) = 1$, $K_X + \sum b_i B_i$ is log canonical, log terminal outside $\sum B_i$ and numerically trivial.

(18.15.$\bar{1}$) $\mathcal{S}_n$(global) is the set of sequences $(b_1, \ldots, b_m)$ such that there is a proper variety X of dimension at most n and a divisor $\sum b_i B_i$ such that $K_X + \sum b_i B_i$ is log canonical, log terminal outside $\sum B_i$ and numerically trivial.

(18.15.2) $\mathcal{S}_n$(local) is the set of sequences $(b_1, \ldots, b_m)$ such that there is a pointed a $\mathbb{Q}$-factorial variety $x \in X$ of dimension at most n and a divisor $\sum b_i B_i$ such that $x \in \cap B_i$ and $K_X + \sum b_i B_i$ is maximally log canonical at x.

(18.15.$\bar{2}$) $\bar{\mathcal{S}}_n$(local) is the set of sequences $(b_1, \ldots, b_m)$ such that there is a $\mathbb{Q}$-factorial variety X of dimension at most n, a closed subset $Z \subset X$ and a divisor $\sum b_i B_i$ such that every B_i intersects Z and $K_X + \sum b_i B_i$ is maximally log canonical near Z.

(18.15.3) $\mathcal{S}_n^0$(local) is the set of sequences $(b_1, \ldots, b_m)$ such that there is a pointed a $\mathbb{Q}$-factorial variety $x \in X$ of dimension at most n and a divisor $B_0 + \sum b_i B_i$ ($B_0 \neq 0$ is reduced but possibly reducible) such that $x \in \cap B_i$, $K_X + B_0$ is purely log terminal and $K_X + B_0 + \sum b_i B_i$ is maximally log canonical at x. (Purely log terminal implies that B_0 is locally irreducible.)

(18.15.$\bar{3}$) $\bar{\mathcal{S}}_n^0$(local) is the set of sequences $(b_1, \ldots, b_m)$ such that there is an a $\mathbb{Q}$-factorial variety X of dimension at most n, a subset $Z \subset X$ and a divisor $B_0 + \sum b_i B_i$ ($B_0 \neq 0$ is reduced but possibly reducible) such that every B_i intersects Z, $Z \subset B_0$, $K_X + B_0$ is purely log terminal and $K_X + B_0 + \sum b_i B_i$ is maximally log canonical near Z.

18.16 Conjecture. *The ascending chain condition holds for any of the six sets in (18.15). (With respect to the ordering given in (18.2.2)).*

For technical reasons we also need the following rather complicated definition. We try to formalize the properties of the different (16.6–7).

18.17 Definition. $\mathcal{S}_n$(local diff) is the set of sequences $(b_1, \ldots, b_m)$ such that there is a pointed a variety $x \in X$ of dimension at most n and a divisor $K + \Delta + \sum b_i B_i$ such that

(18.17.1) $x \in \cap \operatorname{Supp} B_i$,

(18.17.2) $K + \Delta$ is purely log terminal and $K + \Delta + \sum b_i B_i$ is maximally log canonical at x.

(18.17.3) $\Delta = \sum (1 - 1/m_j)\Delta_j$ where Δ_j are irreducible, reduced and the m_j are natural numbers (we allow $m_j = 1$);

(18.17.4) B_i is $\mathbb{Q}$-Cartier for every i and $B_i = \sum_j (s_{ij}/m_j)\Delta_j$ for some integers $s_{ij} \geq 0$ such that $\sum_j s_{ij} > 0$.

It is clear that $\mathcal{S}_n(\text{local}) \subset \mathcal{S}_n(\text{local diff})$.

18.18 Definition. Let $\mathcal{L}$ be a set of sequences. We define two other sets of sequences $\bar{\mathcal{L}}$ and $D^{-1}(\mathcal{L})$ as follows:

(18.18.1) $(b_1, \dots, b_n) \in \bar{\mathcal{L}}$ if and only if for every $1 \le j \le n$ there is a subset $(i_1, \dots, i_{k(j)})$ of $(1, \dots, n)$ containing j such that $(b_{i_1}, \dots, b_{i_{k(j)}}) \in \mathcal{L}$.

(18.18.2) Let $D^{-1}(\mathcal{L})$ be the set of sequences $(b_1, \dots, b_n)$ such that $0 < b_i \le 1$ for every i and the following holds:

There is a natural number k and positive integers r_h, integers $0 \le s_{hi} \le r_h$, and $t_h \in \{0, 1\}$ for every $1 \le h \le k$ and $1 \le i \le n$ such that

$$p_h = \frac{r_h - 1}{r_h} + \sum_{i=0}^{n} \frac{s_{hi}}{r_h} b_i + \frac{t_h}{r_h} \le 1 \quad \text{for every } h;$$

$$(p_1, \dots, p_k) \in \mathcal{L}; \text{ and}$$

$$\max_h \{s_{hi}\} > 0 \quad \text{for every } 1 \le i \le n.$$

(18.18.3) The following two properties are easy to check:

$$\bar{\bar{\mathcal{L}}} = \bar{\mathcal{L}} \quad \text{and} \quad \overline{D^{-1}(\mathcal{L})} = D^{-1}(\bar{\mathcal{L}}).$$

(18.18.4) From (7.4.3) we see that $D^{-1}(D^{-1}(\mathcal{L})) = D^{-1}(\mathcal{L})$.

The barred versions of (18.15) are related to the others in a very simple way:

18.19 Proposition.

(18.19.1) $\bar{\mathcal{S}}_n(\textit{local}) \subset \overline{\mathcal{S}_n(\textit{local})}$;

(18.19.2) $\bar{\mathcal{S}}_n^0(\textit{local}) \subset \overline{\mathcal{S}_n^0(\textit{local})}$;

(18.19.3) $\mathcal{L}$ *satisfies the ascending chain condition iff* $\bar{\mathcal{L}}$ *does.*

Proof. Let $f : Y \to X$ be a log resolution. For every B_j there is a divisor $E_j \subset Y$ as in (18.4.1). Let $x_j \in X$ be the image of the generic point of E_j. Let $i = i_1, \dots, i_k$ be those indices such that $x_j \in B_i$. Then

$$K_X + \sum_{l=1}^{k} B_{i_l}$$

is maximally log canonical at x_j. This proves (18.19.1), and (18.19.2) is proved the same way.

Clearly $\mathcal{L} \subset \bar{\mathcal{L}}$. Assume that $\mathcal{L}$ satisfies the ascending chain condition. Let

$$\mathbf{b}_1 \le \mathbf{b}_2 \le \dots$$

be an infinite ascending chain where

$$\mathbf{b}_i = (b^i_1, \dots, b^i_{n(i)}) \in \bar{\mathcal{L}}.$$

We may assume that $n = n(i)$ is constant. By definition, for every i we have a covering of $(1, \dots, n)$ by n subsets. By passing to a subsequence we may assume that the covering does not depend on i. Thus for every i we get n sequences $\mathbf{b}^j_i \in \mathcal{L}$ such that,

$$\mathbf{b}^j_1 \le \mathbf{b}^j_2 \le \dots \qquad \forall j,$$

and for every i at least one of the inequalities $\mathbf{b}^j_i \le \mathbf{b}^j_{i+1}$ is strict. This is impossible since $\mathcal{L}$ satisfies the ascending chain condition. □

The following was pointed out by Alexeev:

18.19.4 Proposition. *Assum the log MMP for dimension n. Then*

$$\mathcal{S}_n(\mathit{global}) \subset \overline{\mathcal{S}_n(\mathit{fano})}.$$

Proof. Let $(X, \sum b_i B_i) \in \mathcal{S}_n(\text{global})$. As in (8.8.1) let $f : X' \to X$ be a small morphism such that X' is $\mathbb{Q}$-factorial. Then $(X', \sum b_i B'_i) \in \mathcal{S}_n(\text{global})$. We prove by induction on $\dim X'$ and $\operatorname{rank}\operatorname{Pic}(X')$ that $(b_1, \dots, b_m) \in \overline{\mathcal{S}_n(\text{fano})}$. Fix k and consider the $(K + \sum b_i B'_i - \epsilon B'_k)$-MMP. After possibly some flips $X' \dashrightarrow X''$, we perform a divisorial or a Fano contraction $g : X'' \to Z$. B''_k is positive on the extremal ray of g, thus B''_k is not contracted by g in the divisorial case, and intersects the general fiber in the Fano case.

If g is divisorial, then $\operatorname{rank}\operatorname{Pic}(X') = \operatorname{rank}\operatorname{Pic}(X'') > \operatorname{rank}\operatorname{Pic}(Z)$ and we are done by induction on rank Pic. If g is Fano then we can restrict everything to the general fiber of g and conclude by induction on the dimension. □

18.20 Definition. Let $\mathcal{L}$ be a set of sequences. We say that $\mathcal{L}$ has *bounded sums* if there is an M such that $\sum b_i < M$ for every $(b_1, \dots, b_k) \in \mathcal{L}$.

The various cases in (18.15) and (18.17) are related by the following result.

18.21 Theorem. *(Inductive Principle)*

(18.21.1) $\mathcal{S}^0_n(\mathit{local}) \subset \mathcal{S}_{n-1}(\mathit{local\ diff})$.

(18.21.2) Assume that for every n-dimensional log canonical variety $(X, K + D)$ there is a $\mathbb{Q}$-factorial log terminal model $f : Y \to X$. Then

$$\mathcal{S}_n(\mathit{local\ diff}) \subset \overline{D^{-1}(\mathcal{S}_{n-1}(\mathit{global}))}.$$

(18.21.3) If $\mathcal{L}$ has bounded sums then so does $D^{-1}(\mathcal{L})$.

(18.21.4) If $\mathcal{L}$ has the ascending chain condition and has bounded sums then so does $D^{-1}(\mathcal{L})$.

Proof. We start with (18.21.1). Let $(b_1, \cdots, b_k)$ be a sequence in $\mathcal{S}_n^0(\text{local})$. By assumption there exists an n-dimensional variety X and a divisor $S + \sum b_i D_i$, with S reduced and irreducible, such that $K_X + S + \sum b_i D_i$ is maximally log canonical at a point $x \in \cap D_i \subset X$. Set

$$\Delta = \operatorname{Diff}_S(0) \quad \text{and} \quad B_i = i^w \mathcal{O}_X(D_i),$$

where $i : S \to X$ is the natural injection and i^w is defined in (16.3.6). By (16.6) $K + \Delta + \sum b_i B_i$ satisfies the conditions (18.7.3-4) and by (17.8) it also satisfies (18.17.2). Thus $(b_1, \cdots, b_k) \in \mathcal{S}_{n-1}(\text{local diff})$. This proves (18.21.1).

The proof of (18.21.2) is similar. Pick $(b_1, \cdots, b_k) \in \mathcal{S}_n(\text{local diff})$. By hypothesis there exists an n-dimensional pair $(X, K + \Delta + \sum b_i B_i)$ which is maximally log canonical at a point $x \in X$. Let δ_j be the coefficient of Δ_j in $\Delta + \sum b_i B_i$. Let $f : Y \to X$ be a $\mathbb{Q}$-factorial log terminal model. Let $\Delta'_j \subset Y$ be the birational transform of Δ_j and let $E_k \subset Y$ be the exceptional divisors. Pick E_0 such that $x \in f(E_0)$. By (17.5) E_0 is normal. Then

$$0 \equiv f^*(K + \Delta + \sum b_i B_i)|E_0 \equiv K_{E_0} + \operatorname{Diff}(\sum_{k \neq 0} E_k + \sum \delta_j \Delta'_j),$$

where $\equiv$ means numerical equivalence relative to f. By (16.6) and (7.4.3)

$$\operatorname{Diff}_{E_0}(\sum_{k \neq 0} E_k + \sum \delta_j \Delta'_j) = \sum p_h D_h,$$

where $D_h \subset E_0$ are divisors and the coefficients p_h are computed by the formula in (18.18.2) for suitable r_h, s_{hi} and t_h. (The presence of the E_k are the reason of using t_h in (18.18.2).) If E_0 is not proper then replace it with the general fiber of $E_0 \to f(E_0)$. The last assumption of (18.18.2) is not necessarily satisfied since some of the $f_*^{-1}(B_i)$ may not intersect E_0. By (18.4) for every B_i there is an exceptional divisor E_k such that $f_*^{-1}(B_i)$ intersects the general fiber of $E_k \to f(E_k)$.

Let E_0 run through all exceptional divisors such that $x \in f(E_0)$. This proves (18.21.2).

From the formula for p_h it is easy to see that

$$p_h \geq \sum_{\{i : s_{hi} \neq 0\}} b_i.$$

Thus $\sum p_h \geq \sum b_i$, and hence $D^{-1}(\mathcal{L})$ has bounded sums if $\mathcal{L}$ has. This proves (18.21.3).

Finally, consider (18.21.4). Assume that we have an infinite increasing sequence $\mathbf{b}^1 < \mathbf{b}^2 < \dots$. By passing to a subsequence we may assume that they all have the same length $\mathbf{b}^j = (b_1^j, \dots, b_n^j)$.

We use an upper index j to refer to a formula (18.18.2) associated to $\mathbf{b}^j$. The symbols $k^j, p_h^j, r_h^j, s_{hi}^j, t_h^j$ are as in (18.18.2). The numbers b_i^j are bounded from below by $\mu = \min\{b_i^1\} > 0$, and thus $p_h^j \geq \mu$ hence $\sum p_h^j \geq k^j \mu$ which shows that k^j is bounded. Thus by passing to a subsequence we may assume that $k^j = k$ is independent of j.

18.21.5 Claim. *For each fixed index h, $\{p_h^j\}$ has an infinite nondecreasing subsequence.*

Proof. We drop the index h from the notation. By assumption

$$p^j = \frac{r^j - 1}{r^j} + \sum_i \frac{s_i^j}{r^j} b_i^j.$$

Since $p^j \leq 1$ we obtain

$$\sum_i s_i^j b_i^j \leq 1.$$

The numbers b_i^j are bounded from below by $\mu > 0$, hence s_i^j are bounded from above by a constant. By passing to a subsequence we may thus assume that $s_i^j = s_i$ are independent of j. Set $u^j = \sum s_i b_i^j$, then u^j is a nondecreasing sequence of real numbers, and $u^j \leq 1$. By passing to a subsequence we may also assume that r^j is nondecreasing. Thus

$$p^j = \frac{r^j - 1}{r^j} + \frac{u^j}{r^j} = 1 - \frac{1 - u^j}{r^j}$$

is also nondecreasing.

Observe furthermore that p^j is strictly increasing if the sequence r^j is strictly increasing. □

We continue with the proof of (18.21). By passing to a subsequence we may assume that $s_{hi}^j = s_{hi}$ are all independent of j and r_h^j is either constant or increasing for every h. We obtain that $\mathbf{p}^1 \leq \mathbf{p}^2 \leq \dots$ and the sequence is strictly increasing if one of the sequences r_h^j is strictly increasing.

We are left with the case when in addition $r_h^j = r_h$ is also independent of j. Then

$$\begin{aligned} \sum_h p_h^j &= \sum_h \frac{r_h - 1}{r_h} + \sum_i \left(\sum_h \frac{d_{hi}}{r_h} \right) b_i^j \\ &= C_0 + \sum C_i b_i^j \end{aligned}$$

where the C_i are positive and independent of j. Since the sequence $\mathbf{b}^j$ is strictly increasing, the same holds for $\sum_h p_h^j$, hence for the sequence $\mathbf{p}^j$. $\square$

18.22 Theorem. *Let $(X, \sum b_i B_i)$ be log canonical at a point $x \in \cap B_i$. Assume that K_X and B_i are all $\mathbb{Q}$-Cartier at x. Then $\sum b_i \leq \dim X$.*

In particular, $\mathcal{S}_n$(local) has bounded sums.

Proof. The problem is clearly local. The claim is clear if $n = 1$.

By taking repeated cyclic covers we may assume that the $B_i = (f_i = 0)$ are Cartier. Assume that $\sum b_i \geq n = \dim X$. Let $\bar{B} = (\sum c_i f_i = 0)$ for general $c_i \in \mathbb{C}$. Let $g : Y \to X$ be any log resolution of $(X, \bar{B} + \sum B_i)$ with exceptional divisors E_j. By specializing $g^* \bar{B}$ to $g^* B_i$ we obtain

$$g_*^{-1}(\bar{B}) \sim g_*^{-1}(B_i) + \sum e_{ij} E_j,$$

where $e_{ij} \geq 0$. Thus if $0 \leq b_i' \leq b_i$ and $\sum b_i' = 1$ then

$$a(E_j, \bar{B} + \sum (b_i - b_i') B_i) = a(E_j, \sum b_i B_i) + \sum b_i' e_{ij}.$$

Repeating this procedure we eventually obtain an lc pair

$$(X, \bar{B}_1 + \cdots + \bar{B}_n + \Delta)$$

where the $\bar{B}_i$ are general Cartier divisors (with coefficient one) and $\Delta = \sum d_i B_i$ is some other divisor such that $\sum d_i = \sum b_i - n$.

