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A THEORY OF CHARACTERISTIC CURRENTS ASSOCIATED WITH A SINGULAR CONNECTION

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Abstract

A general theory of characteristic currents associated to singular connections is developed. In particular, a Chern-Weil theory for bundle maps is introduced and systematically studied. This theory generalizes the standard one. It associates to a map $\alpha : E \longrightarrow F$ between bundles with connection, singular "pushforward" and "pullback" connections on E and F respectively. Characteristic classes are then shown to be canonically represented by d-closed currents universally constructed from the "curvature" of these singular connections. When $\mathrm{rank}(E) = \mathrm{rank}(F) = n$ and ϕ is an Ad-invariant polynomial on $\mathfrak{gl}_n$, formulas of the type

$$\phi(\Omega_F) - \phi(\Omega_E) \;=\; \mathrm{Res}_\phi \, \mathrm{Div}(\alpha) + dT$$

are derived, where $\mathrm{Div}(\alpha)$ is a rectifiable current canonically associated to the singular structure of α, where Res_ϕ is a smooth form of classical Chern-Weil type computed as a polynomial in the curvatures Ω_E, Ω_F of E and F, and where T is a canonical, functorial trangression form with coefficients in L^1_{loc}. The cases where E and F are complex or quaternion line bundles are examined in detail, and lead to a new proof of the Riemann-Roch Theorem for vector bundles over algebraic curves.

Applications include: A C^∞-generalization of the Poincaré-Lelong Formula to smooth sections of any smooth vector bundle; Universal formulas for the Thom class as an equivariant characteristic form (i.e., canonical formulas for a de Rham representative of the Thom class of a bundle with connection); A Differentiable Grothendieck-Riemann-Roch Theorem at the level of forms and currents (in both the complex and spin cases). Each of these holds in the general setting of atomic bundle maps, as introduced and studied in [HS]. A variety of formulas relating geometry and characteristic classes are deduced as direct consequences of the theory.

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Introduction

The aim of this paper is to lay the foundations of a theory of Chern-Weil-Simons-type for singular connections on a smooth vector bundle. The notion of a singular connection here is quite general, but the focus will be on certain connections which arise naturally from bundle maps. More specifically, suppose that E and F are smooth vector bundles with connection over a manifold X. Then our theory associates to each homomorphism $\alpha : E \longrightarrow F$ a constellation of d-closed characteristic currents on X defined canonically in terms of the curvature of the bundles and the singularities of the map α. This is essentially a "Chern-Weil Theory for bundle maps" which in the special case where $\alpha \equiv 0$ reduces to the usual construction for the bundles themselves.

The theory is two-sided; one can focus attention either on E or on F (and retrieve, when $\alpha \equiv 0$, the standard theory for E or F). Let us suppose the focus is on E, and fix a **characteristic polynomial** ϕ, i.e., an Ad -invariant polynomial on the Lie algebra of the structure group of E. Standard Chern-Weil Theory associates to ϕ a smooth, closed differential form $\phi(\Omega_E)$ on X which is defined canonically in terms of the curvature of E and which represents a certain characteristic class $\phi(E) \in H^*_{\text{de Rham}}(X)$ determined universally by ϕ. Our theory associates to ϕ a d-closed current $\phi(\Omega_{E,\alpha})$ which is defined in terms of curvature and the singular structure of α, and which also represents $\phi(E)$.

The theory also produces a canonically defined, functorial transgression current $T = T(\phi, \alpha)$ with the property that

$$\phi(\Omega_{E,\alpha}) - \phi(\Omega_E) = dT$$

In the special case where $\text{rank}(E) = \text{rank}(F)$, the characteristic current has the form

$$\phi(\Omega_{E,\alpha}) = \phi(\Omega_F) - S$$

where $\phi(\Omega_F)$ is the standard Chern-Weil form associated to the bundle F and where S is a d-closed current supported on the **singularity set** $\Sigma \equiv \{x \in X : \alpha_x \text{ is not invertible}\}$. Thus S is canonically and functorially cohomologous to $\phi(\Omega_F) - \phi(\Omega_E)$. In many important cases it will turn out that Σ is an oriented submanifold (or more generally an integral cycle) and that S can be written in the form

$$S = \text{Res}_\phi[\Sigma]$$

where Res_ϕ is a smooth differential form on X and $[\Sigma]$ is the integral current determined by integration over Σ. This residue form will be expressed in terms of the curvatures of E and F by a universally determined "residue polynomial". Thus we obtain an equation of currents:

$$\phi(\Omega_F) - \phi(\Omega_E) \;=\; \text{Res}_\phi[\Sigma] + dT.$$

One can think of $\text{Res}_\phi[\Sigma]$ as the characteristic current associated to $\alpha : E \longrightarrow F$ which represents the class $\phi(F) - \phi(E) \in H^*_{\text{de Rham}}(X)$.

Detailed formulas cannot, of course, be derived for arbitrary smooth bundle maps $\alpha : E \longrightarrow F$ since the singularities can be quite pathological. Nevertheless, the theory does apply to quite general maps. By work of the first author and Stephen Semmes we know that under weak assumptions on α, there exist certain associated currents with rectifiability properties, which one might call "Thom-Porteous currents". For generic maps these currents are standard singularity sets defined by rank conditions on α. Our general formulas will often be expressed in terms of smooth "residue" forms paired with these currents.

The analytic assumptions we make on our bundle maps are presented in detail here and certain important properties are established. However, the fundamental results concerning the existence and structure of divisors and more general Thom-Porteous currents appear in [HS].

To give a notion of the nature of the results, we present some elementary but important examples. The first is provided by the case where E and F are complex line bundles over an oriented manifold X. Suppose for simplicity that $\alpha : E \longrightarrow F$ vanishes non-degenerately so that the divisor $\text{Div}(\alpha)$ is the current associated to the oriented codimension-2 submanifold $\Sigma = \{x \in X : \alpha_x = 0\}$. Then for each polynomial $\phi(u) \in \mathbf{R}[u]$ in one indeterminate, we obtain the formula

$$(*) \qquad \phi(f) - \phi(e) = \left\{\frac{\phi(f) - \phi(e)}{f - e}\right\} \mathrm{Div}(\alpha) + dT$$

where

$$f = \tfrac{i}{2\pi}\Omega_F \quad \text{and} \quad e = \tfrac{i}{2\pi}\Omega_E$$

are the Chern-Weil representatives of the first Chern classes of F and E respectively (and where $\frac{\phi(f)-\phi(e)}{f-e}$ is the obvious polynomial in e and f.) This formula holds in fact for any α which is **atomic**, that is for which

$$\mathrm{tr}(\alpha^{-1} D\alpha) \in L^1_{\mathrm{loc}},$$

(i.e., for which $\mathrm{tr}(\alpha^{-1} D\alpha)$ has an L^1_{loc}-extension across Σ), where D denotes the induced connection on $\mathrm{Hom}(E, F)$. (Under local trivializations of E and F, α is represented by a complex-valued function a, and atomicity is equivalent to the condition $da/a \in L^1_{\mathrm{loc}}$.) The transgression term T in $(*)$ is given by the formula

$$T = \tfrac{i}{2\pi}\left\{\frac{\phi(f) - \phi(e)}{f - e}\right\} \mathrm{tr}(\alpha^{-1} D\alpha).$$

When $\phi(u) = u$ and E is trivial, equation $(*)$ represents a C^∞ generalization of the classical Poincaré-Lelong formula.

Combining this result with the kernel-calculus of Harvey and Polking [HP] gives a new proof of the Riemann-Roch Theorem for vector bundles over algebraic curves.

Another example of a basic formula coming from the theory is provided by considering a section $\alpha \in \Gamma(V)$ of an even-dimensional vector bundle $V \longrightarrow X$ with spin structure. Assume that α vanishes non-degenerately (or more generally that it is **atomic** in the sense of [HS] below), and let $\mathrm{Div}(\alpha)$ denote its divisor. Now Clifford multiplication by α determines a bundle map $\alpha : \mathcal{S}^+ \longrightarrow \mathcal{S}^-$ between the positive and negative complex spinor bundles canonically associated to V. Consider the function on matrices $\phi(A) = \mathbf{ch}(A) \stackrel{\mathrm{def}}{=} \mathrm{trace}\left\{\exp(\frac{1}{2\pi i}A)\right\}$ which gives the Chern character. Suppose $\mathcal{S}^+$ and $\mathcal{S}^-$ carry connections induced from a riemannian connection on V, and let $\Omega_{\mathcal{S}^\pm}$, Ω_V denote the curvature matrices of these connections (with respect to local orthonormal framings of the bundles). Then we have the formula

$$(**)\qquad \mathbf{ch}\left(\Omega_{\not{S}^+}\right) - \mathbf{ch}\left(\Omega_{\not{S}^-}\right) \;=\; \widehat{\mathbf{A}}(\Omega_V)^{-1}\,\mathrm{Div}(\alpha) + dT$$

where

$$\widehat{\mathbf{A}}(\Omega_V)^{-1} \;=\; \det{}^{\frac{1}{2}}\left\{\frac{\sinh(\frac{1}{4\pi}\Omega_V)}{\frac{1}{4\pi}\Omega_V}\right\}$$

is the series of differential forms on X which canonically represent, via Chern-Weil Theory, the inverse $\widehat{\mathbf{A}}$-class of V, and where T is a canonically defined form with L^1_{loc}-coefficients.

This generalizes immediately to $\alpha\otimes 1 : \not{S}^+\otimes E \longrightarrow \not{S}^-\otimes E$ for any coefficient bundle E with connection. Here we obtain the formula

$$(***)\qquad \mathbf{ch}\left(\Omega_{\not{S}^+\otimes E}\right) - \mathbf{ch}\left(\Omega_{\not{S}^-\otimes E}\right) \;=\; \mathbf{ch}(\Omega_E)\widehat{\mathbf{A}}(\Omega_V)^{-1}\,\mathrm{Div}(\alpha) + dT.$$

When $\mathrm{Div}(\alpha)$ is an oriented submanifold, this equation is precisely a formulation **at the level of differential forms** of the Differentiable Riemann-Roch Theorem of Atiyah and Hirzebruch [AH]. Furthermore it extends this theorem from submanifolds to oriented subcomplexes with "normal bundle", i.e., subcomplexes which arise as divisors of some cross-section of a bundle.

This result emerges naturally from our philosophy of considering pushforward and pullback connections under bundle morphisms. It is also noteworthy that the $\mathbf{ch}\cdot\widehat{\mathbf{A}}^{-1}$-series in this formula falls directly out of our computational calculus. It is not inserted with hindsight and then cleverly verified, as is often the case.

There are formulas analogous to $(***)$ when V is complex or Spin^c. In the complex case, we obtain a version of the classical Grothendieck Theorem at the level of differential forms and currents. To be specific let $j : Y \hookrightarrow X$ be a smooth embedding of compact oriented manifolds. Suppose that the normal bundle to j carries an almost complex structure. If the tangent bundles TY and TX are given connections compatible with this normal complex structure along Y, then we have the following equation of forms and currents on X:

$$\left\{\mathbf{ch}\left(\Omega_{(\Lambda^{\mathrm{even}}F^*)\otimes E}\right) - \mathbf{ch}\left(\Omega_{(\Lambda^{\mathrm{odd}}F^*)\otimes E}\right)\right\}\wedge\mathbf{Todd}\left(\Omega_{TX}\right)$$
$$= \mathbf{ch}(\Omega_E)\wedge\mathbf{Todd}(\Omega_{TY})[Y] + dT$$

for any vector bundle with connection E over Y. By passing to cohomology this formula yields the commutativity of the diagram

$$\begin{tikzcd} K(Y) \arrow[r, "j_!"] \arrow[d, "\mathbf{ch}(\,\cdot\,)\wedge\mathbf{Todd}(Y)"'] & K(X) \arrow[d, "\mathbf{ch}(\,\cdot\,)\wedge\mathbf{Todd}(X)"] \\ H^*(Y) \arrow[r, "j_!"'] & H^*(X) \end{tikzcd}$$

where the $j_!$ represent the Gysin "wrong way"maps in K-theory and cohomology.

In these special Clifford multiplication cases our formalism has some similarities with Quillen's calculus of superconnections [Q] as developed in [MQ], [BV], [BGS*] and elsewhere. However, even in these cases there are substantial differences. We are concerned with convergence questions under weak hypotheses and with the structure of the limiting currents. Our theory also allows for a quite general choice of "approximation mode". As we shall explain in a moment, each mode is determined by a choice of an **approximate one**, i.e., a C^∞-function $\chi : [0, \infty] \longrightarrow [0, 1]$ with $\chi' \geq 0$, $\chi(0) = 0$ and $\chi(\infty) = 1$. Choosing $\chi(t) = 1 - e^{-t}$ puts us closest to Quillen's theory. However, choosing χ with $\chi(t) \equiv 1$ for $t \geq 1$ gives approximations supported in small neighborhoods of the singular set. For many reasons the most natural choice is $\chi(t) = t/(1 + t)$. This form of approximation admits nice compactifications and is closely related to the **Grassmann graph construction** of MacPherson.

A third important application arises when E is the trivial line bundle with trivial connection, and F is real. In this case α corresponds to a cross-section of F. This section is said to be **atomic** if when expressed locally as $\alpha = (a^1, \ldots, a^n)$ it satisfies the condition

$$\frac{da^I}{|a|^{|I|}} \in L^1_{\text{loc}}$$

for all I with $|I| < n \stackrel{\text{def}}{=} \dim F$. Very general elementary criteria for atomicity are given in [HS]. (Roughly speaking, any smooth section which vanishes algebraically on a set of the proper codimension is atomic.) As a special case, it is shown that any real analytic section with zeros of codimension-n is atomic.

One can think of atomicity as a weak criterion which insures the existence of a divisor, i.e., for which the graph of α can be sliced by the zero-section of F. It is proved in [HS] that the vanishing of an atomic section α determines a unique,

d-closed current $\mathrm{Div}(\alpha)$ of codimension-n, called the **divisor** of α. This current is integrally flat, and in particular, when its mass is finite, it is a rectifiable cycle in the sense of Federer [F].

Assume now that F is either complex, or real and oriented, and that ϕ is the top Chern polynomial or the Pfaffian respectively (If $\dim_{\mathbf{R}} F$ is odd, then $\phi \equiv 0$.) We show that if α is atomic, then there exists a canonical L^1_{loc}-form T on X such that the following generalization of the Poincaré-Lelong formula holds:

$$\phi(\Omega_F) - \mathrm{Div}(\alpha) = dT.$$

Furthermore for each approximation mode as above we obtain a smooth family of connections D_s, $0 < s \leq \infty$ on F such that the associated characteristic forms $\tau_s \equiv \phi(\Omega_s)$ have the property that $\tau_\infty = \phi(\Omega_F)$ and

$$\lim_{s \to 0} \tau_s = \mathrm{Div}(\alpha).$$

There is a corresponding family of smooth forms T_s, $0 < s \leq \infty$, such that

$$\tau_s - \mathrm{Div}(\alpha) = d(T - T_s).$$

and $\lim\limits_{s \to 0} T_s = T$ in L^1_{loc}.

It is useful to view this construction universally, that is, on the total space of the bundle itself. Let $\pi : F \longrightarrow X$ be the bundle projection and consider the pullback $\mathbf{F} = \pi^* F$ with the pullback connection. Over F there is a **tautological cross-section** α of $\mathbf{F}$ given by $\alpha(v) = v$.

This section is atomic, in fact, non-degenerate, and the theory applies. For each approximation mode we obtain a smooth family of closed differential forms τ_s, $0 < s \leq \infty$ on F, which are expressed canonically in terms of the connection, and which **represent the Thom class of** F. For example when F is real of dimension $2n$ and we are in the real algebraic approximation mode (where $\chi(t) = 1 - 1/\sqrt{1+t}$), then τ_s is given by the formula

$$\tau_s = \tfrac{1}{(2\pi)^n} \tfrac{s}{\sqrt{|u|^2 + s^2}} \mathrm{Pfaff}\left(\frac{Du^t Du}{|u|^2 + s^2} - \Omega_F \right)$$

where $u = (u_1, \ldots, u_{2n})$ represents the tautological section and where Du is its covariant derivative in the pullback connection (with respect to a local orthonormal frame field).Note that Du can be viewed invariantly as the projection

$Du : TF \cong F \oplus H \longrightarrow F$ along the horizontal subspaces H of the connection. The form above, which falls directly out of the theory, has the following remarkable properties:

(1) τ_s is d-closed,

(2) τ_s extends to a smooth d-closed form on the fibrewise compactification $\mathbf{P}(F \oplus \mathbf{R}) \supseteq F$,

(3) $\pi_* \tau_s = 1$ where π_* denotes integration over the fibre,

(4) $i^* \tau_s = \mathrm{Pfaff}(\Omega_F)$ where $i : X \hookrightarrow F$ is the zero-section,

(5) $\tau_s = \mu_s^*(\tau_1)$ where $\mu_s : F \to F$ denotes scalar multiplication by $\frac{1}{s}$,

(6) $\lim_{s \to 0} \tau_s = [X]$.

In brief, τ_s is a closed $2n$-form on F which dies at infinity, is integrable on the fibres and converges, as $s \to 0$, to the current $[X]$ represented by the zero-section. It is a family of canonical representatives of the Thom class of F (cf. [QM]). It can be written, as in [QM], as the image of a universal class in the equivariant cohomology of $\mathbf{R}^{2n}$.

Choosing other approximation modes yields other universal formulas for the Thom class. In particular, if $\chi(t) \equiv 1$ for $t \geq 1$, then the form τ_s will have support in the tubular neighborhood $X_s \equiv \{v \in F : |v| \leq s\}$. In particular, it has **compact support in each fibre.**

There are many other interesting applications. A particularly nice one concerns the case of an endomorphism $\alpha : E \longrightarrow F$ of **quaternion line bundles**. One finds a very pretty analogy with the complex line bundle case. The classifying space for complex line bundles is the infinite complex projective space $\mathbf{P}^\infty(\mathbf{C})$ whose cohomology is a polynomial ring on one generator $u \in H^2(\mathbf{P}^\infty(\mathbf{C}); \mathbf{Z})$. The classifying space for quaternion line bundles is the infinite quaternion projective space $\mathbf{P}^\infty(\mathbf{H})$ whose cohomology is a polynomial ring on one generator $u \in H^4(\mathbf{P}^\infty(\mathbf{H}); \mathbf{Z})$. If we are given connections on E and F which are compatible with the quaternion structure, and if α is atomic, then for each $\phi \in \mathbf{R}[u]$ there exists an L^1_{loc}-form T with the property that

$$\phi(f) - \phi(e) = \left\{ \frac{\phi(f) - \phi(e)}{f - e} \right\} \mathrm{Div}(\alpha) + dT$$

where

$$f = \tfrac{1}{16\pi^2} \mathrm{tr}\left\{\Omega_F^2\right\} \quad \text{and} \quad e = \tfrac{1}{16\pi^2} \mathrm{tr}\left\{\Omega_E^2\right\}$$

are the canonical representatives of the second Chern class (or instanton class) of E and F. Note the close similarity to the complex case above.

Our general approach to singular connections comes from the following basic observation. Let E and F be smooth vector bundles with connections D_E and D_F respectively over a manifold X. Then given any pair of bundle maps

$$\alpha : E \longrightarrow F \qquad \text{and} \qquad \beta : F \longrightarrow E$$

we can define induced connections:

$$\overleftarrow{D}^{\alpha,\beta} = \beta \circ D_F \circ \alpha \ + \ (1-\beta\alpha) \circ D_E \qquad \text{on } E$$

and

$$\overrightarrow{D}^{\alpha,\beta} = \alpha \circ D_E \circ \beta \ + \ D_F \circ (1-\alpha\beta) \qquad \text{on } F.$$

This allows us to expand the notion of the gauge group. Namely, if we fix D_E and D_F, then to every pair α, β we have the transformed connection $\overrightarrow{D}^{\alpha,\beta}$ on F (and also $\overleftarrow{D}^{\alpha,\beta}$ on E). When E and F are isomorphic (and in particular of the same rank), we can restrict to pairs with $\beta = \alpha^{-1}$ and recover the usual action of the gauge group on the space of connections.

Now it is our intention to study α, so we want to choose β to be naturally adapted to α. Suppose for example that $\text{rank}(E) \leq \text{rank}(F)$ and that α is injective. Then we would choose β to be projection onto $\text{image}(\alpha)$ followed by the inverse. If we introduce metrics on E and F (not necessarily related to the connections), then we can choose $\beta = (\alpha^*\alpha)^{-1}\alpha^*$. This formula breaks down on the singularity set $\Sigma \equiv \{x \ : \ \alpha_x \text{ is not injective}\}$. To remedy this we approximate β by a family β_s, $s > 0$, as follows. Let $\chi : [0,\infty] \longrightarrow [0,1]$ be an approximate one as defined above, and set $\rho(t) = \chi(t)/t$. This is an **approximate reciprocal**, i.e., $\frac{1}{s}\rho(\frac{t}{s}) = (\frac{1}{t})\chi(\frac{t}{s})$ approximates $\frac{1}{t}$ as $s \longrightarrow 0$. For each $s > 0$ we set

$$\beta_s = \rho\left(\frac{\alpha^*\alpha}{s}\right)\frac{\alpha^*}{s}$$

and plug this into the formulas above to obtain a family of connections $\overrightarrow{D}_s$ on F (and $\overleftarrow{D}_s$ on E). Note that $\beta_s \longrightarrow \beta$ as $s \longrightarrow 0$ uniformly on compacta outside the singular set Σ. Note also that if $\chi(t) \equiv 1$ for $t \geq 1$, then $\beta_s = \beta$ outside the "s-tubular neighborhood" $U_s \stackrel{\text{def}}{=} \{x \in X : \alpha_x^* \alpha_x < s\}$ of Σ.

Let $\overrightarrow{\Omega}_s$ denote the curvature of the connection $\overrightarrow{D}_s$, and fix an Ad-invariant polynomial ϕ on the Lie algebra of the structure group of F. We consider the family of smooth forms $\phi(\overrightarrow{\Omega}_s)$ representing the ϕ-characteristic class F. If the limit $\lim_{s \to o} \phi(\overrightarrow{\Omega}_s)$ exists as a current, we say that this limit is the **ϕ-characteristic current associated to the singular "pushforward" connection $\overrightarrow{D}$ on F.** Analogous remarks and definitions hold for the singular "pullback"connection $\overleftarrow{D}$ on E and for the case where $\text{rank}(E) \geq \text{rank}(F)$. However, we stick to the situation above for expository reasons.

Note that on the subset $X - \Sigma$, $\lim_{s \to o} \phi(\overrightarrow{\Omega}_s) \stackrel{\text{def}}{=} \phi(\overrightarrow{\Omega}_0)$ is the smooth characteristic form associated to the smooth connection $\overrightarrow{D}|_{X-\Sigma}$. Hence, the only serious questions concerning this limit arise at points of Σ. Now in the special case that $\text{rank}(E) = \text{rank}(F)$, the connection $\overrightarrow{D}|_{X-\Sigma}$ is gauge equivalent via α to D_E, and so $\phi(\overrightarrow{\Omega}_0) = \phi(\Omega_E)$. In particular, $\phi(\overrightarrow{\Omega}_0)$ has a smooth extension across the singular set Σ. Part of the work of this paper is aimed at finding conditions on α which guarantee, in the case where $\text{rank}(E) < \text{rank}(F)$, that $\phi(\overrightarrow{\Omega}_0)$ also extends across Σ as a d-closed L^1_{loc}-form. We show for example that this always happens in the universal case of the tautological cross-section of $\pi^* \text{Hom}(E, F)$ over the total space of $\text{Hom}(E, F)$. Whenever this does happen we have a decomposition of the ϕ-characteristic current:

$$\lim_{s \to o} \phi(\overrightarrow{\Omega}_s) = \phi(\overrightarrow{\Omega}_0) + S$$

where S is a current on X with the property that

$$dS = 0 \qquad \text{and} \qquad \text{supp}(S) \subseteq \Sigma.$$

Since they are both d-closed, each term in this decomposition represents a de Rham cohomology class on X. These classes can be non-zero even when E and F are trivial bundles. A good example is provided when $\alpha : \mathbf{C} \to F$ represents a cross-section of F. Here the de Rham class of $\phi(\overrightarrow{\Omega}_0)$ is computed from a universal characteristic class (related to ϕ) on the compactification $\mathbf{P}(F \oplus \mathbf{C})$ of F. (See Chapters III and IV.)

The next step in the program is to determine the detailed structure of the singular characteristic current S and to establish its independence of the choice of approximation mode χ. In doing this we consider a family of canonical transgression classes T_s with the property that

$$dT_s = \phi(\Omega_F) - \phi(\overrightarrow{\Omega}_s),$$

and give conditions on α so that $T \stackrel{\text{def}}{=} \lim_{s\to o} T_s$ exists in the space of currents with L^1_{loc}-coefficients. The transgression current T has the property that

$$dT = \phi(\Omega_F) - \phi(\overrightarrow{\Omega}_0) - S.$$

It is also functorial under appropriately transversal maps between manifolds.

In a large number of cases we will show that the singular characteristic current S can be written in the form

$$S = \text{Res}_\phi[\Sigma]$$

where $[\Sigma]$ is the current associated to integration over the singular set Σ, and where Res_ϕ is a smooth differential form on X which is independent of the bundle map α. In many cases this residue form Res_ϕ is proved to be a classical Chern-Weil form, i.e., it is expressed as a universal Ad-invariant polynomial in the curvatures of the given connections on E and F. In particular this residue form is completely determined by computing its associated cohomology class in the universal setting.

The formula above is then written as

$$(\dagger) \qquad \phi(\Omega_F) - \phi(\overrightarrow{\Omega}_0) = \text{Res}_\phi[\Sigma] + dT$$

where Res_ϕ is the canonical residue form and where $[\Sigma]$ is a current canonically determined by the singular structure of α. Formula $(\dagger)$ represents a particularly satisfactory Chern-Weil Theorem for bundle maps. Special cases where this all holds have been discussed above.

A nice feature of formula $(\dagger)$ is that it canonically relates certain characteristic forms to submanifolds and subvarieties which arise in geometric constructions.

In this manner the theory can be applied to a wide range of problems. For example, suppose we are given $k+1$ sections of a complex vector bundle F, i.e., a bundle map $\alpha : \mathbf{C}^{k+1} \to F$ with an appropriate atomicity property (cf. Ch. VI). Assume that F has a complex connection. Then we have a canonical formula

$$c_{n-k}(\Omega_F) - \mathbb{D}_k = dT$$

between the $(n-k)^{\text{th}}$ Chern form of F and the degeneracy current $\mathbb{D}_k$, which is intuitively defined as the set of $x \in X$ where $\alpha(x)$ is not injective, i.e., where $\alpha_1(x), \ldots, \alpha_{k+1}(x)$ are linearly dependent. In fact we will produce a smooth family of mutually cohomologous, d-closed forms Ψ_s, $0 < s \leq \infty$, such that $\Psi_\infty = c_{n-k}(\Omega_F)$ and $\Psi_s \to \mathbb{D}_k$ as $s \to 0$. A related example arises when considering a smooth map $f : M \to \mathbf{C}^{k+1}$ of an oriented riemannian n-manifold M where $n - k = 2\ell > 0$. Here we obtain formulas

$$p_\ell(\Omega_M) = (-1)^\ell \mathbb{C}r(f) + dT$$

where $p_\ell(\Omega_M)$ is the ℓ^{th} Pontrjagin form of M and where $\mathbb{C}r(f)$ is a current associated to the **complex critical set**: $\{x \in M : df : TM \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow \mathbf{C}^{k+1}$ is not surjective$\}$. For example if $\dim M = 4$ and $f : M \longrightarrow \mathbf{C}^3$ is an immersion, then we have the formula

$$p_1(\Omega_M) = -\mathbb{C}r(f) + dT$$

where $\mathbb{C}r(f)$ is the (generically finite) set of complex tangencies to $F(M) \subset \mathbf{C}^3$ taken with appropriate indices.

Another application generalizes the formula for the global Milnor current. The "classical" formula (cf. [Fu, 14.1.5]) concerns the critical locus of a holomorphic map $f : X \longrightarrow C$ of a complex manifold onto a complex curve, and computes the sum of the local Milnor numbers of the singular points in terms of global geometry.

Some interesting formulas in the theory of foliations are derived. Some interesting invariants for pairs of complex structures are introduced, and geometric formulas relating them to characteristic forms are established. It is also possible, using this theory, to rederive and generalize formulas of Sid Webster [W1, 2].

These and other applications will be discussed in [HL1]. It should be remarked that Wolfson's paper [Wo] (cf. [MW]) served as an inspiration for this work.

Much of the work in this paper is devoted to special cases of bundle maps $\alpha : E \longrightarrow F$. However, these cases have much wider applicability than is apparent. Using only the notions of this paper, we introduce general Thom-Porteous currents for each rank $r > 0$. These are cycles associated to a general bundle map $\alpha : E \longrightarrow F$ which encode the condition that rank(α) has dropped at least r. They occur in many contexts. Some indication is given here but full details will appear in a separate article, [HL1].

In the sequel to this paper [HL2] the authors will study some of the more delicate residue formulas associated to bundle maps. These formulas will simultaneously involve several of the degeneracy strata of the map, each paired with its own residue form. There will be further applications given to certain Thom-Porteous classes, also to sections of split bundles, and to multi-foliations.

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Some Notational Conventions

We gather here some notational conventions used throughout the article. The manifolds, bundles, bundle maps, etc. will always be C^∞ unless otherwise stated. If X is a manifold, $\mathcal{E}^r(X)$ denotes the space of differential r-forms on X with the usual C^∞-topology. The topological dual space, denoted $\mathcal{E}^r(X)'$, is the space of r-dimensional currents with compact support on X. Similarly, $\mathcal{D}^r(X)$ denotes r-forms with compact support on X, and $\mathcal{D}^r(X)'$ denotes its dual.

By $L^1_{\text{loc}}(X)$ we shall mean the space of differential forms on X with locally integrable coefficients. When X is orientable, there is a natural embedding $L^1_{\text{loc}}(X) \subset \mathcal{D}^*(X)'$ given by associating to the form φ the current $[\varphi](\psi) = \int \phi\wedge\psi$. The exterior derivative d is well-defined in all of these spaces.

Given a Lie group G with Lie algebra $\mathfrak{g}$, we denote by I_G (or I^*_G) the graded algebra of Ad_G-invariant polynomials on $\mathfrak{g}$.

A connection on a vector bundle $E \longrightarrow X$ is taken here to be a differential operator, i.e., a linear map $D : \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$ such that $D(f\sigma) = df \otimes \sigma + fD\sigma$ for all $f \in C^\infty(X)$. The curvature of D is the operator D^2. If e is a local frame for E, we shall write $De = \omega e$ and $D^2 e = \Omega e$ where ω and Ω are the connection and curvature matrices respectively.

I. Bundle Maps and Singular Connections

1. Chern Currents — The General Approach.

Suppose F is a complex vector bundle on a smooth real manifold X. Suppose Σ is a closed measure zero subset of X, which will be referred to as the **singular set**. Suppose D is a smooth connection on the bundle F over $X - \Sigma$, i.e. outside the singular set Σ. Such a connection D, on F over $X - \Sigma$, will be referred to as a **singular connection D on F over X**. One objective is to compute the "Chern currents" for the singular connection D over X. The general method can be described as follows.

No attempt is made to give meaning to the singular connection D as a notion of differentiation. Instead the singular connection D is understood via approximation by smooth connections. Assume that

$$(1.1) \qquad D_s \quad \text{for} \quad 0 < s \leq \infty$$

is a smooth family of smooth connections on the bundle F over X with the following property:

(1.2) As $s \to 0$, $D_s \to D_0$ as smooth connections on F outside the singular set Σ,

where

$$(1.3) \quad D_0 \equiv D\Big|_{X-\Sigma}.$$

Definition 1.4. For $0 < s \leq \infty$, D_s will be referred to as a **smooth approximation to the singular connection D.**

The **Chern forms** for the connection D_s are defined in the standard manner.

(1.5) $c(D_s) \equiv c(\Omega_s) \equiv \det\left(I + \frac{i}{2\pi}\Omega_s\right)$ (the **total Chern form**),

(1.6) $ch(D_s) \equiv ch(\Omega_s) \equiv \operatorname{trace}\left(e^{\frac{i}{2\pi}\Omega_s}\right)$ (the **Chern character**),

or more generally:

(1.7) $\phi(D_s) \equiv \phi(\Omega_s)$ (the **ϕ-characteristic form**),

for any polynomial ϕ on $\mathfrak{gl}(n)$ which is invariant under the adjoint action. Here $\Omega_s \equiv d\omega_s - \omega_s \wedge \omega_s$ is the **curvature matrix** and ω_s is the **gauge potential** or **connection matrix** for D_s. The gauge potential and the curvature matrix depend on the choice of frame while a Chern form does not depend on the choice of frame and hence is a globally defined differential form on the manifold. The curvature operator $R_s = D_s^2$ is a section of the vector bundle $\Lambda^2T^* \otimes \operatorname{End}(F)$, while the corresponding curvature matrix Ω_s is a local section of $\Lambda^2T^* \otimes M_p(\mathbf{C})$. The connection D_s itself is a section of an affine bundle based on the vector bundle $\Lambda^1T^* \otimes \operatorname{End}(E)$.

Note: Given an invariant polynomial ϕ, such as

$$\phi(\Omega) = \det(I + \Omega) \quad \text{or} \quad \phi(\Omega) = \operatorname{tr} e^{\Omega},$$

it is natural to consider the "renormalized" invariant polynomial $\tilde{\phi}$ defined by

$$\tilde{\phi}(\Omega) = \phi\left(\tfrac{i}{2\pi}\Omega\right)$$

evaluated on the curvature matrix, or alternatively to evaluate ϕ on the normalized curvature

$$\widetilde{\Omega} \equiv \tfrac{i}{2\pi}\Omega.$$

Both of these notations will be employed in this paper.

Definition 1.8. Suppose D_s, $0 < s \le +\infty$, is a smooth approximation to a singular connection D. If the Chern forms $\phi(D_s)$ converge, weakly as currents on X, then the limit will be denoted by $\phi(\!(D)\!)$ and referred to as a **ϕ-characteristic current** for the singular connection D.

In order to compile a list of desirable conditions we start with:

Condition A. The currents $\phi(D_s)$ converge to $\phi((D))$, weakly as currents on X.

A current which can locally be expressed as a differential form with coefficients that are Lebesgue integrable functions will be referred to as an $\mathbf{L^1_{loc}}$ **form.**

Since, outside the singular set Σ, the connections D_s converge to D as $s \to 0$, the ϕ-characteristic forms $\phi(D_s)$ converge to the ϕ-characteristic form of the connection D. That is,

$$\phi(D_s)\Big|_{X-\Sigma} \xrightarrow{C^\infty} \phi(D)\Big|_{X-\Sigma} \qquad \text{as } s \to 0 \qquad \text{on } X - \Sigma. \tag{1.9}$$

The second desirable condition may, at this point, appear somewhat surprising (cf. Remark 2.35).

Condition B. The smooth form $\phi(D)\big|_{X-\Sigma}$ on $X-\Sigma$ extends by zero to an L^1_{loc} form on X which is d-closed. The extension will be denoted by $\phi(D_0) \in L^1_{\text{loc}}(X)$ and referred to as the $\mathbf{L^1_{loc}}$ **part of the ϕ-Chern current $\phi((\mathbf{D}))$.**

Note that if both conditions A and B are satisfied then the ϕ-Chern current splits as:

$$\phi((D)) \equiv \lim_{s\to 0} \phi(D_s) = \phi(D_0) + S, \qquad \text{on } X, \tag{1.10}$$

where S is a current with

$$\operatorname{spt}(S) \subseteq \Sigma \quad \text{and} \quad dS = 0. \tag{1.11}$$

This current S will be referred to as the **singular part of the ϕ-Chern current $\phi((\mathbf{D}))$**. In summary, $\phi(D_0)$ or $\phi(\Omega_0)$ will always denote the L^1_{loc} part of the ϕ-Chern current on X, while $\phi((D))$ will always denote the full ϕ-Chern current (1.10).

Although, in this paper we are interested in transgressions given explicitly, for the sake of completeness we note the following standard result and sketch it's proof.

Proposition 1.12. *If both Conditions A and B are satisfied then there exists a current T satisfying:*

$$dT = \phi(D_\infty) - \phi(D_0) - S \qquad \text{on } X. \tag{1.13}$$

Proof. By assumption (1.1), the family D_s of connections on F converges smoothly to D_∞ as s approaches $+\infty$. Therefore, the standard transgression formula (1.18) for the path of connections joining D_s to D_∞ defines a global transgression form T_s satisfying

$$dT_s = \phi(D_\infty) - \phi(D_s). \tag{1.14}$$

In particular, $\phi(D_\infty) - \phi(D_s)$ belongs to the range of the exterior derivative operator $d : \mathcal{D}' \to \mathcal{D}'$ mapping the space of currents into itself.

It is a standard fact that, for an arbitrary paracompact smooth manifold, the range of d is closed. (We note that the corresponding result for $\overline{\partial}$ is false). Therefore $\phi(D_\infty) - \phi(D_0) - S \equiv \lim_{s\to 0} \phi(D_\infty) - \phi(D_s) \in d\, \mathcal{D}'$. This standard fact that d has closed range is part of the statement of the de Rham duality theorem. The proof follows, via Cech theory, from the fact that all subspaces of the infinite product $\Pi_{j=1}^{\infty} \mathbf{C}$ are closed. □

Let $\operatorname{Reg}\Sigma$ denote the points where Σ is a submanifold. Note that the components of $\operatorname{Reg}\Sigma$ may have various dimensions. Assume that integration of forms over the various components of $\operatorname{Reg}\Sigma$ defines a current with finite mass on X and denote this current by $[\Sigma]$. Further assume that $[\Sigma]$ is d-closed.

Any current with support in an oriented submanifold, say Σ, can be written as a finite sum of currents, each of which is obtained from a current intrinsic to Σ by taking normal derivatives and multiplying by normal one forms. By far the simplest of the currents supported in Σ are those where the intrinsic current is a smooth form ψ and no normal derivatives occur, i.e., currents such as $\psi[\Sigma]$.

Condition C. The singular part S of the Chern current $\phi((D))$ has the form

$$S = \operatorname{Res}_\phi(D)[\Sigma], \tag{1.15}$$

where $\operatorname{Res}_\phi(D)$ is a d-closed smooth form on the manifold $\operatorname{Reg}\Sigma$ called the **residue form** of the singular connection D. (Actually, it is more appropriate to refer to $\operatorname{Res}_\phi(D)$ as the residue form of the current T satisfying (1.13)).

Let $\phi(\Omega_1, \ldots, \Omega_m)$ denote the complete polarization of ϕ (assuming ϕ is homogeneous of degree m). The notation

$$(1.16) \qquad \phi(\sigma\ ;\ \Omega) \equiv \sum_{i=1}^{m} \phi(\Omega, \ldots, \overset{i^{\text{th}}}{\sigma}, \ldots, \Omega) = \frac{d}{dt}\phi(\Omega + t\sigma)\Big|_{t=0}$$

will be useful. The standard transgression formula [BoC] for the family D_s of connections says that:

$$(1.17) \qquad dT_s = \phi(D_\infty) - \phi(D_s),$$

where T_s, the **potential** (or **transgression form**), is given by:

$$(1.18) \qquad T_s = \int_s^\infty \phi\,(\dot{\omega}_t\ ;\ \Omega_t)\,dt.$$

Condition D. The transgression forms T_s converge, as currents on X, to a current T, called the **transgression current** or the **fundamental potential**, i.e.

$$(1.19) \qquad T \equiv \lim_{s \to 0} T_s = \int_0^\infty \phi\,(\dot{\omega}_s\ ;\ \Omega_s)\,ds \quad \text{converges in } \mathcal{D}'.$$

If Condition D is satisfied, then Condition A is automatic and, under Condition B, the transgression equation (1.13) is the limiting form of the standard transgression formula (1.17).

If all four Conditions A–D are satisfied, then the formula

$$(1.20) \qquad \phi(D_\infty) - \phi(D_0) - \mathrm{Res}_\phi(D)[\Sigma] = dT \qquad \text{on } X,$$

is obtained as a limiting form of (1.17), where $\phi(D_0) \in L^1_{\mathrm{loc}}(X)$ is d-closed, and the residue form $\mathrm{Res}_\phi(D)$ is also d-closed.

2. Singular Connections Determined by Bundle Maps.

Suppose that the data

$$E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F \tag{2.1}$$

is given; where E is a rank p complex vector bundle, F is a rank q complex vector bundle, both over the same smooth manifold X, and that α and β are vector bundle maps. Also assume that E is equipped with a connection D_E and that F is equipped with a connection D_F.

Each of the connections D_E and D_F may be transplanted to the other vector bundle.

Lemma 2.2. *Given $E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F$ and D_E, D_F, then:*

(2.3)(**Pushforward**) $$\overrightarrow{D} \equiv \alpha D_E \beta + D_F(1-\alpha\beta) = D_F - (D_F\alpha - \alpha D_E)\beta$$

defines a connection on the bundle F, which will be referred to as the **push forward connection**, *while*

(2.4)(**Pullback**) $$\overleftarrow{D} \equiv \beta D_F \alpha + (1-\beta\alpha) D_E = D_E + \beta(D_F\alpha - \alpha D_E)$$

defines a connection on the bundle E, which will be referred to as the **pullback connection**.

Proof. Suppose s is a section of E and φ is a smooth function. Then

$$\begin{aligned}(D_F\alpha - \alpha D_E)(\varphi s) &= D_F\varphi\alpha(s) - \alpha\left(d\varphi s + \varphi D_E s\right) \\ = d\varphi\alpha(s) + \varphi D_F\alpha(s) - d\varphi\alpha(s) - \varphi\alpha D_E(s) &= \varphi\left(D_F\alpha - \alpha D_E\right)(s).\end{aligned}$$

That is,

$$D_F\alpha - \alpha D_E \in \Gamma\left(\Lambda^1 T^* \otimes \mathrm{Hom}(E,F)\right),$$

so that $\overrightarrow{D}$ and $\overleftarrow{D}$ satisfy the product rule. □

The connections D_E and D_F induce a connection D_H on $\mathrm{Hom}(E, F)$ determined by requiring the product rule

$$D_F(\alpha(s)) \;=\; (D_H(\alpha))(s) + \alpha(D_E s)$$

for all sections s of E. That is

$$D_H(\alpha) \;\equiv\; D_F\alpha - \alpha D_E. \tag{2.5}$$

Thus the connections $\overrightarrow{D}$ and $\overleftarrow{D}$ can be rewritten in the compact form:

(2.6)(**Pushforward**) $\qquad \overrightarrow{D} = D_F - D_H(\alpha)\beta.$

(2.7)(**Pullback**) $\qquad \overleftarrow{D} = D_E + \beta D_H(\alpha).$

The singular connections we wish to study are obtained by choosing $\boldsymbol{\beta}$ to be "**the inverse of** $\boldsymbol{\alpha}$". This is made precise in (2.9) below. Suppose that a single bundle map $E \xrightarrow{\alpha} F$ is given and that E and F are equipped with hermitian metrics denoted by $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ respectively. Also assume that D_E is a connection on E and that D_F is a connection on F, but do not assume that the metrics and connections are compatible. This basic data will be denoted by:

$$\begin{array}{ccc} D_E & & D_F \\ E & \xrightarrow{\alpha} & F \\ \langle\ ,\ \rangle_E & & \langle\ ,\ \rangle_F \end{array} \tag{2.8}$$

Let $\boldsymbol{\Sigma} \equiv \mathbf{sing}\,\boldsymbol{\alpha}$ denote the closed set where the rank of α is less than maximal. On the complement of the singular set Σ let $\mathbf{K} \equiv \mathbf{ker}\,\boldsymbol{\alpha}$ denote the **kernel subbundle** of E and let $\mathbf{T} \equiv \mathbf{im}\,\boldsymbol{\alpha}$ denote the **image** or **target subbundle** of F. Now we can make our choice of β.

(2.9) Let β denote orthogonal projection of F onto T followed by the inverse of the map $\alpha : K^\perp \to T$.

The transplanted connections $\overrightarrow{D}$ and $\overleftarrow{D}$ are singular because the map β is singular on Σ.

Definition 2.10. The **singular pushforward connection associated with α** is the singular connection $\overrightarrow{D}$ on F defined by

$$(2.11)\quad \overrightarrow{D} \equiv \alpha D_E \beta + D_F(1-\alpha\beta) = D_F - (D_F\alpha - \alpha D_E)\beta \;=\; D_F - D_H(\alpha)\beta.$$

The **singular pullback connection associated with α** is the singular connection $\overleftarrow{D}$ on E defined by

$$(2.12)\quad \overleftarrow{D} \;=\; \beta D_F \alpha + (1-\beta\alpha)D_E \;=\; D_E + \beta(D_F\alpha - \alpha D_E) = D_E + \beta D_H(\alpha).$$

In both cases the singular map β is "the inverse of α" precisely defined by (2.9).

Both $\overrightarrow{D}$ and $\overleftarrow{D}$ are smooth connections over the set $X - \Sigma$ which depend on all of the data (2.8). The three distinct expressions for $\overrightarrow{D}$ (or for $\overleftarrow{D}$) given in (2.11) and (2.12) will be useful in what follows.

Remark 2.13. Choices have been made in Definition 2.10. For example,

$$\alpha D_E \beta + (1-\alpha\beta)D_F = D_F + \alpha(D_E\beta - \beta D_F)$$

is an alternative singular pushforward connection on F. The choice (2.11) is preferred. It has the property that, when α is injective, parallel sections of E push forward to $\overrightarrow{D}$-parallel sections of F.

Remark 2.14. In this paper we shall only consider the following two generic cases where either $K = 0$ or $T^{\perp} = 0$.

The **injective case** is where

$$(2.15)\qquad \text{rank } E \leq \text{ rank } F \quad \text{and} \quad E \xrightarrow{\alpha} F \text{ injective on } \; X \sim \Sigma$$

In this case the singular map β is given by

$$(2.16)\qquad \beta \;\equiv\; (\alpha^*\alpha)^{-1}\alpha^*.$$

The **surjective case** is where

$$(2.17)\qquad \text{rank } E \geq \text{ rank } F \quad \text{and} \quad E \xrightarrow{\alpha} F \text{ surjective on } \; X \sim \Sigma.$$

In this case the singular map β is given by

$$(2.18)\qquad \beta \;\equiv\; \alpha^*(\alpha\alpha^*)^{-1}.$$

Here $E \xleftarrow{\alpha^*} F$ denotes the adjoint of α.

Remark 2.19. The Injective Case. This case, where

$$\text{rank } E \leq \text{ rank } F \quad \text{and} \quad \alpha^*\alpha \text{ is invertible on } \sim \Sigma \text{ so that } \beta \equiv (\alpha^*\alpha)^{-1}\alpha^*,$$

further divides into the pushforward and pullback case.

First, note that orthogonal projection $P_T : F \to T$ is given by:

$$P \equiv \alpha(\alpha^*\alpha)^{-1}\alpha^* \tag{2.20}$$

so that

$$\alpha\beta = P \qquad \beta\alpha = 1. \tag{2.21}$$

Injective Pushforward Case: On $X - \Sigma$ the pushforward connection $\overrightarrow{D}$ may be written in block form with respect to $F \equiv T \oplus T^{\perp}$. Since

$$(D_F\alpha - \alpha D_E)\beta = \begin{pmatrix} PD_FP - \alpha D_E\beta & 0 \\ (1-P)D_FP & 0 \end{pmatrix}$$

the pushforward connection $\overrightarrow{D}$ blocks as

$$\overrightarrow{D} = \begin{pmatrix} \alpha D_E\beta & PD_F(1-P) \\ 0 & (1-P)D_F(1-P) \end{pmatrix}, \tag{2.22}$$

which is upper triangular. Therefore, for any invariant polynomial ϕ,

$$\phi(\overrightarrow{D}) = \phi(\alpha D_E\beta \oplus (1-P)D_F(1-P)) \quad \text{on } X \sim \Sigma.$$

Since, when restricted to sections of T, $\alpha D_E\beta = \alpha D_E\alpha^{-1}$ is gauge equivalent to D_E, this proves that

$$\phi(\overrightarrow{D}) = \phi(D_E \oplus D_{T^\perp}) \qquad \text{on } X \sim \Sigma, \tag{2.23}$$

where $D_{T^\perp} \equiv (1-P)D_F(1-P)$ is the connection induced on $T^\perp \subset F$ by D_F.

Injective Pullback Case: Here we have that

$$\overleftarrow{D} \;=\; \beta D_F \alpha \qquad \text{on the bundle } E \text{ over } X \sim \Sigma, \tag{2.24}$$

since $1 - \beta\alpha = 0$ on $X \sim \Sigma$. Therefore,

$$(\overleftarrow{D}, E) \text{ is gauge equivalent to } (D_T,\ T) \quad \text{on } X \sim \Sigma \tag{2.25}$$

where $D_T \equiv PD_FP$ is the connection induced by D_F on the subbundle $T \subset F$. In particular,

$$\phi(\overleftarrow{D}) \;=\; \phi(D_T) \qquad \text{on } X \sim \Sigma. \tag{2.26}$$

Remark 2.27. The Surjective Case. This case, where

$$\text{rank } E \geq \text{ rank } F \quad \text{and} \quad \alpha\alpha^* \text{ is invertible on } \sim \Sigma \text{ so that } \quad \beta \equiv \alpha^*(\alpha\alpha^*)^{-1},$$

further divides into the pullback and pushforward case. First, note that orthogonal projection $P_{K^\perp} : E \to K^\perp$ is given by

$$P \;=\; \alpha^*(\alpha\alpha^*)^{-1}\alpha \tag{2.28}$$

so that

$$\alpha\beta \;=\; 1 \quad \text{and} \quad \beta\alpha \;=\; P. \tag{2.29}$$

Surjective Pullback Case: Since

$$\beta(D_F\alpha - \alpha D_E) \;=\; \begin{pmatrix} \beta D_F\alpha - PD_EP & -PD_E(1-P) \\ 0 & 0 \end{pmatrix}$$

the pullback connection $\overleftarrow{D}$ blocks with respect to $E \equiv K^\perp \oplus K$, outside the singular set Σ, as

$$\overleftarrow{D} \;=\; \begin{pmatrix} \beta D_F\alpha & 0 \\ (1-P)D_EP & (1-P)D_E(1-P) \end{pmatrix} \tag{2.30}$$

in lower triangular form. Consequently, for any invariant polynomial ϕ,

$$\phi(\overleftarrow{D}) \;=\; \phi(D_F \oplus D_K) \qquad \text{on } \; X \sim \Sigma, \tag{2.31}$$

Surjective Pushforward Case: Here we have that

$$\overrightarrow{D} \;=\; \alpha D_E \beta \qquad \text{on the bundle } \; F \text{ over } \; X \sim \Sigma, \tag{2.32}$$

since $1 - \alpha\beta = 0$ on $X \sim \Sigma$. Therefore,

$$\overrightarrow{D},\ F \;\text{ is gauge equivalent to }\; D_{K^\perp},\ K^\perp \subset E \;\text{ over }\; X \sim \Sigma \tag{2.33}$$

where $D_{K^\perp} = PD_EP$ is the connection induced on $K^\perp \subset E$ by the connection D_E. In particular,

$$\phi(\overrightarrow{D}) \;=\; \phi(D_{K^\perp}) \qquad \text{on } \; X \sim \Sigma. \tag{2.34}$$

The injective and surjective cases overlap in the equirank case

Remark 2.35. The Equirank Case. Suppose $E \xrightarrow{\alpha} F$ is a bundle map of equirank bundles so that α is invertible on $X \sim \Sigma$. Then, the singular pushforward connection $\overrightarrow{D}$ on F is given by:

(2.36)(**Pushforward**) $\qquad \overrightarrow{D} = \alpha D_E \beta \;=\; \alpha D_E \alpha^{-1} \qquad \text{on } \; X \sim \Sigma.$

That is, on $X \sim \Sigma$, the connection $\overrightarrow{D}$ on F is gauge equivalent to the connection D_E on E. In particular, if ϕ is any Ad-invariant polynomial on $\mathfrak{gl}_n$, then:

$$\phi(\overrightarrow{D}) \;=\; \phi(\alpha D_E \alpha^{-1}) \;=\; \phi(D_E) \qquad \text{on } \; X \sim \Sigma. \tag{2.37}$$

Consequently, $\phi(\overrightarrow{D})$ on $X \sim \Sigma$ automatically extends to all of X as a d-closed C^∞ form. This proves that $\phi(\overrightarrow{D}_0) = \phi(D_E)$ on X, and so Condition B is satisfied. Now, if furthermore Condition C is satisfied, then the Chern current $\phi((\overrightarrow{D}))$ on X is given by

$$\phi((\overrightarrow{D})) \;=\; \phi(D_E) + \mathrm{Res}_\phi(\overrightarrow{D})[\Sigma]. \tag{2.38}$$

and formula (1.13) becomes

(2.39) $$\phi(D_F) - \phi(D_E) - \mathrm{Res}_\phi(\overrightarrow{D})[\Sigma] \;=\; dT.$$

Similarly, the singular pullback connection is defined by

(2.40)(**Pullback**) $$\overleftarrow{D} \;=\; \alpha^{-1} D_F \alpha \qquad \text{on } X \sim \Sigma,$$

and we see that $\phi(\overleftarrow{D}_0) \;=\; \phi(D_F)$ on X. In particular, Condition B is again automatic. Furthermore, assuming Condition C, we have

(2.41) $$\phi(\!(\overleftarrow{D})\!) \;=\; \phi(D_F) + \mathrm{Res}_\phi(\overleftarrow{D})[\Sigma],$$

and formula (1.13) becomes

(2.42) $$\phi(D_E) - \phi(D_F) - \mathrm{Res}_\phi(\overleftarrow{D})[\Sigma] \;=\; dT.$$

3. The Universal Case.

In this case we will discuss the pullback and pushforward connections associated with the "universal bundle map". In particular, it will be shown that in this case Condition B is always valid, i.e., the limiting characteristic form on $\sim \Sigma$ always extends across the singular set as a d-closed L^1_{loc}-form. Indeed we will show that there is a certain blow up of $\mathrm{Hom}(E, F)$ along Σ where (the lift of) the limiting form always extends smoothly.

Let E and F be complex vector bundles over X with connections D_E and D_F as above. We consider the vector bundle

$$\pi : \mathrm{Hom}(E, F) \longrightarrow X$$

and the pullbacks

$$\mathbf{E} = \pi^* E \quad \text{and} \quad \mathbf{F} = \pi^* F$$

with their pullback connections

$$\mathbf{D_E} \quad \text{and} \quad \mathbf{D_F}$$

respectively. Over $\mathrm{Hom}(E,F)$ there is a **tautological** or **universal homomorphism**

$$\boldsymbol{\alpha} : \mathbf{E} \longrightarrow \mathbf{F} \tag{3.1}$$

which at a point $\alpha \in \mathrm{Hom}(E_x, F_x)$ above $x \in X$ is simply defined to be α. Note that given a smooth bundle map $\alpha : E \longrightarrow F$, i.e., a cross-section of $\mathrm{Hom}(E,F)$, we have that

$$\alpha^*\mathbf{E} = E \quad \text{and} \quad \alpha^*\mathbf{F} = F \tag{3.2}$$

as bundles with connection, and that

$$\alpha^*(\boldsymbol{\alpha}) \;=\; \alpha. \tag{3.3}$$

Thus, every homomorphism is a pullback of the universal one $\boldsymbol{\alpha}$.

Suppose now that there are metrics $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_F$ given on E and F (not necessarily compatible with the connections), and pull them back to $\mathbf{E}$ and $\mathbf{F}$. Then there is also a **universal adjoint homomorphism** $\boldsymbol{\alpha}^* : \mathbf{F} \longrightarrow \mathbf{E}$.

Definition 3.4. Using the universal homomorphism $\boldsymbol{\alpha}$ we define, as in Section 2, the **universal pullback connection** $\overleftarrow{\mathbf{D}}_{\mathbf{E}}(\boldsymbol{\alpha})$ on $\mathbf{E}$ and the **universal pushforward connection** $\overrightarrow{\mathbf{D}}_{\mathbf{F}}(\boldsymbol{\alpha})$ on $\mathbf{F}$.

The Injective Case. Assume that $\operatorname{rank} E \le \operatorname{rank} F$. The singular set $\boldsymbol{\Sigma} \subset \mathrm{Hom}(E,F)$ of the universal homomorphism $\boldsymbol{\alpha} : \mathbf{E} \to \mathbf{F}$ is the complement of $\mathrm{Hom}^\times(E,F) = \{\alpha \in \mathrm{Hom}(E,F) : \alpha \text{ is injective}\}$.

A partial desingularization of $\boldsymbol{\Sigma} \subset \mathrm{Hom}(E,F)$ can be used to verify Condition B. Let

(3.5)

$p_F : \mathrm{Hom}^\times(E,F) \longrightarrow G_p(F)$ be defined by $p_F(\alpha) \equiv \operatorname{im}\alpha$, where $p \equiv \operatorname{rank} E$.

We blow up $\mathrm{Hom}(E,F)$ along $\boldsymbol{\Sigma}$ by setting $\widetilde{\mathrm{Hom}}(E,F)$ equal to the closure of the graph of p_F in $\mathrm{Hom}(E,F) \times G_p(F)$, namely

$$\widetilde{\mathrm{Hom}}(E,F) \;\equiv\; \{(\alpha,P) \in \mathrm{Hom}(E,F) \times G_p(F) : \operatorname{im}\alpha \subset P\}. \tag{3.6}$$

Note that $\widetilde{\mathrm{Hom}}(E,F)$ is a **smooth submanifold** of $\mathrm{Hom}(E,F) \times G_p(F)$, since projection of $\widetilde{\mathrm{Hom}}(E,F)$ onto the second factor $G_p(F)$ is a smooth fiber bundle whose fiber above $P \in G_p(F)$ is the set of all $\alpha \in \mathrm{Hom}(E,F)$ with $\mathrm{im}\,\alpha \subset P$, i.e. the fiber is $\mathrm{Hom}(E,P)$ (cf Lemma 9.9). The projection onto the first factor

$$\rho : \widetilde{\mathrm{Hom}}(E,F) \longrightarrow \mathrm{Hom}(E,F) \tag{3.7}$$

is, of course, a proper map. Also, ρ is a diffeomorphism over $\mathrm{Hom}^\times(E,F)$. Let $\widetilde{\Sigma}$ denote $\rho^{-1}(\Sigma)$. Above a singular point $\alpha \in \Sigma$, the fiber $\rho^{-1}(\alpha) \subset \widetilde{\Sigma}$ consist of all pairs (α, P) with $\mathrm{im}\,\alpha \subset P$.

The universal homomorphism $\mathbf{E} \xrightarrow{\boldsymbol{\alpha}} \mathbf{F}$ can be lifted to $\widetilde{\mathrm{Hom}}(E,F)$ from $\mathrm{Hom}(E,F)$. The lifted universal homomorphism remains singular with new singular set $\widetilde{\Sigma}$ instead of Σ. However, the target bundle T over $\widetilde{\mathrm{Hom}}(E,F) - \widetilde{\Sigma}$ is effectively desingularized. It is just the restriction to $\widetilde{\mathrm{Hom}}(E,F)$ of the pullback to $\mathrm{Hom}(E,F) \times G_p(F)$ of the canonical p-plane bundle over $G_p(F)$. For simplicity of notation we let T denote the canonical bundle on $G_p(F)$ as well as it's pullback to $\mathrm{Hom}(E,F) \times G_p(F)$, as well as it's restriction to $\widetilde{\mathrm{Hom}}(E,F)$.

Proposition 3.8. Injective Case. *Suppose* $\mathrm{rank}\, E \le \mathrm{rank}\, F$.

The lift of $\phi(\overrightarrow{\mathbf{D}})\,|_{\mathrm{Hom}(E,F)\sim\Sigma}$ *to* $\widetilde{\mathrm{Hom}}(E,F) \sim \widetilde{\Sigma}$ *extends across* $\widetilde{\Sigma}$ *to all of* $\widetilde{\mathrm{Hom}}(E,F)$ *as the* d*-closed smooth form* $\phi(\mathbf{D_E} \oplus D_{T^\perp})$.

The lift of $\phi(\overleftarrow{\mathbf{D}})\,|_{\mathrm{Hom}(E,F)\sim\Sigma}$ *to* $\widetilde{\mathrm{Hom}}(E,F) \sim \widetilde{\Sigma}$ *extends across* $\widetilde{\Sigma}$ *to all of* $\widetilde{\mathrm{Hom}}(E,F)$ *as the* d*-closed smooth form* $\phi(D_T)$.

Proof. Over $\mathrm{Hom}(E,F) \sim \Sigma$, we have that $\phi(\overrightarrow{D}) = \phi(D_E \oplus D_{T^\perp})$, by (2.23). Lifting to $\widetilde{\mathrm{Hom}}(E,F) \sim \widetilde{\Sigma}$, the bundles $T(\equiv p_F^*(T))$ and $T^\perp(\equiv p_F^*(T^\perp))$ extend as smooth bundles over all of $\widetilde{\mathrm{Hom}}(E,F)$. In addition,

$$T \oplus T^\perp = \mathbf{F} \qquad \text{over } G_p(F), \tag{3.9}$$

so that the connections D_T and $D_{T^\perp}$ induced on T and $T^\perp$ by the $\mathbf{D_F}$ connection are smooth on all of $G_p(F)$ and hence on all of $\widetilde{\mathrm{Hom}}(E,F) \subset \mathrm{Hom}(E,F) \times G_p(F)$. Thus $\phi(\mathbf{D_E} \oplus D_{T^\perp})$ is a d-closed C^∞ form on all of $\widetilde{\mathrm{Hom}}(E,F)$.

Similarly, by (2.26),

$$\phi(\overleftarrow{D}) = \phi(D_T) \qquad \text{on } \operatorname{Hom}(E,F) \sim \Sigma,$$

and when pulled back to $\widetilde{\operatorname{Hom}}(E,F) \sim \widetilde{\Sigma}$ this form extends to the p_F pullback of the smooth d-closed form $\phi(D_T)$ over $G_p(F)$. $\square$

Theorem 3.10. Injective Case. *The smooth forms $\phi(\overrightarrow{D})$ and $\phi(\overleftarrow{D})$, on $\operatorname{Hom}(E,F) \sim \Sigma$ extend by zero to be d-closed L^1_{loc} forms on all of $\operatorname{Hom}(E,F)$. Namely,*

$$\phi(\overrightarrow{D}_0) = \rho_*\phi(D_E \oplus D_{T^\perp}) \quad \textit{and} \quad \phi(\overleftarrow{D}_0) = \rho_*\phi(D_T) \tag{3.11}$$

are the L^1_{loc}-parts of the ϕ-Chern currents.

Proof. This follows directly from Proposition 3.8 and the following useful fact.

Proposition 3.12. *Let $f : M \longrightarrow N$ be a smooth proper map between oriented manifolds where $m = \dim M \geq \dim N = n$. Assume that the critical set $C = \{x \in M : \operatorname{rank}(df) < n\}$ has measure zero. Fix a differential p-form ψ on M with $L^1_{\text{loc}}(M)$ coefficients, and let $\int_f \psi$ denote the L^1_{loc} $(p-m+n)$-form on $N - f(C)$ obtained by integration over the fibre. Then the coefficients of $\int_f \psi$ are L^1_{loc} across $f(C)$, and thereby $\int_f \psi$ determines a current $\overline{\int_f \psi}$ on all of N. Furthermore, $\overline{\int_f \psi} = f_*(\psi)$, the current push forward of ψ by the map f. Hence, in particular, we have that*

$$d\left(\overline{\int_f \psi}\right) = \overline{\int_f d\psi}.$$

Proof. We assume without loss of generality that N is an open subset of $\mathbf{R}^n$, and that ψ has globally L^1_{loc}-coefficients. Fix $\epsilon > 0$ and let χ_ϵ denote the characteristic function of the open subset $N_\epsilon \stackrel{\text{def}}{=} \{x \in N : \operatorname{dist}(x, f(C)) > \epsilon\}$. Set $\widetilde{\chi}_\epsilon = \chi_\epsilon \circ f$. Then since f is a smooth bundle map over N_ϵ, we have

$$f_*(\widetilde{\chi}_\epsilon \psi) = \chi_\epsilon \int_f \psi \qquad \text{for all } \epsilon > 0.$$

Now since ψ is a current of finite mass, so is $f_*(\psi)$. In particular, the coefficients of the smooth form $\int_f \psi$ on $N - f(C)$ are L^1_{loc} across $f(C)$. Now, since C is closed and has measure zero in M, we know that $\lim_{\epsilon\to 0} \widetilde{\chi}_\epsilon \psi = \psi$ in the mass norm. It follows by continuity that

$$\overline{\int_f \psi} = \lim_{\epsilon\to 0} \chi_\epsilon \int_f \psi = \lim_{\epsilon\to 0} f_*(\widetilde{\chi}_\epsilon \psi) = f_*(\psi). \quad \square$$

The Surjective Case. Assume that $\operatorname{rank} E \geq \operatorname{rank} F$. The singular set $\mathbf{\Sigma} \subset \operatorname{Hom}(E,F)$ of the universal homomorphism $\boldsymbol{\alpha} : \mathbf{E} \to \mathbf{F}$ is the complement of $\operatorname{Hom}^\times(E,F) = \{\alpha \in \operatorname{Hom}(E,F) : \alpha \text{ is surjective}\}$. Let

$$(3.13) \qquad p_E : \operatorname{Hom}^\times(E,F) \longrightarrow G_{p-q}(E) \quad \text{be defined by} \quad p_E(\alpha) = \ker \alpha.$$

We blow up $\operatorname{Hom}(E,F)$ along $\mathbf{\Sigma}$ by setting $\widetilde{\operatorname{Hom}}(E,F)$ equal to the closure of the graph of p_E in $\operatorname{Hom}(E,F) \times G_{p-q}(E)$, namely,

$$(3.14) \qquad \widetilde{\operatorname{Hom}}(E,F) \equiv \{(\alpha, P) \in \operatorname{Hom}(E,F) \times G_{p-q}(E) : \ker \alpha \supset P\}.$$

Projection ρ of $\widetilde{\operatorname{Hom}}(E,F)$ onto the first factor $\operatorname{Hom}(E,F)$ is a proper map which is a diffeomorphism when $\widetilde{\mathbf{\Sigma}} \equiv \rho^{-1}(\mathbf{\Sigma})$ and $\mathbf{\Sigma}$ are excluded.

Lifting the universal homomorphism and the other data from $\operatorname{Hom}(E,F)$ to $\widetilde{\operatorname{Hom}}(E,F)$ has the advantage that the kernel bundle $K \equiv \ker \alpha \subset \mathbf{E}$ defined on $\widetilde{\operatorname{Hom}}(E,F) \sim \widetilde{\mathbf{\Sigma}}$ extends smoothly as a bundle to all of the submanifold $\widetilde{\operatorname{Hom}}(E,F)$. In fact, this extension of K is just the canonical bundle over $G_{p-q}(E)$ pulled back to $\operatorname{Hom}(E,F) \times G_{p-q}(E)$ and then restricted to $\widetilde{\operatorname{Hom}}(E,F)$.

Proposition 3.15. Surjective Case. *Suppose* $\operatorname{rank} E \geq \operatorname{rank} F$.

The lift of $\phi(\overrightarrow{\mathbf{D}})\,|_{\operatorname{Hom}(E,F)\sim\Sigma}$ *to* $\widetilde{\operatorname{Hom}}(E,F) \sim \widetilde{\mathbf{\Sigma}}$ *extends across* $\widetilde{\mathbf{\Sigma}}$ *to all of* $\widetilde{\operatorname{Hom}}(E,F)$ *as the d-closed smooth form* $\phi(D_{K^\perp})$.

The lift of $\phi(\overleftarrow{\mathbf{D}})\,|_{\operatorname{Hom}(E,F)\sim\mathbf{\Sigma}}$ *to* $\widetilde{\operatorname{Hom}}(E,F) \sim \widetilde{\mathbf{\Sigma}}$ *extends across* $\widetilde{\mathbf{\Sigma}}$ *to all of* $\widetilde{\operatorname{Hom}}(E,F)$ *as the d-closed smooth form* $\phi(\mathbf{D_F} \oplus D_K)$.

The proof of this proposition and the next theorem are similar to the corresponding proofs in the injective case and hence are omitted.

Theorem 3.16. Surjective Case. *The smooth forms* $\phi(\overrightarrow{D})\,\big|_{\mathrm{Hom}(E,F)\sim\Sigma}$ *and* $\phi(\overleftarrow{D})\,\big|_{\mathrm{Hom}(E,F)\sim\Sigma}$ *extend by zero to be d-closed* L^1_{loc} *forms on all of* $Hom(E,F)$. *Namely,*

$$\phi(\overrightarrow{D}_0) = \rho_*\phi(D_{K^\perp}) \quad \text{and} \quad \phi(\overleftarrow{D}_0) = \rho_*\phi(D_F \oplus D_K) \tag{3.17}$$

are the L^1_{loc}*-parts of the* ϕ*-Chern currents.*

4. Smoothing Singular Connections.

The key to smoothing a singular connection D is the Lemma 2.2 which says that both of the formulas (2.3) and (2.4) define a connection for any pair of bundle maps α and β. Because of this Lemma it suffices to smooth the bundle map β defined by (2.9). This is most naturally accomplished with an approximation to the identity map.

Definition 4.1. By an **approximate one** we mean a function $\chi : [0,\infty] \to [0,1]$ which is C^∞ on the entire closed interval $[0,\infty]$ and satisfies

$$\chi(0) = 0, \ \chi(\infty) = 1, \ \text{ and } \ \chi' \geqq 0.$$

Note that $\chi_s(t) \equiv \chi\left(\frac{t}{s}\right)$ approximates 1 as $s \to 0$. The smooth function ρ with domain $[0,\infty]$ defined by $\rho(t) \equiv \frac{1}{t}\chi(t)$ will be called an **approximate reciprocal**. It has the property that $\frac{1}{s}\rho\left(\frac{t}{s}\right) = \frac{1}{t}\chi\left(\frac{t}{s}\right)$ approximates $\frac{1}{t}$ as $s \to 0$.

Given a bundle map α, we define the family β_s, $s > 0$ of **approximations to the "inverse of α" based on χ** by setting

$$\beta_s \;=\; \rho\left(\frac{\alpha^*\alpha}{s^2}\right)\frac{\alpha^*}{s^2} \;=\; \frac{\alpha^*}{s^2}\rho\left(\frac{\alpha\alpha^*}{s^2}\right). \tag{4.2}$$

Note that, since $T^\perp \equiv \ker\alpha^*$

$$\beta_s \ \text{ vanishes on } \ T^\perp, \tag{4.3}$$

while on $y = \alpha(x) \in T$,

$$(4.4) \qquad \beta_s y \;=\; \chi\left(\tfrac{\alpha^*\alpha}{s^2}\right) x$$

which vanishes if $x \in K \equiv \ker \alpha$ and converges to x if $x \in K^{\perp}$.

Also note that

$$(4.5) \qquad \alpha\beta_s \;=\; \chi\left(\tfrac{\alpha\alpha^*}{s^2}\right) \qquad \beta_s\alpha \;=\; \chi\left(\tfrac{\alpha^*\alpha}{s^2}\right).$$

In the injective case $\alpha^*\alpha$ is invertible on $X \sim \Sigma$ so that $\beta = (\alpha^*\alpha)^{-1}\alpha^*$. Therefore

$$(4.6) \qquad \beta_s \;=\; \chi\left(\tfrac{\alpha^*\alpha}{s^2}\right)\beta \;=\; \chi\left(\tfrac{\alpha^*\alpha}{s^2}\right)(\alpha^*\alpha)^{-1}\alpha^*.$$

In the surjective case $\alpha\alpha^*$ is invertible on $X \sim \Sigma$ so that $\beta = \alpha^*(\alpha^*\alpha)^{-1}$. Therefore

$$(4.7) \qquad \beta_s \;=\; \beta\chi\left(\tfrac{\alpha\alpha^*}{s^2}\right) \;=\; \alpha^*(\alpha\alpha^*)^{-1}\chi\left(\tfrac{\alpha\alpha^*}{s^2}\right).$$

The smooth behavior of the family of approximations β_s to β can be summarized as follows.

Outside the Singular Set: The family of bundle maps β_s is smooth for $0 \le s \le +\infty$ on $X \sim \Sigma$ with $\beta_0 = \beta$ and $\beta_\infty = 0$. In particular, convergence of β_s to $\beta_0 = \beta$ is in the C^∞ topology on $X \sim \Sigma$.

Across the Singular Set: The family of bundle maps β_s is smooth for $0 < s \le +\infty$ on X with $\beta_\infty = 0$. In particular, convergence of β_s to $\beta_\infty = 0$ is in the C^∞ topology on all of X.

Definition 4.8. The Pushforward Family $\overrightarrow{\mathbf{D}}_{\mathbf{s}}$. The singular pushforward connection $\overrightarrow{D}$ (under α), namely

$$(4.9) \qquad \overrightarrow{D} \equiv \alpha D_E\beta + D_F(1-\alpha\beta) = D_F - (D_F\alpha - \alpha D_E)\beta = D_F - D_H(\alpha)\beta$$

is smoothed by the following smooth family $\overrightarrow{D}_s$, of smooth connections on F, over the entire manifold X including $\Sigma \equiv \operatorname{sing}\alpha$, $(0 < s \le +\infty)$:

$$(4.10) \quad \overrightarrow{D}_s = \alpha D_E\beta_s + D_F(1-\alpha\beta_s) = D_F - (D_F\alpha - \alpha D_E)\beta_s = D_F - D_H(\alpha)\beta_s.$$

Note that

$$(4.11) \qquad \overrightarrow{D}_\infty \;=\; D_F,$$

since $\beta_\infty = 0$, at $s = +\infty$.

The Pullback Family $\overleftarrow{\mathbf{D}}_{\mathbf{s}}$. The singular pull back connection $\overleftarrow{D}$ (under α), namely

$$(4.12)\quad \overleftarrow{D} = \beta D_F\alpha + (1-\beta\alpha)D_E = D_E + \beta(D_F\alpha - \alpha D_E) = D_E + \beta D_H(\alpha),$$

is smoothed by the following smooth family $\overleftarrow{D}_s$ of smooth connections on E, over the entire manifold X including $\Sigma \equiv \operatorname{sing}\alpha$, $(0 < s \leq +\infty)$:

$$(4.13)\quad \overleftarrow{D}_s = \beta_s D_F\alpha + (1-\beta_s\alpha)D_E = D_E + \beta_s(D_F\alpha - \alpha D_E) = D_E + \beta_s D_H(\alpha).$$

Note that

$$(4.14)\qquad \overleftarrow{D}_\infty \ = \ D_E,$$

since $\beta_\infty = 0$, at $s = +\infty$.

We shall adopt the following terminology. The approximate 1, denoted χ, provides an **approximation mode** for the singular connection $\overrightarrow{D}$ or $\overleftarrow{D}$ **based on χ**. There are several specific approximation modes, or examples of an approximate 1 which have specific advantages. By far the most important is algebraic in nature with geometric interpretation which is provided later.

Example 4.15. Algebraic Approximation Mode. Let

$$(4.16)\qquad \chi(t) \ \equiv \ \frac{t}{1+t} \qquad t \in [0, +\infty].$$

Then

$$(4.17)\qquad \chi\left(\frac{\alpha^*\alpha}{s^2}\right) \ = \ \alpha^*\alpha(\alpha^*\alpha + s^2)^{-1} \ = \ (\alpha^*\alpha + s^2)^{-1}\alpha^*\alpha,$$

and

$$(4.18)\qquad \beta_s \ \equiv \ (\alpha^*\alpha + s^2)^{-1}\alpha^* \ = \ \alpha^*(\alpha\alpha^* + s^2)^{-1}.$$

Therefore, we have the following.

Pushforward in Algebraic Approximation Mode.
(4.19)
$$\overrightarrow{D}_s = D_F + (\alpha D_E - D_F\alpha)\,\alpha^* \left(\alpha\alpha^* + s^2\right)^{-1} = \left(s^2 D_F + \alpha D_E \alpha^*\right)\left(\alpha\alpha^* + s^2\right)^{-1}.$$

Pullback in Algebraic Approximation Mode.
(4.20)
$$\overleftarrow{D}_s = D_E + \left(\alpha^*\alpha + s^2\right)^{-1} \alpha^* \left(D_F\alpha - \alpha D_E\right) = \left(\alpha^*\alpha + s^2\right)^{-1}\left(s^2 D_E + \alpha^* D_F \alpha\right).$$

The special formula $t\chi'(t) = \chi(t)(1-\chi(t))$, or

$$\chi'\left(\tfrac{\alpha^*\alpha}{s^2}\right)\tfrac{\alpha^*\alpha}{s^2} \;=\; \chi\left(\tfrac{\alpha^*\alpha}{s^2}\right)\left(1-\chi\left(\tfrac{\alpha^*\alpha}{s^2}\right)\right), \tag{4.21}$$

which is valid in the algebraic approximation mode, is particularly useful.

Example 4.22. Transcendental Approximation Mode. Let

$$\chi(t) \;\equiv\; 1 - e^{-t} \qquad t \in [0,\infty]. \tag{4.23}$$

This approximate 1 does even better at approximating 1 as $t \to +\infty$, and roughly speaking allows certain Chern currents to be approximated by Gaussian distributions.

Example 4.24. Compact Approximation Mode. Assume $\chi \;:\; [0,\infty] \to [0,1]$ is C^∞ and

$$\chi(t) \;\equiv 1 \;\text{ for }\; t \ge t_0 > 0. \tag{4.25}$$

This approximation mode is necessary for obtaining Thom forms. Actually, $1-\chi$ is the cut off function which provides the compact support of the Thom form.

Remark 4.26. Special Compatibility Assumptions. Consider the case where $E \xrightarrow{\alpha} F$ is generically an injective isometry, i.e. $\alpha^*\alpha = 1$. Note that if $\alpha^*\alpha = 1$ on a dense subset, it holds everywhere, and we have $\Sigma = \emptyset$. Then in the algebraic approximation mode,

$$\overrightarrow{D}_s \;=\; D_F + \tfrac{1}{1+s^2}\left(\alpha D_E \alpha^* - D_F \alpha\alpha^*\right),$$

since $\beta_s = \alpha^*(\alpha\alpha^* + s^2)^{-1} = (\alpha^*\alpha + s^2)^{-1}\alpha^* = \frac{1}{1+s^2}\alpha^*$. Note that $P = \alpha(\alpha^*\alpha)^{-1}\alpha^* = \alpha\alpha^* : F \to T \equiv \operatorname{im}\alpha$ is orthogonal projection. In addition, assume that D_E and D_F are compatible, i.e., assume that

$$\alpha D_E \alpha^* = P D_F P.$$

Then

$$\overrightarrow{D}_s = D_F - \tfrac{1}{1+s^2}(1-P)D_F P, \qquad 0 \le s \le +\infty, \tag{4.27}$$

cf. [BoC] page 87.

Remark 4.28. Pushforward and Pullback are Dual. Since a connection D_V on a vector bundle V induces a connection D_{V^*} on the dual bundle V^*, the original data

$$\begin{array}{ccc} D_E & & D_F \\ E & \xrightarrow{\alpha} & F \\ \langle\ ,\ \rangle_E & & \langle\ ,\ \rangle_F \end{array} \tag{4.29}$$

induces dual data.

$$\begin{array}{ccc} D_{F^*} & & D_{E^*} \\ F^* & \xrightarrow{\tilde\alpha} & E^* \\ \langle\ ,\ \rangle_F & & \langle\ ,\ \rangle_E. \end{array} \tag{4.30}$$

Here $\tilde\alpha$ denotes the metric-independent dual map as opposed to the metric-dependent adjoint map α^*.

Consider the induced families of connections on F and on F^* given by

(4.31)(**pushforward**) $$\overrightarrow{D}_{s,F} = D_F - (D_F\alpha - \alpha D_E)\beta_s$$

and

(4.32)(**pullback**) $$\overleftarrow{D}_{s,F^*} = D_{F^*} + \tilde\beta_s(\tilde\alpha D_{F^*} - D_{E^*}\tilde\alpha)$$

where $\tilde\beta_s \equiv \rho\left(\frac{\tilde\alpha^*\tilde\alpha}{s^2}\right)\frac{\tilde\alpha^*}{s^2}$. Since this is dual to β_s, the two connections are dual, i.e.,

$$\left(\overrightarrow{D}_{s,F}\right)^* = \overleftarrow{D}_{s,F^*}. \tag{4.33}$$

5. The Gauge and the Transgression.

This section collects together the explicit local formulas for the gauge potential ω_s and the curvature matrix Ω_s associated with the two approximating families of connections $\overrightarrow{D}_s$ and $\overleftarrow{D}_s$.

Suppose e is a local frame for E and f is a local frame for F, both expressed as columns. Then the equations $D_E e = \omega_E e$ and $D_F f = \omega_F f$ define gauge potentials ω_E for D_E and ω_F for D_F, respectively. The map α determines a matrix a with respect to the given frames; that is $\alpha e = af$ defines a matrix $a \equiv (a_i^j)$ by $\alpha e_i \equiv \sum_j a_i^j f_j$. Similarly, $\beta_s f = b_s e$ defines the matrix b_s for $0 < s \leq \infty$, and $\beta f = be$ defines the matrix b. Note that ba is the matrix form of the map $\alpha\beta$. Finally, let Da denote the matrix form of $D(\alpha)$, i.e., $(D(\alpha))(e) \equiv (Da)f$.

Lemma 5.1. Pushforward. *The singular connection $\overrightarrow{D}$ has gauge potential $\overrightarrow{\omega}$ given by:*

$$(5.2) \quad \overrightarrow{\omega} = b\,\omega_E a + (1 - ba)\omega_F - bda = \omega_F - b(da + a\omega_F - \omega_E a) = \omega_F - bDa$$

while the approximating smooth connection $\overrightarrow{D}_s$ has gauge potential

$$(5.3) \quad \overrightarrow{\omega}_s = b_s\omega_E a + (1 - b_s a)\omega_F - b_s da = \omega_F - b_s(da + a\omega_F - \omega_E a) = \omega_F - b_s Da.$$

Proof. Since $D(\alpha) \equiv D_F\alpha - \alpha D_E \in \Gamma(\Lambda^1 T^* \otimes \mathrm{Hom}(E, F))$, it suffices to compute that

$$(D_F\alpha - \alpha D_E)(e) = D_F(af) - \alpha(\omega_E e) = daf + a\omega_F f - \omega_E af. \quad \square$$

Lemma 5.4. Pullback. *The singular connection $\overleftarrow{D}$ has gauge potential $\overleftarrow{\omega}$ given by*

$$(5.5) \quad \overleftarrow{\omega} = a\omega_F b + \omega_E(1 - ab) + dab = \omega_E + (a\omega_F - \omega_E a + da)\,b = \omega_E + (Da)b$$

while the approximating smooth connection $\overleftarrow{D}_s$ has gauge potential

$$(5.6) \quad \overleftarrow{\omega}_s = a\omega_F b_s + \omega_E(1 - ab_s) + dab_s = \omega_E + (a\omega_F - \omega_E a + da)\,b_s = \omega_E + (Da)b_s.$$

Proof. The argument is analogous to the one above. □

Remark 5.7. In the injective case ($\operatorname{rank} E \le \operatorname{rank} F$ and aa^* invertible on $X \sim \Sigma$)

$$b_s = a^*(aa^*)^{-1}\chi\left(\tfrac{aa^*}{s^2}\right). \tag{5.8}$$

In the surjective case ($\operatorname{rank} E \ge \operatorname{rank} F$ and a^*a invertible on $X \sim \Sigma$)

$$b_s = \chi\left(\tfrac{a^*a}{s^2}\right)(a^*a)^{-1}a^*. \tag{5.9}$$

In the algebraic approximation mode

$$b_s = a^*(aa^* + s^2)^{-1} = (a^*a + s^2)^{-1}a^* \tag{5.10}$$

in both the injective and surjective cases. For the proofs see (4.6) and (4.7).

