AST

Journées de géométrie algébrique d'Orsay - Juillet 1992 - Pages préliminaires

Astérisque, tome 218 (1993), p. 1-7

<http://www.numdam.org/item?id=AST_1993__218__1_0>

218 ASTÉRISQUE

1993

JOURNÉES DE GÉOMÉTRIE ALGÉBRIQUE D'ORSAY

Juillet 1992

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

N^{os} de classification A.M.S. pour chaque exposé
(le premier numéro est le plus significatif)

V. Batyrev : 14M25 ; 81T70
A. Buium : 14K02 ; 14H35
F. Catanese, P. Frediani : 14P25 ; 12E05, 14D20
L. Chiantini, C. Ciliberto : 14E25 ; 14J99
I. Dolgachev, M. Kapranov : 14J60 ; 14J26
R. Donagi : 14D20 ; 58F07
L. Ein, R. Lazarsfeld : 14J99
D. Eisenbud, J. Harris : 14N05 ; 13D40, 14H50, 14H45
H. Esnault, M. Levine : 14C25 ; 14C30
H. Esnault, V. Srinivas, E. Viehweg : 14C25 ; 14C30
D. Morrison : 14J30 ; 14J15, 22E40, 32J05, 81T40
C. Voisin : 14J30 ; 14J28, 81T40

TABLE DES MATIÈRES

INTRODUCTION

Les Journées de géométrie algébrique d'Orsay se sont déroulées du 20 au 26 juillet 1992, dans le bâtiment de Mathématiques de l'Université Paris-Sud ; elles ont réuni près de 300 participants. Leur objet était de faire le point sur l'état des connaissances en géométrie algébrique complexe, en mettant en lumière les perspectives de recherche qui semblent les plus prometteuses.

Dans ce but, les conférences du matin étaient centrées sur 4 grands thèmes de la géométrie complexe : systèmes linéaires, fibrés vectoriels, cycles algébriques, variétés de dimension 3. L'après-midi, des conférences plénières d'une heure ainsi que des conférences de 45 minutes en parallèle ont permis d'aborder des sujets plus spécialisés.

Nous espérons que ces Actes reflètent la vitalité du sujet telle qu'elle nous est apparue lors de ces journées.

Les Journées de géométrie algébrique d'Orsay ont été organisées dans le cadre du Projet européen Science "Geometry of Algebraic Varieties" (AGE) [1], et ont donc bénéficié, directement et indirectement, du soutien de l'Union Européenne (qui s'appelait encore Communauté Européenne). Nous avons d'autre part reçu un soutien financier important du Conseil Général de l'Essonne. Nous tenons à remercier chaleureusement ces deux institutions, sans l'aide desquelles ces Journées n'auraient probablement pas vu le jour. Nous remercions l'Université Paris-Sud et le C.N.R.S. qui ont également contribué au succès de cette manifestation.

Les organisateurs,

A. BEAUVILLE, O. DEBARRE, Y. LASZLO

[1] Contract SCI-0398-C(A).

RÉSUMÉS DES EXPOSÉS

V. BATYREV : *Quantum cohomology ring of toric manifolds*

We compute the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ of an arbitrary d-dimensional smooth projective toric manifold $\mathbf{P}_\Sigma$ associated with a fan Σ. The multiplicative structure of $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ naturally depends on the choice of an element φ in the ordinary cohomology group $H^*(\mathbf{P}_\Sigma, \mathbf{C})$. We check several properties of the quantum cohomology rings $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ which are supposed to be valid for quantum cohomology rings of a wide class of Kähler manifolds.

A. BUIUM : *A finiteness theorem for isogeny correspondences*

Let Y be a curve in the moduli space of principally polarized abelian varieties of a given dimension. An isogeny correspondence on Y is by definition an (irreducible) curve $Z \subset Y \times Y$ such that for any point (y', y'') of Z the abelian varieties corresponding to y' and y'' are isogenous. There are plenty of curves Y which carry infinitely many isogeny correspondences; the union of all these Y 's is dense in the complex topology of the moduli space. However, we prove that for "most" curves Y there exist only finitely many isogeny correspondences. Here "most curves" mean "all curves belonging to a dense open subset of the space of all curves in the moduli space", where the space of curves is given a suitable topology, called the Kolchin topology, defined using algebraic differential equations.

F. CATANESE, P. FREDIANI : *Configurations of real and complex polynomials*

The present paper is devoted to the combinatorial descriptions of the connected components of certain open sets of the space of real or complex polynomials of a fixed degree. One instance is the open set of generic real polynomials (i.e. with distinct critical values). Describing the connected components of the open set of real lemniscate generic polynomials (i.e. with critical values with distinct non-zero absolute values), we give in particular a geometric proof of the equality between the number of connected components of the space $\mathcal{L}_n$ of complex lemniscate generic polynomials of degree $n+1$ and the number of connected components of the space of real monic polynomials of degree $n+1$ with n distinct real critical values, the lemniscate configurations occurring from real polynomials.

L. CHIANTINI, C. CILIBERTO : *A few remarks on the lifting problem*

We start with a projective variety X in $\mathbf{P}^r$ and a family W of projective subvarieties of $\mathbf{P}^r$, parametrized by the space B, such that for any $t \in B$ the corresponding fibre W_t of W is contained in some h-plane L_t and $W_t \supseteq X \cap L_t$; we assume that the L_t 's for variable t fill an open dense subset of the corresponding Grassmannian. We give conditions on the degrees of X and W_t which imply that the varieties W_t glue together to give a variety W (containing X) such that $W_t = W \cap L_t$ for all t. The proofs are based on the classical differential theory of "foci" introduced by C. Segre. Our results generalize the theorems of Laudal and Gruson-Peskine, which deal with the case X is a curve in $\mathbf{P}^3$.

I. DOLGACHEV, M. KAPRANOV : *Schur quadrics, cubic surfaces and rank 2 vector bundles over the projective plane*

Let $\Sigma \subset \mathbf{P}^3$ be a smooth cubic surface. It is known that S contains 27 lines. Out of these lines one can form 36 Schläfli double-sixes, i.e., collections $\{l_1,...,l_6\},\{l'_1,...,l'_6\}$ of 12 lines such that each l_i meets only l'_j, $j \neq i$ and does not meet $l_j, j \neq i$. In 1881 F. Schur proved that any double-six gives rise to a certain unique quadric Q, the *Schur quadric*, characterized as follows : for any i the lines l_i and l'_i are orthogonal with respect to Q.

The aim of the paper is to relate Schur's construction to the theory of vector bundles on $\mathbf{P}^2$. In fact, we show that the whole theory of Hulek of rank 2 vector bundles on $\mathbf{P}^2$ with odd c_1 can be given a "geometric" interpretation involving some natural generalizations of cubic surfaces, double-sixes and Schur quadrics.

R. DONAGI : *Decomposition of spectral covers*

A G-principal Higgs bundle over a variety X (with values in an arbitrary line bundle on X) determines a family of spectral covers $\widetilde{X}_\rho$ of X, one for each irreducible representation ρ of G. We show that each of the $\mathrm{Pic}(\widetilde{X}_\rho)$ is isogenous to the sum, with multiplicities, of a finite collection of abelian varieties, obtained as isotypic pieces for the action of the Weyl group W on $\mathrm{Pic}(\widetilde{X})$, where $\widetilde{X}$ is the cameral, or W-Galois, cover of X, independent of ρ. The piece $\mathrm{Prym}(\widetilde{X})$, corresponding to the reflection representation of W, is distinguished : it occurs in $\mathrm{Pic}(\widetilde{X}_\rho)$ for each ρ (this characterizes Prym for classical G but not for exceptional groups such as G_2, E_6), and is essentially the moduli space of Higgs bundles with spectral data $\widetilde{X}$. Various Prym identities are recovered as the case $X = \mathbf{P}^1$, G simply laced, studied previously by Kanev.

L. EIN, R. LAZARSFELD : *Seshadri constants on surfaces*

Let L be an ample line bundle on a smooth projective variety X of dimension n. Demailly has introduced the *Seshadri constant* $\epsilon(L, x)$ of L at x, which roughly speaking measures how positive L is at x. For example, if L is very ample, then $\epsilon(L,x) \geq 1$ for all $x \in X$. We study these invariants in the first non-trivial case, when X is a smooth surface. We prove (somewhat surprisingly) that in this case $\epsilon(L,x) \geq 1$ for all except perhaps countably many $x \in X$, and moreover if $L \cdot L > 1$ then the exceptional set is finite. On the other hand, simple examples due to Miranda show that $\epsilon(L,x)$ can take on arbitrary small positive values at isolated points. The paper also contains some related examples and open problems.

D. EISENBUD, M. GREEN, J. HARRIS : *Some conjectures extending Castelnuovo theory*

We propose a series of conjectures concerning the Hilbert functions of points (or more generally zero-dimensional subschemes) in projective space. We begin by extending the results of Castelnuovo and others on points in uniform position, and then consider the corresponding problem without the hypothesis of uniform position. A special case is a conjectured extension of the classical Cayley-Bacharach theorem. We prove this conjecture in projective space $\mathbf{P}^r$ for all $r \leq 7$. Finally we make a conjecture extending Macaulay's theorem on the Hilbert function of graded rings, and discuss its relation to the previous conjectures.

H. ESNAULT, M. LEVINE : *Surjectivity of cycle maps*

Let X be a smooth proper complex variety. We consider the cycle map from the Chow ring to the ring of the Deligne cohomology. If this cycle map is injective (modulo torsion), then it has to be surjective as well, and the groups $\mathrm{H}^p(\mathrm{X}, \mathcal{K}_{p+1})$ are generated by constant functions on codimension p cycles (modulo torsion). This generalizes Jannsen's results concerning the cycle map with values in the Betti cohomology.

H. ESNAULT, V. SRINIVAS, E. VIEHWEG : *Decomposability of Chow groups implies decomposability of cohomology.*

Let X be a smooth proper complex n-dimensional variety. We consider the cup product map from the product of the Chow groups (modulo torsion)

$$\mathrm{CH}^{n_1}(\mathrm{V}) \otimes \cdots \otimes \mathrm{CH}^{n_r}(\mathrm{V}) \to \mathrm{CH}^n(\mathrm{X}) ,$$

where $\sum_{i=1}^{i=r} n_i = n$ and V is a non empty Zarisky open set in X. If it is surjective (modulo torsion), then the corresponding map from the "edge" Hodge groups

$$\mathrm{H}^{n_1}(\mathrm{X}, \mathcal{O}_\mathrm{X}) \otimes \cdots \otimes \mathrm{H}^{n_r}(\mathrm{X}, \mathcal{O}_\mathrm{X}) \to \mathrm{H}^n(\mathrm{X}, \mathcal{O}_\mathrm{X})$$

is surjective. We give variants and discuss some problems.

D. MORRISON : *Compactifications of moduli spaces inspired by mirror symmetry*

We study moduli spaces of nonlinear sigma-models on Calabi-Yau manifolds, using the one-loop semiclassical approximation. The data being parameterized includes a choice of complex structure on the manifold, as well as some "extra structure" described by means of classes in H^2. We formulate a simple and compelling conjecture about the action of the automorphism group on the Kähler cone, which would enable the construction of a partial compactification of the moduli space using Looijenga's "semi-toric" method. We then explore the implications which this construction has concerning the properties of the moduli space of complex structures on a "mirror partner" of the original Calabi-Yau manifold.

C. VOISIN : *Miroirs et involutions sur les surfaces K3*

On construit une série d'exemples de "symétrie miroir" en considérant des variétés de Calabi-Yau du type $(\mathrm{E} \times \mathrm{S})/(j, i)$, où S est une surface K3 munie d'une involution i agissant par (-1) sur $\mathrm{H}^{2,0}(\mathrm{S})$, et E une courbe elliptique munie d'une involution j telle que $\mathrm{E}/j \cong \mathbf{P}^1$. On utilise les travaux de Nikulin pour construire l'involution miroir sur $\mathrm{H}^2(\mathrm{S}, \mathbf{Z})$, et le théorème de Torelli pour construire l'application miroir holomorphe

$$\big((\mathrm{E} \times \mathrm{S})/(j, i), \alpha\big) \longmapsto \big((\mathrm{E}' \times \mathrm{S}')/(j', i'), \alpha'\big) .$$

VICTOR V. BATYREV

Quantum cohomology rings of toric manifolds

Astérisque, tome 218 (1993), p. 9-34
<http://www.numdam.org/item?id=AST_1993__218__9_0>

Quantum Cohomology Rings of Toric Manifolds

Victor V. Batyrev

1 Introduction

The notion of *quantum cohomology ring* of a Kähler manifold V naturally appears in theoretical physics in the consideration of the so called *topological sigma model* associated with V ([16], 3a-b). If the canonical line bundle $\mathcal{K}_V$ of V is negative, then one recovers the multiplicative structure of the quantum cohomology ring of V from the intersection theory on the moduli space $\mathcal{I}_\lambda$ of holomorphic mappings f of the Riemann sphere $f : S^2 \cong \mathbf{CP}^1 \to V$ where λ is the homology class in $H_2(V, \mathbf{Z})$ of $f(\mathbf{CP}^1)$.

If the canonical bundle $\mathcal{K}_V$ is trivial, the quantum cohomology ring was considered by C. Vafa as an important tool for explaining the mirror symmetry for Calabi-Yau manifolds [15].

The quantum cohomology ring $QH_\varphi(V, \mathbf{C})$ of a Kähler manifold V, unlike the ordinary cohomology ring, have the multiplicative structure which depends on the class φ of the Kähler $(1,1)$-form corresponding to a Kähler metric g on V. When we rescale the metric $g \to tg$ and put $t \to \infty$, the quantum ring becomes the classical cohomology ring. For example, for the topological sigma model on the complex projective line $\mathbf{CP}^1$ itself, the classical cohomology ring is generated by the class x of a Kahler $(1,1)$-form, where x satisfies the quadratic equation

$$x^2 = 0, \tag{1}$$

while the quantum cohomology ring is also generated by x, but the equation satisfied by x is different:

$$x^2 = \exp(-\int_\lambda \varphi), \tag{2}$$

λ is a non-zero effective 2-cycle. Similarly, the quantum cohomology ring of d-dimensional complex projective space is generated by the element x satisfying the equation

$$x^{d+1} = \exp(-\int_{\lambda} \varphi). \tag{3}$$

The main purpose of this paper is to construct and investigate the quantum cohomology ring $QH^*_{\varphi}(\mathbf{P}_{\Sigma}, \mathbf{C})$ of an arbitrary smooth compact d-dimensional toric manifold $\mathbf{P}_{\Sigma}$ where φ is an element of the ordinary second cohomology group $H^2(\mathbf{P}_{\Sigma}, \mathbf{C})$. Since all projective spaces are toric manifolds, we obtain a generalization of above examples of quantum cohomology rings.

According to the physical interpretation, a quantum cohomology ring is a closed operator algebra acting on the fermionic Hilbert space. For example, the equation (3) one should better write as an equations for a linear operator $\mathcal{X}$ corresponding to the cohomology class x:

$$\mathcal{X}^{d+1} = \exp(-\int_{\lambda} \varphi)id. \tag{4}$$

It is convenient to define quantum rings by polynomial equations among generators.

Definition 1.1 Let

$$h(t,x) = \sum_{n \in \mathcal{N}} c_n(t) x^n$$

be a one-parameter family of polynomials in the polynomial ring $\mathbf{C}[x]$, where $x = \{x_i\}_{i \in I}$ is a set of variables indexed by I, t is a positive real number, $\mathcal{N}$ is a fixed finite set of exponents. We say that the polynomial

$$h^{\infty}(x) = \sum_{n \in \mathcal{N}} c_n^{\infty} x^n$$

is the limit of the family $h(t,x)$ as $t \to \infty$, if the point $\{c_n^{\infty}\}_{n \in \mathcal{N}}$ of the $(\mid \mathcal{N} \mid -1)$-dimensional complex projective space is the limit of the one-parameter family of points with homogeneous coordinates $\{c_n(t)\}_{n \in \mathcal{N}}$.

Definition 1.2 Let R_t be a one-parameter family of commutative algebras over $\mathbf{C}$ with a fixed set of generators $\{r_i\}$, $t \in \mathbf{R}_{>0}$. We denote by J_t the ideal in $\mathbf{C}[x]$ consisting of all polynomial relations among $\{r_i\}$, i.e., the kernel of the surjective homomorphism $\mathbf{C}[x] \to R_t$. We say that the ideal J^{∞} is the limit of J_t as $t \to \infty$, if any one-parameter family of polynomials $h(t,x) \in J_t$ (as in 1.1) has a limit, and J^{∞} is generated as $\mathbf{C}$-vector space by all these limits. The $\mathbf{C}$-algebra

$$R^{\infty} = \mathbf{C}[x]/J^{\infty}$$

will be called the *limit* of R_t.

Remark 1.3 In general, it is not true that if $J^\infty = \lim_{t\to\infty} J_t$, and J_t is generated by a finite set of polynomials $\{h_1(t,x)\ldots,h_k(t,x)\}$, then J^∞ is generated by the limits $\{h_1^\infty(x),\ldots,h_k^\infty(x)\}$. The limit ideal J^∞ is generated by the limits $h_i^\infty(x)$ *only if* the set of polynomials $\{h_i(t,x)\}$ form a *Gröbner-type basis* for J_t.

In this paper, we establish the following basic properties of quantum cohomology rings of toric manifolds:

I : If φ is an element in the interior of the Kähler cone $K(\mathbf{P}_\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{C})$, then there exists a limit of $QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C})$ as $t \to \infty$, and this limit is isomorphic to the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma, \mathbf{C})$ (Corollary 5.5).

II : Assume that two smooth projective toric manifolds $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are isomorphic in codimension 1, for instance, that $\mathbf{P}_{\Sigma_1}$ is obtained from $\mathbf{P}_{\Sigma_2}$ by a flop-type birational transformation. Then the natural isomorphism $H^2(\mathbf{P}_{\Sigma_1}, \mathbf{C}) \cong H^2(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ induces the isomorphism between the quantum cohomology rings

$$QH^*_\varphi(\mathbf{P}_{\Sigma_1}, \mathbf{C}) \cong QH^*_\varphi(\mathbf{P}_{\Sigma_2}, \mathbf{C})$$

(Theorem 6.1). We notice that ordinary cohomology rings of $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are not isomorphic in general.

III : Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ of $\mathbf{P}_\Sigma$ belongs to the closed Kähler cone $K(\mathbf{P}_\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{C})$. Then the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ is isomorphic to the Jacobian ring of a Laurent polynomial $f_\varphi(X)$ such that the equation $f_\varphi(X) = 0$ defines an affine Calabi-Yau hypersurface Z_f in the d-dimensional algebraic torus $(\mathbf{C}^*)^d$ where Z_f is *mirror symmetric* with respect to Calabi-Yau hypersurfaces in $\mathbf{P}_\Sigma$ (Theorem 8.4). Here by the *mirror symmetry* we mean the correspondence, based on the polar duality [6], between families of Calabi-Yau hypersurfaces in toric varieties.

The properties II and III give a general view on the recent result of P. Aspinwall, B. Greene, and D. Morrison [3] who have shown, for a family of Calabi-Yau 3-folds W that their quantum cohomology ring $QH^*_\varphi(W, \mathbf{C})$ does not change under a flop-type birational transformation (see also [1, 2]).

IV: Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ of $\mathbf{P}_\Sigma$ is divisible by r, i.e., there exists an element $h \in H^2(\mathbf{P}_\Sigma, \mathbf{Z})$ such that $c_1(\mathbf{P}_\Sigma) = rh$. Then $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has a natural $\mathbf{Z}/r\mathbf{Z}$-grading (Theorem 5.7). We remark that the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no $\mathbf{Z}$-grading.

The paper is organized as follows. In Sections 2-4, we recall a definition of toric manifolds and standard facts about them. In Section 5, we define the quantum cohomology ring of toric manifolds and prove their properties.

In Section 6, we consider examples of the behavior of quantum cohomology rings under elementary birational transformations such as blow-ups and flops, we also consider the case of singular toric varieties. In Section 7, we give an combinatorial interpretation of the relation between the quantum cohomology rings and the ordinary cohomology rings. In Section 8, we show that the quantum cohomology ring can be interpreted as a Jacobian ring of some Laurent polynomial. Finally, in Section 9, we prove that our quantum cohomology rings coincide with the quantum cohomology rings defined by sigma models on toric manifolds.

Acknowledgements. It is a pleasure to acknowledge helpful discussions with Yu. Manin, D. Morrison, Duco van Straten as well as with S. Cecotti and C. Vafa. I would like to express my thanks for hospitality to the Mathematical Sciences Research Institute where this work was conducted and supported in part by the National Science Foundation (DMS-9022140), and the DFG (Forschungsscherpunkt Komplexe Mannigfaltigkeiten).

2 A definition of compact toric manifolds

Toric varieties were considered in full generality in [9, 11]. For the general definition of toric variety which includes affine and quasi-projective toric varieties with singularities, it is more convenient to use the language of schemes. However, for our purposes, it will be sufficient to have a simplified more classical version of the definition for *smooth and compact toric varieties* over $\mathbf{C}$. This approach to compact toric manifolds was first proposed by M. Audin [4], and developed by D. Cox [8].

In order to obtain a d-dimensional compact toric manifold V, we need a combinatorial object Σ, a *complete fan of regular cones*, in a d-dimensional vector space over $\mathbf{R}$.

Let N, $M = \mathrm{Hom}(N, \mathbf{Z})$ be dual lattices of rank d, and $N_{\mathbf{R}}$, $M_{\mathbf{R}}$ their $\mathbf{R}$-scalar extensions to d-dimensional real vector spaces.

Definition 2.1 A convex subset $\sigma \subset N_{\mathbf{R}}$ is called a *regular k-dimensional cone* ($k \geq 1$) if there exist k linearly independent elements $v_1, \ldots, v_k \in N$ such that

$$\sigma = \{\mu_1 v_1 + \cdots + \mu_k v_k \mid \mu_i \in \mathbf{R}, \mu_i \geq 0\},$$

and $\{v_1, \ldots, v_k\}$ is a subset of some $\mathbf{Z}$-basis of N. In this case, we call $v_1, \ldots, v_k \in N$ the *integral generators of σ*.

The origin $0 \in N_{\mathbf{R}}$ we call the *regular* 0*-dimensional cone.* By definition, the set of integral generators of this cone is empty.

Definition 2.2 A regular cone σ' is called *a face* of a regular cone σ (we write $\sigma' \prec \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.

Definition 2.3 A finite system $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of regular cones in $N_{\mathbf{R}}$ is called *a complete d-dimensional fan* of regular cones, if the following conditions are satisfied:

(i) if $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma' \in \Sigma$;

(ii) if σ, σ' are in Σ, then $\sigma \cap \sigma' \prec \sigma$ and $\sigma \cap \sigma' \prec \sigma'$;

(iii) $N_{\mathbf{R}} = \sigma_1 \cup \cdots \cup \sigma_s$.

The set of all k-dimensional cones in Σ will be denoted by $\Sigma^{(k)}$.

Example 2.4 Choose $d+1$ vectors $v_1, \ldots, v_{d+1}$ in a d-dimensional real space E such that E is spanned by $v_1, \ldots, v_{d+1}$ and there exists the linear relation

$$v_1 + \cdots + v_{d+1} = 0.$$

Define N to be the lattice in E consisting of all integral linear combinations of $v_1, \ldots, v_{d+1}$. Obviously, $N_{\mathbf{R}} = E$. Then any k-element subset $I \subset \{v_1, \ldots, v_{d+1}\}$ $(k \leq d)$ generates a k-dimensional regular cone $\sigma(I)$. The set $\Sigma(d)$ consisting of $2^{d+1} - 1$ cones $\sigma(I)$ generated by I is a complete d-dimensional fan of regular cones.

Definition 2.5 (cf.[5]) Let Σ be a complete d-dimensional fan of regular cones. Denote by $G(\Sigma) = \{v_1, \ldots, v_n\}$ the set of all generators of 1-dimensional cones in Σ ($n = \mathrm{Card}\, \Sigma^{(1)}$). We call a subset $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\} \subset G(\Sigma)$ a *primitive collection* if $\{v_{i_1}, \ldots, v_{i_p}\}$ is not the set of generators of a p-dimensional simplicial cone in Σ, while for all k $(0 \leq k < p)$ each k-element subset of $\mathcal{P}$ generates a k-dimensional cone in Σ.

Example 2.6 Let Σ be a fan $\Sigma(d)$ from Example 2.4. Then there exists the unique primitive collection $\mathcal{P} = G(\Sigma(d))$.

Definition 2.7 Let $\mathbf{C}^n$ be n-dimensional affine space over $\mathbf{C}$ with the set of coordinates $z_1, \ldots, z_n$ which are in the one-to-one correspondence $z_i \leftrightarrow v_i$ with elements of $G(\Sigma)$. Let $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\}$ be a primitive collection in $G(\Sigma)$. Denote by $\mathbf{A}(\mathcal{P})$ the $(n-p)$-dimensional affine subspace in $\mathbf{C}^n$ defined by the equations

$$z_{i_1} = \cdots = z_{i_p} = 0.$$

Remark 2.8 Since every primitive collection $\mathcal{P}$ has at least two elements, the codimension of $\mathbf{A}(\mathcal{P})$ is at least 2.

Definition 2.9 Define the closed algebraic subset $Z(\Sigma)$ in $\mathbf{C}^n$ as follows

$$Z(\Sigma) = \bigcup_{\mathcal{P}} \mathbf{A}(\mathcal{P}),$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$. Put

$$U(\Sigma) = \mathbf{C}^n \setminus Z(\Sigma).$$

Definition 2.10 Two complete d-dimensional fans of regular cones Σ and Σ' are called *combinatorially equivalent* if there exists a bijective mapping $\Sigma \to \Sigma'$ respecting the face-relation "$\prec$" (see 2.2).

Remark 2.11 It is easy to see that the open subset $U(\Sigma) \subset \mathbf{C}^n$ depends only on the combinatorial structure of Σ, i.e., for any two combinatorially equivalent fans Σ and Σ', one has $U(\Sigma) \cong U(\Sigma')$.

Definition 2.12 Let $R(\Sigma)$ be the subgroup in $\mathbf{Z}^n$ consisting of all lattice vectors $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$.

Obvioulsy, $R(\Sigma)$ is isomorphic to $\mathbf{Z}^{n-d}$.

Definition 2.13 Let Σ be a complete d-dimensional fan of regular cones. Define $\mathbf{D}(\Sigma)$ to be the connected commutative subgroup in $(\mathbf{C}^*)^n$ generated by all one-parameter subgroups

$$a_\lambda \;:\; \mathbf{C}^* \to (\mathbf{C}^*)^n,$$
$$t \to (t^{\lambda_1}, \ldots, t^{\lambda_n})$$

where $\lambda \in R(\Sigma)$.

Remark 2.14 Choosing a $\mathbf{Z}$-basis in $R(\Sigma)$, one easily obtains an isomorphism between $\mathbf{D}(\Sigma)$ and $(\mathbf{C}^*)^{n-d}$.

Now we are ready to give the definition of the compact toric manifold $\mathbf{P}_\Sigma$ associated with a complete d-dimensional fan of regular cones Σ.

Definition 2.15 Let Σ be a complete d-dimensional fan of regular cones. Then quotient

$$\mathbf{P}_\Sigma = U(\Sigma)/\mathbf{D}(\Sigma)$$

is called the *compact toric manifold associated with* Σ.

Example 2.16 Let Σ be a fan $\Sigma(d)$ from Example 2.4. By 2.6, $U(\Sigma(d)) = \mathbf{C}^{d+1} \setminus \{0\}$. By the definition of $\Sigma(d)$, the subgroup $R(\Sigma) \subset \mathbf{Z}^n$ is generated by $(1, \ldots, 1) \in \mathbf{Z}^{d+1}$. Thus, $\mathbf{D}(\Sigma) \subset (\mathbf{C}^*)^n$ consists of the elements $(t, \ldots, t)$, where $t \in \mathbf{C}^*$. So the toric manifold associated with $\Sigma(d)$ is the ordinary d-dimensional projective space.

A priori, it is not obvious that the quotient space $\mathbf{P}_\Sigma = U(\Sigma)/\mathbf{D}(\Sigma)$ always exists as the space of orbits of the group $\mathbf{D}(\Sigma)$ acting free on $U(\Sigma)$, and that $\mathbf{P}_\Sigma$ is smooth and compact. However, these facts are easy to check if we take the d-dimensional projective space $\mathbf{P}_{\Sigma(d)}$ as a model example.

There exists a simple open covering of $U(\Sigma)$ by affine algebraic varieties:

Proposition 2.17 *Let σ be a k-dimensional cone in Σ, $\{v_{i_1}, \ldots, v_{i_k}\}$ the set of generators of σ. Define the open subset $U(\sigma) \subset \mathbf{C}^n$ by the conditions $z_j \neq 0$ for all $j \notin \{i_1, \ldots, i_k\}$. Then the open sets $U(\sigma)$ $(\sigma \in \Sigma)$ have the properties:*

(i)

$$U(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma);$$

(ii) *if $\sigma \prec \sigma'$, then $U(\sigma) \subset U(\sigma')$;*

(iii) *for any two cone $\sigma_1, \sigma_2 \in \Sigma$, one has $U(\sigma_1) \cap U(\sigma_2) = U(\sigma_1 \cap \sigma_2)$; in particular,*

$$U(\Sigma) = \bigcup_{\sigma \in \Sigma^{(d)}} U(\sigma).$$

Proposition 2.18 *Let σ be a d-dimensional cone in $\Sigma^{(d)}$, $\{v_{i_1}, \ldots, v_{i_d}\}$ the set of generators of σ. Denote by $u_{i_1}, \ldots, u_{i_d}$ the dual to $v_{i_1}, \ldots, v_{i_d}$ $\mathbf{Z}$-basis of the lattice M, i.e, $\langle v_{i_k}, u_{i_l} \rangle = \delta_{k,l}$, where $\langle *, * \rangle : N \times M \to \mathbf{Z}$ is the canonical pairing between lattices N and M.*

Then the affine open subset $U(\sigma)$ is isomorphic to $\mathbf{C}^d \times (\mathbf{C}^*)^{n-d}$, the action of $\mathbf{D}(\Sigma)$ on $U(\sigma)$ is free, and the space of $\mathbf{D}(\Sigma)$-orbits is isomorphic to the affine space $U_\sigma = \mathbf{C}^d$ whose coordinate functions $x_1^\sigma, \ldots, x_d^\sigma$ are the following Laurent monomials in $z_1, \ldots, z_n$:

$$x_1^\sigma = z_1^{\langle v_1, u_{i_1} \rangle} \cdots z_n^{\langle v_n, u_{i_1} \rangle}, \ \ldots \ , x_d^\sigma = z_1^{\langle v_1, u_{i_d} \rangle} \cdots z_n^{\langle v_n, u_{i_d} \rangle}.$$

The last statement yields a general formula for the local affine coordinates $x_1^\sigma, \ldots, x_d^\sigma$ of a point $p \in U_\sigma$ as functions of its *homogeneous coordinates* $z_1, \ldots, z_n$ (see also [8]).

Compactness of $\mathbf{P}_\Sigma$ follows from the fact that the local polydiscs

$$D_\sigma = \{x \in U_\sigma \ : \mid x_1^\sigma \mid \leq 1, \ldots, \mid x_d^\sigma \mid \leq 1\}, \ \ \sigma \in \Sigma^{(d)}$$

form a finite compact covering of $\mathbf{P}_\Sigma$.

3 Cohomology of toric manifolds

Let Σ be a complete d-dimensional fan of regular cones.

Definition 3.1 A continuous function $\varphi \ : \ N_{\mathbf{R}} \rightarrow \mathbf{R}$ is called *Σ-piecewise linear*, if φ is a linear function on every cone $\sigma \in \Sigma$.

Remark 3.2 It is clear that any Σ-piecewise linear function φ is uniquely defined by its values on elements v_i of $G(\Sigma)$. So the space of all Σ-piecewise linear functions $PL(\Sigma)$ is canonically isomorphic to $\mathbf{R}^n$: $\varphi \rightarrow (\varphi(v_1), \ldots, \varphi(v_n))$.

Theorem 3.3 *The space $PL(\Sigma)/M_{\mathbf{R}}$ of all Σ-piecewise linear functions modulo the d-dimensional subspace of globally linear functions on $N_{\mathbf{R}}$ is canonically isomorphic to the cohomology space $H^2(\mathbf{P}_\Sigma, \mathbf{R})$. Moreover, the first Chern class $c_1(\mathbf{P}_\Sigma)$, as an element of $H^2(\mathbf{P}_\Sigma, \mathbf{Z})$, is represented by the class of the Σ-piecewise linear function $\alpha_\Sigma \in PL(\Sigma)$ such that $\alpha_\Sigma(v_1) = \cdots = \alpha_\Sigma(v_n) = 1$.*

Theorem 3.4 *Let $R(\Sigma)_{\mathbf{R}}$ be the $\mathbf{R}$-scalar extension of the abelian group $R(\Sigma)$. Then the space $R(\Sigma)_{\mathbf{R}}$ is canonically isomorphic to $H_2(\mathbf{P}_\Sigma, \mathbf{R})$.*

Definition 3.5 Let φ be an element of $PL(\Sigma)$, λ an element of $R(\Sigma)_{\mathbf{R}}$. Define the *degree of λ relative to φ* as

$$\deg_\varphi(\lambda) = \sum_{i=1}^{n} \lambda_i \varphi(v_i).$$

It is easy to see that for any $\varphi \in M_{\mathbf{R}}$ and for any $\lambda \in R(\Sigma)_{\mathbf{R}}$, one has $\deg_\varphi(\lambda) = 0$. Moreover, the degree-mapping induces the nondegenerate pairing

$$\deg \ : \ PL(\Sigma)/M_{\mathbf{R}} \times R(\Sigma)_{\mathbf{R}} \rightarrow \mathbf{R}$$

which coincides with the canonical intersection pairing

$$H^2(\mathbf{P}_\Sigma, \mathbf{R}) \times H_2(\mathbf{P}_\Sigma, \mathbf{R}) \rightarrow \mathbf{R}.$$

Definition 3.6 Let $\mathbf{C}[z]$ be the polynomial ring in n variables $z_1, \ldots, z_n$. Denote by $SR(\Sigma)$ the ideal in $\mathbf{C}[z]$ generated by all monomials

$$\prod_{v_j \in \mathcal{P}} z_j,$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$. The ideal $SR(\Sigma)$ is usually called the *Stenley-Reisner ideal* of Σ.

Definition 3.7 Let $u_1, \ldots, u_d$ be any $\mathbf{Z}$-basis of the lattice M. Denote by $P(\Sigma)$ the ideal in $\mathbf{C}[z]$ generated by d elements

$$\sum_{i=1}^{n} \langle v_i, u_1 \rangle z_i, \ldots, \sum_{i=1}^{n} \langle v_i, u_d \rangle z_i.$$

Obviously, the ideal $P(\Sigma)$ does not depend on the choice of basis of M.

Theorem 3.8 *The cohomology ring of the compact toric manifold* $\mathbf{P}_\Sigma$ *is canonically isomorphic to the quotient of* $\mathbf{C}[z]$ *by the sum of two ideals* $P(\Sigma)$ *and* $SR(\Sigma)$:

$$H^*(\mathbf{P}_\Sigma, \mathbf{C}) \cong \mathbf{C}[z]/(P(\Sigma) + SR(\Sigma)).$$

Moreover, the canonical embedding $H^2(\mathbf{P}_\Sigma, \mathbf{C}) \hookrightarrow H^*(\mathbf{P}_\Sigma, \mathbf{C})$ *is induced by the linear mapping*

$$PL(\Sigma) \otimes_{\mathbf{R}} \mathbf{C} \rightarrow \mathbf{C}[z], \;\; \varphi \mapsto \sum_{i=1}^{n} \varphi_i(v_i) z_i.$$

In particular, the first Chern class of $\mathbf{P}_\Sigma$ *is represented by the sum* $z_1 + \cdots + z_n$.

Example 3.9 Let $\mathbf{P}_\Sigma$ be d-dimensional projective space defined by the fan $\Sigma(d)$ (see 2.4). Then

$$P(\Sigma(d)) = < (z_1 - z_{d+1}), \ldots, (z_d - z_{d+1}) >, \;\; SR(\Sigma(d)) = < \prod_{i=1}^{d+1} z_i > .$$

So we obtain

$$\mathbf{C}[z_1, \ldots, z_{d+1}]/(P(\Sigma(d)) + SR(\Sigma(d)) \cong \mathbf{C}[x]/x^{d+1}.$$

4 Line bundles and Kähler classes

Let $\pi : U(\Sigma) \to \mathbf{P}_\Sigma$ be the canonical projection whose fibers are principal homogeneous spaces of $\mathbf{D}(\Sigma)$. For any line bundle $\mathcal{L}$ over $\mathbf{P}_\Sigma$, the pullback $\pi^*\mathcal{L}$ is a line bundle over $U(\Sigma)$. By 2.8, $\pi^*\mathcal{L}$ is isomorphic to $\mathcal{O}_{U(\Sigma)}$. Therefore, the Picard group of $\mathbf{P}_\Sigma$ is isomorphic to the group of all $\mathbf{D}$-linearization of $\mathcal{O}_{U(\Sigma)}$, or to the group of all characters $\chi : \mathbf{D}(\Sigma) \to \mathbf{C}^*$. The latter is isomorphic to the group $\mathbf{Z}^n/M$ where $\mathbf{Z}^n$ is the group of all Σ-piecewise linear functions φ such that $\varphi(N) \subset \mathbf{Z}$.

Proposition 4.1 *Assume that a character χ is represented by the class of an integral Σ-piecewise linear function φ. Then the space $H^0(\mathbf{P}_\Sigma, \mathcal{L}_\chi)$ of global sections of the corresponding line bundle $\mathcal{L}_\chi$, is canonically isomorphic to the space of all polynomials $F(z_1, \ldots, z_n) \in \mathbf{C}[z]$ satisfying the condition*

$$F(t^{\lambda_1} z_1, \ldots, t^{\lambda_n} z_n) = t^{\deg_\varphi \lambda} F(z_1, \ldots, z_n)$$

$$\text{for all } \lambda \in R(\Sigma),\ t \in \mathbf{C}^*.$$

The exponents $(m_1, \ldots, m_n)$ of the monomials satisfying the above condition can be identified with integral points in the convex polyhedron:

$$\Delta_\varphi = \{(x_1, \ldots, x_n) \in \mathbf{R}^n_{\geq 0} : \deg_\varphi \lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n,\ \ \lambda \in R(\Sigma)\}$$

Definition 4.2 A Σ-piecewise linear function $\varphi \in PL(\Sigma)$ is called a *strictly convex support* function for the fan Σ, if φ satisfies the properties

(i) φ is an *upper convex function*, i.e.,

$$\varphi(x) + \varphi(y) \geq \varphi(x + y);$$

(ii) for any two different d-dimensional cones σ_1, $\sigma_2 \in \Sigma$, the restrictions $\varphi|_\sigma$ and $\varphi|_{\sigma'}$ are different linear functions.

Proposition 4.3 *If φ is a strictly convex support function, then the polyhedron Δ_φ is simple (i.e., any vertex of Δ_φ is contained in d-faces of codimension* 1), *and the fan Σ can be uniquely recovered from Δ_φ using the property:*

$$\Delta_\varphi \cong \{x \in M_{\mathbf{R}} : \langle v_i, x\rangle \geq -\varphi(v_i)\}.$$

Definition 4.4 Denote by $K(\Sigma)$ the cone in $H^2(\mathbf{P}_\Sigma, \mathbf{R}) = PL(\Sigma)/M_{\mathbf{R}}$ consisting of the classes of all upper convex Σ-piecewise linear functions $\varphi \in PL(\Sigma)$. We denote by $K^0(\Sigma)$ the interior of $K(\Sigma)$, i.e., the cone consisting of the classes of all strictly convex support functions in $PL(\Sigma)$.

Theorem 4.5 *The open cone $K^0(\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{R})$ consists of classes of Kähler $(1,1)$-forms on $\mathbf{P}_\Sigma$, i.e., $K(\Sigma)$ is isomorphic to the closed Kähler cone of $\mathbf{P}_\Sigma$.*

Next theorem will play the central role in the sequel. Its statement is contained implicitly in [12, 13]:

Theorem 4.6 *A Σ-piecewise linear function φ is a strictly convex support function, i.e., $\varphi \in K^0(\Sigma)$, if and only if*

$$\varphi(v_{i_1}) + \cdots + \varphi(v_{i_k}) > \varphi(v_{i_1} + \cdots + v_{i_k})$$

for all primitive collections $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_k}\}$ in $G(\Sigma)$.

5 Quantum cohomology rings

Definition 5.1 Let φ be a Σ-piecewise linear function with complex values, or an element of the complexified space $PL(\Sigma)_{\mathbf{C}} = PL(\Sigma) \otimes_{\mathbf{R}} \mathbf{C}$. Define the quantum cohomology ring as the quotient of the polynomial ring $\mathbf{C}[z]$ by the sum of ideals $P(\Sigma)$ and $Q_\varphi(\Sigma)$:

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}) := \mathbf{C}[z]/(P(\Sigma) + Q_\varphi(\Sigma))$$

where $Q_\varphi(\Sigma)$ is generated by binomials

$$\exp\left(\sum_{i=1}^n a_i \varphi(v_i)\right) \prod_{i=1}^n z_i^{a_i} - \exp\left(\sum_{j=1}^n b_j \varphi(v_j)\right) \prod_{j=1}^n z_j^{b_j}$$

running over all possible linear relations

$$\sum_{i=1}^n a_i v_i = \sum_{j=1}^n b_j v_j,$$

where all coefficients a_i and b_j are non-negative and integral.

Definition 5.2 Let $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_k}\} \subset G(\Sigma)$ be a primitive collection, $\sigma_{\mathcal{P}}$ the minimal cone in Σ containing the sum

$$v_{\mathcal{P}} = v_{i_1} + \ldots + v_{i_k},$$

$v_{j_1}, \ldots, v_{j_l}$ generators of $\sigma_{\mathcal{P}}$. Let l be the dimension of $\sigma_{\mathcal{P}}$. By 2.3(iii), there exists the unique representation of $v_{\mathcal{P}}$ as an integral linear combination of generators $v_{j_1}, \ldots, v_{j_l}$ with positive integral coefficients $c_1, \ldots, c_l$:

$$v_{\mathcal{P}} = c_1 v_{j_1} + \cdots + c_l v_{j_l},$$

We put

$$E_\varphi(\mathcal{P}) = \exp(\varphi(v_{i_1} + \ldots + v_{i_k}) - \varphi(v_{i_1}) - \ldots - \varphi(v_{i_k}))$$

$$= \exp(c_1\varphi(v_{j_1}) + \cdots + c_l\varphi(v_{j_l}) - \varphi(v_{i_1}) - \ldots - \varphi(v_{i_k})).$$

Theorem 5.3 *Assume that the Kähler cone $K(\Sigma)$ has non-empty interior, i.e., $\mathbf{P}_\Sigma$ is projective. Then the ideal $Q_\varphi(\Sigma)$ is generated by the binomials*

$$B_\varphi(\mathcal{P}) = z_{i_1} \cdots z_{i_k} - E_\varphi(\mathcal{P}) z_{j_1}^{c_1} \cdots z_{j_l}^{c_l},$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$.

Proof. We use some ideas from [14]. Let ϕ be an element in $PL(\Sigma)$ representing an interior point of $K(\Sigma)$. Define the weights $\omega_1, \ldots, \omega_n$ of $z_1, \ldots, z_n$ as

$$\omega_i = \phi(v_i) \ \ (1 \leq i \leq n).$$

We claim that binomials $B_\varphi(\mathcal{P})$ form a reduced Gröbner basis for $Q_\varphi(\Sigma)$ relative to the weight vector

$$\omega = (\omega_1, \ldots, \omega_n).$$

Notice that the weight of the monomial $z_{i_1} \cdots z_{i_k}$ is greater than the weight of the monomial $z_{j_1}^{c_1} \cdots z_{j_l}^{c_l}$, because

$$\phi(v_{i_1}) + \cdots + \phi(v_{i_k}) > \phi(v_{i_1} + \cdots + v_{i_k}) = c_1\phi(v_{j_1}) + \cdots c_l\phi(v_{j_l})$$

(Theorem 4.6). So the initial ideal $init_\omega\langle B_\varphi(\mathcal{P})\rangle$ of the ideal $\langle B_\varphi(\mathcal{P})\rangle$ generated by $B_\varphi(\mathcal{P})$ coincides with the ideal $SR(\Sigma)$. It suffices to show that the initial ideal $init_\omega Q_\varphi(\Sigma)$ also equals $SR(\Sigma)$. The latter again follows from Theorem 4.6. □

Definition 5.4 The tube domain in the cohomology space $H^2(\mathbf{P}_\Sigma, \mathbf{C})$:

$$K(\Sigma)_{\mathbf{C}} = K(\Sigma) + iH^2(\mathbf{P}_\Sigma, \mathbf{R})$$

we call the *complexified Kähler cone of* $\mathbf{P}_\Sigma$.

Corollary 5.5 *Let φ be an element of $H^2(\mathbf{P}_\Sigma, \mathbf{C})$, t a positive real number. Then all generators $B_{t\varphi}(\mathcal{P})$ of the ideal $Q_{t\varphi}(\Sigma)$ have finite limits as $t \to \infty$ if and only if $\varphi \in K(\Sigma)_{\mathbf{C}}$. Moreover, if $\varphi \in K(\Sigma)_{\mathbf{C}}$, then the limit of $QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C})$ is the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma, \mathbf{C})$.*

Proof. Applying Theorem 4.6, we obtain:

$$\lim_{t\to\infty} B_{t\varphi}(\mathcal{P}) = z_{i_1} \cdots z_{i_k}.$$

Thus,

$$\lim_{t\to\infty} Q_{t\varphi}(\Sigma) = SR(\Sigma).$$

By Theorem 3.8,

$$\lim_{t\to\infty} QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C}) = H^*(\mathbf{P}_\Sigma, \mathbf{C}).$$

□

Example 5.6 Consider the fan $\Sigma(d)$ defining d-dimensional projective space (see 2.4). Then we obtain

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}) \cong \mathbf{C}[x]/(x^{d+1} - \exp(-\deg_\varphi \lambda)),$$

where $\lambda = (1, \ldots, 1)$ is the generator of $R(\Sigma(d))$. This shows the quantum cohomology ring $QH^*_\varphi(\mathbf{CP}^d, \mathbf{C})$ coincides with the quantum cohomology ring for $\mathbf{CP}^d$ proposed by physicists.

It is important to remark that the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no any $\mathbf{Z}$-grading, but it is possible to define a $\mathbf{Z}_N$-grading on it.

Theorem 5.7 Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ is divisible by r. Then the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has a natural $\mathbf{Z}/r\mathbf{Z}$-grading.

Proof. A linear relation

$$\sum_{i=1}^n a_i v_i = \sum_{j=1}^n b_j v_j$$

gives rise to an element

$$\lambda = (a_1 - b_1, \ldots, a_n - b_n) \in R(\Sigma).$$

By our assumption,

$$\deg_{\alpha_\Sigma} \lambda = \sum_{i=1}^n a_i - \sum_{j=1}^n b_j$$

is the intersection number of $c_1(\mathbf{P}_\Sigma)$ and $\lambda \in H_2(\mathbf{P}_\Sigma, \mathbf{C})$, i.e., it is divisible by r. This means that the binomials

$$\exp\left(\sum_{i=1}^n a_i\varphi(v_i)\right)\prod_{i=1}^n z_i^{a_i} - \exp\left(\sum_{j=1}^n b_j\varphi(v_j)\right)\prod_{j=1}^n z_j^{b_j}$$

are $\mathbf{Z}/r\mathbf{Z}$-homogeneous. □

Although, the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no any $\mathbf{Z}$-grading, it is possible to define a graded version of this quantum cohomology ring over the Laurent polynomial ring $\mathbf{C}[z_0, z_0^{-1}]$.

Definition 5.8 Let φ be a Σ-piecewise linear function with complex values from the complexified space $PL(\Sigma)_{\mathbf{C}} = PL(\Sigma) \otimes_{\mathbf{R}} \mathbf{C}$. Define the quantum cohomology ring

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}])$$

as the quotient of the Laurent polynomial extension $\mathbf{C}[z][z_0, z_0^{-1}]$ by the sum of ideals $Q_{\varphi,z_0}(\Sigma)$ and $P(\Sigma)$: where $Q_{\varphi,z_0}(\Sigma)$ is generated by binomials

$$\exp\left(\sum_{i=1}^n a_i\varphi(v_i)\right) z_0^{(-\sum_{i=1}^n a_i)}\prod_{i=1}^n z_i^{a_i} - \exp\left(\sum_{j=1}^n b_j\varphi(v_j)\right) z_0^{(-\sum_{j=1}^n b_j)}\prod_{l=1}^n z_j^{b_j}$$

running over all possible linear relations

$$\sum_{i=1}^n a_i v_i = \sum_{j=1}^n b_j v_j$$

with non-negative integer coefficients a_i and b_j.

The properties of the $\mathbf{Z}$-graded quantum cohomology ring

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}])$$

are analogous to the properties of $QH^*(\mathbf{P}_\Sigma, \mathbf{C})$:

Theorem 5.9 *For every binomial $B_\varphi(\mathcal{P})$, take the corresponding homogeneous binomial in variables $z_0, z_1, \ldots, z_n$*

$$B_{\varphi,z_0}(\mathcal{P}) = z_{i_1}\cdots z_{i_k} - E_\varphi(\mathcal{P}) z_{j_1}^{c_1}\cdots z_{j_l}^{c_l} z_0^{(k-\sum_{s=1}^l c_s)}.$$

Then the elements $B_{\varphi,z_0}(\mathcal{P})$ generate the ideal $Q_{\varphi,z_0}(\Sigma)$, and Kähler limits of

$$QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}]),\ \ t \to \infty$$

are isomorphic to the Laurent polynomial extension $H^*(\mathbf{P}_\Sigma, \mathbf{C})[z_0, z_0^{-1}]$ of the odinary cohomology ring.

Finally, if the first Chern class of $\mathbf{P}_\Sigma$ belongs to the Kähler cone, i.e., $\alpha_\Sigma \in PL(\Sigma)$ is upper convex, then it is possible to define the quantum deformations of the cohomology ring of $\mathbf{P}_\Sigma$ over the polynomial ring $\mathbf{C}[z_0]$.

Definition 5.10 Assume that $\alpha_\Sigma \in PL(\Sigma)$ is upper convex. We define the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$ over $\mathbf{C}[z_0]$ as the quotient of the polynomial ring $\mathbf{C}[z_0, z_1, \dots, z_n]$ by the sum of the ideal $P(\Sigma)[z_0]$ and the ideal

$$\mathbf{C}[z_0, z_1, \dots, z_n] \cap Q_{\varphi,z_0}(\Sigma)$$

Theorem 5.11 *The ideal*

$$\mathbf{C}[z_0, z_1, \dots, z_n] \cap Q_{\varphi,z_0}(\Sigma)$$

is generated by homogeneous binomials

$$B_{\varphi,z_0}(\mathcal{P}) = z_{i_1} \cdots z_{i_k} - E_\varphi(\mathcal{P}) z_{j_1}^{c_1} \cdots z_{j_l}^{c_l} z_0^{(k-\sum_{s=1}^l c_s)}$$

where $\mathcal{P}$ runs over all primitive collections $\mathcal{P} \subset G(\Sigma)$. (Notice that convexity of α_Σ implies $k - \sum_{s=1}^l c_s \geq 0$.)

Kähler limits of the quantum cohomology ring

$$QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0]), \quad t \to \infty$$

are isomorphic to the polynomial extension

$$H^*(\mathbf{P}_\Sigma, \mathbf{C})[z_0]$$

of the odinary cohomology ring.

6 Birational transformations

It may look strange that we defined the quantum cohomology rings using infinitely many generators for the ideals $Q_\varphi(\Sigma)$ and $Q_{\varphi,z_0}(\Sigma)$, while these ideals have only finite number of generators indexed by primitive collections in $G(\Sigma)$. The reason for that is the following important theorem:

Theorem 6.1 *Let Σ_1 and Σ_2 be two complete fans of regular cones such that $G(\Sigma_1) = G(\Sigma_2)$, then the quantum cohomology rings $QH^*_\varphi(\mathbf{P}_{\Sigma_1}, \mathbf{C})$ and $QH^*_\varphi(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ are isomorphic.*

Proof. Our definitions of quantum cohomology rings does not depend on the combinatiorial structure of the fan Σ, one needs to know only all lattice vectors $v_1, \ldots, v_n \in G(\Sigma)$, but not the combinatorial structure of the fan Σ. $\square$

Since the equality $G(\Sigma_1) = G(\Sigma_2)$ means that two toric varieties $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are isomorphic in codimension 1, we obtain

Corollary 6.2 *Let $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ be two smooth compact toric manifolds which are isomorphic in codimension 1, then the rings $QH^*_\varphi(\mathbf{P}_1, \mathbf{C})$ and $QH^*_\varphi(\mathbf{P}_2, \mathbf{C})$ are isomorphic.*

Example 6.3 Consider two 3-dimensional fans Σ_1 and Σ_2 in $\mathbf{R}^3$ such that $G(\Sigma_1) = G(\Sigma_2) = \{v_1, \ldots, v_6\}$ where

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1),$$

$$v_4 = (-1,0,0),\ v_5 = (0,-1,0),\ v_6 = (1,1,-1).$$

We define the combinatorial structure of Σ_1 by the primitive collections

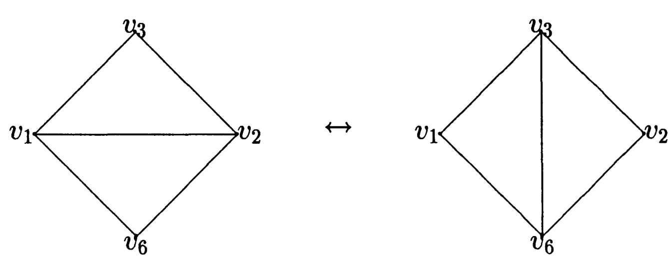
$$\mathcal{P}_1 = \{v_1, v_4\},\ \mathcal{P}_2 = \{v_2, v_5\},\ \mathcal{P}_3 = \{v_3, v_6\},$$

and the combinatorial structure of Σ_2 by the primitive collections

$$\mathcal{P}'_1 = \{v_1, v_4\},\ \mathcal{P}'_2 = \{v_2, v_5\},\ \mathcal{P}'_3 = \{v_1, v_2\},$$

$$\mathcal{P}'_4 = \{v_3, v_5, v_6\},\ \mathcal{P}'_5 = \{v_3, v_4, v_6\}.$$

The flop between two toric manifolds is described by the diagrams:



The ordinary cohomology rings $H^*(\mathbf{P}_{\Sigma_1}, \mathbf{C})$ and $H^*(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ are not isomorphic, because their homogeneous ideals of polynomial relations among $z_1, \ldots, z_6$ have different numbers of minimal generators. There exists the polynomial relation in the quantum cohomology ring:

$$\exp(\varphi(v_1) + \varphi(v_2))z_1 z_2 = \exp(\varphi(v_3) + \varphi(v_6))z_3 z_6.$$

If $\varphi(v_1) + \varphi(v_2) < \varphi(v_3) + \varphi(v_6)$, then we obtain the element $z_3 z_6 \in SR(\Sigma_1)$ as the limit for $t\varphi$, when $t \to \infty$. On the other hand, if $\varphi(v_1) + \varphi(v_2) > \varphi(v_3) + \varphi(v_6)$, taking the same limit, we obtain $z_1 z_2 \in SR(\Sigma_2)$.

Let us consider another simple example of birational tranformation.

Example 6.4 The quantum cohomology ring of the 2-dimensional toric variety F_1 which is the blow-up of a point p on $\mathbf{P}^2$ is isomorphic to the quotient of the polynomial ring $\mathbf{C}[x_1, x_2]$ by the ideal generated by two binomials

$$x_1(x_1 + x_2) = \exp(-\phi_2); \; x_2^2 = \exp(-\phi_1)x_1,$$

where x_1 is the class of the (-1)-curve C_1 on F_1, x_2 is the class of the fiber C_2 of the projection of F_1 on $\mathbf{P}^1$. The numbers ϕ_1 and ϕ_2 are respectively degrees of the restriction of the Kähler class φ on C_1 and C_2.

Remark 6.5 The definition of the quantum cohomology ring for smooth toric manifolds immediatelly extends to the case of singular toric varieties. However, the ordinary cohomology ring of singular toric varieties is not anymore the Kähler limit of the quantum cohomology ring. In some cases, the quantum cohomology ring of singular toric varieties V contains information about the ordinary cohomology ring of special desingularizations V' of V. For instance, if we assume that there exists a projective desingularization $\psi : V' \to V$ such that $\psi^* \mathcal{K}_V = \mathcal{K}_{V'}$. Then for every Kähler class $\varphi \in H^2(V, \mathbf{C})$, one has

$$\dim_{\mathbf{C}} QH^*_{\varphi}(V, \mathbf{C}) = \dim_{\mathbf{C}} H^*(V', \mathbf{C}).$$

7 Geometric interpretation of quantum rings

The spectra of the quantum cohomology ring $\mathrm{Spec}\, QH^*_{\varphi}(\mathbf{P}_\Sigma, \mathbf{C})$, and its two polynomial versions

$$\mathrm{Spec}\, QH^*_{\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}]), \; \mathrm{Spec}\, QH^*_{\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$$

have simple geometric interpretations.

Definition 7.1 Denote by $\Pi(\Sigma)$ the $(n-d)$-dimensional affine subspace in $\mathbf{C}^n$ defined by the ideal $P(\Sigma)$.

Definition 7.2 Choose any isomorphism $N \cong \mathbf{Z}^d$, so that any element $v \in N$ defines a Laurent monomial X^v in d variables $X_1, \ldots, X_d$. Consider the embedding of the d-dimensional torus $T(\Sigma) \cong (\mathbf{C}^*)^d$ in $(\mathbf{C}^*)^n$:

$$(X_1, \ldots, X_d) \to (X^{v_1}, \ldots, X^{v_n}).$$

Denote by $\Theta(\Sigma)$ the $(n-d)$-dimensional algebraic torus $(\mathbf{C}^*)^n/T(\Sigma)$.

Definition 7.3 Denote by *Exp* the analytical exponential mapping

$$Exp\ :\ \mathcal{G} \to \mathbf{G}$$

where $\mathbf{G}$ is a complex analytic Lie group, and $\mathcal{G}$ is its Lie algebra.

For example, one has the exponential mapping

$$Exp\ :\ PL(\Sigma)_{\mathbf{C}} \to (\mathbf{C}^*)^n$$

$$\varphi \mapsto (e^{\varphi(v_1)}, \ldots, e^{\varphi(v_n)})$$

which descends to the exponential mapping

$$Exp\ :\ H^2(\mathbf{P}_\Sigma, \mathbf{C}) \to \Theta(\Sigma).$$

Proposition 7.4 *The $\mathbf{T}(\Sigma)$-orbit $T_\varphi(\Sigma)$ of the point $Exp(\varphi) \in (\mathbf{C}^*)^n$ is closed, and its ideal is canonically isomorphic to $Q_\varphi(\Sigma)$.*

Corollary 7.5 *The scheme* $\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ *is the scheme-theoretic intersection of the d-dimensional subvariety $\overline{T}_\varphi(\Sigma) \subset \mathbf{C}^n$ and the $(n-d)$-dimensional subspace $\Pi(\Sigma)$.*

Definition 7.6 Let $\tilde{N} = \mathbf{Z} \oplus N$. For any $v \in N$, define $\tilde{v} \in \tilde{N}$ as $\tilde{v} = (1, v)$. Define the embedding of the $(d+1)$-dimensional torus $T^\circ(\Sigma) \cong (\mathbf{C}^*)^{d+1}$ in $(\mathbf{C}^*)^{n+1}$:

$$(X_0, X_1, \ldots, X_d) \to (X_0, X^{\tilde{v}_1}, \ldots, X^{\tilde{v}_n}).$$

The quotient $(\mathbf{C}^*)^{n+1}/T^\circ(\Sigma)$ is again isomorphic to $\Theta(\Sigma)$.

Proposition 7.7 *The ideal of the $T^\circ(\Sigma)$-orbit*

$$T^\circ_\varphi(\Sigma) \subset \mathbf{C}^* \times \mathbf{C}^n$$

of the of the point $(1, Exp\,(\varphi)) \in (\mathbf{C}^*)^{n+1}$ is canonically isomorphic to $Q_{\varphi,z_0}(\Sigma)$.

Corollary 7.8 *The scheme* $\operatorname{Spec} QH^*_{\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}])$ *is the scheme-theoretic intersection of the* $(d+1)$*-dimensional subvariety* $T^\circ_\varphi(\Sigma) \subset \mathbf{C}^* \times \mathbf{C}^n$ *and the* $(n-d+1)$*-dimensional subvariety* $\mathbf{C}^* \times \Pi(\Sigma) \subset \mathbf{C}^* \times \mathbf{C}^n$.

Similarly, one obtain the geometric interpretation of $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$, when the first Chern class of $\mathbf{P}_\Sigma$ belongs to the Kähler cone $K(\Sigma)$.

Proposition 7.9 *The scheme* $\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$ *is the scheme-theoretic intersection in* $\mathbf{C}^{n+1}$ *of the* $(d+1)$*-dimensional* $T^\circ(\Sigma)$*-orbit of* $(1, Exp(\varphi))$ *and the* $(n-d+1)$*-dimensional affine subspace* $\mathbf{C} \times \Pi(\Sigma) \subset \mathbf{C}^{n+1}$.

The limits of quantum cohomology rings have also geometric interpretations. One obtains, for instance, the spectrum of the ordinary cohomology ring of $\mathbf{P}_\Sigma$ as the scheme-theoretic intersection of the affine subspace $\Pi(\Sigma)$ with a *toric degeneration* of closures of $\mathbf{T}(\Sigma)$-orbits $\overline{T}_\varphi(\Sigma) \subset \mathbf{C}^n$. Such an interpretation allows to apply methods of M. Kapranov, B. Sturmfels, and A. Zelevinsky (see [10], Theorem 5.3) to establish connection between vertices of Chow polytope (secondary polyhedron) and Kähler limits of quantum cohomology rings.

8 Calabi-Yau hypersurfaces and Jacobian rings

Throughout in this section we fix a complete d-dimensional fan of regular cones, and we assume that $\mathbf{P} = \mathbf{P}_\Sigma$ is a toric manifold whose first Chern class belongs to the closed Kähler cone $K(\Sigma)$, i.e., $\alpha := \alpha_\Sigma$ is a convex Σ-piecewise linear function.

Let $\Delta = \Delta_\alpha$, the convex polyhedron in $M_{\mathbf{R}}$ (see 4.1). For any sufficiently general section S of the anticanonical sheaf $\mathcal{K}_V^{-1}$ on $\mathbf{P}$ represented by homogeneous polynomial $F(z)$, the set $Z = \{\pi(z) \in \mathbf{P} \,:\, F(z) = 0\}$ in $\mathbf{P}$ is a Calabi-Yau manifold ($c_1(\mathcal{K}_V^{-1}) = c_1(\mathbf{P})$).

Since the first Chern class of $\mathbf{P}$ in the ordinary cohomology ring $H^*(\mathbf{P}, \mathbf{C})$ is the class of the sum $(z_1 + \cdots + z_n)$, we obtain:

Proposition 8.1 *The image of* $H^*(\mathbf{P}, \mathbf{C})$ *under the restriction mapping to* $H^*(Z, \mathbf{C})$ *is isomorphic to the quotient*

$$H^*(\mathbf{P}, \mathbf{C}) / \operatorname{Ann}(z_1 + \cdots + z_n),$$

where $\operatorname{Ann}(z_1 + \cdots + z_n)$ *denotes the annulet of the class of* $(z_1 + \cdots + z_n)$ *in* $H^*(\mathbf{P}, \mathbf{C})$.

In general, Proposition 8.1 allows us to calculate only a part of the ordinary cohomology ring of a Calabi-Yau hypersurface Z in toric variety $\mathbf{P}$. If the first Chern class of $\mathbf{P}$ is in the interior of the Kähler cone $K(\Sigma)$, then Z is an ample divisor. For $d \geq 4$, by Lefschetz theorem, the restriction mapping $H^2(\mathbf{P}, \mathbf{C}) \to H^2(Z, \mathbf{C})$ is isomorphism. Thus, using Proposition 8.1, we can calculate cup-products of any $(1,1)$-forms on Z.

Definition 8.2 Denote by Δ^* the convex hull of the set $G(\Sigma)$ of all generators, or equivalently,

$$\Delta^* = \{v \in N_{\mathbf{R}} \mid \alpha(v) \leq 1\}.$$

Remark 8.3 The polyhedron Δ^* is dual to Δ reflexive polyhedron (see [6]).

Theorem 8.4 *There exists the canonical isomorphism between the quantum cohomology ring*

$$QH^*_{\varphi}(\mathbf{P}, \mathbf{C})$$

and the Jacobian ring

$$\mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]/(X_1 \partial f/\partial X_1, \ldots, X_d \partial f/\partial X_d)$$

of the Laurent polynomial

$$f_{\varphi}(X) = -1 + \sum_{i=1}^{n} \exp(\varphi(v_i))^{-1} X^{v_i}.$$

This isomorphism is induced by the correspondence

$$z_i \to X^{v_i}/\exp(\varphi(v_i)) \ \ (1 \leq i \leq n).$$

In particular, it maps the first Chern class $(z_1 + \ldots + z_n)$ *of* $\mathbf{P}$ *to* $f_{\varphi}(X) + 1$.

Proof. Let

$$\mathcal{H} : \mathbf{C}[z_1, \ldots, z_n] \to \mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$$

be the homomorphism defined by the correspondence

$$z_i \to X^{v_i}/\exp(\varphi(v_i)).$$

By 2.3(iii), $\mathcal{H}$ is surjective. It is clear that $Q_{\varphi}(\Sigma)$ is the kernel of $\mathcal{H}$. On the other hand, if we a $\mathbf{Z}$-basis $\{u_1, \ldots, u_d\} \subset M$ which establishes isomorphisms $M \cong \mathbf{Z}^d$ and $N \cong \mathbf{Z}^d$, we obtain:

$$\mathcal{H}(P(\Sigma)) = < X_1 \partial f/\partial X_1, \ldots, X_d \partial f/\partial X_d > .$$

$\square$

Definition 8.5 Let S_{Δ^*} be the affine coordinate ring of the $T^\circ(\Sigma)$-orbit of the point $(1,\ldots,1) \in \mathbf{C}^{n+1}$ (see 7.6).

Definition 8.6 For any Laurent polynomial

$$f(X) = a_0 + \sum_{i=1}^{n} a_i X^{v_i},$$

we define elements

$$F_0, F_1, \ldots, F_d \in S_{\Delta^*}$$

as $F_i = \partial X_0 f(X) \partial X_0$, $(0 \leq i \leq d)$.

Remark 8.7 The ring S_{Δ^*} is a subring of $\mathbf{C}[X_0, X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. There exists the canonical grading of S_{Δ^*} by degree of X_0.

It is easy to see that the correspondence

$$z_0 \to -X_0,$$

$$z_i \to X_0 X^{v_i}/(\exp(\varphi(v_i)))$$

defines the isomorphism

$$\mathbf{C}[z]/Q_\varphi(\Sigma) \cong S_{\Delta^*}.$$

This isomorphism maps $(-z_0 + z_1 + \cdots z_n)$ to F_0.

Theorem 8.8 ([7]) *Let*

$$R_f = S_{\Delta^*} / < F_0, F_1, \ldots, F_d > .$$

Then the quotient

$$R_f / \mathrm{Ann}\,(X_0)$$

is isomorphic to the $(d-1)$*-weight subspace* $W_{d-1}H^{d-1}(Z_f, \mathbf{C})$ *in the cohomology space* $H^{d-1}(Z_f, \mathbf{C})$ *of the affine Calabi-Yau hypersurface in* $T(\Sigma)$ *defined by the Laurent polynomial* $f(X)$.

For any Laurent polynomial $f(X) = a_0 + \sum_{i=1}^{n} a_i X^{v_i}$, we can find an element $\varphi \in PL(\Sigma)_{\mathbf{C}}$ such that

$$\frac{-a_i}{a_0} = \exp(-\varphi(v_i)).$$

A one-parameter family $t\varphi$ in $PL(\Sigma)$ induces the one-parameter family of Laurent polynomials

$$f_t(X) = -1 + \sum_{i=1}^{n} \exp(-t\varphi(v_i)) X^{v_i}.$$

Applying the isomorphism in 8.7 and the statement in Theorem 5.3, we obtain the following:

Theorem 8.9 *Assume that φ is in the interior of the Kähler cone $K(\Sigma)$. Then the limit*

$$R_{f_t}/\mathrm{Ann}(X_0)$$

is isomorphic to

$$H^*(\mathbf{P}, \mathbf{C})/\mathrm{Ann}(z_1 + \cdots + z_n).$$

The last statement shows the relation, established in [3], between the *toric* part of the topological cohomology rings of Calabi-Yau 3-folds in toric varieties and limits of the multiplicative structure on $(d-1)$-weight part of the Jacobian rings of their *mirrors.*

9 Topological sigma models on toric manifolds

So far we have not explained why the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ coincides with the quantum cohomology ring corresponding to the topological sigma model on V. In this section we want to establish the relations between the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ and the quantum cohomology rings considered by physicists.

In order to apply the general construction of the correlation functions in sigma models ([16], 3a), we need the following information on the structure of the space of holomorphic mappings of $\mathbf{CP}^1$ to a d-dimensional toric manifold $\mathbf{P}_\Sigma$.

Theorem 9.1 *Let $\mathcal{I}$ be the moduli space of holomorphic mappings $f : \mathbf{CP}^1 \to \mathbf{P}_\Sigma$. The space $\mathcal{I}$ consists of infinitely many algebraic varieties $\mathcal{I}_\lambda$ indexed by elements*

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in R(\Sigma),$$

where the numbers λ_i are equal to the intersection numbers $\deg_{\mathbf{CP}^1} f^*\mathcal{O}(Z_i)$ with divisors $Z_i \subset \mathbf{P}_\Sigma$ such that $\pi^{-1}(Z_i)$ is defined by the equation $z_i = 0$ in $U(\Sigma)$. Moreover, if all $\lambda_i \geq 0$, then $\mathcal{I}_\lambda$ is irreducible and the virtual dimension of $\mathcal{I}_\lambda$ equals

$$d_\lambda = \dim_{\mathbf{C}} \mathcal{I}_\lambda = d + \sum_{i=1}^{n} \lambda_i.$$

Proof. The first statement follows immediatelly from the description of the intersection product on $\mathbf{P}_\Sigma$ (3.5).

Assume now that all λ_i are non-negative. This means that the preimage $f^{-1}(Z_i)$ consists of λ_i points including their multiplicities. Let $\mathcal{F}_\Sigma$ be the tangent bundle over $\mathbf{P}_\Sigma$. There exists the generalized Euler exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}}^{n-d} \to \mathcal{O}_{\mathbf{P}}(Z_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}}(Z_n) \to \mathcal{F}_\Sigma \to 0.$$

Applying f^*, we obtain the short exact sequence of vector bundles on $\mathbf{CP}^1$.

$$0 \to \mathcal{O}_{\mathbf{CP}^1}^{n-d} \to \mathcal{O}_{\mathbf{CP}^1}(\lambda_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{CP}^1}(\lambda_n) \to f^*\mathcal{F}_\Sigma \to 0.$$

This implies that $h^1(\mathbf{CP}^1, f^*\mathcal{F}_\Sigma) = 0$, and $h^0(\mathbf{CP}^1, f^*\mathcal{F}_\Sigma) = d + \lambda_1 + \cdots + \lambda_n$.

The irreducibility of $\mathcal{I}_\lambda$ for $\lambda \geq 0$ follows from the explicit geometrical construction of mappings $f \in \mathcal{I}_\lambda$:

Choose n polynomials $f_1(t), \ldots, f_n(t)$ such that $\deg f_i(t) = \lambda_i$ $(i = 1, \ldots, n)$. If all $\mid \lambda \mid = \lambda_1 + \cdots + \lambda_n$ roots of $\{f_i\}$ are distinct, then these polynomials define the mapping

$$g \,:\, \mathbf{C} \to U(\Sigma) \subset \mathbf{C}^n.$$

The composition $\pi \circ g$ extends to the mapping f of $\mathbf{CP}^1$ to $\mathbf{P}_\Sigma$ whose homology class is λ. □

Definition 9.2 Let

$$\Phi \,:\, \mathcal{I} \times \mathbf{CP}^1 \to \mathbf{P}_\Sigma$$

be the universal mapping. For every point $x \in \mathbf{CP}^1$ we denote by Φ_x the restriction of Φ to $\mathcal{I} \times x$. Choose the cohomology classes $z_1 = [Z_1], \ldots, z_n = [Z_n]$ of divisors $Z_1, \ldots, Z_n$ on $\mathbf{P}_\Sigma$ in the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma)$. We determine the divisors $W_{z_1}, \ldots, W_{z_n}$ on $\mathcal{I}$ whose cohomology classes are independent of choice of $x \in \mathbf{CP}^1$ as follows

$$W_{z_i} = \Phi_x^{-1}(Z_i) = \{f \in \mathcal{I} \mid f(x) \in Z_i\}.$$

The quantum cohomology ring of the sigma model with the target space $\mathbf{P}_\Sigma$ is defined by the relations

$$W_{\alpha_1} \cdot W_{\alpha_2} \cdot \cdots \cdot W_{\alpha_k} = \sum_{\lambda \in W_{\alpha_1} \cap \cdots \cap W_{\alpha_k}} \exp(-\deg_\varphi \lambda),$$

where α_i are cycles on $\mathbf{P}_\Sigma$ and $W_{\alpha_i} = \Phi_x^{-1}(\alpha_i)$, and the intersection $W_{\alpha_1} \cap \cdots \cap W_{\alpha_k}$ on the moduli space $\mathcal{I}$ is assumed to be of virtual dimension zero.

Theorem 9.3 *Let $\mathbf{P}_\Sigma$ be a d-dimensional toric manifold, $\varphi \in H^2(\mathbf{P}_\Sigma, \mathbf{C})$ a Kähler class. Let $\lambda^0 = (\lambda_1^0, \ldots, \lambda_n^0)$ be a non-negative element in $R(\Sigma)$, $\Omega \in H^{2d}(\mathbf{P}_\Sigma, \mathbf{C})$ the fundamental class of the toric manifold $\mathbf{P}_\Sigma$. Then the intersection number on the moduli space $\mathcal{I}$*

$$(W_\Omega) \cdot (W_{z_1})^{\lambda_1^0} \cdot (W_{z_2})^{\lambda_2} \cdot \cdots \cdot (W_{z_n})^{\lambda_n^0}$$

vanishes for all components $\mathcal{I}_\lambda$ except from $\lambda = \lambda_0$. In the latter case, this number equals

$$\exp(-\deg_\varphi \lambda).$$

Proof. Since the fundamental class Ω is involved in the considered intersection number, this number is zero for all $\mathcal{I}_\lambda$ such that the rational curves in the class λ do not cover a dense Zariski open subset in $\mathbf{P}_\Sigma$. Thus, we must consider only non-negative classes λ. Moreover, the factors $(W_{z_i})^{\lambda_i^0}$ show that we must consider only those $\lambda = (\lambda_1, \ldots, \lambda_n) \in R(\Sigma)$ such that $\lambda_i \geq \lambda_i^0$, i.e., a mapping $f \in \mathcal{I}_\lambda$ is defined by polynomials $f_1, \ldots, f_n$ such that $\deg f_i \geq \lambda_i$.

There is a general principle that non-zero contributions to the intersection product

$$(W_{\alpha_1} \cdot W_{\alpha_2} \cdot \cdots \cdot W_{\alpha_k})_{\mathcal{I}}$$

appear only from the components whose virtual $\mathbf{R}$-dimension is equal to

$$\sum_{i=1} \deg \alpha_i.$$

In our case, the last number is $d + \lambda_1^0 + \ldots + \lambda_n^0$. Therefore, a non-zero contribution appears only if $\lambda = \lambda^0$.

It remains to notice that this contribution equals $\exp(-\deg_\varphi \lambda_0)$. The last statement follows from the observation that the points $f^{-1}(Z_i) \subset \mathbf{CP}^1$ $(i = 1, \ldots, n)$ define the mapping $f : \mathbf{CP}^1 \to \mathbf{P}_\Sigma$ uniquely up to the action of the d-dimensional torus $\mathbf{T} = \mathbf{P}_\Sigma \setminus (Z_1 \cup \cdots \cup Z_n)$, and the weight of the mapping f in the intersection product is $\int_{\mathbf{CP}^1} f^*(\varphi)$. $\square$

Corollary 9.4 *Let $\mathcal{Z}_i$ be the quantum operator corresponding to the class $[Z_i] \in H^2(\mathbf{P}_\Sigma, \mathbf{C})$ $(i = 1, \ldots, n)$ considered as an element of the quantum cohomology ring. Then for every non-negative element $\lambda \in R(\Sigma)$, one has the algebraic relation*

$$\mathcal{Z}_1^{\lambda_1} \circ \cdots \circ \mathcal{Z}_n^{\lambda_n} = \exp(-\deg_\varphi \lambda)\, id.$$

It turns out that the polynomial relations of above type are sufficient to recover the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$:

Theorem 9.5 *Let $A_\varphi(\Sigma)$ be the quotient of the polynomial ring $\mathbf{C}[z]$ by the sum of two ideals: $P(\Sigma)$ and the ideal generated by all polynomials*

$$B_\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n} - \exp(-\deg_\varphi \lambda)$$

where λ runs over all non-negative elements of $R(\Sigma)$. Then $A_\varphi(\Sigma)$ is isomorphic to $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$.

Proof. Let $B_\varphi(\Sigma)$ be the ideal generated by all binomials B_λ. By definition, $B_\varphi(\Sigma) \subset Q_\varphi(\Sigma)$. So it is sufficient to prove that $Q_\varphi(\Sigma) \subset B_\varphi(\Sigma)$.

Let
$$\sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{n} b_j v_j$$
be a linear relation among $v_1, \ldots, v_n$ such that a_i, $b_j \geq 0$. Since the set of all nonnegative elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in R(\Sigma)$ $(\lambda_i \geq 0)$ generates a convex cone of maximal dimension in $H^2(\mathbf{P}_\Sigma, \mathbf{C})$, there exist two nonnegative vectors λ, $\lambda' \in R(\Sigma)$ such that
$$\lambda - \lambda' = (\lambda_1 - \lambda_1', \ldots, \lambda_n - \lambda_n') = (a_1 - b_1, \ldots, a_n - b_n).$$
By definition, two binomials P_λ and $P_{\lambda'}$ are contained in $Q_\varphi(\Sigma)$. Hence, the classes of $z_1, \ldots, z_n$ in $\mathbf{C}[z]/B_\varphi(\Sigma)$ are invertible elements. Thus, the class of the binomial
$$\exp\left(\sum_{i=1}^{n} a_i \varphi(v_i)\right) \prod_{i=1}^{n} z_i^{a_i} - \exp\left(\sum_{j=1}^{n} b_j \varphi(v_j)\right) \prod_{j=1}^{n} z_j^{b_j}$$
is zero in $\mathbf{C}[z]/B_\varphi(\Sigma)$. Thus, $B_\varphi(\Sigma) = Q_\varphi(\Sigma)$. □.

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A finiteness theorem for isogeny correspondences

Astérisque, tome 218 (1993), p. 35-60
<http://www.numdam.org/item?id=AST_1993__218__35_0>

A Finiteness Theorem for Isogeny Correspondences

Alexandru Buium

0. Introduction

Let $A_{g,n}$ be the moduli space of principally polarized abelian varieties over $\mathbb{C}$ of dimension $g \geq 2$ with level n structure, $n \geq 3$; we will view $A_{g,n}$ as an algebraic variety over $\mathbb{C}$. Moreover, let $Y \subset A_{g,n}$ be a curve (by which we will understand an irreducible, closed, possibly singular subvariety of dimension 1). By an isogeny correspondence on Y we will understand an (irreducible, closed, possibly singular) curve $Z \subset Y \times Y$ for which there exists a quasi-finite map $Z' \to Z$ from an irreducible curve Z' with the property that the two abelian schemes over Z' deduced by base change via

$$Z' \to Z \subset Y \times Y \xrightarrow{p_i} Y \qquad i = 1, 2$$

(p_i = i-th projection) are isogenous. Note that two abelian schemes over Z' are called isogenous if there exists a surjective homomorphism between them with kernel finite over Z'; so we do not require our isogenies preserve, say, polarizations.

The question which we address in this paper is: how many isogeny correspondences can exist on a "sufficiently general" curve $Y \subset A_{g,n}$?

It is easy to see that there exist "lots" of curves $Y \subset A_{g,n}$ carrying infinitely many isogeny correspondences: more precisely, the union of all such

Y's in $A_{g,n}(\mathbb{C})$ is dense in the complex topology of $A_{g,n}(\mathbb{C})$ (see the Proposition from Section 1). Nevertheless, our main result here will imply in particular that "most" curves $Y \subset A_{g,n}$ carry at most finitely many isogeny correspondences (see Theorem 1 below).

Indeed, let $C(A_{g,n})$ be the set of all (irreducible, closed, possibly singular) curves in $A_{g,n}$; we will put a natural topology on $C(A_{g,n})$ which we call the Kolchin topology such that $C(A_{g,n})$ becomes an irreducible Noetherian topological space and then we will prove in particular the following:

Theorem 1. *There exists a dense Kolchin open subset C_0 of $C(A_{g,n})$ such that any curve Y belonging to C_0 carries at most finitely many isogeny correspondences.*

Remark. If a curve $Y \subset A_{g,n}$ carries at most finitely many isogeny correspondences Z then any such Z must have only finite orbits.

Let's define in what follows the Kolchin topology on $C(A_{g,n})$. More generally one can define the Kolchin topology on the set $C(A)$ of all (irreducible, closed, possibly singular) curves embedded in a given (irreducible, possibly singular) algebraic variety A over $\mathbb{C}$. Indeed, we consider first the "jet scheme" $\operatorname{jet}(A)$, cf. [B$_1$]; recall that this is by definition an A-scheme with a $\mathbb{C}$-derivation δ of its structure sheaf, characterized by the fact that for any pair (Z, d) consisting of an A-scheme Z and a $\mathbb{C}$-derivation d on $\mathcal{O}_Z$ there is a unique horizontal morphism of A-schemes $Z \to \operatorname{jet}(A)$; "horizontal" here means "commuting with δ and d". For instance, if $A = \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[y_1, \dots, y_n]$ then $\operatorname{jet}(A) = \operatorname{Spec} \mathbb{C}\{y_1, \dots, h_n\}$ where $\mathbb{C}\{y_1, \dots, y_n\}$ is the ring of δ-polynomials in $y_1, \dots, y_n$ with coefficients in $\mathbb{C}$ (which by definition is the ring of polynomials with coefficients in $\mathbb{C}$ in the infinite family of variables $y_j^{(i)}$, $i \geq 0$, $1 \leq j \leq n$, with $\mathbb{C}$-derivation δ sending $y_j^{(i)}$ into $y_j^{(i+1)}$). Now for any Zariski closed subset H of $\mathrm{jet}(A)$ we denote by $C_H(A)$ the set of all curves $Y \in C(A)$ such that the image of the natural horizontal closed immersion $\mathrm{jet}(Y) \to \mathrm{jet}(A)$ is contained in H. One easily checks that the sets $C_H(A)$ are the closed sets of a topology which we call the Kolchin topology (one has to use the non-obvious fact that $\mathrm{jet}(Y)$ is an irreducible scheme which follows from correctly interpreting a theorem of Kolchin, [K] p. 200). We will check in Section 2 below that $C(A)$ with the Kolchin topology is an irreducible Noetherian topological space.

Remark. Intuitively a subset of $C(A)$ is Kolchin closed if it consists of all curves $Y \in C(A)$ which "satisfy a certain system of algebraic differential equations on A". As the proof of Theorem 1 will show, the "system defining" $C(A_{g,n}) \setminus C_0$ has "order 6" (i.e. "comes from jets of order 6") and is highly nonlinear.

Actually we can do much better than in Theorem 1, namely we can "bound asymptotically" (for $Y \in C_0$) the number of isogeny correspondences on Y "counted with certain natural multiplicities" (see Theorem 1′ below). We need more notations. For any curve $Y \subset A_{g,n}$ we denote by $p(Y)$ the genus of a smooth projective model of Y. Moreover, for any isogeny correspondence $Z \subset Y \times Y$ we let $[Z : Y]_i$ denote the degree of the map $Z \subset Y \times Y \xrightarrow{p_i} Y$, $i = 1, 2$ and put $i(Y) = \sum [Z : Y]_1 = \sum [Z : Y]_2 \in \mathbb{N} \cup \{\infty\}$, where Z runs through the set of all isogeny correspondences on Y (we put $i(Y) = 0$ if this set is empty). This $i(Y)$ is the "number of isogeny correspondences counted with multiplicities"; for alternative descriptions of $i(Y)$ we refer to Lemmas 1 and 2 from Section 1. Finally, we shall fix a smooth projective compactification

$\bar{A}_{g,n}$ of $A_{g,n}$ and a very ample line bundle $\mathcal{O}(1)$ on $\bar{A}_{g,n}$; then for any curve $Y \subset A_{g,n}$ we shall denote by $\deg(Y)$ the degree of the Zariski closure of Y in $\bar{A}_{g,n}$ with respect to $\mathcal{O}(1)$.

We can state the following strengthening of Theorem 1:

Theorem 1′. *There exist a dense Kolchin open subset C_0 of $C(A_{g,n})$ and two positive integers m_1, m_2 such that for all $Y \in C_0$ we have*

$$i(Y) \leq m_1 \deg(Y) + m_2 p(Y)$$

Remark. A careful examination of the proof leads to an explicit value for m_2. But determining such a value for m_1 seems much harder.