By (17.2)

$$(\bar{B}_n, \left(\sum_{i=1}^{n-1} \bar{B}_i \right) | \bar{B}_n + \mathrm{Diff}(\Delta))$$

is also lc. Thus $\Delta = 0$ by induction on $\dim X$. $\square$

18.23 Complement. The above argument in fact shows that if the B_i are Cartier and $\sum b_i > \dim X - 1$ then X is smooth at x. Indeed, in this case we can replace

$$\sum_{i=1}^{n} b_i B_i \quad \text{by} \quad \sum_{i=1}^{n-1} \bar{B}_i + \Delta.$$

By induction on the dimension $\bar{B}_{n-1}$ is smooth hence so is X.

Similarly, if the B_i and K_X are Cartier and $\sum b_i > \dim X - 2$ then $x \in X$ is a cDV point.

Combining (18.21) and (18.22) we obtain:

18.24 Corollary. *Let X be an n-dimensional Fano variety with $\rho(X) = 1$ and let $\sum b_iB_i$ be a $\mathbb{Q}$-divisor such that $K_X + \sum b_iB_i$ is lc and numerically trivial. Then $\sum b_i \le \dim X + 1$.*

In particular, $\mathcal{S}_n$(fano) has bounded sums.

Proof. Choose an embedding $X \subset \mathbb{P}^N$ and let $y \in Y$ be the cone over X with vertex y. Let $B'_i \subset Y$ be the cone over B_i. (B'_i is $\mathbb{Q}$-Cartier since $\rho(X) = 1$.) It is easy to see that $(Y, \sum b_iB'_i)$ is lc. Thus (18.22) implies (18.24). □

18.24.1 Remark. $\mathcal{S}_2$(global) does not have bounded sums. Indeed, assume that $D \in |-K_X|$ is reduced with only nodes. Blowing up a node gives $p : X' \to X$ and $p^{-1}(D) \in |-K_{X'}|$ has one more components than D. Thus there are surface examples with arbitrary many reduced components in a member of $|-K_X|$.

18.25 Theorem. *Assume the log MMP in dimension $n-1$. Assume that the ascending chain condition holds for $\mathcal{S}_{n-2}$(fano). Then the ascending chain condition also holds for $\mathcal{S}_{n-1}$(local) and $\mathcal{S}^0_n$(local).*

Proof. By (18.18.3), (18.19.4) and (18.21.2)

$$\mathcal{S}_{n-1}(\text{local diff}) \subset \overline{D^{-1}(\mathcal{S}_{n-2}(\text{global}))} = \overline{D^{-1}(\mathcal{S}_{n-2}(\text{fano}))}.$$

Therefore by (18.21.4) and (18.24) the ascending chain condition holds for $\mathcal{S}_{n-1}$(local diff). The rest follows from (18.21.1) and (18.17). □

18.25.1 Corollary. *The ascending chain condition holds for $\mathcal{S}_1$(global), $\mathcal{S}_2$(local) and $\mathcal{S}^0_3$(local).*

Proof. Consider $\mathcal{S}_1$(global). The only possible X is $\mathbb{P}^1$ and if $K + \sum b_iB_i$ is numerically trivial then $\sum b_i = 2$. Thus if (b_i) and (b'_i) are two sequences of the same length such that $(b_i) \le (b'_i)$ then $(b_i) = (b'_i)$. The rest follows by (18.25). □

18.26 Corollary. *(Dimension three only) Assume that the flip of every special contraction exists.*

Then the flip of every limiting contraction also exists.

Proof. Consider all limiting contractions whose flip does not exist. Consider their types. They give a set $\mathcal{B} \subset \bar{\mathcal{S}}^0_3$(local). By the ascending chain condition, if $\mathcal{B}$ is not empty, it has a maximal element; let $f : (X, K + S + B) \to Z$ be a corresponding contraction. By (18.14) the flip of f exists, a contradiction. □

19. COMPLEMENTS ON LOG SURFACES

DAVID R. MORRISON

One of the key innovations of [Shokurov91] is the notion of n-complement, which we now introduce.

19.1 Definition. Let X be a normal variety and let D be a subboundary (2.2.4) on X. Let S be the smallest effective Weil divisor on X such that $\llcorner D - S \lrcorner \leq 0$, and let $D_0 = D - S$. An *n-complement* of $K_X + D$ is a divisor

$$\overline{D} \in |-nK_X - nS - \llcorner (n+1)D_0 \lrcorner|$$

such that $K_X + D^+$ is log canonical, where

$$D^+ = S + \frac{1}{n}\left(\llcorner (n+1)D_0 \lrcorner + \overline{D}\right).$$

We say that $K_X + D$ is *n-complemented* if an n-complement exists.

Note that nD^+ is an integral divisor belonging to the linear system $|-nK_X|$. The defining properties can be formulated as properties of nD^+, which must satisfy:

(i) $nD^+ - nS - \llcorner (n+1)D_0 \lrcorner$ is effective, and

(ii) $K_X + D^+$ is log canonical.

We start with some easy properties of n-complements.

19.2 Lemma. *If D' is a subboundary, $D \leq D'$, $K_X + D'$ is n-complemented, then $K_X + D$ is n-complemented.*

If $f : Y \to X$ is birational, and $K_X + D$ is n-complemented, then $K_Y + f(D)$ is n-complemented.

Proof. In the first case, set $D^+ = (D')^+$, and in the second case, set $f(D)^+ = f(D^+)$. $\square$

We need a generalization of the notion of n-complement to cover the case in which the variety X is reducible. There are difficulties formulating this in general, so we restrict our attention to curves and surfaces.

A variety X is *semismooth in codimension 1* if all of its codimension 1 singularities are normal crossing points (cf. (12.2.1)). Such an X is Gorenstein in codimension 1, so K_X exists as a Weil divisor (class). When $\dim X = 1$, we call X a *semismooth curve*. (The usual terminology is nodal curve.)

Let X be a semismooth curve, and let $D = \sum d_i D_i$ be an $\mathbb{R}$-Weil divisor supported on the smooth locus of X. The coefficients d_i are allowed to be negative. We say that $K_X + D$ is *semilog canonical (slc)* if $d_i \leq 1$. (Since $\dim X = 1$, there is no need to take a semiresolution before computing discrepancies. The formula

$$K_X = (K_X + D) + \sum (-d_i) D_i$$

shows that $-d_i \geq -1$ is the correct analogue of the lc condition.) Note that $K_X + D$ is slc if and only if D is a subboundary whose support lies in the smooth part of X.

There is also a definition of semi log canonical in the surface case, originally given in [KSB88], and discussed in (12.2). This definition does require taking semiresolutions. We don't repeat it here.

19.3 Definition. Let X be semismooth in codimension 1, and let D be a subboundary whose support lies in the smooth part of X. Suppose that $\dim X \leq 2$. Let S be the smallest Weil divisor on X such that $\llcorner D - S \lrcorner \leq 0$, and let $D_0 = D - S$. An *n-semicomplement* of $K_X + D$ is a divisor

$$\overline{D} \in |-nK_X - nS - \llcorner (n+1) D_0 \lrcorner|$$

such that $K_X + D^+$ is slc, where

$$D^+ = S + \frac{1}{n}\left(\llcorner (n+1) D_0 \lrcorner + \overline{D}\right).$$

(The only place where the restriction on dimension enters is in the definition of slc, which has only been given when $\dim X \leq 2$.)

Shokurov's strategy in studying n-complements is to use inversion of adjunction (16.13, 17.6) to lift an n-complement from S to X. For this to be useful, we need an explicit analysis of complements in low dimension.

19.4 Theorem. *Let X be a semismooth curve, connected but not necessarily complete, and let D be a subboundary whose support is disjoint from* $\operatorname{Sing} X$, *and lies in the union of the complete components of X. Suppose that $\llcorner D \lrcorner \geq 0$ (so that in particular, D is effective, i.e., is a boundary), and that $-(K_X + D)$ is nef on each complete component of X. Then $K_X + D$ is* 1-, 2-, 3-, 4-, *or* 6-*semicomplemented.*

Moreover, if $K_X + D$ is not 1- or 2-semicomplemented, then $X = \mathbb{P}^1$ and $\llcorner D^+ \lrcorner = 0$. In addition, if X contains an incomplete component, and $K_X + D$ is not 1-semicomplemented then D has the form $\frac{1}{2}D_1 + \frac{1}{2}D_2$ for irreducible divisors D_1, D_2.

Proof. The combinatorial ingredients in this proof will seem familiar to those who have studied log canonical surface singularities (cf. Chapter 3), or Kodaira's classification of degenerate elliptic curves. Our proof explicitly gives the divisor D^+ in every case.

Let C be a complete component of X, and let $C \cap \operatorname{Sing} X = \{P_1, \dots, P_k\}$. Then $\deg(K_X|_C) = 2g - 2 + k$. Since $\deg(K_X|_C) \leq 0$, there are four possibilities:

(I) $g = 1$, $k = 0$, $\deg(K_X|_C) = 0$
(II) $g = 0$, $k = 2$, $\deg(K_X|_C) = 0$
(III) $g = 0$, $k = 1$, $\deg(K_X|_C) = -1$
(IV) $g = 0$, $k = 0$, $\deg(K_X|_C) = -2$.

Now D cannot meet components of type (I) or (II), since $\deg(K_X|_C) = 0$. Since X is connected, if it has a component C of type (I) then $X = C$ and $D = 0$. In this case, $K_X + D$ is 1-complemented, with $D^+ = 0$.

Components of type (II), however, can meet other components of the same type, and can meet components of type (III) as well. Since there are only two points of intersection on each component of type (II), the entire curve X must form a chain or a cycle. Chains will be terminated by components of type (III), or by incomplete components.

In the case of a cycle, D is again 0 and $K_X + D$ is 1-semicomplemented with $D^+ = 0$. In the case of a chain, any complete component C of type (III) on the end of the chain will have a divisor $D \cap C = \sum d_i D_i$ with $d_i \leq 1$ and $\sum d_i \leq 1$. If any $d_i = 1$, then $D \cap C = D_1$ which is 1-semicomplemented in a neighborhood of C with $D^+ = D_1$. So we may assume $d_i < 1$. Since $C \cong \mathbb{P}^1$, an n-complement $\overline{D}$ will exist exactly when its degree $n - \deg \llcorner (n+1)(D \cap C) \lrcorner$ is nonnegative. There are only a few possibilities in this case:

(1) $\llcorner 2D \lrcorner = 0$. Then $K_X + D$ is 1-semicomplemented in a neighborhood of C, with $D^+ = \overline{D}$ for some divisor $\overline{D}$ of degree 1.
(2) $\llcorner 2D \lrcorner = D_1$. Then $K_X + D$ is again 1-semicomplemented in a neighborhood of C, with $D^+ = D_1$.
(3) $\llcorner 2D \lrcorner \geq D_1 + D_2$. This implies that $d_1, d_2 \geq \frac{1}{2}$, and hence that $d_1 = d_2 = \frac{1}{2}$. It follows that $\llcorner 3D \lrcorner = D_1 + D_2$, so that $K_X + D$ is 2-semicomplemented in a neighborhood of C, with $D^+ = D \cap C = \frac{1}{2}D_1 + \frac{1}{2}D_2$.

Putting this together from the two ends of the chain, we see that in all cases $K_X + D$ must be 1-semicomplemented or 2-semicomplemented. In addition, if there are any incomplete components in X then $K_X + D$ is 1-semicomplemented *unless* $D = \frac{1}{2}D_1 + \frac{1}{2}D_2$.

If X has any component of type (IV), that component must be the whole of X. So $X \cong \mathbb{P}^1$; we write $D = \sum d_i D_i$ with $1 \geq d_1 \geq d_2 \geq \dots$, and repeatedly use the fact that $\sum d_i \leq 2$. If $d_1 = d_2 = 1$ then $K_X + D$ is 1-complemented with $D^+ = D_1 + D_2$. If $d_1 = 1 > d_2$, then $C - \{D_1\}$ has the same numerical properties as a component of type (III). The analysis given above applies to show that $K_X + D$ is 1- or 2-complemented, with $D^+ = D_1 + \widetilde{D^+}$, where $\widetilde{D^+}$ is the part of D^+ whose support does not contain D_1. $\widetilde{D^+}$ is determined from $\llcorner 2(D - D_1) \lrcorner$ as in (1), (2), and (3) above.

Thus, we may assume $1 > d_1$. Then an n-complement $\overline{D}$ exists if and only if its degree $2n - \deg \llcorner (n+1)D \lrcorner$ is nonnegative. The possibilities are:

(1) $\llcorner 2D \lrcorner = 0$. Then $K_X + D$ is 1-complemented, with $D^+ = \overline{D}$ for some divisor $\overline{D}$ of degree 2.

(2) $\llcorner 2D \lrcorner = D_1$. Then $K_X + D$ is 1-complemented, with $D^+ = D_1 + \overline{D}$ for some divisor $\overline{D}$ of degree 1.

(3) $\llcorner 2D \lrcorner = D_1 + D_2$. Then $K_X + D$ is 1-complemented, with $D^+ = D_1 + D_2$.

(4) $\llcorner 2D \lrcorner \geq D_1 + D_2 + D_3$.
   - (a) $\llcorner 3D \lrcorner = D_1 + D_2 + D_3$. Then $K_X + D$ is 2-complemented, with $D^+ = \frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_3 + \frac{1}{2}\overline{D}$ for some divisor $\overline{D}$ of degree 1.
   - (b) $\llcorner 3D \lrcorner = D_1 + D_2 + D_3 + D_4$. Then $K_X + D$ is 2-complemented, with $D^+ = \frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_3 + \frac{1}{2}D_4$.
   - (c) $\llcorner 3D \lrcorner = 2D_1 + D_2 + D_3$. Then $K_X + D$ is 2-complemented, with $D^+ = D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_3$.
   - (d) $\llcorner 3D \lrcorner = 2D_1 + D_2 + D_3 + D_4$. This implies that $d_1 = \frac{2}{3}$, $d_2 = d_3 = \frac{1}{2}$, $d_4 = \frac{1}{3}$. Thus, $K_X + D$ is 4-complemented, with $D^+ = \frac{3}{4}D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_3 + \frac{1}{4}D_4$.
   - (e) $\llcorner 3D \lrcorner \geq 2D_1 + 2D_2 + D_3$.
     - (i) $\llcorner 4D \lrcorner = 2D_1 + 2D_2 + 2D_3$. Then $K_X + D$ is 3-complemented with $D^+ = \frac{2}{3}D_1 + \frac{2}{3}D_2 + \frac{2}{3}D_3$.
     - (ii) $\llcorner 4D \lrcorner \geq 3D_1 + 2D_2 + 2D_3$.
       - (A) $\llcorner 5D \lrcorner = 3D_1 + 3D_2 + 2D_3$. Then $K_X + D$ is 4-complemented, with $D^+ = \frac{3}{4}D_1 + \frac{3}{4}D_2 + \frac{1}{2}D_3$.
       - (B) $\llcorner 5D \lrcorner = 4D_1 + 3D_2 + 2D_3$. In this case, using $\sum d_i \leq 2$ one can show that $\llcorner 7D \lrcorner = 5D_1 + 4D_2 + 3D_3$. Thus, $K_X + D$ is 6-complemented, with $D^+ = \frac{5}{6}D_1 + \frac{2}{3}D_2 + \frac{1}{2}D_3$.

We leave the verification that this covers all possible cases with $\sum d_i \leq 2$

to the reader. Here is a sample of the type of argument that is required. Suppose that $\llcorner 2D \lrcorner \geq D_1 + D_2 + D_3$ and $\llcorner 3D \lrcorner \geq 2D_1 + 2D_2 + D_3$. Then $d_1 \geq d_2 \geq \frac{2}{3}$, $d_3 \geq \frac{1}{2}$ so that $\llcorner 4D \lrcorner \geq 2D_1 + 2D_2 + 2D_3$. Furthermore, $d_4 \leq 2 - d_1 - d_2 - d_3 \leq \frac{1}{6}$. Thus, $\llcorner 4d_i \lrcorner = 0$ for $i \geq 4$, and $\llcorner 4D \lrcorner$ is supported on $D_1 \cup D_2 \cup D_3$. This justifies the division into cases (i) and (ii).