To summarize the notation, we let

$$\begin{array}{llll} \alpha e = af & \beta f = be & \beta_s f = b_s e & (D\alpha)(e) = (Da)f \\ \alpha^* f = a^* e & \beta^* e = b^* f & \beta_s^* = b_s^* f & (D\alpha^*)(f) = (Da^*)e. \end{array} \tag{5.11}$$

define the matricies a, a^*, b, b^*, b_s, b_s^*, and Da, Da^*. We also let the matrix corresponding to the bundle map $P_s = \alpha\beta_s$ be denoted by P_s, the same symbol as the map, and similarly for $P \equiv \alpha\beta$.

$$P_s f = P_s f \qquad Pf = Pf. \tag{5.12}$$

Our hermitian inner products define matrices

$$h_E \equiv (\langle e_i, e_j\rangle_E) \qquad h_F \equiv (\langle f_i, f_j\rangle_F). \tag{5.13}$$

The following relationships are immediate. (We assume that $\operatorname{rank} E \le \operatorname{rank} F$ in the formulas containing $(aa^*)^{-1}$).

$$a^* = h_F \bar{a}^t h_E^{-1} \qquad b^* = h_E \bar{b}^t h_F^{-1} \qquad b_s^* = h_E \bar{b}_s^t h_F^{-1} \tag{5.14}$$

$$b = a^*(aa^*)^{-1} \qquad b_s = a^*(aa^*)^{-1}\chi\left(\tfrac{aa^*}{s^2}\right) \tag{5.15}$$

$$P = ba = a^*(aa^*)^{-1}a \qquad P_s = b_s a = a^*(aa^*)^{-1}\chi\left(\tfrac{aa^*}{s^2}\right)a \tag{5.16}$$

$$Da = da + a\omega_F - \omega_E a \qquad Da^* = da^* + a^*\omega_E - \omega_F a^*. \tag{5.17}$$

The notation

$$\chi_s \equiv \chi\left(\tfrac{aa^*}{s^2}\right) = ab_s \tag{5.18}$$

will also be used.

Let $a^* \frac{\partial}{\partial a^*}$ denote the vector field that replaces da^* by a^*. This is essentially the $(0,1)$ Euler vector field in Remark 5.24 below. Let $\llcorner$ denote the operation of contraction.

Lemma 5.19. *In both the injective and the surjective cases, one has that*

$$\tfrac{\partial}{\partial s} b_s ds = -a^* \frac{\partial}{\partial a^*} \llcorner\, db_s \tfrac{ds^2}{s^2}. \tag{5.20}$$

Proof. Note that, in the injective case, $a^* \frac{\partial}{\partial a^*} \llcorner\, d\left(a^*(aa^*)^{-1}\right)$

$$= a^* \tfrac{\partial}{\partial a^*} \llcorner \left(da^*(aa^*)^{-1} - a^*(aa^*)^{-1}(daa^* + ada^*)(aa^*)^{-1}\right) = 0.$$

Using power series to differentiate $\chi\left(\frac{aa^*}{s^2}\right)$ in $b_s = a^*(aa^*)^{-1}\chi\left(\frac{aa^*}{s^2}\right)$ yields (5.20). The proof in the surjective case is similar. □

Proposition 5.21. Pushforward. *The transgression integrand can be expressed as*

$$\phi\left(\tfrac{\partial}{\partial s}\omega_s ds \;;\; \overrightarrow{\Omega}_s\right) = -a^* \tfrac{\partial}{\partial a^*} \llcorner\, \phi(\overrightarrow{\Omega}_s) \tfrac{ds^2}{s^2}, \tag{5.22}$$

and hence the transgression form T_s is given by

$$T_s = -\int_s^\infty a^* \tfrac{\partial}{\partial a^*} \llcorner\, \phi(\overrightarrow{\Omega}_s) \tfrac{ds^2}{s^2}. \tag{5.23}$$

Proof. Since

$$a^* \tfrac{\partial}{\partial a^*} \llcorner\, \phi(\overrightarrow{\Omega}_s) = \phi\left(a^* \tfrac{\partial}{\partial a^*} \llcorner\, \overrightarrow{\Omega}_s \;;\; \overrightarrow{\Omega}_s\right)$$

it suffices to prove that

$$-a^* \tfrac{\partial}{\partial a^*} \llcorner\, \overrightarrow{\Omega}_s \tfrac{ds^2}{s^2} = \tfrac{\partial}{\partial s} \overrightarrow{\omega}_s ds.$$

Note that $a^*\frac{\partial}{\partial a^*} \llcorner Da = 0$ and $a^*\frac{\partial}{\partial a^*} \llcorner d(Da) = 0$. Since $\overrightarrow{\omega}_s = \omega_F - b_s Da$ it follows that

$$-a^*\frac{\partial}{\partial a^*} \llcorner \overrightarrow{\Omega}_s \frac{ds^2}{s^2} = \left(a^*\frac{\partial}{\partial a^*} \llcorner db_s\right)(Da)\frac{ds^2}{s^2}$$

which equals

$$-\frac{\partial}{\partial s} b_s(Da)ds = -\frac{\partial}{\partial s}\overrightarrow{\omega}_s ds.$$

by Lemma 5.19. □

Remark 5.24. Euler vector fields. Let $\pi : V \longrightarrow X$ be a smooth complex vector bundle with almost complex structure J. Then there are two real Euler vector fields on the total space of V, ϵ and $J\epsilon$. The field ϵ exists on any real bundle and generates the flow $\varphi_t(v) = e^t v$ given by scalar multiplication by e^t. The field $J\epsilon$ corresponds to the flow $\psi_t(v) = e^{it}v = \cos(t)v + \sin(t)Jv$. These give rise to the complex Euler vector fields:

$$\epsilon_{1,0} = \tfrac{1}{2}(\epsilon - iJ\epsilon) \qquad \epsilon_{0,1} = \tfrac{1}{2}(\epsilon + iJ\epsilon).$$

Suppose that $(f_1(x), \dots, f_n(x))$ is a local framing of V defined in a coordinate patch U on X. Then we have a diffeomorphism $\pi^{-1}(U) \xrightarrow{\approx} U \times \mathbf{C}^n$ given by $\sum z_j f_j(x) \mapsto (x; z_1, \dots, z_n)$. In these coordinates on V we have that

$$\epsilon_{1,0} = z\frac{\partial}{\partial z} \quad \text{and} \quad \epsilon_{0,1} = \bar{z}\frac{\partial}{\partial \bar{z}}.$$

Notice that $\epsilon_{1,0} \llcorner (dz_j) = z_j$ and $\epsilon_{1,0} \llcorner (d\bar{z}_j) = 0$.

Consider now a hermitian inner product h on V. Then h corresponds to a complex antilinear isomorphism

$$\mathbf{h} : V \xrightarrow{\approx} V^*.$$

Since $\mathbf{h}$ commutes with scalar multiplication by real numbers and since $\mathbf{h} \circ J = -J \circ \mathbf{h}$, we see that under this diffeomorphism,

$$\mathbf{h}_*(\epsilon_{1,0}) = \epsilon^*_{0,1} \quad \text{and} \quad \mathbf{h}_*(\epsilon_{0,1}) = \epsilon^*_{1,0}$$

where $\epsilon^*_{1,0}$ and $\epsilon^*_{0,1}$ are the complex Euler vector fields on V^*. Now under $\mathbf{h}$, the local framing $(f_1(x), \dots, f_n(x))$ of V corresponds to a local framing

$(\mathbf{h}_* f_1(x), \ldots, \mathbf{h}_* f_n(x))$ of V^*. Let $(z_1^*, \ldots, z_n^*)$ be the linear complex fibre coordinates on V^* with respect to this framing. Then the equations above can be rewritten as

$$\mathbf{h}_* \left(z \frac{\partial}{\partial z} \right) = \bar{z}^* \frac{\partial}{\partial \bar{z}^*} \quad \text{and} \quad \mathbf{h}_* \bar{z} \frac{\partial}{\partial \bar{z}} = z^* \frac{\partial}{\partial z^*}.$$

To fit this discussion into the above context, let $V = \mathrm{Hom}(E, F)$ and note that $V^* = \mathrm{Hom}(F, E)$.

6. The Fundamental Case - Injective Conformal.

Assume that $\operatorname{rank} E \leq \operatorname{rank} F$ and that $E \xrightarrow{\alpha} F$ is injective outside it's singular set (i.e., we are in the injective case). In addition, assume that:

$$\alpha^* \alpha = |\alpha|^2 \mathrm{Id}_E \tag{6.1}$$

is a scalar multiple, $|\alpha|^2$, of the identity map on sections of E. This situation will be referred to as the **injective conformal case**. It has wide applications and forms the working hypothesis for most of the subsequent discussion in this paper. (There is incidentally a dual "surjective conformal" case which can be treated analogously, but the details of this will not be presented here.) In this section we explicitly compute the curvature for the pullback and pushforward families in this case.

The formulas will be valid for any approximate one χ. For general bundle maps it is often better to restrict to algebraic approximation mode. However, the general approximate one is particularly well suited to the injective conformal case.

In the following we let $R = D^2$ denote the curvature operator associated to a connection D. In a local frame f, we have $Rf = \Omega f$ where Ω is the curvature matrix of 2-forms.

Theorem 6.2. *Suppose $E \xrightarrow{\alpha} F$ is injective conformal and that χ is a general approximate one. Let $\chi_s \equiv \chi\left(\frac{|\alpha|^2}{s^2}\right)$ and $\chi_s' \equiv \chi'\left(\frac{|\alpha|^2}{s^2}\right)$. Consider the pullback* *family* $\overleftarrow{D}_s = D_E + \chi_s \frac{\alpha^*(D\alpha)}{|\alpha|^2}$ *of connections on* E *and the pushforward family* $\overrightarrow{D}_s = D_F - \chi_s \frac{(D\alpha)\alpha^*}{|\alpha|^2}$ *of connections on* F*. Then the curvature is given by:*

$$\begin{aligned} \overleftarrow{R}_s &= (1-\chi_s) R_E + \chi_s \overleftarrow{R}_0 + \chi_s' \tfrac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{\alpha^* D\alpha}{|a|^2} \\ &- \chi_s(1-\chi_s) \frac{\alpha^*(D\alpha)\alpha^*(D\alpha)}{|\alpha|^4}, \end{aligned} \tag{6.3}$$

where

$$\overleftarrow{R}_0 = \frac{\alpha^* R_F \alpha}{|\alpha|^2} + \frac{(D\alpha^*)\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right)(D\alpha)}{|\alpha|^2}, \tag{6.4}$$

and

$$\begin{aligned} \overrightarrow{R}_s &= (1-\chi_s) R_F + \chi_s \overrightarrow{R}_0 - \chi_s' \tfrac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{(D\alpha)\alpha^*}{|a|^2} \\ &- \chi_s(1-\chi_s) \frac{(D\alpha)\alpha^*(D\alpha)\alpha^*}{|\alpha|^4}, \end{aligned} \tag{6.5}$$

where

$$\overrightarrow{R}_0 = \left(R_F + \frac{(D\alpha)(D\alpha^*)}{|\alpha|^2}\right)\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right) + \frac{\alpha R_E \alpha^*}{|\alpha|^2}. \tag{6.6}$$

Remark 6.7. The matrix forms of these equations are as follows (let $|a|^2 \equiv aa^* = |\alpha|^2$)

$$\begin{aligned} \overleftarrow{\Omega}_s &= (1-\chi_s)\Omega_E + \chi_s \overleftarrow{\Omega}_0 - \chi_s' \tfrac{|a|^2}{s^2} \frac{(Da)a^*}{|a|^2} \frac{d|a|^2}{|a|^2} \\ &+ \chi_s(1-\chi_s) \frac{(Da)a^*(Da)a^*}{|a|^4}, \end{aligned} \tag{6.8}$$

where

$$\overleftarrow{\Omega}_0 = \frac{a\Omega_F a^*}{|a|^2} - \frac{(Da)\left(1 - \frac{a^* a}{|a|^2}\right)(Da^*)}{|a|^2}, \tag{6.9}$$

and

$$\begin{aligned} \overrightarrow{\Omega}_s &= (1-\chi_s)\Omega_F + \chi_s \overrightarrow{\Omega}_0 + \chi_s' \tfrac{|a|^2}{s^2} \frac{a^*(Da)}{|a|^2} \frac{d|a|^2}{|a|^2} \\ &+ \chi_s(1-\chi_s) \frac{a^*(Da)a^*(Da)}{|a|^4}, \end{aligned} \tag{6.10}$$

where

$$\overrightarrow{\Omega}_0 = \left(1 - \frac{a^* a}{|a|^2}\right)\left(\Omega_F - \frac{(Da^*)(Da)}{|a|^2}\right) + \frac{a^* \Omega_E a}{|a|^2}. \tag{6.11}$$

The matrix form of $D\alpha$ is

$$Da \equiv da + a\omega_F - \omega_E a \tag{6.12}$$

while

$$Da^* \equiv da^* + a^* \omega_E - \omega_F a^* \tag{6.13}$$

is the matrix form of $D\alpha^*$. The gauges are given by

$$\overleftarrow{\omega}_s = \omega_E + \chi_s \frac{(Da)a^*}{|a|^2} \quad \text{and} \quad \overrightarrow{\omega}_s = \omega_F - \chi_s \frac{a^* Da}{|a|^2}. \tag{6.14}$$

It is possible to give a straightforward verification of these curvature formula in matrix form. This is left to the interested reader.

Proof. First note that

$$\begin{aligned}
\overleftarrow{R}_s &= \overleftarrow{D}_s^2 = \left(D_E + \chi_s \frac{\alpha^*(D\alpha)}{|\alpha|^2}\right)^2 \\
&= D_E^2 + \chi_s \frac{\alpha^*(D^2\alpha)}{|\alpha|^2} + \chi_s \frac{(D\alpha^*)(D\alpha)}{|\alpha|^2} - \chi_s \frac{d|\alpha|^2}{|\alpha|^2} \frac{\alpha^*(D\alpha)}{|\alpha|^2} \\
&\quad + \chi_s' \frac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{\alpha^*(D\alpha)}{|\alpha|^2} + \chi_s^2 \frac{\alpha^*(D\alpha)\alpha^*(D\alpha)}{|\alpha|^4}.
\end{aligned}$$

Substituting

$$\frac{d|\alpha|^2}{|\alpha|^2} = \frac{(D\alpha^*)\alpha}{|\alpha|^2} + \frac{\alpha^*(D\alpha)}{|\alpha|^2}, \tag{6.15}$$

which follows from the conformal hypothesis, and $D^2\alpha = R_H(\alpha)$ where $H \equiv \mathrm{Hom}(E, F)$, into this equation yields the equation (6.17) in the next result.

Proposition 6.16. *Suppose $E \xrightarrow{\alpha} F$ is injective conformal*

$$\begin{aligned}\overleftarrow{R}_s &= R_E + \chi_s \frac{\alpha^* R_H(\alpha)}{|\alpha|^2} + \chi_s \frac{(D\alpha^*)\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right)(D\alpha)}{|\alpha|^2} \\ &\quad + \chi_s' \tfrac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{\alpha^*(D\alpha)}{|\alpha|^2} - \chi_s(1-\chi_s)\frac{\alpha^*(D\alpha)\alpha^*(D\alpha)}{|\alpha|^4}.\end{aligned} \tag{6.17}$$

Setting $\chi_s \equiv 1$ (and hence $\chi_s' \equiv 0$) yields

$$\overleftarrow{R}_0 = R_E + \frac{\alpha^* R_H(\alpha)}{|\alpha|^2} + \frac{(D\alpha^*)\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right)(D\alpha)}{|\alpha|^2} \tag{6.18}$$

so that (6.17) *can be written as* (6.3).

This completes the proof of the pullback portion of Theorem 6.2 except for verifying that the two formula (6.4) and (6.18) for $\overleftarrow{R}_0$ are equal. Since $D_H(\alpha) = D_F\alpha - \alpha D_E$, note that

$$\begin{aligned}R_H(\alpha) &= D_H^2\alpha = D_H(D_F\alpha - \alpha D_E) \\ &= D_F(D_F\alpha - \alpha D_E) + (D_F\alpha - \alpha D_E)D_E \\ &= D_F^2\alpha - \alpha D_E^2 = R_F\alpha - \alpha R_E.\end{aligned}$$

That is,

$$R_H(\alpha) = R_F\alpha - \alpha R_E. \tag{6.19}$$

Therefore

$$\frac{\alpha^* R_F(\alpha)}{|\alpha|^2} = R_E + \frac{\alpha^* R_H(\alpha)}{|\alpha|^2} \tag{6.20}$$

which verifies that (6.4) and (6.18) are equivalent.

Proposition 6.21. *Suppose $E \xrightarrow{\alpha} F$ is injective conformal. Then*

$$\begin{aligned}\overrightarrow{R}_s &= R_F - \chi_s \frac{R_H(\alpha)\alpha^*}{|\alpha|^2} + \chi_s \frac{(D\alpha)(D\alpha^*)}{|\alpha|^2}\left(1 - \tfrac{\alpha\alpha^*}{|\alpha|^2}\right) \\ &\quad - \chi_s' \tfrac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{(D\alpha)\alpha^*}{|\alpha|^2} - \chi_s(1-\chi_s)\frac{(D\alpha)\alpha^*(D\alpha)\alpha^*}{|\alpha|^4}.\end{aligned} \tag{6.22}$$

Setting $\chi_s \equiv 1$ (and $\chi_s' \equiv 0$) yields

$$\overrightarrow{R}_0 = R_F - \frac{R_H(\alpha)\alpha^*}{|\alpha|^2} + \frac{(D\alpha)(D\alpha^*)}{|\alpha|^2}\left(1 - \tfrac{\alpha\alpha^*}{|\alpha|^2}\right), \tag{6.23}$$

so that (6.22) *can be rewritten as* (6.5).

Proof. Similar to the pullback case

$$\begin{aligned}\overrightarrow{R}_s &= \overrightarrow{D}_s^2 = \left(D_F - \chi_s \frac{(D\alpha)\alpha^*}{|\alpha|^2}\right)^2 \\ &= D_F^2 - \chi_s \frac{(D^2\alpha)\alpha^*}{|\alpha|^2} + \chi_s \frac{(D\alpha)(D\alpha^*)}{|\alpha|^2} + \chi_s \frac{d|\alpha|^2}{|\alpha|^2}\frac{(D\alpha)\alpha^*}{|\alpha|^2} \\ &\quad - \chi_s' \tfrac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2}\frac{(D\alpha)\alpha^*}{|\alpha|^2} + \chi_s^2 \frac{(D\alpha)\alpha^*(D\alpha)\alpha^*}{|\alpha|^4}.\end{aligned}$$

Substituting

$$d|\alpha|^2(D\alpha)\alpha^* = -(D\alpha)d|\alpha|^2\alpha^* = -(D\alpha)(D\alpha^*)\alpha\alpha^* - (D\alpha)\alpha^*(D\alpha)\alpha^*$$

and $D^2\alpha = R_H(\alpha)$ into this formula yields (6.22). □

To complete the proof of Theorem 6.2 we must show that

$$\frac{R_H(\alpha)\alpha^*}{|\alpha|^2} = \frac{R_F\alpha\alpha^*}{|\alpha|^2} - \frac{\alpha R_E\alpha^*}{|\alpha|^2} \tag{6.24}$$

to conclude that the two formulas (6.6) and (6.23) for $\overrightarrow{R}_0$ are equivalent. However, (6.24) follows immediately from (6.19). □

Remark 6.25. A third proof of Theorem 6.2 is outlined as follows. To be specific we discuss the pullback case. First verify directly the formula for $\overleftarrow{R}_0$. Then note that $\overleftarrow{D}_s = (1-\chi_s)D_F + \chi_s \overleftarrow{D}_0$ is a convex combination of connections and use the general identity

$$((1-x)B + xC)^2 = (1-x)B^2 + xC^2 - x(1-x)(C-B)^2, \tag{6.26}$$

valid for any pair of non commuting operators B and C and any scalar variable x.

7. Curvature and the Algebraic Approximation Mode.

The curvatures $\overleftarrow{R}_s$ and $\overrightarrow{R}_s$ are computed in this section without the conformal assumption, but assuming the algebraic approximation mode. Actually, the first three propositions are valid for any approximate one χ.

Proposition 7.1. *The connection* $\overleftarrow{D}_s \equiv D_E + \beta_s D_H(\alpha)$ *has curvature*

$$\overleftarrow{R}_s = R_E + \beta_s R_H(\alpha) + \Big(D(\beta_s) + \beta_s(D\alpha)\beta_s\Big)(D\alpha). \tag{7.2}$$

The connection $\overrightarrow{D}_s \equiv D_F - D_H(\alpha)\beta_s$ *has curvature*

$$\overrightarrow{R}_s = R_F - R_H(\alpha)\beta_s + (D\alpha)\Big(D(\beta_s) + \beta_s(D\alpha)\beta_s\Big). \tag{7.3}$$

Recall that in the injective case $\beta_s = \chi\left(\frac{\alpha^*\alpha}{s^2}\right)(\alpha^*\alpha)^{-1}\alpha^*$ *while in the surjective case* $\beta_s = \alpha^*(\alpha\alpha^*)^{-1}\chi\left(\frac{\alpha\alpha^*}{s^2}\right)$.

Proof.

$$\begin{aligned}\overleftarrow{R}_s &= \overleftarrow{D}_s^2 = \Big(D_E + \beta_s(D\alpha)\Big)^2 \\ &= D_E^2 + \beta_s(D^2\alpha) + (D\beta_s)(D\alpha) + \beta_s(D\alpha)\beta_s(D\alpha) \\ &= R_E + \beta_s R_H(\alpha) + (D\beta_s + \beta_s(D\alpha)\beta_s)(D\alpha)\end{aligned}$$

verifies (7.2). The proof of (7.3) is similar and omitted. □

Since $R_H(\alpha) = R_F\alpha - \alpha R_E$ the formulas may be written as follows.

7.4. Injective Case. *Set* $\chi_s \equiv \chi(\frac{\alpha^*\alpha}{s^2}) = \beta_s\alpha$ *and* $P_s \equiv \alpha\beta_s$. *Note that* $\chi_s \to 1$ *and* $P_s \to P \equiv \alpha(\alpha^*\alpha)^{-1}\alpha^*$ *outside the singular locus, as* $s \to 0$. *Then*

$$\overleftarrow{R}_s = (1 - \chi_s)R_E + \beta_s R_F\alpha + \Big(D\beta_s + \beta_s(D\alpha)\beta_s\Big)(D\alpha) \tag{7.5}$$

$$\overrightarrow{R}_s = R_F(1 - P_s) + \alpha R_E\beta_s + (D\alpha)\Big(D\beta_s + \beta_s(D\alpha)\beta_s\Big). \tag{7.6}$$

7.7. Surjective Case. *Set* $\chi_s \equiv \chi\left(\frac{\alpha\alpha^*}{s^2}\right) = \alpha\beta_s$ *and* $P_s = \beta_s\alpha$. *Note that* $\chi_s \to 1$ *and* $P_s \to P \equiv \alpha^*(\alpha\alpha^*)^{-1}\alpha$ *outside the singular locus, as* $s \to 0$. *Then*

$$\overleftarrow{R}_s = (1 - P_s)R_E + \beta_s R_F\alpha + \Big(D\beta_s + \beta_s(D\alpha)\beta_s\Big)(D\alpha). \tag{7.8}$$

$$\overrightarrow{R}_s = R_F(1 - \chi_s) + \alpha R_E\beta_s + (D\alpha)\Big(D\beta_s + \beta_s(D\alpha)\beta_s\Big). \tag{7.9}$$

Lemma 7.10. *In the algebraic approximation mode one has*

$$D\beta_s + \beta_s(D\alpha)\beta_s \;=\; s^2(s^2+\alpha^*\alpha)^{-1}(D\alpha^*)(s^2+\alpha\alpha^*)^{-1}. \tag{7.11}$$

Proof. In the algebraic approximation mode

$$\beta_s \;=\; (s^2+\alpha^*\alpha)^{-1}\alpha^* \;=\; \alpha^*(s^2+\alpha\alpha^*)^{-1}. \tag{7.12}$$

Using the first equality, we have

$$\begin{aligned} D\beta_s + \beta_s(D\alpha)\beta_s &= (s^2+\alpha^*\alpha)^{-1}(D\alpha^*) - (s^2+\alpha^*\alpha)^{-1}(D\alpha^*)\alpha(s^2+\alpha^*\alpha)^{-1}\alpha^* \\ &- (s^2+\alpha^*\alpha)^{-1}\alpha^*(D\alpha)(s^2+\alpha^*\alpha)^{-1}\alpha^* + (s^2+\alpha^*\alpha)^{-1}\alpha^*(D\alpha)(s^2+\alpha^*\alpha)^{-1}\alpha^* \\ &= (s^2+\alpha^*\alpha)^{-1}(D\alpha^*)\left(1-\alpha(s^2+\alpha^*\alpha)^{-1}\alpha^*\right). \end{aligned}$$

However, $1-\alpha(s^2+\alpha^*\alpha)^{-1}\alpha^* = 1-(s^2+\alpha\alpha^*)^{-1}\alpha\alpha^* = s^2(s^2+\alpha\alpha^*)^{-1}$. □

Note that in the algebraic approximation mode

$$\begin{aligned} \alpha\beta_s &= \alpha(s^2+\alpha^*\alpha)^{-1}\alpha^* = \alpha\alpha^*(s^2+\alpha\alpha^*)^{-1} \\ \beta_s\alpha &= (s^2+\alpha^*\alpha)^{-1}\alpha^*\alpha = \alpha^*(s^2+\alpha\alpha^*)^{-1}\alpha. \end{aligned} \tag{7.13}$$

The following main result is an immediate consequence of 7.4, 7.7 and 7.10.

Theorem 7.14. *In the algebraic approximation mode*

$$\begin{aligned} \overleftarrow{R}_s &= s^2(s^2+\alpha^*\alpha)^{-1}R_E + (s^2+\alpha^*\alpha)^{-1}\alpha^*R_F\alpha \\ &\quad + s^2(s^2+\alpha^*\alpha)^{-1}(D\alpha^*)(s^2+\alpha\alpha^*)^{-1}(D\alpha), \end{aligned} \tag{7.15}$$

$$\begin{aligned} \overrightarrow{R}_s &= R_Fs^2(s^2+\alpha\alpha^*)^{-1} + \alpha R_E\alpha^*(s^2+\alpha\alpha^*)^{-1} \\ &\quad + s^2(D\alpha)(s^2+\alpha^*\alpha)^{-1}(D\alpha^*)(s^2+\alpha\alpha^*)^{-1}. \end{aligned} \tag{7.16}$$

To compute the limit as s approaches zero, outside the singular locus, we must distinguish between the injective and the surjective case.

Proposition 7.17. Injective Case. *Let $P \equiv \alpha(\alpha^*\alpha)^{-1}\alpha^*$ denote orthogonal projection from F onto the target bundle $T \equiv \operatorname{im}\alpha$, defined outside the singular set. In the algebraic approximation mode*

$$\overleftarrow{R}_0 = (\alpha^*\alpha)^{-1}\alpha^* R_F \alpha + (\alpha^*\alpha)^{-1}(D\alpha^*)(1-P)(D\alpha) \tag{7.18}$$

and

$$\overrightarrow{R}_0 = R_F(1-P) + (D\alpha)(\alpha^*\alpha)^{-1}(D\alpha^*)(1-P) + \alpha R_E(\alpha^*\alpha)^{-1}\alpha^*. \tag{7.19}$$

Proposition 7.20. Surjective Case. *Let $P \equiv \alpha^*(\alpha\alpha^*)^{-1}\alpha$ denote orthogonal projection from E to $K^\perp$ where $K \equiv \ker\alpha$ is the kernel bundle of α defined outside the singular set. In the algebraic approximation mode*

$$\overleftarrow{R}_0 = (1-P)R_E + (1-P)(D\alpha^*)(\alpha\alpha^*)^{-1}(D\alpha) + \alpha^*(\alpha\alpha^*)^{-1}R_F\alpha \tag{7.21}$$

and

$$\overrightarrow{R}_0 = \alpha R_E \alpha^*(\alpha\alpha^*)^{-1} + (D\alpha)(1-P)(D\alpha^*)(\alpha\alpha^*)^{-1}. \tag{7.22}$$

Of course, in the **equirank case**

$$\overleftarrow{R}_0 = \alpha^{-1}R_F\alpha \quad \text{and} \quad \overrightarrow{R}_0 = \alpha R_E \alpha^{-1}. \tag{7.23}$$

8. Universal Formulae and a Universal Compactification.

In this section we will derive some formulas for the pullback family of connections under the universal homomorphism. It will be shown that in this setting the algebraic approximation mode has a natural "compactification". More specifically, the approximating families of connections will be shown to arise from a simple construction which extends smoothly to a fibrewise compactification of the space $\mathrm{Hom}(E, F)$. As a consequence, we will see that in algebraic approximation mode our families of connections are intimately related to MacPherson's

Grassmann graph construction (cf. [BFM]). This gives some philosophical preference to this particular method of approximation.

We shall adopt the notation of Section 3 where $\mathbf{E} \xrightarrow{\boldsymbol{\alpha}} \mathbf{F}$ denotes the universal homomorphism over $\mathrm{Hom}(E, F)$, $\overrightarrow{\mathbf{D}}$ or $\overrightarrow{\mathbf{D}}_\alpha$ denote the singular pushforward connection on $\mathbf{F}$ induced by $\boldsymbol{\alpha}$, and $\overleftarrow{\mathbf{D}}$ or $\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}}$ denote the singular pullback connection on $\mathbf{E}$ induced by $\boldsymbol{\alpha}$.

Definition 8.1. Using the algebraic approximation mode we define, as in Section 4, the **universal approximating families**

$$
(8.2) \qquad \begin{aligned}
\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} &= (\boldsymbol{\alpha}^*\boldsymbol{\alpha} + s^2)^{-1}(s^2\mathbf{D}_{\mathbf{E}} + \boldsymbol{\alpha}^*\mathbf{D}_{\mathbf{F}}\boldsymbol{\alpha}) \quad \text{on } \mathbf{E} \\
\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} &= (s^2\mathbf{D}_{\mathbf{F}} + \boldsymbol{\alpha}\mathbf{D}_{\mathbf{E}}\boldsymbol{\alpha}^*)(\boldsymbol{\alpha}\boldsymbol{\alpha}^* + s^2)^{-1} \quad \text{on } \mathbf{F}.
\end{aligned}
$$

Using the universal adjoint homomorphism we can define connections $\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*}$ on $\mathbf{E}$ and $\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*}$ on $\mathbf{F}$. In the algebraic approximation mode, we get approximating families

$$
(8.3) \qquad \begin{aligned}
\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s} &= (s^2\mathbf{D}_{\mathbf{E}} + \boldsymbol{\alpha}^*\mathbf{D}_{\mathbf{F}}\boldsymbol{\alpha})(\boldsymbol{\alpha}^*\boldsymbol{\alpha} + s^2)^{-1} \quad \text{on } \mathbf{E} \\
\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s} &= (\boldsymbol{\alpha}\boldsymbol{\alpha}^* + s^2)^{-1}(s^2\mathbf{D}_{\mathbf{F}} + \boldsymbol{\alpha}\mathbf{D}_{\mathbf{E}}\boldsymbol{\alpha}^*) \quad \text{on } \mathbf{F}.
\end{aligned}
$$

Proposition 8.4. *Over Hom(E, F)*

$$
\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} \cong \overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s} \quad \textit{on } \mathbf{E} \quad \textit{and} \quad \overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} \cong \overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*s}, \quad \textit{on } \mathbf{F}.
$$

where "$\cong$" denotes gauge equivalence.

Proof. Define $h : \mathbf{E} \longrightarrow \mathbf{E}$ by $h = \boldsymbol{\alpha}^*\boldsymbol{\alpha} + s^2$. Then $h \circ \overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} \circ h^{-1} = \overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s}$. Similarly, if $\tilde{h} : \mathbf{F} \longrightarrow \mathbf{F}$ is given by $\tilde{h} = \boldsymbol{\alpha}\boldsymbol{\alpha}^* + s^2$, then $\tilde{h}\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s} \circ \tilde{h}^{-1} = \overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}$. □

Remark 8.5. In terms of local frames and local coordinates this universal point of view is equivalent to considering the matrix-valued function a defined by $\alpha e = af$ as an independent variable on the total space of $\mathrm{Hom}(E, F)$. For example, the gauge (see (5.6) and (5.10))

$$
(8.6) \qquad \overleftarrow{\omega}_s = \omega_E + (da + a\omega_F - \omega_E a)a^*(aa^* + s^2)^{-1}
$$

for $\overleftarrow{D}_s$ (where a is matrix-valued function on the base manifold) can now be considered as the gauge for $\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}$ where a is the fiber variable on $\mathrm{Hom}(E, F)$.

This universal construction can be compactified in the vertical directions of the total space of $\mathrm{Hom}(E,F)$ as follows. Let

$$G_p(E\oplus F)\xrightarrow{\pi} X \tag{8.7}$$

denote the bundle over X whose fibre above $x\in X$ is the Grassmann manifold of p-dimensional linear subspaces of $E_x\oplus F_x$. There is a natural inclusion

$$\mathrm{Hom}(E,F)\subset G_p(E\oplus F)$$

which associates to the homomorphism $\alpha: E_x \longrightarrow F_x$ its graph, $P_\alpha \stackrel{\mathrm{def}}{=} \text{graph } \alpha = \{(e,\alpha(e)) : e\in E_x\}\subset E_x\oplus F_x$. Let $\mathbf{E}=\pi^*E$ and $\mathbf{F}=\alpha^*F$ denote the pullbacks over $G_p(E\oplus F)$ of the bundles E and F. (Restricted to $\mathrm{Hom}(E,F)\subset G_p(E\oplus F)$ they agree with the bundles above). Then there is a **tautological** or **universal p-dimensional subbundle** $U\subset \mathbf{E}\oplus\mathbf{F}$ whose fibre at a point $P\in G_p(E_x\oplus F_x)$ above $x\in X$ consists of the vectors in P. Over the open set $\mathrm{Hom}(E,F)\subset G_p(E\oplus F)$, the universal subbundle U is just the graph of the universal homomorphism $\boldsymbol{\alpha}$.

Consider now the flow $\psi_s : E\oplus F\longrightarrow E\oplus F$ defined by

$$\psi_s(e,f) \;=\; (se,f) \qquad \text{for } s\in\mathbf{R}^+. \tag{8.8}$$

This induces a flow $\Psi_s : G_p(E\oplus F)\longrightarrow G_p(E\oplus F)$. For each s, we let

$$U_s \;=\; \Psi_s^*(U)$$

denote the pullback of the universal bundle via Ψ_s. At a p-plane $P\subset E\oplus F$, the fibre of U_s is

$$(U_s)_{\{P\}} \;=\; \Psi_s(\{P\}) \;=\; \psi_s(P).$$

If P lies in the open chart $\mathrm{Hom}(E,F)\subset G_p(E,F)$, i.e. $P\equiv$ graph α, then the fibre of U_s is just graph $\frac{1}{s}\alpha$. Assume as before that there are metrics given on $\mathbf{E}$ and $\mathbf{F}$. Then for each s, there is an orthogonal splitting

$$\mathbf{E}\oplus\mathbf{F} \;=\; U_s\oplus U_s^\perp. \tag{8.9}$$

There are induced connections on these subbundles defined as follows. Let

$$\mathrm{pr}_s : \mathbf{E}\oplus\mathbf{F}\longrightarrow U_s \quad\text{and}\quad \mathrm{pr}_s^\perp : \mathbf{E}\oplus\mathbf{F}\to U_s^\perp \tag{8.10}$$

denote the orthogonal bundle projections.

Definition 8.11. The connections D_{U_s} on U_s and $D_{U_s^\perp}$ on $U_s^\perp$ induced from the direct sum connection on $\mathbf{E} \oplus \mathbf{F}$ are given by

$$D_{U_s} = \mathrm{pr}_s \circ \mathbf{D}_{\mathbf{E}\oplus\mathbf{F}} \quad \text{and} \quad D_{U_s^\perp} = \mathrm{pr}_s^\perp \circ \mathbf{D}_{\mathbf{E}\oplus\mathbf{F}}$$

operating on sections of U_s and $U_s^\perp$ respectively.

Over the open dense subset $\mathrm{Hom}(E,F) \subset G_p(E \oplus F)$, the splitting $U_s \oplus U_s^\perp$ is gauge equivalent to the splitting $\mathbf{E} \oplus \mathbf{F}$. Specifically, consider the family of maps

$$g_s : \mathbf{E} \oplus \mathbf{F} \longrightarrow \mathbf{E} \oplus \mathbf{F}$$

defined over $\mathrm{Hom}(E,F)$ by

$$g_s = \begin{pmatrix} s & -\boldsymbol{\alpha}^* \\ \boldsymbol{\alpha} & s \end{pmatrix}. \tag{8.12}$$

One sees easily that

$$g_s(\mathbf{E} \oplus \{0\}) = U_s \quad \text{and} \quad g_s(\{0\} \oplus \mathbf{F}) = U_s^\perp.$$

Let γ_s denote g_s restricted to $\mathbf{E} \equiv \mathbf{E} \oplus \{0\}$ and let $\tilde{\gamma}_s \equiv g_s$ restricted to $\mathbf{F} \equiv \{0\} \oplus \mathbf{F}$. Straightforward calculation reveals the following important fact

$$\begin{aligned} &g_s^{-1} \circ \begin{pmatrix} \mathbf{D}_\mathbf{E} & 0 \\ 0 & \mathbf{D}_\mathbf{F} \end{pmatrix} \circ g_s = \\ &\begin{pmatrix} \overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} & (\boldsymbol{\alpha}^*\boldsymbol{\alpha} + s^2)^{-1} s(\boldsymbol{\alpha}^*\mathbf{D}_\mathbf{F} - \mathbf{D}_\mathbf{E}\boldsymbol{\alpha}^*) \\ (\boldsymbol{\alpha}\boldsymbol{\alpha}^* + s^2)^{-1} s(\mathbf{D}_\mathbf{F}\boldsymbol{\alpha} - \boldsymbol{\alpha}\mathbf{D}_\mathbf{E}) & \overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s} \end{pmatrix}. \end{aligned} \tag{8.13}$$

Theorem 8.14. *Over the open subset $\mathrm{Hom}(E,F) \subset G_p(E \oplus F)$, consider the bundle isomorphisms*

$$\gamma_s : \mathbf{E} \longrightarrow U_s \quad \textit{and} \quad \tilde{\gamma}_s : \mathbf{F} \longrightarrow U_s^\perp$$

defined for $s > 0$ by

$$\gamma_s(e) = se + \alpha(e) \quad \textit{and} \quad \tilde{\gamma}_s(f) = sf - \alpha^*(f).$$

Then

$$\gamma_s^{-1} \circ \mathbf{D}_{U_s} \circ \gamma_s = \overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s} \quad \textit{and} \quad \tilde{\gamma}_s^{-1} \circ \mathbf{D}_{U_s^\perp} \circ \tilde{\gamma}_s = \overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s}. \tag{8.15}$$

If we set $\delta_s = \gamma_s \circ (\boldsymbol{\alpha}^*\boldsymbol{\alpha} + s^2)^{-1}$ and $\tilde{\delta}_s = \tilde{\gamma}_s \circ (\boldsymbol{\alpha}\boldsymbol{\alpha}^* + s^2)^{-1}$, then

$$\delta_s^{-1} \circ \mathbf{D}_{U_s} \circ \delta_s = \overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s} \quad \textit{and} \quad \tilde{\delta}_s^{-1} \circ \mathbf{D}_{U_s^\perp} \circ \tilde{\delta}_s = \overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}. \tag{8.16}$$

Proof. The first part follows from (8.12) and (8.13). The second follows from 8.4. $\square$

Suppose now that ϕ and ψ are adjoint-invariant polynomials on $\mathfrak{gl}_p$ and $\mathfrak{gl}_q$ respectively (where $q = \operatorname{rank}(F)$). Then by Theorem 8.14 we see that

$$\phi\left(\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}\right) = \phi\left(\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s}\right) = \phi(\mathbf{D}_{U_s})$$

and

$$\psi\left(\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}\right) = \psi\left(\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha}^*,s}\right) = \psi(\mathbf{D}_{U_s^\perp}).$$

Now the bundles U_s and $U_s^\perp$ together with their connections are globally defined on $G_p(\mathbf{E} \oplus \mathbf{F})$. Consequently, we have

Corollary 8.17. *For all invariant polynomials ϕ and ψ as above, and for all $s > 0$, the forms $\phi\left(\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}\right)$ and $\psi\left(\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}\right)$ extend smoothly to all of $G_p(\mathbf{E} \oplus \mathbf{F})$. Consequently, integration over the fibre of the map $\pi : G_p(\mathbf{E} \oplus \mathbf{F}) \longrightarrow X$ gives a well-defined smooth forms $\pi_*\phi\left(\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s})\right)$ and $\pi_*\psi\left(\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}\right)$ on X.*

Theorem 8.14 has an interesting consequence that will be useful in Chapter III.

Theorem 8.18. *For each $s > 0$ there exists a smooth form τ_s on $G_p(\mathbf{E} \oplus \mathbf{F})$ with*

$$c(\mathbf{D_E})c(\mathbf{D_F}) = c(D_{U_s})c(D_{U_s^\perp}) + d\tau_s \quad \text{on} \quad G_p(\mathbf{E} \oplus \mathbf{F}). \tag{8.19}$$

Moreover, on the open subset $Hom(E, F) \subset G_p(\mathbf{E} \oplus \mathbf{F})$, we have

$$c(D_{U_s}) = c(\overleftarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}) \quad \text{and} \quad c(D_{U_s^\perp}) = c(\overrightarrow{\mathbf{D}}_{\boldsymbol{\alpha},s}). \tag{8.20}$$

Proof. The direct sum connection $\mathbf{D_E} \oplus \mathbf{D_F}$ on $\mathbf{E} \oplus \mathbf{F}$ can be blocked with respect to $U_s \oplus U_s^\perp$ in the form

$$\mathbf{D_E} \oplus \mathbf{D_F} = \begin{pmatrix} D_{U_s} & A_{12} \\ A_{21} & D_{U_s^\perp} \end{pmatrix}. \tag{8.21}$$

Since $A_{21} = \mathrm{pr}_s^\perp \circ \mathbf{D_E} \oplus \mathbf{D_F} \circ \mathrm{pr}_s$ is a tensor (i.e., $A_{21}(f\sigma) = f A_{21}(\sigma)$ for smooth functions f on X and smooth sections σ), we see that

$$D(y) = \begin{pmatrix} D_{U_s} & A_{12} \\ yA_{21} & D_{U_s^\perp} \end{pmatrix}$$

defines a smooth family of connections for $0 \le y \le 1$. Let τ_s denote the transgression form for this family with respect to the total Chern polynomial. Then

$$c(\mathbf{D}(1)) - c(\mathbf{D}(0)) = d\tau_s \qquad \text{on } G_p(\mathbf{E} \oplus \mathbf{F}). \tag{8.22}$$

Since $\mathbf{D}(1) = \mathbf{D_E} \oplus \mathbf{D_F}$ and $\mathbf{D}(0)$ is upper triangular with diagonal entries D_{U_s} and $D_{U_s^\perp}$, equation (8.19) follows. The identities (8.20) are a special case of Corollary 8.17. □

We finish this section with two remarks concerning the constructions above.

Remark 8.23. The parameter s in (8.8) can be taken to be complex. If E and F are holomorphic bundles over a complex manifold, then the flow Ψ_s is holomorphic. The discussion and theorems continue to hold in this case with s^2 replaced in the formulas by $|s|^2$.

Remark 8.24. The main constructions above can be reformulated entirely in terms of a change of metric as follows. For each $s > 0$, consider the metric

$$\langle \cdot, \cdot \rangle_s = s^2 \langle \cdot, \cdot \rangle_E + \langle \cdot, \cdot \rangle_F$$

on $\mathbf{E} \oplus \mathbf{F}$ over $G_p(E \oplus F)$. With respect to this there is an orthogonal splitting

$$\mathbf{E} \oplus \mathbf{F} = U \oplus U^{\perp_s} \tag{8.25}$$

where $U = U_1$ is the tautological subbundle. With respect to the blocking (8.25), the direct sum connection can be written in the form:

$$\mathbf{D_E} \oplus \mathbf{D_F} = \begin{pmatrix} D_U^s & B_{12} \\ B_{21} & D_{U^{\perp_s}}^s \end{pmatrix}. \tag{8.26}$$

Theorem 8.27. *Over the open dense subset $Hom(E, F) \subset G_p(E \oplus F)$, there are gauge equivalences:*

$$D_U^s \cong \overleftarrow{\mathbf{D}}_{\alpha,s} \quad \text{and} \quad D_{U^{\perp_s}}^s = \overrightarrow{\mathbf{D}}_{\alpha^*,s}.$$

Proof. The flow Ψ_s, defined above, lifts to a natural action $(\Psi_s)_*$ on the bundle $\mathbf{E} \oplus \mathbf{F}$ given by $(\Psi_s)_*(e, f) = (se, f)$. Note that

(1) $(\Psi_s)_*(U_s) = (\Psi_s)_*((\Psi_s)^*U) = U$,

(2) $(\Psi_s)_*$ preserves the direct sum connection $\mathbf{D_E} \oplus \mathbf{D_F}$,

(3) $\langle(\Psi_s)_*v, (\Psi_s)_*w\rangle = \langle v, w\rangle_s$ for all v, $s \in (\Psi_s)_*$.

Hence, under the connection-preserving automorphism $(\Psi_s)_*$ of $\mathbf{E} \oplus \mathbf{F}$, the splitting $U_s \oplus U_s^\perp$ of Theorem 8.14 is carried to the splitting $U \oplus U^{\perp_s}$. The result now follows from Theorem 8.14. □

The universal setting will play an important role in the general theory. If one can first carry out the program, outlined in Section 1 above, in this universal case, then the general problem reduces to finding analytic-geometric conditions on the section
$\alpha : X \longrightarrow \text{Hom}(E, F)$ so that the current equations established in the universal case (over $\text{Hom}(E, F)$) can be pulled back via α to analogous equations on X. This first serious application of this principle will be given in Chapter III. More general residue theorems will be proved in this manner in a subsequent paper.

9. Universal Desingularization.

In this section we show that the blow-up of $\text{Hom}(E, F)$ introduced in §3 extends to the universal compactification $G_p(E \oplus F) \supset \text{Hom}(E, F)$, and that the lifts of the limiting characteristic forms on $\sim \Sigma$ extend to all of this space as smooth d-closed forms. In particular, Proposition 9.11 below generalizes Theorems 3.10 and 3.16 to the compactification. These results give insight into and control over the limiting characteristic currents.

Let $\mathbf{E}$, $\mathbf{F}$ and $G_p(E \oplus F)$ be as in the previous section. We consider the open dense subsets

$$\begin{aligned} H &= \{P \in G_p(E \oplus F) : P \cap \mathbf{F} = \{0\}\} = \mathrm{Hom}(E, F) \\ \mathcal{R} &= \{P \in G_p(E \oplus F) : P \cap \mathbf{E} = \{0\}\} = G_p(E \oplus F) - \overline{\Sigma}. \end{aligned}$$

Let $H^\times = \mathrm{Hom}^\times(E, F)$ denote $H \cap \mathcal{R}$. This is the set of graphs of all **injective** homomorphisms from $\mathbf{E}$ to $\mathbf{F}$. The complement $H - H^\times = \Sigma$ corresponds to all **singular** homomorphisms.

Note that the natural projection $p_F : \mathbf{E} \oplus \mathbf{F} \longrightarrow \mathbf{F}$ determines a projection

$$p_F : \mathcal{R} \longrightarrow G_p(F), \tag{9.1}$$

with the property that for graph$\alpha \in H^\times \subset \mathcal{R}$, we have $p_F(\mathrm{graph}(\alpha)) = \mathrm{image}(\alpha)$. Over $G_p(F)$ the pullback bundle $\mathbf{F}$ has a canonical splitting

$$\mathbf{F} = T \oplus T^\perp$$

where T is the p-dimensional tautological subbundle.

Lemma 9.2. *Over the subset $\mathcal{R}$ the decomposition $\mathbf{E} \oplus \mathbf{F} = U_s \oplus U_s^\perp$ extends smoothly to $s = 0$, where*

$$U_0 = p_F^*(T) \quad \text{and} \quad U_0^\perp = \mathbf{E} \oplus p_{\mathbf{F}}^*(T^\perp). \tag{9.3}$$

Over the subset H the decomposition $\mathbf{E} \oplus \mathbf{F} = U_s \oplus U_s^\perp$ extends smoothly to $s = \infty$, where

$$U_\infty = \mathbf{E} \quad \text{and} \quad U_\infty^\perp = \mathbf{F}. \tag{9.4}$$

Proof. We show that U_s extends smoothly on $\mathcal{R}$ to $s = 0$. It will suffice to prove this in the case where $X = \{\mathrm{pt}\}$ by the local triviality of the flow on the bundle. The flow Ψ_s is algebraic, and at each point $\{P\} \in G_p(E \oplus F)$ there exists a unique limit

$$\lim_{s \to 0} U_s = p_{\mathbf{F}}(P).$$

It is a general fact that algebraic maps with well-defined limits extend algebraically. This proves the first part of the Lemma 9.2. The second is obvious. □

Corollary 9.5. *Let $\mathbf{D}_T \oplus \mathbf{D}_{T^\perp}$ be the direct sum connection on $\mathbf{F} = T \oplus T^\perp$ over $G_p(F)$ obtained by projecting the pullback connection $\mathbf{D}_\mathbf{F}$ onto the factors. Let ϕ and $\widetilde{\phi}$ be adjoint-invariant polynomials on gl_p and gl_q respectively (where $q = \dim F$). By Corollary 8.17 we know that $\phi(\overleftarrow{\mathbf{D}}_{\alpha,s})$ and $\widetilde{\phi}(\overrightarrow{\mathbf{D}}_{\alpha,s})$ extend to forms Φ_s and $\widetilde{\Phi}_s$ defined smoothly on all of $G_p(E \oplus F)$. Then over the subset $\mathcal{R}$, we have that*

$$\lim_{s\to 0} \Phi_s = p_F^*\{\phi(\mathbf{D}_T)\} \qquad \text{and} \tag{9.6}$$

$$\lim_{s\to 0} \widetilde{\Phi}_s = p_F^*\left\{\widetilde{\phi}(\mathbf{D}_E \oplus \mathbf{D}_{T^\perp})\right\} \tag{9.7}$$

where p_F is the projection (9.1).

Proof. By Theorem 8.14 we know that over the open dense subset $\mathcal{H} \equiv \mathrm{Hom}(E, F)$ both $\Phi_s = \phi(\mathbf{D}_{U_s})$ and $\widetilde{\Phi}_s = \widetilde{\phi}(\mathbf{D}_{U_s^\perp})$ for all $s > 0$, where $\mathbf{D}_{U_s}$ and $\mathbf{D}_{U_s^\perp}$ are the connections defined on U_s and $U_s^\perp$ respectively by projecting $\mathbf{D}_\mathbf{E} \oplus \mathbf{D}_\mathbf{F}$ via the orthogonal decomposition $\mathbf{E} \oplus \mathbf{F} = U_s \oplus U_s^\perp$. Applying Lemma 9.2 gives the result. □

Our next observation is that there exists a natural blow-up $\widetilde{G}_p(E \oplus F)$ of $G_p(E \oplus F)$ along $\overline{\Sigma}$ to which the map p_F extends smoothly. It is defined as usual by taking the closure of the graph of p_F in $G_p(E \oplus F) \times G_p(F)$, namely

$$\widetilde{G}_p(E \oplus F) = \{(P, \Pi) \in G_p(E \oplus F) \times G_p(F) : p_F(P) \subseteq \Pi\}. \tag{9.8}$$

Lemma 9.9. *The subset $\widetilde{G}_p(E \oplus F)$ is a smooth submanifold. Projection onto the first factor in $G_p(E \oplus F) \times G_p(F)$ induces a projection*

$$\rho : \widetilde{G}_p(E \oplus F) \longrightarrow G_p(E \oplus F) \tag{9.10}$$

which is a diffeomorphism over $\mathcal{R}$.

Proof. One checks that projection onto the second factor of $G_p(E\oplus F) \times G_p(F)$ is a smooth fibre bundle $\tilde{p}_\mathbf{F} : \widetilde{G}_p(E \oplus F) \longrightarrow G_p(F)$ whose fibre above Π is the set of all P with $p_\mathbf{F}(P) \subseteq \Pi$. Thus, $\widetilde{G}_p(E \oplus F)$ is a submanifold. The second assertion is obvious. □

Proposition 9.11. *Let ϕ and $\widetilde{\phi}$ be as above and consider the smooth characteristic forms*

$$\phi\left(\overleftarrow{\mathbf{D}}_{\mathbf{E}}\right)\Big|_{H^\times} \quad \text{and} \quad \widetilde{\phi}\left(\overrightarrow{\mathbf{D}}_{\mathbf{F}}\right)\Big|_{H^\times}$$

associated to the universal singular connections on the complement $H^\times$ of Σ in H. In general these forms do not extend smoothly across Σ. However, the liftings of these forms via the projection ρ of (9.10) do extend smoothly to $\widetilde{H} \stackrel{\text{def}}{=} \rho^{-1}(H)$. In fact they extend smoothly to all of $\widetilde{G}_p(E \oplus F)$.

Proof. As seen from Corollary 9.5 (or directly from (9.1) and (9.3)), we have that

$$\phi\left(\overleftarrow{\mathbf{D}}_{\mathbf{E}}\right)\Big|_{H^\times} = p_{\mathbf{F}}^*\{\phi(\mathbf{D}_T)\} \quad \text{and} \quad \widetilde{\phi}\left(\overrightarrow{\mathbf{D}}_{\mathbf{F}}\right)\Big|_{H^\times} = p_{\mathbf{F}}^*\{\widetilde{\phi}(\mathbf{D}_{\mathbf{E}} \oplus \mathbf{D}_{T^\perp})\}.$$

However, the bundles $p_{\mathbf{F}}^*T$, $p_{\mathbf{F}}^*T^\perp$, and $\mathbf{E}$, together with their connections, are defined smoothly over all of $\widetilde{G}_p(E \oplus F)$. □

Theorem 9.12. *The smooth forms $\phi(\overleftarrow{\mathbf{D}}_{\mathbf{E}})\,\big|_{H^\times}$ and $\widetilde{\phi}(\overrightarrow{\mathbf{D}}_{\mathbf{F}})\,\big|_{H^\times}$ extend by 0 to be d-closed forms with L^1_{loc}-coefficients on all of $G_p(E \oplus F)$.*

Proof. This follows directly from Proposition 9.11 because of Proposition 3.12. □

Remark 9.12. Over $H^\times = \operatorname{Hom}^\times(E, F)$, and hence over $\rho^{-1}(H^\times) \subset \widetilde{G}_p(E \oplus F)$, the bundles U_s converge smoothly to the bundle $U_0 = p_F^*(T)$, because of Lemma 9.2. This limiting bundle $p_F^*(T)$ is a smooth bundle on all of $\widetilde{G}_p(E \oplus F)$. Despite these facts the family U_s $0 \leq s \leq \infty$ connecting $U_\infty = \mathbf{E} \times \{0\}$ to $U_0 = p_F^*(T)$ obviously cannot provide a smooth path from $\mathbf{E}$ to T over all of $\widetilde{G}_p(E \oplus F)$. The problem is over $\widetilde{\Sigma} \equiv \rho^{-1}(\Sigma)$. For example, if $\operatorname{rank} E = 1$ then $U_s = \mathbf{E} \times \{0\}$ $0 < s \leq \infty$ over $\widetilde{\Sigma}$, which does not equal $p_F^*(T)$.

10. On the Functoriality and Uniqueness of the Transgression.

Among the main results of this paper are the proofs of the existence and uniqueness of the transgression (including independence of approximation mode) in a wide range of important cases. However, there are also some soft general facts concerning the transgression which are important for the theory. We present these here together with some results on the functoriality of the transgressions (when they exist). At the end of this section, we give a guide to hard results about the transgression which are proved in subsequent sections.

We begin by presenting a very general construction which yields transgressions, double transgressions, etc.

Basic Construction 10.1. Let V be a vector bundle over a manifold X, and let D_y, $y \in Y$ be a family of connections on V smoothly parameterized by a manifold Y. Let $\pi : X \times Y \longrightarrow X$ denote the projection. The the family D_y canonically determines a connection $\widetilde{D}$ on the pull-back bundle $\widetilde{V} = \pi^* V$ as follows. Fix a point $y_0 \in Y$. Let $\widetilde{D}_0$ denote the canonical pull-back of the connection D_{y_0} to $\widetilde{V}$, and let $\widetilde{A}$ be the $\mathrm{Hom}(\widetilde{V}, \widetilde{V})$-valued 1-form on $X \times Y$ which is zero on TY and equals $A_y \stackrel{\text{def}}{=} D_y - D_{y_0}$ on $X \times \{y\}$. Then we set $\widetilde{D} = \widetilde{D}_0 + \widetilde{A}$.

Suppose that ω_y is the gauge of D_y in a local frame for V on X, and let $\tilde{\omega}$ denote the gauge of $\widetilde{D}$ in the pull-back frame on $X \times Y$. Then one sees easily that at a point (x, y)

$$\tilde{\omega} = \omega_y.$$

The corresponding curvature form is given by

$$\widetilde{\Omega} = \Omega_y + d_y(\omega_y)$$

where d_y denotes the Y-component of exterior differentiation.

Suppose now that γ is a compact, oriented smooth arc joining points a to b in Y. Set $\Gamma \equiv [X \times \gamma]$. Then for any invariant polynomial ϕ we have the current equation

$$d(\Gamma \phi(\widetilde{D})) = (\partial \Gamma) \phi(\widetilde{D}). \tag{10.2}$$

Applying the push-forward π_* to this equation we obtain

$$dT = \phi(D_b) - \phi(D_a) \qquad \text{where } T = \pi_*(\Gamma \phi(\widetilde{D})). \tag{10.3}$$

Example 10.4. Let $\gamma = [a,b] \subseteq \mathbf{R} = Y$. By formula (10.2) we have that

$$\widetilde{\Omega} = \Omega_t + \dot{\omega}_t dt$$

where $\cdot$ denotes $\partial/\partial t$. Let χ be the characteristic function of $[a,b]$. Then T can be expressed as the fibre integral

$$T = \pi_*(\chi\phi(\widetilde{D})) = \int_a^b \phi(\dot{\omega}_t \ ; \ \Omega_t)dt.$$

Let us return to the basic construction above, and replace γ by a general rectifiable p-chain Σ in Y. Then setting $\Gamma = X \times \Sigma$ and choosing ϕ as before we see that equation (10.2) continues to hold. Applying π_* to this equation gives

$$dR = \pi_*[\partial\Sigma\phi(\widetilde{D})] \tag{10.5}$$

where $R = \pi_*(\Sigma\phi(\widetilde{D}))$. This procedure allows us for example to relate transgression forms.

Example 10.6. Let $D_{s,t}$ be a 2-parameter family of connections, $0 \le s \le 1$ and $a \le t \le b$, with $D_{s,a} = D_a$ and $D_{s,b} = D_b$ for all $0 \le s \le 1$. Then the two transgressions T_1 and T_0 determined by $D_{1,t}$ and $D_{0,t}$ satisfy

$$T_1 - T_0 = dR \text{ with } R = \int_a^b \int_0^1 \phi\Big(\tfrac{\partial}{\partial s}\omega_{s,t}, \ \tfrac{\partial}{\partial t}\omega_{s,t} \ ; \ \Omega_{s,t}\Big) ds dt$$

where

$$\phi(A, B \ ; \ C) \stackrel{\text{def}}{=} \tfrac{\partial^2}{\partial s \partial t}\phi(C + sA + tB)\Big|_{s=t=0}.$$

Suppose now that we are given basic data

$$\begin{array}{ccc} D_E & & D_F \\ E & \xrightarrow{\alpha} & F \\ \langle\ ,\ \rangle_E & & \langle\ ,\ \rangle_F \end{array}$$

over a manifold X, as in (2.8) above. Let $f : X' \longrightarrow X$ be a smooth map, and consider the induced data

$$\begin{array}{ccc} D_{E'} & & D_{F'} \\ E' & \xrightarrow{\alpha'} & F' \\ \langle\ ,\ \rangle_{E'} & & \langle\ ,\ \rangle_{F'} \end{array}$$

over X', where $E' = f^*E$, $D_{E'} = f^*D_E$, etc.

Lemma 10.7. *Fix an approximation mode χ, and let $\overrightarrow{D}_s$, $\overrightarrow{D}'_s$ be the families of push-forward connection on F and F' respectively which are constructed from the data above. Let $T_{\phi,s}$ and $T'_{\phi,s}$ be the transgression forms associated to an invariant polynomial ϕ. Then*

$$T'_{\phi,s} \;=\; f^*T_{\phi,s}$$

for all $s > 0$. (The analogous result holds for the pull-back connections on E and E'.)

Proof. Fix local frames for E and F and pull them back to local frames for E' and F'. Then $\omega_{E'} = f^*\omega_E$ and $\omega_{F'} = f^*\omega_F$. Hence from the formulas in §5 we see that $\overrightarrow{\omega}'_s = f^*\overrightarrow{\omega}_s$, and so $\overrightarrow{\Omega}'_s = f^*\overrightarrow{\Omega}_s$ for all $s > 0$. The result now follows from the universality of the formula above for the transgression. □

Let $T_{\phi,s}$, $T'_{\phi,s}$ be as above, and denote by Σ the singular set of α and by $\Sigma' = f^{-1}(\Sigma)$ the singular set of α'. Then over $X - \Sigma$ and $X' - \Sigma'$ the limits of $T_{\phi,s}$ and $T'_{\phi,s}$ exist in the C^∞-topology as $s \to 0$. From the above we have that

$$\lim_{s\to 0} T'_{\phi,s}\, \big|_{X'-\Sigma'} \;=\; f^*\left(\lim_{s\to 0} T_{\phi,s}\, \big|_{X-\Sigma}\right).$$

Corollary 10.8. *Suppose $T \equiv \lim_{s\to 0} T_{\phi,s}$ exists in $L^1_{\mathrm{loc}}(X)$ and $T' \equiv \lim_{s\to 0} T'_{\phi,s}$ exists in $L^1_{\mathrm{loc}}(X')$. Then $T' = f^*T$ in the sense that $f^*\left(T\,\big|_{X-\Sigma}\right)$ extends by zero across Σ' to be the L^1_{loc}-form on T'.*

This result has content for us since in subsequent chapters we shall give elementary criteria which assure the hypothesis of 10.7. Specifically, these criteria will always be in terms of the "atomicity" of α. (See II.1, II.7, III.1 for definitions and discussion.) Hence, the maps f with $f^*T \in L^1_{\mathrm{loc}}$ will be those for which $f^*\alpha$ is atomic. This is discussed in III.3.26.

We now consider what happens to the transgression form under a change of approximation mode. Let χ_0 and χ_1 be two approximate-ones. Then $\chi_t = t\chi_1 + (1-t)\chi_0$ is also an approximate-one for all $0 \le t \le 1$, and this gives us a

2-parameter family $\overrightarrow{D}_{s,t}$, $s > 0$ and $0 \le t \le 1$, of push-forward connections on F. For each fixed $s > 0$, consider the smooth form

$$U_s \;=\; (\chi_{0,s} - \chi_{1,s}) \int_0^1 \phi\left(\tfrac{a^* Da}{|a|^2}\ ;\ \Omega_{s,t}\right) dt \tag{10.9}$$

where $\chi_{t,s} = \chi_t(a^*a/s^2)$ and where a represents α in the chosen local frames for E and F.

Lemma 10.10. *Suppose χ_0 and χ_1 are two choices of approximation mode, and let $T_{\chi_0,\phi,s}$ and $T_{\chi_1,\phi,s}$ be the corresponding families of transgressions constructed above using χ_0 and χ_1 respectively. Suppose that the limits $T_0 = \lim_{s\to 0} T_{\chi_0,\phi,s}$ and $T_1 = \lim_{s\to 0} T_{\chi_1,\phi,s}$ exist in the space of currents on X. If $\lim_{s\to 0} U_s = 0$, then there is a current R on X such that*

$$T_1 - T_0 \;=\; dR.$$

Proof. We apply the general form of the basic construction. Let $Y = (0,\infty] \times [0,1]$ and let $\overrightarrow{D}_{s,t}$ be the above family of connections parameterized by Y. Let $\Sigma_s = [s,\infty] \times [0,1] \subset Y$ and $\Gamma_s = X \times \Sigma_s$, and define

$$R_s \;=\; \pi_*(\Gamma_s \phi(\widetilde{D})).$$

Computing the right-hand-side in (10.5) gives

$$dR_s \;=\; T_{\chi_1,\phi,s} - T_{\chi_0,\phi,s} + U_s.$$

By assumption $\lim_{s\to 0} dR_s = T_1 - T_0$. The result now follows from the fact that d has closed range. $\square$

Remark 10.11. (Uniqueness of the transgression). Lemma 10.10 gives a general criterion for establishing that, up to a coboundary, the transgression is independent of the choice of approximation mode. However, in the cases considered below we obtain the much stronger result that the transgression is completely independent of approximation mode — not just up to a coboundary. These results appear in II.5.6, III.3.15, and V.1.37.

II. Complex Line Bundles

In this chapter we illustrate the nature of our results by examining the elementary but important case of complex line bundle maps. This chapter also presents a theory of real codimension-2 divisors which further refines the divisor theory for atomic sections developed in Harvey-Semmes [HS]. An interesting feature of the theory in [HS] is that it even enables one to define a (codimension-2) divisor for certain sections which vanish on sets of real codimension-one.

This chapter contains several interesting applications. There is a C^∞ generalization of the classical Poincaré-Lelong formula. A new proof is given of certain geometric formulas of Sid Webster [W1, 2, 3] in CR geometry, and some new formulas of Levine type are derived and proved. In the last section we combine our results with the kernel-calculus of Harvey-Polking [HP] to obtain a new proof of the Riemann-Roch theorem for vector bundles over algebraic curves.

For clarity of exposition this chapter is self contained.

1. Line Bundles With A Global Atomic Section.

Consider a complex-valued smooth function $g \in C^\infty(X)$ on a real oriented manifold X. In this section, the zero set of g is defined, as a codimension-2 current, under very mild assumptions on g.

Consider coordinates $w \equiv u + iv$ on the complex numbers $\mathbf{C}$. Note that the real and imaginary parts of $\frac{1}{2\pi i} \frac{dw}{w}$ are given by

$$\frac{1}{2\pi i} \frac{dw}{w} = \frac{1}{2\pi}\, d\theta + \frac{1}{2\pi i}\, d\log|w|, \qquad \text{on} \quad \mathbf{C}, \tag{1.1}$$

where

$$d\theta \equiv \frac{-vdu + udv}{u^2 + v^2}. \tag{1.2}$$

The fundamental equation of complex analysis can be written as

$$d\left(\frac{1}{2\pi i}\frac{dw}{w}\right) = [0] \quad \text{on} \quad \mathbf{C}, \tag{1.3}$$

where [0] denotes the δ-current or point mass at the origin. The objective of this section is to give meaning to the pullback, $g^*([0])$, of the current [0]. The strategy is to first give meaning to the pullback $\frac{1}{2\pi i}g^*\left(\frac{dw}{w}\right) = \frac{1}{2\pi i}\frac{dg}{g}$, of the potential $\frac{1}{2\pi i}\frac{dw}{w}$ for [0], as a current and then define the pullback $g^*([0])$ by taking the exterior derivative of $\frac{1}{2\pi i}\frac{dg}{g}$.

A differential form with coefficients that are locally integrable functions will be referred to as a **locally integrable form** or **$\mathbf{L^1_{loc}}$-form**. This is a well defined concept on a smooth manifold.

Definition 1.4. A complex valued function $g \equiv u + iv \in C^\infty(X)$ is **atomic** if

$$\frac{dg}{g} \in L^1_{\text{loc}}(X). \tag{1.5}$$

The function g is **weakly atomic** if

$$g^*(d\theta) \equiv \frac{-vdu + udv}{u^2 + v^2} \in L^1_{\text{loc}}(X) \quad \text{and} \quad \log|g| \in L^1_{\text{loc}}(X) \tag{1.6}$$

See the proof of Lemma 1.2 [HS] for the fact that atomicity implies weak atomicity.

If g is weakly atomic, then

$$\frac{1}{2\pi i}\frac{dg}{g} \equiv \frac{1}{2\pi i}\, d\log|g| + \frac{1}{2\pi}\, g^*(d\theta) \tag{1.7}$$

is well defined as a current on X, and agrees with the smooth form $\frac{1}{2\pi i}\frac{dg}{g}$ outside the zero set of g. When g is weakly atomic, the same expression $\frac{dg}{g}$ will be used to denote the current defined on all of X by (1.7). Note that if g and h are weakly atomic, then the product gh is weakly atomic and

$$\frac{d(gh)}{gh} = \frac{dg}{g} + \frac{dh}{h} \quad \text{as currents on } X. \tag{1.8}$$

The **zero current**, or **zero divisor**, denoted by $\mathbf{Z}_g$, or **Div(g)**, of a weakly atomic function g is defined by

$$Z_g \equiv d\left(\frac{1}{2\pi i}\frac{dg}{g}\right). \tag{1.9}$$

Note that by (1.7)

$$Z_g \quad \text{is a real current of codimension-2.}$$

For weakly atomic functions $g_1, \ldots, g_p$, one has

$$Z_{g_1 \cdots g_p} = Z_{g_1} + \cdots + Z_{g_p} \tag{1.10}$$

by (1.8) above. Furthermore, one has

$$Z_g = 0 \quad \text{if } g \text{ is real valued,} \tag{1.11}$$

because $g^*(d\theta) \equiv 0$ outside the zero set of g, and the zero set of g is a set of measure zero.

Definition 1.12. Suppose E is a smooth complex line bundle on X. A smooth section s of E is **weakly atomic** if for each local frame e for E the function a, defined by $s = ae$, is weakly atomic. The **zero current** or **zero divisor** denoted by $\mathbf{Z}_s$ or **Div(s)**, is defined to be the global current given locally by

$$\mathrm{Div}(s) \equiv d\left(\frac{1}{2\pi i}\frac{da}{a}\right).$$

Since $d\left(\frac{dh}{h}\right) = 0$ for $|h| > 0$, (1.8) implies that $\mathrm{Div}(s)$ is independent of the choice of local frame.

The properties of zero divisors for weakly atomic sections are the same as for zero divisors of functions. For example,

$$\mathrm{Div}(s) \text{ is a real current of codimension-2,}$$

$$\mathrm{Div}(s_1 \otimes \ldots \otimes s_p) \;=\; \mathrm{Div}(s_1) + \cdots + \mathrm{Div}(s_p),$$

and

$\mathrm{Div}(s) = 0$ if s is a real section of the complexification of a real line bundle.

Remark. The Geometric Meaning of Atomicity. The assumption that a function (or section) is atomic is an extremely weak hypothesis. In a later section conditions which insure atomicity are discussed. The assumption of weakly atomic allows codimension-1 zero sets with "folds" as well — see Corollary 7.3 and it's proof.

Remark 1.13. Atomic Bundle Maps. Prescribing a global section s of a complex line bundle F is a special case of prescribing a bundle map α. The bundle map associated to s is the map $\alpha : \underline{\mathbf{C}} \to F$ from the trivial bundle $\underline{\mathbf{C}}$, which is determined by the condition that $\alpha(1) = s$. Conversely, prescribing a bundle map $\alpha : E \to F$ is a special case of prescribing a global section of a complex line bundle, since α is a global section of the complex line bundle $\mathrm{Hom}(E, F)$.

A bundle map $\alpha : E \to F$ is **(weakly) atomic** if for each pair of local frames, e for E and f for F, the function a, defined by $\alpha e = af$, is (weakly) atomic. The **zero current** or **zero divisor**, denoted by $\mathbf{Z}_{\boldsymbol{\alpha}}$ or $\mathbf{Div}(\boldsymbol{\alpha})$ is defined to be the global current given locally by:

$$\mathrm{Div}(\alpha) \;\equiv\; d\left(\tfrac{1}{2\pi i}\,\frac{da}{a}\right). \tag{1.14}$$

If a bundle map $\alpha : E \to F$ is interpreted as a section $\alpha \in \mathrm{Hom}(E, F)$, or if a section s of F is interpreted as a bundle map $\alpha : \underline{\mathbf{C}} \to F$, the notions of atomicity and of zero divisor remain the same. However, the main result presented below is not the same for sections as for bundle maps (cf. Remark 6.11).

2. The Pullback Connection.

Suppose $\alpha : E \to F$ is a bundle map of complex line bundles with connections D_E and D_F respectively.

The **pullback connection** $\overleftarrow{D} \equiv \alpha^{-1} \circ D_F \circ \alpha$ on E is defined outside the singular set of α. If e and f are frames for E and F, respectively, then $D_E e = \omega_E e$, $D_F f = \omega_F f$, and $De = \omega e$ define (local) gauge potentials, and $\alpha e = af$ defines a (local) complex-valued function, a. Since $\left(\alpha^{-1} \circ D_F \circ \alpha\right) e = \left(\frac{da}{a} + \omega_F\right) e$, the gauge potential for the pullback connection $D \equiv \alpha^{-1} \circ D_F \circ \alpha$ is given by:

$$\omega \equiv \frac{da}{a} + \omega_F. \tag{2.1}$$

The difference of the two gauges, $\frac{da}{a} + \omega_F$ and ω_E, namely,

$$\tau \equiv \frac{da}{a} + \omega_F - \omega_E, \tag{2.2}$$

is a well defined global 1-form, outside the singular set of α.

Remark 2.3. This global one form τ has a nice interpretation involving α considered as a section of $H \equiv \mathrm{Hom}(E, F)$. The connections D_E and D_F induce a connection D_H on the line bundle $H \equiv \mathrm{Hom}(E, F)$ by the formula, $D_H \sigma = D_F \circ \sigma - \sigma \circ D_E$, for σ a section of H. One can easily compute that:

$$D_H \alpha = \tau \alpha \tag{2.4}$$

defines the same global one form τ as (2.2).

A current is **Federer flat** if it can be expressed locally as $\alpha + d\beta$ where α and β are in L^1_{loc}, i.e. α and β are forms with locally Lebesgue-integrable coefficients. Using a partition of unity it is easy to see that this is equivalent to requiring that the current be of the global form $\alpha + d\beta$, with α and β in L^1_{loc}.

Proposition 2.5. *If $\alpha : E \to F$ is a weakly atomic line bundle map, then the global one form τ, defined locally by (2.4), extends across the singular set of α as a well defined global current which is Federer flat.*

Proof. By Definition 1.4, the real part of $\frac{1}{2\pi i}\ \frac{da}{a}$ belongs to L^1_{loc}, while the imaginary part of $\frac{1}{2\pi i}\ \frac{da}{a}$ is the exterior derivative of $-\frac{1}{2\pi}\log|a|$, which also belongs to L^1_{loc}. Therefore, the right hand side of (2.2) is a well defined Federer flat current. Since $\frac{da}{a}$ changes by a smooth one-form under a change of the frames e and f, (2.2) defines a global Federer flat current. □

Proposition 2.6. *A bundle map $\alpha : E \to F$ is weakly atomic if and only if both of the globally defined objects*

$$\log \mid \alpha \mid^2 \quad \textit{and} \quad \mathrm{Re}\left(\tfrac{1}{2\pi i}\tau\right) \tag{2.7}$$

belong to $L^1_{\text{loc}}(X)$.

Proof. If $\alpha e = af$ defines a, then $\mid \alpha \mid^2 = \frac{\langle f,f\rangle_F}{\langle e,e\rangle_E} \mid a \mid^2$. Therefore, $\log \mid \alpha \mid^2 \in L^1_{\text{loc}}(X) \iff \log \mid a \mid^2 \in L^1_{\text{loc}}(X)$ and, by applying (2.2) and (1.7) with $g = a$, we see that $\mathrm{Re}\,\frac{1}{2\pi i}\tau \in L^1_{\text{loc}}(X) \iff a^*(d\theta) \in L^1_{\text{loc}}(X)$. □

Remark 2.8. Suppose $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ are hermitian inner products on E and F respectively. There are smooth global one forms α_E and α_F measuring the compatibility of the connections and the metrics. If $p \equiv \langle e, e\rangle_E$ and $q \equiv \langle f, f\rangle_F$, then

$$\alpha_E \ \equiv\ -\frac{dp}{p} + \omega_E + \bar{\omega}_E, \quad \alpha_F \ =\ -\frac{dq}{q} + \omega_F + \bar{\omega}_F. \tag{2.9}$$

Therefore, the imaginary part of $\frac{1}{2\pi i}\tau$ can be expressed as

$$2\,\mathrm{Im}\ \tfrac{1}{2\pi i}\ \tau \ =\ -\tfrac{1}{2\pi}\left(d\log \mid \alpha \mid^2 + \alpha_F - \alpha_E\right). \tag{2.10}$$

3. Smoothing the Pullback Connection.

Let $\alpha : E \to F$ be a map of hermitian line bundles with connections D_E and D_F, and metrics $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$, respectively, and let $\chi : [0,\infty] \to [0,1]$ be any choice of approximate one. (See I.4.1). Then the pullback connection $\overleftarrow{D} \equiv \alpha^{-1} \circ D_F \circ \alpha$ on E can be approximated by a smooth family of connections

$$\overleftarrow{D}_s \ \equiv\ \beta_s \circ D_F \circ \alpha + (1 - \beta_s\alpha) \circ D_E \tag{3.1}$$

where the approximations β_s to α^{-1} are given by

$$\beta_s \equiv \chi\left(\tfrac{\alpha^*\alpha}{s^2}\right)\alpha^{-1}. \tag{3.2}$$

The bundle map $\alpha^*\alpha : E \to E$ multiplies a section of E by a globally defined function. If e and f are frames for E and F respectively, if $p \equiv \langle e, e\rangle_E$, $q \equiv \langle f, f\rangle_F$, and if $\alpha e = af$, then $\alpha^* f = (\bar{a}q/p)e$. Therefore (cf. (2.9)),

$$(\alpha^*\alpha)\, e = (| a |^2 q/p)\, e, \quad \text{or} \quad \alpha^*\alpha = | \alpha |^2 .$$

Consequently the bundle map $\chi\left(\frac{\alpha^*\alpha}{s^2}\right)$ is just multiplication by the function

$$\chi_s \equiv \chi\left(\tfrac{|a|^2 q/p}{s^2}\right). \tag{3.3}$$

This function χ_s will be called the approximation to one based on χ. Since $\chi(0) = 0$, we see that

$$\overleftarrow{D}_s \text{ converges smoothly to } D_E \text{ as } s \to +\infty.$$

Since $\chi(\infty) = 1$, we see that outside the singular set of α,

$$\overleftarrow{D}_s \text{ converges smoothly to } \overleftarrow{D}_0 = \alpha^{-1} D_F \alpha \text{ as } s \to 0.$$

The gauge potential for $\overleftarrow{D}_s$, denoted by ω_s, is given by

$$\omega_s = \omega_E + \tau\chi_s \quad \text{where } \tau \equiv \frac{da}{a} + \omega_F - \omega_E, \tag{3.4}$$

while the curvature is given by

$$\Omega_s = d\omega_s.$$

Example 3.5. Algebraic approximation mode. If $\chi(t) \equiv t/(1+t)$, then

$$\chi_s = \frac{| a |^2 q/p}{| a |^2 q/p + s^2}.$$

Example 3.6. Transcendental approximation mode. If $\chi(t) \equiv 1 - e^{-t}$, then

$$\chi_s \equiv 1 - e^{-\frac{|a|^2 q/p}{s^2}}.$$

4. Chern Currents.

Recall that the Chern form $\phi(D)$ of a complex line bundle E with connection D is defined as follows. Each local frame e for E defines a (local) 1-form ω by the equation

$$De = \omega e.$$

The form ω is called the **gauge potential** or **connection 1-form**. Note that if $e' = be$ is another local frame for E, then

$$De' = \left(\frac{db}{b} + \omega\right) e' \quad \text{or} \quad \omega' = \frac{db}{b} + \omega.$$

Since $\frac{db}{b}$ is d-closed the local expressions $d\omega$ and $d\omega'$ agree on overlapping domains and so the **curvature form**

$$\Omega \equiv d\omega$$

is a globally defined 2-form on M.

Given a polynomial $\phi(t) \in \mathbf{C}[t]$ in one indeterminate, the **ϕ-Chern form**, $\phi(D)$, is defined by

$$\phi(D) \equiv \phi(\Omega).$$

If $\phi(t) = \frac{i}{2\pi}t$ then the normalized curvature $\phi(D) = \frac{i}{2\pi}\Omega$ is called the **first Chern form**. In the following formulae, it is convenient to let

$$\widetilde{\Omega} \equiv \tfrac{i}{2\pi}\Omega$$

denote the first Chern form.

Now suppose $\alpha : E \to F$ is a map of complex hermitian line bundles with connections D_E and D_F, respectively, and set $\Sigma = \{x \in X : \alpha_x = 0\}$. Suppose $\overleftarrow{D}_s$ is the smoothing of the pullback connection $\alpha^{-1} \circ D_F \circ \alpha$ based on the approximate-one χ. Let Ω_s denote the curvature of the connection $\overleftarrow{D}_s$. If $\phi(\overleftarrow{D}_s)$ has a weak limit as currents, then this limit must be of the form

$$\phi((\overleftarrow{D})) = \lim_{s \to 0} \phi(\Omega_s) = \phi(\Omega_F) + S, \tag{4.1}$$

where S is a current with support in the singular set of α, since $\overleftarrow{D}_s$ converges to $\alpha^{-1} \circ D_F \circ \alpha$ outside Σ and $\phi(\alpha^{-1} \circ D_F \circ \alpha) = \phi(D_F)$ extends as a smooth d-closed form across Σ.

Definition 4.2. If the limit in (4.1) exists weakly in the space of currents on X, this limit $\phi((\overleftarrow{D})) \equiv \phi(\Omega_F) + S$ will be called the **ϕ-Chern current** of the pullback connection on E.

5. The Transgression Current.

The standard transgression formula for the family of connections $\overleftarrow{D}_s$ says that

$$dT_s = \phi(\Omega_E) - \phi(\Omega_s), \tag{5.1}$$

where

$$T_s \equiv \int_s^{\infty} \phi(\dot{\omega}_t \,;\, \Omega_t)\, dt,$$

and where $\phi(\alpha \,;\, \Omega)$ is defined by I.1.16. Note that if $\phi(\Omega) \equiv \Omega^n$, then $\phi(\alpha \,;\, \Omega) = n\alpha\Omega^{n-1}$. If the limit $\lim_{s \to 0} T_s$ exist weakly as currents, then this limit will be called the **ϕ-transgression current** or the **ϕ-potential** and will be denoted by T_ϕ.