We close this introduction by giving a consequence of Theorem 1′. To state it note that the set $A_{g,n}(\mathbb{C})$ of $\mathbb{C}$-points of $A_{g,n}$ has a natural equivalence relation on it given by isogeny: two points in $A_{g,n}(\mathbb{C})$ will be called isogenous if the corresponding abelian $\mathbb{C}$-varieties are isogenous. Each isogeny class in $A_{g,n}(\mathbb{C})$ is dense in the complex topology because it contains the image of a $\mathrm{Sp}(2g, \mathbb{Q})$-orbit on the Siegel upper half space. For any $y \in A_{g,n}(\mathbb{C})$ we denote by $I_y \subset A_{g,n}(\mathbb{C})$ the isogeny class of y. Then Theorem 1′ will imply the following:

Theorem 2 *There exist a dense Kolchin open subset C_0 of $C(A_{g,n})$ and two positive integers m_1, m_2 such that for all $Y \in C_0$ and for any point $y \in Y(\mathbb{C})$ outside a certain countable subset of $Y(\mathbb{C})$, the set $Y(\mathbb{C}) \cap I_y$ is finite of cardinality at most $m_1 \deg(Y) + m_2 p(Y)$.*

Remark. As the proof will show, the countable subset of $Y(\mathbb{C})$ appearing in the above statement can be taken simply to be the set of all points in $Y(\mathbb{C})$

whose coordinates lie in the algebraic closure of the smallest field of definition of the embedding $Y \subset A_{g,n}$.

The paper is organized as follows. In Section 1 we make some remarks on isogeny correspondences and we deduce Theorem 2 from Theorem 1′. In Sections 2 – 4 we introduce and review a series of concepts from [B$_1$, B$_2$, B$_3$] and provide complements to that material; a rough sketch of the strategy of the proof of Theorem 1′ is given at the end of Section 2. The main body of the proof of Theorem 1′ is contained in Sections 5 – 7.

Acknowledgement. The author is indebted to P. Deligne for his inspiring comments and suggestions. An earlier version of the present paper was written while the author was visiting the University of Paris VII and (as a Humboldt Fellow) the University of Essen. The final version of the paper was written while the author was a member of the School of Mathematics at the Institute for Advanced Study in Princeton (NSF Grant No. DMS 9304580). Thanks go to all these institutions for hospitality and support. Last but not least thanks go to Dottie Phares for her excellent typing job.

1. Some easy remarks on isogeny correspondences

Let $k \subset F$ be an extension of algebraically closed fields of characteristic zero, $k \neq F$; in applications we shall be interested in both situations when $F = \mathbb{C}$ and $k = \mathbb{C}$.

Let A_k denote the moduli k-scheme of principally polarized abelian varieties over k of dimension $g \geq 2$ with level n structure, $n \geq 3$. For any curve $Y_k \subset A_k$ (i.e. irreducible closed k-subvariety of A_k of dimension 1) we may introduce exactly as in Section 0 the notion of isogeny correspondence $Z_k \subset Y_k \times Y_k$ on Y_k and we may define $i(Y_k)$ similarly. On the other hand, we may consider on the set $A_k(F)$ of F-points of A_k the equivalence relation given by isogeny: two points in $A_k(F)$ are called isogenous if the corresponding abelian F-varieties are isogenous (over F). For $y \in A_k(F)$ we denote by $I_{y,F} \subset A_k(F)$ the isogeny class of y.

Lemma 1. *Let $Y_k \subset A_k$ be a curve and $y \in Y_k(F) \setminus Y_k(k)$. Then we have:*

$$i(Y_k) = \mathrm{card}(Y_k(F) \cap I_{y,F})$$

Proof: Let $L = k(Y_k)$ be the field of rational functions on Y_k and let $\varepsilon_0: L \to F$ be the k-embedding corresponding to y. Since $Y_k(F) \cap I_{y,F} \subset Y_k(F) \setminus Y_k(k)$ each point in $Y_k(F) \cap I_{y,F}$ identifies with a k-embedding $\varepsilon: L \to F$; note that the compositum of the fields $\varepsilon_0 L$ and εL in F is algebraic over both $\varepsilon_0 L$ and εL (because the abelian F-variety corresponding to ε, being isogenous to the one corresponding to ε_0, must be defined over an algebraic extension of $\varepsilon_0 L$). Therefore the ideal $\ker(\varepsilon_0 \otimes \varepsilon: L \otimes_k L \to F)$ in $L \otimes_k L$ is non-zero so it corresponds to a curve $Z_k(\varepsilon) \subset Y_k \times Y_k$ which clearly is an isogeny correspondence. We have constructed a map $\varepsilon \longmapsto Z_k(\varepsilon)$ from the set $Y_k(F) \cap I_{y,F}$ to the set of all isogeny correspondences on Y_k which is clearly surjective and whose fiber at an isogeny correspondence $Z_k \subset Y_k \times Y_k$ has precisely $[Z_k : Y_k]_1$ elements. This closes the proof of the Lemma.

Lemma 2. *Let $Y_k \subset A_k$ be a curve, fix a k-embedding $\varepsilon: L = k(Y_k) \to F$, let X be the abelian F-variety deduced via ε and for any $\sigma \in \mathrm{Aut}\,(F/k)$ denote* *by X^σ the abelian F-variety deduced via σ from X. Consider the groups*

$$G(X) = Aut(F/\varepsilon L)$$
$$G'(X) = \{\sigma \in Aut(F/k);\ X^\sigma \text{ is isogenous to } X\}$$

Then $i(Y_k)$ equals the index $[G'(X):G(X)]$.

Proof: Let $y \in Y_k(F) \setminus Y_k(k)$ be defined by ε. Then clearly $Y_k(F) \cap I_{y,F}$ identifies with the coset set $G'(X)/G(X)$ and conclude by Lemma 1.

Let's show how Theorem $1'$ from Section 0 implies Theorem 2. Denote $A_{g,n}$ simply by A and assume C_0, m_1, m_2 are as in Theorem $1'$. For any $Y \in C_0$ let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition of the embedding $Y \subset A$ and let $Y_k \subset A_k$ be the embedding of k-varieties giving rise to $Y \subset A$; then $Y_k(k)$ is a countable subset of $Y(\mathbb{C})$. Let $y \in Y(\mathbb{C}) \setminus Y_k(k)$; by Lemma 1 (applied to $F = \mathbb{C}$) we have

$$\operatorname{card}(Y(\mathbb{C}) \cap I_y) = i(Y_k) \leq i(Y) \leq m_1 \deg(Y) + m_2 p(Y)$$

which proves Theorem 2.

We close this section by proving the following assertion (which was made in Section 0):

Proposition. *The union in $A_{g,n}(\mathbb{C})$ of all curves carrying infinitely many isogeny correspondences is dense in the complex topology of $A_{g,n}(\mathbb{C})$.*

Proof: Step 1. Note that there exists at least one curve $Y \subset A = A_{g,n}$ carrying infinitely many isogeny correspondences. Indeed, let $E \to S = \mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}$ be the Weierstrass elliptic family, let $X = E \times_S \ldots \times_S E$ (g times) be viewed as a principally polarized abelian scheme over S and make a base change

$S' \to S$, S' some irreducible curve over $\mathbb{C}$, such that $X' := X \times_S S' \to S'$ has a level n structure. Then the closure Y in A of the image of the naturally induced map $S' \to A$ has $i(Y) = \infty$ (e.g. use Lemma 1).

Step 2. Consider any curve $Y \subset A$ with $i(Y) = \infty$ (which exists by Step 1). Then, starting from Y, we shall produce a family of curves Y^z with $i(Y^z) = \infty$ and whose union is dense in the complex topology of $A(\mathbb{C})$. Indeed, let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition for the embedding $Y \subset A$ and let $Y_k \subset A_k$ be the embedding of k-varieties from which $Y \subset A$ is deduced; upon enlarging k we may assume $i(Y_k) = \infty$. Take any point $y \in Y(\mathbb{C}) \setminus Y_k(k)$ and consider the isogeny class $I_y = I_{y,\mathbb{C}}$ of y in $A(\mathbb{C})$. For any point $z \in I_y$ let Y_k^z denote the Zariski closure in A_k of the image of the morphism $\operatorname{Spec}\mathbb{C} \to A_k$ defined by z and let $Y^z \subset A$ be the curve over $\mathbb{C}$ obtained from Y_k^z by base change $k \subset \mathbb{C}$; clearly $z \in Y^z(\mathbb{C})$. We claim that $i(Y_k^z) = \infty$. This will close the proof of the Proposition, for then $i(Y^z) = \infty$ and $I_y \subset \cup Y^z(\mathbb{C})$ the union being taken for all $z \in I_y$; but I_y is already dense in the complex topology of $A(\mathbb{C})$. To check the claim let X_y, X_z be the abelian $\mathbb{C}$-varieties corresponding to y, z; since they are isogenous, $G'(X_y) = G'(X_z)$ (notations as in Lemma 2). Now let L, L_z be the fields of rational funcitons on Y_k, Y_k^z and let $\varepsilon\colon L \to \mathbb{C}$, $\varepsilon_z\colon L_z \to \mathbb{C}$ be the k-embeddings defined by y and z, respectively. Since X_y, X_z are isogenous the compositum of the fields εL and $\varepsilon_z L_z$ in $\mathbb{C}$ is finite over both εL and $\varepsilon_z L_z$. In particular, one of the indices $[G'(X_y)\colon G(X_y)]$ and $[G'(X_z)\colon G(X_z)]$ is finite if and only if the other is so. Now our claim follows from Lemma 2 and our Proposition is proved.

2. Introducing the δ-field U

The most economic way of presenting the proof of Theorem 1′ is to use the setting of δ-fields and the theory of Ritt-Kolchin which goes with them [K] (a δ-field is by definition a field F of characteristic zero with a fixed derivation on it always to be denoted by $\delta: F \to F$).

Instead of dealing with many δ-fields it is still better to deal with one universal δ-field in Kolchin's sense; for convenience we recall the definition of this concept. First there is an obvious notion of morphism of δ-fields and of δ-subfield (morphisms of δ-fields are by definition field homomorphisms which commute with the fixed derivations). If $F_1 \to F_2$ is a morphism of δ-fields we say that F_2 is δ-finitely generated over F_1 if there exist $x_1, \dots, x_n \in F_2$ such that F_2 is generated as a field by F_1 and the elements $\delta^i x_j$, $i \geq 0, 1 \leq j \leq n$. A δ-field U is called universal if for any δ-subfield F_1 of it which is δ-finitely generated over $\mathbb{Q}$ and for any morphism of δ-fields $F_1 \to F_2$ with F_2 δ-finitely generated over F_1 there is a morphism of δ-fields $F_2 \to U$ over F_1. By [K] p. 134 there exists a universal δ-field U whose constant field $\{x \in U;\ \delta x = 0\}$ has the same cardinality as $\mathbb{C}$ hence is isomorphic to $\mathbb{C}$. From now on we fix such an U, identify its constant field with $\mathbb{C}$ and write x', x'', $x''', \dots$ instead of δx, $\delta^2 x$, $\delta^3 x, \dots$ for $x \in U$. For any $\mathbb{C}$-variety A the set $A(U)$ of its U-points has a natural topology called the Kolchin topology defined as follows (cf. [B_1]): for any Zariski closed subset H of $\operatorname{jet}(A)$ let $A_H(U)$ denote the set of all points $\operatorname{Spec} U \to A$ in $A(U)$ whose unique horizontal lifting $\operatorname{Spec} U \to \operatorname{jet}(A)$ has the image contained in H. Then the sets $A_H(U)$ are by definition the closed sets of the Kolchin topology on $A(U)$ (note that this is what Kolchin calls in [K] the δ-$\mathbb{C}$-topology and is slightly different from the topology considered in [B_1] where we do not assume the open sets are "defined over constants"). By [K] p. 200 if A is irreducible in the Zariski topology then $A(U)$ is irreducible in the Kolchin topology. Moreover, by a theorem of Ritt [R] p. 10 the Kolchin topology on $A(U)$ is Noetherian. Note that if we are given a morphism of algebraic $\mathbb{C}$-varieties $A \to B$ then the induced map $A(U) \to B(U)$ is continuous in the Kolchin topology of $A(U)$ and $B(U)$. Moreover, any map $\mathbb{A}^m(U) = U^m \to \mathbb{A}^n(U) = U^n$ whose components are defined by δ-polynomials $f_1, \ldots, f_n \in \mathbb{C}\{y_1, \ldots, y_m\}$ is continuous in the Kolchin topology. Coming back to the set $C(A)$ of curves in A, it is easy to see that the Kolchin open sets of $C(A)$ defined in Section 0 are precisely the sets of the form $C_\Omega(A) = \{Y \in C(A); Y(U) \cap \Omega \neq \emptyset\}$ where Ω is Kolchin open in $A(U)$ (to check this just apply the "δ-Nullstellensatz" [K] p. 148). Noetherianity of Kolchin's topology on $A(U)$ already implies Noetherianity of the Kolchin topology on $C(A)$. Let's check that $C(A)$ is irreducible in the Kolchin topology. It is sufficient to check that $C_\Omega(A) \neq \emptyset$ whenever $\Omega \neq \emptyset$. We may assume A is affine. Then by Noether normalization we may easily assume A is the affine space $\mathbb{A}^n$. Now if $C_\Omega(A)$ is empty for some non-empty $\Omega \subset A(U) = U^n$ there exists a non-zero δ-polynomial $P \in \mathbb{C}\{y_1, \ldots, h_n\}$ such that for any choice of polynomials $f_1, \ldots, f_n \in \mathbb{C}[t]$ we have the equality $P(f_1(t), \ldots, f_n(t)) = 0$. By [K] p. 99 this implies $P = 0$, a contradiction.

Now Lemma 1 (applied to $k = \mathbb{C}$ and $F = U$) shows that Theorem $1'$ is implied by the following:

Theorem $1''$. *There exist a dense Kolchin open subset C_0 of $C(A_{g,n})$ and positive integers m_1, m_2 such that for any curve $Y \in C_0$ and for any isogeny* *class* $I \subset A_{g,n}(U) \setminus A_{g,n}(\mathbb{C})$ *we have:*

$$card(Y(U) \cap I) \leq m_1 \deg(Y) + m_2 p(Y)$$

From now on we concentrate ourselves on Theorem $1''$. The very rough idea of its proof is the following. We will find Kolchin open sets Ω_1, Ω_2 of $A_{g,n}(U)$, positive integers m_1, m_2 and a map $b: \Omega_1 \to U$ which is "constant on isogeny classes" such that for any curve $Y \subset A_{g,n}$ with $Y(U) \cap \Omega_1 \cap \Omega_2 \neq \emptyset$ we have that $Y(U) \cap \Omega_1 = Y(U) \setminus Y(\mathbb{C})$ and the restriction of b to $Y(U) \setminus Y(\mathbb{C})$ is given by a rational function on Y of degree at most $m_1 \deg(Y) + m_2 p(Y)$. A moment's reflection shows that this implies Theorem $1''$ (see the first lines of Section 6 for a few more details). The map b will be constructed in Section 6 as a (quite explicit) differential algebraic invariant of "δ-Hodge structures" of abelian U-varieties. The latter structures will be introduced in Section 3 and morally they are a differential algebraic (simplified) version of usual variations of Hodge structure. An argument different from ours for the existence of the map b was given by P. Deligne in a letter to the author [D].

3. Review of some δ-linear algebra

We shall "recall" and complete some discussion made in [B$_2$] on "δ-Hodge structures". Let $D = U[\delta] = \Sigma U \delta^i$ be the ring of linear differential operators on U generated by U and δ. By a δ-Hodge structure (of weight 1 and dimension g) we understand a pair (V, W) consisting of a D-module V of dimension $2g$ over U and of a U-linear subspace W of V of dimension g. We have an obvious notion of isomoprhism of δ-Hodge structures and we denote by H_g the set of isomorphism classes of such objects. We say that (V, W) has δ-rank g if the U-linear map $W \subset V \xrightarrow{\delta} V \to V/W$ is an isomorphism (where $V \xrightarrow{\delta} V$is the multiplication by δ in the D-module V) and we denote by $H_g^{(g)}$ the set of isomorphism classes of δ-Hodge structures of δ-rank g. There is a natural map $\Phi: H_g^{(g)} \to \mathbb{A}^g(U) = U^g$ defined as follows. For any δ-Hodge structure (V, W) of δ-rank g choose a U-basis $w_1, \dots, w_g$ of W; then $w_1, \dots, w_g$, $\delta w_1, \dots, \delta w_g$ will be a U-linear basis for V hence one can write

$$\delta^2 w + \alpha \delta w + \beta w = 0$$

where $\alpha, \beta \in \mathrm{gl}_g(U)$ are suitable $g \times g$ matrices and w is the transpose of $(w_1, \dots, w_g)$. Then we define Φ by attaching to (V, W) the characteristic polynomial

$$\det(xI_g - \gamma) = x^g + v_1 x^{g-1} + \cdots + v_g$$

of the matrix $\gamma = \beta - \alpha^2/4 - \alpha'/2$ (where we identify polynomials as above with vectors $(v_1, \dots, v_g) \in U^g$). It is trivial to check that changing the basis $w_1, \dots, w_g$ of W amounts to replacing the matrix γ by a matrix conjugated to it so Φ is well defined. Now let

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathrm{gl}_{2g}(U), \qquad M_{ij} \in \mathrm{gl}_g(U)$$

By [K] pp. 420–421, there exists a matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathrm{GL}_{2g}(U)$$

such that $B' = MB$ and B is unique up to right multiplication by a matrix in $\mathrm{GL}_{2g}(\mathbb{C})$. It is trivial to check that $\deg M_{12} \neq 0$ if and only if

$$\det \begin{bmatrix} B_{11} & B_{12} \\ B'_{11} & B'_{12} \end{bmatrix} \neq 0$$

Let $\mathrm{gl}_{2g}(U)^{(g)}$ be the set of all matrices $M \in \mathrm{gl}_{2g}(U)$ with $\det M_{12} \neq 0$. There is a natural map $\Gamma: \mathrm{gl}_{2g}(U)^{(g)} \to H_g^{(g)}$ defined as follows. For any $M \in \mathrm{gl}_{2g}(U)^{(g)}$ let $V = U^{2g}$ (viewed as a D-module via the D-module structure of each factor) and let W be the U-linear subspace of U^{2g} spanned by the rows of the $g \times 2g$ matrix (B_{11}, B_{12}) where $B \in \mathrm{GL}_{2g}(U)$ is such that $B' = MB$. By the above discussion on determinants (V, W) has δ-rank g. By uniqueness of B up to $\mathrm{GL}_{2g}(\mathbb{C})$-action the isomoprhism class of the δ-Hodge structure (V, W) depends only on M and not on the choice of B; so we got a well defined map Γ as desired. It is an easy exercise of linear algebra to compute explicitly the composed map $\Phi \circ \Gamma: \mathrm{gl}_{2g}(U)^{(g)} \to \mathbb{A}^g(U) = U^g$; the result is

$$\begin{aligned}
\Phi(\Gamma(M)) &= \det(xI_g - \gamma) \quad \text{where} \\
\gamma &= \beta - \alpha^2/4 - \alpha'/2 \\
\alpha &= -M_{12}'M_{12}^{-1} - M_{11} - M_{12}M_{22}M_{12}^{-1} \\
\beta &= -M_{11}' + M_{12}'M_{12}^{-1}M_{11} + M_{12}M_{22}M_{12}^{-1}M_{11} - M_{12}M_{21}
\end{aligned}$$

So we see that $\Phi \circ \Gamma$ is defined by g rational fractions whose denominators are powers of $\det M_{12}$ and whose numerators are δ-polynomials with coefficients in $\mathbb{Q}$ in $4g^2$ variables; in particular $\Phi \circ \Gamma$ is continuous in the Kolchin topology. Intuitively it should be viewed as a (highly) non-linear differential operator of order 2.

4. Review of internal versus external Gauss-Manin connection

In this section we review some material from [B2] and [B3], chapter 5; we refer to loc. cit. for details of proof.

Let Y be a smooth $\mathbb{C}$-variety and $TY = \mathrm{Spec}\,(\Omega_Y)$ its tangent bundle;

then there is a natural map $\nabla: Y(U) \to (TY)(U)$ continuous in the Kolchin topology defined as follows.

One first defines it for $Y = \mathbb{A}^N$; here we identify

$$\mathbb{A}^N(U) = U^N, \ (T\mathbb{A}^N)(U) = U^{2N}$$

and put $\nabla(u_1, \ldots, u_N) = (u_1, \ldots, u_N, u_1', \ldots, u_N')$. Next if $Y \subset A^N$ is a closed subvariety and we embed TY naturally into $T\mathbb{A}^N$ then we define $\nabla: Y(U) \to (TY)(U)$ to be the restriction of the analogous map for $\mathbb{A}^N$. Finally if Y is arbitrary we define ∇ be gluing the ∇'s of its affine pieces. Note that if $y \in Y(U)$ then ∇y can be veiwed as vector in the Zariski tangent space $T_y Y_U$ of $Y_U := Y \otimes_{\mathbb{C}} U$ at y. Of course, there is an intrinsic definition of ∇ but we won't need it here.

Now let $X \to Y$ be a smooth projective morphism of smooth $\mathbb{C}$-varieties, let $y \in Y(U)$ be a point and X_y the fiber at y. Then we dispose of an "internal" Kodaira-Spencer map

$$\rho_y^{\text{int}}: \mathrm{Der}_{\mathbb{C}} U \to H^1(X_y, T)$$

associated to the morphism $X_y \to \operatorname{Spec} U$ (here T denotes the tangent sheaf) and also of an "external" Kodaira-Spencer map

$$\rho_y^{\text{ext}}: T_y Y_U \to H^1(X_y, T)$$

associated to the morphism $X_U \to Y_U$. One can easily prove the following formula:

$$(*) \qquad \rho_y^{\text{int}}(\delta) = \rho_y^{\text{ext}}(\nabla y)$$

Similarly we dispose of an "internal" Gauss-Manin connection

$$\nabla_y^{\mathrm{int}}: \mathrm{Der}_{\mathbb{C}}U \to \mathrm{End}_{\mathbb{C}}(H^1_{\mathrm{DR}}(X_y))$$

and (assuming for simplicity that Y is affine) of an "external" Gauss-Manin connection

$$\nabla^{\mathrm{ext}}: \mathrm{Der}_U \mathcal{O}(Y_U) \to \mathrm{End}_U(H^1_{\mathrm{DR}}(X_U/Y_U))$$

See [Ka] for background on Gauss-Manin connection. On the other hand the trivial lifting of δ from U to X_U, Y_U induces an endomorphism $\delta^* \in \mathrm{End}_{\mathbb{C}}(H^1_{\mathrm{DR}}(X_U/Y_U))$. For any $\mathcal{O}(Y_U)$-module E and any $\varphi \in E$ let's agree to denote by $\varphi(y)$ the image of φ in E/m_yE where m_y is the maximal ideal of the local ring of Y_U at y. For instance, if $\varphi \in H^0(Y_U, T)$ is a vector field then $\varphi(y) \in T_yY_U$ is the corresponding tangent vector; if $\varphi \in H^0(Y_U, \Omega)$ is a global 1-form then $\varphi(y)$ is an element in the dual of T_yY_U while if $\varphi \in H^1_{\mathrm{DR}}(X_U/Y_U)$ is a relative de Rham class then $\varphi(y) \in H^1_{\mathrm{DR}}(X_y)$ is the corresponding de Rham class on the fiber. With this convention let $\omega \in H^1_{\mathrm{DR}}(X_U/Y_U)$ and let $\theta_y \in H^0(Y_U, T)$ be such that $\theta_y(y) = \nabla y$. Then one can prove the following formula ([B3] Chapter 5):

$$(**) \qquad \nabla_y^{\mathrm{int}}(\delta)(\omega(y)) = (\delta^*\omega + \nabla^{\mathrm{ext}}(\theta_y)\omega)(y)$$

From now on let X/Y be an abelian scheme of relative dimension $g \geq 1$. The space $H^1(X_y, T)$ naturally identifies with $\mathrm{Hom}_U(H^0(X_y, \Omega), H^1(X_y, \mathcal{O}))$ so for each element of this space we may speak about its determinant which will be a U-linear map between the g-th exterior powers of $H^0(X_y, \Omega)$ and $H^1(X_y, \mathcal{O})$. Then formula $(*)$ easily implies that the set $Y^{(g)}(U)$ of all $y \in Y(U)$ such that $\det \rho_y^{\mathrm{int}}(\delta) \neq 0$ is a Kolchin open subset of $Y(U)$ (which may of course be empty but which is certainly non-empty when $Y = A_{g,n}$ and $X \to Y$ is the universal abelian scheme because in this case any g-fold product of an elliptic curve with j-invariant belonging to $U \setminus \mathbb{C}$ provides an example of y for which $\det \rho_y^{\mathrm{int}}(\delta) \neq 0$).

A key role in what follows will be played by the map $h: Y(U) \to H_g$ defined to attaching to each $y \in Y(U)$ the δ-Hodge structure represented by (V, W) where $V = H^1_{\mathrm{DR}}(X_y)$ (viewed as a D-module via ∇_y^{int}) and $W = H^0(X_y, \Omega)$. Using the relation between "Kodaira-Spencer" and "Gauss-Manin" as explained in [Ka] we see that $h^{-1}(H_g^{(g)}) = Y^{(g)}(U)$ so if the latter is non-empty we dispose of an induced map

$$h: Y^{(g)}(U) \to H_g^{(g)}$$

The map h has the remarkable (easily checked) property that if $y, z \in Y(U)$ are such that X_y and X_z are isogenous then $h(y) = h(z)$. In particular $Y^{(g)}(U)$ is saturated with respect to the isogeny equivalence relation.

Assume in addition that $H^1_{\mathrm{DR}}(X/Y)$ and $H^0(X, \Omega_{X/Y})$ are free $\mathcal{O}(Y)$-modules (this is anyway the case if we replace Y by the Zariski open sets of a covering of it which will be allowed later). Then take an $\mathcal{O}(Y)$-module basis ω of the first module having the form $\omega_1, \ldots, \omega_g$, $\omega_{g+1}, \ldots, \omega_{2g}$ where the first g elements form a basis of the second module. For any vector field $\tau \in \mathrm{Der}_U \mathcal{O}(Y_U)$ on Y_U we may write $\nabla^{\mathrm{ext}}(\tau)\omega = \langle N, \tau \rangle \omega$ where N is a $2g \times 2g$ matrix of 1-forms on Y. The latter defines a morphism of $\mathbb{C}$-varieties $TY \to \mathrm{gl}_{2g}(\mathbb{C})$ which at the level of U-points gives the map still denoted by

$$N: (TY)(U) \to \mathrm{gl}_{2g}(U)$$

sending each tangent vector $t \in T_y Y_U$ into the matrix $\langle N(y), t \rangle$. Denote by $(TY)^{(g)}(U)$ the preimage via N of $\mathrm{gl}_{2g}(U)^{(g)}$; it is a Zariski open subset of $(TY)(U)$. Then one checks using $(**)$ that the map $h: Y^{(g)}(U) \to H_g^{(g)}$ coincides with the composition:

$$\Gamma \circ N \circ \nabla: Y^{(g)}(U) \to (TY)^{(g)}(U) \to \mathrm{gl}_{2g}(U)^{(g)} \to H_g^{(g)}$$

In particular the composition

$$\chi = \Phi \circ h: Y^{(g)}(U) \to H_g^{(g)} \to \mathbb{A}^g(U) = U^g$$

is continuous in the Kolchin topology. Intuitively χ should be viewed as a "third order non-linear differential operator".

5. The case dim Y=1

Assume Y is an irreducible (possible singular) curve over $\mathbb{C}$; we will systematically apply the preparation made in Section 4 to the smooth locus of Y. Choose a non-zero $\mathbb{C}$-derivation τ of the function field $\mathbb{C}(Y)$; then τ induces a U-derivation of the function field $U(Y_U)$ of Y_U so we may (and will) also view τ as a rational vector field on the smooth locus of Y_U. Of course τ has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$ and all singularities of the curve Y_U lie in $Y(\mathbb{C})$.

For any $y \in Y(U) \setminus Y(\mathbb{C})$ we denote by y' the unique element in $U^* = U \setminus \{0\}$ such that $\nabla y = y'\tau(y)$. Moreover we simply denote by $y'', y''', \dots$ the usual derivatives of y' as an element of U. Any element φ in the function field $\mathbb{C}(Y)$ will be systematically viewed as a rational map $\varphi: Y(U) \dashrightarrow \mathbb{A}^1(U) = U$; if $\varphi \neq 0$ then obviously φ has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$. For any $\varphi \in \mathbb{C}(Y)$ and any $y \in Y(U) \setminus Y(\mathbb{C})$ we have the following (easily checked) formula: $\varphi(y)' = y'(\tau\varphi)(y)$.

Assume we are given an abelian scheme X/Y of relative dimension $g \geq 1$. Formula $(*)$ from Section 4 (applied to the smooth locus of Y) says that for any $y \in Y(U) \setminus Y(\mathbb{C})$ we have

$$\rho_y^{\text{int}}(\delta) = \rho_y^{\text{ext}}(y'\tau(y)) = y'\rho^{\text{ext}}(\tau)(y)$$

where $\rho^{\text{ext}}: \mathrm{Der}_U \mathcal{O}(Y) \to H^1(X, T_{X/Y})$ is the natural "external" Kodaira-Spencer map of X/Y, $T_{X/Y}$ = relative tangent sheaf of X/Y. Exactly as in Section 4 we may consider the determinant $\det \rho^{\text{ext}}(\tau)$ which identifies with an element in $\mathbb{C}(Y)$. (Recall we assumed Y small enough, so that the various vector bundles appearing in section 4 are trivial.) If we assume this rational function is non-zero then it has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$; so we get by the above equalities that if $Y^{(g)}(U)$ is non-empty then $Y^{(g)}(U) = Y(U) \setminus Y(\mathbb{C})$. Assume from now on that $Y^{(g)}(U) \neq \emptyset$ (equivalently that $\det \rho^{\text{ext}}(\tau) \neq 0$ in $\mathbb{C}(Y)$). There is a finite set of closed points $S \subset Y$ containing all singular points of Y, all zeroes and poles of τ and of $\det \rho^{\text{ext}}(\tau)$, such that if $Y_1 = Y \setminus S$, X_1 = inverse image of Y_1, we have a basis of the free $\mathcal{O}(Y_1)$-module $H^1_{\mathrm{DR}}(X_1/Y_1)$ of the form

$$\omega_1, \ldots, \omega_g, \nabla^{\text{ext}}(\tau)\omega_1, \ldots, \nabla^{\text{ext}}(\tau)\omega_g$$

where $\omega_1, \ldots, \omega_g$ is a basis of the free $\mathcal{O}(Y_1)$-module $H^0(X_1, \Omega_{X/Y})$. Note that $Y_1(U) \setminus Y_1(\mathbb{C}) = Y(U) \setminus Y(\mathbb{C})$. Then the map $N: (TY)(U) \to \mathrm{gl}_{2g}(U)$ from Section 4 has the form:

$$u\tau(y) \longmapsto \begin{bmatrix} 0 & uI_g \\ uN_{21}(y) & uN_{22}(y) \end{bmatrix} \qquad \text{for all } u \in U,\ y \in Y_1(U)$$

where N_{21}, N_{22} are $g \times g$ matrices with entries in $\mathcal{O}(Y_1)$. Now we compute the image of any $y \in Y(U) \setminus Y(\mathbb{C})$ via the map

$$\chi = \Phi \circ \Gamma \circ N \circ \nabla: Y^{(g)}(U) \to (TY)^{(g)}(U) \to \mathrm{gl}_{2g}(U)^{(g)} \to H_g^{(g)} \to U^g$$

Using the formula of $\Phi \circ \Gamma$ from Section 3 plus the above formula for N, we get that $\chi(y)$ equals the characteristic polynomial of $\gamma = \beta - \alpha^2/4 - \alpha'/2$ where

$$\alpha = -y''(y')^{-1}I_g - y'N_{22}(y)$$
$$\beta = -(y')^2 N_{21}(y)$$

Substituting the expressions of α and β in that of γ, we get

$$\gamma = \sigma(y)I_g + (y')^2 R(y)$$

where $\sigma(y) = y'''/2y' - (3/4)(y''/y')^2$ is the "Schwartzian" of y and R is some $g \times g$ matrix with entries in $\mathcal{O}(Y_1)$. In case $g = 1$ we simply get:

$$\chi(y) = \sigma(y) + (y')^2 R(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

for some regular function $R \in \mathcal{O}(Y_1)$. So if, moreover, $Y = \mathbb{A}^1 \setminus \{0,1\}$ and X is the Weierstrass elliptic curve over Y, then by universality of U the map $\chi: U \setminus \mathbb{C} \to U$ is surjective. Coming back to arbitrary $g \geq 1$ and taking products of g elliptic curves with various j-invariants in $U \setminus \mathbb{C}$, one sees that the map $\chi: A_{g,n}^{(g)}(U) \to U^g$ is surjective too.

6. The basic "fifth order map"

The main idea in what follows is to construct non-empty Kolchin open sets Ω_D and Ω_P of $\Omega := A_{g,n}^{(g)}(U)$ and a map $b: \Omega_D \to \mathbb{A}^1(U) = U$ with the following properties: 1) for any points $y, z \in \Omega_D$ belonging to the same isogeny class in $A_{g,n}(U)$ we have $b(y) = b(z)$ and 2) for any curve $Y \subset A_{g,n}$ with $Y(U) \cap \Omega_D \cap \Omega_P \neq \emptyset$ we have $Y(U) \cap \Omega_D = Y(U) \setminus Y(\mathbb{C})$ and there exists a rational function $s \in \mathbb{C}(Y) \setminus \mathbb{C}$ such that the restrictions of b and s to $Y(U) \setminus Y(\mathbb{C})$ coincide (this makes sense because s viewed as a rational map $Y(U)\dashrightarrow \mathbb{A}^1(U) = U$ has all its poles contained in $Y(\mathbb{C})$). Then we will prove that there exist an effective divisor $W \subset A$ and two positive integers m_1, m_2 such that upon letting $\Omega_W = A_{g,n}(U) \setminus W(U)$ we have that $\deg s \leq m_1 \deg(Y) + m_2 p(Y)$ whenever $Y(U) \cap \Omega_D \cap \Omega_P \cap \Omega_W \neq \emptyset$. This construction will end the proof of Theorem 1″. Indeed, let C_0 be the set of all $Y \in C(A_{g,n})$ such that $Y(U) \cap \Omega_D \cap \Omega_P \cap \Omega_W \neq \emptyset$; if $I \subset A_{g,n}(U) \setminus A_{g,n}(\mathbb{C})$ is any isogeny class then for $Y \in C_0$ we have $I \cap Y(U) = I \cap (Y(U) \setminus Y(\mathbb{C}))$ hence $I \cap Y(U)$ will be contained in a fiber of the restriction of b to $Y(U) \setminus Y(\mathbb{C})$ hence in a fiber of the restriction of s to $Y(U) \setminus Y(\mathbb{C})$, consequently

$$\operatorname{card}(I \cap Y(U)) \leq \deg s \leq m_1 \deg(Y) + m_2 p(Y)$$

and Theorem 1″ will be proved. In this section we construct our "basic map" b; intuitively b will appear as a "non-linear differential operator of order 5". In the next section we will construct W and estimate $\deg s$.

Start by considering the maps $D, T, E \colon \mathbb{A}^g(U) \to \mathbb{A}^1(U)$ defined as follows: for $v = (v_1 \ldots, v_g) \in U^g$,

$$\begin{aligned}
D(v) &= D(x^g + v_1 x^{g-1} + \cdots + v_g) = \operatorname{disc}(x^g + v_1 x^{g-1} + \cdots + v_g)\\
T(v) &= -v_1\\
E(v) &= 4g(g-1)D(v)(D(v)'' - 4(g-1)T(v)D(v)) - (2g-1)^2(D(v)')^2
\end{aligned}$$

where "disc" means "discriminant". Note that D and T are regular maps of algebraic varieties while E is not; yet E is continuous in the Kolchin topology (intuitively E is a non-linear differential operator of order 2). Consider the

Kolchin open set $\Omega_D = \Omega \setminus \chi^{-1}(D^{-1}(0))$ of $A_{g,n}(U)$ (which is non-empty because the map $\chi: \Omega \to U^g$ appearing at the end of Section 4 is surjective) and also consider the Zariski open set $\mathbb{A}^g_D(U) = \mathbb{A}^g(U) \setminus D^{-1}(0)$. Then we may consider the maps $S: \mathbb{A}^g_D(U) \to \mathbb{A}^1(U)$, $P: \mathbb{A}^g(U) \to \mathbb{A}^1(U)$ defined by:

$$S(v) = E(v)^{g(g-1)}/D(v)^{2g(g-1)+1},$$
$$P(v) = (E(v)^{g(g-1)})' D(v)^{2g(g-1)+1} - E(v)^{g(g-1)}(D(v)^{2g(g-1)+1})'$$

The maps S and P are continuous in the Kolchin topology so we may consider the Kolchin open set $\Omega_P = \Omega \setminus \chi^{-1}(P^{-1}(0))$ which is non-empty also by surjectivity of $\chi: \Omega \to U^g$. Now we define our "basic map" to be the composition

$$b = S \circ \chi: \Omega_D \to \mathbb{A}^g_D(U) \to \mathbb{A}^1(U)$$

Clearly b maps each pair of points belonging to the same isogeny class into the same point (because χ has this property). Now let $Y \subset A_{g,n}$ be a curve with $Y(U) \cap \Omega_D \cap \Omega_P \neq \emptyset$; in particular, by Section 5, $Y^{(g)}(U) = Y(U) \setminus Y(\mathbb{C})$. Also by Section 5 we have

$$\chi(y) = \det(xI_g - \sigma(y)I_g - (y')^2 R(y)), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where R is some $g \times g$ matrix with entries in the function field $\mathbb{C}(Y)$. In particular we get:

$$(*)\ T(\chi(y)) = \mathrm{tr}(\sigma(y)I_g + (y')^2 R(y)) = g\sigma(y) + (y')^2 t(y), \quad y \in Y(U) \setminus Y(\mathbb{C})$$

where $t = \mathrm{tr}\, R \in \mathbb{C}(Y)$ and "tr" denotes of course the trace of a matrix. Using the behavior of the discriminant of a polynomial with respect to linear changes of variable, we get

(**)

$$D(\chi(y)) = D(\det((x-\sigma(y))I_g - (y')^2 R(y))) = D(\det(xI_g - (y')^2 R(y))) = \\ = (y')^{2g(g-1)} d(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where $d = D(\det(xI_g - R)) \in \mathbb{C}(Y)$. Since $Y(U) \cap \Omega_D \neq \emptyset$ there exists $y_0 \in Y(U) \setminus Y(\mathbb{C})$ such that $D(\chi(y_0)) \neq 0$; by (**) $d \neq 0$ in $\mathbb{C}(Y)$. Since d viewed as rational function on $Y(U)$ has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$, it follows from (**) that $D(\chi(y)) \neq 0$ for all $y \in Y(U) \setminus Y(\mathbb{C})$ hence $Y(U) \cap \Omega_D = Y(U) \setminus Y(\mathbb{C})$. Now a tedious but straightforward computation with formulae (*) and (**) yields:

$$(***) \qquad E(\chi(y)) = (y')^{4g(g-1)+2} e(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where $e = 4g(g-1)d(\tau^2 d - 4(g-1)td) - (2g-1)^2(\tau d)^2 \in \mathbb{C}(Y)$. From (**) and (***) we get:

$$(****) \qquad b(y) = S(\chi(y)) = s(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where $s = e^{g(g-1)}/d^{2g(g-1)+1} \in \mathbb{C}(Y)$. We claim that $s \notin \mathbb{C}$, equivalently, $\tau s \neq 0$. But, indeed, deriving (****), we get

$$y'(\tau s)(y) = P(\chi(y))/D(\chi(y))^{4g(g-1)+2}, \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

Since $Y(U) \cap \Omega_D \cap \Omega_P \neq \emptyset$, there exists $y_1 \in Y(U) \setminus Y(\mathbb{C})$ such that $P(\chi(y_1)) \neq 0$ and $D(\chi(y_1)) \neq 0$ so we must have $(\tau s)(y_1) \neq 0$ hence $\tau s \neq 0$ as an element of $\mathbb{C}(Y)$ and our claim is proved.

To conclude the proof of Theorem 1″ we need to bound $\deg s$ in terms of $\deg(Y)$ and $p(Y)$ which will be done in the next section.

Remark. The main point in the last step of the proof above was the "miracle" that, for $y \in Y(U) \setminus Y(\mathbb{C})$, both $D(\chi(y))$ and $E(\chi(y))$ were expressible in the form of a product of some power of y' with a sutiable rational function in $\mathbb{C}(Y)$. In case of $D(\chi(y))$, this "miracle" is the reflection of remarkable properties of the discriminant. In a similar way the "miracle" for $E(\chi(y))$ is the reflection of remarkable properties of what may be called the "differential resultant" of two δ-polynomials; this interpretation is of course irrelevant for the proof (but it was quite relevant for the way we were led to the somewhat tricky definition of E). Deligne's arguments in [D] avoid this "miraculous" point in our proof.

7. Bounding deg s

First let's recall various trivial facts related to degrees on curves. Let L be a function field of one variable over $\mathbb{C}$ of genus p. Then for any $f \in L$ we define $\deg f = \deg(f)_\infty$ (where $(f)_\infty$ is the negative part of the principal divisor associated to f of the smooth projective model of L) if $f \neq 0$ and $\deg f = 0$ if $f = 0$. This is of course nothing but the "usual height" of the point $(1 : f)$ on the projective line. Similarly if $\omega \in \Omega^1_{L/\mathbb{C}}$ is a 1-form we let $\deg \omega = \deg(\omega)_\infty$ if $\omega \neq 0$ and $\deg \omega = 0$ if $\omega = 0$. Finally, if $\tau \in \mathrm{Der}_{\mathbb{C}} L$, $\tau \neq 0$ we write $\deg \tau = \deg(\tau)_\infty$. Here $(\omega)_\infty$, $(\tau)_\infty$ have the obvious meaning analogue to $(f)_\infty$. It is trivial to check that:

(i) $\deg\langle \tau, \omega\rangle \leq \deg \tau + \deg \omega$

$\deg f\omega \leq \deg f + \deg \omega$

$\deg \tau f \leq \deg \tau + 2 \deg f$

$\deg(f_1 + f_2) \leq \deg f_1 + \deg f_2$

$$\deg f_1 f_2 \leq \deg f_1 + \deg f_2$$

Note that by Riemann-Roch there always exists $\tau \in \mathrm{Der}_{\mathbb{C}} L$, $\tau \neq 0$ with $\deg \tau \leq 2p$. For any matrix $M = (f_{ij}), f_{ij} \in L$ it will be convenient to denote by deg M the maximum of the numbers $\deg f_{ij}$.

Now let us come back to the set $C(A)$ of curves in $A = A_{g,n}$ and recall that we have fixed a projective compactification $\bar{A}$ of A and a very ample line bundle $\mathcal{O}(1)$ on $\bar{A}$. For any two functions $\varphi, \psi: C(A) \to \mathbb{N}$ and any subset C' of $C(A)$ we write $\varphi(Y) \ll \psi(Y)$, $Y \in C'$ if and only if there exists a constant $m > 0$ such that $\varphi(Y) \leq m\psi(Y)$ for all $Y \in C'$.

After these notational preparations we may proceed to proving the existence of the desired bound for deg s.

Let X/A be the universal abelian scheme over A, let R be the field of rational functions on A and let $X_R = X \times_A \mathrm{Spec}\, R$. Pick any R-basis $\omega_1, \ldots, \omega_{2g}$ of $H^1_{\mathrm{DR}}(X_R/R)$ such that $\omega_1, \ldots, \omega_g$ is an R-basis of $H^0(X_R, \Omega)$ and write

$$\nabla \omega_i = \sum \omega_{ij} \otimes \omega_j, \ \omega_{ij} \in \Omega^1_{R/\mathbb{C}}, \ 1 \leq i, j \leq 2g$$

where ∇ is the Gauss-Manin connection of X_R/R. There clearly exists an integer $N \geq 1$ and a divisor $\bar{W}$ on $\bar{A}$ whose associated line bundle is $\mathcal{O}(N)$ such that, upon letting V be $\bar{A} \setminus \bar{W}$, the following hold:

1) $V \subset A$,

2) $\omega_1, \ldots, \omega_{2g}$ is a basis of the $\mathcal{O}(V)$-module $H^1_{\mathrm{DR}}(X/A)_V$,

3) $\omega_1, \ldots, \omega_g$ is a basis of the $\mathcal{O}(V)$-module of relative regular 1-differentials of X/A over V,

4) ω_{ij} are regular on V,

5) $(\omega_{ij})_\infty \leq \bar{W}$ for all i, j (where $(\omega_{ij})_\infty$ is the divisor of poles of ω_{ij} on $\bar{A}$).

Put $W = \bar{W} \cap A$ and let's check that with this W we have

$$\deg s \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

which will close the proof of Theorem 1″. Recall from Section 6 that we defined C_0 to be the set of all $Y \in C(A_{g,n})$ such that $Y(U) \cap \Omega_D \cap \Omega_P \cap \Omega_W \neq \emptyset$; moreover we defined s by the formula $s = e^{g(g-1)}/d^{2g(g-1)+1} \in \mathbb{C}(Y)$. Now if $Y \in C_0$ and if $\bar{\omega}_{ij}$ are the restrictions of ω_{ij} to Y, then obviously we have

$$\deg \bar{\omega}_{ij} \ll \deg(Y), \qquad Y \in C_0$$

Using formulae (i) we easily check that the matrices N_{21}, N_{22} in Section 5 may be chosen to satisfy

$$\deg N_{21}, \deg N_{22} \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

Clearly the matrix R in Section 5 can be expressed explicitly in terms of N_{21}, N_{22}; using (i) again we get

$$\deg R \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

Finally, if t, d, e, s are as in Section 6, we deduce step by step (using (i)) that

$$\deg t, \deg d, \deg e, \deg s \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

and Theorem 1″ is proved.

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Astérisque

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Astérisque, tome 218 (1993), p. 61-93
<http://www.numdam.org/item?id=AST_1993__218__61_0>

CONFIGURATIONS OF REAL AND COMPLEX POLYNOMIALS

by
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This article is dedicated to the memory of Mario Raimondo.

§0. Introduction.

The purpose of this article is to give a geometric explanation of the surprising equality (cf. [C-P],[Ar1],[Ar3]) between, on one hand, the number of configurations of (complex) lemniscate generic polynomials, and , on the other hand, the number of configurations of real monic Morse polynomials with the maximal number of (real) critical points.

This discovery occurred when Arnold gave a series of talks at the Scuola Normale in 1989 on the subject of catastrophe theory, and there was somehow a bet whether there could be a geometrical correspondence between the two sets.

Afterwards, Arnold developed a quite general theory concerning the ubiquity of Euler, Bernoulli and Springer numbers (cf.[Ar1], [Ar2], [Ar3]) in the realm of singularity theory.

In this article, among other things, we prove the equality of the above two numbers by geometric methods.

It would of course be very interesting to extend the type of correspondence introduced here to a more general context, like the case of spaces of universal deformations of 0-modular isolated singularities. In a different direction, we plan to extend these type of results to the case of real algebraic functions, using the results of [B-C].

Let us explain now in some detail what are our present results.

We adopt here the notation and terminology of [C-P] and [C-W] : given a polynomial $P(z)$ we consider $|P(z)|$ as a (weak) Morse function , and we define the *big lemniscate configuration* of P to be equal to the union of the singular level sets of $|P|$ (the so called lemniscates). P is said to be *lemniscate generic* if P has distinct roots and every level set $\Gamma_c = \{z : |P(z)| = c\}$ has at most one ordinary quadratic singularity. Two big lemniscate configurations Γ_1, Γ_2 are said to be isotopic if there is a path σ in the space of diffeomorphisms of $\mathbb{C}$ such that $\sigma(0)$ is the identity and $\sigma(1)(\Gamma_1) = \Gamma_2$.

One of the main results of [C-P] was that there is a bijective correspondence between isotopy classes of big lemniscate configurations and connected components of the space $\mathcal{L}_n$ of lemniscate generic polynomials. Assume now that $P \in \mathbb{R}[z]$: then, if P is lemniscate generic, automatically all the critical points of P are real; thus, letting $(n+1)$ be the degree of P, P has n distinct real critical values which are different from zero.

Let $\mathcal{L}_n$ be the open set of complex lemniscate generic polynomials of degree $(n+1)$, let $\mathcal{L}_{n,\mathbb{R}}$ be the set of real lemniscate generic polynomials (an open set in the space of real polynomials), let finally $\mathbb{G}\mathcal{M}_n$ (which is called the "Set of generic maximally real polynomials") be the open set of real polynomials with n real and distinct critical values : thus $\mathcal{L}_{n,\mathbb{R}} \subset \mathbb{G}\mathcal{M}_n$, and every component of $\mathbb{G}\mathcal{M}_n$ is the closure (in $\mathbb{G}\mathcal{M}_n$) of a finite number of components of $\mathcal{L}_{n,\mathbb{R}}$.

If P is in $\mathbb{G}\mathcal{M}_n$ and $y_1 < \ldots < y_n$ are the critical points, we associate to P the sequence $u_1 = P(y_1), \ldots, u_n = P(y_n)$, a *snake sequence* (cf. [Da], [Ar3]), what simply means that $(-1)^i(u_i - u_{i+1})$ has constant sign.

If P is lemniscate generic and real, there is another way of ordering the critical values, namely by increasing absolute values : we let $Y_{n,\mathbb{R}} = \{(w_1, \ldots, w_n) \in \mathbb{R}^n : 0 < |w_1| < \ldots < |w_n|\}$ be the space of admissible critical values. Clearly $Y_{n,\mathbb{R}}$ has exactly 2^n connected components homeomorphic to $\mathbb{R}^n$.

Main Theorem.

(a) Each connected component of $\mathcal{L}_n$ contains exactly 2^{n+1} connected components of $\mathcal{L}_{n,\mathbb{R}}$.

(b) The number of connected components of $\mathcal{L}_{n,\mathbb{R}}$ mapping to a fixed component of $Y_{n,\mathbb{R}}$ equals the number of components of $\mathbb{G}\mathcal{M}_n$, whence the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals twice the number K_n of connected components of $\mathcal{L}_n$; the number instead of components of $\mathbb{G}\mathcal{M}_n \cap \{\text{monic polynomials}\}$ equals K_n.

(c) (cf. Arnold [Ar1]) The number of components of $\mathbb{G}\mathcal{M}_n$ equals the number of *snake sequences* (this means, for fixed $w_1, ..., w_n$, the number of snake sequences $u_1, ..., u_n$ that can be obtained by permuting $w_1, ..., w_n$).

(d) (cf. [Ar1],[C-P]) The number of components b_n of $\mathcal{L}_{n,\mathbb{R}}$ gives rise to the following exponential generating function :

$$2\Sigma_n(b_n/n!)t^n = \int 4/(1-\sin(2t)) = 2(\sec(2t)+\tan(2t)).$$

e) the number of snake sequences equals the number of isotopy classes of lemniscate configurations multiplied by 2.

The above result is related to a curious rediscovery of Riemann's existence theorem, done by Thom in 1960 ([Thom]) In fact, in 1957 C. Davis ([Da]) showed in particular that for each choice of n distinct real numbers there is a real polynomial of degree $(n+1)$ having those as critical values (in fact, up to affine transformations in the source, a unique one for each snake sequence formed with those numbers), and a similar question was asked for complex polynomials.

Thom remarked that by Riemann's existence theorem the answer is that for each choice of n distinct complex numbers and an equivalence class of admissible monodromy there exists exactly one polynomial,up to affine transformations in the source, having those points as critical values and the given monodromy.

In this paper we link the two answers by describing explicitly, even when the branch points are not all real, the monodromies which come from real polynomials.

In fact, in [C-P] it was shown also that every big lemniscate configuration occurs for some real polynomial for which the monodromy tree (cf.[C-W]) is linear (that is, homeomorphic to a segment).

Here, in a similar vein, we establish another result (which is essential in order to establish our main theorem), which allows us to understand the lemniscate configurations which come from real polynomials as the ones obtained from "snake" linear trees (theorem B stated below is an abridged version of theorems 2.1, 2.3 and 2.12) :

Theorem B.

Given $w_1, ..., w_n \in \mathbb{R}$ with $0 < |w_1| < ... < |w_n|$, there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ such that the real lemniscate generic polynomials P having $w_1, ..., w_n$ as critical values, correspond exactly to the monodromy trees which are "snake" linear trees.

Also, for each fixed choice of $w_1, ..., w_n$ as above, if $n \geq 4$ there is some lemniscate configuration which cannot be obtained with a real polynomial.

To get the flavour of the second statement one should remark that the monodromies which come from real monic polynomials, (whose number is $K_n \sim O((2/\pi)^n(n)!))$ are quite few compared with all the possible monodromies, whose number is $(n+1)^{n-2}$. Nevertheless, since the number of lemniscate configurations is exactly K_n, we initially hoped that there would be a bijection between the set of real monodromies and the set of lemniscate configurations.

From theorem 2.1 it is then easily seen that, fixing the (real) critical values, and a linear tree in the canonical basis, the snake condition is equivalent to the condition that the associated polynomial is real.

In this way part b) of the main theorem is proven.

Finally, the proof of a) of the main theorem is a straightforward consequence of Lefschetz' fixed points theorem, while c) follows from the quoted result of Davis, which we reprove (in 2.3) with a small precision, for the sake of completeness.

Parts d),e) follow then from a),b),c) and the results of [Ar1],[C-P].

Section 2 contains also other miscellaneous results.

In the third section we employ the branch points map used by several authors ([Da],[Lo],[Ly],[C-W],[C-P],[Ar3]) in order to give a quick proof of a generalization of Davis' theorem along the same lines. Later on, we prove in theorem 3.7 a much more precise result, namely that the monodromies of real generic polynomials are given, in a canonical basis, by trees obtained from a snake linear trees by adding, in a symmetric way, pairs of isomorphic trees

(see section 3 for a more precise statement). From this result it is possible to calculate the number of connected components of the space of real generic polynomials of degree equal to $n+1$, but we have not yet found a simple formula for it.
The proof that we give of 3.7 is completely algebraic, implies in particular a third proof of the quoted theorem of Davis (after the ones given in [Da], [Ar2], and in 2.1, 2.3), and is susceptible of generalizations.

Acknowledgements.

The present cooperation took place in the framework of the SCIENCE contract n. SCI-0398-C(A). The first author is a member of G.N.S.A.G.A. and of the M.U.R.S.T. 40% Program "Geometria Algebrica".

1. Notation, set-up, preliminary results.

(1.1) Definition.
a) Let $P \in \mathbb{R}[z]$ be a polynomial of degree $(n+1)$: P is said to be maximally real if all the critical points of P (the roots $y_1, ..., y_n$ of its derivative) are real. We let $\mathcal{M}_n$ be the closed set of maximally real polynomials. Its interior $\mathcal{M}'_n$ corresponds to the polynomials with real distinct critical points and contains the open set $\mathbb{G}\mathcal{M}_n$ of the maximally real polynomials which are also generic, i.e., are such that the branch points of P, wiz., the real numbers $u_i = P(y_i)$, are distinct.
b) If P is maximally real there is a standard ordering $y_1 \leq ... \leq y_n$ of the critical points, hence we have, for P as above, also a canonical ordering $u_1 = P(y_1), ..., u_n = P(y_n)$ of the branch points, which we shall call the *source ordering.*

(1.2) Definition-remark.
i) A sequence $u_1, ..., u_n$ of real numbers is said to be a *weak up-down sequence* if $(-1)^i(u_i - u_{i+1}) \leq 0$, a *weak down-up sequence* if $(-1)^i(u_i - u_{i+1}) \geq 0$, a *weak snake sequence* if one of the two above holds. A *snake sequence* will be a weak snake sequence where $u_i \neq u_{i+1}$ for each i.

ii) if P is a maximally real polynomial, then its branch points $u_1 = P(y_1), \ldots, u_n = P(y_n)$, taken with the source ordering, yield a weak snake sequence.

(1.3) Definition.

a) A polynomial $P \in \mathbb{C}[z]$ of degree $(n+1)$ is said to be *generic* iff it has n distinct branch points.

b) P is moreover said to be lemniscate-generic if the branch points have n distinct absolute values different from 0.

c) If P is lemniscate generic, then there is a standard ordering for the branch points, by which we get another sequence $w_1, \ldots, w_n$ with $0 < |w_1| < \ldots < |w_n|$.

(1.4) Remark.

A polynomial $P \in \mathbb{R}[z]$ which is lemniscate generic is automatically maximally real, and there are three distinct orderings for the set of its branch points, the source, the standard and the target ordering (the first never coincides with the last).

We want to define the *Hurwitz space* $\mathcal{H}_n$ of polynomials. To do this, we consider the notion of source equivalence.

(1.5) Definition.

i) Two polynomials $P, Q \in \mathbb{C}[z]$ are said to be *source equivalent* $(P \sim Q)$ iff there exists an isomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ $(\varphi \in A(1, \mathbb{C}))$ such that $Q = P \circ \varphi$.

ii) The Hurwitz space $\mathcal{H}_n$ of polynomials is the quotient $V_n / A(1, \mathbb{C})$ of the space V_n of polynomials of degree $(n+1)$ in $\mathbb{C}[z]$, by the relation of source equivalence.

We want now to define the real part of the Hurwitz space.
In order to do it, let us observe that the operation $P \to \bar{P}$ of complex conjugation of coefficients of P passes to the quotient, since if $Q = P \circ \varphi$, then $\bar{Q} = \bar{P} \circ \bar{\varphi}$, as it is easy to verify.
The fixed locus for complex conjugation is given by $V_n \cap \mathbb{R}[z]$, and the next proposition determines the fixed locus inside $\mathcal{H}_n$.

(1.6) Proposition.

If $Q \in \mathbb{C}[z]$ is (source-) equivalent to $\bar{Q}$, then Q is equivalent to a real polynomial $P \in \mathbb{R}[z]$. Whence the real part $\mathcal{H}_{n,\mathbb{R}}$ of the Hurwitz space $\mathcal{H}_n$ is indeed the image to the quotient of $V_n \cap \mathbb{R}[z]$.

Proof. We can clearly replace Q by any other polynomial which is equivalent to Q, and therefore we can assume that Q is of the form

$$Q = z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0.$$

If $\varphi(z) = \alpha z + \beta$, and $\bar{Q} = Q \circ \varphi$, then we immediately get $\alpha^{n+1} = 1$, and $\beta = 0$.

Let a be a square root of α, so that $\alpha = a/\bar{a}$, and set $P(z) = Q(az)$.

Then $\bar{P}(z) = \bar{Q}(\bar{a}z) = Q(\alpha\bar{a}z) = Q(az) = P(z)$.

Q.E.D.

Unfortunately, it is not true that two real polynomials P,Q are equivalent iff there exists a φ in $A(1,\mathbb{R})$ with $Q = P \circ \varphi$, as it is shown by the example of $P = z^4 + z^2 + 1$, $Q = z^4 - z^2 + 1$.
But this holds true if the polynomials are generic :

(1.7) Proposition.

If $P, Q \in \mathbb{R}[z]$ are $A(1,\mathbb{C})$ equivalent, then they are $A(1,\mathbb{R})$ equivalent if they cannot be written as a composition of two polynomial maps of strictly lower degree. In particular, if U_n is the open set of generic polynomials, then the image to the quotient of $U_n \cap \mathbb{R}[z]$ is the quotient $(U_n \cap \mathbb{R}[z])/A(1,\mathbb{R})$.

Proof. We can clearly assume, replacing P and Q by $A(1,\mathbb{R})$ equivalent polynomials, that P and Q are of the following *"Tschirnhausen"* form

$$P = \pm z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0,$$

$$Q = \pm z^{n+1} + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \dots\dots + b_0.$$

Since there are $\alpha \neq 0$, $\beta \in \mathbb{C}$ such that $Q(z) = P(\alpha z + \beta)$ we get as before $\beta = 0, \alpha^{n+1} = \pm 1$.

Then $b_i = \alpha^i a_i$, whence $\alpha^i \in \mathbb{R}$ whenever $a_i \neq 0$.

If $\alpha \in \mathbb{R}$, we are done, else, there is a minimal m such that $\alpha^m \in \mathbb{R}$, and $a_i = 0$ if i is not divisible by m. In the latter case there is a polynomial R of degree $(n+1)/m$ such that $P(z) = R(z^m)$.
The proof is over, since a polynomial of the form $R(z^m)$ can only be generic if $m = 2$ and R is linear : but in this case $\alpha \in \mathbb{R}$. Q.E.D.

(1.8) Corollary.

The generic real Hurwitz space of polynomials, that is, the quotient $(U_n \cap \mathbb{R}[z])/A(1,\mathbb{R})$, is isomorphic to the quotient of the space $T_{n,\mathbb{R}}$ of generic real Tschirnhausen polynomials

$$\{P | P(z) = \pm z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0 \ , \ a_i \in \mathbb{R},$$
$$\text{and } P \text{ is a generic polynomial}\}$$

by the involution ι which sends $P(z)$ to $P(-z)$. In particular, for n even, the generic real Hurwitz space is isomorphic to the space of monic generic real Tschirnhausen polynomials.
The quotient $\mathbb{R}[z]/A(1,\mathbb{R})$ is also isomorphic to the quotient of the space $N_{n,\mathbb{R}}$ of normalized real polynomials

$$\{P | P(z) = \pm z^{n+1} + a_n z^n + a_{n-1}z^{n-1} + \dots\dots + a_1 z, a_i \in \mathbb{R}\}$$

by the involution ι which sends $P(z)$ to $P(-z)$.

Proof.

The first assertion was already proven, the second follows in an entirely similar way.

(1.9) Remark.

$U_n/A(1,\mathbb{C})$ is an open set in $V_n/A(1,\mathbb{C}) \cong T_n/\mu_{n+1}$, where T_n is the space of complex (Tschirnhausen) polynomials of the form

$$P(z) = z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0,$$

and where μ_{n+1} is the group of $(n+1)^{\text{th}}$ roots of unity in $\mathbb{C}$. The difference $T_n - U_n$ is called the bifurcation hypersurface Δ_n (cf. [C–W]).
The group extension associated to the Galois cover $T_n - \Delta_n \to U_n/A(1,\mathbb{C}) \cong (T_n - \Delta_n)/\mu_{n+1}$ is described in the main theorem of [C-W].

(1.10) Definition-remark.

i) Let us denote by $\mathcal{H}'_n$ be the generic Hurwitz space, i.e., the quotient $U_n/A(1,\mathbb{C})$, and by $\mathcal{H}'_{n,\mathbb{R}}$ its real part. If we set moreover $U_{n,\mathbb{R}} = U_n \cap \mathbb{R}[z]$, then $\mathcal{H}'_{n,\mathbb{R}} \cong U_{n,\mathbb{R}}/A(1,\mathbb{R})$.

ii) Inside $U_{n,\mathbb{R}}$ we let $\mathbb{G}\mathcal{M}_n$ be the subset of those generic polynomials for which the critical points (or, equivalently, the critical values) are real. It is clear that $\mathbb{G}\mathcal{M}_n$ is a union of connected components of $U_n \cap \mathbb{R}[z]$, and that each component of $\mathbb{G}\mathcal{M}_n$ is made up of $A(1,\mathbb{R})$–orbits. The polynomials in $\mathbb{G}\mathcal{M}_n$ are said to be *maximally real and generic.*

iii) Let $\mathcal{L}_n$ be the open set in U_n consisting of lemniscate generic complex polynomials, and let $\mathcal{L}_{n,\mathbb{R}}$ be its real part. Clearly these open sets are made of equivalence classes, whence one can define the lemniscate generic Hurwitz space $\mathcal{L}\mathcal{H}_n$, and similarly its real part $\mathcal{L}\mathcal{H}_{n,\mathbb{R}}$.

In [C-W] and [C-P] (where, though, $\mathcal{H}'_n$ was denoted Z_n) a key importance had the study of the critical value fibration, associating to a generic polynomial P the unordered set of its n critical values :

(1.11)

$$\psi_n : \mathcal{H}'_n \to W_n = \{B = \{w_1, ..., w_n\} | w_i \in \mathbb{C}, \text{ and } w_i \neq w_j \text{ for } i \neq j\}.$$

We recall some definitions and results from the two cited papers, which are a consequence of Riemann's existence theorem

(1.12) Results and definitions concerning the critical value fibration.

a) $\psi_n : \mathcal{H}'_n \to W_n$ is an unramified covering space whose fibre over B is the set of conjugacy classes $[\mu]$ of monodromies $\mu : \pi_1(\mathbb{C} - B, x_0) \to \mathcal{S}_{n+1}$, such that the image of μ is a transitive subgroup, and each element of a geometric basis of $\pi_1(\mathbb{C} - B, x_0)$ is mapped to a transposition. Here, two homomorphisms μ and μ' as above are said to be in the same conjugacy class iff there exists an inner automorphism φ of $\mathcal{S}_{n+1}$, such that $\mu = D\varphi \circ \mu' \circ \varphi^{-1}$; and a geometric basis is a basis of n loops γ_i $(i = 1, ..., n)$ formed by a segment joining x_0 with a small circle around w_i.

b) Since the group $\mathcal{B}_n = \pi_1(W_n, \{1, ..., n\})$, called Artin's braid group, acts (cf.[Bir]) as a group of automorphisms of $\pi_1(\mathbb{C} - \{1, ..., n\}, x_0)$, the monodromy of ψ_n is such that σ sends $[\mu]$ to $[\mu \circ \sigma^{-1}]$.

c) the elements in a fibre of ψ_n, once a geometric basis $\gamma_1, ..\gamma_n$ for $\pi_1(\mathbb{C} - B, x_0)$ has been fixed, can be put in a bijective correspondence with E_n, the set of isomorphism classes of edge labelled trees with n edges (we take $(n+1)$ unlabelled vertices which represent the set $P^{-1}(x_0)$, and we adjoin n edges, labelled by an integer from 1 to n, corresponding to the transpositions $\mu(\gamma_i)$, and joining the two vertices moved by the above transposition).

d) Let $Y_n \subset W_n$ be the subset $\{\{w_1, ..., w_n\} | 0 < |w_1| < ... < |w_n|\}$, so that $\mathcal{L}_n = \psi_n^{-1}(Y_n)$.
Let Λ_n be the image of $\pi_1(Y_n, \{1, ..., n\}) \to \pi_1(W_n, \{1, ..., n\})$: then the connected components of $\mathcal{L}_n$ correspond to the Λ_n-orbits on E_n.

(1.13) Remark.
i) Writing $r_i = |w_i| - |w_{i-1}|$ and $\eta_i = w_i/|w_i|$, we see that Y_n is homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, hence $\pi_1(Y_n) \cong \mathbb{Z}^n$. The images T_j of the generators of $\pi_1(Y_n)$ are the braids, which keep fixed the points $1, ..., n$ different from j, and move j in a circle around the origin $(t \mapsto e^{2\pi i t} j)$.

ii) each connected component of $\mathcal{LH}_n$, being a finite connected covering of Y_n, is also homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$.

iii) the real part of $Y_n, Y_{n,\mathbb{R}} = \{(w_1, ..., w_n) \in \mathbb{R}^n : 0 < |w_1| < ... < |w_n|\}$ is homeomorphic to $\{-1, +1\}^n \times (\mathbb{R}^+)^n$.

iv) ψ_n commutes with complex conjugation.

From the last part of the previous remark it follows that ψ_n carries the real part $\mathcal{LH}_{n,\mathbb{R}}$ of the lemniscate generic Hurwitz space to $Y_{n,\mathbb{R}}$, but we are going to see soon that $\mathcal{LH}_{n,\mathbb{R}}$ is far from being the full inverse image of $Y_{n,\mathbb{R}}$, which consists of $(n+1)^{n-2}2^n$ disjoint copies of $\mathbb{R}^+$.

(1.14) Lemma.
Each connected component A of $\mathcal{LH}_n$ contains exactly 2^n connected components of $\mathcal{LH}_{n,\mathbb{R}}$.

Proof. Each connected component A of $\mathcal{LH}_n$ is homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, and for each point $r = (r_1, ..., r_n) \in (\mathbb{R}^+)^n$, the set $A_r \cong (S^1)^n \times \{r\}$, which is invariant by conjugation, contains only a finite number of self conjugate points.

We apply now the Lefschetz's fixed point formula to f = complex conjugation on A_r.
Since A_r is a covering of $(S^1)^n$, and f induces, via the covering projection, the standard conjugation on $(S^1)^n$, we see immediately that f acts as -1 on $H_1(A_r, \mathbb{Z})$.
Thus, the number of fixed points of f on A_r is exactly 2^n.
Now, the real part of A is a closed submanifold of A, whence, it union of components of $A \cap \psi_n^{-1}(Y_{n,\mathbb{R}})$, which is a trivial covering of $Y_{n,\mathbb{R}}$. Our assertion follows then immediately.

Q.E.D.

(1.15) Corollary

For each connected component A of $\mathcal{LH}_n$ the restriction φ_n of ψ_n to the real part of A is injective to $Y_{n,\mathbb{R}}$ if and only if it maps surjectively to $Y_{n,\mathbb{R}}$.

§2. Statement and proof of the main theorems.

In this section, before giving a proof of the main theorem, we will give a characterization of the monodromies of maximally real polynomials as "snake" linear trees.

This will be done geometrically, whereas a second proof, of algebraic nature, will be given in section three, where we will more generally characterize the monodromies of real generic polynomials (identifying them as self conjugate monodromies).

(2.1) Theorem.

Let $(w_1, ..., w_n) \in Y_{n,\mathbb{R}}$ (thus $w_i \in \mathbb{R}$, and $0 < |w_1| < ... < |w_n|$). Then there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ such that for each real lemniscate generic polynomial P having $w_1, ..., w_n$ as critical values, the edge labelled monodromy tree $\mathcal{T}$ of P can be determined as follows. Let $y_1 < ... < y_n$ be the critical points of P, let $u_1 = P(y_1), ..., u_n = P(y_n)$ be the snake of its critical values, and let moreover σ be the permutation such that $u_i = w_{\sigma(i)}$.
Then the tree $\mathcal{T}$ is a linear tree consisting of n consecutive segments with

labels (from left to right) $\sigma(1), ..., \sigma(n)$.

(2.2) Choice of the canonical basis (see figure 1)

Let Ξ be the planar graph consisting of the union of n circumpherences χ_i, of radius $\epsilon << 1$ and with centres in the n points w_i, together with the complement in $\mathbb{R}$ of n open intervals of radius ϵ centered around the n points w_i.

Clearly $\mathbb{C} - \{w_1, ..., w_n\}$ is homotopy equivalent to Ξ, thus it suffices to choose the geometric basis inside $\pi_1(\Xi, 0)$.

Let γ_i be the loop based at 0 which consists of a *"right turning"* symplicial path δ_i from 0 to $P_i = (w_i - \epsilon[\operatorname{sign} w_i])$, followed by χ_i run counterclockwise, and finally followed by the inverse of δ_i. Here, a symplicial path is said to be right turning if, whenever the path, after following an edge, comes to a node, then takes as next edge the one to the right. We might observe that the inverse of a right turning path is left turning.

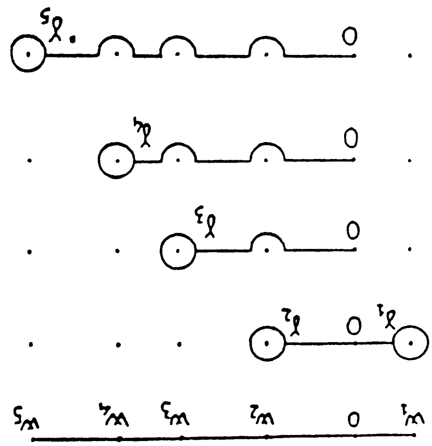

Figure 1 : Choice of the canonical basis for $\pi_1(\mathbb{C} - \{w_1, ..., w_5\}, 0)$

Proof of theorem 2.1

Consider P as a map of $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself, and consider the graph $\Theta = P^{-1}(\mathbb{P}^1_{\mathbb{R}})$. We consider $\mathbb{P}^1_{\mathbb{R}}$ as a graph with vertices $w_1, ..., w_n$ and ∞, and therefore also $(n+1)$ edges. Letting as usual the weight of a vertex be the number of edges stemming from it,Θ has one vertex (∞) of weight $2(n+1)$, n vertices of weight 4 at the critical points $y_1, ..., y_n$ and all the other vertices

of weight 2.
Whence, an easy calculation yields the number $(-2n)$ for the topological Euler-Poincare's characteristic $\chi(\Theta)$.
Let us now disregard the vertices of weight 2 in Θ, and remark that Θ contains $\mathbb{P}^1_{\mathbb{R}}$. We are left then with $(3n+1)$ edges, $(n+1)$ of which are intervals in $\mathbb{P}^1_{\mathbb{R}}$. The remaining $2n$ are in conjugate pairs, each contained either in the upper or in the lower half plane. Therefore through each critical point y_i passes exactly one edge E_i contained in the upper half plane.
Claim : We contend that the other end point of E_i must be ∞ (compare figure 2).
In fact, otherwise, the other end point of E_i should be a critical point y_j, with $i \neq j$. We can clearly assume $i < j$, and we shall see that if $j = i+1$ we have a contradiction. In fact, in this case we would have three edges, namely E_i, its conjugate, and the interval $[y_i, y_j]$ mapping to the interval with ends u_i, u_j and not containing ∞. But this contradicts the local structure of the map P at the simple critical point y_i (the local degree is 2). If instead, $j > i+1$, since the y_i's and ∞ are the only singular points of yΘ, the other end point of E_{i+1} must be a critical point y_k, with $i+1 < k < j$. By induction on $|j-i|$, we finally find a contradiction.

Q.E.D for the claim.

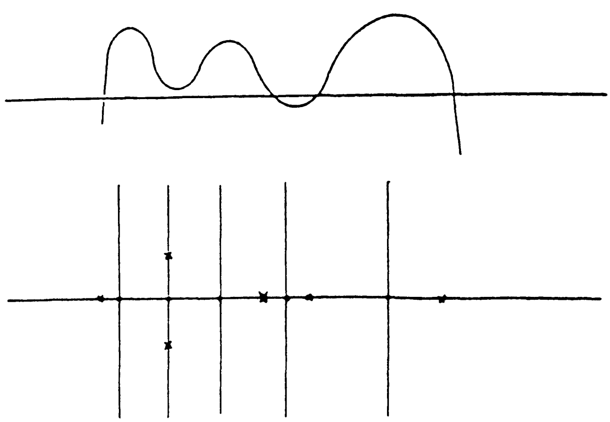

Figure 2: A polynomial of degree 6 and its graph $\Theta = P^{-1}(\mathbb{P}^1_{\mathbb{R}})$.

The critical points of P are partitioned into two sets : the set of local minima for $|P|_{|\mathbb{R}}$, and the set of local maxima for $|P|_{|\mathbb{R}}$. If y_i is a local minimum, then the edge E_i maps bijectively to the interval with ends u_i and ∞ which contains 0 ; if instead y_i is a local maximum, then the edge E_i maps bijectively to the interval with ends u_i and ∞ which does not contain 0. We clearly have a pair of conjugate roots of P for each local minimum of $|P|_{|\mathbb{R}}$, and all the remaining roots are real.

Observe moreover that we have a real root exactly in each interval in $\mathbb{R}$ between two consecutive maxima of $|P|_{|\mathbb{R}}$, and that one cannot have two consecutive minima.

In order to describe the monodromy μ of P, we want to determine explicitly the transposition τ_i of the roots of P obtained by the liftings of the path γ_i described above.

Clearly a lifting of γ_i is contained in the graph $P^{-1}(\Xi)$ (see figure 3) . We remark that since P is orientation preserving, the lifting of a right turning path will be right turning too.

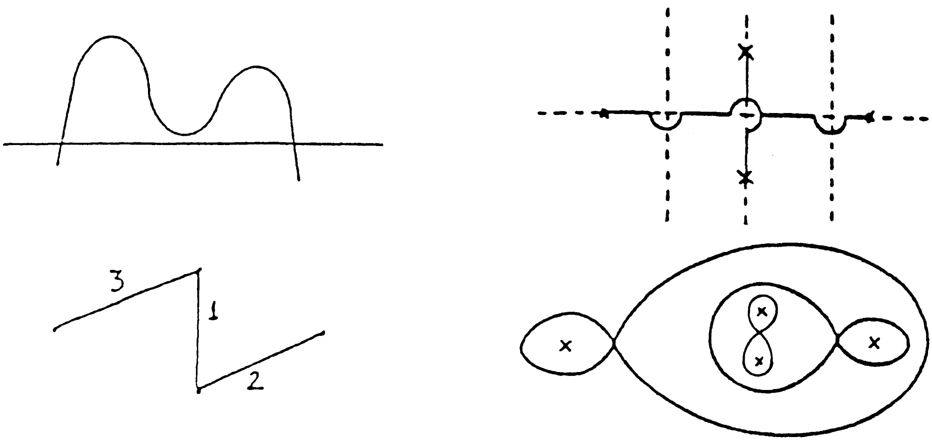

Figure 3 : A polynomial of degree 4, part of the graph $P^{-1}(\Xi)$, the associated monodromy tree, the lemniscate configuration.

(2.2) Sublemma

1) If y_i is a local minimum of $|P|_{|\mathbb{R}}$, the corresponding $\tau_{\sigma(i)}$ permutes the pair of conjugate roots of P lying on E_i and its conjugate.

2) If y_i is a local maximum, as well as y_{i+1}, y_{i-1}, $\tau_{\sigma(i)}$ permutes the pair of real roots lying in the two intervals with endpoint y_i.

3) If y_i is a local maximum, but y_{i+1}, y_{i-1} are local minima, $\tau_{\sigma(i)}$ permutes the non real root in E_{i+1} with the non real root in the conjugate of E_{i-1}.

4) In the remaining cases, $\tau_{\sigma(i)}$ permutes the neighbouring real root with the non real root in the union of E_{i+1} with the conjugate of E_{i-1}.

Proof.

Let ξ_i be the small circle around y_i which is the local inverse image of χ_i. Then $P^{-1}(\Xi)$ has four nodes on ξ_i which partition it into 4 arcs, each mapping to a semicircle in χ_i.
These nodes are called upper, left, lower, right, with obvious meaning.

1) Lifting the path δ_i with initial point the root on E_i, we end up to the upper node, then lifting χ_i we end up in the lower node, finally the lifting of the inverse of δ_i gives the conjugate of the first part of the path, therefore the end point is the conjugate root of the one we started with.

2) Lifting the path δ_i with initial point the real root on the left of y_i, we end up to the left node, then lifting χ_i we end up in the right node, finally lifting the inverse of δ_i we get to the real root on the right of y_i.
In the remaining two cases the situation changes since we have to lift some semicircles to a neighbourhood of a critical point, whence the lifts will be one of the above mentioned 4 arcs around the critical points.

3) We lift the path δ_i with initial point the root on the conjugate of E_{i-1},thus we end in the left node around y_i, since when we approach y_{i-1} we have to turn right, then we proceed to the right node : when we approach y_{i+1} we have to turn left, thus we end up to the non real root in E_{i+1}.

4) The proof is similar to case 3 : if we approach y_{i-1} we have to turn right, if instead we approach y_{i+1} we have to turn left.

Q.E.D. for the Sublemma.

In order to finish the proof of theorem 2.1, we recall that the roots of our polynomial P are partitioned as follows :

a) a conjugate pair is associated to critical points which are local minima of

$|P|_{|\mathbb{R}}$,

b) a single real root is associated to a sequence of consecutive (in the source ordering) local maxima of $|P|_{|\mathbb{R}}$,

c) a single real root is associated to any local maximum of $|P|_{|\mathbb{R}}$, which is either y_1 or y_n.
Recall also that one cannot have two consecutive local minima. We associate to P the linear edge labelled tree $\mathcal{T}$ consisting of n consecutive segments with labels (from left to right) $\sigma(1), ..., \sigma(n)$.
We take a bijection of the roots of P with the vertices of $\mathcal{T}$ as follows

β) assume that y_i, y_{i+1} are local maxima for $|P|_{|\mathbb{R}}$: then to the root corresponding according to b) we associate the vertex v lying between the edges labelled $\sigma(i)$ and $\sigma(i+1)$

α) if y_i is a local minimum for $|P|_{|\mathbb{R}}$, we take any bijection between the two roots associated according to a) and the two vertices of the edge labelled $\sigma(i)$

γ) if y_1 (resp. y_n) is a local maximum for $|P|_{|\mathbb{R}}$, the root corresponding according to c) will be associated to the end of $\sigma(1)$ (resp. : $\sigma(n)$).
According to the meaning of the monodromy graph, and by sublemma 2.2 the monodromy of $\gamma_{\sigma(i)}$ is the transposition $\tau_{\sigma(i)}$ permuting the two vertices of the edge labelled $\sigma(i)$.

Q.E.D.

For the reader's convenience, we reformulate in our context the result of Davis quoted in the introduction (with essentially the same proof).

(2.3) Theorem (C.Davis, cf. [Da])

For each weak snake sequence $u_1, ..., u_n$ of real numbers, there exists exactly one maximally real Tschirnhausen polynomial, and exactly one (maximally real) normalized polynomial whose snake sequence of critical values is the given one.
In particular, if $u_1, ..., u_n$ are "lemniscate generic" (i.e., there is a permutation σ such that if $u_i = w_{\sigma(i)}$, then $0 < |w_1| < ... < |w_n|$) each linear monodromy

tree as in theorem 2.1 comes from a real polynomial.

Proof. Let us prove the assertion first in the case where we have a snake sequence (thus u_i, u_{i+1} are distinct).
A first remark is that the snake sequence associated to $P(-z)$ is the reverse of the snake sequence associated to P (that is, $u_n, ..., u_1$) whence the snake sequence is only $A^+(1, \mathbb{R})$–invariant (moreover, by corollary 1.8, every normalized polynomial is $A^+(1, \mathbb{R})$–equivalent to a unique Tschirnhausen one.
Remark that the space of normalized maximally real polynomials has two components, mapping to the space of up-down, respectively down-up sequences.
The crucial point is that a maximally real polynomial determines a natural source ordering of the critical points $y_1, ..., y_n$ thus the space of monic normalized maximally real polynomials is isomorphic to a subspace of the space of complex monic normalized polynomials taken together with an ordering of the critical points.
More precisely, we have

$$\mathcal{C} = \{(y_1, ..., y_n) | y_i \in \mathbb{R} \ , \ y_1 < y_2 ... < y_n\} \cong \mathbb{R} \times (\mathbb{R}^+)^{n-1}$$

inside $\{(y_1, ..., y_n) | y_i \in \mathbb{C}\} \cong \mathbb{C}^n$.
There is a surjective polynomial map, homogeneous of degree $(n+1)$, $\beta : \mathbb{C}^n \to \mathbb{C}^n$ associating to $(y_1, ..., y_n)$ the branch points $u_i = P_y(y_i)$ of the normalized polynomial $P_y = \int (\prod_{i=1,..n} (z - y_i))$. The claim is that β (or $-\beta$, if n is even) maps the above $\mathcal{C} = \mathbb{R} \times (\mathbb{R}^+)^{n-1}$ to the space $\mathcal{V}$ of up-down sequences, which is again $\mathbb{R} \times (\mathbb{R}^+)^{n-1}$. A first remark is that β is unramified on the open set of $\mathbb{C}^n$ where all the y_i's are distinct . This follows from Riemann's existence theorem, which shows indeed more, as follows.
If all the y_i's are distinct, once a geometric basis for $\pi_1(\mathbb{C} - \{u_1, ..., u_n\})$ is fixed, to each y_i is associated a transposition τ_i, and if $u_i = u_j$, the transpositions τ_i and τ_j are disjoint. Conversely, given $(u_1^\circ, ..., u_n^\circ)$, we can give in a continuous way, for each $(u_1, ..., u_n)$ in a neighbourhood of $(u_1^\circ, ..., u_n^\circ)$, a homomorphism of a free group in n elements to $\pi_1(\mathbb{C} - \{u_1, ..., u_n\})$ taking the i-th generator to a geometric loop around u_i.
For each choice of the τ_i's as above, we can use the monodromy determined by the products of the τ_i's, to construct a continuous family (parametrized

by the points $(u_1, ..., u_n)$) of Riemann surfaces isomorphic to $\mathbb{C}$, together with source classes of pointed polynomial maps (this means, with a choice of a fixed point over the base point for the fundamental group), and an ordering of the (distinct) critical points.

Therefore the intersection of the inverse image of the given neighbourhood U of $(u_1^\circ, ..., u_n^\circ)$ with the open set where the y_i's are distinct is homeomorphic to a product of U with a finite discrete space. A second remark is that β is closed (in fact, being homogeneous, it induces a map between the corresponding projective spaces, and we can use the compactness of projective space).

A third remark is that β is finite, as it follows from Riemann's existence theorem.

Since β is unramified on $\mathcal{C}, \beta(\mathcal{C})$ is an open set in $\mathcal{V}$. Since β is closed, the image of the closure of $\mathcal{C}$ is closed. If $\beta(\mathcal{C})$ would not be the entire $\mathcal{V}$, there would be a point in the closure of $\mathcal{C}$ mapping to the interior of $\mathcal{V}$. But this is a contradiction, since obviously if u_i, u_{i+1} are distinct, also y_i, y_{i+1} are distinct.

We have thus proven that $\beta : \mathcal{C} \to \mathcal{V}$ is unramified, surjective, closed, whence it is a covering map. Since $\mathcal{V}$ is simply connected, $\beta : \mathcal{C} \to \mathcal{V}$ is a homeomorphism.

In the general case when some u_i, u_{i+1} are not distinct, observe that since β is closed, β maps the closure of $\mathcal{C}$, $\bar{\mathcal{C}}$ to the closure $\bar{\mathcal{V}}$ of $\mathcal{V}$; thus surjectivity is proven in general. Unicity follows since $\bar{\mathcal{C}}$ maps surjectively via a proper and finite map to $\bar{\mathcal{V}}$: the general fibre is one point, thus connected, therefore any fibre is connected, thus reduced to one point.

The last assertion follows immediately from theorem 2.1.

Q.E.D.

(2.4) Definition.

Given distinct real numbers $t_1 < t_2 < ... < t_n$ we consider the number K_n of up–down sequences that can be formed out of $t_1, ..., t_n$. It is easy to see that this number is independent of the choice of $t_1, ..., t_n$. In fact there is a bijection of the above up–down sequences with the set of permutations σ of $\{1, ..., n\}$ such that $\sigma(1) > \sigma(2) < \sigma(3) > ...$, which we will call *up-down (abstract) snakes*. Similarly we can define down-up snakes, snakes, and then the number for down-up sequences formed with $t_1, ...t_n$ is equal to K_n, and the number of snakes is $2K_n$.

(2.5) Remark.
The number K_n of up-down snakes equals the number of connected components of the open set $\mathcal{W}$, in the space $\mathcal{V}$ ($\mathcal{V} \cong (\mathbf{R} \times (\mathbf{R}^+)^{n-1})$) of up–down sequences, given by the sequences $u_1, ..., u_n$ where all the u_i's are distinct.