The last statements in the theorem are clear. □

The following corollary is immediate.

19.5 Corollary. *Let $D = \sum d_i D_i$ be a subboundary on $\mathbb{P}^1$. Suppose that each d_i has the form $d_i = (m_i - 1)/m_i$ for some integer $m_i > 1$, and that $\deg(K_{\mathbb{P}^1} + D) < 0$. Then the integers m_i must fall into one of the following cases:*

(1) (m_1) *or* (m_1, m_2),
(2) $(2, 2, m_3)$,
(3) $(2, 3, 3)$,
(4) $(2, 3, 4)$,
(5) $(2, 3, 5)$.

Moreover, $K_{\mathbb{P}^1} + D$ is 1-, 2-, 3-, 4-, or 6-complemented in cases (1), (2), (3), (4), or (5), respectively. □

We now begin the analysis which relates complements on X to complements on S. The first step can be done in arbitrary dimension.

19.6 Theorem. *Let X be a smooth variety, let Z be a normal variety, and let $h : X \to Z$ be a proper morphism with connected fibers. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-subboundary on X (i.e., a subboundary with $d_i \in \mathbb{Q}$), whose support is a divisor with normal crossings. Assume that $-(K_X + D)$ is h-nef and h-big.*

Write $D = S + D_0$ with S the smallest effective Weil divisor such that $\llcorner D_0 \lrcorner \leq 0$, and suppose that either S is irreducible, or $\dim X \leq 3$ and S is semismooth in codimension 1. Given an n-(semi)-complement $\overline{D}_S$ of $K_S +$ $\mathrm{Diff}(D_0)$, then in a neighborhood of any fiber of h meeting S, there exists a divisor $\overline{D} \in |-nK_X - nS - \llcorner (n+1)D_0 \lrcorner|$ such that $\mathrm{Diff}(\overline{D}) = \overline{D}_S$.

If $K_X + S$ is plt or $\dim X \leq 2$ then $\overline{D}$ is an n-complement. Moreover, if $K_S + (D_S)^+$ is plt then so is $K_X + D^+$.

Proof. Divisors from the linear system $|-nK_X - nS - \llcorner (n+1)D_0 \lrcorner|$ on X restrict to divisors in the linear system $|-nK_S - \mathrm{Diff}(\llcorner (n+1)D_0 \lrcorner)|$ on S, which is the system containing $\overline{D}_S$. A failure of surjectivity of the restriction map would be detected by

$$\begin{aligned} &R^1 h_*(\mathcal{O}_X(-nK_X - (n+1)S - \llcorner (n+1)D_0 \lrcorner) \\ &\quad = R^1 h_*(\mathcal{O}_X(K_X + \ulcorner -(n+1)(K_X + D) \urcorner). \end{aligned}$$

But this latter sheaf is 0 by Kawamata–Viehweg vanishing [KMM87,1-2-3]. Thus, the divisor $\overline{D}$ exists in a neighborhood of any fiber of h intersecting S.

$K + D^+$ is lc near S by (17.7). Since $K + D^+ \equiv 0$, (17.4) shows that $K + D^+$ is lc in a neighborhood of any fiber of h intersecting S. □

In order to apply this to surfaces, we need a lemma.

19.7 Lemma. *Let X be a smooth surface, let Z be a normal surface, and let $h : X \to Z$ be a birational morphism. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-subboundary on X. Assume that*

(19.7.1) if $d_i < 0$ then $h(D_i)$ is a point in Z; and

(19.7.2) $-(K_X + D)$ is h-nef.

Write $D = S + D_0$ with S the smallest effective Weil divisor such that $\llcorner D_0 \lrcorner \leq 0$, and suppose that S is a semismooth curve. Then every component of D_0 which meets S has nonnegative multiplicity in D_0.

Proof. Write $D_0 = D_+ - D_-$, with D_+ and D_- effective such that $\mathrm{Supp}(D_+)$ and $\mathrm{Supp}(D_-)$ have no common components. By (19.7.1), $\mathrm{Supp}(D_-)$ is contained in the exceptional locus of h. If $D_- \neq 0$, let E be a component of D_- with $D_- \cdot E > 0$. (This exists by negative-definiteness of the intersection matrix of a contractible curve.) Then since $D_+ \cdot E \geq 0$ and $-(K_X + S + D_0)$ is h-nef, $(K_X + S) \cdot E < 0$. It follows that E is a -1-curve disjoint from S. Thus, blowing down E preserves the assumptions of the lemma.

The lemma now follows by induction on the number of components of $\mathrm{Supp}(D_-)$. □

We can now apply (19.4) and (19.6) to classify n-complements on surfaces.

19.8 Theorem. *Let X be a smooth surface, let Z be a normal surface, and let $h : X \to Z$ be a birational morphism. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-subboundary on X whose support is a divisor with normal crossings. Assume that*

(19.8.1) if $d_i < 0$ then $h(D_i)$ is a point in Z;

(19.8.2) $-(K_X + D)$ is h-nef; and

(19.8.3) $K_X + D$ is log canonical.

Write $D = S + D_0$ with S the smallest effective Weil divisor such that $\llcorner D_0 \lrcorner \leq 0$, and suppose that S is non-empty. Then $K_X + D$ is 1-, 2-, 3-, 4-, or 6-complemented in a neighborhood of a fiber of h.

Moreover, if $K_X + D$ is not 1- or 2-complemented, then $S = \mathbb{P}^1$ and $\llcorner D^+ - S \lrcorner = 0$ in a neighborhood of a fiber of h. In addition, if there is a component of S which is not contained in a fiber of h, and if $K_X + D$ is not 1-complemented, then in a neighborhood of any fiber, $\mathrm{Diff}_S(D_0) = \frac{1}{2}P_1 + \frac{1}{2}P_2$ for some points $P_1, P_2 \in S$.

Proof. Since $\mathrm{Supp}\, D$ has normal crossings, S is a semismooth curve. By adjunction (16.9), since $K_X + S + D_0$ is lc, $K_S + \mathrm{Diff}\, D_0$ is slc. The normal crossing assumption implies that $\mathrm{Diff}(D_0)$ is supported on the smooth locus of S. Moreover, $-(K_S + \mathrm{Diff}\, D_0)$ is nef on every complete component of S. By (19.7), $\llcorner \mathrm{Diff}\, D_0 \lrcorner = 0$. Thus we may apply (19.4) and conclude that $K_S + \mathrm{Diff}\, D_0$ is 1-, 2-, 3-, 4-, or 6-semicomplemented and that it is 1- or 2-semicomplemented unless $S = \mathbb{P}^1$.

If $\overline{D}_S$ is the n-semicomplement of $K_S + \mathrm{Diff}\, D_0$, then by (19.6), in a neighborhood of a fiber of h there is a divisor $\overline{D}$ with $\mathrm{Diff}(\overline{D}) = \overline{D}_S$ which is an n-complement of $K_X + D$.

Suppose that $K_X + D$ is not 1- or 2-complemented. Then by (19.4), $\llcorner D^+ - S \lrcorner = 0$ in a neighborhood of $S = \mathbb{P}^1$. Let $g : Y \to X$ be a blowup on which $\mathrm{Supp}\, g^{-1}(D^+)$ has normal crossings. Write $K_Y = g^*(K_X + D) + A - F$ with all multiplicities of components of A being greater than -1 and all multiplicities of components of F being greater than or equal to 1; by the connectedness theorem (17.4), F is connected in a neighborhood of a fiber of h. But then in that neighborhood, the union of all components of D^+ other than S which have multiplicity 1 in D^+ would necessarily meet S. Since S already contains all components of that kind in a neighborhood of itself, there can be no such components.

The last statement follows immediately from (19.4). □

19.9 Definition. Let X be a normal variety and let D be a subboundary on X. An *exceptional n-complement* of $K_X + D$ is an n-complement $\overline{D}$ such that there is exactly one divisor E of $\mathbb{C}(X)$ such that $a(E, X, \overline{D}) = -1$. $K_X + D$ is *exceptionally n-complemented* if there exists an exceptional n-complement.

19.10 Corollary. *Let X and Z be normal surfaces, and let $h : X \to Z$ be a birational morphism. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-subboundary on X. Assume that*

(19.10.1) if $d_i < 0$ then $h(D_i)$ is a point in Z;
(19.10.2) $-(K_X + D)$ is h-nef; and
(19.10.3) $K_X + D$ is log canonical.

Then in a neighborhood of a fiber of h,
either $K_X + D$ is 1- or 2-complemented,
or $K_X + D$ is exceptionally 3-, 4-, or 6-complemented.

Proof. Fix $P \in Z$, and let H be a general hyperplane section of Z through P. Let λ be the largest nonnegative number such that $K_X + D + \lambda h^*(H)$ is log canonical.

We first replace D by $\widetilde{D} = D + \lambda h^*(H)$ and then replace X by a resolution of singularities $g : Y \to X$ on which the support of the birational transform Δ of $\widetilde{D}$ has normal crossings. Note that there is at least one component of multiplicity 1 in Δ, for if not one could increase λ. Thus, we can apply (19.8) to $K_Y + \Delta$ and obtain an n-complement in a neighborhood of $g^{-1}(h^{-1}(P))$.

By (19.2), $D^+ = g(\Delta^+)$ will determine an n-complement of $K_X + D$ in a neighborhood of $h^{-1}(P)$. Furthermore, if $\llcorner \Delta^+ - S \lrcorner = 0$ then the induced n-complement of D is exceptional. The corollary follows. □

19.11 Corollary. *Let X and Z be normal surfaces, and let $h : X \to Z$ be a birational morphism. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-subboundary on X. Assume that*

(19.11.1) if $d_i < 0$ then $h(D_i)$ is a point in Z;

(19.11.2) $-(K_X + D)$ is h-ample; and

(19.11.3) $K_X + D$ is log terminal.

Suppose in addition that there is a reduced component S_0 of D not contained in a fiber of h. Then in a neighborhood of any fiber of h meeting S_0, $K_X + D$ is 1-complemented.

Proof. We proceed as in the previous proof, replacing (X, D) by (Y, Δ). Write $\Delta = S + \Delta_0$ with S the smallest effective Weil divisor such that $\llcorner \Delta_0 \lrcorner \leq 0$. Note that the birational transform of S_0 is an incomplete component of S in a neighborhood of any fiber of $g \circ h$ which it meets. Since $K_Y + \Delta$ is lt and $-(K_Y + \Delta)$ is $(g \circ h)$-ample, we may replace Δ by $S + (1 + \varepsilon)\Delta_0$ (for small $\varepsilon > 0$) without disturbing the assumptions. But now it is impossible for $\mathrm{Diff}((1 + \varepsilon)\Delta_0)$ to be $\frac{1}{2}P_1 + \frac{1}{2}P_2$, independent of ε. It then follows from the last statement in (19.8) that $K_Y + (1 + \varepsilon)\Delta$ (and hence $K_X + D$) is 1-complemented. □

20. COVERING METHOD AND EASY FLIPS

JÁNOS KOLLÁR

In this chapter we construct some log flips by reducing their existence to the case of flops. The reduction relies on the following:

20.1 Proposition. *Let $f : (C \subset X) \to (P \in Z)$ be a small contraction of threefolds. Assume that there exists a three dimensional log terminal singularity $0 \in Y$ and a finite morphism $(0 \in Y) \to (P \in Z)$. Then finite generation holds for Z (4.4). In particular, if $-H$ is an f-ample divisor, then the opposite of f with respect to H exists.*

Proof. By (4.6) and (6.14) finite generation holds for Z. (4.2) gives a small modification $f^+ : X^+ \to Z$ such that H^+ is f^+-ample. This is the opposite (or flip) of f. □

Thus the question ahead is to find conditions which ensure that $P \in Z$ is covered by a log terminal point. Such conditions are given after some preparatory remarks about ramified covers.

20.2 Proposition. *Let $h : U \to V$ be a finite and dominant morphism between irreducible normal schemes of characteristic zero. Let $B = \sum b_i B_i$ be a divisor on V such that $\sum B_i$ contains the branch locus of h. (We allow $b_i = 0$, so that the latter condition is easy to satisfy.) Let $\operatorname{red} h^{-1}(\sum B_i) = \sum D_{ij}$ where $h(D_{ij}) = B_i$. Let e_{ij} be the ramification index of h at the generic point of D_{ij}. Then*

$$h^*(K_V + B) \equiv K_U + \sum (1 - (1 - b_i)e_{ij}) D_{ij}.$$

Proof. Codimension two subsets do not affect the claim, and hence we may assume that U, V are smooth. There is a natural morphism $h^*K_V \to K_U$, so that the verification of (20.2) reduces to computing ramifications at the generic point of D_{ij} for every i, j. By localizing we are reduced to the case when U and V are one dimensional regular schemes, and this case is straightforward. □

20.3 Proposition. *Notation and assumptions as in (20.2). Let* $h^*(K_V + B) = K_U + \bar{B}$. *Then*

(20.3.1) $\operatorname{logdiscrep}(U, \bar{B}) \geq \operatorname{logdiscrep}(V, B) \geq \frac{1}{\deg(U/V)} \operatorname{logdiscrep}(U, \bar{B})$, *and*

(20.3.2) $K_V + B$ *is lc (resp. plt) iff* $K_U + \bar{B}$ *is lc (resp. plt).*

Proof. Note first that even if B is effective, $\bar{B}$ is not necessarily so. In the definition of lc and plt (2.10) and (2.13) it is not important that B be effective. We use this more general case in the proof. In most applications however we only use the case when B and $\bar{B}$ are effective.

Let $g : W \to V$ be a proper modification with W normal (e.g., a resolution of singularities). Let W_U be the normalization of $W \times_V U$. We have a diagram

$$\begin{array}{ccc} W_U & \xrightarrow{p} & W \\ {\scriptstyle g_U}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{h} & V. \end{array}$$

Let $D = \sum B_i$. We may assume that g^{-1} is an isomorphism outside D. Rewriting (2.5) we get that

$$K_W + \operatorname{red} g^{-1}(D) \equiv g^*(K_V + B) + \sum a_\ell(E_i, B) E_i,$$

where $\operatorname{Supp} E_i \subset \operatorname{Supp} g^{-1}(D)$. Applying (20.2) to $p : W_U \to W$ we obtain

$$\begin{aligned} K_{W_U} + \operatorname{red}(g \circ p)^{-1}(D) &\equiv p^*(K_W + \operatorname{red} g^{-1}(D)) \\ &\equiv p^* g^*(K_V + B) + p^* \sum a_\ell(E_i, B) E_i \\ &\equiv g_U^*(K_U + \bar{B}) + p^* \sum a_\ell(E_i, B) E_i. \end{aligned}$$

If $p^* E_i = \sum e_{ij} F_{ij}$ then $a_\ell(F_{ij}, \bar{B}) = e_{ij} a_\ell(E_i, B)$. □

It is worthwhile to mention the special case when $B = 0$:

20.3.3 Corollary. *Let* $f : X \to Y$ *be a finite and dominant morphism between normal varieties. Assume that* K_X *and* K_Y *are* $\mathbb{Q}$*-Cartier. If* X *is lt (resp. lc) then* Y *is lt (resp. lc).* □

For ease of reference we mention three special cases of (20.2–3):

20.4 Corollary. *Let* $h : U \to V$ *be as in (20.3).*

(20.4.1) Assume that h *is étale in codimension one. Then* $K_V + B$ *is lc (resp. plt) iff* $K_U + h^* B$ *is lc (resp. plt).*

(20.4.2) Let $S \subset V$ be the branch locus of h and assume that $B = S + D$. Then $K_V + S + D$ is lc (resp. plt) iff $K_U + \operatorname{red} h^{-1}(S) + h^*D$ is lc (resp. plt).

(20.4.3) Let $S \subset V$ be the branch locus of h. Assume that h is a double cover and $B = (1/2)S + D$. Then $K_V + (1/2)S + D$ is lc (resp. plt) iff $K_U + h^*D$ is lc (resp. plt). $\square$

20.5 Proposition. *Let X be a normal singularity and B an effective $\mathbb{Q}$-divisor. Assume that $K_X + B$ is plt and has index 2 or 1 (i.e. $2(K + B) \sim 0$). Then there is a double cover $p : Z \to X$ such that Z is canonical of index one.*

If $\dim X = 3$ then finite generation holds for every Weil divisor E on X.