Lemma 5.2.

$$T_s = -\tau \frac{\phi(\Omega_E + (\Omega_F - \Omega_E)\chi_s) - \phi(\Omega_E)}{\Omega_F - \Omega_E} \tag{5.3}$$

where τ is given by (2.2) *above.*

Proof. Since both sides of (5.3) are smooth across the singular set of α, it suffices to verify (5.3) on the set where $\alpha \neq 0$. By (3.4), we have

$$\dot{\omega}_t = \tau\chi_t', \tag{5.4}$$

and

$$\Omega_t = \Omega_E + (\Omega_F - \Omega_E)\,\chi_t - \tau d\chi_t. \tag{5.5}$$

Therefore,

$$
\begin{aligned}
T_s &= \int_s^\infty \phi(\dot{\omega}_t \; ; \; \Omega_t)\, dt \\
&= \int_s^\infty \phi\left(\tau\chi_t' \; ; \; \Omega_E + (\Omega_F - \Omega_E)\chi_t - \tau d\chi_t\right)\, dt \\
&= \tau \int_s^\infty \phi\left(\chi_t' \; ; \; \Omega_E + (\Omega_F - \Omega_E)\chi_t\right)\, dt
\end{aligned}
$$

since $\tau \wedge \tau = 0$. Now under the change of variables $x \equiv \chi_t$ we have

$$\int_s^\infty \phi\left(\chi_t' \; ; \; \Omega_E + (\Omega_F - \Omega_E)\chi_t\right)\, dt \;=\; \int_{\chi_s}^0 \phi(1 \; ; \; \Omega_E + (\Omega_F - \Omega_E)x)\, dx.$$

This integral equals $\phi\left(\Omega_E + (\Omega_F - \Omega_E)x\right)\Big|_{\chi_s}^{0}$ "divided by" $\Omega_F - \Omega_E$, yielding (5.3). $\square$

Theorem 5.6. *Suppose $\alpha : E \to F$ is a bundle map of hermitian line bundles with connections D_E and D_F respectively. Suppose $\overleftarrow{D}_s$ is a smoothing of the pullback connection $\overleftarrow{D} = \alpha^{-1} \circ D_F \circ \alpha$ based on the approximate one χ. Let $\phi(t) \in \mathbf{C}[t]$ be any polynomial. Then if the bundle map α is weakly atomic, the ϕ-transgression current T_ϕ exists. In fact, T_s converges to T_ϕ in the Federer flat topology and*

$$T_\phi \;\equiv\; -\tau\, \frac{\phi(\Omega_F) - \phi(\Omega_E)}{\Omega_F - \Omega_E}. \tag{5.7}$$

In particular, T_ϕ is independent of the choice of metrics, $\langle \; , \; \rangle_E$ and $\langle \; , \; \rangle_F$, and of the choice of smoothing family $\overleftarrow{D}_s$ (i.e., independent of the choice of approximate one χ).

Proof. By (5.3) T_s is equal to τ times a polynomial in χ_s whose coefficients are smooth forms and whose constant term vanishes. Hence it suffices to prove that for $k \geq 1$,

$$\tau\, \chi_s^k \quad \text{converges to} \quad \tau$$

in the Federer flat topology, as s approaches zero. Observe now that the properties required of $\chi(t)$ in I.4.1 are also valid for $\chi(t)^k$. Therefore, it suffices to consider the case $k = 1$.

By the hypothesis that α is weakly atomic we see that $\operatorname{Re} \frac{1}{2\pi i}\tau \in L^1_{\text{loc}}(X)$. Therefore, by the Lebesgue Dominated Convergence Theorem

$$\operatorname{Re} \tfrac{1}{2\pi i}\tau\chi_s \quad \text{converges to} \quad \operatorname{Re} \tfrac{1}{2\pi i}\tau \;\text{ in }\; L^1_{\text{loc}}(X),$$

as s approaches zero. It remains to prove the analogous statement for the imaginary part.

To do this we first note that by the hypothesis on α we have $\log \mid \alpha \mid^2 \in L^1_{\text{loc}}(X)$ where $\mid \alpha \mid^2 \equiv \mid a \mid^2 q/p$. By equation (2.10), $\operatorname{Im} \frac{1}{2\pi i}\tau$ and $-\frac{1}{4\pi} d\log \mid \alpha \mid^2$ differ by a (global) smooth form. Therefore, it suffices to show that

$$\chi_s d\log \mid \alpha \mid^2 \quad \text{converges to} \;\; d\log \mid \alpha \mid^2,$$

in the Federer flat topology. Set $\varphi = \mid \alpha \mid^2$ and note that if $h(t)$ satisfies

$$h'(t) \;=\; \chi(t) \cdot \frac{1}{t},$$

then by (3.3)

$$d\left(h\left(\tfrac{\varphi}{s^2}\right)\right) \;=\; \chi\left(\tfrac{\varphi}{s^2}\right) \frac{d\varphi}{\varphi} \;=\; \chi_s \frac{d\varphi}{\varphi} \;=\; \chi_s d\log\varphi.$$

Let us choose

$$h(t) \;\equiv\; \int_0^t \chi(t)\, \frac{dt}{t}.$$

Then

$$\begin{aligned} h\left(\tfrac{\varphi}{s^2}\right) &= \int_0^{\frac{\varphi}{s^2}} \chi(t)\, \frac{dt}{t} \;=\; \int_0^1 \chi(t)\, \frac{dt}{t} + \int_1^{\frac{\varphi}{s^2}} \chi(t)\, \frac{dt}{t} \\ &= \int_0^1 \chi(t)\, \frac{dt}{t} - \int_1^{\frac{\varphi}{s^2}} (1 - \chi(t))\, \frac{dt}{t} + \log \tfrac{\varphi}{s^2} \\ &= \int_0^1 \chi(t)\, \frac{dt}{t} - \int_1^{\infty} (1 - \chi(t))\, \frac{dt}{t} - \log s^2 + \log\varphi + \int_{\frac{\varphi}{s^2}}^{\infty} (1 - \chi(t))\, \frac{dt}{t}. \end{aligned}$$

Consequently,

$$h_s \equiv \log\varphi + \int_{\frac{\varphi}{s^2}}^{\infty} (1-\chi(t))\,\frac{dt}{t}, \tag{5.8}$$

has the same exterior derivative as $h\left(\frac{\varphi}{s^2}\right)$, i.e.

$$dh_s = \chi_s\, d\log\varphi. \tag{5.9}$$

(One can also easily verify directly that (5.8) implies (5.9)).

Since h_s decreases to $\log\varphi \in L^1_{\rm loc}(X)$, the Monotone Convergence Theorem implies that h_s converges to $\log\varphi$ in $L^1_{\rm loc}(X)$. Hence $dh_s = \chi_s d\log\varphi$ converges to $d\log\varphi$ in the Federer flat topology. □

Remark. This proof could have been shortened slightly by only considering unitary frames, i.e., $p = q = 1$. Then $|\,\alpha\,|^2 = |\,a\,|^2$, for example.

Remark 5.10. Functoriality of the Transgression Current. Suppose $f : X' \longrightarrow X$ is a smooth map between manifolds, and let $E' = f^*E$, $D_{E'} = f^*D_E$, $F' = f^*F, \ldots, \alpha' = f^*\alpha$ be the pullbacks of the line bundles, connections, etc. given over X. Fix any approximate one χ. Then the family of smooth transgression forms T'_s for this induced family is given by $T'_s = f^*T_s$ for $s > 0$ (cf. I.5.25). Suppose now that α is weakly atomic and assume that the induced section $\alpha' = f^*\alpha$ is also weakly atomic. Then the transgression currents $T = \lim_{s\to 0} T_s$ and $T' = \lim_{s\to 0} T'_s$ satisfy

$$T' = f^*T \tag{5.11}$$

whenever this equation makes sense. For example suppose α and α' are both atomic, and let $\Sigma = \{x \in X : \alpha_x = 0\}$ and $\Sigma' = \{x' \in X' : \alpha'_{x'} = 0\} = f^{-1}(\Sigma)$. In this case T and T' are $L^1_{\rm loc}$-extensions of well defined, smooth forms in $X - \Sigma$ and $X' - \Sigma'$ across Σ and Σ' respectively. Equation (5.11) asserts the equality of two smooth forms in $X' - \Sigma'$ (which possess an $L^1_{\rm loc}$ extension to X').

6. The Main Results—First Version.

The ϕ-Chern current $\phi((\overleftarrow{D}))$ for the pullback connection $\overleftarrow{D} \equiv \alpha^{-1} \circ D_F \circ \alpha$ can be computed from Theorem 5.6.

Theorem 6.1. Pullback. *Under the hypotheses of Theorem 5.6, the ϕ-Chern current, $\phi((\overleftarrow{D}))$, of the pull back connection $\overleftarrow{D} \equiv \alpha^{-1} \circ D_F \circ \alpha$ exists (in fact, $\phi((\overleftarrow{D})) = \lim_{s\to 0} \phi(\overleftarrow{D}_s)$ exists in the Federer flat topology) and equals:*

$$\phi((\overleftarrow{D})) \equiv \lim_{s\to 0} \phi(\overleftarrow{D}_s) = \phi(\Omega_F) + 2\pi i \frac{\phi(\Omega_F) - \phi(\Omega_E)}{\Omega_F - \Omega_E} \operatorname{Div}(\alpha). \tag{6.2}$$

Furthermore,

$$\phi((\overleftarrow{D})) - \phi(\Omega_E) = -d T_\phi$$

or equivalently,

$$\phi(\Omega_F) - \phi(\Omega_E) + 2\pi i \frac{\phi(\Omega_F) - \phi(\Omega_E)}{\Omega_F - \Omega_E} \operatorname{Div}(\alpha) = -d T_\phi, \tag{6.3}$$

In particular, $\phi(t) \equiv \frac{i}{2\pi} t$ yields

$$\tfrac{i}{2\pi}\Omega_E - \tfrac{i}{2\pi}\Omega_F + \operatorname{Div}(\alpha) = \tfrac{1}{2\pi i}\, d\tau, \tag{6.4}$$

where

$$\tau \equiv \frac{da}{a} + \omega_F - \omega_E \quad \textit{is a global current.} \tag{6.5}$$

Proof. To begin note that (6.4) is just the special case of (6.3) where $\phi(t) \equiv \frac{i}{2\pi} t$ and $T_\phi = \frac{1}{2\pi i}\tau$. This special case, (6.4), is obtained by taking the exterior derivative of (6.5). Define

$$\operatorname{Res}_\phi(\overleftarrow{D}) \equiv -2\pi i \frac{\phi(\Omega_F) - \phi(\Omega_E)}{\Omega_F - \Omega_E}. \tag{6.6}$$

Then (5.7) says that $T_\phi = \frac{1}{2\pi i}\tau \operatorname{Res}_\phi(\overleftarrow{D})$. Taking the exterior derivative of this equation and using (6.4) yields the general case (6.3).

Observe now that by (5.1) the ϕ-Chern current is given by

$$\phi((\overleftarrow{D})) = \lim_{s\to 0} \phi(\overleftarrow{D}_s) = \phi(\Omega_E) - dT_\phi.$$

Since the ϕ-Chern current exists, $\phi((\overleftarrow{D})) = \phi(\Omega_F) + S$ (see Definition 4.2). Now (6.3) implies that $S = -\operatorname{Res}_\phi(\overleftarrow{D}) \operatorname{Div}(\alpha)$, completing the proof of (6.2). $\square$

Remark 6.7. Pushforward. The **pushforward connection** on F

$$\overrightarrow{D} \equiv \alpha \circ D_E \circ \alpha^{-1}$$

can be analyzed in a similar manner, where

$$\overrightarrow{D}_s \equiv \alpha \circ D_E \beta_s + D_F \circ (1 - \alpha\beta_s)$$

provides the smoothing family (based on χ as before). The local gauge potential for $\overrightarrow{D}_s$ is given by

$$\overrightarrow{\omega}_s \equiv \omega_F - \tau\chi_s,$$

with the same τ given by (6.5) above. The curvature of $\overrightarrow{D}_s$ is given by

$$\Omega_s = \Omega_F - (\Omega_F - \Omega_E)\,\chi_s + \tau\, d\chi_s$$

If α is atomic then the ϕ-Chern current exists and is given by

$$\phi((\overleftarrow{D})) = \phi(\Omega_E) - 2\pi i \frac{\phi(\Omega_F) - \phi(\Omega_E)}{\Omega_F - \Omega_E} \operatorname{Div}(\alpha).$$

Furthermore

$$\phi((\overleftarrow{D})) - \phi(\Omega_F) = dT_\phi$$

with the same transgression current T_ϕ as above. Combining these equations gives a second derivation of equation (6.3).

Remark 6.8. Atomic Sections. Suppose s is a global weakly atomic section of a complex line bundle F with connection D_F. Then the one form τ, defined by $D_F s = \tau s$ outside the zero set of s, extends as a Federer flat current, also denoted by τ.

$$\text{Note that } \tau = \frac{da}{a} + \omega_F, \quad \text{and let } T \equiv \tfrac{1}{2\pi i}\tau\widetilde{\Omega}_F^{n-1}.$$

Then as in (5.7) with $E \equiv \underline{\mathbf{C}}$ trivial and $\phi(t) = \left(\frac{i}{2\pi}t\right)^n$ we have

$$-dT = \widetilde{\Omega}_F^n - \widetilde{\Omega}_F^{n-1} \operatorname{Div}(\alpha). \tag{6.9}$$

This formula can be verified directly by differentiating T using the fact that

$$-\tfrac{1}{2\pi i}\, d\tau = \widetilde{\Omega}_F - \operatorname{Div}(\alpha). \tag{6.10}$$

Note that (6.10) is the case of (6.9) when $n = 1$.

Remark 6.11. Bundle Maps Versus Sections. If a bundle map $\alpha : E \to F$ is considered to be a section of $H \equiv \operatorname{Hom}(E, F)$, then

$$\widetilde{\Omega}_H = \widetilde{\Omega}_F - \widetilde{\Omega}_E$$

and (6.9) becomes

$$(6.12) \qquad -dT = \left(\widetilde{\Omega}_F - \widetilde{\Omega}_E\right)^n - \left(\widetilde{\Omega}_F - \widetilde{\Omega}_E\right)^{n-1} \operatorname{Div}(\alpha).$$

This is not the formula (6.3) of the Main Theorem. That is, Theorem 6.1 for bundle maps is not a special case of the result (6.12) for cross-sections.

Remark 6.13. Global self intersections. Considerations purely of local analysis might tempt one to define

$$[a = 0] \wedge da = [a = 0] \wedge d\bar{a} = 0,$$

since $[a = 0] = \delta_0(a)\frac{i}{2} da \wedge d\bar{a}$. However, this would imply that $\operatorname{Div}(\alpha) \wedge \operatorname{Div}(\alpha) = 0$, i.e., that all global self intersections vanish. In this context note the following. First, the limit

$$\lim_{s \to 0} \widetilde{\Omega}_s = \widetilde{\Omega}_F - \operatorname{Div}(\alpha),$$

does not involve the connection D_E on E. Moreover,

$$\lim_{s \to 0} \widetilde{\Omega}_s^2 = \widetilde{\Omega}_F^2 - \left(\widetilde{\Omega}_E + \widetilde{\Omega}_F\right) \wedge \operatorname{Div}(\alpha),$$

so that formally

$$\operatorname{Div}(\alpha) \cdot \operatorname{Div}(\alpha) = \left(\widetilde{\Omega}_F - \widetilde{\Omega}_E\right) \cdot \operatorname{Div}(\alpha),$$

which need not be zero.

Remark 6.14. Residues. Formula (5.7) expresses the transgression current as $T_\phi \equiv \frac{da}{a} \wedge \beta + \gamma$ where β and γ are smooth forms. Whenever we are given such a decomposition, we can define the **residue** of T_ϕ to be the form $2\pi i \beta \,\big|_{a=0}$. (See Chapter III, Section 1.) This justifies the terminology adopted in (6.6).

Remark 6.15. Approximate identities and safety disks. Given any approximate one χ as in I.4.1, an **approximate identity** $[0]_s$ can be constructed for the origin in the complex plane. Let $\chi_s \equiv \chi\left(\frac{|a|^2}{s^2}\right)$. Define $[0]_s \equiv d\left(\frac{1}{2\pi i}\frac{da}{a}\chi_s\right)$. Then $[0]_s \equiv d\left(\frac{1}{2\pi i}\frac{da}{a}\chi_s\right) = -\frac{1}{2\pi i}\frac{da}{a}\,d\chi_s = \frac{i}{2\pi}\chi'\left(\frac{|a|^2}{s^2}\right)\frac{dad\bar{a}}{s^2} = \frac{1}{\pi}\varphi\left(\frac{|a|^2}{s^2}\right)\frac{i}{2}\frac{dad\bar{a}}{s^2}$, where $\varphi(t) \equiv \chi'(t)$.

Let $\mu_s : \mathbf{C} \to \mathbf{C}$ be the map $\mu_s(z) = z/s$. Then $[0]_s \equiv \mu_s^*\left(\frac{1}{\pi}\varphi(|a|^2)\frac{i}{2}\,da \wedge d\bar{a}\right)$ is the standard formula for the **approximate identity at the origin in C based on the function** $\frac{1}{\pi}\varphi(|a|^2)$. Note that $\int_{\mathbf{C}}\frac{1}{\pi}\varphi(|a|^2)\frac{i}{2}\,dad\bar{a} = \int_0^\infty\int_0^{2\pi}\frac{1}{\pi}\varphi(r^2)rdrd\theta = \int_0^\infty \varphi(r^2)2rdr = \chi(\infty) - \chi(0) = 1$. The standard fact about approximate identities is that $[0]_s$ converges to $[0]$.

For example, if $\chi(t) \equiv \frac{t}{1+t}$, then $[0]_\epsilon \equiv \frac{1}{\pi}\frac{\epsilon^2\frac{i}{2}dad\bar{a}}{(|a|^2+\epsilon^2)^2}$ while if $\chi(t) \equiv 1 - e^{-t}$, then $[0]_\epsilon = \frac{1}{\pi}e^{-\frac{|a|^2}{\epsilon^2}}\frac{i}{2}\frac{da\wedge d\bar{a}}{\epsilon^2}$.

Since $\frac{1}{2\pi i}\frac{da}{a}\chi_\epsilon$ converges to $\frac{1}{2\pi i}\frac{da}{a}$ in $L^1_{\mathrm{loc}}(\mathbf{C})$, it's exterior derivative $[0]_\epsilon$ must converge to $d\left(\frac{1}{2\pi i}\frac{da}{a}\right)$ as currents. This proves the following equation of currents on $\mathbf{C}$:

$$d\left(\frac{1}{2\pi i}\frac{da}{a}\right) = [0] \tag{6.16}$$

Consider now the function

$$\chi(t) = \begin{cases} 1 & \text{for } t \geq 1 \\ 0 & \text{for } 0 \leq t < 1. \end{cases}$$

Then the associated current $[0]_\epsilon$ is given on a test function ψ by the mean value $[0]_\epsilon(\psi) = \frac{1}{2\pi}\int_0^{2\pi}\psi(\epsilon e^{i\theta})d\theta$. Again one obtains a proof of (6.16) using this choice of χ. In fact this is just the classical proof of the Cauchy Integral formula for a disk, based on the usual safety disk argument. However, **non-smooth functions of this type are not acceptable as approximate ones** in our theory. We can explain this as follows. Let χ be any (smooth) approximate one and let $[0]_\epsilon$ be the associated current as above. Then for each $k \geq 1$ we have that

$$\lim_{\epsilon\to 0}\chi_\epsilon^{k-1}[0]_\epsilon = \frac{1}{k}[0]. \tag{6.17}$$

Using this formula, one can give a direct proof of the formula for the ϕ-Chern current $\phi(D)$ of the pullback connection. However, for the discontinuous choice of χ given above we have $\chi^k = \chi$ so that (6.17) is not valid. Thus the classical geometric arguments employing a "safety disk" or "tubular neighborhood" of $a = 0$ can not be used to compute general Chern currents.

As it stands the results of this section are incomplete. Two important additional ingredients are needed to provide more substance. First, we will give geometric conditions which ensure that a section is atomic or (equivalently) that a bundle map is atomic. Then we will discuss conditions which ensure that the divisor is just integration over "submanifolds with multiplicities".

7. Conditions Which Insure Atomicity.

Let E be a complex line bundle over a connected manifold X. In this section we give some easily verified geometric conditions on a section s of E, which guarantee that s is atomic. The results apply immediately to maps between line bundles. We shall consider three cases: where s is holomorphic, real analytic and C^∞.

For the first result we assume that E is a holomorphic line bundle over a complex manifold X.

Proposition 7.1. The Holomorphic Case. *A holomorphic section s is atomic if (and only if) s is not identically zero.*

First Proof. Suppose s is a non-trivial holomorphic section. Choose a local holomorphic frame e so that $s = ae$ defines a local holomorphic function a. Then $\log|a|^2$ is pluri-subharmonic. It is a standard fact that $\log|a|$ and all first partial derivatives are locally integrable. In particular, $\frac{da}{a} = \partial \log|a|^2$ is locally integrable, so that s is atomic. $\square$

It is remarkable that a similar result holds in the real analytic case. Here we must, of course assume that E and X are real analytic.

Theorem 7.2. *(Harvey-Semmes* [HS]*) Suppose g is a (complex valued) real analytic function.*

a) If g is not identically zero, then $\log|g| \in L^1_{\text{loc}}$.

b) If $\operatorname{codim}_{\mathbf{R}} g^{-1}(0) \geq 2$, then $\frac{dg}{g} \in L^1_{\text{loc}}$ (i.e., g is atomic).

Corollary 7.3. The Real Analytic Case. *A real analytic section s is weakly atomic if (and only if) s is not identically zero.*

Proof of Corollary. Consider a complex-valued real analytic function g. Locally g can be factored into prime factors. It suffices to show that each factor is weakly atomic because of (1.10). Thus we may assume that g is prime and that the real codimension of $g^{-1}(0)$ is one by 7.2.a. By 7.2.b it suffices to prove that $g^*(d\theta) \in L^1_{\text{loc}}$. Let $\tilde{g}$ denote the holomorphic extension of g to the complexification of the domain of g. Then $\tilde{g}$ is prime so that its zero set Z is irreducible. In particular, if $\tilde{h}$ vanishes on Z then $\tilde{h}$ is a multiple of $\tilde{g}$. Now let $g \equiv u + iv$ with u, v real-valued. Since u vanishes on $g^{-1}(0)$ and $g^{-1}(0)$ has codimension-one, the holomorphic extension $\tilde{u}$ of u must vanish on Z. Thus $\tilde{u}$ is a multiple of $\tilde{g}$, say $\tilde{u} = \phi\tilde{g}$. Therefore $(1-\phi)\,u = i\phi v$, which implies that either u is a multiple of v or v is a multiple of u. The multiple must be a unit since g is prime. Consequently, replacing g by a unit times g we may assume that g is real-valued. Therefore $g^*(d\theta)$ is identically zero and hence it is certainly in L^1_{loc} (note that $\text{Div}_g = 0$ by (1.7) and (1.9)). □

Second Proof of Proposition 7.1. Apply Corollary 7.3. □

Theorem 7.2 follows from the next result. This result can be considered the C^∞ case. Under very mild and geometrically reasonable restrictions on the vanishing of a smooth function g, it insures that g is atomic.

Lemma 7.4. $\mathbf{C^\infty}$ Case. *(Harvey-Semmes* [HS]*) Suppose $g \in C^\infty(U)$ is a complex-valued function defined on an open subset U in $\mathbf{R}^m$ and set $Z \equiv \{x \epsilon U : g(x) = 0\}$. Assume that:*

(i) **g vanishes algebraically,** *i.e., for each compact set $K \subset U$ there exist constants $c > 0$ and N such that:*

$$|g(x)| \geq c\,\text{dist}(x, Z)^N \qquad \text{for all } \; x \epsilon K.$$

(ii) **Z is of Minkowski codimension strictly greater than one** *in the sense that for each compact set $K \subset U$ there exists an $\epsilon > 0$ such that the upper Minkowski content of $Z \cap K$ in dimension $m - 1 - \epsilon$ is finite.*

Then g is atomic. In fact, $\frac{dg}{g}$ is locally integrable.

Proof of Theorem 7.2. If g is real analytic, then g must vanish algebraically. This important result is due to Lojasiewicz [L]. Furthermore, an irreducible real-analytic subvariety has Minkowski codimension equal to its codimension as a real-analytic variety. See, for example, Federer [F] for a proof. $\square$

8. Divisors of Atomic Sections.

Suppose Z is the zero set of a smooth function g or a smooth section s. Let

$$\operatorname{Reg} Z \;=\; \{x : Z \text{ is a codimension-2 Lipschitz submanifold near } x\}$$

denote the set of of **regular points** of Z, and let

$$\operatorname{Sing} Z \;\equiv\; Z \sim \operatorname{Reg} Z$$

denote the set of **singular points**. Let $\{Z_j\}$ denote the family of connected components of $\operatorname{Reg} Z$.

Proposition 8.1. *Suppose that s is an atomic section. Then*

$$\operatorname{Div}(s) \;=\; \sum_{j=1}^{\infty} n_j[Z_j], \qquad \text{on} \quad X \sim \operatorname{Sing} Z \tag{8.2}$$

where each n_j is an integer and where $[Z_j]$ is the current defined by choosing a continuous orientation on Z_j.

Remark 8.3. Orientation. First note that if $n_j = 0$ then Z_j need not be orientable. We shall adopt the following conventions. The orientation of a component Z_j with $n_j \neq 0$ is chosen so that $n_j > 0$ is positive except in the following two cases.

Each zero-dimensional submanifold, (i.e., a point p) is canonically oriented by defining $[p]$ to be "evaluation of degree-zero forms at the point p." Thus

Point Divisors $\operatorname{Div}(s) \;=\; \sum n_j\,[p_j^+] - \sum m_j\,[p_j^-]$ with $n_j > 0$ and $m_j > 0$.

If X is a complex manifold then each complex submanifold of X has a canonical orientation. Thus if each Z_j is a complex submanifold, then

Holomorphic Divisors

$$\operatorname{Div}(s) = \sum n_j[Z_j^+] - m_j[Z_j^-] \quad \text{with } n_j > 0 \text{ and } m_j > 0.$$

Proof of 8.1. A current of the form $\alpha + d\beta$ with α, $\beta \in L^1_{\text{loc}}$ is said to be **Federer flat**. These currents have many exciting properties. First, note that if g is atomic then $\frac{dg}{g}$ has been defined as a Federer flat current. Therefore the exterior derivative is also Federer flat. Thus $\text{Div}(g)$ is a Federer flat current. In a neighborhood of a regular point of Z, the codimension-two flat current $\text{Div}(g)$ is supported in a codimension-two submanifold Z. By a theorem of Federer [F] this implies that the current $\text{Div}(g)$ must be of the form $\phi[Z]$ where ϕ is an L^1_{loc} function on Z. However, since $\text{Div}(g)$ is d-closed, the function ϕ must be d-closed on M and hence a constant. Thus $\text{Div}(g) = c[Z]$ near a regular point of Z.

To show that c is an integer first consider the case where the ambient manifold is of dimension two. Then $\text{Div}(g) = c[p]$ where p is a point. In this case it is standard that c is the multiplicity of g considered as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$. The general case can be reduced to this case by considering real two-planes transverse to Z. □

Proposition 8.4. *Suppose that s is an atomic section. Assume that the singular set of the zeros of s is neglible in the sense that the codimension-2 Hausdorff measure of* $\text{Sing}\, Z$ *vanishes. Assume also that the current $\sum n_j[Z_j]$ given by* (8.2) *on $X \sim$* $\text{Sing}\, Z$ *has locally finite mass in X. Then*

$$\text{Div}(s) = \sum n_j[Z_j] \qquad on \quad X.$$

Proof. The mass of $\sum n_j[Z_j]$ on a compact set $K \subset X$ is $\sum |n_j| \,\text{vol}(Z_j \cap K)$ which is assumed to be finite. Thus, the current $\sum n_j[Z_j]$ has a natural extension by zero to all of X. This current $\sum n_j[Z_j]$ on X is Federer flat (see [F]). Since $\text{Div}(s)$ is Federer flat, the difference $\text{Div}(s) - \sum n_j[Z_j]$ is also a Federer flat current. This current is of codimension two and supported in the set $\text{Sing}\, Z$ whose Hausdorff measure in codimension-2 is zero. A useful theorem of Federer [F; 4.1.20] says that this difference must vanish. □

If g is a real analytic function, then (see [F] for example)

$$\begin{gathered}\text{at most a finite number of components } \ Z_j \text{ of } \operatorname{Reg} Z_g \\ \text{intersect a given compact set } \ K \subset X,\end{gathered}$$

and

$$\text{each } Z_j \text{ has locally finite volume in } X.$$

In particular $\sum n_j[Z_j]$ has locally finite mass in X, for any choice of integer multiplicities n_j associated with the oriented components Z_j of $\operatorname{Reg} Z_g$. Therefore the previous Proposition applies to prove the following.

Corollary 8.5. *If s is a real analytic section which is not identically zero, then*

$$\operatorname{Div}(s) = \sum n_j[Z_j] \text{ is a real analytic chain of real codimension-2.} \tag{8.6}$$

In applications, it is important to remember that many of the components of the manifold points of the zero set of g make no contribution to the zero divisor of g. For example, any component of real codimension-one does not occur on the right hand side of (8.6). Furthermore, no component which is non-orientable can appear on the right hand side of (8.6), even if it has codimension-2.

9. The Main Results—Second Version.

The main results are obtained by summarizing the previous sections. For simplicity attention is restricted to the polynomial $\varphi(t) = \frac{i}{2\pi}t$ yielding the first Chern form.

Theorem 9.1. C^∞ Case. *Suppose s is a smooth section of a complex line bundle E and let D_E be a smooth connection on E. Assume the following:*

(i) *The section s vanishes algebraically (i.e., to finite order).*
(ii) *The zero set Z of s has Minkowski codimension greater than one.*
(iii) *The set $\operatorname{Sing} Z$ has Hausdorff measure zero in codimension-two.*
(iv) *The current $\operatorname{Div}(s)$ has locally finite mass.*

Then $\operatorname{Div}(s)$ is of the form $\operatorname{Div}(s) = \sum n_j[Z_j]$, where the n_j's are integers and the Z_j's are oriented codimension-2 components of $\operatorname{Reg} Z$; and we have the equation

$$\frac{1}{2\pi i}\, d\tau = \sum n_j[Z_j] - c_1(D_E) \tag{9.2}$$

where τ is the locally integrable 1-form defined by $Ds = \tau s$.

Theorem 9.3. Real-Analytic Case. *Suppose s is a real analytic section of a complex line bundle E with smooth connection D_E over a connected manifold X. If s is not identically zero, then $D_E s = \tau s$ defines a Federer flat current τ of degree one on X such that*

$$\frac{1}{2\pi i}\, d\tau \;=\; \sum n_j[Z_j] - c_1(D_E), \tag{9.4}$$

where $\sum n_j[Z_j] \equiv \mathrm{Div}(s)$ and $Z_1,\ Z_2, \cdots$ are the irreducible components of the zero set which are orientable and of real codimension-two.

The holomorphic case is a subcase of the real analytic case. However, it is well known and much easier to establish. Just recall (Proposition 7.1) that s is atomic and that $\mathrm{Div}(s) = \sum n_j[Z_j]$ is a holomorphic chain.

On a holomorphic line bundle E each hermitian metric $<,>_E$ determines a hermitian connection as follows. Given a local holomorphic frame e, let $h = |e|_E^2 =< e, e >_E$, and set

$$\omega \;=\; \partial \log h \;=\; \frac{\partial h}{h}.$$

If $e' = ce$ is another local holomorphic frame, then $h' = |c|^2 h$ and hence $\omega' = \partial \log h' = \omega + \frac{dc}{c}$. Hence, our metric on E determines a global connection D by requiring that $De = \omega e$. The next result is sometimes called the Poincaré-Lelong formula.

Theorem 9.5. Holomorphic Case. *Suppose s is a holomorphic section of a holomorphic line bundle E with hermitian metric $<,>_E$ and associated connection D. Then*

$$\frac{i}{2\pi}\; \partial\bar{\partial} \log |s|_E^2 \;=\; \sum n_j[Z_j] - c_1(D). \tag{9.6}$$

Proof. It suffices to show that the global 1-form τ defined by $Ds = \tau s$ satisfies $\tau = \partial \log |s|_E^2$. By (6.9), $\tau = \frac{da}{a} + \omega$. On the other hand, $\partial \log |s|^2 = \partial \log |a|^2 h = \frac{da}{a} + \omega$ since $s = af$, $|s|_E^2 = |a|^2 h$, $h \equiv |f|_E^2$, and $\omega = \partial \log h$. $\square$

Remark 9.7. Smooth Sections with Poles. Each of the three results: 1) Theorem 9.1 for C^∞ sections, 2) Theorem 9.3 for real analytic sections, and 3) Theorem 9.5 for holomorphic sections can be easily generalized to include "meromorphic" sections. That is, if a section s can be expressed locally as $\frac{a}{b}e$ where both a and b satisfy the same hypothesis as in one of these three results, then the conclusion holds with $\mathrm{Div}(s) = \mathrm{Div}(a) - \mathrm{Div}(b)$ since the logarithmic derivative of $\frac{a}{b}$ equals $\frac{da}{a} - \frac{db}{b}$.

In the real analytic case (as in the holomorphic case) a and b are uniquely determined up to a never vanishing factor by requiring that a and b be relatively prime. Consequently, the **codimension-2 zero divisor** $\sum n_j[Z_j], n_j \in \mathbf{Z}$ and the **codimension-2 polar divisor** $\sum m_j[P_j], m_j \in \mathbf{Z}$ are each globally defined.

Theorem 9.8. *Suppose s is a (real-analytic) meromorphic section of a complex line bundle E with smooth connection D_E. Then we have the equation*

$$\frac{1}{2\pi i}d\tau = \sum n_j[Z_j] - \sum m_j[P_j] - c_1(D_E). \tag{9.9}$$

where the n_j's and m_j's are integers and where the Z_j's and P_j's are oriented components of real codimension-2 of the zero set and polar set of s respectively.

Remark 9.10. The Argument Principle. Applications will be discussed elsewhere. However, we briefly mention the simplest possible case of Theorem 9.1. Suppose the bundle E is trivial and $D \equiv d$ is the trivial flat connection so that $c_1(D_E) = 0$. Then a section is just a smooth complex-valued function f on the manifold X. Suppose X is a real oriented surface, compact with boundary ∂X. Further, assume that f has isolated zeros of finite order n_j at p_j for $j = 1, \ldots, N$. Then we have (cf. Lemma 7.4 and (6.9)) that

$$\tau \equiv \frac{df}{f} \in L^1_{\mathrm{loc}}(X),$$

and equation (9.2) becomes

$$d\left(\frac{1}{2\pi i}\frac{df}{f}\right) = \sum_{j=1}^{N} n_j[p_j] \qquad \text{on } X \tag{9.11}$$

Green's Theorem transforms the local formula (9.14) into the global formula

$$\frac{1}{2\pi i}\int_{\partial X}\frac{df}{f} = \sum_{j=1}^{N} n_j, \tag{9.12}$$

for a smooth function f with isolated zeros of finite order. (This, of course, is just the usual argument principle when f is a holomorphic function.)

For a real analytic function f the Argument Principle (9.12) holds in great generality, as a consequence of Theorem 9.8. As long as f is not identically zero the local version (9.11) holds. Assume that none of the **divisor points** p_j lie on the boundary of X. If f does not vanish identically on any boundary component, and ∂X is real analytic (this can be weakened), then one can show that $\frac{df}{f}$ restricts to ∂X as a current (which is Federer flat) and that the Argument Principle (9.12) is valid. Note that the zero set of f is even allowed to have components of real codimension one, but these "folds" do not contribute to the divisor.

10. Some Applications to Complex and CR Geometry.

The results in this chapter on line bundles have many interesting applications. A number of these are in fact concerned with bundles of higher rank. We present here two such examples which illustrate well the possibilities and techniques and which also have independent interest.

Our first application is related to the construction of Levine forms in complex geometry. It shows how our theory for line bundles and divisors can be applied through the process of "blowing-up". Let $E \to X$ and $F \to X$ be a pair of smooth hermitian vector bundles of rank p and q respectively, and consider the inclusion of projectivized bundles $\mathbf{P}(E) \subset \mathbf{P}(E \oplus F)$. Note that the linear projection $\pi : E \oplus F \longrightarrow F$ with kernel E does not descend to a map of projectivized bundles. However it does extend over the blow up

$$\hat{\mathbf{P}}(E \oplus F) \stackrel{\text{def}}{=} \{(A, B) \in \mathbf{P}(E \oplus F) \times_X \mathbf{P}(F) \ : \ \pi(A) \subseteq B\}$$

which was introduced in I.9.8. Here "$\times_X$" denotes the fibre product over X. This space is easily seen to be a manifold, and the map

$$p : \hat{\mathbf{P}}(E \oplus F) \longrightarrow \mathbf{P}(E \oplus F),$$

defined by projection onto the first factor, induces a diffeomorphism over the subset $\mathbf{P}(E \oplus F) \sim \mathbf{P}(E)$.

Let U denote the universal line bundle over $\mathbf{P}(E \oplus F)$ and L the universal line bundle over $\mathbf{P}(F)$. We denote by $\mathbf{U}$ and $\mathbf{L}$ the pullbacks of U and L to $\hat{\mathbf{P}}(E \oplus F)$. Note that at a point $(A, B) \in \hat{\mathbf{P}}(E \oplus F)$, A is the fibre of $\mathbf{U}$ and B is the fibre of $\mathbf{L}$. Since $\pi(A) \subset B$ over $\hat{\mathbf{P}}(E \oplus F)$, the projection $\pi : A \longrightarrow B$ defines a map of line bundles

$$\boldsymbol{\pi} : \mathbf{U} \longrightarrow \mathbf{L}.$$

which vanishes exactly to first order on $\hat{\mathbf{P}}(E) \stackrel{\text{def}}{=} p^{-1}(\mathbf{P}(E)) = \mathbf{P}(E) \times_X \mathbf{P}(F)$. In particular $\boldsymbol{\pi}$ is atomic and $\operatorname{Div}(\boldsymbol{\pi}) = [\hat{\mathbf{P}}(E)]$.

Suppose connections D_E, D_F are given on E and F respectively. These connections induce connections $\mathbf{D_U}$ and $\mathbf{D_L}$ on $\mathbf{U}$ and $\mathbf{L}$. Let $u = \frac{i}{2\pi}\Omega_{\mathbf{U}}$ and $\ell = \frac{i}{2\pi}\Omega_{\mathbf{L}}$ denote the first Chern forms of $\mathbf{D_U}$ and $\mathbf{D_L}$. Then by Theorem 6.1 we have that

$$\ell - u - [\hat{\mathbf{P}}(E)] \; = \; \tfrac{i}{2\pi} d\tau \tag{10.1}$$

and

$$\ell^n - u^n - \frac{\ell^n - u^n}{\ell - u}[\hat{\mathbf{P}}(E)] \; = \; \tfrac{i}{2\pi} d\left(\frac{\ell^n - u^n}{\ell - u}\tau\right) \tag{10.2}$$

for all n. Let us consider the case where $n = q = \operatorname{rank}(F)$. Note that the form

$$\frac{\ell^q - u^q}{\ell - u}\tau \in L^1_{\text{loc}}(\hat{\mathbf{P}}(E \oplus F))$$

is smooth on $\hat{\mathbf{P}}(E \oplus F) \sim \hat{\mathbf{P}}(E) \cong \mathbf{P}(E \oplus F) \sim \mathbf{P}(E)$ and has an L^1_{loc}-extension to $\mathbf{P}(E \oplus F)$ which we also denote by $\frac{\ell^q - u^q}{\ell - u}\tau$. Note also that the fibre integral of

$$\frac{\ell^q - u^q}{\ell - u} = \ell^{q-1} + \ldots$$

over the fibres of $p_0 : \hat{\mathbf{P}}(E) = \mathbf{P}(E) \times_X \mathbf{P}(F) \longrightarrow \mathbf{P}(E)$, given by restricting p, is the same as the fibre integral of ℓ^{q-1}, and

$$(p_0)_*(\ell^{q-1}) = (-1)^{q-1}.$$

Thus the pushforward by the projection $p : \hat{\mathbf{P}}(E\oplus F) \longrightarrow \mathbf{P}(E\oplus F)$ of the current equation (10.2) yields

$$\ell^q - c_1(D_U)^q - (-1)^{q-1}[\mathbf{P}(E)] \;=\; \tfrac{i}{2\pi} d\left(\frac{\ell^q - u^q}{\ell - u}\tau\right) \tag{10.3}$$

as an equation of currents on $\mathbf{P}(E \oplus F)$. Of course $u = c_1(D_U)$ is smooth on $\mathbf{P}(E \oplus F)$.

In the case where F and D_F are trivial we have that

$$\ell^q = 0,$$

and with $c_1(D_{U^*}) = -c_1(D_U)$, equation (10.3) becomes

$$c_1(D_{U^*})^q - [\mathbf{P}(E)] \;=\; (-1)^{q-1}\tfrac{i}{2\pi} d\left(\frac{\ell^q - u^q}{\ell - u}\tau\right)$$

If E and F are holomorphic bundles, and if D_E and D_F are holomorphic connections induced by the hermitian metrics, then for $x \in E$, $y \in F$, τ can be written as

$$\tau = \partial \log\left\{\frac{|y|^2}{|x|^2 + |y|^2}\right\}.$$

Hence in this holomorphic case (10.3) yields the equation

$$\ell^q - c_1(D_U)^q - (-1)^{q-1}[\mathbf{P}(E)] \;=\; \partial\bar{\partial}\Lambda$$

where

$$\Lambda \;\stackrel{\text{def}}{=}\; -\tfrac{i}{2\pi}\frac{\ell^q - u^q}{\ell - u}\log\left\{\frac{|y|^2}{|x|^2 + |y|^2}\right\} \;\in\; L^1_{\text{loc}}(\mathbf{P}(E \oplus F))$$

is the **generalized Levine form** and q is the codimension of $\mathbf{P}(E)$ in $\mathbf{P}(E\oplus F)$ (See [GSII] where this is discussed for the case where X is a point.)

Note. Downstairs on $\mathbf{P}(E\oplus F)$ the closed 2-form ℓ is only L^1_{loc}. Pushing forward equation (10.2) by p shows that, for each $n < q$, ℓ^n is cohomologous to $c_1(D_U)^n$. However, for ℓ^q this is not true as we have just seen. When taking powers $\geq q$ of ℓ, the singularities enter non-trivially into the calculation of the cohomology class of the form.

Our second application is concerned with questions in CR geometry. It stems from work of Lai [Lai], Webster [W1,2,3] and of Wolfson [Wo] on (real) surfaces in complex 2-manifolds and related matters.

To begin suppose that $j : V \hookrightarrow F$ is a real subbundle of rank n in a complex bundle F of rank n over a manifold X. Let J denote the complex structure on the fibres of F. Then the set of **totally real** points is defined as $\{x \in X : V_x \cap J_x V_x = \{0\}\}$. The complementary set

$$\Sigma_{\mathrm{CR}} = \{x \in X : \dim(V_x \cap J_x V_x) > 0\},$$

will be called the set of **CR-singularities**. We shall construct a complex line bundle map whose zero set is precisely Σ_{CR}.

Let $i_{\mathbf{C}} : V_{\mathbf{C}} \longrightarrow F$ denote the complexification of the inclusion of V into F given by

$$i_{\mathbf{C}}(u + iv) = u + Jv.$$

where $V_{\mathbf{C}} = V \otimes \mathbf{C}$. Note that the map $i_{\mathbf{C}}$ has non-trivial kernel precisely along Σ_{CR}. Equivalently, the divisor of the n^{th} exterior power

$$\lambda \stackrel{\text{def}}{=} \Lambda^n i_{\mathbf{C}} : \Lambda^n_{\mathbf{C}} V_{\mathbf{C}} \longrightarrow \Lambda^n F \tag{10.4}$$

has

$$\operatorname{spt}(\operatorname{Div}(\lambda)) = \Sigma_{\mathrm{CR}}.$$

Proposition 10.5. *Let $i : V \hookrightarrow F$ be a real rank n subbundle of a complex rank n bundle F over a manifold X. Suppose that V and F are provided with (real and complex) connections D_V and D_F respectively. Assume that the induced bundle map λ in (10.4) is weakly atomic. Then*

$$c_1(D_F) - \operatorname{Div}(\lambda) = d\sigma$$

with $\sigma \in L^1_{\mathrm{loc}}(X)$.

Remark. Rather than assume that λ is weakly atomic, the geometric hypotheses of Theorem 9.1 will suffice.

Proof. Since $\Lambda^n_{\mathbf{C}} V_{\mathbf{C}} = (\Lambda^n V) \otimes_{\mathbf{R}} \mathbf{C}$, the first Chern class of the bundle $\Lambda^n_{\mathbf{C}} V_{\mathbf{C}}$ is exact. Now the Proposition follows from Theorem 6.1. $\square$

This result is a localized version of a result of Webster [W3]. Moreover, the type of singularities allowed by the hypothesis of weak atomicity is vastly more general than those considered in [W3] where one assumes regular first order vanishing.

Remark. The current $\mathrm{Div}(\lambda)$ does not depend on a choice of orientation on V or even on the orientability of V.

Example. Surfaces. Suppose $f : X \longrightarrow M$ is an immersion of a real oriented surface X into a complex surface M. Let $V = TX$ and $F = f^*TM$, and consider the bundle embedding

$$df : TX \longrightarrow f^*TM.$$

given by the differential of f. In this case the Σ_{CR} consists of the points of complex tangency, i.e.,

$$\Sigma_{\mathrm{CR}} = \{x \in X \,;\, f_* T_x X \ \text{ is a complex subspace}\}.$$

The divisor of the map $\lambda = \Lambda^2(df)_{\mathbf{C}}$, which is supported in Σ_{CR}, can be computed geometrically. To simplify matters let us make the following

Assumption 10.6. At each point of complex tangency the given orientation on X agrees with the canonical complex one, i.e., all the tangency points are "complex" as opposed to "anticomplex".

Suppose now that $p \in X$ is an isolated point in the support of $\mathrm{Div}(\lambda)$ with multiplicity m. In a neighborhood of p the submanifold X can be considered as the graph of a function $w(z)$ where $(z, w) = (x + iy, u + iv)$ are complex coordinates on M with p corresponding to $(0, 0)$. By 10.6 the given orientation on X agrees with the one induced by considering X as the graph of $w(z)$. In the frame $(1, \partial w/\partial x)$, $(i, \partial w/\partial y)$ for TX, the map $i_{\mathbf{C}}$ has the matrix form

$$\begin{pmatrix} 1 & i \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix}$$

so that $\lambda = \Lambda^2 i_{\mathbb{C}}$ has matrix form $-i\partial w/\partial\bar{z}$. Therefore

$$\mathrm{Div}\,(\lambda) = \mathrm{Div}\left(\frac{\partial w}{\partial \bar{z}}\right).$$

For example if $w = |z|^2$, then p is a point of multiplicity one and is called an **elliptic point**, while if $w(z) = x^2 - y^2$, then p is a point of multiplicity -1 and p is called a **hyperbolic point**. In particular, if λ is atomic with isolated zeros, and if N_{ellip} and N_{hyp} denote the number of elliptic and hyperbolic points counted to multiplicity, then under Assumption 10.6 we have that

$$\int_X c_1(D_F) \;=\; N_{\text{ellip}} - N_{\text{hyp}}$$

generalizing a result of Webster [W3].

Remark. At anticomplex tangencies the same calculation holds with a reversal of sign. This yields the general geometric formula for the divisor.

Remark. With the addition of a hermitian structure on $F \supset V$ one can construct many other complex line bundle maps and apply the main results of this chapter. This yields other Webster formulas (cf. [MW], [W1], [W2], [W3], and [Wo]).

11. Riemann Roch Theorem.

As an application of Theorem 9.5 we give a local proof of the classical Riemann-Roch Theorem in the spirit of Sibner and Sibner [SS] but avoiding the use of Čech Theory. The special case of Theorem 9.5 required for Riemann-Roch is described as follows.

Consider a Riemann surface X and the diagonal $\Delta \subset X \times X$. Let $L = L_\Delta$ denote the holomorphic line bundle associated with the divisor Δ in the product space $X \times X$. Let σ denote a holomorphic section of L on $X \times X$ with divisor Δ. (Recall that σ is unique up to multiplication by a never vanishing holomorphic function on $X \times X$). Choose a hermitian metric $\|\ \|_L$ on L. The Poincaré-Lelong formula says that

$$(11.1) \qquad \bar{\partial} T \;=\; [\Delta] - c_1(\|\ \|_L) \quad \text{on} \quad X \times X,$$

where $c_1(\| \; \|_L)$ is the 1st Chern form of the hermitian connection on L and

$$(11.2) \qquad T \equiv \frac{1}{2\pi i}\, \partial \log \|\sigma\|_L^2.$$

The derivative of the map $\sigma : \Delta \to L$ can be used to establish that the normal bundle to Δ is isomorphic to $L \,|_\Delta$ and hence under the natural embedding i of X as the diagonal Δ in $X \times X$,

$$T_X \text{ is isomorphic to } L \,|_\Delta \, .$$

In particular, the hermitian metric $\| \; \|_L$ on L can be considered as an extension of a given metric on the Riemann surface X, and

$$i^* c_1(\| \; \|_L) = c_1(\| \; \|_{T_X})$$

only depends on the metric on X. Let $K_X = T_X^*$ denote the canonical bundle on X.

We must show how (11.1) implies

Theorem 11.3. Riemann-Roch. *Suppose V is a holomorphic vector bundle of rank n on a compact Riemann surface X. Then the* **analytic index**

$$\chi(\mathcal{O}_V) \equiv \dim H^0(X, \mathcal{O}_V) - \dim H^1(X, \mathcal{O}_V)$$

is equal to the following combination of Chern numbers

$$c_1(V) - \frac{n}{2}\, c_1(K_X).$$

Proof. Suppose P and Q are operators on $\Gamma(\Lambda^{*,*} \otimes V)$ of bidegree -1, 0 and 0, 0 respectively which satisfy

$$\overline{\partial}_V \circ P + P \circ \overline{\partial}_V = I - Q.$$

Assume Q is trace class and can be written $Q \equiv \Sigma Q^{p,q}$ where $Q^{p,q}$ maps sections of $\Lambda^{p,q} \otimes V$ into sections of $\Lambda^{p,q} \otimes V$. Then the alternating sum formula (cf. Proposition 2.4 in [AB]) says that the analytic index is given by

$$(11.4) \qquad \chi(\mathcal{O}_V) = \Sigma(-1)^q \operatorname{Tr} Q^{0,q}.$$

Here $\operatorname{Tr} Q$ is defined, for operators Q having a smooth kernel $A(x,y)$, to be the integral of $A(x,y)$ over the diagonal.

Now the idea of the proof can be explained quite simply as follows. Use $\overline{\partial} T = [\Delta] - c_1(\|\ \|_L)$ to construct P and Q satisfying $\overline{\partial}_V \circ P + P \circ \overline{\partial}_V = I - Q$ with Q "sufficiently topological" so that we can deduce

$$(11.5) \qquad \Sigma(-1)^q \operatorname{Tr} Q^{0,q} = c_1(V) - \frac{n}{2}\, c_1(K_X)$$

by integration over the diagonal.

Remark 11.6. Kernels. Given an operator Q from sections of $\Lambda^p \otimes V$ to sections of $\Lambda^p \otimes V$ there are two different ways of using a kernel on the product space $X \times X$ to induce the operator. Both ways are crucial to this proof. First we describe the kernel $A(y,x)$ used to compute $\operatorname{Tr} Q$ in the alternating sum formula (11.4). Let Y denote the first copy of X in the product $X \times X$. Let $\pi_Y : Y \times X \to Y$ denote projection onto the first factor and let $\pi_X : Y \times X \to X$ denote projection onto the second factor. Let $W \equiv \Lambda^p \otimes V$. The kernel $A(y,x)$ is a distributional section of the bundle $\pi_Y^* \Lambda^m \otimes \operatorname{Hom}(\pi_Y^* W, \pi_X^* W)$, where $m \equiv \dim_{\mathbf{R}} X$. Then

$$Q(s) = \int_Y A(y,x) s(y)$$

and

$$\operatorname{Tr} Q = \int_\Delta \operatorname{trace} A(x,x),$$

where trace A is with respect to $\operatorname{Hom}(W,W)$.

The second way of using a kernel $B(y,x)$ to induce the operator Q is motivated by the desire to have the operator equation

$$(11.7) \qquad \overline{\partial} \circ P + P \circ \overline{\partial} = I - Q$$

correspond to the kernel equation

$$(11.8) \qquad \overline{\partial} T = [\Delta] - B \qquad \text{on } X \times X,$$

(The equation (11.8) has certain significant advantages over (11.7) — even though they are equivalent.) The kernel $B(y,x)$ is a distributional section of the bundle

$\Lambda^*(Y \times X) \otimes \mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$. By definition, B determines an operator on a section s of $W \equiv \Lambda^p \otimes V$ by formal integration over the fibre of the map π_X

$$Q(s) \;=\; \int_{y \in Y} B(y,x) \wedge s(y).$$

This is rigorously defined to be the current pushforward of $B(y,x) \wedge s(y)$ under the projection π_X. For example, if $B \equiv kh$ with k a distributional section of $\Lambda^*(Y \times X)$ and h a smooth section of $\mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$ then

$$Q(\varphi \otimes v)(x) \;=\; \left(\int_{y \in Y} k(y,x) \wedge \varphi(y) \right) (hv)(x),$$

where the section s has been chosen of the form $s = \varphi \otimes v$ with φ a section of Λ^* and v a section of V. By convention all differentials in the x variables must be moved to the far right before the y integration is preformed. See Harvey-Polking [HP] for a complete development of this kernel calculus. Since $W \equiv \Lambda^p \otimes V$, the bundle $\pi_Y^* \Lambda^m \otimes \mathrm{Hom}(\pi_Y^* W, \pi_X^* W)$ is bundle isomorphic to the bundle

$$\pi_Y^* \Lambda^m \otimes (\pi_Y^* \Lambda^p)^* \otimes \pi_X^* \Lambda^p \otimes \mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$$

which in turn is isomorphic to

$$\pi_Y^* \Lambda^{m-p} \otimes \pi_X^* \Lambda^p \otimes \mathrm{Hom}(\pi_Y^* V, \pi_X^* V).$$

However, the kernels $A(y,x)$ and $B(y,x)$ are not the same under this canonical isomorphism, but are related by

$$B(y,x) \;=\; (-1)^p A(y,x). \tag{11.9}$$

In particular, the alternating sum formula becomes

$$\chi(\mathcal{O}_V) \;=\; \int_\Delta \operatorname{trace} B. \tag{11.10}$$

The proof of (11.9) is omitted.

Note the following two facts: (A) The kernel corresponding to the identity operator on forms is $[\Delta]$, integration over the diagonal. (B) If B is a kernel on

$X \times X$, of bidegree $(k, k-1)$ where $k \equiv \dim_{\mathbb{C}} X$, and Q is the associated operator, then the kernel $\overline{\partial} B$ corresponds to the operator $\overline{\partial} \circ Q + Q \circ \overline{\partial}$.

Given a kernel $B(y, x)$ which for each fixed p maps sections of $\sum_q \Lambda^{p,q} \otimes V$ into itself let $B^p(y, x)$ denote the part of the kernel inducing this action on $\sum_q \Lambda^{p,q} \otimes V$. Also, let I_V denote the identity section of the bundle $\mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$ over the diagonal Δ.

This second version, (11.10), of the alternating sum formula combines with (11.13) of the next lemma to complete the proof of Riemann Roch for X a Riemann surface.

Lemma 11.11. *There exists a fundamental kernel T_V satisfying the current equation*

$$\overline{\partial} T_V \;=\; [\Delta] \otimes I_V - K_V \quad \text{on} \;\; X \times X, \tag{11.12}$$

where K_V is a smooth kernel of bidegree 1, 1 which is $\mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$-valued. Moreover, $K_V^0 \equiv K_V^{0,0} + K_V^{0,1}$ satisfies

$$i^*(\operatorname{trace} K_V^0) \;=\; c_1(\| \; \|_V) - \frac{n}{2}\, c_1(\| \; \|_{K_X}), \tag{11.13}$$

where c_1 denotes the first Chern form.

Proof. The section I_V of $\mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$ over the diagonal $\Delta \subset X \times X$ can be extended to a section $\widetilde{I}$ over $X \times X$ using the metric on the bundle V over X, as follows. Let

$$U \;\equiv\; \{(y, x) \in X \times X : \operatorname{dist}(y, x) < \rho\}$$

where ρ is the convexity radius of the metric. For each $(y, x) \in U$ let $\gamma_{y,x}$ denote the unique arc-length geodesic on X from y to x, and let $\tau(y, x) : V_y \to V_x$ for $(y, x) \in U$ denote parallel translation along $\gamma_{y,x}$. We then let $\widetilde{I}(y, x)$ denote any smooth section of $\mathrm{Hom}(\pi_Y^* V, \pi_X^* V)$ over $X \times X$ which agrees with $\tau(y, x)$ on a neighborhood of the diagonal.

We define

$$T_V \;\equiv\; T \otimes \widetilde{I}$$

with T given by (11.1). The formula (11.2) implies that

$$\overline{\partial}\, T_V \;=\; [\Delta]\otimes I_V - K_V \quad \text{on} \quad X\times X$$

where

$$K_V \;=\; T\otimes\overline{\partial}\,\widetilde{I}+c_1(L)\otimes\widetilde{I}$$

since $[\Delta]\otimes\widetilde{I}=[\Delta]\otimes I_V$ is independent of the extension $\widetilde{I}$.

It remains to show two things. The first is that $(c_1(\|\ \|_L)\otimes\widetilde{I})^0$, the part of $c_1(L)\otimes\widetilde{I}$ acting on $\Lambda^{0,*}\otimes V$, satisfies

$$i^*\Big(\mathrm{trace}(c_1(\|\ \|_L)\otimes\widetilde{I})^0\Big) \;=\; -\frac{n}{2}\, c_1(\|\ \|_{K_X}).$$

The second is that $T\otimes\overline{\partial}\widetilde{I}$ extends smoothly across Δ with

$$i^*\Big(\mathrm{trace}(T\otimes\overline{\partial}\widetilde{I})^0\Big) \;=\; c_1(\|\ \|_V). \tag{11.14}$$

Now we observe that

$$\begin{aligned}
i^*\Big(\mathrm{trace}\, c_1(\|\ \|_L)\otimes\widetilde{I}\Big) &= n\ i^*(c_1(\|\ \|_L)) \;=\; -n\ c_1(\|\ \|_{K_X}),\\
i^*\Big(\mathrm{trace}(c_1(\|\ \|_L)\otimes\widetilde{I})^0\Big) &= n\ i^*(c_1(\|\ \|_L)^0)\\
i^*\Big(\mathrm{trace}(c_1(\|\ \|_L)\otimes\widetilde{I})^1\Big) &= n\ i^*(c_1(\|\ \|_L)^1)
\end{aligned}$$

where the isomorphism $T_X\cong L\,\big|_\Delta$ is used in the first equation. Thus it suffices to prove that

$$i^*(c_1(\|\ \|_L)^0) \;=\; i^*(c_1(\|\ \|_L)^1) \tag{11.15}$$

in order to conclude that $i^*(\mathrm{trace}(c_1(\|\ \|_L)\otimes\widetilde{I})^0 = -\frac{n}{2}c_1(\|\ \|_{K_X})$ as desired. We may choose the metric on L invariant under the switch s. Then $s^*c_1(\|\ \|_L) = c_1(\|\ \|_L)$. Then one can show that $c_1(\|\ \|_L)^1 = s^*(c_1(\|\ \|_L)^0)$ (say using local holomorphic coordinates) which implies (11.15).

Let Ω_V denote the curvature of the bundle V, i.e., $c_1(\|\ \|_V)=\frac{i}{2\pi}\operatorname{Tr}\Omega_V$. The second equation (11.14) follows from

$$i^*\Big((T\otimes\overline{\partial}\,\widetilde{I})^0\Big) \;=\; \frac{i}{2\pi}\Omega_V. \tag{11.16}$$

Pick a local holomorphic coordinate x for X. Then

$$T \equiv \frac{1}{2\pi i}\frac{d(y-x)}{y-x} - \frac{i}{2\pi}\frac{\partial\rho}{\rho}$$

with ρ never zero.

We will show that $\overline{\partial}\widetilde{I}$ vanishes on Δ and hence need only consider the term

$$\frac{1}{2\pi i}\frac{d(y-x)}{y-x} \otimes \overline{\partial}\widetilde{I} \quad \text{in } T \otimes \overline{\partial}\widetilde{I}.$$

First, note that

$$\frac{1}{2\pi i}\left(\frac{d(y-x)}{y-x} \otimes \overline{\partial}\widetilde{I}\right)^0 = \frac{1}{2\pi i}\frac{dy}{y-x} \otimes \overline{\partial}\widetilde{I}. \tag{11.17}$$

Choose a local holomorphic frame $v_1, \ldots, v_n$ for V. Then (11.17) applies to v can be rewritten as

$$\frac{1}{2\pi i}\overline{\partial}\left(\frac{dy}{y-x} \otimes \left(\widetilde{I}(y,x)v(y) - v(x)\right)\right)$$

Therefore, the desired equation (11.16) follows from the next result.

Lemma 11.18.

$$\frac{dy}{y-x} \otimes \left(\widetilde{I}(y,x)v(y) - v(x)\right)$$

extends smoothly across Δ and pulls back by the inclusion $i : X \to \Delta \subset X \times X$ to

$$Dv = \omega v.$$

Proof. Fix any point $x \in X$ and a real tangent vector $\xi \in T_xX$. Let $\gamma(t)$ denote the arc-length geodesic on X with $\gamma(0) = x$ and $\gamma'(0) = \xi$. Our first assertion is that

$$D_\xi v = \lim_{t\to 0}\frac{v(\gamma(t)) - \widetilde{I}(x,\gamma(t))v(x)}{t} = \lim_{t\to 0}\frac{\widetilde{I}(\gamma(t),x)v(\gamma(t)) - v(x)}{t}. \tag{11.19}$$

To see this set

$$w(t) = \widetilde{I}(x,\gamma(t))v(x)$$

and write $v(\gamma(t)) = A(t)w(t)$ where $A(t) \in \mathrm{GL}_n(\mathbf{C})$ for each t. Then

$$D_\xi v \;=\; \frac{Dv}{dt}\bigg|_{t=0} \;=\; \left\{\frac{dA}{dt}w + A\frac{Dw}{dt}\right\}\bigg|_{t=0} \;=\; \frac{dA}{dt}(0)v(x)$$

since by the definition of w, $\frac{Dw}{dt} \equiv 0$. However,

$$\lim_{t\to 0}\frac{1}{t}(v(\gamma(t)) - w(t)) \;=\; \lim_{t\to 0}\frac{1}{t}(A(t) - I)w(t) = \frac{dA}{dt}(0)v(x).$$

This proves the first equation in (11.19). For the second we apply $\widetilde{I}(x,\gamma(t))\widetilde{I}(g(t),x) = \mathrm{Id}v_{\gamma(t)}$ to the first and regroup before taking the limit.

We now consider the V_x-valued function $f(y) = \widetilde{I}(y,x)v(y) - v(x)$ defined for y in a neighborhood of x. Equation (11.19) immediately implies that

$$(11.20) \qquad Dv \;=\; d_y\left\{\widetilde{I}(y,x)v(y) - v(x)\right\}\bigg|_{y=x}$$

where d_y denotes the exterior derivative with respect to the y-variables.

Since D is a canonical hermitian connection, the complex linear extension of Dv to $T_xX \otimes \mathbf{C} = T^{1,0} \oplus T^{0,1}$ is purely of type $(1,0)$, i.e., (11.20) can be rewritten as

$$(11.21) \qquad Dv \;=\; \partial_y\left\{\widetilde{I}(y,x)v(y) - v(x)\right\}\bigg|_{y=x}.$$

We now consider a coordinate neighborhood $U \subset X$ and let $x = x_1 + ix_2$ and $y = y_1 + iy_2$ denote the local holomorphic coordinate on U. Then (11.21) can be written as

$$Dv \;=\; \frac{\partial}{\partial y}\left\{\widetilde{I}(y,x)v(y) - v(x)\right\}\bigg|_{y=x} dy.$$

The result now follows from Taylor's Theorem applied to the function $f(y)$ at the point $y = x$. $\square$

III. Sections of Vector Bundles

In this chapter we develop our theory in the case where E and F are complex vector bundles and $\mathrm{rank}(E) = 1$. When $E = \underline{\mathbf{C}}$ is trivial, bundle maps $\underline{\mathbf{C}} \to F$ correspond to cross-sections of F. We begin with this case. In Section 1 we present the concept, introduced in [HS], of an atomic cross-section of a vector bundle. If the bundle is oriented, then each atomic section μ has an associated divisor $\mathrm{Div}(\mu)$, which is a d-closed integrally flat current on the base manifold. If μ vanishes non-degenerately, $\mathrm{Div}(\mu)$ is the current associated to the manifold of zeros of μ. The majority of Section 1 is devoted to proving a basic technical result concerning residues of forms, which plays an important role throughout the rest of the paper.

In Section 2 we fix a complex bundle with connection $F \to X$ and examine the singular pushforward connection under an atomic section μ. It is shown that if μ is atomic, then for every Ad-invariant polynomial ϕ on $\mathfrak{gl}_n(\mathbf{C})$, the Chern current $\phi((\overrightarrow{D}))$ exists and has the form

$$\phi((\overrightarrow{D})) \;=\; \phi(\overrightarrow{\Omega}_0) + \mathrm{Res}_\phi(\overrightarrow{D})\,\mathrm{Div}(\mu),$$

where $\phi(\overrightarrow{\Omega}_0)$ is L^1_{loc} across the singular set of μ and satisfies

$$d\phi(\overrightarrow{\Omega}_0) \;=\; 0 \qquad \text{on} \quad X,$$

and where $\mathrm{Res}_\phi(\overrightarrow{D})$ is a smooth d-closed form on X, called the **residue form**, which is independent of μ.

In Section 3 we show that the characteristic current $\phi((\overrightarrow{D}))$ *is independent of the choice of approximation mode.* We also show that *the residue form* $\mathrm{Res}_\phi(\overrightarrow{D})$

is in fact a Chern-Weil characteristic form, i.e., it is written as a universally determined polynomial in the curvature of the given connection on F. Detailed computations of this residue form are carried out in Section 7. The results of Section 3 are proved by careful computation of the transgression forms for the family $\phi(\overrightarrow{D}_s)$ and their limit as $s \to 0$.

In Section 4 we examine the case of the top Chern class. This leads to a C^∞ generalization of the Poincaré-Lelong formula. More specifically, for each atomic section μ there is a family of transgression forms $r_s \in L^1_{\text{loc}}(X)$ such that

$$c_n(\overrightarrow{D}_s) - \mathrm{Div}(\mu) \;=\; dr_s$$

and

$$\lim_{s \to 0} r_s \;=\; 0 \qquad \text{in } \; L^1_{\text{loc}}(X).$$

Thus the Chern form $c_n(D_F)$ and the divisor of μ are cohomologous. Note also that as $s \to 0$, $c_n(\overrightarrow{D}_s) \to \mathrm{Div}(\mu)$ as flat currents on X.

Here our theory also produces, for each approximation mode, a family $\boldsymbol{\tau}_{\mathbf{s}}$, $s > 0$ of *canonical representatives of the Thom class of* F. In fact we produce explicit universal classes $\mathfrak{T}_{\mathbf{s}}$ in the equivariant cohomology of $\mathbf{C}^n$ such that $\boldsymbol{\tau}_{\mathbf{s}} = w(\mathfrak{T}_{\mathbf{s}})$ where w is the equivariant Chern-Weil homomorphism. The family $\boldsymbol{\tau}_{\mathbf{s}}$ has the nice properties:

$$\lim_{s \to 0} \boldsymbol{\tau}_{\mathbf{s}} \;=\; [X] \;\; \text{(The zero-section of } F\text{)}$$

and

$$\boldsymbol{\tau}_{\mathbf{s}} \big|_X \;=\; c_n(D_F) \;\; \text{(The Chern Euler form) for all } \; s.$$

In compact approximation mode the support of the Thom forms is compact in each fibre. In algebraic approximation mode, the Thom form extends to $\mathbf{P}(\mathbf{C} \oplus F)$ and determines the Thom class as an element in $H^{2n}(\mathbf{P}(\mathbf{C} \oplus F), \mathbf{P}(F))$.

In Section 5 we prove a C^∞ version of the Grothendieck-Riemann-Roch Theorem at the level of differential forms. As above, the construction is completely canonical in each approximation mode. Furthermore, the result holds for a large class of subcomplexes of X, not just for smooth submanifolds.

Section 8 extends our results to general atomic maps $\alpha : E \to F$ where $\mathrm{rank}(E) = 1$. Interestingly, these results cannot be obtained from the section case by twisting with E^*. The formulas at the level of forms and currents are different.

In an appendix we discuss generalized Bochner-Martinelli kernels from the point of view of Chern-Weil theory (cf. Berndtsson [Be]) and use them to prove some basic results for atomic bundle maps.

1. Atomic Sections and Divisors.

Suppose V is a smooth real vector bundle of rank m over X. A smooth section ν of V is locally of the form $\nu = ue$ with $e^t \equiv (e_1, \ldots, e_m)$ a local frame for V and $u = (u_1, \ldots, u_m)$ a smooth $\mathbf{R}^m$-valued function.

Definition 1.1. $(m > 1)$ A smooth section ν is said to be **atomic** if for each local frame e the vector valued function u satisfies:

$u^* \left(\frac{dy^I}{|y|^p} \right)$ has an L^1_{loc} extension across the zero set of u, to all of X, for all $p = |I| \le m - 1$.

This extension is unique because the zero set of u has Lebesgue measure zero ([HS; 1.2]).

Now let

$$\theta(y) \equiv \sum_{j=1}^{m} (-1)^{j-1} \frac{y_j dy_1 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_m}{|y|^m} \tag{1.2}$$

denote the **solid angle kernel on $\mathbf{R}^m$**, and let γ_m denote the volume of the unit sphere in $\mathbf{R}^m$.

In order to correctly formulate the notion of the divisor of a section of V, the bundle V must be oriented and only frames compatible with the orientation are allowed.

Definition 1.3. Suppose ν is an atomic section of V, so that, for each $\mathbf{R}^m$-valued function u determined by $\nu = ue$ from a local frame e compatible with the orientation,

$$u^*(\theta(y)) \in L^1_{\mathrm{loc}}(X).$$

The **divisor of $\boldsymbol{\nu}$**, denoted by $\mathbf{Div}(\boldsymbol{\nu})$, is defined by the current equation

$$\text{(1.4)} \qquad \mathrm{Div}(\nu) \;\equiv\; d\left(\gamma_m^{-1}u^*(\theta)\right) \qquad \text{on } X,$$

i.e., the exterior derivative of the potential $\gamma_m^{-1}u^*(\theta) \in L^1_{\mathrm{loc}}(X)$.

This notion of divisor is independent of the choice of local frame compatible with the orientation on V (Theorem 2.11 [HS]).

The remainder of this section is devoted to the proof of a result (Theorem 1.10) that is of crucial importance for our understanding of Chern currents.

A smooth form on $V \sim X$ which is invariant under multiplication by positive scalars in the fibers of V will be called **homogeneous (of degree zero)**. In local coordinates $y \in \mathbf{R}^m \cong V_x$ in the fiber, a homogeneous form is the sum of terms of the form $\varphi\left(x, \frac{y}{|y|}\right)\frac{dy^I}{|y|^p}$ with $p = |I|$ and φ a smooth family of forms on U parametrized by $\frac{y}{|y|} \in S^{m-1}$ on $U \times S^{m-1}$, $U^{\mathrm{open}} \subset X$.

Lemma 1.5. *Suppose T is a homogeneous form on $V \sim X$ which is of degree $\leq m-1$ in the fiber differentials. Suppose that ν is an atomic section of V. Let $Z \equiv \{x \in X : \nu(x) = 0\}$. Then:*

a) *T extends (uniquely) as an $L^1_{\mathrm{loc}}(V)$ form across $X \subset V$.*

b) *$\nu^*(T)$ extends (uniquely) as an $L^1_{\mathrm{loc}}(X)$ form across the zero set $Z \subset X$.*

Proof. a) Note that $|y|^{-p} \in L^1_{\mathrm{loc}}(\mathbf{R}^m)$ if $p \leq m-1$.

b) This follows directly from the definition of an atomic section and the fact that Z has Lebesgue measure zero. $\square$

Remark 1.6. Because of this Lemma, we shall use the notations $T \in L^1_{\mathrm{loc}}(V)$ for the unique extension of $T \in C^\infty(V \sim X)$, and $\nu^*(T) \in L^1_{\mathrm{loc}}(X)$ for the unique extension of $\nu^*(T) \in C^\infty(X \sim Z)$.

The objective of the remainder of this section is to compute the exterior derivative of the currents $T \in L^1_{\mathrm{loc}}(V)$ and $\nu^*(T) \in L^1_{\mathrm{loc}}(X)$.

Since $T \in C^\infty(V \sim X)$ is homogeneous, the form $L \in C^\infty(V \sim X)$ defined by $L = dT$ on $V \sim X$ is also homogeneous. Moreover, if T has degree $\leq k$ in the fiber differentials then $L = dT$ has degree $\leq k+1$ in the fiber differentials.

Under the hypothesis that both T and $L \equiv dT$ are of degree $\leq m-1$ in the fiber differentials, both T and L have unique $L^1_{\mathrm{loc}}(V)$ extensions across $X \subset V$, so that (letting T and L denote these unique extensions):

$$(1.7) \qquad dT = L + S \qquad \text{on } V,$$

where S is a current with support in $X \subset V$. The current $L \in L^1_{\mathrm{loc}}(V)$ will be called the $\mathbf{L^1_{loc}}$ **part of dT** and S will be called the **singular part of dT**.

First we consider the easy case of codegree at least two, where there is no singular part.

Lemma 1.8. *Suppose T is a homogeneous form on $V \sim X$ of degree $\leq m-2$ in the fiber differentials.*

a) The L^1_{loc} extensions, $T \in L^1_{\mathrm{loc}}(V)$ of $T \in C^\infty(V \sim X)$ and $L \in L^1_{\mathrm{loc}}(V)$ of $L \equiv dT \in C^\infty(V \sim X)$ satisfy the current equation

$$dT = L \qquad \text{on } V.$$

b) If ν is an atomic section then the L^1_{loc} extensions $\nu^*(T) \in L^1_{\mathrm{loc}}(X)$ of $\nu^*(T) \in C^\infty(X - Z)$ and $\nu^*(L) \in L^1_{\mathrm{loc}}(X)$ of $\nu^*(dT) \in C^\infty(X \sim Z)$ satisfy the current equation

$$d\nu^*(T) = \nu^*(L) \qquad \text{on } X.$$

Proof. For the purpose of clarity in this proof let $\widetilde{T} \in L^1_{\mathrm{loc}}(V)$ denote the extension of $T \in C^\infty(V \sim X)$ etc. Choose an approximate one $\chi(t)$ which vanishes identically in a neighborhood of $0 \in \mathbf{R}$. Then $\widetilde{T} = \lim_{\epsilon \to 0} \chi\left(\frac{|y|}{\epsilon}\right) T$ converges in $L^1_{\mathrm{loc}}(V)$ by the Lebesgue dominated convergence theorem. Therefore,

$$d\widetilde{T} = \lim_{\epsilon \to 0} \left(\chi\left(\tfrac{|y|}{\epsilon}\right) dT + \chi'\left(\tfrac{|y|}{\epsilon}\right) \tfrac{|y|}{\epsilon} \tfrac{d|y|}{|y|} T \right).$$

Note that dT and $\frac{d|y|}{|y|} T$ are homogeneous of degree $\leq m-1$ while $\chi\left(\frac{|y|}{\epsilon}\right)$ and $\chi'\left(\frac{|y|}{\epsilon}\right)\frac{|y|}{\epsilon}$ are bounded independent of y and ϵ. Therefore $\lim_{\epsilon \to 0} \chi\left(\frac{|y|}{\epsilon}\right) dT = \widetilde{dT}$ converges in $L^1_{\mathrm{loc}}(V)$ and $\lim_{\epsilon \to 0} \chi\left(\frac{|y|}{\epsilon}\right) \frac{|y|}{\epsilon} \frac{d|y|}{|y|} T = 0$ converges in $L^1_{\mathrm{loc}}(V)$. That is $d\widetilde{T} = \widetilde{dT}$, or in different notation $dT = L$ on V.

The proof of part b) is similar and omitted. $\square$

Now we turn to the more difficult case where T has top degree $m-1$ in the fiber differentials. Pick a metric on V and let S denote the unit sphere bundle in V, with projection $\pi : S \to X$.

Definition 1.9. The **residue of T along** $X \subset V$ is defined by

$$\operatorname{Res}(T) \;=\; \pi_*(T),$$

or in terms of fiber integration

$$\operatorname{Res}(T) \;=\; \int_{\pi^{-1}} T.$$

Note that $\operatorname{Res}(T)$ is a smooth form on X. We shall prove that $\operatorname{Res}(T)$ does not depend on the metric chosen for V.

The next result extends Lemma 1.8 by allowing the higher degree $m-1$, however the special case Lemma 1.8 will be used repeatedly in the proof.

Theorem 1.10. *Suppose T is a homogeneous form on $V \sim X$ of degree $\le m-1$ in the fiber differentials. In addition, assume the homogeneous form $L \equiv dT$ on $V \sim X$ is also of degree $\le m-1$ in the fiber differentials. Then*

a) $$dT = L + \operatorname{Res}(T)[X] \qquad \textit{on } V.$$

If ν is an atomic section of V then

b) $$d(\nu^*(T)) = \nu^*(L) + \operatorname{Res}(T)\operatorname{Div}(\nu) \qquad \textit{on } X.$$

In particular, note that Part a) implies that the residue of T independent of the metric chosen for V.

Proof. It suffices to prove a) and b) locally in the base X. Thus we may assume that $V \equiv X \times \mathbf{R}^m = \underline{\mathbf{R}}^m$ is trivial. Recall the standard fact that

$$(1.11) \qquad d\left(\frac{1}{\gamma_m}\widetilde{\theta(y)}\right) \;=\; [X] \quad \text{on} \quad X \times \mathbf{R}^m,$$

and the definition that

$$d\left(\frac{1}{\gamma_m}\widetilde{u^*(\theta)}\right) \equiv \operatorname{Div}(u) \tag{1.12}$$

is the **divisor** of the atomic section u of the trivial bundle $\underline{\mathbf{R}}^m$. Suppose we can prove that

$$T - \operatorname{Res}(T)\frac{1}{\gamma_m}\theta = d\alpha + \beta \quad \text{on } V \sim X \tag{1.13}$$

with α and β homogeneous, and both α and β of degree $\leq m-2$ in the fiber differentials $dy_1, \ldots, dy_m$. Then Lemma 1.8 is applicable and equation (1.13) extends across X to the equation of $L^1_{\text{loc}}(V)$ currents on V,

$$\widetilde{T} - \operatorname{Res}(T)\frac{1}{\gamma_m}\tilde{\theta} = d\tilde{\alpha} + \tilde{\beta} \quad \text{on } V, \tag{1.14}$$

and, by part b) of Lemma 1.13, this equation pulls back to an equation of $L^1_{\text{loc}}(X)$ currents on X

$$\widetilde{u^*(T)} - \operatorname{Res}(T)\frac{1}{\gamma_m}\widetilde{u^*(\theta)} = d\left(\widetilde{u^*(\alpha)}\right) + \widetilde{u^*(\beta)} \tag{1.15}$$

Finally, taking the exterior derivative of (1.14) yields

$$d\widetilde{T} - \operatorname{Res}(T)[X] \in L^1_{\text{loc}}(V). \tag{1.16}$$

Here we use (1.11) to compute the exterior derivative of $\operatorname{Res}(T)\frac{1}{\gamma_m}\widetilde{u^*(\theta)}$ modulo $L^1_{\text{loc}}(V)$ and use Lemma 1.8 to compute that $d\tilde{\beta} = \widetilde{d\beta} \in L^1_{\text{loc}}(V)$.

Note that part a) of Theorem 1.10 is an immediate consequence of (1.16). Thus it remains to prove (1.13). The homogeneous form

$$\phi \equiv T - \operatorname{Res}(T)\frac{1}{\gamma_m}\theta$$

satisfies the same two conditions as T, namely that both ϕ and $d\phi$ are of degree $\leq m-1$ in the differentials $dy_1, \ldots, dy_m$. In addition, ϕ satisfies the condition

$$\operatorname{Res}(\phi) \equiv \int_{\pi^{-1}(x)} \phi = 0. \tag{1.17}$$

These three conditions on ϕ will be used in the proof of (1.13).

The form ϕ can be expressed as

$$\phi \;=\; \frac{d|y|^2}{|y|^2} \wedge \eta + \psi \tag{1.18}$$

where $\eta \equiv y \cdot \frac{\partial}{\partial y} \llcorner \phi$ and $\psi \equiv y \cdot \frac{\partial}{\partial y} \llcorner \left(\frac{d|y|^2}{|y|^2} \wedge \phi\right)$. We shall prove (1.13) by verifying that both ψ and $\frac{d|y|^2}{|y|^2} \wedge \eta$ can be expressed as $d\alpha + \beta$ with α, β of degree $\leq m-2$. Since η and ψ are both homogeneous and killed by contraction with the euler vector field $y \cdot \frac{\partial}{\partial y}$ they both are pull backs of smooth forms $\bar{\eta}$ and $\bar{\psi}$ on the sphere bundle $X \times S^{m-1}$ under the projection $y \mapsto \frac{y}{|y|}$ of $\mathbf{R}^m - \{0\}$ onto the unit sphere S^{m-1}.

First, we show that ψ is of the form $d\alpha + \beta$ with degree α and $\beta \leq m-2$. For $x \in X$ fixed, both ϕ and ψ have the same integral over the unit sphere since $d|y|^2$ restricts to zero. Thus $\int_{S^{m-1}} \psi_x = 0$. Consequently, we can solve the equation

$$d_{S^{m-1}} \bar{\alpha} \;=\; \bar{\psi} \quad \text{on} \quad X \times S^{m-1}$$

with a (smooth) solution $\bar{\alpha}$. Here $d_{S^{m-1}}$ denotes the S^{m-1} partial exterior derivative.

This implies that

$$\bar{\psi} \;=\; d\bar{\alpha} + \bar{\beta} \quad \text{on} \quad X \times S^{m-1}$$

where $\bar{\beta}$ is smooth and of degree $\leq m-2$ in the differentials on S^{m-1}. Pulling this equation back to $X \times (\mathbf{R}^m - \{0\})$ yields

$$\psi \;=\; d\alpha + \beta \quad \text{on} \quad X \times (\mathbf{R}^m \sim \{0\})$$

where α and β are homogeneous and of degree $\leq m-2$ in $dy_1, \ldots, dy_m$.