(2.6) Main Theorem.

(a) Each connected component of $\mathcal{L}_n$ contains exactly 2^{n+1} connected components of $\mathcal{L}_{n,\mathbf{R}}$.

(b) The number of connected components of $\mathcal{L}_{n,\mathbf{R}}$ mapping to a fixed component of $Y_{n,\mathbf{R}}$ equals the number $2K_n$ of snakes, whence the number of connected components of $\mathcal{L}_n$ equals K_n.

(c) (cf. Arnold [Ar1]) the number of connected components of $\mathbb{G}\mathcal{M}_n \cap$ {monic polynomials} equals the number K_n of up-down snakes ; the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals $2K_n$.

(d) (cf. [Ar1],appendix to [C-P]) the number of components b_n of $\mathcal{L}_{n,\mathbb{R}}$ gives rise to the following exponential generating function :

$$2\Sigma_n(b_n/n!)t^n = \int 4/(1 - \sin(2t)) = 2(\sec(2t) + \tan(2t)).$$

(e) (cf. appendix to [C-P]) the number K_n of up-down snakes (which by b) equals the number of components of $\mathcal{L}_n$) is equal to the number of sequences $x_0, ..., x_{n-2}$, such that x_i is an integer with $0 \le x_i \le i$, and such that for each integer m there are at most two i's with $x_i = m$.

Proof. Recall that the number of connected components of $\mathcal{L}_n$ equals the number of connected components of the quotient $\mathcal{LH}_n$, whereas the inverse image of any connected component of $\mathcal{LH}_{n,\mathbf{R}}$ consists (cf. 1.8) of 2 connected components of $\mathcal{L}_{n,\mathbf{R}}$.

Therefore a) is an immediate consequence of lemma (1.14).

To prove b), it suffices to show that the number of connected components of $\mathcal{LH}_{n,\mathbb{R}}$ mapping to a fixed component of $Y_{n,\mathbb{R}}$ equals K_n. But this follows from the last assertion of theorem (2.3), since two snake sequences yield isomorphic linear trees (according to 2.1) if and only if they are the reverse of each other (although not needed, we recall that if $P(z)$ yields a snake sequence, $P(-z)$ yields the reverse snake sequence, and that $P(z)$ and $P(-z)$ are source equivalent).
Then the number α_n of connected components of $\mathcal{L}_n$ equals K_n since by a) the number of connected components of $\mathcal{L}_{n,\mathbb{R}}$ equals $2^{n+1}\alpha_n$, while, by what we have just seen, it equals $2K_n$ times the number of components of $Y_{n,\mathbb{R}}$, which is 2^n. To prove c), recall that Davis' theorem 2.3 shows that the map which associates to a polynomial with distinct real critical points its snake sequence of branch points yields a homeomorphism of the quotient $\mathbb{G}\mathcal{M}_n/A^+(1,\mathbb{R})$ with the space $\mathcal{W}'$ of snake sequences formed of n distinct points. Whence, the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals the number of connected components of $\mathcal{W}'$. But $\mathcal{W}'$ is just given by two disjoint copies of the open set $\mathcal{W}$ considered in remark 2.5, which has K_n components. Thus c) is proven.
d) and e) follow immediately from a), b),c) and the cited papers. To avoid confusion, we only remark that K_n is denoted by a_{n-1} in [C-P], where it is proven that $\Sigma_n(a_n/n!)t^n = 1/(1-\sin(t))$, whereas [Ar1] shows that $\Sigma_n(K_n/n!)t^n = \sec(t)+\tan(t)$: a baby calculus verification shows d).

Q.E.D.

We want now to consider, for a given choice of critical values, the lemniscate configurations that can be obtained from real polynomials. Before doing this, we recall the connection between monodromy trees and lemniscate configurations.
The big lemniscate configuration Γ_P of a polynomial $P:\mathbb{C}\to\mathbb{C}$ is the union of the preimages of 0 under P together with the singular level sets of $|P|$. Denoting by $\Delta_c = \{z\in\mathbb{C} : |P(z)| = c\}$, we have

$$\Gamma_P = P^{-1}(0) \cup \bigcup_{i=1,\dots,k} \Delta_{|w_i|}, \tag{2.7}$$

where $w_1,..w_k$ are the critical values of P. If p_i is a critical point of multiplicity m_i-1, the lemniscate $\Delta_{|w_i|}$ has a singularity consisting of m_i smooth curves intersecting with angles π/m_i. In the case $m_i = 2$ this singularity is called a *node*.

If P is lemniscate generic, let $w_1, ..., w_n$ be the critical values of P with the usual order $0 < |w_1| < ... < |w_n|$. We have a monodromy edge labelled tree once a geometric basis $\gamma_1, ..., \gamma_n$ of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ is fixed.

In [C-P] (cf. also [B-C]) it was proven that the isotopy class of the embedding of Γ_P in $\mathbb{C}$ is completely determined by a rooted (connected) tree g whose vertices correspond to the connected components of Γ_P (the root corresponds to $\Delta_{|w_n|}$), and whose edges correspond to the connected components of $\cup_{i=0,...,k}\Delta_{|w_i|+\epsilon}$ (if we set $w_0 = 0$, and we choose $\epsilon > 0$ a sufficiently small real number such that $\epsilon < |w_1|$ and $|w_i| + \epsilon < |w_{i+1}|$).

The main theorem of [C-P] would in particular describe the class of graphs obtained from lemniscate configurations and show that there is a bijection between connected components of $\mathcal{L}_n$ and the isomorphism class of such trees.

To describe abstractly the correspondence associating to an edge labelled tree $\mathcal{T}$ the associated lemniscate rooted tree g, it was convenient (cf. ibidem) to give the following

(2.8) Definition

Given an edge labelled tree $\mathcal{T}$ with n edges, the k–skeleton $\mathcal{T}_k$ of $\mathcal{T}$ is the subgraph of $\mathcal{T}$ with the same vertices and with the edges whose label is $\leq k$.

To $\mathcal{T}$ one associates a rooted graph g, whose vertices correspond to the connected components $\mathcal{C}$ of the various skeleta $\mathcal{T}_k$, with $\mathcal{T}$ corresponding to the root, and with an edge connecting $\mathcal{C}$ and $\mathcal{C}'$ if $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{C}$ is a component of $\mathcal{T}_k$, $\mathcal{C}'$ is a component of $\mathcal{T}_{k+1}$.

The k–partition $\mathcal{P}_k$ of $\mathcal{T}$ is the partition of $\{1, ..k\}$ determined by the components of $\mathcal{T}_k$ of dimension 1.

(2.9) Remark.

Given two edge labelled trees $\mathcal{T}, \mathcal{T}'$ with n edges, they determine the same lemniscate tree g if and only if they determine, for each k, the same k–partition $\mathcal{P}_k$ (the proof of this statement is phrased in slightly different terms in [C-P], pages 630-631).

We restrict from now on to linear trees $\mathcal{T}$.

(2.10) Remarks.

1) There is a natural correspondence which associates to a permutation τ the linear edge labelled tree $\mathcal{T}$ with n edges labelled $\tau(1), ..., \tau(n)$ from left to right. This correspondence induces a bijection between the set of isomorphism classes of linear edge labelled trees with n edges and the family of left cosets in the symmetric group $\mathcal{S}_n$ for the subgroup of order two generated by the reflection r sending i to $n - i$.

2) Let $w_1, ..., w_n$ be "lemniscate generic" real numbers taken with the standard order $(0 < |w_1| < ... < |w_n|)$. Then there is a permutation ψ giving the target ordering of the given numbers $(t_1 < t_2 < ... < t_n)$, and $t_i = w_{\psi(i)}$.

Notice that ψ depends only upon the sign of w_i. The condition that the monodromy associated to τ comes from a real polynomial can thus be phrased by the condition that $\sigma = \psi^{-1} \circ \tau$ *is an abstract snake* (cf. 2.4).

(2.11) Corollary.

For each sequence of critical values $w_1, ..., w_n$ such that $0 < |w_1| < ... < |w_n|$, the lemniscate configurations of real polynomials are the image of the map from the set of n-snakes to the set of those nested partitions $\mathcal{P}_1, ...\mathcal{P}_n$ coming from lemniscate trees g, which associates to a snake σ (cf. 2.4) the nested partitions corresponding as in 2.8 to $\tau = \psi \circ \sigma$ (ψ is the permutation, as in 2.10)2, comparing the standard with the target ordering) .

In the statement of 2.11 we did not bother so much about specifying the image set : the main reason for this is that we know a priori that the above map factors through the equivalence relation $\sigma \sim \sigma \circ r$, thus we can view it as a map between $\{n$–snakes modulo reflection$\} \rightarrow \{$lemniscate configurations$\}$, where we know that both sets have cardinality K_n. Thus the lack of surjectivity will be measured by the lack of injectivity (1.15 states the same principle from the opposite point of view of fixing the lemniscate configuration and asking whether all choices of signs are achieved by a real polynomial yielding the given configuration).

(2.12) Theorem.

For $n \geq 4$ and for each sequence of real numbers $w_1, ..., w_n$ such that $0 < |w_1| < ... < |w_n|$, the lemniscate configurations of real polynomials having

$w_1, ..., w_n$ as critical values are not all the possible lemniscate configurations.

Proof. It will suffice, by the remarks we have just made, to exhibit two non-isomorphic edge labelled linear trees $\mathcal{T}, \mathcal{T}'$ yielding the same nested partitions. The rest of the proof follows by several steps :

2.13) Define an *inner reflection* to be the operation associating to a linear tree f as above the tree $\mathcal{T}'$ obtained by picking up a 1-dimensional component B of the k-skeleton and reversing it (that is, if B is a segment with labels $h_1, ..., h_b$ from left to right, B^r will be the segment with labels $h_b, ..., h_1$ from left to right).
Define an inner reflection to be even iff the number b is even, odd otherwise.

2.14) an inner reflection does not affect the associated nested partitions (whence, the associate lemniscate configuration remains the same)

2.15) We claim that if $\psi^{-1} \circ \tau$ is a snake σ, applying an inner reflection we get τ' and then $\psi^{-1} \circ \tau' = \sigma'$ is a snake if and only if the reflection is odd. In fact, defining $\psi^{-1}(B)$ as the segment with labels $\psi^{-1}(h_1), ..., \psi^{-1}(h_b), \psi^{-1}(B^r) = \psi^{-1}(B)^r$, therefore if $\psi^{-1}(B)$ is a snake also $\psi^{-1}(B)^r$ is a snake. The only problem to check whether σ' is a snake comes by comparing $\psi^{-1}(h_0)$ with $\psi^{-1}(h_1)$, and $\psi^{-1}(h_b)$ with $\psi^{-1}(h_{b+1})$, where h_0, h_{b+1} are the labels respectively preceding and following B. But since B is a component of the k skeleton, h_0, h_{b+1} are $> k$, whence for instance $\psi^{-1}(h_0)$ is either bigger than all of $\psi^{-1}(h_1), ..., \psi^{-1}(h_b)$, or smaller. Thus we have a snake if and only if either $\psi^{-1}(B)$ and $\psi^{-1}(B)^r$ are both up-down, or they are both down-up. But this clearly holds if and only if b is odd.

2.16) For each j with $1 \leq j \leq n$, there exists a snake σ with $\sigma(j) = n$. In fact w_n is either the biggest or the smallest of $w_1, ..., w_n$, and it suffices to observe that for each i, given arbitrary i distinct real numbers, it is possible to form with them an up-down sequence and also a down-up sequence (this applies after dividing $w_1, ..., w_{n-1}$ in two sets of respective cardinalities $(j-1)$ and $(n-j)$).

2.17) Let j be even, take τ such that the associated snake σ has $\sigma(j) = n$, and operate on the corresponding $\mathcal{T}$ the odd inner reflection corresponding to the segment B of $\mathcal{T}_{n-1}$ lying to the left of n. The resulting $\mathcal{T}'$ is obviously non isomorphic to $\mathcal{T}$, but it yields a snake.

Q.E.D.

(2.18) Remark. For $n = 3$ and for each sequence of real numbers $w_1, ..., w_n$ such that $0 < |w_1| < ... < |w_n|$, the 2 possible configurations are achieved by real polynomials.
For $n = 4$ we get 4 out of 5 for each choice of the signs of $w_1, ..., w_n$. The missing configuration varies.
For $n = 5$ we get 11 out of 16 configurations for each choice of the signs.
For $n = 6$ there is a choice for which we get 34 configurations out of 61, and a choice for which we get 37 ones.

From 1.15 and 2.12 follows immediately the following

(2.19) Remark. For $n \geq 4$ there exist lemniscate configurations g such that the signs of real numbers $w_1, ..., w_n$ (with $0 < |w_1| < ... < |w_n|$) which are the critical values of a real polynomial yielding the given configuration g are subject to some restrictions.
For $n \leq 5$ there exist configurations g such that no such restriction occurs.

Question : for which n does there exist a configuration g for which all the possible signs for $w_1, ..., w_n$ can be realized?

(2.20) Example. Given a lemniscate generic real polynomial P, we can compose P with a real affinity in the target. Clearly, if we replace P by $aP (a \in \mathbb{R}^*)$, then aP remains lemniscate generic and with the same lemniscate configuration. If instead we replace P by $P + c$ $(c \in \mathbb{R})$, we remain in the same component of $\mathcal{M}_n$, but $P + c$ is lemniscate generic if and only if, assuming without loss of generality that the critical values of P are positive, $c \neq -w_i$, or, for $i < j, c \neq 1/2(w_i - w_j)$.
Therefore, it is easy to see that we range in $(1/2)n(n+1)+1$ distinct components of $\mathcal{L}_{n,\mathbb{R}}$, thus a natural question is whether one hits $(1/2)n(n+1)+1$ distinct components of $\mathcal{L}_n$.

The answer is negative, as it is shown by the case where $n = 4$, the critical values are 1,2,3,4, and the snake linear tree is $-1-4-2-3-$. In fact, for $c = -(2-\epsilon)$, we get the same configuration as for P.

(2.21) Definition. A real polynomial of degree $n+1$ is said to be ***totally real*** if it has $n+1$ distinct real roots. Clearly, a totally real polynomial is maximally real.

(2.22) Proposition. The space of totally real lemniscate generic polynomials of degree $n+1$ has $2(m!)^2$ components for $n = 2m$, and $2(m!)((m+1)!)$ components for $n = 2m+1$.

Proof. Remark that a lemniscate generic polynomial is totally real if and only if the associated snake of critical values $u_1, ..., u_n$ has alternating signs. Therefore the critical values $w_1, ..., w_n$ must be partitioned according to their sign into two disjoint sets of respective cardinalities $m, n-m$. It is easy now to count the number of snakes obtainable by $w_1, ..., w_n$, and we conclude by theorem 2.6.

Q.E.D.

§3. Components of the space of real generic polynomials.

We start this section by generalizing the theorem of Davis to the case of non maximally real polynomials.

We begin by setting up some notation.
Assume that P is a polynomial with k real critical points $y_1 < ... < y_k$ and m pairs $(\zeta_1, \bar{\zeta}_1)...(\zeta_m, \bar{\zeta}_m)$ of complex conjugate critical points $(n = k + 2m)$. As usual, the $A^+(1, \mathbb{R})$ source -equivalence class of P (or of $-P$) is uniquely represented by the normalized polynomial

$$(3.1) \qquad P_{y,\zeta} = \int \left(\prod_{i=1,..k} (z - y_i) \prod_{j=1,..m} (z - \zeta_j)(z - \bar{\zeta}_j) \right).$$

We consider the critical values $u_i = P_{y,\zeta}(y_i)$, which form a weak snake sequence, and the conjugate pairs of critical values $(v_j, \bar{v}_j) = (P_{y,\zeta}(\zeta_j), P_{y,\zeta}(\bar{\zeta}_j))$. Let $\mathbb{H}$ be the upper half plane in $\mathbb{C}$, and $\bar{\mathbb{H}}$ its closure. Naturally, conjugate pairs of complex numbers are parametrized by points of $\bar{\mathbb{H}}$.
Similarly to the proof of 2.3, we let

$$\mathcal{C}'' = \{(y_1, ..., y_k) | y_i \in \mathbf{R}, y_1 \leq y_2 ... \leq y_k\} \times \bar{\mathbb{H}}^m \cong \mathbf{R} \times (\mathbf{R}^{\geq 0})^{k-1} \times \bar{\mathbb{H}}^m$$

embedded inside $\mathbb{C}^n$, by associating to $(y_1, ..., y_k)(\zeta_1, ..., \zeta_m)$ the n–tuple $(y_1, ..., y_k, \zeta_1, \bar{\zeta}_1, ..., \zeta_m, \bar{\zeta}_m)$. We consider again the branch point map, the surjective polynomial map, homogeneous of degree $(n+1)$, $\beta : \mathbb{C}^n \to \mathbb{C}^n$ associating to $(y_1, ..., y_k, \zeta_1, \bar{\zeta}_1, ..., \zeta_m, \bar{\zeta}_m)$ the ordered set $(u_1, ..., u_k, v_1, \bar{v}_1, ..., v_m, \bar{v}_m)$.

Finally, we let $\beta'' : \mathcal{C}'' = \mathbf{R} \times (\mathbf{R}^{\geq 0})^{k-1} \times \bar{\mathbb{H}}^m \to \mathcal{V}'' = \mathbf{R} \times (\mathbf{R}^{\geq 0})^{k-1} \times \bar{\mathbb{H}}^m$ the composition with the projection associating to $(u_1, ..., u_k, v_1, \bar{v}_1, ..., v_m, \bar{v}_m)$ the point $[(u_1, ..., u_k), \{v_1, \bar{v}_1\}, ... \{v_m, \bar{v}_m\}]$, where we view now $(u_1, ..., u_k)$ as a point of the space $\mathcal{V}$ of weak up-down sequences (down-up if n is odd).
We have the following analogue of the theorem of C. Davis (except for unicity, which does not hold) :

(3.2) Proposition. The map $\beta'' : \mathcal{C} \to \mathcal{V}''$ is surjective.

Proof. The map $\beta'' : \mathcal{C}'' \to \mathcal{V}''$ is closed and finite and the boundary of $\mathcal{C}''$ maps to the boundary of $\mathcal{V}''$. Let $\mathcal{C}'$ be the open set in $\mathcal{C}''$ where $y_1 < y_2 ... < y_k, \zeta_j \notin \mathbf{R}$, $\zeta_i \neq \zeta_j$, for all i, j.
Define similarly $\mathcal{V}'$. Then $\mathcal{C}'' - \mathcal{C}'$ maps to $\mathcal{V}'' - \mathcal{V}'$. Moreover, as we know, β'' is unramified, whence open on $\mathcal{C}'$. If the open set $\beta''(\mathcal{C}')$ would not contain $\mathcal{V}'$, there would be a point in $\mathcal{V}'$ which belongs to $\beta''(\mathcal{C}'')$ =closure of $\beta''(\mathcal{C}')$, a contradiction again.
Therefore $\beta''(\mathcal{C}'')$ =closure of $\beta''(\mathcal{C}')$ contains the closure of $\mathcal{V}'$, that is, $\mathcal{V}''$.

Q.E.D.

(3.3) Remark. The $A^+(1, \mathbf{R})$ source -equivalence classes of generic monic real polynomials with exactly k real critical values correspond to the inverse image $\beta''^{-1}(\mathcal{V}^*)$, where $\mathcal{V}^*$ is the open set in $\mathcal{V}'$ where all the u_i's are distinct. We shall not pursue this point of view, since we shall determine the connected components of $\beta''^{-1}(\mathcal{V}^*)$ by a different method.

(3.4) Remark. A necessary condition for an algebraic function $f : C \to \mathbb{P}^1$ to be real is that the branch locus B is self conjugate (hence, the branch points will be k real critical values $w_1, ...w_k$ and m pairs $(v_1, \bar{v}_1)...(v_m, \bar{v}_m)$ of complex conjugate critical values where v_i lies in the upper half plane). If B is self conjugate, moreover, it is easy to see that f is real if and only if complex conjugation on $\mathbb{P}^1$ lifts to C. This means that complex conjugation sends the class of the monodromy μ to itself (of course we have to express both monodromies in a fixed basis of π_1).
Assuming that 0 is not a critical value, we choose a geometric basis of $\pi_1(\mathbb{P}^1 - B, 0)$ by choosing loops $\gamma_1, .., \gamma_k$ around the w_i's as in 2.2, and by choosing pairs of self conjugate loops $(\delta_j, \bar{\delta}_j)$ around the pairs $(v_j, \bar{v}_j)$.

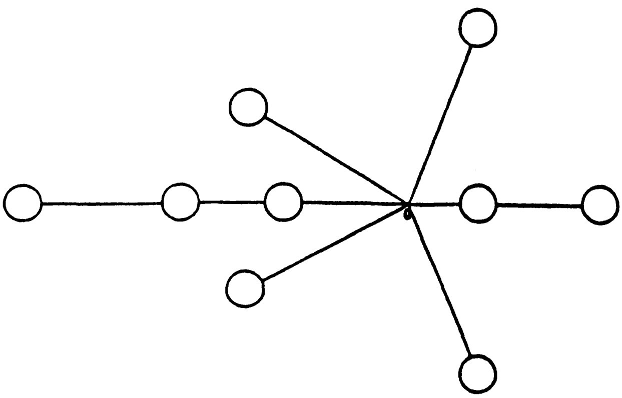

Figure 4 : choice of the canonical basis for a real polynomial

For use in the calculation, we observe that, if we separate the real branch points into the set of negative ones $w_s^- < .. < w_1^- < 0$ and the set of positive ones $0 < w_1^+ < .. < w_r^+$, we have

$$(3.5) \qquad \bar{\gamma}_i^+ = (\gamma_1^+)^{-1}(\gamma_2^+)^{-1}....(\gamma_{i-1}^+)^{-1}(\gamma_i^+)\gamma_{i-1}^+....\gamma_1^+$$

and similarly for the $\bar{\gamma}_i^-$'s. We can thus rephrase 3.4 as follows :

(3.6) μ is the monodromy of a real algebraic function if and only if there exists a permutation α of period 2 (induced by conjugation on $f^{-1}(0)$) such that, setting $\tau_i = \mu(\gamma_i^+), \tau_i' = \mu(\gamma_i^-), \nu_j = \mu(\delta_j), \nu_j' = \mu(\bar{\delta}_j), \rho_{i-1} = \tau_1^{-1}\tau_2^{-1}....\tau_{i-1}^{-1}$, and similarly ρ_{i-1}', we have :

$$\alpha\tau_i\alpha = \rho_{i-1}\tau_i\rho_{i-1}^{-1} \ , \ \alpha\tau_i'\alpha = \rho_{i-1}'\tau_i'\rho_{i-1}'^{-1} \ , \ \alpha\nu_j\alpha = \nu_j'.$$

We can now characterize the monodromies of generic real polynomials

(3.7) Theorem.

Let $w_s^- < .. < w_1^- < 0 < w_1^+ < .. < w_r^+$, be distinct real numbers $\neq 0$, and let $(v_1, \bar{v}_1)...(v_m, \bar{v}_m)$ m distinct pairs of conjugate complex numbers with v_i in the upper half plane.
Set $k = s + r, n = 2m + k, B = \{w_s^-, .., w_1^-, w_1^+, .., w_r^+\} \cup \{v_1, \bar{v}_1\} ... \cup \{v_m, \bar{v}_m\}$.

Then there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - B, 0)$ (as in 3.4), such that the edge labelled monodromy trees $\mathcal{T}$ (in E_n, and with the branch points as labels) coming from generic real polynomials are exactly those obtained as follows.
Take a snake linear edge labelled tree $\mathcal{T}'$ in E_k, having $w_s^-, .., w_1^-, w_1^+, .., w_r^+$, as labels (snake with respect to the ordering of the $w_i^{\pm}$'s in $\mathbf{R}$), and let α' be the canonical permutation on the vertices of $\mathcal{T}'$ which is the product of all the transpositions corresponding to the "local minima" edges, i.e., the edges which have a label of the same sign of its neighbours .

Then $\mathcal{T}$ is made out of a subtree isomorphic to $\mathcal{T}'$ and of the union $\mathcal{T}^*$ of an unordered pair of edge labelled graphs $\mathcal{T}_1^*, \mathcal{T}_2^*$, (simply connected but not necessarily connected), with respective labels obtained by choosing m among the labels $v_1, ..v_m \ , \bar{v}_1, ..., \bar{v}_m$, which are isomorphic under the natural isomorphism α^* which exchanges the edge v_i and the edge $\bar{v}_i$ in such a way that α^* agrees with α' on the common vertices of the subgraphs $\mathcal{T}', \mathcal{T}^*$ (thus α' and α^* together define an involution α on $\mathcal{T}$).

(3.8) Remark.

A more efficient way to label $\mathcal{T}$ is to use the target ordering for the edges of the snake linear tree (hence those labels are numbers from 1 to k), and numbers i'' for v_i, numbers i' for $\bar{v}_i$ (cf. figure 5).

Proof of theorem 3.7.

Let $\mathcal{T}$ be the monodromy tree associated to μ, let E^+ be the subgraph consisting of the edges labelled by the w_j^+'s, define E^- analogously and let finally $\mathcal{T}' = E^+ \cup E^-$.
We shall show later that $\mathcal{T}'$ is a tree.
Let moreover S_i be the the subgraph consisting of the edges labelled by the w_j^+'s with $j \leq i$ (resp. S_i' for the w_j^-'s with $j \leq i$). Define moreover, for a subgraph $\mathcal{S}$, supp($\mathcal{S}$) as the union of the vertices of $\mathcal{S}$.

By the formulae 3.6. α carries supp (S_i) into itself, and since μ is the monodromy of a polynomial, ρ_i acts on S_i as a product of cycles corresponding to the supports of the connected components of S_i. In particular, 3.6 implies that the support of the connected component of S_j containing the edge w_j^+ is sent to itself, thus by induction α leaves the support of every component of S_j invariant for each j.

Recall that the edge w_j^+ corresponds to the transposition τ_j and let $\{a, b\} =$ supp (τ_j). We have three cases :

1) $\{a, b\} \cap$ supp $(S_{j-1}) = \emptyset$,

2) $\{a, b\} \cap$ supp $(S_{j-1}) = \{b\}$

3) $\{a, b\} \subset$ supp (S_{j-1}).

Since by 3.6 we have an equality $\alpha\{a, b\} = \rho_{i-1}\{a, b\}$, in case 1) $\alpha(\{a, b\}) = \{a, b\}$, in case 2) $\alpha(a) = a, \alpha(b) = \rho_{i-1}(b) \neq b$, in case 3) a and b belong to different components of S_{j-1}, whence by our previous remark $\alpha(a) = \rho_{i-1}(a), \alpha(b) = \rho_{i-1}(b)$.
Let a be such that $\alpha(a) = c \neq a$. If j is minimum such that $a \in$ supp (S_j), we must be in case 1), and then $\alpha(a) = b = \tau_j(a)$.
Conversely, if case 1) holds, and the edge τ_j is not a component of E^+, then τ_j appears in the cycle decomposition of α (in fact, if a, b are the vertices of τ_j, we can assume then that there is a smallest $i > j$ such that a belongs to supp (S_i)), and then $\alpha(a) = \rho_{i-1}(a) = \tau_j(a) = b$.
In order to consider the case where the edge τ_j is a component of E^+ (note that the argument for E^- is completely analogous) we first prove that $\mathcal{T}'$ is

connected. In fact, we saw that α preserves the connected components of E^+, E^-,whence if A, B are two connected components of $\mathcal{T}'$, they are left invariant by α and there exists an edge $\nu_j = \mu(\delta_j)$ connecting A and B : but then also $\nu'_j = \mu(\bar{\delta}_j)$ connects A and B, contradicting the fact that $\mathcal{T}$ is a tree.

If now the edge τ_j is a component of E^+, then there exists a component A of E^- intersecting the edge τ_j in a vertex a, which must then be a fixed point for α.

The conclusion is that α acts on supp $(\mathcal{T}')$ as the product of those transpositions τ_j, τ'_j such that the edge corresponding to τ_j is a connected component of S_j but not of E^+ (similarly for τ'_j).

We prove now that $\mathcal{T}'$ is a snake linear tree.

E^+ is a union of disjoint linear trees : else, there is b belonging to edges τ_j, τ_h, τ_k, with $j < h < k$, and j, h, k minimal with this property.

But then,$\alpha(b)$ must equal $\rho_{h-1}(b)$ and $\rho_{k-1}(b)$. By our choice of k, $\rho_{k-1}(b) = \rho_{h-1}\tau_h(b)$, thus $b = \tau_h(b)$, a contradiction.

Using that the transpositions giving the cycle decomposition of α are disjoint, we immediately see that the components of E^+ are snake linear trees. In fact, if the edge τ_i intersects S_{i-1}, then either both of its vertices lie in S_{i-1}, or α fixes one of two vertices.

$\mathcal{T}'$ is linear : otherwise, since the intersection points of E^+ and E^- are left fixed by α, if a vertex a would belong to, say, two edges of E^+ and one edge of E^-, then α would act as the identity on the vertices of two adjacent edges of E^+, what is easily seen to be impossible.

Since if an edge of E^+ intersects E^- then its vertices are left fixed by α, it follows that $\mathcal{T}'$ is also snake linear.

We set then $\mathcal{T}^*$ to be the union of the edges of $\mathcal{T}$ not in $\mathcal{T}'$.

If A is a connected component of $\mathcal{T}^*$, then A intersects $\mathcal{T}'$ in a vertex a. Assume that $\alpha(A) = A$: then, since $\alpha(a) = a$ in this case, α would have a fixed edge in A, contradicting 3.6.

Therefore $\alpha(A)$ and A have disjoint edges, are clearly canonically isomorphic, and $\alpha(A)$ intersects $\mathcal{T}'$ in $\alpha(a)$.

The rest of the proof is now straightforward.

In fact, conversely, a tree $\mathcal{T}$ with the stated properties defines an involution α satisfying 3.6, and we conclude by remark 3.4. Q.E.D.

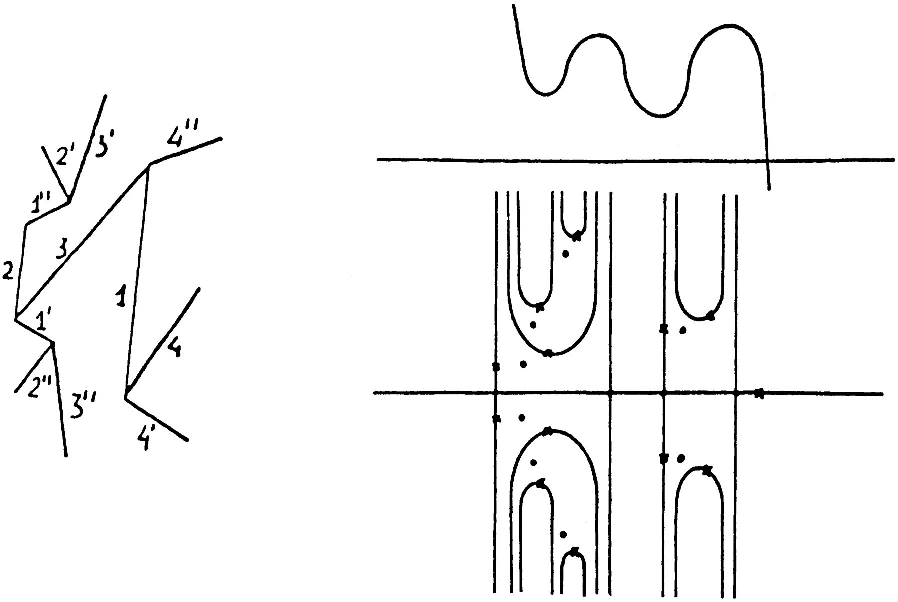

Figure 5 : A generic polynomial of degree 13, its monodromy tree , and its graph $\Theta = P^{-1}(\mathbb{P}^1_{\mathbb{R}})$.

(3.9) Remark.

A first observation is that for each generic real polynomial P, there is an equivalent polynomial $Q(z) = P(z+c)$ such that all the real critical values are positive. Therefore the connected components of the open set of real generic polynomials of degree $n+1$ with k real critical values correspond to the set of orbits of the braid group $\mathcal{B}_m(2m+k=n)$ on the isomorphism classes of edge labelled trees $\mathcal{T}$ as in theorem 3.7 (where, though, the role of $\mathcal{T}_1^*, \mathcal{T}_2^*$ cannot be interchanged). Here the braid group acts in the standard way (cf. [C-W]) on the labels i' and i'' (that is, the standard generators $\sigma_j, j = 1,..m-1$, of $\mathcal{B}_m$ act by letting ν_j become ν_{j+1}, whereas the new ν_j is the old ν_{j+1} conjugated by the old ν_j, and similarly for ν'_j, ν'_{j+1}). For each subgraph , say $\mathcal{T}_1^*$, we have two more subgraphs, $\mathcal{T}_1^{*'}, \mathcal{T}_1^{*''}$, whose connected components (which can be reduced to a vertex) correspond to the connected components of the complement of $P^{-1}(\mathbb{P}^1_{\mathbb{R}})$ which are contained in the upper half plane and map to the lower half plane (resp.: to the upper half plane). Notice that in this case the roles of $\mathcal{T}_1^*, \mathcal{T}_2^*$, are distinguished since we only look at $A^+(1,\mathbb{R})$–orbits.

The geometric picture is illustrated in figure 5.

It is clear that the action of the braid group respects the subtrees given by these connected components , and that it can transform any such tree to any other with the same number of edges.

Using the above remarks one can find , for each snake linear tree $\mathcal{T}'$ with k edges, the number of the braid group orbits on the set of trees $\mathcal{T}$ which have $\mathcal{T}'$ as the "snake" part.

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Astérisque

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Astérisque, tome 218 (1993), p. 95-109
<http://www.numdam.org/item?id=AST_1993__218__95_0>

A FEW REMARKS ON THE LIFTING PROBLEM

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0. Introduction

Let X be a reduced, non-degenerate variety of dimension n in $\mathbf{P}^r$, the projective space of dimension r over an algebraically closed field k of characteristic zero. If W is an irreducible variety of dimension n+m and degree s containing X, then for a general point t in the grassmannian Grass(h,r) of the h-planes in $\mathbf{P}^r$, with $h+n \geq r$, the corresponding h-plane L_t intersects X along a subvariety $X_t = X \cap L_t$ lying on the irreducible variety $W_t = W \cap L_t$ of dimension h+n+m-r and degree s.

Conversely, assume we have the following situation:

(0.1) Let X be a reduced, non-degenerate variety of dimension n in $\mathbf{P}^r$, let B be a smooth scheme and f: $B \rightarrow$ Grass(h,r) a dominant smooth morphism, $h+n \geq r$. For any $t \in B$ we let L_t be the h-plane corresponding to the point $f(t) \in$ Grass(h,r). Let W in $B \times \mathbf{P}^r$ be a family of projective varieties flat over B. For $t \in B$ we let W_t be the fibre of W over t. We suppose that the general fibre W_t of W is irreducible of dimension h+n+m-r and degree s, and that for $t \in B$ one has $L_t \supseteq W_t \supseteq X_t = X \cap L_t$.

In such a situation it is not true in general that there is a variety W of dimension n+m and degree s containing X and such that $W_t = W \cap L_t$ for $t \in B$: e.g. a general plane section of an irreducible curve of degree five in $\mathbf{P}^3$ lies on a conic, whereas there are such quintic curves lying in no quadrics.

The lifting problem consists in looking for suitable conditions on the variety X and the family W ensuring the existence of the variety W such that $W_t = W \cap L_t$ for $t \in B$.

* Both authors have been supported by MURST and CNR of Italy. The research has been performed in the framework of Europroj's project "Hyperplane sections".

The problem has been first considered for the case of curves in $\mathbf{P}^3$, i.e. n=1, r=3, by Laudal [5], who gave a solution later refined by Gruson and Peskine [3]. Gruson-Peskine's result asserts that if X is a reduced, irreducible curve of degree d in $\mathbf{P}^3$, whose general plane section lies on some curve Γ of degree s, and if $d>s^2+1$, then X lies on a surface of degree s whose general plane section is Γ. Curves arising from sections of a null-correlation bundle show that the bound $d>s^2+1$ is sharp (see [3], [12]). More results on the lifting problem, especially for curves in $\mathbf{P}^3$, have been found by Strano with a purely algebraic approach relating the lifting problem to the syzigies of the resolution of the ideal of X_t (see [11], [9] and [6]).

Inspired by Gruson-Peskine's result mentioned above, we will restrict ourselves to the search of a function D(s,h,r,n,m) such that, if (0.1) holds, the lifting problem has a positive answer for d>D(s,h,r,n,m). And one could be so optimistic to try to find an optimal such function, i.e. a function D(s,h,r,n,m) with the above properties and such that there are counterexamples to the lifting problem for d≤D(s,h,r,n,m), e.g. in Gruson-Peskine's case (h=2, r=3, n=m=1) the optimal function is $D(s)=s^2+1$. The question, if one puts in this form, makes sense only if dim W_t=dim X_t+1, i.e. only if m=1 (see however § 3). Consider in fact the following:

Example: Let V be a smooth projection of the Veronese surface in $\mathbf{P}^4$, which is known to be not contained in any quadric 3-fold. Let X be an irreducible curve cut out on V by a hypersurface of degree d>3. By the theorem of Bezout X does not lie on any quadric 3-fold in $\mathbf{P}^4$, whereas its general hyperplane section is contained on a quartic rational curve, hence it does lie on a quadric surface in $\mathbf{P}^3$.

Hence in the present paper we will mainly restrict our attention to the case m=1, and we will determine a function D(s,h,r,n) such that if X has dimension n and degree d>D(s,h,r,n), and if there is a family $\mathcal{W}$ as in (0.1) with m=1, then there is a variety W of dimension n+1 such that $W_t=W\cap L_t$ for $t\in U$. The proof makes use of the differential-geometric concepts of foci and of focal locus for families of projective varieties, a classical notion firstly systematised by C. Segre [10] for families of linear subspaces and recently extended in [1] to any family of projective varieties. Similar ideas are already present in implicit form in [3]. We collect in § 1 all basic facts about foci and focal loci of a family which we need in the sequel. In § 2 we show that, if the lifting problem for X and the family $\mathcal{W}$ as in (0.1) with m=1 has a negative answer, then the points of X either lie in the focal locus of $\mathcal{W}$ or X_t lies in the singular locus of W_t for t a general point in B. Then by estimating the degrees of these loci, we prove the following:

Theorem (0.2).- Let X be a reduced, non-degenerate, projective subvariety of dimension n and degree d in $\mathbf{P}^r$ and let us suppose there is a family W as in (0.1) with m=1. If

$$d>D(s,h,r,n):=(r+h-3)s+k(k-1)(r-n-1)+2ek-2$$

where $s-1=k(r-n-1)+e$, $0\leq e<r-n-1$, then the image W of W in $\mathbf{P}^r$ is a variety of dimension n+1 and degree s, containing X and such that $W_t=W\cap L_t$ for $t\in B$.

Our function D(s,h,r,n) is not optimal in general. Slight improvements can be obtained in some cases with a more detailed analysis in the same vein of our proof below: for example the case of codimension two, n=r-2, has been recently carefully investigated by E. Mezzetti [7], whose result fully generalizes Gruson-Peskine's theorem to the case $r\leq5$. She also makes a nice conjecture on the optimal function D(s,r-1,r,r-2). However we point out that, although in general not optimal, our function D(s,h,r,n) is asymptotically optimal. Indeed for instance in the case of curves we have that $D(s,r):=D(s,r-1,r,1)=[s^2/(r-2)]+o(s)$ and we find in § 3 curves X in $\mathbf{P}^r$ of degree $d=d(s)>>0$ with $d<D(s,r)$ but with $D(s,r)=d(s)+o(s)$, for which the lifting fails. These curves, as well as the curves in $\mathbf{P}^3$ achieving Gruson-Peskine's bound, are obtained as sections of suitable rank two vector bundles on certain rational normal scrolls. At the end of § 3 we will also briefly discuss an extension of theorem (0.2) to the case m>1.

In conclusion we want to mention that our approach via the focal loci has unexpected close relationships with Strano's algebraic approach mentioned above. We do not exploit this in the present paper, but we hope to come back on this subject in the future.

1. Generalities on foci.

In this section we let:

B be a non singular scheme of dimension b

W inside $B\times\mathbf{P}^r$ be a family, flat over B, of irreducible projective varieties of dimension w

V be a desingularization of W

After having shrinked B we may assume that V is flat over B, with smooth and irreducible fibres. Indeed, we may assume that for $t\in B$, the fibre V_t of $\mathsf{V}\to B$ over t is a desingularization of the corresponding fibre W_t of $\mathsf{W}\to B$.

The natural morphism u: $\mathsf{V}\to B\times\mathbf{P}^r$ yields the map of sheaves du: $T_{\mathsf{V}}\to u^*T_{B\times\mathbf{P}^r}$ which is generically injective, and therefore injective, since T_{V} is locally free. The cokernel of du is, by definition, the normal sheaf N_u to the map u, thus we have the exact sequence

(1.1) $$0\to T_{\mathsf{V}}\to u^*T_{B\times\mathbf{P}^r}\to N_u\to 0$$

and we notice that, in general, N_u is not necessarily torsion free.

We let p: $B\times\mathbf{P}^r\to B$ and q: $B\times\mathbf{P}^r\to\mathbf{P}^r$ be the projections. Then we have another natural map dq: $u^*T_{B\times\mathbf{P}^r}\to u^*q^*T_{\mathbf{P}^r}$ which is surjective. The kernel of dq is a locally free sheaf T(q) of rank b on V and we have the exact sequence

(1.2) $$0\to T(q)\to u^*T_{B\times\mathbf{P}^r}\to u^*q^*T_{\mathbf{P}^r}\to 0$$

The above sequences (1.1) and (1.2) fit into the commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & T(q) \arrow[r, "\lambda"] \arrow[d] & N_u \arrow[d, equal] & \\
0 \arrow[r] & T_V \arrow[r] \arrow[d, "\partial"'] & u^*T_{B\times\mathbf{P}^r} \arrow[r] \arrow[d] & N_u \arrow[r] & 0 \\
 & u^*q^*T_{\mathbf{P}^r} \arrow[r, equal] & u^*q^*T_{\mathbf{P}^r} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where ∂ is the differential of the map $q\circ u$, λ is the characteristic map for the family V and L is the kernel of λ.

Since we are in characteristic 0, q is smooth at the general point of W. So if we set $w_0=\dim q(W)$, then we have

$$\mathrm{rk}\ \partial=w_0,\ \mathrm{rk}\ L=\mathrm{rk}\ T_V-w_0=b+w-w_0,\ \mathrm{rk}\ \lambda=w_0-w$$

where of course $w_0-w=\dim q(W)-w\geq 0$.

Next we consider the restriction of λ to a general fibre of $V\to B$. Take $t\in B$ and let V_t be the corresponding fibre of $V\to B$. Let U be an affine open neighborhood of t in B over which T_B trivializes. Then over $p^{-1}(U)$ the map dq: $T_{B\times\mathbf{P}^r}\to q^*T_{\mathbf{P}^r}$ has a trivial kernel. Accordingly T(q) also trivializes over $V=u^{-1}p^{-1}(U)$, hence we have an isomorphism

(1.3) $$T(q)_{|V}\cong\mathcal{O}_V^{\,b}$$

Now we denote by N_t the normal sheaf to the induced map $u_t=q\circ u_{|V_t}: V_t\to\mathbf{P}^r$, and we prove the following basic:

<u>Proposition</u> (1.4).- One has $N_{u\,|\,V_t}\cong N_t$.

<u>Proof</u>. Consider the following commutative exact diagram:

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & T_{V_t} & \to & u^*q^*T_{Pr\,|\,V_t} & \to & N_t \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & T_{V\,|\,V_t} & \to & u^*T_{B\times Pr\,|\,V_t} & \to & N_{u\,|\,V_t} \to 0 \\
 & & \alpha\downarrow & & \downarrow & & \\
 & & N_{V_t,V} & \xrightarrow{\beta} & u^*N_{Pr,B\times Pr\,|\,V_t} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

Since $V \to B$ is smooth, we have $N_{V_t,V} \cong T_{B,t} \otimes \mathcal{O}_{V_t}$. Similarly one has $u^*N_{Pr,B\times Pr\,|\,V_t} \cong T_{B,t} \otimes \mathcal{O}_{V_t}$. A straightforward local computation shows now that β is in fact an isomorphism. The assertion follows then by the diagram. q.e.d.

In view of the isomorphism (1.3) and by proposition (1.4), we may interpret the restriction λ_t of the characteristic map λ to a fibre V_t as a map

$$\lambda_t: \mathcal{O}_{V_t}{}^b \to N_t$$

We notice that on a suitable dense open subset A_t of V_t the kernel of λ_t coincides with $L_t = L_{|V_t}$. Hence at a general point $p \in V_t$ we have

$$\mathrm{rk}\, \lambda_t = w_o - w$$

Futhermore if $p \in V$ is a general point, then we have

$$\dim ((q \circ u)^{-1}((q \circ u)(p))) = b + w - w_o$$

and the map

$$T_{V,p} \to u^*T_{B\times Pr,p}$$

is injective.

Now we are in position to give the following:

Definition (1.5).- A point $p \in V$ is called:

i) a focus, or a focal point, if the map

$$\lambda_p: T(q) \otimes k(p) \to N_u \otimes k(p)$$

has rank $r < w_o - w$;

ii) a fundamental point if the fibre $(q \circ u)^{-1}((q \circ u)(p))$ has dimension $\delta > b + w - w_o$;

iii) a cuspidal point, if the map

$$T_{V,p} \to u^*T_{B\times Pr,p}$$

is not injective.

The focal (resp. fundamental, cuspidal) locus is the set of all focal (resp. fundamental, cuspidal) points of V. V_t is a focal (resp. fundamental, cuspidal) fibre if it is contained in the focal (resp. fundamental, cuspidal) locus.

Remark (1.6).- i) The cuspidal locus is the set of all points $p \in V$ such that $Tor^1(N_u,k(p)) \neq 0$, hence it is the torsion locus of N_u, thus it is Zariski closed. Notice that if p is a cuspidal point, then $p'=u(p)$ is a singular point of W, otherwise N_u would be locally free at p. Accordingly p' is singular in the fibre of $W \to B$ in which it sits.

ii) The focal locus is closed off the cuspidal locus. Indeed it is then defined as the set of points where the map

$$\Lambda^\rho \lambda: \Lambda^\rho T(q) \to \Lambda^\rho N_u, \quad (\rho = \mathrm{rk}\, \lambda)$$

drops rank.

Proposition (1.7).- A fundamental point is either a focal or a cuspidal point.

Proof. Consider the commutative exact diagram

$$\begin{tikzcd} & 0 \arrow[d] & \\ & T(q)_p \arrow[r, "\lambda_p"] \arrow[d] & N_{u,p} \arrow[d, equal] & \\ T_{V,p} \arrow[r] \arrow[d, "\partial_p"'] & u^*T_{B\times Pr,p} \arrow[r] \arrow[d] & N_{u,p} \arrow[r] & 0 \\ u^*q^*T_{Pr,p} \arrow[r, equal] & u^*q^*T_{Pr,p} \arrow[d] & \\ & 0 & \end{tikzcd}$$

and set $D_p = \ker \partial_p$. By assumption, since p is a fundamental point, we have dim $D_p > b+w-w_0$. If p is not cuspidal, then the map

$$T_{V,p} \to u^*T_{B\times Pr,p}$$

is injective, hence D_p is nothing but the kernel of λ_p. q.e.d.

In the next two examples the reader will find an easy application of the above definitions and propositions and the description of a situation which shows that in general the behaviour of the focal and cuspidal loci can be rather tricky.

Example (1.8).- The classical trisecant lemma [5] says that a general chord of a reduced, non-degenerate space curve C is not a trisecant. An easy proof of this fact follows by proposition (1.7).

Let C' be the regular locus of C and let Δ be the diagonal in $C' \times C'$. We set $B = C' \times C' - \Delta$, and cosider the incidence correspondence

$$V = \{(x,y,z) \in B \times \mathbf{P}^3 : z \in \text{line joining x and y}\}$$

V is a smooth familiy of lines defined over B, and we use for it the notation introduced above. Since C is non-degenerate, we have that $q: V \to \mathbf{P}^3$ is dominant. Hence we have

$$b = 2 \quad , \quad w = 1 \quad , \quad w_0 = 3$$

thus L has rank 0, i.e. L=0. Let $t\in B$ be a general point. On the corresponding line V_t we have the map

$$\lambda_t: \mathcal{O}_{V_t}{}^2 \to N_t \cong {}_{V_t}(1)^2$$

which is given by a 2×2 matrix Λ_t of linear forms. Since the general fibre V_t is not focal because rk $\lambda_t = w_o - w = 2$, then the focal locus on V_t consists of at most two points, defined by the quadratic equation $\det(\Lambda_t)=0$, unless V_t is a focal fibre. On the other hand the points in $q^{-1}(q(V_t)\cap C')$ are clearly fundamental points of V and since V is smooth they are focal points. Hence $q(V_t)$ intersects C at two distinct points, where the intersection has to be transverse, since the tangent lines to C form a 1-dimensional system only.

Essentially the same argument can be applied more generally to control the dimension of the family of (n+2)-secant lines to a variety of dimension n in $\mathbf{P}^{n+2}$, thus proving a theorem of Z. Ran's [8], whose approach is based on differential geometric ideas which are very close to the ones we introduce in the present paper.

Note that if C is a smooth complete intersection of a quadric cone Q and of a smooth quadric, then the vertex of the cone gives rise to a fundamental point of V. So any line V_t such that $q(V_t)$ is contained in Q has at least three focal points, thus is a focal fibre. A general point on such a line is a focal point which is not a fundamental point.

Example (1.9).- We sketch now an example which shows the existence of fundamental points which are not foci and an example of a focal locus which is not Zariski closed.

Let C be a smooth conic in $\mathbf{P}^2$. Let $p_1,\dots,p_8$ be general points on C and $p_9,\dots, p_{12}$ be four more general points in $\mathbf{P}^2$. Let us consider the rational map f: $\mathbf{P}^2 \to \mathbf{P}^2$ determined by the two-dimensional linear system of all quartics through the points $p_1,\dots,p_{12}$. Consider now the smooth family p: $V \to \mathbf{P}^1$ given by the pencil of lines through $y=p_{12}$. The map f induces a map q: $V \to \mathbf{P}^2$ and accordingly a map $u=p\times q$: $V \to \mathbf{P}^1\times\mathbf{P}^2$. Since f contracts C, every point x of C, regarded as a point of the line xy of V, is a fundamental point for q: $V \to \mathbf{P}^2$. Furthermore there are two points x, x' of C such that the lines xy, x'y are tangent to C. The points x, x', regarded as points of the lines xy, x'y, hence as points of V, are clearly cuspidal point with respect to q: $V \to \mathbf{P}^2$. In view of proposition (1.7), the general point of C is a focus, but x and x' are not foci. In fact with our usual notation, we have b=w=1, $w_0=2$ but the map

$$\lambda_x: T(q)\otimes k(x) \to N_u\otimes k(x)$$

is non-zero, since $N_u\otimes k(x)$ has dimension 2, acquiring a 1-dimensional torsion summand, and the image of λ_x is exactly the torsion summand. The same holds for x'.

2. <u>Bounds for the degree of the focal locus</u>.

In this section we prove the theorem (0.2) stated in the introduction, by giving a bound for the degree of the focal locus of a family $\mathbf{W}$ as in (0.1), with m=1. The bound will be derived from Castelnuovo's bound on the genus of projective curves.

Let V be a reduced, irreducible, non-degenerate, projective subvariety of degree s and dimension n in $\mathbf{P}^r$ and let π: $V' \rightarrow V$ be a desingularization of V. We denote by S(V) the reduced variety formed by the union of all codimension one irreducible components of Sing(V) and by s(V) its degree. We let H' be the pull-back by π of a general hyperplane section of V. Then H' is smooth and irreducible by Bertini's theorem. For all divisors D of V', we define the <u>degree</u> of D to be

$$\deg(D):=D \cdot H'^{n-1}$$

In particular we have deg(H')=s. Notice that the degree can be interpreted as a homomorphism of Pic(V') in $\mathbf{Z}$.

Notice that if Y is a reduced subvariety of V of pure dimension n-1 and D is an effective divisor on V' such that $\pi(D) \cup S(V) \supseteq Y$, then $\deg(Y) \leq \deg(D)+s(V)$. Indeed if Y' is the union of all components of Y not contained in S(V), then $\deg(Y') \leq \deg(\pi(D))=\deg(D)$.

<u>Proposition</u> (2.1).- Let V be as above. Then

$$s(V) \leq [k(k-1)/2] \cdot (r-n)+ke$$

where $s-1=k(r-n)+e$, $0 \leq e < r-n$. Moreover, if K' is the canonical class of V', one has

$$\deg(K') \leq (1-n)s-2s(V)+k(k-1)(r-n)+2ek-2$$

<u>Proof</u>. A curve C' which is the pull-back via π of a general curve section C of V is smooth and irreducible. Indeed $\pi_{|C'}$: $C' \rightarrow C$ is the normalization morphism for C. We denote by g (resp. g') the arithmetic genus of C (resp. C'). Of course every point of $C \cap S(V)$ is singular for C, hence

$$0 \leq g' \leq g-s(V)$$

Since C is a non-degenerate curve in $\mathbf{P}^{r-n+1}$, Castelnuovo's bound yields

$$g \leq [k(k-1)/2] \cdot (r-n)+ke$$

whence the estimate for s(V) easily follows. Furthermore the adjunction formula yields

$$2g'-2=K' \cdot C'+(n-1)H' \cdot C'=\deg(K')+(n-1)s$$

whence the estimate for deg(K') follows. q.e.d.

<u>Remark</u> (2.2).- The first part of proposition (2.1) can be extended as follows. Let V be a reduced variety of degree s and of pure dimension $n \geq 1$ in $\mathbf{P}^r$ and let $S_i(V)$ be the Zariski closure of the locus of singular points of codimension i in V. Then

$$\deg(S_i(V)) < s^{2i}$$

In fact consider a general $\mathbf{P}^{r-n-2}$ and project V from this $\mathbf{P}^{r-n-2}$ into a $\mathbf{P}^{n+1}$ as a hypersurface V'. Take a general polar of V' in $\mathbf{P}^{n+1}$. The cone from the original $\mathbf{P}^{r-n-2}$ over V" is a hypersurface of degree s-1 passing through all singular points of V. Now it is easy to see that the cones over V from the $\mathbf{P}^{r-n-2}$'s of $\mathbf{P}^r$ cut out V set theoretically and indeed scheme theoretically along the smooth points of V. This yields that the family of hypersurfaces like V" above has no base points on V except at the singular points. Hence if we take n-1 general such hypersurfaces, their intersection with V contains a one-dimensional component C passing through all isolated singularities of V and singular there. By Fulton's version of Bezout's theorem [F, pg. 223], we have

$$\deg(C) \leq s(s-1)^{n-1} < s^n$$

On the other hand the theorem clearly holds for n=i=1, whence the assertion.

We now go back to consider our original reduced, irreducible, non-degenerate variety X of dimension n in $\mathbf{P}^r$ with the family W as in (0.1), with m=1. For this family W we use the notation we introduced in § 2, e.g. V is a desingularization of W, etc. In particular we have the morphism q: $\mathsf{W} \to \mathbf{P}^r$ and we denote by W the Zariski closure of q(W), which is an irreducible subvariety of $\mathbf{P}^r$.

<u>Proposition</u> (2.3).- One has dim W≥n+1 and if dim W=n+1, then deg(W)=s.
<u>Proof</u>. For a general h-plane L_t corresponding to a general point $t \in B$, $W \cap L_t$ is irreducible and it contains W_t which has dimension h+n-r+1. Hence clearly dim W≥n+1 and if the equality holds, then $W \cap L_t = W_t$, whence the assertion. q.e.d.

Assume now dim W ≥ n+2. Consider then a general projection π of W onto $\mathbf{P}^{n+2}$. We denote by W ' the image of W via the map $p \times (\pi \circ q)$: $\mathsf{W} \to B \times \mathbf{P}^{n+2}$. We may assume, after perhaps having shrinked B, that:

i) π: $W \to \mathbf{P}^{n+2}$ is dominant;
ii) π maps X birationally onto its image;
iii) if h≤n+2 then π restricts to an isomorphism to W_t for all $t \in B$;
iv) if h>n+2 then π restricts to a birational map to W_t for all $t \in B$; furthermore, since dim W_t=h+n-r+1≤n, then all components of $S(\pi(W_t))$ are birational projections of components of $S(W_t)$ and $s(W_t)=s(\pi(W_t))$, for all $t \in B$;
v) W '$\to$B is flat.

Then we may look at V as a desingularization of W ' and we denote by u' the obvious map $\mathsf{V} \to B \times \mathbf{P}^{n+2}$ and by q' the composition of u' with the projection onto the second factor.

<u>Proposition</u> (2.4).- Every point $x \in q'^{-1}(\pi(X))$ is a fundamental point.

Proof. Since q' is dominant, then a general fibre of q' has dimension b+h-r-1. Pick $x\in V$ such that $y=q(x)$ is a general point of X. The Shubert cycle G_y of h-planes of $\mathbf{P}^r$ containing y has codimension r-h in Grass(h,r). By the construction of W the projection on B of the fibre $q^{-1}(y)$ contains $f^{-1}(G_y)$. Hence $\dim q^{-1}(q(x))\geq b+h-r$, and then for such a $x\in V$ one a fortiori has $\dim q'^{-1}(q'(x))\geq b+h-r$, whence the assertion. q.e.d.

We are now in position to conclude the:

Proof of theorem (0.2). We keep the above notation. If dim W=n+1, we are done by proposition (2.3). Assume that dim W $\geq$n+2. Let t be a general point in B and let V_t be the corresponding fibre of $V\to B$ and let F_t be the focal locus of V_t in relation with the family W'. Since $\pi(X_t)$ has codimension one in $\pi(W_t)$, propositions (1.7) and (2.4) yield

$$q'(F_t)\cup S(\pi(W_t))\supseteq\pi(X_t)$$

so that

$$d=\deg(X)=\deg(\pi(X))=\deg(\pi(X_t))\leq\deg(q'(F_t))+s(\pi(W_t))$$

Look now at the characteristic map

$$\lambda_t\colon \mathcal{O}_{V_t}{}^b\to N_t$$

relative to the family W'. Since q' is dominant, λ_t is generically surjective. Hence, off the cuspidal locus, F_t is contained in some effective divisor D whose first Chern class is $c_1(N_t)$ defined by a non-zero section of $\mathcal{O}_{V_t}(c_1(N_t))$ given by $\Lambda^{r-h+1}\lambda_t$. One has

$$c_1(N_t)=K_t+(n+3)H_t$$

where K_t is the canonical class of V_t and H_t is the pull-back of a hyperplane of $\mathbf{P}^{n+2}$ via the map q'. Therefore

$$d\leq\deg(K_t)+(n+3)s+s(\pi(W_t))$$

Then proposition (2.1) yields

$$d\leq(r-h+3)s+k(k-1)(r-n-1)+2ek-2=D(s,h,r,n)$$

a contradiction. q.e.d.

3. Comments, examples and extensions.

In this section we collect a few remarks and an example which shows that theorem (0.2) is asymptotically sharp. At the end of the section we briefly discuss an extension to the case m$\geq$2 of theorem (0.2).

Remark (3.1).- In the case of curves n=1, one has to take h=r-1, and our function D(s,h,r,n) becomes a function $D(s,r)=[s^2/(r-2)]+o(s)$. In particular for r=3 one has

$D(s,3)=s(s+1)$, and we thus recover Laudal's theorem [5] later extended by Gruson and Peskine [3] (see the Introduction).

Example (3.2).- Let M be a smooth threefold of degree r-2 in $\mathbf{P}^r$, $r\geq 5$. Such threefolds are described in [4]: M is a scroll in planes over a rational curve and Pic(M) is freely generated by the class F of a plane and by the hyperplane class H. The canonical class of M is

$$K_M = -3H+(r-4)F$$

In what follows we will need the:

Lemma (3.3).- $h^1(\mathcal{O}_M(aH+bF))=0$ for any $a\in Z$ and for any $b\geq 0$.

Proof. It is well known that the assertion holds for b=0. We proceed by induction on b. The exact sequence

$$0\to\mathcal{O}_M(aH+bF)\to\mathcal{O}_M(aH+(b+1)F)\to\mathcal{O}_F(aH)\to 0$$

shows that $h^1(\mathcal{O}_M(aH+(b+1)F))=0$, since $h^1(\mathcal{O}_M(aH+bF))=0$ by induction and $h^1(\mathcal{O}_F(aH))=0$. q.e.d.

We will assume from now on that the class H-(r-4)F is effective on M, representing a smooth irreducible quadric surface Q inside M.

Let Y be a union of r-1 disjoint lines in M, each contained in a plane of M. We will assume Y to be general under the above conditions. We make the following:

Claim (3.4).- Let S be a surface in M containing Y then $\deg(S)\geq r-1$.

Proof of the claim. It goes by induction on r, the case r=5 being trivial. Assume $r\geq 6$ and let S be a surface of minimal degree containing Y. Then perform a projection of M from a point of one of the lines of Y to a scroll M' in $\mathbf{P}^{r-1}$. Then the remaining lines of Y are projected to a set Y' of r-2 general lines of M', contained in the projection S' of S. Then by induction $\deg(S)-1\geq\deg(S')\geq r-2$, whence $\deg(S)\geq r-1$. q.e.d.

Let ω_Y be the dualizing sheaf of Y and let I_Y the ideal sheaf of Y in M. Then

$$\mathcal{O}_Y \cong \mathcal{O}_Y(-2H-K_M-H-F) \cong \omega_Y(-H-F-K_M) \cong$$
$$\cong \mathcal{E}xt^2(\mathcal{O}_Y,\mathcal{O}_M(K_M))(-H-F-K_M) \cong \mathcal{E}xt^2(\mathcal{O}_Y,\mathcal{O}_M(-H-F)) \cong \mathcal{E}xt^1(I_Y(H+F),\mathcal{O}_M)$$

The map

$$\mathrm{Ext}^1(I_Y(H+F),\mathcal{O}_M)\to H^0(\mathcal{E}xt^1(I_Y(H+F),\mathcal{O}_M)) \cong H^0(\mathcal{O}_Y) \cong k$$

is surjective, since its cokernel sits inside

$$H^2(\mathcal{H}om(I_Y(H+F),\mathcal{O}_M)) \cong H^2(\mathcal{O}_M(-H-F)) = 0$$

So a constant in $k \cong H^0(\mathcal{O}_Y) \cong H^0(\mathcal{E}xt^1(I_Y(H+F),\mathcal{O}_M))$ lifts to an extension

$$0\to\mathcal{O}_M\to E\to I_Y(H+F)\to 0$$

with E locally free of rank two: indeed $\mathcal{E}xt^1(E,\mathcal{O}_M)$ turns out to be zero since $\mathcal{E}xt^1(\mathcal{O}_M,\mathcal{O}_M)$ is such and $\mathcal{O}_M \cong \mathcal{H}om(\mathcal{O}_M,\mathcal{O}_M) \to \mathcal{E}xt^1(\mathcal{I}_Y(H+F),\mathcal{O}_M) \cong \mathcal{O}_Y$ is surjective by construction. Furthermore $c_1(E)=H+F$ and $c_2(E) = \deg(Y)=r-1$. Let X be a curve which is the 0-locus of some section of E(aH), for a>>0, which is clearly non-degenerate. We may also assume X to be smooth and irreducible. Its degree is

$$d=\deg(X)=H\cdot c_2(E(aH))=a^2(r-2)+(a+1)(r-1)$$

Let X_o be a general hyperplane section of X.

<u>Claim</u> (3.5).- X_o is contained in some curve of degree $s=(a+1)(r-2)$.
<u>Proof of the claim</u>. In fact X_o sits on the general hyperplane section M_o of M, and we have the exact sequence

$$0\to\mathcal{O}_{M_o}\to E_o\to\mathcal{I}_{Y_o}(H+F)\to 0$$

where Y_o is the general hyperplane section of Y and E_o is $E_{|M_o}$. Note that Y_o consists of r-1 points in $\mathbf{P}^{r-1}$, hence Y_o is degenerate, i.e. $h^0(\mathcal{I}_{Y_o}(H))\neq 0$. Hence $h^0(E_o(-F))\neq 0$ since $h^1(\mathcal{O}_{M_o}(-F)))=0$. From the exact sequence

$$0\to\mathcal{O}_{M_o}\to E_o(aH)\to\mathcal{I}_{X_o}((2a+1)H+F)\to 0$$

we have $h^0(\mathcal{I}_{X_o}((a+1)H))\neq 0$ proving the claim. q.e.d.

We remark now that $h^0(E)\neq 0$ yields $h^0(\mathcal{I}_X((a+1)H+F))\neq 0$, hence X lies on surfaces of degree $(a+1)(r-2)+1$. On the other hand we make the following:

<u>Claim</u> (3.6).- If $r\geq 7$ then X is not contained on any surface of degree $\sigma\leq(a+1)(r-2)$.
<u>Proof of the claim</u>. We argue by contradiction. Let S be such a surface and assume it has minimal degree, so that it is reduced and irreducible. By the theorem of Bezout S has to lie on M since a>>0, hence $h^0(\mathcal{I}_X(S))\neq 0$.

Notice that $S-F-(a+1)H$ has negative degree, hence $h^0(\mathcal{O}_M(S-F-(a+1)H))=0$. Thus if $h^0(E(S-F-(a+1)H))\neq 0$, then $h^0(\mathcal{I}_Y(S-aH))\neq 0$ contradicting the claim (3.4), since $\deg(S-aH)=\sigma-a(r-2)\leq r-2$. Hence we have $h^0(E(S-F-(a+1)H))=0$ which yields $h^1(\mathcal{O}_M(S-F-(2a+1)H))\neq 0$. Let $S=\alpha H+\beta F$ in Pic(M) hence $S-F-(2a+1)H=(\alpha-2a-1)H+(\beta-1)F$. By lemma (3.3) we must have $\beta\leq 0$. Then by the Kodaira vanishing theorem we must have $\alpha\geq 2a+1$.

Remark now that

$$S\cdot Q\cdot H=(\alpha H^2+\beta F\cdot H)\cdot(H-(r-4)F)=2\alpha+\beta$$

Since S and Q are irreducible and distinct, it is clear that $2\alpha+\beta\geq 0$. But then, since

$$\deg(S)=\alpha(r-2)+\beta\leq(a+1)(r-2)$$

we should have

$$(2a+1)(r-4)\leq\alpha(r-4)\leq(a+1)(r-2)$$

i.e. $r\leq 6$, a contradiction. q.e.d.

Finally we notice that we have

$$s-1=a(r-2)+(r-3)$$

hence with the notation of theorem (0.2), we have $k=a$, $e=r-3$. Hence

$$D(s,r)=(r-2)(a^2+5a+4)-2(a+1)$$

and therefore

(3.7) $$D(s,r)-d=(r-2)(5a+4)-(a+1)(r+1)=o(a)=o(s)$$

If $r\geq 7$ then, according to theorem (0,2), we have $D(s,r)>d$, but (3.7) shows that indeed the optimal function differs from our $D(s,r)$ by a function $\delta(s,r)=o(s)$.

Remark (3.8).- In the proof of theorem (0.2) an important role is played by the hypothesis that the general fibre W_t of $\mathcal{W}$ is irreducible. Sometimes this assumption can be replaced by the assumption that X itself is irreducible. For example if $n+m=r-1$, i.e. W_t is a hypersurface in L_t and s is the minimal degree of such a hypersurface containing X_t, then W_t is clearly irreducible if $n\geq 2$, and the same happens by monodromy if $n=1$.

Remark (3.9).- Let us go back to the proof of theorem (0.2). One of the main points there is the fact that the focal locus F_t is contained in some effective divisor D whose first Chern class is $c_1(N_t)$. This follows from the consideration of the map of generically maximal rank

$$\lambda_t: \mathcal{O}_{V_t}{}^b \to N_t$$

where $b=\dim B\geq \dim \mathrm{Grass}(h,r)=(h+1)(r-h)>\mathrm{rk}\, N_t=r-h+1$. Let us then consider the map

$$\Lambda^{r-h+1}\lambda_t: \Lambda^{r-h+1}\mathcal{O}_{V_t}{}^b \to \det(N_t)$$

whose image gives rise to a linear system δ of divisors in the linear system $|\mathcal{O}_{V_t}(\det(N_t))|$, the so called focal linear system introduced in [7]. If $\dim \delta$ is sufficiently large, then one has better estimates for the degree of the focal locus, thus improving the estimate for the function $D(s,h,r,n)$. This idea, exploited in [7], is very useful in the case $n=r-2$. However it does not seem equally useful in the case of varieties of high codimension, in particular for curves.

Remark (3.10).- Let we weaken the hypotheses in (0.1) in the following way: $f: B\to \mathrm{Grass}(h,r)$ is no more necessarily dominant, but the union of the h-planes parametrized by the points of $f(B)$ is dense in $\mathbf{P}^r$ and $b\geq r-h+1$. Then the map λ_t is still generically surjective and the proof of theorem (0.2) still goes through, except that proposition (2.3) could fail to hold.

For instance let us take for X the disjoint union of three lines on a smooth quadric surface W in $\mathbf{P}^3$ and let us take for B the set of all tangent planes to W. For all $t\in B$ the corresponding plane L_t meets X at three points on a line W_t and these lines form a two-dimensional flat family $\mathcal{W}$ verifying the assumptions of (0.1), modified as above. Of course proposition (2.3) does not hold for such a family, inasmuch as for a general point $t\in B$, the corresponding plane L_t is such that $W\cap L_t$ is reducible.

In order to let theorem (0.2) still work if f: $B\rightarrow Grass(h,r)$ is not dominant we must therefore make the following assumptions:

i) the union of the h-planes parametrized by the points of f(B) is dense in $\mathbf{P}^r$ and $b\geq r-h+1$;

ii) for a general point $t\in B$, the corresponding plane L_t is such that $W\cap L_t$ is irreducible, where as usual W is the Zariski closure of q(W).

Condition ii) is rather unpleasant. However it is automatically verified if, for instance, f(B) is dense in some Schubert cycle.

In conclusion we want to briefly point out the following extension of theorem (0.2) to the case $m\geq 2$:

Proposition (3.11).- Let X be a reduced, irreducible, non-degenerate variety of dimension $n\geq 2$ in $\mathbf{P}^r$, and suppose we have a situation like in (0.1) with $h+n>r$. Suppose that $d>(2rs)^{2^m}$. Then there is a variety Y containing X, with dim Y=n+m-i and $\deg(Y)\leq(2rs)^{2^i}$ such that for $t\in B$ general, one has $L_t\supseteq W_t\supseteq Y_t\supseteq X_t$.

Proof. We proceed by induction on m. The case m=1 follows by theorem (0.2). Let $m\geq 2$. Now we use the notation introduced in § 2. As in proposition (2.3) we see that dim $W\geq n+m$ and if the eqauality holds then deg(W)=s. So we may assume dim $W\geq n+m+1$ and we make a general projection π to $\mathbf{P}^{n+m+1}$. The statement of proposition (2.4) still holds. Hence as in the proof of theorem (0.1) we have

$$q'(F_t)\cup \mathrm{Sing}(\pi(W_t))\supseteq\pi(X_t)$$

Since dim $X\geq 2$ and therefore X_t is irreducible, we have either $\mathrm{Sing}(\pi(W_t))\supseteq\pi(X_t)$ or $q'(F_t)\supseteq\pi(X_t)$. Now we claim that in either case $\pi(X_t)$ is contained in some irreducible subvariety of $\pi(W_t)$ of codimension one and of "low" degree. In fact in the first case one can prove, with an argument already used in remark (2.2), that $(\mathrm{Sing}(\pi(W_t))$ is certainly contained in some hypersurface of degree s-1 not containing $\pi(W_t)$. In the latter case we notice that the map λ_t: $\mathcal{O}_{V_t}{}^b\rightarrow N_t$ relative to the family W ' is generically of maximal rank. Hence we can consider the focal linear system inside $|\mathcal{O}_{V_t}(\det(N_t))|=|K_{V_t}+(n+m+2)H_{V_t}|$ (see remark (3.9)), and by proposition (2.1) we have

$$\deg(K_{V_t}+(n+m+2)H_{V_t})\leq 2rs^2$$

Now, after may be a base change, we have a new family of varieties verifying (0.1) with m replaced by m-1. Futrthermore since

$$(2rs^2)^{2^{m-1}}<(2rs)^{2^m}<d$$

by induction we have that there is a variety Y containing X, with dim Y=n+m-1-i and

$$\deg(Y)\leq[2r(2rs^2)]^{2^i}=(2rs)^{2^{i+1}}$$

such that for $t \in B$ general, one has $L_t \supseteq W_t \supseteq Y_t \supseteq X_t$. This proves our assertion. q.e.d.

It is useless to say that the hypothesis $d > (2rs)^{2^m}$ is very rough and could be refined as well as the bound for the degree of Y. It is also possible that the hypothesis dim $X \geq 2$, which we introduced for technical reasons, could be dropped.

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Astérisque

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Schur quadrics, cubic surfaces and rank 2 vector bundles over the projective plane

Astérisque, tome 218 (1993), p. 111-144

<http://www.numdam.org/item?id=AST_1993__218__111_0>

SCHUR QUADRICS, CUBIC SURFACES AND RANK 2 VECTOR BUNDLES OVER THE PROJECTIVE PLANE

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Let $\Sigma \subset P^3$ be a smooth cubic surface. It is known that S contains 27 lines. Out of these lines one can form 36 *Schläfli double - sixes* i.e., collections $\{l_1, ..., l_6\}, \{l'_1, ..., l'_6\}$ of 12 lines such that each l_i meets only l'_j, $j \neq i$ and does not meet $l_j, j \neq i$, see n.0.1 below. In 1881 F. Schur proved **[S]** that any double - six gives rise to a certain quadric Q , called *Schur quadric* which is characterized as follows: for any i the lines l_i and l'_i are orthogonal with respect to (the quadratic form defining) Q.

The aim of the present paper is to relate Schur's construction to the theory of vector bundles on P^2 and to generalize this construction along the lines of the said theory.

Let us describe the vector bundle interpretation of the Schur quadric. Note that the first six lines $\{l_1, ..., l_6\}$ of a double - six on Σ define a blow-down $\pi : \Sigma \to P^2$ which takes the lines l_i into some points $p_i \in P^2$. These points are in general position i.e. no three of them lie on a line. Let $\check{P}^2$ be the dual projective plane and $H_i \subset \check{P}^2$ be the lines corresponding to p_i. The union $\mathcal{H}$ of these lines is a divisor with normal crossing in $\check{P}^2$. Let $E(\mathcal{H}) = \Omega^1_{\check{P}_2}(\log \mathcal{H})$ be the corresponding vector bundle (locally free sheaf) of logarithmic 1-forms on $\check{P}^2$. The twisted bundle $E = E(\mathcal{H})(-2)$ is a stable rank 2 bundle on $\check{P}^2$ with Chern classes $c_1 = -1, c_2 = 4$ (see **[DK]**). For such bundles K.Hulek **[Hu1]** has defined the notion of a *jumping line of the second kind* (shortly JLSK). This is a line $l \subset \check{P}^2$ such that the restriction of E to the first infinitesimal neigborhood $l^{(1)}$ of l is not isomorphic to $\mathcal{O}_{l^{(1)}} \oplus \mathcal{O}_{l^{(1)}}(-1)$. Hulek has shown that such lines form a

* Research of both authors supported in part by NSF.

curve $C(E)$ in the projective plane of lines in $\check{P}^2$ i.e. in P^2. Now the result is as follows.

Theorem 1. *The space P^3 containing the cubic surface Σ is naturally identified with the projectivization of $H^1(\check{P}^2, E(-1))^*$. Under this identification the Schur quadric Q becomes dual to the zero locus of the quadratic form given by the cup-product*

$$H^1(\check{P}^2, E(-1)) \otimes H^1(\check{P}^2, E(-1)) \to H^2(\check{P}^2, \bigwedge^2(E(-1))) = H^2(\check{P}^2, \mathcal{O}(-3)) = \mathbf{C}.$$

The intersection $\Sigma \cap Q$ is mapped, under the projection $\pi : \Sigma \to P^2$, to the curve of JLSK $C(E)$.

More generally, the whole theory of Hulek **[Hul]** of rank 2 vector bundles on P^2 with odd c_1 can be given a "geometric" interpretation involving some natural generalizations of cubic surfaces, double - sixes and Schur quadrics. This is done in §2 of the paper. This interpretation implies Theorem 1.

The outline of the paper is as follows. In §0 we recall some known (and less known) facts about cubic surfaces and Schur quadrics. In §1 we give a short overview of Hulek's theory of monads corresponding to vector bundles with $c_1 = -1$. In §2 we give an interpretation of Hulek's theory mentioned above. In §3 we consider bundles of logarithmic 1-forms corresponding to arrangements of $2d$ lines in P^2 in general position. The main result of this section is that all these bundles satisfy certain condition of Σ - genericity in the sense defined in §2, which makes working with bundles satisfying this condition easier. Finally, in §4 we consider various examples of the previous constructions corresponding to some special types of vector bundles.

§0. Cubic surfaces.

0.1. Here we recall some standard known facts about cubic surfaces. All the proofs can be found either in **[H]**, Ch.V, §4 or in **[M]** or can be easily reconstructed by the reader. Let $p_1, ..., p_6$ be six distinct points in the projective plane P^2. Assume that no three of these points lie on a line. Denote by Z the union of the points p_i and by $\mathcal{J}_Z \subset \mathcal{O}_{P(V)}$ the sheaf of ideals of Z. The linear system $P\big(H^0(\mathcal{J}_Z(3))\big)$ of cubic curves through Z is of dimension 3 and defines a rational map

$$f : P^2 \to P(H^0(\mathcal{J}_Z(3)^*) = P^3$$

whose image is a cubic surface, denoted Σ. The rational map f comes from a regular map $f' : \mathrm{Bl}_Z(P^2) \to P^3$ where $\mathrm{Bl}_Z(P^2)$ is the blow up of Z. Let $\pi : \mathrm{Bl}_Z(P^2) \to P^2$ be the projection. If we further assume that the points p_i do not lie on a conic then f' is an isomorphism and Σ is nonsingular. If p_i do lie on a conic then Σ is singular and f' blows down this conic to a singular point of Σ.

Suppose Σ is nonsingular. Then Σ has 27 lines on it. They can be grouped into three subsets:

$$\{l_1, ..., l_6\},\ \{l'_1, ..., l'_6\},\ \{m_{ij},\ 1 \le i < j \le 6\}. \tag{0.1}$$

The lines l_i are the images under f' of the exceptional lines $\pi^{-1}(p_i)$. The lines l'_i are images under f' of proper transforms of the conics $C_i \subset P^2$ passing through $Z - \{p_i\}$. Finally the lines m_{ij} are images of the proper transforms of the lines $< p_i, p_j >$ joining the points p_i and p_j.

The first two groups of lines form a *double - six* which means that

$$l_i \cap l_j = \emptyset, \quad l'_i \cap l'_j = \emptyset, \quad l_i \cap l'_j \neq \emptyset \qquad \text{iff} \quad i \neq j. \tag{0.2}$$

Every set of 6 disjoint lines on Σ can be included in a unique double - six from which Σ can be reconstructed uniquely. There are 36 double - sixes of Σ. Every double - six defines two regular birational maps $\pi_1 : \Sigma \to P^2$, $\pi_2 : \Sigma \to P^2$, each blowing down one of the two sixes (sixtuples of disjoint lines) of the double - six.

The birational map $\pi_2 \circ \pi_1^{-1} : P^2 \to P^2$ is given by the linear system of quintics with double points at p_i. The two collections of 6 points in P^2 given by $\{\pi_1(l_i)\}$ and $\{\pi_2(l'_i)\}$ are associated to each other in the sense of Coble (cf.**[DO]**,**[DK]**).

0.2. Here we shall discuss somewhat less known facts about the determinantal representation of a cubic surface **[B]**. A modern treatment of this can be found in **[G]**,**[Gi]**. Consider the homogeneous ideal of the subscheme Z i.e.

$$I_Z = \bigoplus_{n \geq 0} H^0(P^2, \mathcal{J}_Z(n)) \tag{0.3}$$

in the graded ring $R = \mathbf{C}\,[T_0, T_1, T_2]$. It is easy to see that the ring R/I_Z is Cohen - Macaulay hence of homological dimension 1. Any four linearly independent cubic forms vanishing on Z represent a minimal set of generators of I_Z. According to the Hilbert-Burch theorem (see **[No]**,7.5) the ideal I_Z is generated by the maximal minors of some 3×4 matrix of homogeneous linear forms. In other words, we have a resolution

$$0 \to R(-4)^3 \to R(-3)^4 \to I_Z \to 0.$$

This resolution gives the resolution of the sheaf $\mathcal{J}_Z(3)$:

$$0 \to \mathcal{O}_{P(V)}(-1)^3 \to \mathcal{O}^4_{P(V)} \to \mathcal{J}_Z(3) \to 0.$$

We can rewrite this resolution in the form

$$0 \to \mathcal{O}_{P^2}(-1) \otimes I^* \xrightarrow{\gamma} \mathcal{O}_{P^2} \otimes L^* \to \mathcal{J}_Z(3) \to 0 \tag{0.4}$$

where vector spaces I^* and L^* of respective dimensions 3 and 4 are defined intrinsically as follows:

$$L^* = H^0(P^2, \mathcal{J}_Z(3)); \tag{0.5}$$

$$I^* = \mathrm{Ker}\{H^0(P^2, \mathcal{O}(1) \otimes L^*) \to H^0(P^2, \mathcal{J}_Z(4))\}. \tag{0.6}$$

Note that one can also obtain (0.4) from the Beilinson spectral sequence applied to the sheaf $\mathcal{J}_Z(3)$. It gives also an isomorphism

$$I^* \cong H^1(P^2, \mathcal{J}_Z(1)).$$

It will be convenient for us to regard henceforth our projective plane P^2 as $P(V^*)$ where V is a 3-dimensional vector space. With this choice of notation, the map γ in (0.4) is given by a linear map $I^* \otimes V^* \to L^*$. We shall be more interested in the transpose of this map which we denote by

$$g : L \longrightarrow I \otimes V = \mathrm{Hom}(V^*, I). \tag{0.7}$$

Choosing bases in V, I we can regard g as a 3 by 3 matrix of linear forms on L. Here is the classical result on the determinantal representation.

0.3. Proposition. *The map g is an embedding. The locus*

$$\Sigma = \{x \in P(L) : \ \text{rank } g(x) \le 2\} \tag{0.8}$$

is a nonsingular cubic surface in $P(L) = P^3$ isomorphic to $\mathrm{Bl}_Z(P(V^*)$. *An explicit blow-down $\pi_1 : \Sigma \to P(V^*)$ takes $x \in \Sigma$ into* Ker $g(x) \in P(V^*)$. *It is isomorphism outside the set $Z = \{p_1, ., p_6\} \subset P(V^*) = P^2$ (see n. 0.1). The dual blow-down $\pi_2 : \Sigma \to P(I^*)$ takes $x \in \Sigma$ into* $(\mathrm{Im} g(x))^\perp \in P(I^*)$. *It is an isomorphism outside a six - element set $Z^{as} = \{q_1, ..., q_6\} \subset P(I^*)$ (this is the set associated to Z).*

Note that a given cubic surface $\Sigma \subset P^3$ has many non-equivalent determinantal representations corresponding to different ways of blowing down Σ onto a P^2 (i.e. to different choices of a double - six).

0.4. All the other attributes of the cubic surface Σ can be easily found from the map g. For example, the set Z can be recovered in terms of g as follows. Consider the partial transposes of (0.7):

$$g_V : V^* \to I \otimes L^* = \mathrm{Hom}(L, I).$$

$$g_I : I^* \to V \otimes L^* = \mathrm{Hom}(L, V).$$

Then

$$Z = \{z \in P(V^*) : \ \text{rank } g_V(z) \le 2\}. \tag{0.9}$$

The 12 lines of the double - six can be written in the form $A_z = P(\mathbf{A}_z), A'_z = P(\mathbf{A}'_z)$, $z \in Z$ where $\mathbf{A}_z$ and $\mathbf{A}'_z$ are 2-dimensional vector subspaces in L defined for $z \in Z$ as follows:

$$\mathbf{A}_z = Ker(g_V(z)); \tag{0.10}$$

$$\mathbf{A}'_z = Ker(g_I(z)), \tag{0.11}$$

where $z^\perp \subset V$ is the 2-plane orthogonal to $z \in P(V^*)$. Thus if $Z = \{p_1, ..., p_6\}$ then the line A_{p_i} is what was denoted in n.0.1 by l_i and the line A'_{p_i} is l'_i.

The classical theorem of F. Schur **[S]** can be stated as follows.

0.5. Theorem. *There exists a unique, up to a scalar factor, symmetric bilinear form $C(x,y)$ on L with the following property: $C(x,y) = 0$ whenever $x \in A_z, y \in A'_z$ for some $z \in Z$ (i.e. the corresponding lines of the double - six are orthogonal with respect to C). This form is non-degenerate.*

Proof. a) Non-degeneracy: Suppose such a form C exists and is degenerate. Let K be the kernel of C. Suppose dim $K = 1$. Then for any 2-dimensional subspace $\Lambda \subset L$ not meeting K its orthogonal (with respect to C) is a 2-subspace containing K. Since $P(A_z)$, $P(A'_z)$ form a double - six, K can lie on no more than one among the A_z and no more than one among the A'_z. Hence there is a 4-element subset $Z_0 \subset Z$ such that for $z \in Z_0$ both A_z and A'_z do not contain K. For such z the space A'_z should coincide with $A_z^\perp$ and hence contain K. Hence for $z_1 \neq z_2 \in Z_0$ we have $A'_{z_1} \cap A'_{z_2} \neq \{0\}$ which is a contradiction. The cases dim $K = 2,3$ are similar and left to the reader.

b) Uniqueness: If there are two non-proportional forms C_1, C_2 with the required property then for any λ, μ the linear combination $\lambda C_1 + \mu C_2$ also satisfies this property. However, there will be always such λ, μ that the linear combination is non-zero but degenerate. This contradicts a).