Proof. Let D be a general member of the linear system $|2B|$. By Bertini theorems, D is irreducible and reduced. By (2.33), $K + (1/2)D$ is plt. Since $2(-K) \sim D$, we can construct a double cover $p : Z \to X$ which ramifies along D. By (20.4.3) K_Z is plt. K_Z is also Cartier, hence Z is canonical of index one. In dimension three finite generation holds by (20.1). $\square$

20.6 Proposition. *Let X be a normal singularity and D an effective $\mathbb{Q}$-divisor. Assume that $K_X + D$ is lc and has index 2 (or 1). Assume furthermore that $\llcorner D \lrcorner$ is LSEPD and $K + D$ is plt outside $\llcorner D \lrcorner$. Then there is a finite cover $p : Z \to X$ such that Z is canonical of index one.*

If $\dim X = 3$ then finite generation holds for every Weil divisor E on X.

Proof. The required cover is constructed in two steps. By assumption there is a regular function s such that $\operatorname{Supp}(s = 0) = \llcorner D \lrcorner$. Let $(s = 0) = \sum m_i D_i$ and let m be a natural number which is divisible by every m_i. Let $h : X' \to X$ be the normalization of an irreducible component of $\operatorname{Spec}_X \mathcal{O}_X[t]/(t^m - s)$. By (20.2)

$$K_{X'} + \operatorname{red} h^{-1}(\llcorner D \lrcorner) + h^*\{D\} = h^*(K_X + D).$$

Set $D' = \operatorname{red} h^{-1}(\llcorner D \lrcorner) + h^*\{D\}$. (X', D') has index 2, is lc and plt outside $\llcorner D' \lrcorner$. Furthermore,

$$\llcorner D' \lrcorner = \operatorname{red} h^{-1}(\llcorner D \lrcorner) = (t = 0)$$

is Cartier.

Since $2(-K - \llcorner D' \lrcorner) \sim 2\{D'\}$, we can construct a double cover $p : Z \to X'$ ramified along $\operatorname{Supp}\{D'\}$. By (20.4.3)

$$K_Z + p^{-1}(\llcorner D' \lrcorner) = K_Z + (t \circ p = 0)$$

is Cartier, lc and plt outside $\operatorname{Supp} p^{-1}(\llcorner D' \lrcorner)$. Thus K_Z is also Cartier and plt (2.17), hence Z is canonical of index one. In dimension three the flip of f exists by (20.1). $\square$

The following result shows that flips exist if the boundary has at least two components intersecting the flipping curve. Such flips are used repeatedly in Chapters 21 and 22.

20.7 Theorem. *Let X be a $\mathbb{Q}$-factorial threefold. Let $D = S + B$ be a $\mathbb{Q}$-divisor, S reduced and $\llcorner B \lrcorner = \emptyset$. Let $f : (C \subset X) \to (P \in Z)$ be a small contraction with $\rho(X/Z) = 1$. Assume that*

(20.7.1) $K + D$ is log canonical and numerically nonpositive with respect to f;

(20.7.2) S has at least 2 irreducible components S^+ and S^- meeting C such that $S^- \cdot C < 0$ and $S^+ \cdot C > 0$.

Then the flip of f exists.

If $C \cdot (K + D) = 0$ then strictly speaking we cannot talk about the flip of f. However by (4.5) Z has at most one other small, normal and projective modification. By slight abuse of terminology we call it the flip of f. It can also be defined as the flip with respect to $K + D - S^+$.

The proof is done in several steps. First we prove a weaker version:

20.8 Lemma. *Let X be a $\mathbb{Q}$-factorial threefold. Let $D = S + B$ be a $\mathbb{Q}$-divisor, S reduced and $\llcorner B \lrcorner = \emptyset$. Let $f : (C \subset X) \to (P \in Z)$ be a small contraction with $\rho(X/Z) = 1$. Assume that*

(20.8.1) $K + D$ is log terminal and numerically negative with respect to f;

(20.8.2) S has at least 2 irreducible components S^+ and S^- meeting C such that $S^- \cdot C < 0$ and $S^+ \cdot C \geq 0$.

Then $K_X + D$ is 1-complemented in a neighborhood of C and the flip of f exists.

Proof. First we prove that $K + D$ is 1-complemented. By (17.5) S^- is normal. By (16.9.2)

$$K_{S^-} + \mathrm{Diff}(D - S^-) = (K + D)|S^-$$

is lt and f-negative. Assume that $S^+ \cap S^- \subset C$. Then

$$S^+ \cdot_X C = (S^+ \cap S^-) \cdot_{S^-} C < 0$$

since $C \subset S^-$ is contractible; a contradiction. Thus there exists an irreducible component of $S^+ \cap S^-$ intersecting C but not contained in it. Therefore by (19.11) and (19.6) $K + D$ is 1-complemented. That is, there exists a reduced divisor $D^+ \geq \llcorner D \lrcorner$ such that $K + D^+$ is lc and numerically 0 relative to f.

I claim that D^+ is LSEPD. This is clear if $S^+ \cdot C > 0$. If $S^+ \cdot C = 0$ then $C \subset S^+$. Since both S^- and S^+ contain C, no other component of $S + B$ can contain C, hence they all have nonnegative intersection with C. Thus

$$(K + S^+ + S^-) \cdot C \leq (K + S + B) \cdot C < 0,$$

thus in D^+ there is a component which has positive intersection with C.

Let $D_Z = f(D^+)$. Then $K_Z + D_Z$ is $\mathbb{Q}$-Cartier, lc and plt outside $\operatorname{Supp} D_Z$. $S^+, S^- \subset D^+$, therefore D^+ and D_Z are LSEPD. Thus the flip exists by (20.6). □

20.9 Corollary. *Let $(X, K+D)$ be a $\mathbb{Q}$-factorial threefold, not necessarily log canonical. Then in a neighborhood of $S = \llcorner D \lrcorner$ there exists a $\mathbb{Q}$-factorial log terminal model for $K+D$.*

Proof. Let $(Y, K_Y + D_Y)$ be a log resolution (where D_Y is as in (2.7)) and apply the minimal model program relative to the morphism $f : Y \to X$ in a neighborhood of S. In doing so we might encounter a small contraction

$$f_k : Y_k \xrightarrow{g_k} Z \to X$$

with respect to $K_{Y_k} + D_{Y_k}$. Let C be the exceptional curve of g_k. By hypothesis $f_k^*(S) \cdot C = 0$ and $C \subset f_k^*(S)$. Then $0 = f_k^*(S) \cdot C = \sum c_i B_i \cdot C$, where the sum is taken over the i such that $B_i \cap C \neq \emptyset$. Note that $\sum c_i B_i \neq 0$, because $C \subset f_k^*(S)$. This shows that there exists an irreducible component of $\llcorner D_{Y_k} \lrcorner$ meeting C and nef on it.

Let H be an ample divisor on $f_k : Y_k \to X$. Then $H = f_k^* f_k(H) - \sum a_i E_i$ for some $a_i > 0$ and the E_i are exceptional for f_k. Then $\sum a_i E_i \cdot C < 0$, and thus there exists an index i such that $E_i \cdot C < 0$. By definition $E_i \subset \llcorner D_{Y_k} \lrcorner$. Therefore the flip exists by (20.8) and termination was proved in (7.1). Thus the $(K_Y + D_Y)$-MMP terminates and gives the required $\mathbb{Q}$-factorial log terminal model. $\square$

20.10 Lemma. *Notation and assumptions as in (20.7).*

(20.10.1) Assume in addition that $2(K+D) \sim 0$. Then the flip of f exists.

(20.10.2) Assume in addition that $K + D \equiv 0$. Assume furthermore that flips of contractions as in (20.7) exist if $K + D \equiv 0$ and $K + D - S^+$ is lt. Then the flip of f exists.

Proof. Let $g : (Y, D_Y) \to (X, D)$ be a $\mathbb{Q}$-factorial log terminal model. Let $S_1^+ \subset Y$ be the birational transform of S^+. By assumption

$$\begin{aligned} 2(K_Y + D_Y) &\sim g^*(2(K_X + D)) \sim 0 \quad \text{in case } 1, \\ K_Y + D_Y &\equiv g^*(K_X + D) \qquad \equiv 0 \quad \text{in case } 2. \end{aligned}$$

By assumption $\llcorner D \lrcorner$ is LSEPD with respect to f, thus $\llcorner D_Y \lrcorner$ is LSEPD with respect to $f \circ g$. For $0 < \epsilon \ll 1$ apply the $(K_Y + D_Y - \epsilon S_1^+)$-MMP to $(f \circ g) : Y \to Z$. We successively construct objects $h_k : Y_k \to Z$ such that

(20.10.3.1) $\llcorner D_{Y_k} \lrcorner$ is LSEPD with respect to h_k;

(20.10.3.2) $K_{Y_k} + D_{Y_k} - \epsilon S_k^+$ is lt where $S_k^+ \subset Y_k$ be the birational transform of S^+;

(20.10.3.3)

$$\begin{aligned} 2(K_{Y_k} + D_{Y_k}) &\sim 0 \quad \text{in case } 1, \\ K_{Y_k} + D_{Y_k} &\equiv 0 \quad \text{in case } 2. \end{aligned}$$

Assume that in the process we encounter a small contraction $g_k : Y_k \to Z_k$. Let $C_k \subset Y_k$ be the flipping curve. Then $C_k \cdot S_k^+ > 0$, hence $\llcorner D_{Y_k} \lrcorner$ has another irreducible component which intersects C_k negatively. Furthermore, by (20.10.3.2) $K_{Y_k} + D_{Y_k}$ is lc, lt outside $\operatorname{Supp} \llcorner D_{Y_k} \lrcorner$ and $K_{Y_k} + D_{Y_k} - S_k^+$ is lt.

In the first case $K_{Z_k} + g_k(D_{Y_k})$ has index two on Z_k and is plt outside the LSEPD divisor $\llcorner g_k(D_{Y_k}) \lrcorner$. Thus the flip of g_k exists by (20.6). In the second case the flip of g_k exists by assumption. By Chapter 7 the sequence of flips terminate. Therefore the program stops with

$$\bar{h} : \bar{Y} \to Z$$

such that $K_{\bar{Y}} + D_{\bar{Y}} - \epsilon \bar{S}^+$ is $\bar{h}$-nef. (2.32.2) implies that the flip of f with respect to $K_X + D - \epsilon S^+$ exists. □

Proof of (20.7). Let H be a sufficiently general and sufficiently f-ample divisor. Then for a suitable $1 > \epsilon \geq 0$, $K + D + \epsilon H$ is numerically f-trivial and satisfies all the assumptions of (20.7). Thus we may assume that $K + D \equiv 0$. By (20.10.2) it is sufficient to consider the case when in addition $K + D - S^+$ is lt. As in the proof of (20.8) we see that there exists an irreducible component of $S^+ \cap S^-$ intersecting C but not contained in it. Therefore by (19.10) and (19.6) $K + D$ is 1- or 2-complemented. Thus by (20.10.1) the flip exists. □

The following result applies every time in dimension three when the opposite exists. However in practice it is usually very difficult to find the divisors S_i required in the assumptions.

20.11 Theorem. *(Mori, unpublished) Let $f : X \to Z$ be a small morphism with exceptional set $C \subset X$. Let $S_1, S_2 \subset X$ be effective divisors such that $S_1 \cap S_2 = C$. Assume that $m_1 S_1$ and $m_2 S_2$ are linearly equivalent for some $m_1, m_2 > 0$. Then the opposite of f with respect to S_1 exists and $S_1^+ \cap S_2^+ = \emptyset$.*

Proof. The pencil $\langle m_1 S_1, m_2 S_2 \rangle$ is base point free outside C; denote by $p : X \dashrightarrow \mathbb{P}^1$ the corresponding rational map. Then the opposite of f is the normalization of the closure of the image of the map $p \times f : X \dashrightarrow \mathbb{P}^1 \times Z$. □

20.12 Corollary. *Let $f : X \to Z$ be a small morphism with exceptional set $C \subset X$. Assume that*

(20.12.1) $\rho(X/Z) = 1$;

(20.12.2) there is a divisor D such that (X, D) is klt and $-(K_X + D)$ is f-nef; and

(20.12.3) there are effective divisors $S_1, S_2 \subset X$ such that $S_1 \cap S_2 = C$.

Then the opposite of f exists and $S_1^+ \cap S_2^+ = \emptyset$.

Proof.

$$S_i \cdot_X C = (S_i \cap S_{3-i}) \cdot_{S_{3-i}} C < 0$$

since $C \subset S_{3-i}$ is contractible. Therefore by the base point free theorem [KMM87,3-1-2], suitable positive multiples of S_i are linearly equivalent. Thus (20.11) applies. $\square$

20.13 Corollary. *Let X be a $\mathbb{Q}$-factorial threefold. Let $f : (X, K + S_1 + S_2) \to Z$ be a small contraction with $\rho(X/Z) = 1$. Assume that $C \cdot (K + S_1 + S_2) < 0$, $C \cdot S_1 < 0$, $C \cap S_2 \neq \emptyset$, $K + S_1 + S_2$ is lc and $K + S_1$ is lt. Then the flip of f exists.*

Proof. If $C \cdot S_2 > 0$ then (20.7) applies. If $C \cdot S_2 \leq 0$ then $C \subset S_1 \cap S_2$. If equality holds then (20.12) applies, otherwise there is a 1-complement B by (19.11) and we can apply (20.7) with $S^+ = B$. $\square$

21. SPECIAL FLIPS

JÁNOS KOLLÁR

The aim of the next two chapters is to investigate flips of special contractions (18.8). The importance of these is clear from (18.9). For a special flip B is also reduced, in fact usually it is empty. We change notation and write S for what used to be $S + B$. This is important, since (following Shokurov) from (21.3) on B is used for something different.

21.1 Notation. Let X be a normal $\mathbb{Q}$-factorial threefold and $S \subset X$ an integral Weil divisor. Assume that $K_X + S$ is lt. Let $f : X \to Z$ be a small $(K + S)$-extremal contraction; i.e., $K + S$ is f-negative and $\rho(X/Z) = 1$. Thus there is a proper curve $C \subset X$ and a finite subset $P \subset Z$ such that $f : X - C \to Z - P$ is an isomorphism. The existence of flips is local on Z; we may pick a point $0 \in P \subset Z$ and assume that Z is a small neighborhood of 0. Therefore we may assume that $C = f^{-1}(0)$ is connected, but in general C may be reducible.

We assume that $C \cdot S_i < 0$ for every irreducible component $S_i \subset S$. In particular, every S_i contains C.

We call $f : (X, S) \to Z$ a *special contraction.* By a slight abuse of language, the flip of f is called a *special flip.*

Our aim is to construct the flip of f. This is done in several steps. First we construct the flip in certain special cases. For the remaining cases, we prove that they exist provided index two flips exist. Index two flips turn out to be the hardest, they are discussed in the next chapter.

21.2 Proposition. *If S is reducible, the flip exists.*

Proof. As we remarked, $C \subset S_i$ for every i. Since $K + S$ is lt, $S_1 \cap S_2$ is a locally irreducible curve (16.9). Thus $S_1 \cap S_2 = C$ (and C is irreducible). Thus the flip exists by (20.12). □

21.2.1 Convention. For the rest of the chapter we always assume that S is irreducible.

21.3 Definition. Assumptions as above.

(21.3.1) By (19.6,19.8) there is an $n \in \{1,2,3,4,6\}$, called the *index* of $K+S$, such that $K+S$ is *n-complemented.* I.e. there is a $\mathbb{Q}$-divisor B such that nB is an integral divisor, $K+S+B$ is lc and $n(K+S+B) \sim 0$. (This B has nothing to do with the B occurring in the definitions (18.1-8).)

(21.3.2) We say that $K+S+B$ is *exceptional* if $(S, \operatorname{Diff}_S(B))$ is exceptional in the sense of (19.9). Observe that this may depend on the choice of B. Sometimes the contraction f itself is called exceptional if the choice of B is already agreed upon. The same applies to the flip of f.

21.4 Proposition. *Index one flips exist.*

Proof. Index one means that B is an integral Weil divisor. Since

$$C \cdot B = -C \cdot (K+S) > 0,$$

$S+B$ and $f(S+B)$ are reduced LSEPD divisors, and the flip exists by (20.7). □

21.5 Proposition. *If $K+S+B$ has index two and $K+S+B$ is lt, then the flip exists.*

Proof. Assume first that $\llcorner B \lrcorner \neq \emptyset$ and let $S' \subset \llcorner B \lrcorner$. If $C \cdot S' > 0$ then the flip exists by (20.7). If $C \cdot S' \leq 0$ then $C \cdot (K+S+S') \leq C \cdot (K+S) < 0$; the flip exists by (20.13). If $\llcorner B \lrcorner = \emptyset$ then $K+S+B$ is plt by (2.16.3) hence (20.5) gives the flip. □

Next we apply the Backtracking Method (6.4–5). The notation and conventions of (6.4–5) are used throughout.