Now consider $\eta \equiv y \cdot \frac{\partial}{\partial y} \llcorner \phi$. Note η is homogeneous and $y \cdot \frac{\partial}{\partial y} \llcorner \eta = 0$ so that η is the pullback of a form $\bar{\eta}$ on the sphere bundle S. The form η has degree $\leq m-2$ in the fiber differentials $dy_1, \ldots, dy_m$. The equation

$$0 \;=\; \mathcal{L}_{y \cdot \partial/\partial y}(\phi) \;=\; d\left(y \cdot \frac{\partial}{\partial y} \llcorner \phi\right) + y \cdot \frac{\partial}{\partial y} \llcorner d\phi$$

implies that

$$d\eta \;=\; -y \cdot \frac{\partial}{\partial y} \llcorner d\phi \quad \text{is of degree} \quad \leq m-2 \quad \text{in} \quad dy_1, \ldots, dy_m.$$

Let H denote the harmonic projection operator on S^{m-1} and let K denote the operator that satisfies

$$d \circ K + K \circ d \;=\; I - H$$

operating on differential forms on the sphere S^{m-1}, based on the L^2 decomposition of a form. Then

$$\bar{\eta} \;=\; d(K(\bar{\eta})) + K(d\bar{\eta}) + H(\bar{\eta}). \tag{1.19}$$

Note that $K(\bar{\eta})$ and $K(d\bar{\eta})$ are smooth differential forms of degree $\leq m-3$. Let γ denote the pullback of $K(\bar{\eta})$ to $V \sim X$ and let δ denote the pullback of $K(d\bar{\eta})$ to $V \sim X$. Since $\bar{\eta}$ has degree $\leq m-2$ the only possible harmonic is a constant. Thus (1.19) pulls back to

$$\eta \;=\; d\gamma + \delta + c \qquad \text{on } V \sim X, \tag{1.20}$$

with γ and δ of degree $\leq m-3$ in $dy_1, \ldots, dy_m$. Therefore

$$\tfrac{d|y|^2}{|y|^2} \wedge \eta \;=\; d\left(\tfrac{d|y|^2}{|y|^2} \wedge \gamma + c \log |y|^2\right) + \tfrac{d|y|^2}{|y|^2} \wedge \delta. \tag{1.21}$$

Both of the homogeneous forms $\frac{d|y|^2}{|y|^2} \wedge \gamma$ and $\frac{d|y|^2}{|y|^2} \wedge \delta$ are of degree $\leq m-2$ in $dy_1, \ldots, y_m$. This completes the proof of a modified form of (1.13), namely

$$\phi \;=\; d\alpha + \beta + c\tfrac{d|y|^2}{|y|^2}. \tag{1.13$'$}$$

The extra term $c\frac{d|y|^2}{|y|^2}$ is of degree $\leq m-2$ in $dy_1, \ldots, dy_m$ and can be absorbed into β unless $m = 2$. In this special case the extra term $cd\log|y|^2$ can be absorbed into α since $\log|y|^2 \in L^1_{\text{loc}}(V)$. This completes the proof of part a). The proof of part b) is similar and omitted. The fact that $u^*(\log|y|^2) \in L^1_{\text{loc}}(X)$ must be used. This fact is a consequence of the fact that it's exterior derivative $\nu^*\left(\frac{d|y|^2}{|y|^2}\right) \in L^1_{\text{loc}}(X)$. $\square$

Corollary 1.22. *The current $L \in L^1_{\text{loc}}(V)$, extending $dT \in C^\infty(V \sim X)$, is d-closed on V if and only if* $\operatorname{Res}(dT) = d\operatorname{Res}(T) = 0$ *on X, in which case $\nu^*(L) \in L^1_{\text{loc}}(X)$ is also d-closed.*

Proof. Note that $L \in C^\infty(V \sim X)$ is homogeneous and of degree $\leq m-1$ in the fiber differentials; and that $dL \in C^\infty(V \sim X)$ is equal to zero. Therefore, Theorem 1.10 can be applied with T replaced by L to obtain

$$dL = \operatorname{Res}(L)[X] \quad \text{on } V. \tag{1.23}$$

Since d commutes with pushforward, $\operatorname{Res}(dT) = \pi_*(dT) = d\pi_*(T) = d\operatorname{Res}(T)$. That is

$$\operatorname{Res}(L) = \operatorname{Res}(dT) = d\operatorname{Res}(T). \tag{1.24}$$

Finally, if one of the equivalent conditions, say $\operatorname{Res}(L) = 0$, is valid and ν is atomic then by Theorem 1.10

$$d(\nu^*(L)) = \operatorname{Res}(L)\operatorname{Div}(\nu) = 0 \quad \text{on } X. \quad \square \tag{1.25}$$

The example where $V = \underline{\mathbf{R}}^m = X \times \mathbf{R}^m$ and $T \equiv \gamma_m^{-1}\pi^*(\alpha) \wedge \theta(y)$, where α is an arbitrary form on the base, is instructive. Here both of the homogeneous forms T and $dT = \gamma_m^{-1}\pi^*(d\alpha) \wedge \theta(y)$ are of degree $m-1$ in the fiber differentials, while $\operatorname{Res}(T) = \alpha$ and $\operatorname{Res}(dT) = d\alpha$.

This corollary will be used later to show that in some important cases $\operatorname{Res}(T)$ is d-closed even though dT is not identically zero.

Remark 1.26. Note that the condition $\operatorname{Res}(dT) = 0$ is independent of the section ν, but implies that the $L^1_{\text{loc}}(X)$ part of $d\nu^*(T)$ is d-closed for all atomic sections ν.

2. The Singular Pushforward Connection $\overrightarrow{\mathbf{D}}$ on F.

Suppose μ is a smooth section of a complex vector bundle F of rank n. Note that μ can be equivalently described as a bundle map $\underline{\mathbf{C}} \xrightarrow{\alpha} F$ where $\alpha 1 \equiv \mu$. Assume that F has a connection D_F and a metric $\langle\ ,\ \rangle_F$; and consider the trivial connection d on $\underline{\mathbf{C}}$. The push forward singular connection (see (I.2.11)),

$$\overrightarrow{D} \equiv \alpha \circ d \circ \beta + D_F \circ (1 - \alpha\beta),$$

is a smooth connection on F outside the zero set of μ. Here $\beta \equiv \frac{\alpha^*}{|\alpha|^2}$ where $|\alpha|^2 \equiv \alpha^*\alpha$ and $\alpha^*(\nu) = \langle \nu, \mu \rangle$. Note that $\vec{D}$ may also be expressed on a section ν by the formula

$$\vec{D}\nu \equiv D_F\nu - \frac{\langle \nu, \mu \rangle}{|\mu|^2} D_F\mu.$$

Note in particular that μ is $\vec{D}$-parallel.

Let $\chi(t)$ denote a given approximate one. Throughout this chapter we shall use the notation $\chi_s \equiv \chi\left(\frac{|\mu|^2}{s^2}\right) = \chi\left(\frac{|u|^2}{s^2}\right)$ and $\chi_s' = \chi'\left(\frac{|\mu|^2}{s^2}\right) = \chi'\left(\frac{|u|^2}{s^2}\right)$. Recall from (I.4.9) and (I.4.6) the family $(0 < s \le +\infty)$ of smooth connections

$$\vec{D}_s \;=\; \alpha \circ d \circ \beta_s + D_F \circ (1 - \alpha\beta_s)$$

where $\beta_s \equiv \chi_s\beta$. On a section ν, $\vec{D}_s$ can be reexpressed by,

$$\vec{D}_s\nu \;=\; D_F\nu - \chi_s\frac{\langle \nu, \mu \rangle}{|\mu|^2} D_F\mu.$$

We shall adopt the following notation. Given a local frame f for F, let $\mu = uf$ define $u \equiv (u_1, \ldots, u_n)$, and let $u^* \equiv h_F\bar{u}^t$. Set $|u|^2 = uu^*$. (This is more convenient). Also let

$$Du \;\equiv\; du + u\omega_F \quad \text{and} \quad Du^* \;\equiv\; du^* - \omega_F u^*.$$

The local gauge for $\vec{D}$ is given by

$$\vec{\omega} \;\equiv\; \omega_F - \frac{u^*}{|u|^2}(u\omega_F + du) \;=\; \omega_F - \frac{u^*Du}{|u|^2}, \tag{2.1}$$

because of (I.5.2) and (I.5.15). The local gauge for $\vec{D}_s$ is given by

$$\vec{\omega}_s \;\equiv\; \omega_F - \chi_s\frac{u^*}{|u|^2}(u\omega_F + du) \;=\; \omega_F - \chi_s\frac{u^*Du}{|u|^2}, \tag{2.2}$$

because of (I.5.3) and (I.5.15).

The case we are considering, of a section $\mu = \alpha 1$, is a special case of the fundamental injective conformal case of Section I.6. Therefore, the following formula for the curvature $\vec{\Omega}_s$ of the connection $\vec{D}_s$ is a special case of a formula in Remark I.6.7. Alternatively the formula for Ω_s can be computed directly using the formula (2.2) for the gauge ω_s.

Lemma 2.3. *The curvature $\overrightarrow{\Omega}_s$ of $\overrightarrow{D}_s$ is given by:*

$$(2.4)\quad \overrightarrow{\Omega}_s = (1-\chi_s)\Omega_F + \chi_s \overrightarrow{\Omega}_0 - \chi'_s \tfrac{|u|^2}{s^2} \frac{d|u|^2}{|u|^2} \frac{u^*Du}{|u|^2} + \chi_s(1-\chi_s)\frac{u^*Duu^*Du}{|u|^4},$$

where

$$(2.5)\quad \Omega_0 = \left(1 - \frac{u^*u}{|u|^2}\right)\left(\Omega_F - \frac{Du^*Du}{|u|^2}\right).$$

Moreover, $\lim_{s\to 0} \Omega_s = \Omega_0$, *outside the singular (zero) set of μ, in the C^∞ topology.*

Suppose that ϕ is an invariant polynomial. Let $\phi(\overrightarrow{D}_s) = \phi(\Omega_s)$ denote the ϕ-Chern form of the connection $\overrightarrow{D}_s$. Recall the standard transgression formula for the family of smooth connections $\overrightarrow{D}_s$ $(0 < s \le +\infty)$,

$$(2.6)\quad \phi(D_F) - \phi(\overrightarrow{D}_s) = dT_s,$$

where

$$(2.7)\quad T_s \equiv \int_s^\infty \phi(\dot\omega_s \,;\, \overrightarrow{\Omega}_s)ds = -\int_s^\infty \phi\left(\frac{u^*Du}{|u|^2} \,;\, \overrightarrow{\Omega}_s\right)\frac{\partial}{\partial s}\chi_s ds.$$

The next result provides the key to the main Theorem 2.30 of this Section.

Proposition 2.8. *Suppose μ is an atomic section. The transgression current $T \equiv \lim_{s\to 0} T_s$ converges in $L^1_{\mathrm{loc}}(X)$. That is the integral*

$$(2.9)\quad T \equiv -\int_0^\infty \phi\left(\frac{u^*Du}{|u|^2} \,;\, \overrightarrow{\Omega}_s\right)\frac{\partial}{\partial s}\chi_s ds$$

converges in $L^1_{\mathrm{loc}}(X)$.

Proof 1. First we sketch a direct proof. It suffices to dominate the first factor,

$$(2.10)\quad \phi\left(\frac{u^*Du}{|u|^2} \,;\, \overrightarrow{\Omega}_s\right),$$

of the transgression integrand in (2.7), by an $L^1_{\mathrm{loc}}(X)$ form independent of s because then this factor can be discarded from the integrand leaving the positive second factor $-\frac{\partial}{\partial s}\chi\left(\frac{|u|^2}{s^2}\right)$ which integrates to

$$(2.11)\quad -\int_s^\infty \frac{\partial}{\partial s}\chi\left(\tfrac{|u|^2}{s^2}\right)ds = \chi\left(\tfrac{|u|^2}{s^2}\right).$$

Note that $\chi\left(\frac{|u|^2}{s^2}\right)$ is bounded and converges to one almost everywhere.

Since both $\chi_s \equiv \chi\left(\frac{|u|^2}{s^2}\right)$ and $\chi'\frac{|u|^2}{s^2} = \chi'\left(\frac{|u|^2}{s^2}\right)\frac{|u|^2}{s^2}$ are bounded independent of u and s, the formula for the curvature given in Lemma 2.3 can be used in conjunction with the hypothesis of atomicity to dominate (2.10). This works because $\phi\left(\frac{u^*Du}{|u|^2}\,;\,\overrightarrow{\Omega}_s\right)$ is of degree $\leq 2n-1$ in the fiber differentials $du_1,\ldots,du_n$, $d\bar{u}_1,\ldots,d\bar{u}_n$. To verify this fact we need Lemma 2.20 below. This Lemma implies that

$$u^*\cdot\frac{\partial}{\partial u^*}\llcorner\,\phi\left(\frac{u^*Du}{|u|^2}\,;\,\overrightarrow{\Omega}_s\right) = 0,$$

so that the degree of $\phi\left(\frac{u^*Du}{|u|^2}\,;\,\overrightarrow{\Omega}_s\right)$ must be $\leq 2n-1$. $\square$

Proof 2. This proof is in essence the same as the first proof but takes advantage of an explicit formula for T_s in Proposition 3.17 which shows that T_s is a polynomial in $\chi_s = \chi\left(\frac{|u|^2}{s^2}\right)$ with coefficients that are $L^1_{\text{loc}}(X)$ forms independent of s. Since χ_s is bounded uniformly in u and s and χ_s converges to one almost everywhere, the Lebesgue Dominated Convergence Theorem implies that T_s converges to T in $L^1_{\text{loc}}(X)$, as $s\to 0$. $\square$

This explicit formula for T_s yields an explicit formula for T as well, which shows that T is independent of the choice of approximate one χ (see Theorem 3.15).

The two proofs just presented rely on facts presented below. These facts are most easily derived by first working in the universal case and then "pulling down" by the atomic section μ.

Remark 2.12. The Universal Case. All the calculations above are valid of course in the universal case introduced in I.3. Via the canonical isomorphism $\text{Hom}(\underline{\mathbf{C}}, F)\cong F$, there are certain simplifications here of the universal construction. Let $\pi: F\to X$ denote the bundle projection and consider the pull backs: $\mathbf{F} = \pi^*F$ and $\mathbf{D_F} = \pi^*D_F$ of F and its given connection to the total space of F.

Thus we have the fibre square

$$\begin{array}{ccc} \mathbf{F} = \pi^*F & \longrightarrow & F \\ \big\downarrow & & \big\downarrow \\ F & \longrightarrow & X \end{array}$$

Let $X \subset F$ denote the set of 0-vectors in F. Then the bundle $\mathbf{F}$ comes equipped with a "tautological" atomic cross-section $\boldsymbol{\mu}$ which vanishes to first order on X and has

$$\mathrm{Div}(\boldsymbol{\mu}) \;=\; [X].$$

This section $\boldsymbol{\mu}$ is defined at a point $v \in F_x = \pi^{-1}(x)$ by

$$\boldsymbol{\mu}(v) \;=\; v$$

under the identification $\mathbf{F}_v = (\pi^*F)_v = F_{\pi(v)} = F_x$.

We denote by $\overrightarrow{\mathbf{D}}$ the singular pushforward connection on $\mathbf{F}$ associated to the tautological cross-section $\boldsymbol{\mu}$, and we let $\overrightarrow{\mathbf{D}}_{\mathbf{s}}$ denote the smoothing family constructed via the fixed approximate one χ. The formulas derived above are valid in this case. In fact they can be pulled down via any atomic cross-section $\mu : X \to F$ to give the corresponding formulas on X.

For what follows we begin by working in the universal case. Thus we have the equation

$$\phi(\mathbf{D}_{\mathbf{F}}) - \phi(\overrightarrow{\mathbf{D}}_{\mathbf{s}}) \;=\; d\mathbf{T}_{\mathbf{s}} \quad \text{on } F$$

and each of the formulas above are valid for the gauge $\overrightarrow{\boldsymbol{\omega}}_s$, the curvature $\overrightarrow{\boldsymbol{\Omega}}_{\mathbf{s}}$ and the transgression $\mathbf{T}_{\mathbf{s}}$ on the total space of F. Note however that $u = (u_1, \ldots, u_n)$ can now be interpreted as linear fibre coordinates on F (with respect to a local framing).

Fix $\epsilon > 0$ and let ρ denote the restriction of $\pi : F \to X$ to the ϵ-sphere bundle in F, so that for each $x \in X$,

$$\rho^{-1}(x) \;=\; \{v \in F_x : |v| \;=\; \epsilon\}.$$

Definition 2.13. The **residue form** $\mathrm{Res}_\phi(\vec{D})$ is defined to be minus the residue of the transgression potential $\mathbf{T}$, i.e.,

$$\mathrm{Res}_\phi(\vec{D}) = -\int_{\rho^{-1}} \mathbf{T} = -\mathrm{Res}(\mathbf{T}).$$

This fiber integral is the same as the current pushforward

$$\mathrm{Res}_\phi(\vec{D}) = -\rho_*(\mathbf{T}).$$

Universal Theorem 2.14. *The ϕ-Chern current $\phi((\vec{\mathbf{D}}))$ of the pushforward singular connection $\vec{\mathbf{D}}$ associated with the canonical section $\boldsymbol{\mu}$ of $\mathbf{F} = \pi^* F$ is given by*

$$\phi((\vec{\mathbf{D}})) = \phi(\vec{\boldsymbol{\Omega}}_0) + [X]\,\mathrm{Res}_\phi(\vec{D}) \qquad \text{on } F. \tag{2.15}$$

The equation

$$\phi(\mathbf{D_F}) - \phi(\vec{\boldsymbol{\Omega}}_0) - [X]\,\mathrm{Res}_\phi(\vec{D}) = d\mathbf{T} \qquad \text{on } F \tag{2.16}$$

is the limiting form, as s approaches zero, of the equation

$$\phi(\mathbf{D_F}) - \phi(\vec{\mathbf{D}}_s) = d\mathbf{T_s}. \qquad \text{on } F \tag{2.17}$$

The L^1_{loc}-form $\phi(\vec{\boldsymbol{\Omega}}_0)$ is d-closed on F, and the residue $\mathrm{Res}_\phi(\vec{D})$ is a smooth d-closed form on X.

Remark 2.18. Compact Support. These equations imply that

$$\phi(\vec{\mathbf{D}}_s) - \phi(\vec{\boldsymbol{\Omega}}_0) - [X]\,\mathrm{Res}_\phi(\vec{D}) = d\mathbf{R_s} \qquad \text{where } \mathbf{R_s} \equiv \mathbf{T} - \mathbf{T_s},$$

and

$$\mathbf{R_s} = \mathbf{T} - \mathbf{T_s} \text{ converges to zero in } L^1_{\mathrm{loc}}(F).$$

Note that if χ is an approximate one with compact support, i.e., $\chi(t) \equiv 1$ for t large, then $\mathbf{R_s}$, and hence $d\mathbf{R_s}$ are compactly supported in the fibers of F.

Before giving the proof some lemmas are required. First, note that (2.17) is just the standard C^∞ transgression formula. Also note that, on $F \sim X$,

$$\mathbf{T} = -\int_0^\infty \phi\left(\frac{u^*Du}{|u|^2}\ ;\ \vec{\boldsymbol{\Omega}}_s\right)\frac{\partial}{\partial s}\chi_s ds$$

converges as an integral of C^∞ forms and that on $F \sim X$, equation (2.16) becomes

$$\phi(\mathbf{D_F}) - \phi(\vec{\boldsymbol{\Omega}}_0) = d\mathbf{T} \qquad \text{on } F \sim X, \tag{2.19}$$

which is just another special case of the standard C^∞ transgression formula.

Lemma 2.20. *The transgression integrand is given by*

$$\phi\left(\frac{u^*Du}{|u|^2}\ ;\ \vec{\boldsymbol{\Omega}}_s\right)\frac{\partial}{\partial s}\chi_s ds = u^* \cdot \frac{\partial}{\partial u^*} \llcorner \phi(\vec{\boldsymbol{\Omega}}_s)\tfrac{ds^2}{s^2}.$$

Proof. Note that

$$u^*\frac{\partial}{\partial u^*} \llcorner \phi(\vec{\boldsymbol{\Omega}}_s) = \phi\left(u^*\frac{\partial}{\partial u^*} \llcorner \vec{\boldsymbol{\Omega}}_s\ ;\ \vec{\boldsymbol{\Omega}}_s\right)$$

and that, by the formula (2.4) for $\vec{\boldsymbol{\Omega}}_s$

$$u^*\frac{\partial}{\partial u^*} \llcorner \vec{\boldsymbol{\Omega}}_s = -\tfrac{|u|^2}{s^2}\chi_s'\frac{u^*Du}{|u|^2}.$$

Since

$$\frac{\partial}{\partial s}\chi_s ds = -\tfrac{|u|^2}{s^2}\chi_s'\tfrac{ds^2}{s^2}$$

the lemma follows. □

Corollary 2.21. *On $F \sim X$, the transgression $\mathbf{T_s}$ is a homogeneous form which is of degree $\leq 2n-1$ in the fiber differentials.*

In fact because of Theorem 3.15 in the next section $\mathbf{T_s}$ is the sum of terms of bidegree k, $k-1$ in the differential one forms $Du_1, \dots, Du_n\ ;\ Du_1^*, \dots, Du_n^*$.

Lemma 2.22. *Both*

$$u\,\frac{\partial}{\partial u} \llcorner \phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}}) = 0 \quad \text{and} \quad u^*\,\frac{\partial}{\partial u^*} \llcorner \phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}}) = 0$$

so that

$$\phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}}) \text{ is of degree } \leq 2n-2 \text{ in the fiber one forms } du, d\bar{u}.$$

Proof. Consulting the formula (2.5) for $\vec{\boldsymbol{\Omega}}_{\mathbf{0}}$ we see that $\vec{\boldsymbol{\Omega}}_{\mathbf{0}}$ is of the form $P\Sigma$, where $P \equiv 1 - \frac{u^*u}{|u|^2}$, and where $\Sigma \equiv \vec{\boldsymbol{\Omega}}_{\mathbf{F}} - \frac{Du^*Du}{|u|^2}$. Thus $u^*\frac{\partial}{\partial u^*} \llcorner \Sigma = -\frac{u^*Du}{|u|^2}$. Therefore $u^*\frac{\partial}{\partial u^*} \llcorner \vec{\boldsymbol{\Omega}}_{\mathbf{0}} = 0$ and hence $u^*\frac{\partial}{\partial u^*} \llcorner \phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}}) = 0$. Since ϕ is an invariant polynomial $\phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}}) = \phi(P\Sigma) = \phi(\Sigma P)$. The proof that $u\frac{\partial}{\partial u} \llcorner \phi(\Sigma P) = 0$ is similar and omitted. □

Corollary 2.23. *On $F \sim X$,*

$$d\mathbf{T} \;=\; \phi(\mathbf{D_F}) - \phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}})$$

is a homogeneous form of degree $\leq 2n-2$ in the fiber differentials.

Proof of Theorem. Because of Corollary 2.21, Theorem 1.10 applies to $\mathbf{T}$ yielding the current equation (2.16) on tot F. Because $\mathbf{T_s}$ converges to $\mathbf{T}$ in $L^1_{\text{loc}}(F)$ by Proposition 2.8, $d\mathbf{T_s}$ must converge, weakly as currents, to $d\mathbf{T}$. Therefore, (2.17) and (2.16) imply that the ϕ-Chern current $\phi((\vec{\mathbf{D}})) \equiv \lim_{s\to 0} \phi(\vec{\mathbf{D}}_{\mathbf{s}})$ exists and is given by (2.15).

Because of Corollary 1.22, it remains to prove that

$$\text{Res}(d\mathbf{T}) \;=\; 0$$

which is an immediate consequence of Corollary 2.23. □

Remark 2.24. Alternate Formula for $\text{Res}_\phi(\vec{\mathbf{D}})$. Recall Remark 2.18. Applying π_* to the equation in this Remark we obtain

$$\text{Res}_\phi(\vec{\mathbf{D}}) \;=\; \pi_*\left(\phi(\vec{\mathbf{D}}_{\mathbf{s}}) - \phi(\vec{\boldsymbol{\Omega}}_{\mathbf{0}})\right), \tag{2.25}$$

since $\pi_*(d\mathbf{R_s}) = d\pi_*(\mathbf{R_s})$, and $\pi_*(\mathbf{R_s}) = 0$ because $\mathbf{R_s} = \mathbf{T} - \mathbf{T_s}$ is of degree $< 2n$ in the fiber variables. In terms of fiber integration this yields

$$\text{(2.26)} \qquad \operatorname{Res}_\phi(\overrightarrow{D}) = \int_{\pi^{-1}} \phi(\overrightarrow{\mathbf{D}}_s),$$

since $\phi(\overrightarrow{\boldsymbol{\Omega}}_0)$ is of degree $< 2n$ in the fiber variables.

Note that the residue form $\operatorname{Res}_\phi(\overrightarrow{D})$ is independent of

1. the radius ϵ of the sphere bundle in Definition 2.13,
2. the parameter $s > 0$ in equation (2.25) or (2.26),
3. the choice of approximate one χ.

The main result of this section, Theorem 2.30 below, can be viewed as the "pullback of the Universal Theorem 2.14 by an atomic section μ of F". Of course the equation

$$\phi(\mathbf{D_F}) - \phi(\overrightarrow{\mathbf{D}}_s) = d\mathbf{T}_s \qquad \text{on } F$$

of smooth forms pulls back by any smooth section μ of F to the equation

$$\phi(D_F) - \phi(D_s) = dT_s \qquad \text{on } X,$$

also of smooth forms. Similarly

$$\phi(\mathbf{D_F}) - \phi(\overrightarrow{\boldsymbol{\Omega}}_0) = d\mathbf{T} \qquad \text{on } F \sim X$$

is an equation of smooth forms which pulls back to the equation

$$\text{(2.27)} \qquad \phi(D_F) - \phi(\overrightarrow{\Omega}_0) = dT \qquad \text{on } X \sim Z.$$

If μ is atomic then by the key Proposition 2.8, $T \equiv \lim_{s\to 0} T_s$ converges in $L^1_{\text{loc}}(X)$. Therefore, the ϕ-Chern current $\phi((\overrightarrow{D})) = \lim_{s\to 0} \phi(\overrightarrow{D}_s)$ exists and satisfies

$$\text{(2.28)} \qquad \phi(D_F) - \phi((\overrightarrow{D})) = dT,$$

as an equation of currents on X.

Since the homogeneous forms $\mathbf{T}$ and $d\mathbf{T} = \phi(\mathbf{D_F}) - \phi(\overrightarrow{\boldsymbol{\Omega}}_0)$ on $F \sim X$ are both of degree $\leq 2n-1$ in the fiber differentials (Corollary 2.21 and Lemma 2.22) the exterior derivative of $T = \mu^*\mathbf{T}$ can, alternatively, be computed using Theorem 1.10, yielding the equation

$$\phi(D_F) - \phi(\overrightarrow{\Omega}_0) + \mathrm{Res}(\mathbf{T})\,\mathrm{Div}(\mu) \;=\; dT \quad \text{of currents on } X. \tag{2.29}$$

Combining (2.28) and (2.29) gives formula (2.32) below and completes the proof of the following main theorem.

Theorem 2.30. *Given an atomic section μ of a hermitian bundle F with connection D_F, the ϕ-Chern current $\phi((\overrightarrow{D})) \equiv \lim_{s\to 0}(\phi(\overrightarrow{D}_s))$ of the singular connection $\overrightarrow{D}$ of F (obtained by pushing forward the trivial connection d on $\underline{\mathbf{C}}$) is given by*

$$\phi((\overrightarrow{D})) \;=\; \phi(\overrightarrow{\Omega}_0) + \mathrm{Res}_\phi(\overrightarrow{D})\,\mathrm{Div}(\mu) \qquad \text{on } X, \tag{2.31}$$

where

$$\phi(\overrightarrow{\Omega}_0) \ \text{ belongs to } \ L^1_{\mathrm{loc}} \ \text{ and is } \ d\text{-closed on all of } \ X.$$

Therefore,

$$\phi(D_F) - \phi(\overrightarrow{\Omega}_0) - \mathrm{Res}_\phi(\overrightarrow{D})\,\mathrm{Div}(\mu) \;=\; dT \qquad \text{on } X. \tag{2.32}$$

The residue form $\mathrm{Res}_\phi(\overrightarrow{D})$ is a global smooth d-closed form on X which is independent of the atomic section μ.

This equation (2.32) of currents on X is the limiting form, as $s \to 0$, of the equation

$$\phi(D_F) - \phi(\overrightarrow{D}_s) \;=\; dT_s \tag{2.33}$$

of smooth forms on X.

Note. We shall prove in the next section that the form Res_ϕ is in fact a polynomial in the curvature of F. It is therefore entirely determined by its cohomology class, which will be computed in section 5.

3. The Transgression.

In this section we derive an explicit formula for the transgression potential T for the pushforward connection $\overrightarrow{D}$, given an invariant polynomial ϕ. It is convenient in formal calculations to adopt the universal point of view, which we shall do throughout.

Recall that by definition,

$$T_s \equiv \int_s^\infty \phi\left(\dot{\omega}_s \ ; \ \overrightarrow{\Omega}_s\right) ds. \tag{3.1}$$

The transgression integrand can be rewritten in many forms. For example, substituting for $\dot{\omega}_s$ we obtained (2.7)

$$\phi\left(\dot{\omega}_s \ ; \ \overrightarrow{\Omega}_s\right) = -\phi\left(\frac{u^*Du}{|u|^2} \ ; \ \overrightarrow{\Omega}_s\right)\frac{\partial}{\partial s}\chi_s, \tag{3.2}$$

while in Lemma 2.20 we obtained

$$\phi\left(\frac{u^*Du}{|u|^2} \ ; \ \overrightarrow{\Omega}_s\right)\frac{\partial}{\partial s}\chi_s ds = u^*\frac{\partial}{\partial u^*} \llcorner \phi(\overrightarrow{\Omega}_s)\tfrac{ds^2}{s^2}. \tag{3.3}$$

In order to obtain an even more convenient formula for the transgression integrand we need a lemma.

Lemma 3.4. *Let*

$$\alpha = \frac{Duu^*}{|u|^2} \quad \text{and} \quad \bar{\alpha} \equiv \frac{uDu^*}{|u|^2}$$

denote scalar one forms and φ and ψ any scalar functions. Suppose $u^*\frac{\partial}{\partial u^*} \llcorner A = 0$. Then

$$\phi\left(\tfrac{u^*Du}{|u|^2} \ ; \ A + (\varphi\alpha + \psi\bar{\alpha})\frac{u^*Du}{|u|^2}\right) = \phi\left(\tfrac{u^*Du}{|u|^2} \ ; \ A\right). \tag{3.5}$$

Proof. Since $\varphi\alpha + \psi\bar{\alpha}$ is a scalar one form,

$$\phi\left(A + (\varphi\alpha + \psi\bar{\alpha})\tfrac{u^*Du}{|u|^2}\right) = \phi(A) + (\varphi\alpha + \psi\bar{\alpha})\phi\left(\tfrac{u^*Du}{|u|^2} \ ; \ A\right). \tag{3.6}$$

Contracting $u^*\frac{\partial}{\partial u^*}$ into the L.H.S. of (3.6) yields ψ times the L.H.S. of (3.5) while contracting $u^*\frac{\partial}{\partial u^*}$ into the R.H.S. of (3.6) yields ψ times the R.H.S. of (3.5). $\square$

To apply this Lemma define

$$(3.7)\quad A(x) \equiv (1-x)\Omega_F + x\left(1 - \tfrac{u^*u}{|u|^2}\right)\left(\Omega_F - \frac{Du^*Du}{|u|^2}\right) + x(1-x)\frac{u^*Duu^*Du}{|u|^4}$$

and note that

$$(3.8)\qquad u^*\frac{\partial}{\partial u^*} \llcorner A(\chi_s) = 0$$

because $\left(1 - \frac{u^*u}{|u|^2}\right) u^* = 0$. The curvature can be expressed as

$$(3.9)\qquad \overrightarrow{\Omega}_s = A(\chi_s) - \chi_s' \tfrac{|u|^2}{s^2} \frac{d|u|^2}{|u|^2} \frac{u^*Du}{|u|^2}.$$

Corollary 3.10.

$$\phi\left(\tfrac{u^*Du}{|u|^2} \,;\, \overrightarrow{\Omega}_s\right) = \phi\left(\tfrac{u^*Du}{|u|^2} \,;\, A(\chi_s)\right).$$

Now define

$$(3.11)\qquad \Lambda(x) \equiv \left(1 - x\tfrac{u^*u}{|u|^2}\right)\Omega_F - x\tfrac{Du^*Du}{|u|^2}.$$

Then

$$(3.12)\qquad \begin{aligned} \Lambda(\chi_s) &= A(\chi_s) - \chi_s \frac{u^*uDu^*Du}{|u|^4} - \chi_s(1-\chi_s)\frac{u^*Duu^*Du}{|u|^4} \\ &= A(\chi_s) - \chi_s\bar{\alpha}\frac{u^*Du}{|u|^2} - \chi_s(1-\chi_s)\alpha\frac{u^*Du}{|u|^2} \end{aligned}$$

so that Lemma 3.4 implies that

$$(3.13)\qquad \phi\left(\tfrac{u^*Du}{|u|^2} \,;\, \Lambda(\chi_s)\right) = \phi\left(\tfrac{u^*Du}{|u|^2} \,;\, A(\chi_s)\right).$$

Therefore, Lemma 3.3 has the following consequence.

Corollary 3.14.

$$\phi\left(\tfrac{u^*Du}{|u|^2}\ ;\ \overrightarrow{\Omega}_s\right) \;=\; \phi\left(\tfrac{u^*Du}{|u|^2}\ ;\ \Lambda(\chi_s)\right).$$

Theorem 3.15. *Suppose μ is an atomic section of F. The transgression T is given by the formal integral*

$$T \;=\; \int_0^1 \phi\left(\frac{u^*Du}{|u|^2}\ ;\ \Lambda(x)\right) dx$$

where

$$\Lambda(x) \;\equiv\; \left(1 - x\frac{u^*u}{|u|^2}\right)\Omega_F - x\frac{Du^*Du}{|u|^2}.$$

In particular, the transgression T is independent of the choice of approximate one χ.

Proof. Corollary 3.14 yields the formula

$$(3.16) \qquad T_s \;=\; -\int_s^\infty \phi\left(\tfrac{u^*Du}{|u|^2}\ ;\ \Lambda(\chi_s)\right) \frac{\partial}{\partial s}\chi_s ds$$

for the transgression at time $s > 0$.

The change of variables $s \mapsto x \equiv \chi\left(\frac{|u|^2}{s^2}\right)$ in the integrand for T_s yields the next

Proposition 3.17. *The transgression T_s is given by the formal integral*

$$T_s \;=\; \int_0^{\chi_s} \phi\left(\tfrac{u^*Du}{|u|^2}\ ;\ \left(1 - x\tfrac{u^*u}{|u|^2}\right)\Omega_F - x\tfrac{Du^*Du}{|u|^2}\right) dx.$$

Now the integrand is a polynomial in x and this is the sense in which the integral is formal. Therefore T_s is a polynomial in χ_s. The coefficients of this polynomial are homogeneous of degree zero in u and independent of s.

We now recall Corollary 2.21 which states that

$$(3.18) \qquad T_s \text{ is of degree } < 2n \text{ in the fiber one forms } du_i,\ d\bar{u}_i.$$

The hypothesis that μ is atomic, combined with (3.19) ensures that when T_s is considered as a polynomial in χ_s, the coefficients of this polynomial are $L^1_{\text{loc}}(X)$ forms. Since $\chi_s \equiv \chi\left(\frac{|u|^2}{s^2}\right)$ is bounded and converges to one almost everywhere, the Lebesgue dominated convergence theorem implies that T_s converges to T in $L^1_{\text{loc}}(X)$ as $s \to 0$, where T is given as in Theorem 3.15. This completes the proof of Theorem 3.15. $\square$

Remark 3.19. The Chern Forms $\phi(\overrightarrow{\mathbf{D}}_{\mathbf{s}})$. The formulas in this section can also be used to simplify the Chern forms $\phi(\overrightarrow{D}_s)$. Note that with α and $\bar{\alpha}$ defined as in Lemma 3.4

(3.20)
$$\begin{aligned}\overrightarrow{\Omega}_s &= A(\chi_s) - \chi_s' \tfrac{|u|^2}{s^2} \tfrac{d|u|^2}{|u|^2} \tfrac{u^* Du}{|u|^2} = \Lambda(\chi_s) + \chi_s \bar{\alpha} \tfrac{u^* Du}{|u|^2} - \chi_s(1-\chi_s)\alpha \tfrac{u^* Du}{|u|^2} \\ &\quad - \chi_s' \tfrac{|u|^2}{s^2} \tfrac{d|u|^2}{|u|^2} \tfrac{u^* Du}{|u|^2}.\end{aligned}$$

Therefore,

$$(3.21)\qquad \begin{aligned}\phi(\overrightarrow{\Omega}_s) &= \phi(A(\chi_s)) - \chi_s' \tfrac{|u|^2}{s^2} \tfrac{d|u|^2}{|u|^2} \phi\left(\tfrac{u^* Du}{|u|^2}\ ;\ A(\chi_s)\right) \\ &= \phi(A(\chi_s)) - \chi_s' \tfrac{|u|^2}{s^2} \tfrac{d|u|^2}{|u|^2} \phi\left(\tfrac{u^* Du}{|u|^2}\ ;\ \Lambda(\chi_s)\right),\end{aligned}$$

because of (3.13). Consequently, because of (3.6),

(3.22)
$$\phi(\overrightarrow{\Omega}_s) = \phi(\Lambda(\chi_s)) + \left(\chi_s \tfrac{uDu^*}{|u|^2} - \chi_s(1-\chi_s)\tfrac{Duu^*}{|u|^2} - \chi_s' \tfrac{|u|^2}{s^2} \tfrac{d|u|^2}{|u|^2}\right) \phi\left(\tfrac{u^* Du}{|u|^2}\ ;\ \Lambda(\chi_s)\right).$$

Theorem 3.23. *The residue form $\mathrm{Res}_\phi(\overrightarrow{D})$ defined in 2.13 is a polynomial in the curvature of F. Thus, the residue map factors as*

$$\begin{array}{ccc} I_{\mathrm{GL}_n(\mathbf{C})} & \xrightarrow{\mathrm{Res}} & \mathcal{E}^*(X) \\ & r \searrow \quad \nearrow W & \\ & I_{\mathrm{GL}_n(\mathbf{C})} & \end{array}$$

where r is an additive homomorphism of degree $-n$ and where W is the standard Weil homomorphism.

Note. Determination of the map r amounts to a determination of the cohomology class of the residue universally, i.e., over $B\mathrm{GL}_n(\mathbf{C})$.

Proof. From 2.13 and 3.15 we see that Res_ϕ is given pointwise on X by the following universal invariant function applied to the curvature and connection of F:

$$(3.24)\qquad \mathrm{Res}_\phi(\overrightarrow{D}) = \int_{|u|=1} \int_0^1 \phi(u^* Du\ ;\ \Lambda(x))dx.$$

We want to show that in fact this can be expressed purely in terms of the curvature. Let $(Du)_1, \ldots, (Du)_n$, denote the components of $Du = du + u\omega_F$. Then the integrand in (3.24) can be written in the form

$$\phi(u^*Du\ ;\ \Lambda(x)) \;=\; \sum_{|I|,|J|\le n} c_{I,J}(Du)_I(Du^*)_J, \tag{3.25}$$

where the sum is over strictly increasing multi-indices and where each coefficient $c_{I,J}$ is a polynomial in Ω_F and x. From Corollary 2.21 we conclude that the coefficient $c_{(1,\ldots,n),(1,\ldots,n)}$ of top degree is zero. Hence

$$\begin{aligned}\int_{|u|=1} \phi(u^*Du\ ;\ \Lambda(x)) \;&=\; \sum_{|I|+|J|\le 2n-1}\int_{|u|=1} c_{I,J}(Du)_I(Dv)_J \\ &=\; \sum_{|I|+|J|=2n-1}\int_{|u|=1} c_{I,J}(du)_I(du)_J,\end{aligned}$$

since in each $(Du)_I(Du^*)_J$ with $|I| + |J| = 2n - 1$, only the leading term $(du)_I(du^*)_J$ is picked off by the fibre integral. Consequently, no terms in ω_F occur in this expression, except for the curvature terms in the $c_{I,J}$. □

Remark 3.26. Functoriality of the Transgression. Suppose $f : X' \longrightarrow X$ is a smooth map between manifolds, and let $F' = f^*F$, $D_{F'} = f^*D_F$, $\mu' = f^*\mu$ be the induced bundle, connection and cross-section. Suppose that both μ and μ' are atomic. Then the transgression current is also induced, i.e.,

$$T' \;=\; f^*T$$

in the sense that on $X' \sim \operatorname{spt}(\operatorname{Div}(\mu'))$ this is an equation of smooth forms having an L^1_{loc}-extension across $\operatorname{spt}(\operatorname{Div}(\mu'))$ (cf. II.5.10).

Note that given an atomic section μ, there are conditions on f required for $f^*\mu$ to be atomic. For example, if f and μ are real analytic, then it suffices that the zeros of $\mu' = \mu \circ f$ have the proper dimension. This means that codim $(f^{-1}Z) = \operatorname{codim}(Z)$ where Z is the zero set of μ. In the case where f and μ are only smooth, one must require that f be sufficiently transversal to Z. In particular if every point of Z is a regular value of f, then $f^*\mu$ is atomic.

4. The Top Chern Current and Universal Thom-Chern Forms.

In this section we examine our main theorems in the special case of the top Chern class. This immediately yields several interesting results. For each atomic section μ, the associated Chern current will be the divisor of μ, and we obtain a C^∞ Poincaré-Lelong formula ((4.4) below). When applied to the tautological section of $\mathbf{F} = \pi^* F$ over F we obtain, for each choice of approximation mode and each $s > 0$, a representative $\boldsymbol{\tau}_{\mathbf{s}}$ of the Thom class of F which is written *canonically and universally* in terms of the connection on F. In fact for each χ and s we produce an explicit class in the equivariant cohomology of $\mathbf{C}^n$ whose image under the equivariant Weil homomorphism is $\boldsymbol{\tau}_{\mathbf{s}}$. The family $\boldsymbol{\tau}_{\mathbf{s}}$, $s > 0$ turns out to differ only by pullback under homotheties and so depends only midly on s. However, as $s \to 0$ we show that $\boldsymbol{\tau}_{\mathbf{s}} \to [X]$ which is of course the canonical singular representative of the Thom class. The family also has the pleasant property that for each $s > 0$, the restriction of τ_s to $X \subset F$ is the Chern-Weil representative of the top Chern class in the given connection.

To begin we assume as above that F, $\langle\ ,\ \rangle_F$ is a hermitian bundle of rank n with connection D_F (which is *not* assumed to be compatible with the metric), and we fix an approximate one χ. Throughout this section we shall focus exclusively on the invariant polynomial

$$c_n \;=\; \widetilde{\det} \;=\; \left(\tfrac{i}{2\pi}\right)^n \det,$$

and for any atomic section μ on F we shall call the associated Chern current

$$\widetilde{\det}((\overrightarrow{D})) \;=\; \lim_{s\to 0} \widetilde{\det}(\overrightarrow{D}_s) \;=\; \lim_{s\to 0} \det\left(\tfrac{i}{2\pi}\overrightarrow{\Omega}_s\right)$$

the **top Chern current** of the singular pushforward connection. Our first series of results concerning this current is as follows. Proofs are postponed until after the statements and discussion.

Theorem 4.1. *The top Chern current $\widetilde{\det}((\overrightarrow{D}))$ of the singular connection $\overrightarrow{D}$ determined by an atomic section μ of F is equal to the divisor $\mathrm{Div}(\mu)$ of the section. That is,*

$$\widetilde{\det}((\overrightarrow{D})) \;=\; \mathrm{Div}(\mu).$$

This result implies, in particular, that the top Chern current depends only on the section μ, it is independent of the connection D_F on F, the metric $\langle\ ,\ \rangle_F$ on F, and the approximate one χ.

Theorem 4.2. *The standard transgression formula gives the equation*

$$\widetilde{\det}(D_F) - \widetilde{\det}(\overrightarrow{D}_s) \;=\; d\sigma_s \tag{4.3}$$

of smooth forms on X. These potentials σ_s converge in $L^1_{\mathrm{loc}}(X)$ to an L^1_{loc} form σ, and hence in the limit equation (4.3) *becomes*

$$\widetilde{\det}(D_F) - \mathrm{Div}(\mu) \;=\; d\sigma. \tag{4.4}$$

These two Theorems are particularly important in the universal case as they provide a universal formula (cf. [MQ]) for a Thom form. The canonical section $\boldsymbol{\mu}$ of the pullback bundle $\mathbf{F}$ over F is atomic.

Definition 4.5. For $s > 0$ fixed, the **Thom-Chern Form** $\boldsymbol{\tau}_{\mathbf{s}}$ is defined by

$$\boldsymbol{\tau}_{\mathbf{s}} \equiv \widetilde{\det}(\overrightarrow{\mathbf{D}}_{\mathbf{s}})$$

where $\overrightarrow{\mathbf{D}}_{\mathbf{s}}$ is the connection family associated with the canonical section of $\mathbf{F}$ on the total space of F. This form is determined by the connection $\mathbf{D_F}$, the metric $\langle\ ,\ \rangle_{\mathbf{F}}$, and the approximate one χ.

Consulting the local formula (2.4) for $\overrightarrow{\boldsymbol{\Omega}}_{\mathbf{s}}$ and setting $u = 0$ yields

$$i^*\boldsymbol{\tau}_{\mathbf{s}} \;=\; \widetilde{\det}(D_F) \tag{4.6}$$

where $i : X \to F$ is the inclusion map. That is, the restriction of each Thom Form $\boldsymbol{\tau}_{\mathbf{s}}$ to the zero section $X = i(X) \subset F$ is the **top Chern form** or **Euler form** $\widetilde{\det}(D_F)$ of (F, D_F) on X.

This form $\boldsymbol{\tau}_{\mathbf{s}}$ has the usual properties justifying the label "Thom form". Namely, in addition to (4.6) we have that

$$\boldsymbol{\tau}_{\mathbf{s}} \ \text{ is } \ d\text{-closed,}$$

and in Lemma 4.13 below we shall prove that each fiber integral is convergent and

$$\int_{\pi^{-1}} \boldsymbol{\tau}_{\mathbf{s}} = 1,$$

or equivalently, the pushforward by $\pi : F \to X$ of $\boldsymbol{\tau}_{\mathbf{s}}$ (considered as a current) is equal to the degree-zero current 1 on the base manifold X. That is, $\pi_*(\boldsymbol{\tau}_{\mathbf{s}}) = 1$.

Because of Remark 2.24, this fiber integral is just the residue $\mathrm{Res}_\phi(\overrightarrow{D})$ when $\phi = \widetilde{\det}$.

Remark 4.7. If the approximate one χ is chosen to be of compact type, i.e., if $\chi(t) \equiv 1$ for t large, then the Thom form $\boldsymbol{\tau}_{\mathbf{s}}$ is **compactly supported in the fibers** of F.

Theorem 4.1 and Theorem 4.2 applied to the canonical section $\boldsymbol{\mu}$ of $\mathbf{F} = \pi^* F$ yield a universal equation of currents on the total space F. Let $[X]$ denote the current of integration over the zero section $X \subset F$.

Theorem 4.8. *The canonical section $\boldsymbol{\mu}$ of the bundle $\mathbf{F} = \pi^* F$ over F is atomic. Moreover,*

$$\widetilde{\det}(\mathbf{D}_{\mathbf{F}}) - [X] = d\boldsymbol{\sigma} \qquad \text{on } F$$

is obtained as the limiting form of the equation

$$\widetilde{\det}(\mathbf{D}_{\mathbf{F}}) - \boldsymbol{\tau}_{\mathbf{s}} = d\boldsymbol{\sigma}_{\mathbf{s}} \qquad \text{on } F$$

where $\boldsymbol{\tau}_{\mathbf{s}}$ is the Thom-Chern form

$$\boldsymbol{\tau}_{\mathbf{s}} \equiv \widetilde{\det}\left(\overrightarrow{\mathbf{D}}_{\mathbf{s}}\right).$$

Corollary 4.9. *The zero section $[X]$ and the Thom-Chern form $\boldsymbol{\tau}_{\mathbf{s}} \equiv \det\left(\frac{i}{2\pi}\boldsymbol{\Omega}_{\mathbf{s}}\right)$ are cohomologous, that is, setting $\mathbf{r}_s = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathbf{s}}$,*

$$\boldsymbol{\tau}_{\mathbf{s}} - [X] = d\mathbf{r}_{\mathbf{s}} \qquad \text{on } F,$$

and

$$\lim_{s \to 0} \mathbf{r}_{\mathbf{s}} = 0 \qquad \text{in } L^1_{\mathrm{loc}}(F).$$

Remark. The main results (Theorem 4.1 and Theorem 4.2) can be interpreted as follows. Any smooth section $\mu : X \to F$ can be used to pullback the second equation in Theorem 4.8 on F to the equation

$$\widetilde{\det}(D_F) - \det\left(\frac{i}{2\pi}\overrightarrow{\Omega}_s\right) \;=\; d\sigma_s \quad \text{on } X.$$

If μ is atomic, then as $s \to 0$ this equation has limit

$$\widetilde{\det}(D_F) - \operatorname{Div}(\mu) \;=\; d\sigma \;\text{ on }\; X.$$

One may also take this limiting equation (4.4) as a definition of the term by term pulling back of the first universal current equation in Theorem 4.8.

Remark 4.10. The Thom isomorphism. These results yield canonical representations, at the level of differential forms, of various versions of the Thom isomorphism. Suppose for example that the approximate one χ is chosen so that $\chi(t) = 1$ for $t \geq 1$. Then $\boldsymbol{\sigma}_{\mathbf{s}}$ has support in the s-ball bundle $U_s(F) = \{\nu \in F : |\nu| \leq s\}$. Suppose X is compact and let $\pi : F \to X$ be the bundle projection. Then the map

$$i_{!,s} : \mathcal{E}^*(X) \longrightarrow \mathcal{E}^{*+2n}_{\text{cpt}}(F)$$

from forms on X to forms with compact support on F, given by

$$i_{!,s}(\varphi) \;=\; \pi^*\varphi \wedge \boldsymbol{\tau}_{\mathbf{s}}$$

induces an isomorphism

$$i_! : H^*(X) \longrightarrow H^{*+2n}_{\text{cpt}}(F).$$

Integration over the fibre π_* clearly inverts this map since $\pi_*(\boldsymbol{\tau}_{\mathbf{s}}) = 1$. Letting s go to zero yields the canonical version of this map

$$i_{!,0} : \mathcal{E}^*(X) \longrightarrow \mathcal{E}^{*+2n}_{\text{cpt}}(F)',$$

now into currents with compact support on F, given by

$$i_{!,0}(\varphi) \;=\; \varphi[X].$$

If on the other hand we choose the algebraic approximation mode $\chi(t) = t/(1+t)$, then by Theorem I.8.14 and its corollary we see that $\boldsymbol{\tau}_{\mathbf{s}}$ extends to a smooth d-closed form on $\mathbf{P}(\underline{\mathbf{C}} \oplus F) = F \cup \mathbf{P}(F)$. Furthermore at "infinity", $\mathbf{P}(F)$, this form is zero. (To see this note that along $\mathbf{P}(F)$, $\boldsymbol{\tau}_{\mathbf{s}} = \det(D_{\underline{\mathbf{C}} \oplus V^{\perp}})$ where $\underline{\mathbf{C}} \oplus V^{\perp}$ has the direct-sum connection, trivial on $\underline{\mathbf{C}}$, and where $\pi^* F = V \oplus V^{\perp}$ is the canonical splitting over $\mathbf{P}(F)$.) Hence, $\boldsymbol{\tau}_{\mathbf{s}}$ represents a class in $H^{2n}(\mathbf{P}(\underline{\mathbf{C}} \oplus F), \mathbf{P}(F))$ which, as we have shown, is cohomologous to the zero section $[X]$.

Remark 4.11. The Gysin map. Consider an embedding $j : Y \hookrightarrow X$ of a compact oriented-manifold Y into our manifold X. Then there is a natural map $j_! : \mathcal{E}^*(Y) \to \mathcal{E}^{*+m}_{\mathrm{cpt}}(X)'$, from forms on Y to currents with compact support on X, given by

$$j_!(\varphi) \;=\; \varphi[Y].$$

(Here $m = \dim X - \dim Y$.) Suppose now that the normal bundle N to Y carries an almost complex structure and give it a complex connection. Let $\boldsymbol{\tau}_{\mathbf{s}}$ be the family of Thom forms in compact approximation mode (as in 4.10). Identify N with a tubular neighborhood of Y. Since $\boldsymbol{\tau}_{\mathbf{s}}$ has compact support, it extends by zero to a d-closed form with compact support on X, and we have $\boldsymbol{\tau}_{\mathbf{s}} - [Y] = d\mathbf{r}_{\mathbf{s}}$ where $\mathbf{r}_{\mathbf{s}} \to 0$ as $s \to 0$. Thus we get a smooth Gysin map $j_{!,s} : \mathcal{E}^*(Y) \to \mathcal{E}^{*+m}_{\mathrm{cpt}}(X)$ defined by

$$j_{!,s}(\varphi) \;=\; \pi^*\varphi \wedge \boldsymbol{\tau}_{\mathbf{s}}$$

where $\pi : N \to Y$ is the retraction of the tubular neighborhood. Note that $j_{!,s} \to j_!$ as $s \to 0$ and that $j_{!,s}$ and $j_!$ induce the same maps

$$j_! : H^*(Y) \longrightarrow H^{*+m}_{\mathrm{cpt}}(X)$$

in de Rham cohomology.

Remark 4.12. A generalized Gysin map. The standard Gysin map now generalizes as follows. Let α be a atomic section of a complex bundle $F \to X$ with connection. Assume α vanishes on a compact set $Z = \mathrm{spt}(\mathrm{Div}\,\alpha)$. Choose χ with $\chi(t) = 1$ for $t \geq 1$ and let $\tau_s = \widetilde{\det}(\overrightarrow{D}_s)$ as above. Then for any neighborhood

U of Z there is an s_0 with $\operatorname{supp} \tau_s \subset\subset U$ for all $s < s_0$. Hence we get a map $j_{!,s} : \mathcal{E}^*(U) \to \mathcal{E}^{*+2n}_{\text{cpt}}(X)$ (where $n = \dim_{\mathbf{C}} F$) given by

$$j_{!,s}(\varphi) \;=\; \varphi \wedge \tau_s.$$

Letting $s \to 0$ gives a map $j_! : \mathcal{E}^*_Z(X) \to \mathcal{E}^{*+2n}_{\text{cpt}}(X)'$, from forms germed on Z to currents on X, defined by

$$j_!(\varphi) \;=\; \varphi \cdot \operatorname{Div}(\alpha).$$

By 4.2 they induce the same **generalized Gysin map**

$$j_! : H^*_Z(X) \longrightarrow H^{*+2n}_{\text{cpt}}(X).$$

In the next chapter we shall generalize the constructions in 4.10, 4.11 and 4.12 by replacing the complex bundles with any real oriented bundle with an orthogonal connection.

Proofs of the main theorems. Theorems 4.1, 4.2 and 4.8 are immediate special cases of Theorems 2.14 and 2.30, once the following two Lemmas are established.

Lemma 4.13. *If* $\phi = \widetilde{\det}$, *then* $\operatorname{Res}_\phi(\overrightarrow{D}) = 1$.

Proof. Let $\lambda(u) \equiv \frac{i}{2} du_1 \wedge d\bar{u}_1 \wedge \cdots \wedge \frac{i}{2} du_n \wedge d\bar{u}_n$ denote the standard volume form on $\mathbf{C}^n$. The part of $\det\left(\frac{i}{2\pi}\overrightarrow{\Omega}_s\right)$ of degree $2n$ in du, $d\bar{u}$ is given by

$$\deg_{2n} \det\left(\frac{i}{2\pi}\overrightarrow{\Omega}_s\right) \;=\; \frac{n!}{\pi^n}\chi_s^{n-1}\chi_s'\frac{|u|^2}{s^2}\frac{d\lambda(u)}{|u|^{2n}}.$$

See, for example, equation (3.21). Since $\operatorname{vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$, polar coordinates yield

$$\int_{u\in\mathbf{C}^n} \det\left(\frac{i}{2\pi}\overrightarrow{\Omega}_s\right) \;=\; \int_0^\infty n\chi(r^2)^{n-1}\chi'(r^2)\,dr^2 \;=\; 1, \tag{4.14}$$

so that by Remark 2.24

$$\operatorname{Res}_{\widetilde{\det}}(\overrightarrow{D}) \;=\; 1. \quad \square$$

Lemma 4.15. *The top Chern current* $\widetilde{\det}((\overrightarrow{D}))$ *has no* $L^1_{\rm loc}$ *part, i.e.,*

$$\widetilde{\det}(\overrightarrow{\Omega}_0) \;=\; 0.$$

Proof. Note that $\det\left(1 - \frac{u^*u}{|u|^2}\right) = 0$ and consult the formula (2.5) for $\overrightarrow{\Omega}_0$. □

The geometric secret of this proof is that outside the zero set, $\overrightarrow{D}$ is a smooth connection which admits a parallel cross section.

Theorem 4.2 and Corollary 4.9 have many applications. For example they can be used to compute the residue $\mathrm{Res}_\phi(\overrightarrow{D})$ whenever ϕ has $\widetilde{\det}$ as a factor. This has important consequences described in the next section.

Proposition 4.16. *Suppose* $\phi = \psi\cdot\widetilde{\det}$ *with* ψ *a Ad-invariant polynomial. Then the* $L^1_{\rm loc}$ *part of the* ϕ*-Chern current vanishes and*

$$\mathrm{Res}_\phi(\overrightarrow{D}) \;=\; \psi(D_F).$$

Moreover,

$$\phi(\overrightarrow{D}_s) - [X]\psi(D_F) \;=\; d(\psi(\overrightarrow{D}_s)r_s). \tag{4.17}$$

Proof. The equation $\tau_s - [X] = dr_s$, when multiplied by $\psi(\overrightarrow{D}_s)$, yields equation (4.17) since $\psi(\overrightarrow{D}_s)\,\big|_X = \psi(D_F)$. Since $\phi(\overrightarrow{\Omega}_0) = \psi(\overrightarrow{\Omega}_0)\widetilde{\det}(\overrightarrow{\Omega}_0) = 0$ by Lemma 4.15, the formula $\mathrm{Res}_\phi(\overrightarrow{D}) = \pi_*(\phi(\overrightarrow{D}_s) - \phi(\overrightarrow{\Omega}_0))$ reduces to $\mathrm{Res}_\phi(\overrightarrow{D}) = \pi_*(\phi(\overrightarrow{D}_s))$. Therefore (4.17) can be used to compute $\mathrm{Res}_\phi(\overrightarrow{D})$.

Proposition 3.17 implies that r_s is the sum of terms of bidegree $k, k-1$ in the differential one forms $Du_1, \ldots, Du_n$; $Du_1^*, \ldots, Du_n^*$. Also, $\psi(D_s)$ is a sum of terms of bidegree k, k. In particular, $\psi(D_s)r_s$ cannot have a part of bidegree n, n and hence must have fiber integral zero over the fibers of $F \xrightarrow{\pi} X$. Therefore pushing forward the current equation (4.17) by π yields the formula for $\mathrm{Res}_\phi(\overrightarrow{D})$. □

The next part of this section is devoted to obtaining some elegant explicit algebraic expressions for the Thom-Chern form $\boldsymbol{\tau}_{\mathrm{s}}$, and the various potentials $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_{\mathrm{s}}$ and $\mathbf{r}_{\mathrm{s}}$.

Theorem 4.18. *The Thom-Chern form τ_s for F determined by D_F, $\langle\ ,\ \rangle$ and χ is given explicitly by*

$$\begin{aligned}\tau_s \;&=\; \left(\tfrac{i}{2\pi}\right)^n (1-\chi_s)\det\left(\Omega_F - \chi_s \frac{Du^*Du}{|u|^2}\right)\\ &\quad + \left(\tfrac{i}{2\pi}\right)^n \left(\chi_s(1-\chi_s) - \chi_s' \tfrac{|u|^2}{s^2}\right) \tfrac{d|u|^2}{|u|^2} \det\left(\frac{u^*Du}{|u|^2}\ ;\ \Omega_F - \chi_s \frac{Du^*Du}{|u|^2}\right).\end{aligned}$$

Proof. Consider, the frame $f_1, \ldots, f_n$ for F and the dual frame $f_1^*, \ldots, f_n^*$ as elements of the grassmannian algebra $\Lambda(F^* \oplus F)$. Let $\lambda(f) \equiv f_1^* \wedge f_1 \wedge \cdots \wedge f_n^* \wedge f_n$ denote the volume form. Then, for any matrix A, the determinant can be computed from

$$\frac{1}{n!}(f^*Af)^n \;=\; \det(A)\lambda(f).$$

Consequently, the equation

$$\det(A\ ;\ B)\lambda(f) \;=\; \tfrac{1}{(n-1)!}(f^*Af)(f^*Bf)^{n-1},$$

can be used to compute $\det(A\ ;\ B)$. Let $\nu \equiv \sum u_j f_j$ denote the tensor that contracts into a form by replacing each f_j^* by u_j. The identity

$$\begin{aligned}\tfrac{f^*u^*}{|u|^2}(\nu \llcorner f^* \overrightarrow{\Omega}_s f) \;&=\; (1-\chi_s)\tfrac{f^*u^*}{|u|^2}\left(\nu \llcorner \left(f^*\Omega_F f - \chi_s \frac{f^*Du^*Duf}{|u|^2}\right)\right)\\ &\quad + \left(\chi_s(1-\chi_s) - \chi_s' \tfrac{|u|^2}{s^2}\right) \tfrac{d|u|^2}{|u|^2} \frac{f^*u^*Duf}{|u|^2}\end{aligned} \tag{4.19}$$

follows from the formula (2.4) for $\overrightarrow{\Omega}_s$. This identity (4.19) can be used to prove the Theorem 4.18, because

$$\left(f^*\overrightarrow{\Omega}_s f\right)^n \;=\; \tfrac{f^*u^*}{|u|^2}(\nu \llcorner (f^*\overrightarrow{\Omega}_s f)^n) = n\tfrac{f^*u^*}{|u|^2}(\nu \llcorner (f^*\overrightarrow{\Omega}_s f))(f^*\overrightarrow{\Omega}_s f)^{n-1},$$

and, modulo the 1-form $f^*u^* \in \Lambda^1(F^*)$,

$$f^*\overrightarrow{\Omega}_s f \;=\; f^*\Omega_F f - \chi_s \frac{f^*Du^*Duf}{|u|^2}. \quad \square$$

Corollary 4.20. Algebraic Approximation Mode. *In the algebraic approximation mode the family of Thom-Chern form $\boldsymbol{\tau}_{\mathbf{s}}$ on F determined by D_F and $\langle\ ,\ \rangle_F$ is given explicitly by*

$$\boldsymbol{\tau}_{\mathbf{s}} \;=\; \frac{s^2}{|u|^2+s^2}\widetilde{\det}\left(\boldsymbol{\Omega}_{\mathbf{F}} - \frac{Du^*Du}{|u|^2+s^2}\right),$$

Equivalently,

$$\boldsymbol{\tau}_{\mathbf{s}}\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \tfrac{s^2}{|u|^2+s^2}\left(f^*\boldsymbol{\Omega}_{\mathbf{F}}f - \frac{f^*Du^*Duf}{|u|^2+s^2}\right)^n. \tag{4.21}$$

If a and b are elements of a ring, let $\frac{a^n-b^n}{a-b}$ denote the expression $\sum\limits_{k=0}^{n-1} a^k b^{n-1-k}$.

Theorem 4.22. *The potentials $\boldsymbol{\sigma}_{\mathbf{s}}$, $\boldsymbol{\sigma}$ and $\mathbf{r}_{\mathbf{s}} \equiv \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathbf{s}}$ are given explicitly by the algebraic formulas:*

$$\boldsymbol{\sigma}_{\mathbf{s}}\lambda \;=\; -\tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \frac{f^*u^*Duf}{|u|^2}\;\frac{(f^*\boldsymbol{\Omega}_{\mathbf{F}}f - \chi_s f^*\frac{Du^*Du}{|u|^2}f)^n - (f^*\boldsymbol{\Omega}_{\mathbf{F}}f)^n}{f^*\frac{Du^*Du}{|u|^2}f} \tag{4.23}$$

and,

$$\boldsymbol{\sigma}\lambda \;=\; -\tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \frac{f^*u^*Duf}{|u|^2}\;\frac{(f^*\boldsymbol{\Omega}_{\mathbf{F}}f - f^*\frac{Du^*Du}{|u|^2}f)^n - (f^*\boldsymbol{\Omega}_{\mathbf{F}}f)^n}{f^*\frac{Du^*Du}{|u|^2}f} \tag{4.24}$$

and,

$$\mathbf{r}_{\mathbf{s}}\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \frac{f^*u^*Duf}{|u|^2}\;\frac{(A+(1-\chi_s)B)^n - A^n}{B} \tag{4.25}$$

where $A \equiv f^*\Omega_F f - f^*\frac{Du^*Du}{|u|^2}f$ and $B \equiv f^*\frac{Du^*Duf}{|u|^2}f$.

First, we express $\boldsymbol{\sigma}_{\mathbf{s}}$ and $\boldsymbol{\sigma}$ as formal integrals.

Lemma 4.26. *The potentials $\boldsymbol{\sigma}_{\mathbf{s}}$ and $\mathbf{s}$ are given explicitly by*

$$\boldsymbol{\sigma}_{\mathbf{s}} \;=\; \int_0^{\chi_s} \widetilde{\det}\left(\tfrac{u^*Du}{|u|^2}\;;\;\boldsymbol{\Omega}_{\mathbf{F}} - x\tfrac{Du^*Du}{|u|^2}\right)dx \tag{4.27}$$

$$\boldsymbol{\sigma} \;=\; \int_0^{1} \widetilde{\det}\left(\tfrac{u^*Du}{|u|^2}\;;\;\boldsymbol{\Omega}_{\mathbf{F}} - x\tfrac{Du^*Du}{|u|^2}\right)dx. \tag{4.28}$$

Proof. Because of Proposition 3.17

$$\boldsymbol{\sigma}_{\mathbf{s}} = \int_0^{\chi_s} \widetilde{\det}\left(\tfrac{u^*Du}{|u|^2}\ ;\ \boldsymbol{\Omega}_{\mathbf{F}} - x\tfrac{Du^*Du}{|u|^2} - x\tfrac{u^*u}{|u|^2}\boldsymbol{\Omega}_{\mathbf{F}}\right) dx.$$

Therefore, to prove (4.27) it suffices to show that

(4.29)

$$\det\left(\tfrac{u^*Du}{|u|^2}\ ;\ \boldsymbol{\Omega}_{\mathbf{F}} - x\tfrac{Du^*Du}{|u|^2} - x\tfrac{u^*u}{|u|^2}\boldsymbol{\Omega}_{\mathbf{F}}\right) = \det\left(\tfrac{u^*Du}{|u|^2}\ ;\ \boldsymbol{\Omega}_{\mathbf{F}} - x\tfrac{Du^*Du}{|u|^2}\right).$$

The completely polarized polynomial det($,\dots,$) has the special property that if P is orthogonal projection onto a one-dimensional subspace then $\det(PA, PB, C, \dots) \equiv 0$. It suffices to consider $P \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $1 \times (n-1)$ block form; in which case the verification is straightforward. Applying this fact with $P = \frac{u^*u}{|u|^2}$ and noting that $u^* = Pu^*$ yields (4.29) completing the proof of (4.27). The limiting form of (4.27) is (4.28). □

Proof of Theorem 4.22. The transgression integrand in (4.27), times $\lambda(f)$, equals

$$\tfrac{1}{(n-1)!}\,\frac{f^*u^*Duf}{|u|^2}\,\left(f^*\boldsymbol{\Omega}_{\mathbf{F}}f - xf^*\frac{Du^*Du}{|u|^2}f\right)^{n-1}.$$

Now the formulas in Theorem 4.22 follow easily. □

Remark 4.30. Universal Thom forms in equivariant cohomology over $\mathbf{GL_n(C)}$. The formulas above can be succinctly expressed, as in [MQ], by stating that they represent universal Thom forms in equivariant cohomology. Our first reformulation, however, will differ in spirit from [MQ]. Recall that our construction employs a metric, but applies to any $\mathrm{GL}_n(\mathbf{C})$-connection, not just metric-compatible ones. This will be reflected in what follows. In the subsequent remark we will specialize to the case of unitary connections.

Our main observation here is that the formulas in 4.18 and the interesting special case 4.20 determine elements in the equivariant cohomology group $H_G^{2n}((\mathbf{C}^n)^* \times \mathbf{C}^n)$ where $G = \mathrm{GL}_n(\mathbf{C})$. More explicitly, let $W = (\Lambda\mathfrak{g}^*) \otimes (S\mathfrak{g}^*)$ denote the Weil algebra of $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{C})$ with standard generators ω_{ij} and Ω_{ij} for $1 \le i,\ j \le n$. Let $(u_1, \dots, u_n)$ denote standard coordinates on $\mathbf{C}^n$ and $(u_1^*, \dots, u_n^*)$ the dual coordinates on $(\mathbf{C}^n)^*$. Write

$$u = (u_1, \dots, u_n) \quad \text{and} \quad u^* = \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix}$$

as always, and let G act from the right on $\mathbf{C}^n$ and $(\mathbf{C}^n)^*$ by $u \mapsto ug$ and $u^* \to g^{-1}u^*$ respectively. Set $Du_i = du_i + \sum_{j=1}^{n} u_j w_{ji}$ and $Du_i^* = du_i^* - \sum_{j=1}^{n} w_{ij}u_j^*$ in $W \otimes \mathcal{E}^*((\mathbf{C}^n)^* \times \mathbf{C}^n)$ and as above write

$$Du = du + uw \quad \text{and} \quad Du^* = du^* - wu^*.$$

We now recall briefly the de Rham model of equivariant cohomology ([AB], [MQ], [BV]). Let Y be a manifold with a smooth right G-action. Then the algebra $W \otimes \mathcal{E}^*(Y)$ has a natural Weil structure extending the standard one on W, where by an **algebra with Weil structure** over G we mean a graded differential algebra $(\mathcal{A}, d)$ with a Lie group homomorphism $L : G \to \mathrm{Aut}(\mathcal{A}, d)$ and a graded Lie algebra homomorphism $i : \mathfrak{g} \to \mathrm{Der}_{\deg=-1}(\mathcal{A})$ satisfying the standard identity

$$d \circ i_V + i_V \circ d \;=\; L_V$$

for $V \in \mathfrak{g}$. The "contraction operators" i_V on $\mathcal{E}^*(Y)$ are given by standard contraction with the associated generating vector field $\widetilde{V}$ on Y. For a general algebra $\mathcal{A}$ with Weil structure over G, let $\mathcal{A}^G \subset \mathcal{A}$ denote the G-fixed elements and set $\mathcal{A}_{\text{basic}} = \{a \in \mathcal{A}^G : i_V a = 0 \text{ forall } V \in \mathfrak{g}\}$. Note that $d(\mathcal{A}_{\text{basic}}) \subseteq \mathcal{A}_{\text{basic}}$. Then by the **equivariant forms on Y** we mean the graded differential algebra

$$\mathcal{E}_G^*(Y) \stackrel{\text{def}}{=} \{W \otimes \mathcal{E}^*(Y)\}_{\text{basic}},$$

and by the **equivariant deRham cohomology** $H_G^*(Y)$ of Y we mean the cohomology of $\mathcal{E}_G^*(Y)$.

Returning to $Y = (\mathbf{C}^n)^* \times \mathbf{C}^n$, we consider the G-invariant function $|u|^2 = |\Sigma u_i u_i^*| = |uu^*|$ and fix an approximate one $\chi(t)$ on $[0, \infty]$. For each $s > 0$ we define

$$\omega_s \;=\; \omega - \chi\left(\tfrac{|u|^2}{s^2}\right) \frac{u^* Du}{|u|^2} \in W \otimes \mathcal{E}^*((\mathbf{C}^n)^* \times \mathbf{C}^n)$$

and set $\Omega_s = d\omega_s - \frac{1}{2}[\omega_s, \omega_s]$.

Proposition 4.31. Universal Thom forms. *For any choice of approximate one χ, and any $s > 0$, the determinant of Ω_s gives an* **equivariant cocycle**

$$\mathfrak{T}_s \stackrel{\text{def}}{=} \widetilde{\det}(\Omega_s) \in \mathcal{E}^*_{\mathrm{GL}_n(\mathbf{C})}((\mathbf{C}^n)^* \times \mathbf{C}^n). \tag{4.32}$$

These forms are mutually cohomologous. Each is given explicitly by the formula in 4.20 by replacing $\boldsymbol{\omega}_{\mathbf{F}}$ with ω and $\boldsymbol{\Omega}_{\mathbf{F}}$ with Ω.

In particular, if $\chi(t) = t/(1+t)$, then

$$\mathfrak{T}_s \;=\; \frac{s^2}{|u|^2+s^2}\widetilde{\det}\left(\Omega - \frac{Du^*Du}{|u|^2+s^2}\right). \tag{4.33}$$

Proof. Note that $\mathfrak{T}_s$ is manifestly G-invariant and that the standard argument shows that $d\mathfrak{T}_s = 0$. Hence to see that $\mathfrak{T}_s$ is basic it remains to show that $i_V\mathfrak{T}_s = 0$ for $V \in \mathfrak{g}$. Since $i_V\Omega = 0$ by definition, it suffices to note that $i_V Du = 0$ and $i_V Du^* = 0$ and then apply formula 4.18. However, $i_V Du = du(V) + u\omega(V) = \frac{d}{dt}u(\exp(-tV))\big|_{t=0} + u\omega(V) = -u\omega(V) + u\omega(V) = 0$. (Recall that $\omega(V)$ is the realization of V as an $n\times n$-matrix.) The equation $i_V(Du^* - \omega u^*) = 0$ is similar.

To see that $\mathfrak{T}_s$ and $\mathfrak{T}_{s'}$ are cohomologous, consider the universal version of σ_s given by setting $\boldsymbol{\Omega}_{\mathbf{F}} = \Omega$ in the formula (4.23). Then

$$d(\sigma_s - \sigma_{s'}) \;=\; \mathfrak{T}_{s'} - \mathfrak{T}_s. \quad \square$$

Suppose now that $F \to X$ is a smooth n-plane bundle with connection D_F and write F as the standard quotient $F = P(F) \times \mathbf{C}^n/\mathrm{GL}_n(\mathbf{C})$ where $P(F)$ is the complex frame bundle of F. Then D_F determines a homomorphism

$$W \otimes \mathcal{E}^*((\mathbf{C}^n)^* \times \mathbf{C}^n) \longrightarrow \mathcal{E}^*(P(F) \times (\mathbf{C}^n)^* \times \mathbf{C}^n)$$

of algebras with Weil structure over $\mathrm{GL}_n(\mathbf{C})$. This induces a **g.d.a. homomorphism** on basic forms

$$\mathcal{E}^*_{\mathrm{GL}_n(\mathbf{C})}((\mathbf{C}^n)^* \times \mathbf{C}^n) \longrightarrow \mathcal{E}^*(F^* \oplus F). \tag{4.34}$$

A choice of hermitian metric on F corresponds to choosing a complex anti-linear isomorphism $h : F \xrightarrow{\approx} F^*$. Let

$$\Gamma_h : F \longrightarrow F^* \oplus F$$

denote the graphing map: $\Gamma_h(\nu) = (h(\nu), \nu)$. Pulling back forms induces a g.d.a. homomorphism

$$\mathcal{E}^*(F^* \oplus F) \longrightarrow \mathcal{E}^*(F). \tag{4.35}$$

Composing (4.34) and (4.35) gives a g.d.a. homomorphism

$$\mathcal{E}^*_{\mathrm{GL}_n(\mathbf{C})}((\mathbf{C}^n)^* \times \mathbf{C}^n) \xrightarrow{\omega} \mathcal{E}^*(F) \tag{4.36}$$

called the **Weil homomorphism** associated to the metric h and the (not necessarily compatible) connection D_F. This extends the usual Weil homomorphism: $\mathcal{E}^*_{\mathrm{GL}_n(\mathbf{C})}(\mathrm{pt}) \longrightarrow \mathcal{E}^*(X)$.

We now have the following, whose proof is evident.

Proposition 4.37. *For any choice of approximate-one, the image of $\mathfrak{T}_s$ under the Weil homomorphism is the Chern-Thom form of Proposition 4.17. In particular, if $\chi(t) = 1$ for $t \geq 1$, then $\mathfrak{T}_s$ determines a form in $\mathcal{E}^{2n}_{\mathrm{cpt}}(F)$ which represents the Thom class in $H^{2n}_{\mathrm{cpt}}(F)$. Furthermore, if $\chi(t) = t/(t+1)$, then $\mathfrak{T}_s$ determines a form which extends to $\mathbf{P}(\mathbf{C} \oplus F) = F \cup \mathbf{P}(F)$ and vanishes on $\mathbf{P}(F)$. It thereby determines the Chern-Thom form in $\mathcal{E}^{2n}(\mathbf{P}(\mathbf{C} \oplus F), \mathbf{P}(F))$.*

Note. There is an advantage in formulating the Thom class in $\mathcal{E}^*(F^* \times F)$. One can restrict the class to any family of cones $\Gamma \subset F^* \times F$. Examples are given by graphs of other bilinear and sesquilinear forms on F. However, the theory of kernels in several complex variables provides other examples which are even more interesting.

Remark 4.38. Universal Thom forms in equivariant cohomology over $\mathbf{U}_n$. In the special case where the connection is metric compatible, the above discussion simplifies and our formulas are seen to determine universal Thom classes in $H^*_{U_n}$ as follows. Fix the standard hermitian inner product on $\mathbf{C}^n$ and let $\mathbf{C}^n \hookrightarrow (\mathbf{C}^n)^* \times \mathbf{C}^n$ be the graph of the metric as above. This is compatible with $U_n \hookrightarrow \mathrm{GL}_n(\mathbf{C})$ and we get a restriction homomorphism

$$\mathcal{E}^*_{\mathrm{GL}_n(\mathbf{C})}((\mathbf{C}^n)^* \times \mathbf{C}^n) \longrightarrow \mathcal{E}^*_{U_n}(\mathbf{C}^n). \tag{4.39}$$

For $s > 0$ let $B_s = \{v \in \mathbf{C}^n : |v| < s\}$.

Proposition 4.40. *Let χ be any approximate-one such that $\chi(t) = 1$ for $t \geq 1$. Then under the restriction (4.39) the universal Thom class $\mathfrak{T}_s$ of Proposition 4.31 determines an equivariant cocycle*

$$\mathfrak{T}_s = \widetilde{\det}(\Omega_s) \in \mathcal{E}^{2n}_{U_n}(\mathbf{C}^n, \mathbf{C}^n - B_s).$$

Given any complex n-plane bundle $F \to X$ with a unitary connection, the Weil homomorphism

$$\mathcal{E}^*_{U_n}(\mathbf{C}^n, \mathbf{C}^n - B_s) \longrightarrow \mathcal{E}^*(F, F - B_s(F))$$

(where $B_s(F) = \{v \in F : |v| < s\}$) carries $\mathfrak{T}_s$ to the Chern-Thom form of 4.15 which represents the Thom class

$$[\mathfrak{T}_s] \in H^{2n}(F, F - B_s(F)).$$

Proof. In light of the discussion above it suffices to observe that $\operatorname{spt}\mathfrak{T}_s \subseteq B_s$. □

Proposition 4.41. *Let $\chi(t) = t/(1+t)$. Then the formula in 4.17 determines an equivariant cocycle*

$$\mathfrak{T}_s = \frac{s^2}{|u|^2 + s^2} \widetilde{\det}\left(\Omega - \frac{Du^* Du}{|u|^2 + s^2}\right) \in \mathcal{E}^{2n}_{U_n}(\mathbf{C}^n).$$

This form extends to an equivariant cocycle on $\mathbf{P}^n \supset \mathbf{C}^n$ which vanishes on $\mathbf{P}^{n-1}$ and so determines a relative cocycle

$$\mathfrak{T}_s \in \mathcal{E}^{2n}_{U_n}\left(\mathbf{P}^n, \mathbf{P}^{n-1}\right).$$

Given any complex bundle $F \to X$ with a unitary connection, the Weil homomorphism

$$\mathcal{E}^*_{U_n}\left(\mathbf{P}^n, \mathbf{P}^{n-1}\right) \longrightarrow \mathcal{E}^*\left(\mathbf{P}(\mathbf{C} \oplus F), \mathbf{P}(F)\right)$$

carries $\mathfrak{T}_s$ to the Chern-Thom form 4.15 which represents the Thom class

$$[\boldsymbol{\tau}_{\mathbf{s}}] \in H^{2n}\left(\mathbf{P}(\mathbf{C} \oplus F), \mathbf{P}(F)\right).$$

In a similar vein our formulas for $\boldsymbol{\sigma}_{\mathbf{s}}$ and $\mathbf{r}_{\mathbf{s}}$ determine universal equivariant currents. The details are straightforward.

5. The Rectifiable Grothendieck-Riemann-Roch Theorem—Version 1.

In this section we use our results on the Chern-Thom form to prove a version of the Differentiable Grothendieck-Riemann-Roch Theorem for Embeddings (cf. [AH]) at the level of differential forms. We also extend this result from submanifolds to certain embeddings of subcomplexes. Not only do our methods take place at the level of differential forms, but we actually produce for each embedding $j : Y \hookrightarrow X$ and bundle $E \to Y$ with connection, a *canonical family* of closed differential forms representing $\mathrm{ch}(j_!E)$ which converge to the current $j_!(\mathrm{ch}(E) \cup \mathrm{Todd}^{-1}(N)) \equiv \mathrm{ch}(E) \wedge \mathrm{Todd}^{-1}(N)[Y]$, where N is the normal bundle to j. The support of this family squeezes down to Y, and the Todd^{-1}-factor falls naturally out of the residue computation.

In this section we shall only deal with complex normal bundles and the standard Thom class in K-theory. The more general results for Spin- and Spin^c-embeddings (and their corresponding Thom classes in KO- and K-theory) will be treated in the next chapter.

To begin we assume as above that $\pi : F \longrightarrow X$ is an n-dimensional complex vector bundle with connection D over an oriented manifold X. Let $R = R^F$ denote the curvature operator D^2 determined by this connection. We define extensions $\Lambda^k R$ and $\lambda^k R$ of this operator to $\Lambda^k F$ by setting

$$\begin{aligned}(\Lambda^k R)(v_1 \wedge \cdots \wedge v_k) &= Rv_1 \wedge \cdots \wedge Rv_k \qquad \text{and} \\ (\lambda^k R)(v_1 \wedge \cdots \wedge v_k) &= \sum_{j=1}^{k} v_1 \wedge \cdots \wedge Rv_j \wedge \cdots \wedge v_k\end{aligned}$$

for $v_1, \ldots, v_k \in \Gamma(F)$.

We extend D to the full tensor algebra of sections of F, as usual, by requiring that D be a derivation that commutes with contractions. Then the corresponding curvature operator $R^{\Lambda^k F}$ on $\Lambda^k F$ is easily seen to be equal to $\lambda^k R$. Furthermore, we have $R^{F^*} = -(R)^*$, so the curvature of $\Lambda^k F^*$ is given by

$$R^{\Lambda^k F^*} = -\lambda^k R^* = -\left(\lambda^k R\right)^* \tag{5.1}$$

The following classical algebraic identity is of fundamental importance. For completeness, a proof is given at the end of this section.