0.6. It remains to prove the existence part of Theorem 0.5. To do this, let us take the second symmetric power of the map g in (0.7) and use the natural decomposition

$$S^2(I \otimes V) = \left(\bigwedge^2 I \otimes \bigwedge^2 V\right) \quad \oplus \quad (S^2 I \otimes S^2 V). \tag{0.12}$$

By projecting $S^2 g$ to the first summand, we get a linear map

$$S^2 L \longrightarrow S^2(I \otimes V) \longrightarrow \left(\bigwedge^2 I \otimes \bigwedge^2 V\right). \tag{0.13}$$

Note that dim $S^2 L = 10$, and $\dim(\bigwedge^2 I \otimes \bigwedge^2 V) = 9$. Hence the map (0.13) has non-trivial kernel. (We shall see later that this kernel is in fact 1-dimensional).

0.7. Proposition. *If B is a non-zero form from the kernel of (0.13) then $B : L^* \to L$ is invertible and $C = B^{-1} : L \to L^*$ is a bilinear form on L satisfying the conditions of Theorem 0.5.*

We shall concentrate on the proof of this proposition.

0.8. A form $B \in S^2L$ lying in the kernel of (0.13) is classically called "apolar to all the quadratic forms given by 2×2 minors of g", cf. **[B]**. In general, if E is a vector space then quadratic forms $G \in S^2E$, $H \in S^2E^*$ are called *apolar* if $(G, H)_2 = 0$ where $(\cdot, \cdot)_2$ is the natural pairing $S^2E \otimes S^2E^* \to \mathbf{C}$. Note the particular case when G has rank 2 i.e. $G = e \cdot f$ is the symmetric product of two vectors $e, f \in E$. In this case the apolarity of G and H means that $H(e, f) = 0$.

We shall need a different description of the map dual to (0.13). Let us denote this map by

$$\delta : \bigwedge^2 I^* \otimes \bigwedge^2 V^* \longrightarrow S^2L^*. \tag{0.14}$$

Let us chose volume forms on V and I. Then we can write $\bigwedge^2 V^* = V$, $\bigwedge^2 I^* = I$. It is immediate to see that there are identifications

$$\bigwedge^2 V^* \cong V \quad = \quad H^0(P(V^*), \mathcal{O}_{P(V^*)}(1)); \tag{0.15}$$

$$\bigwedge^2 I^* \cong I \quad = \quad H^0(P(V^*), \mathcal{J}_Z^2(5)); \tag{0.16}$$

$$S^2L^* \quad = \quad H^0(P(L), \mathcal{O}(2)) \quad \cong \quad H^0(P(V^*), \mathcal{J}_Z^2(6)). \tag{0.17}$$

Indeed, (0.15) follows by definition of $\mathcal{O}(1)$; the identification (0.16) expresses the fact that the Cremona transformation $\pi_2 \circ \pi_1^{-1} : P(V^*) \to P(I^*)$ is given by the linear system of quintics with singular points p_i, see n. 0.1. Finally, to see (0.17) we note that the embedding of the cubic surface Σ into $P(L) = P^3$ is given by the linear system of cubics in $P(V^*)$ through p_i, so L^* is the space of cubic polynomials on V^* vanishing at p_i. The second symmetric power of this space maps therefore to the space of polynomials of degree 6 vanishing at p_i together with their first derivatives i.e, to the RHS of (0.17); this map is easily seen to be an isomorphism.

0.9. Lemma. *Under identifications (0.15) - (0.17) the map δ corresponds to the multiplication map*

$$H^0(P(V^*), \mathcal{O}(1)) \otimes H^0(P(V^*), \mathcal{J}_Z^2(5)) \to H^0(P(V^*), \mathcal{J}_Z^2(6)).$$

In other words, quadrics in $P(L)$ are identified with sextics in $P(V^*)$ with double points at $p_i \in Z$ and quadrics from the image of δ correspond to sextics containing a line.

Proof of the lemma: We have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{Sq}} & S^2 L \\ g\downarrow & & \downarrow \\ \mathrm{Hom}(V^*, I) & \xrightarrow{\lambda} & \mathrm{Hom}(\bigwedge^2 V^*, \bigwedge^2 I) \end{array} \tag{0.18}$$

where the map Sq takes $x \mapsto x^2$, the map λ takes $\phi \mapsto \bigwedge^2 \phi$ and the map on the right is the same as in (0.13). We keep the volume forms in I and V and identify correspondingly the spaces $\bigwedge^2 I$ and $\bigwedge^2 V$ with V^* and I^*. For any $\phi \in \mathrm{Hom}(V^*, I)$ of rank 2 the second exterior power $\bigwedge^2 \phi \in I^* \otimes V^*$ is a tensor of rank 1. Hence it can be written in the form $i^* \otimes v^*$ for some $i^* \in I^*$, $v^* \in V^*$. This shows that the restriction of the map $\lambda \circ g$ to the cubic surface $\Sigma \subset P(L)$ coincides with the composition

$$\Sigma \xrightarrow{\pi_1 \times \pi_2} P(I^*) \times P(V^*) \xrightarrow{\mathrm{Segre}} P(I^* \otimes V^*) \tag{0.19}$$

where π_j are the blow-downs from n. 0.3.

The map $\lambda \circ g : P(L) \to P(I^* \otimes V^*)$ is given by the linear system $\mathcal{Q}$ of quadrics which is the projectivization of the image of the linear map δ from (0.14). The system $\mathcal{Q}$ is spanned by the 2×2 minors of the matrix of linear forms on L defining the determinantal representation of Σ. In other words, the preimage of the linear system of hyperplane sections of $P(I^* \otimes V^*)$ under $\lambda \circ g$ is the linear system of quadric sections on Σ which is (the projectivization of) the image of the canonical pairing

$$H^0(P(I^*), \mathcal{O}(1)) \otimes H^0(P(V^*), \mathcal{O}(1)) \longrightarrow H^0(\Sigma, \mathcal{O}(2)).$$

By Theorem 0.3, we can make an identification of the projective spaces $P(I^*)$ and

$P(H^0(P(V^*), \mathcal{J}_Z^2(5))^*)$. Under the rational map $P(V^*) \to P(I^*)$ given by the linear system of sections of $\mathcal{J}_Z^2(5)$, zeroes of these sections are preimages of the lines in $P(I^*)$ and the resulting map

$$H^0(P(V^*),\mathcal{O}(1))\otimes H^0(P(V^*),\mathcal{J}_Z^2(5)) \to H^0(\Sigma,\mathcal{O}(2)) = H^0(P(V^*),\mathcal{J}_Z^2(6))$$

coincides with the natural multiplication map from the assertion of the lemma. So the lemma is proven.

0.10. We continue to prove Theorem 0.5 and shall now use Lemma 0.9. Let us consider some particular sextics with double points at $Z = \{p_1, ..., p_6\}$. Let C_i be the unique conic through $Z - \{p_i\}$. We can take a sextic curve which is the union of two lines $< p_i, p_j >$, $< p_k, p_s >$ and two conics C_k, C_s. By means of (0.17) this sextic corresponds to some quadric $Q_{ij,ks}$. Moreover, since the quintic $< p_k, p_s > \cup C_k \cup C_s$ belongs to the linear system of quintics singular at points of Z, the quadric $Q_{ij,ks}$ lies in the image of the map δ from (0.14). Now let us take $j = s$. Then our sextic can be represented as the union of two cubic curves through Z namely

$$< p_i, p_j > \cup\ C_j \quad \text{and} \quad < p_k, p_j > \cup\ C_k.$$

Since such cubics correspond to hyperplanes in $P(L)$, we conclude that the quadric $Q_{ij,kj}$ is in fact the union of two planes, say H_{ij} and H_{kj}. Moreover, H_{ij} cuts out the cubic surface Σ along 3 lines l_i, l'_j, m_{ij} (see n. 0.1). The plane H_{jk} cuts out the lines l_j, l'_k, m_{kj} on Σ. Since the (quadratic form defining the) quadric $Q_{ij,kj} = H_{ij} \cup H_{kj}$ is apolar to our chosen $B \in S^2L$, we conclude that the equations of H_{ij} and H_{jk} (belonging to L^*) are B - orthogonal.

0.11. Let us now prove Proposition 0.7 and hence Theorem 0.5. The form B is a linear map $L^* \to L$. For any linear subspace $U \subset L$ we define its *polar subspace* (with respect to B) to be

$$U_B^\perp = B(U^\perp),$$

where $U^\perp$ denotes the orthogonal subspace of U in L^*. If B is non-degenerate then $U_B^\perp$ is the orthogonal complement of U in L with respect to the inverse form $B^{-1} \in S^2L^*$. If B is degenerate and $K \subset L^*$ is its kernel then $U_B^\perp$ is contained in $K^\perp$ for any U.

We shall apply the previous notation for projective subspaces in $P(L)$. In particular, if $H \subset P(L)$ is a hyperplane whose equation does not lie in $K = \mathrm{Ker}\ B$ then $H_B^{\perp}$ is a point called the *pole* of H.

Let us prove that B is non-degenerate. Let the double - six be $\{l_1, ..., l_6\}$, $\{l'_1, ..., l'_6\}$. Assume first that B is of rank at least 3. Then at most one hyperplane H_{ij} belongs to the kernel of B. Without loss of generality we may assume that all planes H_{ij} are not in the kernel except maybe H_{56}. Consider the plane H_{12} spanned by lines l_1 and l'_2 (which intersect). Its equation (in V^*) is orthogonal with respect to B to equations of similar planes H_{21}, H_{23}, H_{31} (see n. 0.10). Hence $(H_{12})_B^{\perp} = H_{21} \cap H_{23} \cap H_{31}$ and this intersection is easily seen to be the point $l'_1 \cap l_2$. In this way we show that each $l'_i \cap l_j$ is the pole of some plane H_{ji}, where $(i, j) \neq (5, 6)$. Since these points obviously span $P(L)$, the form B must be non-degenerate. Now assume that B is of rank at most 2. Since the planes $H_{12}, H_{13}, H_{24}, H_{25}$ are linearly independent, at least one of them is not in the kernel of B. Let it be H_{12}. Similarly we find that H_{34} and H_{56} are not in the kernel. Their three poles $l'_1 \cap l_2, l'_3 \cap l_4, l'_5 \cap l_6$ are not on a line. This contradicts the assumption that B is of rank at most 2.

It remains to show that $l_i^{\perp} = l'_i$. We have already seen that the point $l'_1 \cap l_2$ is the pole of the plane H_{12} spanned by l_1 and l'_2. Similarly, $l'_1 \cap l_3$ is the pole of $H_{13} = \mathrm{Span}(l_1, l'_3)$. Hence $l'_1 = \mathrm{Span}(l'_1 \cap l_2, l'_1 \cap l_3)$ is the orthogonal complement of $H_{12} \cap H_{23} = l_1$. Similarly we prove that $l'_i = l_i^{\perp}$ for other i.

Theorem 0.5 is completely proven. The reader should compare this rather cumbersome proof with a more straightforward one based on the theory of vector bundles (Theorem 2.17 below).

0.12. Definition. *The quadric $Q \subset P(L)$ defined by $C(x, x) = 0$ where C is the quadratic form given by Theorem 0.5, is called the Schur quadric (associated with the double - six $\{A_z, A'_z\}$).*

0.13. Example. Let us consider the following 4-dimensional space L:

$$L = \left\{(x_1, ..., x_5) \in \mathbf{C}^5 : \quad \sum x_i = 0\right\}$$

and define the cubic surface $\Sigma \subset P(L)$ by the equation $\sum x_i^3 = 0$ (the Clebsch diagonal surface). The symmetric group S_5 acts on $\mathbf{C}^5$ by permutations of coordinates and preserves L and Σ. The line

$$l = \left\{ x \in P(L) : \quad x_1 + \frac{1+\sqrt{5}}{2} x_2 + x_3 = x_2 + \frac{1+\sqrt{5}}{2} x_3 + x_4 = 0 \right\}$$

lies on Σ and so do all the lines obtained from l by the action of S_5. It is known **[Bu]** that the S_5 - orbit of l consists of 12 lines which form a double - six. Their equations can be found in **[B]**, p.168. The two sextuples of lines constituting this double - six are orbits of the alternating group $A_5 \subset S_5$. So one sextuple is the A_5 - orbit of l and the other is the A_5 - orbit of the line

$$l' = \left\{ x : \quad x_1 + x_2 + \frac{1-\sqrt{5}}{2} x_4 = \frac{1-\sqrt{5}}{2} x_1 + x_3 + x_4 = 0 \right\}$$

So l' is line of the second sextuple corresponding to l (because $l \cap l' = \emptyset$). The lines l and l' are orthogonal with respect to the bilinear form $C(x,y) = \sum_{i=1}^{5} x_i y_i$ on L. By symmetry, all the other corresponding pairs of lines of our double - six are also orthogonal with respect to C. Thus the Schur quadric Q associated to this double - six is given by the equation $\sum_{i=1}^{5} x_i^2 = 0$.

§1. An overview of Hulek's theory.

1.1. Let E be a stable rank 2 vector bundle on $P^2 = P(V)$ with $c_1(E) = -1$, $c_2(E) = n$. According to Le Potier **[L]** and Hulek **[Hu1]**, the bundle E can be realized as the middle cohomology of a monad

$$H \otimes \mathcal{O}_{P(V)}(-1) \xrightarrow{\alpha} M \otimes \Omega^1_{P(V)}(1) \xrightarrow{\beta} H' \otimes \mathcal{O}_{P(V)}. \tag{1.1}$$

where

$$H = H^1(E(-2)) \cong \mathbf{C}^{n-1}, \quad M = H^1(E(-1)) \cong \mathbf{C}^n, \quad H' = H^1(E) \cong \mathbf{C}^{n-1} \tag{1.2}$$

and the maps α and β are defined as follows. Let $\Omega^1(1)$ be identified with $\Theta(-2)$ where Θ is the tangent bundle of $P(V)$. Let $t : V \otimes \mathcal{O}_{P(V)}(-1) \to \Omega^1(1)$ be the Euler homomorphism twisted by $\mathcal{O}(-1)$ (see **[OSS]**). It allows one to identify

$$\mathrm{Hom}\big(H \otimes \mathcal{O}_{P(V)}(-1), M \otimes \Omega^1(1)\big) \quad \cong \quad \mathrm{Hom}_{\mathbf{C}}(H \otimes V^*, M). \tag{1.3}$$

The map α is induced by the cup - product

$$a : H^1(E(-2))\otimes V^* \;=\; H^1(E(-2))\otimes H^0(\mathcal{O}_{P(V)}(1)) \longrightarrow H^1(E(-1)). \qquad (1.4)$$

Similarly, we have a map $t^* : \Omega^1(1) \to V^* \otimes \mathcal{O}_{P(V)}$ which allows us to identify

$$\mathrm{Hom}(M \otimes \Omega^1(1), H' \otimes \mathcal{O}_{P(V)}) \;\cong\; \mathrm{Hom}_{\mathbf{C}}(M \otimes V^*, H'). \qquad (1.5)$$

After this identification the map β is induced by the cup - product

$$b : H^1(E(-1))\otimes V^* \;=\; H^1(E(-1))\otimes H^0(\mathcal{O}_{P(V)}(1)) \longrightarrow H^1(E). \qquad (1.6)$$

The cup - product pairing

$$B : M \otimes M \;=\; H^1(E(-1))\otimes H^1(E(-1)) \longrightarrow H^2\Big(\Big(\bigwedge^2 E\Big)(-2)\Big) \;=$$

$$= \; H^2(\mathcal{O}_{P(V)}(-3)) \;=\; \mathbf{C} \qquad (1.7)$$

is a symmetric non - degenerate bilinear form on M. We regard it as an isomorphism

$$B : M \to M^*. \qquad (1.8)$$

The spaces H and H' are dual to each other by means of the Serre duality and the isomorphism

$$E \;\cong\; E^* \otimes \bigwedge^2 E \;\cong\; E^*(-1).$$

With respect to the constructed pairings the monad (1.1) is self - dual in the sense that $\beta = \alpha^*(-1)$. Equivalently, if $\lambda \in V^*$ and $a(\lambda) : H \to M$ is the linear map defined by the pairing a and similarly $b(\lambda) : M \to H'$ is the map defined by b then

$$b(\lambda) = a(\lambda)^* \circ B.$$

This shows that the monad (1.1) is completely determined by the pairing (1.4) and the symmetric bilinear form B. The pairing must satisfy the following properties (cf. **[Hu1]**):

(α1) The map $a(\lambda)$ is injective for generic $\lambda \in V^*$.

(α2) For any $h \in H$ the map $a_H(h) : V^* \to M$ defined by the pairing a is of rank ≥ 2.

(α3) For any $\lambda, \lambda' \in V^*$ we have $b(\lambda') \circ a(\lambda) = b(\lambda) \circ a(\lambda')$ where $b(\lambda) = a(\lambda)^* \circ B$ and similarly for $b(\lambda')$.

Note that by a theorem of Grauert-Mülich, the last two properties imply the first one.

1.2. Theorem. *Let V, H, M be linear spaces of respective dimensions $3, n-1$ and n and $n \geq 2$. Let us fix a non-degenerate symmetric bilinear form B on M. By assigning to each $a \in \mathrm{Hom}(H \otimes V^*, M)$ satisfying $(\alpha 1)$ – $(\alpha 3)$ the map*

$$\alpha = \big(\mathrm{Id} \otimes t\big) \circ \big(a \otimes \mathrm{Id}\big) \colon \;\; H \otimes \mathcal{O}_{P(V)}(-1) \longrightarrow M \otimes \Omega^1(1),$$

we get a bijective correspondence between equivalence classes of self - dual monads (1.1) modulo action of the group $O(M, B) \times GL(H)$ and isomorphism classes of stable rank 2 vector bundles E on $P(V)$ with $c_1(E) = -1$ and $c_2(E) = n$.

1.3. Let l be a line in P^2 and E be a stable bundle as in Theorem 1.2. Let $\lambda \in V^*$ be a linear form defining l. We have a canonical exact sequence

$$0 \longrightarrow E(-1) \xrightarrow{\lambda} E \longrightarrow E|_l \longrightarrow 0,$$

which together with the fact $H^0(E) = 0$ which follows from the stability of E, gives an isomorphism

$$H^0(E|_l) = \mathrm{Ker}\big\{H^1(E(-1)) \to H^1(E)\big\} = \mathrm{Ker}\big\{a(\lambda) : M \to H\big\}. \tag{1.9}$$

Since $E|_l \cong \mathcal{O}(p) \oplus \mathcal{O}(q)$ with $p + q = -1$, we obtain that

$$E|_l \cong \mathcal{O} \oplus \mathcal{O}(-1) \qquad \Longleftrightarrow \qquad \mathrm{rank} a(\lambda) = n - 1.$$

A line l is called a *jumping line* if $E|_l \neq \mathcal{O} \oplus \mathcal{O}(-1)$. It follows from the Grauert - Mülich theorem **[OSS]** that the set of jumping lines is a proper Zariski closed subset of the dual plane $P(V^*)$. This set is known to be 0 -dimensional for a generic E.

1.4. In **[Hu1]** the notion of a *jumping line of the second kind* (shortly JLSK) was introduced. Let $l^{(1)}$ be the first infinitesimal neighborhood of l in $P(V)$. We use the exact sequence

$$0 \to \mathcal{O}_{P(V)}(-2) \xrightarrow{\lambda^2} \mathcal{O}_{P(V)} \to \mathcal{O}_{l^{(1)}} \to 0 \tag{1.10}$$

to obtain

$$H^0(E|_{l^{(2)}}) = \mathrm{Ker}\big\{s(\lambda) : H^1(E(-2)) \to H^1(E)\big\}. \tag{1.11}$$

Here the map $s(\lambda)$ corresponds to the canonical pairing

$$S^2(V^*) \otimes H^1(E(-2)) \quad \longrightarrow \quad H^1(E))$$

evaluated at λ^2. In the notation of the previous subsections, $s(\lambda)$ is the composition

$$a(\lambda)^* \circ B \circ a(\lambda): \; H \to M \to M^* \to H^*.$$

We say that l is a JLSK if $s(\lambda)$ is not bijective. Since the source and target of $s(\lambda)$ have the same dimension, l is a JLSK if and only if $H^0(E|_{l(1)}) \neq 0$.

Let us introduce a rational quadratic map

$$\gamma : P(V^*) \to P(S^2 H^*), \quad \lambda \mapsto s(\lambda)$$

By property (α1), for a generic line $l \in P(V^*)$ the value $\gamma(\lambda)$ is well defined and is an non-degenerate quadric in $P(H)$. We denote by $C(E)$ the set of all JLSK of E. Thus outside a finite set of points in $P(V^*)$ the set $C(E)$ is equal to the preimage, under γ, of the locus of degenerate quadrics in $P(H)$. So we get that $C(E)$ is a closed subscheme in $P(V^*)$ defined by the equation $\det \gamma(l) = 0$. We shall consider $C(E)$ as a closed subscheme of $P(V^*)$ defined by this equation. So $C(E)$ is a (possibly reducible) curve of degree $2n-2$ containing the set of jumping lines of E in the usual sense.

1.5. One can give another interpretation of the curve $C(E)$. Consider the rational map

$$\sigma : P(V^*) \to P(M^*), \quad \lambda \mapsto \mathrm{Im}(a(\lambda))^{\perp} \subset M^*.$$

It is defined on the complement of the set of jumping lines of E. A non-jumping line l is a JLSK of and only if the hyperplane $\sigma(l) \subset P(M)$ is tangent to the quadric defined by $B(m,m) = 0$. Let us denote by Q the dual quadric in $P(M^*)$ (which parametrizes the hyperplane tangent to $\{B(m,m) = 0\}$; so it is given by the inverse quadratic form $C = B^{-1}$). Then

$$l \text{ is a JLSK} \quad \text{if and only if} \quad \sigma(l) \in Q.$$

§2. Generalized Schur quadrics and cubic surfaces.

2.1. Let E be a stable rank 2 vector bundle on $P^2 = P(V)$ with $c_1 = -1$, $c_2 = n$. As we mentioned in the previous section, its monad (1.1) defines (and is uniquely defined by) the following linear algebra data: a linear map (tensor)

$$a : H \otimes V^* \to M \tag{2.1}$$

and a quadratic form (the cup - product)

$$B : M \otimes M \to \mathbf{C}. \tag{2.2}$$

Our aim in this section is the study of the geometry of some algebraic varieties naturally associated to a and B (and hence to E).

2.2. We denote by $Q \subset P(M^*)$ the quadric defined by the equation $C(m, m) = 0$ where C is the quadratic form on M^* inverse to B, see n.1.5. We shall call Q the *Schur quadric* of E. We shall see later in this section how the classical Schur quadric of a double - six is a particular case of this construction.

2.3. By taking various partial transposes of the tensor a, we construct the following linear operators:

$$a_M : M^* \to H^* \otimes V = \mathrm{Hom}(H, V); \tag{2.3}$$

$$a_V : V^* \to H^* \otimes M = \mathrm{Hom}(H, M); \tag{2.4}$$

$$a_H : H \to M \otimes V = \mathrm{Hom}(M^*, V). \tag{2.5}$$

These operators define determinantal varieties in $P(M^*), P(V^*), P(H)$ consisting of points whose images (under the corresponding a) are operators not of maximal rank. Before going into details, let us recall some well known facts about varieties of matrices of given rank.

Let L_1, L_2 be vector spaces of respective dimensions n_1, n_2. We denote by $\mathrm{Hom}(L_1, L_2)_r \subset \mathrm{Hom}(L_1, L_2)$ the variety of linear maps of rank $\leq r$. We assume that $r \leq \min(n_1, n_2)$. Then the following is true **[ACGH]**,**[R]**.

2.4. Proposition.

a) The codimension of $\mathrm{Hom}(L_1, L_2)_r$ *in* $\mathrm{Hom}(L_1, L_2)$ *is equal to* $(n_1 - r)(n_2 - r)$.

b) $\mathrm{Hom}(L_1, L_2)_r$ *is irreducible and Cohen - Macaulay;*

c) The degree of (the projectivization of) $\mathrm{Hom}(L_1, L_2)_r$ *is equal to*

$$\prod_{i=0}^{n_1-r-1} \frac{(n_2+i)!\ i!}{(r+i)!\ (n_2-r-i)!}.$$

d) Let $\phi \in \mathrm{Hom}(L_1, L_2)_r$ *be a linear map of rank* $k \leq r$. *Then the multiplicity of* $\mathrm{Hom}(L_1, L_2)_r$ *at* ϕ *is given by*

$$\mathrm{mult}_\phi(\mathrm{Hom}(L_1, L_2)_r) = \prod_{i=0}^{n_1-r-1} \frac{(n_2-k-i)!\ i!}{(r-k-1)!(n_2-r-i)!}.$$

2.5. Let us return to the situation of n. 2.3. We define the variety $\Sigma \subset P(M^*)$ as follows

$$\Sigma = \{\mu \in P(M^*) : \mathrm{rank}\ a_M(\mu) \leq 2\}. \tag{2.6}$$

This is an analog of a cubic surface in P^3, cf. Proposition 0.3.

Note that $\dim M = n$, $\dim H = n-1$, $\dim V = 3$. Therefore, by Proposition 2.4, the variety $\mathrm{Hom}(H, V)_2$ has codimension $n-3$ in $\mathrm{Hom}(H, V)$ and so $\dim \Sigma \geq 2$. Generically, one would expect that $\dim \Sigma = 2$.

We shall call the tensor a (and the bundle E) Σ - *generic* if for any $\mu \in \Sigma$ the rank of $a_M(\mu)$ is exactly 2. We shall see in section 3 that if n is a square then Σ - generic bundles exist. Since being Σ - generic is an open condition, this will imply that such bundles form an open dense subset in the moduli space. We shall also see that for (some other) open dense subset in the moduli space the variety Σ is indeed a surface. However, there are important particular cases when Σ is reducible and contains components of higher dimension, see n. 3.5 below.

2.6. Consider now the partial transpose a_V of the tensor a given in (2.4). We define the determinantal variety $Z \subset P(V^*)$ by

$$Z = \{\lambda \in P(V^*) : \quad \mathrm{rank}\ a_V(\lambda) \leq n-2\}. \tag{2.7}$$

It will be important for us to consider Z as a scheme with the scheme structure given naturally by (2.7). This means that we choose bases in H and M and regard a_V as a $(n-1) \times n$ -matrix whose entries are linear forms in λ. The n maximal minors of this matrix are taken to be the equations of the subscheme Z.

Since $\mathrm{Hom}(H, M)_{n-2}$ has codimension 2 in $\mathrm{Hom}(H, M)$, generically one expects Z to be 0-dimensional and reduced. If this is indeed the case, we shall call the tensor a (and the bundle E) Z - *generic*. It follows from **[Hu1]** that Z -generic bundles exist for any values of n. Namely, the so-called Hulsbergen bundles will be Z -generic (see also §4 for discussion of these bundles). Thus Z -generic bundles form an open dense subset in the moduli space.

If a is Z-generic then, by Proposition 2.4. c), the degree of the 0- dimensional scheme Z equals deg $\mathrm{Hom}(H, M)_{n-2} = \binom{n}{2}$. Moreover, the multiplicity of any point $\lambda \in Z$ in Z is at least $\binom{n-r(\lambda)}{2}$ where $r(\lambda) = \mathrm{rank}\ a_V(\lambda)$

The meaning of Z is as follows.

2.7. Lemma. *The support of the scheme Z is precisely the set of jumping lines of E.*

Proof: This immediately follows from considerations of n.1.3.

2.8. Let $\mathcal{J}_Z \subset \mathcal{O}_{P(V^*)}$ be the sheaf of ideals of the subscheme Z. By construction of Z (see n. 2.6), maximal minors of the $(n-1)\times n$ - matrix a_V are global sections of $\mathcal{J}_Z(n-1)$. In invariant terms, we consider the linear map

$$a_V^* : H \otimes M^* \to V \tag{2.9}$$

and, by taking its $(n-1)$ -st symmetric power, we get a linear map

$$\bigwedge^{n-1} H \otimes \bigwedge^{n-1} M^* \hookrightarrow S^{n-1}(H \otimes M^*) \longrightarrow S^{n-1}V = H^0(P(V^*), \mathcal{O}(n-1)) \tag{2.10}$$

whose image is contained in $H^0(P(V^*), \mathcal{J}_Z(n-1))$. It will be convenient for us to rewrite (2.10) as

$$A : M \otimes \left(\bigwedge^{n-1} H \otimes \bigwedge^{n} M^* \right) \quad \longrightarrow \quad H^0(P(V^*), \mathcal{J}_Z(n-1)). \tag{2.11}$$

The 1-dimensional vector space $\bigwedge^{n-1} H \otimes \bigwedge^n M^*$ can be chased away by choosing bases in H and M.

2.9. Proposition. *If* $\dim Z = 0$ *then the operator* A *in (2.11) is an isomorphism. In other words, the linear system of curves of degree* $n-1$ *through* Z *is generated by maximal minors of* a_V.

Proof: We associate to a_V, in a standard way, a morphism $\tilde{a}$ of sheaves on $P(V^*)$ and denote its cokernel by $\mathcal{F}$:

$$0 \to H \otimes \mathcal{O}_{P(V^*)}(-1) \xrightarrow{\tilde{a}} M \otimes \mathcal{O}_{P(V^*)} \to \mathcal{F} \to 0 \qquad (2.12)$$

(the fact that $\tilde{a}$ is injective, follows from $\dim Z = 0$). We claim the following:

2.10. Lemma. $\mathcal{F}$ *is isomorphic to* $\mathcal{J}_Z(n-1)$. *Under this isomorphism the natural map* $M \to H^0(\mathcal{J}_Z(n-1))$ *corresponds, up to a scalar multiple, to the map* A *from (2.11).*

Clearly, Lemma 2.10 implies our proposition in virtue of the exact cohomological sequence of (2.12).

Proof of the lemma: The assertion follows from the well-known resolution of Eagon-Northcott of the ideal of a determinant variety defined by maximal minors (see **[No]**, Appendix C.1). However we prefer to give an elementary proof here. We choose a bases $h_1, ..., h_{n-1} \in H$ and $m_1, ..., m_n \in M$. This makes it possible to speak about the determinant $\det[v_1, ..., v_n]$ of a system of n vectors in M (this is just $|b_{ij}|$ where $v_i = \sum b_{ij} m_j$). We define a morphism of sheaves $\psi : \mathcal{F} \to \mathcal{J}_Z(n-1)$ i.e. a morphism $\Psi : M \otimes \mathcal{O}_{P(V^*)} \to \mathcal{J}_Z(n-1)$ vanishing on $\mathrm{Im}(\tilde{a})$, as follows. Let $m = m(\lambda)$ be a local section of $M \otimes \mathcal{O}_{P(V^*)}$ i.e. an M - valued function in λ homogeneous of degree 0. We put $\Psi(m)$ to be the homogeneous (of degree $n-1$) function

$$\lambda \mapsto \det\big[m(\lambda), a_V(\lambda)(h_1), ..., a_V(\lambda)(h_{n-1})\big].$$

This defines ψ. It is clear that ψ is injective. The fact that ψ is surjective follows by comparing Chern classes of $\mathcal{F}$ and $\mathcal{J}_Z(n-1)$. The rest of the lemma is obvious.

2.11. We continue to assume that $\dim Z = 0$. Let S be the blow up of $P(V^*)$ along Z and $\pi_S : S \to P(V^*)$ be the canonical projection. In virtue of Proposition 2.9 the linear system of curves of degree $n-1$ through Z defines a regular map $p : S \to P(M^*)$. A generic point $s = \pi_S^{-1}(\lambda) \in S$, $\lambda \in P(V^*)$ goes under p into the hyperplane in $P(M)$ consisting of m such that $A(m) \in H^0(P(V^*), \mathcal{J}_Z(n-1))$ vanishes at λ as well. Here A is as in (2.11). The interpretation of A in Lemma 2.10 shows that $p(S)$ is contained in the determinantal variety $\Sigma \subset P(M^*)$ as an irreducible component. We shall denote variety $p(S)$ (typically a surface) by $\Sigma' \subset \Sigma$.

2.12. Suppose that our bundle E is Σ - generic. Then we have a regular map

$$\pi_\Sigma : \Sigma \to P(V^*)$$

which takes $\mu \in \Sigma \subset P(M^*)$ to the linear subspace $\mathrm{Im}(a_M(\mu)) \subset V$ (this subspace has dimension 2 by the assumption of Σ -genericity). The map π_Σ is the analog of the blow-down of a cubic surface onto a plane.

If the bundle E is not Σ -generic, the map π_Σ will be defined on the open part $\Sigma_0 \subset \Sigma$ consisting of μ such that $a_M(\mu)$ has rank exactly 2.

For $\lambda \in P(V^*)$ the fiber $\pi_\Sigma^{-1}(\lambda)$ is the projective space $P(\mathrm{Ker}\ a_V(\lambda)^*)$. The dimension of this fiber is equal to $n-\mathrm{rank}\ a_V(\lambda)-1$. Hence π_Σ is an isomorphism over the complement of $\mathrm{Supp}(Z)$. On the other hand, if the rank of $a_V(\lambda)$ is small the fiber $\pi_\Sigma^{-1}(\lambda)$ will have dimension ≥ 2 and the variety Σ will be reducible. We shall see in §3 that such situations do occur for stable bundles.

2.13. Proposition. *Assume that E is Z - generic and no $n-1$ points of Z lie on a line. Then:*
(a) The map $p : S \to \Sigma$ is an isomorphism (so, in particular, $\Sigma' = \Sigma$);
(b) Σ is a projectively Cohen - Macaulay surface in $P(M^*)$ of degree $(n-1)^2 - \binom{n}{2}$.

Proof: Introducing the Hilbert function $H(Z,t) = h^0(\mathcal{O}_{P(V^*)}(t)) - h^0(\mathcal{J}_Z(t))$, and applying exact sequence (2.12), we have

$$H(Z, n-1) = (1/2)n(n+1) - n = (1/2)n(n-1) =$$

$$H(Z, n-2) > H(Z, n-3) = (1/2)(n-1)(n-2).$$

This gives

$$n-1 = \min\{t : H(Z,t) = H(Z,t-1)\}.$$

By **[DG]**, this implies that the linear system of curves of degree $n-1$ through Z maps $S = \mathrm{Bl}_Z(P^2)$ isomorphically into $P(H^0(\mathcal{J}_Z(n))^*) = P(M^*)$. By **[Gi]** the image of this map i.e., the variety Σ', is projectively Cohen - Macaulay. Recall that this means that the projective coordinate ring of Σ' is Cohen - Macaulay. In particular, we get that Σ' is projectively normal i.e., for any $k \geq 0$ the restriction map

$$H^0(P(M^*), \mathcal{O}(k)) \longrightarrow H^0(\Sigma', \mathcal{O}(k))$$

is surjective. Since the rational map $P(V^*) \to \Sigma$ is given by the linear system of curves of degree $n-1$ through Z, we obtain the assertion about the degree of Σ'. Since E is Z - generic, the fiber of the map $\pi_\Sigma : \Sigma \to P(V^*)$ over each point $z \in Z$ is isomorphic to P^1. Since Σ' and Σ coincide outside the union of the fibers $\pi_\Sigma^{-1}(z)$, $z \in Z$, this implies that $\Sigma' = \Sigma$. Q.E.D.

2.14. Let $z \in Z \subset P(V^*)$. We denote the fiber

$$\pi_\Sigma^{-1}(z) = P\big(\mathrm{Ker}(a_V(z)^*) \subset P(M^*) \quad \text{by} \quad A_z.$$

The corresponding linear subspace $\mathrm{Ker}(a_V(z)^*) \subset M^*$ of which A_z is the projectivization, will be denoted by $\mathbf{A}_z$.

Consider the space

$$H_z = \mathrm{Ker}\ a_V(z) \subset H.$$

We also consider the linear subspace

$$\mathbf{A}'_z = \bigcap_{h \in H_z} \mathrm{Ker}\ a_H(h) \quad \subset \quad M^*$$

and denote its projectivization by $A'_z \subset P(M^*)$.

The collection of projective subspaces A_z, A'_z, $z \in Z$, forms the analog of a Schläfli double - six on a cubic surface in P^3.

In our case A'_z lies on Σ but A_z does not, in general, do so. Indeed, the typical situation (see Proposition 2.13) is that Σ is a surface, that for any $z \in Z$ we have $\mathrm{rk}(a(z)) = n-2$ and so $\dim A_z = 1$, $\dim A'_z = n-3$. So for $n \geq 5$ A'_z cannot lie on Σ. The relation of A'_z with the component $\Sigma' = p(S) \subset \Sigma$ is as follows.

2.15. Proposition. *Assume that E is Z - generic. Then A'_z is a subspace of codimension 2 in $P(M^*)$ which intersects the surface Σ' along a curve. The image* *of this curve under the projection* $\pi_\Sigma : \Sigma \to P(V^*)$ *is the unique curve of degree* $n-2$ *which passes through the points* $z' \in Z \setminus \{z\}$.

For the case $n = 4$ we get the standard description of the second sextuple of lines of the double - six on the cubic surface as the inverse image of quadrics containing some 5 of the 6 points of Z. For $n \geq 5$ instead of the property that A'_z lies on Σ' we have that $A'_z \cap \Sigma'$ is a curve (instead of a set of isolated points, as one would expect by dimension count).

Proof: Since the rank of $a_V(z)$ equals $n-2$, we have $\dim(H_z) = 1$. Thus $\mathbf{A}'_z$ is the kernel of the map $a_V(h) : M^* \to V$ where h is any non-zero vector from H_z. Note that the rank of this map equals 2. In fact, otherwise Z would contain a line as an irreducible component. This shows that $\dim(\mathbf{A}_z) = n-2$. Now let us observe that $A'_z = P(\mathbf{A}'_z)$ intersects each $A_{z'}$ for $z' \neq z$. Indeed, the sum of linear subspaces $\mathbf{A}'_z + \mathbf{A}_{z'}$ is contained in the hyperplane of zeroes of the linear form $a(h, z') \in M = (M^*)^*$, where a is as in (2.1).

Let $\{H(\lambda)\}_{\lambda \in P^1}$ be the pencil of hyperplanes in $P(M^*)$ which contain the subspace A'_z. It cuts out a pencil $\mathcal{P}$ of curves on Σ with the base locus $A'_z \cap \Sigma$. For each $z' \neq z$ one of the hyperplanes $H(\lambda)$ contains the line $A_{z'}$. Thus each $A_{z'}$ contains one of the base points of the pencil $\mathcal{P}$. Under the rational map $P(V^*) \to P(M^*)$ (given by curves of degree $n-1$ through Z) the preimage of the pencil $\{H(\lambda)\}$ is some pencil of curves of degree $n-1$ passing through Z. Let $\mathcal{C}$ be its moving part and F be its fixed curve. Let d be the degree of F (zero if $F = \emptyset$). Curves of the pencil $\mathcal{C}$ have degree $n-1-d$. Suppose that they pass through some m points say, $z_1, ..., z_m$ of Z. Then, since z_i remain basic for $\mathcal{C}$ after the blow - up, curves from $\mathcal{C}$ have the same tangent direction at each $z_i \neq z$. The curve F passes through the remaining $(1/2)n(n-1) - m$ points of Z. Consider a typical curve $C \in \mathcal{C}$. Let $\tilde{C}$ be its proper transform in $S = \mathrm{Bl}_Z(P(V^*))$. Since $\tilde{C}$ moves, its self - intersection index is non - negative so we get

$$0 \leq \tilde{C}^2 \leq (n-d-1)^2 - 2(m-1) - 1 = (n-d)(n-d-1) - 2m - (n-d-2).$$

If $n-d-2 \geq 0$, we obtain that $(n-d)(n-d-1) - 2m \geq 0$ thus there exists a plane curve of degree $n-d-2$ passing through $z_1, ..., z_m$. Together with the curve F, it defines a curve of degree $n-2$ passing through all the points of Z. But Lemma 2.10 and the exact sequence (2.12) show that this is impossible. So we must have $d = n-2$ and hence $m = 1$, so F is a curve of degree $n-2$ which passes through all the points of Z except z. If there is another curve, say, F', with this property then we would have a pencil of curves of degree $n-2$ through $Z - \{z\}$. This pencil must then contain a curve passing also through z. This, as we have just seen, is impossible.

2.17. Up until now we worked exclusively with the tensor a from (2.1). Now we take into account the non-degenerate quadratic form $B \in S^2M^*$ from (2.2). Let $C = B^{-1}$ be the inverse quadratic form on M^*. The following result justifies the name "Schur quadric" for the quadric defined by C.

2.17. Theorem. *Let $z \in \mathrm{Supp}(Z)$. Then A_z is contained in the orthogonal complement $(A'_z)^{\perp}_{C}$ of A'_z with respect to C . If, moreover, rk $a_V(z) = n-2$ then we have equality $A'_z = (A_z)^{\perp}_{C}$.*

Proof: For any $\lambda \in V^*$ let

$$b(\lambda) = a(\lambda)^* \circ B : \quad M \longrightarrow H^*$$

where $a(\lambda)$ is the map induced by the a from (2.1). Then

$$B^{-1}(\mathbf{A}'_z) = \{m \in M : (b(\lambda)(m), h) = 0, \quad \forall \lambda \in V^*, h \in H_z = \mathrm{Ker}\ (a_V(z))\}.$$

For any $m \in \mathbf{A}_z^{\perp} = a_V(z)(H)$ we write $m = a(z)(h')$ for some $h' \in H$ and obtain

$$(b(\lambda)(a_V(z)(h')), h) \quad = \quad (b(\lambda)(a_V(z)(h), h') = (0, h') = 0.$$

Here we use the property $(\alpha 3)$ from n.1.1. Thus we obtain

$$\mathbf{A}_z^{\perp} \subset B^{-1}(\mathbf{A}'_z).$$

If rank $(a(z)) = n-2$ then dim $\mathbf{A}_z = 2$, dim $H_z = 1$ and dim $\mathbf{A}'_z = n-2$. Thus the dimensions of the spaces $\mathbf{A}_z^{\perp}$ and $B^{-1}(\mathbf{A}'_z)$ are the same so these spaces are equal. Theorem is proven.

2.18. Remark. Let Z be any set of $\binom{n}{2}$ points in P^2 such that no curve of degree $n-2$ contains Z and no lines pass through $n-1$ points of Z. The linear system of curves of degree $n-1$ through Z defines a rational map of P^2 into P^{n-1}

whose image is a nonsingular surface X classically known as a *White surface* **[R]**. If $n = 4$, this is a cubic surface. The surface X is given by vanishing of maximal minors of a $3 \times (n-1)$ matrix of linear forms. A modern proof of these results can be found in **[DG]** and **[Gi]**.

Every White surface comes equipped with a set of $\binom{n}{2}$ lines $E_z, z \in Z$ corresponding to exceptional curves of the blow - up $\mathrm{Bl}_Z(P^2)$ and a set of $\binom{n}{2}$ curves C_z of degree $(n-2)(n-4)/2+1$. The curve C_z is the image of the (unique) plane curve of degree $n-2$ passing through $Z - \{z\}$. Each curve C_z spans a subspace E'_z of codimension 2 in P^{n-1}. We have $E_z \cap E'_z = \emptyset$ but $E_z \cap E'_{z'} \neq \emptyset$ for $z' \neq z$. This situation is analogous to a configuration of a double - six on a cubic surface.

Propositions 2.13 and 2.15 imply that for a Σ - generic stable bundle E the variety Σ is a White surface. However, by counting constants it follows that not every White surface comes in this way, as soon as $n \geq 5$. Although one can reconstruct a linear map

$$a : H \otimes V^* \cong \mathbf{C}^{n-1} \otimes \mathbf{C}^3 \longrightarrow M = \mathbf{C}^n$$

from a determinantal representation of X, there does not exist, in general, a quadratic form B on M such that a satisfies the property $(\alpha 3)$ from n.1.1. By Theorem 1.2 the existence of such a B is necessary and sufficient in order that $X = \Sigma$ for some Z - generic bundle E. It seems likely that these conditions are equivalent to the existence of a "Schur quadric" for the "double - six" $\{E_z, E'_z\}$ i.e., a quadric Q in P^{n-1} such that E_z and E'_z are orthogonal with respect to the (quadratic form defining) Q.

2.20. The role of the Schur quadric Q (see n.2.2) in the description of jumping lines of the second kind is given by the following remark.

2.21. Proposition. *Let $\Sigma_0 \subset \Sigma$ be the open set of μ such that the rank of $a_V(\mu)$ equals 2 (so $\Sigma_0 = \Sigma$ if the bundle is Σ - generic). Let $\pi_\Sigma : \Sigma_0 \to P(V^*)$ be the projection defined in n. 2.12. Then the curve $C(E)$ of jumping lines of second kind coincides with the closure of $\pi_\Sigma(Q \cap \Sigma_0)$.*

In particular, when the bundle E is Σ - generic, we have $C(E) = \pi_\Sigma(Q \cap \Sigma)$

Proof: This is a reformulation of what has been done in n.1.5.

As an application of our formalism of Schur quadrics let us prove a statement about the singular tangent lines of the curves of JLSK which strengthens, under assumptions of genericity, a theorem of Hulek. More precisely, Hulek **[Hu1]** has proven the following fact.

2.22. Theorem. *Let $l \in C(E)$ be a JLSK of E. Suppose that $E|_l \cong \mathcal{O}(-1-k) \oplus \mathcal{O}(k)$ with $k \geq 1$. Then l is a singular point of the curve $C(E)$ of multiplicity $2k$ and for any line T in the tangent cone of $C(E)$ at l the intersection index of $C(E)$ and T at l is at least $2k+2$.*

Assume that E is Z - generic. Then every singular point l of $C(E)$ is a double point (a node or, possibly, a cusp of type $y^2 = x^r$). Theorem 2.22 gives that in this case there exist at least one line T through the point l with intersection index ≥ 4.

We claim that the case of the cusp does not occur for Z-generic E. Call an ordinary double point p of a plane curve C a *biflexnode* if each of the two branches has a flex at this point i.e. each of the two tangents has the intersection index ≥ 4 with C at p.

2.23. Theorem. *Assume that the bundle E is Z - generic. Then every singular point of $C(E)$ corresponding to a jumping line is a biflexnode.*

Proof: Let $z \in Z$ be a singular point of $C(E)$ corresponding to a jumping line. Then $z \in Z$. The branches of $C(E)$ at z correspond to the points of intersection of the line A_z and the Schur quadric Q. Note that Q cannot be tangent to A_z since otherwise we would have $A_z \cap A'_z \neq \emptyset$ which contradicts Proposition 2.15. This proves that the point z is an ordinary node.

Although Theorem 2.22 allows us to finish the proof, we prefer to give an independent proof based on the properties of the Schur quadric.

Now let x be one of the two points of $Q \cap A_z$ and let Π be a hyperplane in $P(M^*)$ which is spanned by the point x and the codimension 2 subspace A'_z. Let $Q(z)$ denote the quadric in A'_z cut out by Q. For any point $y \in Q(z)$ the line $< x, y >$ is contained in Q. This implies that Π is tangent to Q at x. Let $\tilde{C}(E) = \Sigma \cap Q$ be the proper inverse transform of the curve $C(E)$ in Σ, under the blow-down $\pi_\Sigma : \Sigma \to P(V^*)$. Let τ be the tangent line to $C(E)$ at z at the branch corresponding to x and let $\tilde{\tau}$ be its proper inverse transform on Σ.

Under the correspondence between hyperplanes in $P(M^*)$ and curves of degree $n-1$ in $P(V^*)$ through Z, the hyperplane Π corresponds to the reducible curve $\tau + C_z$ where C_z is the plane curve of degree $n-2$ passing through $Z - \{z\}$. This implies that $\tilde{\tau} \subset \Pi$ and so

$$T_x(\tilde{\tau}) = \Pi \cap T_x(\Sigma) = T_x(Q) \cap T_x(\Sigma) = T_x(\tilde{C}(E)).$$

This shows that $\tilde{\tau}$ is tangent to $\tilde{C}(E)$ at the point x. Obviously this implies that τ is a flex tangent at the branch of $C(E)$ at z corresponding to x. Theorem is proven.

§3. Logarithmic bundles.

3.1. Consider a projective plane $P^2 = P(V)$, dim $V = 3$. Let $\mathcal{H} = (H_1, ..., H_m)$ be an arrangement of m lines in $P(V)$ in general position (i.e., no three of these lines have a common point). Let $E(\mathcal{H}) = \Omega^1_{P(V)}(\log \mathcal{H})$ be the sheaf of 1-forms on $P(V)$ with logarithmic poles along H_i. Since $\mathcal{H}$ is a divisor with normal crossings, $E(\mathcal{H})$ is locally free i.e. we can and will regard it as a rank 2 vector bundle. It was proven in **[DK]** that this bundle is stable.

We further suppose that the number of lines is even: $m = 2d$. In this case $c_1 E(\mathcal{H}) = 2d - 3$. The normalized bundle $E_{\text{norm}}(\mathcal{H}) = E(\mathcal{H})(-d+1)$ is a stable bundle with $c_1 = -1, c_2 = (d-1)^2$. In this section we apply considerations of §§1,2 to bundles $E_{\text{norm}}(\mathcal{H})$.

3.2. It was shown in **[DK]** that the bundle $E(\mathcal{H})$ has a resolution of the form

$$0 \to I \otimes \mathcal{O}_{P(V)}(-1) \xrightarrow{\tau} W \otimes \mathcal{O}_{P(V)} \to E(\mathcal{H}) \to 0. \tag{3.1}$$

In (3.1) the space W is defined as

$$W = \left\{ (a_1, ..., a_{2d}) \in \mathbf{C}^{2d} : \sum a_i = 0 \right\}. \tag{3.2}$$

The space I is defined as follows. Let $f_i \in V^*$ be a linear equation of the line H_i. Then I is the space of relations among $(f_1, ..., f_{2d})$ i.e.,

$$I = \left\{ (a_1, ..., a_{2d}) \in \mathbf{C}^{2d} : \sum a_i f_i = 0 \right\}. \tag{3.3}$$

The map τ is induced by the canonical map

$$t : I \otimes V \to W, \quad (a_1, ..., a_{2d}) \otimes v \mapsto \big(a_1 f_1(v), ..., a_{2d} f_{2d}(v)\big) \tag{3.4}$$

called the fundamental tensor of $\mathcal{H}$.

3.3. By twisting the resolution (3.1) with $\mathcal{O}(-d+1)$ we get a resolution for $E_{\mathrm{norm}}(\mathcal{H}) = E(\mathcal{H})(-d+1)$. From this it is immediate to find the data defining the Hulek's monad for $E_{\mathrm{norm}}(\mathcal{H})$ (see §1). To formulate the answer neatly, let again $f_j \in V^*$ be the equation of H_j. For any $m \geq 1$ denote by $\partial/\partial f_j : S^m V \to S^{m-1} V$ the derivation corresponding to f_j regarded as a constant vector field on V^*. We define the following map

$$t_{(m)} : S^m V \otimes I \to S^{m-1} V \otimes W, \tag{3.5}$$

$$p \otimes (a_1, ..., a_{2d}) \mapsto \left(a_1 \frac{\partial p}{\partial f_1}, ., a_{2d} \frac{\partial p}{\partial f_{2d}} \right) \tag{3.6}$$

where we regard $S^{m-1} V \otimes W$ as the space of collections $(q_1, ..., q_{2d})$ of polynomials $q_j \in S^{m-1} V$ summing up to 0.

Now the vector spaces in the monad for $E_{\mathrm{norm}}(\mathcal{H})$ have the form

$$H = H^1\big(E_{\mathrm{norm}}(\mathcal{H})(-2)\big) = H^1\big(E(\mathcal{H})(-d-1)\big) = \mathrm{Ker}\ (t_{(d-1)}); \tag{3.7}$$

$$M = H^1\big(E(\mathcal{H})(-d)\big) = \mathrm{Ker}\ (t_{(d-2)}); \tag{3.8}$$

$$H' = H^1\big(E(\mathcal{H})(-d+1)\big) = \mathrm{Ker}\ (t_{(d-3)}), \tag{3.9}$$

as it follows immediately from the resolution (3.1). For example, the map $t_{(d-1)} : S^{d-1} V \otimes I \to S^{d-2} V \otimes W$ in (3.7) appears as the map

$$H^2\big(P(V), \mathcal{O}(-d-2) \otimes I\big) \to H^2\big(P(V), \mathcal{O}(-d-1) \otimes W\big)$$

in the long exact sequence of cohomology of the resolution (3.1) tensored with $\mathcal{O}(-d-1)$.

As regards maps in the monad (1.1), we shall only need the explicit form of the operator

$$b_M : M \to V \otimes H' \tag{3.10}$$

defined by the map b in (1.1). Namely, b_M is induced by

$$\psi \otimes \mathrm{Id}_I : S^{d-2} V \otimes I \to V \otimes S^{d-3} V \otimes I \tag{3.11}$$

where $\psi : S^{d-2} V \to V \otimes S^{d-3} V$ is the canonical $GL(V)$ - equivariant embedding. The map a in (1.1) is dual to b by means of the form B.

The following is the main result of this section.

3.4. Theorem. *Any bundle $E_{\mathrm{norm}}(\mathcal{H})$ is Σ -generic (see n. 2.5).*

Proof: In the notation of §2 we have to prove that

$$a_M(M^*) \quad \cap \quad \mathrm{Hom}(V^*, H^*)_1 \quad = \quad \{0\}. \tag{3.12}$$

We have a commutative diagram

$$\begin{array}{ccc} M^* & \xrightarrow{a_M} & \mathrm{Hom}(V^*, H^*) \\ \downarrow & & \downarrow \\ M & \xrightarrow{b_M} & \mathrm{Hom}(V^*, H') \end{array}$$

where the left vertical arrow is induced by the form B and the right vertical arrow — by the isomorphism $H^* = H'$ (see n. 1.2). It is enough therefore to prove that

$$b_M(M) \cap \mathrm{Hom}(V^*, H')_1 = \{0\}. \tag{3.13}$$

Let $m = \sum p_i \otimes x_i$ be an element of $M \subset S^{d-2}V \otimes I$, so $p_i \in S^{d-2}V, x_i \in I$. The element m is mapped by b_M into an element of $\mathrm{Hom}(V^*, H')_1$ if and only if there is $v \in V$ such that each p_i equals vq_i for some $q_i \in S^{d-3}V$ and also $\sum q_i \otimes x_i \in H'$. Each $x_i \in I$ is in fact a vector $x_i = (x_i^{(1)}, ..., x_i^{(2d)})$ such that $\sum_{j=1}^{2d} x_i^{(j)} f_j = 0$. Since m belongs to $M = \mathrm{Ker}\,(t_{(d-1)})$, we have, by (3.5) and (3.6):

$$\sum_i x_i^{(j)} \frac{\partial(vq_i)}{\partial f_j} = 0, \quad j = 1, ..., 2d. \tag{3.14}$$

By applying Leibnitz' rule for $\partial/\partial f_j$ and taking into account the fact that $\sum_i q_i \otimes x_i \in H' = \mathrm{Ker}(t_{(d-3)})$, we get the equalities

$$f_j(v) \sum_i x_i^{(j)} q_i = 0, \quad j = 1, ..., 2d. \tag{3.15}$$

We claim that these equalities imply that $q_i = 0$ for all i. Indeed, let $\lambda : S^{d-3}V \to \mathbf{C}$ be any linear functional. Consider the vector

$$y = \sum_i \lambda(q_i) x_i \in I.$$

If we write y in terms of its components: $y = (y^{(1)}, ..., y^{(2d)})$ then (3.15) implies that

$$f_j(v) y^{(j)} = 0, \quad j = 1, ..., 2d.$$

Let $J = \{j : f_j(v) = 0\}$. Since the lines $\{f_j = 0\}$ are in general position, $|J| \leq 2$. For $j \notin J$ we have therefore $y^{(j)} = 0$. Since $y \in I$, we have

$$0 = \sum_{j=1}^{2d} y^{(j)} f_j = \sum_{j \in J} y^{(j)} f_j$$

which means that we have a nontrivial linear relation among $|J| \leq 2$ elements of $\{f_1, ..., f_{2d}\}$. This contradicts the general position of $\{f_i = 0\}$ so the vector $y \in I$ is zero. In other words, for any linear functional $\lambda : S^{d-3}V \to \mathbf{C}$ we have $\sum \lambda(q_i)x_i = 0$ in I. This means that $\sum q_i \otimes x_i = 0$ in $S^{d-3}V \otimes I$ and Theorem 3.4 is proven.

3.5. Let Z be the subscheme of jumping lines of $E_{\text{norm}}(\mathcal{H})$. As was shown in **[DK]** (Proposition 7.4), the lines H_i belong to Z. Moreover,

$$E_{\text{norm}}(\mathcal{H})|_{H_i} = \mathcal{O}_{H_i}(1-d) \oplus \mathcal{O}_{H_i}(d-2). \tag{3.16}$$

Denote, as usual, by $f_i \in V^*$ the equation of H_i. The equality (3.16) means that the matrix $a_V(f_i)$ (see formula (2.4)) has rank $n-d-1$. By Proposition 2.4 d) this implies that the multiplicity of each H_i as a point of Z is at least $(d-1)(d-2)/2$. The total degree of Z, however, equals to $\binom{n}{2}$ where $n = c_2(E_{\text{norm}}(\mathcal{H})) = (d-1)^2$. Thus one expects that for $d \geq 4$ there will be many other jumping lines apart from $H_1, ..., H_{2d}$.

Let us also note that the fibers of the map $\pi_\Sigma : \Sigma \to P(V^*)$ introduced in n. 2.12 over points $H_i \in P(V^*)$ are projective spaces of dimension $d-2$. This means that for $d \geq 4$ the determinantal variety Σ ("cubic surface") will be always reducible.

§4. Examples.

4.1. In this section we shall illustrate geometric constructions of §2 on some particular classes of bundles. The example with cubic surfaces and Schur quadrics (which motivated the present paper) will be considered in n.4.4.

In each of the examples below we shall indicate the value of $n = c_2$ (we assume $c_1 = -1$) and describe the following geometric objects (all introduced in §2):

a) The subscheme $Z \subset P(V^*)$ of jumping lines. If dim $Z = 0$ then deg $Z = \binom{n}{2}$.

b) The determinantal variety $\Sigma \subset P(M^*)$ (the analog of the cubic surface). It comes with a natural map $p : \mathrm{Bl}_Z P(V^*) \to \Sigma$ whose image is a component of Σ. The map p is given by the linear system of curves of degree $n-1$ in $P(V^*)$ through Z.

c) The Schur quadric $Q \subset P(M^*)$.

d) The curve $C(E) \subset P(V^*)$ of JLSK. Its degree is $2n-2$. It can be described as $\overline{\pi_\Sigma(\Sigma_0 \cap Q)}$ where $\pi_\Sigma : \Sigma_0 \to P(V^*)$ is the projection of the generic part of Σ introduced in n. 2.12.

e) The projective subspaces $A_z, A'_z, z \in Z$ (the analog of the double - six).

By $M(-1,n)$ we shall denote the moduli space of stable rank 2 vector bundles on P^2 with $c_1 = -1,\ c_2 = n$. It is an irreducible variety of dimension $4n-4$, see **[Hu1]**,**[OSS]**.

4.2. The case $n = 2$. This case was considered in **[Hu1]**. The features are as follows:

a) Z consists of just one point $z_0 \in P(V^*)$. This point corresponds to the 1-dimensional kernel of

$$a_V : V^* \to \mathrm{Hom}(H, M) = \mathbf{C}^2.$$

b) The determinantal variety $\Sigma \subset P(M^*) = P^1$ coincides with $P(M^*)$. The regular map $p : \mathrm{Bl}_Z P(V^*) \to \Sigma$ is the natural projection $\mathrm{Bl}_{z_0} P^2 \to P^1$.

c) The Schur quadric $Q \subset P(M^*) = P^1$ consists of two distinct points.

d) The curve $C(E)$ is $\pi(p^{-1}(Q))$ where $\pi : \mathrm{Bl}_Z P(V^*) \to P(V^*)$ is the projection. In other words, $C(E)$ is the union of two distinct lines through z_0.

e) The "double - six" is as follows: $A_{z_0} = P(M^*), A'_{z_0} = \emptyset$.

4.3. The case $n = 3$. There may be several possibilities for Z which were also listed by Hulek **[Hu1]**. We shall consider only the most generic case when Z consists of three distinct non-collinear points. In this case the features are as follows:

b)The variety $\Sigma \subset P(M^*)$ again coincides with $P(M^*) = P^2$. The regular map $\mathrm{Bl}_Z P(V^*) \to P(M^*) = \Sigma$ resolves the standard Cremona transformation $c : P(V) = P^2 \to P^2 = P(M^*)$ defined by quadrics through Z (three points). If we choose homogeneous coordinates x_i in $P(V^*)$ in which Z consists of points $(1,0,0),(0,1,0),(0,0,1)$ then c is given by the formula $t_0 = x_1x_2, t_1 = x_0x_2, t_2 = x_0x_1$ where t_i are appropriate coordinates in $P(M^*)$.

c) The Schur quadric $Q \in P(M^*)$ is the conic $t_0^2 + t_1^2 + t_2^2 = 0$.

d) The curve $C(E)$ is the inverse image of this conic under the Cremona transformation defined in b). In other words, the equation of $C(E)$ is $x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 = 0$.

e) The subspaces A_z are coordinate lines $\{t_i = 0\}$ in $P(M^*)$, the subspaces A'_z are the opposite points of the coordinate triangle i.e., points $\{t_i = t_j = 0\}$.

4.4. The case $n = 4$. The moduli space $M(-1,4)$ has dimension 12. As shown in **[DK]**, an open dense subset in $M(-1,4)$ is provided by normalized logarithmic bundles

$$E_{\mathrm{norm}}(\mathcal{H}) = \Omega^1_{P(V)}(\log \mathcal{H}) \otimes \mathcal{O}(-2)$$

where $\mathcal{H} = (H_1, ..., H_6)$ is an arrangement of 6 lines in $P(V) = P^2$ in general position. We consider only such bundles E. Let $p_i \in P(V^*)$ be points corresponding to lines $H_i \subset P(V)$. We first assume that p_i do not lie on a conic (i.e., H_i are not all tangent to a conic). In this case:

a) $Z = \{p_1, ..., p_6\}$.

b) The variety $\Sigma \subset P(M^*) = P^3$ is the cubic surface obtained by blowing up Z.

c) The quadric Q is the classical Schur quadric associated with the double - six $\{l_i = A_{p_i},\ l'_i = A'_{p_i}\}$ (see §0). This follows from Theorem 2.17.

d) The curve $C(E)$ is the image under $\pi_\Sigma : \Sigma \to P(V^*)$ of the intersection $\Sigma \cap Q$. The intersection is non singular of degree 6 and genus 4; the projection will have nodes at p_i since each the six lines $l_i \subset \Sigma$ blown down to p_i by π_Σ meets Q twice.

e) The subspaces A_{p_i}, A'_{p_i} form the standard double - six associated to the blow-down π_Σ.

If all p_i do lie on a conic $\Gamma \subset P(V^*)$, the situation changes. In this case $E(\mathcal{H})$ is the Schwarzenberger bundle associated to Γ (see **[Sch1,Sch2]**, **[DK]**) and the features are as follows:

a) Z equals the conic Γ (so dim $Z = 1$).

b) The variety Σ is the union of a smooth quadric surface Q and a plane Π. The projection $\pi_\Sigma : \Sigma \to P(V^*)$ maps Π bijectively to $P(V^*)$ and projects $Q = P^1 \times P^1$ to one of its P^1 - factors which is then being embedded into $P(V^*)$ as the conic Γ.

c) The "Schur quadric" is the surface Q from n. b).

d) The curve $C(E)$ coincides with Γ (taken three times).

e) For any $z \in Z = \Gamma$ the lines A_z and A'_z both coincide with the generator of $Q = P^1 \times P^1$ mapped into z by π_Σ, see n.b).

4.5. Bring's curve as $C(E)$. Consider the situation of Example 0.13: the cubic surface Σ is given by equation $x_1^3 + ... + x_5^3 = 0$ where x_i are linear functions on M^* constrained by $\sum x_i = 0$. The Schur quadric corresponding to double - six described in n. 0.13 is given by $\sum x_i^2 = 0$. The intersection $C = \Sigma \cap Q$ i.e. the curve given in P^4 by equations

$$\sum x_i = \sum x_i^2 = \sum x_i^3 = 0$$

is known as *Bring's curve* **[K]** **[Hu2]**. The blow-down of the first six lines of the double - six described in n. 0.13 gives 6 points $p_1, ..., p_6$ in P^2 forming an orbit of the alternating group A_5 **[Hu2]**. These points will be the nodes of the sextic curve $\pi_\Sigma(C) \subset P^2$ i.e., of the projection of C to P^2, which is also called Bring's curve. The equation of $\pi_\Sigma(C)$ can be found in **[Hu2]**, p. 82.

Thus Bring's curve can be represented as the curve of JLSK of a certain bundle on $\check{P}^2$: the (normalized) logarithmic bundle of the configuration of lines dual to p_i.

4.6. Hulsbergen bundles. Let $q_1, ..., q_n$ be n points in general position in $P(V)$. There exists an $n-1$ -dimensional family of stable rank 2 bundles E on $P(V)$ with $c_1 = -1, c_2 = n$ such that $\{q_1, ..., q_n\}$ is the set of zeros of a section of $E(1)$ (see **[Hul]**). They are called Hulsbergen bundles. For such E the subscheme Z of jumping lines of E is reduced and consists of $\binom{n}{2}$ lines $< q_i, q_j >$. We denote by f_j linear functions on V^* corresponding to $q_i \in P(V)$. The linear system of curves of degree $n-1$ through Z has a basis formed by the curves

$$F_j = \prod_{i \neq j} f_i = 0.$$

This system maps the surface $S = \mathrm{Bl}_Z(P(V^*))$ to the surface $\Sigma \subset P(M^*) = P^{n-1}$ given, in natural homogeneous coordinates $(t_1, ..., t_n)$, by equations

$$\left(\prod_{i=1}^{n} t_i\right)\left(\sum_{i=1}^{n} \frac{a_{ji}}{t_i}\right) = 0, j = 1, ..., n-3$$

where $(a_{j1}, ..., a_{jn})$, $j = 1, ..., n-3$ is a basis of the space of linear relations among the vectors f_i. In the coordinates t_i the "Schur quadric" Q is given by the equation $\sum c_i t_i^2 = 0$ so the curve of JLSK in $P(V^*)$ has the equation

$$\sum_{i=1}^{n} c_i F_i^2 = 0.$$

(cf. **[Hul]** n. 10.5). Note that $p : S \to \Sigma$ blows down the proper transforms of the lines l_i to singular points of Σ which have the coordinates $(1, 0, ..., 0), ...,$ $(0, ..., 1)$. These points belong to $a_M^{-1}(\mathrm{Hom}\ (V^*, H^*)_1)$. So Hulsbergen bundles are not Σ - generic in the sense of n. 2.5, although they are Z - generic.

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Astérisque

RON DONAGI

Decomposition of spectral covers

Astérisque, tome 218 (1993), p. 145-175

<http://www.numdam.org/item?id=AST_1993__218__145_0>

DECOMPOSITION OF SPECTRAL COVERS

Ron Donagi

Contents

1 Introduction

To a vector bundle $E \to X$ and an endomorphism $\varphi : E \to E$ one associates a spectral cover $\pi : \widetilde{X} \to X$, whose fibers $\pi^{-1}(x)$, $x \in X$, are given by the eigenvalues of φ_x. If φ is, more generally, a K-valued endomorphism (a "Higgs bundle") $\varphi : E \to E \otimes K$, where K is a line bundle on X, we still get a cover $\pi : \widetilde{X} \to X$, but now $\widetilde{X}$ is contained in the total space $|K|$ of K, since the eigenvalues live in K. The eigenspaces of φ give a sheaf L on $\widetilde{X}$, which is a line bundle if φ is regular [BNR,B]. (Even more generally, K can be allowed to be a vector bundle on X, as long as a symmetry condition (trivial in case K is a line bundle) is imposed on φ: the case where K is the cotangent bundle of X arises in [S]. In this work we will consider only the case of a line bundle K.) One way to construct these objects is to let $\pi_K : |K| \to X$ denote the natural projection, and let τ be the tautological section of $\pi_K^* K$. Then $\pi_K^* \varphi - \tau$ is a $\pi_K^* K$-valued endomorphism of $\pi_K^* E$. Now L is the cokernel of $\pi_K^* \varphi - \tau$, considered as a sheaf on its support $\widetilde{X} := Supp(L) \subset |K|$.

This situation arises frequently in the study of completely integrable Hamiltonian systems on a manifold M which can be written as a Lax equation depending on parameters [AvM,B,G,H,K]; here X is the parameter space, often the affine line or $\mathbf{P}^1$, and the flow of the system is linearized on the Picard variety $Pic\widetilde{X}$ (or the Jacobian, when X is a curve). The linearization map typically gives an isogeny from the Liouville tori of the completely integrable system to (an Abelian subvariety of) $Pic\widetilde{X}$, by sending a point of M where the Lax equation is regular to the eigen line bundle computed at that point.

The vector bundle $E \to X$ often has G-structure, where G is some reductive Lie group. In other words, E is associated to a principal G-bundle $\mathcal{V} \to X$ via a representation $\rho : G \to GL(V)$ of G. The endomorphism φ then becomes a section of $\mathbf{ad}\mathcal{V} \otimes K$, where $\mathbf{ad}\mathcal{V}$ is the associated bundle of Lie algebras $\mathcal{V} \times_G \mathbf{g}$. In [AvM], Adler and van Moerbeke raised the question of the dependence of the resulting cover $\widetilde{X}_\rho$ on the representation ρ. If the situation comes from a completely integrable system as above, then the Liouville torus, which depends on the differential equation but not on the particular Lax equations or on the representation ρ, should occur, up to isogeny, as a subvariety of $Pic\widetilde{X}_\rho$, for all ρ. One may therefore expect to find a natural, Prym-type subvariety of each $Pic\widetilde{X}_\rho$, together with correspondences between pairs $\widetilde{X}_\rho$, $\widetilde{X}_{\rho'}$ whose images in the Picard varieties should be isogenous to this generalized Prym. More generally, one may wish to describe all correspondences acting on each $\widetilde{X}_\rho$ (or between pairs) over the base X, and to find the isogeny decomposition of $Pic\widetilde{X}_\rho$

into isotypic pieces under this action. One of these isotypic pieces should be common to all $\widetilde{X}_\rho$, and this would be the generalized Prym.

Several special cases of this situation, arising from orthogonal groups, are well known in Prym theory, e.g. Recillas' trigonal construction [R], my tetragonal construction [D1,D2], and Pantazis' bigonal construction [P]. The case of the exceptional group G_2 is discussed in [KP]. Other examples, related to the geometry of families of Del Pezzo surfaces, are given in [K]. In that work, Kanev gives a solution of Adler-van Moerbeke's question, under a few hypotheses: the base X is $\mathbf{P}^1$, the principal bundle $\mathcal{V}$ is trivial, the Lie algebra $\mathbf{g}$ is simple of type A_n, D_n, or E_n. Under these assumptions he constructs, for each $\widetilde{X}_\rho$, a Prym-Tyurin variety $Prym(\widetilde{X}_\rho/X) \subset Jac(\widetilde{X}_\rho)$ and a correspondence whose image is $Prym(\widetilde{X}_\rho/X)$. The Prym-Tyurin varieties for different representations are isogenous, and even isomorphic if both representations are minuscule.

The purpose of this work is to analyze the decomposition of the Picard varieties of general spectral covers for a reductive group G. We will show (Theorem 8.1) that there is a distinguished isotypic component of $Pic\widetilde{X}_\rho$, corresponding to the reflection representation Λ of the Weyl group W. When G is one of the classical simple groups, this is the unique piece common to $Pic\widetilde{X}_\rho$ for all non-trivial representations ρ of G. For some exceptional groups the uniqueness fails, as we see in Sections 10,11.

Our approach throughout is based on the observation that the geometry of the spectral covers reflects not so much the representations of G as those of its Weyl group W. Various questions about a spectral cover $\widetilde{X}_\rho$ simplify considerably when the emphasis is placed on the action of W rather than on the way $\widetilde{X}_\rho$ sits inside K. Here is what we do in more detail:

The spectral covers $\widetilde{X}_\rho$ decompose into subcovers $\widetilde{X}_\lambda$, indexed by W-orbits of weights λ. There are infinitely many distinct covers $\widetilde{X}_\rho$ or $\widetilde{X}_\lambda$, but they fall into only a finite number (2^r, where $r = rank_{ss}(G)$) of birational classes, cf. lemma (3.3). In section 2 we construct an abstract W-Galois cover $\widetilde{X} \to X$ which dominates all $\widetilde{X}_\lambda$. In good cases, points of $\widetilde{X}$ over $x \in X$ parametrize chambers in the dual of the unique Cartan subalgebra $\mathbf{t}(\varphi(x))$ containing $\varphi(x)$, so we call $\widetilde{X} \to X$ the *cameral* cover. With very few exceptions (listed in (4.3)), the spectral covers $\widetilde{X}_\lambda$ are forced to be singular as soon as X contains a compact curve, while the cameral cover $\widetilde{X}$ and its quotients by the parabolic subgroups serve as natural desingularizations, as long as the endomorphism φ remains regular. For example, this happens for $\mathbf{g} = \mathbf{so}(2n)$ and *any* non-trivial representation. (For the standard, $2n$-dimensional representation of $\mathbf{so}(2n)$, Hitchin notes these accidental singularities in [H], and attributes them to the vanishing of the Pfaffian.) In particular, it is unrealistic to hope that the eigensheaf L will "generically" be a line bundle on $\widetilde{X}_\rho$ or $\widetilde{X}_\lambda$: In typical situations we get torsion free sheaves on $\widetilde{X}_\lambda$, which come from line bundles on the cameral $\widetilde{X}$. (The original situation, where $G = GL(n)$ and ρ is the standard representation, is thus quite atypical!)

The ring of natural correspondences on $\widetilde{X}_\lambda$ is described in §6 in terms of the Weyl group W and the parabolic subgroup W_P determined by λ. The question of decomposing the spectral Picards is translated to decomposition of the permutation representation $\mathbf{Z}[W/W_p]$ as W-module. Some general results, based on Springer's representation and the work of [BM], are reviewed in Section 9. These results clarify the general form of the decomposition, but do not seem to imply the uniqueness of the common component. We thus work out the uniqueness for classical groups, and the non-uniqueness for some exceptional groups, by direct computations, in Sections 8, 10 and 11.

In this group-theoretic context, actually writing down the decomposition in any given case is very easy. In §12, we write down some formulas for the projection of a spectral Picard onto any generalized Prym. In the case of the projection to the distinguished Prym we recover Kanev's formulas (with minor modifications, which we explain). Kanev's construction, which is very geometric, is motivated by the interpretation of certain Weyl groups as symmetries of line configurations on rational surfaces. Our point is that similar formulas work much more generally, and require only elementary group theory. J.Y. Merindol informed me, during the Orsay conference, that he has also obtained projection formulas (onto the distinguished Prym) for arbitrary reductive groups, removing Kanev's restriction to "simply laced" groups, of types A_n, D_n, E_n.

For our purpose in this paper, we can take G to be any complex reductive group, but the resulting spectral and cameral covers depend only on the semisimple part G_{ss} of G, as does the distinguished Prym. There is however a more natural subvariety of $Pic\widetilde{X}$, consisting up to isogeny of $Prym(\widetilde{X})$ together with a number (equal to the dimension of the center of G) of copies of $PicX$. This corresponds to the reflection representation of W on the weights of G, which decomposes up to isogeny into the weights of G_{ss} and a trivial representation. In a sequel to this work [D3] we will describe this enlarged Prym in terms of W-equivariant bundles on $\widetilde{X}$, and interpret it as a moduli space of generalized Higgs bundles on X with given spectral invariants. Combined with work of Markman on the existence of Poisson structures [M], this leads to an algebraically completely integrable Hamiltonian system, generalizing those of Hitchin, Jacobi-Mumford-Beauville [B], and so on. The construction extends to Higgs bundles with values in a vector bundle K, as in [S], where K is the cotangent bundle of X.

My interest in these questions arose from conversations with M. Adler and P. van Moerbeke, P. Griffiths, and V. Kanev, about their works [AvM,G,K], followed by discussions with L. Katzarkov and T. Pantev about the G_2 case, which they analyzed in [KP], and with E. Markman about the general version of Hitchin's system. Conversations with C. Curtis and N. Spaltenstein provided valuable information about Weyl group representations. I also enjoyed and benefitted from discussions with A. Beauville, A. Kouvidakis, R. Lazarsfeld and E. Previato.

2 Cameral covers.

Given a principal Higgs bundle $(\mathcal{V}, \varphi)$ on X, we are going to construct a W-Galois cover $\widetilde{X} \to X$, which we call the cameral cover of $(\mathcal{V}, \varphi)$. It is independent of the choice of a representation. For each representation $\rho : G \to GL(V)$, the spectral cover $\widetilde{X}_\rho$ will break into pieces indexed by W-orbits of weights of ρ. Each of these pieces will be the image, under an appropriate morphism, of the cameral cover $\widetilde{X}$, in fact of a certain parabolic quotient $\widetilde{X}/W_P$. If the Higgs bundle is regular, we also have for each weight λ a line bundle L_λ on $\widetilde{X}$. It descends to a line bundle on the quotient $\widetilde{X}/W_P$, but only to torsion-free sheaves on $\widetilde{X}_\lambda, \widetilde{X}_\rho$, "usually" of rank 1 on $\widetilde{X}_\lambda$.

We start with some elementary observations on components of the spectral covers $\widetilde{X}_\rho$. First, if ρ is reducible:

$$(V, \rho) = \oplus (V_i, \rho_i),$$

then the spectral cover $\widetilde{X}_\rho \to X$ is just the union of the covers $\widetilde{X}_{\rho_i} \to X$. We may thus restrict attention to irreducible ρ. Next, consider the weight decomposition of V with respect to a maximal torus $T \subset G$:

$$V = \oplus_{\lambda \in D} V_\lambda = \oplus_{\lambda \in D \cap C} \oplus_{\mu \in W\lambda} V_\mu, \tag{2.1}$$

where Λ_G is the lattice of weights of G, $D \subset \Lambda_G$ is the set of weights of ρ, and C is the closed Weyl chamber (determined by a Borel $B \supset T$). There is a corresponding decomposition of the spectral cover:

$$\widetilde{X}_\rho = \sum_{\lambda \in D \cap C} m_\lambda \widetilde{X}_\lambda. \tag{2.2}$$

Here $m_\lambda = dim V_\lambda$ are the multiplicities, and the $\widetilde{X}_\lambda$ are constructed as follows:

Recall that Chevalley's theorem [Hu1] says that the restriction map

$$\mathbf{C}[\mathbf{g}]^G \to \mathbf{C}[\mathbf{t}]^W,$$

from **ad**-invariant polynomial functions on the Lie algebra **g** to W-invariant polynomial functions on the Cartan subalgebra **t**, is an isomorphism. This implies that for any weight λ, there is a unique **ad**-invariant polynomial function

$$P_\lambda : \mathbf{g} \to \mathbf{C}[x]$$

(the values are polynomials in one variable x), whose restriction to the Cartan **t** is the W-invariant function

$$\prod_{\mu \in W\lambda} (x - \mu) : \ \mathbf{t} \to \mathbf{C}[x].$$

The **ad**-invariance implies that P_λ makes sense on the bundle of algebras $\mathbf{ad}(\mathcal{V})$. The quantity $P_\lambda(\varphi)$ then gives a morphism between the total spaces of the line bundles:

$$P_\lambda(\varphi) : |K| \to |K^N|,$$

where $N = \#(W\lambda)$.

Definition 2.3. *The spectral cover $\widetilde{X}_\lambda$ determined by the Higgs bundle $(\mathcal{V}, \varphi)$ and the weight λ is the inverse image by $P_\lambda(\varphi)$ of the 0-section.*

By construction, $\widetilde{X}_\lambda$ is a subscheme of $|K|$, finite of degree N over X. The decomposition (2.2) now follows from (2.1) and the definitions of $\widetilde{X}_\rho$, $\widetilde{X}_\lambda$; in fact,

$$\prod_{\lambda \in D \cap C} (P_\lambda(\varphi))^{m_\lambda} = char(\rho(\varphi)).$$

From now on, instead of the (usually reducible) cover $\widetilde{X}_\rho$, we consider the collection of spectral covers $\widetilde{X}_\lambda$. We note that an irreducible ρ determines an "extremal" $\widetilde{X}_\lambda$, corresponding to the W-orbit of extremal weights for ρ. It occurs with multiplicity 1 in $\widetilde{X}_\rho$. Equality $\widetilde{X}_\rho = \widetilde{X}_\lambda$ holds if and only if the representation ρ is minuscule. In general, Weyl's character formula gives an explicit way of reconstructing $\widetilde{X}_\rho$ from this extremal piece $\widetilde{X}_\lambda$.

Although in general there are infinitely many non-isomorphic covers $\widetilde{X}_\lambda$, they fall into only a finite number of birational equivalence classes. Next we construct an object $\widetilde{X}$ which dominates all of them.

Using Chevalley's theorem again, we have an injective ring homomorphism

$$\mathbf{C}[\mathbf{t}]^W \approx \mathbf{C}[\mathbf{g}]^G \hookrightarrow \mathbf{C}[\mathbf{g}].$$

Taking Spec, we find a surjective, G-invariant morphism of affine varieties

$$\mathbf{g} \twoheadrightarrow \mathbf{t}/W. \tag{2.4}$$

We can then form the fiber product

$$\tilde{\mathbf{g}} = \mathbf{g} \times_{\mathbf{t}/W} \mathbf{t}. \tag{2.5}$$

The projection $\pi : \tilde{\mathbf{g}} \to \mathbf{g}$ is a finite morphism which is W-Galois; we call it the cameral cover of the Lie algebra $\mathbf{g}$. A regular semisimple element $g \in \mathbf{g}$ is contained in a unique Cartan subalgebra $\mathbf{t}$. The fiber $\pi^{-1}(g)$ can be identified with the set of Borels containing $\mathbf{t}$, or equivalently with the set of chambers in $\mathbf{t}^*$.

Given a Higgs bundle $(\mathcal{V}, \varphi)$ on X, we relativize the previous construction to define $\widetilde{X}$: Since (2.4) is G-invariant and $\mathbf{C}^*$-equivariant, it extends to a morphism

$$|\mathbf{ad}(\mathcal{V}) \otimes K| \to |(\mathbf{t} \otimes K)|/W$$

so we can form the fiber product with $\mathbf{t} \otimes K$ as in (2.5), then pull back to X via φ:

Definition (2.6). *The cameral cover determined by the principal Higgs bundle $(\mathcal{V}, \varphi)$ is given by $\pi : \widetilde{X} \to X$, where*

$$\widetilde{X} = \varphi^*(|\mathbf{ad}(\mathcal{V}) \otimes K| \times_{|\mathbf{t}\otimes K|/W} |\mathbf{t} \otimes K|)$$

and π is the projection on the first factor.

We can describe the fiber $\pi^{-1}(x)$ over $x \in X$ in several ways, e.g. as the set of prints (in $\mathbf{t} \otimes K_x$) which are conjugate (via elements of $\mathcal{V}_x$) to the semisimple part of $\varphi(x) \in \mathbf{ad}(\mathcal{V}_x) \otimes K_x$. If $\varphi(x)$ is regular semisimple, hence contained in a unique Cartan $\mathbf{t}_x \subset \mathbf{ad}(\mathcal{V}_x) \otimes K_x$, the fiber can be more simply described as the set of Borels through $\mathbf{t}_x$, or chambers in $\mathbf{t}_x^*$. When $\mathbf{g} = \mathbf{gl}(n)$ on $\mathbf{sl}(n)$, a point of the fiber is given by an ordering of the eigenvalues of $\varphi(x)$.

For each λ, consider the morphism

$$j_\lambda : \tilde{\mathbf{g}} = \mathbf{g} \times_{\mathbf{t}/W} \mathbf{t} \to \mathbf{g} \times \mathbf{C}$$

sending

$$(g, t) \mapsto (g, \lambda(t)).$$

Clearly $P_\lambda \circ j_\lambda = 0$, so j_λ factors through a morphism

$$i_\lambda : \tilde{\mathbf{g}} \to \tilde{\mathbf{g}}_\lambda = \{P_\lambda = 0\} \subset \mathbf{g} \times \mathbf{C}.$$

Since the covers $\widetilde{X}$, $\widetilde{X}_\lambda$ are (locally) obtained as pullbacks via φ of $\tilde{\mathbf{g}}, \tilde{\mathbf{g}}_\lambda$ respectively, this globalizes to a morphism

$$i_\lambda : \widetilde{X} \to \widetilde{X}_\lambda. \tag{2.7}$$

Finally, for each λ, the line bundle L_λ on $\widetilde{X}$ is induced from the corresponding Borel-Weil-Bott line bundle on the flag variety G/B. These bundles are the main object of study in [D3], and will not be further discussed here.

3 Parabolic subgroups.

Fix a Cartan subalgebra $\mathbf{t} \subset \mathbf{g}$ and a Borel subalgebra $\mathbf{b} \supset \mathbf{t}$, with corresponding maximal torus T and Borel subgroup B in G. Let $R \subset \mathbf{t}^*$ be the root system, R^+ the positive roots with respect to $\mathbf{b}$, $S = \{\alpha_1, \cdots, \alpha_r\}$ the simple roots, and C the closed Weyl chamber. We recall (e.g. [Hu2], §30) that there is a natural bijection between the following sets,

(3.1)

(1) Parabolic subalgebras $\mathbf{p} \supset \mathbf{b}$

(2) Parabolic subgroups $P \supset B$.

(3) Subsets $S_P \subset S$

(4) Subgroups $W_P \subset W$ generated by reflections in simple roots.

(5) Faces C_P of C.

This goes as follows: to the subset S_P associate the reductive (Levi) subalgebra $\mathbf{l}_P$ spanned by the root spaces $\mathbf{g}_\alpha$ with α in

$$R_P := R \cap span(S_P),$$

and then

$$\mathbf{p} := \mathbf{l}_P \mathbf{b} = \mathbf{l}_P \mathbf{u}_P \subset \mathbf{g}$$

where $\mathbf{u}_P$ is spanned by $\mathbf{g}_\alpha$ with α in $R^+ - R_P$. We can also get the parabolic group directly as

$$P := BW_PB$$

where W_P is the subgroup of W generated by the reflections σ_α in simple roots $\alpha \in S_P$. Conversely, P determines W_P as its Weyl group, i.e. it determines the normalizer

$$N_P := N_P(T) = P \cap N_G(T),$$

hence also its image $W_P := N_P/T$ in $W = N_G(T)/T$. Now W_P determines

$$S_P := \{\alpha \in S | \sigma_\alpha \in W_P\}.$$

Finally, we define the face

$$C_P := (Span(S_P))^\perp = \{\text{fixed points of } W_P \text{ in } C\}.$$

The subgroup W_P is recovered as the stabilizer of (all, or any one of) the points in the interior C_P^0 of C_P.