21.6 Construction of $q_1 : Y_1 \to X_0$. Assume that $K+S+B$ is not lt. Let $h : X^t \to X$ be a $\mathbb{Q}$-factorial lt model (20.9). By assumption h is not an isomorphism. If $E^t \subset X^t$ is the reduced exceptional divisor then $K_{X^t} + E^t + S^t + B^t = h^*(K+S+B)$ is lt.

21.6.1 Lemma. *For any irreducible component $E \subset E^t$ there is a unique projective morphism $q_1 : Y_1 \to X_0$ with the following properties*

(21.6.1.1) Y_1 is $\mathbb{Q}$-factorial and $\rho(Y_1/X) = 1$.

(21.6.1.2) The exceptional set of q_1 is an irreducible divisor $E_1 \subset Y_1$ such that under the birational map $Y_1 \to X \leftarrow X^t$ the birational transform of E_1 is E.

(21.6.1.3) q_1 is a log crepant morphism, i.e.

$$K + E_1 + S_1 + B_1 = q_1^*(K+S+B)$$

and is lc.

Proof. Uniqueness of Y_1 follows from (6.2).

The existence follows from the $(K_{X^t} + E^t + S^t + B^t - \epsilon E)$-MMP applied to $X^t \to X$. We need to check that all flips exist and any sequence of them terminates. Let $C \subset X^t$ be a flipping curve. Since $K_{X^t} + E^t + S^t + B^t$ is numerically h-trivial, $C \cdot E > 0$. As in the proof of (20.9) we find another exceptional divisor E_k such that $C \cdot E_k < 0$. Thus the flip exists by (20.7) and termination follows from (7.1).

At the end we obtain a morphism $q' : Y' \to X$ and $E' \subset Y'$, which is the birational transform of E. Furthermore,

$$-\epsilon E' \equiv K_{Y_1} + (E^t + S^t + B^t)' - \epsilon E'$$

is q'-nef. $\operatorname{Supp}(E^t + S^t + B^t)' = \operatorname{Supp} q'^*(S+B)$ is LSEPD with respect to q', hence base point freeness applies to $-E'$ (2.32.2). Thus we obtain a morphism

$$Y' \xrightarrow{r} Y_1 \xrightarrow{q_1} X,$$

such that $-E_1$ (the birational transform of $-E'$) is q_1-ample. Thus E_1 contains the exceptional set of q_1. If D is a Weil divisor on Y_1 then $D = q_1^*(q_1(D)) + c(D)E_1$ for some $c(D) \in \mathbb{Q}$, and hence Y_1 is $\mathbb{Q}$-factorial. □

From now on we always assume that $q_1 : Y_1 \to X_0$ is chosen as in (21.6.1).

21.6.2 Lemma. *Notation as in (6.4).*

(21.6.2.1) $K + S_j + E_j + B_j$ is lc for every j;

(21.6.2.2) There are $c_e, c_b > 0$ such that $S_j + c_e E_j + c_b B_j$ is numerically trivial on Y_j/Z for every j.

Proof. By (21.6.1.3) the first part holds for $j = 1$. Since $S \cdot C < 0$ and $B \cdot C > 0$, there is a $c_b > 0$ such that $(S + c_b B) \cdot C = 0$. Let $S_1 + c_e E_1 + c_b B_1 = q_1^*(S + c_b B)$. Both of these properties are stable under flips and flops. □

21.7 Existence of the contractions r_i.

Choose $0 \le a_b < 1$ such that the coefficient of E_1 in $q_1^*(K_X + S + a_b B) = K + S_1 + a_b B_1 + a_e' E_1$ is positive.

21.7.1 Proposition. *Assume that $1 > a_e > a_e'$ is sufficiently close to a_e'. Then*

(21.7.1.1) $R_j \cdot (K + S_j + a_b B_j + a_e E_j) < 0$ for every j.

(21.7.1.2) $K + S_j + a_b B_j + a_e E_j$ is plt for every j.

(21.7.1.3) R_j can be contracted.

(21.7.1.4) There are $b_s, b_e > 0$ such that $R_j \cdot (b_s S_j + b_e E_j) < 0$ for every j.

Proof. (6.5.2) proves (21.7.1.1).

By assumption $K_X + S$ is plt. Since q_1 is an isomorphism outside E_1, this implies that $K + S_1$ is plt outside $E_1 \cup B_1$. By (21.6.2.1) $K + S_1 + E_1 + B_1$

is lc, thus $K + S_1 + a_bB_1 + a_eE_1$ is plt along $E_1 \cup B_1$ since $a_b, a_e < 1$. Thus $K + S_1 + a_bB_1 + a_eE_1$ is plt. By (21.7.1.1) plt is preserved under flips. The first two claims imply the third.

Both $K + S + a_bB$ and S are negative on C, thus $K + S + a_bB \equiv cS$ for some $c > 0$. Thus

$$\begin{aligned} K + S_1 + a_bB_1 + a_eE_1 &\equiv q_1^*(K + S + a_bB) + (a_e - a_e')E_1 \\ &\equiv q_1^*(cS) + (a_e - a_e')E_1 \equiv b_sS_1 + b_eE_1 \end{aligned}$$

for some $b_s, b_e > 0$. This equivalence is preserved by subsequent flips. □

Another useful general result is the following:

21.7.2 Lemma. *Notation as above. Then either $R_j \cdot S_j > 0$ or $Q_j \cdot S_j > 0$.*

Proof. S_j intersects the exceptional set but does not contain it. Thus S_j cannot be seminegative on Y_j/Z. □

21.8 Three Kinds of Flips of the Backtracking Method.

The sequence of flips in the backtracking method can be broken into three parts. Some easy flips in the beginning, some hard flip (hopefully at most one) in the middle and then again a sequence of easy flips. (Any of these may be empty in a given situation.)

21.8.1 Beginning Flips.

In the first step $Q_1 \cdot E_1 < 0$. If $R_1 \cdot E_1 \leq 0$ then there is no beginning flip. In general however $R_1 \cdot E_1 > 0$. Assume more generally that $R_{i-1} \cdot E_{i-1} > 0$. Then $Q_i \cdot E_i < 0$. If $R_i \cdot E_i \leq 0$ then the beginning sequence is finished. If $R_i \cdot E_i > 0$ then $R_i \cdot S_i < 0$ since by (21.7.1.4) $R_i \cdot (b_sS_i + b_eE_i) < 0$. Thus the contracted curve is contained in S_i and intersects E_i. The flip of r_i exists by (21.6.2.1) and (20.7). Since the flipping curve is contained in S_i, the beginning sequence of flips terminates (7.1). Thus we eventually get a divisorial contraction (and we are finished) or reach $E_m \subset Y_m$ such that E_m is seminegative on Y_m/Z. Therefore E_m contains every Y_m/Z-exceptional curve.

21.8.2 Middle Flips.

The flipping of $r_m : Y_m \to X_m$ is the hardest step. We distinguish several cases. We use $\mathrm{Locus}(R_m)$ to denote the exceptional set of r_m.

(21.8.2.1) $\mathrm{Locus}(R_m) = S_m \cap E_m$.

The flip exists by (20.11) and S_{m+1} and E_{m+1} are disjoint. Since E_m contains every Y_m/Z-exceptional curve, $S_m \cap E_m \subset S_m$ is the only $S_m \to f(S)$-exceptional curve. Thus S_{m+1} does not contain any exceptional curves, hence S_{m+1} is nef relative to $Y_{m+1} \to Z$. $Q_{m+1} \cdot S_{m+1} > 0$, so that $R_{m+1} \cdot S_{m+1} = 0$, and therefore $r_{m+1} : Y_{m+1} \to X_{m+1}$ contracts E_{m+1}. Thus X_{m+1} is the flip of f. We are finished.

(21.8.2.2) Locus(R_m) is a proper subset of $S_m \cap E_m$.

I do not know any useful general result in this case. An important special case is treated in [Shokurov91, 6.11].

(21.8.2.3) Locus(R_m) is disjoint from S_m.

In this case we need to argue that $(r_m : Y_m \to X_m, K + E_m + B_m)$ is in some sense "simpler" than $(f : X \to Z, K+S+B)$ and use induction. Assume that r_m can be flipped. Then $Q_{m+1} \cdot S_{m+1} = 0$. By (21.7.2) $R_{m+1} \cdot S_{m+1} > 0$.

(21.8.2.4) $R_m \cdot S_m > 0$. This belongs to the next case.

21.8.3 Final Flips.

These are the flips of type (21.8.2.4) or any flip following a flip of type (21.8.2.3–4).

21.8.3.1 Lemma. *Assume that $R_j \cdot S_j > 0$. Then $R_j \cdot E_j < 0$, the flip of r_j exists and $R_{j+1} \cdot S_{j+1} > 0$.*

Proof. $R_j \cdot E_j < 0$ follows from (21.7.1.4). The flip of r_j exists by (20.7).

By assumption $Q_{j+1} \cdot S_{j+1} < 0$. Thus $R_{j+1} \cdot S_{j+1} > 0$ by (21.7.2). □

21.8.3.2 Corollary. *Final flips exist and terminate.* □

First we give an easy application of the backtracking method.

21.9 Proposition. *Assume that $K_X + S + B$ is not lt outside* Supp S. *Then the flip exists.*

Proof. Assume first that $\llcorner B \lrcorner$ is not empty. Let E be an irreducible component of $\llcorner B \lrcorner$. If $E \cdot C > 0$ then (20.7) applies. Thus assume that $E \cdot C \leq 0$. Then

$$(K + S + E) \cdot C \leq (K + S) \cdot C < 0,$$

and therefore the $(K + S + E)$-flip exists by (20.13).

If $\llcorner B \lrcorner = \emptyset$ then there is an irreducible curve $D \subset X$, not contained in S such that $K + S + B$ is not lt along D. Let $h : X^t \to X$ be a $\mathbb{Q}$-factorial lt model of $(X, K + S + B)$ in a neighborhood of S. This exists by (20.9). By assumption there is an exceptional divisor $E \subset X^t$ such that $h(E) = D$. Using E construct $q_1 : Y_1 \to X_0$ as in (21.6). Assume that we already constructed $r_i : Y_i \to X_i$. Using (21.6.2.1) the following claim implies that r_i can be flipped:

21.9.1 Claim. *If r_i is small then E_i intersects every curve in the extremal ray R_i and $R_i \cdot S_i < 0$.*

Proof. Let P_1, P_2 be the two extremal rays. $Y_i \to Z$ maps E_i to $f(D)$. Let $F_i \subset E_i$ be a general fiber of $E_i \to f(D)$. Then $F_i \cdot E_i < 0$. We can specialize

F_i to the central fiber to conclude that E_i is negative on at least one of the extremal rays of $\overline{NE}(Y_i/Z)$, say $P_1 \cdot E_i < 0$. If E_i contains the whole exceptional set U of Y_i/Z then E_i intersects every exceptional curve. U is connected, so if $U \not\subset E_i$ then there is a curve $D \subset U$ such that $D \cdot E_i > 0$. In this case necessarily $P_2 \cdot E_i > 0$, thus E_i intersects every curve in P_2 as well.

$R_i \cdot S_i < 0$ is proved by induction. First let $i = 1$ and let D be a curve in R_1. Then $q_1(D)$ is an irreducible component of C, and hence

$$S_1 \cdot D = q_1^* S \cdot D = S \cdot q_1(D) < 0.$$

Assume now that $R_{i-1} \cdot S_{i-1} < 0$. Then $Q_i \cdot S_i > 0$. Thus if $R_i \cdot S_i \geq 0$ then S_i is Y_i/Z-nef. Y_i is $\mathbb{Q}$-factorial, hence $S_i \cap E_i \subset E_i$ is a divisor which lies entirely in the central fiber of $E_i/f(D)$. Therefore it can not be nef unless it is empty. If $S_i \cap E_i = \emptyset$ and S_i is Y_i/Z-nef then $S_i \cdot R_i = 0$ and $r_i : Y_i \to X_i$ contracts the whole divisor E_i. □

If $E_i \cdot R_i > 0$ then the flip of r_i exists by (20.7). Otherwise $E_i \cdot R_i \leq 0$, hence by (21.6.2.2) $R_i \cdot B_i > 0$. Thus $(K + S_i + E_i) \cdot R_i < 0$ and the flip of r_i exists by (20.13).

The above claim also implies that the exceptional locus of r_i is contained in S_i, and hence the sequence of flips terminates by (7.1). □

The second application of the backtracking method requires more delicate considerations.

21.10 Theorem. *Assume that index two special flips exist. Then all special flips exist.*

Using (19.6) and (19.8) this is a direct consequence of two propositions (21.12–13) whose formulation requires a definition:

21.11 Definition. Consider an extremal contraction with $K + S + B$ of index n. For certain values of $s \geq 1$ we can write

$$B = \sum_{i=1}^{s} \frac{1}{n} B^i$$

where the B^i are nonzero effective integral Weil divisors and $C \cdot B^i \geq 0$. One such way is $B = (1/n)(nB)$, but there may be others. The maximum value of s for which this is possible is called the type of $(f : X \to Z, K + S + B)$. (This has nothing to do with the type defined in (18.2) and no confusion is possible.)

21.12 Proposition. *Fix n and $t \geq 1$. Assume that index n exceptional special flips of type at least t exist whenever $K + S + B$ is lt. Then all exceptional special flips of index n and type t exist.*

21.12.1 Corollary. *Index 2 exceptional special flips exist.*

Proof. By (21.5) index two flips exist if $K+S+B$ is lt. $\square$

The following is a reformulation of [Shokurov91,7.4]. The proof is a case by case analysis.

21.13 Proposition. *Let $(f : X \to Z, K+S+B)$ be an exceptional special contraction of index n and type t such that $K+S+B$ is lt. Then one can find a B' such that one of the following holds:*

(21.13.1) $K+S+B'$ has index 1 or 2;

(21.13.2) $(f : X \to Z, K+S+B')$ is an exceptional special contraction of index n' and type t' and in the following diagram (n', t') lies to the right of (n, t).

$$\begin{aligned}&(6, \geq 1) \to (4, \geq 1) \to (6,2) \to (6, \geq 3) \to \\ &\qquad \to (3, \geq 1) \to (4,2) \to (4,3) \to (4, \geq 4). \quad \square\end{aligned}$$

21.14 Proof of (21.12). Let $(f : X \to Z, K+S+B)$ be an exceptional special contraction of index n and type t. If it is lt, there is nothing to prove. Otherwise let $h : X^t \to X$ be a $\mathbb{Q}$-factorial lt model with exceptional divisor $E^t \subset X^t$. The proof proceeds by induction on the number of irreducible components of E^t. To be more precise, we consider the minimum of the number of irreducible components of E^t_i where $h_i : X^t_i \to X$ runs through all $\mathbb{Q}$-factorial lt models. (Usually there are infinitely many.) We call this number the *minimal number of log crepant divisors.* In what follows we let $h : X^t \to X$ be a $\mathbb{Q}$-factorial lt model where the minimum is achieved.

If $f(E^t) \not\subset S$ then the flip exists by (21.9). Thus assume from now on that $f(E^t) \subset S$. Let S^t (resp. B^t) be the birational transform of S (resp. B) on X^t. Then

$$(h|S^t)^*(K+S+B|S) = h^*(K+S+B)|S^t = K_{X^t} + S^t + E^t + B^t|S^t.$$

Since f is exceptional, $K+S+B|S$ is exceptional, and therefore on S^t there is at most one curve with log discrepancy zero. Every curve in $E^t \cap S^t$ appears with log discrepancy zero, hence $E^t \cap S^t$ is an irreducible curve. Thus there is a unique component $E \subset E^t$ which intersects S^t and $D = E \cap S^t$ is an irreducible curve. By (21.6), E determines $q_1 : Y_1 \to X$. However, we need a direct construction of Y_1 which provides additional information.