Lemma 5.2. *Let V be a finite dimensional complex vector space. To each $A \in \mathrm{End}(V)$, let $\lambda^k A \in \mathrm{End}(\Lambda^k V)$ denote the induced derivation. Then*

$$\mathrm{tr}_{\Lambda^{\mathrm{even}}}\left\{e^{-\lambda^* A}\right\} - \mathrm{tr}_{\Lambda^{\mathrm{odd}}}\left\{e^{-\lambda^* A}\right\} = \det\left\{\frac{I - e^{-A}}{A}\right\}\det A \tag{5.3}$$

for all $A \in \mathrm{End}(V)$.

This identity signals the importance of the Todd polynomial defined by

$$\mathrm{Todd}(A) = \det\left\{\frac{\frac{i}{2\pi}A}{1 - e^{-\frac{i}{2\pi}A}}\right\}.$$

Corollary 5.4. *Let $R^{\Lambda F^*}$ be the curvature of the bundle $\Lambda F^* = \oplus_{k=0}^{n}\Lambda^k F^*$ induced from the curvature R of F as above. This map preserves the splitting $\Lambda F^* = \Lambda^{\mathrm{even}} \oplus \Lambda^{\mathrm{odd}}$ into even- and odd-degree forms, and the following formula holds*

$$\mathrm{tr}_{\Lambda^{\mathrm{even}}}\{e^{R^{\Lambda F^*}}\} - \mathrm{tr}_{\Lambda^{\mathrm{odd}}}\{e^{R^{\Lambda F^*}}\} = \det\left\{\frac{I - e^{-R}}{R}\right\}\det(R) \tag{5.5}$$

Equivalently, the following identity between characteristic forms holds on X

$$ch(D_{\Lambda^{\mathrm{even}}F^*}) - ch(D_{\Lambda^{\mathrm{odd}}F^*}) = \mathrm{Todd}^{-1}(D_F)\,c_n(D_F). \tag{5.6}$$

Furthermore, let E be any other complex vector bundle with connection D_E and curvature $R^E = (D_E)^2$ over X. Let $R^{(\Lambda F^*)\otimes E}$ denote the curvature of the tensor product connection on $(\Lambda F^*) \otimes E$.This map also preserves the splitting $(\Lambda F^*) \otimes E = \Lambda^{\mathrm{even}} \oplus \Lambda^{\mathrm{odd}}$ into even- and odd-degree E-valued forms, and the following identity of differential forms holds on X

(5.7)
$$\mathrm{tr}_{\Lambda^{\mathrm{even}}}\left\{e^{R^{(\Lambda F^*)\otimes E}}\right\} - \mathrm{tr}_{\Lambda^{\mathrm{odd}}}\left\{e^{R^{(\Lambda F^*)\otimes E}}\right\} = \mathrm{tr}\left\{e^{R^E}\right\}\det\left\{\frac{I - e^{-R}}{R}\right\}\det(R),$$

or equivalently

$$ch\left(D_{(\Lambda^{\mathrm{even}}F^*)\otimes E}\right) - ch\left(D_{(\Lambda^{\mathrm{odd}}F^*)\otimes E}\right) = ch(D_E)\,\mathrm{Todd}^{-1}(D_F)\,c_n(D_F). \tag{5.8}$$

Proof. Note that

$$\mathrm{tr}_{\Lambda^k}\left\{e^{R^{\Lambda F^*}}\right\} = \mathrm{tr}_{\Lambda^k}\left\{e^{-(R^{\Lambda F})^*}\right\} = \mathrm{tr}_{\Lambda^k}\left\{e^{-(R^{\Lambda F})}\right\}.$$

where the first equality is just 5.1. □

Using these beautiful formulae we can now apply our Chern current theory to derive generalizations of the Grothendieck-Riemann-Roch Theorem and certain formulas of [BGS1, 2] for smooth embeddings.

Theorem 5.9. Rectifiable Grothendieck-Riemann-Roch — Version 1. *Let F be a complex vector bundle with connection over an oriented manifold X. For each atomic section μ of F, the following identity of d-closed forms and currents holds on X:*

$$(5.10) \qquad ch(D_{\Lambda^{\text{even}} F^*}) - ch(D_{\Lambda^{\text{odd}} F^*}) = \text{Todd}^{-1}(D_F)\,\text{Div}(\mu) + dT$$

where T is the L^1_{loc}-form given by $T \stackrel{\text{def}}{=} \text{Todd}^{-1}(D_F)\,\sigma$, where σ is the Chern-Thom transgression from the previous section. Moreover, for any "auxiliary" complex bundle E with connection D_E over X, there is the identity

(5.11)

$$ch\left(D_{(\Lambda^{\text{even}} F^*)\otimes E}\right) - ch\left(D_{(\Lambda^{\text{odd}} F^*)\otimes E}\right) = ch(D_E)\,\text{Todd}^{-1}(D_F)\,\text{Div}(\mu) + dT'$$

where $T' = ch(D_E)\,\text{Todd}^{-1}(\mathbf{D}_F)\,\sigma$.

Furthermore, suppose that $\overrightarrow{D}_s$ is the family of pushforward connections on F associated to a choice of approximate one χ, and let $\overrightarrow{D}_{s,\Lambda^* F^*}$ be the family of connections induced on $\Lambda^* F^*$ by $\overrightarrow{D}_s$. Then

(5.12)

$$ch(\overrightarrow{D}_{s,(\Lambda^{\text{even}} F^*)\otimes E}) - ch(\overrightarrow{D}_{s,(\Lambda^{\text{odd}} F^*)\otimes E}) = ch(D_E)\,\text{Todd}^{-1}(D_F)\,\text{Div}(\mu) + dR_s$$

where R_s is a family of L^1_{loc}-forms on X such that

$$\lim_{s\to 0} R_s = 0 \qquad \textit{and} \qquad \lim_{s\to\infty} R_s = T \qquad \textit{in } L^1_{\text{loc}}(X).$$

In particular,

$$(5.13) \qquad \lim_{s\to 0}\left\{ch(\overrightarrow{D}_{s,\Lambda^{\text{even}} F^*}) - ch(\overrightarrow{D}_{s,\Lambda^{\text{odd}} F^*})\right\} = \text{Todd}^{-1}(D_F)\,\text{Div}(\mu).$$

If $\chi(t) = 1$ for all $t \geq 1$, then

$$(5.14) \qquad spt\left\{ch(\overrightarrow{D}_{s,\Lambda^{\text{even}} F^*}) - ch(\overrightarrow{D}_{s,\Lambda^{\text{odd}} F^*})\right\} \subset \overline{U}_s$$

and $spt(R_s) \subset \overline{U}_s$, where $\overline{U}_s = \{x \in X : |\mu(x)| \leq s\}$ for all $s > 0$.

Proof. Applying (5.8) to the connection $\vec{D}_s$ gives the equation

$$\text{(5.15)} \qquad \mathrm{ch}(\vec{D}_{s,(\Lambda^{\mathrm{even}}F^*)\otimes E}) - \mathrm{ch}(\vec{D}_{s,(\Lambda^{\mathrm{odd}}F^*)\otimes E}) \;=\; \mathrm{ch}(D_E)\,\mathrm{Todd}^{-1}(\vec{D}_s)\tau_s$$

where $\tau_s = \widetilde{\det}(\vec{D}_s)$. As remarked after Corollary 4.9

$$\tau_s \;=\; \mathrm{Div}(\mu) + dr_s$$

where $r_s = \sigma - \sigma_s$ is an L^1_{loc}-form on X. Theorem 4.2 and Proposition 4.16 now apply to yield the main part of the theorem. In particular, if we define

$$\text{(5.16)} \qquad R_s \;=\; \mathrm{ch}(D_E)\,\mathrm{Todd}^{-1}(\vec{D}_s)r_s,$$

then

$$\lim_{s\to 0} R_s = 0 \qquad \text{and} \qquad \lim_{s\to\infty} R_s = \mathrm{ch}(D_E)\,\mathrm{Todd}^{-1}(D_F)\sigma \stackrel{\mathrm{def}}{=} T$$

in $L^1_{\mathrm{loc}}(X)$. Finally, if $\chi(t) = 1$ for $t \ge 1$, then

$$\mathrm{spt}(\tau_s) \subset \overline{U}_s$$

for all $s > 0$. This proves (5.14). □

Remark 5.17. Localization at Div(α). The main formula in Theorem 5.9 can be reformulated purely in terms of the zeros of α as follows. Suppose that Z is an integral current on X with the property that $Z = \mathrm{Div}(\alpha)$ for an atomic section μ of a complex bundle F **defined only over some neighborhood** U **of** $|Z| = \mathrm{spt}(Z)$ in X. Fix an approximate-one χ such that $\chi(t) = 1$ for $t \ge 1$, and choose a metric on F so that $\overline{U}_s \subset U$ for all $s < 1$. Then the family of forms

$$\mathrm{K}(s) \stackrel{\mathrm{def}}{=} \mathrm{ch}(\vec{D}_{s,\Lambda^{\mathrm{even}}F^*}) - \mathrm{ch}(\vec{D}_{s,\Lambda^{\mathrm{odd}}F^*})$$

has support in $\overline{U}_1$ and extends by zero to a family of smooth d-closed forms defined on all of X. It has the property that

$$\lim_{s\to 0} \mathrm{K}(s) \;=\; \mathrm{Todd}^{-1}(D_F)Z.$$

Suppose now that the embedding

$$j : |Z| \hookrightarrow U$$

admits a retraction $p : U \to |Z|$ which is proper on $\overline{U}_1 \subset U$. Then we can define a **Gysin map in K-theory**

$$j_! : K_{\mathrm{cpt}}(|Z|) \longrightarrow K_{\mathrm{cpt}}(X)$$

by

$$j_!(u) \;=\; i_*(\Lambda_{-1}F^* \cdot p^*u)$$

where $i_* : K_{\mathrm{cpt}}(U) \to K_{\mathrm{cpt}}(X)$ is the map induced by inclusion $i : U \hookrightarrow X$, and where

$$\Lambda_{-1}F^* \;=\; \pi^*\Lambda^{\mathrm{even}}F^* - \pi^*\Lambda^{\mathrm{odd}}F^* \;\in\; K_{\mathrm{cpt}}(U)$$

is determined by identifying $\Lambda^{\mathrm{even}}F^*$ with $\Lambda^{\mathrm{odd}}F^*$ over $U - |Z|$ by the map $(\mu\wedge\cdot)+(\mu\wedge\cdot)^*$. Recall that there is also a **Gysin map in cohomology**

$$j_! : H^*_{\mathrm{deRham}}(|Z|) \longrightarrow H^{*+2n}_{\mathrm{deRham}}(X)$$

from the cohomology of forms germed an $|Z|$ to the cohomology of currents on X, given by

$$j_!(\varphi) \;=\; \varphi Z.$$

From the preceeding discussion and Theorem 5.9 we have the following

Corollary 5.18. Localization. *Let Z be a current on X which arises as the divisor of a section of a complex vector bundle F defined in a neighborhood U of $|Z| = sptZ$ which admits a retraction $p : U \to |Z|$. Then*

$$ch(j_!(1)) \;=\; \left[ch\left(\overrightarrow{D}_{s,\Lambda^{\mathrm{even}}F^*}\right) - ch\left(\overrightarrow{D}_{s,\Lambda^{\mathrm{odd}}F^*}\right)\right]$$

for all $s < 1$ (with χ and D as above). Furthermore, for any complex bundle E on $|Z|$, pulled back over U and endowed with a complex connection, we have

$$ch(j_!(E)) \;=\; \left[ch\left(\overrightarrow{D}_{s,(\Lambda^{\mathrm{even}}F^*)\otimes E}\right) - ch\left(\overrightarrow{D}_{s,(\Lambda^{\mathrm{odd}}F^*)\otimes E}\right)\right].$$

Taking the limit as $s \to 0$ of this family of d-closed forms in X gives the equation

$$(5.19) \qquad ch(j_!(E)) \;=\; j_!\left(ch(E)\,\mathrm{Todd}^{-1}(F)\right).$$

Remark 5.20. An alternative method for localizing the main formula in 5.9 is the following. Let $U \supset |Z|$ be as above and choose a neighborhood U_0 of $|Z|$ such that $\overline{U}_0 \cap (X - U) = \emptyset$. Let $\psi \in C_0^\infty(U)$ be any smooth function such that $\psi \equiv 1$ on U_0 and $\psi \equiv 0$ on $X - U$. Let T' be the transgression from Theorem 5.9. Taking d of $\psi T'$ then gives a formula

$$\mathrm{ch}_{\mathrm{cpt}}(\Lambda_{-1}F^* \otimes E) \;\equiv\; \mathrm{ch}(D_E)\,\mathrm{Todd}^{-1}(D_F)\,\mathrm{Div}(\alpha) + d(\psi T')$$

where $\mathrm{ch}_{\mathrm{cpt}}(\Lambda_{-1}F^* \otimes E)$ is a smooth differential form **with compact support in** U such that

$$\mathrm{ch}_{\mathrm{cpt}}(\Lambda_{-1}F^* \otimes E)\,|_{U_0} \;=\; \left\{\mathrm{ch}\left(D_{(\Lambda^{\mathrm{even}}F^*)\otimes E}\right) \;-\; \mathrm{ch}\left(D_{(\Lambda^{\mathrm{odd}}F^*)\otimes E}\right)\right\}|_{U_0}.$$

Remark 5.21. The relation to Grothendieck-Riemann-Roch. Corollary 5.18 constitutes a promotion to the level of differential forms, of Atiyah and Hirzebruch's "Differentiable Riemann-Roch" Theorem for embeddings with complex normal bundle. Their result can be recovered by considering the current $Z = [Y]$ associated to a compact oriented smooth submanifold $Y \subset X$ whose normal bundle $p : N \to Y$ carries an almost complex structure. Under this assumption we can identify N with a tubular neighborhood Y, and after writing Y as the divisor of the tautological cross-section of p^*N, the theory applies.

We recall for the reader how this result can be rewritten to resemble the theorem of Grothendieck. Suppose that the manifolds X and Y are compact and *almost complex*. From the Grothendieck viewpoint we should begin with a smooth embedding

$$j : Y \hookrightarrow X$$

which respects the almost complex structure. Then the normal bundle N to Y is complex and there is a splitting

$$TX|_Y \;=\; TY \oplus N.$$

We fix a direct sum of complex connections on $TY \oplus N$ and extend it to a complex connection on all of TX. From the multiplicative property of the Todd series with respect to direct sums, we have that

$$\mathrm{Todd}(D_{TX})\,|_Y \;=\; \mathrm{Todd}(D_{TY})\,\mathrm{Todd}(D_N)$$

or equivalently that

$$\mathrm{Todd}\,(D_{TX})\,|_Y\,\mathrm{Todd}^{-1}\,(D_N) \;=\; \mathrm{Todd}\,(D_{TY})$$

We now fix a tubular neighborhood U of Y in X, and choose an identification

$$U \;\cong\; N \xrightarrow{p} Y$$

of this neighborhood with the normal bundle. Under this identification Y becomes the divisor of the tautological section of $\pi^* N$, and we can apply the theory. Namely, Let $D = \overrightarrow{D}_1$ be the pushforward connection with support in $\overline{U}_1 \subset U$ as above. Then given any bundle E with connection over Y, we pull E back to U via p. Multiplying (5.11) by Todd(D_{TX}) then gives the following equation

$$\begin{aligned} &\left\{\mathrm{ch}\left(D_{(\Lambda^{\mathrm{even}}F^*)\otimes E}\right) - \mathrm{ch}\left(D_{(\Lambda^{\mathrm{odd}}F^*)\otimes E}\right)\right\} \wedge \mathrm{Todd}\,(D_{TX}) \\ &\qquad\qquad = \mathrm{ch}(D_E) \wedge \mathrm{Todd}(D_{TY})[Y] + dT \end{aligned} \tag{5.22}$$

where T is a canonically defined L^1_{loc}-form on X.

Now associated to the proper embedding $j : Y \hookrightarrow X$ there are the Gysin maps

$$j_! : K(Y) \longrightarrow K(X) \quad \text{and} \quad j_! : H^*(Y) \longrightarrow H^*(X)$$

defined above. Let $\mathrm{ch} : K \longrightarrow H^*$ be the transformation of multiplicative theories given by the Chern character. Then in passing from currents to cohomology, equation (5.22) becomes

$$\mathrm{ch}\,(j_! E) \cup \mathrm{Todd}(X) \;=\; j_!\,\{(\mathrm{ch} E) \cup \mathrm{Todd}(Y)\} \tag{5.23}$$

which is the C^∞-form of Grothendieck's Theorem for j. It can be restated by asserting that the following diagram

$$\begin{tikzcd} K(Y) \arrow[r, "j_!"] \arrow[d, "\mathrm{ch}(\,\cdot\,)\wedge\mathrm{Todd}(Y)"'] & K(X) \arrow[d, "\mathrm{ch}(\,\cdot\,)\wedge\mathrm{Todd}(X)"] \\ H^*(Y) \arrow[r, "j_!"'] & H^*(X) \end{tikzcd}$$

commutes. For this reason it is sensible to consider Theorem 5.9 as a generalization of this result to divisors of atomic sections. In particular, using the localization argument above we get a **Grothendieck-Riemann-Roch Theorem for any subcomplex $Y \subset X$ of a smooth triangulation of X which can be expressed as the divisor of a section of a complex vector bundle defined in some neighborhood of Y.**

In the next chapter we will generalize this to embeddings with spin or Spin^c-structures on the normal bundle as in [AH].

Proof of Lemma 5.2. We first note that for all $A \in \mathrm{Hom}(V, V)$ we have that

$$e^{\lambda^k A} = \Lambda^k \left(e^A\right). \tag{5.24}$$

To prove this set $L_t = e^{tA}$ in $\mathrm{Hom}(V, V)$. Both $\phi_t = \Lambda^k L_t$ and $\psi_t = e^{t\lambda^k A}$ are 1-parameter groups in $\mathrm{GL}(\Lambda^k V)$, and $d\phi_t/dt|_{t=0} = d\psi_t/dt|_{t=0} = \lambda^k A$.

The case dim$_{\mathbf{C}}$V=1. Suppose that $\dim_{\mathbb{C}} V = 1$ and $A = a\mathrm{Id}$ on $V = \Lambda^1 V$. By definition, $\lambda^0 A = 0$ on $\mathbf{C} = \Lambda^0 V$. Thus, $\Lambda^0 e^A = e^{\lambda^0 A} = \mathrm{I}$ and $\Lambda^1 e^A = e^{\lambda^1 A} = e^a \mathrm{I}$. In particular,

$$\mathrm{tr}\left(e^{\lambda^0 A}\right) - \mathrm{tr}\left(e^{\lambda^1 A}\right) = \mathrm{tr}\left(\Lambda^0 e^A\right) - \mathrm{tr}\left(\Lambda^1 e^A\right) = 1 - e^a.$$

which completes the proof when $\dim_{\mathbb{C}} V = 1$.

Definition 5.25. Given $L \in \mathrm{Hom}(V, V)$ and $t \in \mathbb{R}$, we set

$$\mathrm{Tr}_t(L) = \sum_{k \geq 0} \mathrm{tr}\left(\Lambda^k L\right) t^k.$$

Lemma 5.26. *Given $L_1 : V_1 \longrightarrow V_1$ and $L_2 : V_2 \longrightarrow V_2$, we have for all t that*

$$\mathrm{Tr}_t(L_1 \oplus L_2) = \mathrm{Tr}_t(L_1)\,\mathrm{Tr}_t(L_2).$$

Proof. Note that

$$\Lambda^k(V_1 \oplus V_2) = \sum_{i+j=k} (\Lambda^i V_1) \otimes (\Lambda^j V_2)$$

and therefore

$$\operatorname{Tr}\left(\Lambda^k(L_1 \oplus L_2)\right) = \sum_{i+j=k} \operatorname{Tr}(\Lambda^i L_1)\operatorname{Tr}(\Lambda^j L_2). \qquad \square$$

Suppose now that A is diagonalizable with eigenvalues $a_1, \ldots, a_n$. Applying 5.28 to the eigenspace decomposition $V = \bigoplus V_i$, we see that

$$\operatorname{Tr}_t\left(e^A\right) = \prod_{i=1}^{n}(1 + e^{a_i}t) = \det\left(1 + te^A\right),$$

and in particular, if $\det A \neq 0$,

$$\operatorname{Tr}_{-1}\left(e^{-A}\right) = \det\left(1 - e^{-A}\right) = \det\left\{\frac{\mathrm{Id} - e^{-A}}{A}\right\}\det A.$$

Since invertible diagonalizable endomorphisms are dense in $\operatorname{Hom}(V, V)$, equation (5.3) is established and Lemma 5.2 is proved. $\square$

6. Bundle Maps $\mathbf{E} \xrightarrow{\alpha} \mathbf{F}$ where Rank $\mathbf{E} = 1$.

The results of this chapter easily extend from the case $\underline{\mathbf{C}} \xrightarrow{\alpha} F$ to the case $E \xrightarrow{\alpha} F$ where rank $E = 1$. We present here brief statements of the modified results. Most of the proofs are omitted.

Fix F, D_F and χ as above. Consider a bundle map $\alpha : E \to F$ where E is a complex line bundle with connection D_E. Given local frames e for E and f for F, $u = (u_1, \ldots, u_n)$ is defined by $\alpha(e) = uf$, and α is **atomic** is each such u is atomic. The divisor of α is defined by $\operatorname{Div}(\alpha) = \operatorname{Div}(u)$.

Let $\overrightarrow{D}_s$ be the family of smooth approximations to the push forward connection on F, and let $\overrightarrow{\Omega}_s$, T_s, etc. be defined as usual.

Proposition 6.1. *Suppose $E \xrightarrow{\alpha} F$ is atomic. The transgression is given by*

$$T_s = \int_0^{\chi_s} \phi\left(\tfrac{u^*Du}{|u|^2} \,;\, \Omega_F + x\left(\frac{u^*\Omega_E u}{|u|^2} - \tfrac{u^*u}{|u|^2}\Omega_F - \tfrac{Du^*Du}{|u|^2}\right)\right)dx, \tag{6.2}$$

and converges in $L^1_{\text{loc}}(X)$ to

$$T = \int_0^1 \phi\left(\tfrac{u^*Du}{|u|^2}\ ;\ \Omega_F + x\left(\frac{u^*\Omega_E u}{|u|^2} - \tfrac{u^*u}{|u|^2}\Omega_F - \tfrac{Du^*Du}{|u|^2}\right)\right) dx. \tag{6.3}$$

In particular, the transgression T is independent of the choice of approximate one χ.

Formula (2.4) is valid for $\vec{\Omega}_s$ provided that in this formula $\vec{\Omega}_0$ is replaced by

$$\vec{\Omega}_0 \ = \ \left(1 - \tfrac{u^*u}{|u|^2}\right)\left(\Omega_F - \tfrac{Du^*Du}{|u|^2}\right) + \frac{u^*\Omega_E u}{|u|^2}, \tag{6.4}$$

i.e., the term $\frac{u^*\Omega_E u}{|u|^2}$ is added to $\vec{\Omega}_0$. The proof of Proposition 6.1 follows exactly as the proof of Proposition 2.8. The extra term $\frac{u^*\Omega_E u}{|u|^2}$ is harmless.

Define the **residue form** on X as before by setting

$$\text{Res}_\phi(\vec{D}) \ = \ -\int_{p^{-1}} \mathbf{T} \ = \ -\rho_*(\mathbf{T}), \tag{6.5}$$

where p denotes the restriction of $\pi : \text{Hom}(E,F) \to X$ to the ϵ-sphere bundle and $\mathbf{T}$ is the transgression in the universal case. Note that with this definition $\text{Res}_\phi(\vec{D})$ is smooth.

In the universal case we adopt the notation of Section 3 in Chapter I. In particular, recall the blow up

$$\rho : \widetilde{\text{Hom}}(E,F) \longrightarrow \text{Hom}(E,F)$$

and the target bundle $T \subset F$ over $\widetilde{\text{Hom}}(E,F)$, obtained by pulling back the universal line bundle over $\mathbf{P}(F)$ to $\widetilde{\text{Hom}}(E,F)$.

Universal Theorem 6.6. *The ϕ-Chern current $\phi((\vec{\mathbf{D}}))$ of the pushforward singular connection associated with the canonical bundle map $\mathbf{E} \xrightarrow{\alpha} \mathbf{F}$ over* $\text{Hom}(E,F)$ *has L^1_{loc}-part equal to $\rho_*\phi(\mathbf{D_E} \oplus D_{T^\perp})$ and singular part equal to* $\text{Res}_\phi(\vec{D})[X]$ *where the residue form is smooth and d-closed. The equation*

$$\phi(\mathbf{D_F}) - \rho_*\phi(\mathbf{D_E} \oplus D_{T^\perp}) - \text{Res}_\phi(\vec{D})[X] \ = \ d\mathbf{T}, \quad \text{over } \text{Hom}(E,F) \tag{6.7}$$

is the limiting form, as $s \to 0$, of the equation

$$(6.8) \qquad \phi(\mathbf{D_F}) - \phi(\overrightarrow{\mathbf{D}}_s) \;=\; d\mathbf{T}_s \qquad \text{over } \operatorname{Hom}(E,F)$$

and the potentials $\mathbf{T}_s$ converge to $\mathbf{T}$ in $L^1_{\mathrm{loc}}(\operatorname{Hom}(E,F))$.

Furthermore, the residue form $\operatorname{Res}_\phi(\overrightarrow{D})$ is a universally determined polynomial in the curvatures of E and F, i.e., the strict analogue of Theorem 3.23 holds in this case.

The residue can be computed by utilizing the algebraic approximation mode. Consider $\operatorname{Hom}(E,F)$ as an open dense chart in the compactification $\mathbf{P}(E \oplus F)$ and let π also denote projection from $\mathbf{P}(E \oplus F)$ to X. Then it will be proved in the next section that

$$(6.9) \qquad \operatorname{Res}_\phi(\overrightarrow{D}) \;=\; \pi_*\phi(D_{U^\perp})$$

where U is the universal bundle on $\mathbf{P}(E \oplus F)$.

If $E \xrightarrow{\alpha} F$ is atomic, then the results of the universal Theorem 6.6 can be successfully pulled back from $\operatorname{Hom}(E,F)$ to the base manifold X. As before $[X]$ pulls back to $\operatorname{Div}(\alpha)$ and the residue form $\operatorname{Res}_\phi(\overrightarrow{D})$ is independent of the bundle map α. Moreover, the L^1_{loc}-form $\phi(\overrightarrow{\Omega}_0) = \rho_*\phi(\mathbf{D_E} \oplus D_{T^\perp})$ on $\operatorname{Hom}(E,F)$ pulls back to a d-closed L^1_{loc}-form on X, which we also denote by $\phi(\overrightarrow{\Omega}_0) = \rho_*\phi(D_E \oplus D_{T^\perp})$.

Note that because of (6.4) and I.2.19, the L^1_{loc}-part of the ϕ Chern current can be expressed as

$$(6.10) \quad \rho_*\phi(D_E \oplus D_{T^\perp}) \;=\; \phi\left(\Omega_E \oplus \left(1 - \tfrac{u^*u}{|u|^2}\right)\left(\Omega_F - \tfrac{Du^*Du}{|u|^2}\right)\left(1 - \tfrac{u^*u}{|u|^2}\right)\right).$$

Atomic Theorem 6.11. *Suppose $E \xrightarrow{\alpha} F$ is atomic. The equation*

$$(6.12) \qquad \phi(D_F) - \phi(\overrightarrow{D}_s) \;=\; dT_s \qquad \text{on } X$$

has limiting form (as $s \to 0$)

$$(6.13) \qquad \phi(D_F) - \rho_*\phi(D_E \oplus D_{T^\perp}) - \operatorname{Res}_\phi(\overrightarrow{D})\operatorname{Div}(\alpha) \;=\; dT \quad \text{on } X,$$

so that the ϕ-Chern current exists and has $L^1_{\mathrm{loc}}(X)$-part equal to $\rho_*\phi(D_E \oplus D_{T^\perp})$ and singular part equal to $\operatorname{Res}_\phi(\overrightarrow{D})\operatorname{Div}(\alpha)$.

Remark 6.14. The Total Chern Form. The special case of the results of this section where ϕ is the total Chern polynomial, denoted c, are summarized in this remark. First, one calculates that

$$\operatorname{Res}_c(\overrightarrow{D}) = 1 \tag{6.15}$$

exactly as in the case $E = \underline{\mathbf{C}}$. The universal equation (6.7) can be written as

$$c(\mathbf{D_F})c(\mathbf{D_E})^{-1} - \rho_* c(D_{T^\perp}) - c(\mathbf{D_E})^{-1}[X] = d\left(c(\mathbf{D_E})^{-1}\sigma\right) \tag{6.16}$$

on $\operatorname{Hom}(E, F)$ where the degree $2n$ part is given by

$$c_n(\mathbf{D_F}) - c_{n-1}(\mathbf{D_F})c_1(\mathbf{D_E}) + \ldots (-1)^n c_1(\mathbf{D_E})^n - [X] = d\gamma \tag{6.17}$$

on $\operatorname{Hom}(E, F)$. If $E \xrightarrow{\alpha} F$ is atomic then (6.13) can be written as

$$c(D_F)c(D_E)^{-1} - \rho_* c(D_{T^\perp}) - c(D_E)^{-1}\operatorname{Div}(\alpha) = d\left(c(D_E)^{-1}\alpha^*(\sigma)\right) \tag{6.18}$$

on X, and hence

$$c_n(D_F) - c_{n-1}(D_F)c_1(D_E) + \ldots (-1)^n c_1(D_E)^n - \operatorname{Div}(\alpha) = d(\alpha^*(\gamma)). \tag{6.19}$$

The equations

$$c(\mathbf{D_F}) - c(\overrightarrow{\mathbf{D}}_s) = d\boldsymbol{\sigma}_\mathbf{s} \quad \text{on} \quad \operatorname{Hom}(E, F)$$
$$c(\mathbf{D_F}) - c(\mathbf{D_E})c(D_{T^\perp}) - [X] = d\boldsymbol{\sigma} \quad \text{on} \quad \operatorname{Hom}(E, F)$$

yield

$$c(\overrightarrow{\mathbf{D}}_s) - c(\mathbf{D_E})c(D_{T^\perp}) - [X] = d\mathbf{r_s},$$

with $\mathbf{r_s} \equiv \boldsymbol{\sigma} - \boldsymbol{\sigma}_\mathbf{s}$. Equivalently,

$$c(\overrightarrow{\mathbf{D}}_s)c(\mathbf{D_E})^{-1} - c(D_{T^\perp}) - c(\mathbf{D_E})^{-1}[X] = d(\mathbf{r_s}c(\mathbf{D_E})^{-1}). \tag{6.20}$$

Taking the degree $2n$-part of this equation yields an equation

$$\boldsymbol{\tau}_s - [X] = d\boldsymbol{\gamma}_\mathbf{s} \quad \text{on} \quad \operatorname{Hom}(E, F) \tag{6.21}$$

where $\boldsymbol{\gamma}_\mathbf{s}$ is the degree $2n-1$ part of $\mathbf{r_s}c(\mathbf{D_E})^{-1}$ and where

$$\boldsymbol{\tau}_s \equiv c_n(\overrightarrow{\mathbf{D}}_s) - c_{n-1}(\overrightarrow{\mathbf{D}}_s)c_1(\mathbf{D_E}) + \ldots (-1)^n c_1(\mathbf{D_E})^n, \tag{6.22}$$

since $c(D_{T^\perp})$ has no degree $2n$ part. This form $\boldsymbol{\tau}_s$ is a **Thom-Chern** form for $\operatorname{Hom}(E, F)$. One can establish that

$$\alpha^*\boldsymbol{\tau}_s \ \text{converges to} \ \operatorname{Div}(\alpha) \quad \text{on} \quad X \tag{6.23}$$

if α is an atomic bundle map from E to F.

Remark 6.24. It is natural to define the total Chern form of the difference $D_F - D_E$ to be

$$c(D_F - D_E) \;=\; c(D_F)c(D_E)^{-1}. \tag{6.25}$$

If, more generally, ϕ is any polynomial in n variables $c_1, \ldots, c_n$ then it is natural to define

$$\phi(D_F - D_E) \;\equiv\; \phi(c_1(D_F - D_E), \ldots, c_n(D_F - D_E)). \tag{6.26}$$

This yields a definition of the ϕ-Chern form $\phi(D_F - D_E)$ for all Ad-invariant forms ϕ.

Remark 6.27. The Top Chern Form for H = Hom(E, F) — the Direct Approach. Consider $H \equiv \mathrm{Hom}(E, F)$ as a hermitian vector bundle with connection D_H (induced by the connections on E and F). The results of Sections 2, 3, and 4 can be applied directly to H, yielding an easy proof of the results of Remark 6.14, independent of the constructions of this section.

Let $\overrightarrow{D}_{s,F}$ denote the push forward family — from **E** to **F**, and let $\overrightarrow{D}_{s,H}$ denote the push forward family — from $\underline{\mathbf{C}}$ to **H**, all over $H \equiv \mathrm{Hom}(E, F)$. Local frames e for E and f for F induce a local frame for H with fiber coordinates given by $u \equiv (u_1, \ldots, u_n)$ defined by $\alpha(e) = uf$ at a point $\alpha \in H$. The gauge for D_H is given by

$$\omega_H \;=\; \omega_F - \omega_E \tag{6.28}$$

where, in this equation, ω_E is a one form times the identity $m \times n$ matrix, since we are assuming that E is of rank one. One can easily check that

$$\omega_{s,H} \;=\; \overrightarrow{\omega}_H - \chi_s \frac{u^* Du}{|u|^2}, \tag{6.29}$$

by (2.2). Therefore

$$\overrightarrow{\omega}_{s,H} \;=\; \overrightarrow{\omega}_{s,F} - \omega_E. \tag{6.30}$$

Since $\omega_E \wedge \overrightarrow{\omega}_{s,F} + \overrightarrow{\omega}_{s,F} \wedge \omega_E = 0$ (because E has rank one),

$$\overrightarrow{\Omega}_{s,H} \;=\; \overrightarrow{\Omega}_{s,F} - \Omega_E. \tag{6.31}$$

As an immediate consequence the Thom form τ_s for $H = \mathrm{Hom}(E, F)$ is given by

$$(6.32) \qquad \tau_s = \left(\tfrac{i}{2\pi}\right)^n \det(\overrightarrow{\Omega}_{F,s} - \Omega_E) = \sum_{j=0}^{n} (-1)^j c_{n-j}(\overrightarrow{D}_{s,F}) c(D_E)^j .$$

In particular, this Thom form for H, obtained as a direct application of Section 4, equals the Thom form of (6.22). The results of Sections 2, 3, and 4 yield

Theorem 6.33. *If $E \xrightarrow{\alpha} F$ is atomic then*

$$c_n(D_F - D_E) - \mathrm{Div}(\alpha) = d\alpha^*(\sigma)$$

with $\alpha^*(\sigma) \in L^1_{\mathrm{loc}}(X)$. Moreover, this is the limiting form of the pull back by α of the universal equation

$$(6.34) \qquad c_n(D_F - D_E) - \tau_s = d\sigma_s$$

on $H \equiv \mathrm{Hom}(E, F)$. Here

$$(6.35) \qquad \sigma_s = \left(\tfrac{i}{2\pi}\right)^n \int_0^{\chi_s} \det\left(\tfrac{u^* Du}{|u|^2} \;;\; \Omega_F - \Omega_E - x\tfrac{Du^* Du}{|u|^2}\right) dx.$$

Proof. The proof of the formula for σ_s is all that remains. Use the formula (6.31) for $\overrightarrow{\Omega}_{s,H}$, replacing Ω_F by $\Omega_F - \Omega_E$ in Proposition 3.17 and in (4.29). $\square$

7. Residues.

In this section we carry out explicit computations of the residue form $\mathrm{Res}_\phi(\overrightarrow{D})$ discussed in previous sections. We already know from 3.23 and 6.6 that $\mathrm{Res}_\phi(\overrightarrow{D})$ is written as a universal polynomial in the curvatures of E and F. The polynomial is determined by the cohomology class of the residue (on, say, the classifying space of the bundles). Therefore the main point of this section will be to give explicit universal computations on this residue class.

At the end of the section we shall prove a general theorem concerning the decomposition of characteristic forms in terms of the tautological splitting of a bundle V when pulled-back over its projectivization. The main point of this result is the explicit form of the transgression. When $V = E \oplus F$, the theorem gives direct and parallel proofs of the main results in this chapter without recourse to 3.23 or 6.6. This theorem also has independent interest.

We now fix E and F as in the previous section and we work with pullbacks $\mathbf{E}$ and $\mathbf{F}$ of these bundles to the total space of $\mathrm{Hom}(E, F)$. Since $\mathrm{Res}_\phi(\overrightarrow{D})$ is independent of the choice of approximation mode, we shall use the algebraic one. This enables us to use the results of Section I.6 on compatification.

The residue form is defined by

$$\mathrm{Res}_\phi(\overrightarrow{D}) \;=\; -\rho_*(\mathbf{T}) \;=\; -\int_{\rho^{-1}} \mathbf{T} \tag{7.1}$$

where ρ is the restiction of $\pi : \mathrm{Hom}(E, F) \longrightarrow X$ to the unit sphere bundle. One also has the alternative formula (cf. Remark 2.24).

$$\mathrm{Res}_\phi(\overrightarrow{D}) \;=\; \pi_*\left(\phi(\overrightarrow{D}_s) - \phi(\overrightarrow{\Omega}_0)\right) \;=\; \int_{\pi^{-1}} \phi(\overrightarrow{D}_s) \tag{7.2}$$

for any $s > 0$. Now consider the compactification, $\mathbf{P}(E \oplus F)$, of $\mathrm{Hom}(E, F)$ and let U denote the universal line bundle on $\mathbf{P}(E \oplus F)$. From Chapter I.6, we know that, over the subset $\mathrm{Hom}(E, F) \subset \mathbf{P}(E \oplus F)$, there is an isomorphism

$$(\mathbf{F}, \overrightarrow{D}_1) \;\cong\; (U^\perp, D_{U^\perp}) \tag{7.3}$$

of bundles with connection. Thus

$$\phi(\overrightarrow{D}_1) \;=\; \phi(D_{U^\perp}). \tag{7.4}$$

Since the form $\phi(D_{U^\perp})$ extends to be a smooth form on $\mathbf{P}(E \oplus F)$ and since $\mathbf{P}(E \oplus F) \sim \mathrm{Hom}(E, F)$ is a subset of measure zero, this proves the next result.

Theorem 7.5. *Consider the bundle map $\pi : \mathbf{P}(E \oplus F) \to X$. Let*

$$\mathbf{E} \oplus \mathbf{F} = U \oplus U^{\perp}$$

denote the direct sum decomposition given in I.6. Furnish U and $U^{\perp}$ with the connections induced from the connection $D_E \oplus D_F$ on $\mathbf{E} \oplus \mathbf{F}$. Let D_U and $D_{U^{\perp}}$ denote these induced connections. Then for any invariant polynomial ϕ

$$\text{(7.6)} \qquad \mathrm{Res}_{\phi}(\overrightarrow{D}) = \pi_* \left(\phi(D_{U^{\perp}})\right) = \int_{\pi^{-1}} \phi(D_{U^{\perp}}).$$

In particular, the cohomology class of the residue on X is given by the formula

$$\text{(7.7)} \qquad \left[\mathrm{Res}_{\phi}(\overrightarrow{D})\right] = \pi_! \phi(U^{\perp})$$

where $\phi(U^{\perp}) \in H^*(\mathbf{P}(E \oplus F); \mathbf{R})$ is the ϕ-characteristic class of $U^{\perp}$ over $\mathbf{P}(E \oplus F)$.

We shall say that ϕ is **integral** if it corresponds to an integral cohomology class under the canonical identification $I_{\mathrm{GL}_n(\mathbf{C})} \cong H^*(\mathrm{BGL}_n(\mathbf{C}); \mathbf{R})$ given by the Chern-Weil homomorphism.

Corollary 7.8. *If ϕ is integral, then the residue class $[\mathrm{Res}_{\phi}(\overrightarrow{D})]$ is an integral cohomology class, i.e., it lies in the image of $H^*(X; \mathbf{Z})$ in $H^*(X; \mathbf{R})$.*

The remainder of this section is devoted to computing $\pi_! \phi(U^{\perp})$ explicitly in terms of ϕ, $c(E)$ and $c(F)$. (This will yield exact formulas for $\mathrm{Res}_{\phi}(\overrightarrow{D})$ in terms of Ω_E and Ω_F.)

We begin with a general algorithm for computing $\pi_!(U^{\perp})$. (In most important cases this algorithm can be considerably simplified, as we shall see). From the equation

$$\text{(7.9)} \qquad \pi^*(E \oplus F) = U \oplus U^{\perp}$$

we have that

$$\text{(7.10)} \qquad c(U^{\perp}) = c(U)^{-1} \pi^*(c(E)c(F)).$$

Let us set

$$t \stackrel{\text{def}}{=} -c_1(U)$$

so that

$$(7.11) \qquad c(U)^{-1} = (1-t)^{-1} = \sum_{k=0}^{\infty} t^k.$$

Then (7.10) implies that

$$(7.12) \qquad c_\ell(U^\perp) = \sum_{i+j+k=\ell} c_i(\mathbf{E})c_j(\mathbf{F})t^k$$

for $\ell = 1, \ldots, n$, where $\mathbf{E} = \pi^* E$ and $\mathbf{F} = \pi^* F$. We now express $\phi(U^\perp)$ as a polynomial in the Chern classes:

$$\phi(U^\perp) = \Phi(c_1(U^\perp), \ldots, c_n(U^\perp)).$$

Using (7.12) this can be expanded in the form

$$\phi(U^\perp) = \sum_{k\geq 0} A_k(c(\mathbf{E}), c(\mathbf{F}))t^k$$

where each A_k is a polynomial in $c_1(\mathbf{E})$, $c_1(\mathbf{F}), \ldots, c_n(\mathbf{F})$. It follows that

$$(7.13) \qquad \pi_!\phi(U^\perp) = \sum_{k\geq 0} A_k(c(E), c(F))\pi_!(t^k).$$

To compute the general residue formula, it remains only to compute $\pi_!(t^k)$.

Proposition 7.14. *One has that*

(i) $\pi_!(t^k) = 0$ *if* $k < n$,

(ii) $\pi_!(t^n) = 1$ *and*

(iii) $\pi_!(t^{n+k}) =$ *the component of* $c(E)^{-1}c(F)^{-1}$ *in degree* $2k$.

Proof. Equation (i) is trivial since $\pi_!$ is zero in degrees $< 2n$. Equation (ii) is a direct consequence of the elementary fact that

$$c_1(\lambda^*)^n[\mathbf{P}^n] = 1$$

where λ is the tautological line bundle over $\mathbf{P}^n$. To prove (iii) note from (7.10) and (7.11) that

$$\sum_{k=0}^{\infty} t^k = \pi^*(c(E)^{-1}c(F)^{-1})c(U^{\perp}).$$

Applying $\pi_!$ gives

$$\sum_{k=0}^{\infty} \pi_!(t^{n+k}) = c(E)^{-1}c(F)^{-1}\pi_!(c(U^{\perp}))$$

since $\pi_!((\pi^* a)\cdot b) = a\pi_! b$. Applying (ii) to the component in degree zero in this equation gives

$$\pi_! c_n(U^{\perp}) = 1. \tag{7.15}$$

Hence,

$$\sum_{k=0}^{\infty} \pi_!(t^{n+k}) = c(E)^{-1}c(F)^{-1}. \qquad \square \tag{7.16}$$

We have proved the following.

Theorem 7.17. *For any Ad-invariant polynomial ϕ on the Lie algebra $\mathfrak{gl}_n$, the residue form $\mathrm{Res}_\phi(\overrightarrow{D})$ is given pointwise as the polynomial in Ω_E and Ω_F computed by combining* (7.13) *and* 7.14 *above.*

This result extends, of course, to formal series of Ad-invariant homogeneous polynomials on $\mathfrak{gl}_n$. Note that we have recovered the basic facts that

$$\mathrm{Res}_{c_n}(\overrightarrow{D}) \equiv \mathrm{Res}_{(c_1)^n}(\overrightarrow{D}) \equiv 1$$

In many important cases the function ϕ has properties which make the calculation of $\mathrm{Res}_\phi(\overrightarrow{D})$ considerably easier. The first example is that of the Chern character. Given $\psi \in \mathcal{E}^*(X)$, let $\{\psi\}_m$ denote the component of degree m, i.e., $\psi = \sum\{\psi\}_m$ with $\{\psi\}_m \in \mathcal{E}^m(X)$.

Theorem 7.18.

$$\mathrm{Res}_{\mathrm{ch}}(\overrightarrow{D}) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(n+k)!} \left\{c(D_E)^{-1}c(D_F)^{-1}\right\}_{2k}$$

Proof. Applying ch to (7.9) gives

$$\pi^*(\mathrm{ch}E) + \pi^*(\mathrm{ch}F) = \mathrm{ch}(U) + \mathrm{ch}(U^\perp),$$

by the additivity property of the Chern character. Applying $\pi_!$ then gives

$$\pi_!\mathrm{ch}(U^\perp) = -\pi_!\mathrm{ch}(U) = \sum_{k\geq 0} \frac{(-1)^{k+1}}{k!}\pi_!(t^k).$$

Applying 7.14 and 6.6 gives the result. □

Theorem 7.19. *Let $\tau_k(A) = \mathrm{tr}\{(\frac{i}{2\pi}A)^k\}$ for $A \in gl_n(\mathbf{C})$ be the k^{th} "trace-power" function. Then*

$$\mathrm{Res}_{\tau_k}(\overrightarrow{D}) = (-1)^{k+1}\{c(D_E)^{-1}c(D_F)^{-1}\}_{2k}$$

Suppose now that $\mathbb{H} = \mathbb{H}(c)$ is a multiplicative series of Chern classes associated to the formal power series

$$h(x) = 1 + a_1x + a_2x^2 + \ldots \in \mathbf{R}[[x]].$$

(See [Hi], [MS] or [LM] for definitions and discussion.) Let

$$h^{-1}(-x) = 1 + b_1x + b_2x^2 + \ldots \in \mathbf{R}[[x]]$$

be the modified inverse series determined by $h(x)h^{-1}(x) = 1$.

Theorem 7.20.

$$\mathrm{Res}_{\mathbb{H}}(\overrightarrow{D}) = \mathbb{H}(D_E)\mathbb{H}(D_F)\sum_{k\geq 0} b_{n+k}\{c(D_E)^{-1}c(D_F)^{-1}\}_{2k}$$

Proof. By the multiplicative property of $\mathbb{H}$ we have

$$\pi^*\{\mathbb{H}(E)\mathbb{H}(F)\} = \mathbb{H}(U^\perp)\mathbb{H}(U) = \mathbb{H}(U^\perp)h(-t).$$

Hence,

$$\mathbb{H}(U^\perp) = \pi^*\{\mathbb{H}(E)\mathbb{H}(F)\}\sum_{k=0}^{\infty} b_k(-t)^k.$$

Applying $\pi_!$ gives the result. □

Note that $\mathbb{H}(D_E) = h(c_1(D_E)) = h(\frac{i}{2\pi}\Omega_E)$.

A basic example is given by the Todd series Todd which is associated to the polynomial

$$\operatorname{td}(x) = \frac{x}{1-e^{-x}}.$$

and so

$$\operatorname{td}^{-1}(-x) = \frac{1-e^x}{-x} = \sum_{k=1}^{\infty} \frac{1}{k!} x^{k-1}.$$

Corollary 7.21. *Let $u = c_1(D_E) = \frac{i}{2\pi}\Omega_E$. Then*

$$\begin{aligned}
\operatorname{Res}_{\mathrm{Todd}}(\overrightarrow{D}) &= \frac{1-e^u}{-u} \operatorname{Todd}(D_F) \sum_{k=0}^{\infty} \tfrac{1}{(n+k+1)!} \left\{(1-u)^{-1} c(D_F)^{-1}\right\}_{2k} \\
&= \frac{1-e^u}{-u} \operatorname{Todd}(D_F) \sum_{k=0}^{\infty} \tfrac{1}{(n+k+1)} \{\operatorname{Res}_{\mathrm{ch}}(\overrightarrow{D})\}_{2k}
\end{aligned}$$

A second example is given by $\mathbb{H} = e^{c_1}$ which has associated formal power series $h(x) = e^x$. Note that in this case $h^{-1}(-x) = e^x$.

Another set of examples for which the residue is easily computable comes from the following. Consider a polynomial

$$\phi(\cdot, t) \in I_{\mathrm{GL}_n}[t]$$

in one indeterminate t with coefficients in the ring of Ad-invariant polynomials on $\mathfrak{gl}_n$. Given bundles E, $F \longrightarrow X$ with connection as above, we can construct the Chern-Weil form

$$\phi(\Omega_F, \Omega_E) \in H^*(X).$$

All the results of this chapter hold with ϕ replaced by $\phi(\cdot, \Omega_E)$.

Suppose now that T, $T^\perp$ are the bundles over the blow-up $\rho : \widetilde{\mathbf{P}}(E \oplus F) \longrightarrow \mathbf{P}(E \oplus F)$ defined in I.3. Then we have the following generalization of Theorem 4.16.

Theorem 7.22. *Let $\phi(\cdot,t) \in I_{\mathrm{GL}_n}[t]$ be as above. Suppose that whenever $F \longrightarrow X$ is the tautological bundle over the Grassmannian of n-planes in $\mathbf{C}^N$, for some N, the class*

$$\phi(E \oplus T^{\perp}, E) \;=\; 0$$

in $H^*(\widetilde{\mathbf{P}}(E \oplus F))$. Then for any pair of bundles with connection F, $E \longrightarrow X$ over any manifold X (where $\operatorname{rank}(E) = 1$ and $\operatorname{rank}(F) = n$), we have that

$$\operatorname{Res}_{\psi(\cdot,\Omega_E)\phi(\cdot,\Omega_E)} \;=\; \psi(\Omega_F, \Omega_E)\operatorname{Res}_{\phi(\cdot,\Omega_E)}$$

for all $\psi(\cdot,t) \in I_{\mathrm{GL}_n}[t]$

Remark 7.23. Consider the example

$$c(\cdot,t) \stackrel{\text{def}}{=} \sum_{k=0}^{n} c_{n-k}(-t)^k,$$

where c_{n-k} is the polynomial corresponding to the k^{th} Chern class. This has the property that

$$c(F,E) \;=\; c_n(E^* \otimes F).$$

It follows that

$$c(E \oplus T^{\perp}, E) \;=\; c_n(E^* \otimes (E \oplus T^{\perp})) \;=\; c_n(\mathbf{C} \oplus T^{\perp}) \;=\; 0$$

and the Theorem applies. In particular when E is the trivial line bundle, we recover Proposition 4.16.

Proof of 7.22. On $\widetilde{\mathbf{P}}(E \oplus F)$ we have the equation of smooth forms

$$\begin{aligned} &\rho^*\{\psi(\overrightarrow{D}_s, D_E)\phi(\overrightarrow{D}_s, D_E)\} - \psi(D_E \oplus D_{T^{\perp}}, D_E)\phi(D_E \oplus D_{T^{\perp}}, D_E) \\ (7.24) \qquad &= \rho^*\{\psi(\overrightarrow{D}_s, D_E)\phi(\overrightarrow{D}_s, D_E)\} - \psi(D_E \oplus D_{T^{\perp}}, D_E)dA \\ &= \rho^*\{\psi(\overrightarrow{D}_s, D_E)\phi(\overrightarrow{D}_s, D_E)\} - d\tilde{A} \end{aligned}$$

where $\tilde{A} = \psi(D_E \oplus D_{T^{\perp}}, D_E)A$. Applying ρ_* to the top line gives $\operatorname{Res}_{\psi\phi}[X] + dS$. Hence equation (7.24) shows that $\operatorname{Res}_{\psi\phi}[X]$ is cohomologous to $\psi(\overrightarrow{D}_s, D_E)\phi(\overrightarrow{D}_s, D_E)$ on $\mathbf{P}(E \oplus F)$. On the other hand, our hypothesis that

$\phi(D_E \oplus D_{T^\perp}, D_E) = dA$ implies that $\phi(\overrightarrow{\Omega}_0, \Omega_E) = \rho_*\phi(D_E \oplus D_{T^\perp}, D_E) = d\rho_* A \stackrel{\text{def}}{=} dA'$. Hence the equation

$$\phi(\overrightarrow{D}_s, D_E) - \phi(\overrightarrow{\Omega}_0, \Omega_E) = \mathrm{Res}_\phi[X] + dS'$$

and the fact that

$$\psi(\overrightarrow{D}_s, D_E)\Big|_X = \psi(D_F, D_E)$$

imply that

$$\psi(\overrightarrow{D}_s, D_E)\phi(\overrightarrow{D}_s, D_E) = \psi(D_F, D_E)\,\mathrm{Res}_\phi[X] + d\tilde{S}$$

where $\tilde{S} = \psi(\overrightarrow{D}_s, D_E)(S' + A')$. We conclude that

$$[\mathrm{Res}_{\psi\phi}] = [\psi(D_F)][\mathrm{Res}_\phi]$$

in $H^*(X)$. The result now follows from 6.6. $\square$

The detailed results above on residue forms can be given a direct proof without using Theorems 3.23 or 6.6. To do this one begins with Theorem 7.25 below, sets $V = E \oplus F$ and carries through the formal arguments above. (See Remark 7.29).

This next result concerns the splitting of characteristic forms of a vector bundle V with connection, when they are lifted to the projectivization. The main point of the theorem is the detailed structure of the transgression current. The result is of independent interest and quite useful.

Theorem 7.25. *Let V be a complex vector bundle with connection D_V over a manifold X and let $\pi : \mathbf{P}(V) \longrightarrow X$ denote the projectivized bundle. Suppose V is furnished with a hermitian metric (not necessarily related to the connection). Denote by U the tautological line bundle over $\mathbf{P}(V)$ with connection D_U induced from D_V via the splitting*

$$\pi^* V = U \oplus U^\perp.$$

Then for any Ad-invariant polynomial ϕ on $\mathfrak{gl}_n(\mathbf{C})$ there exists a transgression form S on $\mathbf{P}(V)$ satisfying

$$\phi(D_V) - \phi(D_U \oplus D_{U^\perp}) = dS \qquad \text{on } \mathbf{P}(V) \tag{7.26}$$

with the property that S can be expressed, as a sum over k, of forms which are of bidegree k, $k-1$ in the fiber 1-forms Du_i, Du_j^* for i, $j = 1, \ldots, n$.

Consequently, if ψ is any form on $\mathbf{P}(V)$ which is of type k, k in these fibre 1-forms, then

$$\pi_*(\psi S) = 0. \tag{7.27}$$

Corollary 7.28. (cf. Bott [Bo]). *Under the hypotheses of Theorem 7.25,*

$$\pi_*(c(D_U)^{-1}) = c(D_V)^{-1}.$$

Proof. Note that by (7.26),

$$c(D_V) - c(D_U)c(D_U^\perp) = dS.$$

Multiply the equation by $c(D_U)^{-1}c(D_V)^{-1}$, take π_*, and apply (7.27). □

Remark 7.29. Using Theorem 7.25 one can give direct proofs of the results 7.17—7.21 above. To do this, set $V = E \oplus F$ and apply argument parallel to those above. We leave this to the reader.

Proof of Theorem 7.25. Let $P : \mathbf{V} \to U$ denote orthogonal projection. The family of connections on $\mathbf{V}$ given by

$$D_x \equiv D_V - x(1-P)D_V P \qquad \text{for } \ 0 \le x \le 1$$

has initial connection D_V and terminal connection $D_1 = D_V - (1-P)D_V P$. Blocking D_1 with respect to the decomposition $\mathbf{V} = U \oplus U^\perp$ yields

$$D_1 = \begin{pmatrix} D_U & PD_V(1-P) \\ 0 & D_{U^\perp}. \end{pmatrix}$$

Since D_1 is in upper triangular form

$$\phi(D_1) = \phi(D_U \oplus D_{U^\perp})$$

for any invariant polynomial ϕ. Consequently,

$$\phi(D_V) - \phi(D_U \oplus D_{U^\perp}) \;=\; dS$$

with

$$S \;=\; -\int_0^1 \phi\left(\dot{D}_x \;;\; R_x\right) dx.$$

To complete the proof we calculate this formula for S.

Write $D_x = D_V - xB$ where $B = (1-P)D_V P$, and note that $B^2 = 0$ since $P(1-P) = 0$. Therefore

$$R_x \;=\; (D_V - xB)^2 \;=\; R_V - x(BD_V + D_V B).$$

Note that

$$B \;=\; (1-P)D_V P \;=\; (1-P)(D_V P - PD_V) \;=\; (1-P)D(P),$$

where by definition $D(P) = D_V P - PD_V$. Since $P = P^2$, we have $D(P) = D(P)P + PD(P)$, and therefore

$$B \;=\; D(P)P.$$

This implies

$$BD_V \;=\; D(P)PD_V.$$

Since

$$\begin{aligned} D_V B \;&=\; D_V(1-P)D_V P \\ &=\; \{D_V(1-P) - (1-P)D_V\}D_V P + (1-P)R_V P \\ &=\; -D(P)D_V P + (1-P)R_V P, \end{aligned}$$

we have

$$BD_V + D_V B = (1-P)R_V P - D(P)D(P).$$

This proves

Lemma 7.30.

$$R_x = R_V - x(1-P)R_V P + xD(P)D(P)$$

and

$$\dot{D}_x = -(1-P)D(P).$$

To prove Theorem 7.25 it will suffice to show that $\dot{D}_x$ is of bidegree 1, 0 and that R_x is a sum of terms of bidegree k, k in the fibre 1-forms Du_i, Du_j^*. We shall calculate in homogeneous coordinates $V \sim Z \to \mathbf{P}(V)$. Over $V \sim Z$, the bundle $\mathbf{V}$ has a canonical section c. Consider the associated bundle map $\underline{\mathbf{C}} \xrightarrow{\alpha} \mathbf{V}$ defined by $\alpha(1) = c$. The image of α is the subbundle U of $\mathbf{V}$. The orthogonal projection P is given by $P = \frac{\alpha\alpha^*}{|\alpha|^2}$ where $|\alpha|^2 = \alpha^*\alpha$. To prove our assertions it suffices to show that

$$B = (1-P)D(P) = (1-P)\frac{(D\alpha)\alpha^*}{|\alpha|^2} \tag{7.31}$$

and that

$$D(P)D(P) = \frac{\alpha(D\alpha^*)}{|\alpha|^2}(1-P)\frac{(D\alpha)\alpha^*}{|\alpha|^2} + (1-P)\frac{(D\alpha)(D\alpha^*)}{|\alpha|^2}(1-P). \tag{7.32}$$

Both follow from the formula

$$D(P) = D_V P - PD_V = (1-P)\frac{(D\alpha)\alpha^*}{|\alpha|^2} + \frac{\alpha(D\alpha^*)}{|\alpha|^2}(1-P)$$

which is an easy consequence of $P = \frac{\alpha\alpha^*}{|\alpha|^2}$. This completes the proof of Theorem 7.25. $\square$

Remark 7.33. Setting $x = 1$ in (7.30), and using (7.32) yields

$$R_U = P\left(R_V + \frac{\alpha(D\alpha^*)}{|\alpha|^2}(1-P)\frac{(D\alpha)\alpha^*}{|\alpha|^2}\right)P,$$

and

$$R_{U^\perp} = (1-P)\left(R_V + \frac{(D\alpha)(D\alpha^*)}{|\alpha|^2}\right)(1-P).$$

Remark 7.34. If u is a homogeneous coordinate for $\mathbf{P}(V)$ with respect to a local frame for V, then S is given by the following formula.

(7.35)

$$S = \int_0^1 \phi\Bigg(\tfrac{u^* Du}{|u|^2}\left(1 - \tfrac{u^* u}{|u|^2}\right) \ ; \ \Omega_V - x\tfrac{u^* u}{|u|^2}\Omega_V\left(1 - \tfrac{u^* u}{|u|^2}\right)$$

$$- x\tfrac{u^*(Du)}{|u|^2}\left(1 - \tfrac{u^* u}{|u|^2}\right)\tfrac{(Du^*)u}{|u|^2} - x\left(1 - \tfrac{u^* u)}{|u|^2}\right)\tfrac{(Du^*)(Du)}{|u|^2}\left(1 - \tfrac{u^* u)}{|u|^2}\right)\Bigg) dx.$$

If we apply Theorem 7.25 to the case where $V = E \oplus F$ with direct sum connection, then formula (7.26) becomes

$$\phi(D_E \oplus D_F) - \phi(D_U \oplus D_{U^\perp}) = dS \tag{7.36}$$

on $\mathbf{P}(E \oplus F)$. Here we allow E to be of any rank. When $\phi = c^{-1}$, this gives the following.

Proposition 7.37. *Let c denote the total Chern polynomial. If* $\operatorname{rank} F > 1$, *then*

$$\pi_*\{c(D_{U^\perp})^{-1}\} = 0.$$

In particular, when $\operatorname{rank} E = 1$ *and* $\operatorname{rank}(F) > 1$ *we have that*

$$\operatorname{Res}_{c^{-1}}(\overrightarrow{D}) = 0.$$

Proof. By 7.25 we have

$$c(D_E)c(D_F) - c(D_U)c(D_{U^\perp}) = dS$$

which implies that

$$c(D_{U^\perp})^{-1} - c(D_U)c(D_E)^{-1}c(D_F)^{-1} = d\left(c(D_{U^\perp})^{-1}c(D_E)^{-1}c(D_F)^{-1}S\right) \stackrel{\text{def}}{=} dT.$$

By 7.27 we have that

$$\pi_* dT = c(D_E)^{-1}c(D_F)^{-1}d\pi_*\left(c(D_{U^\perp})^{-1}S\right) = 0.$$

Furthermore, since $\dim \mathbf{P}(E \oplus F) = \operatorname{rank}(E) + \operatorname{rank}(F) - 1 > \operatorname{rank}(E) = \operatorname{rank}(U)$, we have that

$$\pi_*\{c(D_U)c(D_E)^{-1}c(D_F)^{-1}\} = c(D_E)^{-1}c(D_F)^{-1}\pi_* c(D_U) = 0.$$

Hence, $\pi_*\{c(D_{U^\perp})^{-1}\} = 0$. $\square$

8. The Singular PullBack Connection $\overleftarrow{\mathbf{D}}$ on E.

Throughout this section assume that $E \xrightarrow{\alpha} F$ is an atomic bundle map of hermitian bundles with connections and that $\operatorname{rank}(E) = 1$. The special case $\underline{\mathbf{C}} \xrightarrow{\alpha} F$ where $\alpha(1) = \mu$ is a section of F is of particular importance. The objective is to compute the Chern currents associated with the singular pullback connection $\overleftarrow{D}$ on the line bundle E. One consequence is that the divisor of μ can be expressed as $(-1)^n$ times the singular part of the Chern current $c_1^n((\overleftarrow{D}))$ (See Corollary 8.18 below).

The pullback family of connections $\overleftarrow{D}_s$ on E is defined by

$$\overleftarrow{D}_s = D_E + \chi_s \frac{\alpha^* D\alpha}{|\alpha|^2}, \tag{8.1}$$

with

$$\overleftarrow{D} = D_E + \frac{\alpha^* D\alpha}{|\alpha|^2}. \tag{8.2}$$

As we shall see this case is formally like the line bundle case of Chapter II. The power functions $\phi_m(t) = \left(\frac{i}{2\pi}t\right)^m$ generate all invariant polynomials so we need only consider $\left(\frac{i}{2\pi}\overleftarrow{\Omega}_s\right)^m$. Equivalently we need only consider $c^{-1}(t) = \left(1 + \frac{i}{2\pi}t\right)^{-1}$, the inverse of the total Chern polynomial $c(t) = 1 + \frac{i}{2\pi}t$. The residue has a particularly nice form with this choice.

Theorem 8.3. *Suppose $E \xrightarrow{\alpha} F$ is an atomic bundle map from a line bundle E to a rank n bundle F. The Chern current $c^{-1}((\overleftarrow{D})) = \lim_{s\to 0} c^{-1}(\overleftarrow{D}_s)$ of the singular pullback connection $\overleftarrow{D}$ on E is given by*

$$c^{-1}((\overleftarrow{D})) = \left(1 + \tfrac{i}{2\pi}\overleftarrow{\Omega}_0\right)^{-1} + c(D_F)^{-1}c(D_E)^{-1}\operatorname{Div}(\alpha), \tag{8.4}$$

where the L^1_{loc} part $\left(1 + \frac{i}{2\pi}\overleftarrow{\Omega}_0\right)^{-1} \in L^1_{\mathrm{loc}}(X)$ is d-closed. The transgressions T_s converge in $L^1_{\mathrm{loc}}(X)$ to $T \in L^1_{\mathrm{loc}}(X)$. The current equation

$$c(D_E)^{-1} - c(\overleftarrow{\Omega}_0)^{-1} - c(D_F)^{-1}c(D_E)^{-1}\operatorname{Div}(\alpha) = dT \tag{8.5}$$

is the limiting form of the standard transgression formula

$$c(D_E)^{-1} - c(\overleftarrow{D}_s)^{-1} = dT_s. \tag{8.6}$$

Proof. The singular pullback connection $\overleftarrow{D}$ on E has gauge

$$\overleftarrow{\omega} = \omega_E + \frac{Duu^*}{|u|^2} = \omega_E + \frac{(u\omega_F - \omega_E u + du)u^*}{uu^*} = \frac{(u\omega_F + du)u^*}{uu^*}$$

in terms of local frames e for E and f for F where $\alpha e = uf$ defines $u \equiv (u_1, \ldots, u_n)$. Since u is atomic the singular gauge $\overleftarrow{\omega}$ has an L^1_{loc} extension to X which will be denoted by

$$\overleftarrow{\omega}_0 = \frac{(u\omega_F + du)u^*}{uu^*}.$$

The matrix form of $\alpha^* D\alpha$ is the global $L^1_{\text{loc}}(X)$ one form

$$\tau = \frac{Duu^*}{|u|^2} \equiv \frac{(u\omega_F - \omega_E u + du)u^*}{uu^*} = \omega_0 - \omega_E$$

just as $|\alpha|^2 = \langle \alpha, \alpha \rangle_{\text{Hom}} = uu^* = |u|^2$ is a global C^∞ function. The pullback family of connections $\overleftarrow{D}_s$ on E has local gauge

$$\overleftarrow{\omega}_s = \omega_E + \chi_s \tau = \omega_E + \chi_s(\omega_0 - \omega_E). \tag{8.7}$$

The curvature at time $s = 0$ is given by

$$\overleftarrow{\Omega}_0 = \frac{u\Omega_F u^*}{|u|^2} - \frac{Du\left(1 - \frac{u^* u}{|u|^2}\right) Du^*}{|u|^2}, \tag{8.8}$$

while the curvature at time $s > 0$ is given by

$$\overleftarrow{\Omega}_s = \chi_s(1 - \chi_s)\Omega_E + \chi_s \overleftarrow{\Omega}_0 + \chi_s' \tfrac{|u|^2}{s^2} \tfrac{d|u|^2}{|u|^2} \frac{Duu^*}{|u|^2}. \tag{8.9}$$

It will suffice to work in the universal case. We set

$$e = c(D_E),\ e_s = c(\overleftarrow{D}_s) \quad \text{and} \quad \ell = 1 + \tfrac{i}{2\pi}\overleftarrow{\Omega}_0$$

and note that $e = e_\infty$. Taking the exterior derivative in (8.7) gives

$$e_s = e + \chi_s(\ell - e) + \tfrac{i}{2\pi} d\chi_s \tau. \tag{8.10}$$

Since $\dot{\omega}_s = \dot{\chi}_s \tau$ the formula for the transgression can be integrated. We first carry this out for the m^{th} power function $\phi_m(t) = \left(\frac{i}{2\pi}t\right)^m$ and then take a sum.

Let $\bar{e}_s$ denote $c_1(\overleftarrow{D}_s)$, $\bar{e}$ denote $c_1(D_E)$, and $\bar{\ell}$ denote $\frac{i}{2\pi}\Omega_0$. The transgression T_m for ϕ_m is computed as follows. $T_m =$

$$
(8.11)\qquad
\begin{aligned}
\left(\tfrac{i}{2\pi}\right)^m m \int\limits_s^\infty \dot{\omega}_s \overleftarrow{\Omega}_s^{m-1} ds &= \tfrac{i}{2\pi}\tau m \int\limits_s^\infty \bar{e}_s^{m-1}\dot{\chi}_s ds \\
&= -\tfrac{i}{2\pi}\tau m \int\limits_0^{\chi_s} (\bar{e} + x(\bar{\ell} - \bar{e}))^{m-1} dx \\
&= -\tfrac{i}{2\pi}\tau \,\frac{(\bar{e} + \chi_s(\bar{\ell} - \bar{e}))^m - \bar{e}^m}{\bar{\ell} - \bar{e}}.
\end{aligned}
$$

Consequently when $\phi = c^{-1}$, one derives the equation

$$
(8.12)\qquad T_s = -\tfrac{i}{2\pi}\tau \,\frac{(e + \chi_s(\ell - e))^{-1} - e^{-1}}{\ell - e}.
$$

Using the equation $\chi_s(\ell - e) = (e + \chi_s(\ell - e)) - e$ one can rewrite (8.12) as

$$
(8.13)\qquad T_s = \tfrac{i}{2\pi}\chi_s \tau e^{-1}(e + \chi_s(\ell - e))^{-1}.
$$

Since $e_s = e + \chi_s(\ell - e)$ modulo τ this simplifies to

$$
T_s = \tfrac{i}{2\pi}\chi_s \tau e^{-1} e_s^{-1}.
$$

Hence, the limiting transgression, as $s \to 0$, is given by

$$
(8.14)\qquad T = \tfrac{i}{2\pi}\tau e^{-1}\ell^{-1},
$$

and the residue is defined by

$$
(8.15)\qquad \mathrm{Res}_{c^{-1}}(\overleftarrow{D}) = -\rho_*(T).
$$

The proof of the Theorem is easily completed using the formulas presented above, except for the residue calculation

$$
(8.16)\qquad \mathrm{Res}_{c^{-1}}(\overleftarrow{D}) = c(D_E)^{-1} c(D_F)^{-1}.
$$

In order to prove this it is convenient to use an alternate to the residue formula (8.15).

Remark 8.17. Alternate formulas for Res($\overleftarrow{\mathbf{D}}$). First note that

$$\left(\tfrac{i}{2\pi}\overleftarrow{\Omega}_s\right)^m - \left(\tfrac{i}{2\pi}\overleftarrow{\Omega}_0\right)^m - [X]\,\mathrm{Res}_m(\overleftarrow{D}) \;=\; dR_s \quad \text{on} \quad \mathrm{Hom}(E,F)$$

where $\mathrm{Res}_m \stackrel{\mathrm{def}}{=} \mathrm{Res}_{\phi_m}$. Assume that χ is a compactly supported approximate one (i.e. $\chi(t) \equiv 1$ for t large). Then R_s and $\Omega_s^m - \Omega_0^m$ vanish for $\frac{|u|^2}{s^2}$ large. Consequently we may use $\pi : \mathrm{Hom}(E,F) \to X$ to push forward this current equation on $\mathrm{Hom}(E,F)$ to a current equation on X. Since $\pi_*(R_s) = 0$ and $\pi_*([X]\,\mathrm{Res}_m(\overleftarrow{D})) = \mathrm{Res}_m(\overleftarrow{D})$ we obtain the following alternate formula for the residue form

$$\mathrm{Res}_m(\overleftarrow{D}) \;=\; \pi_*\left\{\left(\tfrac{i}{2\pi}\overleftarrow{\Omega}_s\right)^m - \left(\tfrac{i}{2\pi}\overleftarrow{\Omega}_0\right)^m\right\}.$$

Using the fact that $\int_{\pi^{-1}} \left(\frac{i}{2\pi}\Omega_0\right)^m = 0$, we can rewrite this in terms of fibre integration as

$$\mathrm{Res}_m(\overleftarrow{D}) \;=\; \int_{\pi^{-1}} \left(\tfrac{i}{2\pi}\overleftarrow{\Omega}_s\right)^m .$$

Actually, this alternate description is equally valid for any approximate one χ. One must verify that the fiber integral of $\left(\frac{i}{2\pi}\overleftarrow{\Omega}_s\right)^m - \left(\frac{i}{2\pi}\overleftarrow{\Omega}_0\right)^m$ converges at infinity in $\mathrm{Hom}(E,F)$.

Let U denote the tautological line bundle over $\mathbf{P}(E\oplus F)$. Since E, $\overleftarrow{D}_s$ and U, D_U are isomorphic over $\mathrm{Hom}(E,F) \subset \mathbf{P}(E\oplus F)$ as bundles with connections (see I.6.14),

$$\left(\tfrac{i}{2\pi}\Omega_s\right)^m \;=\; \left(\tfrac{i}{2\pi}\Omega_U\right)^m$$

extends as a smooth form to all of $\mathbf{P}(E\oplus F)$ and

$$\mathrm{Res}_m(\overleftarrow{D}) \;=\; \pi_*(c_1(D_U)^m).$$

where $\pi : \mathbf{P}(E\oplus F) \to X$.

Combining this formula and Proposition 7.14 yields the residue formula 8.16, and completes the proof of Theorem 8.3. $\square$

Corollary 8.18. *For $m < n$, the residues* $\mathrm{Res}_m(\overleftarrow{D}) = 0$ *and, for $m = n$, the residue* $\mathrm{Res}_n(\overleftarrow{D}) = (-1)^n$ *so that*

$$(m < n) \qquad c_1(D_E)^m - c_1(\overleftarrow{\Omega}_0)^m = dT_m,$$

and

$$(m = n) \qquad c_1(D_E)^n - c_1(\overleftarrow{\Omega}_0)^n - (-1)^n[X] = dT_n,$$

as equations of currents on $\mathrm{Hom}(E, F)$. *Moreover, for all $m \geq 0$,*

$$T_m = -\tfrac{i}{2\pi}\tau \frac{\ell^m - e^m}{\ell - e}.$$

Proof. $\mathrm{Res}_m(\overleftarrow{D})$ is equal to $(-1)^m$ times the degree $2m - 2n$ part of $c(D_E)^{-1}$ $c(D_F)^{-1}$, for all $m = 0, 1, \ldots$, by (8.16). The formula for T_m is just (8.11). □

Appendix A The Bochner-Martinelli Kernel and Chern-Weil Theory.

Suppose $u \equiv (u_1, \ldots, u_n) \in \mathbf{C}^n$ are coordinates for $\mathbf{C}^n$ and let $\lambda(u) \equiv \frac{i}{2}du_1 \wedge d\bar{u}_1 \wedge \cdots \wedge \frac{i}{2}du_n \wedge d\bar{u}_n$ denote the standard volume form on $\mathbf{C}^n$. The **Bochner-Martinelli potential or kernel** is defined to be

$$(\mathrm{A.1}) \qquad B = \tfrac{(n-1)!}{\pi^n} \frac{\bar{u}\frac{\partial}{\partial \bar{u}} \llcorner \lambda(u)}{|u|^{2n}}.$$

If $n = 1$ this is just the Cauchy kernel $\frac{1}{2\pi i}\frac{du}{u}$.

Suppose $u \equiv (u_1, \ldots, u_n)$ is a smooth function on a manifold X with values in $\mathbf{C}^n$. The pullback of B via u will be denoted by $B(u)$, or sometimes simply by B as well. This will also be called the **Bochner-Martinelli potential** (or **kernel**). Note that if u is atomic, then $B(u) \in L^1_{\mathrm{loc}}(X)$, i.e., $B(u)$ has a unique extension across the zero set of u as an L^1_{loc} form.

Alternate Expressions for Bochner-Martinelli Related to the n^{th} Power of the First Chern Form c_1

Consider the pullback case of Section 8. Suppose D_E and D_F are flat, i.e., there exist local frames which are parallel. In these frames we have

$$\overleftarrow{\omega}_s = \chi_s \tfrac{duu^*}{|u|^2}. \tag{A.2}$$

Let

$$\beta \equiv \tfrac{duu^*}{|u|^2}$$

so that

$$d\beta = -\tfrac{dudu^*}{|u|^2} + \tfrac{duu^*udu^*}{|u|^4} \equiv -\tfrac{dudu^*}{|u|^2} \quad (\text{mod } \beta).$$

Then $\overleftarrow{\omega}_s = \chi_s\beta$, $\dot{\omega}_s = \dot{\chi}_s\beta$, and

$$\Omega_s = \chi_s d\beta + d\chi_s\beta \equiv \chi_s d\beta \quad (\text{mod } \dot{\omega}_s).$$

Therefore the transgression T_s for the n^{th} power, is given by

$$T_s = \left(\tfrac{i}{2\pi}\right)^n \int_s^\infty \dot{\omega}_s \Omega_s^{n-1} ds = \left(\tfrac{i}{2\pi}\right)^n \chi_s\beta(d\beta)^{n-1}$$

and

$$T = \left(\tfrac{i}{2\pi}\right)^n \beta(d\beta)^{n-1}.$$

Definition A.3. Suppose h is a smooth function on X which takes values in the set of positive definite hermitian $n \times n$ matrices. Define $u^* = h\bar{u}^t$ and $|u|^2 \equiv uu^*$. The transgression obtained above,

$$B(u, u^*) = \left(\tfrac{i}{2\pi}\right)^n \beta(d\beta)^{n-1}, \tag{A.4}$$

will be called the **Bochner-Martinelli kernel based on the metric h**.

If the metric h is the identity matrix, so that $u^* = \bar{u}^t$ we define

$$\alpha = \tfrac{du\bar{u}^t}{|u|^2}.$$

Lemma A.5. *The Bochner-Martinelli kernel $B(u)$ can be written in the form $B(u, \bar{u}^t)$, i.e.,*

$$B(u) \;=\; -\left(\tfrac{i}{2\pi}\right)^n \frac{duu^t(dudu^t)^{n-1}}{|u|^{2n}} \;=\; \left(\tfrac{1}{2\pi i}\right)^n \alpha(d\alpha)^{n-1}.$$

Proof. First note the standard fact

$$n!\lambda(u) \;=\; \left(\tfrac{i}{2}dud\bar{u}^t\right)^n.$$

This implies

$$B(u) \;=\; \frac{1}{n}\bar{u}\frac{\partial}{\partial\bar{u}} \llcorner \left(\tfrac{i}{2\pi}\frac{dud\bar{u}^t}{|u|^2}\right)^n,$$

and the Lemma follows since $d\alpha \equiv -\frac{dud\bar{u}^t}{|u|^2}$ (mod α). $\square$

Remark. Utilizing the complex structure on $\mathbf{C}^n$, the Bochner-Martinelli kernel B can also be written as

$$B \;=\; \left(\tfrac{i}{2\pi}\right)^n \partial\log|u|^2(\overline{\partial}\partial\log|u|^2)^{n-1}, \tag{A.6}$$

since $\alpha = \partial\log|u|^2$. Note that $\frac{i}{2\pi}\overline{\partial}\partial\log|u|^2 = \frac{i}{2\pi}d\alpha$ is real.

Remark A.7. The Solid Angle Kernel. If $u \equiv x + iy \in \mathbf{C}^n$ has real and imaginary parts x and y, then $2\,\mathrm{Re}\,\bar{u}\frac{\partial}{\partial\bar{u}} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the real euler vector field on $\mathbf{R}^{2n} = \mathbf{C}^n$. Consequently (A.1) implies that the real part of the Bochner-Martinelli potential is equal to the **normalized solid angle kernel**

$$\mathrm{Re}\,B \;=\; \mathrm{vol}(S^{n-1})^{-1}\Theta.$$

Here

$$\Theta \;\equiv\; \sum_{j=1}^{2n}(-1)^{j-1}\,\frac{t_j dt_1 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_n}{|t|^{2n}}$$

defines the **solid angle kernel** in real coordinates $(t_1, \ldots, t_{2n}) \in \mathbf{R}^{2n}$. The imaginary part of the Bochner-Martinelli kernel is exact. That is

$$\mathrm{Im}\,B \;=\; -d\left(\tfrac{1}{4\pi}\log|u|^2\left(\tfrac{i}{2\pi}\partial\overline{\partial}\log|u|^2\right)^{n-1}\right) \;=\; -d\left(\tfrac{1}{4\pi}\log|u|^2\left(\tfrac{i}{2\pi}d\alpha\right)^{n-1}\right).$$

Because of Corollary 8.18, if we assume X is point then

$$dB(u,u^*) = [0] \quad \text{on } \mathbf{C}^n \tag{A.8}$$

or if we assume that $u(x)$ is an atomic function then $B(u,u^*) \in L^1_{\mathrm{loc}}(X)$ and

$$dB(u,u^*) = \mathrm{Div}(u) \quad \text{on } X \tag{A.9}$$

In particular, $B(u,u^*)-B(u)$, considered as an $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ form, is d-closed. Therefore, it is d-exact. There is an explicit formula for a form whose exterior derivative is $B(u,u^*) - B(u)$. Before exhibiting this formula we make an important generalization.

The Bochner-Martinelli kernel can be replaced by other kernels such as the Cauchy-Fantappia kernel. This fact is of considerable importance in complex analysis.

Let $v^* \in (\mathbf{C}^n)^*$ denote a column vector whose transpose is $(v_1^*, \ldots, v_n^*)$. Let

$$B(u,v^*) = \tfrac{(n-1)!}{\pi^n} \frac{v^* \frac{\partial}{\partial v^*} \llcorner \lambda(u,v^*)}{(uv^*)^n} \tag{A.10}$$

and let

$$\lambda(u,v^*) \equiv \frac{i}{2} du_1 \wedge dv_1^* \wedge \cdots \wedge \frac{i}{2} du_n \wedge dv_n^*.$$

We have simply replaced $\bar{u}^t$ by v^* in the Bochner-Martinelli kernel. (Here v^* is an independent set of variables, however, v^* may also be considered another n-vector valued function.)

Remark A.11. Formulas for $\mathbf{B(u,v^*)}$. The various alternate expressions for $B(u,u^*)$ have counterparts for $B(u,v^*)$.

The expression

$$n!\lambda(u,v^*) = \left(\frac{i}{2} du dv^*\right)^n$$

for $\lambda(u,v^*)$ yields

$$B(u,v^*) = -\left(\tfrac{i}{2\pi}\right)^n \frac{duv^*(dudv^*)^{n-1}}{(uv^*)^n}. \tag{A.12}$$

It follows that

$$B(u,v^*) = \tfrac{1}{(2\pi i)^n} \alpha (d\alpha)^{n-1} \quad \text{with } \alpha \equiv \frac{duv^*}{uv^*}, \tag{A.13}$$

since

$$d\alpha \equiv -\frac{dudv^*}{uv^*} \pmod{\alpha}.$$

The next lemma and it's corollary provide the foundation for the kernel approach to the study of several complex variables. See, for example, Harvey-Polking [HP].