Thus $\mathbf{g}, G$ correspond to the subset S and the group W, and the face C_G is the vertex; $\mathbf{b}, B$ correspond to $\varnothing$, (1), C; minimal parabolics correspond to singletons $S_{P_i} = \{\alpha_i\}$, subgroups $W_{P_i} = (\sigma_{\alpha_i})$, and to walls of C; maximal parabolics to $S \setminus \{\alpha_i\}$ and to edges of C (if $\mathbf{g}$ is semisimple; otherwise, C_P modulo the center is an edge).

Returning to the cameral cover, we define the intermediate cover $\widetilde{X}_P \to X$ corresponding to a parabolic $P \supset B$ (or subset $S_P \subset S$) as the quotient

$$\widetilde{X}_P = \widetilde{X}/W_P. \tag{3.2}$$

Thus $\widetilde{X}_G = X$ and $\widetilde{X}_B = \widetilde{X}$. We see immediately:

Lemma (3.3). *The map $i_\lambda : \widetilde{X} \to \widetilde{X}_\lambda$ of (2.7) factors through $\widetilde{X}_P$ if (and generically, only if) λ is in the face C_P.*

(One interpretation of the generic statement is that over the whole $\mathbf{g}$, $\tilde{\mathbf{g}}_P \to \tilde{\mathbf{g}}_\lambda$ is a birational morphism whenever λ is in the interior of face C_P.)

4 Accidental singularities.

Fix a parabolic P and a weight λ in the interior C_P^0 of the corresponding face. What is the expected behavior of the birational morphism $i_\lambda : \widetilde{X}_P \to \widetilde{X}_\lambda$ of (3.3)? Consider the simplest case: $\mathbf{g} = \mathbf{sl}(n)$, with its standard representation ρ, so λ is the fundamental weight ω_1, and $\mathbf{p}$ the corresponding maximal parabolic. In this case both $\widetilde{X}_P$ and $\widetilde{X}_\lambda$ parameterize the eigenvalues of $\rho(\varphi(x))$, for $x \in X$, so i_λ is an isomorphism. (These are the standard spectral covers considered, e.g. in [BNR].)

We would like to point out that this situation is quite atypical. In fact, for any $\mathbf{g}$ and almost any λ, the birational map

$$i_\lambda : \tilde{\mathbf{g}}_P \to \tilde{\mathbf{g}}_\lambda$$

will fail to be an isomorphism over a non-empty divisor in $\mathbf{g}$, consisting of elements $g \in \mathbf{g}$ at which distinct weights λ, $w\lambda$ accidentally take the same value. Most points of this divisor will actually be regular semisimple. As a result, we expect $i_\lambda : \widetilde{X}_P \to \widetilde{X}_\lambda$ to fail to be an isomorphism as soon as X (contains, or) is a complete curve, and no regularity requirement on $\varphi(x)$, $x \in X$ will improve this situation.

Let $t \in \mathbf{t}$ be regular, i.e. $\alpha(t) \neq 0$ for each root $\alpha \in R$. Then i_λ has an accidental singularity at t iff $(\lambda - w\lambda)t = 0$ for some $w \in W \setminus W_P$. Hence:

$$i_\lambda \text{ has no accidental singularities at regular semisimple points } \Leftrightarrow$$

$$\forall w \in W \setminus W_p,\ \lambda - w\lambda \text{ vanishes only at singular points} \Leftrightarrow$$

$$\forall w \in W,\ \lambda - w\lambda \text{ is a multiple of some root.} \tag{4.1}$$

This is therefore a necessary condition for $i_\lambda = \tilde{\mathbf{g}}_P \to \tilde{\mathbf{g}}_\lambda$ to be an isomorphism.

Lemma (4.2). *Condition (4.1) implies that* $\mathbf{p}$ *is maximal parabolic, so* λ *equals a multiple of a fundamental weight, modulo the center.*

To see this, we may as well divide by the center and assume that $\mathbf{g}$ is semisimple. Let $\{\alpha_i\}$ be the simple roots, $\{\omega_i\}$ the fundamental weights, and $\sigma_i \in W$ the reflection perpendicular to α_i. Write $\lambda = \sum m_i\omega_i$, $m_i \geq 0$. Then $\lambda - \sigma_i\lambda = m_i\alpha_i$, so $m_i\alpha_i - m_j\alpha_j$ should be a root multiple for every i, j. By the definition of a simple set of roots, m_i and $-m_j$ must have the same sign.

We conclude that at most one m_i is $\neq 0$, i.e. λ is a multiple of ω_i as claimed. Q.E.D.

For each Dynkin diagram there are therefore, up to homothety, only finitely many possibilities for λ such that i_λ is an isomorphism (on regular semisimple elements, or equivalently, everywhere). For the classical algebras and for G_2, we see easily that the possibilities are:

$$\begin{array}{lcl} A_n & : & \omega_1, \omega_n \\ B_2 & : & \omega_1, \omega_2 \\ B_n & : & \omega_1 \quad (n \geq 3) \\ C_n & : & \omega_1 \quad (n \geq 3) \\ D_n & : & none \\ G_2 & : & \omega_1, \omega_2 \end{array}$$

We note that the spectral curves considered by Hitchin[H] are the $\widetilde{X}_\rho$ for the classical algebras and for the standard representation ρ, of highest weight ω_1. These are minuscule for types A, C, D but have the additional weight 0 for B_n, and i_λ is an isomorphism for types A, B, C but has accidental singularities for D_n. Accordingly, his spectral curves are (generically) non-singular for A_n, C_n; always singular for D_n; and reducible for B_n.

5 Isotypic decomposition of Pic.

Consider, in this and the next section, the general situation of a finite group W acting faithfully on a variety $\widetilde{X}$, with quotient X. We get actions of $\mathbf{Z}[W]$ on $\widetilde{X}$, hence on $H_*(\widetilde{X}, \mathbf{Z})$ and on $Pic(\widetilde{X})$. Given an irreducible $\mathbf{Z}[W]$-module Λ, we consider its equivariant maps to $Pic\widetilde{X}$:

$$(5.1) \qquad Prym_\Lambda(\widetilde{X}) := Hom_W(\Lambda, Pic\widetilde{X}).$$

this is an algebraic group, and an abelian variety if $Pic\widetilde{X}$ is, e.g. if $\widetilde{X}$ is non-singular and projective.

For each $e \in \Lambda$ we get an evaluation map

$$eval_e : Prym_\Lambda(\widetilde{X}) \to Pic\widetilde{X}.$$

The kernel of $eval_e$ is finite, for any $e \neq 0$, since $\mathbf{Z}[W]e$ has finite index in the irreducible module Λ. Up to isogeny we may therefore think of $Prym_\Lambda(\widetilde{X})$ as an algebraic subgroup of $Pic\widetilde{X}$; but this is usually unnatural, since changing e can result in a different (isogenous) copy of $Prym_\Lambda(\widetilde{X})$ inside $Pic\widetilde{X}$.

A W-submodule $\Lambda' \subset \Lambda$ of finite index determines a restriction map

$$Res: \ Prym_\Lambda \widetilde{X} \to Prym_{\Lambda'} \widetilde{X}$$

of finite kernel and cokernel. Therefore, $Prym_\Lambda \widetilde{X}$ makes sense up to isogeny if Λ is only an irreducible $\mathbf{Q[W]}$-module.

From now on, we will assume that all irreducible representations of W are defined over $\mathbf{Q}$. We can then choose a set $\{\Lambda_i\}$, $i \in W$, of irreducible $\mathbf{Z}[W]$-modules whose complexifications give all the irreducible representations. $\mathbf{Z}[W]$ then decomposes as a two-sided W module, up to isogeny:

$$\mathbf{Z}[W] \sim \oplus_i \Lambda_i \otimes \Lambda_i^*, \tag{5.2}$$

where W acts on the left on Λ_i and on the right on Λ_i^*. We obtain the corresponding isotypic isogeny decomposition:

$$Pic\widetilde{X} \sim \oplus_i \Lambda_i \otimes_{\mathbf{Z}} Prym_{\Lambda_i} \widetilde{X}. \tag{5.3}$$

6 Subgroups, subcovers, correspondences.

Fix a subgroup W_P of W. The action of W on $\widetilde{X}$ restricts to an action of W_P, giving an intermediate cover $\widetilde{X}_P$:

$$\widetilde{X} \xrightarrow{\pi_P} \widetilde{X}_P \xrightarrow{\pi^P} X.$$

There is no natural action of W on $\widetilde{X}_P$ or on $Pic(\widetilde{X}_P)$, but $Pic(\widetilde{X}_P)$ can still be decomposed into its isotypic components with respect to the action of the ("Hecke") ring $\mathcal{C}_P$ of correspondences on $\widetilde{X}_P$ over X. This has the following elementary description:

(6.1) A correspondence on $\widetilde{X}_P$ over X is a top dimensional cycle, or linear combination of components, of

$$\widetilde{X}_P \times_X \widetilde{X}_P = (\widetilde{X}/W_P) \times_X (\widetilde{X}/W_P).$$

The set of components is given by the quotient

$$W\backslash(W \times W)/W_P \times W_P \approx W_P\backslash W/W_P,$$

i.e. by double W_P-cosets in W. Each double-coset $C = W_P w W_P$, $w \in W$, gives an effective correspondence:

$$I_C = I_w := \{(xW_P, xw'W_P) | w' \in C/W_P,\ x \in \widetilde{X}\}.$$

These correspondences are independent (as long as the action of W is faithful), and if $\widetilde{X}$ is irreducible, these I_w form a $\mathbf{Z}$-basis for all correspondences. So, as a group, $\mathcal{C}_P \approx \mathbf{Z}[W_P\backslash W/W_P]$.

(6.2) When $W_P = (1)$, the ring of correspondences is just the integral group ring $\mathcal{C}_1 \approx \mathbf{Z}[W]$, acting naturally on $\widetilde{X}$ and hence on $Pic(\widetilde{X})$. In general, there is an injective pullback map

$$\begin{array}{rcl} \pi_P^* : \mathcal{C}_P & \to & \mathcal{C}_1 \approx \mathbf{Z}[W] \\ I_C & \mapsto & \sum_{w \in C} I_w \end{array}$$

satisfying

$$\pi_P^* I_{c_1} \cdot \pi_P^* I_{c_2} := \#(W_P) \cdot \pi_P^*(I_{c_1} \cdot I_{c_2}).$$

We can thus identify $\mathcal{C}_P$ with the subring of $\mathbf{Q}[W]$ generated (as abelian group) by

$$i_C := \frac{1}{\#(W_P)} \sum_{w \in C} I_w$$

as C runs over the double W_P cosets in W. (The image will usually not contain the identity element.)

We can now describe several ways of decomposing $Pic(\widetilde{X}_P)$ into natural components:

(6.3) The ring of correspondences $\mathcal{C}_P$ acts naturally on $Pic(\widetilde{X}_P)$, so every integral representation of $\mathcal{C}_P$ determines a generalized Prym variety as in (5.1), and these can be grouped into isotypic components as in (5.3).

(6.4) We can map $Pic(\widetilde{X}_P)$ to $Pic(\widetilde{X})$ via π_P^*, and intersect the image with the isotypic components $\Lambda_i \otimes Prym_{\Lambda_i}\widetilde{X}$ (with respect to the W action) in $Pic(\widetilde{X})$, defined in (5.3).

(6.5) The direct image sheaf $\pi_*^P \mathbf{Z}$ on X is associated to the W-cover $\widetilde{X}$ by the permutation representation $\mathbf{Z}[W_P\backslash W]$:

$$\pi_*^P \mathbf{Z} \approx \mathbf{Z}[W_P\backslash W] \times_W \widetilde{X}.$$

The decomposition of $\mathbf{Z}[W_P\backslash W]$ into irreducible W-representations (over $\mathbf{Q}$) determines a decomposition of $\pi_*^P\mathbf{Z}$, hence of $H^1(\widetilde{X},\mathbf{Z})$, and hence an isogeny decomposition of $Pic(\widetilde{X}_P)$.

We see that all these decompositions are essentially the same (up to isogeny, i.e. the connected components agree). In terms of the decomposition (5.3) of $Pic\widetilde{X}$, the image of π_P^* in (6.4) is given up to isogeny by

$$\oplus_i M_i \otimes Prym_{\Lambda_i}\widetilde{X} \subset \oplus_i \Lambda_i \otimes Prym_{\Lambda_i}\widetilde{X}, \tag{6.6}$$

Where the multiplicity space M_i is given by the W_P-invariants $(\Lambda_i)^{W_P}$. By Frobenius reciprocity, this corresponds exactly to the decomposition of $\mathbf{Q}[W_P\backslash W]$:

$$\mathbf{Q}[W_P\backslash W] \approx \oplus_i \Lambda_i^{W_P} \otimes \Lambda_i^* \otimes \mathbf{Q},$$

so (6.4) and (6.5) agree.

To compare with the action of the ring of correspondences, we note that

$$\begin{aligned}\mathbf{Q}\otimes\mathcal{C}_P &= \mathbf{Q}[W_P\backslash W/W_P] = End_{\mathbf{Q}[W]}\mathbf{Q}[W_P\backslash W]\\ &= End_{\mathbf{Q}[W]}(\oplus_i M_i \otimes \Lambda_i^* \otimes \mathbf{Q})\\ &= \oplus_i End_{\mathbf{Q}}(M_i\otimes\mathbf{Q}) = \oplus_i(End M_i)\otimes\mathbf{Q},\end{aligned}$$

and the action on $Pic\widetilde{X}_P \sim \oplus_i M_i \otimes Prym_{\Lambda_i}(\widetilde{X})$ is consistent with this decomposition, i.e. $End M_i$ acts as 0 on $M_j \otimes Prym_{\Lambda_j}\widetilde{X}$ if $j \neq i$, and through its action on M_i if $j = i$. So the generalized Pryms obtainable from (6.3) are precisely those $Prym_{\Lambda_i}(\widetilde{X})$ for which $M_i = \Lambda_i^{W_P}$ is non-zero, and the isotypic decomposition is the same as the one obtained from (6.4) or (6.5).

Let $\mathbf{1}_{W_i}^W$ denote the permutation representation of W on W_i cosets, or its character. We note for subsequent use the following corollary of Frobenius reciprocity:

Lemma (6.7). *If W_i, W_j are subgroups of W, then*

$$(\mathbf{1}_{W_i}^W, \mathbf{1}_{W_j}^W) = dim(\mathbf{1}_{W_j}^W)^{W_i} = \#(W_j\backslash W/W_i),$$

where the left side denotes the inner product of characters, and the right side is the number of two sided cosets.

7 An example.

Consider the symmetric group $W = S_3$. It has 3 subgroups of order 2, generated by (23), (13), and (12), and a normal subgroup A_3, of order 3. The W-cover $\widetilde{X} \to X$ has corresponding intermediate covers $X_1, X_2, X_3, \overline{X}$. W has character table:

	1	C_2	C_3
$\mathbf{1}$	1	1	1
ε	1	-1	1
ρ	2	0	-1

where C_i is the conjugacy class of elements of order i; 1 is the trivial character, ε the sign character, and ρ the character of the 2-dimensional reflection representation Λ:

$$\Lambda = \mathbf{Z}e \oplus \mathbf{Z}f$$

$$\begin{array}{ll} (12)\ e = e & (12)\ f = -e - f \\ (23)\ e = -e - f & (23)\ f = f \end{array}$$

We obtain the decompositions,

$$\begin{array}{lllllllll}
Pic\widetilde{X} & \sim & PicX & + & Prym_\varepsilon\widetilde{X} & + & \Lambda & \otimes & Prym_\Lambda\widetilde{X} \\
Pic\overline{X} & \sim & PicX & + & Prym_\varepsilon\widetilde{X} & & & & \\
PicX_1 & \sim & PicX & + & & & e & \otimes & Prym_\Lambda\widetilde{X} \\
PicX_2 & \sim & PicX & + & & & (-e-f) & \otimes & Prym_\Lambda\widetilde{X} \\
PicX_3 & \sim & PicX & + & & & f & \otimes & Prym_\Lambda\widetilde{X}
\end{array}$$

8 The distinguished Prym.

As explained in the introduction, the linearization of algebraically completely integrable systems via spectral covers suggests that there should be a unique, or at least a distinguished, nontrivial irreducible representation $\Lambda \neq 1$ of W such that the Prym variety $Prym_\Lambda$ occurs in $\widetilde{X}_P$ for all proper subgroups $W_P \neq W$. As the example in §7 shows, this is not true for all finite groups W, in fact not even for Weyl groups if we allow W_P to be an arbitrary subgroup. Restricting attention only to the Weyl subgroups, we are left in the above example with

$\widetilde{X}$, X_1 and X_3, each of which contains the trivial piece $PicX$ and at least one copy of $Prym_\Lambda \widetilde{X}$.

This picture generalizes as follows. For any Weyl group W of a reductive Lie group G, all representations are defined over $\mathbf{Q}$. We can describe three natural irreducible representations: the trivial representation $\mathbf{1}$, the sign representation ε, and the reflection representation of W acting on the weight lattice Λ of the semisimple part of G. Of these, $\mathbf{1}$ occurs in $\mathbf{1}_{W_P}^W$ for any $W_P \subset W$, and ε does not occur in $\mathbf{1}_{W_P}^W$ for any proper Weyl subgroup W_P.

Theorem (8.1). *(1) The Prym variety $Prym_\Lambda \widetilde{X}$ corresponding to the reflection representation Λ occurs with positive multiplicity in $Pic(\widetilde{X}_P)$ for any proper Weyl subgroup $W_p \neq W$.*

(2) For the classical groups, Λ is the only nontrivial irreducible representation of W with this property.

Proof.

$$\begin{aligned}(1)\ mult(Prym_\Lambda \widetilde{X}, Pic\widetilde{X}_P) &= mult(\Lambda, \mathbf{1}_{W_P}^W) = && \text{(by Frobenius)}\\ &= dim(\Lambda)^{W_P} = && \text{(compare (3.1))}\\ &= dim(C_P/\text{center}) > 0. &&\end{aligned}$$

(2) For a given Weyl group W, the question is: Find all irreducible W-representations V such that $V^{W_P} \neq (0)$ for each proper Weyl subgroup $W_P \neq W$.

For type A_n, i.e. $G = SL(n+1)$ and $W = S_{n+1}$, take $W_{P_0} = S_n$ corresponding to a Dynkin subdiagram of type A_{n-1}. Then over $\mathbf{Q}$:

$$\mathbf{1}_{W_{P_0}}^W = \mathbf{1} \oplus \Lambda,$$

so no representation other then ($\mathbf{1}$ and) Λ is common to all $\mathbf{1}_{W_P}^W$, as required.

Consider the Weyl group W of type B_n, with the nodes labeled as in [Bo] so that $\alpha_1, \cdots, \alpha_{n-1}$ are long and α_n is short. Let W_i denote the Weyl subgroup obtained by deleting α_i. In the standard permutation representation of W on the $2n$ vectors $\pm\varepsilon_i$, $1 \leq i \leq n$, the stabilizer of ε_1 is W_1, so we get the decomposition

$$\mathbf{1}_{W_P}^W = \mathbf{1} \oplus \Lambda \oplus \Lambda'$$

where Λ is the n-dimensional reflection representation of W, and Λ' is the $(n-1)$-dimensional (reflection) representation of S_n, pulled back to W.

More generally, W_i is the stabilizer in W of $\varepsilon_1 + \cdots + \varepsilon_i$. We see from (6.7) that

$$(\mathbf{1}_{W_1}^W, \mathbf{1}_{W_i}^W) = \#(W_1 \backslash W / W_i) = \begin{cases} 3 & i = 1, 2, \cdots, n-1 \\ 2 & i = n \end{cases}.$$

In particular, for $i = n$ we have by part (1) a decomposition

$$\mathbf{1}_{W_n}^W = \mathbf{1} \oplus \Lambda \oplus V$$

for some representation V satisfying $0 = (V, \mathbf{1}_{W_1}^W)$, and in particular $0 = (V, \Lambda')$. So again, $\mathbf{1}$ and Λ are the only irreducible representations common to the $\mathbf{1}_{W_i}^W$ for all i. The same argument works with no change for type C_n. For D_n, the only change is that W_{n-1} is the stabilizer of $\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n$; but W_n is still the stabilizer of $\varepsilon_1 + \cdots + \varepsilon_n$, so we still have

$$(\mathbf{1}_{W_1}^W, \mathbf{1}_{W_n}^W) = 2,$$

and this case follows as well. Q.E.D.

9 Remarks on Springer's correspondence.

In fact, we can say much more about the decompositon of $\mathbf{1}_{W_i}^W$, or more generally of $\mathbf{1}_{W_P}^W$, into irreducibles. The picture is clearest for type A_n; in this case, the irreducible representations V_λ of $W = S_{n+1}$ are parametrized by partitions λ of $n+1$, as are the conjugacy classes (in $GL(n+1)$) of Levi subgroups, the conjugacy classes in W of Weyl sugbroups W_λ, and the unipotent conjugacy classes in $GL(n+1)$ (or the nilpotent classes in $\mathbf{gl}(n+1)$). These classes are partially ordered (e.g. by inclusion of unipotent class closures), and Young's rule says that the decomposition matrix $(m_{\lambda,\mu})$, giving the multiplicity of V_λ in $\mathbf{1}_{W_\mu}^W$, is a triangular matrix with $1's$ on the diagonal.

In fact, $m_{\lambda\mu}$ is the Kostka number [J], defined as the number of semistandard tableaux on μ of type λ. The uniqueness of Λ of course follows from the triangularity of the decomposition matrix. Explicitly, the representations $\mathbf{1}$, Λ and ε correspond to the partitions (n), $(n-1,1)$, and (1^n). The partitions which occur in the $\mathbf{1}_{W_i}^W$ are those with at most two parts:

(9.1) $$\mathbf{1}_{W_i}^W = \oplus_{j=0}^i V_{n-j,j} \qquad \text{if} \quad 2i \le n.$$

For other groups, the picture is somewhat more complicated. To a Levi subgroup $L \subset G$ (e.g. to the Levi $L(P)$ of a parabolic P) we can associate the (unipotent) conjugacy class $\mathcal{O}_u$ of a regular unipotent element $u \in L$. In general, this correspondence may no longer be bijective. We say that a conjugacy class is of parabolic type if it comes from some $L(P)$. For a unipotent u, consider the Springer fiber

$$\mathcal{B}_u := \{\text{Borel subgroups of G containing } u\}.$$

Springer and others construct an action of W on $\mathcal{B}_u$, hence on its cohomology $H^*(\mathcal{B}_u)$. Alvis and Lusztig [AL] identify $H^*(\mathcal{B}_u)$ with $\mathbf{1}_{W_p}^W$, in case u is regular unipotent in $L(P)$ as above. On the other hand, the top cohomology $H^{top}(\mathcal{B}_u)$ decomposes into irreducible W- representations $S_{u,\ell}$ indexed by the irreducible local systems ℓ on $\mathcal{O}_u$. (In case A_n, $\mathcal{O}_u$ is simply connected so ℓ is trivial.) The Springer correspondence

$$(\mathcal{O}_u, \ell) \mapsto S_{u,\ell}$$

gives all irreducible W-representations.

The triangularity part of Young's rule has an analogue for arbitrary W, due to Borho and MacPherson[BM]: any component of $H^*(\mathcal{B}_u)$ (i.e. of $\mathbf{1}_{W_p}^W$, by [AL]) is of the form $S_{v,\ell}$ for some unipotent v (and local system ℓ on $\mathcal{O}_v$) such that $\overline{\mathcal{O}}_v \supset \mathcal{O}_u$.

Since the Springer correspondence has been completely determined (see, e.g. [Ca] §13.3), the triangularity result provides a powerful tool for analyzing the decomposition of permutation representations of W. Yet we do not see how to use it for our purposes, since it only provides block-triangularity. Two things can go wrong:

- The largest non regular (="subregular") unipotent class corresponds to the reflection representation. But it may fail to be simply connected, and may thus contribute to more than a single irreducible W-representation which cannot be excluded; or

- There may be unipotent classes, strictly smaller (= in the closure of) the subregular, but containing in their closure all unipotent classes of parabolic type.

We will see below that for the exceptional groups both problems do occur.

10 Decomposition for G_2.

In this section and the next we show that the uniqueness (part (2) of Theorem (8.1))

can fail for some exceptional groups. We do this by explicit calculation.

The Weyl group $W(G_2)$ is isomorphic to the dihedral group $Dihed_6$, of order 12, generated by a rotation r, of order 6, and a reflection s. The character table is, in the notation of [Se]:

	ψ_1	ψ_2	ψ_3	ψ_4	χ_1	χ_2
1	1	1	1	1	2	2
r, r^{-1}	1	1	-1	-1	1	-1
r^2, r^{-2}	1	1	1	1	-1	-1
r^3	1	1	-1	-1	-2	2
s, sr^2, sr^{-2}	1	-1	1	-1	0	0
sr, sr^3, sr^{-1}	1	-1	-1	1	0	0

Here $\mathbf{1} = \psi_1$, $\Lambda = \chi_1$, $\varepsilon = \psi_2$. There are only two non-trivial Weyl subgroups, say $W_1 = (s)$ and $W_2 = (sr)$. We see then that the characters of $\mathbf{1}_{W_1}^W$ and $\mathbf{1}_{W_2}^W$ are, respectively, $(6,0,0,0,2,0)$ and $(6,0,0,0,0,2)$. The decomposition is thus:

$$\text{(10.1)} \qquad \mathbf{1}_{W_1}^W = \psi_1 + \psi_3 + \chi_1 + \chi_2, \quad \mathbf{1}_{W_2}^W = \psi_1 + \psi_4 + \chi_1 + \chi_2.$$

This can also be seen very explicitly: let H denote $W_1 = (1, s)$, then $\mathbf{Q}[W/H] =: \mathcal{U}_1$ decomposes into:

$$\begin{aligned}
\psi_1 &\sim \mathbf{Q}[H + rH + r^2H + r^3H + r^{-2}H + r^{-1}H] \\
\psi_3 &\sim \mathbf{Q}[H - rH + r^2H - r^3H + r^{-2}H - r^{-1}H] \\
\chi_1 &\sim \mathbf{Q}[H - rH - r^3H + r^{-2}H, H - r^2H - r^3H + r^{-1}H] \\
\chi_2 &\sim \mathbf{Q}[H - rH + r^3H - r^{-2}H, H - r^2H + r^3H - r^{-1}H].
\end{aligned}$$

The decomposition for $\mathcal{U}_2 := \mathbf{1}_{W_2}^W$ is obtained similarly.

Theorem (8.1) is thus <u>false</u> for G_2: there are two non-trivial pieces common to all $Pic(\widetilde{X_P})$, namely $\Lambda = \chi_1$ and χ_2.

These Pryms can be described more explicitly as follows: The cover $\widetilde{X}^1 = \widetilde{X}/W_1$, of degree 6 over X, is the fiber product of its two intermediate covers:

$$X' = \widetilde{X}^1/(r^2), \quad X'' = \widetilde{X}^1/(r^3),$$

of degrees $2,3$ respectively over X. From the decomposition of permutation representations:

$$\begin{aligned} \mathbf{1}^W_{(s,r^2)} &= \psi_1 + \psi_3 \\ \mathbf{1}^W_{(s,r^3)} &= \psi_1 + \chi_2 \end{aligned}$$

We find

$$\begin{aligned} PicX' &\sim PicX + Prym_{\psi_3}\widetilde{X} \\ PicX'' &\sim PicX + Prym_{\chi_2}\widetilde{X}, \end{aligned}$$

hence

$$\begin{aligned} Prym_{\chi_2}\widetilde{X} &\sim Prym(X''/X) \\ Prym_{\chi_1}\widetilde{X} + Prym_{\chi_2}\widetilde{X} &\sim Prym(\widetilde{X}^1/X') \\ Prym_{\chi_1}\widetilde{X} &\sim Prym(\widetilde{X}^1/(X',X'')). \end{aligned}$$

We note that $W(G_2)$ has the outer automorphism $s \mapsto sr$, $r \mapsto r$, which exchanges W_1 and W_2, ψ_3 and ψ_4, and fixes $\psi_1, \psi_2, \chi_1, \chi_2$; so the above decomposition of $Pic\widetilde{X}^1$ is transformed to the corresponding decomposition for $Pic\widetilde{X}^2$.

Starting with arbitrary X and (branched) covers X', X'' of degrees $2,3$, we construct $X''!$, of degree 6, and then set

$$\widetilde{X}^1 = X' \times_X X'', \quad \widetilde{X} = X' \times_X X''!,$$

recovering the previous situation. If X is a curve of genus g, and X', X'' have respectively $2n, 2m$ simple ramification points over disjoint branch loci in X, we find for the Pryms of types $\psi_1, \psi_3, \chi_1, \chi_2$ the dimensions

$$g, g-1+n, 2g-2+m+2n, 2g-2+m.$$

In particular, we see that the three components $Prym_{\psi_1}$, $Prym_{\chi_1}$, $Prym_{\chi_2}$ common to $Pic\widetilde{X}^1$ and $Pic\widetilde{X}^2$ have different dimensions, and in general there will be no nontrivial maps between them. If we take $X = \mathbf{P}^1$, the common piece $Prym_{\chi_2}$ becomes the Jacobian of the trigonal curve X''. But the distinguished piece, $Prym_{\chi_1}$, still seems to be neither a Jacobian nor a classical Prym.

It is also interesting to compare this with the explicit description of the Springer correspondence, in [Ca]. There are 5 unipotent conjugacy classes in G_2, denoted there G_2 (the regular unipotents), $G_2(a_1)$ (the subregulars), $\tilde{A}_1$, A_1 and 1. These are in descending order (the partial order is a total order for G_2). The Springer correspondence then sends:

$$\begin{array}{rcl} G_2 & \mapsto & \psi_1 \\ G_2(a_1) & \mapsto & \psi_4, \chi_1 \\ \tilde{A}_1 & \mapsto & \chi_2 \\ A_1 & \mapsto & \psi_3 \\ 1 & \mapsto & \psi_2, \end{array}$$

where χ_1 comes from the trivial local system on the subregular orbit, and ψ_4 from the nontrivial rank-2 local system (the fundamental group is S_3). In the terminology of §9, the unipotent class associated to (the Levi of) W_2 is $\tilde{A}_1$, and all characters allowed by Borho-MacPherson's triangularity do occur. But the unipotent class associated to W_1 is A_1, and one character (ψ_4) allowed by [BM] is missing.

11 The decomposition for E_6.

We use Schläfli's description of the exceptional Weyl group $W := W(E_6)$, as incidence-preserving permutations of the 27 lines on a general cubic surface (cf.[CCNPW], p.26). For the "lines" we take the $27 = 6 + 6 + 15$ objects:

$$a_i \quad , \quad b_j \quad , \quad c_{ij} = c_{ji} \qquad (i, j = 1, \cdots, 6, \ i \neq j).$$

Two lines are incident if they lie in one of the $45 = 30 + 15$ "tritangent planes":

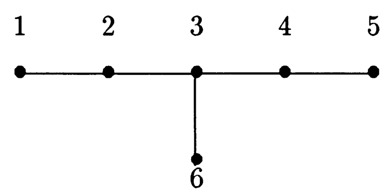
$$(a_i, b_j, c_{ij}) \quad , \quad (c_{ij}, c_{k\ell}, c_{mn}) \qquad (i, \cdots, n \text{ distinct } \in \{1, \cdots, 6\}).$$

There are 72 "sixes" of pairwise non-incident lines, arranged in $36 = 1+20+15$ "double-sixes" in which each line of one of the sixes meets all but one of the lines in the other six:

$$\begin{array}{llllllll}
s & = & \{a_1, & a_2, & a_3, & a_4, & a_5, & a_6\} \\
s' & = & \{b_1, & b_2, & b_3, & b_4, & b_5, & b_6\} \\
\\
s_{ijk} & = & \{a_i, & a_j, & a_k, & c_{\ell m}, & c_{\ell n}, & c_{mn}\} \\
s'_{ijk} & = & \{c_{jk}, & c_{ik}, & c_{ij}, & b_n, & b_m, & b_\ell\ \} \\
\\
s_{ij} & = & \{a_i, & b_i, & c_{jk}, & c_{j\ell}, & c_{jm}, & c_{jn}\} \\
s'_{ij} & = & \{a_j, & b_j, & c_{ik}, & c_{i\ell}, & c_{im}, & c_{in}\} \quad = s_{ji}
\end{array}$$

Each "six" s_I determines an involution $\sigma_I \in W$ which exchanges s_I with s'_I and fixes the remaining 15 lines.

We label the six fundamental weights ω_i as follows:



(This seems simpler than the notation in [Bou].) The corresponding reflections σ_i, $1 \leq i \leq 6$, generate W. Explicitly, for $1 \leq i \leq 5$, σ_i corresponds to the double-six $s_{i,i+1}, s'_{i,i+1}$; it preserves the partition of the lines into $a's$, $b's$ and $c's$, and exchanges indices $i, i+1$. The last reflection, σ_6, corresponds to s_{123}, s'_{123}.

The Weyl subgroup W_i is generated by $\{\sigma_j | j \neq i\}$, and is the stabilizer in W of ω_i. It has a simple description as stablilizer of a set of lines:

W_1 : the line b_1

W_2 : the disjoint pair $\{b_1, b_2\}$

W_3 : the disjoint triple $\{b_1, b_2, b_3\}$ (or $\{a_4, a_5, a_6\}$)

W_4 : the disjoint pair $\{a_5, a_6\}$

W_5 : the line a_6

W_6 : the disjoint "six" s (or s').

The graph automorphism takes the permutation representation $U_i := \mathbf{1}_{W_i}^W$ to U_{6-i}, $1 \leq i \leq 5$. We are thus concerned with the four fundamental representations U_1, U_2, U_3, U_6, of dimensions $27, 216, 720, 72$ respectively. We note that U_1 is the permutation representation of W on the 27 lines, and U_6 is the permutation representation on the 72 roots.

The character inner products $U_i \cdot U_j$ are given in the following table:

(11.1)

	U_1	U_2	U_3	U_6
U_1	3	4	5	3
U_2	4	10	17	6
U_3	5	17	36	9
U_6	3	6	9	5

The computation can be done by a straightforward application of lemma (6.7). For example, $U_1 \cdot U_i$ is given by the number of orbits of W_i on the 27 lines. These orbits are:

$$
\begin{array}{rcll}
U_1 & = & b_1 & \\
& & \text{10 lines incident to } b_1 & (= a_i,\ c_{1i},\ i \neq 1) \\
& & \text{16 non-incident lines} & (= a_1, b_i, c_{ij},\ i,j \neq 1) \\
U_2 & = & b_1, b_2 & \\
& & \text{5 lines meeting both} & (= a_i, i \neq 1,2, \text{and } c_{12}) \\
& & \text{10 lines meeting one of } b_1, b_2 & (= a_1, a_2, c_{1i}, c_{2i}, i \neq 1,2) \\
& & \text{10 lines meeting neither} & (= b_i, c_{ij}, i,j \neq 1,2) \\
U_3 & = & b_1, b_2, b_3 & \\
& & \text{3 lines meeting all 3} & (= a_4, a_5, a_6) \\
& & \text{6 lines meeting 2} & (= a_1, a_2, a_3, c_{12}, c_{13}, c_{23}) \\
& & \text{9 lines meeting 1} & (= c_{ij}, 1 \leq i \leq 3, 4 \leq j \leq 6) \\
& & \text{6 lines meeting none} & (= b_4, b_5, b_6, c_{45}, c_{46}, c_{56}) \\
U_6 & = & 6a's & \\
& & 6b's & \\
& & 15c's &
\end{array}
$$

Other rows are similarly computed as numbers of W_i-orbits on pairs, triples and sixes of disjoint lines. Here are the orbits on sixes:

Orbits of	Orbit size	Orbit description			
w_1	16	$s_I \ni b_1$			
	16	$s'_I \ni b_1$			
	40	neither			
w_2		$\#(s_I \cap \{b_1, b_2\})$	$\#(s'_I \cap \{b_1, b_2\})$		
	5	2	0		
	2	1	1		
	20	1	0		
	5	0	2		
	20	0	1		
	20	0	0		
w_3		$\#(s_I \cap \{b_1, b_2, b_3\})$	$\#(s_I \cap \{a_4, a_5, a_6\})$	$\#(s'_I \cap \{b_1, b_2, b_3\})$	$\#(s'_I \cap \{a_4, a_5, a_6\})$
	2	0	3	3	0
	2	3	0	0	3
	2	0	0	0	0
	18	0	1	1	0
	18	1	0	0	1
	9	0	2	2	0
	9	2	0	0	2
	6	1	0	1	0
	6	0	1	0	1
w_6	1	s			
	20	s_{ijk}			
	30	s_{ij}			
	20	s'_{ijk}			
	1	s'			

The analogue of (8.1) in this case is:

Proposition (11.2).

(i) $U_1 = \mathbf{1} \oplus \Lambda \oplus \Xi$, *where* Λ *is the* 6*-dimensional reflection representation of* $W(E_6)$, *and* Ξ *is an irreducible* 20*-dimensional representation.*

(ii) *Each* U_i *contains* $\mathbf{1}$ *and* Λ *with multiplicity* 1. *The multiplicity of* Ξ *in* U_1, U_2, U_3, U_6 *is, respectively,* $1, 2, 3, 1$.

Proof. The multiplicities of $\mathbf{1}, \Lambda$ in U_i are always 1. The irreducibility of the remaining piece Ξ in U_1 and its multiplicity in the U_i then follow from (the first row of) (11.1). Q.E.D.

In particular, we again have two different non-trivial Pryms, P_Λ and P_Ξ, which occur in all spectral Picards $Pic(\widetilde{X}_\rho)$. Using the rest of the information in (11.1), we can work out the complete decomposition of the U_i.

We find in the Atlas [CCNPW] that the simple group

$$W^+ := ker(\varepsilon : W \to \mathbf{Z}/2)$$

has 20 irreducible characters. Of these, 10 merge in pairs to give 5 charcters of W which we denote by their dimensions: $\mathbf{10}, \mathbf{20}, \mathbf{60}, \mathbf{80}, \mathbf{90}$. (These are the characters which vanish on $W \setminus W^+$.) Each of the 10 remaining irreducible characters of W^+ splits into a pair of irreducible characters of W; the dimensions are $1, 6, 15, 15, 20, 24, 30, 60, 64, 81$. We denote each of these 20 characters by its dimension followed by a $+$ or $-$ according to its sign on the reflections σ_i (which are all in the same conjugacy class in W, class $2c$ in Atlas notation). The two 15-dimensional pairs are separated by their values on products (in W^+) of two commuting σ_i, σ_j: we write $\mathbf{15}^{\pm}$ (respectively $\mathbf{15}'^{\pm}$) for the pair where these values are positive (respectively negative), lifting the character χ_8 (respectively χ_7) of W^+, in Atlas notation.

Proposition (11.3). *The decomposition of the permutation representations U_i into irreducibles is given by:*

$$\begin{aligned}
U_1 &= \mathbf{1} \oplus \Lambda \oplus \Xi \qquad (\Lambda = \mathbf{6}^+, \quad \Xi = \mathbf{20}^+) \\
U_6 &= \mathbf{1} \oplus \Lambda \oplus \ \Xi \oplus \mathbf{15}^+ \oplus \mathbf{30}^+ \\
U_2 &= \mathbf{1} \oplus \Lambda \oplus 2\Xi \oplus \mathbf{15}^+ \oplus \mathbf{30}^+ \oplus \mathbf{60}^+ \oplus \mathbf{64}^+ \\
U_3 &= \mathbf{1} \oplus \Lambda \oplus 3\Xi \oplus 2 \cdot \mathbf{15}^+ \oplus 2 \cdot \mathbf{30}^+ \oplus 3 \cdot \mathbf{60}^+ \oplus 2 \cdot \mathbf{64}^+ \oplus \mathbf{60} \oplus \mathbf{90} \oplus \\
&\quad\ \mathbf{24}^+ \oplus \mathbf{81}^+
\end{aligned}$$

Proof.

We claim that the decomposition above is the only one consistent with Table (11.1), with the known dimensions of the U_i, and with the values of their characters χ_i on a simple reflection σ, say the one which exchanges $a's$ and $b's$ (i.e. corresponding to the double six s, s', or the root α_6):

$$\chi_i(\sigma) \ = \ \#(W_i\text{–cosets fixed by } \sigma)$$

$$= \quad 15, 30, 66, 140 \quad \text{for} \quad i = 1, 6, 2, 3 \text{ respectively.}$$

These numbers count the i-tuples of disjoint lines which are preserved by σ. For example, the 140 triples, for $i = 3$, are of the form:

$$\begin{array}{ll} < a_i, b_i, c_{jk} > & (60) \\ < c_{ij}, c_{ik}, c_{i\ell} > & (60) \\ < c_{ij}, c_{ik}, c_{jk} > & (20) \end{array}$$

Here are the main steps:

- We must have $\Lambda = \mathbf{6}^+$ or $\mathbf{6}^-$ and $\Xi = \mathbf{20}^+$, $\mathbf{20}^-$ or $\mathbf{20}$; since $\chi_{\mathbf{1}}(\sigma) = 15$, $\mathbf{6}^{\pm}(\sigma) = \pm 4$, $\mathbf{20}^{\pm}(\sigma) = \pm 10$, $\mathbf{20}(\sigma) = 0$, we must have $\Lambda = \mathbf{6}^+$, $\Xi = \mathbf{20}^+$.

- There are two new components in U_6, by (11.1). The dimensions should add up to 45 and the values on σ to 15, so one must be $\mathbf{30}^+$ and the other either $\mathbf{15}^+$ or $\mathbf{15}'^+$. The 36-dimensional representation $\mathbf{1}_{W_6}^{W^+}$ of W^+ on double-sixes is decomposed in the Atlas as $\mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}$. Since $U_{\mathbf{6}}$ must contain a lift of these, it decomposes as stated.

- We have already seen the multiplicities of $1, \Lambda, \Xi$ in U_2, U_3. The difference $U_2 - 1 - \Lambda - 2\Xi$ has inner product 4 with itself and 9 with U_3, so it must be the sum of 4 distinct characters. Since the inner product with U_6 is 2, two of these must be $\mathbf{15}^+$, $\mathbf{30}^+$. Let γ, δ be the remaining two characters. We have

$$\gamma(1) + \delta(1) = 216 - 1 - 6 - 2 \cdot 20 - 15 - 30 = 124$$

$$\gamma(\sigma) + \delta(\sigma) = \ 66 - 1 - 4 - 2 \cdot 10 - 5 - 10 = \ 26$$

so these characters must be $\mathbf{60}^+$ and $\mathbf{64}^+$, as claimed.

- Write U_3 as $\mathbf{1} \oplus \Lambda \oplus 3\Xi \oplus k \cdot \mathbf{15}^+ \oplus (4-k) \cdot \mathbf{30}^+ \oplus \ell \cdot \mathbf{64}^+ \oplus (5-\ell) \cdot \mathbf{60}^+ \oplus \sum m_a \varepsilon_a$, where the ε_a are new characters, the m_a non-negative integers, and the coefficients $4 - k$, $5 - \ell$ are determined by (11.1). Evaluating the self-product and values on $1, \sigma$, we find:

$$\begin{array}{lcl} \sum m_a^2 & = & 25 - (k^2 + (4-k)^2) - (\ell^2 + (5-\ell)^2) \\ \sum m_a \varepsilon_a(1) & = & 233 + 15k - 4\ell \\ \sum m_a \varepsilon_a(\sigma) & = & 15 + 5k - 6\ell. \end{array}$$

The first equation gives

$$\sum m_a^2 \leq 4,$$

the second gives, after some fiddling, that $k = 2$, there are exactly four ε_a's, with all $m_a = 1$, and that $\ell = 2$ or 3, which yields respectively

$$\begin{array}{lll} \sum_{a=1}^4 \varepsilon_a(1) = 255 & \text{or} & 251 \\ \sum_{a=1}^4 \varepsilon_a(\sigma) = 13 & \text{or} & 7. \end{array}$$

The only solution is $\ell = 3$ with the ε_a's equal $\mathbf{90}, \mathbf{81}^+, \mathbf{60}, \mathbf{24}^+$.

Q.E.D.

12 Projection formulas.

We conclude by writing down explicitly some correspondences on spectral covers which induce on the spectral Picards the projection to the spectral Pryms. The method is very general, so we return to the setting of §6. W is an arbitrary finite group, V an irreducible representation of W, $v_0 \in V$ a vector fixed by a subgroup $W_P \subset W$. We then have a natural projection

$$\begin{array}{rcl} pr: \ U_P := \mathbf{1}_{W_P}^W = \mathbf{C}[W/W_P] & \to & V \\ w & \mapsto & wv_0. \end{array}$$

Assume now that V is either real or quaternionic, so there is a W-invariant, non-degenerate bilinear form $<,>$ on V. We then get a W-equivariant transpose map:

$$\begin{array}{rcl} i := (pr)^t : \ V & \to & U_P \\ v & \mapsto & \displaystyle\sum_{w \in W/W_P} < v, wv_0 > w. \end{array}$$

The composite:

$$\begin{array}{rcl} c = c_{P,V} = i \circ pr : \ U_P & \to & U_P \\ w_0 & \mapsto & \sum_{w \in W/W_P} < w_0 v_0, wv_0 > w \end{array}$$

is then the desired correspondence on U_P giving the projection to the V-factor. It satisfies

$$c^2 = qc$$

for a constant q which depends on our choice of $<,>$. It can be computed directly:

$$q \cdot dimV = Trace(c) = \sum_{w \in W/W_P} < wv_0, wv_0 >= \frac{\#(W)}{\#(W_P)} |v_0|^2,$$

so

$$q = \frac{dimU_P}{dimV} |v_0|^2. \tag{12.1}$$

It can also be computed by considering

$$\begin{array}{rcl} c' := pr \circ i : \; V & \to & V \\ v & \mapsto & \sum_{w \in W/W_P} < v, wv_0 > wv_0. \end{array}$$

By Schur's lemma, c' is multiplication by the scalar q, which is determined by:

$$q < v_0, v_0 >=< v_0, c'v_0 >= \sum_{w \in W/W_P} < v_0, wv_0 >^2,$$

so

$$q = \sum_{w \in W/W_P} \frac{< v_0, wv_0 >^2}{< v_0, v_0 >}. \tag{12.2}$$

(The compatibility of (12.1) and (12.2) amounts to the identity:

$$\text{Average}_{w \in W} \frac{< v_0, wv_0 >^2}{< v_0, v_0 >^2} = \frac{1}{dimV}.)$$

In the ring $\mathcal{C}_P$ of correspondences (6.1), we have

$$c = \sum_{w \in W_P \backslash W/W_P} < v_0, wv_0 > I_w. \tag{12.3}$$

When the representation V is rational it is therefore natural to choose $<,>$ so that the coefficients $< v_0, wv_0 >$ will be integers.

When W is a Weyl group, all irreducible representations V are rational, so the above applies. The integral correspondence $c = c_{P,V}$ acts on the spectral Picard, $Pic(\widetilde{X}_P)$, projecting it to a copy of $Prym_V(\widetilde{X})$. Projections to different copies of $Prym_V(\widetilde{X})$ are obtained by varying the initial vector v_0 within the fixed subspace $(V)^{W_P}$. When V is the reflection representation Λ, these v_0 can be taken to be the fundamental weights ω_i in the face C_P of the Weyl chamber determined by the subgroup W_P, cf. (3.1).

In [K], Kanev obtains (essentially) formula (12.3) for the projection and analogues of (12.1), (12.2) for the eigenvalue q, in case the base X is $\mathbf{P}^1$, the Lie algebra is of type A_n, D_n or E_n, and the representation V of W is the reflection representation Λ. (But W_P is an arbitrary Weyl subgroup, which by our preliminary observations in §§2,3 is equivalent to considering spectral covers $\widetilde{X}_\rho$ for arbitrary representations ρ of G. In other words, Kanev considers the distinguished $Prym$, $Prym_\Lambda(\widetilde{X}_\rho)$, in $Pic(\widetilde{X}_\rho)$, arbitrary ρ.) His approach is based on the construction, for each $\lambda \in \Lambda$ (corresponding to our choice of $v_0 \in V$), of a lattice $N(\Lambda, \lambda)$ with bilinear pairing $(,)$. For $\mathbf{g}$ of type A_n, D_n, E_n, these lattices are interpreted as cohomology of an appropriate rational surface, with Λ recovered as the primitive cohomology. Kanev's correspondence differs from our c by (a sign, since the primitive cohomology is negative definite, and) a translation by a multiple of $\sum I_w$ (= projection onto $\mathbf{1}$). When $PicX$ is trivial (e.g. under his assumption that $X = \mathbf{P}^1$), this translation is immaterial, and yields an effective representative of the correspondence. (In general it would map $Pic(\widetilde{X}_P)$ to the sum of $PicX$ and $Prym_\Lambda(\widetilde{X})$.)

For example, when $G = GL(n)$, $W = S_n$, the fundamental spectral covers are $\widetilde{X}_i$, $1 \leq i \leq n-1$, of degree $\binom{n}{i}$ over X. A $\mathbf{Z}$-basis for the correspondences on $\widetilde{X}_i$ is given by I_j, $0 \leq j \leq i$, sending an i-tuple to all other i-tuples intersecting it with cardinality j. Kanev's formula for the projection of $Pic(\widetilde{X}_i)$ to $Prym_\Lambda(\widetilde{X})$ is

$$c = \sum_{j=0}^{i} (i-1-j) I_j,$$

while ours gives:

$$c = \sum_{j=0}^{i} (j - \frac{i^2}{n}) I.$$

At the Orsay meeting, I was informed by J.Y. Merindol that he had also obtained extensions of Kanev's results, similar in spirit to the formulas in this section. It seems that he still considers only the distinguished $Prym$, $Prym_\Lambda \widetilde{X}$ (and takes $X = \mathbf{P}^1$), but removes the restrictions on the type of the reductive Lie algebra $\mathbf{g}$.

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Astérisque

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Seshadri constants on smooth surfaces

Astérisque, tome 218 (1993), p. 177-186
<http://www.numdam.org/item?id=AST_1993__218__177_0>

SESHADRI CONSTANTS ON SMOOTH SURFACES

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Introduction

Let X be a smooth complex projective variety of dimension n, and let L be a numerically effective line bundle on X. Following Demailly [**De2**], one defines the *Seshadri constant* of L at a point $x \in X$ to be the real number

$$\epsilon(L,x) = \inf_{C \ni x} \frac{L \cdot C}{m_x(C)} \quad ,$$

where the infimum is taken over all irreducible curves C passing through x, and $m_x(C)$ is the multiplicity of C at x. It is profitable to view $\epsilon(L,x)$ as a local measure of how positive L is at x. For example if L is very ample, then $\epsilon(L,x) \geq 1$; on a surface X the same is true more generally if $L = \mathcal{O}_X(D)$ for an ample effective divisor $D \ni x$ which is smooth at x. In general, if $f : Bl_x(X) \longrightarrow X$ denotes the blowing up of X at x and $E = f^{-1}(x)$ is the exceptional divisor, then for $\epsilon > 0$ the $\mathbb{R}$-divisor $f^*L - \epsilon \cdot E$ is nef if and only if $\epsilon \leq \epsilon(L,x)$. (Consult [**De2**, §6] for other interpretations.) Similarly, one defines the global Seshadri constant

$$\epsilon(L) = \inf_{x \in X} \epsilon(L,x).$$

Thus Seshadri's criterion for ampleness states that $\epsilon(L) > 0$ if and only if L is ample.

Recent interest in Seshadri constants stems from the fact that they govern a simple method for producing sections of adjoint bundles $K_X + kL$ (c.f. [**De2**, (6.8)]). In brief, by means of vanishing theorems on the blow-up $Bl_x(X)$, a lower bound on $\epsilon(L,x)$ yields an explicit value of k such that $K_X + kL$ has a section which is non-zero at x (see (3.4) below). We shall see in §3 that Seshadri

*Partially supported by NSF Grant DMS 91-05183
**Partially supported by NSF Grant DMS 89-02551

constants alone cannot account for the known results on global generation and very ampleness of adjoint bundles ([**Rdr**], [**Del**], [**EL**]). However they remain very interesting in their own right as measures of local positivity. The subtlety of these invariants is reflected in the fact, pointed out by Demailly, that they are already rather difficult to compute on surfaces.

The purpose of this note is to study Seshadri constants in this first non-trivial case, when X is a smooth projective surface. One might anticipate that in general $\epsilon(L,x)$ could become small on fairly arbitrary algebraic subsets of X. Somewhat surprisingly, our main result shows that this is not the case:

THEOREM. *Let L be an ample line bundle on a smooth complex projective surface X. Then $\epsilon(L,x) \geq 1$ for all except perhaps countably many points $x \in X$, and moreover if $c_1(L)^2 > 1$, then the set of exceptional points is in fact finite. More generally, given an integer $e > 1$, suppose that*

$$c_1(L)^2 \geq 2e^2 - 2e + 1 \quad \text{and} \quad c_1(L)\cdot\Gamma \geq e \text{ for every irreducible curve } \Gamma \subset X.$$

Then $\epsilon(L,x) \geq e$ for all but finitely many $x \in X$.

On the other hand, simple examples (constructed by Miranda) show that $\epsilon(L,x)$ can take on arbitrarily small values at isolated points. We hope that this gives some sense of the kind of picture one might hope for in higher dimensions.

The proof of the theorem is completely elementary, the essential point being simply to view the question variationally. Specifically, suppose that L is an ample line bundle, and $C = C_0 \subset X$ is a curve with $m = m_x(C) > C\cdot L$ for some point $x = x_0 \in C$. By combining a simple computation in deformation theory (§1) with the Hodge index theorem, we show that (C,x) cannot move in a non-trivial one-parameter family (C_t, x_t) with $m_{x_t}(C_t) \geq m$ for all t. In other words, pairs (C,x) forcing $\epsilon(L,x) < 1$ are rigid, and the first statement of the Theorem follows at once. We were inspired in this argument by work of G. Xu [**Xu**], who uses related but much more elaborate calculations to study geometric genera of subvarieties of general hypersurfaces in projective space. We present some examples and open questions in §3.

We have benefitted from discussions with J. Kollár, W. Lang, R. Miranda, Y.-T. Siu, H. Tsuji, E. Viehweg, G. Xiao, and G. Xu.

§1. Deformations of Singular Curves on a Surface

This section is devoted to a proof, in the spirit of [**Xu**], of an elementary lemma concerning the deformation theory of singular curves on a surface. While the result in question is certainly well known in the folklore, we include an argument here for lack of a suitable reference and for the convenience of the reader.

We consider the following situation. X is a smooth complex projective surface, and we suppose given a one-parameter family

$$\{ C_t \ni x_t \}_{t\in\Delta}$$

consisting of curves $C_t \subset X$ plus a point $x_t \in C_t$, parametrized by a smooth curve or small disk Δ. Setting $C = C_0$ and $x = x_0$ for $0 \in \Delta$, the deformation determines a Kodaira-Spencer map

$$\rho : T_0\Delta \longrightarrow H^0(C, N),$$

where $N = \mathcal{O}_C(C)$ is the normal bundle to C in X.

LEMMA 1.1. *Assume that $m_{x_t}(C_t) \geq m$ for all $t \in \Delta$. Then $\rho(\frac{d}{dt}) \in H^0(C, N)$ vanishes to order $\geq (m-1)$ at x.*

REMARK. We say that a section $s \in H^0(C, N)$ vanishes to order $\geq k$ at a (possibly singular) point $y \in C$ if s is actually a section of the subsheaf $N \otimes \mathfrak{m}_y^k \subset N$, where $\mathfrak{m}_y$ is the maximal ideal sheaf of y.

PROOF OF LEMMA 1.1: We simply make an explicit computation. Specifically, the assertion is local on C and Δ, so we can assume that Δ is a small disk with coordinate t, and that C lies in an open subset U of $\mathbb{C}^2$ with coordinates (z, w), and $x = (0,0)$. The total space $\mathcal{C} \subset U \times \Delta$ of the deformation is then defined by a power series $F(z, w, t) = f_t(z, w)$ where $C_t = \{f_t = 0\}$. We may suppose that $x_t = (a(t), b(t))$ for suitable power series $a(t), b(t)$. Then the curve defined by

$$\phi_t(z, w) =_{\text{def}} F(z + a(t), w + b(t), t)$$

has multiplicity $\geq m$ at $(0,0)$ for all $t \in \Delta$. Expanding $\phi_t(z, w) = \sum \phi_i(z, w)t^i$ as a power series in t, it follows that $\phi_i \in (z, w)^m$ for all i. On the other hand,

$$\phi_1(z, w) = \frac{\partial f_0}{\partial z}(z, w) \cdot a'(0) + \frac{\partial f_0}{\partial w}(z, w) \cdot b'(0) + \frac{\partial F}{\partial t}(z, w, 0),$$

and since $\frac{\partial f_0}{\partial z}(z, w), \frac{\partial f_0}{\partial w}(z, w) \in (z, w)^{m-1}$, we find that

$$\frac{\partial F}{\partial t}(z, w, 0) \in (z, w)^{m-1}.$$

But $\frac{\partial F}{\partial t}|C$ is the local expression for $\rho(\frac{d}{dt}) \in H^0(C, N)$, and the lemma follows. ∎

COROLLARY 1.2. *In the situation of the Lemma, assume in addition that C is reduced and irreducible, and that the Kodaira-Spencer deformation class $\rho(\frac{d}{dt}) \in H^0(C,N)$ is non-zero. Then $C \cdot C \geq m(m-1)$.*

PROOF: This follows from the Lemma plus the fact that $c_1(N)$ represents $C \cdot C$. In more detail, let $f : Y \longrightarrow X$ be the blowing-up of X at x, with exceptional divisor $E \subset Y$. Then $f^*C = C' + kE$, where $C' \subset Y$ is the proper transform of C, and $k = m_x(C) \geq m$. Note that C' is the blowing-up of C at x. Put $s = \rho(\frac{d}{dt})$, so that $0 \neq s \in H^0(C, \mathfrak{m}_x^{m-1} \otimes \mathcal{O}_C(C))$. Then s induces a non-zero section

$$s' \in H^0(C', f^*(\mathcal{O}_C(C)) \otimes \mathcal{O}_Y((1-m)E)|_{C'}).$$

This implies that $\deg f^*(\mathcal{O}_C(C))|_{C'} \geq (m-1)E \cdot C' = k(m-1)$. It follows that

$$C \cdot C = \deg \mathcal{O}_C(C) = \deg f^*(\mathcal{O}_C(C))|_{C'} \geq k(m-1) \geq m(m-1),$$

as claimed. ∎

§2. Proof of the Theorem

We now give the proof of the theorem stated in the Introduction.

As in the statement, let L be an ample line bundle on the smooth surface X. Then there are only finitely many algebraic families of reduced irreducible (i.e. integral) curves on X of bounded degree with respect to L. Therefore for fixed $d > 0$ the set

$$S_d = \left\{ (C,x) \;\middle|\; x \in C \subset X \text{ an integral curve },\ m_x(C) > C \cdot L,\ C \cdot L \leq d \right\}$$

is parametrized by a finite union of irreducible quasi-projective varieties. Consequently

$$S = \left\{ (C,x) \;\middle|\; x \in C \subset X \text{ a reduced irreducible curve },\ m_x(C) > C \cdot L \right\}$$

consists of at most countably many algebraic families. The first statement of the theorem will follow if we prove that each of these families is discrete.

Suppose to the contrary that there exists a non-trivial continuous family $\{\ (C_t, x_t)\ \}_{t \in \Delta}$ of reduced irreducible curves $C_t \subset X$, plus points $x_t \in C_t$, with

$$(*) \qquad m_t =_{\text{def}} mult_{x_t}(C_t) > C_t \cdot L \qquad \text{for all } t \in \Delta.$$

Without loss of generality we may assume here that Δ is a smooth irreducible curve (or a disk). Since each C_t is reduced, we have $m_y(C_t) = 1$ for all but finitely many $y \in C_t$. So it follows from (*) that the curves $\{C_t\}$ must themselves move in a non-trivial family. Hence for general $t^* \in \Delta$ the corresponding Kodaira-Spencer map

$$T_{t^*}\Delta \longrightarrow H^0(C_{t^*}, N_{C_{t^*}/X})$$

is non-zero. Let $C = C_{t^*}$ and $m = m_{t^*}$ for such a point $t^* \in \Delta$. Corollary 1.2 then implies that $C \cdot C \geq m(m-1)$. On the other hand, $(C^2)(L^2) \leq (C \cdot L)^2$ thanks to the Hodge index theorem, and since $C \cdot L \leq m - 1$ by assumption, we find:

$$m(m-1) \leq (C^2)(L^2) \leq (C \cdot L)^2 \leq (m-1)^2.$$

This is a contradiction when $m > 1$, which proves the first statement of the Theorem.

Suppose next that $L^2 \geq 2$. To prove the finiteness of the exceptional points, it is enough to show that $S = S_d$ for some d, i.e. that any reduced irreducible curve C with $m = m_x(C) > C \cdot L$ for some $x \in X$ has bounded L-degree. To this end observe first that there exists a large integer N with the property that for any point $y \in X$ there is a divisor $D_y \in |N \cdot L|$ with $m_y(D_y) \geq N$. Indeed, it follows from Riemann-Roch that for $n \gg 0$:

$$h^0(X, nL) \sim \frac{n^2 L^2}{2} \geq n^2,$$

whereas it is only $\binom{n+1}{2} \sim \frac{n^2}{2}$ conditions to impose an n-fold point at $y \in X$. Suppose now that C is a reduced irreducible curve with $m = m_x(C) > C \cdot L$ for some $x \in X$. Setting $D = D_x$, we claim next that C must appear as a component of D. In fact, if C were to meet D properly, then

$$m \cdot N \leq m_x(C) \cdot m_x(D) \leq C \cdot D = N(C \cdot L),$$

whence $m \leq C \cdot L$, a contradiction. But once we know that C appears as a component of $D_x \in |N \cdot L|$, we find that

$$C \cdot L \leq D_x \cdot L = N \cdot L^2,$$

which gives the required bound.

Finally, fix $e \geq 2$, and assume that $L^2 \geq 2e^2 - 2e + 1$ and that $\Gamma \cdot L \geq e$ for all curves $\Gamma \subset X$. Suppose that $C \subset X$ is an integral curve such that $m = m_x(C) > \frac{C \cdot L}{e}$. If (C, x) moves in a non-trivial family satisfying this same condition, then the lower bound on $C \cdot L$ shows that C itself must move. Then one argues as above that

$$(2e^2 - 2e + 1)m(m-1) \leq (L^2)(C^2) \leq (C \cdot L)^2 \leq (em-1)^2.$$

But we claim this is a contradiction when $m \geq 2$. In fact the function

$$f(m) = (2e^2 - 2e + 1)m(m-1) - (em-1)^2$$

is increasing for $m \geq 1$, and $f(2) > 0$. Hence pairs (C, x) with $m_x(C) > \frac{C \cdot L}{e}$ are rigid. The finiteness of the exceptional points is similarly proved much as before.

This completes the proof of the Theorem.

§3. Complements, Examples and Open Problems

We collect in this section some applications, examples and open questions.

We begin with an example, given by Miranda, to show that $\epsilon(L, x)$ can take on arbitrarily small values at isolated points. Miranda's construction improves and simplifies a more cumbersome example we had produced where $\epsilon(L, x) \leq \frac{1}{2}$.

EXAMPLE 3.1. Let $D \subset \mathbf{P}^2$ be an irreducible plane curve of degree d with a point $x \in D$ of multiplicity m. Let D' be a second irreducible curve of degree d, meeting D transversely. Choosing D' generally, we may suppose that all the curves in the pencil spanned by D and D' are irreducible. Blow up the base-points of the pencil to obtain a surface X, admitting a map $f : X \longrightarrow \mathbf{P}^1$ with irreducible fibres, among them $D \subset X$. Observe that f has a section $S \subset X$ meeting D transversely at one point. Fix an integer $a \geq 2$. It follows from the Nakai criterion that the divisor $L = aD + S$ on X is ample. But $L \cdot D = 1$ whereas $m_x(D) = m$, so $\epsilon(L, x) \leq \frac{1}{m}$. Note that by taking suitable a we can make L^2 arbitrary large, and by taking L to be a multiple of $aD + S$ we can arrange that $L \cdot \Gamma$ be bounded below by any preassigned integer. ∎

As Viehweg points out, once one has an example of a surface where $\epsilon(L, x)$ is small at isolated points, one gets examples of higher dimensional varieties where the Seshadri constant becomes small on a codimension two subset:

EXAMPLE 3.2. Let (X,L) be as in Example (3.1), and for $n \geq 3$ let $Y = X \times \mathbf{P}^{n-2}$ and put $N = p_1^*(L) \otimes p_2^*(\mathcal{O}_{\mathbf{P}}(1))$. By taking curves in $X \times \{z\}$, one sees that

$$\epsilon(N, (x, z)) \leq \epsilon(L, x) \quad \text{for all } z \in \mathbf{P}^{n-2}.$$

In particular $\epsilon(N, y)$ can be arbitrarily small in codimension two. ∎

It would be very interesting to understand whether Seshadri constants are otherwise well-behaved:

PROBLEM 3.3. Let L be an ample line bundle on a smooth projective variety X. Does there always exist a point $x \in X$ at which $\epsilon(L, x) \geq 1$? If $L^n \gg 0$ is $\epsilon(L, x) \geq 1$ off a subset of codimension two?

Unfortunately the elementary methods of the present paper do not seem to shed much light on this question.

As noted in the Introduction, bounds on Seshadri constants lead to statements on the existence of sections of adjoint bundles. On surfaces, adjoint bundles are well understood thanks to the celebrated theorem of Reider **[Rdr]**. It is interesting to compare Reider's results with the statements obtained from our main Theorem. To this end recall first the well-known:

PROPOSITION 3.4. *Let X be a smooth complex projective variety of dimension n, and let L be an ample (or nef and big) line bundle on X. Fix a point $x \in X$ and a positive integer $k \geq \frac{n}{\epsilon(L,x)}$. If $L^n > (\frac{n}{k})^n$, then $K_X + kL$ has a section which does not vanish at x.*

SKETCH OF PROOF: Let $f : Y \longrightarrow X$ be the blowing up of X at x, and denote by $E \subset Y$ the exceptional divisor. Setting $\epsilon = \epsilon(L, x)$ we have the linear equivalence of $\mathbb{R}$-divisors:

$$k \cdot f^*L - nE \equiv \frac{n}{\epsilon}(f^*L - \epsilon E) + (k - \frac{n}{\epsilon})f^*L,$$

and therefore $k \cdot f^*L - nE$ is nef and big. On the other hand $K_Y = f^*K_X + (n-1)E$, whence $f^*(K_X + kL) - E \equiv K_Y + (k \cdot f^*L - nE)$. Kawamata-Viehweg vanishing then gives

$$H^1(Y, \mathcal{O}(f^*(K_X + kL) - E) = 0$$

which in turn implies the existence of the required section. ∎

In particular, taking $e = 2$ in the main theorem implies:

COROLLARY 3.5. *Let X be a smooth complex projective surface and let L be an ample line bundle on X such that $L^2 \geq 5$ and $\Gamma \cdot L \geq 2$ for all irreducible curves $\Gamma \subset X$. Then at all but finitely many points $x \in X$, $K_X + L$ has a section which is non-vanishing at x.*

On the other hand, it is a consequence of Reider's theorem that under the hypotheses of (3.5), $K_X + L$ is in fact globally generated. Hence we may view our main theorem here as a sort of local Reider-type result, which however holds only off a finite set. A proof of the global generation of $K_X + L$ using vanishing theorems for $\mathbb{Q}$-divisors appears in [**EL**, §1].

While the results of the present paper give a fairly complete picture of the behavior of the Seshadri constants $\epsilon(L, x)$ for a given line bundle L on a smooth surface X, it is less clear what happens as L varies. The essential question here, which is in effect posed by Demailly [**De2**, (6.9)], is the following:

PROBLEM 3.6. Let X be a smooth projective variety, and for an ample line bundle L consider the global Sheshadri constant $\epsilon(L)$ defined in the Introduction. As L varies are these constants bounded away from zero? In other words, setting

$$\epsilon(X) =_{\text{def}} \inf\{\epsilon(L) \mid L \text{ ample on } X\},$$

is it always the case that $\epsilon(X) > 0$?

Our sense is that there may well exist surfaces where $\epsilon(X) = 0$, although we have been unable to construct any. This ties in with the following considerations.

Given an ample line bundle L on a smooth projective variety X, define $\nu(L)$ to be the least integer ν such that νL is very ample. Note that if X is a curve of genus g, then $\nu(L) \leq 2g + 1$ for all ample L. In general, if there is a fixed ν such that $\nu(L) \leq \nu$ for every ample line bundle L on X, then $\epsilon(X) \geq \frac{1}{\nu}$. On the other hand, the following example, due to Kollár, shows that it need not be the case in general that $\nu(L)$ is bounded from above.

EXAMPLE 3.7. [Kollár]. We give an example of a surface X carrying a family of ample line bundles L_n such that $\nu(L_n) \to \infty$ with n.

We start with an elliptic curve E, and put $Y = E \times E$. Fix a point $P \in E$, and define on Y the divisors:

$$h = pr_1^*(P), \quad v = pr_2^*(P), \quad \delta = \text{ diagonal } \subset E \times E.$$

Next, given a positive integer $n \geq 2$ consider the divisor

$$M_n = n \cdot h + (n^2 - n + 1) \cdot v - (n-1)\delta.$$

Then $M_n^2 = 2$ and $M_n \cdot v > 0$, and consequently M_n is ample. [Proof: The inequalities imply by Riemann Roch that M_n has a section, and since Y is homogeneous it follows that M_n is in any event nef. If $M_n \cdot C = 0$ for some effective curve C, then the Hodge index theorem shows that $C^2 < 0$, which is absurd. Hence the Nakai criterion applies.] Finally, let $R = v + h$, and let $B \in |2R|$ be a smooth divisor.

For our surface X we take the double cover $f : X \longrightarrow Y$ of Y branched along B. Let $L_n = f^*(M_n)$. Then L_n is ample and we claim that the natural inclusion

$$(*) \qquad H^0(Y, \mathcal{O}_Y(n \cdot M_n)) \longrightarrow H^0(X, \mathcal{O}_X(n \cdot L_n))$$

is an isomorphism. It follows that $n \cdot L_n$ cannnot very ample, and hence $\nu(L_n) > n$. For the claim, observe that $f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-R)$, and therefore

$$f_*(\mathcal{O}_X(n \cdot L_n)) = \mathcal{O}_Y(n \cdot M_n) \oplus \mathcal{O}_Y(n \cdot M_n - R).$$

So to verify that the map in (*) is bijective, it suffices to prove that $H^0(Y, \mathcal{O}_Y(n \cdot M_n - R)) = 0$. But this follows from the computation that $(n \cdot M_n - R)^2 < 0$. [Note that the specific choices we have made are relatively unimportant; the essential point is simply that $M_n \cdot R$ grows much more quickly than $M_n \cdot M_n$.] ∎

Finally, we note that the definition of the Seshadri constant of a line bundle at a point can be generalized to measure positivity along a subvariety. Let X be a smooth projective variety, and let $V \subset X$ be a subvariety, say smooth to fix ideas. Let $f : Bl_V(X) \longrightarrow X$ be the blowing up of X along V, with exceptional divisor $E \subset Bl_V(X)$. Given an ample line bundle L on X, define the Seshadri constant of L along V to be

$$\epsilon(L, V) = \sup\{\epsilon \mid f^*L - \epsilon \cdot E \text{ is nef}\}.$$

Paoletti [**P**] has investigated these invariants when V is a curve in $\mathbf{P}^3$ (or a general smooth threefold X), and $L = \mathcal{O}_{\mathbf{P}^3}(1)$. In this case $\epsilon(L, V)$ detects such classical information as the presence of multisecant lines, but it seems to be a more delicate invariant. Paoletti proves the striking result that under suitable numerical hypotheses, $\epsilon(L, V)$ governs the gonality of the curve V. It would be interesting to see what other concrete geometric properties are influenced by these invariants. It would also be useful to develop some techniques for computing or estimating $\epsilon(L, V)$; some first steps in this direction appear in [**P**].

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Astérisque, tome 218 (1993), p. 187-202
<http://www.numdam.org/item?id=AST_1993__218__187_0>

HIGHER CASTELNUOVO THEORY

David Eisenbud, Mark Green, Joe Harris[1]

0. Introduction.

In this paper and others to follow, we intend to set out a series of conjectures concerning the Hilbert functions of points (or more generally, zero-dimensional subschemes) in projective space; or, more generally still, the Hilbert functions of graded Artinian rings. We were first led to make some of these conjectures in Eisenbud-Harris [1982] in the course of our work on Castelnuovo theory. A special case of these was proved independently by us in that paper and by Miles Reid – though as Ciliberto later noted [1987] we were both anticipated by G. Fano [1894]. Recently, we saw how our conjectures might be generalized; and in this form they relate to a number of other areas: for example, another special case is equivalent to a conjectured generalization of the classical Cayley-Bacharach theorem (as we will also discuss here); another to the Kruskal-Katona and Clements-Lindström theorems of combinatorics (see, for example, Kleitman-Green [1978]); and still others, which we intend to describe in a later paper, to questions about the existence of exceptional linear series on complete intersection curves.

Good references for unexplained terminology are Arbarello-Cornalba-Griffiths-Harris [1985] or Eisenbud-Harris [1982].

[1]The authors are grateful to the NSF – the second through grant number DMS 88-02020 for partial support during the preparation of this work

1. Castelnuovo theory.

Recall that a set of points in projective space is in *uniform position* if the Hilbert function (= postulation) of a subset depends only on the cardinality of the subset. Castelnuovo theory is concerned with the possible Hilbert functions of points in uniform position. Its origins are classical: Castelnuovo first used estimates on the Hilbert functions of points to derive his upper bound on the genus of an irreducible nondegenerate curve C in projective space $\mathbf{P}^r$ in terms of the degree d of C. Castelnuovo's argument has been reproduced too many times to repeat in detail here (see, for example, Eisenbud-Harris [1982] or Arbarello-Cornalba-Griffiths-Harris [1985]), but briefly what he shows first, by completely elementary means, is that if $\Gamma \subset \mathbf{P}^{n-1}$ is a general hyperplane section of C then

$$g(C) \leq \sum_{\ell=1}^{\infty} h^1(\mathbf{P}^{n-1}, \mathcal{I}_\Gamma(\ell))$$

or, in other words, the genus of C is bounded by the sum over all ℓ of the failure of Γ to impose independent conditons on hypersurfaces of degree ℓ. Curves of maximal genus for their degree therefore are likely to be those whose hyperplane sections Γ have the smallest possible Hilbert function h_Γ. Next, Castelnuovo shows that among all configurations Γ of $d \geq 2n+1$ points in uniform position in $\mathbf{P}^{n-1}$, the ones with minimal Hilbert function are exactly those lying on rational normal curves; he calculates his bound $\pi(d,n)$ on the genus of a curve accordingly. Finally, since if Γ is a subset of a rational normal curve any quadric containing Γ will contain the rational normal curve, he shows that if C is a curve achieving his bound the quadrics containing C must cut out in $\mathbf{P}^n$ a surface whose hyperplane section is a rational normal curve (in particular, a surface of degree $n-1$, the minimum possible degree for a nondegenerate surface in $\mathbf{P}^n$).

In Eisenbud-Harris [1982], we undertook to extend the results of Castelnuovo – in particular, his characterization of curves of maximal genus for their degree as lying on rational normal scrolls – to curves of high, but not maximal genus. This involved asking, for example, "What is the second smallest possible Hilbert function of a collection of points?" and in general, "What configurations of points have small Hilbert function?" What emerged was the following philosophy: The way to achieve a configuration $\Gamma \subset \mathbf{P}^r$ in uniform position having small Hilbert function is to put Γ on a positive-dimensional variety with small Hilbert function – in effect, on a curve of smallest possible degree, and of largest possible genus given that degree – which is the intersection of the hypersurfaces of low degree containing Γ.

To be specific, let $\Gamma \subset \mathbf{P}^r$ be a nondegenerate collection of d points in uniform position; let h_Γ be its Hilbert function, so that for example $h_\Gamma(2)$ is the number of conditions imposed by Γ on quadrics. Castelnuovo says that if $d \geq 2r+3$, then Γ must impose at least $2r+1$ conditions on quadrics; and if $h_\Gamma(2) = 2r+1$ exactly, then Γ must lie on a rational normal curve. Extending this, it turned out that if $d \geq 2r+5$ and if $h_\Gamma \geq 2r+2$ then necessarily Γ had to lie on an elliptic normal curve (Fano [1894],Eisenbud-Harris [1982], Reid [unpublished]). We deduced in particular that if a curve $C \subset \mathbf{P}^n$ had genus exceeding a bound $\pi_1(d,n)$ (substantially lower than $\pi(d,n)$), then the quadrics containing C have to cut out a surface of degree n in $\mathbf{P}^n$, which allowed us to classify such curves. Both we and Miles Reid went on to conjecture that this pattern would persist, at least for a while: for $\alpha < r$, we conjectured, under the hypothesis $d \geq 2r+2\alpha+1$ we could conclude that either $h_\Gamma \geq 2r+\alpha+1$ or Γ lay on a curve of degree $r+\alpha-1$ or less in $\mathbf{P}^r$.

In all of these cases, the latter conclusion – that Γ lay on a curve of small degree – would follow immediately if one knew that the intersection of the quadrics containing Γ was in fact positive dimensional. This observation last year suggested to us a seemingly trivial restatement. If we hypothesize that Γ is cut out by quadrics, we can ask: given $h_\Gamma(2)$, what is the largest possible d? In other words, *What is the largest number $d(h)$ of points of intersection of a linear system of quadrics of codimension h in the space of all quadrics in $\mathbf{P}^r$, given that the intersection of those quadrics is zero-dimensional?* In these terms, we may summarize the state of our knowledge as of 1981 (and its origins) as follows:

$$
\begin{array}{ll}
d(r+1) = r+1 & \text{(elementary)} \\
d(r+2) = r+2 & \text{(elementary)} \\
\vdots & \\
d(2r-1) = 2r-1 & \text{(elementary)} \\
d(2r) = 2r & \text{(elementary)} \\
d(2r+1) = 2r+2 & \text{(Castelnuovo)} \\
d(2r+2) = 2r+4 & \text{(Fano, Eisenbud-Harris, Reid)}
\end{array}
$$

The conjectures mentioned above extend this pattern to:

$$d(2r+3) = 2r+6$$

$$\vdots$$

$$d(3r-3) = 4r-6$$

$$d(3r-2) = 4r-4.$$

Note that this conjectured bound on the number of points is sharp, if it holds: for $h \leq 2r$, of course, any configuration of h points in linear general position will be cut out by quadrics and will impose independent conditions on quadrics; and for $2r+2 \leq h = 2r+\alpha \leq 3r-2$ we can take Γ the interseciton of a linearly normal curve of degree $r+\alpha$ – that is, a curve of degree $r+\alpha$ and (maximal) genus α – with another quadric. Note, moreover, that in the last case – $d(3r-2) = 4r-4$ – there is also another example we can use to show that the bounds is sharp: we can take Γ the intersection of a rational normal scroll $X \subset \mathbf{P}^r$ with two more quadrics.

This last example suggests that at this point the pattern of $d(h)$ increasing by 2 each time stops. Indeed, corresponding to the two examples above in case $h = 3r-2$ there are two examples to suggest that the next value of d should be

$$d(3r-1) = 4r.$$

On the one hand, the maximal genus of a curve of degree $r+\alpha$ in $\mathbf{P}^r$ increases by 2 from $\alpha = r-1$ to $\alpha = r$, with the result that a curve of degree $2r-1$ and genus $r-1$ in $\mathbf{P}^r$ will lie on the same number of quadrics as a curve of degree $2r$ and genus $r+1$ (that is, a canonical curve). Thus we can take Γ the intersection of a canonical curve in $\mathbf{P}^r$ with a quadric to arrive at a configuration of $4r$ points imposing only $3r-1$ conditions on quadrics. On the other hand, in the latter example, if we replace the rational normal surface scroll S, which has degree $r-1$, with a linearly normal surface of one larger degree r (for example, a del Pezzo surface or a cone over an elliptic normal curve), the intersection of our surface with two quadrics will again have degree $4r$ and impose $3r-1$ conditions on quadrics.

Similar examples indicate that for the next $r-3$ steps $d(h)$ will increase by 4 each time we increase h: by way of an example, we can take Γ the intersection of a surface of degree $r-1+\beta$ with two further quadrics. When we get to the case $h = 4r-5$, however, we get a new example: the intersection of a threefold rational normal scroll X with three additional quadrics, and thereafter we can increase the degree of Γ by 8 while increasing $h_\Gamma(2)$ by only 1 by increasing the degree of X by 1.

By now the pattern seems relatively clear, and we may state the

CONJECTURE. *Starting with the value $d(r+1) = r+1$, the successive differences of the function $d(h)$ are:*

$$\begin{array}{ll}
1, 1, \ldots\ldots\ldots\ldots\ldots, 1 & (r-1 \text{ times}); \\
2, 2, \ldots\ldots\ldots\ldots, 2 & (r-2 \text{ times}); \\
4, 4, \ldots\ldots\ldots, 4 & (r-3 \text{ times}); \\
\vdots & \\
2^{k-1}, \ldots, 2^{k-1} & (r-k \text{ times}); \\
\vdots & \\
2^{r-3}, 2^{r-3}, & \\
2^{r-2}. &
\end{array}$$

Where do we wind up at the end of this string? Here we have our first surprise: the last predicted value of d is

$$d\left(\frac{r^2+r+2}{2}\right) = 2^r,$$

that is to say, the largest possible number of isolated points of intersectin of r quadrics in $\mathbf{P}^r$ is 2^r. The fact that the terminal case of the conjecture is simply the Bézout theorem is striking. But more intriguing is the next case:

$$d\left(\frac{r^2+r}{2}\right) = 2^r - 2^{r-2},$$

or, in other words,

(*)
The largest number of points of a complete intersection of quadrics in $\mathbf{P}^r$ that another independent quadric can contain is $2^r - 2^{r-2}$.

Let us express this conjecture in closed form. It will also be useful to replace the variable h, corresponding to the number of conditions imposed by a set of points Γ on quadrics, with the absolute number m of independent quadrics containing Γ – that is, $\binom{r+2}{2} - h$.

First, some notation: given r, any number $m \geq r+1$ can be uniquely written in the form

$$m = (r+1) + \binom{b}{2} + c, \qquad b > c \geq 0.$$

With this notation, we make the

CONJECTURE ($I_{m,r}$). *If Γ is any nondegenerate collection of d points in uniform position in $\mathbf{P}^r$ lying on m independent quadrics whose intersection is zero-dimensional, then*

$$d \leq (2b - c + 1) \cdot 2^{r-b-1}.$$

In particular, the statement (*) above is simply the case ($I_{r+1,r}$) of this conjecture.

As suggested above, examples show that this bound, if indeed it holds, is sharp: for m quadrics, we can take Γ the intersection of $r-b-1$ quadrics with a linearly normal variety of degree $2b-c+1$ and dimension $r-b-1$ in $\mathbf{P}^r$ (for example, the divisor residual to $c+1$ planes in the intersection of a rational normal $(r-b)$-fold scroll in $\mathbf{P}^r$ with a quadric).

Conjecture (I) remains an open problem in general, though we have been able to verify it for all $r \geq 5$ (note that all cases with $r \leq 4$ are covered by existing theorems of Castelnuovo, Fano, Eisenbud-Harris and Reid). We have also been able to verify the special case ($I_{r+1,r}$) for all $r \leq 6$; we will give a proof in §3 below.

2. The algebraic case.

What happens if we omit the hypothesis of uniform position from our basic Conjecture (I), or for that matter if we allow arbitrary (nondegenerate) zero-dimensional subschemes of $\mathbf{P}^r$? This problem is one that has a purely algebraic formulation. Passing to the homogeneous coordinate ring of the configuration $\Gamma \subset \mathbf{P}^r$ modulo a general linear form, it becomes the question: what is the largest possible length $e(m)$ of an Artinian ring of the form

$$R = k[x_1, \ldots, x_r]/I$$

where I is the ideal generated by an m-dimensional vector space of homogeneous quadric polynomials in $x_1, \dots, x_r$?

The extreme cases $m = \binom{r+1}{2}$ and $m = r$ are exactly the same as before: the corresponding values of e are $r+1$ and 2^r. In between, though, the successive differences are quite different: they are expressed in the

CONJECTURE. *Starting with the value $e(r) = 2^r$, the (negative) successive differences of the function $e(m)$ are:*

$$\begin{aligned}
&2^{r-2}, 2^{r-3}, 2^{r-4}, \dots\dots, 4, 2, 1,\\
&2^{r-3}, 2^{r-4}, \dots\dots\dots, 4, 2, 1,\\
&\vdots\\
&4, 2, 1,\\
&2, 1,\\
&1.
\end{aligned}$$

In other words, they are the same successive differences as the function d, in a different order.

As strange as the conjecture may sound, it also has been completely verified for $r \leq 5$ (including cases where the value of e differs from that of d). It should also be noted that the conjectured last two values of the function e before the Bézout case ($e(r+1) = 2^r - 2^{r-2}$, $e(r+2) = 2^r - 2^{r-2} - 2^{r-3}$) are the same as for the function d; and these two values have also been verified for $r \leq 6$.

As in the geometric case, it will be useful to have a form of the conjecture that applies to individual values of the function e. To do this, we write an arbitrary $m \leq \binom{r+1}{2}$ in the form

$$m = \binom{r+1}{2} - \binom{u}{2} - v, \qquad u > v \geq 0.$$

With this notation, we make the

CONJECTURE ($II_{m,r}$). *Let $\Gamma \subset \mathbf{P}^r$ be any nondegenerate, zero-dimensional subscheme of degree d. If Γ lies on m quadrics whose intersection is zero-dimensional, then*

$$d \leq 2^u + 2^v + r - u - 1.$$

Equivalently, if R is an Artinian ring of the form

$$R = k[x_1, \dots, x_r]/I,$$

where I is the ideal generated by an m-dimensional vector space of homogeneous quadric polynomials in $x_1, \dots, x_r$, then

$$dim_k(R) \leq 2^u + 2^v + r - u - 1.$$

As in the case of Conjecture (I), this bound is sharp, if indeed it holds. To construct examples, let $\Lambda \subset \Omega \subset \mathbf{P}^r$ be subspaces of dimension v and $u+1$, respectively. Let Γ_1 be a 0-dimensional complete intersection of quadrics in Λ, consisting of 2^v points. Let C be a curve in Ω given as a complete intersection of quadrics and containing Γ_1 (that is, choose a regular sequence of quadrics in Ω restricting to the quadrics in Λ cutting out Γ_1 and add $u - v$ more quadrics in Ω containing Λ to form a regular sequence of length u. Let H be a hyperplane section of C. Let $p_1, \dots, p_{r-u-1}$ be $r - u - 1$ additional points in $\mathbf{P}^r$ that, together with Ω, span $\mathbf{P}^r$; and set

$$\Gamma = H \cup \Gamma_1 \cup \{p_1, \dots, p_{r-u-1}\}.$$

3. Cayley-Bacharach theory.

There is another way to interpret the statement (*) (equivalently, $(II_{r+1,r})$) above, which is as an extension of the classical Cayley-Bacharach theorem.