21.14.1 Claim. *S^t is h-nef.*

Proof. The only curve where this may fail is D. If $h(D)$ is a curve then we do not have to consider D. Thus assume that $h(D)$ is a point (this is the typical case). Let H be a sufficiently ample divisor on X disjoint from $h(D)$. D is the only curve with the property that $D \cdot S^t < 0$, thus $[D]$ is an extremal ray. Since $D = E \cap S^t$, the flip exists by (20.13). After the flip $(S^t)^+$ becomes nef relative to $h^+ : (X^t)^+ \to X$. Furthermore $(S^t)^+$ and $(E^t)^+$ are disjoint. Applying base point freeness to $(S^t)^+$ (2.32.2) we obtain a morphism

$$(X^t)^+ \xrightarrow{p} U \xrightarrow{s} X$$

where p contracts $(E^t)^+$. Thus $s : U \to X$ is small. s is not an isomorphism since

$$D^+ \cdot (S^t)^+ > 0$$

hence $p(D^+) \subset U$ is a curve contracted by s. Since X is $\mathbb{Q}$-factorial, this is impossible. $\square$

21.14.2 Construction of $q_1 : Y_1 \to X$. Since S^t is h-nef, we can apply base point freeness to obtain

$$X^t \xrightarrow{p} Y_1 \xrightarrow{q_1} X.$$

p contracts $E^t - E$, thus q_1 has at most one exceptional divisor, the image of E. There is a curve $A \subset E$ such that $h(A)$ is a point and A intersects S^t positively. Thus $p(A)$ is not contracted by q_1. Since X is $\mathbb{Q}$-factorial, q_1 is not small, thus $E_1 = p(E)$ is the exceptional divisor of q_1.

Let $W_1 \subset Y_1$ denote the closed subset where p^{-1} is not defined. By construction $W_1 \cap S_1 = \emptyset$. Since

$$K + S^t + E^t + B^t = p^*(K + S_1 + E_1 + B_1),$$

we see that $K + S_1 + E_1 + B_1$ is lt outside W_1. $p : X^t \to Y_1$ is a $\mathbb{Q}$-factorial lt model which has one fewer exceptional divisors than $h : X^t \to X$.

21.14.3 Applying the Backtracking Method.

21.14.3.1 Claim. *Suppose that $i \leq m$ (i.e., we performed only beginning flips). Then*

(21.14.3.1.1) $Y_1 \dashrightarrow Y_i$ is an isomorphism in a neighborhood of W_1. Let $W_i \subset Y_i$ be the image of W_1.

(21.14.3.1.2) $K + S_i + E_i + B_i$ is plt outside W_i and the generic point of $S_i \cap E_i$.

Proof. By construction $K + S_1 + E_1 + B_1$ is lt outside W_1. Therefore it is plt outside the generic point of $S_1 \cap E_1$ since there are no triple intersections. Assume that the claim holds for $i - 1$. Let $D_{i-1} \subset Y_{i-1}$ be the locus of R_{i-1}. As we showed in (21.8.1), $D_{i-1} \subset S_{i-1}$ and $D_{i-1} \cdot E_{i-1} > 0$. Therefore

$$D_{i-1} \cap W_{i-1} \subset S_{i-1} \cap W_{i-1} = \emptyset.$$

Therefore $Y_{i-1} \dashrightarrow Y_i$ is an isomorphism in a neighborhood of W_{i-1} and (21.14.3.1.1) is clear.

$S_{i-1} \cap E_{i-1}$ is a contractible curve in S_{i-1}, hence

$$(S_{i-1} \cap E_{i-1}) \cdot E_{i-1} < 0.$$

Therefore $D_{i-1} \not\subset E_{i-1}$ and so $K + S_{i-1} + E_{i-1} + B_{i-1}$ is plt along D_{i-1}. Therefore $K + S_i + E_i + B_i$ is plt along D^+_{i-1} which proves (21.14.3.1.2) for i. $\square$

Now consider middle flips. (21.8.2.1) finishes the backtracking method. (21.8.2.2) is impossible since $S_m \cap E_m$ is irreducible by (21.14.3.1.2).

In case (21.8.2.3) the flip of r_m is provided by induction in view of the following:

21.14.3.2 Claim.

(21.14.3.2.1) $(r_m : Y_m \to X_m, K + E_m + B_m)$ has the same index as $(f : X \to Z, K + S + B)$.

(21.14.3.2.2) The type of $(r_m : Y_m \to X_m, K + E_m + B_m)$ is not smaller than the type of $(f : X \to Z, K + S + B)$.

(21.14.3.2.3) The minimal number of log crepant divisors of $(r_m : Y_m \to X_m, K + E_m + B_m)$ is smaller than the minimal number of log crepant divisors of $(f : X \to Z, K + S + B)$.

(21.14.3.2.4) $(r_m : Y_m \to X_m, K + E_m + B_m)$ is either lt or an exceptional special neighborhood.

Proof. By definition of index, $n(K_X + S + B)$ is a principal divisor. Thus $n(K + S_1 + E_1 + B_1) = q_1^* n(K_X + S + B)$ is also a principal divisor. This property is preserved under flips, and hence $n(K + S_m + E_m + B_m)$ is a principal divisor. Since S_m is disjoint from $\mathrm{Locus}(R_m)$, $n(K + E_m + B_m)$ is a principal divisor in a neighborhood of $\mathrm{Locus}(R_m)$.

Let $B = \sum (1/n) B^i$ be the decomposition giving the type. Since $C \cdot B^i \geq 0$, there is an $s^i \geq 0$ such that

$$C \cdot (s^i S + B^i) = 0.$$

Pulling it back to Y_1, we obtain positive numbers e^i such that

$$s^i S_1 + e^i E_1 + B_1^i$$

is numerically trivial on Y_1/Z. This property is preserved by flips, and therefore

$$s^i S_m + e^i E_m + B_m^i$$

is numerically trivial on Y_m/Z. Since D_m is disjoint from S_m and $D_m \cdot E_m < 0$, we conclude that $D_m \cdot B^i_m > 0$ for every i. Thus

$$B_m = \sum \frac{1}{n} B^i_m$$

shows that the type of $(r_m : Y_m \to X_m, K + E_m + B_m)$ is at least as large as the type of $(f : X \to Z, K + S + B)$.

Let $W_i \subset U_i \subset Y_i$ $(i = 1, m)$ be open neighborhoods such that $Y_m \dashrightarrow Y_1$ is an isomorphism between U_m and U_1. Then patching $Y_m - W_m$ and $p^{-1}(U_1) \subset X^t$ gives a $\mathbb{Q}$-factorial lt model of Y_m with one less crepant divisors than in $X^t \to X$.

We still need to show that $(r_m : Y_m \to X_m, K + E_m + B_m)$ is exceptional. Let $F' = E_m$ and let $K_{F'} + D' = \operatorname{Diff}(K + E_m + B_m)$. Then $K + D'$ is lc. Let $b : F \to F'$ be a log terminal model and $K_F + D = b^*(K_{F'} + D')$. If $K + E_m + B_m$ is lt along $\operatorname{Locus}(R_m)$ then there is nothing to prove. Otherwise $\llcorner D \lrcorner$ has at least two connected components: one is the birational transform of $S_m \cap E_m$ and the other lives over $\operatorname{Locus}(R_m)$. Thus $(r_m : Y_m \to X_m, K + E_m + B_m)$ is either lt or exceptional by (12.3.2). $\square$

After the middle flip, we have only final flips left, and these always exist by (21.8.3.2). $\square$

22. INDEX TWO FLIPS

TIE LUO

In this chapter we outline some of the steps of Shokurov's proof of the existence of log-flips in the case where $f : C \subset X \to 0 \in Z$ is a special nonexceptional index 2 extremal contraction. The proof given in [Shokurov 92] is long (about 35 pages) and we can not claim to have understood all of it.

The assumptions are:

(22.1.1) $K + S + B$ is lc and $2(K + S + B) \sim 0$ in a neighborhood of C;
(22.1.2) $S \cdot C < 0$;
(22.1.3) $K + S$ is plt and X is $\mathbb{Q}$-factorial;
(22.1.4) $(K + S) \cdot C < 0$;
(22.1.5) $K + S + B$ is nonexceptional in a neighborhood of C (by (21.12.1));
(22.1.6) $K + S + B$ is lt outside $\operatorname{Supp} S$ (by (21.9));
(22.1.7) there is a unique irreducible component $L \subset \llcorner \operatorname{Diff}_S(B) \lrcorner$ which is not contained in C (this follows from (22.2)).

22.2 Lemma. *(22.2.1) Assume that all index two flips satisfying (22.1.1–7) exist. Then all index two flips satisfying (22.1.1–6) exist.*

(22.2.2) Assume that $f : X \to Z$ satisfies (22.1.1–6) and $\llcorner \operatorname{Diff}_S(B) \lrcorner \not\subset C$. Pick a component $L \subset \llcorner \operatorname{Diff}_S(B) \lrcorner$ which is not in C. Let $L^c \subset \operatorname{Supp} \operatorname{Diff}_S(B)$ be the connected component of $\operatorname{Supp} \operatorname{Diff}_S(B)$ containing L. If L^c contains another noncontracted curve then (X, S) is 1-complemented, and the flip exists.

Proof. The first part is essentially the statement that if we apply the backtracking method to an arbitrary index two flip then the middle flip satisfies (22.1.7), (21.9) or else it is exceptional.

In case (22.2.2) it is easy to see that one can find an effective divisor M such that M is f-nef and $\operatorname{Supp} M = \operatorname{Supp} L^c - L$. Thus $(S, \operatorname{Diff}(B) - \epsilon M)$ satisfies the assumptions of (19.11). Hence $(S, \operatorname{Diff}(B) - \epsilon M)$ is 1-complemented, and therefore so is $(S, \operatorname{Diff}(\emptyset))$. $\square$

22.3 Further subdivison of cases. By passing to the analytic category we may assume that our flipping curve C is irreducible. We consider four types of extremal contractions. These four cases exhaust all possibilities if C is irreducible, but not if C is reducible. Because of certain technical details of the inductive proof, in case (22.3.c) we allow the flipping curve to be reducible.

(22.3.a) C is irreducible and $C \not\subset \operatorname{Supp}(B)$. In particular, $K+S+B$ is lt at the generic point of C;

(22.3.b$'$) C is irreducible, $K+S+B$ is not lt along C and there is a point $Q \notin L$ such that $(S, \operatorname{Diff}(B))$ is not lt at Q. In this case $C \subset \operatorname{Supp}(B)$.

(22.3.b$''$) C is irreducible, $K+S+B$ is not lt along C and $(S, \operatorname{Diff}(B))$ is lt outside L. In this case $C \subset \operatorname{Supp}(B)$;

(22.3.c) C is possibly reducible, $K+S+B$ is lt at every generic point of C and $C \subset \operatorname{Supp}(B)$.

The proof of the existence of flips proceeds by induction on two numbers: the height of a Shokurov flower and the S-log difficulty. These are defined shortly.

22.4 Definition – Proposition. Assume that X is lt and (X, S) is lc. By (6.6) there are only finitely many exceptional divisors E such that $\operatorname{discrep}(E, X) \leq 0$. We define the *S-log difficulty* of (X, S) (denoted by $\delta(X, S)$ or simply by δ) to be the number of exceptional divisors E such that

$$\operatorname{discrep}(E, X) \leq 0 \quad \text{and} \quad \operatorname{discrep}(E, X, S) = -1. \tag{22.4.1}$$

If $f: Y \to X$ is a proper birational morphism then set $f^*S = f_*^{-1}(S) + \sum d(E_i)E_i$. By definition $d(E_i) = \operatorname{discrep}(E_i, X, B) - \operatorname{discrep}(E_i, X, S+B)$. Thus (22.4.1) can be rewritten as

$$d(E) \leq 1 \quad \text{and} \quad \operatorname{discrep}(E, X, S+B) = -1. \tag{22.4.2}$$

22.5 Definition. An extremal contraction $g: Y \to X$ is a *good extraction* if it is log crepant and satisfies the following conditions:

(22.5.1) $K+S+E$ is lt;

(22.5.2) $D = S \cap E \cong \mathbb{P}^1$; (By (22.5.1) S and E cross normally generically along D.)

(22.5.3) For double adjunctions, we have

$$\operatorname{Diff}_D(\operatorname{Diff}_S(E+B)) = \operatorname{Diff}_D(\operatorname{Diff}_E(S+B)) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + P,$$

where P is the unique non-lt point for $K+S+B+E$ on D.

22.6 Lemma. *There is a finite chain of good extremal extractions in cases (22.3.a) and (22.3.b″) such that in the last extraction there is only one non-lt point (as P in (22.5.3)) on the exceptional divisor.*

Proof. In either case the boundary of $K_S + \mathrm{Diff}(B)$ has exactly one component passing through the non-lt point. We construct the chain inductively. Taking a lt model for $K+S+B$, $h : X^t \to X$, we apply the $(K^t+S^t+B^t+E^t-\epsilon B^t)$-MMP. Along the way back to X, there is a flip or divisorial contraction, after which we get the neighborhood of $Q = C \cap L$ in case (a) ($Q \in C \neq C \cap L$ in case (b″)). This step is not a flip since the modifications are done over X. So it is a divisorial contraction. It must be a contraction of E to the point Q because the non-exceptional assumption. Let us call the contraction $g_1 : (X_1, E_1 = E) \to (X_0 = X, Q)$. g_1 is log-crepant and $K + g_1^{-1}S + E$ is lt by assumptions. It implies that $g_1^{-1}S$ and E cross normally and $E \cap (g_1)_*^{-1}S \cong \mathbb{P}^1$. Also

$$\left(K + (g_1)_*^{-1}S + E + (g_1)_*^{-1}B|g_1^{-1}S\right)|E \cap (g_1)_*^{-1}S = K_{\mathbb{P}^1} + \frac{1}{2}P_1 + \frac{1}{2}P_2 + P,$$

where $P = (g_1)_*^{-1}S \cap E \cap (g_1)_*^{-1}L$. g_1 is a good extraction. Assume that we have constructed a chain of good extractions that starts at X and ends with:

$$g_i : (X_i, E_i) \to (X_{i-1}, Q_{i-1} \in E_{i-1}).$$

We are done if there is no non-lt point on E_i except along $E_i \cap (g_i)_*^{-1}E_{i-1}$. Otherwise let Q_i be the non-lt point. There is an irreducible L_i from the reduced part of $g^*(K+S+B)|E_i$ (where g is the composite of all the g_i). $Q_i \in L_i$. Moreover L_i is the only locus where $g_*^{-1}B$ intersects E_i in a neighborhood of Q_i. Looking again from the $\mathbb{Q}$-factorial lt model X^t using $(K^t + S^t + B^t + E^t - \epsilon B^t)$-MMP, we claim that the last step of the modification that gives the neighborhood of Q_i is a divisorial contraction to Q_i. We show that this step cannot be a flip. Let C' be the flipped curve passing Q_i. By assumption $C' \cdot g_*^{-1}B < 0$. $g^*B \cdot C' = 0$ implies that there is an exceptional divisor E' such that $E' \cdot C' > 0$. There is no exceptional locus passing through Q_i and $g_i^*E_{i-1} \cdot C' = 0$, $C' \cdot (g_i)_*^{-1}E_{i-1} < 0$.

So C' lies in $(g_i)_*^{-1}E_{i-1}$.

$$\mathrm{Diff}_{(g_i)_*^{-1}E_{i-1}}((g_i)_*^{-1}B + E_i)$$

is not klt along $E'|(g_i)_*^{-1}E_{i-1}$ and at Q_i. Hence by (12.3.1) it is not lt along C'. $K+(g_i)_*^{-1}E_{i-1}+E_i$ is lt before the flip and $K+(g_i)_*^{-1}E_{i-1}+(g_i)_*^{-1}B+E_i$ is not lt in the neighborhood of the flipping curve. Therefore $(g_i)_*^{-1}B$ intersects

E_i more than L_i around Q_i. This is a contradiction. The nonexceptional assumption prevents the modification being a contraction to a curve. So this modification is a divisorial contraction to Q_i. We check as before that it is a good contraction. The process stops after finitely many steps since the number of exceptional divisors in the lt model is finite. □

The above lemma justifies the following:

22.7 Definition. We define the minimum number of good extremal extractions needed in the lemma *the height of the Shokurov flower,* and denote it by λ.

We remark that the above construction does not work in the cases (22.3.b′) and (22.3.c) since condition (22.5.3) for a good extraction would fail.

22.8 Proposition. *For the cases (22.3.b′) and (22.3.c) we have the following:*
(22.8.1) either there is a good extraction of $g : (Y, E) \to (X, C)$ *such that* $d(E) \leq 1$*;*
(22.8.2) or the flip of f *exists.*

Comments. This formulation (taken from [Shokurov92, 8.8]) does not make much sense since a posteriori flips always exist. The claim is that if the construction given in [ibid, 8.8] fails to yield a good extraction then the end result of the construction can be used to produce the flip.