Lemma A.14. *Suppose α and β are arbitrary smooth one forms. Then the following* **fundamental identity** *holds:*
(A.15)

$$\alpha(d\alpha)^{n-1}-\beta(d\beta)^{n-1} = d\left(\alpha\beta \sum_{j+k=n-2} (d\alpha)^j(d\beta)^k\right)-(\beta-\alpha) \sum_{j+k=n-1} (d\alpha)^j(d\beta)^k.$$

This can be written symbolically as:
(A.15)′

$$\alpha(d\alpha)^{n-1} - \beta(d\beta)^{n-1} = d\left(\alpha\beta\frac{(d\alpha)^{n-1} - (d\beta)^{n-1}}{d\alpha - d\beta}\right) - (\beta-\alpha)\frac{(d\alpha)^n - (d\beta)^n}{d\alpha - d\beta}.$$

First Proof. Note that

$$\begin{aligned}
&d\left(\alpha\wedge\beta\wedge \sum_{j+k=n-2} (d\alpha)^j(d\beta)^k\right)\\
&= d\Big(\alpha\wedge\beta\wedge((d\alpha)^{n-2}+(d\alpha)^{n-3}(d\beta)+\cdots+(d\beta)^{n-2})\Big)\\
&= \beta\wedge\Big((d\alpha)^{n-1}+(d\alpha)^{n-2}(d\beta)+\cdots+(d\alpha)(d\beta)^{n-2}\Big)\\
&\qquad -\alpha\wedge\Big((d\alpha)^{n-2}d\beta+\cdots+(d\beta)^{n-1}\Big)\\
&= \alpha(d\alpha)^{n-1}-\beta(d\beta)^{n-1}+(\beta-\alpha)\sum_{j+k=n-1}(d\alpha)^j(d\beta)^k. \quad \square
\end{aligned}$$

Second Proof. Let $\sigma_t \equiv (1-t)\alpha + t\beta$. Then

$$\begin{aligned}\frac{d}{dt}\left(\sigma_t(d\sigma_t)^{n-1}\right) &= \dot{\sigma}_t(d\sigma_t)^{n-1} + (n-1)\sigma_t d\dot{\sigma}_t(d\sigma_t)^{n-2} \\ &= n\dot{\sigma}_t(d\sigma_t)^{n-1} - (n-1)d\Big(\sigma_t\dot{\sigma}_t(d\sigma_t)^{n-2}\Big) \\ &= n(\beta-\alpha)(d\sigma_t)^{n-1} - (n-1)d\Big(\alpha\wedge\beta(d\sigma_t)^{n-2}\Big)\end{aligned}$$

Now integrate both sides using

$$\int_0^1 m(d\sigma_t)^{m-1}dt \;=\; \frac{(d\beta)^m - (d\alpha)^m}{d\beta - d\alpha}. \quad \square$$

Third Proof. The identity

$$\begin{aligned}& d\Big(\alpha\beta\Big((d\alpha)^{n-1} - (d\beta)^{n-1}\Big)\Big) \\ &\quad = (\beta-\alpha)\Big((d\alpha)^n - (d\beta)^n\Big) + (d\alpha - d\beta)\Big(\alpha(d\alpha)^{n-1} - \beta(d\beta)^{n-1}\Big)\end{aligned}$$

can be divided by $d\alpha - d\beta$ to yield (1.15′). $\square$

Corollary A.16. An Algebraic Identity Fundamental for $\overline{\partial}$. *Suppose u, v^*, and w^* are arbitrary n-tuples of functions with u a row and v^*, w^* columns. Set*

$$\alpha \;=\; \frac{duv^*}{uv^*} \quad \text{and} \quad \beta \;=\; \frac{duw^*}{uw^*}.$$

Then

$$\alpha(d\alpha)^{n-1} - \beta(d\beta)^{n-1} \;=\; d\left(\alpha\wedge\beta\wedge\sum_{j+k=n-2}(d\alpha)^j\wedge(d\beta)^k\right) \tag{A.17}$$

on the set where uv^* and uw^* are non-zero.

Actually, to obtain the usual formula take the bidegree n, $n-1$ part of this formula, or equivalently, replace the exterior derivative d by $\overline{\partial}$ wherever d occurs in the formula.

Proof. It remains to show that

$$(\beta - \alpha) \wedge (d\alpha)^j \wedge (d\beta)^k = 0, \quad \text{if } j + k = n - 1. \tag{A.18}$$

The **complex euler vector field**

$$\theta \equiv u\frac{\partial}{\partial u}$$

has the property that

$$\theta \llcorner \alpha = \theta \llcorner \beta = 1, \quad \text{and} \quad \theta \llcorner d\alpha = \theta \llcorner d\beta = 0.$$

Now

$$\alpha \wedge \beta \wedge (d\alpha)^j \wedge (d\beta)^k = 0$$

since it is of degree $n+1$ in $du_1, \ldots, du_n$. Contracting this equation with the euler vector field θ yields (A.18).

Alternatively, note that

$$\theta \llcorner (d\alpha)^n = 0 \qquad \text{implies } (d\alpha)^n = 0$$

since $(d\alpha)^n$ is of top degree n in du and apply (1.15′). $\square$

This Lemma A.14 can be used to give an alternate proof that the notion of divisor of a section is independent of the choice of frame, at least when the bundle is complex. First we repeat some definitions in slightly altered form to suit the complex bundle case so that what follows may be considered as an independent second approach to divisors.

Suppose μ is a smooth section of a complex vector bundle F of rank n.

Definition A.19. The section μ is said to be **atomic** if, for each local frame f, the vector valued function u, defined by $\mu \equiv uf$, is an atomic function.

Remark A.20. One can easily show that if u, defined by $\mu = uf$, is atomic then $\tilde{u}$ defined by $\mu = \tilde{u}\tilde{f}$ under a change of frame $f = g\tilde{f}$ is also atomic.

Definition A.21. Suppose that μ is an atomic section of a complex vector bundle F. The **zero divisor**, or **zero current** of μ, denoted by $\operatorname{Div}(\mu)$, is locally defined to be the zero divisor of the vector valued function u determined by a local frame f, i.e., $\operatorname{Div}(\mu) \equiv dB(u)$.

Because of Remark A.7, concerning the real and imaginary parts of the Bochner-Martinelli potential, this notion of divisor is the same as that in Definition A.21, so that the independence proof in [HS] is applicable.

Proposition A.22. *Suppose μ is an atomic section. The zero divisor of μ is independent of the choice of local frame.*

A lemma is needed before proving this Proposition.

Lemma A.23. *Suppose u is an atomic function and that h, g are $GL_n(\mathbf{C})$ valued smooth functions with h positive definite. Then each of the three potentials*

i) $\quad B(u) = \left(\frac{1}{2\pi i}\right)^n \alpha(d\alpha)^{n-1} \quad \text{where } \alpha \equiv \dfrac{du\bar{u}^t}{|u|^2}$

ii) $\quad B(u,u^*) = \left(\frac{1}{2\pi i}\right)^n \beta(d\beta)^{n-1} \quad \text{where } \beta \equiv \dfrac{duh\bar{u}^t}{uh\bar{u}^t} = \dfrac{duu^*}{uu^*} \text{ and } u^* \equiv h\bar{u}^t$

iii) $\quad B(v) = \left(\frac{1}{2\pi i}\right)^n \gamma(d\gamma)^{n-1} \quad \text{where } \gamma \equiv \dfrac{dv\bar{v}^t}{|v|^2} = \dfrac{d(ug)\bar{g}^t\bar{u}^t}{ug\bar{g}^t\bar{u}^t} \text{ and } v \equiv ug,$

belongs to L^1_{loc} and has the same exterior derivative, namely, $\text{Div}(u)$, *the divisor of u.*

Proof of Proposition A.22. Suppose f is a local frame and $f = g\tilde{f}$ is a change of frame. Then $\mu = uf$ defines u and $\mu = v\tilde{f}$ defines $v = ug$. Lemma A.23 says that $B(u)$ and $B(v)$ have the same exterior derivative. $\square$

Proof of Lemma. First note that, since u is atomic,

$$\alpha \wedge \beta \sum_{j+k=n-2} (d\alpha)^j (d\beta)^k \qquad \in L^1_{\text{loc}}(X),$$

$$\alpha(d\alpha)^{n-1} \in L^1_{\text{loc}}(X), \quad \text{and} \quad \beta(d\beta)^{n-1} \in L^1_{\text{loc}}(X).$$

To show that $\alpha(d\alpha)^{n-1}$ and $\beta(d\beta)^{n-1}$ have the same exterior derivative it suffices to verify that

$$\alpha(d\alpha)^{n-1} - \beta(d\beta)^{n-1} = d\left(\alpha\beta \sum_{j+k=n-2} (d\alpha)^j (d\beta)^k\right) \tag{A.24}$$

holds as an equation of currents on the entire manifold X. Of course, outside the set $u = 0$, this is just equation (A.17).

Choose an approximate one, say χ. Since u is atomic, the Lebesgue dominated convergence theorem implies that

$$\lim_{s\to 0} \chi\left(\tfrac{|u|^2}{s^2}\right)^n \alpha(d\alpha)^{n-1} = \alpha(d\alpha)^{n-1} \quad \text{in } L^1_{\text{loc}}(X)$$

$$\lim_{s\to 0} \chi\left(\tfrac{|u|^2}{s^2}\right)^n \beta(d\beta)^{n-1} = \beta(d\beta)^{n-1} \quad \text{in } L^1_{\text{loc}}(X)$$

$$\lim_{s\to 0} \chi\left(\tfrac{|u|^2}{s^2}\right)^n \alpha\beta \sum_{j+k=n-2} (d\alpha)^j(d\beta)^k \;=\; \alpha\beta \sum_{j+k=n-2} (d\alpha)^j(d\beta)^k \quad \text{in } L^1_{\text{loc}}(X).$$

Therefore, if we can show that

$$\text{(A.25)} \qquad \lim_{s\to 0} d\left(\chi\left(\tfrac{|u|^2}{s^2}\right)\right) \alpha\beta \sum_{j+k=n-2} (d\alpha)^j(d\beta)^k = 0 \quad \text{in } L^1_{\text{loc}}(X),$$

then (A.24) is valid. However, the equation $d\left(\chi\left(\tfrac{|u|^2}{s^2}\right)\right) = \chi'\left(\tfrac{|u|^2}{s^2}\right)\tfrac{|u|^2}{s^2}\tfrac{d|u|^2}{|u|^2}$, implies that the limit (A.25) is zero a.e. (since $\lim_{t\to\infty} t\chi'(t) = 0$), as well as implying that the convergence is dominated.

To show that

$$\beta(d\beta)^{n-1} \in L^1_{\text{loc}}(X) \quad \text{and} \quad \gamma(d\gamma)^{n-1} \in L^1_{\text{loc}}(X)$$

have the same exterior derivative we must use the full formula (A.15) with α replaced by γ. Exactly as in the above proof one can establish this as an equation of currents on the entire manifold X. Note that

$$\gamma \equiv \beta + \frac{udgg^t\bar{u}^t}{uh\bar{u}^t} \quad \text{if } h \equiv g\bar{g}^t.$$

In this case the error term

$$\sigma \equiv (\beta - \gamma) \sum_{j+k=n-1} (d\gamma)^j(d\beta)^k \in L^1_{\text{loc}}(X)$$

does not vanish. However,

$$d\left(\gamma(d\gamma)^{n-1}\right) - d\left(\beta(d\beta)^{n-1}\right) \;=\; d\sigma$$

so it suffices to prove that $d\sigma = 0$. Again, using an approximate one χ, and the fact that $\beta - \gamma = \frac{udg\bar{g}^t\bar{u}^t}{uh\bar{u}^t}$ one can show that

$$d\sigma \in L^1_{\text{loc}}(X).$$

However, outside the set of measure zero $\{u = 0\}$,

$$d\sigma = (d\gamma)^n - (d\beta)^n.$$

Finally, outside $u = 0$, both

$$(d\beta)^n = 0 \quad \text{and} \quad (d\gamma)^n = 0,$$

because, for example

$$v \cdot \frac{\partial}{\partial v} \llcorner (d\gamma)^n = 0,$$

where $v \equiv ug$. □

Alternate Expressions for Bochner-Martinelli Related to the Top Chern Form $\mathbf{c_n}$.

Recall that

$$\det(A\ ;\ B) \;=\; \frac{d}{dt}\det(tA + B)\,\Big|_{t=0} \;=\; n\det(A, B, \ldots, B)$$

where $\det(A_1, \ldots, A_n)$ is the completely polarized determinant.

Lemma A.26. *The Bochner-Martinelli kernel may be expressed in the form*

$$\text{(A.27)} \qquad B \;=\; \left(\tfrac{1}{2\pi i}\right)^n \frac{1}{n} \det\left(\frac{\bar{u}^t du}{|u|^2}\ ;\ \frac{d\bar{u}^t du}{|u|^2}\right).$$

Proof. The volume form may be expressed in terms of the matrix $d\bar{u}^t du$ by

$$\text{(A.28)} \qquad n!\lambda(u) \;=\; \det\left(-\frac{i}{2} d\bar{u}^t du\right),$$

and we have

$$\bar{u}\frac{\partial}{\partial \bar{u}} \llcorner \det\left(d\bar{u}^t du\right) \;=\; \det\left(\bar{u}^t du\ ;\ d\bar{u}^t du\right).$$

Hence, formula (A.27) for B is a consequence of the definition (A.1). □

In a similar fashion the expression

$$n!\lambda(u, v^*) \;=\; \det\left(-\tfrac{i}{2} dv^* du\right)$$

for $\lambda(u, v^*)$ yields

$$B(u, v^*) \;=\; \tfrac{1}{(2\pi i)^n} \frac{1}{n} \det\left(\frac{v^* du}{uv^*} \,;\, \frac{dv^* du}{uv^*}\right). \tag{A.29}$$

In particular, the **Bochner-Martinelli kernel based on the metric** $\langle\ ,\ \rangle_{\mathbf{F}}$ can be written as

$$B(u, u^*) \;=\; \tfrac{1}{(2\pi i)^n} \frac{1}{n} \det\left(\frac{u^* du}{|u|^2} \,;\, \frac{du^* du}{|u|^2}\right). \tag{A.30}$$

Compare this with Theorem 4.22 and its proof to see that $-B(u, u^*)$ is the transgression potential for c_n in the flat case.

IV. Real Vector Bundles

This chapter is the analogue of Chapter III for the case of real vector bundles with orientation. Given such a bundle $V \longrightarrow X$ and a choice of approximation mode, we associate to each orthogonal connection on V a canonically defined family of Thom forms $\boldsymbol{\tau}_{\mathbf{s}}$, $s > 0$ such that

$$\boldsymbol{\tau}_{\mathbf{s}} \big|_X = \begin{cases} \chi(D_V) & \text{when } \operatorname{rank}(V) \text{ is even} \\ 0 & \text{when } \operatorname{rank}(V) \text{ is odd,} \end{cases}$$

and

$$\lim_{s \to 0} \boldsymbol{\tau}_{\mathbf{s}} = [X].$$

As in Chapter III there is a family of L^1_{loc} transgressions $\mathbf{r}_{\mathbf{s}}$ such that $\boldsymbol{\tau}_{\mathbf{s}} - [X] = d\mathbf{r}_{\mathbf{s}}$ and $\mathbf{r}_{\mathbf{s}} \to 0$ as $s \to 0$. When rank(V) is even, the form $\boldsymbol{\tau}_{\mathbf{s}}$ is constructed via the Pfaffian. When rank(V) is odd the construction is more subtle.

As in Chapter III, our methods produce explicit formulas for universal Thom classes in the equivariant de Rham complex $\mathcal{E}^*_{SO_m}(\mathbf{R}^m)$. For each choice of an orthogonal connection on a bundle V as above, these classes map to our Thom forms under the equivariant Chern-Weil homomorphism. The formulas for these classes are simple and quite pretty.

An interesting fact concerning approximation modes emerges in this Chapter. For real orthogonal bundles, the "most natural" choice of approximate one is

$$\chi(t) = 1 - \frac{1}{\sqrt{1+t}}.$$

This mode has several nice features. Let μ be a section of V and suppose D_V is Riemannian. Then the family of pushforward connections $\overrightarrow{D}_s$ in this approximation mode is Riemannian for all s. Furthermore, we have $\overrightarrow{D}\frac{\mu}{|\mu|} \equiv 0$ i.e., $\frac{\mu}{|\mu|}$

is parallel in its singular pushforward connection. It is also this approximate one which allows us to make the link between Thom forms in even and odd dimensions. Finally, this approximate one is related to the Grassmann graph construction just as $\chi(t) = t/(1+t)$ is in the complex case.

In Section 5 we establish the Atiyah-Hirzebruch "Differential Riemann-Roch" Theorem canonically at the level of differential forms. In this formulation the $\widehat{A}$-form drops straightforwardly out of the calculations. The result is extended to singular subcomplexes with "spin normal bundle".

1. The Pfaffian and a Universal Thom Form.

Suppose $\pi : V \to X$ is an oriented real rank m vector bundle with metric compatible connection D_V, $\langle , \rangle$. The purpose of this section is to describe a method of constructing a family of Thom forms for V which only depends on the metric connection D_V, $\langle , \rangle$ (cf. [MQ]); and then to show that the pullback of this family of Thom forms by an atomic section has initial value the divisor of the section and terminal value the Chern-Euler form associated with the metric connection D_V, $\langle , \rangle$.

To begin we recall some definitions. For each local oriented orthonormal frame e for V the local gauge ω_V determined by $D_V e \equiv \omega_V e$ and the local curvature Ω_V, determined by $\Omega_V \equiv d\omega_V - \omega_V \wedge \omega_V$, are both skew matricies. Also note that

(1.1) $e^t \Omega_V e$ is a globally defined section of the bundle $\Lambda^2 V \otimes \Lambda^2 T^* X$ on X,

In fact, $e^t \Omega_V e$ is just the curvature operator $R_V \equiv D_V^2$ if the metric is used to identify $\Lambda^2 V$ with the skew part of End V.

Consider the even case $m = 2n$. Suppose A is a skew $2n \times 2n$ matrix. Let $e_1, \ldots, e_{2n}$ denote an **oriented** orthonormal (local) frame for V and let $\lambda \equiv e_1 \wedge \cdots \wedge e_{2n} \in \Lambda^{2n} V$ denote the unit volume element. Recall the definition of the **pfaffian** of A

$$\mathrm{Pf}(A)\lambda = \frac{1}{n!}\left(\tfrac{1}{2} e^t A e\right)^n . \tag{1.2}$$

Since V is oriented the unit volume form λ for V is globally defined, and hence $\mathrm{Pf}(A)$ is globally defined independently of the choice of oriented orthonormal frame e.

The **Chern-Euler form** of the orthogonal connection D_V will be denoted by $\chi(D_V))$. When $m = \operatorname{rank} V = 2n$, it is defined by

$$(1.3) \qquad \chi(D_V) \;=\; \mathrm{Pf}\left(\tfrac{-1}{2\pi}\Omega_V\right) \quad \text{or equivalently} \quad \chi(D_V)\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{-1}{4\pi}e^t\Omega_V e\right)^n.$$

For $m = \operatorname{rank} V = 2n-1$ we set $\chi(D_V) \equiv 0$.

Normalization is different in the real case. Here, we let $\widetilde{\mathrm{Pf}} \equiv \chi$, i.e., the normalized Pfaffian is defined to be the Euler polynomial.

A **Thom form** τ on the total space of V is conventionally defined to be any smooth form of degree m which is d-closed, has compact support in the fibers of V, and satisfies

$$(1.4) \qquad \pi_*\tau \;=\; \int_{\pi^{-1}} \tau = 1.$$

We find it convenient to relax the condition of compact support in the fibers of V and only require that

(1.5)

$$\text{Along each fiber } \pi^{-1}(x) \text{ the form } \tau \text{ belongs to } L^1(\pi^{-1}(x)) \text{ and } \int_{\pi^{-1}(x)} \tau = 1.$$

The Thom form τ restricted to the base $X \equiv Z \subset V$, is denoted by χ and called the **euler form for V determined by τ**.

Let s denote the homothety by $s \in \mathbf{R}$ in the fibers of V. Each Thom form τ determines a **family of Thom forms** by $(s > 0)$

$$(1.6) \qquad \tau_s \;=\; \left(\tfrac{1}{s}\right)^*(\tau)$$

This is a global geometrization of the notion of an approximate identity (or approximate point mass) in the fibers of V.

We will construct such a family of Thom forms τ_s whose associated Euler form is exactly the Chern-Euler form of D_V.

Now suppose that μ is a global section of V. The formula for the singular connection $\overrightarrow{D}$ in Section 2 of Chapter III becomes $\overrightarrow{D} = D_V - A$ with $A\nu =$

$\frac{\langle\nu,\mu\rangle}{|\mu|^2} D_V\mu$. This connection is not metric compatible. The skew symmetrization $D_V - \frac{1}{2}(A - A^t)$ is metric compatible. However, we will consider the singular connection $D_V - (A - A^t)$ which has the property that it is both metric compatible and has a parallel section, namely $\frac{\mu}{|\mu|}$. Specifically, we define the **Riemannian singular pushforward connection** on V by setting

$$\overrightarrow{D}\nu \equiv D_V\nu - \frac{\langle\nu,\mu\rangle}{|\mu|^2} D_V\mu + \frac{\langle D_V\mu,\nu\rangle}{|\mu|^2}\mu. \tag{1.7}$$

One easily checks that $\overrightarrow{D}(\mu/|\mu|) = 0$ outside the zero set $Z = \{x : \mu(x) = 0\}$. Consequently, the Chern-Euler form satisfies

$$\chi((\overrightarrow{D})) \equiv 0 \quad \text{outside} \quad Z. \tag{1.8}$$

If e is a local orthonormal frame for V, then the singular gauge ω, defined outside Z, is given by

$$\omega \equiv \omega_V - \frac{u^t Du}{|u|^2} + \frac{Du^t u}{|u|^2}, \tag{1.9}$$

where $\mu \equiv ue$ defines the $\mathbf{R}^m$-valued function $u = (u_1, \ldots, u_m)$, u^t denotes the transpose, and

$$Du \equiv du + u\omega_V \quad \text{and} \quad Du^t = du^t - \omega_V u^t.$$

Let $\chi(t)$ be any approximate one (see Definition I.4.1) and let $\chi_s \equiv \chi\left(\frac{|u|^2}{s^2}\right)$. Then for $0 < s \leq +\infty$,

$$\overrightarrow{D}_s\nu = D_V\nu - \chi_s \frac{\langle\nu,\mu\rangle}{|\mu|^2} D_V\mu + \chi_s\frac{\langle D_V\nu,\mu\rangle}{|\mu|^2}\mu, \tag{1.10}$$

with gauge

$$\overrightarrow{\omega}_s = \omega_V - \chi_s\left(\frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2}\right), \tag{1.11}$$

defines a family of connections on V which are smooth across the zero set Z. The family is also smooth across $s = \infty$ with $D_\infty = D_V$.

The standard transgression formula applied to this smooth family now says that

$$\chi(D_V) - \chi(\overrightarrow{D}_s) = d\sigma_s \tag{1.12}$$

where

$$(1.13) \qquad \sigma_s \equiv (-1)^n \int_s^\infty \mathrm{Pf}\left(\tfrac{1}{2\pi}\tfrac{\partial \vec{\omega}_s}{\partial s}\,;\,\tfrac{1}{2\pi}\vec{\Omega}_s\right) ds.$$

Let $\mathbf{V}$ denote the pullback of the bundle V over itself. All of the formulas above remain valid with u now interpreted as a linear fibre coordinate on the total space of V.

Definition 1.14. ($m = 2n$ even). The forms $\tau_{\mathbf{s}} \equiv \chi(\vec{\mathbf{D}}_{\mathbf{s}})$ on the total space of V defined by

$$(1.15) \qquad \tau_{\mathbf{s}}\lambda = \tfrac{1}{n!}\left(-\tfrac{1}{4\pi}e^t\vec{\Omega}_s e\right)^n$$

will be referred to as the **family of Thom forms associated with the metric connection** D_V, $\langle,\rangle$ on V.

Theorem 1.16. The Thom Form. *Consider the total space V as the base manifold of the bundle $\mathbf{V}$ obtained by pulling V back over itself. The family, $\tau_{\mathbf{s}}$, of Thom forms associated with the metric connection D_V, $\langle,\rangle$ on V is given explicitly by:*

$$(1.17) \qquad \tau_{\mathbf{s}}\lambda = \tfrac{1}{n!}\left(\tfrac{-1}{4\pi}\right)^n(1-\chi_s)\left(e^t\Omega_V e - 2\chi_s\left(1-\tfrac{\chi_s}{2}\right)\frac{(Due)^2}{|u|^2}\right)^n +$$

$$\tfrac{2}{(n-1)!}\left(\tfrac{-1}{4\pi}\right)^n\left(\chi_s(1-\chi_s)\left(1-\tfrac{\chi_s}{2}\right)-\chi_s'\tfrac{|u|^2}{s^2}\right)\tfrac{d|u|^2}{|u|^2}\tfrac{(ue)(Due)}{|u|^2}\left(e^t\Omega_V e-2\chi_s\left(1-\tfrac{\chi_s}{2}\right)\tfrac{(Due)^2}{|u|^2}\right)^{n-1}.$$

Note that $\tau_{\mathbf{s}}$ is compactly supported in the fibers of V if χ is chosen to be a compactly supported approximate one, i.e., if $\chi(t) \equiv 1$ for $t \geq t_0$.

The solution to the differential equation

$$(1.18) \qquad x(1-x)\left(1-\tfrac{x}{2}\right) = x' \cdot t$$

given by

$$(1.19) \qquad \chi(t) \equiv 1 - \frac{1}{\sqrt{t+1}} = \frac{t}{t+1+\sqrt{t+1}}$$

is an approximate one which will be referred to as the **real algebraic approximation** mode. This choice has three different motivations. The first motivation is formula (1.17). The second and third motivations will be discussed later.

Corollary 1.20. Real Algebraic Approximation Mode. *If χ is given by (1.19), then the Thom form is given by*

$$(1.21)\qquad \boldsymbol{\tau}_{\mathbf{s}}\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{-1}{4\pi}\right)^n \frac{s}{\sqrt{|u|^2+s^2}}\left(e^t\Omega_V e - \frac{(Due)^2}{|u|^2+s^2}\right)^n .$$

The family of transgression forms $\boldsymbol{\sigma}_{\mathbf{s}}$ defined by (1.13) in the universal case will be called an **approximate spherical kernel** and the limit $\boldsymbol{\sigma} \equiv \lim_{s\to 0} \boldsymbol{\sigma}_{\mathbf{s}}$ (which will be shown to exist) will be called **the spherical kernel.**

Theorem 1.22. ($m = 2n$ even). *Each form $\boldsymbol{\tau}_{\mathbf{s}}$ has the properties of a Thom form, in fact*

$$(1.23)\qquad \boldsymbol{\tau}_{\mathbf{s}} \text{ is } d\text{-closed and } \int_{\pi^{-1}} \boldsymbol{\tau}_{\mathbf{s}} = 1,$$

and the Euler form associated with $\boldsymbol{\tau}_{\mathbf{s}}$ is the Chern-Euler form $\chi(D_V)$ of D_V. The approximate spherical forms $\boldsymbol{\sigma}_{\mathbf{s}}$ converge in $L^1_{\mathrm{loc}}(V)$ to $\boldsymbol{\sigma}$ and

$$(1.24)\qquad d\boldsymbol{\sigma} \;=\; \chi(D_V) - [X] \qquad \text{on } V$$

is the limiting form of the equation (as $s \to 0$)

$$d\boldsymbol{\sigma}_{\mathbf{s}} \;=\; \chi(D_V) - \boldsymbol{\tau}_{\mathbf{s}} \qquad \text{on } V.$$

In particular,

$$d(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathbf{s}}) \;=\; \boldsymbol{\tau}_{\mathbf{s}} - [X] \quad \text{and} \quad \lim_{s\to 0} \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathbf{s}} \;=\; 0 \;\text{ in }\; L^1_{\mathrm{loc}}(V)$$

so that

$$(1.25)\qquad \lim_{s\to 0} \boldsymbol{\tau}_{\mathbf{s}} \;=\; [X].$$

Before proving the previous two Theorems we explicitly compute the spherical kernel.

Lemma 1.26. ($m = 2n$ even). *The transgression form $\boldsymbol{\sigma}_s$ is given by*

$$(1.27)\quad \boldsymbol{\sigma}_s\lambda \;=\; \tfrac{2}{(n-1)!}\left(\tfrac{-1}{4\pi}\right)^n \tfrac{(ue)(Due)}{|u|^2}\int_0^{\chi_s}\left(e^t\Omega_V e - 2x\left(1-\tfrac{x}{2}\right)\frac{(Due)^2}{|u|^2}\right)^{n-1} dx.$$

Proof. Since $\mathrm{Pf}(\vec{\Omega}_s)\lambda = \frac{1}{2^n n!}\,(e^t\vec{\Omega}_s e)^n$,

$$(1.28)\qquad \mathrm{Pf}\left(\tfrac{\partial\vec{\omega}_s}{\partial s}\;;\;\vec{\Omega}_s\right)\lambda \;=\; \tfrac{1}{2^n(n-1)!}\left(e^t\tfrac{\partial\vec{\omega}_s}{\partial s}e\right)(e^t\vec{\Omega}_s e)^{n-1}.$$

Therefore, the transgression form $\boldsymbol{\sigma}_s$ given by (1.13) equals

$$(1.29)\qquad \boldsymbol{\sigma}_s\lambda \;=\; (-1)^n\,\tfrac{1}{2^n\cdot 2^n(n-1)!\pi^n}\int_s^\infty (e^t\tfrac{\partial\vec{\omega}_s}{\partial s}e)(e^t\vec{\Omega}_s e)^{n-1}ds.$$

Now the formula (1.11) for $\vec{\omega}_s$ implies

$$(1.30)\qquad \frac{\partial\vec{\omega}_s}{\partial s} \;=\; \left(-\frac{u^tDu}{|u|^2}+\frac{Du^tu}{|u|^2}\right)\frac{\partial\chi}{\partial s}$$

so that

$$(1.31)\qquad e^t\,\frac{\partial\vec{\omega}_s}{\partial s}e \;=\; -2\,\frac{(ue)(Due)}{|u|^2}\,\frac{\partial\chi}{\partial s}.$$

Consequently, we need only compute $e^t\vec{\Omega}_s e$ modulo the 1-vector ue. Direct calculation yields

$$(1.32)\qquad e^td\vec{\omega}_s e \;=\; e^td\omega_V e - 2\chi_s\frac{(due)(Due)}{|u|^2}\qquad \mathrm{mod}\ ue$$

and

$$(1.33)\quad e^t\vec{\omega}_s\wedge\vec{\omega}_s e \;=\; e^t\omega_V\wedge\omega_V e - \frac{2\chi u\omega_V e(Due)}{|u|^2} + \frac{\chi^2}{|u|^2}\,(Due)^2\qquad \mathrm{mod}\ ue,$$

so that

$$(1.34)\qquad e^t\vec{\Omega}_s e \;=\; e^t\Omega_V e - 2\chi\left(1-\tfrac{\chi}{2}\right))\;\frac{(Due)^2}{|u|^2}\qquad \mathrm{mod}\ ue.$$

Inserting (1.31) and (1.34) into the equation (1.29) for $\boldsymbol{\sigma}_s$ and then using the substitution $x \equiv \chi\left(\frac{|u|^2}{s^2}\right)$ in the integral (1.29) yields the Lemma. □

Since $\int_0^1 \left(2x\left(1-\frac{x}{2}\right)\right)^p dx = 2^{2p}\frac{(p!)^2}{(2p+1)!}$, Lemma 1.26 has the following corollary.

Corollary 1.35. The Spherical Kernel. *$\lim\limits_{s\to 0} \boldsymbol{\sigma}_{\mathbf{s}} = \boldsymbol{\sigma}$ converges in $L^1_{\mathrm{loc}}(V)$ with the spherical kernel $\boldsymbol{\sigma}$ given explicitly by*

(1.36)
$$\boldsymbol{\sigma}\lambda = \frac{1}{\pi^n}\sum_{p=0}^{n-1}(-1)^{n-p}\frac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-1}}\frac{(ue)(Due)^{2p+1}}{|u|^{2p+2}}\left(e^t\Omega_V e\right)^{n-p-1}.$$

Thus the part of $\boldsymbol{\sigma}$ of top degree $2n-1$ in the 1-forms $du_1, \dots, du_{2n}$ is

$$\boldsymbol{\sigma}_{2n-1} = \mathrm{vol}(S^{2n-1})^{-1}\theta(u) \tag{1.37}$$

where

$$\theta(u) \equiv \sum_{k=1}^{2n}(-1)^{k-1}\frac{u_k du_1\wedge\cdots\wedge\widehat{du_k}\wedge\cdots\wedge du_{2n}}{|u|^{2n}}$$

denotes the **solid angle kernel** *on $\mathbf{R}^{2n}$.*

Moreover, the potential $\sigma - \sigma_s$ (as well as τ_s) is compactly supported in the fibers of V if χ is chosen to be a compactly supported approximate one.

Proof. Note that

$$\mathrm{vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$$

and that

$$\frac{(ue)(due)^{2n-1}}{|u|^{2n}} = (2n-1)!\theta(u)\lambda.$$

Since $\sigma - \sigma_s$ is an integral from $1-\chi_s$ to 1 it follows that $\sigma - \sigma_s$ is compactly supported in the fibers of V. $\square$

The homogeneous form $\boldsymbol{\sigma}$ was discovered by Chern [C1, 2]. Note that $\boldsymbol{\sigma}$ is independent of the choice of approximate one χ.

Let $\rho_\epsilon : S_\epsilon(V) \to X$ denote the ϵ-sphere bundle contained in the vector bundle $\pi : V \to X$. Corollary 1.35 implies the first half of the next result

Lemma 1.38.

$$(\rho_\epsilon)_*(\boldsymbol{\sigma}) = \int_{\rho_\epsilon^{-1}} \boldsymbol{\sigma} = -1 \tag{1.39}$$

$$\pi_*(\boldsymbol{\tau}_{\mathbf{s}}) = \int_{\pi^{-1}} \boldsymbol{\tau}_{\mathbf{s}} = 1 \tag{1.40}$$

Proof. The second equation follows from the first as in Remark 2.24 of Chapter III. □

Proof of Theorem 1.22. The previous Lemma shows that $\boldsymbol{\tau}_{\mathbf{s}}$ has the properties of a Thom form. Since $\overrightarrow{D}_s$ restricts to $X \subset V$ to be the connection D_V, the form $\boldsymbol{\tau}_{\mathbf{s}} \equiv \chi(\overrightarrow{D}_s)$ restricts to $X \subset V$ to be $\chi(D_V)$.

As noted in Corollary 1.35, the explicit formula (1.20) for $\boldsymbol{\sigma}_{\mathbf{s}}$ implies that $\boldsymbol{\sigma}_{\mathbf{s}}$ converges in $L^1_{\mathrm{loc}}(V)$ to $\boldsymbol{\sigma}$. The remainder of the proof of Theorem 1.22 is similar to previous proofs in Chapter III and is omitted. □

In preparation for proving Theorem 1.16 we compute the curvature of the connection $\overrightarrow{D}_s$.

Lemma 1.41. *The curvature $\overrightarrow{\Omega}_s$ of the connection $\overrightarrow{D}_s$ is given by*

$$\begin{aligned} \overrightarrow{\Omega}_s &= \Omega_V - \chi_s\left(\tfrac{u^t u}{|u|^2}\,\Omega_V + \Omega_V \tfrac{u^t u}{|u|^2}\right) + 2\chi_s\left(1 - \tfrac{\chi_s}{2}\right)\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2}\right)^2 \\ &\quad - \chi_s' \tfrac{|u|^2}{s^2}\,\tfrac{d|u|^2}{|u|^2}\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2}\right). \end{aligned} \tag{1.42}$$

Proof. The gauge $\overrightarrow{\omega}_s$ of $\overrightarrow{D}_s$ is given by

$$\overrightarrow{\omega}_s = \omega_V - \chi\boldsymbol{\sigma} \quad \text{where} \quad \sigma \equiv \tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2}. \tag{1.43}$$

Therefore,

$$\overrightarrow{\Omega}_s = \Omega_V - \chi_s(d\boldsymbol{\sigma} - [\omega_V, \boldsymbol{\sigma}]) - \chi_s^2\boldsymbol{\sigma}^2 - d\chi_s\boldsymbol{\sigma}. \tag{1.44}$$

Now

$$(1.45) \qquad d\chi_s \;=\; \chi_s' \tfrac{|u|^2}{s^2}\, \tfrac{d|u|^2}{|u|^2}.$$

To complete the proof of (1.42) we must show that

$$(1.46) \qquad d\boldsymbol{\sigma} - [\omega_V, \boldsymbol{\sigma}] \;=\; \tfrac{u^t u \Omega_V}{|u|^2} + \tfrac{\Omega_V u^t u}{|u|^2} - 2\boldsymbol{\sigma}^2.$$

Since $dDu = u\Omega_V + Du\omega_V$ and $dDu^t = -\Omega_V u^t + \omega_V Du^t$, we have

$$\begin{aligned} d\boldsymbol{\sigma} - \omega_V\boldsymbol{\sigma} - \boldsymbol{\sigma}\omega_V \;&= \\ = \; -\tfrac{d|u|^2}{|u|^2}\,\boldsymbol{\sigma} + \tfrac{du^t Du}{|u|^2} + \tfrac{Du^t du}{|u|^2} + \tfrac{u^t(u\Omega_V + Du\omega_V)}{|u|^2} - \tfrac{(-\Omega_V u^t + \omega_V Du^t)u}{|u|^2} \\ - \tfrac{\omega_V u^t Du}{|u|^2} + \tfrac{\omega_V Du^t u}{|u|^2} - \tfrac{u^t Du\omega_V}{|u|^2} + \tfrac{Du^t u\omega_V}{|u|^2} \\ = \; \tfrac{u^t u\Omega_V}{|u|^2} + \tfrac{\Omega_V u^t u}{|u|^2} - \tfrac{d|u|^2}{|u|^2}\,\boldsymbol{\sigma} + 2\tfrac{Du^t Du}{|u|^2} \;=\; \tfrac{u^t u\Omega_V}{|u|^2} + \tfrac{\Omega_V u^t u}{|u|^2} - 2\boldsymbol{\sigma}^2. \end{aligned}$$

To verify the last equality note that $d|u|^2 = duu^t = udu^t = Duu^t = uDu^t$ since $u\omega_V u^t = 0$, and use the fact that $DuDu^t = 0$. $\square$

Corollary 1.47.

$$(1.48) \qquad \begin{aligned} e^t \overrightarrow{\Omega}_s e \;&= e^t\Omega_V e - 2\chi_s \tfrac{(ue)(u\Omega_V e)}{|u|^2} - 2\chi_s\left(1 - \tfrac{\chi_s}{2}\right)\tfrac{(Due)^2}{|u|^2} \\ &+ 2\left(\chi_s\left(1 - \tfrac{\chi_s}{2}\right) - \chi_s' \tfrac{|u|^2}{s^2}\right)\tfrac{d|u|^2}{|u|^2}\,\tfrac{(ue)(Due)}{|u|^2}. \end{aligned}$$

Proof. Use (1.42) and

$$e^t(u^t Du - Du^t u)e \;=\; 2(ue)(Due),$$

and

$$e^t(u^t Du - Du^t u)^2 e \;=\; d|u|^2(ue)(Due) - |u|^2(Due)^2. \quad \square$$

Proof of Theorem 1.16. Let

$$(1.49) \qquad A \;\equiv\; e^t\Omega_V e - 2\chi_s\left(1 - \tfrac{\chi_s}{2}\right)\tfrac{(Due)^2}{|u|^2}.$$

The previous corollary implies that

$$e^t \overrightarrow{\Omega}_s e \;=\; A \qquad \mathrm{mod}\;\; ue. \tag{1.50}$$

Now

$$\begin{aligned}(e^t \overrightarrow{\Omega}_s e)^n &= \tfrac{ue}{|u|^2}(ue \llcorner (e^t \overrightarrow{\Omega}_s e)^n) = n\tfrac{ue}{|u|^2}(ue \llcorner e^t \overrightarrow{\Omega}_s e)(e^t \overrightarrow{\Omega}_s e)^{n-1} \\ &= n\tfrac{ue}{|u|^2}(ue \llcorner e^t \overrightarrow{\Omega}_s e)A^{n-1}.\end{aligned}$$

The proof is completed by verifying that

$$\tfrac{ue}{|u|^2}(ue \llcorner e^t \overrightarrow{\Omega}_s e) = \tfrac{ue}{|u|^2}(ue \llcorner A) + 2\left(\chi_s(1-\chi_s)\left(1-\tfrac{\chi_s}{2}\right)-\chi_s' \tfrac{|u|^2}{s^2}\right)\tfrac{d|u|^2}{|u|^2}\tfrac{(ue)(Due)}{|u|^2}. \qquad \square$$

A section μ of a real vector bundle is said to be atomic if, for each local oriented frame e, the $\mathbf{R}^m$ valued function $u(x)$ defined by $\mu \equiv ue$ is atomic, i.e., if $\frac{du^I}{|u|^{|I|}} \in L^1_{\mathrm{loc}}$ for each I with $|I| < m$. The divisor of μ, denoted $\mathrm{Div}(\mu)$, is defined to be the exterior derivative of the L^1_{loc} potential $c_m u^*(\theta)$, where $\theta \equiv \sum_{j=1}^{m}(-1)^{j-1}\frac{du_1\wedge\cdots\wedge\widehat{du_j}\wedge\cdots\wedge du_m}{|u|^m}$ is the solid angle kernel and c_m^{-1} is the volume of the unit sphere in $\mathbf{R}^m$. In [HS] it is shown that this is a meaningful notion of divisor, i.e., $\mathrm{Div}(\mu)$ is independent of the choice of oriented frame e. Note that the frame e is not required to be orthonormal (cf. Remark 1.52).

Theorem 1.51. **($m = 2n$ even).** *Suppose μ is an atomic section of a real vector bundle V of rank $m = 2n$. For each choice of metric and metric-compatible connection D_V the associated Chern-Euler current $\chi((\overrightarrow{D})) \equiv \lim_{s\to 0}\chi(\overrightarrow{D}_s)$ exists and equals $\mathrm{Div}(\mu)$. For each s, $\chi(\overrightarrow{D}_s) = \mu^*(\boldsymbol{\tau}_{\mathbf{s}})$ is the pullback of the Thom form by the section μ. Moreover, the transgression currents $\mu^*(\boldsymbol{\sigma}_{\mathbf{s}})$ converge in $L^1_{\mathrm{loc}}(X)$ to $\mu^*(\boldsymbol{\sigma})$ and the equation*

$$\chi(D_V) - \mathrm{Div}(\mu) \;=\; d\mu^*(\boldsymbol{\sigma}) \qquad \text{on } X \tag{1.52}$$

is the limiting form of the equation

$$\chi(D_V) - \chi(\overrightarrow{D}_s) \;=\; d\mu^*(\boldsymbol{\sigma}_{\mathbf{s}}) \qquad \text{on } X. \tag{1.53}$$

The proof of this Theorem is similar to earlier proofs and hence is omitted.

Remark 1.54. Divisors and Chern-Euler Currents. Let μ be an atomic section of a real oriented bundle V with metric $\langle\ ,\ \rangle_V$ and metric-compatible connection D_V. Then (without using the results of the paper [HS]) one can use the formulas above to show that with respect to an oriented orthonormal frame the Chern-Euler current exists and equals

$$\lim_{s\to 0}\chi(\overrightarrow{D}_s) \;=\; c_m d(u^*(\theta)). \tag{1.55}$$

However, given another pair D_V', $\langle\ ,\ \rangle_V'$ it is not automatic that the Chern-Euler current of the family D_s' is the same.

Pick $g\in\Gamma(\mathrm{Hom}^+(V,V))$ a global section so that $\langle\ ,\ \rangle_V \equiv \langle g(\ ),g(\ )\rangle_V'$. Then $\widetilde{D}_V \equiv g^{-1}D_V' g$ is compatible with the original metric $\langle\ ,\ \rangle_V$ and the gauge $\tilde{\omega}_V$ for $\widetilde{D}_V$ in the frame e is exactly the same as the gauge ω_V' for D_V' in the frame $e'\equiv ge$. Consequently, $\chi(D_V') = \chi(\widetilde{D}_V)$. However, the family $g^{-1}D_s' g$ is not the same as the family $\widetilde{D}_s$ except at $s=+\infty$. The construction of the family of connections depends not only on the metric (which is the same for D_V and $\widetilde{D}_V$) but also on the section μ. Consider the section $\nu\equiv g^{-1}\mu$ as well as μ. One can check that the family for $\widetilde{D}_V$, based on ν and $\langle\ ,\ \rangle_V$, is the same as the family $g^{-1}D_s' g$, where D_s' is the family for D_V', based on μ and $\langle\ ,\ \rangle_V'$. Hence $\lim_{s\to 0}\chi(\overrightarrow{D}_s) = \lim_{s\to 0}\chi(D_s')$ if and only if

$$d(u^*(\theta)) \;=\; d(v^*(\theta)) \tag{1.56}$$

where $v\equiv uG$, and $g^{-1}e\equiv Gf$ defines the matrix G. This equality is precisely the result proved in Section 1 in [HS].

Remark 1.57. Real Rank Two Bundles/Complex Line Bundles. Let $\omega_V \equiv \begin{pmatrix} 0 & -\rho \\ \rho & 0\end{pmatrix}$. Then $Du = du + u\omega_V = (du_1 + u_2\rho, du_2 - u_1\rho)$, and hence

$$\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2} \;=\; \begin{pmatrix} 0 & \theta-\rho \\ \rho-\theta & 0\end{pmatrix}.$$

Therefore, if $\overrightarrow{\omega}_s = \begin{pmatrix} 0 & -\rho_s \\ \rho_s & 0\end{pmatrix}$ defines ρ_s, we have that

$$\rho_s \;=\; (1-\chi_s)\rho + \chi_s\theta \tag{1.58}$$

since $\overrightarrow{\omega}_s = \omega_v - \chi_s\left(\frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2}\right)$. Let $\overrightarrow{\Omega}_s = \begin{pmatrix} 0 & -\kappa_s \\ \kappa_s & 0 \end{pmatrix}$ and $\Omega_V = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}$ define κ_s and κ, i.e., $\kappa_s \equiv d\rho_s$ and $\kappa = d\rho$. Note that $\chi(D_V) = \frac{1}{2\pi}\kappa$. Now (1.58) implies that the Thom form $\tau_s = \frac{1}{2\pi}\kappa_s$ is given by

$$\tau_s = (1-\chi_s)\frac{1}{2\pi}\kappa - \chi_s' \frac{|u|^2}{s^2}\frac{1}{2\pi}\frac{d|u|^2}{|u|^2}(\rho - \theta). \tag{1.59}$$

The equation (1.58) can be rewritten in terms of the spherical kernels

$$\sigma_s = \frac{1}{2\pi}\chi_s(\rho - \theta) \quad \text{and} \quad \sigma = \frac{1}{2\pi}(\rho - \theta), \tag{1.60}$$

which are globally defined. Namely

$$\rho_s = \rho - \chi_s \sigma. \tag{1.58$'$}$$

The equation (1.59) becomes

$$\tau_s = (1-\chi_s)\chi(D_V) - d\chi_s\sigma \tag{1.59$'$}$$

with all entries globally defined. Of course, each oriented real rank 2 bundle V with an inner product is naturally equipped with a complex line bundle structure J. Note that $Je_1 = e_2$, $Je_2 = -e_1$ or $J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that the above formulas can be reinterpreted as exactly those occuring in the complex line bundle case.

Now we present the second motivation for the real algebraic approximation mode $\chi(t) = 1 - \frac{1}{\sqrt{t+1}}$. Note that in this case

$$\chi_s = \frac{|u|^2}{|u|^2 + s^2 + s\sqrt{|u|^2+s^2}}$$

so that the connection $\overrightarrow{D}_s$ defined by (1.10) is given by

$$\overrightarrow{D}_s = D_V - \frac{D\alpha\alpha^* - \alpha D\alpha^*}{|\alpha|^2 + s^2 + s\sqrt{|\alpha|^2+s^2}} \tag{1.61}$$

where $\underline{\mathbf{R}} \xrightarrow{\alpha} V$ is defined by $\alpha(1) = \mu$ and $\underline{\mathbf{R}} \xleftarrow{\alpha^*} V$ is the adjoint map. Recall that $\gamma(v) \equiv (-\alpha^*(v), sv)$ defines a bundle isomorphism $\gamma : \mathbf{V} \to U_s^\perp$ over the set $V \equiv \mathrm{Hom}(\underline{\mathbf{R}}, V)$. Here U_s is the homothety of the universal line bundle over $\mathbf{P}(\underline{\mathbf{R}} \oplus V)$ defined by (I.8.8) and $U_s^\perp$ is the orthogonal hyperplane bundle. Let

$$g \equiv (\alpha\alpha^* + s^2)^{\frac{1}{2}}. \tag{1.62}$$

In the next theorem and its proof let U, $U^\perp$ abbreviate U_s, $U_s^\perp$.

Theorem 1.63. Relation to the Grassmann Graph Construction. *Over the open chart* $V = \mathrm{Hom}(\underline{\mathbf{R}}, V) \subset \mathbf{P}(\underline{\mathbf{R}} \oplus V)$ *the bundle map* $\gamma \circ g^{-1} : \mathbf{V} \to U^{\perp}$ *is a bundle isometry which pulls back the projected connection* $D_{U^{\perp}}$ *to the connection* $\overrightarrow{D}_s$ *given by* (1.61). *That is*

$$\overrightarrow{D}_s \;=\; g\gamma^{-1}(D_{U^{\perp}})\gamma g^{-1}. \tag{1.64}$$

Proof. First note that

$$\gamma g^{-1}(v) \;=\; (-\alpha^*(\alpha\alpha^* + s^2)^{-\frac{1}{2}}v,\; -s(\alpha\alpha^* + s^2)^{-\frac{1}{2}}v) \in \underline{\mathbf{R}} \oplus V$$

has norm squared equal to $|v|^2$ so that γg^{-1} is an isometry. Recall from Chapter I that

$$g^2\gamma^{-1}D_{U^{\perp}}\gamma g^{-2} \;=\; D_V - \tfrac{D\alpha\alpha^*}{|\alpha|^2+s^2}.$$

Therefore, since $\frac{D\alpha\alpha^*}{|\alpha|^2+s^2} = D\alpha(\alpha^*\alpha + s^2)^{-1}\alpha^* = D\alpha\alpha^*(\alpha\alpha^* + s^2)^{-1} = D\alpha\alpha^* g^{-2}$,

$$g\gamma^{-1}D_{U^{\perp}}\gamma g^{-1} = g^{-1}D_V g - g^{-1}D\alpha\alpha^* g^{-1} \;=\; D_V + g^{-1}Dg - g^{-1}D\alpha\alpha^* g^{-1}. \tag{1.65}$$

Next we calculate that

$$\begin{aligned} g^{-1}Dg - g^{-1}D\alpha\alpha^* g^{-1} \;&=\; -\frac{D\alpha\alpha^* - \alpha D\alpha^*}{\sqrt{|\alpha|^2 + s^2}(\sqrt{|\alpha|^2 + s^2} + s)} \\ &\quad + \frac{1}{2}\frac{\alpha(a^*D\alpha - D\alpha^*\alpha)\alpha^*}{(|\alpha|^2 + s^2)(\sqrt{|\alpha|^2 + s^2} + s)^2} \end{aligned} \tag{1.66}$$

and that

$$\alpha^*D\alpha - D\alpha^*\alpha \;=\; 0. \tag{1.67}$$

These equations, (1.66) and (1.67), combined with (1.65) immediately imply that

$$g\gamma^{-1}(D_{U^{\perp}})\gamma g^{-1} \;=\; D_V - \frac{D\alpha\alpha^* - \alpha D\alpha^*}{\sqrt{|\alpha|^2 + s^2}(\sqrt{|\alpha|^2 + s^2} + s)} \tag{1.68}$$

as desired.

The matrix form of the expression $\alpha^*D\alpha - D\alpha^*\alpha$ is $Duu^t - Du^tu = (du + u\omega_V)u^t - u(du^t - \omega_V u^t) = duu^t - udu^t - 2u\omega_v u^t$, and we have that $duu^t = udu^t$ and $u\omega_v u^t = 0$. This proves (1.67).

To prove (1.66) first note that $\alpha\alpha^* + s^2 = s^2\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right) + (|\alpha|^2 + s^2)\frac{\alpha\alpha^*}{|\alpha|^2}$ so that

$$g = s\left(1 - \tfrac{\alpha\alpha^*}{|\alpha|^2}\right) + \sqrt{|\alpha|^2+s^2}\,\tfrac{\alpha\alpha^*}{|\alpha|^2} = s + \frac{1}{\sqrt{|\alpha|^2+s^2}+s}\alpha\alpha^*. \tag{1.69}$$

Therefore

$$\begin{aligned} Dg &= \frac{D\alpha\alpha^* + \alpha D\alpha^*}{\sqrt{|\alpha|^2+s^2}+s} - \frac{1}{2}\frac{d|\alpha|^2\alpha\alpha^*}{\sqrt{|\alpha|^2+s^2}(\sqrt{|\alpha|^2+s^2}+s)^2} \\ &= \frac{D\alpha\alpha^* + \alpha D\alpha^*}{\sqrt{|\alpha|^2+s^2}+s} - \frac{1}{2}\frac{\alpha\alpha^* D\alpha + \alpha D\alpha^*\alpha\alpha^*}{\sqrt{|\alpha|^2+s^2}(\sqrt{|\alpha|^2+s^2}+s)^2}. \end{aligned} \tag{1.70}$$

By (1.69),

$$g^{-1} = \frac{1}{s}\left(1 - \tfrac{\alpha\alpha^*}{|\alpha|^2}\right) + \frac{1}{\sqrt{|\alpha|^2+s^2}}\,\frac{\alpha\alpha^*}{|\alpha|^2} = \frac{1}{s} - \frac{\alpha\alpha^*}{s\sqrt{|\alpha|^2+s^2}(\sqrt{|\alpha|^2+s^2}+s)}. \tag{1.71}$$

Therefore, by direct calculation

$$D\alpha\alpha^* g^{-1} = \frac{D\alpha\alpha^*}{\sqrt{|\alpha|^2+s^2}}. \tag{1.72}$$

Finally, using (1.70), (1.71) and (1.72) a tedious direct calculation verifies (1.66), completing the proof of the Theorem. □

2. Odd Rank Real Vector Bundles.

Now consider the case of an oriented real vector bundle V of odd rank $m = 2n-1$. The Pfaffian no longer exists, and it is natural to define the Chern-Euler form $\chi(D_V)$ of any connection to be identically zero. Nevertheless, for each orthogonal connection D_V on V there does exist a canonically constructed family of Thom forms τ_s as in the even-rank case above. The purpose of the section is to construct this family.

Consider the bundle $\widetilde{V} \equiv \underline{\mathbf{R}} \oplus V$ equipped with the direct sum metric and connection. Let e_0 denote the global parallel frame 1 for $\underline{\mathbf{R}}$. Consider the spherical potential $\tilde{\sigma}$ for $\widetilde{V}$ (of even rank $2n$). Since $\widetilde{V}$ has the global section e_0, the Chern-Euler form $\chi(D_{\widetilde{V}}) = 0$ so that $\tilde{\sigma}$ is d-closed outside of the zero set $X \subset \underline{\mathbf{R}} \oplus V$. Moreover, the fiber integral of $\tilde{\sigma}$ over the ϵ-sphere bundle contained in $\underline{\mathbf{R}} \oplus V$ is -1. Let se_0 denote a general point in the fiber of $\underline{\mathbf{R}}$.

Definition 2.1. ($m = 2n-1$ odd). Suppose $s > 0$ is fixed. First restrict $-2\tilde{\sigma}$ as a differential form to the affine subbundle $\{s\} \times V \subset \widetilde{V}$. Then pull back to V using the obvious identification of $\{s\} \times V$ with V. Let τ_s denote this smooth $2n-1$ form on V. This family $\{\tau_s\}$ will be called the **family of Thom forms on V associated with the metric connection D_V.**

This terminology is justified by the next result. Recall that $\mathbf{P}(\widetilde{V})$ is oriented since the fibres are odd dimensional projective spaces.

Lemma 2.2. *τ_s extends to be a smooth d-closed form on the compactification $\mathbf{P}(\underline{\mathbf{R}} \oplus V)$ of V. Moreover, with $\pi : V \to X$ extended to $\pi : \mathbf{P}(\underline{\mathbf{R}} \oplus V) \to X$, one has that*

$$\pi_*(\tau_s) \;=\; \int\limits_{V_x} \tau_s = 1.$$

Proof. The transgression form $-2\tilde{\sigma}$ on $\widetilde{V} \equiv \underline{\mathbf{R}} \oplus V$ is homogeneous of degree zero in the fiber and even, i.e., fixed by the antipodal map. Therefore $-2\tilde{\sigma}$ is the pullback of a smooth d-closed form τ_1 on $\mathbf{P}(\widetilde{V})$.

In the affine chart $\{s = 1\} \times V \subset \mathbf{P}(\widetilde{V})$, the form τ_1 is obtained by restricting the homogeneous form $-2\tilde{\sigma}$ to the hyperplane $s = 1$, so that this τ_1 is just the τ_1 of Definition 2.1. Thus τ_1 has a smooth d-closed extension to the compactification $\mathbf{P}(\underline{\mathbf{R}} \oplus V)$ of V. The integral of τ_1 over the fiber of V, or equivalently over the fiber of $\mathbf{P}(\underline{\mathbf{R}} \oplus V)$, equals $\frac{1}{2}$ the integral of τ_1, over the fiber of the sphere bundle in $\widetilde{V}$. Finally, in homogeneous coordinates, $\frac{1}{2}\tau_1 = -\tilde{\sigma}$, and $-\tilde{\sigma}$ has integral one over the sphere in the fiber of $\widetilde{V}$. $\square$

Now, we define a spherical potential σ for the family τ_s. Let $\underline{\mathbf{R}}_s$ denote the submanifold (with boundary) consisting of all te_0 with $s \le t$. Let $\rho_s : \underline{\mathbf{R}}_s \oplus V \to V$ denote projection onto the second factor V of $\underline{\mathbf{R}}_s \oplus V$. Define σ_s by the fiber integral

$$\sigma_s \;\equiv\; -2 \int\limits_{\rho_s^{-1}} \tilde{\sigma} \;=\; -2(\rho_s)_*(\tilde{\sigma}), \tag{2.3}$$

and define the **spherical kernel $\boldsymbol{\sigma}$ based on the metric connection D_V** by

$$\sigma = -2 \int_{\rho_0^{-1}} \tilde{\sigma} = -2(\rho_0)_*(\tilde{\sigma}). \tag{2.4}$$

We shall see that this integral converges in $L^1_{\mathrm{loc}}(V)$, i.e., $\lim_{s\to 0} \boldsymbol{\sigma}_{\mathbf{s}} = \boldsymbol{\sigma}$ in $L^1_{\mathrm{loc}}(V)$.

Theorem 2.5. ($m = 2n-1$ odd). *The family of forms $\boldsymbol{\tau}_{\mathbf{s}}$ and $\boldsymbol{\sigma}_{\mathbf{s}}$ defined above satisfy:*

(1) $\quad \boldsymbol{\tau}_{\mathbf{s}}\big|_X = \chi(D_V) \stackrel{\mathrm{def}}{=} 0$ *or equivalently* $\lim_{s\to\infty} \boldsymbol{\tau}_{\mathbf{s}} = \chi(D_V) \stackrel{\mathrm{def}}{=} 0$

(2) $\quad \lim_{s\to 0} \boldsymbol{\tau}_{\mathbf{s}} = [X]$ *since* $\int_{V_x} \boldsymbol{\tau}_{\mathbf{s}} = 1$

(3) $\quad d\boldsymbol{\sigma}_{\mathbf{s}} = \chi(D_V) - \boldsymbol{\tau}_{\mathbf{s}} \qquad$ *on V*

(4) $\quad \lim_{s\to 0} \boldsymbol{\sigma}_{\mathbf{s}} = \boldsymbol{\sigma}$ *converges in $L^1_{\mathrm{loc}}(V)$.*

The limiting form of equation (3) is

$$d\boldsymbol{\sigma} = \chi(D_V) - [X] \qquad \text{on } V. \tag{5}$$

Also note that

$$d(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathbf{s}}) = \boldsymbol{\tau}_{\mathbf{s}} - [X] \qquad \text{on } V. \tag{6}$$

Proof. Inspection shows that $\boldsymbol{\tau}_{\mathbf{s}}$ is odd, i.e., $\boldsymbol{\tau}_{\mathbf{s}}$ pulls back to $-\boldsymbol{\tau}_{\mathbf{s}}$ under the map minus the identity on the fibers of V. Thus, the euler form $\boldsymbol{\tau}_{\mathbf{s}}\big|_Z$ associated to $\boldsymbol{\tau}_{\mathbf{s}}$, vanishes, or equivalently, $\lim_{s\to\infty} \boldsymbol{\tau}_{\mathbf{s}}$ vanishes. This proves 1). Condition 2) follows from Lemma 2.2 and the fact that $\lim_{s\to 0} \boldsymbol{\tau}_{\mathbf{s}} = 0$ outside $X \subset V$.

Condition 3) is immediate from the definitions of $\boldsymbol{\sigma}_{\mathbf{s}}$ and $\boldsymbol{\tau}_{\mathbf{s}}$ (and $\chi(D_V) = 0$) and Stokes theorem applied to the fiber, $s \le t \le \infty$, of ρ_s.

Consulting the formula (1.34) (or 1.20) for $\tilde{\sigma}$, the Lebesgue dominated convergence theorem implies that $\boldsymbol{\sigma}_{\mathbf{s}} = -2 \int_{\rho_s^{-1}} \tilde{\sigma}$ converges to $-2 \int_{\rho_0^{-1}} \tilde{\sigma}$ in $L^1_{\mathrm{loc}}(V)$. □

Now we explicitly compute $\boldsymbol{\tau}_{\mathbf{s}}$ and $\boldsymbol{\sigma}$. With u replaced by $(s,u) = (s,u_1,\ldots,u_{2n-1})$ and e replaced by the column obtained by transposing $(e_0,e) = (e_0,e_1,\ldots,e_{2n-1})$ the formulas (1.27) and (1.36) for $\tilde{\sigma}$ become

(2.6) $\quad \tilde{\sigma}\tilde{\lambda} =$

$$\frac{2}{(n-1)!}\left(\frac{-1}{4\pi}\right)^n \frac{(se_0+ue)(dse_0+Due)}{|u|^2+s^2}\int_0^1 \left(e^t\Omega_V e - 2x\left(1-\frac{x}{2}\right)\frac{(dse_0+Due)^2}{|u|^2+s^2}\right)^{n-1} dx$$

$$= \frac{1}{\pi^n}\sum_{p=0}^{n-1}(-1)^{n-p}\frac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-1}}\,\frac{(se_0+ue)(dse_0+Due)^{2p+1}}{(|u|^2+s^2)^{p+1}}(e^t\Omega_V e)^{n-p-1}.$$

Restriction of $-2\tilde{\sigma}$ to $u_0 = s$ constant yields two formulas for $\boldsymbol{\tau}_{\mathbf{s}}$. Let $\lambda \equiv e_1\wedge\cdots\wedge e_{2n-1}$ denote the unit oriented volume element for V. Then $\tilde{\lambda} = e_0\lambda$ is the unit oriented volume element for $\widetilde{V}$.

$$\boldsymbol{\tau}_{\mathbf{s}}\lambda = -\frac{4}{(n-1)!}\left(\frac{-1}{4\pi}\right)^n \frac{sDue}{|u|^2+s^2}\int_0^1 \left(e^t\Omega_V e - 2x\left(1-\frac{x}{2}\right)\frac{(Due)^2}{|u|^2+s^2}\right)^{n-1} dx$$

(2.7)

$$= \frac{1}{\pi^n}\sum_{p=0}^{n-1}(-1)^{n-p-1}\frac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-2}}\,\frac{s(Due)^{2p+1}(e^t\Omega_V e)^{n-p-1}}{(|u|^2+s^2)^{p+1}}.$$

Setting $u_0 = s$ and integrating over the fiber $0 < s < +\infty$ of ρ_0 by using the formula (2.6) for $\tilde{\sigma}$ yields

(2.8) $\quad \boldsymbol{\sigma}\tilde{\lambda} = -2\displaystyle\int_{\rho_0^{-1}} \tilde{\sigma}\tilde{\lambda} =$

$$\frac{-1}{\pi^n}\sum_{p=0}^{n-1}(-1)^{n-p}\frac{2p!(2p+1)(-1)}{(n-p-1)!(2p+1)!2^{2n-2p-1}}\,\frac{(ue)(Due)^{2p}}{|u|^{2p+1}}(e^t\Omega_V e)^{n-p-1}\int_0^\infty \frac{|u|^{2p+1}dse_0}{(|u|^2+s^2)^{p+1}}.$$

Since $\int_0^\infty \frac{dt}{(t^2+1)^{p+1}} = \frac{1}{2}\frac{(p-\frac{1}{2})\ldots\frac{1}{2}\pi}{p!} = \frac{(2p)!\pi}{(p!)^2 2^{2p+1}}$ equation (2.8) implies that the spherical kernel for odd rank $m = 2n-1$ is given by

$$\boldsymbol{\sigma}\lambda = \frac{-1/2}{(n-1)!}\left(\frac{-1}{4\pi}\right)^{n-1}\frac{ue}{|u|}\left(e^t\Omega_V e - \frac{(Due)^2}{|u|^2}\right)^{n-1} \tag{2.9}$$

Again, for odd rank, this kernel σ was discovered by Chern [C2].

Example 2.10. Real Line Bundles.

$$2\tilde{\sigma} \;=\; \frac{1}{\pi}\frac{uds - sdu}{|u|^2 + s^2}$$
$$\boldsymbol{\tau}_{\mathbf{s}} \;=\; -2\tilde{\sigma}\,\Big|_s \;=\; \frac{1}{\pi}\frac{sdu}{|u|^2 + s^2}$$
$$\boldsymbol{\sigma} \;=\; -\int_{0\le s\le\infty} 2\tilde{\sigma} \;=\; -\frac{1}{2}\frac{u}{|u|}\frac{2}{\pi}\int_0^\infty \frac{|u|ds}{|u|^2+s^2} \;=\; -\frac{1}{2}\frac{u}{|u|}.$$

Remark 2.11. Compactly Supported Thom Forms. As noted in the last Section, for even rank bundles, the spherical kernel $\boldsymbol{\sigma}$ is independent of the choice of approximation mode χ, but the Thom form $\boldsymbol{\tau}_s$ depends on the choice of the χ. In particular, a choice of compactly supported χ gives a compactly supported Thom form (see Theorem 1.16). In this Section, there is no reference to an approximation mode χ. The Thom form $\boldsymbol{\tau}_s$ of Theorem 2.5 does not have compact support in the fibers of V. (However, note by Theorem 1.63 that $\boldsymbol{\tau}_s$ extends to the bundle $\mathbf{P}(\underline{\mathbf{R}} \oplus V)$ obtained by compactifying the fibers of V.)

A new Thom form with compact support can be constructed as follows. Choose $\psi(t) \in C^\infty_{\text{cpt}}(\mathbf{R})$, $\psi(t) \ge 0$, and $\psi(t) \equiv 1$ in a neighborhood of zero. Define $\psi_s \equiv \psi\left(\frac{|u|^2}{s^2}\right)$. Then $\boldsymbol{\sigma}'_s \equiv \psi_s\boldsymbol{\sigma}_{\mathbf{s}} + (1-\psi_s)\boldsymbol{\sigma}$ is smooth. Since $\boldsymbol{\sigma} - \boldsymbol{\sigma}'_{\mathbf{s}} = \psi_s(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathbf{s}})$ has compact support the new Thom Form $\boldsymbol{\tau}'_{\mathbf{s}} = -d\boldsymbol{\sigma}'_{\mathbf{s}}$ has compact support. Note that $\boldsymbol{\sigma} = \lim_{s\to 0} \boldsymbol{\sigma}'_{\mathbf{s}}$ and that

$$d(\boldsymbol{\sigma} - \boldsymbol{\sigma}'_{\mathbf{s}}) \;=\; \boldsymbol{\tau}'_{\mathbf{s}} - [X].$$

So if we replace $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_s$, $\boldsymbol{\tau}_s$ by $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'_{\mathbf{s}}$, $\boldsymbol{\tau}'_{\mathbf{s}}$ in Theorem 2.5 the Theorem remains valid.

Remark 2.12. Alternate Approach to Compactly Supported Thom Forms. Choose χ to be a compactly supported approximate one, i.e. $\chi(t) \equiv 1$ for $t \ge t_0$. If V is of odd rank, consider $\widetilde{V} = \mathbf{R} \oplus V$ of even rank and the formula

$$d(\tilde{\sigma} - \tilde{\sigma}_s) = \tilde{\tau}_s - [X] \tag{2.13}$$

on the total space of $\widetilde{V}$. Let $\pi : \widetilde{V} \longrightarrow V$ denote the natural projection. Push the current equation (2.13) forward to V. That is if we define

$$r_s \equiv \pi_*(\tilde{\sigma} - \tilde{\sigma}_s) \quad \text{and} \quad \tau_s = \pi_*(\tilde{\tau}_s).$$

Then

$$dr_s = \tau_s - [X] \quad \text{on} \quad V \tag{2.14}$$

as desired. The explicit calculation of $\pi_*(\tilde{\tau}_s)$ as a fiber integral using formula (1.17) for $\tilde{\tau}_s$ yields a formula for τ_s similar to the Thom form τ_s' obtained in Remark 2.11, but not with $\chi = 1 - \psi$ as one might hope but with a complicated expression for ψ. Since $\pi_*(\tilde{\tau}_s)$ yields a complicated explicit formula for τ_s we have emphasized the approach of Definition 2.1 and Remark 2.11 in this Section on odd rank.

Remark 2.15. A third justification for distinguishing the real algebraic approximation mode is provided by the inductive procedure of this section. Suppose V is of even real rank $2n$ with metric connection D_V. Then *the spherical kernel for* $\underline{\mathbf{R}} \oplus V$ of odd rank given by (2.9), *restricts to the hyperplane* $u_0 = s$ *to yield a Thom form in even rank. This Thom form is precisely the real algebraic Thom form of Corollary 1.20.*

General $\mathrm{SO_m}$ Invariant Polynomials

For the sake of completeness the remainder of this section is devoted to briefly examining the ϕ Chern current for a general SO_m-invariant polynomial ϕ (m even or m odd). As a corollary of Lemma 1.41 we have

Corollary 2.16. *If ϕ is any SO_m-invariant polynomial on $\mathfrak{so}_m$, then*

$$\phi(\overrightarrow{\Omega}_s) \;=\; \phi(A(\chi_s)) - \chi_s' \tfrac{|u|^2}{s^2}\, \tfrac{d|u|^2}{|u|^2} \phi\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2}\; ;\; A(\chi_s)\right) \tag{2.17}$$

where

$$A(x) \;=\; \Omega_V - x\left(\tfrac{u^t u \Omega_V}{|u|^2} + \Omega_V \tfrac{u^t u}{|u|^2}\right) + 2x\left(1 - \tfrac{x}{2}\right)\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2}\right)^2 . \tag{2.18}$$

As a consequence of Corollary 2.16, we have that

$$u \cdot \tfrac{\partial}{\partial u} \llcorner \phi(\overrightarrow{\Omega}_s) \;=\; -2\chi_s' \tfrac{|u|^2}{s^2} \phi\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2}\; ;\; A(\chi_s)\right) \tag{2.19}$$

because

$$u \cdot \tfrac{\partial}{\partial u} \llcorner A(x) = 0 \quad \text{and} \quad u \cdot \tfrac{\partial}{\partial u} \llcorner \left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2} \right) = 0. \tag{2.20}$$

Since

$$u \cdot \tfrac{\partial}{\partial u} \llcorner \overrightarrow{\Omega}_s \tfrac{ds^2}{s^2} \;=\; \tfrac{\partial \overrightarrow{\omega}_s}{\partial s} ds \tag{2.21}$$

the transgression integrand satisfies

$$\phi\left(\tfrac{\partial \overrightarrow{\omega}_s}{\partial s} ds \;;\; \overrightarrow{\Omega}_s \right) \;=\; 2\chi_s' \tfrac{|u|^2}{s^2} \phi\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2} \;;\; A(\chi_s) \right) \tfrac{ds^2}{s^2}. \tag{2.22}$$

Therefore, using the change of variables $x \equiv \chi\left(\frac{|u|^2}{s^2}\right)$ the transgression T_s can be expressed as follows.

Proposition 2.23.

$$T_s \;=\; -2 \int_0^{\chi_s} \phi\left(\tfrac{u^t Du}{|u|^2} - \tfrac{Du^t u}{|u|^2} \;;\; A(x) \right) dx$$

Using this formula for the transgression T_s it is easy to verify that $\lim_{s \to 0} T_s = T$ converges in $L^1_{\text{loc}}(X)$ if the section μ of V is atomic, and to deduce other results paralleling the results for a section of a complex bundle which were described before. The statements and proofs are left to the reader.

However, the next related result will be used to prove the second version of the rectifiable Grothendieck-Riemann-Roch theorem presented in Section 5.

Theorem 2.24. *Suppose ϕ and ψ are SO_{2n}-invariant polynomials on $\mathfrak{so}_{2n}$, related by $\phi = \psi \cdot \chi$, where $\chi(\Omega) = \operatorname{Pf}\left(-\frac{1}{2\pi}\Omega\right)$ is the Euler polynomial. Then*

$$\operatorname{Res}_\phi(\overrightarrow{D}) \;=\; \psi(D_V), \tag{2.25}$$

$$\text{the } L^1_{\text{loc}}(X) \text{ part of } \phi((\overrightarrow{D})) \text{ vanishes, and} \tag{2.26}$$

$$\phi(\overrightarrow{D}_s) - \psi(D_V)\operatorname{Div}(\mu) \;=\; d(\psi(\overrightarrow{D}_s) r_s) \tag{2.27}$$

where $R_s \equiv \psi(\overrightarrow{D}_s) r_s = \psi(\overrightarrow{D}_s)(\sigma - \sigma_s)$ converges to zero in $L^1_{\text{loc}}(X)$ as s approaches zero.

Proof. Note first that by Theorem 1.22 with $\mathbf{r_s} \equiv \boldsymbol{\sigma} - \boldsymbol{\sigma}_\mathbf{s}$, one has

$$\begin{aligned}\phi(\overrightarrow{\mathbf{D}}_\mathbf{s}) &= \psi(\overrightarrow{\mathbf{D}}_\mathbf{s})\chi(\overrightarrow{D}_s) = \psi(\overrightarrow{\mathbf{D}}_\mathbf{s})\boldsymbol{\tau}_\mathbf{s} \\ &= \psi(\overrightarrow{\mathbf{D}}_\mathbf{s})[X] + d(\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})\mathbf{r_s}) = \psi(D_V)[X] + d(\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})\mathbf{r_s}).\end{aligned}$$

Then note that $\mathbf{r_s}$, and hence also $\mathbf{r_s}\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})$, converge to zero on $V \sim X$. Inspection of the formula (1.27) for $\boldsymbol{\sigma}_\mathbf{s}$ and $\boldsymbol{\sigma}$ shows that $\mathbf{r_s}$ is of odd degree in $Du_1, \ldots, Du_{2n}$. Inspection of the formula (1.42) for $\overrightarrow{\boldsymbol{\Omega}}_\mathbf{s}$ shows that each entry in this matrix is of even degree in $Du_1, \ldots, Du_{2n}$. Thus $\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})$ is of even degree in $Du_1, \ldots, Du_{2n}$. Therefore, $\mathbf{r_s}\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})$ is of odd degree in $Du_1, \ldots, Du_{2n}$. In particular, $\mathbf{r_s}\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})$ is of degree $< 2n$ in $Du_1, \ldots, Du_n$. Thus by atomicity, $\mathbf{r_s}\psi(\overrightarrow{\mathbf{D}}_\mathbf{s})$ is $L^1_{\text{loc}}(V)$ dominated and therefore converges to zero in $L^1_{\text{loc}}(V)$.

This proves that the Chern current

$$\phi((\overrightarrow{D})) = \lim_{s \to 0} \phi(\overrightarrow{D}_s) = \psi(D_V)[X].$$

Now the Theorem follows. □

3. Universal Thom Forms in Equivariant de Rham Theory over $\mathbf{SO_m}$.

The formulas derived above can be reinterpreted as determining universal equivariant forms in $\mathcal{E}^*_{SO_m}(\mathbf{R}^m)$ (as in Chapter III.4). These results improve on [MQ] in that they apply to all dimensions, not just even ones, and they naturally give forms with compact support. The terminology and derivation which follow are in strict analogy with those of the development in Remark III.4.30 and Remark III.4.38, and so we shall pass rapidly to the statements.

Let $W = (\Lambda\mathfrak{g}^*) \otimes (S\mathfrak{g}^*)$ denote the Weil algebra of $\mathfrak{so}_m$ with standard generators $\omega_{ij} = -\omega_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$ for $1 \le i < j \le n$. Let $u = (u_1, \ldots, u_m)$ denote the standard coordinates on $\mathbf{R}^m$. Let SO_m act from the right by $u \mapsto ug$ and $u^t \to g^{-1}u^t$. Set

$$Du = du + u\omega.$$

Then given an approximate-one χ and a number $s > 0$, we define

$$\overrightarrow{\omega}_s = \omega - \chi\left(\tfrac{|u|^2}{s^2}\right)\left\{\frac{u^*Du}{|u|^2} - \frac{Du^t u}{|u|^2}\right\} \in W \otimes \mathcal{E}^*(\mathbf{R}^n)$$

and define

$$\vec{\Omega}_s = d\vec{\omega}_s - \tfrac{1}{2}[\vec{\omega}_s, \vec{\omega}_s].$$

Proposition 3.1. (m = 2n). *If $m = 2n$, then for each χ and $s > 0$ the Pfaffian of $\vec{\Omega}_s$ gives an equivariant cocycle*

$$\mathfrak{T}_s \stackrel{\text{def}}{=} \operatorname{Pf}\left(-\tfrac{1}{2\pi}\vec{\Omega}_s\right) \in \mathcal{E}^{2n}_{SO_{2n}}(\mathbf{R}^{2n}).$$

These forms are mutually cohomologous. In fact $\mathfrak{T}_s = \operatorname{Pf}\left(-\frac{1}{2\pi}\Omega\right) - d\sigma_s$ where

$$\sigma_s \stackrel{\text{def}}{=} (-1)^n \int_s^\infty \operatorname{Pf}\left(\tfrac{1}{2\pi}\dot{\omega}_t \; ; \; \tfrac{1}{2\pi}\Omega_t\right) dt \in \mathcal{E}^{2n-1}_{SO_{2n}}(\mathbf{R}^{2n})$$

with $\dot{\omega}_t = \partial\omega_t|\partial t$. Explicit formulas for $\mathfrak{T}_s$ and σ_s in terms of Ω, ω and χ are given in 1.17 and 1.27. These forms have the following properties. Let $i : \mathrm{pt} \mapsto \mathbf{R}^{2n}$ and $\pi : \mathbf{R}^{2n} \to \mathrm{pt}$ be the obvious equivariant maps and consider the induced maps

$$i^* : \mathcal{E}^*_{SO_{2n}}(\mathbf{R}^{2n}) \longrightarrow \mathcal{E}^*_{SO_{2n}}(\mathrm{pt}) = W \quad \textit{and} \quad \pi_* : \mathcal{E}^*_{SO_{2n}}(\mathbf{R}^{2n}) \longrightarrow \mathcal{E}^{*-2n}_{SO_{2n}}(\mathrm{pt}) = W.$$

Then

(i) $i^*\mathfrak{T}_s = \operatorname{Pf}\left(-\frac{1}{2\pi}\Omega\right)$ *for all s,*
(ii) $\lim_{s\to 0} \mathfrak{T}_s = [0]$ *in equivariant currents on $\mathbf{R}^{2n}$,*
(iii) $\pi_*\mathfrak{T}_s = 1$,
(iv) $\lim_{s\to 0} \sigma_s \stackrel{\text{def}}{=} \sigma$ *exists in equivariant forms with L^1_{loc}-coefficients.*

Proof. The proof that $\mathfrak{T}_s$ is an equivariant cocycle follows exactly the lines of the proof of Proposition III.4.31. The properties (i)–(iv) are translations of the properties established in §1. The proofs given there carry over directly. □

Notice that the equivariant form σ is smooth outside the origin in $\mathbf{R}^{2n}$. It is given succintly by

$$\sigma = (-1)^n \int_0^\infty \operatorname{Pf}\left(\tfrac{1}{2\pi}\dot{\omega}_t \; ; \; \tfrac{1}{2\pi}\Omega_t\right) dt \tag{3.2}$$

and explicitly by formula 1.36 reinterpreted in this universal context. We have that

$$d\sigma \;=\; \mathrm{Pf}\left(-\tfrac{1}{2\pi}\Omega\right) - [0] \quad \text{in} \;\; \mathcal{E}^*_{SO_{2n}}(\mathbf{R}^{2n}).$$

Proposition 3.3. **(m = 2n − 1).** *If $m = 2n-1$ consider the embeddings $j_s : \mathbf{R}^{2n-1} \hookrightarrow \mathbf{R}^{2n}$ given by $j_s(x) = (s, x)$ and define*

$$\mathfrak{T}_s \overset{\text{def}}{=} j_s^*(\sigma) \in \mathcal{E}^{2n-1}_{SO_{2n-1}}(\mathbf{R}^{2n-1})$$

where σ is the spherical kernel in $\mathbf{R}^{2n}$ given by 3.2 or explicitly by 1.36. Then $\mathfrak{T}_s$ is a family of mutually cohomologous equivariant cocycles. In fact, let $\rho_s : [s, \infty) \times \mathbf{R}^{2n-1} \to \mathbf{R}^{2n-1}$ denote projection and set

$$\sigma_s \overset{\text{def}}{=} -2(\rho_s)_*(\sigma) \in \mathcal{E}^{2n-1}_{SO_{2n-1}}(\mathbf{R}^{2n-1}).$$

Then $d\sigma_s = -\mathfrak{T}_s$ for all $s > 0$. Furthermore if $i : \mathrm{pt} \hookrightarrow \mathbf{R}^{2n-1}$ and $\pi : \mathbf{R}^{2n-1} \to$ pt are the obvious equivariant maps, then

(i) $i^*(\mathfrak{T}_s) = 0$ *for all s,*

(ii) $\lim_{s\to 0} \mathfrak{T}_s = [0]$ *in equivariant currents on $\mathbf{R}^{2n-1}$,*

(iii) $\pi_* \mathfrak{T}_s s = 1$,

(iv) $\lim_{s\to 0} \sigma_s = \sigma_0$ *exists in equivariant forms with L^1_{loc}-coefficients.*

Proof. One argues as in the proof of 3.1. $\square$

We now fix m (even or odd) and consider some special choices of χ. If $\chi(t) = 1$ for $t \geq 1$, then $\mathrm{spt}(\mathfrak{T}_s) \subseteq \overline{B}_s = \{v \in \mathbf{R}^m : |v| \leq 1\}$. Furthermore, the smooth forms $r_{s,t} = \sigma_t - \sigma_s$ for $0 < t < s$, and the L^1_{loc}-forms $r_s = \sigma_0 - \sigma_s$ each have support on $\overline{B}_s$. Note that $dr_{s,t} = \mathfrak{T}_s - \mathfrak{T}_t$ and $dr_s = \mathfrak{T}_s - [0]$. In particular, we have the following

Proposition 3.4. *Suppose $\chi(t) = 1$ for $t \geq 1$ and let m be arbitrary. Then the forms $\mathfrak{T}_s$ from Propositions 2.1 and 2.3 each determine an equivariant cocycle*

$$\mathfrak{T}_s \in \mathcal{E}^m_{SO_m}(\mathbf{R}^m, \mathbf{R}^m - B_s)$$

where $B_s = \{v \in \mathbf{R}^m : |v| < s\}$. Furthermore, given an oriented, real m-plane bundle $V \to X$ with an orthogonal connection, the Weil homomorphism

$$\mathcal{E}^*_{SO_m}(\mathbf{R}^m, \mathbf{R}^m - B_s) \longrightarrow \mathcal{E}^*(V, V - B_s(V)),$$

where $B_s(V) = \{\nu \in V : |\nu| < s\}$, carries $\mathfrak{T}_{\mathbf{s}}$ to the Euler Thom forms of 1.16 and 2.5, which represent the Thom class

$$[\tau_s] \in H^m(V, V - B_s(V)).$$

Similarly the Weil homomorphism carries the spherical transgression forms $\sigma_s \in \mathcal{E}^{m-1}_{SO_m}(\mathbf{R}^m)$ over to the corresponding forms in $\mathcal{E}^{m-1}(V)$.