We start by reviewing the statement of the modern Cayley-Bacharach theorem. If Γ is a zero-dimensional scheme and $\Gamma' \subset \Gamma$ a closed subscheme, we define the *residual subscheme* of Γ' in Γ to be the subscheme Γ'' of Γ defined by the sheaf of ideals

$$\mathcal{I}_{\Gamma''} = Ann(\mathcal{I}_{\Gamma'}/\mathcal{I}_{\Gamma}).$$

In English: Γ'' is the smallest subscheme of Γ such that any product of functions vanishing on Γ' and Γ'' vanishes on Γ. For example, if Γ is reduced then Γ'' is the complement of Γ' in Γ.

In general, however, it is not true that the degree of Γ'' is the difference $\deg(\Gamma) - \deg(\Gamma')$ (nor is either inequality valid); and the residual of the residual of a subscheme $\Gamma' \subset \Gamma$ will not in general equal Γ'. One circumstance, however, in which residuation does behave well is if Γ is locally a complete intersection (or even locally Gorenstein): in this case, by liaison (Peskine-Szpiro [1974]) we do have

$$\deg(\Gamma') + \deg(\Gamma'') = \deg(\Gamma),$$

and the residual of the residual of Γ' is again Γ'. In this case, we say that Γ' and Γ'' are residual to each other in Γ.

Note also that if the ideal of Γ' in Γ is locally principal, then the equality on degrees holds (though it is not in general true in this case that the residual of the residual of Γ' is Γ').

With this, we may state a

MODERN CAYLEY-BACHARACH THEOREM. *(Davis-Geramita-Orecchia [1985]): Let $\Gamma \subset \mathbf{P}^r$ be a complete intersection of hypersurfaces $X_1, \ldots, X_r$ of degrees $d_1, \ldots, d_r$, and let $\Gamma', \Gamma'' \subset \Gamma$ be closed subschemes residual to one another. Set*

$$m = -r - 1 + \sum d_i.$$

Then, for any $\ell \geq 0$, we have

$$h^0(\mathbf{P}^r, \mathcal{I}_{\Gamma'}(\ell)) - h^0(\mathbf{P}^r, \mathcal{I}_{\Gamma}(\ell)) = h^1(\mathbf{P}^r, \mathcal{I}_{\Gamma''}(m - \ell)).$$

In English: the number of hypersurfaces of degree ℓ containing Γ' (modulo the ideal of Γ) is exactly the failure of Γ'' to impose independent conditions on hypersurfaces of degree $m - \ell$.

According to Semple and Roth [1949], p. 98, the classical Cayley-Bacharach Theorem concerns the special case of where Γ is a reduced complete intersection of points in the plane. It asserts that if

$$\text{degree } \Gamma'' = \binom{m - \ell + 2}{2}$$

and the right hand side of the above equality is 0, then the left hand side is as well. This was asserted by Cayley without the hypothesis that the right hand side is 0 (which is automatic if $m - \ell = 0$ and degree $\Gamma'' = 1$), and corrected by Bacharach (Math. Annalen 26, p. 275). The most commonly stated form of the Theorem is this:

CLASSICAL CAYLEY-BACHARACH THEOREM. *Let $\Gamma \subset \mathbf{P}^r$ be a reduced complete intersection of hypersurfaces $X_1, \ldots, X_r$ of degrees $d_1, \ldots, d_r$. Then any hypersurface of degree $m = \sum d_i - r - 1$ containing a closed subscheme of Γ of degree $\prod d_i - 1$ contains Γ.*

This Cayley-Bacharach theorem says in particular that if $\Omega \subset \mathbf{P}^r$ is a complete intersection of quadrics, then any hypersurface $X \subset \mathbf{P}^r$ of degree $r-1$ containing all but one point of Γ contains Γ. We could ask more generally: Suppose Ω is the complete intersection of r quadrics in $\mathbf{P}^r$. What is the largest degree $g(k)$ of a subscheme of Ω that a hypersurface of degree k, not containing Ω, can contain? By Bézout in $\mathbf{P}^{r-1}$, a hyperplane can contain at most 2^{r-1}, so that $g(1) = 2^{r-1}$, while Bézout in $\mathbf{P}^r$ says that $g(r-1) = 2^r - 2$ and $g(r) = 2^r - 1$. These two remarks are the cases $k = 1$ and $k = r - 1$ of

CONJECTURE $(III_{k,r})$. *(Generalized Cayley-Bacharach). Let $\Omega \subset \mathbf{P}^r$ be a complete intersection of quadrics. Any hypersurface of degree k that contains a subscheme $\Gamma \subset \Omega$ of degree strictly greater than $2^r - 2^{r-k}$ must contain Ω.*

There is an appealing boundary case:

CONJECTURE $(III_{k,r}$, BOUNDARY CASE$)$. *Moreover, if X is a hypersurface of degree k with $\deg(X \cap \Omega) = 2^r - 2^{r-k}$, the scheme residual to $X \cap \Omega$ in Ω is a complete intersection of quadrics in a subspace $\mathbf{P}^{r-k}$.*

Note that the inequality in case $(III_{2,r})$ of this conjecture is exactly the conjecture $(II_{r+1,r})$ above.

We will prove below the conjecture $(III_{k,r})$ for all k and $r \leq 6$. To do this, it will be useful to introduce yet another conjecture:

CONJECTURE (IV_m). *Let $\Gamma \subset \mathbf{P}^r$ be any subscheme of a zero-dimensional complete intersection of quadrics, let $d = \deg(\Gamma)$, and suppose that Γ fails to impose independent conditions on hypersurfaces of degree m – that is, $h^1(\mathbf{P}^r, \mathcal{I}_\Gamma(m)) \neq 0$. Then*

$$d \geq 2^{m+1}.$$

Note that this conjecture is independent of the dimension r of the ambient projective space (in particular, we do not assume that Γ spans $\mathbf{P}^r$).

CONJECTURE $(IV_m$, BOUNDARY CASE$)$. *Equality holds in Conjecture (IV_m) if and only if Γ is itself a complete intersection of quadrics in $\mathbf{P}^{m+1}$.*

THEOREM 1. *The following are equivalent:*

a. *$(III_{k,r})$ for all k and r;*

b. *(IV_m) for all m.*

In particular, either one implies $(III_{2,r})$, and hence $(II_{r+r,r})$, for all r. Moreover, for any value of m, (IV_m) implies $(III_{k,r})$ for all k and r with $r-k-1=m$ and in particular $(II_{r+1,r})$ for $r=m+3$. (The same is true for the boundary cases of (IV_m) and $(III_{k,r})$).

PROOF: We first prove that (IV_m) implies $(III_{r-m-1,r})$. We apply the modern Cayley-Bacharach Theorem. To begin with, assume (IV_m), and let $\Omega \subset \mathbf{P}^r$ be a complete intersection of quadrics. Let X be any hypersurface of degree $k=r-m-1$ not containing Ω, and let Γ be the subscheme of Ω residual to the intersection $\Omega \cap X$. By Cayley-Bacharach Γ must fail to impose independent conditions on hypersurfaces of degree $m=r-1-k$. By assumption, $\deg(\Gamma) \geq 2^{r-k}$ and correspondingly $\deg(X \cap \Omega) \leq 2^r - 2^{r-k}$. (The boundary case of (IV_m) easily implies the boundary case of $(III_{r-m-1,r})$ as well.)

Now assume $(III_{k,r})$ for all r. Let Γ be any subscheme of a complete intersection of quadrics and suppose that Γ fails to impose independent conditions on hypersurfaces of degree k. Assuming that Γ spans a projective space $\mathbf{P}^n$, take Ω a complete intersection of quadrics in $\mathbf{P}^n$ containing Γ, and let $\Gamma' \subset \Omega$ be the subscheme of Ω residual to Γ. By Cayley-Bacharach, Γ' lies on a hypersurface of degree $n-1-k$ not containing Ω; it follows that $\deg(\Gamma') \leq 2^n - 2^{n-1-k}$ and hence that $\deg(\Gamma) \geq 2^{k+1}$. Moreover, if we have equality in the last inequality, then Γ is itself a complete intersection of quadrics. ∎

As promised, we will prove $(III_{k,r})$ for all k and $r \leq 6$ by establishing:

THEOREM 2. *Conjecture (IV_m) holds for $m \leq 3$.*

We will make use of the following simple result several times.

LEMMA. *Let $\Omega \subset \mathbf{P}^r$ be a finite subscheme, and let m be a nonnegative integer. Suppose that every form of degree m vanishing on a codegree 1 subscheme of Ω (that is, on a subscheme of degree one less than Ω) vanishes on all of Ω. If $H \subset \mathbf{P}^r$ is any hypersurface of degree k, and Θ is the subscheme residual to $H \cap \Omega$, then any form of degree $m-k$ vanishing on a codegree 1 subscheme of Θ vanishes on all of Θ.*

PROOF OF THE LEMMA: To say that a form G vanishes on a codegree 1 subscheme of Θ is to say that $((G)+\mathcal{I}_\Theta)/\mathcal{I}_\Theta$ is a 1-dimensional vector space, or equivalently $G \in (\mathcal{I}_\Theta : \mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}_\Theta$.

Now let F be the form of degree k defining H, let $\Gamma = H \cap \Omega$, and let Θ be the subscheme residual in Ω to Γ. If G is a form of degree $m-k$ vanishing on a codegree 1 subscheme of Θ, then $G \in (\mathcal{I}_\Theta : \mathfrak{m})$, so $FG \in (\mathcal{I}_\Theta : \mathfrak{m})$ and FG vanishes on a codegree 1 subscheme of Ω. Since FG has degree m it follows from our hypothesis on Ω that FG vanishes on Ω, and thus $G \in (\mathcal{I}_\Theta : F) = \mathcal{I}_\Theta$ – that is, G vanishes on Θ. ∎

PROOF OF THEOREM 2: First we show that (IV_m) holds for any m in case the linear span of the scheme Γ is a projective space of dimension $n \leq m+2$. The modern Cayley-Bacharach Theorem implies that a complete intersection of quadrics in $\mathbf{P}^n$ imposes independent conditions of hypersurfaces of degree n, and any proper subscheme of it imposes independent conditions on hypersurfaces of degree $n-1$, from which we get the case $n \leq m+1$. If, on the other hand, $n = m+2$, let Ω be a complete intersection of quadrics in $\mathbf{P}^{m+2}$ containing Γ, and let $\Gamma' \subset \Omega$ be the subscheme residual to Γ in Ω. By Cayley-Bacharach the subscheme Γ' lies in a hyperplane $\mathbf{P}^{m+1} \subset \mathbf{P}^{m+2}$. We thus have

$$\deg(\Gamma') \leq 2^{m+1}$$

and hence

$$\deg(\Gamma) \geq 2^{m+1}.$$

Note, moreover, that if equality holds in the last inequality, then Γ' must be a complete intersection of $m+1$ quadrics in $\mathbf{P}^{m+1}$. It follows that the restriction map

$$H^0(\mathbf{P}^{m+2}, \mathcal{I}_\Omega(2)) \longrightarrow H^0(\mathbf{P}^{m+1}, \mathcal{I}_{\Gamma'}(2))$$

must have a kernel. In other words, the linear system of quadrics cutting out Ω contains a reducible element $Q_0 = H_0 \cup L_0$, with $L_0 = \mathbf{P}^{m+1} \supset \Gamma'$. Since Ω is a complete intersection, Γ is residual to Γ', and thus L_1 vanishes on Γ, contradicting the hypothesis that Γ spanned $\mathbf{P}^{m+2}$.

Conjecture (IV_m) is immediate for $m = 0$ or 1; we will deal with the remaining two cases in turn. By what we have just done we may assume that Γ spans a space of dimension $n > m+2$, and we wish to show that $\deg \Gamma > 2^{m+1}$. We may as well assume that Γ is minimal among schemes failing to impose independent conditions on hypersurfaces of degree m and thus that any hypersurface of degree m containing a subscheme of Γ of codegree 1 in Γ must in fact contain Γ.

Case i. $m = 2$. Suppose that Γ spans a space of dimension $r \geq 5$. Then we can find a proper subscheme of Γ of degree at least r contained a hyperplane H in $\overline{\Gamma} = \mathbf{P}^r$; let $\Gamma' = H \cap \Gamma$ be the degree of Γ'. Let Θ be the residual scheme to Γ' in Γ.

By the Lemma, Θ fails to impose independent conditions on hyperplanes. By the case $m = 1$ of our conjecture we have

$$\deg(\Theta) \geq 4,$$

so

$$\begin{aligned} d = \deg(\Gamma) &\geq \deg(\Gamma') + 4 \\ &\geq r + 4 \\ &\geq 9, \end{aligned}$$

as desired.

Case ii. $m = 3$. Say Γ spans a linear space $\mathbf{P}^r$ of dimension $r \geq 6$ and fails to impose independent conditions on cubics. We must show that $\deg \Gamma \geq 17$. By Castelnuovo theory for schemes (see Eisenbud-Harris [1992]) any subscheme of $\mathbf{P}^r$ in linearly general position imposes independent conditions on m-ics if $d \leq mr + 1$. If Γ were in linearly general position, then taking $m = 3$ and $r = 6$ we find $\deg \Gamma > 3r + 1 = 19$, and we would be done. Thus we may assume that there is a hyperplane $H \subset \mathbf{P}^r$ intersecting Γ in a subscheme $\Gamma' = H \cap \Gamma$ of degree $s \geq r + 1 \geq 7$; we suppose that s is the maximal degree of such a subscheme. Let $\Theta \subset \Gamma$ be the subscheme residual to Γ' in Γ. Since the ideal of Γ' in Γ is principal, we have $\deg \Theta + \deg \Gamma' = \deg \Gamma$, so we must show that $s +$ degree $\Theta \geq 17$. Thus we may assume that degree $\Theta \leq 9$.

By the Lemma, Θ fails to impose independent conditions on quadrics. By the case $m = 2$ we must have $\deg(\Theta) \geq 8$. If degree $\Theta = 8$, then by case $m = 2$, Θ must be contained in a $\mathbf{P}^3$. It follows that some subscheme of length > 8 containing Θ is contained in a hyperplane in $\mathbf{P}^r$. Thus $s \geq 9$, and we are done.

It remains to treat the case where degree $\Theta = 9$. If Θ lies in a hyperplane, then $s \geq$ degree $\Theta = 9$, so we are done. If Θ were in linearly general position in $\mathbf{P}^r$ then since Θ imposes dependent conditions on quadrics it follows as above that $\deg\Theta > 2r+1$. Since $r \geq 6$, this contradicts the assumption $\deg\Theta = 9$. Thus we may find a hyperplane section $\Theta' = H' \cap \Theta \subset \Theta$ of degree $t \geq r+1 \geq 7$.

Let Ξ be the subscheme of Θ residual to Θ'. By the Lemma, Ξ fails to impose independent conditions on hyperplanes, so degree $\Xi \geq 3$. Since Θ' is cut out in Θ by just one equation, $\deg\,\Theta = \deg\Theta' + \deg\Xi \geq 7+3 = 10$. ∎

4. A stepwise formulation.

Another way of approaching Hilbert functions is to ask, simply: suppose we know the value $h(m)$ of the Hilbert function of a graded ring in degree m. What can we say about the value in degree $m+1$? In this generality, the answer was supplied by Macaulay, who proved that if we wrote

$$h(m) = \binom{a_m}{m} + \binom{a_{m-1}}{m-1} + \ldots + \binom{a_1}{1}$$

with $a_m > a_{m-1} > \ldots > a_1 \geq 0$, then $h(m+1)$ satisfied the inequality

$$h(m+1) = \binom{a_m+1}{m+1} + \binom{a_{m-1}+1}{m} + \ldots + \binom{a_1+1}{2}.$$

Moreover, this bound is sharp. In line with what we have suggested above, however, we now ask what the estimate should be if we assume in addition that the ring is of the form

$$R = k[x_1, \ldots, x_r]/I$$

where I contains a regular sequence of length r in degree 2. Based on examples and some partial proofs, we make the

CONJECTURE (V_m). *Under this hypothesis, if $h(m)$ is as above, the value $h(m+1)$ of the Hilbert function of R satisfies the inequality*

$$h(m+1) = \binom{a_m}{m+1} + \binom{a_{m-1}}{m} + \ldots + \binom{a_1}{2}.$$

This is sharp, if true; an example would be the ideal generated by the squares of the variables together with the lexicographical ideal of appropriate size in degree m. Moreover, if we sum up the estimates for $h(m)$ over all m, we arrive at the same estimate for the length of R in terms of $h(2)$ given in Conjecture III; thus Conjecture (V) in general implies Conjecture (III).

Moreover, Conjecture (V) is true if the ideal of I contains the squares of the variables. This follows from the Kruskal-Katona Theorem (see Kleitman-Green [1978]), which is equivalent to the monomial case, and a deformation argument. Of course it follows in turn from this that the Theorem is true if I contains a "sufficiently general" regular sequence of quadrics.

In this setting, the hypothesis that the ideal $I \subset k[x_1, \ldots, x_r]$ defining R contains a regular sequence specifically of quadrics is artificial. Conjecture (V) generalizes directly to the case where we assume just that I contains a regular sequence $(f_1, \ldots, f_2)$ of homogeneous polynomials of arbitrary degrees. The case where the f_i are powers of the variables then follows from the Clements-Lindström Theorem, also treated in Kleitman-Green [1978]; we intend to devote a future paper to this and the cases of it that we can prove.

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Surjectivity of cycle maps

Astérisque, tome 218 (1993), p. 203-226
<http://www.numdam.org/item?id=AST_1993__218__203_0>

Surjectivity of cycle maps

Hélène Esnault and Marc Levine

Introduction

The complicated nature of the theory of cycles of codimension two and higher became apparent with Mumford's paper [M], which showed that $p_g = 0$ is a necessary condition for the representability of the group of zero-cycles on a smooth projective surface over $\mathbb{C}$. This was generalized by Roitman [R] when he showed that the vanishing of all the groups $H^0(\Omega^q)$, $q > 1$, is necessary for the representability of the group of zero-cycles on a smooth projective variety over $\mathbb{C}$. Bloch, Kas and Lieberman [BKL] investigated the zero-cycles on surfaces with $p_g = 0$, showing that the group of zero- cycles was in fact representable, at least if the surface is not of general type; Bloch [Bl] has conjectured that $p_g = 0$ is sufficient for the representability of the zero-cycles on a smooth projective surface. The case of surfaces of general type is still an open problem, although there has been some progress, most recently by Voisin [V].

Bloch's proof in [Bl] of Mumford's infinite dimensionality theorem views the diagonal in $X \times X$ as a family of zero-cycles on X, parametrized by X, and goes on to consider the consequences of the generic triviality of this family. This may be the first appearance of this point of view. Coombes and Srinivas used this idea in [CS] to get a decomposability result for $H^1(\mathcal{K}_2)$ of a surface. Bloch and Srinivas [BS] push this approach further, making a study of the cycle groups on a smooth variety X which relies on a partial decomposition of the diagonal in $X \times X$. They have applied this method to give some examples for which certain cycle groups are representable. This approach was recently used by Paranjape [P] in his discussion of the cycle groups of subvarieties of projective space of small degree and small codimension. Schoen [S] has also applied this method to give generalizations of the Mumford-Roitman criterion for non-representability to the Chow groups of cycles of positive dimension. Jannsen [J] used the ideas of Bloch and Srinivas in his discussion of smooth projective varieties X for which the rational topological cycle maps

$$\mathrm{CH}^p(X) \otimes \mathbb{Q} \to H_B^{2p}(X, \mathbb{Q})$$

are injective. For such a variety, Jannsen shows that the diagonal in $X \times X$ decomposes in $\mathrm{CH}^*(X \times X)_{\mathbb{Q}}$ into a sum of product cycles

$$\Delta = A_0 \times B^0 + A_1 \times B^1 + \ldots + A_d \times B^d$$

where A_i is a dimension i cycle, B^i is a codimension i cycle, and $d = \dim(X)$. One consequence of this decomposition is that the total cycle map

$$\bigoplus_{p=0}^{d} \mathrm{CH}^p(X) \otimes \mathbb{Q} \to \bigoplus_{q=0}^{2d} H^q_{\mathcal{B}}(X, \mathbb{Q})$$

is an isomorphism; in particular, X has no odd cohomology.

In this paper, we prove an analog of Jannsen's result, considering the cycle map to rational Deligne cohomology rather than Betti cohomology. Assuming injectivity of the Deligne cycle maps, we arrive at a decomposition of the diagonal into a sum of codimension one cycles on products of the form $\Gamma_{i+1} \times D^i$, with $\dim(\Gamma_{i+1}) = i+1$, $\mathrm{cod}(D^i) = i$ (see Theorem 1.2 for a more precise statement). The consequences of this decomposition are a surjectivity statement for certain cycle maps to Deligne cohomology and some other related maps (Theorem 2.5), a vanishing result for certain Hodge numbers (Theorem 3.2), and a decomposability result for the K-cohomology (Theorem 4.1). If we assume that *all* the rational cycle class maps for a smooth projective variety X are injective, then

(1) all the rational Hodge cycles on X are algebraic (Corollary 2.6)

(2) the Abel-Jacobi maps

$$cl^n \colon \mathrm{CH}^n(X)_{alg} \to J^n(X)$$

are all surjective (Corollary 3.3)

(3) the Hodge numbers $h^{p,q}(X)$ all vanish for $|p-q| > 1$.

(4) the maps

$$\mathrm{CH}^p(X) \otimes \mathbb{C}^\times \to H^p(X, \mathcal{K}_{p+1})$$

are all surjective.

The results on the Hodge numbers are a direct generalization of the results of Mumford-Roitman mentioned above. This points the way to some possible generalizations of Bloch's conjecture to a conjecture on the representability of cycle groups of higher dimension (see Questions 1 and 2 in §3). What is novel about the situation is that it involves *all* the groups of cycles of dimension 0 to s rather than the cycles of a single dimension s. Schoen has raised similar questions in his paper [S], from a slightly different point of view, replacing the injectivity assumption with an assumption that the generalized Hodge conjecture holds, and that the group of dimension s cycles is representable; we haven't attempted to reconcile these two points of view.

We would like to thank Uwe Jannsen and Kapil Paranjape for sending us preliminarly version of their manuscripts, which have greatly influenced this work. This joint paper arose out of conversations while the second author was visiting at the University of Essen; he would like to thank the University of Essen for its gracious hospitality and the DFG Schwerpunkt "Komplexe Mannigfaltigkeiten" for its generous support.

§1. Decomposition of the diagonal

In this section, we show how the injectivity of the cycle map to Deligne cohomology leads to a decomposition of the diagonal. If X is a smooth projective variety, we let $\mathcal{Z}^n(X)$ denote the group of codimension n cycles on X, $\mathrm{CH}^n(X)$ the group of cycles modulo rational equivalence. We let $\mathcal{Z}_n(X)$ and $\mathrm{CH}_n(X)$ denote the group of dimension n cycles and cycle classes. If X is defined over $\mathbb{C}$, we have the cycle class map

$$cl^n\colon \mathcal{Z}^n(X) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

This map passes to rational equivalence, giving the map

$$cl^n\colon \mathrm{CH}^n(X) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

We refer to an element of $\mathcal{Z}^n(X)_{\mathbb{Q}}$ as a $\mathbb{Q}$-cycle. We also denote by cl^n the maps induced by cl^n after extending the coefficient ring. For the basic properties of Deligne cohomology and the cycle map, we refer the reader to [B].

Let $Hg^n(X)$ denote the group of codimension n *Hodge cycles* on X:

$$Hg^n(X) := \{x \in H^{2n}(X, \mathbb{Z}(n)) \mid x \otimes 1 \in F^n H^{2n}(X, \mathbb{C})\}.$$

We have the exact sequence describing $H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ as an extension:

$$0 \to \frac{H^{2n-1}(X, \mathbb{C})}{H^{2n-1}(X, \mathbb{Z}(n)) + F^n H^{2n-1}(X, \mathbb{C})} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \to Hg^n(X) \to 0.$$

The n^{th} intermediate Jacobian, $J^n(X)$, is the complex torus on the left-hand side of the above sequence.

Lemma 1.1. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class map*

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is injective. Let D be a pure codimension $i = d - n$ closed subset of X, and let γ be a codimension d $\mathbb{Q}$-cycle on $X \times X$, supported on $X \times D$. Then there are closed subsets D' and Γ of X, codimension d $\mathbb{Q}$-cycles $\gamma_?$ and $\gamma^?$ on $X \times X$ such that

(1) D' has pure codimension $i + 1$ and Γ has pure dimension $i + 1$.

(2) $\gamma_?$ is supported on $\Gamma \times D$ and $\gamma^?$ is supported on $X \times D'$.

(3) $\gamma = \gamma_? + \gamma^?$ in $CH^d(X \times X)_{\mathbb{Q}}$.

Proof. If D has irreducible components $D_1, \ldots, D_s$, we can write γ as a sum

$$\gamma = \gamma^1 + \ldots + \gamma^s$$

with γ^j supported on $X \times D_j$. Thus we may assume that D is irreducible. Write γ as a sum, $\gamma = \gamma' + \gamma''$, such that each irreducible component of the support of γ' dominates D, and no irreducible component of the support of γ'' dominates D. Since γ'' is supported on $X \times p_2(supp(\gamma''))$, and $p_2(supp(\gamma''))$ has codimension at least $i+1$ on X, we may assume that $\gamma = \gamma'$. We may then find a smooth projective variety $\tilde{D}$, mapping birationally to D by $p: \tilde{D} \to D$, and a $\mathbb{Q}$-cycle $\tilde{\gamma}$ on $X \times \tilde{D}$ such that

(i) for each $y \in \tilde{D}$, $X \times y$ and $\tilde{\gamma}$ intersect properly on $X \times \tilde{D}$.

(ii) $(id_X \times p)_*(\tilde{\gamma}) = \gamma$.

Indeed, for a resolution of singularities $r: E \to D$, and a subvariety Z of $X \times D$, there is a subvariety W of $X \times E$ which is generically finite over Z. Thus each cycle γ as above can be lifted to a $\mathbb{Q}$-cycle γ_E on $X \times E$. Having done this, we may further blow-up E via $\tilde{D} \to E$ so that each component of γ_E has proper transform to $X \times \tilde{D}$ which is flat over $\tilde{D}$, giving us the desired resolution $\tilde{D}$ and $\mathbb{Q}$-cycle $\tilde{\gamma}$.

For a point $y \in \tilde{D}$, let γ_y be the $\mathbb{Q}$- cycle $p_{X*}((X \times y) \cdot \tilde{\gamma})$. Each γ_y has codimension n on X. Fix a point $0 \in \tilde{D}$. Since $\tilde{D}$ is connected, the cycles γ_0 and γ_y are homologous on X, for each y in $\tilde{D}$. Thus $cl^n(\gamma_y - \gamma_0)$ is in $J^n(X)_{\mathbb{Q}}$, for each $y \in \tilde{D}$. Let $cl: \tilde{D} \to J^n(X)_{\mathbb{Q}}$ be the map

$$cl(y) = cl^n(\gamma_y - \gamma_0).$$

In similar fashion, we have the map $ch: \tilde{D} \to \mathrm{CH}^n(X)_{\mathbb{Q}}$ defined by

$$ch(y) = \gamma_y - \gamma_0 \quad \text{mod rational equivalence.}$$

Both ch and cl extend by linearity to maps

$$\begin{aligned} ch&: \mathrm{CH}_0(\tilde{D}) \to \mathrm{CH}^n(X)_{\mathbb{Q}} \\ cl&: \mathrm{CH}_0(\tilde{D}) \to J^n(X)_{\mathbb{Q}}. \end{aligned}$$

The map cl factors further through the Albanese map

$$\alpha_{\tilde{D}}: CH_0(\tilde{D}) \to \mathrm{Alb}(\tilde{D}).$$

Clearly we have $cl^n \circ ch = cl$; since the map cl^n is injective by hypothesis, this implies that ch factors through $\mathrm{Alb}(\tilde{D})$ as well.

Take an embedding of $\tilde{D}$ in a $\mathbb{P}^N$, and let C be a smooth linear section of $\tilde{D}$ of dimension one; we assume that C contains 0. By the weak Lefschetz theorem, the map $\mathrm{Alb}(C) \to \mathrm{Alb}(\tilde{D})$ is surjective; in particular, this implies that, for each $y \in \tilde{D}$, there is a $\mathbb{Q}$-zero cycle a_y, supported on C, such that $cl(y) = cl(a_y)$. As the map ch factors through $\mathrm{Alb}(\tilde{D})$, we have $ch(y) = ch(a_y)$.

Take y to be a geometric generic point of $\tilde{D}$ over $\mathbb{C}$, so $\mathbb{C}(y) = \mathbb{C}(\tilde{D}) = \mathbb{C}(D)$. The zero-cycle a_y is defined over some finitely generated field extension of $\mathbb{C}(\tilde{D})$; by specializing a_y and changing notation, we may assume that the zero-cycle a_y is defined over a finite extension L of $\mathbb{C}(\tilde{D})$, of degree say M. Let b_y be the zero cycle $\frac{1}{M} \cdot Nm_{L/\mathbb{C}(\tilde{D})}(a_y)$. Then b_y is defined over $\mathbb{C}(\tilde{D})$, b_y is supported on C and $ch(y) = ch(b_y)$. In particular, there is a unique $\mathbb{Q}$-cycle $\tilde{\gamma}_?$ on $X \times \tilde{D}$ such that

(iii) $p_{X*}((X \times y) \cdot \tilde{\gamma}_?) = p_{X*}((X \times b_y) \cdot \tilde{\gamma})$, for y a geometric generic point of $\tilde{D}$ over $\mathbb{C}$.

(iv) each irreducible component of $\mathrm{supp}(\tilde{\gamma}_?)$ dominates $\tilde{D}$.

Let $S = p_X(\mathrm{supp}(\tilde{\gamma}) \cap X \times C)$. Since the fibers of $\mathrm{supp}(\tilde{\gamma})$ over $\tilde{D}$ all have dimension i, S has dimension at most $i+1$. By (iii) and (iv), $\tilde{\gamma}_?$ is supported on $S \times \tilde{D}$. Since $ch(y) = ch(b_y)$, (iii), together with the localization sequence for the Chow groups, implies there is a codimension one closed subset $\tilde{D}'$ of $\tilde{D}$, and a cycle $\tilde{\gamma}^? \in CH^{d-i}(X \times \tilde{D})$, supported on $X \times \tilde{D}'$, such that

(v) $\tilde{\gamma} = \tilde{\gamma}_? + \gamma_0 \times \tilde{D} + \tilde{\gamma}^?$ in $\mathrm{CH}^{d-i}(X \times \tilde{D})_{\mathbb{Q}}$.

Let Γ be a pure dimension $i+1$ closed subset of X containing S and $\mathrm{supp}(\gamma_0)$, let D' be a pure codimension $i+1$ closed subset of X containing $p(\tilde{D}')$. Take $\gamma_? = (id_X \times p)_*(\tilde{\gamma}_? + \gamma_0 \times \tilde{D})$, $\gamma^? = (id_X \times p)_*(\tilde{\gamma}^?)$. Since $(id_X \times p)_*(\tilde{\gamma}) = \gamma$, we have

$$\gamma = \gamma_? + \gamma^? \text{ in } \mathrm{CH}^d(X \times X)_{\mathbb{Q}}$$
$$\gamma^? \text{ is supported on } X \times D'$$
$$\gamma_? \text{ is supported on } \Gamma \times D,$$

as desired. □

Theorem 1.2. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d, and let Δ be the class of the diagonal in $CH^d(X \times X)_{\mathbb{Q}}$. Suppose the $\mathbb{Q}$-cycle class maps*

$$cl^n: CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$, for some integer s, $0 \le s \le d-2$. Then there are closed subsets $X = D^0, D^1, \ldots, D^{s+1}$, $\Gamma_1, \ldots, \Gamma_{s+1}$, and cycles $\gamma_1, \ldots, \gamma_s, \gamma^{s+1} \in CH^d(X \times X)_{\mathbb{Q}}$ such that

(1) D^i has pure codimension i, Γ_i has pure dimension i.

(2) γ_i is supported on $\Gamma_{i+1} \times D^i$, for $i = 0, \ldots, s$.

(3) γ^{s+1} is supported on $X \times D^{s+1}$.

(4) $\Delta = \gamma_0 + \ldots + \gamma_s + \gamma^{s+1}$ in $CH^d(X \times X)_{\mathbb{Q}}$.

Proof. We first apply Lemma 1.1 to the cycle Δ on $X \times X$, with $n = d$, $i = 0$ and $D = X$. This gives us the $\mathbb{Q}$-cycles γ_0 and γ^1, a codimension one closed subset D^1 and a dimension one closed subset Γ_1 with γ_0 supported on $\Gamma_1 \times X$, γ^1 supported on $X \times D^1$ and with $\Delta = \gamma_1 + \gamma^1$ in $\mathrm{CH}^d(X \times X)_{\mathbb{Q}}$. This proves the case $s = 0$. The general case follows by induction on s, applying Lemma 1.1 to the cycle γ^{s+1} supported on $X \times D^{s+1}$. □

Note. We have systematically indexed our cycle groups by *codimension* rather than dimension for notational convenience. However, it seems instructive to view the hypotheses of Theorem 1.2 as requiring the injectivity of the rational cycle maps for cycles of *dimension* 0 to s.

§2. Surjectivity

In this section, we use the decomposition of the diagonal given in §1 to study the surjectivity of the cycle map.

Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Let γ be in $\mathrm{CH}^d(X \times X)_{\mathbb{Q}}$, supported on a product $\Gamma \times D$, with $\Gamma \subset X$ of pure dimension j, $D \subset X$ of pure codimension i. Let $p: \tilde{\Gamma} \to \Gamma$, $q: \tilde{D} \to D$ be birational maps, with $\tilde{\Gamma}$ and $\tilde{D}$ smooth and projective. If Z is a subvariety of $\Gamma \times D$, then there is a subvariety W of $\tilde{\Gamma} \times \tilde{D}$, with $(p \times q)(W) = Z$, and with W generically finite over Z. In particular, there is a cycle $\tilde{\gamma} \in \mathrm{CH}^{j-i}(\tilde{\Gamma} \times \tilde{D})_{\mathbb{Q}}$ with $(p \times q)_*(\tilde{\gamma}) = \gamma$.

The cycle γ determines the homomorphisms

$$\gamma_*: H^a_{\mathcal{D}}(X, \mathbb{Q}(b)) \to H^a_{\mathcal{D}}(X, \mathbb{Q}(b))$$

by

$$\gamma_*(\eta) = p_{2*}(p_1^*(\eta) \cup cl^d(\gamma)), \quad \text{for } \eta \in H^a_{\mathcal{D}}(X, \mathbb{Q}(b)).$$

Let $f: \tilde{\Gamma} \to X$, $g: \tilde{D} \to X$ be the obvious maps, and let $p_{\tilde{D}}: \tilde{\Gamma} \times \tilde{D} \to \tilde{D}$, $p_{\tilde{\Gamma}}: \tilde{\Gamma} \times \tilde{D} \to \tilde{\Gamma}$ denote the projections. The cycle $\tilde{\gamma}$ determines homomorphisms $\tilde{\gamma}_*: H^a_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(b)) \to H^{a-2i}_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(b-i))$ by

$$\tilde{\gamma}_*(\eta) = p_{\tilde{D}*}(p_{\tilde{\Gamma}}^*(\eta) \cup cl^{j-i}(\gamma)), \quad \text{for } \eta \in H^a_{\mathcal{D}}(\Gamma, \mathbb{Q}(b)).$$

Lemma 2.1. *Let $\eta \in H^a_{\mathcal{D}}(X, \mathbb{Q}(b))$. Then*

$$\gamma_*(\eta) = f_*(\tilde{\gamma}_*(g^*(\eta))).$$

Proof. We have

$$\begin{aligned}
\gamma_*(\eta) &= p_{2*}(p_1^*(\eta) \cup cl^d(\gamma)) \\
&= p_{2*}(p_1^*(\eta) \cup cl^d((g \times f)_*(\tilde{\gamma}))) \\
&= p_{2*}(p_1^*(\eta) \cup (g \times f)_*(cl^{j-i}(\tilde{\gamma}))) \\
&= p_{2*}((g \times f)_*((g \times f)^*(p_1^*(\eta)) \cup cl^{j-i}(\tilde{\gamma}))) \quad \text{(projection formula)} \\
&= f_*(p_{\tilde{D}*}(p_{\tilde{\Gamma}}^*(g^*(\eta)) \cup cl^{j-i}(\tilde{\gamma}))) \\
&= f_*(\tilde{\gamma}_*(g^*(\eta))).
\end{aligned}$$

□

The Deligne cohomology groups $H^0_{\mathcal{D}}$ and $H^1_{\mathcal{D}}$ of a point $*$ are easily computed; we give here a partial computation:

For $k \geq 0$, we have

$$H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) = \mathbb{Q}(-k)$$
$$H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) = \mathbb{C}/\mathbb{Q}(k)$$

Let $p_X: X \to *$ be the projection to a point. Using the cycle class map cl^n, we obtain the maps

$$cl^n_{0,-k}: \mathrm{CH}^n(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))$$
$$cl^n_{1,k}: \mathrm{CH}^n(X) \otimes H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) \to H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k)),$$

defined by

$$cl^n_{0,-k}(\eta \otimes \alpha) = cl^n(\eta) \cup p_X^*(\alpha)$$
$$cl^n_{1,k}(\eta \otimes \beta) = cl^n(\eta) \cup p_X^*(\beta),$$

for $\alpha \in H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))$, $\beta \in H^1_{\mathcal{D}}(X, \mathbb{Q}(1+k))$ and $\eta \in \mathrm{CH}^n(X)$.

Lemma 2.2. *Let Y be a smooth irreducible projective variety over $\mathbb{C}$ of dimension d_Y. Then, for $k \geq 0$, we have*

$$H^0_{\mathcal{D}}(Y, \mathbb{Q}(-k)) = \mathbb{Q}(-k)$$
$$H^1_{\mathcal{D}}(Y, \mathbb{Q}(1+k)) = \mathbb{C}/\mathbb{Q}(1+k).$$

The map

$$cl^{d_Y}_{0,0}: CH^{d_Y}(Y) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(0)) \to H^{2d_Y}_{\mathcal{D}}(Y, \mathbb{Q}(d_Y))$$

is surjective. If $\iota : * \to Y$ is a point of Y, the maps

$$\iota_*: H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2d_Y}_{\mathcal{D}}(Y, \mathbb{Q}(d_Y - k)), \quad k > 0$$

and

$$\iota_*: H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) \to H^{2d_Y+1}_{\mathcal{D}}(Y, \mathbb{Q}(d_Y + 1 + k)), \quad k \geq 0$$

are isomorphisms.

Proof. The computation of $H^0_{\mathcal{D}}$ and $H^1_{\mathcal{D}}$ follow directly from the isomorphism

$$H^0_{\mathcal{D}}(Y, \mathbb{Q}(-k)) \to H^0(Y, \mathbb{Q}(-k)) \cap F^{-k} H^0(Y, \mathbb{C})$$

and the short exact sequence

$$0 \to \frac{H^0(Y,\mathbb{C})}{H^0(Y,\mathbb{Q}(1+k)) + F^{1+k}H^0(Y,\mathbb{C})}$$

$$\to H^1_{\mathcal{D}}(Y,\mathbb{Q}(1+k)) \to H^1(Y,\mathbb{Q}(1+k)) \cap F^{1+k}H^1(Y,\mathbb{C}) \to 0,$$

together with the identities (for $k \geq 0$)

$$\begin{aligned} F^{-k}H^0(Y,\mathbb{C}) &= H^0(Y,\mathbb{C}) \\ F^{1+k}H^0(Y,\mathbb{C}) &= 0 \\ F^{1+k}H^1(Y,\mathbb{C}) &= 0. \end{aligned}$$

For the surjectivity statement, we have the exact sequence

$$0 \to \frac{H^{2d_Y-1}(Y,\mathbb{C})}{H^{2d_Y-1}(Y,\mathbb{Z}(d_Y-k)) + F^{d_Y-k}H^{2d_Y-1}(Y,\mathbb{C})} \to H^{2d_Y}_{\mathcal{D}}(Y,\mathbb{Z}(d_Y-k))$$

$$\to H^{2d_Y}(Y,\mathbb{Z}(d_Y-k)) \cap F^{d_Y-k}H^{2d_Y}(Y,\mathbb{C}) \to 0.$$

For $k = 0$, this is just the exact sequence

$$0 \to \mathrm{Alb}(Y) \to H^{2d_Y}_{\mathcal{D}}(Y,\mathbb{Z}(d_Y)) \to H^{2d_Y}(Y,\mathbb{Z}(d_Y)) \to 0;$$

and the cycle class map cl^{d_Y} breaks up into degree map to $H^{2d_Y}(Y,\mathbb{Z}(d_Y)) = \mathbb{Z}$ and the Albanese map $\alpha\colon \mathrm{CH}_0(Y)_0 \to \mathrm{Alb}(Y)$. As both these maps are surjective, $cl^{d_Y}_{0,0}$ is surjective as well. For $k < 0$, we have

$$H^{2d_Y}_{\mathcal{D}}(X,\mathbb{Q}(d_Y-k)) = H^{2d_Y}(Y,\mathbb{Q}(d_Y-k)).$$

As this latter group is isomorphic to $\mathbb{Q}(-k)$, generated by the class of a point, the map ι_* is an isomorphism as claimed. The computation of the group $H^{2d_Y+1}_{\mathcal{D}}(X,\mathbb{Q}(d_Y+1+k))$ is similar. □

Lemma 2.3. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d, let Γ be a closed subset of pure dimension $i+1$, D a closed subset of pure codimension i, and let $\gamma \in CH^d(X \times X)_{\mathbb{Q}}$ be a $\mathbb{Q}$-cycle supported on $\Gamma \times D$. Then, for all $n, k \geq 0$, $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k)))$ is contained in the image of $cl^n_{0,-k}$, and $\gamma_*(H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k)))$ is contained in the image of $cl^n_{1,k}$.*

Proof. As in the paragraph preceeding Lemma 2.1, we let $p: \tilde{\Gamma} \to \Gamma$, $q: \tilde{D} \to D$ be birational maps, with $\tilde{\Gamma}$ and $\tilde{D}$ smooth and projective. Let $g: \tilde{\Gamma} \to X$, $f: \tilde{D} \to X$ be the obvious maps, and let $\tilde{\gamma} \in \mathrm{CH}^1(\Gamma \times \tilde{D})_{\mathbb{Q}}$ be a $\mathbb{Q}$-cycle with $(g \times f)_*(\tilde{\gamma}) = \gamma$. By Lemma 2.1, we have

$$\gamma_*(\eta) = g_*(\tilde{\gamma}_*(f^*(\eta)))$$

for $\eta \in H^a_{\mathcal{D}}(X, \mathbb{Q}(b)))$. Also, the homomorphism $\tilde{\gamma}_* \circ g^*$ maps $H^a_{\mathcal{D}}(X, \mathbb{Q}(b)))$ to $H^{a-2i}_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(b-i)))$, and g^* maps $H^a_{\mathcal{D}}(X, \mathbb{Q}(b)))$ to $H^a_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(b)))$. Since $H^a_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(b))) = 0$ for $a > 2i+3$, and $H^{a-2i}_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(b-i))) = 0$ for $a < 2i$, we need only consider four cases:

(1) $a = 2n = 2i$, $b = n - k$;

(2) $a = 2n + 1 = 2i + 1$, $b = n + 1 + k$;

(3) $a = 2n = 2i + 2$, $b = n - k$;

(4) $a = 2n + 1 = 2i + 3$, $b = n + 1 + k$.

For cases (1) and (2), it follows from Lemma 2.2 that $f_*(H^0_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(-k))$ is in the image of $cl^i_{0,-k}$, and that $f_*(H^1_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(1+k))$ is in the image of $cl^i_{1,k}$. For case (3) , it follows from Lemma 2.2 that $H^{2i+2}_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(i+1-k))$ is generated by $cl^{i+1}_{0,-k}(\mathrm{CH}^{i+1}(\tilde{\Gamma}) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))$, i.e., by the classes of points of any dense Zariski open subset of $\tilde{\Gamma}$. If x is a point of $\tilde{\Gamma}$, let $\tilde{\gamma}_x$ be the divisor $p_{\tilde{D}*}(\tilde{\gamma} \cdot x \times \tilde{D})$, when the intersection $\tilde{\gamma} \cap x \times \tilde{D}$ has codimension one on $\tilde{\Gamma} \times D$. Then $\tilde{\gamma}_*(x)$ is the class in $H^2_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(1)))$ of $\tilde{\gamma}_x$, when the latter is defined; using the projection formula, we see that

$$\tilde{\gamma}_*(H^{2i+2}_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(i+1-k))) \subset cl^1_{0,-k}(\mathrm{CH}^1(\tilde{D}) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))).$$

Following $\tilde{\gamma}_*$ by f_*, and using the compatibility of cycle classes with proper pushforward, we see that

$$\gamma_*(H^a_{\mathcal{D}}(X, \mathbb{Q}(b))) \subset cl^{i+1}_{0,-k}(\mathrm{CH}^{i+1}(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))).$$

Case (4) is similar, and is left to the reader. $\square$

Lemma 2.4. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d, let D be a closed subset of pure codimension $s+1$, and let $\gamma \in CH^d(X \times X)_{\mathbb{Q}}$ be a $\mathbb{Q}$-cycle supported on $X \times D$. Then*

(i) $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))) = \gamma_*(H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k))) = 0$, *for $n < s+1$, and for all $k \geq 0$.*

(ii) $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k)))$ *is contained in the image of $cl^n_{0,-k}$, and* $\gamma_*(H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k)))$ *is contained in the image of $cl^n_{1,k}$, for $n = s+1$, and for all $k \geq 0$.*

(iii) $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n)))$ *is contained in the image of $cl^n_{0,0}$, for $n = s+2$.*

Proof. The proofs of (i) and (ii) are similar to the argument in the proof of the preceeding lemma, and are left to the reader. For (iii), let $\tilde{D} \to D$ be a resolution of singularities, and let $f: \tilde{D} \to X$ be the obvious map. Arguing as in the preceeding lemma, we see that $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n)))$ is contained in $f_*(H^2_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(1)))$. Since the cycle class map $cl^1: \mathrm{CH}^1(\tilde{D}) \to H^2_{\mathcal{D}}(\tilde{D}, \mathbb{Z}(1)))$ is an isomorphism, we find that $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n)))$ is contained $f_*(\mathrm{CH}^1(\tilde{D}))$, proving (iii). □

Theorem 2.5. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer s, with $0 \leq s \leq d-2$, such that the $\mathbb{Q}$-cycle class maps*

$$cl^n: CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$. Then the maps

$$cl^n_{0,-k}: CH^n(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))$$

and

$$cl^n_{1,k}: CH^n(X) \otimes H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) \to H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k))$$

are surjective for $n = 0, \ldots, s+1$ and for all $k \geq 0$. The map

$$cl^n_{0,0}: CH^n(X) \otimes \mathbb{Q} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is surjective for $n = s+2$. In particular, if the $\mathbb{Q}$-cycle class maps cl^n are injective for all $n \geq 0$, then the maps $cl^n_{0,-k}$ and $cl^n_{1,k}$ are surjective for all $n \geq 0$ and for all $k \geq 0$.

Proof. This follows from Theorem 1.2, and Lemmas 2.3 and 2.4, noting the the map Δ_* is the identity. □

Corollary 2.6. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class maps*

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for all n. Then the group $Hg^n(X) \otimes \mathbb{Q}$ of rational Hodge cycles of X is generated by the classes of algebraic cycles for all n.

Proof. The surjectivity of the rational cycle class map

$$\mathrm{CH}^n(X)_{\mathbb{Q}} \to Hg^n(X) \otimes \mathbb{Q}$$

follows directly from Theorem 2.5. □

Remark. We will show in the next section that the injectivity of the cycle maps implies that the intermediate Jacobians of X are generated by the classes of algebraic cycles which are algebraically equivalent to zero.

§3. Hodge numbers and the failure of injectivity of the cycle map

We proceed to examine some consequences of Theorem 1.2 for the Hodge numbers of a smooth projective variety, and derive a criterion for ensuring that the cycle class maps are *not* injective. This can be viewed as a generalization of the theorems of Mumford-Roitman ([M], [R]) on the non-representability of the group of zero cycles on smooth projective varieties with non-trivial holomorphic p-forms for $p > 1$. What is novel in this setting is that it is not clear *which* cycle group is contributing to the lack of injectivity, although there is an obvious question one can pose (see Question 1 below).

For a smooth projective variety X over $\mathbb{C}$, we let $H^{p,q}(X)$ denote (p,q)-component in the Hodge decomposition of $H^*(X, \mathbb{C})$, and let $h^{p,q}(X) = \dim_{\mathbb{C}}(H^{p,q}(X))$. Let $cl^{n,n}(\gamma)$ denote the cohomology class in $H^{n,n}(X)$ of $\gamma \in \mathrm{CH}^n(X)_{\mathbb{Q}}$. If Y and Z are smooth projective varieties over $\mathbb{C}$, with Z of dimension a, and if γ is in $\mathrm{CH}^b(Y \times Z)$, we have the homomorphism

$$\gamma_* \colon H^{p,q}(Y) \to H^{p+b-a,q+b-a}(Z)$$

defined by $\gamma_*(\eta) = p_{2*}(p_1^*(\eta) \cup cl^{b,b}(\gamma))$.

Lemma 3.1. *Let X, D and Γ be smooth projective varieties over $\mathbb{C}$, with maps $f\colon D \to X$, $g\colon \Gamma \to X$. Let $\tilde{\gamma}$ be in $CH^b(\Gamma \times D)$, and let $\gamma = (g \times f)_*(\tilde{\gamma})$. Then $\gamma_* = f_* \circ \tilde{\gamma}_* \circ g^*$.*

Proof. The proof is the same as the proof of Lemma 2.1. □

Let $CH^n(X)_{hom}$ denote the group of cycles homologous to zero, modulo rational equivalence, and let $CH^n(X)_{alg}$ denote the group of cycles algebraically equivalent to zero, modulo rational equivalence.

Theorem 3.2. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer s, $0 \leq s \leq d-2$ such that the $\mathbb{Q}$-cycle class maps*

$$cl^n\colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$. Then the Hodge numbers $h^{p,q}(X)$ vanish if

(i) $p + q \leq 2s + 2$ and $|p - q| > 1$,

or if

(ii) $p + q > 2s + 2$ and $p < s + 1$.

In particular, if the $\mathbb{Q}$-cycle class maps cl^n are injective for all $n \geq 0$, then the Hodge numbers $h^{p,q}(X)$ vanish if $|p - q| > 1$. In addition, the cycle class map cl^n induce a surjection

$$cl^n\colon CH^n(X)_{alg} \to J^n(X)$$

for $n \leq s + 2$.

Proof. For (i), first suppose $p + q = 2n$ is even. By Theorem 2.5, the map

$$cl^n_{0,-k}\colon CH^n(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))$$

is surjective for all $k \geq 0$. On the other hand, for $k = n$, we have

$$H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k)) = H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(0)) = H^{2n}(X, \mathbb{Q}),$$

and the map $cl^n_{0,-n}$ is the usual topological cycle class map to singular cohomology (after twisting by $\mathbb{Q}(-n)$). Since the topological cycle class map lands in $H^{n,n}(X)$, the surjectivity of $cl^n_{0,-n}$ forces the vanishing of the Hodge numbers $h^{p,q}(X)$ if $p \neq q$. This proves (i) for $p+q$ even.

For $p+q = 2n-1$ odd, consider the groups $CH^n(X)_{hom}$ and $CH^n(X)_{alg}$. As the difference of two cycles belonging to the same connected component of a family of cycles on X goes to zero in the quotient group

$$CH^n(X)_{hom}/CH^n(X)_{alg},$$

this latter group is generated by the connected components of the union of the Chow varieties of degree t cycles of codimension n on X, for varying t. In particular, $CH^n(X)_{hom}/CH^n(X)_{alg}$ is a countably generated group. On the other hand, $cl^n(CH^n(X)_{alg})$ is an abelian subvariety A of $J^n(X)$, with tangent space $T_0(A)$ contained in the the subspace $H^{n-1,n}(X)$ of $T_0(J^n(X))$. By Theorem 2.5, the restriction of cl^n to $CH^n(X)_{hom}$ gives a surjective map

$$CH^n(X)_{hom} \otimes \mathbb{Q} \to J^n(X) \otimes \mathbb{Q}.$$

Thus, the complex torus $J^n(X)/A$ is a countably generated group, which is impossible unless $J^n(X) = A$. But, as

$$T_0(J^n(X)) = H^{0,n}(X) \oplus H^{1,n-1}(X) \oplus \ldots \oplus H^{n-1,n}(X),$$

the Hodge numbers $h^{p,q}(X)$ vanish if $|p-q| > 1$, completing the proof of (i). The same argument, using the surjectivity of

$$cl^n \colon \mathrm{CH}^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

for $n \leq s+2$, as given by Theorem 2.5, shows that

$$cl^n \colon CH^n(X)_{alg} \to J^n(X)$$

is surjective for $n \leq s+2$.

For (ii), we use the decomposition

$$\Delta = \gamma_0 + \ldots + \gamma_s + \gamma^{s+1}$$

of the diagonal Δ given by Theorem 1.2, with γ_i supported on $\Gamma_{i+1} \times D^i$. Take resolutions of singularities $\tilde{D}^i \to D^i$, $\tilde{\Gamma}_i \to \Gamma_i$, and let $g^i: \tilde{\Gamma}_i \to X$, $f^i: \tilde{D}^i \to X$ be the obvious maps. Take $\mathbb{Q}$-cycles $\tilde{\gamma}_i$ on $\tilde{\Gamma}_i \times D^{i-1}$ with $(g_i \times f^{i-1})_*(\tilde{\gamma}_i) = \gamma_i$. We note that $g_i^*(H^{p,q}(X)) = 0$ if $p + q > 2i$, for dimensional reasons. Applying Lemma 3.1, we see that $\Delta_* = \gamma_*^{s+1}$ as endomorphisms of $H^{p,q}(X)$, for $p + q > 2s + 2$. Let $D = D^{s+1}$, let $\tilde{D} \to D$ be a resolution of singularities of D, and let $f: \tilde{D} \to X$ be the obvious map. Take a $\mathbb{Q}$-cycle $\tilde{\gamma}$ on $X \times \tilde{D}$ such that $\gamma^{s+1} = (id_X \times f)_*(\tilde{\gamma})$; applying Lemma 3.1 again, we see that

$$H^{p,q}(X) = \Delta_*(H^{p,q}(X)) = \gamma_*^{s+1}(H^{p,q}(X)) \subset f_*(H^{p-s-1,q-s-1}(\tilde{D})),$$

the second equality being valid for $p + q > 2s + 2$. In particular, we have $H^{p,q}(X) = 0$ if $p + q > 2s + 2$ and $p < s + 1$, proving (ii). □

Corollary 3.3. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose that the $\mathbb{Q}$-cycle class maps*

$$cl^n: CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for all n. Then the Hodge numbers $h^{p,q}(X)$ vanish if $|p - q| > 1$, and the cycle class maps

$$cl^n: CH^n(X)_{alg} \to J^n(X)$$

are surjective for all n.

Proof. This follows directly from Theorem 3.2. □

If we adjoin the identities $h^{p,q}(X) = h^{q,p}(X) = h^{d-p,d-q}(X)$ to the information supplied by Theorem 3.2, we obtain a nice picture of the Hodge diamond of X, assuming that the $\mathbb{Q}$-cycle maps cl^n are injective for $n =$

$d, d-1, \ldots, d-s$. Here the stars represent all the coordinates (p, q) where it is possible that $h^{p,q}(X) \neq 0$; in this example $d = 20$, $s = 5$.

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$$

Theorem 3.2, taken in the light of Bloch's conjecture that the zero-cycles on a smooth projective surface with $p_g = 0$ should be detected by the Albanese map, leads to the following:

Question 1. Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer $s \geq 0$ such that the Hodge numbers $h^{p,q}(X)$ vanish if

(i) $p + q \leq 2s + 2$ and $|p - q| > 1$,

and if

(ii) $p + q > 2s + 2$ and $p < s + 1$.

Then are the cycle class maps

$$cl^p\colon \mathrm{CH}^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

injective for $p = d, d-1, \ldots, d-s$? If not, are at least the $\mathbb{Q}$-cycle class maps

$$cl^p\colon \mathrm{CH}^p(X) \otimes \mathbb{Q} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))$$

injective for $p = d, d-1, \ldots, d-s$?

In light of the proof of Theorem 3.2, it might be better to replace (ii) with

(ii)′ There are smooth projective varieties $Y_1, \ldots, Y_s$ of dimension $d_X - s - 1$ and morphisms $Y_i \to X$ inducing a surjection of $\mathbb{Q}$-Hodge structures

$$\oplus_i H^*(Y_i, \mathbb{C}) \otimes \mathbb{Q}(-s-1) \to \oplus_{n=2s+2}^{2d_X} H^n(X, \mathbb{C}),$$

or even

(ii)″ For each $n > 2s+2$, there is a pure $\mathbb{Q}$- motive (i.e. a compatible collection of Galois representations, together with Hodge and Betti realizations, in the sense of Deligne [D] and Jannsen [J2]) M_n of weight $n - 2s - 2$ and an isomorphism of $\mathbb{Q}$-motives $M_n \otimes \mathbb{Q}(-s-1) \to H^n(X)$. □

As far as we know, the integral question is unsettled even for torsion cycles, except for zero-cycles (Roitman [R2], Bloch [Bl]) and for codimension two cycles (Murre [M]).

In any case, the contrapositive of Theorem 3.2 gives a criterion for the *failure* of the injectivity of the cycle map.

Corollary 3.4. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer s, $0 \leq s \leq d-2$, such that some Hodge number $h^{p,q}(X)$ is non-zero, with*

(i) $p+q \leq 2s+2$ *and* $|p-q| > 1$,

or with

(ii) $p+q > 2s+2$ *and* $p < s+1$.

Then there is an integer n, $d-s \leq n \leq d$ such that the $\mathbb{Q}$-cycle class map

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is not injective. □

Nori [N] has given examples of projective varieties with $\mathrm{CH}^n(X)_h \otimes \mathbb{Q} \neq 0$, but with $J^n(X) = 0$ as *generic* complete intersections of sufficiently high degree in certain smooth quadrics. It would be interesting to check the Hodge numbers of these varieties, to see if similar non-injectivity results could be obtained by applying Corollary 3.4. With reference to Question 1, one could ask if the minimal s satisfying the conditions of Corollary 3.4 points to precisely the cycle group of highest codimension for which the cycle class map fails to be injective, i.e.,

Question 2. Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Let s be the minimal integer such that some Hodge number $h^{p,q}(X)$ is non-zero, with

(i) $p+q \leq 2s+2$ and $|p-q| > 1$,

or with

(ii) $p+q > 2s+2$ and $p < s+1$

(supposing such an s exists). Then does the $\mathbb{Q}$-cycle class map

$$cl^n \colon \mathrm{CH}^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

have a non-trivial kernel for $n = d-s$? □

§4. Relations with K-theory

The injectivity of the cycle maps, and the ensuing decomposition of the diagonal given by Theorem 1.2, have consequences for higher K-theory, most notably K_1, although one can say something about the other K-groups as well. This leads to a generalization of a result of Coombes and Srinivas [CS], who showed that the map

$$\mathrm{CH}^1(X)\otimes K_1(\mathbb{C})\to H^1(X,\mathcal{K}_2)$$

is surjective, assuming that the group of zero-cycles modulo rational equivalence on X is representable.

Using the Gersten resolution (see [Q]) of the K-sheaves $\mathcal{K}_p$ on a smooth variety X over a field k, one arrives at the exact sequence

$$0\to H^0(X,\mathcal{K}_p)\to K_p(k(X))\to\oplus_{x\in X^{(1)}}K_{p-1}(k(x)),$$

where $X^{(p)}$ is the set of codimension p points of X. In particular, the map $H^0(X,\mathcal{K}_p)\to K_p(k(X))$ is injective; thus, if $p\colon Y\to X$ is a proper birational map of smooth varieties, the maps

$$p_*\colon H^0(Y,\mathcal{K}_p)\to H^0(X,\mathcal{K}_p);\quad p^*\colon H^0(X,\mathcal{K}_p)\to H^0(Y,\mathcal{K}_p)$$

are inverse isomorphisms. If we require X to be smooth and projective, the group $H^0(X,\mathcal{K}_p)$ is thus a birational invariant (assuming resolution of singularities for varieties over k). In particular, we may define the group $K_p(X)^{gen}$ for X an arbitrary projective variety over $\mathbb{C}$ by setting $K_p(X)^{gen}=H^0(\tilde{X},\mathcal{K}_p)$, where $\tilde{X}\to X$ is a resolution of singularities. We have

$$\begin{aligned}K_0(X)^{gen}&=\mathbb{Z};\\K_1(X)^{gen}&=\mathbb{C}^\times,\end{aligned}$$

for X an arbitrary projective variety over $\mathbb{C}$. The groups $K_p(X)^{gen}$ for $p>1$ are more mysterious, and in general contain $K_p(\mathbb{C})$ as a proper summand.

The cup product in K-theory gives rise to the natural maps

$$K_0(X) \otimes K_q(\mathbb{C}) \to K_q(X)$$
$$H^p(X, \mathcal{K}_p) \otimes K_q(X)^{gen} \to H^p(X, \mathcal{K}_{p+q}),$$

we call the image of these maps the *decomposable* part of $K_q(X)$ or of $H^p(X, \mathcal{K}_{p+q})$, respectively. There is a possibly larger subgroup of $H^p(X, \mathcal{K}_{p+q})$, which we now describe.

Let $\mathcal{Z}^p(X, q)$ be the group

$$\mathcal{Z}^p(X, q) = \bigoplus_{x \in X^{(p)}} K_q(\bar{x})^{gen},$$

where $\bar{x}$ is the closure of x in X. Via the Gersten resolution for $\mathcal{K}_{p+q}$, we have the natural map

$$\mathcal{Z}^p(X, q) \to H^p(X, \mathcal{K}_{p+q}).$$

We call the image of this map the *geometrically decomposable* part of $H^p(X, \mathcal{K}_{p+q})$. For $q = 0, 1$, the decomposable part and geometrically decomposable part of $H^p(X, \mathcal{K}_{p+q})$ agree; in general, the geometrically decomposable part contains the decomposable part. We extend the definition of the decomposable and geometrically decomposable parts to the rational versions $K_q(X)_{\mathbb{Q}}$ and $H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}}$ in the obvious way.

Theorem 4.1. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class maps*

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$, for some integer s, $0 \leq s \leq d-2$. Then the groups $H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}}$ are geometrically decomposable for $0 \leq p \leq s+1$. In particular, the map

$$CH^p(X) \otimes \mathbb{C}^\times \otimes \mathbb{Q} \to H^p(X, \mathcal{K}_{p+1})_{\mathbb{Q}}$$

is surjective for $0 \leq p \leq s+1$.

Proof. The bi-graded ring $\oplus_{p,q} H^p(X, \mathcal{K}_q)_{\mathbb{Q}}$ satisfies the Bloch-Ogus axioms [BO] for a twisted duality theory; in particular, if γ is a codimension d cycle on $X \times X$, γ gives rise to the endomorphism $\gamma_*: H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}} \to H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}}$, and the obvious analog of Lemmas 2.1 and 3.1 hold. We apply Theorem 1.2, retaining the notation of that theorem. The vanishing of $H^p(Y, \mathcal{K}_{p+q})$ for $p > \dim(Y)$ and for $p < 0$, together with the decomposition of the diagonal

$$\Delta = \gamma_0 + \ldots + \gamma_s + \gamma^{s+1}$$

implies that, on $H^p(X, \mathcal{K}_{p+q})$,

$$\Delta_* = \begin{cases} \gamma_{p-1*} + \gamma_{p*}; & \text{if } 0 \leq p \leq s \\ \gamma_{s*} + \gamma_*^{s+1}; & \text{if } p = s+1 \end{cases}$$

For Y smooth of dimension d_Y, the map

$$\mathrm{CH}^{d_Y}(Y) \otimes K_q(\mathbb{C}) \to H^{d_Y}(Y, \mathcal{K}_{d_Y+q})$$

is surjective; arguing as in the proof of Lemma 2.3, we see that the image $\gamma_{p-1*}(H^p(X, \mathcal{K}_{p+q}))$ is in the decomposable part of $H^p(X, \mathcal{K}_{p+q})$. Similarly, the argument of Lemma 2.3 shows that $\gamma_{p*}(H^p(X, \mathcal{K}_{p+q}))$ is in the geometrically decomposable part of $H^p(X, \mathcal{K}_{p+q})$. Finally, arguing as in the proof of Lemma 2.4, we see that $\gamma_*^{s+1}(H^p(X, \mathcal{K}_{p+q}))$ is in the geometrically decomposable part of $H^p(X, \mathcal{K}_{p+q})$. This proves the theorem. □

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2^{nd} author partially supported by the NSF and the DFG

Astérisque

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Decomposability of Chow groups implies decomposability of cohomology

Astérisque, tome 218 (1993), p. 227-241
<http://www.numdam.org/item?id=AST_1993__218__227_0>

Decomposability of Chow groups implies decomposability of cohomology

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Let X be an n-dimensional complete irreducible smooth variety defined over the field $\mathbb{C}$ of complex numbers. For any Zariski open subset V of X, we have the following graded rings.

(i) $\bigoplus_{i=0}^{n} CH^i(V)_{\mathbb{Q}}$, where $CH^i(V)_{\mathbb{Q}}$ is the Chow group of algebraic cycles of codimension i on V with rational coefficients, modulo rational equivalence (see [F], Chapter 8, Prop. 8.3).

(ii) $\bigoplus_{i=0}^{n} H^i(V)/N^1H^i(V)$, where $H^i(V) = H^i(V_{an}, \mathbb{Q})$ is the singular cohomology of the underlying complex manifold V_{an}, and

$$N^aH^i(V) = \varinjlim_{\operatorname{codim} Z \geq a} \ker\Big(H^i(V) \longrightarrow H^i(V - Z)\Big)$$

defines Grothendieck's coniveau filtration (here Z runs over the Zariski closed subsets of V of codimension $\geq a$).

(iii) $\bigoplus_{i=0}^{n} H^0(V, \mathcal{H}^i_V)$, where $\mathcal{H}^i_V$ is the sheaf for the Zariski topology associated to the presheaf

$$U \longmapsto H^i(U) = H^i(U_{an}, \mathbb{Q}).$$

(iv) We also have a graded ring associated to X: $\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X))$, where

$$\begin{aligned} H^i(\mathbb{C}(X)) := {} & \varinjlim_{V \subset X} H^i(V) = \varinjlim_{V \subset X} H^i(V)/N^1H^i(V) \\ = {} & \varinjlim_{V \subset X} H^0(V, \mathcal{H}^i_V) \end{aligned}$$

Here the direct limits are over the non-empty Zariski open sets V in X, and $\mathbb{C}(X)$ denotes the function field of X. The first equality defines the cohomology of the function field; the right side of the equality is clearly a birational invariant of X.

In (ii), (iii), (iv) above, we consider only cohomology in degrees up to n, since the singular cohomology of an affine variety of dimension n vanishes in degrees larger than n, by the weak Lefschetz theorem (this implies that for any variety V of dimension n, we have $H^i(V) = N^1H^i(V)$ for $i > n$).

Theorem 1 *Let X be a smooth complete variety of dimension n over $\mathbb{C}$. Suppose there exists a non empty Zariski open subset $V \subset X$, and positive integers $n_1, \dots, n_r$ with $\sum_i n_i = n$, such that one of the following product maps is surjective:*

(i) $CH^{n_1}(V)_{\mathbb{Q}} \otimes \cdots \otimes CH^{n_r}(V)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$

(ii) $H^{n_1}(V)/N^1H^{n_1}(V) \otimes \cdots \otimes H^{n_r}(V)/N^1H^{n_r}(V) \longrightarrow H^n(V)/N^1H^n(V)$

(iii) $H^0(V, \mathcal{H}_V^{n_1}) \otimes \cdots \otimes H^0(V, \mathcal{H}_V^{n_r}) \longrightarrow H^0(V, \mathcal{H}_V^n)$

(iv) $H^{n_1}(\mathbb{C}(X)) \otimes \cdots \otimes H^{n_r}(\mathbb{C}(X)) \longrightarrow H^n(\mathbb{C}(X))$

Then the cup product map for the coherent cohomology

$$H^{n_1}(X, \mathcal{O}_X) \otimes H^{n_2}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X) \qquad (*)$$

is surjective.

The proof of (i) is motivated by Bloch's proof [B] of Mumford's theorem that for surfaces X with $H^2(X, \mathcal{O}_X) \neq 0$, the Chow group of 0-cycles $CH^2(X)$ is not 'finite dimensional' (see also the 'metaconjecture' in Chapter 1 of [B2]). Many other variants of Bloch's method have been considered by several authors. The method involves the action of correspondences on the cohomology. At the referee's suggestion, we try to make this argument with some care, though this type of reasoning is well known to experts.

The proofs of (ii), (iii) and (iv) are a consequence of the mixed Hodge structure on the cohomology of the open sets V (see [D]). For $V = X$, the surjectivity of the map (ii) trivially implies that $(*)$ is surjective, using the Hodge decomposition on cohomology, since the ring $\oplus H^i(X, \mathcal{O}_X)$ is a graded quotient of $\oplus(H^i(X)/N^1H^i(X)) \otimes \mathbb{C}$.

The proof of the theorem

We first discuss (i). Let $C = X - V$, and let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition of X, C and V. Let X_0, C_0, V_0 be the corresponding models over k, and for any extension L of k, let $X_L = X_0 \times_k L$, etc. We embed $k(X_0) \hookrightarrow \mathbb{C}$ as a k-subalgebra, and consider the generic point of X_0 as a closed point $\eta \in X_{k(X_0)}$, hence as an element of $CH^n(X_{k(X_0)})_{\mathbb{Q}}$. By assumption, its image under the composite

$$CH^n(X_{k(X_0)})_{\mathbb{Q}} \longrightarrow CH^n(X)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

decomposes as

$$\sum_{\text{finite}} m_{n_1} \cdot \cdots \cdot m_{n_r}$$

where $m_{n_i} \in CH^{n_i}(V)_{\mathbb{Q}}$. The m_{n_i} are defined over a subfield $L \subset \mathbb{C}$ which is finitely generated over $k(X_0)$, and (see [B2], Lecture 1, Appendix, Lemma 3) the natural map

$$CH^n(V_L)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

is injective, so

$$\sum_{\text{finite}} m_{n_1} \cdot \cdots \cdot m_{n_r} = [\eta] \tag{1}$$

holds in $CH^n(V_L)_{\mathbb{Q}}$.

Let F be the algebraic closure of $k(X_0)$ in L; since L is finitely generated over $k(X_0)$, F is a finite algebraic extension of $k(X_0)$. We can find a non-singular affine F-variety W with function field L. The graded ring

$$\bigoplus_{i \geq 0} CH^i(V_L)$$

is the direct limit of the graded rings

$$\bigoplus_{i \geq 0} CH^i(V_F \times_F W'),$$

where W' runs over the non-empty Zariski open sets in W (see [B2], Lecture 1, Appendix, Lemma 1). So after replacing W by a nonempty open subset, we may assume given classes $m_{n_i} \in CH^{n_i}(V_F \times_F W)$ such that (1) holds in

$$CH^n(V_F \times_F W)_{\mathbb{Q}},$$

where $[\eta]$ now denotes the image in $CH^n(V_F \times_F W)_{\mathbb{Q}}$ of the earlier class

$$[\eta] \in CH^n(V_{k(X_0)})_{\mathbb{Q}} \subset CH^n(V_F)_{\mathbb{Q}}.$$

Let $P \in W$ be a closed point. Then there is a homomorphism of rings

$$f^* : \bigoplus_{i \geq 0} CH^i(V_F \times_F W) \to \bigoplus_{i \geq 0} CH^i(V_F \times_F \operatorname{Spec} F(P)),$$

where $f : V_F \times_F \operatorname{Spec} F(P) \to V_F \times_F W$ is induced by the inclusion of P into W (f is a morphism of non-singular F-varieties, hence by [F], Prop. 8.3, such a homomorphism f^* exists). Then $f^*[\eta]$ is just $[\eta]$ considered as an element of $CH^n(V_{k(X_0)})_{\mathbb{Q}} \subset CH^n(V_{F(P)})_{\mathbb{Q}}$. Hence

$$\sum_{\text{finite}} f^*(m_{n_1}) \cdot \cdots \cdot f^*(m_{n_r}) = [\eta] \tag{2}$$

holds in $CH^n(V_{F(P)})_{\mathbb{Q}}$, where $f^*(m_{n_i}) \in CH^{n_i}(V_{F(P)})_{\mathbb{Q}}$.

Hence, we are reduced to the situation when (1) holds, where L is a finite algebraic extension of $k(X_0)$, and $m_{n_i} \in CH^i(V_L)_{\mathbb{Q}}$.

By resolution of singularities, we can find a projective non-singular k-variety Z_0, together with a k-morphism $\sigma_0 : Z_0 \to X_0$, such that the induced map on function fields is the given inclusion $k(X_0) \to L$. Since L is a finite extension of $k(X_0)$, the morphism σ_0 is generically finite.

The (flat) k-morphism $\operatorname{Spec} L \to Z_0$ given by the inclusion of the generic point gives rise to a natural surjective homomorphism of graded rings

$$Cl : \bigoplus_{i \geq 0} CH^i(X_0 \times_k Z_0)_{\mathbb{Q}} \to \bigoplus_{i \geq 0} CH^n(V_L)_{\mathbb{Q}},$$

such that if $[\Delta_{\sigma_0}] \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}$ is the class of the transposed graph of σ_0, then $Cl([\Delta_{\sigma_0}])$ is just $[\eta] \in CH^n(V_L)_{\mathbb{Q}}$. The kernel of

$$CH^n(X_0 \times_k Z_0)_{\mathbb{Q}} \to CH^n(V_L)$$

consists of the subgroup generated by the classes supported on subsets of the form $(C_0 \times_k Z_0) \cup (X_0 \times_k D_0)$, as D_0 runs over all proper subvarieties of Z_0 (see [B2], Lecture 1, Appendix, Lemma 1, and [F], Prop. 1.8). Thus we have an equation

$$[\Delta_{\sigma_0}] - \sum M_{n_1} \cdot \cdots \cdot M_{n_r} = \gamma_0 + \delta_0$$

in $CH^n(X_0 \times_k Z_0)$, where for some divisor $D_0 \subset Z_0$, we have

$$\begin{aligned} M_{n_i} &\in CH^{n_i}(X_0 \times_k Z_0)_{\mathbb{Q}}, & M_{n_i} &\mapsto m_{n_i} \in CH^{n_i}(V_L)_{\mathbb{Q}} \\ \gamma_0 &\in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}, & \operatorname{supp} \gamma_0 &\subset C_0 \times_k Z_0 \\ \delta_0 &\in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}, & \operatorname{supp} \delta_0 &\subset X_0 \times_k D_0 \end{aligned}$$

Thus if $Z = Z_0 \times_k \mathbb{C}$, $\sigma : Z \longrightarrow X$ the induced map, $M'_{n_i} = (M_{n_i})_{\mathbb{C}}$, $\gamma = (\gamma_0)_{\mathbb{C}}$, $\delta = (\delta_0)_{\mathbb{C}}$, $C = (C_0)_{\mathbb{C}}$, $D = (D_0)_{\mathbb{C}}$, then

$$[\Delta_\sigma] - \sum M'_{n_1} \cdot \cdots \cdot M'_{n_r} = \gamma + \delta \tag{3}$$

in $CH^n(X \times Z)_{\mathbb{Q}}$, where γ is supported on $C \times Z$, and δ is supported on $X \times D$ (in the rest of this proof, $\times$ denotes $\times_{\mathbb{C}}$).

Elements of $CH^n(X \times Z)_{\mathbb{Q}}$ act on $H^n(X)$ as follows. First, there is a cycle class homomorphism of graded rings

$$\bigoplus_{i \geq 0} CH^i(X \times Z) \to \bigoplus_{i \geq 0} H^{2i}(X \times Z)$$

(see [F], Chapter 19, Cor. 19.2(b)). By [F], Prop. 16.1.2 and Example 19.2.7, an element $\alpha \in CH^n(X \times Z)_{\mathbb{Q}}$ yields mappings

$$\alpha_* : CH^i(X)_{\mathbb{Q}} \to CH^i(Z)_{\mathbb{Q}}, \qquad \alpha^* : CH^i(Z)_{\mathbb{Q}} \to CH^i(X)_{\mathbb{Q}}$$

on Chow groups, and

$$\alpha_* : H^i(X) \to H^i(Z), \qquad \alpha^* : H^i(Z) \to H^i(X)$$

on cohomology, where if $p : X \times Z \to X$, and $q : X \times Z \to Z$ are the projections, then $\alpha_*(x) = q_*(p^*(x) \cup \alpha)$, and $\alpha^*(y) = p_*(q^*(y) \cup \alpha)$. Since X, Z are proper and smooth over $\mathbb{C}$, the required operations exists on cohomology as well as Chow groups. Further, if α is the class of the transposed graph of a morphism $f : Z \to X$, then $\alpha_* = f^*$, and $\alpha^* = f_*$, where f^* is the natural map on cohomology, and f_* is the Gysin map (see [F], Prop. 16.1.2 and Example 19.2.7).

On the level of cohomology, the Gysin (push forward) map

$$q_* : H^m(X \times Z) \to H^{m-2n}(Z)$$

is defined via Poincaré duality. As we see below, an equivalent (up to sign) description of q_* is as follows: one may use the Künneth isomorphism

$$H^m(X \times Z) \cong \bigoplus_{i+j=m} H^i(X) \otimes H^j(Z)$$

to project onto the summand $H^{2n}(X) \otimes H^{m-2n}(Z)$, and then use the canonical isomorphism $\deg_X : H^{2n}(X) \xrightarrow{\cong} \mathbb{Q}$ (for any non-singular projective variety T over $\mathbb{C}$ of dimension d, let $\deg_T : H^{2d}(T) \xrightarrow{\cong} \mathbb{Q}$ denote the natural isomorphism). The map p_* is defined similarly.

To see that the two procedures for defining q_* are equivalent up to sign, note that the natural isomorphism (induced by $\deg_{X\times Z}$)

$$H^{2n}(X) \otimes H^{2n}(Z) = H^{4n}(X \times Z) \xrightarrow{\cong} \mathbb{Q}$$

is the tensor product of the natural isomorphisms

$$H^{2n}(X) \xrightarrow{\cong} \mathbb{Q}, \qquad H^{2n}(Z) \xrightarrow{\cong} \mathbb{Q}$$

(this is because a similar assertion is valid for integral cohomology – now we are comparing two isomorphisms $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, which are equal because the natural orientation on $X \times Z$ is the product orientation of those on X and Z). Now if $x \in H^m(X \times Z)$, then $q_*(x)$ defined via Poincaré duality is the unique element of $H^{m-2n}(Z)$ such that for any $x' \in H^{4n-m}(Z)$, we have

$$\deg_{X\times Z}(x \cup q^*(x')) = \deg_Z(q_*(x) \cup x').$$

But $x \cup q^*(x')$ depends only on the Künneth component of x in

$$H^{2n}(X) \otimes H^{m-2n}(Z).$$

If this Künneth component of x is $\sum_j p^*x_j \cup q^*y_j$, then

$$x \cup q^*(x') = \pm \sum_j p^*(x_j) \cup q^*(y_j \cup x'),$$

so that

$$\deg_Z(q_*(x) \cup x') = \deg_{X\times Z}(x \cup q^*(x')) = \pm \sum_j \deg_X(x_j) \deg_Z(y_j \cup x').$$

On the other hand, the second procedure for defining $q_*(x)$ yields the element $\sum_j \deg_X(x_j) y_j$, whose cup product with x' is $\sum_j \deg_X(x_j)(y_j \cup x')$, which thus has the same image under $\deg_Z$ as $q_*(x) \cup x'$, up to sign.

The cup product on the cohomology of $X \times Z$ is compatible up to signs with the Künneth decomposition, and the cup products on the cohomology of X and Z respectively. This is because we may view the Künneth component

$$H^i(X) \otimes H^j(Z) \subset H^{i+j}(X \times Z)$$

as image of the mapping given by

$$x \otimes y \mapsto p^*x \cup q^*y.$$

Now our assertion follows because the cup product is functorial, associative, and commutative up to sign.

In particular, the action of $\alpha \in H^{2n}(X \times Z)$ on $H^n(X)$ (via α_*) or on $H^n(Z)$ (via α^*) is determined by the Künneth component of α in $H^n(X) \otimes H^n(Z)$. If $\alpha = \sum_i p^* x_i \otimes q^* y_i$, then

$$\alpha^*(z) = \pm \sum_i \deg_Z(y_i \cup z) x_i,$$

where $y_i \cup z \in H^{2n}(Z)$, and $\deg_Z : H^{2n}(Z) \to \mathbb{Q}$ is the natural isomorphism.

The Künneth decomposition, as well as the action of classes of elements of $CH^n(X \times Z)_{\mathbb{Q}}$ on cohomology, are compatible with the Hodge decompositions on the various cohomology groups. Hence for $\alpha \in CH^n(X \times Z)_{\mathbb{Q}}$, the map

$$\alpha^* : H^n(Z, \mathbb{C})/F^1 H^n(Z, \mathbb{C}) \to H^n(X, \mathbb{C})/F^1 H^n(X, \mathbb{C})$$

depends only on the image of the class of α under the composite

$$CH^n(X \times Z)_{\mathbb{Q}} \to H^{2n}(X \times Z, \mathbb{C}) \to H^n(X, \mathbb{C}) \otimes H^n(Z, \mathbb{C}) \to \\ H^n(X, \mathcal{O}_X) \otimes H^0(Z, \Omega^n_{Z/\mathbb{C}}).$$

Here the last map is a tensor product of projections onto appropriate summands of the Hodge decompositions. This is because if $y \in H^n(X, \mathbb{C})$ is of Hodge type (p, q), $z \in H^n(X, \mathcal{O}_X)$ (*i.e.* is of type $(0, n)$), and $x \in H^n(X, \mathcal{O}_X)$, then $\deg_Z(y \cup z)x$ is 0, unless y has type $(n, 0)$. Let

$$\sum_j t_j \otimes x_j \in H^n(X) \otimes H^n(X)$$

be the Künneth component of type (n, n) of the diagonal of $X \times X$, whose inverse image

$$\sum_i t_i \otimes \sigma^*(x_i) \in H^n(X) \otimes H^n(Z)$$

is the Künneth component of type (n, n) of Δ_σ. Then

$$\sigma_* = [\Delta_\sigma]^* : H^n(Z, \mathbb{C}) \to H^n(X, \mathbb{C})$$

is given by

$$z \mapsto \sum_j \deg_Z(\sigma^* x_j \cup z) t_j.$$

On the other hand, if $\alpha_{n_i} \in H^{2n_i}(X \times Z)$ is the cohomology class of M'_{n_i}, then

$$[M'_{n_1} \cdot \cdots \cdot M'_{n_r}]^* : H^n(Z) \to H^n(X)$$

is determined by the (n,n)th Künneth component of the cohomology class $\alpha_{n_1} \cup \cdots \cup \alpha_{n_r}$. Further, the map on the Hodge components of type $(0,n)$

$$\xi = [M'_{n_1} \cdot \cdots \cdot M'_{n_r}]^* : H^n(Z, \mathcal{O}_Z) \to H^n(X, \mathcal{O}_X)$$

depends only on the Hodge component of type $(0,n) \otimes (n,0)$ in

$$H^n(X, \mathbb{C}) \otimes H^n(Z, \mathbb{C})$$

of $\alpha_{n_1} \cup \cdots \cup \alpha_{n_r}$. This is just $\alpha'_{n_1} \cup \cdots \cup \alpha'_{n_r}$, where α'_{n_i} is the Hodge component of α_{n_i} in $H^{n_i}(X, \mathcal{O}_X) \otimes H^0(Z, \Omega^{n_i}_{Z/\mathbb{C}})$. Hence ξ is expressible in the form

$$\xi(z) = \sum_{\text{finite}} \deg_Z(y_{n_1} \cup \cdots \cup y_{n_r} \cup z) z_{n_1} \cup \cdots \cup z_{n_r},$$

for suitable $y_{n_j} \in H^0(Z, \Omega^{n_j}_{Z/\mathbb{C}})$, and $z_{n_j} \in H^{n_j}(X, \mathcal{O}_X)$. In particular,

$$\operatorname{image} \xi \subset \operatorname{image} \left(H^{n_1}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X) \right).$$

The correspondence γ_* maps $H^n(Z)$ into $N^a H^n(X)$, where a is the codimension of C in X, whereas δ_* maps $H^n(Z)$ into $N^1 H^n(X)$ (see [B], and [J], proof of (10.1)).

Hence on the Hodge components of type $(0,n)$, the map

$$\sigma_* : H^n(Z, \mathbb{C}) \to H^n(X, \mathbb{C})$$

maps $H^n(Z, \mathcal{O}_Z)$ into

$$\operatorname{image} \left(H^{n_1}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X) \right).$$

Finally, we note that $\sigma_* \circ \sigma^* : H^n(X, \mathbb{C}) \to H^n(X, \mathbb{C})$ is multiplication by the degree of σ; hence it is an isomorphism. Hence σ_* is surjective, *i.e.*,

$$H^{n_1}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X)$$

is surjective.