The original version [Shokurov91, 8.8] claimed that one always has the first case. It is not clear if the second case is really necessary.

The following easy lemma (whose proof is left to the reader) is used repeatedly in the proof of the final theorem.

22.9 Lemma. *Let* $f : S \dashrightarrow T$ *be a birational map between normal surfaces and* D *an effective ample divisor on* S*. If* $f^{-1}(\mathrm{Supp}(D))$ *is irreducible then* $f^{-1}(\mathrm{Supp}(D))$ *is nef.* □

The attached flow chart at the end of this chapter outlines the proof of the last theorem. Here $g : Y \to X$ is a good extraction as constructed for the cases (22.3.a,b′,b″,c). The flow chart ignores the possibility that at some step we ended up in case (22.8.2), when the flip is known to exist.

22.10 Theorem. *Index two flips exists.*

Proof. As we remarked earlier, we may assume that all the assumptions of (22.1) are satisfied and we need to consider only the cases (22.3.a,b′,b″,c).

We reduce the existence of the required flip to that of the exceptional cases when either λ or δ is zero, for which the result is known.

Let $g : Y \to X$ be the good extraction according to cases (22.3.a–c). (If a good extraction does not exist then the flip exists by (22.8).) We have $\rho(Y/Z) = 2$. There are two extremal rays, and R_1 corresponds to g. The contraction of R_2 is small. We consider the first flip accordingly. In the following we identify L and S with their birational transforms on Y and $D = S \cap E$.

Assume f is of case (22.3.a). On Y the locus of R_2 is C, the proper trasform of C in X. $E \cdot R_2 > 0$ and $S \cdot R_2 < 0$. The first flip exists by (20.7). If $P = D \cap L$ is not on C, $K + S + B + E$ is lt in a neighborhood of C. After the flip, on X_1, $K + S_1 + B_1 + E_1$ is lt around the flipped C^+ in E_1. Let R_2 be the new flipping ray on X_1. The locus C_1 of R_2 is in E_1. As in (21.8) we treat only the case where C_1 is apart from S_1 and $K + S_1 + B_1 + E_1$ is not plt along C_1. Otherwise the existence of flip is known and is followed by a divisorial contraction to X^+, the required flip of f. When that is so, since

$$D_1 = E_1 \cap S_1 \subset \llcorner \mathrm{Diff}_{E_1}(S_1 + B_1) \lrcorner,$$

there is an $L' \subset \llcorner \mathrm{Diff}_{E_1}(S_1 + B_1) \lrcorner$ not in C_1 intersecting C_1 at a point Q and D_1 at P^+. This implies L' irreducible. $\mathrm{Supp}(B_1)$ does not contain C_1. If C_1 were reducible, we contract an irreducible component of C_1 and then take an extremal ray R of E_1 (after contraction of a component of C_1) such that $R \cdot D_1 > 0$. Notice that L' becomes ample by (22.9). The existence of such a ray is guaranteed by $(D_1)^2 > 0$. If cont_R contracts a curve F then (12.3.1) forces

$$F \cap D_1 \cap L' = P^+.$$

By induction on $\rho(E_1)$, we may assume that cont_R is of fiber type over a curve after all because $C_1 \cap D_1 = \emptyset$. By (12.3.1) $\mathrm{Diff}_{E_1}(S_1 + B_1)$ has only $P^+ = L' \cap D_1$ as non-plt point on L' which contradicts our starting assumption. So C_1 is irreducible. This technique is used later. We are again in Case (22.3.a) with smaller λ. If $P \in C$, we have $R_2 \cdot E > 0$, $R_2 \cdot B > 0$ and $R_2 \cdot S < 0$. The flip in C exists by (20.7). $C^+ \subset E_1 \cap \mathrm{Supp}(B_1)$. As before we consider only the case where the locus C_1 of the new flipping curve R_2 is away from S_1 and $(K + S_1 + B_1 + E_1)|E_1$ has LCS along C_1. It implies the flipped C^+ is irreducible and $C^+ \cap D_1 = P^+$, which is the only point where B_1 passes D_1. So

$$S_1 \cdot (\mathrm{Supp}(B_1|E_1) - C^+) = 0.$$

It means $\mathrm{Supp}(B_1|E_1) - C^+$ is in C_1. In fact they are equal, for otherwise we contract $\mathrm{Supp}(B_1|E_1) - C^+$ in C_1, C^+ becomes ample by (22.9). We use the same method as we did in the first part by looking at contractions on E_1 to get a contradiction. If $K + S_1 + B_1 + E_1$ is lt along C_1 we go to case (22.3.c). Otherwise C_1 has to be irreducible and we have case (22.3.b$'$).

Let f be of case (22.3.b$'$). Let C be the locus of R_2. If $C = D = S \cap E$. The flip in C exists and S and E are separated. It is followed by a divisorial contraction of E^+. We are done. Otherwise $C \cap S = \emptyset$. We treat only the case when $K+S+B+E$ has LCS along C. Then there is an irreducible L' as a fiber of g in E, $L' \cap D = P$ and $L' \cap C_1 = Q$. P is the only point where B intersects D. That is to say

$$S \cdot (\operatorname{Supp}(B|E) - L') = 0.$$

It implies $\operatorname{Supp}(B|E) - L'$ is in C. If we contract $\operatorname{Supp}(B|E) - L'$, L' becomes ample. As before

$$\operatorname{Supp}(B|E) - L' = C.$$

If $K+S+B+E$ is lt along C we are in case (22.3.c). Otherwise C is irreducible and we are in case (22.3.b$'$) with smaller δ.

f is in case (22.3.b$''$). The locus of R_2 is C. $E \cdot R_2 > 0$ and $S \cdot R_2 < 0$. The flip to X_1 in C exists by (20.7). By the proof of (22.8) we may consider only the case when $D_1 = S_1 \cap E_1$ is irreducible. The locus C_1 of the new flipping ray R_2 is away from S_1. We treat only the case when $K + S_1 + B_1 + E_1$ is not lt along C_1. There is an irreducible $L' \subset \llcorner \operatorname{Diff}_{E_1}(S_1 + B_1) \lrcorner$ such that $L' \cap D_1 = P$ and $L' \cap C_1 = Q$. Indeed $L' = C^+$. P is the only point where B_1 intersects D_1. We check as before that

$$\operatorname{Supp}(B_1|E_1) - L' = C_1.$$

If $K + S_1 + B_1 + E_1$ is lt along C_1 we are in case (22.3.c). Otherwise C_1 is irreducible and we are in case (22.3.b$'$).

If f is of case (22.3.c). The locus C of R_2 may not be connected. $D = E \cap S$ is irreducible. $P = L \cap D$. If C has components passing through P_1,P_2 on D, $K+S+B+E$ is lt in neighborhoods of those components. Hence the flips in these curves exist. After the flips on X_1 we may assume $P^+ = L_1 \cap D_1$ is the only point on D_1 where B_1 intersects D_1. We consider only the case when the locus C_1 of the new flipping ray R_2 is away from S_1 and $K+S_1+B_1+E_1$ is not lt along C_1. Then there is an $L' \subset \llcorner \operatorname{Diff}_{E_1}(S_1+B_1) \lrcorner$ such that $L' \cap D_1 = P^+$ and $L' \cap C_1 = Q$, the non-lt point on C_1. As before, we can check that

$$\operatorname{Supp}(B_1|E_1) - L' = C_1.$$

$K + S_1 + B_1 + E_1$ is klt at some point of C_1. We are in the case (22.3.c) with smaller δ.

This is the end of the induction. $\square$

The flowchart ignores easy flips and (22.8.2) outcome of the procedure.

The Flow Chart

$\boxed{f : X \to Z \text{ is of case (a), (b) or (c)}}$

$\boxed{\text{Good contraction exists}} \quad \to \quad \Downarrow$

$\boxed{g : Y \to X \text{ is good}}$

$\boxed{\text{First flip exists}} \quad \to \quad \Downarrow$

$\boxed{\text{Done}} \quad \leftarrow \quad \boxed{\text{Yes}} \quad \leftarrow \quad \boxed{\text{Is } R \cdot S > 0 \text{ or } K + S + E + B \text{ lt?}}$

$\downarrow$

$\boxed{\text{No}}$

$\Downarrow$

$\boxed{\text{Four cases: (a), } (b'), (b''), \text{ (c)}}$

$\Downarrow$

$$\overbrace{\boxed{\text{Case (a) flip in } R, \lambda \text{ drops}} \quad \text{or} \quad \boxed{\text{Case (b) or (c) flip in } R}}^{\text{Case (a)}}$$

$$\overbrace{\boxed{\text{Case } (b') \text{ flip in } R, \delta \text{ drops}} \quad \text{or} \quad \boxed{\text{Case (c) flip in } R}}^{\text{Case } (b')}$$

$$\overbrace{\boxed{\text{Case } (b'') \text{ flip in } R, \lambda \text{ drops}} \quad \text{or} \quad \boxed{\text{Case } (b') \text{ or (c) flip in } R}}^{\text{Case } (b'')}$$

$$\overbrace{\boxed{\text{Case (c) flip in } R, \delta \text{ drops}}}^{\text{Case (c)}}$$

23. UNIRATIONALITY OF THE GENERAL COMPLETE INTERSECTION OF SMALL MULTIDEGREE

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It is well-known that a quadric hypersurface with a rational point is rational. Similarly, a cubic hypersurface of dimension at least two is unirational once it contains a rational line; over an algebraically closed field this latter condition is always satisfied. These results were generalized by [Morin40], who showed that the general hypersurface of degree d and dimension sufficiently large is unirational once it contains a linear space of sufficiently large dimension defined over the given field; this latter condition being always true over an algebraically closed field. This was further generalized by [Predonzan49] to include the case of complete intersections.

The papers [Morin40,Predonzan49] are quite hard to locate and the only easily available account is in the book of [Roth55], where one finds a sketch of the proof for the result of Morin. Analysing this proof it is easy to recover a proof of the result of Predonzan. We present here a proof of these results and some related results. After this paper was written we came to know of a recent paper [Ramero90] where the bounds obtained by Predonzan have been improved.

23.1 An illustrative example. We illustrate the proof in the general case by showing how to deduce the unirationality of a general quartic of sufficiently large dimension using as inductive starting point the following well known

23.1.1 Fact. A smooth cubic hypersurface $X \subset \mathbb{P}^n_k$ of dimension at least two ($n \geq 3$) which contains a line $P^1_k \subset X \subset \mathbb{P}^n_k$, is unirational over k.

The proof is in several steps.

(23.1.2). We choose n sufficiently large so that a general quartic hypersurface in $\mathbb{P}^n_k$ contains a linear subspace $\mathbb{P}^3_k$ (this choice of dimension is dictated by the ambient dimension for the case of cubics), for k an algebraically closed field.

To do this consider the incidence locus

$$\begin{array}{ccc} Z & \hookrightarrow & G \times \mathbb{P}^N \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

where $G = G(\mathbb{P}^3, \mathbb{P}^n)$ is the Grassmanian, $\mathbb{P}^N$ is the space of all quartic hypersurfaces in $\mathbb{P}^n$ and Z consists of pairs (L, X), with $L \cong \mathbb{P}^3 \subset \mathbb{P}^n$ a linear subspace contained in a quartic hypersurface X. Then Z is a projective subbundle of $G \times \mathbb{P}^n$ of codimension $\binom{3+4}{4} = 35$. Hence, if

$$(\dim G =) \quad (n-3)(3+1) \geq 35$$

then $\dim Z \geq \dim \mathbb{P}^N$; so that we can expect the map $Z \to \mathbb{P}^N$ to be surjective. This is the case as shown in (23.2.3). Now in X is a point of $\mathbb{P}^N$, we can find a point of Z lying over it if k is algebraically closed.

(23.1.3). Assume that we have a general pair (L, X) in Z defined over some field k (not neccessarily algebraically closed). The collection of all $\mathbb{P}^4$ which contain $L \cong \mathbb{P}^3_k$ form a $\mathbb{P}^{n-4}_k$. The intersection $\mathbb{P}^4 \cap X$ is the union of $\mathbb{P}^3_k$ and $Y \subset \mathbb{P}^4$, which is a cubic hypersurface. Moreover $Y \cap \mathbb{P}^3$ is a cubic surface.

Let $\tilde{X}_L \to X$ be the blow up of X along L and let E be the exceptional divisor. We have a natural map $\tilde{X}_L \to \mathbb{P}^{n-4}_k$ which is a fibration by cubic hypersurfaces. Moreover, we have a natural diagram

$$\begin{array}{ccc} E & \hookrightarrow & \mathbb{P}^3_k \times \mathbb{P}^{n-4}_k \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^{n-4}_k & \end{array}$$

which makes the map $E \to \mathbb{P}^{n-4}_k$ a fibration by cubic surfaces.

Let $G' = G(\mathbb{P}^1_k, \mathbb{P}^3_k)$ be the Grassmannian of lines in $\mathbb{P}^3$ and let $I \subset G' \times \mathbb{P}^{n-4}_k$ be the incidence locus of pairs (M, t) such that the fibre of E over t contains the line M. We already know that I dominates $\mathbb{P}^{n-4}_k$ so suppose that

(23.1.4)

there is a component of I that dominates $\mathbb{P}^{n-4}$ and is rational over k.

Let K be the function field of this component and (M, t) the corresponding point of I. Then t is a generic point of $\mathbb{P}^{n-4}_k$ and M is contained in the fibre E_t of E over $t \in \mathbb{P}^{n-4}$. Thus, M is contained in the fibre Y_t of $\tilde{X}_L$ over $t \in \mathbb{P}^{n-4}$, making Y_t a cubic hypersurface of dimension three which contains a line. Thus Y_t is unirational over K by induction. Now in the Cartesian diagram

$$\begin{array}{ccc} Y_t & \hookrightarrow & \tilde{X}_L \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \hookrightarrow & \mathbb{P}^{n-4}_k \end{array}$$

the horizontal arrows are dominant and so the unirationality of Y_t over K and the rationality over k of K imply that $\tilde{X}_L$ is unirational over k.

In order to ensure that condition (23.1.4) holds we note that *if $I \to G'$ is dominant*, then the generic fibre of this map is a linear projective subspace of $\mathbb{P}^{n-4}$; in particular, this generic fibre is rational over k. Thus in order to complete the inductive argument we must choose our n in step one so that $Z \to G$ is also dominant. This is achieved by the condition (23.2.4) below.

23.2 Linear spaces in Complete Intersections. Let $\mathbf{d} = (d_1, \dots, d_r)$ be an r-tuple of positive integers, and n, k be any positive integers such that one of the following conditions hold:

(23.2.1) If $r = 1$ and $d = 2$ then $n > 2k$.

(23.2.2) If $r > 1$ or there is i with $d_i > 2$, and $d_i > 1$ for all i, then

$$(n-k)(k+1) \geq \sum_{i=1}^{r} \binom{k+d_i}{d_i}.$$

23.2.3 Lemma. *Let n, k, $d_1, \dots, d_r$ be positive integers satisfying one of the conditions above. Let H_i be hypersurfaces of degree d_i in $\mathbb{P}^n$. There is a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$ which is contained in the intersection of all the H_i.*

Proof. Let $V = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and G be the Grassmannian of linear subspaces of dimension k in $\mathbb{P}^n = \mathbb{P}(V)$. We have the universal short exact sequence

$$0 \to S \to V \times G \to Q \to 0$$

of vector bundles on G, where Q is of rank $k+1$. This yields a filtration on $\operatorname{Sym}^{d_i}(V) \times G$ such that

$$(F^0/F^1)(\operatorname{Sym}^{d_i}(V) \times G) = \operatorname{Sym}^{d_i}(Q)$$

and we have a surjection

$$D_i : F^1(\operatorname{Sym}^{d_i}(V) \times G) \twoheadrightarrow S \otimes \operatorname{Sym}^{d_i - 1}(Q).$$

The incidence locus

$$Z = \{(F_1, \dots, F_r, L) \in (\oplus_{i=1}^{r} \operatorname{Sym}^{d_i}(V)) \times G \mid F_i \text{ vanishes on } L, \text{ for all } i\}$$

can alternatively be described as the direct sum

$$Z = \oplus_{i=1}^{r} F^1(\mathrm{Sym}^{d_i}(V) \times G).$$

We need to show that the projection $\pi : Z \to \oplus_{i=1}^{r} \mathrm{Sym}^{d_i}(V)$ is surjective. If we knew that the top Chern class of $\oplus_{i=1}^{r} \mathrm{Sym}^{d_i}(Q)$ is nonzero this would follow easily, but there seems to be no direct way of proving this nonvanishing statement.