Proposition 3.5. *Let $\chi(t) = 1 - 1/\sqrt{1+t}$. Then the equivariant cocycle $\mathfrak{T}_{\mathbf{s}} \in \mathcal{E}^m_{SO_m}(\mathbf{R}^m)$ extends to an equivariant cocycle on $\mathbf{P}(\mathbf{R} \oplus \mathbf{R}^m) = \mathbf{R}^m \cup \mathbf{P}(\mathbf{R}^m)$ which vanishes on $\mathbf{P}(\mathbf{R}^m)$. It thereby determines a relative equivariant cocycle*

$$\mathfrak{T}_{\mathbf{s}} \in \mathcal{E}^m_{SO_m}\left(\mathbf{P}(\mathbf{R} \oplus \mathbf{R}^m),\ \mathbf{P}(\mathbf{R}^m)\right).$$

For a bundle V with connection as above, the Weil homomorphism carries $\mathfrak{T}_{\mathbf{s}}$ to a cocycle

$$\tau_s \in \mathcal{E}^m\left(\mathbf{P}(\mathbf{R} \oplus V),\ \mathbf{P}(V)\right)$$

which represents the Thom class

$$[\tau_s] \in H^m\left(\mathbf{P}(\mathbf{R} \oplus V),\ \mathbf{P}(V)\right).$$

When $m = 2n$, each $\mathfrak{T}_{\mathbf{s}}$ is given explicitly by the formula

$$\mathfrak{T}_{\mathbf{s}} = \left(\tfrac{-1}{2\pi}\right)^n \frac{s}{\sqrt{|u|^2 + s^2}} \operatorname{Pf}\left(\Omega - \frac{Du^* Du}{|u|^2 + s^2}\right) \tag{3.6}$$

Proofs of 3.4 and 3.5 are straightforward and details are omitted.

Note. With a proper choice of embedding $V \xrightarrow{\approx} B_1(V) \subset V$ the Thom form corresponding to any approximate-one χ extends to a closed form on all of V with support in $B_1(V)$.

4. Thom Isomorphisms and Gysin Maps.

The Thom forms constructed in §§1 and 2 produce canonical representations, at the level of differential forms, of various forms of the Thom isomorphism for a real oriented m-plane bundle $\pi : V \to X$ with an orthogonal connection. As noted in 3.4, if $\chi(t) = 1$ for $t \geq 1$, then $\mathrm{spt}(\tau_s) \subseteq B_s(V)$ and we have a **Thom map**

$$i_{!,s} : \mathcal{E}^*(X) \longrightarrow \mathcal{E}^{*+m}(V, V - B_s(V)) \tag{4.1}$$

given by

$$i_{!,s}(\varphi) \;=\; \pi^*(\varphi) \wedge \tau_s$$

which induces the **Thom isomorphism**

$$i_! : H^*(X) \xrightarrow{\approx} H^*(V, V - B_s(V)). \tag{4.2}$$

Integration over the fibre π_* inverts this map since $\pi_*(\tau_s) = 1$. Letting $s \to 0$ gives the canonical version of this map

$$i_{!,0} : \mathcal{E}^*(X) \hookrightarrow \mathcal{E}^{*+m}(V, V - X)' \tag{4.3}$$

defined by

$$i_{!,0}(\varphi) \;=\; \varphi \cdot [X]. \tag{4.4}$$

If we choose $\chi(t) = 1 - 1/\sqrt{1+t}$, then we get a Thom map

$$i_! : \mathcal{E}^*(X) \longrightarrow \mathcal{E}^{*+m}\left(\mathbf{P}(\mathbf{R} \oplus V),\ \mathbf{P}(V)\right) \tag{4.5}$$

defined by $i_!(\varphi) = \pi^*\varphi \wedge \tau_s$ where τ_s is the extended Thom form defined in 3.5. It induces the isomorphism

$$i_! : H^*(X) \longrightarrow H^{*+m}\left(\mathbf{P}(\mathbf{R} \oplus V),\ \mathbf{P}(V)\right). \tag{4.6}$$

Suppose now that $j : Y \hookrightarrow X$ is a compact oriented manifold Y embedded into an oriented manifold X, and let

$$j_! : \mathcal{E}^*(Y) \longrightarrow \mathcal{E}^{*+m}_{\mathrm{cpt}}(X)'$$

be the Gysin map defined as in III.4.11 by $j_!(\varphi) = \varphi[Y]$. Then just as in III.4.11 we can: identify the normal bundle to Y with a tubular neighborhood of Y in X, transplant the tautological cross-section, choose χ with $\chi(t) = 1$ for $t \geq 1$, and obtain a smooth Gysin map

$$j_{!,s} : \mathcal{E}^*(Y) \longrightarrow \mathcal{E}^{*+m}_{\mathrm{cpt}}(X)$$

which converges to $j_!$ as $s \to 0$. Each element of the family induces the Gysin homomorphism

$$j_! : H^*(Y) \longrightarrow H^{*+m}_{\mathrm{cpt}}(X).$$

Just as in III.4.12, this discussion of Gysin maps generalizes to a current which can be written as the divisor of an atomic section of an oriented real m-plane bundle defined in some neighborhood of its support.

5. The Rectifiable Grothendieck-Riemann-Roch Theorem—Version 2.

In this section we use our construction of the Thom form to prove the general version of the Grothendieck-Riemann-Roch Theorem [AH] at the level of differential forms. More specifically suppose $j : Y \hookrightarrow X$ is a proper smooth embedding of real manifolds such that $j^* w_2(X) = w_2(Y)$ (where $w_2 =$ the second Stiefel Whitney class). Then for any complex bundle E with connection over Y, our methods directly produce a family of smooth, d-closed forms K_s, $0 < s < 1$, which represent $\mathrm{ch}(j_!(E))$. Furthermore the family $\mathrm{spt}(\mathrm{K}_s)$ forms a shrinking system of neighborhoods of Y in X and

$$\lim_{s \to 0} \mathrm{K}_s \;=\; \mathrm{ch}(D_E)\widehat{\mathbb{A}}^{-1}(D_N)[Y] \;=\; j_!(\mathrm{ch}(D_E)\widehat{\mathbb{A}}(D_N))$$

where N is the normal bundle to Y in X. Thus K_s gives a homotopy through cohomologous closed forms from the Chern character of $j_!(E)$ to the canonical representative of $j_!(\mathrm{ch}(E)\widehat{\mathbb{A}}(N))$. The residue $\mathrm{ch}(D_E)\widehat{\mathbb{A}}(D_N)$ falls automatically out of the calculation.

The process is sufficiently natural that it generalizes to embeddings of complexes into X with "spin normal bundle".

To begin, let $\pi : F \to X$ be a $2n$-dimensional Riemannian vector bundle with spin structure. Assume F is provided with a compatible (orthogonal) connection D with curvature $R = D^2$. Let $\$(F)$ denote the complex spinor bundle canonically associated to F and let $D_{\$}$ be the connection on $\$$ induced from the one on F (cf. [H], [LM]). Via the metric and spin structure, we have

$$\mathrm{Skew\,End}(F) \cong \Lambda^2 F \hookrightarrow \Lambda^*_{\mathbf{C}} F \hookrightarrow C\ell(F) \;=\; \mathrm{End}_{\mathbf{C}}(\$(F)).$$

That is, any skew-symmetric endomorphism $R : F \to F$ canonically determines an endomorphism $\mathbf{R} : \$ \to \$$ via Clifford multiplication by the corresponding 2-form. Then the curvature $R^{\$} = (D_{\$})^2$ of $\$$ in its induced connection is given by the formula

$$R^{\$} \;=\; \tfrac{1}{2}\mathbf{R} \tag{5.1}$$

(cf. [LM, pg 110]).

We now present an identity which is contained in [MQ]. A proof will be given at the end of the section. To state it we need the following definition. Let V be an oriented real inner product space of dimension $2n$. To each $A \in \mathrm{Skew\,End}(V) \cong \Lambda^2 V$, there is an oriented, orthonormal basis $e_1, \ldots, e_{2n}$ with respect to which A can be diagonalized as

$$A \;=\; \sum_{j=1}^{n} a_j e_{2j-1} \wedge e_{2j}$$

where $a_j \in \mathbf{R}$ for each j. Then

$$\mathrm{Pf}(A) \;=\; a_1 \ldots a_n$$

and we define

$$\widehat{\mathbb{A}}(A) \;=\; \prod_{j=1}^{n} \frac{a_j/4\pi}{\sinh(a_j/4\pi)}. \tag{5.2}$$

This function can be rewritten as a power series of $\mathrm{Ad}_{O_{2n}}$-invariant polynomials on $\mathrm{Skew\,End}(V)$, and thereby extends to all of $\mathrm{End}(V)$. Let $C\ell(V)$ be the Clifford algebra of V and denote by $\$(V)$ the irreducible complex "spinor" module over $C\ell(V)$ of complex dimension 2^n. There is a canonical decomposition

$$\$(V) \;=\; \$^+ \oplus \$^- \tag{5.3}$$

where the complex volume form

$$\omega = (-i)^n e_1 \dots e_{2n}$$

acts by

$$\omega \big|_{\$^+} = \mathrm{Id}_{\$^+} \qquad \text{and} \qquad \omega \big|_{\$^-} = -\mathrm{Id}_{\$^-}.$$

Note. This splitting differs from the one in [LM] where ω is chosen to be $i^n e_1 \dots e_{2n}$.

Using the canonical identification $\Lambda^* V \cong C\ell(V)$, we associate to each $A \in \mathrm{Skew\,End}(V) \cong \Lambda^2 V$, the endomorphism $\mathbf{A} : \$ \to \$$ as above.

Lemma 5.4. *Set $\widehat{\mathbb{A}}_0(A) = \widehat{\mathbb{A}}(-2\pi A)$. Then for each $A \in \mathrm{Skew\,End}(V)$ we have that*

$$\mathrm{tr}_{\$^+}\left(e^{\frac{1}{2i}\mathbf{A}}\right) - \mathrm{tr}_{\$^-}\left(e^{\frac{1}{2i}\mathbf{A}}\right) = \widehat{\mathbb{A}}_0^{-1}(A)\,\mathrm{Pf}(A) \tag{5.5}$$

Corollary 5.6. *Let (F, D) and $(\$(F), D_{\$})$ be as above. Then the curvature $R^{\$}$ respects the splitting $\$(F) = \$^+ \oplus \$^-$ determined as in (5.3), and we have the identity*

$$\mathrm{tr}_{\$^+}\left\{e^{-iR^{\$}}\right\} - \mathrm{tr}_{\$^-}\left\{e^{-iR^{\$}}\right\} = \widehat{\mathbb{A}}_0^{-1}(R)\,\mathrm{Pf}(R). \tag{5.7}$$

In particular, we have the following identity of characteristic forms:

$$ch(D_{\$^+}) - ch(D_{\$^-}) = \widehat{\mathbb{A}}^{-1}(D)\chi(D). \tag{5.8}$$

Furthermore if E is any complex vector bundle with complex connection D_E, we let $D_{\$\otimes E} \stackrel{\text{def}}{=} D_{\$} \otimes 1 + 1 \otimes D_E$ denote the tensor product connection on $\$ \otimes E$. Then the following identity

$$ch\left(D_{\$^+\otimes E}\right) - ch\left(D_{\$^-\otimes E}\right) = ch(D_E)\widehat{\mathbb{A}}^{-1}(D)\chi(D) \tag{5.9}$$

holds in the space of differential forms on X.

Proof. Plugging (5.1) into (5.5) gives the first equation. Replacing R by $-\frac{1}{2\pi}R$ gives the second equation. For the third we simply recall that for the tensor product connection we have $\mathrm{ch}\left(D_{\$^\pm} \otimes D_E\right) = \mathrm{ch}\left(D_{\$^\pm}\right)\mathrm{ch}(D_E)$. $\square$

We can now apply our theory of Chern currents to derive a generalization of the Differentiable Riemann-Roch Theorem for embeddings.

Theorem 5.10. Rectifiable Grothendieck-Riemann-Roch—Version 2. *Let $\pi : F \to X$ is a smooth real vector bundle of rank $2n$ provided with a spin structure and an orthogonal connection D. Let $\$^\pm$ denote the canonical complex spinor bundles associated to F and provided with the induced connections $D_{\$^\pm}$. Let α be any atomic section of F. Then for any complex vector bundle E with connection over X, the following identity of d-closed forms and currents holds on X:*

$$(5.11) \qquad ch\left(D_{\$^+\otimes E}\right) - ch\left(D_{\$^-\otimes E}\right) \;=\; ch(D_E)\widehat{\mathbb{A}}^{-1}(D)\,\mathrm{Div}(\alpha) + dT$$

where T is a canonically defined L^1_{loc}-form on X.

Furthermore, fix any choice of approximate one χ and let $\overrightarrow{D}_s$ denote the family of Riemannian pushforward connections on F as in (1.10)*. Let $\overrightarrow{D}_s^+$ and $\overrightarrow{D}_s^-$ denote the connections induced on $\$^+$ and $\$^-$ respectively by $\overrightarrow{D}_s$, and set $\overrightarrow{D}_{s,E}^\pm = \overrightarrow{D}_s^\pm \otimes D_E$ on $\$^\pm \otimes E$. Then*

$$(5.12) \qquad ch(\overrightarrow{D}_{s,E}^+) - ch(\overrightarrow{D}_{s,E}^-) - ch(D_E)\widehat{\mathbb{A}}^{-1}(D)\,\mathrm{Div}(\alpha) \;=\; dR_s$$

where R_s is a family of $L^1_{\mathrm{loc}}(X)$-forms such that

$$\lim_{s\to 0} R_s = 0 \qquad \text{and} \qquad \lim_{s\to\infty} R_s = T \quad \text{in } L^1_{\mathrm{loc}}(X).$$

In particular,

$$(5.13) \qquad \lim_{s\to 0}\left\{ch(\overrightarrow{D}_s^+) - ch(\overrightarrow{D}_s^-)\right\} \;=\; \widehat{\mathbb{A}}^{-1}(D)\,\mathrm{Div}(\alpha).$$

If $\chi(t)$ is chosen so that $\chi(t) = 1$ for all $t \geq 1$, then

$$(5.14) \qquad spt\left\{ch(\overrightarrow{D}_{s,E}^+) - ch(\overrightarrow{D}_{s,E}^-)\right\} \subset \overline{U}_s$$

and $spt(R_s) \subset \overline{U}_s$, where $U_s = \{x \in X : |\alpha(x)| < s\}$ for all $s > 0$.

Proof. Applying (5.9) to the connection $\overrightarrow{D}_s$ gives the equation

$$\mathrm{ch}(\overrightarrow{D}^+_{s,E}) - \mathrm{ch}(\overrightarrow{D}^-_{s,E}) = \mathrm{ch}(D_E)\widehat{\mathbb{A}}^{-1}(\overrightarrow{D}_s)\chi(\overrightarrow{D}_s). \tag{5.15}$$

By Theorem 1.51 we can write

$$\chi(\overrightarrow{D}_s) = \mathrm{Div}(\alpha) + dr_s \tag{5.16}$$

where r_s is an L^1_{loc}-form on X with

$$\lim_{s\to 0} r_s = 0 \quad \text{and} \quad \lim_{s\to\infty} r_s = \sigma \ \text{ in } \ L^1_{\mathrm{loc}}(X).$$

where σ denotes the pullback by α of the singular Chern-Euler transgression form $\boldsymbol{\sigma}$. Substituting (5.16) into (5.15) gives equation (5.12) with

$$R_s \stackrel{\text{def}}{=} \mathrm{ch}(D_E)\widehat{\mathbb{A}}^{-1}(\overrightarrow{D}_s)r_s.$$

As $s \to \infty$, we know that $\widehat{\mathbb{A}}^{-1}(\overrightarrow{D}_s) \to \widehat{\mathbb{A}}^{-1}(D)$ in the C^∞ topology. Hence

$$\lim_{s\to\infty} R_s = \mathrm{ch}(D_E)\widehat{\mathbb{A}}^{-1}(D)\sigma \stackrel{\text{def}}{=} T.$$

On the other hand by Theorem 2.21 we know that

$$\lim_{s\to 0} \widehat{\mathbb{A}}^{-1}(\overrightarrow{D}_s)r_s = 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(X)$$

and so $R_s \to 0$ as $s \to 0$ as claimed. Finally, if $\chi(t) = 1$ for all $t \geq 1$, then

$$\mathrm{spt}(\chi(\overrightarrow{D}_s)) \subseteq \overline{U}_s$$

for all $s > 0$ and therefore

$$\mathrm{spt}\left\{\mathrm{ch}(\overrightarrow{D}^+_s) - \mathrm{ch}(\overrightarrow{D}^-_s)\right\} \subseteq \overline{U}_s$$

for all $s > 0$ by (5.15). Multiplying by $\mathrm{ch}(D_E)$ gives (5.14). $\square$

Note that for each choice of approximation mode, we have produced a smooth family

$$\mathrm{K}_\alpha(s) \stackrel{\text{def}}{=} \mathrm{ch}(D^+_s) - \mathrm{ch}(D^-_s), \quad 0 < s \leq \infty$$

such that

$$\mathrm{K}_\alpha(\infty) \;=\; \mathrm{ch}(D^+_{\$}) - \mathrm{ch}(D^-_{\$}) \quad \text{and} \quad \lim_{s\to 0} \mathrm{K}_\alpha(s) \;=\; \widehat{\mathbb{A}}^{-1}(D)\,\mathrm{Div}(\alpha).$$

That is, $\mathrm{K}_\alpha(s)$ gives a canonical homotopy from the fixed Chern-Weil representative of the Chern character of $\$^+ - \$^-$ to the canonical residue $\widehat{\mathbb{A}}^{-1}(D)$ times the divisor of α.

This homotopy is compatible with multiplication by $\mathrm{ch}(D_E)$ for any complex bundle E with connection D_E over X. In particular, the family $\mathrm{ch}(D_E)\mathrm{K}_\alpha(s)$, $0 < s \le \infty$, gives a homotopy through smooth forms on X from the fixed Chern-Weil representative of the Chern character of $E \otimes (\$^+ - \$^-)$ to the residue $\mathrm{ch}(D_E)\widehat{\mathbb{A}}^{-1}(D)$ times the divisor of α.

Remark 5.17. Localization at Div(α). Choose an approximate one $\chi(t)$ such that $\chi(t) = 1$ for $t \ge 1$. Then by (5.14)

$$\mathrm{spt}\left\{\mathrm{ch}(\overrightarrow{D}^+_s) - \mathrm{ch}(\overrightarrow{D}^-_s)\right\} \subseteq \overline{U}_s$$

for all $s > 0$. In other words the family $\mathrm{K}_\alpha(s)$ is supported in smaller and smaller "tubular neighborhoods" of $\mathrm{spt}\,\mathrm{Div}(\alpha)$. This allows us to localize the construction as follows.

Suppose that Z is an integral current on X with the property that $Z = \mathrm{Div}(\alpha)$ for an atomic section α of a spin bundle F **defined only over some neighborhood** U of $\mathrm{spt}(Z)$ in X. Choose a metric on F so that $\overline{U}_s \subset U$ for all $s < 1$. Then the family of forms

$$\mathrm{K}_\alpha(s) \stackrel{\mathrm{def}}{=} \mathrm{ch}(\overrightarrow{D}^+_s) - \mathrm{ch}(\overrightarrow{D}^-_s)$$

has support in $\overline{U}_1$ and extends by zero to a family of smooth d-closed forms defined on all of X. It has the property that

$$\lim_{s\to 0} \mathrm{K}_\alpha(s) \;=\; \widehat{\mathbb{A}}^{-1}(D)Z.$$

Suppose now that there exists a neighborhood

$$|Z| \xhookrightarrow{j} U \subset X$$

of $|Z| \equiv \mathrm{spt} Z$ in X which admits a retraction $p : U \rightarrow |Z|$. Then we can define a map

$$j_! : K_{\mathrm{cpt}}(|Z|) \longrightarrow K_{\mathrm{cpt}}(U) \subset K_{\mathrm{cpt}}(X)$$

by

$$j_!(E) \;=\; (p^*E) \cdot (\$^+ - \$^-)$$

where $\$^+$ and $\$^-$ are identified over $U - |Z|$ by α. Recall that we have a natural generalized Thom homomorphism

$$j_! : H^*_{\mathrm{deRham}}(|Z|) \longrightarrow H^{*+2n}_{\mathrm{deRham}}(X)$$

from the cohomology of forms germed an $|Z|$ to the cohomology of currents on X, given by

$$j_!(\varphi) \;=\; \varphi Z.$$

From the preceeding discussion and 5.10 we have the following

Corollary 5.18. *Let Z be a current on X which arises as the divisor of a section of a spin vector bundle F defined in a neighborhood U of $|Z| = \mathrm{spt} Z$ which admits a retraction $p : U \rightarrow |Z|$. Fix D and χ as above. Then*

$$ch(j_!(1)) \;=\; [ch(D_s^+) - ch(D_s^-)]$$

for all $s < 1$ (where $[\gamma]$ denotes the cohomology class of γ). Furthermore, for any complex bundle E on $|Z|$, pulled back over U and endowed with a complex connection, we have

$$ch(j_!(E)) \;=\; [ch(D_{s,E}^+) - ch(D_{s,E}^-)].$$

Taking the limit as $s \rightarrow 0$ of this family of d-closed forms in X gives the equation

$$ch(j_!(E)) \;=\; j_!\left(ch(E)\widehat{\mathbb{A}}^{-1}(F)\right). \tag{5.19}$$

To relate this to the more classical result, suppose that $Z = [Y]$ is the current associated to a compact oriented submanifold $j : Y \hookrightarrow X$ whose normal bundle N carries a spin structure. Fix a tubular neighborhood U of Y and an identification

$U \cong N \xrightarrow{p} Y$. Then the tautological cross-section α of p^*N over $N \cong U$ has $\mathrm{Div}(\alpha) = [Y]$ and the theory applies. In this case (5.19) in the Atiyah-Hirzebruch formula for embeddings $j : Y \hookrightarrow X$ with $j^*w_2(X) = w_2(Y)$ [AH]. Note that we have obtained this formula **at the level of differential forms.**

In this case of a spin embedding $j : X \hookrightarrow Y$ $(w_2(N) = 0)$ the formula can be rewritten slightly. Choose a Riemannian direct sum connection on $TX\,\big|_Y = TY \oplus N$ and extend to a connection on TX over all of X. Then we have that

$$\widehat{\mathbb{A}}(D_{TX})[Y] \;=\; \widehat{\mathbb{A}}(D_{TX})\,\big|_Y\,[Y] \;=\; \widehat{\mathbb{A}}(D_{TY})\widehat{\mathbb{A}}(D_N)[Y].$$

Hence, multiplying (5.12) by the form $\widehat{\mathbb{A}}(D_{TX})$ and passing to cohomology gives the equation

$$\mathrm{ch}(j_!E)\widehat{\mathbb{A}}(X) \;=\; j_!\left(\mathrm{ch}E \cdot \widehat{\mathbb{A}}(Y)\right).$$

This can be rephrased by saying that the diagram

$$\begin{array}{ccc} K(Y) & \xrightarrow{\;j_!\;} & K_{\mathrm{cpt}}(X) \\ {\scriptstyle \mathrm{ch}(\cdot)\widehat{\mathbb{A}}(Y)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ch}(\cdot)\widehat{\mathbb{A}}(X)} \\ H^{2*}(Y) & \xrightarrow[\;j_!\;]{} & H^{2*}_{\mathrm{cpt}}(X) \end{array}$$

commutes.

Remark 5.20. If in all of the above we assume that $\dim_{\mathbb{R}} F \equiv 0 \pmod 8$, then $\$^{\pm}$ are complexifications of real spinor bundles $\$^{\pm}_{\mathbb{R}}$ and $[\$^{+}_{\mathbb{R}}] - [\$^{-}_{\mathbb{R}}]$ represents the Thom class for $KO_{\mathrm{cpt}}(F)$. One can tensor by real bundles with connection and everything goes through. However, at the level of real cohomology little is gained.

Remark 5.21. The Spin^c case. Suppose we are given an oriented real $2n$-dimensional bundle F with Spin^c-structure, and let $\$(F) = \$^+ \oplus \$^-$ be the associated complex spinor bundle. Associated to $\$(F)$ is an auxiliary complex line bundle $\lambda \to X$ with $w_2(F) \equiv c_1(\lambda) \pmod 2$. Choosing an orthogonal connection on F and a hermitian connection D_λ on λ canonically determines a hermitian connection on $\$(F)$. Given an atomic section α of F and an approximate-one χ, we obtain the family of connections $\overrightarrow{D}_s$ on F. With fixed D_λ, we get a family of connections $\overrightarrow{D}_s^{\pm}$ on $\mathbb{S}^{\pm}$. Applying the arguments as above leads to the following analogue of (5.12). Let E be any complex bundle with connection over X and let $D_{E,s}^{\pm}$ be the tensor product of $\overrightarrow{D}_s^{\pm}$ and D_E on $\mathbb{S}^{\pm}\otimes E$. Then we have

$$\mathrm{ch}(\overrightarrow{D}_{s,E}^{+}) - \mathrm{ch}(\overrightarrow{D}_{s,E}^{-}) - \mathrm{ch}(D_E)\mathrm{ch}\left(\tfrac{1}{2}D_\lambda\right)\widehat{\mathbb{A}}^{-1}(D)\,\mathrm{Div}(\alpha) \;=\; dR_s \tag{5.22}$$

where R_s is a family of L^1_{loc}-forms on X such that $\lim_{s\to 0} R_s = 0$.

Proof of Lemma 5.4. Write V as an orthogonal direct sum $V = \bigoplus_{j=1}^{n} V_j$, of oriented 2-dimensional subspaces, each invariant under A and such that with respect to an oriented orthonormal basis (e_{2j-1}, e_{2j}) of V_j we have $A(e_{2j-1}) = a_j e_{2j}$ and $A(e_{2j}) = -a_j e_{2j-1}$. It is a standard fact that

$$\mathbb{S}(V) \;=\; \bigoplus_{j=1}^{n} \mathbb{S}(V_j)$$

and also that

$$\mathrm{tr}_{\mathbb{S}^+}(e^{\mathbf{A}}) - \mathrm{tr}_{\mathbb{S}^-}(e^{\mathbf{A}}) \;=\; \prod_{j=1}^{n}\left\{\mathrm{tr}_{\mathbb{S}_j^+}(e^{\mathbf{A}_j}) - \mathrm{tr}_{\mathbb{S}_j^-}(e^{\mathbf{A}_j})\right\}$$

where $\mathbf{A}_j = \mathbf{A}\,\big|_{\mathbb{S}(V_j)}$ and $\mathbb{S}(V_j) = \mathbb{S}_j^+ \oplus \mathbb{S}_j^-$ is the canonical decomposition with respect to the volume form $e_{2j-1}e_{2j}$. (See [LM]). Hence it suffices to consider the 2-dimensional case. Now if V has oriented bases (e_1, e_2), then Clifford multiplication by $\omega = -ie_1e_2$ is $+1$ on $\mathbb{S}^+$ and -1 on $\mathbb{S}^-$. Hence,

$$(e_1\wedge e_2)\,\big|_{\mathbb{S}^+} = i \quad\text{and}\quad (e_1\wedge e_2)\,\big|_{\mathbb{S}^-} = -i.$$

Writing $A = \sum a_j e_{2j-1}\wedge e_{2j}$ and applying the above, we see that

$$\begin{aligned}\mathrm{tr}_{\mathbb{S}^+}\left(e^{\frac{1}{2i}\mathbf{A}}\right) - \mathrm{tr}_{\mathbb{S}^-}\left(e^{\frac{1}{2i}\mathbf{A}}\right) \;&=\; \prod_{j=1}^{n}\left\{e^{\frac{1}{2}a_j} - e^{-\frac{1}{2}a_j}\right\}\\ &=\; \prod_{j=1}^{n}\left\{\frac{\sinh(a_j/2)}{a_j/2}\right\}a_1\ldots a_n \;=\; \widehat{\mathbb{A}}_0^{-1}(A)\,\mathrm{Pf}(A). \quad \square\end{aligned}$$

V. Cases of Basic Interest

In this chapter we focus attention on some cases of special importance. These include quaternion line bundles, where the theory is in striking analogy with that of Chapter II, and the case of generalized spinor bundles, with homomorphisms given by Clifford multiplication. This latter case gives a different perspective on work in Chapters III and IV. In particular, the natural construction of the Thom class (via the pushforward connection) will extend directly to give an alternate proof of the Differentiable Riemann-Roch Theorem at the level of differential forms.

All the cases considered here fall under a general umbrella of hypotheses which are presented in Section 1. The consequences of these hypotheses are also established there. The subsequent sections then specialize to the diverse cases of interest.

1. The General Rubric.

Throughout this section we shall operate under a set of standing assumptions. The cases of real and complex bundles are parallel here, and our discussion will largely apply to either case. As always, we fix vector bundles $E \to X$ and $F \to X$ with connections D_E and D_F and with metrics, over a manifold X.

When E and F are complex, the connections are assumed to respect the complex structures but not (necessarily) the hermitian metrics. When E and F are only real, the connections are assumed to be orthogonal, i.e., to be compatible with the given bundle metrics.

We assume that $\text{rank}(E) \leq \text{rank}(F)$ and set

$$\text{IC}(E,F) \;=\; \{\alpha \in \text{Hom}(E,F) : \alpha^*\alpha = |\alpha|^2 \text{Id}_E\}.$$

This is a bundle whose fibre over $x \in X$ is the quadratic cone of injective conformal (linear) maps $E_x \to F_x$. We are interested in studying the following situation.

Basic Assumption 1.1. We suppose that there exists an oriented vector subbundle $V \subset \text{Hom}(E,F)$ of real dimension m with the property that

$$V \;\subset\; \text{IC}(E,F).$$

(In the complex case we assume that V is a *complex* subbundle of rank $n = m/2$.)

The given metrics on E and F determine a companion subbundle $V^* = \{v^* \in \text{Hom}(F,E) : v \in V\}$ of $\text{Hom}(F,E)$. Furthermore the connections on E and F canonically determine connections on $\text{Hom}(E,F)$ and on $\text{Hom}(F,E)$. Recall that a vector subbundle $V \subset H$ of a vector bundle H with connection D_H is said to **totally geodesic** if for all $\alpha \in \Gamma(V) \subset \Gamma(H)$, we have $D_H\alpha \in \Gamma(T^*X \otimes V) \subset \Gamma(T^*X \otimes H)$, i.e., if covariant differentiation in H maps sections of V to sections of V.

There is an important family of cases for which the embeddings of V and V^* are totally geodesic, namely those with Property 1.2 which is defined below. All the results of this chapter will be shown to have a particularly beautiful form whenever this property holds.

Property 1.2. *The embeddings $V \hookrightarrow \text{Hom}(E,F)$ and $V^* \hookrightarrow \text{Hom}(F,E)$ are given by a universal construction.*

By a **universal construction** we mean the following. Consider, for simplicity of notation, the case where V, E and F are real bundles of ranks m, M and N respectively. (The complex case is analogous). The data for a universal construction consists of a linear embedding $\lambda : \mathbf{R}^m \hookrightarrow \text{Hom}(\mathbf{R}^M, \mathbf{R}^N)$ and a Lie group homomorphism

$$\varphi \;=\; (\eta,\xi) : \text{GL}_m \longrightarrow \text{GL}_M \times \text{GL}_N$$

so that the diagram

$$\begin{array}{ccc} \mathbf{R}^m & \overset{\lambda}{\hookrightarrow} & \mathrm{Hom}(\mathbf{R}^M, \mathbf{R}^N) \\ g\Big\downarrow & & \Big\downarrow\varphi_g \\ \mathbf{R}^m & \overset{\lambda}{\hookrightarrow} & \mathrm{Hom}(\mathbf{R}^M, \mathbf{R}^N) \end{array}$$

commutes for all $g \in \mathrm{GL}_m$ (where φ_g acts in the standard way on $\mathrm{Hom}(\mathbf{R}^M, \mathbf{R}^N)$). From this data we associate to any m-plane bundle V with connection D_V, a pair of bundles with connection:

$$E = P(V) \times_\eta \mathbf{R}^M \quad \text{and} \quad F = P(V) \times_\xi \mathbf{R}^M$$

where $P(V)$ denotes the frame bundle of V. The equivariant embedding λ determines a totally geodesic, linear embedding $V \hookrightarrow \mathrm{Hom}(E, F)$. Whenever (E, D_E) and (F, D_F) are determined by (V, D_V) in this manner, we say that they are given by the **universal** (λ, φ) **construction.**

Note that under our general set-up a cross-section of V determines a bundle map $E \to F$. Maps occuring in this fashion are well adapted to atomic theory [HS], and many cases of interest do occur this way.

Suppose now that $\alpha \in \Gamma(V)$ is a smooth cross-section of V. We consider α as a bundle map $\alpha : E \to F$ and let $\alpha^* : F \to E$ denote its adjoint. Then by Assumption 1.1 we have

$$\alpha^*\alpha \;=\; |\alpha|^2 \mathrm{Id}_E \tag{1.3}$$

and consequently

$$(D\alpha^*)\alpha + \alpha^*(D\alpha) \;=\; d|\alpha|^2 \mathrm{Id}_E, \tag{1.4}$$

where $D\alpha = D_F \circ \alpha - \alpha \circ D_E$ and $D\alpha^* = D_E \circ \alpha^* - \alpha^* \circ D_F$ are the canonically determined connections on $\mathrm{Hom}(E, F)$ and $\mathrm{Hom}(F, E)$ respectively.

We want to consider the pullback and pushforward connections on E and F given respectively by the formulae: (cf. I.2.7)

$$\overleftarrow{D} = D_E + \frac{\alpha^* D\alpha}{|\alpha|^2} \quad \text{and} \quad \overrightarrow{D} = D_F - \frac{(D\alpha)\alpha^*}{|\alpha|^2}. \tag{1.5}$$

Let $\chi(t)$ be a general approximate one. As before we set $\chi_s \equiv \chi\left(\frac{|\alpha|^2}{s^2}\right)$ and $\chi'_s \equiv \chi'\left(\frac{|\alpha|^2}{s^2}\right)$. The approximating families of smooth connections are then given by

$$\overleftarrow{D}_s = D_E + \chi_s \frac{\alpha^* D\alpha}{|\alpha|^2} \quad \text{and} \quad \overrightarrow{D}_s = D_F - \chi_s \frac{(D\alpha)\alpha^*}{|\alpha|^2}. \tag{1.6}$$

Theorem 1.7. *Suppose V satisfies Assumption 1.1. Fix $\alpha \in \Gamma(V)$ as above and let χ be any approximate one. Consider the pull-back family $\overleftarrow{D}_s$ of connections on E and the pushforward family $\overrightarrow{D}_s$ of connections on F. Then the associated curvatures $\overleftarrow{R}_s = \overleftarrow{D}_s^2$ and $\overrightarrow{R}_s = \overrightarrow{D}_s^2$ are given by the following formulae:*

$$\begin{aligned} \overleftarrow{R}_s &= (1-\chi_s)R_E + \chi_s \overleftarrow{R}_0 + \chi'_s \frac{|\alpha|^2}{s^2} \frac{(D\alpha^*)\alpha\alpha^*(D\alpha)}{|\alpha|^4} \\ &\quad - \left(\chi_s(1-\chi_s) - \chi'_s \frac{|\alpha|^2}{s^2}\right) \frac{\alpha^*(D\alpha)\alpha^*(D\alpha)}{|\alpha|^4} \end{aligned} \tag{1.8}$$

where

$$\overleftarrow{R}_0 = R_E + \frac{\alpha^* R_H(\alpha)}{|\alpha|^2} + \frac{(D\alpha^*)\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right)(D\alpha)}{|\alpha|^2} \tag{1.9}$$

and

$$\begin{aligned} \overrightarrow{R}_s &= (1-\chi_s)R_F + \chi_s \overrightarrow{R}_0 + \chi'_s \frac{|\alpha|^2}{s^2} \frac{(D\alpha)(D\alpha^*)\alpha\alpha^*}{|\alpha|^4} \\ &\quad - \left(\chi_s(1-\chi_s) - \chi' \frac{|\alpha|^2}{s^2}\right) \frac{(D\alpha)\alpha^*(D\alpha)\alpha^*}{|\alpha|^4} \end{aligned} \tag{1.10}$$

where

$$\overrightarrow{R}_0 = R_F - \frac{R_H(\alpha)\alpha^*}{|\alpha|^2} + \frac{(D\alpha)(D\alpha^*)}{|\alpha|^2}\left(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\right) \tag{1.11}$$

and where $R_H(\alpha) = R_F\alpha - \alpha R_E$ denotes the curvature of the bundle $H =$ $\mathrm{Hom}(E,F)$.

Proof. Equations (1.8) and (1.10) follow from I.6.3 and I.6.5 by using the relation 1.4 above. Equations (1.9) and (1.11) come from I.6.18 and I.6.23 respectively. □

Observation 1.12.

(a) If V has Property 1.2, then $D\alpha$ and $D\alpha^*$ use the connections on V and V^*, and

$$R_H = R_V.$$

(b) If $\chi(t) = \frac{t}{1+t}$ (algebraic approximation mode) then

$$\chi_s(1-\chi_s) - \chi_s' \frac{|\alpha|^2}{s^2} \equiv 0.$$

(c) If $\operatorname{rank} E = \operatorname{rank} F$, then $\alpha\alpha^* = |\alpha|^2$ and we have

$$\overleftarrow{R}_0 = \frac{\alpha^* R_F \alpha}{|\alpha|^2} \quad ; \quad \overrightarrow{R}_0 = \frac{\alpha R_E \alpha^*}{|\alpha|^2}.$$

Note 1.13. Via (1.4) the equations (1.8) and (1.10) are equivalent to the following. (This is the form in which they appear in I.6.2.)

(1.14)
$$\overleftarrow{R}_s = (1-\chi_s)R_E + \chi_s \overleftarrow{R}_0 + \chi_s' \frac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{\alpha^*(D\alpha)}{|\alpha|^2} - \chi_s(1-\chi_s)\frac{\alpha^*(D\alpha)\alpha^*(D\alpha)}{|\alpha|^4}$$

(1.15)
$$\overrightarrow{R}_s = (1-\chi_s)R_F + \chi_s \overrightarrow{R}_0 + \chi_s' \frac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2} \frac{(D\alpha)\alpha^*}{|\alpha|^2} - \chi_s(1-\chi_s)\frac{(D\alpha)\alpha^*(D\alpha)\alpha^*}{|\alpha|^4}.$$

We can now begin our study of characteristic forms and currents in this context. For the moment we restrict attention to the pull-back case.

Let ϕ be an Adjoint-invariant polynomial on $gl_n(K)$ where $n = \operatorname{rank}(E)$ and $K = \mathbf{R}$ or $\mathbf{C}$ depending on whether the bundles are real or complex. We want to study the characteristic form $\phi(\overleftarrow{D}_s)$ and transgression forms T_s' with $dT_s' = \phi(\overleftarrow{D}_s) - \phi(D_E)$. To do this we shall first pass to the "universal case" by pulling back over V. Then before computing transgressions we shall replace our connection family $\overleftarrow{D}_s$ by a gauge equivalent family $\overleftarrow{D}'_s$ which, of course, leaves $\phi(\overleftarrow{D}_s)$ unchanged but produces a better behaved transgression, denoted T_s' rather than T_s.

We now pass to the total space of the vector bundle $\pi : V \to X$. Set $\mathbf{E} = \pi^* E$, $\mathbf{F} = \pi^* F$, $\mathbf{V} = \pi^* V \subset \operatorname{Hom}(\mathbf{E}, \mathbf{F})$ and furnish these bundles with the pullback connections and metrics. Let $\mathbf{v} \in \Gamma(\mathbf{V})$ denote the **tautological cross-section** defined by $\mathbf{v}_v = v$. Note that if $\alpha : X \to V$ is a cross-section of V, then

$$\alpha^*(\mathbf{v}) \;=\; \alpha.$$

We let $\overleftarrow{\mathbf{D}}_{\mathbf{s}}$ denote the pull-back family of connections on $\mathbf{E}$ associated to the tautological cross-section $\mathbf{v}$ by the approximate one $\chi(t)$. Note that

$$\overleftarrow{\mathbf{D}}_{\mathbf{s}} \;=\; \mathbf{D}_E + \chi_s \frac{\mathbf{v}^*\mathbf{D}\mathbf{v}}{|\mathbf{v}|^2} \tag{1.16}$$

where $\mathbf{D}\mathbf{v}$ denotes the covariant derivative of $\mathbf{v}$ as a section of $\mathrm{Hom}(\mathbf{E}, \mathbf{F})$. By (1.4) we have the equation

$$d|\mathbf{v}|^2 \;=\; (\mathbf{D}\mathbf{v}^*)\mathbf{v} + \mathbf{v}^*\mathbf{D}\mathbf{v}. \tag{1.17}$$

We now perform, for each $s > 0$, a global **scalar** gauge transformation of the bundle $\mathbf{E}$ by multiplication by a function of the form e^{r_s} for $r_s \in C^\infty(V)$. This gives a transformed family of connections

$$\overleftarrow{\mathbf{D}}'_{\mathbf{s}} \;=\; e^{r_s} \circ \overleftarrow{\mathbf{D}}_{\mathbf{s}} \circ e^{-r_s} \;=\; \overleftarrow{\mathbf{D}}_{\mathbf{s}} - dr_s, \tag{1.18}$$

and of course

$$\overleftarrow{\mathbf{R}}'_{\mathbf{s}} \overset{\text{def}}{=} (\overleftarrow{\mathbf{D}}'_{\mathbf{s}})^2 \;=\; (\overleftarrow{\mathbf{D}}_{\mathbf{s}})^2 \;=\; \overleftarrow{\mathbf{R}}_{\mathbf{s}} \tag{1.19}$$

for all s. We set

$$r_s(v) \;=\; r(|v|^2/s^2)$$

where

$$r(t) \;=\; \tfrac{1}{2}\int_0^t \frac{1}{\tau}\chi(\tau)d\tau.$$

Note that if $\chi(t) = t/(1+t)$, then $e^{r_s} = \frac{1}{s}\sqrt{|v|^2 + s^2}$.

A straightforward calculation using (1.17) shows that

$$\overleftarrow{\mathbf{D}}'_{\mathbf{s}} \;=\; \mathbf{D}_E + \chi_s \frac{1}{2|\mathbf{v}|^2}(\mathbf{v}^*\mathbf{D}\mathbf{v} - (\mathbf{D}\mathbf{v}^*)\mathbf{v}). \tag{1.20}$$

Lemma 1.21. *The family of connections $\overleftarrow{\mathbf{D}}'_{\mathbf{s}}$ which is conformally gauge equivalent to the pull-back family $\overleftarrow{\mathbf{D}}_{\mathbf{s}}$ (and therefore has the same curvature tensor), has the following property. If $\mathbf{D}_E$ and $\mathbf{D}_F$ are metric compatible connections, then $\overleftarrow{\mathbf{D}}'_{\mathbf{s}}$ is also a metric compatible connection for every $s > 0$.*

Proof. Under this assumption we have that $\mathbf{Dv}^* = (\mathbf{Dv})^*$. Therefore by (1.20) we have that $\overleftarrow{\mathbf{D}}'_s - \mathbf{D}_E$ is skew-adjoint. $\square$

We now consider the (real) Euler vector field ϵ on V which generates the scalar multiplication flow $\varphi_t(v) = e^t v$. If e is a local framing for V over $\mathbf{R}$ and $v = (v_1, \ldots, v_m)$ are the linear fibre coordinates with respect to this framing, then $\epsilon = v \cdot \frac{\partial}{\partial v} = \sum v_j \frac{\partial}{\partial v_j}$.

Lemma 1.22.

$$\epsilon \llcorner (\mathbf{Dv}) = \mathbf{v} \quad \textit{and} \quad \epsilon \llcorner (\mathbf{Dv}^*) = \mathbf{v}^*.$$

Proof. Fix a local real orthonormal framing $e = (e_1, \ldots, e_m)$ of V, and let $v = (v_1, \ldots, v_m)$ be the linear fibre coordinates on V with respect to this framing. Extend the framing $e_1, \ldots, e_m$ to a local orthonormal framing $e_1, \ldots, e_M$ of $\mathrm{Hom}(E, F) \supset V$. Let $w_{ij} = \langle De_i, e_j \rangle_{\mathbf{R}}$ be the matrix of connection 1-forms for $\mathrm{Hom}(E, F)$ on X with respect to this framing, and let $\tilde{w}_{ij} = \pi^* w_{ij}$ denote the lift of these forms back over V. Let $\tilde{e}_i = \pi^* e_i$ for all i, so that $\mathbf{v} = \sum v_i \tilde{e}_i$. Then

$$\mathbf{Dv} = \mathbf{D}\left(\sum_{i=1}^{m} v_i \tilde{e}_i\right) = \sum_{i=1}^{m} dv_i \tilde{e}_i + \sum_{i=1}^{m} \sum_{j=1}^{M} v_i \tilde{w}_{ij} \tilde{e}_j.$$

Recalling that $\epsilon = v \cdot \frac{\partial}{\partial v} = \sum v_i \frac{\partial}{\partial v_i}$ we see that $\epsilon \llcorner \mathbf{Dv} = \mathbf{v}$.

For the second equation we consider the $\mathbf{R}$-linear ($\mathbf{C}$-antilinear) identification $V \xrightarrow{(\ )^*} V^*$ which is the restriction of the adjoint map $(\cdot)^* : \mathrm{Hom}(E, F) \to \mathrm{Hom}(F, E)$. Then $e_1^*, \ldots, e_M^*$ is a local orthonormal frame field for $\mathrm{Hom}(F, E)$ such that $e_1^*, \ldots, e_m^*$ spans V^* pointwise. Let $w_{ij}^* = \langle De_i^*, e_j^* \rangle$, and set $\tilde{e}_i^* = \pi^* e_i^*$ and $\tilde{w}_{ij}^* = \pi^* w_{ij}^*$ as before. Then $\mathbf{v}^* = \sum v_i \tilde{e}_i^*$ and

$$\mathbf{Dv}^* = \mathbf{D}\left(\sum_{i=1}^{m} v_i \tilde{e}_i^*\right) = \sum_{i=1}^{m} dv_i \tilde{e}_i^* + \sum_{i=1}^{m} \sum_{j=1}^{M} v_i \tilde{w}_{ij}^* \tilde{e}_j^*$$

and we see that $\epsilon \llcorner \mathbf{Dv}^* = \mathbf{v}^*$ as claimed. $\square$

It follows immediately from (1.9), with α^* replaced by $\mathbf{v}^*$, that $\epsilon \llcorner \overleftarrow{\mathbf{R}}_0 = 0$. Note also that $\epsilon \llcorner (\mathbf{v}^*(\mathbf{D}\mathbf{v})\mathbf{v}^*(\mathbf{D}\mathbf{v})) = 0$ and that $\epsilon \llcorner d|\mathbf{v}|^2 = 2|\mathbf{v}|^2$. Set

$$\mathbf{A}(x) \stackrel{\text{def}}{=} (1-x)\mathbf{R}_E + x\overleftarrow{\mathbf{R}}_0 - x(1-x)\frac{\mathbf{v}^*(\mathbf{D}\mathbf{v})\mathbf{v}^*(\mathbf{D}\mathbf{v})}{|\mathbf{v}|^4}$$

for an indeterminate x, and note that

$$\epsilon \llcorner \mathbf{A}(x) \;=\; 0.$$

From (1.14) and (1.4) we have

$$\begin{aligned} \overleftarrow{\mathbf{R}}_s &= \mathbf{A}(\chi_s) + \chi'_s \frac{d|\mathbf{v}|^2}{|\mathbf{v}|^2}\frac{1}{s^2}\mathbf{v}^*(\mathbf{D}\mathbf{v}) \\ &= \mathbf{A}(\chi_s) + \chi'_s \frac{d|\mathbf{v}|^2}{2|\mathbf{v}|^2}\frac{1}{s^2}\left\{\mathbf{v}^*(\mathbf{D}\mathbf{v}) - (\mathbf{D}\mathbf{v}^*)\mathbf{v}\right\}. \end{aligned} \tag{1.23}$$

We now define

$$\omega_s \;=\; \overleftarrow{\mathbf{D}}'_s - \mathbf{D}_E = \chi_s \frac{1}{2|\mathbf{v}|^2}\left\{\mathbf{v}^*(\mathbf{D}\mathbf{v}) - (\mathbf{D}\mathbf{v}^*)\mathbf{v}\right\}$$

and note that

$$\epsilon \llcorner \omega_s \;=\; 0.$$

From (1.23) we see that

$$\epsilon \llcorner \overleftarrow{\mathbf{R}}_s \;=\; \chi'_s \frac{1}{s^2}\left\{\mathbf{v}^*(\mathbf{D}\mathbf{v}) - (\mathbf{D}\mathbf{v}^*)\mathbf{v}\right\} \;=\; -s\frac{\partial}{\partial s}(\omega_s). \tag{1.24}$$

Consequently

$$\left(\tfrac{-1}{s}\right)\epsilon \llcorner \phi(\overleftarrow{\mathbf{R}}_s) \;=\; \phi\left(\frac{\partial \omega_s}{\partial s}\;;\;\overleftarrow{\mathbf{R}}_s\right), \tag{1.25}$$

the standard transgression integrand. In part, we have the transgression form

$$\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}_E) \;=\; d\int_s^\infty \epsilon \llcorner \phi(\overleftarrow{\mathbf{R}}_t)\frac{dt}{t}. \tag{1.26}$$

We now make an important simplification

Lemma 1.27. *Set*

$$\Theta = -\frac{1}{2|\mathbf{v}|^2}\{\mathbf{v}^*(\mathbf{D}\mathbf{v}) - (\mathbf{D}\mathbf{v}^*)\mathbf{v}\}.$$

Then

$$\tfrac{1}{s}\epsilon \lrcorner\, \phi(\overleftarrow{\mathbf{R}}_s) = \phi(\Theta; \mathbf{A}(\chi_s))\,\frac{\partial}{\partial s}(\chi_s).$$

Proof. Note that $\overleftarrow{\mathbf{R}}_s = \mathbf{A}(\chi_s) - \frac{1}{s^2}\chi_s' d|\mathbf{v}|^2\Theta$, and since $d|\mathbf{v}|^2$ is a scalar 1-form

$$\phi(\overleftarrow{\mathbf{R}}_s) = \phi(\mathbf{A}(\chi_s)) - \frac{1}{s^2}\chi_s' d|\mathbf{v}|^2 \wedge \phi(\Theta; \mathbf{A}(\chi_s)).$$

Contract this equation with ϵ. Noting that $\epsilon \lrcorner\, A(\chi_s) = \epsilon \lrcorner\, \Theta = 0$ and that $\epsilon \lrcorner\, d|\mathbf{v}|^2 = 2|\mathbf{v}|^2$ gives the lemma. □

Proposition 1.28. *For each $s > 0$, the smooth form*

$$\mathbf{T}_s' \stackrel{\text{def}}{=} \int_0^s \phi(\Theta\ ;\ \mathbf{A}(x))dx \tag{1.29}$$

on V satisfies the equation

$$d\mathbf{T}_s' = \phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}_E). \tag{1.30}$$

The limit

$$\mathbf{T}' \stackrel{\text{def}}{=} \lim_{s \to 0} \mathbf{T}_s' \tag{1.31}$$

exists in $L^1_{\text{loc}}(V)$ and satisfies the equation

$$d\mathbf{T}' = \phi(\overleftarrow{\mathbf{R}}_0) - \phi(\mathbf{R}_E) + \operatorname{Res}(\mathbf{T})[X] \qquad \text{on } V \tag{1.32}$$

where $\operatorname{Res}_\phi \stackrel{\text{def}}{=} \operatorname{Res}(\mathbf{T}')$ is defined as in III.1.9.

When restricted to the subset $V - X$, each of $\mathbf{T}'$, $d\mathbf{T}'$ and $\phi(\overleftarrow{\mathbf{R}}_0)$ is a smooth form which is homogeneous (i.e., invariant under multiplication by positive scalars on V) and has degree $\leq m - 1$ in the fibre differentials.

Proof. Note that $\mathbf{T}'_s$ can be written as

$$\mathbf{T}'_s = -\int_s^\infty \phi(\Theta; \mathbf{A}(\chi_s))\,(\tfrac{\partial}{\partial s}\chi_s)\,ds.$$

Applying (1.26) and 1.27 gives the first assertion.

It is evident from the definitions of $\mathbf{A}$ and Θ that the integrand $\phi(\Theta; \mathbf{A}(x))$ in (1.29) is a polynomial in x whose coefficients are homogeneous forms on V. Hence $\mathbf{T}'_s$ can be written as a finite sum

$$\mathbf{T}'_s = \sum_{k=1}^{N} \psi_k\,(\chi_s)^k$$

where each ψ_k is a homogeneous form (which is smooth on $V - X$). Furthermore, by 1.27 we see that

$$\epsilon \llcorner \phi\,(\Theta; \mathbf{A}(x)) = 0. \tag{1.33}$$

Hence, each coefficient ψ_k is of degree $\leq m - 1$ in the fibre differentials. In particular, ψ_k is in L^1_{loc} (cf. III.1.5). Note that $0 \leq \chi_s \leq 1$ and $\chi_s \to 1$ a.e. on V. Therefore by the Lebesgue Dominated Convergence Theorem, the limit $\lim_{s\to 0} \mathbf{T}'_s = \mathbf{T}'$ exists in L^1_{loc} on V.

Observe that $\mathbf{T}'_s \to \mathbf{T}'$ and $\overleftarrow{\mathbf{R}}_s \to \overleftarrow{\mathbf{R}}_0$ in the C^∞-topology on $V - X$. Hence we have the equation

$$d\mathbf{T}' = \phi(\overleftarrow{\mathbf{R}}_0) - \phi(\mathbf{R}_E)$$

of smooth forms on $V - X$. Now from (1.9) we see that $\phi(\overleftarrow{\mathbf{R}}_0)$ is homogeneous and (since $\epsilon \llcorner \phi(\overleftarrow{\mathbf{R}}_0) = 0$) has degree $\leq m - 1$ in the fibre differentials. Of course $\phi(\mathbf{R}_E)$ also has these properties since it is the pull-back of a form on X. Consequently $\mathbf{T}'$ satisfies the hypotheses of Theorem III.1.10, and equation (1.32) follows from part a) of that result. □

Proposition 1.34. *The smooth form $\phi(\overleftarrow{\mathbf{R}}_0)$ on $V - X$ has an L^1_{loc}-extension across X which satisfies*

$$d\phi(\overleftarrow{\mathbf{R}}_0) = 0 \qquad \text{on } V$$

Consequently, the smooth form Res_ϕ *satisfies*

$$d\operatorname{Res}_\phi = 0 \qquad \text{on } X. \tag{1.35}$$

Proof. Since $\phi(\overleftarrow{\mathbf{R}}_0)$ is homogeneous and of degree $\leq m-1$ in the fibre differentials, we know from III.1.5 that it extends across X as an L^1_{loc}-form on V. We denote this L^1_{loc}-extension by $\phi(\overleftarrow{\mathbf{R}}_0)^{\sim}$. Of course $d\phi(\overleftarrow{\mathbf{R}}_0) = 0$ on $V - X$, so to prove that $d(\phi(\overleftarrow{\mathbf{R}}_0)^{\sim}) = 0$ it suffices to prove that the residue of $\phi(\overleftarrow{\mathbf{R}}_0)$ is zero (cf. III.1.10).

We first suppose that V is complex of dimension $n = m/2$. Fix a local frame field for V and let $(Dv)_j$, $(Dv^*)_j$, $j = 1, \dots, n$ denote the scalar 1-forms on the total space of V which appear as the components of $\mathbf{Dv}$ and $\mathbf{Dv}^*$ with respect to (the pull-back of) this frame field. Then at each point of $V - X$ the $\phi(\overleftarrow{\mathbf{R}}_0)$ can be written in the form

$$\phi(\overleftarrow{\mathbf{R}}_0) \;=\; \sum_{\substack{|I|,|J|\leq n \\ |I|+|J|\leq 2n-1}} r_{I,J}(Dv)_I(Dv^*)_J$$

where the sum is over strictly ascending multi-indices and where each $r_{I,J}$ is a polynomial in Ω_E, Ω_F. Observe that since $\deg\phi(\overleftarrow{\mathbf{R}}_0)$, $\deg\Omega_E$, and $\deg\Omega_F$ are all even integers, we must have $r_{I,J} = 0$ whenever $|I| + |J|$ is odd. In particular, $\phi(\overleftarrow{\mathbf{R}}_0)$ is of degree $\leq 2n - 2 = m - 2$ in the fibre differentials. It now follows from III.1.8 that $d(\phi(\overleftarrow{\mathbf{R}}_0)^{\sim}) = 0$. This completes the argument in the complex case.

The argument for real bundles of even rank is similar. Note that metric compatibility implies that $\mathbf{Dv}^* = (\mathbf{Dv})^* = (\sum(Dv)_j\epsilon_j)^* = \sum(Dv)_j\epsilon_j^*$, and so $(Dv^*)_j = (Dv)_j$ for $j = 1, \dots, m$. Consequently $\phi(\overleftarrow{\mathbf{R}}_0)$ has an expansion

$$\phi(\overleftarrow{\mathbf{R}}_0) \;=\; \sum_{|I|\leq m-1} r_I(Dv)_I$$

analogous to the one above, where each r_I is a polynomial in Ω_E and Ω_F. Again for reasons of parity we must have $r_I = 0$ for $|I|$ odd. Hence if m is even, then $\phi(\overleftarrow{\mathbf{R}}_0)$ is of degree $\leq m-2$ in the fibre differentials, and III.1.8 applies as above.

It remains to consider the case where V is of real and of odd rank. (Here the argument must be different.) Let $S(V) = \{v \in V : |v| = 1\}$ denote the sphere bundle with projection $\rho : S(V) \longrightarrow X$. Then by III.1.10 it will suffice to prove that

$$\int_\rho \phi(\overleftarrow{\mathbf{R}}_0) \;=\; 0.$$

This follows from the fact that $\mu^*\phi(\overleftarrow{\mathbf{R}}_0) = \phi(\overleftarrow{\mathbf{R}}_0)$, where $\mu : V \longrightarrow V$ is given by $\mu(v) = -v$, and the fact that μ reverses the orientation of the fibres of ρ.

In this last case an alternative argument can be given by observing that, when lifted to the blow-up $f : \widetilde{V} \longrightarrow V$ of V along the zero-section, $\phi(\overleftarrow{\mathbf{R}}_0)$ extends to a smooth, d-closed form, say $\hat{\phi}$. (See I.3.) Since $\widetilde{V}$ is orientable when $\dim V$ is odd, we can then apply Proposition I.3.12 to conclude that $df_*\hat{\phi} = d(\phi(\overleftarrow{\mathbf{R}}_0)^\sim) = 0$. □

Remark 1.36. We introduced the conformal gauge change above precisely in order to prove that $\mathbf{T}'$ is homogeneous and of degree $\leq m - 1$ in the fibre 1-forms. This fact enabled us to apply the general residue theorem III.1.10 for atomic sections.

This brings us to the main result of this section.

Theorem 1.37. *Suppose $\pi : V \to X$ is a bundle which satisfies Assumption 1.1. Let $\alpha \in \Gamma(V)$ be an atomic section of V, and let ϕ be an invariant polynomial as above. Let $A(x) = \alpha^*\mathbf{A}(x)$ and $\Theta = \alpha^*(\Theta)$ be the forms obtained by substituting α for $\mathbf{v}$ in the universal expressions above, and for each $s > 0$ define*

$$T'_s \;=\; \int_0^{\chi_s} \phi(\Theta; A(x))dx.$$

Then

$$dT'_s \;=\; \phi(\overleftarrow{R}_s) - \phi(R_E).$$

Furthermore $T' = \lim_{s\to 0} T'_s$ exists in $L^1_{\mathrm{loc}}(X)$ and is given by the formula

$$T' \;=\; \int_0^1 \phi(\Theta; A(x))\,dx \;=\; \alpha^*(\mathbf{T}') \tag{1.38}$$

which is independent of χ. In particular, the characteristic current $\phi((\overleftarrow{D})) = \lim_{s\to 0} \phi(\overleftarrow{R}_s) = dT + \phi(R_E)$ exists and is independent of the choice of the approximate-one χ used in the limiting process. Furthermore,

$$\phi(\overleftarrow{R}_0) - \phi(R_E) + \mathrm{Res}_\phi \operatorname{Div}(\alpha) \;=\; dT \tag{1.39}$$

where $\phi(\overleftarrow{R}_0)$ is in $L^1_{\rm loc}(X)$ and where Res_ϕ *is the smooth, d-closed differential form on X given as follows. Set*

$$\mathbf{T}' = \int_0^1 \phi(\Theta; \mathbf{A}(x))\, dx, \tag{1.40}$$

and let $\rho : S(V) \longrightarrow X$ be the projection of the unit sphere bundle $S(V) = \{v \in V : ||v|| = 1\}$. Then the residue is given by the fibre integral

$$\mathrm{Res}_\phi = \int_\rho \mathbf{T}'. \tag{1.41}$$

In the special case where $\mathrm{rank}(E) = \mathrm{rank}(F)$, *we have the equation of currents and forms on X*

$$\phi(R_F) - \phi(R_E) + \mathrm{Res}_\phi\, \mathrm{Div}(\alpha) = dT. \tag{1.42}$$

Proof. Proposition 1.28 and the hypothesis that α is atomic together imply that T'_s is a polynomial in $\chi_s = \chi(|\alpha|^2/s^2)$ whose coefficients are $L^1_{\rm loc}$-forms on X. Since χ_s is bounded and converges to 1 almost everywhere, we have that $T'_s \to T'$ in $L^1_{\rm loc}$ as $s \to 0$ by the Lebesgue Dominated Convergence Theorem. This establishes the first assertion.

We now recall that the universal form $\phi(\overleftarrow{\mathbf{R}}_0)$ on $V - X$ is homogeneous and of degree $\leq m-1$ in the fibre differentials on V. Since α is atomic, it follows that $\phi(\overleftarrow{R}_0) = \alpha^*\phi(\overleftarrow{\mathbf{R}}_0)$ is an $L^1_{\rm loc}$-form on X (cf. III.1.5). Furthermore by (1.35) and (1.39) we conclude that $d\phi(\overleftarrow{R}_0) = 0$.

The remainder of the theorem now follows directly from 1.28 and III.1.10. □

Observation 1.43. The residue Res_ϕ given in (1.41) is a smooth, d-closed differential form which is given pointwise on X by a **universal invariant function applied to the curvature and connection of** V. Specifically, at each $p \in X$

$$(\mathrm{Res}_\phi)_p = \int\limits_{\substack{|v|=1 \\ v \in V_x}} \int_0^1 \phi(\Theta;\, \mathbf{A}(x))_{top}\, dx \tag{1.44}$$

where $\phi(\Theta;\ \mathbf{A}(x))_{top}$ denotes the coefficient of $\epsilon \llcorner (dv_1 \wedge \cdots \wedge dv_m)$ in $\phi(\Theta; \mathbf{A}(x))$. This residue defines a linear map

$$I^*_{\mathrm{GL}_M} \xrightarrow{\mathrm{Res}} \mathcal{E}^*_X \tag{1.45}$$

from the Ad_{GL_M}-invariant polynomials on the Lie algebra $\mathfrak{gl}_M$ (where $M = \mathrm{rank}(E)$) into the closed differential forms on X. We have the following fundamental result.

Theorem 1.46. *The residue (1.41) is a Chern-Weil characteristic form associated universally to the bundle $E \oplus F$ with its given connection. In fact there exists a linear map of degree $-m$:*

$$\rho : I^*_{\mathrm{GL}_M} \longrightarrow I^*_{\mathrm{GL}_M} \otimes I^*_{\mathrm{GL}_{M'}},$$

where $M' = \mathrm{rank}(F)$, such that the residue map (1.45) is given as the composition

$$I^*_{\mathrm{GL}_M} \xrightarrow{\rho} I^*_{\mathrm{GL}_M} \otimes I^*_{\mathrm{GL}_{M'}} \xrightarrow{W^{E\oplus F}} \mathcal{E}^*_X \tag{1.47}$$

where $W^{E\oplus F}$ is the usual Chern-Weil homomorphism.

Furthermore, if the embedding $V \subset \mathrm{Hom}(E, F)$ comes from a universal construction (i.e., if Property 1.2 holds), then the residue is a Chern-Weil characteristic form for V, i.e., the residue map (1.45) factors as above

$$I^*_{\mathrm{GL}_M} \xrightarrow{\rho'} I^*_{\mathrm{GL}_m} \xrightarrow{W^V} \mathcal{E}^*_X \tag{1.48}$$

where W^V is the Chern-Weil homomorphism for V.

The linear maps ρ and ρ' are topologically determined, and so therefore is the residue map itself.

Proof. Suppose that V is complex of dimension $n = m/2$ and that the connection respects the complex structure. Fix a local frame field for V and let $(Dv)_j$, $(Dv^*)_j$, $j = 1, \ldots, n$ denote the scalar 1-forms on the total space of V which appear as the components of $\mathbf{Dv}$ and $\mathbf{Dv}^*$ with respect to (the pull-back of) this frame field. Then at each point of $V - X$ the integrand in (1.44) can be written in the form

$$\phi(\Theta; \mathbf{A}(x)) = \sum_{|I|,|J|\le n} c_{I,J}(Dv)_I(Dv^*)_J \tag{1.49}$$

where the sum is over strictly ascending multi-indices and where each $c_{I,J}$ is a polynomial in Ω_E, Ω_F, and x. By (1.33) the coefficient $c_{(1,\dots,n),(1,\dots,n)}$ of top degree is zero. Consequently, the fibre integral

$$\begin{aligned}\int_{|v|=1} \phi(\Theta;\ \mathbf{A}(x))\,dx \;&=\; \sum_{|I|+|J|\le m-1} \int_{|v|=1} c_{I,J}(Dv)_I(Dv^*)_J \\ &=\; \sum_{|I|+|J|=m-1} \int_{|v|=1} c_{I,J}(dv)_I(dv^*)_J.\end{aligned} \tag{1.50}$$

contains no terms involving ω_E or ω_F. In other words the fibre integral (1.44) is a universal Ad-invariant function in Ω_E and Ω_F. Hence it is produced in the standard way from a series of homogeneous Ad-invariant polynomials on the Lie algebra $\mathfrak{gl}_M \times \mathfrak{gl}_{M'}$. We have now established the existence of the linear map ρ. This map is topologically determined since the universal Weil map $I^*_{\mathrm{GL}_M} \otimes I^*_{\mathrm{GL}_{M'}} \longrightarrow H^*(BGL_m \times BGL_{M'})$ is injective. This completes the argument in the case of complex connections.

The argument for the case of real bundles with metric-compatible connections is similar. To begin we note that metric compatibility implies that $\mathbf{Dv}^* = (\mathbf{Dv})^* = (\sum (Dv)_j \epsilon_j)^* = \sum (Dv)_j \epsilon_j^*$, and so $(Dv^*)_j = (Dv)_j$ for $j = 1, \dots, m$. Consequently we have an expansion

$$\phi(\Theta; \mathbf{A}(x)) = \sum_{|I|\le m} c_I (Dv)_I$$

analogous to (1.49) above. Proceeding from this point the argument is completely analogous. □

Theorem 1.6 asserts that the residue form (1.44) is uniquely determined by a topological determination of class at the universal level. In subsequent sections we shall carry this out explicitly for a number of important cases.

We observe now that in the universal case (over the total space of V) equation (1.39) can be rewritten as

$$\phi(\overleftarrow{\mathbf{R}}_0) - \phi(\mathbf{R}_E) + \mathrm{Res}_\phi[X] \;=\; d\mathbf{T}'. \tag{1.51}$$

Using the equation $d\mathbf{T}'_s = \phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}_E)$ from 1.28 gives

$$\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\overleftarrow{\mathbf{R}}_0) \;=\; \mathrm{Res}_\phi[X] + d(\mathbf{T}'_s - \mathbf{T}') \tag{1.52}$$

where $\mathbf{T}'_s - \mathbf{T}' \to 0$ in L^1_{loc} as $s \to 0$. We now observe that the forms $\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\overleftarrow{\mathbf{R}}_0)$ and $\mathbf{T}'_s - \mathbf{T}'$ can be integrated over the fibres of $\pi : V \to X$. This follows from Corollary I.6.17 by restricting to the compactification of $V \subset \text{Hom}(E, F)$ in $G_p(E \oplus F)$. Applying π_* to (1.52) gives the following.

Proposition 1.53. Alternative Formula for the Residue. *The residue form in the universal case can be written as*

$$\text{Res}_\phi \;=\; \pi_*\{\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\overleftarrow{\mathbf{R}}_0)\}.$$

Proof. Note that $\pi_* d(\mathbf{T}'_s - \mathbf{T}') = d\pi_*(\mathbf{T}'_s - \mathbf{T}')$, and that $\pi_*(\mathbf{T}'_s - \mathbf{T}') = 0$ because $\mathbf{T}'_s - \mathbf{T}'$ is of degree $< m$ in the fibre 1-forms $dv_1, \ldots, dv_m$. $\square$

This Proposition has an important interpretation at the cohomology level. Let $H^*_{\text{cpt}}(V\,;\,\mathbf{R})$ denote the deRham cohomology of V with compact supports in the fibre directions. If V is oriented, there is a **Thom isomorphism**

$$i_! : H^*(X\,;\,\mathbf{R}) \longrightarrow H^{*+m}(V\,;\mathbf{R})$$

whose **inverse**

$$(i_!)^{-1} \;=\; \pi_! : H^{*+m}(V\,;\mathbf{R}) \longrightarrow H^*(X\,;\,\mathbf{R})$$

is given by integration over the fibre. Suppose now that $\text{rank}\, E = \text{rank}\, F$ and choose an approximate-one χ so that $\chi(t) \equiv 1$ for $t \geq 1$. Then $\overleftarrow{\mathbf{D}}_s$ is a connection on $\mathbf{E} = \pi^* E$ with the property that

$$\overleftarrow{\mathbf{D}}_s \;=\; \overleftarrow{\mathbf{D}}_0 = \mathbf{v}^{-1} \circ (\mathbf{D}_F) \circ \mathbf{v}$$

outside the s-neighborhood $(X)_s \overset{\text{def.}}{=} \{v \in V : |v| < s\}$ of the zero section. Consequently

$$\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}_F) \equiv 0 \quad \text{on} \quad V - (X)_s$$

and thereby determines a class

$$\Big[\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}_F)\Big] \in H^*_{\text{cpt}}(V\,;\mathbf{R}).$$

This class is an invariant of the triple $(\pi^*E, \pi^*F; \mathbf{v})$ in the following sense. Suppose $\mathbf{w} : \pi^*E \longrightarrow \pi^*F$ is a bundle isomorphism defined in $V - (X)_t$ for some $t > 0$, and homotopic over $V - (X)_t$ to the map $\mathbf{v}$. Suppose D' and D'' are connections in π^*E and π^*F respectively with the property that $D' = \mathbf{w}^{-1} \circ D'' \circ \mathbf{w}$ in $V - (X)_t$. Let $R^{(k)} = (D^{(k)})^2$. Then the standard Chern-Weil argument shows that

$$[\phi(R') - \phi(R'')] \;=\; [\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}_F)] \quad \text{in} \quad H^*_{\text{cpt}}(V\ ;\ \mathbf{R}).$$

We denote this class simply by $[\phi(\mathbf{R}_E) - \phi(\mathbf{R}_F)]$.

Proposition 1.53 has the following important consequence

Corollary 1.54. *The cohomology class of the residue form on X is given by*

$$[\mathrm{Res}_\phi] \;=\; \pi_![\phi(\mathbf{R}_E) - \phi(\mathbf{R}_F)].$$

If furthermore Property 1.2 holds, then both E and F are associated to the bundle V, and the class $\phi(R_E) - \phi(R_F)$ is universally divisible by the Euler class $\chi(V)$ of V. In this case, the residue form is given precisely by the formula

$$\mathrm{Res}_\phi \;\equiv\; \frac{\phi(R_E) - \phi(R_F)}{\chi(V)},$$

where the right hand side denotes the Chern-Weil representative of this universal class for the connection on V.

Proof. The first assertion follows immediately from 1.53. To prove the second assertion we consider the case of the universal bundle with some connection $V_0 \longrightarrow X_0 \sim \mathrm{BGL}_m$ over a finite-dimensional approximation to the classifying space (i.e., the tautological m-plane bundle over the Grassmannian of oriented m-planes in $\mathbf{R}^{m+n}$ for n large). The universal construction comes from an embedding $\mathbf{R}^m \hookrightarrow \mathbf{R}^M \times \mathbf{R}^N$ equivariant with respect to a homomorphism $y = (\eta, \xi)$: $\mathrm{GL}_m \longrightarrow \mathrm{GL}_M \times \mathrm{GL}_N$. We define the associated bundles $E_0 = P(V_0) \times_\eta \mathbf{R}^M$ and $F_0 = P(V_0) \times_\eta \mathbf{R}^N$ with connections determined by the one already chosen on the frame bundle $P(V_0)$ of V_0.

Now over the total space of V_0 we have already established the formula

$$[\phi(\mathbf{R}_{E_0}) - \phi(\mathbf{R}_{F_0})] \;=\; i_![\mathrm{Res}_\phi]$$

where $i_! = (\pi_!)^{-1}$ is the Thom isomorphism and $i : X_0 \hookrightarrow V_0$ denotes the inclusion of the zero section. A basic formula for the Thom isomorphism is that

$$i^* i_! [\mathrm{Res}_\phi] \;=\; \chi(V_0)[\mathrm{Res}_\phi].$$

Hence

$$[\phi(R_{E_0}) - \phi(R_{F_0})] \;=\; \chi(V_0)[\mathrm{Res}_\phi]$$

in $H^*(X_0\ ; \mathbf{R})$. Since $H^*(\mathrm{BGL}_m\ ;\ \mathbf{R})$ is a polynomial algebra, and since the inclusion $X_0 \hookrightarrow \mathrm{BGL}_m$ can be assumed to be k-connected for k arbitrarily large, we conclude that there is a universal class

$$[\mathrm{Res}_\phi]_0 \;=\; \frac{[\phi(R_{F_0}) - \phi(R_{E_0})]}{\chi(V_0)} \in H^*(\mathrm{BGL}_m\ ;\ \mathbf{R}) \cong I^*_{\mathrm{GL}_m}(gl_m).$$

Any embedding of bundles $V \hookrightarrow \mathrm{Hom}(E, F)$ which comes from the φ-universal construction is topologically a pull-back of the one given by the bundles V_0, E_0 and F_0 over BGL_m by the map $f_V : X \to \mathrm{BGL}_m$ which classifies V. Hence the associated residue has cohomology class $f_V^*[\mathrm{Res}_\phi]_0$. By Theorem 1.46 we know that Res_ϕ is actually given pointwise on X by the Chern-Weil representative of this class, i.e., by applying the Ad_{GL_m}-invariant function corresponding to $[\mathrm{Res}_\phi]_0$ to the curvature of the connection on V. □

Remark 1.55. The non-orientable case. Suppose that V is not orientable and let $p : \widetilde{X} \longrightarrow X$ be the 2-sheeted covering space corresponding to the first Stiefel-Whitney class

$$w_1(V) \in H^1(X; \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}_2).$$

The pull-back bundle $\widetilde{V} = p^*V$ on $\widetilde{X}$ is orientable and the deck transformation $g : \widetilde{X} \longrightarrow \widetilde{X}$ lifts to a bundle map

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{g_V} & \widetilde{V} \\ \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow{g} & \widetilde{X} \end{array}$$

which is orientation-reversing on the fibres.

Any smooth cross-section $\alpha \in \Gamma(V)$ lifts to a g-invariant cross-section $\tilde{\alpha} \in \Gamma(\widetilde{V})$, and it is easy to see that

$$g_* \operatorname{Div}(\tilde{\alpha}) = -\operatorname{Div}(\tilde{\alpha}). \tag{1.56}$$

We can now apply Theorem 1.37 to the pull-back bundles p^*E and p^*F to obtain an equation of currents

$$\phi(\overleftarrow{R}_0) - \phi(R_E) + \operatorname{Res}_\phi \operatorname{Div}(\tilde{\alpha}) = dT' \tag{1.57}$$

on $\widetilde{X}$. For any $\phi \in I^*_{\mathrm{GL}_m}$ the forms $\phi(\overleftarrow{R}_0)$ and $\phi(R_E)$ are constructed without regard to orientations on either $\widetilde{X}$ or $\widetilde{V}$. Hence,

$$g^*\phi(\overleftarrow{R}_0) = \phi(\overleftarrow{R}_0) \qquad \text{and} \qquad g^*\phi(R_E) = \phi(R_E),$$

and for similar reasons

$$g^*T' = T'.$$

Consequently, we have that

$$g^*\left(\operatorname{Res}_\phi \operatorname{Div}(\tilde{\alpha})\right) = \operatorname{Res}_\phi \operatorname{Div}(\tilde{\alpha}) \tag{1.58}$$

and in particular by (1.56) that

$$g^* \operatorname{Res}_\phi = -\operatorname{Res}_\phi . \tag{1.59}$$

It follows that **the formula**

$$\phi(\overleftarrow{R}_0) - \phi(R_E) + \operatorname{Res}_\phi \operatorname{Div}(\alpha) = dT' \tag{1.60}$$

holds down on X, where the term $\operatorname{Res}_\phi \operatorname{Div}(\alpha)$ is defined via the g-invariance (1.58).

Remark 1.61. It should be noted that the transgression forms T'_s in Theorem 1.37 are produced out of the canonically modified family of connections $\overrightarrow{D}'_s$. Let us consider the transgression T_s given by the unmodified family $\overrightarrow{D}_s$. Since

$\vec{R}'_s = \vec{R}_s$ and $\vec{D}'_s - \vec{D}_s = -dr_s$, we see directly from the standard formulas (e.g., I.1.18) that

$$(1.62) \qquad T'_s - T_s = \int_s^\infty \phi(d\dot{r}_t\ ;\ \Omega_t)dt = d\int_s^\infty \phi(\dot{r}_t\ ;\ \Omega_t)dt.$$

Since $dr_s = \chi_s d\log|v|$ this can be rewritten as

$$(1.63) \qquad T'_s - T_s = \int_s^\infty d\log|v|\phi(1\ ;\ \Omega_t)\dot{\chi}_t dt.$$

From (1.63) we see that

$$(1.64) \qquad T'_s \equiv T_s \quad \text{if} \quad \phi(1\ ;\ \Omega_t) \equiv 0.$$

We also conclude that if α is atomic then $T = \lim_{s\to 0} T_s$ exists and, since d has closed range, there exists a current R with

$$(1.65) \qquad T' - T = dR.$$

2. Quaternionic Line Bundles.

In this section we consider the quaternionic analogue of the case discussed in Chapter II. The results are in striking parallel with those of the complex case.

Throughout this section we suppose that E and F are quaternion line bundles over a smooth manifold X (with scalar multiplication from the left) and that E and F carry connections, D_E and D_F respectively, with respect to which scalar multiplication by any quaternion is parallel.

We consider the bundle

$$V \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbf{H}}(E, F) \subset \mathrm{Hom}_{\mathbf{R}}(E, F)$$

of quaternion-linear maps from E to F. V is a real bundle of rank 4. However it does not in general carry a quaternionic structure, nor even a natural complex structure. Note that for $\alpha \in \Gamma(V)$, we have that $D\alpha \stackrel{\text{def}}{=} D_F \circ \alpha - \alpha \circ D_E$ is also a section of V by our assumption of quaternion compatibility for D_E and D_F.

Suppose $\alpha \in \Gamma(V)$ is an atomic section, i.e., $E \xrightarrow{\alpha} F$ is an atomic quaternionic bundle map. Consider the $L^1_{\text{loc}}(X)$ one form with values in $\text{End}_{\mathbf{H}}(E)$ given by

$$\tau \equiv \alpha^{-1} \circ D\alpha. \tag{2.1}$$

Similar to the complex line bundle case we may consider

$$D\alpha \;=\; \alpha\tau$$

as defining τ. However, in the complex case $\text{End}_{\mathbf{C}}(E)$ is canonically isomorphic to $\mathbf{C}$.

If E and F are furnished with metrics with respect to which scalar multiplication by unit quaternions is a pointwise isometry then one easily checks that

$$\alpha^*\alpha = |\alpha|^2 \text{Id}_E \quad \text{and} \quad \alpha\alpha^* = |\alpha|^2 \text{Id}_F, \tag{2.2}$$

so that $\alpha^{-1} = \alpha^*|\alpha|^{-2}$. Thus

$$\tau \;=\; \frac{\alpha^* \circ (D\alpha)}{|\alpha|^2}. \tag{2.3}$$

For some of the results we must assume that the connections D_E and D_F are metric compatible. The pull back family of connections on E is given by

$$\overleftarrow{D}_s \equiv D_E + \chi_s\tau. \tag{2.4}$$

Note that for each fixed $0 < s \leq \infty$ the connection $\overleftarrow{D}_s$ is quaternionic.

For the sake of simplicity we shall only consider the pullback family. The pushforward family is defined by $\overrightarrow{D}_s = D_F - \chi_s\tau'$ where τ' is defined by $\tau' = (D\alpha)\alpha^{-1} = \frac{(D\alpha)\alpha^*}{|\alpha|^2}$, or $D\alpha = \tau'\alpha$, and is a one form with values in $\text{End}_{\mathbf{H}}(F)$.

For quaternion line bundles there is essentially one topological invariant. The classifying space is $B\mathbf{H}^\times = \mathbf{P}^\infty(\mathbf{H}) = \lim_{n\to\infty} \mathbf{P}^n(\mathbf{H})$, the infinite-dimensional quaternion projective space (with the weak limit topology). The cohomology is a polynomial ring

$$H^*(\mathbf{P}^\infty(\mathbf{H})\ ;\ \mathbf{Z}) \ = \ \mathbf{Z}[\mathbf{u}]$$

where the generator $\mathbf{u} \in H^4(\mathbf{P}^\infty(\mathbf{H})\ ;\ \mathbf{Z})$, called the **instanton class**, arises as

$$\mathbf{u} = \chi(\xi_{\mathbf{H}}) = c_2(\xi_{\mathbf{H}}) \ = \ -\tfrac{1}{2}p_1(\xi_{\mathbf{H}}), \tag{2.5}$$

the Euler, second Chern, and $-\frac{1}{2}$ (first Pontrjagin) classes respectively of the universal quaternion line bundle $\xi_{\mathbf{H}} \longrightarrow \mathbf{P}^\infty(\mathbf{H})$. Now Chern-Weil theory gives an explicit isomorphism of $H^*(\mathbf{P}^\infty(\mathbf{H})\ ;\ \mathbf{R}) = \mathbf{R}[u]$ with the $Ad_{\mathbf{H}^\times}$-invariant polynomials on $\mathrm{Hom}_{\mathbf{H}}(\mathbf{H},\mathbf{H}) \cong \mathbf{H}$. The integral generator above corresponds to the polynomial

$$u(Y) \ = \ \frac{1}{16\pi^2}\mathrm{tr}(Y^2) \tag{2.6}$$

for $Y \in \mathrm{Hom}_{\mathbf{H}}(\mathbf{H},\mathbf{H})$.