This proves that if the map (i) is surjective, so is the map $(*)$. Hence to complete the proof of the theorem, it suffices to show that if any of the maps (ii), (iii) or (iv) is surjective, so is $(*)$. From Hodge theory (see [D]), there is a surjection

$$\theta_V : H^i(V) \otimes \mathbb{C} \longrightarrow H^i(X, \mathcal{O}_X)$$

for any non empty Zariski open set $V \subset X$, which is compatible with cup products; this is just the quotient modulo the subspace $F^1(H^i(V) \otimes \mathbb{C})$, where $F^j(H^i(V) \otimes \mathbb{C})$ is the Hodge filtration for Deligne's mixed Hodge structure on $H^i(V)$. Further, for any inclusion of Zariski open sets $W \subset V \subset X$, the triangle

$$\begin{tikzcd} H^i(V) \otimes \mathbb{C} \arrow[rr] \arrow[dr, "\theta_V"'] & & H^i(W) \otimes \mathbb{C} \arrow[dl, "\theta_W"] \\ & H^n(X, \mathcal{O}_X) & \end{tikzcd}$$

commutes, by functoriality of the mixed Hodge structure.

Hence there is a commutative diagram of graded rings

$$\begin{tikzcd} \bigoplus_{i=0}^{n} (H^i(V)/N^1H^i(V)) \otimes \mathbb{C} \arrow[r] \arrow[dr, "\alpha"'] & \bigoplus_{i=0}^{n} H^0(V, \mathcal{H}^i_V) \otimes \mathbb{C} \arrow[r] \arrow[d, "\beta"] & \bigoplus_{i=0}^{n} H^i(\mathbb{C}(X)) \otimes \mathbb{C} \arrow[dl, "\gamma"] \\ & \bigoplus_{i=0}^{n} H^i(X, \mathcal{O}_X) & \end{tikzcd}$$

where α, β and γ are induced by the θ_W for all open $W \subset V$, and are all surjective (incidentally the horizontal maps are known to be injective by [BO]). The surjections α, β and γ immediately imply that if the maps in (ii), (iii) or (iv) respectively are surjective, then so is $(*)$. $\square$

From the formulation of the proof, it appears that (ii), (iii) and (iv) are directly related to $(*)$ via the maps α, β and γ, while the relation between (i) and $(*)$ is indirect. It is possible to give another proof (which is really more or less a reformulation of the old one) which looks more like the proof in the other three cases, as follows.

We make use of the existence of a cycle class homomorphism

$$\bigoplus_{i=0}^{n} CH^i(X)_{\mathbb{Q}} \longrightarrow \bigoplus_{i=0}^{n} H^i(X, \Omega^i_{X/\mathbb{Z}}),$$

where $\Omega^i_{X/\mathbb{Z}}$ is the sheaf of absolute Kähler i-forms (see [S], for example; the proof below is motivated by the proof in [S] of the infinite dimensionality theorem for zero cycles). If k, X_0 are as in the proof above, this induces a ring homomorphism

$$\bigoplus_{i=0}^{n} CH^i(X)_{\mathbb{Q}} \longrightarrow \bigoplus_{i=0}^{n} H^i(X, \Omega^i_{X/X_0}) = H^i(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^i_{\mathbb{C}/k}.$$

Suppose $l : H^n(X_0, \mathcal{O}_{X_0}) \longrightarrow k$ is a non-zero linear functional such that the composite

$$H^{n_1}(X_0, \mathcal{O}_{X_0}) \otimes \cdots \otimes H^{n_r}(X_0, \mathcal{O}_{X_0}) \longrightarrow H^n(X_0, \mathcal{O}_{X_0}) \xrightarrow{l} k$$

is zero. Then the induced homomorphism

$$\begin{tikzcd} CH^{n_1}(X)_{\mathbb{Q}} \otimes \cdots \otimes CH^{n_r}(X)_{\mathbb{Q}} \arrow[r] & CH^n(X)_{\mathbb{Q}} \arrow[r] \arrow[dr, "\mu"'] & H^n(X_0,\mathcal{O}_{X_0}) \otimes_k \Omega^n_{\mathbb{C}/k} \arrow[d, "l\otimes 1"] \\ & & \Omega^n_{\mathbb{C}/k} \end{tikzcd}$$

clearly vanishes. We claim that

(a) for any $P \in C$, $[P] \in CH^n(X)_{\mathbb{Q}}$ lies in the kernel of the map (defined above using the functional l)

$$\mu : CH^n(X)_{\mathbb{Q}} \longrightarrow \Omega^n_{\mathbb{C}/k}$$

(b) if $k(X_0) \hookrightarrow \mathbb{C}$ yields the point $\eta \in X$, corresponding to the generic point of X_0, then $\mu([\eta]) \neq 0$.

These properties follow from certain properties of the cycle map

$$CH^n(X)_{\mathbb{Q}} \longrightarrow H^n(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^n_{\mathbb{C}/k},$$

discussed below. If $P \in X$ has ideal sheaf I, then there is an exact sequence

$$I/I^2 \xrightarrow{\psi} \Omega^1_{X/X_0} \otimes \mathcal{O}_P \longrightarrow \Omega^1_{P/X_0} \longrightarrow 0. \tag{4}$$

The image of $\bigwedge^n \psi$ under the composite

$$\begin{aligned} \mathrm{Hom}(\textstyle\bigwedge^n I/I^2, \Omega^n_{X/X_0} \otimes \mathcal{O}_P) \cong \mathrm{Ext}^n_X(\mathcal{O}_P, \Omega^n_{X/X_0}) \longrightarrow H^n_P(X, \Omega^n_{X/X_0}) \longrightarrow \\ \longrightarrow H^n(X, \Omega^n_{X/X_0}) = H^n(X_0, \mathcal{O}_{X_0}) \otimes \Omega^n_{\mathbb{C}/k} \end{aligned}$$

is the cycle class of P (this follows from the definition of the cycle class given in [S]). If $Q \in X_0$ is the image of P (note that Q need not be a closed point), then the sequence (4) may be rewritten as

$$\mathbb{C}^n \xrightarrow{\psi} \Omega^1_{\mathbb{C}/k} \xrightarrow{\chi} \Omega^1_{\mathbb{C}/k(Q)} \longrightarrow 0$$

where χ is the natural surjection. Thus

$$\operatorname{rank} \psi = \text{tr.deg.}\,(k(Q)/k).$$

Hence if $P \in C$, so that $Q \in C_0 \subset X_0$ but $C_0 \neq X_0$, then rank $\psi < n$, and $\bigwedge^n \psi = 0$. This proves (a).

Secondly the linear functional

$$l : H^n(X_0, \mathcal{O}_{X_0}) \longrightarrow k$$

is determined, via Serre duality, by a unique $\omega \in H^0(X_0, \Omega^n_{X_0/k})$. The embedding $k(X_0) \hookrightarrow \mathbb{C}$ used to determine $\eta \in X$ also yields an embedding

$$H^0(X_0, \Omega^n_{X_0/k}) \hookrightarrow \Omega^n_{k(X_0)/k} \hookrightarrow \Omega^n_{\mathbb{C}/k},$$

and it is shown in [S] that $\mu(\eta)$ is the image of ω under this map. In particular it is non-zero.

Further remarks

1. The theorem has been stated in the present form, as urged by the referee. However, possible applications would seem to be in the direction that if the cup product on coherent cohomology is *not* surjective, then none of the other products (i)-(iv) is surjective. This is because it is presumably easier to directly compute the cup product on coherent cohomology than to compute any of the products (i)-(iv), in most situations.

2. One might hope (this is consistent with the philosophy outlined in [B2], Chapter 1) that if

$$H^{n_1}(X)/N^1H^{n_1}(X) \otimes \cdots \otimes H^{n_r}(X)/N^1H^{n_r}(X) \longrightarrow H^n(X)/N^1H^n(X)$$

is not surjective, then for any non empty open set $V \subset X$,

$$CH^{n_1}(V)_{\mathbb{Q}} \otimes \cdots \otimes CH^{n_r}(V)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

is not surjective. In an earlier version of the paper, the authors had claimed to prove this, but the argument was found to be incomplete. This statement is purely algebraic, and suggests an analogous theorem in arbitrary characteristics, if we interpret $H^i(X)$ as a suitable l-adic cohomology group, equipped with Grothendieck's coniveau filtration, defined as before.

However, note that if $(*)$ is surjective, then the image of

$$H^{n_1}(X) \otimes \cdots \otimes H^{n_r}(X) \to H^n(X)$$

is a $\mathbb{Q}$-Hodge substructure, which after tensoring with $\mathbb{C}$, maps onto $H^{(0,n)}(X)$. Hence this image maps onto the smallest quotient Hodge structure with the same space $H^{(0,n)}$. According to Grothendieck's generalized Hodge conjecture, this smallest quotient is just $H^n(X)/N^1H^n(X)$. Thus the surjectivity of the map in (ii) for $V = X$ is conjecturally equivalent to that of $(*)$.

3. Of course, it would be very interesting to have information in the converse direction to the theorem. For example, for $n = 2$ and surfaces of general type for which $H^2(X, \mathcal{O}_X) = 0$, one also knows that $H^1(X, \mathcal{O}_X) = 0$, so that $CH^1(X) = \mathrm{Pic}\,(X)$ is a finitely generated abelian group. Now the implication $(i) \Longrightarrow (*)$ is equivalent to Bloch's conjecture that $CH^2(X) = \mathbb{Z}$. (Here and below, by '(i)' or '$(*)$' we mean the surjectivity of the corresponding map, for some choices of $n_1, \ldots, n_r$; these choices will be fixed in each discussion.) This is because the subgroup of $CH^2(X)$ of cycles of degree 0 is a divisible group ([B2], Lemma 1.3), so if it is finitely generated, it must be 0. Note that $(*)$ and (ii) are equivalent for surfaces; a generalisation of Bloch's conjecture is the assertion that $(ii) \Longrightarrow (i)$.

However, $(*) \Longrightarrow (iii)$ is false in general. If X is the Jacobian of a general curve of genus 3, then the natural map

$$H^i(X, \mathbb{Q}) \longrightarrow H^0(X, \mathcal{H}^i_X)$$

is surjective for $i \leq 2$, while the cokernel for $i = 3$ is the Griffiths group of codimension 2 cycles (with rational coefficients) homologous to 0 modulo algebraic equivalence, by results of [BO]. But Ceresa [C] has shown that this Griffiths group is a non-zero $\mathbb{Q}$-vector space. Hence the map (iii) is not surjective, while $(*)$ (and even (i)) is always surjective on an abelian variety (see [B3]).

We do not know an example where the map (iv) is known to be surjective.

4. In contrast to the situation in (iv), Bloch (see [B2], 5.12) wonders whether the graded ring

$$\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z}) = \varinjlim_{V \subset X} \bigoplus_{i=0}^{n} H^i(V_{an}, \mathbb{Z}/m\mathbb{Z})$$

is generated by $H^1(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z})$ as a $\mathbb{Z}/m\mathbb{Z}$-algebra. If $n = 2$, this is known, from the Merkurjev-Suslin theorem, and Bloch (*loc. cit.*) states that

$$H^1(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z})^{\otimes n} \longrightarrow H^n(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z})$$

is always surjective. More generally, Kato has conjectured that for any field K containing a primitive l^{th} root of unity, the Galois cohomology ring with $\mathbb{Z}/l\mathbb{Z}$ coefficients, $l \neq \operatorname{char} K$, is generated by $H^1(K, \mathbb{Z}/l\mathbb{Z})$.

One may be tempted to argue using inverse limits that in view of the above conjectures, one should expect that

$$\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X), \mathbb{Q}_l) = \bigoplus_{i=0}^{n} H^i(\mathbb{C}(X), \mathbb{Q}) \otimes \mathbb{Q}_l$$

is generated by $H^1(\mathbb{C}(X), \mathbb{Q}_l)$ as a $\mathbb{Q}_l$-algebra. However, the inverse systems

$$\{H^i(\mathbb{C}(X), \mathbb{Z}/l^m\mathbb{Z})\}_{m \geq 1}$$

do not satisfy the Mittag-Leffler condition, so the surjectivity of multiplication maps need not be preserved under taking inverse limits.

5. If $R = \bigoplus_{i=0}^{n} R_i$ is a graded $\mathbb{Q}$-algebra, define $x \in R_n$ to be *r-decomposable* if there is an expression

$$x = \sum_{i=1}^{r} x_i y_i$$

where the $x_i, y_i \in R$ are homogeneous of degree > 0. If x is not r-decomposable, we say that x is *r-indecomposable.*

Nori [N] has shown that if X is a proper smooth variety of dimension n over $\mathbb{C}$ with $H^n(X, \mathcal{O}_X) \neq 0$, then for any non-empty open subset $V \subset X$ and any $r > 0$, $CH^n(V)_{\mathbb{Q}}$ contains elements which are r-indecomposable in $\bigoplus_i CH^i(V)_{\mathbb{Q}}$. Nori's proof involves an argument analogous to the second proof of $(i) \Rightarrow (*)$ using the cycle class.

In a similar vein, suppose X is a smooth proper variety of dimension n over a universal domain Ω, such that $H^n_{\text{ét}}(X, \mathbb{Q}_l) \neq N^1 H^n_{\text{ét}}(X, \mathbb{Q}_l)$. Then one may raise the following questions.

(1) For any non-empty open set $V \subset X$ and any $r > 0$, does $CH^n(V)_{\mathbb{Q}}$ contain r-indecomposable elements?

(2) Does $H^n_{\text{ét}}(\Omega(X), \mathbb{Q}_l)$ contain elements which are r-indecomposable in

$$\bigoplus_i H^i_{\text{ét}}(\Omega(X), \mathbb{Q}_l),$$

for each $r > 0$? Is this true at least when $\Omega = \mathbb{C}$ and $H^n(X, \mathcal{O}_X) \neq 0$?

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Compactifications of moduli spaces inspired by mirror symmetry

Astérisque, tome 218 (1993), p. 243-271

<http://www.numdam.org/item?id=AST_1993__218__243_0>

Compactifications of moduli spaces inspired by mirror symmetry

David R. Morrison

The study of moduli spaces by means of the period mapping has found its greatest success for moduli spaces of varieties with trivial canonical bundle, or more generally, varieties with Kodaira dimension zero. Now these moduli spaces play a pivotal rôle in the classification theory of algebraic varieties, since varieties with nonnegative Kodaira dimension which are not of general type admit birational fibrations by varieties of Kodaira dimension zero. Since such fibrations typically include singular fibers as well as smooth ones, it is important to understand how to compactify the corresponding moduli spaces (and if possible, to give geometric interpretations to the boundary of the compactification). Note that because of the possibility of blowing up along the boundary, abstract compactifications of moduli spaces are far from unique.

The hope that the period mapping could be used to construct compactifications of moduli spaces was given concrete expression in some conjectures of Griffiths [25, §9] and others in the late 1960's. In particular, Griffiths conjectured that there would be an analogue of the Satake-Baily-Borel compactifications of arithmetic quotients of bounded symmetric domains, with some kind of "minimality" property among compactifications. Although there has been much progress since [25] in understanding the behavior of period mappings near the boundary of moduli, compactifications of this type have not been constructed, other than in special cases.

In the case of algebraic K3 surfaces, the moduli spaces themselves are arithmetic quotients of bounded symmetric domains, so each has a minimal (Satake-Baily-Borel) compactification. In studying the moduli spaces for K3 surfaces of low degree in the early 1980's, Looijenga [35] found that the Satake-Baily-Borel compactification needed to be blown up slightly in order to give a good geometric interpretation to the boundary. He introduced a class of compactifications, the *semi-toric compactifications*, which includes the ones with a good geometric interpretation.

In higher dimension, the moduli spaces are not expected to be arithmetic quotients of symmetric domains, so different techniques are needed. The study of these moduli spaces has received renewed attention recently, due to the discovery by theoretical physicists of a phenomenon called "mirror symmetry". One of the predictions of mirror symmetry is that the moduli space for a variety with trivial canonical bundle, which parameterizes the possible complex structures on the underlying differentiable manifold, should *also* serve as the parameter space for a very different kind of structure on a "mirror partner"—another variety with trivial canonical bundle. This alternate description of the moduli space turns out to be well-adapted to analysis by Looijenga's techniques; we carry out that analysis here.

In the physicists' formulation, one fixes a differentiable manifold X which admits complex structures with trivial canonical bundle (a "Calabi-Yau manifold"), and studies something called nonlinear sigma-models on X. Such an object can be determined by specifying both a complex structure on X, and some "extra structure" (cf. [40]); the moduli space of interest to the physicists parameterizes the choice of both. The rôles of the "complex structure" and "extra structure" subspaces of this parameter space are reversed when X is replaced by a mirror partner.

Most aspects of mirror symmetry must be regarded as conjectural by mathematicians at the moment, and in this paper we conjecture much more than we prove. In a companion paper [41], we consider formally degenerating variations of Hodge structure near normal crossing boundary points of the moduli space, and describe a conjectural link to the numbers of rational curves of various degrees on a mirror partner. In the present paper, we extend these considerations to boundary points which are *not* of normal crossing type, and formulate a mathematical mirror symmetry conjecture in greater generality. In addition, we find that when studied from the mirror perspective, a "minimal" partial compactification of the moduli space—analogous to the Satake-Baily-Borel compactification—appears very natural, provided that several conjectures about the mirror partner hold.

One of our conjectures is a simple and compelling statement about the Kähler cone of Calabi-Yau varieties. If true, it clarifies the rôle of some of the "infinite discrete" structures on such a variety, which nevertheless seem to be finite modulo automorphisms. We have verified this conjecture in a nontrivial case in joint work with A. Grassi [21].

The plan of the paper is as follows. In the first several sections, we review Looijenga's compactifications, describe a concrete example, and add a refinement to the theory in the form of a flat connection on the holomorphic cotangent bundle of the moduli space. We then turn to the description of the larger moduli spaces of interest to physicists, and analyze certain boundary points of those spaces. Towards the end of the paper, we explore the mathematical implications of mirror symmetry in constructing compactifications of moduli spaces. We close by discussing some evidence for mirror symmetry which (in hindsight) was available in 1979.

1 Semi-toric compactifications

The first methods for compactifying arithmetic quotients of bounded symmetric domains were found by Satake [46] and Baily-Borel [5]. The compactification produced by their methods, often called the *Satake-Baily-Borel compactification*, adds a "minimal" amount to the quotient space in completing it to a compact complex analytic space. This minimality can be made quite precise, thanks to the Borel extension theorem [10] which guarantees that for a given quotient of a bounded symmetric domain by an arithmetic group, any compactification whose boundary is a divisor with normal crossings will map to the Satake-Baily-Borel compactification (provided that the arithmetic group is torsion-free).

Satake-Baily-Borel compactifications have rather bad singularities on their boundaries, so they are difficult to study in detail. Explicit resolutions of singularities for these compactifications were constructed in special cases by Igusa [30], Hemperly [27], and Hirzebruch [28]; the general case was subsequently treated by Satake [47] and Ash et al. [1]. The methods of [1] produce what are usually called *Mumford compactifications*—these are smooth, and have a divisor with normal crossings on the boundary, but unfortunately many choices must be made in their construction. The Satake-Baily-Borel compactification, on the other hand, is canonical.

Some years later, Looijenga [35] generalized both the Satake-Baily-Borel and the Mumford compactifications by means of a construction which can be applied widely, not just in the case of arithmetic quotients of bounded symmetric domains. Looijenga's construction gives partial compactifications of certain quotients of tube domains by discrete group actions. A *tube domain* is the set of points in a complex vector space whose imaginary parts are constrained to lie in a specified cone. Whereas Ash et al. [1] had only considered homogeneous self-adjoint cones, Looijenga showed that analogous constructions could be made in a more general context.

The starting point is a free $\mathbb{Z}$-module L of finite rank, and the real vector space $L_{\mathbb{R}} := L \otimes \mathbb{R}$ which it spans. A convex cone σ in $L_{\mathbb{R}}$ is *strongly convex* if $\sigma \cap (-\sigma) \subset \{0\}$. A convex cone is *generated* by the set S if every element in the cone can be written as a linear combination of the elements of S with nonnegative coefficients. And a convex cone is *rational polyhedral* if it is generated by a finite subset of the rational vector space $L_{\mathbb{Q}} := L \otimes \mathbb{Q}$.

Let $\mathcal{C} \subset L_{\mathbb{R}}$ be an open strongly convex cone, and let $\Gamma \subset \operatorname{Aff}(L)$ be a group of affine-linear transformations of L which contains the translation subgroup L of $\operatorname{Aff}(L)$. If the linear part $\Gamma_0 := \Gamma/L \subset \operatorname{GL}(L)$ of Γ preserves the cone $\mathcal{C}$, then the group Γ acts on the tube domain $\mathcal{D} := L_{\mathbb{R}} + i\,\mathcal{C}$. We wish to partially compactify the quotient space $\mathcal{D}/\Gamma$, including limit points for all paths moving out towards infinity in the tube domain.

Looijenga formulated a condition which guarantees the existence of partial compactifications of this kind. Let $\mathcal{C}_+$ be the convex hull of $\overline{\mathcal{C}} \cap L_{\mathbb{Q}}$. Following [35], we say that $(L_{\mathbb{Q}}, \mathcal{C}, \Gamma_0)$ is *admissible* if there exists a rational polyhedral cone $\Pi \subset \mathcal{C}_+$ such that $\Gamma_0.\Pi = \mathcal{C}_+$. Given an admissible triple $(L_{\mathbb{Q}}, \mathcal{C}, \Gamma_0)$, the (somewhat cumbersome) data needed to specify one of Looijenga's partial compactifications is as follows.[1]

DEFINITION 1 [35] *A* locally rational polyhedral decomposition *of $\mathcal{C}_+$ is a collection $\mathcal{P}$ of strongly convex cones such that*

(i) $\mathcal{C}_+$ is the disjoint union of the cones belonging to $\mathcal{P}$,

(ii) for every $\sigma \in \mathcal{P}$, the $\mathbb{R}$-span of σ is defined over $\mathbb{Q}$,

(iii) if $\sigma \in \mathcal{P}$, if τ is the relative interior of a nonempty face of the closure of σ, and if $\tau \subset \mathcal{C}_+$, then $\tau \in \mathcal{P}$, and

(iv) if Π is a rational polyhedral cone in $\mathcal{C}_+$, then Π meets only finitely many members of $\mathcal{P}$.

(The decomposition $\mathcal{P}$ is called *rational polyhedral* if all the cones in $\mathcal{P}$ are relative interiors of rational polyhedral cones. This is the same notion which appears in toric geometry [19, 43], except that the cones appearing in $\mathcal{P}$ as formulated here are the relative interiors of the cones appearing in that theory.)

For each Γ_0-invariant locally rational polyhedral decomposition $\mathcal{P}$ of $\mathcal{C}_+$, there is a partial compactification of $\mathcal{D}/\Gamma$ called the *semi-toric (partial) compactification associated to $\mathcal{P}$*. This partial compactification has the form $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$, where $\widehat{\mathcal{D}}(\mathcal{P})$ is the disjoint union of certain strata $\mathcal{D}(\sigma)$ associated to the cones σ in the decomposition. The complex dimension of the stratum $\mathcal{D}(\sigma)$ coincides with the real codimension of the cone σ in $L_{\mathbb{R}}$; in particular, the open cones in $\mathcal{P}$ correspond to the 0-dimensional strata in $\widehat{\mathcal{D}}(\mathcal{P})$. The delicate points in the construction are the specification of a topology on $\widehat{\mathcal{D}}(\mathcal{P})$,

[1] We have modified Looijenga's definition slightly, so that the use of the term "face" is the standard one (cf. [45]): a subset $\mathcal{F}$ of a convex set $\mathcal{S}$ is a *face* of $\mathcal{S}$ if every closed line segment in $\mathcal{S}$ which has one of its relative interior points lying in $\mathcal{F}$ also has both endpoints lying in $\mathcal{F}$.

and the proof that the quotient space $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ has a natural structure of a normal complex analytic space. For more details, we refer the reader to [35] or [50].

The construction has the property that if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then there is a dominant morphism $\widehat{\mathcal{D}}(\mathcal{P}')/\Gamma \to \widehat{\mathcal{D}}(\mathcal{P})/\Gamma$. Blowups of the boundary can be realized in this way.

A bit more generally, we can partially compactify finite covers $\mathcal{D}/\Gamma'$ of $\mathcal{D}/\Gamma$, built from $L' \subset L$ of finite index, $\Gamma_0' \subset \mathrm{GL}(L') \cap \Gamma_0$ of finite index in Γ_0, and $\Gamma' := L' \rtimes \Gamma_0'$, by specifying a Γ_0'-invariant locally rational polyhedral decomposition $\mathcal{P}'$ of $\mathcal{C}_+$.

There are two extreme cases of a semi-toric compactification. The *Satake-Baily-Borel decomposition* $\mathcal{P}_{\mathrm{SBB}}$ consists of all relative interiors of nonempty faces of $\mathcal{C}_+$. The resulting (partial) compactification $\widehat{\mathcal{D}}(\mathcal{P}_{\mathrm{SBB}})/\Gamma$ is the *Satake-Baily-Borel-type compactification of* $\mathcal{D}/\Gamma$. This is "minimal" among semi-toric compactifications in an obvious combinatorial sense; I do not know whether a more precise analogue of the Borel extension theorem holds in this context. The strata added to $\mathcal{D}/\Gamma$ include a unique 0-dimensional stratum $\mathcal{D}(\mathcal{C})$, which serves as a distinguished boundary point.

At the other extreme, if every cone $\sigma \in \mathcal{P}$ is the relative interior of a rational polyhedral cone $\overline{\sigma}$ which is generated by a subset of a basis of L, then the associated partial compactification is smooth, and the compactifying set is a divisor with normal crossings. We call this a *Mumford-type* semi-toric compactification. We will spell out the structure of the compactification more explicitly in this case, giving an alternative description of $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$.

We can think of producing a Mumford-type semi-toric compactification in two steps. In the first step, we construct a partial compactification $\widehat{\mathcal{D}}(\mathcal{P})/L$ of $\mathcal{D}/L$ which is Γ_0-equivariant; in the second step we recover $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ as the quotient of $\widehat{\mathcal{D}}(\mathcal{P})/L$ by Γ_0.

The first step is done one cone at a time. Given $\sigma \in \mathcal{P}$, there is a basis $\ell^1, \ldots, \ell^r$ of L such that

$$\sigma = \mathbb{R}_{>0}\ell^1 + \cdots + \mathbb{R}_{>0}\ell^k \quad \text{for some } k \le r.$$

Let $\{z_j\}$ be complex coordinates dual to $\{\ell^j\}$, so that $z = \sum z_j \ell^j$ represents a general element of $L_{\mathbb{C}}$. Consider the set $\mathcal{D}_\sigma := L_{\mathbb{R}} + i\,\sigma$. Translations by the lattice L preserve $\mathcal{D}_\sigma$, and coordinates on the quotient $\mathcal{D}_\sigma/L \subset L_{\mathbb{C}}/L$ can be given by $w_j = \exp(2\pi i\, z_j)$. In terms of those coordinates, $\mathcal{D}_\sigma/L$ can be described as

$$\mathcal{D}_\sigma/L = \{w \in \mathbb{C}^r : 0 < |w_j| < 1 \text{ for } j \le k, |w_j| = 1 \text{ for } j > k\}.$$

We partially compactify this to

$$(\mathcal{D}_\sigma/L)^- := \{w \in \mathbb{C}^r : 0 \le |w_j| < 1 \text{ for } j \le k, |w_j| = 1 \text{ for } j > k\}.$$

(We have suppressed the σ-dependence of ℓ^j, z_j, w_j to avoid cluttering up the notation.) We call any $w \in (\mathcal{D}_\sigma/L)^-$ with $w_j = 0$ for $j \le k$ a *distinguished limit point* of $\mathcal{D}_\sigma/L$. Note that any path in $\mathcal{D}_\sigma$ along which $\mathrm{Im}(z_j) \to \infty$ for all $j \le k$, maps to a path in $\mathcal{D}_\sigma/L$ which approaches such a distinguished limit point. The set $\mathrm{DLP}(\sigma)$ of distinguished limit points is a subset of the stratum $\widehat{\mathcal{D}}(\sigma)$, and is a compact real torus of dimension $\dim_{\mathbb{R}} \mathrm{DLP}(\sigma) = r - k = \dim_{\mathbb{C}} \widehat{\mathcal{D}}(\sigma)$. When $k = r$, the distinguished limit point is unique, and it coincides with the 0-dimensional stratum $\mathcal{D}(\sigma)$ of $\widehat{\mathcal{D}}(\mathcal{P})$.

The partial compactification $\widehat{\mathcal{D}}(\mathcal{P})/L$ can now be described as a disjoint union of the $(\mathcal{D}_\sigma/L)^-$'s, with $(\mathcal{D}_\tau/L)^-$ lying in the closure of $(\mathcal{D}_\sigma/L)^-$ whenever τ is the relative interior of a face of $\overline{\sigma}$. This space $\widehat{\mathcal{D}}(\mathcal{P})/L$ is smooth and simply-connected, and the induced action of Γ_0 on it has no fixed points. The action of Γ_0 permutes the various $(\mathcal{D}_\sigma/L)^-$'s, a finite number of which serve to cover $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ after we take the quotient by Γ_0. We have thus achieved an alternative description of Mumford-type compactifications.

Later in this paper, we will be concerned with recognizing when a complex analytic space has the structure of a semi-toric compactification. We can take a first step in that direction by formalizing the structure of $(\mathcal{D}_\sigma/L)^-$ near the distinguished limit point when $k = r$ in the following way. For a complex manifold T, we say that p is a *maximal-depth normal crossing point of* $B \subset T$ if there is an open neighborhood U of p in T and an isomorphism $\varphi : U \to \Delta^r$ such that $\varphi(U \cap (T - B)) = (\Delta^*)^r$ and $\varphi(p) = (0, \dots, 0)$, where Δ is the unit disk, and $\Delta^* := \Delta - \{0\}$. There are thus r local components $B_j := \varphi^{-1}(\{v_j = 0\})$ of $B \cap U$, with $p = B_1 \cap \cdots \cap B_r$, where v_j is a coordinate on the j^{th} disk.

2 Cusps of Hilbert modular surfaces

We now give an example to illustrate the construction in the previous section: the cusps of Hilbert modular surfaces, as analyzed by Hirzebruch [28] and by Mumford in the first chapter of [1]. Let $\mathrm{PGL}^+(2, \mathbb{R}) = \mathrm{PSL}(2, \mathbb{R})$ act by fractional linear transformations on the upper half plane $\mathfrak{H}$. Let K be a real quadratic field with ring of integers $\mathfrak{O}_K$, and let $\mathrm{PGL}^+(2, K)$ be the group of invertible 2×2 matrices with entries in K whose determinant is mapped to a positive number under both embeddings of K into $\mathbb{R}$, modulo scalar multiples of the identity matrix. The map $\Phi : K \to \mathbb{R}^2$ given by the two embeddings of K into $\mathbb{R}$ induces an action of $\mathrm{PGL}^+(2, K)$ on $\mathfrak{H} \times \mathfrak{H}$.

A *Hilbert modular surface* is an algebraic surface of the form $\mathfrak{H} \times \mathfrak{H}/\Gamma$ for some arithmetic group $\Gamma \subset \mathrm{PGL}^+(2, K)$ (that is, a group commensurable with $\mathrm{PGL}^+(2, \mathfrak{O}_K)$), often assumed to be torsion-free. The Satake-Baily-

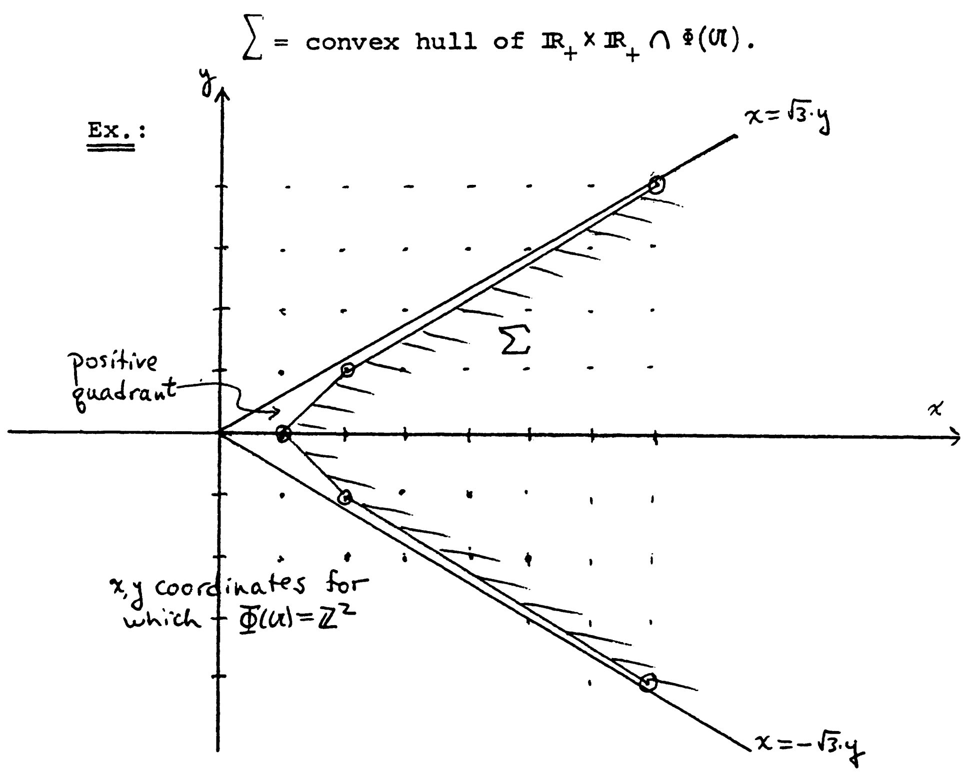

Figure 1.

Borel compactification of a Hilbert modular surface adds a finite number of compactification points, called *cusps.* Small deleted neighborhoods of such points have inverse images in $\mathfrak{H} \times \mathfrak{H}$ whose Γ-stabilizer is a parabolic subgroup Γ_{par} of the form

$$\Gamma_{\text{par}} = \{\begin{pmatrix} \varepsilon^k & a \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}, a \in \mathfrak{A}\},$$

where $\mathfrak{A} \subset \mathfrak{O}_K$ is an ideal, and $\varepsilon \in \mathfrak{O}_K^\times$ is a totally positive unit such that $\varepsilon\,\mathfrak{A} = \mathfrak{A}$. We can analyze a neighborhood of a cusp by studying appropriate partial compactifications of $\mathfrak{H} \times \mathfrak{H}/\Gamma_{\text{par}}$.

The elements in Γ_{par} with $k = 0$ form the translation subgroup, which we identify with $\mathfrak{A}$. This is a free abelian group of rank 2. Let (α, α'), (β, β') be a $\mathbb{Z}$-basis of $\Phi(\mathfrak{A})$. Define a map $\mathfrak{H} \times \mathfrak{H} \to \mathbb{C}^2$ by

$$(w_1, w_2) \mapsto \frac{1}{\alpha\beta' - \alpha'\beta}(\beta' w_1 - \beta w_2, -\alpha' w_1 + \alpha w_2),$$

and let $\mathcal{D}$ denote the image of $\mathfrak{H} \times \mathfrak{H}$ in $\mathbb{C}^2$. Under this map, $\Phi(\mathfrak{A})$ is sent to the standard lattice $L := \mathbb{Z}^2$, and $\Phi(\Gamma_{\text{par}})$ is sent to a subgroup of $\text{Aff}(L)$ with the translation subgroup $\mathfrak{A}$ of Γ_{par} mapped to the translation subgroup L of $\text{Aff}(L)$. As in section 1, we form the quotient in two steps: first take the quotient $\mathfrak{H} \times \mathfrak{H}/\mathfrak{A} = \mathcal{D}/L$, and then take the quotient of the resulting space by the group $\Gamma_0 := \Gamma_{\text{par}}/\mathfrak{A}$.

Mumford shows how to partially compactify the space $\mathcal{D}/L \subset L \otimes \mathbb{C}^* = (\mathbb{C}^*)^2$ in a Γ_0-equivariant way, so that the quotient by Γ_0 gives the desired partial compactification of $\mathfrak{H} \times \mathfrak{H}/\Gamma_{\text{par}}$. The map of $\mathfrak{H} \times \mathfrak{H} \to \mathbb{C}^2$ was designed so that the image would be a tube domain $\mathcal{D} := \mathbb{R}^2 + i\,\mathcal{C}$, where $\mathcal{C}$ is the cone

$$\mathcal{C} = \{(y_1, y_2) : \alpha y_1 + \beta y_2 > 0, \alpha' y_1 + \beta' y_2 > 0\}.$$

The boundary lines of the closure $\overline{\mathcal{C}}$ have irrational slope, and in fact $\mathcal{C}_+ = \mathcal{C}$ is an open convex cone. To construct a Γ_0-invariant rational polyhedral decomposition $\mathcal{P}$, let Σ be the convex hull of $\mathcal{C} \cap \Phi(\mathfrak{A})$. The vertices of Σ form a countable set $\{v_j\}_{j\in\mathbb{Z}}$ which can be numbered so that the edges of Σ are exactly the line segments $\overline{v_j v_{j+1}}$. If we let σ_j be the relative interior of the cone on $\overline{v_j v_{j+1}}$, and let τ_j be the relative interior of the cone on v_j, then $\mathcal{P} := \{\sigma_j\}_{j\in\mathbb{Z}} \cup \{\tau_j\}_{j\in\mathbb{Z}}$ is a Γ_0-invariant rational polyhedral decomposition. An explicit example of this construction is illustrated on p. 52 of [1], reproduced as figure 1 of this paper.

The resulting partial compactification of $\mathcal{D}/L$ adds a point p_j for each σ_j, and a curve $B_j \cong \mathbb{P}^1$ for each τ_j, with $B_j \cap B_{j+1} = p_j$. This can be pictured as an "infinite chain" of $\mathbb{P}^1$'s, as in the top of figure 2 (which is also reproduced from [1], p. 46). The generator $[\text{diag}(\varepsilon, 1)]$ of $\Gamma_0 = \Gamma_{\text{par}}/\mathfrak{A}$ acts by sending v_j to v_{j+m} for some fixed m. Taking the quotient by Γ_0 leaves us with a "cycle" of rational curves, of length m (as depicted in the bottom of figure 2). We arrive at Hirzebruch's description of the resolution of the cusps.

Conversely, suppose we are given a normal surface singularity $p \in \overline{S}$ (with $\overline{S}$ a small neighborhood of p) which has a resolution of singularities $f : T \to \overline{S}$ such that $B := f^{-1}(p)$ is a cycle of rational curves, that is, $B = B_1 + \cdots + B_m$ is a divisor with normal crossings such that B_j only meets $B_{j\pm1}$, with subscripts calculated mod m. Much of the structure above can be recovered from this information alone. In fact, by a theorem of Laufer [33] these singularities are *taut*, which means that the isomorphism type is determined by the resolution data. We will work out in detail some aspects of this tautness, in preparation for a general construction in the next section.

The starting point is Wagreich's calculation [54] of the local fundamental group $\pi_1(\overline{S} - p)$ for such singularities, which goes as follows. Let $S := \overline{S} - p = T - B$. The natural map $\iota : \pi_1(S) \to \pi_1(T)$ induced by the inclusion $S \subset T$

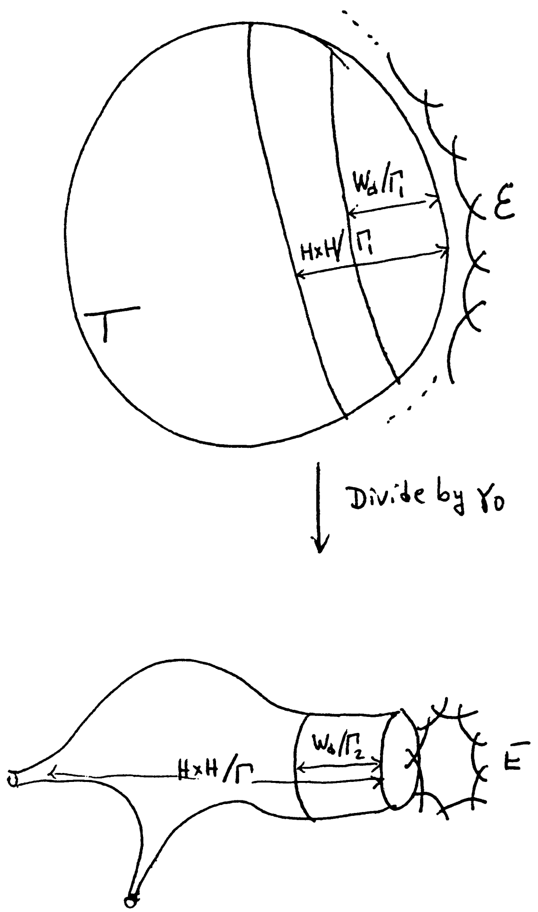

Figure 2.

is surjective. Since T retracts onto a cycle of $\mathbb{P}^1$'s, the group $\pi_1(T) \cong \pi_1(S^1)$ is infinite cyclic, and the universal cover $\widehat{T}$ of T contains an infinite chain $\widehat{B} = \cdots + \widehat{B}_j + \widehat{B}_{j+1} + \cdots$ of $\mathbb{P}^1$'s lying over the cycle B. The kernel of ι is $\pi_1(\widehat{T} - \widehat{B})$, and by a result of Mumford [42] this is a free abelian group generated by loops around any pair of adjacent components $\widehat{B}_j$, $\widehat{B}_{j+1}$ of $\widehat{B}$.

In this way, we recover the two steps of the quotient construction, and the compactification $\widehat{T}$ of the intermediate quotient $\widehat{T} - \widehat{B}$. Let $\widehat{S}$ be the universal cover of S (and of $\widehat{T} - \widehat{B}$). To complete the discussion of tautness, we should exhibit an isomorphism between $\widehat{S}$ and an open subset of $\mathfrak{H} \times \mathfrak{H}$, which descends to a $\pi_1(T)$-equivariant map $(\widehat{T} - \widehat{B}) \to (\mathfrak{H} \times \mathfrak{H})/\mathfrak{A}$. The easiest way to do this is to consider an extra piece of structure on $p \in \overline{S}$: a flat connection on the holomorphic cotangent bundle Ω^1_S. We discuss this structure, and how to use it to determine the mapping from $\widehat{S}$ to $\mathfrak{H} \times \mathfrak{H} = \mathcal{D}$, in the next section. (To give a complete proof of Laufer's tautness result along these lines, we would also need to show how the connection is to be constructed; we will not attempt to do that here.)

3 The toric connection

Let $(L_{\mathbb{Q}}, \mathcal{C}, \Gamma_0)$ be an admissible triple, with associated tube domain $\mathcal{D} = L_{\mathbb{R}} + i\,\mathcal{C}$ and discrete group $\Gamma = L \rtimes \Gamma_0 \subset \mathrm{Aff}(L)$. We will define a flat connection on the holomorphic cotangent bundle of the quotient space $\mathcal{D}/\Gamma$.

The intermediate quotient space $\mathcal{D}/L$ is an open subset of the algebraic torus $L_{\mathbb{C}}/L = L \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^{\mathrm{rk}(L)}$. We identify the dual of the Lie algebra $\mathrm{Lie}(L_{\mathbb{C}}/L)^*$ of that torus with the space of right-invariant one-forms on the group $L_{\mathbb{C}}/L$. Any basis of $\mathrm{Lie}(L_{\mathbb{C}}/L)^*$, when regarded as a subset of the space of global sections of the sheaf $\Omega^1_{L_{\mathbb{C}}/L}$, freely generates that sheaf at any point. We can therefore define a connection ∇_{toric} on $\Omega^1_{L_{\mathbb{C}}/L}$, the *toric connection*, by the requirement that $\nabla_{\mathrm{toric}}(\alpha) = 0$ for every $\alpha \in \mathrm{Lie}(L_{\mathbb{C}}/L)^*$. Since the group $L_{\mathbb{C}}/L$ is abelian, the connection ∇_{toric} is flat.

The action of $\mathrm{Aff}(L)$ on $L_{\mathbb{C}}$ descends to an action of $\mathrm{GL}(L)$ on $L_{\mathbb{C}}/L$ which preserves the space of right-invariant one-forms. In particular, the $\mathrm{GL}(L)$-action will be compatible with the toric connection. Thus, if we restrict ∇_{toric} to $\mathcal{D}/L$, it commutes with the action of Γ_0 and induces a connection on the holomorphic cotangent bundle of $(\mathcal{D}/L)/\Gamma_0 = \mathcal{D}/\Gamma$, still denoted by ∇_{toric}.

Let $\sigma \subset L_{\mathbb{R}}$ be the relative interior of a rational polyhedral cone which is generated by a basis $\ell^1, \dots, \ell^r$ of L, and let $z_1, \dots, z_r$ be the coordinates on $L_{\mathbb{C}}$ dual to $\{\ell^j\}$. The one-forms $d\log w_j := 2\pi i\, dz_j$ are right-invariant one-forms on $L_{\mathbb{C}}/L$ which serve as a basis of $\mathrm{Lie}(L_{\mathbb{C}}/L)^*$. If we compactify the open set $\mathcal{D}_\sigma/L \subset L_{\mathbb{C}}/L$ to $U_\sigma := (\mathcal{D}_\sigma/L)^-$, then the forms $d\log w_j$ extend to meromorphic one-forms on U_σ with poles along the boundary $B_\sigma := (\mathcal{D}_\sigma/L)^- - (\mathcal{D}_\sigma/L)$. In fact, the forms $d\log w_1, \dots, d\log w_r$ freely generate the sheaf $\Omega^1_{U_\sigma}(\log B_\sigma)$ as an $\mathcal{O}_{U_\sigma}$-module. The flat connection ∇_{toric} therefore extends to a flat connection on $\Omega^1_{U_\sigma}(\log B_\sigma)$ for which the $d\log w_j$ are flat sections. Note that the connection does *not* acquire singularities along the boundary, but extends as a regular connection to the sheaf of logarithmic differentials.

If $\mathcal{P}$ is a rational polyhedral decomposition of $\mathcal{C}_+$, we get in this way an extension of the flat connection ∇_{toric} from $\Omega^1_{\mathcal{D}/L}$ to the sheaf of logarithmic differentials on $\widehat{\mathcal{D}}(\mathcal{P})/L$ with poles on the boundary $(\widehat{\mathcal{D}}(\mathcal{P})/L) - (\mathcal{D}/L)$. As this extended connection still commutes with Γ_0, there is an induced extension of ∇_{toric} from $\Omega^1_{\mathcal{D}/\Gamma}$ to $\Omega^1_{\widehat{\mathcal{D}}(\mathcal{P})/\Gamma}(\log \mathcal{B})$, where $\mathcal{B} := (\widehat{\mathcal{D}}(\mathcal{P})/\Gamma) - (\mathcal{D}/\Gamma)$. This holds for any Mumford-type semi-toric compactification.

The existence of this toric connection on $\mathcal{D}/\Gamma$ depends in an essential way on Γ being a group of affine-linear transformations of L. If $\mathcal{D}$ admits an action by a larger group Γ_{big} which includes discrete symmetries that do not lie in $\mathrm{Aff}(L)$, then ∇_{toric} may fail to descend to the quotient $\mathcal{D}/\Gamma_{\text{big}}$. For example, if $L = \mathbb{Z}$ acts on the upper half plane $\mathfrak{H}$ by translations, then the associated flat connection ∇_{toric} has the property that $\nabla_{\text{toric}}(d\tau) = 0$, where τ is the standard coordinate on $\mathfrak{H}$. The flat section $d\tau$ is invariant under translations $\tau \mapsto \tau + n$, but if we apply ∇_{toric} to the pullback of the flat section $d\tau$ under the inversion $\tau \mapsto -1/\tau$ we get

$$\nabla_{\text{toric}}(\tau^{-2}\, d\tau) = -2\tau^{-3}\, d\tau \otimes d\tau,$$

which is not 0. In particular, the connection ∇_{toric} does not descend to the j-line $\mathfrak{H}/\,\mathrm{SL}(2,\mathbb{Z})$.

We now want to explain how the abstract knowledge of the flat connection ∇_{toric} and of a Mumford-type semi-toric compactification of $\mathcal{D}/\Gamma$ can be used to recover the structure of $\mathcal{D}$ and of Γ. Suppose we are given a complex manifold T, a divisor with normal crossings B on T, and a flat connection ∇ on $\Omega^1_T(\log B)$. By the usual equivalence between flat connections and local systems [16], the flat sections of ∇ determine a local system $\mathbb{E}$ on T. Such a local system is specified by giving its fiber E at a fixed base point $\star$ (which we choose to lie in $T - B$), together with a representation of $\pi_1(T, \star)$ in $\mathrm{GL}(E)$.

We first restrict the connection and the local system to $T - B$. If we pass to the universal cover $\widehat{S}$ of $T - B$, the flat sections give a global trivialization of the bundle $\mathbb{E} \otimes \mathcal{O}_{\widehat{S}} = \Omega^1_{\widehat{S}}$. There is a natural map $\mathrm{int}_\star : \widehat{S} \to E^*$ which sends $s \in \widehat{S}$ to the functional

$$\alpha \mapsto \int_\star^s \widehat{\alpha},$$

where $\widehat{\alpha}$ is the unique flat section of $\mathbb{E}$ (a holomorphic 1-form on $\widehat{S}$) such that $\widehat{\alpha}|_\star = \alpha \in E$. (Notice that if we vary the basepoint $\star$, we simply shift the image of the map by some constant vector in E^*.)

On the other hand, if we consider ∇ on T and pass to the universal cover $\widehat{T}$ of T, the flat sections of $\mathbb{E}$ will trivialize the bundle $\Omega^1_{\widehat{T}}(\log \widehat{B})$, where $\widehat{B}$ is a divisor with normal crossings in $\widehat{T}$, the inverse image of $B \subset T$. We once again encounter the intermediate quotient space $\widehat{T} - \widehat{B}$, and its partial compactification $\widehat{T}$.

At any maximal-depth normal crossing point p of $\widehat{B} \subset \widehat{T}$, let $v_j = 0$ define the j^{th} local component B_j of the boundary at p. There is a unique flat section $\widehat{\alpha}_j$ of $\Omega^1_{\widehat{T}}(\log \widehat{B})$, defined locally near p, such that $\widehat{\alpha}_j - d \log v_j$ vanishes at p. It follows that $\widehat{\alpha}_j - d \log v_j$ is a holomorphic one-form in a neighborhood of p, and so that $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r$ is a basis for (flat) local sections of $\Omega^1_{\widehat{T}}(\log \widehat{B})$. Using the global trivialization, we may regard each $\alpha_j := \widehat{\alpha}_j|_\star$ as an element of E. We let $L_p \subset E^*$ be the lattice spanned by the dual basis $\ell^1, \ldots, \ell^r$ to $\alpha_1, \ldots, \alpha_r$, and let $\sigma_p \subset L_p \otimes \mathbb{R}$ be the relative interior of the cone generated by $\ell^1, \ldots, \ell^r$.

If we are to recover the structure of the semi-toric compactification, we need a certain compatibility among the L_p's and the σ_p's: they should be related to a common lattice and a common cone, independent of p. We formalize this as follows.

Definition 2 *We call* (T, B, ∇) compatible *provided that*

1. *each component of B contains at least one maximal-depth normal crossing point,*

2. *the lattices L_p for maximal-depth normal crossing points p all coincide with a common lattice $L \subset E^*$,*

3. *the natural map* $\mathrm{int}_\star : \widehat{S} \to E^* = L_{\mathbb{C}}$ *descends to a map* $(\widehat{T} - \widehat{B}) \to (L_{\mathbb{C}}/L)$ *which induces an isomorphism of fundamental groups, and*

4. *the collection $\mathcal{P}$ of relative interiors of faces of the σ_p's is a locally rational polyhedral decomposition of a strongly convex cone $\mathcal{C}_+$.*

Suppose that (T, B, ∇) is compatible, let $\mathcal{C}$ be the interior of $\mathcal{C}_+$, and let $\mathcal{D} = L_{\mathbb{R}} + i\,\mathcal{C}$. The action of $\pi_1(T)$ on $L_{\mathbb{C}}$ permutes the set of maximal-depth normal crossing points of $B \subset T$, and so preserves $\mathcal{P}$ and $\mathcal{C}$. Thus, $\Gamma := \pi_1(T - B)$ acts on $\mathcal{D}$, and there is an induced map $(T - B) \to (\mathcal{D}/\Gamma)$.

We can now recover the compactification T from this data (or at least its structure in codimension one). For any maximal-depth normal crossing boundary point p of $\widehat{B} \subset \widehat{T}$, there is a neighborhood U_p of p in $\widehat{T}$ and a natural extension of the induced map $U_p \cap (\widehat{T} - \widehat{B}) \to L_{\mathbb{C}}/L$ to a map $U_p \to \widehat{\mathcal{D}}(\mathcal{P})/L$. We cannot tell from the behavior of these extensions what happens at "interior" points of boundary components (those which do not lie in any U_p), but we *can* conclude that there is a meromorphic map $\widehat{T} \to \widehat{\mathcal{D}}(\mathcal{P})/L$ which does not blow down any boundary components. This map is $\pi_1(T)$-equivariant, so it descends to a map $T \to \widehat{\mathcal{D}}(\mathcal{P})/\Gamma$.

4 Moduli spaces of sigma-models

A *Calabi-Yau manifold* is a compact connected orientable manifold X of dimension $2n$ which admits Riemannian metrics whose (global) holonomy is contained in $\mathrm{SU}(n)$.[2] Given such a metric, there exist complex structures on X for which the metric is Kähler. The holonomy condition is equivalent to requiring that this Kähler metric be Ricci-flat and that there exist a nonzero holomorphic n-form on X (cf. [7]). On the other hand, if we are given a complex structure on a Calabi-Yau manifold, then by the theorems of Calabi [11] and Yau [60], for each Kähler metric $\widetilde{g}$ there is a unique Ricci-flat Kähler metric g whose Kähler form is in the same de Rham cohomology class as that of $\widetilde{g}$. (We have implicitly used the topological consequence of Ricci-flatness: Calabi-Yau manifolds have vanishing first Chern class.)

Examples of Calabi-Yau manifolds are provided by the differentiable manifolds underlying smooth complex projective varieties with trivial canonical bundle. One can apply Yau's theorem to a Kähler metric coming from a projective embedding in order to produce a metric with holonomy contained in $\mathrm{SU}(n)$, where n is the complex dimension of the variety. As explained in [7], if the Hodge numbers $h^{p,0}$ vanish for $0 < p < n$ and if the manifold is simply-connected, then the holonomy of this metric is precisely $\mathrm{SU}(n)$.

Physicists have constructed a class of conformal field theories called nonlinear sigma-models on Calabi-Yau manifolds X (cf. [22, 29]). We consider here an approximation to those theories, which should be called "one-loop semiclassical nonlinear sigma-models". Such an object is determined by the data of a Riemannian metric g on X whose holonomy is contained in $\mathrm{SU}(n)$ together with the de Rham cohomology class $[b] \in H^2(X,\mathbb{R})$ of a real closed 2-form b on X.

Two such pairs (g,b) and (g',b') will determine isomorphic conformal field theories if there is a diffeomorphism $\varphi : X \to X$ such that $\varphi^*(g') = g$, and $\varphi^*([b']) - [b] \in H^2(X,\mathbb{Z})$. It is therefore natural to regard the class of $[b]$ in $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$ as the fundamental datum. We denote this class by $[b] \bmod \mathbb{Z}$.

The set of all isomorphism classes of such pairs we call the *one-loop semiclassical nonlinear sigma-model moduli space*, or simply the *sigma-model moduli space* (for short). This may differ from the actual *conformal field theory moduli space*, for several reasons: first, there may be additional isomorphisms of conformal field theories which are not visible in this geometric interpretation, second, there may be deformations of the nonlinear sigma-model as

[2] There is some confusion in the literature about whether "Calabi-Yau" should mean that the holonomy is precisely $\mathrm{SU}(n)$, or simply contained in $\mathrm{SU}(n)$. In this paper, we adopt the latter interpretation.

a conformal field theory which do not have a sigma-model interpretation on X (cf. [2, 59]), and third, the putative conformal field theory may fail to converge for some values of the input data (g, b) (although it is believed to converge whenever the volume of the metric is sufficiently large). For our present purposes, we ignore these more delicate questions about the conformal field theory moduli space, and concentrate on the sigma-model moduli space we have defined above.

We focus attention in this paper on the case in which the holonomy of the metric g is precisely $SU(n)$, $n \neq 2$. For each such metric, there are exactly two complex structures on X for which the metric is Kähler (complex conjugates of each other).[3] Thus, there is a natural map from a double cover of the sigma-model moduli space to the usual "complex structure moduli space", given by assigning to (g, b) one of the two complex structures for which g is Kähler. The fibers of this map can be described as follows. If we fix a complex structure on X, then the corresponding fiber consists of all $B + i\,J \bmod \mathbb{Z} \in H^2(X, \mathbb{C})/H^2(X, \mathbb{Z})$ (modulo diffeomorphism) with B denoting the class $[b]$, for which J is the cohomology class of a Kähler form. (The metric g is uniquely determined by J, by Calabi's theorem.) This quantity $B+i\,J \bmod \mathbb{Z}$ describes the "extra structure" S which was alluded to in [40]. This is often called the *complexified Kähler structure* on X determined by (g, b).

The natural double cover of the sigma-model moduli space will be locally a product near (g, b), with the variations of complex structure and of complexified Kähler structure describing the factors in the product, provided that neither the Kähler cone nor the group of holomorphic automorphisms "jumps" when the complex structure varies. (The non-jumping of the Kähler cone was shown to hold by Wilson [55] in the case of holonomy SU(3), when the complex structure is generic.) We will tacitly assume this local product structure, and separately study the parameter spaces for the variations of complexified Kähler structure and of complex structure.

With a fixed complex structure on X, the parameter space for complexified Kähler structures on X can be described in terms of the Kähler cone $\mathcal{K}$ of X, and the lattice $L = H^2(X, \mathbb{Z})/(\text{torsion})$. We must identify any pair of complexified Kähler structures which differ by a diffeomorphism that fixes the complex structure, that is, by an element of the group $\Gamma_0 = \operatorname{Aut}(X)$ of holomorphic automorphisms. The natural parameter space for pairs (g, b) such that g is Kähler for the given complex structure thus has the form $\mathcal{D}/\Gamma$, where $\mathcal{D} = \{B + i\,J : J \in \mathcal{K}\}$ and $\Gamma = L \rtimes \Gamma_0$ is the extension of Γ_0 by the lattice translations. This is exactly the kind of space encountered in the

[3] More generally, as we will show elsewhere, if $h^{2,0}(X) = 0$ there are only a finite number of complex structures for which g is Kähler. The number depends on the decomposition of the holonomy representation into irreducible pieces.

first part of this paper: a tube domain modulo a discrete symmetry group of affine-linear transformations which includes a lattice acting by translations.

A common technique in the physics literature is to consider what happens along paths $\{tz \bmod \Gamma\}_{t\to\infty}$, which go from $z \in \mathcal{D}$ out towards infinity in the tube domain. Many aspects of the conformal field theory can be analyzed perturbatively in t along such paths. It seems reasonable to hope that such limits can be described in a common framework, based on a single partial compactification of $\mathcal{D}/\Gamma$. This hope (together with a bit of evidence, discussed below) leads us to conjecture that $(L_{\mathbb{Q}}, \mathcal{K}, \mathrm{Aut}(X))$ is an admissible triple, in order that Looijenga's methods could be applied to construct compactifications of $\mathcal{D}/\Gamma$. We formulate this conjecture more explicitly as follows.

THE CONE CONJECTURE *Let X be a Calabi-Yau manifold on which a complex structure has been chosen, and suppose that $h^{2,0}(X) = 0$. Let $L := H^2(X,\mathbb{Z})/\mathrm{torsion}$, let $\mathcal{K}$ be the Kähler cone of X, let $\mathcal{K}_+$ be the convex hull of $\overline{\mathcal{K}} \cap L_{\mathbb{Q}}$, and let $\mathrm{Aut}(X)$ be the group of holomorphic automorphisms of X. Then there exists a rational polyhedral cone $\Pi \subset \mathcal{K}_+$ such that $\mathrm{Aut}(X).\Pi = \mathcal{K}_+$.*

The Kähler cone of X can have a rather complicated structure, analyzed in the case $n = 3$ by Kawamata [31] and Wilson [55]. Away from classes of triple-self-intersection zero, the closed cone $\overline{\mathcal{K}}$ is locally rational polyhedral, but the rational faces may accumulate towards points with vanishing triple-self-intersection. The cone conjecture predicts that while the closed cone $\overline{\mathcal{K}}$ of X may have infinitely many edges, there will only be finitely many $\mathrm{Aut}(X)$-orbits of edges. Other finiteness predictions which follow from the cone conjecture include finiteness of the set of fiber space structures on X, modulo automorphisms.

Many of the large classes of examples, such as toric hypersurfaces, have Kähler cones $\mathcal{K}$ such that $\mathcal{K}_+ = \overline{\mathcal{K}}$ is a rational polyhedral cone. For these, the cone conjecture automatically holds. A nontrivial case of the cone conjecture—Calabi-Yau threefolds which are fiber products of generic rational elliptic surfaces with section (as studied by Schoen [49])—has been checked by Grassi and the author [21]. In addition, Borcea [9] has verified the finiteness of $\mathrm{Aut}(X)$-orbits of edges of $\overline{\mathcal{K}}$ in another nontrivial example, and Oguiso [44] has discussed finiteness of $\mathrm{Aut}(X)$-orbits of fiber space structures in yet another example.[4] All three examples involve cones with an infinite number of edges.

For any X for which the cone conjecture holds, the Kähler parameter space $\mathcal{D}/\Gamma$ will admit both a Satake-Baily-Borel-type "minimal" compactification,

[4]Neither of these constitutes a complete verification of the cone conjecture for the threefold in question.

and smooth compactifications of Mumford type built out of many cones $\sigma \subset \mathcal{K}$ as above.

5 Additional structures on the moduli spaces

Of particular interest to the physicists studying nonlinear sigma-models has been the "large radius limit" in the Kähler parameter space. This is typically analyzed in the physics literature as follows (cf. [56, 57]). The quantities of physical interest will be invariant under translation by L. Many such quantities vary holomorphically with parameters, and their Fourier expansions take the form

$$\sum_{\eta \in L^*} c_\eta \, e^{2\pi i \, z \cdot \eta}. \tag{$*$}$$

The coefficients c_η for $\eta \neq 0$ are called *instanton contributions* to the quantity (*), and in many cases they can be given a geometric interpretation which shows that they vanish unless η is the class of an effective curve on X. A "large radius limit" should be a point at which instanton contributions to quantities like (*) are suppressed [24, 3].

If we pick a basis $\ell^1, \dots, \ell^r$ of L consisting of vectors which lie in the closure of the Kähler cone, write $\eta = \sum \eta^j \ell_j$ in terms of the basis $\{\ell_j\}$ of L^* dual to $\{\ell^j\}$, and express (*) as a power series in $w_j := \exp(2\pi i \, z_j)$, where $\{z_j\}$ are coordinates dual to $\{\ell^j\}$, then the series expansion

$$\sum_{\eta \in L^*} c_\eta \, w_1^{\eta^1} \cdots w_r^{\eta^r} \tag{$**$}$$

involves only terms with nonnegative exponents [4]. If convergent,[5] this will define a function on $(\mathcal{D}_\sigma/L)^-$, where σ is the relative interior of the cone generated by $\ell^1, \dots, \ell^r$. Thus, approaching the distinguished limit point of $\mathcal{D}_\sigma/L$ (where all w_j's approach 0) suppresses the instanton contributions, so the distinguished limit point is a good candidate for the large radius limit. We can repeat this construction for any cone $\sigma \subset \mathcal{K}$ which is the relative interior of a cone generated by a basis of L, obtaining partial compactifications which include large radius limit points for paths that lie in various cones σ.

Among the "quantities of physical interest" to which this analysis is applied are a collection of multilinear maps of cohomology groups called *three-point functions.* These maps should depend on the data (g, b), and should vary holomorphically with both complex structure and complexified Kähler

[5] From a rigorous mathematical point of view, the Fourier coefficients c_η can often be defined and calculated, but no convergence properties of the series (*) or (**) are known.

structure parameters. Certain of these three-point functions (related to Witten's "A-model" [58]) would depend only on the complexified Kähler structure, while others (related to Witten's "B-model") would depend only on the complex structure. The B-model three-point functions can be mathematically interpreted in terms of the variation of Hodge structure, or period mapping, induced by varying the complex structure on the Calabi-Yau manifold [15, 40, 23].

In [41], we discuss a mathematical version of the A-model three-point functions, expressed as formal power series near the distinguished limit point associated to the relative interior σ of a rational polyhedral cone generated by a basis of L. (The coefficients c_η of this power series are derived from the numbers of rational curves on X of various degrees.) The choice of σ is an additional piece of data in the construction which we call a *framing*.

These formal power series representations of A-model three-point functions can be regarded as defining a formal degenerating variation of Hodge structure, which we call the *framed A-variation of Hodge structure* with framing σ. Now there are manipulations of these formal series which suggest that the underlying convergent three-point functions (if they exist) will not depend on the choice of σ and will be invariant under the action of $\mathrm{Aut}(X)$.

We must refer the reader to [41] for the precise definition of framed A-variation of Hodge structure. But for reference, we would like to state here a conjecture which suggests how the various framed A-variations of Hodge structure will fit together, along the lines being discussed in this paper.

THE CONVERGENCE CONJECTURE *Suppose that X is a Calabi-Yau manifold with $h^{2,0}(X) = 0$, endowed with a complex structure, which satisfies the cone conjecture. Let $L := H^2(X, \mathbb{Z})/\mathrm{torsion}$, let $\mathcal{K}$ be the Kähler cone of X, let $\mathcal{D} := L_{\mathbb{R}} + i\,\mathcal{K}$ be the associated tube domain, and let $\Gamma := L \rtimes \mathrm{Aut}(X)$. Then there is a neighborhood U of the 0-dimensional stratum $\widehat{\mathcal{D}}(\mathcal{K})$ in the Satake-Baily-Borel-type compactification $\widehat{\mathcal{D}}(\mathcal{P}_{\mathrm{SBB}})/\Gamma$, and a variation of Hodge structure on $U \cap (\mathcal{D}/\Gamma)$, such that for any $\sigma \subset \mathcal{K}$ which is the relative interior of a rational polyhedral cone $\overline{\sigma} \subset \mathcal{K}_+$ generated by a basis of L, the induced formal degenerating variation of Hodge structure at the distinguished limit point of $\mathcal{D}_\sigma/L$ agrees with the framed A-variation of Hodge structure with framing σ.*

If this variation of Hodge structure exists, we call it the *A-variation of Hodge structure* associated to X.

6 Maximally unipotent boundary points

In the previous section, we discussed how to let the complexified Kähler parameter $B + i\,J$ approach infinity, analyzing certain partial compactifications and boundary points of the sigma-model moduli space in the $B + i\,J$ directions. We now turn to compactifications and boundary points in the transverse directions—the directions obtained by varying the complex structure on the Calabi-Yau manifold. We consider what happens when the complex structure degenerates.

The local moduli spaces of complex structures on Calabi-Yau manifolds are particularly well-behaved, thanks to a theorem of Bogomolov [8], Tian [51], and Todorov [52], which guarantees that all first-order deformations are unobstructed. In particular, there will be a local family of deformations of a given complex structure for which the Kodaira-Spencer map is an isomorphism. More generally, we consider arbitrary families $\pi : \mathcal{Y} \to S$ of complex structures on a fixed Calabi-Yau manifold Y, by which we mean: π is a proper and smooth map between connected complex manifolds, and all fibers $Y_s := \pi^{-1}(s)$ are diffeomorphic to Y. We will often assume that the Kodaira-Spencer map is an isomorphism at every point $s \in S$, so that S provides good local moduli spaces for the fibers Y_s.

To study the behavior when the complex structure degenerates, we partially compactify the parameter space S to $\overline{S}$. There is a class of boundary points on $\overline{S}$ of particular interest from the perspective of conformal field theory. According to the interpretation of [40, 41], these points can be identified by the monodromy properties of the associated variation of Hodge structure[6] near $p \in \overline{S}$. We first review from [41] these monodromy properties for normal crossing boundary points, and then extend the definition to a wider class of compactifications and boundary points.

Let p be a maximal-depth normal crossing point of $B \subset \overline{S}$, where $B := \overline{S} - S$ is the boundary, assumed for the moment to be a divisor with normal crossings. Let U be a small neighborhood of p in $\overline{S}$, and write $B \cap U$ in the form $B_1 + \cdots + B_r$. If we fix a point $s \in U - B$, then each local divisor B_j gives rise to an monodromy transformation $T^{(j)} : H^n(Y_s, \mathbb{Q}) \to H^n(Y_s, \mathbb{Q})$, which is guaranteed to be quasi-unipotent by the monodromy theorem [32].

DEFINITION 3 *A maximal-depth normal crossing point p of $B \in \overline{S}$ is called a* maximally unipotent point[7] *under the following conditions.*

1. *The monodromy transformations $T^{(j)}$ around local boundary components B_j near p are all unipotent.*

[6] The variation of Hodge structure in question is the usual geometric one (cf. [26]) associated to a variation of complex structure. These might be called "B-variations of Hodge structure" by analogy with the previous section.

[7] When $\dim(S) = 1$, this definition is equivalent to the one given in [40].

2. *Let $N^{(j)} := \log T^{(j)}$, let $N := \sum a_j N^{(j)}$ for some $a_j > 0$, and define*

$$\begin{aligned} W_0 &:= \operatorname{Im}(N^n) \\ W_1 &:= \operatorname{Im}(N^{n-1}) \cap \operatorname{Ker} N \\ W_2 &:= \left(\operatorname{Im}(N^{n-2}) \cap \operatorname{Ker}(N)\right) + \left(\operatorname{Im}(N^{n-1}) \cap \operatorname{Ker}(N^2)\right). \end{aligned}$$

Then $\dim W_0 = \dim W_1 = 1$ *and* $\dim W_2 = 1 + \dim(S)$.

3. *Let $g^0, g^1, \ldots, g^r$ be a basis of W_2 such that g^0 spans W_0, and define m^{jk} by $N^{(j)} g^k = m^{jk} g^0$ for $1 \le j, k \le r$. Then $m := (m^{jk})$ is an invertible matrix.*

(The spaces W_0 and W_2 are independent of the choice of coefficients $\{a_j\}$ [14, 17], *and the invertibility of m is independent of the choice of basis $\{g^k\}$.)*

Given a maximally unipotent point $p \in \overline{S}$, we define the *canonical logarithmic one-forms* $d\log q_j \in \Gamma(U, \Omega^1_{\overline{S}}(\log B))$ at p by

$$\frac{1}{2\pi i} d\log q_j := d\left(\frac{\sum_{k=1}^r \langle g^k | \omega \rangle m_{kj}}{\langle g^0 | \omega \rangle}\right)$$

where (m_{kj}) is the inverse matrix of (m^{jk}), and ω is a section of the sheaf $\Omega^n_{\mathcal{Y}/S}$ of relative holomorphic n-forms on the family of complex structures parameterized by S. The elements $g^k \in H^n(Y_s, \mathbb{Q})$ have been implicitly extended to multi-valued sections of the local system $R^n\pi^*(\mathbb{Q}_{\mathcal{Y}})$ in order to evaluate $\langle g^k | \omega \rangle$; the monodromy measures the multi-valuedness of the resulting (locally defined) holomorphic functions $\langle g^k | \omega \rangle$. The fact that each $d\log q_j$ as defined above has a single-valued meromorphic extension to U follows from the nilpotent orbit theorem [48]. In [41] we show that the canonical one-forms are independent of the choice of basis $\{g^k\}$, and also of the choice of relative n-form ω; that for any local defining equation $v_j = 0$ of B_j, the one-form $d\log q_j - d\log v_j$ extends to a regular one-form on U; and that $d\log q_1, \ldots, d\log q_r$ freely generate the locally free sheaf $\Omega^1_{\overline{S}}(\log B)$ near p.

The canonical logarithmic one-forms can be integrated to produce *quasi-canonical coordinates* $q_1, \ldots, q_r$ near p, but due to constants of integration, these coordinates are not unique. That is, if we attempt to define

$$q_j = \exp\left(2\pi i \frac{\sum_{k=1}^r \langle g^k | \omega \rangle m_{kj}}{\langle g^0 | \omega \rangle}\right)$$

we find that changing the basis $\{g^k\}$ will alter the q_j's by multiplicative constants (cf. [39]). To specify truly canonical coordinates, further conditions on the basis $\{g^k\}$ must be imposed, as discussed in [40, 41]. For example, by demanding that g^0 span $W_0 \cap H^n(Y_s, \mathbb{Z})/\text{torsion}$ and that $g^0, \ldots, g^r$ span $W_2 \cap H^n(Y_s, \mathbb{Z})/\text{torsion}$ we can reduce the ambiguity in the q_j's to a finite number of choices.

With no ambiguity, we can use the canonical logarithmic one-forms to produce a (canonical) flat connection ∇ on the holomorphic vector bundle $\Omega^1_U(\log B)$ by declaring $d\log q_1, \ldots, d\log q_r$ to be a basis for the ∇-flat sections, that is, $\nabla(d\log q_j) = 0$. Notice that the connection ∇ is regular along the boundary divisor B. This connection is what we will use to extend the definition of maximally unipotent to a more general case.

We now consider partial compactifications $\overline{S}$ of S which are not necessarily smooth, and whose boundary is not necessarily a divisor with normal crossings.

DEFINITION 4 *Let $\Xi \subset \overline{S} - S$ be a connected subset of the boundary. We say that Ξ is* maximally unipotent *if there is a neighborhood V of Ξ in $\overline{S}$ and a flat connection ∇_{unip} on $\Omega^1_{V \cap S}$ such that for some resolution of singularities $f : U \to V$ which is an isomorphism over $V \cap S$, we have*

1. *the new boundary $B := U - f^{-1}(V \cap S)$ on U is a divisor with normal crossings,*
2. *the flat connection ∇_{unip} extends to a connection on $\Omega^1_U(\log B)$ (also denoted by ∇_{unip}),*
3. *for every maximal-depth normal crossing point p of $B \subset U$, we have $\nabla_{\mathrm{unip}}(d\log q_j) = 0$ for each canonical logarithmic one-form $d\log q_j$ at p, and*
4. *$(U, B, \nabla_{\mathrm{unip}})$ is compatible in the sense of definition 2.*

We call ∇_{unip} the maximally unipotent connection determined by Ξ.

Note that $d\log q_1, \ldots, d\log q_r$ is a basis for the vector space of local solutions of $\nabla_{\mathrm{unip}} e = 0$ near p. By analytic continuation of solutions, the connection ∇_{unip} is unique if it exists. The requirement of compatibility is quite strong, essentially guaranteeing that the structure of $\overline{S}$ near Ξ resembles that of a semi-toric compactification.

7 Implications of mirror symmetry

Mirror symmetry [18, 34, 13, 24] predicts that Calabi-Yau manifolds should come in pairs,[8] with the rôles of variation of complex structure and of com-

[8] The most recent results [2, 59] suggest that it is *birational equivalence classes* of Calabi-Yau manifolds which come in pairs.

plexified Kähler structure being reversed between mirror partners. We wish to formulate a precise mathematical version of these mirror symmetry predictions, taking into account the semi-toric compactification structure we have studied in this paper. The resulting statements are unfortunately rather technical, but they appear to be completely general. We hope the reader will bear with the technicalities.

Our conjectures involve the A-variations of Hodge structure introduced in [41], whose essential ingredients are the numbers of rational curves of various degrees on a Calabi-Yau manifold. Our first mathematical conjecture about mirror symmetry is carefully formulated in [41], and can be stated as follows.

THE MATHEMATICAL MIRROR SYMMETRY CONJECTURE (Normal Crossings Case) *Let Y be a Calabi-Yau manifold with $h^{2,0}(Y) = 0$, and let $\pi : \mathcal{Y} \to S$ be a family of complex structures on Y such that the Kodaira-Spencer map is an isomorphism at every point. Let $S \subset \overline{S}$ be a partial compactification whose boundary is a divisor with normal crossings. To each maximally unipotent normal crossing boundary point p in $\overline{S}$ there is associated the following:*

1. *a Calabi-Yau manifold X with $h^{2,0}(X) = 0$,*
2. *a lattice L of finite index*[9] *in $H^2(X, \mathbb{Z})/\text{torsion}$,*
3. *the relative interior $\sigma \subset H^2(X, \mathbb{R})$ of a rational polyhedral cone $\overline{\sigma}$ which is generated by a basis $\ell^1, \ldots, \ell^r$ of L, and*
4. *a map μ from a neighborhood of p in $\overline{S}$ to $((H^2(X,\mathbb{R}) + i\,\sigma)/L)^-$, determined up to constants of integration by the requirement that $\mu^*(d\log w_j)$ is the canonical logarithmic one-form $d\log q_j$ on $\overline{S}$ at p (as defined in section 6), where $z_1, \ldots, z_r$ are coordinates dual to $\ell^1, \ldots, \ell^r$, and $w_j := \exp(2\pi i\, z_j)$,*

such that

a. *σ is contained in the Kähler cone for some complex structure on X, and*
b. *μ induces an isomorphism between the formally degenerating geometric variation of Hodge structure at p and the A-variation of Hodge structure with framing σ associated to X.*

[9]The reason for allowing such an L rather than insisting on $H^2(X, \mathbb{Z})/\text{torsion}$ itself is that our basic defining condition on the family S—that the Kodaira-Spencer map be an isomorphism at every point—is invariant under finite unramified base change. So we must allow finite unramified covers of the parameter spaces.

Put more concretely, if we calculate the geometric variation of Hodge structure near $p \in \overline{S}$ using appropriate quasi-canonical coordinates q_j, we should produce power series expansions for B-model three-point functions (for Y) whose coefficients agree with the c_η which are derived from the numbers of rational curves on X. This is precisely the type of calculation pioneered by Candelas, de la Ossa, Green, and Parkes [12] in the case of the quintic threefold.

This first version of our mathematical mirror symmetry conjecture depends rather explicitly on the choice of a maximally unipotent (normal crossing) boundary point. And unfortunately, if we move from point to point along the boundary of $\overline{S}$, or if we vary the compactification $\overline{S}$ by blowing up the boundary, we can produce many such boundary points. On the other hand, if X is a mirror partner of Y for which the cone and convergence conjectures hold, there are many framed A-variations of Hodge structure (with different framings) associated to X. In fact, given framings σ and σ' which belong to rational polyhedral decompositions $\mathcal{P}$ and $\mathcal{P}'$, respectively, there is always a common refinement $\mathcal{P}''$ of these decompositions. Geometrically, the corresponding compactification $\widehat{\mathcal{D}}(\mathcal{P}'')/\Gamma$ is a blowup of both $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ and $\widehat{\mathcal{D}}(\mathcal{P}')/\Gamma$. Analytic continuation on the common blowup $\widehat{\mathcal{D}}(\mathcal{P}'')/\Gamma$ from a point in the inverse image of $\mathcal{D}(\sigma)$ to one in the inverse image of $\mathcal{D}(\sigma')$ will give an isomorphism of the A-variations of Hodge structure.

Each of the various maximally unipotent normal crossing boundary points will conjecturally lead to a mirror isomorphism. We wish to fit these various mirror isomorphisms together, thus removing the dependence of the conjectures on an arbitrary choice of boundary point. In fact, the mirror symmetry isomorphism is expected by the physicists to extend to an isomorphism between the full conformal field theory moduli spaces, and so, presumably, to compactifications as well. Thus, the structure of the semi-toric compactifications which is natural from the point of view of variation of complexified Kähler structure on X should be reflected in the structure of compactifications of the complex structure moduli space $\mathcal{M}_Y$ of Y.

This philosophy suggests two things about the compactified parameter spaces $\overline{S}$ of complex structures on Y. First, there should be a compatibility between compactification points whose mirror families are associated to the *same* space X, and the *same* Kähler cone $\mathcal{K}$. In fact, we should be able to extend our mathematical mirror symmetry conjecture to arbitrary maximally unipotent subsets of the boundary for *any* compactification, not just ones whose boundary is a divisor with normal crossings. And second, there should be some kind of *minimal* compactification of the coarse moduli space $\mathcal{M}_Y$ of complex structures on Y, whose mirror compactified family would be the

Satake-Baily-Borel-type compactification of $\mathcal{D}/\Gamma$.

The compatibility between compactifications can be recognized by means of the flat connection ∇_{unip} which we used to identify maximally unipotent subsets of the boundary. We extend our mirror symmetry conjecture to the general case as follows.

THE MATHEMATICAL MIRROR SYMMETRY CONJECTURE (General Case) *Let Y be a Calabi-Yau manifold with $h^{2,0}(Y) = 0$, let $\pi : \mathcal{Y} \to S$ be a family of complex structures on Y such that the Kodaira-Spencer map is an isomorphism at every point, and let $S \subset \overline{S}$ be a partial compactification. To each maximally unipotent connected subset Ξ of the boundary $\overline{S} - S$ there is associated the following:*

1. *a Calabi-Yau manifold X satisfying the cone and convergence conjectures,*
2. *a subgroup $\Gamma \subset \mathrm{Aff}(H^2(X,\mathbb{R}))$ whose translation subgroup L is a lattice of finite index in $H^2(X,\mathbb{Z})/\mathrm{torsion}$,*
3. *a locally rational polyhedral decomposition $\mathcal{P}$ of a cone $\mathcal{C}_+$ (which coincides with the convex hull of $\overline{\mathcal{C}_+} \cap L_{\mathbb{Q}}$) that is invariant under the group $\Gamma_0 := \Gamma/L$, and*
4. *a map μ from a neighborhood U of Ξ in $\overline{S}$ to $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$, determined up to constants of integration by the requirement that the flat connection ∇_{toric} on $\mathcal{D}/\Gamma$ pulls back to ∇_{unip} on $U \cap S$, where ∇_{unip} is the maximally unipotent connection determined by Ξ,*

such that

a. *for some complex structure on X, the interior $\mathcal{C}$ of $\mathcal{C}_+$ is contained in the Kähler cone and Γ_0 is contained in the group of holomorphic automorphisms, and*

b. *μ induces an isomorphism between the geometric variation of Hodge structure over $U \cap S$ and the A-variation of Hodge structure associated to X.*

A priori, the map μ determined by compatibility of the connections would only be a meromorphic map; we are asserting that it is in fact regular, and a local isomorphism.

There is one further refinement of this conjecture which could be made: we could demand that the map μ also respect the quasi-canonical coordinates determined by choosing integral bases $g^0, \ldots, g^r$. This would reduce the ambiguity in the choice of μ to a finite number of choices, but would require a compatibility among such integral quasi-canonical coordinates at various boundary points.

Finally, suppose that $\mathcal{M}_Y$ is the coarse moduli space for complex structures on a Calabi-Yau variety Y such that $h^{2,0}(Y)=0$. (This coarse moduli space is known to exist as a quasi-projective variety, once we have specified a polarization, thanks to a theorem of Viehweg [53].) In this case, we conjecture the existence of a Satake-Baily-Borel-style compactification, as follows.

THE MINIMAL COMPACTIFICATION CONJECTURE *There is a partial compactification* $(\overline{\mathcal{M}_Y})_{\mathrm{SBB}}$ *of the coarse moduli space* $\mathcal{M}_Y$ *with distinguished boundary points* $p_1, \ldots, p_k$ *which are maximally unipotent, such that the data associated by the mathematical mirror symmetry conjecture to* p_j *consists of: (1) a Calabi-Yau manifold* X_j *(with a complex structure specified that determines the group* $\mathrm{Aut}(X_j)$ *of holomorphic automorphisms and the Kähler cone* $\mathcal{K}_j$ *of* X_j*), (2) the group*

$$\Gamma_j := (H^2(X_j, \mathbb{Z})/\text{torsion}) \rtimes \mathrm{Aut}(X_j),$$

and (3) the locally rational polyhedral decomposition $\mathcal{P}_j$ *which is the Satake-Baily-Borel decomposition* $\mathcal{P}_{\mathrm{SBB}}$ *of the cone* $(\mathcal{K}_j)_+$ *(the convex hull of* $\overline{\mathcal{K}}_j \cap H^2(X_j, \mathbb{Q})$*).*

A related conjecture has been made independently by Batyrev [6].

8 Mumford cones and Mori cones

In the fall of 1979, Mori lectured at Harvard on his then-new results [36] on the cone of effective curves. In order to show that his theorem about local finiteness of extremal rays fail when the canonical bundle is numerically effective, he gave an example. (A similar example appears in a Japanese expository paper he wrote a few years later, which has since been translated into English [37].) The example was of an abelian surface with real multiplication, that is, one whose endomorphism algebra contains the ring of integers $\mathfrak{O}_K$ of a real quadratic field K. For such a surface X, the Néron-Severi group $L := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is a lattice of rank 2. The Kähler cone of X lies naturally in $L_{\mathbb{R}}$, and is an open cone $\mathcal{K}$ bounded by two rays whose slopes are irrational numbers in the field K (cf. [38, 20]). Rays through classes of ample divisors $[D] \in L \cap \mathcal{K}$ can be found which are arbitrarily close to the boundary, but the boundary is never reached. This phenomenon indicated that Mori's results on the structure of the dual cone $\mathcal{K}^\vee \subset H_2(X,\mathbb{R})$ could not be extended to the case of abelian surfaces. The picture Mori drew for this example was remarkably similar to figure 1.

The Hilbert modular surfaces in fact serve as moduli spaces for abelian surfaces with endomorphisms of this type (cf. [20, Chap. IX]), although a bit more data must be specified, which determines the group Γ. Now Mumford's figure 1 was drawn in some auxiliary space being used to describe this "complex structure moduli space", while Mori's version of figure 1 depicted the Kähler cone in $H^{1,1}$, and so is related to "complexified Kähler moduli" of the surfaces. The setting is not quite the same as the one in the present paper, since $h^{2,0} \neq 0$. However, mirror symmetry for complex tori does predict that each cusp in the complex structure moduli space will be related to the Kähler moduli space for the abelian varieties parametrized by some $\mathfrak{H} \times \mathfrak{H}/\Gamma$, with the Γ determined by the cusp. (This is not completely clear from the literature; I will return to this point in a subsequent paper.) In fact, under this association the Mumford cone from figure 1 corresponds precisely to the (dualized) Mori cone.

Mirror symmetry might have been anticipated by mathematicians had anyone noticed the striking similarity between these two pictures back in 1979!

Acknowledgments

I would like to thank P. Aspinwall, R. Bryant, P. Deligne, G. Faltings, A. Grassi, B. Greene, R. Hain, S. Katz, R. Plesser, L. Saper and E. Witten for useful discussions. I also thank the referee for constructive comments. This work was partially supported by NSF Grant DMS-9103827 and by an American Mathematical Society Centennial Fellowship.