For each point $z = (\mathbf{F}, L)$ in Z we have a linear inclusion of the fibre Z_L of $Z \to G$ at L, into $\oplus_{i=1}^{r} \mathrm{Sym}^{d_i}(V)$. Thus, in order to show the surjectivity of $d\pi$ at z it is enough to show the surjectivity of the induced map

$$\psi_z : T_{G,L} = T_{Z,z}/Z_L \to \oplus_{i=1}^{r} \mathrm{Sym}^{d_i}(Q_L) = (\oplus_{i=1}^{r} \mathrm{Sym}^{d_i}(V))/Z_L.$$

Writing $T_{G,L} = S_L^* \otimes Q_L$ we check that the ψ_z is the composite

$$S_L^* \otimes Q_L \xrightarrow{\phi_z \otimes \mathrm{Id}} \oplus_{i=1}^{r} \mathrm{Sym}^{d_i - 1}(Q_L) \otimes Q_L \xrightarrow{product} \oplus_{i=1}^{r} \mathrm{Sym}^{d_i}(Q_L)$$

where ϕ_z is the map induced by the image of z under the map

$$\oplus D_i : Z \twoheadrightarrow S \otimes \oplus_{i=1}^{r} \mathrm{Sym}^{d_i - 1}(Q).$$

Since the product homomorphism is surjective we would have surjectivity of ψ_z if we knew the surjectivity of ϕ_z. This in turn would follow for a suitable choice of z if we have the stronger condition

$$\dim S_L = n - k \geq \sum_{i=1}^{r} \binom{k + d_i - 1}{d_i - 1} \tag{23.2.4}$$

Once we have the surjectivity of $d\pi$ at some z, we get that π is dominant. Since G is complete, π is proper and thus we get surjectivity of π as required.

Since we need the lemma only for the stronger hypothesis (23.2.4) we defer the proof of the general cases (23.2.1–2) to (23.6).

23.3 Definition. We define, by induction on the positive integers $r, d_1, \dots, d_r$, the positive integers $n(d_1, \dots, d_r)$ and $k(d_1, \dots, d_r)$ as follows

(23.3.1) If $r = 1$ and $d_1 = 1$ then $n(1) = 1$ and $k(1) = 0$.

(23.3.2) If $r > 1$, $d_i = 1$ for some i and $\mathbf{d}' = (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_r)$, then we define $n(\mathbf{d}) = n(\mathbf{d}') + 1$ and $k(\mathbf{d}) = k(\mathbf{d}')$

(23.3.3) If $d_i > 1$ for all i, let $\mathbf{d} - \mathbf{1} = (d_1 - 1, \dots, d_r - 1)$. We define $k(\mathbf{d}) = n(\mathbf{d} - \mathbf{1})$ and

$$n(\mathbf{d}) = k(\mathbf{d}) + \sum_{i=1}^{r} \binom{k(\mathbf{d}) + d_i - 1}{d_i - 1}$$

Note that we obtain the inequality

$$n(\mathbf{d}) - k(\mathbf{d}) \geq \sum_{i=1}^{r} \binom{k(\mathbf{d}) + d_i - 1}{d_i - 1}$$

in all of the above cases, i.e. (23.2.4) is always satisfied if we take $n \geq n(\mathbf{d})$ and $k = k(\mathbf{d})$.

23.4 Theorem. *Let (X, L) be a general pair, where $X = H_1 \cap \ldots \cap H_r$ is the complete intersection of hypersurfaces H_i in $\mathbb{P}^n$ of degree d_i respectively, L is a linear space of dimension k contained in X which is smooth along L and irreducible. Then if $n \geq n(\mathbf{d})$ and $k = k(\mathbf{d})$, X is unirational.*

Proof. We prove this result by induction on the positive integers r, $d_1, \ldots, d_r$ and we require the following more precise statement.

Let the notation be as in (23.1). For each $z = (\mathbf{F}, L)$ in Z, let $H_i(z)$ be the hypersurface in $\mathbb{P}^n$ defined by F_i and X_z be the intersection of these hypersurfaces. Let $U(n, \mathbf{d})$ be the open subset of Z consisting of points $z = (\mathbf{F}, L)$ satisfying the following conditions

(23.4.1) X_z is irreducible and the complete intersection of the $H_i(z)$.

(23.4.2) X_z is smooth along L.

(23.4.3) ϕ_z is surjective.

23.5 Theorem. *If $n \geq n(\mathbf{d})$ and $k = k(\mathbf{d})$, then for each $z \in U(n, \mathbf{d})$, X_z is unirational.*

Since we have shown that $Z \to \oplus_{i=1}^{r} \operatorname{Sym}^{d_i}(V)$ is dominant, this implies (23.4).

Proof. We proceed by induction on the positive integers r, $d_1, \ldots, d_r$.

(23.5.1) $r = 1$ and $d_1 = 1$.

In this case X_z is a linear space and hence it is rational.

(23.5.2) $r > 1$ and $d_i = 1$ for some i.

Let $V' = V/\langle F_i \rangle$ so that $H_i(z) = \mathbb{P}(V') \subset \mathbb{P}(V) = \mathbb{P}^n$. Then $F_i \in S_L$ gives the i-th projection of

$$\phi_z : S_L^* \to \oplus_{j=1}^{r} \operatorname{Sym}^{d_j - 1}(Q_L).$$

Since ϕ_z is a surjection, we have an induced surjection

$$\phi' : (S_L / < F_i >)^* \to \oplus_{j \neq i} \operatorname{Sym}^{d_j - 1}(Q_L).$$

Let $G' \subset G$ be the sub-Grassmannian of k-dimensional linear subspaces of $H_i(z)$ and $Z' \subset \oplus_{j \neq i} \operatorname{Sym}^{d_i}(V') \times G'$ be the locus of pairs $(\mathbf{F}', L)$, where F_j' vanish along L. Then, if we take

$$\mathbf{F}' = (F_1 |_{H_i(z)}, \ldots, F_{i-1} |_{H_i(z)}, F_{i+1} |_{H_i(z)}, \ldots, F_r |_{H_i(z)})$$

and $z' = (\mathbf{F}', L)$. Then z' lies in Z'. Note that $\dim V' = n - 1 \geq n(\mathbf{d}) - 1 = n(\mathbf{d}')$ and $k = k(\mathbf{d}) = k(\mathbf{d}')$ and $X_{z'} = X_z$. Further, $\phi_{z'} = \phi'$ is a surjection so that we have the result by induction in this case.

(23.5.3) $d_i > 1$ for all i.

Choose a splitting of the sequence

$$0 \to S_L \to V \to Q_L \to 0$$

Then if $P = \mathbb{P}(S_L)$, the blow up $\widetilde{\mathbb{P}(V)_L}$ of $\mathbb{P}(V) = \mathbb{P}^n$ along $L = \mathbb{P}(Q_L)$ is alternatively described by

$$\widetilde{\mathbb{P}(V)_L} = \mathbb{P}_P(Q_L \times P \oplus \mathcal{O}_P(1))$$

The surjection (induced by the splitting chosen above) $V \times P \twoheadrightarrow Q_L \times P \oplus \mathcal{O}_P(1)$ gives an inclusion $\widetilde{\mathbb{P}(V)_L} \subset \mathbb{P}^n \times P$. Further, the element $F_i \in \mathrm{Sym}^{d_i}(V)$ goes to the kernel of

$$\mathrm{Sym}^{d_i}(Q_L \times P \oplus \mathcal{O}_P(1)) \to \mathrm{Sym}^{d_i}(Q_L) \times P$$

which is $\mathrm{Sym}^{d_i - 1}(Q_L \times P \oplus \mathcal{O}_P(1)) \otimes \mathcal{O}_P(1)$. Denote these images by $\tilde{F}_i$.

The subvariety defined by the vanishing of all the $\tilde{F}_i$ is the birational transform of X in $\widetilde{\mathbb{P}(V)_L}$; this strict transform is just the blow up $\tilde{X}_L$ of X along L. Since X is smooth along L, this is an irreducible variety and the exceptional locus, which is its intersection with $\mathbb{P}(Q_L) \times P$, is also smooth. In particular, the generic fibre of $\tilde{X}_L \to P$ is irreducible and smooth along its intersection with $\mathbb{P}(Q_L)$. Further, this fibre is a complete intersection defined by the simultaneous vanishing of the equations $\tilde{F}_i$ which are of degree $d_i - 1$.

Since $k = k(\mathbf{d}) = n(\mathbf{d} - \mathbf{1})$, we can repeat the constructions of section 2 with $V_1' = Q_L$ and G_1' the Grassmannian of $h = k(\mathbf{d} - \mathbf{1})$ dimensional linear subspaces of $\mathbb{P}(V_1') = L$. Let $Z_1' \subset \oplus_{i=1}^r \mathrm{Sym}^{d_i - 1}(V_1') \times G_1'$ denote the corresponding incidence locus and

$$0 \to S_1 \to V_1' \times G_1' \to Q_1 \to 0$$

be the universal sequence on G_1'.

We have a surjection $\phi_z : S_L^* \to \oplus_{i=1}^r \mathrm{Sym}^{d_i - 1}(V_1')$, so that we can form the base change

$$Z_1'' = Z_1' \times_{\oplus_{i=1}^r \mathrm{Sym}^{d_i - 1}(V_1')} S_L^*$$

Then Z_1'' is a vector bundle over the Grassmannian G_1' and a rational variety. Let $z_1' : \mathrm{Spec}\, K \to Z_1''$ denote its generic point. From the lemma we deduce that $\operatorname{Spec} K \to S_L^*$ is dominant. Since the natural map $S_L^* - \{0\} \to \mathbb{P}(S_L)$ is surjective we see that the map $\operatorname{Spec} K \to \mathbb{P}(S_L)$ is also dominant. Let Y be the pullback of $\tilde{X}_L \to \mathbb{P}(S_L)$ to $\operatorname{Spec} K$. Then Y is a complete intersection variety in $\mathbb{P}(Q_L \otimes K \oplus K)$ which is defined by the equations $\tilde{F}_1, \dots, \tilde{F}_r$ which are of degrees $d_1 - 1, \dots, d_r - 1$ respectively. Further, if $L_1 \in G_1'(K)$ is the image of z_1', then

$$L_1 \subset \mathbb{P}(Q_L \otimes K) \subset \mathbb{P}(Q_L \otimes K \oplus K) \subset Y$$

Further, Y is smooth along its intersection with $\mathbb{P}(Q_L \otimes K)$ hence in particular along L_1.

By the genericity of z_1', the induced morphism

$$\phi_{z_1'} : S_{1,L_1}^* \to \oplus_{i=1}^r \operatorname{Sym}^{d_i - 1}(Q_{1,L_1})$$

is surjective. But now, if G_1 is the Grassmannian of h-dimensional subspaces of $\mathbb{P}(V_1' \otimes K \oplus K)$ and $Z_1 \subset \oplus_{i=1}^r \operatorname{Sym}^{d_i-1}(V_1' \otimes K \oplus K) \times G_1$ is as before, let $z_1 = (\tilde{\mathbf{F}}, L_1) \in Z_1$. Then the map for z_1 is

$$\phi_{z_1} : S_{1,L_1}^* \oplus K \to \oplus_{i=1}^r \operatorname{Sym}^{d_i-1}(Q_{1,L_1})$$

which restricts to $\phi_{z_1'}$ and hence is also surjective.

By the induction hypothesis, $Y = X_{z_1}$ is unirational over the field K. But K is the function field of a rational variety and since $\operatorname{Spec} K \to \mathbb{P}(S_L)$ is dominant $Y \to X$ is dominant. Thus X is unirational. □

23.6 Proof of (23.2.3). The result is trivial in the case (23.2.1), so we only need to show the surjectivity of ψ_z for a suitable choice of z in the case (23.2.2). Since the map $F^1(\operatorname{Sym}^{d_i}(V) \times G) \twoheadrightarrow S \otimes \operatorname{Sym}^{d_i-1}(Q)$ is surjective this follows from the following proposition; taking Q to be Q_L and U to be S_L^*, the map ψ can be thought of as an element of $S_L \otimes \oplus_{i=1}^r \operatorname{Sym} d_i - 1(Q_L)$ which can be lifted to a point $z \in Z$, hence $\psi_z = \psi$.

23.6.1 Proposition. *Let n, k and $\mathbf{d} = (d_1, \dots, d_r)$ be chosen satisfying (23.2.2). Let Q be a vector space of dimension $k+1$. For any space U of dimension $n-k$ and there exists a map $\psi : U \to \oplus_{i=1}^r \operatorname{Sym}^{d_i-1}(Q)$ such that the induced map*

$$U \otimes Q \xrightarrow{\psi \otimes \mathrm{Id}} \oplus_{i=1}^r \operatorname{Sym}^{d_i-1}(Q) \otimes Q \xrightarrow{\text{product}} \oplus_{i=1}^r \operatorname{Sym}^{d_i}(Q)$$

is surjective.

Proof. Since the product homomorphism $\pi : \operatorname{Sym}^{d_i-1}(Q) \otimes Q \to \operatorname{Sym}^{d_i}(Q)$ is surjective, we may assume that $u = n - k < \dim \oplus_{i=1}^r \operatorname{Sym}^{d_i-1}(Q)$. Let X be the Grassmannian of $(n-k)$-dimensional linear subspaces of $\oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q)$ and $U \subset \oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q) \otimes \mathcal{O}_X$ be the universal subbundle. The composite homomorphism

$$U \otimes Q \to \oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q) \otimes Q \otimes \mathcal{O}_X \to \oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q) \otimes \mathcal{O}_X$$

is not surjective at some point of X if and only if there is a one dimensional quotient of $\oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q)$ where the image of the composite goes to zero.

Let $Y = \mathbb{P}(\oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q))$; we have a surjection

$$\oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q) \otimes \mathcal{O}_Y \to \mathcal{O}_Y(1)$$

and thus a composite homomorphism

$$U \otimes Q \otimes \mathcal{O}_Y \to \oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q) \otimes Q \otimes \mathcal{O}_{X\times Y} \to \mathcal{O}_X \otimes \mathcal{O}_Y(1)$$

This composite is zero at all the "bad" pairs $(x, y) \in X \times Y$.

Let $\mathcal{F}$ be the cokernel of the natural homomorphism on Y

$$Q \otimes \mathcal{O}_Y \to (\oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q))^* \otimes \mathcal{O}_Y(1)$$

The locus of "bad" pairs, $Z \subset X \times Y$ is then the Grassmanian of rank u quotients of $\mathcal{F}$. We need to show that $Z \to X$ is not surjective.

Let $Y = \coprod Y_m$ be the flattening stratification for $\mathcal{F}$. We have an exact sequence of vector bundles on Y_m

$$0 \to E_m \to Q \otimes \mathcal{O}_{Y_m} \to (\oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q))^* \otimes \mathcal{O}_{Y_m}(1) \to \mathcal{F}\mid_{Y_m} \to 0$$

where E_m has rank m. Thus for all $y \in Y_m$, the one dimensional quotient of $\oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q)$ is zero on $(E_m)_y \cdot \oplus_{i=1}^r \mathrm{Sym}^{d_i-1}(Q)$. Thus it is induced from a one dimensional quotient of $\oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q/(E_m)_y)$.

Let A_m be the Grassmanian of m-dimensional subspaces of Q and let $E_m \hookrightarrow Q \otimes \mathcal{O}_{A_m}$ be the universal subbundle. Let B_m be the projective bundle $\mathbb{P}_{A_m}(\oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q \otimes \mathcal{O}_{A_m}/E_m))$. We have a natural morphism $B_m \to Y$ whose image contains Y_m as seen above. Thus, we can bound the dimension of Y_m and thus also $Z\mid_{Y_m}$.

$$\dim Z\mid_{Y_m} \leq \dim B_m + u(\dim \mathcal{F}\mid_{Y_m} - u)$$

Comparing with $\dim X = u(\dim \oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q) - u)$ we see that we would be done if

$$(*_m) \qquad \dim \oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q/E_m) \leq (u-m)(\dim Q - m)$$

Since the conditions of the proposition give us

$$(*_0) \qquad \dim \oplus_{i=1}^r \mathrm{Sym}^{d_i}(Q) \leq u \cdot \dim Q$$

we have to show that $(*_m)$ implies $(*_{m+1})$. But then, replacing Q by Q/E_m we need only show that $(*_0)$ implies $(*_1)$. This is easily checked by calculation. □

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A blank space or a $*$ stands for a variable symbol in the notation.