The standard transgression applied to the family $\overleftarrow{D}_s$ on E yields

$$u(\overleftarrow{R}_s) - u(R_E) \ = \ dT_s \tag{2.7}$$

where

$$8\pi^2 T_s \ = \ -\int_s^\infty \mathrm{tr}(\overleftarrow{D}_t\ ;\ \overleftarrow{R}_t)dt. \tag{2.8}$$

Theorem 2.9. *Let E and F be quaternionic hermitian line bundles over a manifold X, furnished with quaternionic hermitian connections D_E and D_F. Let $e = u(R_E)$ and $f = u(R_F)$ be the Chern-Weil representatives of the instanton classes of these bundles. Suppose $\alpha : E \to F$ is an atomic section of* $\mathrm{Hom}_{\mathbf{H}}(E,F)$.

Then $T \equiv \lim_{s\to 0} T_s$ converges in $L^1_{\mathrm{loc}}(X)$. The **instanton transgression** *T satisfies*

$$f - e - \mathrm{Div}(\alpha) \ = \ dT, \tag{2.10}$$

and is independent of χ and the metrics.

Fix any polynomial $\phi(u) \in \mathbf{R}[u]$. Then the $\phi(u)$-characteristic current of the pullback connection exists and equals

$$\lim_{s \to 0} \phi(u(\overleftarrow{R}_s)) = \phi(f) - \mathrm{Div}(\alpha) \wedge \left\{ \frac{\phi(f) - \phi(e)}{f - e} \right\} \tag{2.11}$$

where the limit exists in the Federer flat norm topology and is independent of the function χ as well as the metrics $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ used to define this approximating family $\overleftarrow{D}_s$. Furthermore,

$$\lim_{s \to 0} \phi(u(\overleftarrow{R}_s)) - \phi(e) = dT_\phi \tag{2.12}$$

where T_ϕ is the L^1_{loc}-form on X defined by

$$T_\phi = \frac{\phi(f) - \phi(e)}{f - e} T \tag{2.13}$$

In particular, we have the current equation

$$\phi(f) - \phi(e) - \mathrm{Div}(\alpha) \wedge \left\{ \frac{\phi(f) - \phi(e)}{f - e} \right\} = dT_\phi. \tag{2.14}$$

Remark 2.15. Quaternionic Poincaré-Lelong. If E is trivialized and given the corresponding flat connection, then $\mathrm{Hom}_{\mathbf{H}}(E, F) \cong \Gamma(F)$. In this case, for a given section $\alpha \in \Gamma(F)$, Equation (2.10) gives a quaternionic analogue of the Poincaré-Lelong Formula, namely,

$$f - \mathrm{Div}(\alpha) = dT.$$

Proof. Once we have verified that the general rubric of Section 1 applies then by Theorem 1.37

$$\lim_{s \to 0} \phi(u(\overleftarrow{R}_s)) = \phi(f) - \mathrm{Div}(\alpha)\,\mathrm{Res}_\phi\,. \tag{2.16}$$

In particular, with $\phi(u) = u$

$$f - e + \mathrm{Div}(\alpha)\,\mathrm{Res}_u = dT \tag{2.17}$$

where Res_u is (locally) constant. We shall give three different proofs that the instanton residue equals minus one,

$$\mathrm{Res}_u = -1. \tag{2.18}$$

This will verify equation (2.10).

Differentiating (2.13) and using (2.10) immediately yields (2.14). The equation (2.11) follows from (2.16) and the fact that

$$\mathrm{Res}_\phi = -\frac{\phi(f)-\phi(e)}{f-e}. \tag{2.19}$$

If T'_ϕ denotes the Chern-Weil transgression of Section 1 then by equation (1.42)

$$\phi(f)-\phi(e)+\mathrm{Div}(\alpha)\,\mathrm{Res}_\phi = dT'_\phi \tag{2.20}$$

Comparing (2.14) and (2.20) yields that Res_ϕ and $-\frac{\phi(f)-\phi(e)}{f-e}$ belong to the same cohomology class. Because of Theorem 1.46 this is enough to prove (2.19).

Consequently, the proof will be complete once we have verified that Section 1 applies and that the instanton residue equals minus one.

As noted above if $\alpha \in \Gamma(V)$ then $D\alpha \in \Gamma(V)$ is also a section of $V \equiv \mathrm{Hom}_{\mathbf{H}}(E,F) \subset \mathrm{Hom}_{\mathbf{R}}(E,F)$. Consequently, V is totally geodesic in $\mathrm{Hom}_{\mathbf{R}}(E,F)$. Since $\alpha^*\alpha = |\alpha|^2\mathrm{Id}_E$ and $\alpha\alpha^* = |\alpha|^2\mathrm{Id}_F$, our Basic Assumption 1.1 is satisfied. In fact we have the following.

Lemma 2.21. The universal construction. *Property 1.2 holds under the hypotheses stated above.*

Proof. Let $\mathbf{H}$ denote the quaternions and take the standard identification $\mathbf{H} = \mathbf{R}^4$ by the basis $(1,i,j,k)$. Let $\mathbf{H}$ act on itself by scalar multiplication from the left. Define

$$\lambda : \mathbf{H} \longrightarrow \mathrm{Hom}(\mathbf{R}^4,\mathbf{R}^4) \tag{2.22}$$

by associating to $q \in \mathbf{H}$ the linear map $\lambda_q : \mathbf{H} \to \mathbf{H}$ given by $\lambda_q(r) = r \cdot q$. This identifies $\mathbf{H}$ with $\mathrm{Hom}_{\mathbf{H}}(\mathbf{H},\mathbf{H}) \subset \mathrm{Hom}(\mathbf{R}^4,\mathbf{R}^4)$.

The restriction of $\lambda^{-1} = 1/\lambda$ to the multiplicative group $\mathbf{H}^\times \overset{\text{def}}{=} \mathbf{H} - \{0\}$ gives a Lie group homomorphism $\lambda^{-1} : \mathbf{H}^\times \xrightarrow{\approx} \mathrm{GL}_1(\mathbf{H}) \subset \mathrm{GL}_4(\mathbf{R})$. Consider the product homomorphism

$$\varphi = (\lambda^{-1}, \lambda^{-1}) : \mathbf{H}^\times \times \mathbf{H}^\times \longrightarrow \mathrm{GL}_1(\mathbf{H}) \times \mathrm{GL}_1(\mathbf{H}). \tag{2.23}$$

The natural action of $\mathrm{GL}_1(\mathbf{H}) \times \mathrm{GL}_1(\mathbf{H})$ on $\mathbf{H} = \mathrm{Hom}_{\mathbf{H}}(\mathbf{H}, \mathbf{H})$ induces a homomorphism

$$\psi : \mathbf{H}^\times \times \mathbf{H}^\times \longrightarrow \mathrm{GL}_4$$

given by

$$\psi_{(q_1,q_2)}(q) = q_1 q q_2^{-1}$$

for $q \in \mathbf{H} \cong \mathbf{R}^4$. This action factors through an embedding

$$\tilde{\psi} : \mathbf{H}^\times \times \mathbf{H}^\times / \mathbf{R}^\times \hookrightarrow \mathrm{GL}_4$$

whose image is the **conformal group** in 4-dimensions.

It is easy to check that for any $g = (q_1, q_2) \in \mathbf{H}^\times \times \mathbf{H}^\times$, the diagram

$$\begin{array}{ccc} \mathbf{H} & \overset{\lambda}{\hookrightarrow} & \mathrm{Hom}(\mathbf{R}^4, \mathbf{R}^4) \\ {\scriptstyle \psi_g}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\varphi}_g} \\ \mathbf{H} & \overset{\lambda}{\hookrightarrow} & \mathrm{Hom}(\mathbf{R}^4, \mathbf{R}^4) \end{array}$$

commutes, where $\tilde{\varphi}_g(T) = \lambda_{q_2}^{-1} \circ T \circ \lambda_{q_1}$ is the induced action on $\mathrm{Hom}(\mathbf{R}^4, \mathbf{R}^4)$. Hence, the embedding (2.22) and the homomorphism (2.23) constitute the data for a universal construction.

Suppose now that we are given a principal $\mathbf{H}^\times \times \mathbf{H}^\times$ bundle with connection. Then $E \oplus F$ is the bundle associated to the representation φ and $V \subset \mathrm{Hom}(E, F)$ is the bundle associated to ψ. Similarly, the bundle $V^* = \mathrm{Hom}_{\mathbf{H}}(F, E)$ is associated to the opposite representation $\tilde{\psi}$. Hence, our set-up comes from a universal construction as claimed. □

Because of Lemma 2.13 the results of Section 1 are applicable. Our first proof of the remaining lemma is topological.

Lemma 2.24. $\quad \mathrm{Res}_u = -1.$

Proof. Since Property 1.2 is satisfied, this lemma follows directly from the universal topological formula

$$\chi(V) = [f] - [e]. \tag{2.25}$$

This formula is rather well-known (See [BL] for example). It follows from the observation that up to homotopy equivalence we may replace the homomorphism $\psi : \mathbf{H}^\times \times \mathbf{H}^\times \longrightarrow \mathrm{GL}_4^+$ by its restriction

$$\psi_0 : Sp_1 \times Sp_1 = \mathrm{Spin}_4 \longrightarrow SO_4.$$

(This is the usual 2-fold covering). Restricting further to the maximal tori yields the map

$$\psi_0\left(e^{i\theta}, e^{i\varphi}\right) = \left(e^{i(\theta+\varphi)}, e^{i(\theta-\varphi)}\right).$$

One now considers the induced map $B\psi_0 : BS^1 \times BS^1 \to BS^1 \times BS^1$ and the pullback in cohomology to complete the calculation of (2.25), which comes down essentially to:

$$\chi(V) = \frac{1}{16\pi^2}X_1X_2 = \frac{1}{16\pi^2}(x_1+x_2)(x_1-x_2) = \frac{1}{16\pi^2}(x_1^2 - x_2^2) = [f] - [e].$$

This proves the lemma and completes the proof of the Theorem if the transgression T is interpreted as the transgression T' of the modified family $\overleftarrow{D}'_s$ of Section 1. However, Remark 1.61 and equation (1.64) are applicable so that the transgression T'_s of Section 1 equals the transgression T_s defined by (2.8). Note that if $\phi(\Omega_s) \equiv (\mathrm{tr}(\Omega_s^2))^p$ then $\phi(1\ ;\ \Omega_s) = 2p\,\mathrm{tr}\Omega_s(\mathrm{tr}(\Omega_s^2))^{p-1}$ which vanishes since $\mathrm{tr}\Omega_s = 0$ by metric compatibility. □ □

In order to explicitly describe the instanton transgression T it is convenient to use the form $\tau' \equiv (D\alpha)\alpha^{-1}$ as well as $\tau = \alpha^{-1}(D\alpha)$.

Theorem 2.26. The Instanton Transgression Form. *For any atomic quaternionic line bundle map $E \xrightarrow{\alpha} F$, the instanton transgression form T is given by the formula*

$$16\pi^2 T = -tr(\tau R_E) - tr(\tau' R_F) + \tfrac{1}{3}tr(\tau^3). \tag{2.27}$$

Proof. We must compute

$$16\pi^2 T_s = -\int_s^\infty 2\mathrm{tr}\,\dot{\overleftarrow{D}}_t \overleftarrow{R}_t \, dt. \tag{2.28}$$

Obviously $\dot{\overleftarrow{D}}_s = \dot{\chi}_s \tau$, and by I.6.3 (or V1.15 and V1.12c) the curvature $\overleftarrow{R}_s$ is given by

$$\overleftarrow{R}_s = (1-\chi_s)R_E + \chi_s \frac{\alpha^* R_F \alpha}{|\alpha|^2} - \chi_s(1-\chi_s)\tau^2 + \chi_s' \tfrac{|\alpha|^2}{s^2} \tfrac{d|\alpha|^2}{|\alpha|^2}\tau. \tag{2.29}$$

To complete the proof we need a lemma.

Lemma 2.30. *Suppose τ is an arbitrary $End_{\mathbf{H}}(E)$-valued 1-form. Then*

$$tr(\tau^2) = 0.$$

Proof. Locally, we may assume that τ is an $\mathbf{H}$-valued one form since, locally, $\mathrm{End}_{\mathbf{H}}(E) \cong \mathbf{H}$ as algebras. Also, the trace equals four times the real part. Now $\tau = \tau_0 + i\tau_1 + j\tau_2 + k\tau_3$ with τ_0, τ_1, τ_2, τ_3 one forms implies that $\mathrm{Re}\,\tau^2 = 0$. $\square$

Computing (2.28) directly, using (2.29) and $\dot{\overleftarrow{D}} = \dot{\chi}_s \tau$, gives

$$\begin{aligned} 16\pi^2 T_s &= 2\int_0^{\chi_s} \mathrm{tr}\Big(\tau\Big((1-x)R_E + x\alpha^{-1}R_F\alpha - x(1-x)\tau^2\Big)\Big)dx \\ &= 2(\chi_s - \tfrac{1}{2}\chi_s^2)\mathrm{tr}(\tau R_E) + \chi_s^2 \mathrm{tr}(\tau\alpha^{-1}R_F\alpha) - 2(\tfrac{1}{2}\chi_s^2 - \tfrac{1}{3}\chi_s^3)\mathrm{tr}(\tau^3). \end{aligned} \tag{2.31}$$

The $\mathrm{tr}\dot{\chi}_s \tau \chi_s' \frac{|\alpha|^2}{s^2} \frac{d|\alpha|^2}{|\alpha|^2}\tau$ term drops by Lemma 2.30. The term $\mathrm{tr}(\tau\alpha^{-1}R_F\alpha)$ can be rewritten as $\mathrm{tr}(\tau' R_F)$.

The atomic hypothesis implies that T_s converges, in $L^1_{\mathrm{loc}}(X)$, to T defined by (2.27). $\square$

Second Proof of Lemma 2.24. Consider the total space of $V \equiv \mathrm{Hom}_{\mathbf{H}}(E,F)$. Let ρ denote projection from the unit sphere subbundle of $\mathrm{Hom}_{\mathbf{H}}(E,F)$ to the base manifold. Because of the formula (2.27) for the instanton potential T_s

$$\mathrm{Res}_u(\overleftarrow{D}) \;=\; \int\limits_{\rho} T = \frac{1}{48\pi^2}\int\limits_{|\alpha|=1} \mathrm{tr}(\boldsymbol{\tau}^3). \tag{2.32}$$

Here $\boldsymbol{\tau} = \mathbf{v}^{-1}\mathbf{D}\mathbf{v}$ is the form corresponding to the tautological section $\mathbf{v}$ of the pullback bundle $\mathbf{V}$ of $V \equiv \mathrm{Hom}_{\mathbf{H}}(E,F)$ over itself.

The form $\mathrm{tr}(\boldsymbol{\tau}^3)$ has a particularly nice expression. Let e and f denote frames for E and F and let $\alpha(e) = vf$ define $v \in \mathbf{H}$. This gives the local identification $V = \mathrm{Hom}_{\mathbf{H}}(E,F) \cong \mathbf{H}$. Note that $\alpha^{-1}(f) = v^{-1}e$. Let $\alpha_1(e) = f$, $\alpha_2(e) = if$, $\alpha_3(e) = jf$ and $\alpha_4(e) = kf$ be the basic associated orthonormal frame for V.

Lemma 2.33. *Let $\epsilon = v\frac{\partial}{\partial v}$ be the Euler vector field on the total space of V, and let $\boldsymbol{\tau} = \mathbf{v}^{-1}D\mathbf{v}$ be the form corresponding to the tautological section $\mathbf{v} = \sum_{i=1}^{4} v_i\alpha_i$. Then $D\mathbf{v} = \sum_{i=1}^{4}(Dv)_i\alpha_i$ defines local 1-forms $(Dv)_i = dv_i + \sum v_k w_{ki}$, and*

$$\tfrac{1}{24}\mathit{tr}(\boldsymbol{\tau}^3) \;=\; \epsilon \,\llcorner \left(\frac{(Dv)_1 \wedge (Dv)_2 \wedge (Dv)_3 \wedge (Dv)_4}{|v|^4}\right).$$

Proof. In replacing endomorphism-valued forms by matrix-valued forms, minus signs are sometimes introduced. For example, dvv^{-1} is the matrix form of τ while $-(dvv^{-1})^2$ is the matrix form of τ^2. Also, τ^3 has matrix form $-(dvv^{-1})^3$ (cf. I.6.3 and I.6.8). For the sake of convenience, but only in this proof, we shall let $\tilde{\tau} \equiv \mathbf{v}^{-1}\mathbf{D}\mathbf{v}$ denote the operator version and let $\tau = Dvv^{-1}$ denote the matrix version. In addition, for the sake of clarity, we shall give the proof with $(Dv)_i$ replaced by dv_i. In fact, as a slight digression we shall describe several quaternion valued differential form identities on $\mathbf{H}$. Define $\tau = dvv^{-1}$ for $v \in \mathbf{H}$. Since trace equals four times the real part we must show that

$$\frac{1}{6}\,\mathrm{Re}\,\tau^3 \;=\; -\epsilon \,\llcorner \left(\frac{dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4}{|v|^4}\right). \tag{2.34}$$

where τ is the matrix version $\tau = dvv^{-1}$. Now $\mathrm{Re}\,\tau = \frac{1}{2}(\tau + \bar{\tau}) = \frac{1}{2}\frac{dv\bar{v}+vd\bar{v}}{|v|^2} = d\log|v|$ is exact. Define $\sigma \equiv \mathrm{Im}\,\tau = \frac{1}{2}(\tau - \bar{\tau})$, or equivalently $\tau = d\log|v| + \sigma$.

(The reader may wish to verify that $d\sigma = d\tau = \sigma^2 = \tau^2 = \bar{\tau}^2$ and that $\frac{dv\wedge d\bar{v}}{|v|^2} = \tau\bar{\tau} = -2d\log|v| \wedge \sigma - \sigma^2$, $\tau\bar{\tau} = 2d\log|v| \wedge \sigma - \sigma^2$.)

Direct calculation yields

$$dv \wedge d\bar{v} \wedge dv \wedge d\bar{v} \;=\; -24 dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4.$$

(Use that fact that $e_i\bar{e}_j + e_j\bar{e}_i = 2\delta_{ij}$ with $e_1 = 1$, $e_2 = i$, $e_3 = j$, $e_4 = k$. Also note that $dvd\bar{v}dvd\bar{v}$ is real.) However,

$$\begin{aligned} -\epsilon \llcorner \left(\frac{d\bar{v} \wedge dv \wedge dv \wedge d\bar{v}}{|v|^4} \right) &= \bar{\tau}\tau\bar{\tau} - \tau^2\bar{\tau} + \tau(\bar{\tau})^2 - \tau\bar{\tau}\tau \\ &= 4\sigma^3. \end{aligned}$$

This proves that

$$\frac{1}{6}\sigma^3 \;=\; \frac{-\epsilon \llcorner dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4}{|v|^4}. \qquad (2.35)$$

Finally, note that $\sigma^3 = \operatorname{tr}(\tau^3)$ since $\sigma^2 = d\sigma$ is purely imaginary. $\square$

The proof that $\operatorname{Res}_u(\overleftarrow{D}) = -1$ can now be completed using this lemma.

$$\operatorname{Res}_u(\overleftarrow{D}) = \frac{-1}{48\pi^2} \int\limits_{|v|=1} \operatorname{tr}(\tau^3) = -\frac{1}{2\pi^2} \int\limits_{|v|=1} \frac{\epsilon \llcorner (dv_1 \wedge \cdots \wedge dv_4)}{|v|^4} = -1,$$

because of (2.32), Lemma 2.33, and the fact that $\operatorname{vol}(S^3) = 2\pi^2$. $\square$

Note. Each choice of frames e, f induces a local isomorphism $V \cong \mathbf{H}$ via $\alpha(e) = vf$, $v \in \mathbf{H}$. This local isomorphism is easily seen to be unique up to a conformal map of $\mathbf{H}$ to $\mathbf{H}$. Thus V is naturally equipped with a conformal structure. Moreover, for each choice of metrics $\langle\ .\ \rangle_E$, $\langle\ .\ \rangle_F$, compatible with the quaternion structures, the induced metric on $V \subset \operatorname{Hom}_{\mathbf{R}}(E, F)$ is always in this conformal class. The form $\operatorname{tr}(\tau^3)$, restricted to the fibers of V is conformally invariant.

The metrics on E and F are necessary in order to define the pullback family $\overleftarrow{D}_s = D_E + \chi_s\alpha^{-1}D\alpha$ because the norm $|\alpha|^2$ in the expression $\chi_s \equiv \chi(|\alpha|^2/s^2)$ depends on the metrics. However, the assumption that the connections D_E and D_F are metric compatible is not needed in the proof of Theorem 2.26. That is,

$\lim_{s\to o} T_s = T$ converges in $L^1_{\text{loc}}(X)$ to the instanton transgression T given by (2.27). Since τ and τ' are well defined without the use of metrics, the instanton transgression T is also metric independent.

The next result is in complete parallel with the complex case. In particular, it is metric independent.

Theorem 2.36. *Suppose E and F are quaternionic line bundles over a manifold X furnished with quaternionic connections D_E and D_F. Suppose $E \xrightarrow{\alpha} F$ is an atomic section of* $\operatorname{Hom}_{\mathbf{H}}(E, F)$*. Then the instanton transgression T defined by* (2.27) *satisfies*

$$u(R_E) - u(R_F) - \operatorname{Div}(\alpha) \;=\; dT,$$

as an equation of currents on X.

Proof. Because of the results of Chapter III.1, it suffices to calculate that the residue of T is -1 directly from the formula for T. This was done in the second proof of Lemma 2.24 presented above. $\square$

Remark 2.37. Twisted Quaternionic Scalars. Motivated by examples such as the tangent bundle to $\mathbf{P}^1(\mathbf{H}) = S^4$ it is natural to weaken the definition of quaternionic line bundle by replacing $\mathbf{H}$ by a bundle A of $\mathbf{H}$-algebras, which is not necessarily trivial. An **A-quaternionic line bundle** E is a real rank 4 vector bundle E equipped with an action of A, i.e., A is embedded as a subbundle of algebras of the endomorphism bundle $\operatorname{End}_{\mathbf{R}}(E)$. Consider a pair E, F of A-quaternionic line bundles. The subbundle of $\operatorname{Hom}_{\mathbf{R}}(E, F)$ consisting of A-quaternionic linear maps (on each fiber) will be denoted

$$V \equiv \operatorname{Hom}_A(E, F).$$

The natural invariant to use in extending the previous results is the Euler form. Consequently, we must assume metric compatibility. That is assume E and F are equipped with A-quaternionic metric compatible A-quaternionic connections D_E, $\langle\ ,\ \rangle_E$ and D_F, $\langle\ ,\ \rangle_F$. Moreover, assume that the embeddings $A \subset \operatorname{End}_{\mathbf{R}}(E)$ and $A \subset \operatorname{End}_{\mathbf{R}}(F)$ induce the same connection D_A on A and induce the canonical metric $\langle\ ,\ \rangle_A$ on A determined by the algebraic structure of A. Let D_V, $\langle\ ,\ \rangle_V$ denote the induced metric compatible connection on $V \equiv \operatorname{Hom}_A(E, F) \subset \operatorname{Hom}_{\mathbf{R}}(E, F)$.

Theorem 2.9 remains valid if the standard notion of quaternionic is replaced by the twisted notion of A-quaternionic and the compatibility described above is valid. The instanton forms must, of course, be replaced by the Euler forms, i.e., $e \equiv \chi(D_E)$ and $f \equiv \chi(D_F)$. In particular, the formula

$$\chi(D_F) - \chi(D_E) - \mathrm{Div}(\alpha) \;=\; dT \tag{2.38}$$

extends the quaternionic Poincaré-Lelong formulas in Equation (2.7) and Remark 2.15.

Actually, for the parts of the statement of Theorem 2.9 involving the family of connections D_s, it is natural to replace the family (2.4) by a metric compatible family, as was done in Chapter IV.

Because of the similarities with the standard quaternionic case the details of the A-quaternionic case are omitted, except for a discussion of a key fact. Namely, the Euler class of V is just the difference

$$\chi(V) \;=\; \chi(F) - \chi(E), \tag{2.39}$$

in the A-quaternionic case. The topological argument used to establish (2.25) can also be used to verify (2.39). In fact, this result is true on the level of forms, yielding a direct proof of (2.38) and a third proof of Lemma 2.24.

Proposition 2.40. *Suppose E and F are A-quaternion line bundles equipped with metric compatible A-quaternionic connections. Then*

$$Pf(\Omega_V) \;=\; Pf(\Omega_F) - Pf(\Omega_E). \tag{2.41}$$

Proof. It is convenient to have matrices acting on the left of frames, so we assume that the A-quaternions act on E and F on the right. Choose a local orthonormal frame I, J, K for $\mathrm{Im}\, A$. A fibre of E should have the orientation induced by the complex structure I. Consequently, we choose oriented orthonormal frames e, eI, eJ, $-eK$ for E; and f, fI, fJ, $-fK$ for F. Note that $I(eJ) = eJI = -eK$. Set

$$\Omega_A \;=\; \begin{pmatrix} 0 & \Omega_A^3 & \Omega_A^2 \\ -\Omega_A^3 & 0 & -\Omega_A^1 \\ -\Omega_A^2 & \Omega_A^1 & 0 \end{pmatrix},$$

i.e.,

$$R_A(I) = \Omega_A^3 J + \Omega_A^2 K \qquad \text{etc.}$$

Let

$$R_F f = \Omega_F^1 f I + \Omega_F^2 f J - \Omega_F^3 f K$$

define the first row of the curvature matrix Ω_F. The remaining rows are determined by using the fact that

$$R_A(a) = R_F \circ a - a \circ R_F \qquad \text{for } a \in \Gamma(A).$$

For example, with $a = I$ we have

$$\begin{aligned} R_F(fI) &= (R_A(I))(f) + I(R_F f) = \Omega_A^3 f J + \Omega_A^2 f K + (\Omega_F^1 f I + \Omega_F^2 f J - \Omega_F^3 f K) I \\ &= -\Omega_F^1 f + \left(\Omega_A^3 - \Omega_F^3\right) f J + \left(\Omega_A^2 - \Omega_F^2\right) f K. \end{aligned}$$

Similarly, one computes $R_F(fJ)$. Therefore,

$$\Omega_F = \begin{pmatrix} 0 & \Omega_F^1 & \Omega_F^2 & \Omega_F^3 \\ -\Omega_F^1 & 0 & \Omega_A^3 - \Omega_F^3 & -\left(\Omega_A^2 - \Omega_F^2\right) \\ -\Omega_F^2 & -\left(\Omega_A^3 - \Omega_F^3\right) & 0 & \Omega_A^1 - \Omega_F^1 \\ -\Omega_F^3 & \Omega_A^2 - \Omega_F^2 & -\left(\Omega_A^1 - \Omega_F^1\right) & 0 \end{pmatrix}.$$

The curvature Ω_E is exactly the same as Ω_F except Ω_F^i should be replaced by Ω_E^i $i = 1, 2, 3$.

Now we calculate Ω_V. Consider the local oriented orthonormal frame α_0, α_1, α_2, α_3 for $V \equiv \mathrm{Hom}_A(E, F)$ defined by $\alpha_0(e) = f$, $\alpha_1(e) = fI$, $\alpha_2(e) = fJ$, $\alpha_3(e) = fK$. Again, recall that $R_V(\alpha) = R_F \circ \alpha - \alpha \circ R_E$. Therefore,

$$\begin{aligned} (R_V(\alpha_0))(e) &= R_F f - \alpha_0(R_E e) \\ &= \Omega_F^1 f I + \Omega_F^2 f J - \Omega_F^3 f K - \alpha_0\left(\Omega_E^1 e I + \Omega_E^2 e J - \Omega_E^3 e K\right) \\ &= \left(\Omega_F^1 - \Omega_E^1\right) f I + \left(\Omega_F^2 - \Omega_E^2\right) f J - \left(\Omega_F^3 - \Omega_E^3\right) f K. \end{aligned}$$

Similarly,

$$R_V(\alpha_1) = -\left(\Omega_F^1 - \Omega_E^1\right)\alpha_0 + \left(\Omega_A^3 - \Omega_F^3 - \Omega_E^3\right)\alpha_2 + \left(-\Omega_A^2 + \Omega_F^2 + \Omega_E^2\right)\alpha_3,$$

and $R_V(\alpha_2) = \cdots + \left(-\Omega_A^1 + \Omega_F^1 + \Omega_E^1\right)\alpha_3$. Thus

$$\Omega_V = \begin{pmatrix} 0 & \Omega_F^1 - \Omega_E^1 & \Omega_F^2 - \Omega_E^2 & \Omega_F^3 - \Omega_E^3 \\ -(\Omega_F^1 - \Omega_E^1) & 0 & \Omega_A^3 - \Omega_F^3 - \Omega_E^3 & -(\Omega_A^2 - \Omega_F^2 - \Omega_E^2) \\ -(\Omega_F^2 - \Omega_E^2) & -(\Omega_A^3 - \Omega_F^3 - \Omega_E^3) & 0 & \Omega_A^1 - \Omega_F^1 - \Omega_E^1 \\ -(\Omega_F^3 - \Omega_E^3) & \Omega_A^2 - \Omega_F^2 - \Omega_E^2 & -(\Omega_A^1 - \Omega_F^1 - \Omega_E^1) & 0 \end{pmatrix}.$$

If

$$\Omega = \begin{pmatrix} 0 & \Omega_1 & \Omega_2 & \Omega_3 \\ -\Omega_1 & 0 & \Sigma_3 & -\Sigma_2 \\ -\Omega_2 & -\Sigma_3 & 0 & \Sigma_1 \\ -\Omega_3 & \Sigma_2 & -\Sigma_1 & 0 \end{pmatrix}$$

then

$$\mathrm{Pf}(\Omega) = \Omega_1\Sigma_1 + \Omega_2\Sigma_2 + \Omega_3\Sigma_3.$$

Consequently,

$$\begin{aligned} \mathrm{Pf}(\Omega_V) &= (\Omega_F^1 - \Omega_E^1)(\Omega_A^1 - \Omega_F^1 - \Omega_E^1) + \cdots \\ &= \Omega_F^1(\Omega_A^1 - \Omega_F^1) - \Omega_E^1(\Omega_A^1 - \Omega_E^1) + \cdots \\ &= \mathrm{Pf}(\Omega_F) - \mathrm{Pf}(\Omega_E). \quad \square \end{aligned}$$

3. Dirac Morphisms.

In this section we shall examine the important family of bundle morphisms which arise from Clifford multiplication. They appear naturally in various constructions of the Thom isomorphism in K-theory, and they arise as the principal symbols of the standard families of elliptic operators. In this context, our theory is particularly nice. The formalism is elegant. The results are canonical and the limit currents are independent of any choice of approximation mode.

Given a vector bundle $\pi : V \to X$ which is, say, complex or spin, there are Thom isomorphisms $i_! : K(X) \longrightarrow K_{\mathrm{cpt}}(V)$ and $i_! : H^*(X) \longrightarrow H^*_{\mathrm{cpt}}(V)$, and a diagram

$$\begin{tikzcd} K(X) \arrow[r, "i_!"] \arrow[d, "ch"'] & K_{\mathrm{cpt}}(V) \arrow[d, "ch"] \\ H^*(X) \arrow[r, "i_!"] & H^*_{\mathrm{cpt}}(V) \end{tikzcd}$$

which commutes up to a factor. In our study of the Euler class we established a canonical "Chern-Weil" representation of this diagram with K-Theory replaced by bundles-with-connection and cohomology replaced by closed differential forms. (See III.5 and IV.5.) However, in that approach the theorems and their proofs were pulled magically out of a hat.

In this section we shall derive these results as a natual consequence of our philosophy, by looking at the Chern character of the pullback connection under Clifford multiplication. The commutativity factor in the diagram above will come out as a fundamental residue.

We begin with some definitions. Throughout this section $\pi : V \to X$ will denote a smooth, real vector bundle of rank m equipped with a metric $\langle \cdot, \cdot \rangle$ and an orthogonal connection D. We shall denote by $\mathrm{C}\ell(V) \to X$ the associated Clifford bundle of V with its induced metric and connection (cf. [LM]). Let $S = S^+ \oplus S^-$ be a bundle of $\mathbf{Z}_2$-graded modules over $\mathrm{C}\ell(V)$, and assume S is furnished with a direct sum metric and connection $D_{S^+} \oplus D_{S^-}$.

By definition of $\mathrm{C}\ell(V)$ we have

$$v \cdot v \; = \; -|v|^2 \mathbf{1}$$

for all $v \in V \subset \mathrm{C}\ell(V)$. Hence the composition $V \subset \mathrm{C}\ell(V) \longrightarrow \mathrm{Hom}(S^+, S^-)$ satisfies Assumption 1.1. We should assume further that the metric and connection on S are adapted to this Clifford multiplication as follows.

Definition 3.1. The bundle S is called a **Dirac bundle** if

$$\text{(a)} \qquad \langle vs, s' \rangle + \langle s, vs' \rangle \; = \; 0 \quad \forall v \in V \;\text{ and }\; \forall s, s' \in S,$$

$$\text{(b)} \qquad D_S(\varphi\sigma) \; = \; (D\varphi)\sigma + \varphi(D_S\sigma)$$

for all $\varphi \in \Gamma(\mathrm{C}\ell(V))$ and $\sigma \in \Gamma(S)$. (See [LM] for more details.)

Example 3.2. Let $S = \mathrm{C}\ell(V)$ with its metric and connection, and with the standard even-odd grading $\mathrm{C}\ell(V) = \mathrm{C}\ell^{\mathrm{even}}(V) \oplus \mathrm{C}\ell^{\mathrm{odd}}$.

To begin we shall work on the total space of V. As in section 1 we set $\mathbf{V} = \pi^* V$ and $\mathbf{S}^{\pm} = \pi^* S^{\pm}$, and we let $\mathbf{D}$, $\mathbf{D}_{\mathbf{S}^{\pm}}$ denote the pullback connections on these bundles. Note that there is a canonical embedding

$$\text{(3.3)} \qquad \mathbf{V} \subset TV$$

as the subbundle tangent to the fibres of $\pi : V \to X$. The connection on V determines a complement to this, i.e., a splitting

$$TV = \mathbf{V} \oplus \mathbf{H} \tag{3.4}$$

Let $\mathbf{v} \in \Gamma(\mathbf{V})$ denote the tautological section as in §1, and consider the $\mathbf{V}$-valued 1-form $\mathbf{Dv}$. As a linear map, $\mathbf{Dv} : TV \to \mathbf{V}$ is just projection along $\mathbf{H}$ onto $\mathbf{V}$, i.e.,

$$\mathbf{Dv}\,\big|_{\mathbf{V}} = \mathrm{Id}_{\mathbf{V}} \quad \text{and} \quad \ker \mathbf{Dv} = \mathbf{H}. \tag{3.5}$$

If we choose a local orthogonal normal frame field $(e_1, \ldots, e_m)$ for V and lift it to a frame field $(\mathbf{e}_1, \ldots, \mathbf{e}_m)$ for $\mathbf{V}$, then $\mathbf{v} = \sum v_j \mathbf{e}_j$ where $(v_1, \ldots, v_m)$ are the linear fibre coordinates. Writing $De_j = \sum_k \omega_{jk} e_k$, we get the expression

$$\mathbf{Dv} = \sum_{k=1}^{m} \left(dv_k + \sum_{j=1}^{m} v_j \omega_{jk} \right) \mathbf{e}_k \overset{\text{def}}{=} \sum_{k=1}^{m} Dv_k \mathbf{e}_k \tag{3.6}$$

in terms of the local gauge ω. Of course, by definition we have

$$\mathbf{D}^2\mathbf{v} = \mathbf{R}(\mathbf{v}) \tag{3.7}$$

where $\mathbf{R}$ is the (pullback) of the curvature of V. In terms of the local frame above

$$\mathbf{D}^2\mathbf{v} = \sum_{j,k=1}^{m} v_j \Omega_{jk} \mathbf{e}_k. \tag{3.8}$$

We recall that $V \subset C\ell(V)$ is totally geodesic. Hence, by assumption 3.1.b the inclusion

$$\mathbf{V} \mapsto \mathrm{Hom}(\mathbf{S}^+, \mathbf{S}^-) \text{ is totally geodesic.} \tag{3.9}$$

From assumption 3.1.a we have that

$$\mathbf{v}^* = -\mathbf{v}. \tag{3.10}$$

Hence we have canonical connection-preserving identification $\mathbf{V} \xrightarrow{\cong} \mathbf{V}^*$ given by $-\mathrm{Id}_{\mathbf{V}}$, and in particular,

$$(\mathbf{Dv})^* \;=\; \mathbf{Dv}^*. \tag{3.11}$$

Whenever our map $\mathbf{v} : \mathbf{S}^+ \to \mathbf{S}^-$ is given by a standard Clifford constructions, Property 1.2 will hold.

We now fix an approximate 1 and consider the families of connections

$$\overleftarrow{\mathbf{D}}_s = \mathbf{D}_{S^+} + \chi_s \mathbf{v}^{-1}(\mathbf{Dv}) \quad \text{and} \quad \overrightarrow{\mathbf{D}}_s = \mathbf{D}_{S^-} - \chi_s(\mathbf{Dv})\mathbf{v}^{-1} \tag{3.12}$$

where $\mathbf{v}^{-1} = -\frac{1}{|\mathbf{v}|^2}\mathbf{v}$. The corresponding curvatures are given by

$$\begin{aligned} \overleftarrow{\mathbf{R}}_s &= \mathbf{R}^+ + \chi_s \mathbf{v}^{-1}\mathbf{R}(\mathbf{v}) - \chi_s(1-\chi_s)\boldsymbol{\tau}^2 + \chi_s' \frac{d|\mathbf{v}|^2}{s^2}\boldsymbol{\tau} \\ \overrightarrow{\mathbf{R}}_s &= \mathbf{R}^- - \chi_s \mathbf{R}(\mathbf{v})\mathbf{v}^{-1} - \chi_s(1-\chi_s)(\boldsymbol{\tau}^{-1})^2 + \chi_s' \frac{d|\mathbf{v}|^2}{s^2}\boldsymbol{\tau}^{-1} \end{aligned} \tag{3.13}$$

where $\mathbf{R}^\pm \overset{\text{def}}{=} \mathbf{R}_{S^\pm}$ and where

$$\boldsymbol{\tau} \overset{\text{def}}{=} -\frac{1}{|\mathbf{v}|^2}\mathbf{v}(\mathbf{Dv}) = \mathbf{v}^{-1}(\mathbf{Dv}) \quad \text{and} \quad \boldsymbol{\tau}^{-1} \overset{\text{def}}{=} -\frac{1}{|\mathbf{v}|^2}(\mathbf{Dv})\mathbf{v} = (\mathbf{Dv})\mathbf{v}^{-1}. \tag{3.14}$$

The expressions (3.13) can be rewritten using the equation

$$\mathbf{R}(\mathbf{v}) \;=\; \mathbf{R}^- \circ \mathbf{v} - \mathbf{v} \circ \mathbf{R}^+. \tag{3.15}$$

We now consider the expressions in the indeterminant x given by

$$\begin{cases} \mathbf{A}^+(x) &= \mathbf{R}^+ + x\mathbf{v}^{-1}\mathbf{R}(\mathbf{v}) - x(1-x)\boldsymbol{\tau}^2 \\ \mathbf{A}^-(x) &= \mathbf{R}^- - x\mathbf{R}(\mathbf{v})\mathbf{v}^{-1} - x(1-x)\boldsymbol{\tau}^{-2}. \end{cases} \tag{3.16}$$

These can be rewritten via (3.15) as

$$\begin{cases} \mathbf{A}^+(x) &= (1-x)\mathbf{R}^+ + x\mathbf{v}^{-1} \circ \mathbf{R}^- \circ \mathbf{v} - x(1-x)\boldsymbol{\tau}^2 \\ \mathbf{A}^-(x) &= (1-x)\mathbf{R}^- + x\mathbf{v} \circ \mathbf{R}^+ \circ \mathbf{v}^{-1} - x(1-x)\boldsymbol{\tau}^{-2}. \end{cases} \tag{3.17}$$

Note the close relationship between $\mathbf{A}^+(x)$ and $\mathbf{v}^{-1}\mathbf{A}^-(x)\mathbf{v}$ since $\boldsymbol{\tau} = \mathbf{v}^{-1}\boldsymbol{\tau}^{-1}\mathbf{v}$. From the main results in §1 we have the following.

Theorem 3.18. *Let $\pi : V \to X$ be an oriented Riemannian vector bundle with connection and suppose $S = S^+ \oplus S^-$ is a Dirac bundle for V as above. Fix an approximate-one χ and consider the families (3.12) of pullback and pushforward connections associated to Clifford multiplication $\mathbf{v} : \mathbf{S}^+ \to \mathbf{S}^-$ (where $\mathbf{S}^\pm = \pi^* S^\pm$) by the tautological section defined over the total space of V. For each Ad-invariant polynomial ϕ defined on gl_N (where $N = \operatorname{rank} S^+ = \operatorname{rank} S^-$), the characteristic currents $\lim_{s\to 0} \phi(\overleftarrow{\mathbf{R}}_s^+)$ and $\lim_{s\to 0} \phi(\overrightarrow{\mathbf{R}}_s^-)$ exist and are independent of χ. In fact we have equations*

$$\left\{ \begin{array}{lcl} \phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}^-) - \mathrm{Res}_\phi[X] & = & d\mathbf{r}_s^+ \\ \phi(\mathbf{R}^+) - \phi(\overrightarrow{\mathbf{R}}_s) - \mathrm{Res}_\phi[X] & = & d\mathbf{r}_s^- \end{array} \right. \tag{3.19}$$

where

$$\mathbf{r}_s^\pm \stackrel{\text{def}}{=} \int_{\chi_s}^{1} \phi(\boldsymbol{\tau}^{\pm 1}\ ;\ \mathbf{A}^\pm(x))dx \tag{3.20}$$

are L^1_{loc}-forms which converge to zero in L^1_{loc} on V as $s \to 0$. In particular we have the current equation

$$\phi(\mathbf{R}^+) - \phi(\mathbf{R}^-) - \mathrm{Res}_\phi[X] = dT \tag{3.21}$$

where

$$\mathbf{T} \stackrel{\text{def}}{=} \int_0^1 \phi(\boldsymbol{\tau}\ ;\ \mathbf{A}(x))dx \tag{3.22}$$

is an L^1_{loc} form on V which is smooth on $V - X$. The residue is a smooth form on X given explicitly by the fibre integral

$$\mathrm{Res}_\phi = \int_\rho \mathbf{T} \tag{3.23}$$

where $\rho : \{v \in V : |v| = 1\} \to X$ is the bundle projection. If S is associated to V by a universal construction (i.e., Property 1.2 holds) then Res_ϕ is exactly the Chern-Weil form

$$\mathrm{Res}_\phi = \frac{\phi(R^+) - \phi(R^-)}{\chi(R)} \tag{3.24}$$

representing the universal class

$$\frac{\phi(S^+) - \phi(S^-)}{\chi(V)} \tag{3.25}$$

defined in $H^*(BSO_m)$.

Remark 3.26. Suppose χ is chosen with the property that $\chi(t) = 1$ for $t \geq 1$. Then $\phi(\overleftarrow{\mathbf{R}}_s) - \phi(\mathbf{R}^-)$, $\phi(\mathbf{R}^+) - \phi(\overrightarrow{\mathbf{R}}_s)$, and $\mathbf{r}_s^\pm$ all have support in the s-tubular neighborhood

$$B_s(V) \stackrel{\text{def}}{=} \{v \in V : |v| \leq s\} \subset V.$$

Consequently if X is compact, then (3.19) is an equation between forms with compact support.

Proof. The pullback case is a direct consequence of 1.37 and 1.50, with the exception of the formulae (3.20) and (3.22) where $\mathbf{\Theta}$ has been replaced by $\boldsymbol{\tau}$. This substitution is permitted because $\boldsymbol{\tau} \equiv \mathbf{\Theta}$ (mod $d|\mathbf{v}|^2$).

By Remark I.4.28, the pushforward connection on $\mathbf{S}^-$ corresponds to the pullback via the map $\mathbf{v}^* = -\mathbf{v} : \mathbf{S}^- \to \mathbf{S}^+$. (Note that since our connections are orthogonal, the canonical isomorphisms $(\mathbf{S}^\pm)^* = \mathbf{S}^\pm$, given by the metrics, are connection preserving.) Applying the pullback case to this map gives everything but the explicit formula (3.20). For this we observe that $\phi(\boldsymbol{\tau}^{-1}\ ;\ \mathbf{A}^-(x)) = \phi(\mathbf{v}^{-1}\boldsymbol{\tau}^{-1}\mathbf{v}\ ;\ \mathbf{v}^{-1}\mathbf{A}^-(x)\mathbf{v}) = \phi(\boldsymbol{\tau}\ ;\ \widetilde{\mathbf{A}}^+(x))$ where $\widetilde{\mathbf{A}}^+(x)$ denotes the expression obtained from $\mathbf{A}^+(x)$ in (3.17) by interchanging $\mathbf{R}^+$ and $\mathbf{R}^-$. This completes the proof. □

Theorem 3.27. *Let $\pi : V \to X$ and $S = S^+ \oplus S^-$ be as in Theorem 3.18, and suppose that α is an atomic section of V. Fix an approximate-one χ and consider the families of pullback and pushforward connections associated to Clifford multiplication $\alpha : S^+ \to S^-$ by α. For each Ad-invariant polynomial ϕ defined on gl_N, the characteristic currents $\phi((\overleftarrow{D}_s^+)) \equiv \lim_{s\to 0} \phi(\overleftarrow{R}_s^+)$ and $\phi((\overrightarrow{D}_s^-)) \equiv \lim_{s\to 0} \phi(\overrightarrow{R}_s^-)$ exist, are independent of χ, and satisfy the equations*

$$\begin{aligned}
\phi(\overleftarrow{R}_s) - \phi(R^-) - \mathrm{Res}_\phi \,\mathrm{Div}(\alpha) &= dr_s^+ \\
\phi(R^+) - \phi(\overrightarrow{R}_s) - \mathrm{Res}_\phi \,\mathrm{Div}(\alpha) &= dr_s^-
\end{aligned}$$

where $r_s^\pm$ are L^1_{loc}-forms which converge to zero in L^1_{loc} on X as $s \to 0$. In particular we have the current equation

$$\phi(R^+) - \phi(R^-) - \mathrm{Res}_\phi \,\mathrm{Div}(\alpha) = dT$$

where $T = \alpha^*\mathbf{T}$ is an L^1_{loc}-form on X and where Res_ϕ is the canonical residue form defined in Theorem 3.18.

Proof. This is also a direct consequence of Theorem 1.37. □

Remark 3.28. If χ has the property that $\chi(t) = 1$ for $t \geq 1$, then $\phi(\overleftarrow{R}_s) - \phi(R^-)$, $\phi(R^+) - \phi(\overrightarrow{R}_s)$, and $r_s^{\pm}$ all have support in the s-neighborhood

$$U_s(\Sigma) \overset{\text{def}}{=} \{x \in X : |\alpha(x)| \leq s\}.$$

We now examine some important cases.

Corollary 3.29. The Spin Case. *Let $\pi : V \to X$ be an oriented real $2n$-plane bundle with orthogonal connection. Suppose V carries a spin structure and let $\not{S}^{\pm}(V)$ denote the complex spinor bundles with connection canonically associated to V. Let E be any complex bundle with a unitary connection, and consider the Dirac bundle $S = S^+ \oplus S^-$ where $S^{\pm} = \not{S}^{\pm}(V) \otimes_{\mathbb{C}} E$ with the tensor product connections. Then Theorems 3.18 and 3.27 apply. In particular, for any atomic section α of V we have the current equation*

$$ch\left(R_{\not{S}_{\mathbb{C}}^+ \otimes E}\right) - ch\left(R_{\not{S}_{\mathbb{C}}^- \otimes E}\right) = ch(R_E)\widehat{\mathbb{A}}^{-1}(R_V)\,\mathrm{Div}(\alpha) + dT \tag{3.30}$$

on X, where $ch(u) = tr\left(\exp\left(\frac{i}{2\pi}u\right)\right)$ and where $\widehat{\mathbb{A}}$ denotes the $\widehat{A}$-series (cf. IV.5).

Proof. Formula (3.30) is a direct consequence of the standard fact from topology that

$$\frac{\mathrm{ch}(\not{S}^+(V) \otimes E) - \mathrm{ch}(\not{S}^-(V) \otimes E)}{\chi(V)} = \mathrm{ch}(E)\widehat{\mathbb{A}}^{-1}(V)$$

which was established, for example, in IV.5 above. □

Remark 3.31. Corollary 3.29 is a restatement of the Rectifiable Grothendieck-Riemann-Roch Theorem proved in IV.5. Notice that in this case the statement and proof come naturally from our philosophy of considering the characteristic currents associated to pullback connections. By contrast, finding the arguments given in Chapter IV required considerable hindsight.

Example 3.32. Consider the spin case in the universal setting of 3.18 where $\alpha = \mathbf{v}$. Suppose $\chi(t) = 1$ for $t \geq 1$ and X is compact. Let $\phi = \mathrm{ch}$ as above. Then by Remark 3.28 each equation (3.19) can be written

$$\mathrm{ch}_{\mathrm{cpt},s} \;=\; \mathrm{ch}(\mathbf{R}_E)\widehat{\mathbb{A}}^{-1}(\mathbf{R}_V)[X] + d\mathbf{r}_s \tag{3.33}$$

where $\mathrm{ch}_{\mathrm{cpt},s} \stackrel{\mathrm{def}}{=} \mathrm{ch}(\overleftarrow{\mathbf{R}}_s) - \mathrm{ch}(\mathbf{R}_{\not{S}^-})$ (or $\mathrm{ch}(\mathbf{R}_{\not{S}^+}) - \mathrm{ch}(\overrightarrow{\mathbf{R}}_s)$) and $\mathbf{r}_s$ are forms with compact support on V. Thus $\mathrm{ch}_{\mathrm{cpt},s}$ for $s > 0$ is a canonical family of Chern-Weil representatives of the class

$$\mathrm{ch}\left[\not{S}_{\mathbf{C}}^{+} \otimes E,\ \not{S}_{\mathbf{C}}^{-} \otimes E\ ;\ \mathbf{v}\right] \in H^*_{\mathrm{cpt}}(V)$$

where $\left[\not{S}_{\mathbf{C}}^{+} \otimes E, \not{S}_{\mathbf{C}}^{-} \otimes E\ ;\ \mathbf{v}\right] \in K_{\mathrm{cpt}}(V)$ is the element determined by the identifying $\not{S}_{\mathbf{C}}^{+} \otimes E$ with $\not{S}_{\mathbf{C}}^{-} \otimes E$ outside a compact set via the map $\mathbf{v}$. As $s \to 0$, the forms $\mathbf{r}_s$ converge to 0 in L^1_{loc}. Thus we have convergence $\mathrm{ch}_{\mathrm{cpt},s} \to \mathrm{ch}(\mathbf{R}_E)\widehat{\mathbb{A}}^{-1}(\mathbf{R}_V)[X]$, as $s \to 0$, in the flat topology on V. This gives the direct proof of the commutativity of the diagram (up to a factor) mentioned at the beginning of this section.

Example 3.34. If $\chi(t) = t/(1+t)$ (i.e., algebraic approximation mode) each equation in (3.19) can be written, in analogy with (3.33), as

$$\mathrm{ch}_{L^1,s} \;=\; \mathrm{ch}(\mathbf{R}_E)\widehat{\mathbb{A}}^{-1}(\mathbf{R}_V)[X] + d\mathbf{r}_s \tag{3.35}$$

where $\mathrm{ch}_{L^1,s}$ and $\mathbf{r}_s$ are integrable when restricted to each fibre. This family is absolutely canonical. As shown in Chapter I, the forms in (3.35) extend smoothly to the compactification $\mathbf{P}(\mathbf{C} \oplus V) \supset V$.

Example 3.36. If $\chi(t) = 1 - e^{-t}$ (i.e., transcendental approximation mode), the comments of 3.34 also apply. Here the characteristic forms look "Gaussian" in each fibre. This family is closely related to the one constructed by Quillen via superconnections [Q].

Remark 3.37. (The real case) Corollary 3.29 could be restated for a bundle $\pi : V \to X$ of real dimension m, where $S = S^+ \oplus S^-$ is the real Dirac bundle associated to a $\mathbf{Z}_2$-graded $C\ell_m$-module (cf. [LM]) and where E is a real vector bundle with connection. If $m \not\equiv 0 \pmod 4$ the residue term is always zero. If $m = 4 \pmod 8$, the fundamental real spinor bundles coincide with the complex ones. In fact they carry a quaternionic structure. When $m \equiv 0 \pmod 8$, the fundamental real spinor bundles complexify to give the complex ones. They determine a Thom isomorphism $i_! : KO(X) \to K_{\mathrm{cpt}}(V)$ in KO-theory which under complexification becomes the Thom isomorphism in K-theory. Since the Pontrjagin classes are defined in terms of the complexification of the bundle, the residues here are computable from the complex case.

Corollary 3.38. The Spin^c Case. *Let $\pi : V \to X$ be an oriented real $2n$-plane bundle with orthogonal connection, and suppose V carries a* Spin^c*-structure. Let $\$^{\pm}_{\mathbf{C}}(V)$ denote the fundamental spinor bundles with connection associated to V, and let λ denote the complex line bundle with connection associated to the* Spin^c*-structure (cf.* [LM]*). Consider a complex bundle E with unitary connection and construct a Dirac bundle $S = S^+ \oplus S^-$ with $S^{\pm} = \$^{\pm}_{\mathbf{C}}(V) \otimes E$ as in 3.29. Then Theorems 3.18 and 3.27 apply. In particular for any atomic section α of V we have the current equation on X:*

$$(3.39)\quad ch\left(R_{\$^+_{\mathbf{C}} \otimes E}\right) - ch\left(R_{\$^-_{\mathbf{C}} \otimes E}\right) = ch(R_E)\, e^{\frac{1}{2}c_1(\lambda)} \widehat{\mathbb{A}}^{-1}(R_V)\, \mathrm{Div}(\alpha) + dT.$$

Proof. This is proved in direct analogy with 3.29. □

Remark 3.40. The complex case. A basic special case of Corollary 3.38 occurs when $\pi : V \to X$ is a complex vector bundle with unitary connection. The Spin^c-structure is then canonical, and we have that

$$\$^+_{\mathbf{C}} \cong \bigoplus_{k \geq 0} \Lambda^{2k}_{\mathbf{C}} V^* \stackrel{\mathrm{def}}{=} \Lambda^{\mathrm{even}}_{\mathbf{C}} V^* \quad \text{and} \quad \$^-_{\mathbf{C}} \cong \bigoplus_{k \geq 0} \Lambda^{2k+1}_{\mathbf{C}} V^* \stackrel{\mathrm{def}}{=} \Lambda^{\mathrm{odd}}_{\mathbf{C}} V^*.$$

Clifford multiplication by the complex vector v is given by

$$v \cdot \varphi = v^* \wedge \varphi - v \llcorner \varphi$$

where $(\cdot)^* : V \to V^*$ is the hermitian metric. In this case the residue in (3.39) has the special form

$$\mathrm{Res}_{\mathrm{ch}} = \mathrm{ch}(R_E)\mathrm{Todd}^{-1}(R_V)$$

where Todd denotes the polynomial corresponding to the total Todd class. (The derivation of this used the standard identity: $\widehat{\mathbb{A}}^{-1}(V) = \exp(c_1(V))\mathrm{Todd}^{-1}(V)$. Hence Corollary 3.38 gives a direct derivation and proof of the Rectifiable GRR Theorem III.5.9 (in analogy with Remark 3.31 above).

Theorem 3.18 is equally interesting when the bundle V is not spin or Spin^c.

Corollary 3.41. The signature case. *Let $\pi : V \to X$ be an oriented $2n$-plane bundle with orthogonal connection, and let $\omega_{\mathbf{C}} \in \Gamma(C\ell(V) \otimes \mathbf{C})$ denote the* **volume element** *of V given by*

$$\omega_{\mathbf{C}} = (-i)^n e_1 \dots e_{2n}$$

where $e_1, \dots, e_{2n}$ is any local oriented orthonormal frame field for V. Set

$$C\ell^{\pm} = (1 \pm \omega_{\mathbf{C}})C\ell(V) \otimes \mathbf{C},$$

then ± 1 eigenbundles for left Clifford multiplication by $\omega_{\mathbf{C}}$. Then Theorems 3.18 and 3.27 apply to any Dirac bundle of the form $S = S^+ \oplus S^-$ where

$$S^{\pm} = C\ell^{\pm} \otimes E$$

and where E is any complex bundle with unitary connection. In this case we have that

$$\mathrm{Res}_{\mathrm{ch}} = 2^n ch(R_E)\widehat{\mathbf{L}}(R_V)\widehat{\mathbb{A}}^{-2}(R_V)$$

where $\widehat{\mathbf{L}}$ represents multiplicative series of Pontrjagin classes associated to the formal power series

$$\frac{x/2}{\tanh(x/2)}.$$

In particular for any atomic section α of V we have the current equation on X:

$$ch(R_{\mathbf{C}\ell^+\otimes E}) - ch(R_{\mathbf{C}\ell^-\otimes E}) = 2^n ch(R_E)\,\widehat{\mathbf{L}}(R_V)\widehat{\mathbb{A}}^{-2}(R_V)\,\mathrm{Div}(\alpha) + dT.$$

Note. $\widehat{L}_n(R_V) = \left(\frac{1}{4}\right)^n L_n(R_V)$ where L_n is the classical Hirzebruch L-polynomial.

Proof. This follows directly from 3.18 and the formulae in [LM, III.12]. □

Note. We have changed the convention for the sign of the volume form here from that used in [LM]. Specifically,

$$\omega_{\mathbf{C}} \;=\; (-1)^n \tilde{\omega}_{\mathbf{C}}$$

where $\tilde{\omega}_{\mathbf{C}}$ denotes the volume form appearing in [LM]. As a consequence there is a change in the sign (by $(-1)^n$) in our residue formulas from the formulas one would expect from directly [LM].

Recall that Example 3.2 gives a standard construction of a Dirac bundle associated to any real m-dimensional bundle V as follows. We decompose

$$\mathrm{C}\ell(V) \;=\; \mathrm{C}\ell^{\mathrm{even}}(V) \oplus \mathrm{C}\ell^{\mathrm{odd}}(V) \cong \Lambda^{\mathrm{even}}(V) \oplus \Lambda^{\mathrm{odd}}(V)$$

where $\mathrm{C}\ell^{\mathrm{even}}(V)$ and $\mathrm{C}\ell^{\mathrm{odd}}$ are the $+1$ and -1 eigenbundles respectively for the automorphism $\alpha : \mathrm{C}\ell(V) \to \mathrm{C}\ell(V)$ generated by $-\mathrm{Id} : V \to V$. Let E be any complex bundle with unitary connection over X and set

$$S^+ \;=\; \mathrm{C}\ell^{\mathrm{even}}(V) \otimes_{\mathbf{R}} E\;;\; S^- \;=\; \mathrm{C}\ell^{\mathrm{odd}}(V) \otimes_{\mathbf{R}} E. \tag{3.42}$$

Corollary 3.43. The even-odd form case. *Consider the Dirac bundle $S = S^+ \oplus S^-$ given by (3.42) above. Then Theorems 3.18 and 3.27 apply to this bundle. If V is oriented and $\dim_{\mathbf{R}}(V) = 2n$, then $\mathrm{Res}_{\mathrm{ch}}$ is the canonical Chern-Weil representative of*

$$[\mathrm{Res}_{\mathrm{ch}}] \;=\; (-1)^n \chi(V) ch(E) \widehat{\mathbb{A}}^{-2}(V).$$

If $\dim_{\mathbf{R}}(V)$ is odd, then $\mathrm{Res}_{\mathrm{ch}} = 0$.

In particular for any atomic section α of V we have the current equation on X:

$$ch\left(R_{\Lambda^{\mathrm{even}}(V)\otimes E}\right) - ch\left(R_{\Lambda^{\mathrm{odd}}(V)\otimes E}\right) = (-1)^n ch(R_E)\, \chi(R_V) \widehat{\mathbb{A}}^{-2}(R_V)\, \mathrm{Div}(\alpha) + dT.$$

when $\dim_{\mathbf{R}}(V) = 2n$, and

$$ch\left(R_{\Lambda^{\mathrm{even}}(V)\otimes E}\right) - ch\left(R_{\Lambda^{\mathrm{odd}}(V)\otimes E}\right) \;=\; dT.$$

when $\dim_{\mathbf{R}}(V)$ is odd.

Proof. To compute the residue formula we apply the splitting principle and formally write V as a sum of oriented 2-plane bundles $V = V_1 \oplus \ldots \oplus V_n$ with $x_i \stackrel{\text{def}}{=} \chi(V_i)$. Let $\lambda_{-1}(V \otimes_{\mathbf{R}} \mathbf{C}) = \mathrm{C}\ell^{\text{even}}(V) \otimes_{\mathbf{R}} \mathbf{C} - \mathrm{C}\ell^{\text{odd}}(V) \otimes_{\mathbf{R}} \mathbf{C}$. Then we have

$$\begin{aligned}
\mathrm{ch}(S^+) - \mathrm{ch}(S^-) &= \mathrm{ch}(\lambda_{-1}(V \otimes_{\mathbf{R}} \mathbf{C}) \otimes_{\mathbf{C}} E) \\
&= \mathrm{ch}(\lambda_{-1}(V \otimes_{\mathbf{R}} \mathbf{C}))\mathrm{ch}(E) \\
&= \prod_{i=1}^{n} (1 - e^{x_i})(1 - e^{-x_i})\mathrm{ch}(E) \\
&= \prod_{i=1}^{n} (e^{-x_i/2} - e^{x_i/2})(e^{x_i/2} - e^{-x_i/2})\mathrm{ch}(E) \\
&= (-1)^n (x_1 \ldots x_n)^2 \left(\prod_{i=1}^{n} \frac{\sinh(x_i/2)}{x_i/2} \right)^2 \mathrm{ch}(E) \\
&= (-1)^n \chi(V)^2 \widehat{\mathbb{A}}^{-2}(V)\mathrm{ch}(E).
\end{aligned}$$

We now apply 1.54 to compute the residue form.

If $\dim_{\mathbf{R}} V$ is odd, we replace V by $V \oplus \mathbf{R}$ with the direct sum connection. The residue is unchanged but can now be computed by the formula above. Since $\chi(\mathbf{R}_{V \oplus \mathbf{C}}) \equiv 0$ we get the result. $\square$

The residue in 3.43 could be rewritten by using the identity: $\widehat{\mathbb{A}}^2(V) = \mathrm{Todd}(V \otimes_{\mathbf{R}} \mathbf{C})$.

VI. Further Applications and Future Directions

Much of this paper is devoted to quite specialized cases of bundle maps. However, these cases have much wider applicability than is apparent at first. In this chapter we shall briefly indicate how these applications are made. A full development of the resulting geometric formulae is given in a separate article [HL1]. A deeper analysis of the general theory for the case of arbitrary bundle maps will be done in the sequel [HL2] to this paper.

1. The Top Degeneracy Current and Chern Classes.

Here we study the geometry associated to a linear family of cross-sections of a bundle. To begin, fix a smooth complex vector bundle $F \longrightarrow X$ of rank n with a connection D_F. Suppose we are given a set of $k+1 \leq n$ cross-sections $\mu_0, \ldots, \mu_k$ of F. This is equivalent to being given a bundle map

$$\alpha : \underline{\mathbf{C}}^{k+1} \longrightarrow F \tag{1.1}$$

from the trivial $(k+1)$-plane bundle, where $\mu_j = \alpha(e_j)$ and e_j is the standard j^{th} basis element. Let $\underline{\mathbf{P}} = \mathbf{P}(\underline{\mathbf{C}}^{k+1}) = \mathbf{P}(\mathbf{C}^{k+1}) \times X$ be the projectivization of the trivial bundle $\underline{\mathbf{C}}^{k+1}$ over X, and denote by $U \longrightarrow \underline{\mathbf{P}}$ the tautological line bundle with its standard connection (pulled back from the case where X is a point). Let

$$\pi : \underline{\mathbf{P}} \longrightarrow X \tag{1.2}$$

denote the projection onto the base X. The bundle map α determines a bundle map

$$\tilde{\alpha} : U \longrightarrow \mathbf{F}, \tag{1.3}$$

(where $\mathbf{F} \stackrel{\text{def}}{=} \pi^* F$) by composing α with the canonical inclusion $U \longrightarrow \underline{\mathbf{C}}^{k+1}$.

Definition 1.4. The family of cross-sections α is said to be **k-atomic** if the associated map $\tilde{\alpha}$ is atomic. Under this hypothesis the associated **degeneracy current** of the family is defined by

$$\mathbb{D}_k(\alpha) \;=\; \pi_* \operatorname{Div}(\tilde{\alpha}).$$

Note that

$$\operatorname{spt}\mathbb{D}_k(\alpha) \;\subset\; \{x \in X : \mu_0(x), \ldots, \mu_k(x) \;\text{ are linearly dependent}\}.$$

One considers $\mathbb{D}_k$ to be the current which appropriately measures the degeneration of linear independence in the family α (just as $\operatorname{Div}(\mu)$ is the appropriate measure of the vanishing of a section μ).

Theorem 1.5. *Let α be a family of $k+1$ cross-sections of F which is k-atomic. Then there is a L^1_{loc} current σ on X, canonically defined for each choice of hermitian metric on F, such that*

$$c_{n-k}(D_F) - \mathbb{D}_k(\alpha) \;=\; d\sigma.$$

Moreover for each approximation mode there exist families of smooth forms ψ_s and σ_s, $0 < s \le \infty$, with

$$c_{n-k}(D_F) - \psi_s \;=\; d\sigma_s,$$

such that

$$\lim_{s\to\infty} \sigma_s \;=\; 0 \quad \text{and} \quad \lim_{s\to 0} \sigma_s \;=\; \sigma \;\text{ in }\; L^1_{\mathrm{loc}}(X).$$

Proof. From equation (6.19) in Chapter III we know that there is a canonically defined L^1_{loc}-form T on $\underline{\mathbf{P}}$ such that

$$\sum_{j=0}^{n} (-1)^j c_{n-j}(D_F) c_1(D_U)^j - \operatorname{Div}(\tilde{\alpha}) \;=\; dT.$$

Applying π_* to this equation and using the fact (Proposition III 3.17) that

$$\pi_*\{c_1(D_U)^j\} = \begin{cases} (-1)^k & \text{if } j = k \\ 0 & j < k \end{cases}$$

gives the first assertion. For the second assertion, one applies parallel reasoning to the family of pushforward connections $\overrightarrow{D}_s$ induced by the bundle map $U \xrightarrow{\tilde{\alpha}} \mathbf{F}$ over $\underline{\mathbf{P}}$. $\square$

2. Thom-Porteous Currents — General Degeneracy Currents.

Let $\alpha : E \longrightarrow F$ be a map of smooth bundles with connections where $\text{rank}(E) = m$ and $\text{rank}(F) = n$. For each positive integer $k = 0, 1, \ldots, \min\{m, n\}$, let

$$\pi : G_{m-k}(E) \longrightarrow X$$

denote the bundle whose fibre at $x \in X$ is the Grassmannian of $(m-k)$-planes in E_x (oriented in the real case). Let

$$U \longrightarrow G_{m-k}(E)$$

denote the tautological $(m-k)$-plane bundle. Note that there is a natural inclusion

$$j : U \hookrightarrow \mathbf{E}$$

as a subbundle of $\mathbf{E} \stackrel{\text{def}}{=} \pi^* E$.

Definition 2.1. Given a bundle map $\alpha : E \longrightarrow F$, let

$$\tilde{\alpha} : U \longrightarrow \mathbf{F}$$

be the associated map defined by $\tilde{\alpha} = \alpha \circ j$. Then α is said to be **k-atomic** if $\tilde{\alpha}$ is an atomic section of the bundle $\text{Hom}(U, \mathbf{F})$ over $G_{m-k}(E)$. Under this hypothesis there is a **Thom-Porteous (or degeneracy) current of level k** defined on X by

$$\mathbb{D}_k(\alpha) \;=\; \pi_* \,\text{Div}(\tilde{\alpha}).$$

Note that the support of $\mathbb{D}_k(\alpha)$ is contained in the k^{th} degeneracy locus

$$D_k(\alpha) \stackrel{\text{def}}{=} \{x : \text{rank}(\alpha_x) \le k\}.$$

Remark 2.2. The bundle $\text{Hom}(U, \mathbf{F})$ has $\text{rank}(m-k)n$, and so the current $\text{Div}(\tilde{\alpha})$ has codimension $2(m-k)n$ (in the complex case and $(m-k)n$ in the real case) on $G_{m-k}(E)$. It follows that $\mathbb{D}_k(\alpha)$ is a current of codimension $2(m-k)(n-k)$ (in the complex case and $(m-k)(n-k)$ in the real case) on X.

Remark 2.3. When E and F are algebraic bundles over a complex projective manifold, and α is an algebraic bundle map, the current $\mathbb{D}_k(\alpha)$ is a positive holomorphic chain, and hence an effective algebraic cycle. The class determined by $\mathbb{D}_k(\alpha)$ in the Chow ring $A^*(X)$ is exactly the Thom-Porteous class defined by Fulton [Fu].

Applying the theory in Chapter III to the section $\tilde{\alpha}$ and then pushing forward by π_* gives a geometric formulae down on X. These formulae are examined in detail in [HL1] and [HL2].

A case of fundamental interest occurs when one considers a smooth map $f : X \longrightarrow Y$ between manifolds. Associated to this is the bundle map

$$df : TX \longrightarrow f^*TY.$$

Suppose TX and TY are furnished with connections and let k be as above. Then our formulas are of the form

$$\phi(D_{TX}, D_{TY}) - \mathbb{D}_k(\alpha) \;=\; d\sigma$$

where ϕ is a universal polynomial in the curvatures of TX and f^*TY and where σ is a canonically defined L^1_{loc}-form on X. A special case is given in the next section.

3. Milnor Currents.

Let $f : X \longrightarrow \Sigma$ be a proper holomorphic map of a complex n-manifold onto a complex curve, and consider the complex bundle map

$$(df)^* : f^*T^*\Sigma \longrightarrow T^*X.$$

This map is atomic provided that the zero set $Z = \{x \in X : df_x = 0\}$ has dimension zero (cf. Chapter III, Sections 7, 8 and 9). In this case $Z = \{x \in X : df_x = 0\} = \{x_1, x_2, x_3, \dots\}$ is a discrete subset of X and the **Milnor current** of f is of the form

$$\mathbb{M} \overset{\text{def}}{=} \operatorname{Div}((df)^*) \;=\; \Sigma m_j[x_j] \;\text{ with }\; m_j \in \mathbf{Z}.$$

Suppose now that X and Σ are provided with complex connections. Then applying the theory to the top Chern class gives a canonically defined L^1_{loc}-form σ on X such that

$$c_n(D_{T^*X}) - c_{n-1}(D_{T^*X})c_1(D_{T^*\Sigma}) - \mathbb{M} \ = \ d\sigma \tag{3.1}$$

where $c_1(D_{T^*\Sigma})$ really means $f^*c_1(D_{T^*\Sigma})$. If we choose hermitian metrics on X and Σ, and use them to identify T^*X with $\overline{TX}$ and $T^*\Sigma$ with $\overline{T\Sigma}$, formula (3.1) becomes

$$(-1)^n\{c_n(D_{TX}) - c_{n-1}(D_{TX})c_1(D_{T\Sigma})\} - \mathbb{M} \ = \ d\sigma. \tag{3.2}$$

The integers m_i are the local **Milnor numbers**

$$m_i \ = \ \dim\left\{\mathcal{O}_{x_i}\Big/\left\langle\frac{\partial f}{\partial t_1},\ldots,\frac{\partial f}{\partial t_n}\right\rangle\right\}.$$

This discussion parallels Fulton's in [Fu; 14.1.5] for the algebraic case. An important point is that the hypothesis that f be holomorphic can be considerably relaxed here. For example, assume f is C^∞ almost complex, then the above discussion remains valid if df vanishes to finite order at isolated points.

4. CR-Singularities and Pontrjagin Forms.

Let X be a smooth oriented manifold of (real) dimension n, and consider a smooth map

$$f : X \longrightarrow \mathbf{C}^{k+1}$$

into complex euclidean $(k+1)$-space where $k < n$. Suppose X is provided with a metric and a (not necessarily compatible) connection D_{TX}. The differential of f gives a bundle map $df : TX \longrightarrow \underline{\mathbf{C}}^{k+1}$ which extends naturally to a complex linear map

$$df_{\mathbf{C}} : TX \otimes \mathbf{C} \longrightarrow \underline{\mathbf{C}}^{k+1}$$

of the complexification of TX. Via the metric we get an adjoint map

$$(df_{\mathbf{C}})^* : \underline{\mathbf{C}}^{k+1} \longrightarrow TX \otimes \mathbf{C}. \tag{4.1}$$

We define the **complex critical set** of f to be the set

$$Cr(f) \;=\; \{x \in X : (df_{\mathbf{C}})^*_x \text{ is not surjective}\}.$$

If $(df_{\mathbf{C}})^*$ is k-atomic, then the degeneracy current

$$\mathbb{C}r(f) \;=\; \mathbb{D}_k$$

is defined as in Section 1. Note that $\mathrm{spt}\mathbb{C}r(f) = Cr(f)$.

Theorem 4.2. *Let $f : X \longrightarrow \mathbf{C}^{k+1}$ be a smooth map of a real oriented n-dimensional manifold where $n - k = 2l > 0$. Suppose that the associated map (4.1) is k-atomic. Then for each connection D_{TX} on X, there is a canonically associated L^1_{loc}-form σ with the property that*

$$p_l(D_{TX}) \;=\; (-1)^l\mathbb{C}r(f) + d\sigma$$

where $p_l(D_{TX})$ is the l^{th} Pontrjagin form of the connection.

Proof. This follows immediately from Theorem 1.5 and the fact that $c_{2l}(D_{TX\otimes\mathbf{C}}) = (-1)^l p_l(D_{TX})$. $\square$

Example. Consider an immersion $j : X \longrightarrow \mathbf{C}^3$ of an oriented 4-manifold for which $(df_{\mathbf{C}})^*$ is 2-atomic. Then

$$p_1(\Omega_{TX}) \;=\; \Sigma n_i[x_i] + d\sigma$$

where $x_1, x_2, \ldots$ are the points of **complex tangency** of X, i.e., where j_*TX is a complex subspace of $\mathbf{C}^3$. The integer multiplicities n_i are computed from the local geometry of the immersion. (See [HL1].)

5. Foliations.

Let F be a real n-dimensional foliation of a complex n-manifold X, and consider the associated bundle injection

$$j : TF \hookrightarrow TX.$$

This map extends to a complex linear map

$$j_{\mathbf{C}} : T_{\mathbf{C}}F \longrightarrow TX$$

of the complexification $T_{\mathbf{C}}F \stackrel{\mathrm{def}}{=} TF \otimes \mathbf{C}$. Suppose that the associated map of line bundles

$$\lambda \stackrel{\mathrm{def}}{=} \Lambda^n j_{\mathbf{C}} : \Lambda^n T_{\mathbf{C}}F \longrightarrow \Lambda^n TX \tag{5.1}$$

is atomic. Then we have the current

$$\mathbb{C}r(F) \stackrel{\mathrm{def}}{=} \mathrm{Div}(\lambda)$$

of **complex degeneracies of the foliation** F, whose support is

$$\begin{aligned} \mathrm{spt}\mathbb{C}r(F) &= \{x \in X : T_xF \text{ is not totally real}\} \\ &= \{x \in X : \dim_{\mathbf{C}}(T_xF \cap iT_xF) > 0\}. \end{aligned}$$

Theorem 5.2. *Let F be a real n-dimensional foliation of a complex n-manifold X and suppose the associated map λ in (5.1) is atomic. Then for each complex connection on X there is an L^1_{loc}-form σ such that*

$$c_1(D_{TX}) - \mathbb{C}r(F) = d\sigma.$$

Further formulas can be obtained by considering other Thom-Porteous currents associated to $j_{\mathbf{C}}$.

6. Invariants for pairs of complex structures.

Let $V \longrightarrow X$ be a smooth real vector bundle of dimension $2n$, and suppose that J_1 and J_2 are two almost complex structures on V. Associated to these we have two splittings

$$V \otimes \mathbf{C} = V_1 \oplus \overline{V}_1 = V_2 \oplus \overline{V}_2$$

where $V_k = \{v - iJ_k v \in V \otimes \mathbf{C} : v \in V\}$ for $k = 1, 2$. Consider the complex bundle map

$$p : V_1 \longrightarrow V_2$$

given by restricting the projection $V_2 \oplus \overline{V}_2 \longrightarrow V_2$ to the subbundle V_1. Let

$$\lambda = \Lambda^n_{\mathbf{C}} p : \Lambda^n_{\mathbf{C}} V_1 \longrightarrow \Lambda^n_{\mathbf{C}} V_2 \tag{6.1}$$

be the associated map of complex line bundles. Then assuming that λ is atomic we can define the characteristic current

$$\mathbb{C}r(J_1, J_2) \stackrel{\text{def}}{=} \operatorname{Div}(\lambda) \tag{6.2}$$

which is supported in the set

$$\mathbf{C}r(J_1, J_2) = \{x \in X : \ker(J_1 + J_2) \neq \{0\}\}$$

of points where there is a non-trivial subspace which is simultaneously J_1-complex and J_2-anticomplex. Applying the theory in Chapter II gives the following result.

Theorem 6.3. *Let $V \longrightarrow X$ be a smooth vector bundle with two complex structures J_1 and J_2. Let D_1 and D_2 be connections such that $D_k(J_k) = [D_k, J_k] = 0$ for $k = 1, 2$. Then if the map λ defined in (6.2) is atomic, there exists $\sigma \in L^1_{\text{loc}}(X)$ such that the following equation of forms and currents holds on X*

$$c_1(D_2) - c_1(D_1) = \mathbb{C}r(J_1, J_2) + d\sigma.$$

Thus given any $\phi \in \mathbf{C}[t]$, there exists $\sigma_\phi \in L^1_{\text{loc}}$ such that

$$\phi(e_2) - \phi(e_1) = \frac{\phi(e_2) - \phi(e_1)}{e_2 - e_1} \mathbb{C}r(J_1, J_2) + d\sigma_\phi$$

where $e_1 = c_1(D_1)$ and $e_2 = c_1(D_2)$.

Similar formulas can be derived by considering higher Thom-Porteous currents associated to the map $p : V_1 \longrightarrow V_2$.

REFERENCES

[AB] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes*, I, Ann. Math. **86** (1967), 374–407; *II*, Ann. Math. **88** (1968), 451–491.

[AH] M.F. Atiyah and F. Hirzebruch, *The Riemann-Roch theorem for analytic embeddings*, Topology **1** (1962), 151–166.

[BaFM] P. Baum, W. Fulton and R. MacPherson, *Riemann-Roch for singular varieties*, Publ. Math. IHES **45** (1975), 101–146.

[Be] B. Berndtsson, *Cauchy-Leray forms and vector bundles*, Ann. Scient. Ec. Norm Sup. **24** (1991), 319–337.

[BGS1] J.-M. Bismut, H. Gillet, and C. Soulé, *Analytic torsion and holomorphic determinant bundles, I, II, III*, Comm. Math. Physics **115** (1988), 49–78, 79–126, 301–351.

[BGS2] ———, *Bott-Chern currents and complex immersions*, Duke Journal **60** (1990), 255–284.

[BL] J.-P. Bourguignon and H.B. Lawson, *Yang Mills theory: its physical origins and differential aspects in seminar on Differential Geometry* edited by S.-T. Yau, Ann. Math. Studies **102**, Princeton Univ. Press (1982), 395–421.

[Bo] R. Bott, *A residue formula for holomorphic vector fields*, J. Diff. Geom. **1** (1967), 311–330.

[BoC] R. Bott and S.S. Chern, *Hermitian vector bundles and the equidistribution of the zeros of their holomorphic sections*, Acta. Math. **114** (1968), 71–112.

[BV] N. Berline and M. Vergne, *A computation of the equivariant index of the Dirac operator*, Bull. Soc. Math. France **113** (1985), 305–345.

[C1] S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. Math. **45** (1944), 747–752.

[C2] ———, *On the curvature integra in a Riemannian manifold*, Ann. Math. **46** (1945), 674–684.

[F] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969.

[Fu] W. Fulton, *Intersection Theory*, Ergobnisse der Mathematik und ihrer Grenzgebiete 3, Folge, Band 2, Springer Verlag, Berlin-Heidelberg, 1984.

[GS] H. Gillett and C. Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metrics, I, II*, Annals of Math. **131** (1990), 163–203, 205–238.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[H] F.R. Harvey, *Spinors and Calibrations*, Perspectives in Mathematics, vol. 9, Academic Press, San Diego, CA, 1990.

[HL1] F.R. Harvey and H. Blaine Lawson, *Geometric Residue Theorems*, to appear.

[HL2] ———, *A Theory of Characteristic Currents Associated with a Singular Connection — Part II*, to appear.

[HP] F.R. Harvey and J. Polking, *Fundamental solutions in complex analysis Part I*, Duke Math. J. **46** (1979), 253–300.

[HS] F.R. Harvey and S. Semmes, *Zero divisors of atomic functions*, Ann. Math. **135** (1992), 567–600.

[Hi] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.

[LM] H.B. Lawson and M. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, New Jersey, 1989.

[L] Lojasiewicz, *Sur de problème de la division*, Warsaw Rozprawy Mathematyezne **22** (1961).

[MQ] V. Mathai and D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986), 85–110.

[MS] J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Math. Studies no. 76 (1974), Princeton Univ. Press.

[MW] M. Micallef and J. Wolfson, *The second variation of area of minimal surfaces in four-manifolds*, to appear in Math. Ann.

[Q] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985), 89–95.

[SS] L.M. Sibner and R.J. Sibner, *A constructive proof of the Riemann-Roch theorem for curves,* Contributions to Analysis (a collection of papers dedicated to Lipman Bers), 401–405, Academic Press, New York, 1974.

[W1] S. Webster, *Minimal surfaces in a Kähler surface*, J. Diff. Geom. **20** (1984), 463–470.

[W2] ———, *The Euler and Pontrjagin numbers of an n-manifold in* $\mathbf{C}^n$, Comment. Math. Helv. **60** (1985), 193–216.

[W3] ———, *On the relation between Chern and Pontrjagin numbers*, Contemp. Math. **49** (1986), 135–143.

[Wo] J. Wolfson, *Minimal surfaces in Kähler surfaces and Ricci curvature*, J. Diff. Geom. **29** (1989), 281–294.