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Astérisque

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Astérisque, tome 218 (1993), p. 273-323
<http://www.numdam.org/item?id=AST_1993__218__273_0>

Miroirs et involutions sur les surfaces $K3$

Claire Voisin*

§0 - Introduction

0.1. On propose dans ce travail une construction explicite de la "mirror symmetry" prédite par les physiciens et portée récemment à l'attention des mathématiciens par le travail de D. Morrison ([21]), pour un certain nombre de familles de variétés de Calabi-Yau, construites à l'aide de surfaces $K3$ munies d'une involution. Les numéros 0.2 à 0.5 sont une tentative de description des idées des physiciens sur le sujet (voir aussi [21]).

0.2. La prédiction du phénomène de miroirs entre variétés de Calabi-Yau provient de la théorie des supercordes et des σ-modèles et de la recherche d'un modèle consistant mathématiquement et physiquement, rendant compte de différents types d'interactions, refletées dans l'allure générale de l'action S, et quantifiable. L'action classique en théorie des cordes (se propageant dans une variété Riemannienne (M, g)) associe à une surface de Riemann avec bords (Σ, γ) et à une application $\varphi : \Sigma \to M$ son énergie $S(\varphi) = \int_\Sigma \|d\varphi\|^2 \, dA_\Sigma$; les solutions classiques sont des extrémales de S, par rapport à φ et γ. L'introduction de fermions (variables anticommutatives à considérer comme des sections tordues de φ^*TM), permet d'ajouter à cette action des termes fermioniques, où interviennent la connexion de Levi-Civita de M, et la courbure de M.

* Avec le support partiel du projet Science "Geometry of Algebraic varieties", Contrat SCI-0398-C(A)

L'action peut enfin être modifiée par l'ajout d'une terme du type $S_\omega(\varphi) = \int_\Sigma \varphi^*\omega$, où ω est une 2-forme fermée sur M. La somme de ces trois actions est invariante sous certaines transformations. L'action classique est invariante par difféomorphisme de Σ et changements conformes de métrique γ. L'introduction des variables fermioniques permet de définir au moins localement la supersymetrie, et l'existence de deux supersymetries dont le supercommutateur engendre les transformations conformes, recherchée pour des raisons physiques, conduit à prédire l'existence d'une structure complexe sur M.

Les physiciens cherchent à quantifier la théorie, c'est-à-dire à calculer des valeurs probables de certaines fonctionnelles (les "observables") sur l'espace des applications $\varphi : \Sigma \to M$ (et d'autres données comme les variables fermioniques), ayant des valeurs fixées sur le bord de Σ. L'instrument principal est fourni par les intégrales de Feynman.

Pour préserver les symétries de l'action lors de ce processus de quantification, les physiciens sont menés à imposer certaines conditions à la variété M, dont $\dim_{\mathbb{R}} M = 10$, et en supposant, dans une théorie à la Kaluza-Klein, que $M = \mathbb{R}^4 \times K$ avec K compacte de dimension 6, la préservation de la sypersymétrie mène à imposer que (K, g_K) soit une variété Kählerienne à courbure de Ricci nulle, c'est-à-dire une variété de Calabi-Yau. (Ce résultat qui résulte d'un calcul perturbatif au deuxième ordre, est d'ailleurs contredit par le calcul des termes d'ordre supérieur).

En admettant la possibilité de quantifier rigoureusement la propagation des supercordes dans une variété de Calabi-Yau de dimension trois, on est amené à associer à la donnée d'une telle variété X, d'une forme de Kähler η (correspondant à une métrique de Kähler Einstein) et d'une classe réelle λ dans $H^2(X)$ (correspondant au terme $\displaystyle\int_\Sigma \varphi^*\lambda$ de l'action), une théorie $N = 2$ surperconforme des champs. $\alpha = \eta + i\lambda$ est alors un élément de $H^1(\Omega_X)$ tel que $\operatorname{Re}\alpha$ soit une classe de Kähler.

0.3. Gepner [14] a conjecturé que cette correspondance est bijective à condition de ne considérer que les théories conformes à charges $U(1)$ entières et à charge centrale $c = 9$. D'autre part, il existe une involution naturelle sur l'espace des théories $N = 2$ surperconformes, qui consiste à considérer la même théorie conforme des champs sous-jacente, mais à changer le signe de certaines "charges" q, determinées comme les valeurs propres d'un opérateur de l'algèbre superconforme, représentée sur l'espace de Hilbert de la théorie.

Si X est une variété de Calabi-Yau, et $\alpha \in H^1(\Omega_X)$, l'espace tangent à la variété paramétrant les déformations de (X, α) se scinde naturellement en $H^1(T_X) \oplus H^1(\Omega_X)$. Witten [24] a expliqué comment construire un isomorphisme entre $H^1(T_X) \oplus H^1(\Omega_X)$ et les champs primaires "chiraux" (correspondant à $H^1(T_X)$) ou "antichiraux" (correspondant à $H^1(\Omega_X)$) de charge conforme $h = 2$ de la théorie conforme associée, qui décrivent l'espace tangent aux déformations (générateurs) de la théorie conforme. L'effet de l'involution mentionnée ci-dessus est le suivant: les champs primaires "chiraux" satisfont la condition $h = 2q$, tandis que les "antichiraux" satisfont $h = -2q$. A supposer que la théorie superconforme obtenue par involution provienne d'une donnée (X', α'), on doit donc avoir des isomorphismes $H^1(T_X) \simeq H^1(\Omega_{X'}), H^1(\Omega_X) \simeq H^1(T_{X'})$; (X', α') est appelé le miroir de (X, α). Notons que les isomorphismes ci-desus doivent être obtenus comme la différentielle de l'application miroir, dont l'existence résulte de la conjecture de Gepner.

0.4. Finalement, un des aspects les plus fascinants de cette application miroir réside dans la formule précise, annoncée par les physiciens, comparant l'accouplement de Yukawa sur $H^1(T_X)$ et la forme d'intersection sur $H^1(\Omega_{X'})$ où (X', α') est le miroir de (X, α). L'accouplement de Yakawa sur $H^1(T_X)$ est la forme cubique ψ donnée par l'application naturelle $S^3 H^1(T_X) \rightarrow H^3(\Lambda^3 T_X)$. Ce dernier espace est rendu isomorphe à $\mathbb{C}$ par le choix d'une section de $K_X^{\otimes 2}$. La formule est alors la suivante: soit $u \in H^1(T_X)$ et $v \in H^1(\Omega_{X'})$ l'élément correspondant à u par l'isomorphisme : $H^1(T_X) \simeq$

$H^1(\Omega_{X'})$. Alors:

$$\psi(u) = \int_{X'} v^3 + \sum_{f:\mathbb{P}^1 \to X} e^{-\int_{\mathbb{P}^1} \alpha'} n(f) \left(\int_{\mathbb{P}^1} v \right)^3 \tag{0.4.1}$$

où la somme est effectuée sur toutes les composantes de l'ensemble des applications holomorphes $f : \mathbb{P}^1 \to X$, et où $n(f)$ est un entier. Aspinwall et Morrison [4], ont montré, en utilisant la définition précise de $n(f)$ (cf [24]) que $n(f) = 1$ pour une immersion $\mathbb{P}^1 \subset X'$ avec fibré normal $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, ainsi que pour les familles données par les revêtements ramifiés d'une telle courbe.

0.5. Les exemples connus de phénomène de miroir sont essentiellement fournis par les intersections complètes "du type Fermat" dans les espaces projectifs anisotropes, et leurs quotients par des sous-groupes du groupe d'isomorphimes agissant sur celles ci en préservant "la" forme holomorphe.

Cela tient à la construction de Gepner, qui constitue une des évidences pour la conjecture de Gepner, et qui produit une série de théories superconformes satisfaisant les conditions de 0.3, essentiellement obtenues par produits tensoriels de modèles E_k (connus) formant une série discrète indicée par les entiers. Le k^{ieme} modèle de la série discrète a une charge centrale $c_k = 3k/k + 2$. Pour obtenir, en composant les modèles $E_{k_1}, \cdots, E_{k_5}$, une théorie conforme $E_{(k)}$ de charge centrale $c = 9$, on doit imposer la condition $3 \sum_1^5 k_i / (k_i + 2) = 9$, ce qui est équivalent au fait que l'hypersurface de Fermat $M_{(k)}$ définie par $\sum_{i=1}^5 X_i^{k_i+2} = 0$ dans l'espace projectif inhomogène $\mathbb{P}(d/(k_1+2)), \cdots, (d/(k_5+2))$, où $d = PPCM(k_i + 2)$, est à fibré canonique trivial.

Gepner (1) suppose alors que $E_{(k)}$ est la théorie conforme des champs

(1) En fait, Gepner n'a noté cette correspondance que pour les hypersurfaces de Fermat dans l'espace projectif usuel. L'extension au cas anisotrope est due Greene-Vafa-Warner [16].

associée à la propagation des cordes dans $M_{(k)}$ et donne les justifications suivantes: $H^1\left(TM_{(k)}\right)$ et $H^1\left(\Omega_{M_{(k)}}\right)$ ont la dimension de l'espace des champs primaires "chiraux" et "antichiraux" respectivement de charge conforme $h = 2$ de $E_{(k)}$ (cf. 0.3). De plus $M_{(k)}$ et $E_{(k)}$ ont le même groupe d'automorphismes, représenté de façon isomorphe sur les générateurs de $E_{(k)}$ et sur $H^1\left(TM_{(k)}\right)$ et $H^1\left(\Omega_{M_{(k)}}\right)$.

Enfin les sous-groupes agissant trivialement sur la (3,0)-forme de $M_{(k)}$ d'une part, de façon compatible avec la supersymétrie d'autre part sont identiques.

Dans [15], Greene et Plesser ont montré comment l'involution de 0.3 peut-être realisée concrètement sur les modèles $E_{(k)}$ construits par Gepner, et ceux qui en sont déduits en prenant l'espace des invariants par un sous-groupe H des isomorphismes de $E_{(k)}$ compatibles avec la supersymétrie. La théorie conforme miroir est obtenue en prenant l'espace des invariants de $E_{(k)}$ par un sous-groupe H', dual natural de H. Cela suggère que le miroir géométrique (0.3) de $M_{(k)}/H$ (probablement munie de la forme α la plus naturelle du point de vue géométrique) est $M_{(k)}/H'$. Roan [23] a montré rigoureusement que $M_{(k)}/H$ et $M_{(k)}/H'$ admettent des désingularisations qui sont des variétés de Calabi-Yau et a exhibé un isomorphisme naturel:

$$H^1\left(T_{M_{(k)}/H}\right) \simeq H^1\left(\Omega_{M_{(k)}/H'}\right).$$

0.6. Les variétés que l'on considère dans ce travail sont obtenues par désingularisation de quotients $X = E \times S/(j,i)$ où j est l'involution standard d'une courbe elliptique, de quotient $E/j \simeq \mathbb{P}^1$, et i est une involution sur une surface $K3$ S, de quotient $T = S/i$ rationnelle.

En utilisant les résultats de Nikulin on montre comment (avec quelques exceptions) on peut associer à i, caractérisée par son action $H(i)$ sur $H^2(S,\mathbb{Z})$, une involution miroir $H(i')$. On construit dans la section 2 une correspondance bijective entre les structures complexes i invariantes marquées de S et les formes α invariantes sous $H(i')$, satisfaisant la condition $(\mathrm{Re}\,\alpha)^2 > 0$.

Dans la première section on calcule les nombres de Hodge de X et ses accouplements de Yukawa. On voit facilement alors que $X = \widetilde{E \times S}/(j,i)$ et $X' = \widetilde{E \times S'}/(j,i')$, où (S',i') est une surface $K3$ munie d'une involution i' agissant comme $H(i')$ sur $H^2(S',\mathbb{Z})$, satisfont:

$$b_2(X) = h^{2,1}(X') \quad , \quad b_2(X') = h^{2,1}(X).$$

De plus, la construction de la section 2, ainsi que la contruction des miroirs pour les courbes elliptiques, [10], fournissent l'application miroir $(X,\alpha) \to (X',\alpha')$, mais seulement sur un sous-espace de la famille paramètrant les données (X,α).

Dans la troisième section, on calcule le comportement asymptotique des accouplements de Yakawa de X, lorsque S dégénère, et l'on montre que le résultat obtenu confirme la formule 0.4.1, lorsque l'on fait tendre la partie réelle de α' vers l'infini.

§1 - Construction de variétés de Calabi-Yau

1.1. Soit S une surface $K3$ munie d'une involution i holomorphe, agissant par -1 sur la deux forme holomorphe $w \in H^{2,0}(S)$. Le lieu fixe de i est alors contitué d'une union disjointe de courbes lisses $C_1,\cdots,C_N$ de genres respectifs $g_1,\cdots,g_N$.

Par le théorème de l'indice de Hodge, et par la relation $C_i^2 = 2g_i - 2, C_iC_j = 0$, on voit qu'il existe au plus un entier i tel que $g_i > 1$, et que si un tel entier existe, C_j est rationnelle pour $j \neq i$. Si d'autre part toutes les courbes C_i sont de genre 0 ou 1, on a les quatre possibilités suivantes:

1.1.1.

o) $N = 0$

i) Toutes les C_i sont rationnelles.

ii) L'une des courbes C_i est elliptique et les autres sont rationnelles.

iii) $N = 2$ et C_1, C_2 sont elliptiques.

En effet, notons T la surface quotient $S/i, U = T \backslash \cup C_i$, et $V = S \backslash \cup C_i$. Notons $\varphi : V \to U, \varphi : S \to T$ les applications quotients. Par le théorème de Castelnuovo, T est rationnelle, donc simplement connexe, dès que $N > 0$. Supposons $N \neq 0$; on a la suite exacte de faisceaux de $\mathbb{Z}/2\mathbb{Z}$ modules sur U:

$$(1.1.1) \qquad 0 \to (\mathbb{Z}/2\mathbb{Z})_U \to \varphi_*\left((\mathbb{Z}/2\mathbb{Z})_V\right) \to (\mathbb{Z}/2\mathbb{Z})_U \to 0,$$

qui montre que $\operatorname{Ker}\varphi^* : H^1(U, \mathbb{Z}/2\mathbb{Z}) \to H^1(V, \mathbb{Z}/2)$ est isomorphe à $\mathbb{Z}/2\mathbb{Z}$. Par ailleurs on a le diagramme commutatif suivant de suites exactes de cohomologie relative :

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(U, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \oplus H^0\left(C_i, \mathbb{Z}/2\mathbb{Z}\right) & \xrightarrow{\alpha_T} & H^2(T, \mathbb{Z}/2\mathbb{Z}) \\
 & & {\scriptstyle \varphi^*}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi^*} \\
0 & \longrightarrow & H^1(V, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \oplus H^0\left(C_i, \mathbb{Z}/2\mathbb{Z}\right) & \xrightarrow[\alpha_S]{} & H^2(S, \mathbb{Z}/2\mathbb{Z})
\end{array}$$

où les applications α_S et α_T s'identifient par la dualité de Poincaré aux applications $\oplus H_2\left(C_i, \mathbb{Z}/2\mathbb{Z}\right) \to H_2\left(S, \mathbb{Z}/2\mathbb{Z}\right)$ et $\oplus H_2\left(C_i, \mathbb{Z}/2\mathbb{Z}\right) \to H_2\left(T, \mathbb{Z}/2\mathbb{Z}\right)$ induites par les inclusions de C_i dans S et T.

Ceci montre que le noyau α_T est de rang 1 et necessairement engendré par $\oplus\ [C_i]$.

Si au moins deux des courbes C_i sont elliptiques, soit C_1 et C_2, elles sont homologues dans S, par le théorème de l'indice, donc aussi dans T, puisque $H^2(T, \mathbb{Z})$ n'a pas de torsion, et $[C_1] + [C_2]$ est dans $\operatorname{Ker}\alpha_T$. On a donc $[C_1] + [C_2] = \sum_i [C_i]$ ce qui entraîne iii).

1.2. Fixons une courbe elliptique E, munie d'une involution j telle que le quotient E/j soit isomorphe à $\mathbb{P}^1$. Le lieu fixe de j est alors constitué de quatre points $p_1, \cdots, p_4$. Soit $k = (j, i)$ l'involution agissant sur le produit $E \times S$. k a pour lieu fixe les courbes disjointes $p_t \times C_s, r = 1, \cdots, 4, s = 1, \cdots, N$.

Les variétés que l'on considérera sont obtenues par éclatement des quotients $E \times S/k$ le long de $\bigcup_{(r,s)} p_r \times C_s$. Ce que l'on notera: $X = (\widetilde{E \times S/k})$. On a:

1.3. Lemme: X est lisse, à fibré canonique trivial et est simplement connexe dès que $N > 0$.

Démonstration: La lissité est facile à montrer; X peut-être définie de manière équivalente comme le quotient $\widetilde{E \times S}/\widetilde{k}$ de l'éclatement de $E \times S$ le long des courbes $p_r \times C_s$, par l'involution $\widetilde{k}$ agissant naturellement sur $\widetilde{E \times S}$. Comme $\widetilde{k}$ a au moins un point fixe 0 dès que $N > 0$, et agit par -1 sur $\pi_1(\widetilde{E \times S}, 0)$, on a $\pi_1(X) = 0$. Donc $H^2(X, \mathbb{Z})$ n'a pas de torsion et il suffit de montrer que le pull-back $\varphi^*(c_1(K_X))$ est nul dans $H^2(\widetilde{E \times S}, \mathbb{Z})$ où $\varphi : \widetilde{E \times S} \to X$ est l'application quotient. Soit D le diviseur exceptionnel de l'éclatement $\tau : \widetilde{E \times S} \to E \times S$, on a $\varphi^* K_X = K_{\widetilde{E \times S}} - D$, et $K_{\widetilde{E \times S}} = \tau^* K_{E \times S} + D$. Comme $K_{E \times S}$ est trivial, on en déduit que $\varphi^* K_X$ est aussi trivial.

1.4. Par le Lemme 1.3 on voit donc qu'on a, pour chaque type d'involution i comme en 1.1 (avec $N > 0$) un type de déformations de variétés de Calabi-Yau de dimension trois simplement connexe. La suite de cette section est consacrée à la description de la structure de Hodge, de l'accouplement de Yukawa, et de la forme d'intersection de ces variétés.

1.5. On commence d'abord par calculer les nombres de Hodge de X. En utilisant la représentation $X = \widetilde{E \times S}/\widetilde{k}$, on voit que $H^2(X, \mathbb{Q}) = H^2(\widetilde{E \times S}, \mathbb{Q})^+$ est l'espace des invariants sous $\widetilde{k}$ de $H^2(\widetilde{E \times S}, \mathbb{Q})$. De même $H^3(X, \mathbb{Q}) = H^3(\widetilde{E \times S}, \mathbb{Q})^{\mathrm{inv}}$, cette égalité étant un isomorphisme de structures de Hodge. Il vient alors:

1.6 Lemme: $H^2(X, \mathbb{Q})$ est engendré librement par les classes des diviseurs exceptionnels $D_{r,s} = \tau^{-1}(p_r \times C_s), H^2(E, \mathbb{Q})$ et $H^2(T, \mathbb{Q})$. De plus on a: rang $H^2(T, \mathbb{Q}) = 10 + N - N'$, où $N' = \Sigma g_i$.

Démonstration: k agit sur $H^2(\widetilde{E \times S}, \mathbb{Q}) = \bigoplus_{(r,s)} [D_{r,s}] \cdot \mathbb{Q} \oplus H^2(S, \mathbb{Q}) \oplus$

$H^2(E,\mathbb{Q})$ en laissant fixe les $[D_{r,s}]$, et comme (j,i) sur la somme $H^2(E) \oplus H^2(S)$. j agit trivialement sur $H^2(T)$, et l'espace des invariants sous i de $H^2(S,\mathbb{Q})$ est isomorphe à $H^2(T,\mathbb{Q})$. $b_2(T) = \operatorname{rang} H^2(T,\mathbb{Q})$ se calcule par la formule de Noether: On a $\chi(\mathcal{O}_T) = 1 = \dfrac{c_1^2(T)+c_2(T)}{12}$ avec $c_2 = b_2 + 2$.

Enfin, le diviseur $\sum C_i$ de T est un membre du système linéaire $|-2K_T|$, puisque $0 = K_S = \varphi^* K_T + \frac{1}{2}\varphi^*\left(\sum C_i\right)$ et que $H^2(T,\mathbb{Z})$ est sans torsion; on a donc: $4K_T^2 = \left(\sum C_i\right)_T^2 = \frac{1}{2}\left(\sum \varphi^* C_i\right)_S^2 = 2\left(\sum C_i\right)_S^2$, où dans le dernier terme C_i est considéré comme une courbe de S. Finalement comme les C_i sont disjointes et que K_S est trivial, on trouve: $\left(\sum C_i\right)_S^2 = 2\left(\sum g_i\right) - 2N$, d'où $K_T^2 = N' - N$ et $b_2(T) = 10 - K_T^2 = 10 + N - N'$.

Pour le calcul du nombre de Hodge $h^{2,1}(X) = \dim H^1\left(\Omega_X^2\right)$ on a:

1.7 Lemme: $H^{2,1}(X)$ est engendré librement par les $j_{r,s*}\left(\tau_{r,s}^*\left(H^0\left(\Omega_{C_s}\right)\right)\right)$ où $\tau_{r,s} : D_{r,s} \to p_r \times C_s$ est la restriction de τ et $j_{r,s} : D_{r,s} \hookrightarrow \widetilde{E \times S}$ est l'inclusion, et par $H^0\left(\Omega_E\right) \otimes H^1\left(\Omega_S\right)^- \oplus H^1\left(\mathcal{O}_E\right) \otimes H^0\left(\Omega_S^2\right)$.

Démonstration: On a $H^{2,1}(X) = H^{2,1}(\widetilde{E \times S})^+$ et

$$H^{2,1}(\widetilde{E \times S}) = \bigoplus_{(r,s_*)} j_{r,s}\left(\tau_{r,s}^*\left(H^0\left(\Omega_{C_s}\right)\right)\right) \oplus H^0\left(\Omega_E\right)$$
$$\otimes H^1\left(\Omega_S\right) \oplus H^1\left(\mathcal{O}_E\right) \otimes H^0\left(\Omega_S^2\right).$$

k agit trivialement sur le premier terme et agit comme (j,i) sur les deux derniers. Comme j agit par -1 sur $H^1(E)$ (j,i) agit par $-1 \otimes H(i)$ sur $H^1(E) \otimes H^2(S)$, ce qui donne immédiatement le résultat.

1.8 Corollaire: On a $b_2(X) = 11 + 5N - N'$ et $h^{2,1}(X) = 11 + 5N' - N$.

Démonstration:

$$b_2(X) = b_2(E) + b_2(T) + \sharp\{(r,s)\} = 1 + 10 + N - N' + 4N = 11 + 5N - N'$$

$$h^{2,1}(X) = \sum_{r,s} h^0\left(\Omega_{C_s}\right) + 1 + h^1\left(\Omega_S\right)^- = 4N' + 1 + 10 - N + N' = 11 + 5N' - N,$$

où l'égalité $h^1(\Omega_S)^- = 10 - N + N'$ vient de $h^1(\Omega_S) = 20 = h^1(\Omega_S)^- + b_2(T)$, et du Lemme 1.6.

1.9. La "mirror symmetry" $X_1 \leftrightarrow X_2$ est supposée échanger les nombres b_2 et $h^{2,1}$, i.e. $b_2(X_1) = h^{2,1}(X_2), b_2(X_2) = h^{2,1}(X_1)$ et d'après le corollaire 1.8, on voit qu'à supposer que le miroir X_2 de $X_1 = \widetilde{E \times S_1}\left(\widetilde{j, i_1}\right)$ soit encore de la forme $X_2 = \widetilde{E \times S_2}/\left(\widetilde{j, i_2}\right)$, cela revient à échanger les nombres N et N', i.e.: $N_1 = N_2', N_2 = N_1'$. Ceci bien sûr n'est possible que si $N_1' > 0$ (on exclut désormais le cas $N = 0$). On montrera dans la section suivante en utilisant les travaux de Nikulin, comment étant donné une involution i_1 sur une surface K_3 S_1 agissant par -1 sur $H^0\left(\Omega^2_{S_1}\right)$ on peut construire un second type i_2 d'involution sur une surface K_3, satisfaisant la même condition, à condition que $N_1' > 0$ et à l'exception d'un cas (cf. 2.17), et telle que $N_2' = N_1, N_1' = N_2$.

1.10. On va décrire maintenant la forme d'intersection sur $H^2(X, \mathbb{Q})$, dans la base décrite en 1.6. Notons $(d + \alpha + \beta) \in H^2(X, \mathbb{Q}) \simeq \bigoplus_{r,s} \langle D_{r,s} \rangle \mathbb{Q} \oplus H^2(T, \mathbb{Q}) \oplus H^2(E, \mathbb{Q})$. On a alors:

1.11 Lemme: $(d + \alpha + \beta)^3_X = \left(d^3\right)_X + 3\left(d^2\alpha\right)_X + 3\left(\alpha^2\beta\right)_X$. De plus pour $d = \sum d_{r,s} D_{r,s}$ on a $\left(d^3\right)_X = \sum d^3_{r,s} D^3_{r,s}$, avec $D^3_{r,s} = 8 - 8g_s$, et $\left(d^2\alpha\right)_X = -2\sum d^2_{r,s}\left(C_s \cdot \alpha\right)_T$. Enfin $\left(\alpha^2\beta\right)_X = \left(\alpha^2\right)_T \cdot \int_E \beta$.

Démonstration: Soit $\varphi : \widetilde{E \times S} \to X$ l'application quotient, $\tau : \widetilde{E \times S} \to E \times S$ l'éclatement et p_1, p_2 les projections de $E \times S$ sur E et S respectivement. On utilise aussi la notation $\varphi : S \to T$ pour l'application quotient.

On a alors

$$(d + \alpha + \beta)^3_X = \frac{1}{2}\left(\varphi^*(d + \alpha + b)\right)^3_{\widetilde{E \times S}} = \frac{1}{2}\left(\varphi^* d + p_1^*\beta + p_2^*\varphi^*\alpha\right)^3.$$

Comme $\varphi^* d$ est supporté sur les diviseurs exceptionnels de τ et que les courbes éclatées sont contenues dans des fibres de p_1, on voit facilement qu'on a les relations suivantes dans l'anneau de cohomologie de $\widetilde{E \times S}$: $\left(p_1^*\beta\right)^2 = 0, \quad \forall \beta \in H^2(E), p_1^*\beta \cdot d = 0, \ \forall \beta \in H^2(E), \forall d$ supporté sur les diviseurs exceptionnels de $\tau, d \cdot \left(p_2^*\gamma\right)^2 = 0, \quad \forall \gamma \in H^2(S), \forall d$ supporté sur les diviseurs exceptionnels. Cela montre immédiatement la première assertion du Lemme. Comme les diviseurs exceptionnels sont disjoints on a d'autre part $D_{r,s} \cdot D_{r',s'} = 0$ pour $(r,s) \neq (r',s')$ et donc $\left(d^3\right)_X = \sum d^3_{r,s} \left(D_{r,s}\right)^3_X$, $\left(d^2\alpha\right)_X = \sum d^2_{r,s} \left(D^2_{r,s} \cdot \alpha\right)_X$. Finalement $\left(D_{r,s}\right)^3_X = \frac{1}{2} \left(\varphi^* D_{r,s}\right)^3_{\widetilde{E \times S}} = 4 \left(D_{r,s}\right)^3_{\widetilde{E \times S}}$ où dans le dernier terme $D_{r,s}$ est considéré comme le diviseur de $\widetilde{E \times S}$ au dessus de $p_r \times C_s$. Le fibré normal de $p_r \times C_s$ dans $E \times S$ étant égal à $\mathcal{O}_{C_s} \oplus K_{C_s}$ on a $D_{r,s} \simeq \mathbb{P}\left(\mathcal{O}_{C_s} \oplus K^{-1}_{C_s}\right), D_{r,s|D_{r,s}} = \mathcal{O}_{D_{r,s}}(-1)$, soit $\left(D_{r,s}\right)^3_{\widetilde{E \times S}} = \left(\mathcal{O}_{D_{r,s}}(1)\right)^2_{D_{r,s}} = 2 - 2g_s$, et $\left(D_{r,s}\right)^3_X = 8 - 8g_s$.

Enfin $\left(D^2_{r,s} \cdot \alpha\right)_X = \frac{1}{2} \left(\left(\varphi^* D_{r,s}\right)^2 \cdot p_2^* \circ \varphi^* \alpha\right)_{\widetilde{E \times S}} = 2 \left(D^2_{r,s} \cdot p_2^* \circ \varphi^* \alpha\right)_{\widetilde{E \times S}} = 2 \left(p_{2*} \left(D_{r,s}\right)^2 \cdot \alpha\right)_S = -2 \left(C_s \cdot \varphi^* \alpha\right)_S = -2 \left(C_s \cdot \varphi^* \alpha\right)_T$ où l'on a utilisé l'égalité
$p_{2*} \left(D_{r,s}\right)^2_{\widetilde{E \times S}} = -C_s$ dans $H^2(S)$. On a aussi $\left(\alpha^2 \beta\right)_X = \frac{1}{2} \left(\left(\varphi^* \alpha\right)^2 \cdot \varphi^* \beta\right)_{\widetilde{E \times S}}$ $= \frac{1}{2} \left(p_1^* \beta \cdot \left(p_2^* \varphi^* \alpha\right)^2\right)_{\widetilde{E \times S}} = \frac{1}{2} \left(\int_E \beta\right) \cdot \left(\varphi^* \alpha\right)^2_S = \int_E \beta \cdot \left(\alpha^2\right)_T$, ce qui termine la preuve du lemme.

1.12. On va calculer maintenant l'accouplement de Yukawa sur $H^1\left(T_X\right)$. Cet accouplement est une forme cubique dépendant du choix d'une section ω non nulle de K_X, et peut se définir en termes de variations infinitésimales de structure de Hodge de la façon suivante: la variation infinitésimale de structure de Hodge de X est décrite par une application $\varphi = \bigoplus_{p+q=3} \varphi^{p,q} : H^1\left(T_X\right) \to \bigoplus_{(p,q)} \operatorname{Hom}\left(H^q\left(\Omega^p_X\right), H^{q+1}\left(\Omega^{p-1}_X\right)\right)$. Le composé $\varphi^{1,2} \circ \varphi^{2,1} \circ \varphi^{3,0}$ donne alors une application $\psi : S^3 H^1\left(T_X\right) \to \operatorname{Hom}\left(H^{3,0}(X), H^{0,3}(X)\right)$ qui est la forme cubique cherchée moyennant l'isomorphisme $\operatorname{Hom}\left(H^{3,0}(X), H^{0,3}(X)\right) \simeq \mathbb{C}$ donnée par $\omega^{\otimes 2}$.

1.12.1. Par les résultats de Griffiths décrivant la variation infinitésimale de structure de Hodge en termes d'accouplements de Yoneda, on peut aussi définir ψ comme le produit $S^3 H^1\left(T_X\right) \to H^3\left(\Lambda^3 T_X\right)$, le dernier terme étant isomorphe à $\mathbb{C}$ par la multiplication avec $\omega^{\otimes 2}$.

1.12.2. On utilisera aussi la variante suivante: ω fournit, via $\varphi^{3,0}$ un isomorphisme $H^1(\Omega_X^2) \simeq H^1(T_X)$. La forme cubique ψ, vue sur $H^1(\Omega_X^2)$ est alors obtenue comme le produit $S^3H^1(\Omega_X^2) \to H^3(\Lambda^3(\Omega_X^2))$ suivi de l'isomorphisme: $\omega^{-1} : H^3(\Lambda^3(\Omega_X^2)) \xrightarrow{\sim} H^3(K_X) \simeq \mathbb{C}$.

D'après le lemme 1.7 on a une décomposition naturelle:

$$H^1(\Omega_X^2) \simeq \bigoplus_{(r,s)} H^0(\Omega_{C_s}) \oplus H^1(\Omega_S)^- \otimes H^0(\Omega_E) \oplus H^0(\Omega_S^2) \oplus H^1(\mathcal{O}_E).$$

En utilisant l'isomorphisme $H^1(T_X) \simeq H^1(\Omega_X^2)$ de 1.12.2, on obtient une décomposition naturelle:

$$H^1(T_X) = \bigoplus_{(r,s)} H^0(\Omega_{C_s}) \oplus H^1(T_S)^+ \oplus H^1(T_E).$$

Dans cette decomposition, il est clair que le sous-espace $W = H^1(T_S)^+ \oplus H^1(T_E)$
correspond aux déformations de X données par une déformation de $E \times S$ préservant l'involution k.

Sur W, la variation de structure de Hodge $\oplus\varphi^{p,q}$ préserve la décomposition de la structure de Hodge de X en somme de $H^3(E \times S)^{\text{inv}}$ et $\bigoplus_{(r,s)} H^1(C_s)$, et l'on a donc: $\varphi^{2,1}_{|W} = \varphi'^{2,1} + \varphi''^{2,1}$, où $\varphi'^{2,1}$ décrit la variation de structure de Hodge de $\bigcup_{(r,s)} p_r \times C_s$, lorque S varie infinitésimalement en préservant l'involution i et $\varphi''^{2,1}$ décrit la variation de structure de Hodge de $E \times S$ lorsque E et S varient.

Soit :

$$\varphi'^{2,1} : W \to \bigoplus_{(r,s)} \operatorname{Hom}\left(H^0(\Omega_{C_s}), H^1(\mathcal{O}_{C_s})\right)$$

et $\varphi''^{2,1}$:

$$W \to \operatorname{Hom}\Big(H^0(\Omega_E) \otimes H^1(\Omega_S)^- \oplus H^1(\mathcal{O}_E) \otimes H^0(\Omega_S^2),$$

$$H^1(\mathcal{O}_E)\otimes H^1(\Omega_S)^-\oplus H^0(\Omega_E)\otimes H^2(\mathcal{O}_S)\Big).$$

La flèche $\varphi''^{2,1}$ décrit la restriction de l'accouplement de Yukawa à W, modulo l'identification donnée par ω:

$$W\simeq H^0(\Omega_E)\otimes H^1(\Omega_S)^-\oplus H^1(\mathcal{O}_E)\otimes H^0(\Omega_S^2)$$
$$W^*\simeq H^1(\mathcal{O}_E)\otimes H^1(\Omega_S)^-\oplus H^0(\Omega_E)\otimes H^2(\mathcal{O}_S),$$

par la relation: $\psi(u)=\left\langle u,\varphi''_{2,1}(u)(u)\right\rangle$, pour $u\in W$. Finalement, si on écrit $u=(u_E,u_S)$ pour $u\in W$, avec $u_E\in H^1(T_E),u_S\in H^1(T_S)^+$, il est clair que $\psi(u_E,u_S)=\psi'(u_E,u_S)$, où ψ' est l'accouplement de Yukawa pour la variété $E\times S$, restreint à $H^1(T_E)\times H^1(T_S)^+\subset H^1(T_{E\times S})$. D'après 1.12.1 on a: $\psi':S^3\big(p_1^*H^1(T_E)+p_2^*H^1(T_S)\big)\to\big(\Lambda^3(p_1^*T_E\oplus p_2^*T_S)\big)$ est donnée par le produit et on voit immédiatement que $\psi'(u_E,u_S)=3\psi_1(u_E)\otimes\psi_2(u_E)$ où $\psi_1(u_E)=u_E\in H^1(T_E),\psi_2(u_S)=u_S^2\in H^2(\Lambda^2T_S)$. On a donc montré:

1.13 Lemme: Sur W, l'accouplement de Yukawa est décrit à un coefficient près par $\psi(u_E,u_S)=3\psi_1(u_E)\psi_2(u_S)$, où ψ_1 est une forme linéaire non nulle sur $H^1(T_E)$ et ψ_2 est une forme quadratique non dégénérée sur $H^1(T_S)^+$. Si l'on choisit une 2-forme $\omega_S\in H^{2,0}(S)$, fournissant un isomorphisme $H^1(T_S)^+\simeq H^1(\Omega_S)^-,\psi_2$ s'identifie à la forme d'intersection sur $H^1(\Omega_S)^-$.

1.14. On revient à l'application $\varphi'^{2,1}$ qui décrit les accouplements de Yukawa du type $\psi(w,\eta,\gamma)$, pour $w\in W,\eta\in\bigoplus_{(r,s)}H^0(\Omega_{C_s}),\gamma\in H^1(\Omega_X^2)$ où ψ est la forme trilinéaire symétrique correspondant à la forme cubique ψ. Le fait que $\varphi'^{2,1}$ soit à valeurs dans $\bigoplus_{(r,s)}\mathrm{Hom}\left(H^0(\Omega_{C_s}),H^1(\mathcal{O}_{C_s})\right)$ est équivalent à l'annulation des termes $\psi(w,\eta,w)$ où $w\in W$, $\eta\in\bigoplus_{(r,s)}H^0(\Omega_{C_s})$, et des termes $\psi(w,\eta_{r,s},\eta_{r',s'})$ pour $w\in W,\eta_{r,s}\in H^0(\Omega_{p_r\times C_s})$ $\eta_{r',s'}\in H^0\left(\Omega_{p_{r'}\times C_{s'}}\right)$ et $(r,s)\neq(r',s')$.

D'autre part, comme $\varphi'^{2,1}$ décrit la variation de structure de Hodge de $\bigcup_s C_s$ lorsque S varie infinitésimalement (en préservant i), elle est obtenue par le composé de la projection $W \to H^1(T_S)^+$, de l'application ν : $H^1(T_S)^+ \to \bigoplus_s H^1(T_{C_S})$, différentielle de l'application naturelle $\mathrm{Def}(S,i) \to \mathrm{Def}\left(\bigcup_s C_s\right)$, et de l'application décrivant la variation de structure de Hodge de $\bigcup_s C_s$: $\bigoplus_s H^1(T_{C_s}) \to \bigcup_s \mathrm{Hom}\left(H^0(\Omega_{C_s}), H^1(\mathcal{O}_{C_s})\right)$. Il est bien connu que cette dernière flèche est duale de produit: $\bigoplus_s H^0(\Omega_{C_s})^{\otimes 2} \to \bigcup_s H^0\left(\Omega_{C_s}^{\otimes 2}\right)$. Finalement la flèche $\nu : H^1(T_S)^+ \to \bigcup_s H^1(T_{C_s})$ se dualise, modulo le choix d'une forme $\omega_S \in H^{2,0}(S)$, en une flèche:

1.14.1. $\nu' : \oplus H^0\left(\Omega_{C_s}^{\otimes 2}\right) \to H^1(\Omega_S)^-$, qui permet de réécrire les accouplements $\psi(w,\eta,\gamma)$ sous la forme $\psi(w,\eta,\gamma) = \langle w, \nu'(\eta\gamma)\rangle_S$ pour $w \in H^1(T_S)^+ \simeq H^1(\Omega_S)^-$
$\eta, \gamma \in \bigoplus_{(r,s)} H^0(\Omega_{p_r \times C_s})$. On utilisera dans la section 3 l'interprétation suivante de l'application ν' :

1.15 Lemme: Fixons $\omega \in H^0(K_S), \sigma \in H^0(-2K_T)$ une equation pour $C = \cup C_s$; pour $P \in H^0(-2K_T)$, la forme méromorphe $\frac{P\omega}{\sigma}$, à pôles d'ordre 2 le long de $\cup C_s$ est sans résidus sur C. Sa classe dans $H^2(S,\mathbb{C})$ est en fait dans $F^1H^2(S)^-$, et son image dans $H^1(\Omega_S)^-$ ne dépend que de la restriction $P_{|C} \in \oplus H^0\left(\Omega_{C_s}^{\otimes 2}\right)$. Ceci fournit l'application ν' à un coefficient près.

Cette correspondance, qui est une variation due à Clemens ([8] de la construction de Griffiths [17] est décrite précisement dans [8]. On résume ici l'argument: $\frac{P\omega}{\sigma}$ étant antiinvariante sous i n'a pas de résidu sur C. Elle définit donc une classe dans $H^2(S,\mathbb{C})$. Comme elle est à pôle d'ordre au plus 2 le long de C, la théorie de la filtration par l'ordre du pôle pour la construction de la structure de Hodge mixte sur $H^2(S \backslash C)$ montre que $\frac{P\omega}{\sigma}$ définit une classe dans $F^1H^2(S,\mathbb{C})^-$, donc dans $F^1H^2(S)^-$. Si P s'annule le long de C, $\frac{P\omega}{\sigma}$ n'a pas de pôle et définit un multiple de ω. Donc la classe de $\frac{P\omega}{\sigma}$ dans $H^1(\Omega_S)^-$ ne dépend que de $P_{|C}$. Notons finalement que la suite exacte $0 \to \mathcal{O}_T \xrightarrow{\sigma} -2K_T \to \Omega_C^{\otimes 2} \to 0$ et $H^1(\mathcal{O}_T) = 0$ montrent la surjectivité de la restriction $H^0(2K_T) \to H^0(\Omega_C^{\otimes 2})$.

Il reste à montrer que l'application ainsi construite est bien égale à ν'. Mais il suffit pour cela de noter que $P \in H^0(-2K_{T|C})$ détermine une déformation infinitésimale ε_p de C dans T, et donc une déformation infinitésimale ε'_p i-invariante de S; on a donc une application $\nu'' : H^0(-2K_{T|C}) \to H^1(T_S)^+$. On peut montrer alors, comme dans le cas des hypersurfaces [7], que $\varphi^{2,0}(\nu''(P)(\omega)$ est égale à la classe de $P\omega/\sigma$ dans $H^1(\Omega_S)^-$, où $\varphi^{2,0} : H^1(T_S)^+ \to \mathrm{Hom}(H^{2,0}(S), H^1(\Omega_S)^-)$ est la variation infinitésimale de structure de Hodge de S.

Il reste alors à voir que $\nu'' : H^0(-2K_{T|C}) \to H^1(T_S)^+$ est duale de $\nu : H^1(T_S)^+ \to H^1(T_C)$ (modulo le choix de ω).

1.15.1. On montre d'abord que ν'' est donné par le cobord associé à la suite exacte:

1.15.2. $$0 \longrightarrow T_S \xrightarrow{\varphi_*} \varphi^* T_T \longrightarrow N_C T \longrightarrow 0\,.$$

Considérons une déformation infinitésimale de C dans T :

$$\begin{array}{ccc} C_\varepsilon & \subset & T \times \mathbb{C}_\varepsilon \\ \downarrow & & \downarrow \\ \mathbb{C}_\varepsilon & = & \mathrm{Spec}\mathbb{C}[\varepsilon]/\varepsilon^2 \end{array}$$

correspondant à $P \in H^0(N_C T)$.

La déformation infinitésimale

$$\begin{array}{ccc} S_\varepsilon & \xrightarrow{\varphi} & T \times \mathbb{C}_\varepsilon \\ \downarrow & & \\ \mathbb{C}_\varepsilon & & \end{array}$$

de S qui lui est associée par ν'' est le revêtement double de $T \times \mathbb{C}_\varepsilon$ ramifié le long de C_ε. Le diagramme

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_S & \longrightarrow & T_{S_\varepsilon|S} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varphi^* T_T & \longrightarrow & \varphi^* T_{T\times \mathbb{C}_\varepsilon | T} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_C T & = & N_C T & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

où la seconde ligne est scindée et où la classe d'extension de la première ligne est $\nu''(P)$, tandis que la section de $N_C T$ obtenue grâce au scindage est P, donne maintenant 1.15.1. En dualisant 1.15.2 on obtient :

1.15.3. $$0 \to \varphi^*\Omega_E \to \Omega_S \xrightarrow{\alpha} N_C S^* \to 0$$

et l'on en déduit que le dual de ν'' est donné par l'application $H^1(T_S)^+ \simeq H^1(\Omega_S)^- \to H^1(T_C)^+$ induite par α, où α est donné par le scindage $\Omega_{S|C} = N_C S^* \oplus \Omega_C$. De façon équivalente, le dual de ν'' est l'application $\nu''' : H^1(T_S)^- \to H^1(T_C)$ induite par la décomposition $T_{S|C} \simeq T_C \oplus N_C S$.

1.15.4. On montre enfin l'égalité de ν et ν''': On considère une déformation

infinitésimale $\begin{array}{ccccc} S_\varepsilon & \longrightarrow & T_\varepsilon & \longleftarrow & C_\varepsilon \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}_\varepsilon & & \mathbb{C}_\varepsilon & & \mathbb{C}_\varepsilon \end{array}$, avec paramètre infinitésimaux $u \in H^1(T_C), v \in H^1(T_S)$. On a le diagramme commutatif suivant:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_S & \longrightarrow & T_{S_\varepsilon|S} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{S|C} & \longrightarrow & T_{S_\varepsilon|C} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_C & \longrightarrow & T_{C_\varepsilon|C} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0
\end{array}$$

où les classes d'extension de la première et dernière ligne sont respectivement v et u, et la flèche verticale $T_{S_\varepsilon|C} \to T_{C_\varepsilon|C}$ est donnée par le scindage naturel $T_{S_\varepsilon|C} \simeq T_{C_\varepsilon|C} \oplus N_C S$. On en déduit immédiatement que $u = \nu'''(v)$, et que le lemme 1.15 est prouvé.

Le calcul de l'accouplement de Yukawa se termine enfin par la preuve de:

1.16. Lemme: La restriction de l'accouplement de Yukawa au sous-espace $\bigoplus_{r,s} H^0(\Omega_{C_s})$ de $H^1(\Omega_X^2) \underset{\omega}{\simeq} H^1(T_X)$ est nulle.

Démonstration: On utilise la troisième description suivante de X. Soit $E/j = \mathbb{P}^1, \varphi_1 : E \to \mathbb{P}^1$ l'application quotient, de lieu de branchement $A = \{p_1, \cdots p_4\} \subset \mathbb{P}^1$. De même, soit $T = S/i, \varphi_2 : S \to T$ l'application quotient, ramifiée le long de $C = UC_s \subset T$. L'application $(\varphi_1, \varphi_2) E \times S \to \mathbb{P}^1 \times T$ fournit une application $(\widetilde{\varphi_1, \varphi_2}) : \widetilde{E \times S} \to \widetilde{\mathbb{P}^1 \times T}$ où $\widetilde{E \times S}$ et $\widetilde{\mathbb{P}^1 \times T}$ sont les éclatés de $E \times S$ et $\mathbb{P}^1 \times T$ respectivement le long de $\bigcup_{(r,s)} p_r \times C_s$.

Cette application descend en une application $\varphi : X \to \widetilde{\mathbb{P}^1 \times T}$ et fait de X le revêtement double de $Y = \widetilde{\mathbb{P}^1 \times T}$, ramifié le long du transformé propre de $A \times T \cup \mathbb{P}^1 \times C$. On notera I l'évolution agissant sur X au dessus de Y. Comme Y est l'éclaté de $\mathbb{P}^1 \times T$ le long de $\bigcup_{(r,s)} p_r \times C_s$ on a une inclusion

$\oplus j'_{r,s} \circ \tau'^{*}_{r,s} = \bigoplus_{(r,s)} H^0\left(\Omega_{C_s}\right) \to H^1\left(\Omega_Y^2\right)$, où $j'_{r,s} : E_{r,s} \to Y$ est l'inclusion du diviseur exceptionnel au dessus de $p_r \times C_s$ et $\tau'_{r,s} : E_{r,s} \to C_s$ est la restriction de $\tau' : Y \to \mathbb{P}^1 \times T$. On a alors: $\bigoplus_{(r,s)} j_{r,s*} \circ \tau_{r,s} = \varphi^* \circ \left(\bigoplus_{(r,s)} j'_{r,s*} \circ \tau'_{r,s}\right)$: $\bigoplus_{(r,s)} H^0\left(\Omega_{C_s}\right) \to H^1\left(\Omega_X^2\right)$.

Soit $\eta \in \bigoplus_{(r,s)} H^0\left(\Omega_{C_s}\right)$; alors

$$\left(\bigoplus_{(r,s)} j_{r,s*} \circ \tau^*_{r,s}(\eta)\right)^3 = \varphi^*\left(\left(\left(\bigoplus_{(r,s)} j'_{r,s} \circ \tau'^{*}_{r,s}\right)(\eta)\right)^3\right)$$

dans $H^3\left(\Lambda^3\left(\Omega_X^2\right)\right)$ où: $\varphi^* : H^3\left(\Lambda^3\left(\Omega_Y^2\right)\right) \to H^3\left(\Lambda^3\left(\Omega_X^3\right)\right)$ est induite par $\varphi^* : \varphi^*\Omega_Y \to \Omega_X$.

Mais $\Lambda^3\left(\Omega_Y^2\right) = K_Y^{\otimes 2}$ et $\pi_*\left(\Lambda^3\left(\Omega_X^2\right)\right) = K_Y \oplus \mathcal{O}_Y$, et comme l'application $\pi^*\left(\Lambda^3\left(\Omega_Y^2\right)\right) \xrightarrow{\pi^*} \Lambda^3\left(\Omega_Y^2\right)$ s'annule doublement le long du lieu de ramification de π, on voit que l'application induite $\Lambda^3\left(\Omega_Y^2\right) \to \pi *\left(\Lambda^3\left(\Omega_X^2\right)\right)$ se factorise par $\Lambda^3\left(\Omega_Y^2\right) \xrightarrow{\sigma} \mathcal{O}_Y$, où σ est l'équation du diviseur de ramification dans Y. Comme $H^3\left(\mathcal{O}_Y\right) = 0$, l'application $\varphi^* : H^3\left(\Lambda^3\left(\Omega_Y^2\right)\right) \to H^3\left(\Lambda^3\left(\Omega_X^2\right)\right)$ est nulle.

Utilisant la définition 1.12.1 de l'accouplement de Yukawa, on obtient donc le Lemme 1.16.

L'accouplement de Yukawa de X est maintenant complètement calculé, et les numéros 1.12 à 1.16 se résument de la façon suivante:

1.17 Proposition: Soit $(w + u_S + u_E) \in \bigoplus_{(r,s)} H^0\left(\Omega_{C_s}\right) \oplus H^1\left(\Omega_S\right)^- \oplus H^1\left(\Omega_E\right) \simeq H^1\left(T_X\right)$ où l'isomorphisme dépend du choix de $\omega_S \in H^{2,0}(S)$, $\omega_E \in H^0\left(\Omega_E\right)$. Alors $\psi\left(w + u_S + u_E\right) = 3\left(\nu'(w^2) \cdot u_S\right) + 3\int_E u_E \cdot \int_S u_S^2$ où

$\nu' : \bigoplus_{(r,s)} H^0\left(\Omega_{C_s}^{\otimes 2}\right) \to H^1\left(\Omega_S\right)^-$ est décrite en 1.14.1 et 1.15.

1.18. Il y a évidemment une analogie entre la forme d'intersection cubique sur $H^1\left(\Omega_X\right)$ (Lemme 1.11) et la forme cubique de la proposition 1.17.

Dans la section suivante on construira le miroir $X_2 = \widetilde{E_2 \times S_2} / \left(\widetilde{j_2, i_2}\right)$ de $X_1 = \widetilde{E_1 \times S_1} / \left(\widetilde{j_1, i_1}\right)$, et on montrera qu'on a: $\dim H^1\left(\Omega_{T_1}\right) = \dim H^1\left(\Omega_{S_2}\right)^-$,
$\dim H^1\left(\Omega_{T_2}\right) = \dim H^1\left(\Omega_{S_1}\right)^-$, et le rang du sous espace de $H^1\left(\Omega_{X_1}\right)$ engendré par les diviseurs exceptionnels de X_1 est égal au rang du sous espace de $H^1\left(\Omega_{X_2}^2\right)$ engendré par la cohomologie des diviseurs exceptionnels de X_2. Cependant la forme d'intersection du Lemme 1.11, calculée sur $H^1\left(\Omega_{X_1}\right)$, n'est pas une spécialisation de l'accouplement de Yukawa sur $H^1\left(\Omega_{X_2}^2\right)$, à cause du terme en d^3 qui est non nul. Dans la 3ème section de cet article, on suggère une interprétation possible de ce défaut, en spéculant sur la formule 0.4.1.

§2 - Miroirs

2.1. On va utiliser dans cette section les résultats de Nikulin ([11], [1], [22]) sur les involutions i sur les surfaces $K3$ S, agissant par -1 sur la deux forme holomorphe ω_S de S. L'involution i agit par une isométrie $H(i)$ sur $H^2(S, \mathbb{Z})$ et $H^2(S, \mathbb{C})$, et par hypothèse la forme ω_S est dans $H^2(S, \mathbb{C})^-$. Par le théorème de l'indice de Hodge appliqué aux surfaces S et $T = S/i$ (T est lisse, projective et $H^{2,0}(T) = 0$), on voit que $H^2(S, \mathbb{Z})^-$ muni de la forme d'intersection de S est de signature $(2, b_2^- - 2)$. Par le théorème de Torelli pour les surfaces K3, et la surjectivité de l'application des périodes, on peut associer à chaque involution $H(i)$ sur $H^2(S, \mathbb{Z})$ satisfaisant cette condition sur la signature de H^{2-}, et définie à conjugaison près par le groupe des isométries de $H^2(S, \mathbb{Z})$ une famille de surfaces $K3$ S_t munies d'une involution i agissant par $H(i)$ sur $H^2(S, \mathbb{Z})$.

2.1.1. On considère en effet $D = \{\omega \in \mathbb{P}(H^2(S, \mathbb{C})^-) / \omega \cdot \omega = 0 \text{ et } \omega \cdot \overline{\omega} > 0\}$

et $U \subset D$, défini par la condition : $\nexists \alpha \in H^2(S,\mathbb{Q})^-, \alpha \neq 0$ et $\alpha \cdot \omega = 0$. Alors U a deux composantes connexes isomorphes et chaque point t de U paramètre une surface $K3$ S_t, projective car $H^2(S_t,\mathbb{Z})^+$ est orthogonal à $H^{2,0}(S_t)$ donc de type (1,1), et contient un élément de self-intersection > 0. Comme S_t est projective, il existe une classe de Kähler entière $c \in H^2(S_t,\mathbb{Z}) \cap H^{1,1}(S_t)$ et comme $\omega \in U, c \in H^2(S_t,\mathbb{Z})^+$. Comme $H(i)$ agit trivialement sur $H^2(S_t,\mathbb{Z})^+$, $H(i)$ préserve c, et est donc une isométrie de structure de Hodge préservant une classe de Kähler. Donc par [27], Th. 11.1, $H(i)$ est induite par une involution i sur S_t.

2.1.2. On travaillera en fait sur D, pour $H(i)$ fixé. Les points de $D\backslash U$ correspondent à des surfaces $K3$ S munies d'une involution i' n'agissant pas comme $H(i)$ sur $H^2(S,\mathbb{Z})$. Par exemple lorsque la courbe de ramification de l'application quotient $S_t \rightarrow S_t/i$ acquiert un nœud, par une dégénération de Lefschetz, on peut construire une résolution simultanée de la famille $(S_{\sqrt{t}})$ mais l'involution agissant sur la fibre centrale n'a plus le même type topologique que i_t. Cela est dû au fait que l'espace total de la résolution simultanée est une petite résolution d'une variété de dimension trois avec un point double, et l'involution agissant sur cette variété de dimension trois ne se prolonge pas à la petite résolution; elle se prolonge en un isomorphisme de l'une des petites résolutions sur l'autre.

Ceci étant dit, on voit que ces différentes familles sont essentiellement caractérisées par la classe de conjugaison de l'involution $H(i)$ et le résultat de Nikulin que l'on utilisera est le suivant (cf [11],[22]):

2.2 Théorème :

i) La classe de conjugaison de $H(i)$ (qui est déterminée par la classe de l'immersion primitive $H^2(S,\mathbb{Z})^- \subset H^2(S,\mathbb{Z})$ ne dépend que de la classe d'isométrie du réseau $H^2(S,\mathbb{Z})^-$ muni de la forme d'intersection induite.

ii) Cette classe d'isométrie est sujette aux conditions:

a) Sign $H^2(S,\mathbb{Z})^- = (2, b^{2-} - 2)$

b) Le conoyau K de l'application $H^2(S,\mathbb{Z})^- \to H^2(S,\mathbb{Z})^{-*}$ donnée par la forme d'intersection est un groupe de 2-torsion.

iii) La classe d'isométrie du réseau $H^2(S,\mathbb{Z})^-$ est déterminée par trois invariants (b_2^-, k, δ) où $b_2^- = \text{rang } H^2(S,\mathbb{Z})^-$, $k = \text{rang}_{\mathbb{Z}/2\mathbb{Z}} K$ et $\delta \in \{0,1\}$ est l'entier déterminé de la façon suivante:

La forme d'intersection sur $H^2(S,\mathbb{Z})^-$ se prolonge uniquement en une forme quadratique à valeurs dans $\mathbb{Q}$ sur $H^2(S,\mathbb{Z})^{-*}$. On peut construire alors une fonction quadratique $\overline{q}$ sur K à valeurs dans $\mathbb{Q}/2\mathbb{Z}$, en posant $\overline{q}(\overline{x}) = q(x) \pmod{2\mathbb{Z}}$. On pose $\delta = 0$ si $\overline{q}$ est à valeurs dans $\mathbb{Z}/2\mathbb{Z}, \delta = 1$ sinon.

2.3. On revient maintenant à la situation géométrique $\varphi : S \to T = S/i$ du §1. On rappelle que N = nombre de composantes de la courbe de ramification et N' = somme des genres des composantes de cette courbe. On commence par calculer les invariants b_2^-, k en fonction des nombres N et N'. On a:

2.4 Lemme:

i) $b_2^- = 12 - N + N'$

ii) $k = 12 - N - N'$

Démonstration:

i) a été montré en 1.6 (compte-tenu de $b_2^- = 22 - b_2^+$).

ii) 2^k est le discriminant de la forme d'intersection induite sur $H^2(S,\mathbb{Z})^-$, donc aussi le discriminant de la forme d'intersection induite sur $H^2(S,\mathbb{Z})^+$.

D'autre part, on a une suite exacte:

$$0 \to H^2(T) \xrightarrow{\varphi^*} H^2(S,\mathbb{Z})^+ \to (\mathbb{Z}/2\mathbb{Z})^{N-1} \to 0 \tag{2.4.1}$$

En effet, reprenant la notation $C = \bigcup_{s=1}^{N} C_s$ pour le lieu de ramification de φ, et notant $U = T\backslash C, V = S\backslash C$, on a le diagramme commutatif suivant :

$$\begin{array}{ccccccc}
H^2(S,\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\varphi_*} & H^2(T,\mathbb{Z}/2\mathbb{Z}) & & & & \\
\downarrow & & \downarrow & & & & \\
H^2(V,\mathbb{Z}/2\mathbb{Z}) & \xrightarrow[\beta]{} & H^2(U,\mathbb{Z}/2\mathbb{Z}) & \rightarrow & H^3(U,\mathbb{Z}/2\mathbb{Z}) & \rightarrow & H^3(V,\mathbb{Z}/2\mathbb{Z}) \\
\downarrow & & \downarrow & & \rightarrow H^3(U,\mathbb{Z}/2\mathbb{Z}) \rightarrow 0 & & \\
\bigoplus_s H^1(C_s,\mathbb{Z}/2\mathbb{Z}) & = & \bigoplus_s H^1(C_s,\mathbb{Z}/2\mathbb{Z}) & & & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

où la ligne du milieu est induite par la suite exacte 1.1.1. On en déduit immédiatement que le conoyau de φ_* est égal au conoyau de β, et comme $H^3(V,\mathbb{Z}/2\mathbb{Z})$ et $H^3(U,\mathbb{Z}/2\mathbb{Z})$ sont de dimension (sur $\mathbb{Z}/2\mathbb{Z}$) $N-1$, on a coker $\varphi_* \simeq (\mathbb{Z}/2\mathbb{Z})^{N-1}$. D'autre part les applications de φ_* de 2.4.2 et φ^* de 2.4.1 sont reliées par la relation suivante:

Considérons le composé

$$\psi : H^2(S,\mathbb{Z}) \underset{\text{Poincare}}{\simeq} H^2(S,\mathbb{Z})^* \rightarrow H^2(S,\mathbb{Z})^{+*} \overset{(\varphi^*)^{\text{dual}}}{\rightarrow} H^2(T)^* \underset{\text{Poincare}}{\simeq} H^2(T)$$

alors la réduction de $\psi \bmod 2$ est égale à φ_*. Comme la flèche $H^2(S,\mathbb{Z})^* \rightarrow H^2(S,\mathbb{Z})^{+*}$ est surjective et que l'on sait que φ^* a un conoyau de 2-torsion on obtient immédiatement coker $\varphi^* \simeq (\mathbb{Z}/2\mathbb{Z})^{N-1}$ et donc 2.4.1. est démontré.

Sur $H^2(T) \subset H^2(S,\mathbb{Z})^+$ la forme d'intersection de S est égale à 2 fois celle de T qui est unimodulaire. Donc son discriminant est égal à $2^{b_2(T)}$. La suite exacte 2.4.1 montre alors que: $\operatorname{discr} H^2(S,\mathbb{Z})^+ = 2^{b_2(T)-2(N-1)}$, ce qui donne $k = 10 + N - N' - 2(N-1) = 12 - N - N'$.

Les entiers N et N' déterminent donc b_2^- et k. Le Lemme suivant permet de construire topologiquement la "mirror symmetry" échangeant les nombres N et N', compte-tenu du théorème de Nikulin:

2.5 Lemme: Soit i une involution sur S agissant par (-1) sur $H^{2,0}(S)$ et d'entiers N, N', δ associés satisfaisant $N > 0, N' > 0$ et $(N, N', \delta) \neq (5,1,0)$. Alors $H^2(S, \mathbb{Z})^-$ contient un plan hyperbolique.

Démonstration:

2.5.1. Supposons d'abord $N' \geq 2$. Soit $\overline{L} \subset H^2(S,\mathbb{Z})^- \otimes \mathbb{Z}/2\mathbb{Z}$ un sous espace isotropique maximal pour la forme d'intersection réduite modulo 2 $\langle \overline{\cdot} \rangle$. Soit $L \subset H^2(S, \mathbb{Z})^-$ l'image réciproque de $\overline{L}$ par l'application de réduction mod 2. La forme $\langle\ \rangle$ restreinte à L est divisible par 2, soit $\langle\ \rangle_{|L} = 2\langle\ \rangle_L$ et $\langle\ \rangle_L$ est unimodulaire. $\langle\ \rangle_L$ étant indéfinie et n'étant pas paire, (on peut toujours le supposer par un choix adéquat de L), $\langle\ \rangle_L$ est diagonale, soit dans une base convenable (e_i), $\langle x, x\rangle_L = x_1^2 + x_2^2 - \sum\limits_{i>2} x_i^2$. Notons que de $k = 12 - N - N' \geq 0$ et $N' \geq 2$ on tire $N \leq 10$ et $b_2^- \geq 4$ ce qui montre bien que $\langle\ \rangle_L$ est indéfinie. On voit alors facilement que l'ensemble $\{x \in L, \langle x, x\rangle_L = 0\}$ engendre modulo 2 l'hyperplan $\{\sum \overline{x_i} = 0 \bmod 2\}$. On en déduit qu'il existe $x \in L$ tel que $\langle x, x\rangle = 0$ et l'image de x dans $H^2(S,\mathbb{Z})^-$ a une réduction mod 2 qui n'est pas dans Ker $\langle \overline{\ } \rangle$ car les éléments satisfaisant cette dernière condition engendrent un espace de codimension $N' \geq 2$ dans $L \otimes \mathbb{Z}/2\mathbb{Z}$. On peut supposer que x est primitif dans L, et alors il existe $y \in H^2(S,\mathbb{Z})^-$ tel que $\langle x, y\rangle = 1$. Comme $\langle y, y\rangle$ est pair x et y engendrent un plan hyperbolique dans $H^2(S, \mathbb{Z})$.

2.5.2 Supposons maintenant $N' = 1$: de $k = 12 - N - N' \geq 0$ on tire $N \leq 11$. Si $N = 11$ on a $k = 0$ et $b_2^- = 2$ et la forme d'intersection sur H^{2-} est paire, unimodulaire et définie positive, ce qui est absurde. Donc $N \leq 10$ et $b_2^- \geq 3$. Reprenons maintenant la construction de 2.5.1: on a $L \subset H^2(S,\mathbb{Z})^-$ avec $\langle\ \rangle_L$ indéfinie de la forme $\langle x, x\rangle_L = x_1^2 + x_2^2 - \sum\limits_{i>2} x_i^2$.

2.5.3 Supposons qu'il existe $x \in L$ primitif tel que $\langle x, x\rangle_L = 0$ et x est divisible par 2 dans $H^2(S,\mathbb{Z})^-$; alors $x = 2y$, et la projection de y dans $H^2(S,\mathbb{Z})^- \otimes \mathbb{Z}/2\mathbb{Z}$ n'est pas dans Ker $\langle \overline{\ } \rangle$ donc il existe $z \in H^2(S,\mathbb{Z})^-$ tel que $\langle z, y\rangle = 1$ et comme en 2.5.1, $H^2(S,\mathbb{Z})^-$ contient un plan hyperbolique.

On sait d'autre part que le noyau de $L \otimes \mathbb{Z}/2\mathbb{Z} \to H^2(S,\mathbb{Z})^- \otimes \mathbb{Z}/2\mathbb{Z}$

est de rang 1, représenté par un élément $x = \sum x_i e_i \in L$, avec $x_i = 0$ ou 1. Comme $x = 2y, y \in H^2(S,\mathbb{Z})^-$ on a: $\langle x,x\rangle = 4\langle y,y\rangle = 2\langle x,x\rangle_L$, et comme $\langle y,y\rangle$ est divisible par 2, on a $\langle x,x\rangle_L$ divisible par 4. Si $\langle x,x\rangle_L = 0$, on a fini par ce qui précède. Comme $b_2^- = 13 - N \leq 12$ on voit que $\langle x,x\rangle_L = -4$ ou -8 sont les possibilités restantes. Si $\langle x,x\rangle_L = -8$, et $x_1 = x_2 = 0$ on prend $x' = x + 2e_1 + 2e_2$ et alors $\langle x',x'\rangle_L = 0$.

Si $\langle x,x\rangle_L = -8$ et $x_1 = 1$ on prend $x' = x + e_1$ et alors $\langle x',x'\rangle_L = 0$. De même si $x_2 = 1, x_1 = 0$. Dans tous les cas on est ramené à la situation 2.5.3 et donc $H^2(S,\mathbb{Z})^-$ contient un plan hyperbolique.

Supposons donc $\langle x,x\rangle_L = -4$. Si l'une des coordonnées x_1, x_2 est nulle, soit x_1, il suffit de prendre $x' = x + 2e_1$, et alors $\langle x',x'\rangle_L = 0$. Si $x_1 = x_2 = 1$, et l'une des coordonnées $x_i, i > 2$ de x s'annule, soit x_3, il suffit de prendre $x' = x + 2e_1 + 2e_3$ pour obtenir $\langle x',x'\rangle_L = 0$.

En conclusion, si 2.5.3 n'est pas satisfait, on doit avoir: toutes les coordonnées de x sont égales à 1. Comme $\langle x,x\rangle_L = -4$ on a donc rang $L = 8$, soit encore $12 - N + N' = 8$ et $N = 5$. De plus on a:

2.5.4. $H^2(S,\mathbb{Z})^-$ est engendré par $e_1 \cdots e_8$ et $y = \frac{1}{2}\sum_1^8 e_i$, la forme d'intersection sur $H^2(S,\mathbb{Z})^-$ étant donnée par $\langle x,x\rangle = 2\left(x_1^2 + x_2^2 - \sum_{i>2} x_i^2\right)$ dans la base (sur $\mathbb{Q}$) de $H^2(S,\mathbb{Q})^-$ donnée par les e_i.

Le dual $H^2(S,\mathbb{Z})^{-*}$ est engendré par les $\sum x_i e_1^*$ tels que $\sum x_i = 0 \bmod 2$, et l'application ψ donnée par $\langle\ \rangle$: ψ : $H^2(S,\mathbb{Z})^- \to H^2(S,\mathbb{Z})^{-*}$ est determinée par les conditions:

$\psi(e_i) = 2e_i^*$, pour $i = 1,2; \psi(e_i) = -2e_i^*$ pour $i > 2$. La forme $\langle\ \rangle_*$ obtenue par prolongement de $\langle\ \rangle$ à $H^2(S,\mathbb{Z})^{-*}$ est donc telle que $\langle \sum x_i e_i^*, \sum x_i e_i^*\rangle_* = \frac{1}{4}\left\langle \sum_i \varepsilon_i x_i e_i, \sum_i \varepsilon_i x_i e_i\right\rangle$ où $\varepsilon_i = 1,\ i \leq 2,\ \varepsilon_i = -1,$

$i > 2$. Ce qui donne encore $\langle \sum x_i e_i^*, \sum x_i e_i^* \rangle_* = \frac{1}{2}\left\langle x_1^2 + x_2^2 - \sum_{i>2} x_i^2 \right\rangle$. Mais $\sum x_i e_i^* \in H^2(S,\mathbb{Z})^- \iff \sum x_i \equiv 0 \bmod 2 \iff x_1^2 + x_2^2 - \sum_{i>2} x_i^2 \equiv 0 \bmod 2$. Donc la fonction $\langle x, x \rangle_*$ est à valeurs dans $\mathbb{Z}/2\mathbb{Z}$ et δ est nul.

Le cas peut être exceptionnel 2.5.4 est donc caractérisé par les invariants $N = 5, N' = 1, \delta = 0$ et le lemme est démontré.

2.6. Construction de la "mirror symmetry" topologique:

Supposons que $H^2(S,\mathbb{Z})^-$ contienne un plan hyperbolique P. Soit r_P la réflexion par rapport à P, définie sur $H^2(S,\mathbb{Z})$ par la condition $r_{P|P} = \mathrm{Id}_P, r_{P|P^\perp} = -\mathrm{Id}_{p^\perp}$. Considérons l'involution $i' = r_P \circ H(i)$. Pour i', on a $H^2(S,\mathbb{Z})^{+(i)'} = H^2(S,\mathbb{Z})^- \cap P^\perp$. Donc $H^2(S,\mathbb{Z})^{-(i)'}$ a la signature correcte $(2, b^{2-(i')} - 2)$. De plus, on a clairement $k' = k$, $\delta' = \delta$, où k' et δ' sont les invariants de 2.2 iii associés à i', puisque k' et δ' sont determinés par la forme discriminante $\overline{q}$ de 2.2 iii, qui est la même pour $H^2(S,\mathbb{Z})^-$ et $H^2(S,\mathbb{Z})^- \cap P^\perp$, donc aussi pour $H^2(S,\mathbb{Z})^-$ et $H^2(S,\mathbb{Z})^{-(i')} = (H^2(S,\mathbb{Z})^- \cap P^\perp)^\perp$.

Ecrivant $i' = H(i_1)$ pour une involution i_1 agissant sur une surface $K3$ S_1 (cf 2.1), $H(i_1)$ agissant sur (-1) sur $H^{2,0}(S_1)$, on a les entiers N_1 et N_1' définis comme en 2.3, pour (S_1, i_1) et par le Lemme 2.4 appliqué à S_1 et S, on a:

2.6.1:

$$12 - N_1 + N_1' = b_2^-(S_1) = b_2^+(S) + 2 = 12 + N - N'$$
$$k(S_1) = k(S) = 12 - N_1 - N_1' = 12 - N - N'$$

On en déduit immédiatement: $N_1 = N', N_1' = N$.

On appellera $\left(H^2(S_1,\mathbb{Z}), H(i_1)\right)$ (ou par abus (S_1, i_1)) le miroir topologique de $(H^2(S,\mathbb{Z}), H(i))$ (ou par abus (S,i)): le théorème de Nikulin justifie ceci puisqu'il montre que la classe de conjugaison de $H(i_1)$ ne dépend pas du choix du plan P. Remarquons que l'opération $(S,i) \to (S_1, i_1)$ est clairement involutive.

2.7. Cette construction montre aussi que le cas $(N, N', \delta) = (5, 1, 0)$ est bien une exception au Lemme 2.5, compte-tenu de la classification complète des invariants (N, N', δ) possibles données par Nikulin ([22]). En effet, d'une part il existe bien une involution d'invariants (5, 1, 0) associés, d'autre part, il n'existe pas d'involution d'invariants (1, 5, 0), donc, par la construction 2.6, pour l'involution d'invariants associés (5.1.0), $H^2(S, \mathbb{Z})^-$ ne contient pas de plan hyperbolique.

2.8. Soit maintenant donnés $\left(H^2(S_1, \mathbb{Z}), H(i_1)\right)$ et son miroir topologique $\left(H^2(S_2, \mathbb{Z}), H(i_2)\right)$, où $H(i_1)$ et $H(i_2)$ satisfont la condition 2.2 ii) a): On va construire précisément un isomorphisme entre le domaine D_1 des périodes marquées pour S_1, et l'ensemble D_2' des formes $\eta \in H^2(S_2, \mathbb{C})^+$ telles que $(\operatorname{Re} \eta)^2 > 0$.

2.9. Ce qu'on appelle domaine des périodes marquées pour (S_1, i_1) est l'ensemble $D_1 = \{\omega \in \mathbb{P}(H^2(S_1, \mathbb{C})^-)/\omega^2 = 0, \omega\overline{\omega} > 0\}$. Un point général de D_1 détermine une structure complexe sur S_1, avec une involution i_1 agissant comme $H(i_1)$ sur $H^2(S_1, \mathbb{Z})$ (cf 2.1.1), ainsi qu'un marquage de $\left(H^2(S_1, \mathbb{Z}), H(i_1)\right)$, mais d'après 2.1.2 ces données différent en général. Ceci étant noté, la construction est la suivante:

2.10. Le plan hyperbolique $P \subset H^2(S_1)^-$ est fixé. P possède deux éléments $\alpha, \beta \in P_{\mathbb{Z}}$ bien définis au signe près, tels que $\alpha^2 = 2, \beta^2 = -2$, et $\alpha\beta = 0$. Soit $\omega \in D_1$(ω définie à un coefficient près); alors $\omega \notin P \otimes \mathbb{C}$ car sinon on aurait : $P \otimes \mathbb{R} = \langle \operatorname{Re} \omega, \operatorname{Im} \omega \rangle \subset H^2(S_1, \mathbb{R})^-$, ce qui contredit le fait que la forme d'intersection sur $P \otimes \mathbb{R}$ est indéfinie, tandis que celle de $\langle \operatorname{Re} \omega, \operatorname{Im} \omega \rangle$ est définie positive. Notons $Q_\omega \subset H^2(S_1)^-$ le sous-espace complexe de dimension trois engendré sur $\mathbb{C}$ par P et ω.

Comme la forme d'intersection de $H^2(S_1, \mathbb{C})^-$ a une restriction non dégénérée sur P, Q_ω se scinde en la somme directe orthogonale: $Q_\omega = P \perp \Delta$, où Δ est la droite complexe $Q_\omega \cap P^\perp$. On a alors:

2.11 Lemme: La forme d'intersection de $H^2(S_1, \mathbb{C})^-$, restreinte à Q_ω, est non dégénérée.

Démonstration: Soit $u \in P \otimes \mathbb{C}$, tel que $\omega + u$ engendre Δ. Il suffit de montrer que $(\omega + u)^2 \neq 0$.

Comme $\omega + u$ est orthogonal à P, on a: $\forall v \in P, (\omega + u) \cdot v = 0 = \omega v + uv$. En particulier $(\omega + u)^2 = \omega^2 + u^2 + 2\omega u = -u^2$, puisque $\omega^2 = 0$.

Si $(\omega + u)^2 = 0, u^2 = 0$, et donc $u = \lambda u'$, où $\lambda \in \mathbb{C}$ et $u' \in P \otimes \mathbb{R}$. On a $u \neq 0$, car sinon ω serait orthogonale à P; ceci entraînerait que $\mathrm{Re}\,\omega$ et $\mathrm{Im}\,\omega$ sont orthogonales à P, et contredirait le fait que la forme d'intersection de $H^2(S_1, \mathbb{R})^-$, restreinte à $P^\perp$ est la signature $(1, b_2^- - 3)$, tandis qu'elle est définie positive sur $\langle \mathrm{Re}\,\omega, \mathrm{Im}\,\omega \rangle$.

Maintenant, de $\omega u + u^2 = 0, u^2 = 0$, on tire $\omega u' = 0$, avec u' réel $\neq 0$, d'où $Re\omega \cdot u' = 0 = \mathrm{Im}\,\omega . u' = 0$, ce qui contredit le fait que $u'^2 = 0$, tandis que $\langle \ \rangle$ est définie négative sur l'orthogonal de $\langle \mathrm{Re}\,\omega, \mathrm{Im}\,\omega \rangle$ dans $H^2(S_1)^-$. L'hypothèse $(\omega + u)^2 = 0$ est donc absurde.

2.12. Soit $\alpha' = \frac{1}{\sqrt{2}}\alpha, \beta' = \frac{1}{\sqrt{2}}\beta$, et $\chi \in \Delta$ tel que $\chi^2 = 1$, où α, β sont définis au signe près en 1.10 et χ est défini au signe près.

(α', β', χ) fournissent une base de Q_ω, et on peut écrire: $\omega = \lambda\alpha' + \mu\beta' + \nu\chi$, où (λ, μ, ν) sont des coordonnées homogènes, pour $\mathbb{P}(Q_\omega)$. On a $\nu \neq 0$ puisque $\omega \neq P$ (cf 2.20). On a alors:

2.13. Lemme: Soit $\omega \in \mathbb{P}(Q_\omega)$ telle que $\omega^2 = 0$, soit $\eta = i\left(\frac{\lambda - \mu}{\nu}\right)\chi \in \Delta$. Alors on a l'équivalence des deux conditions suivantes:

i) $\omega\overline{\omega} > 0$

ii) $(Re\eta)^2 > 0$

Démonstration: Comme α' et β' sont réels, on a $\overline{w} = \overline{\lambda}\alpha' + \overline{\mu}\,\beta' + \overline{\nu}\,\overline{\chi}$. Comme χ est orthogonal à P, qui est réel, $\overline{\chi}$ est orthogonal à P, et l'on a:

2.13.1.

$$\omega\overline{\omega} = \lambda\overline{\lambda} - \mu\overline{\mu} + \nu\overline{\nu}\chi\overline{\chi}$$

$$\omega^2 = \lambda^2 - \mu^2 + \nu^2 = 0$$

Quitte à remplacer ω par $\frac{\omega}{\nu}$ et λ par $\lambda' = \frac{\lambda}{\nu}$, μ par $\mu' = \frac{\mu}{\nu}$, on peut supposer $\nu = 1$.

On a alors $\eta = i(\lambda' - \mu')\chi$. 2.13.1 devient

2.13.2.

i)$\omega\overline{\omega} = \lambda'\overline{\lambda}' - \mu'\overline{\mu}' + \chi\overline{\chi}$

ii)$\omega^2 = \lambda'^2 - \mu'^2 + 1 = 0$

Maintenant $\mathrm{Re}\,\eta = \frac{1}{2}(\eta + \overline{\eta}) = \frac{1}{2}\left(i(\lambda' - \mu')\chi - i(\overline{\lambda}' - \overline{\mu}')\overline{\chi}\right)$ d'où $(\mathrm{Re}\,\eta)^2 = \frac{1}{4}(-(\lambda' - \mu')^2 - (\overline{\lambda}' - \overline{\mu}')^2 + 2(\lambda' - \mu')(\overline{\lambda}' - \overline{\mu}')\chi\overline{\chi})$. D'après 2.13. 2 ii) on a $(\lambda' - \mu')(\overline{\lambda}' - \overline{\mu}') > 0$ et il vient: $2(\mathrm{Re}\,\eta)^2/(\lambda' - \mu')(\overline{\lambda}' - \overline{\mu}') = -\frac{1}{2}\left(\frac{\lambda' - \mu'}{\overline{\lambda}' - \overline{\mu}'} + \frac{\overline{\lambda}' - \overline{\mu}'}{\lambda' - \mu'}\right) + \chi\overline{\chi}$. Utilisant 2.13.2 ii) sous la forme $\overline{\lambda}' + \overline{\mu}' = \frac{-1}{\overline{\lambda}' - \overline{\mu}'}$ il vient : $2(\mathrm{Re}\,\eta)^2/(\lambda' - \mu')(\overline{\lambda}' - \overline{\mu}') = \mathrm{Re}((\lambda' - \mu')(\overline{\lambda}' + \overline{\mu}')) + \chi\overline{\chi} = \lambda'\overline{\lambda}' - \mu'\overline{\mu}' + \chi\overline{\chi} = \omega\overline{\omega}$. Le Lemme est donc prouvé.

2.14. Notons finalement que $\eta \in \Delta \subset H^2(S_1, \mathbb{C})^- \cap P^\perp = H^2(S_2, \mathbb{C})^+$ ne dépend pas du choix de χ à ± 1 près, et ne dépend que du point projectif défini par ω dans $\mathbb{P}\left(H^2(S_1, \mathbb{C})^-\right)$. On a donc bien construit une application M_1, clairement holomorphe: $M_1 : D_1 \to D_2'$. Il reste à voir que M_1 est un isomorphisme. On a d'abord:

2.15 Lemme: Soit $\eta \in D_2'$; alors $\eta^2 \neq 0$.

Démonstration: Ecrivons $\eta = \mathrm{Re}\,\eta + i\,\mathrm{Im}\,\eta$; alors $\eta^2 = 0$ entraîne $\mathrm{Re}\,\eta \cdot \mathrm{Im}\,\eta = 0$, et $(\mathrm{Re}\,\eta)^2 = (\mathrm{Im}\,\eta)^2$. Comme $(\mathrm{Re}\,\eta)^2 > 0$, si $\eta^2 = 0, H^2(S_2, \mathbb{R})^+$ contient un plan réel sur lequel la forme d'intersection est définie positive, ce qui contredit le fait que la signature de $H^2(S_2, \mathbb{R})^+$ est $(1, b_2^+(S_2) - 1)$. Par l'isomorphisme $H^2(S_2, \mathbb{R})^+ \simeq H^2(S_1, \mathbb{R})^- \cap P^\perp$, voyons $\eta \in D_2'$ comme un élément de $H^2(S_1, \mathbb{R})^-$, orthogonal à P. Soit Q_η l'espace de dimension 3 complexe engendré par P et η. Ecrivons enfin $\eta = \varepsilon\chi$ avec $\varepsilon \in \mathbb{C}^*$, et $\chi^2 = 1$ (Lemme 2.15); Le Lemme suivant est évident:

2.16 Lemme: Pour $\varepsilon \neq 0$ fixé, il existe un unique couple (λ, μ) de complexes tels que $\lambda^2 - \mu^2 + 1 = 0$ et $i(\lambda - \mu) = \varepsilon$.

Posant alors $\omega = \lambda\alpha' + \mu\beta' + \chi$, on note que le point projectif défini par ω dans $\mathbb{P}(Q_\eta)$ ne dépend pas du choix de χ, et satisfait $\omega^2 = 0$, et $\omega\overline{\omega} > 0$, par le Lemme 2.13. On a donc construit $M_1^{-1} : D_2' \to D_1$.

Echangeant S_1 et S_2, on dispose de $M_2 : D_2 \to D_1'$ et $M_2^{-1} : D_1' \to D_2$. Le couple $\left(M_1, M_2^{-1}\right) : D_1 \times D_1' \to D_2' \times D_2$ fournit une application "miroir" holomorphe et bijective. Ces résultats se résument de la façon suivante:

Théorème: Soit S_1 une surface $K3$ munie d'une involution i_1 telle que $H(i_1)$ agit par (-1) sur $H^{2,0}(S_1)$. Supposons que les invariants (N_1, N_1', δ) associés satisfont $N_1' > 0$ et $(N_1, N_1', \delta_1) \neq (5, 1, 0)$. Fixons un plan P hyperbolique dans $H^2(S_1, \mathbb{Z})^-$. Alors il existe un miroir $\left(H^2(S_2, \mathbb{Z}), H(i_2)\right)$ topologique de $\left(H^2(S_1, \mathbb{Z}), H(i_1)\right)$, qui ne dépend pas du choix de P, et satisfait $N_2 = N_1', N_2' = N_1, \delta_2 = \delta_1$, et un isomorphisme de domaines de périodes marquées: $\left(M_1, M_2^{-1}\right) : D_1 \times D_1' \to D_2' \times D_2$ (qui dépend de P).

2.18 Remarques:

a) Une direction possible d'investigations sur la structure de l'application miroir serait l'analyse de la compatibilité avec l'action des groupes orthogonaux, de manière à supprimer autant que possible le marquage.

b) Il y a évidemment d'autres choix possibles pour M_1: pour tout réel $\lambda \neq 0$, on pourrait aussi bien définir $M_1^\lambda(w) = \lambda M_1(w)$. Peut-être l'étude de a) fournirait elle des raisons pour privilégier l'une de ces applications.

c) Comme noté en 2.1.2 il existe un ouvert U_1' de D_1, avec $U_1 \subset U_1' \subset D_1$ (où U_1 est défini en 2.1.1) sur lequel la structure complexe sur S_1 paramétrée par $t \in U_1'$ est effectivement compatible avec une involution i_1 sur S_1, agissant comme $H(i_1)$ sur $H^2(S_1, \mathbb{Z})$. Il serait intéressant de décrire précisément U_1' et son image $M_1(U_1') \subset D_2'$ pour comprendre le domaine de définition de l'application miroir géométrique (0.3, 2.2.1); il y a en effet une incertitude sur le domaine de définition de la forme α de 0.2, pour laquelle le miroir (X', α')

serait défini; la condition "Re α Kähler" est probablement trop contraignante; cependant la condition $(\mathrm{Re}\,\eta)^2 > 0$ de 2.8 est trop faible.

(Cette dernière remarque suppose la lecture des numéros suivants, où l'on contruit à l'aide de 2.17 la "mirror symmetry" pour les variétés X de la section 1).

2.19. On revient maintenant aux variétés de Calabi–Yau $X_1 = \widetilde{E_1 \times S_1}/\left(\widetilde{j_1, i_1}\right)$ construites dans la section 1. On suppose que le miroir topologique de (S_1, i_1) existe. On supposera donné un marquage de

2.19.1 $$H^3(X_1, \mathbb{Z}) \subset \bigoplus_{(r,s)} H^1(C_s, \mathbb{Z}) \oplus H^1(E_1) \oplus H^2(S_1, \mathbb{Z})^-,$$

incluant un marquage de chacun des termes intervenant dans cette décomposition. L'inclusion de 2.19.1 induit le $\mathbb{Q}$ isomorphisme de 1.5 : $H^3(X, \mathbb{Q}) \simeq H^3(\widetilde{E \times S}, \mathbb{Q})^{\mathrm{inv}}$. De même on se donne un marquage de

$$H^2(X_1, \mathbb{Z}) \subset H^2(E_1, \mathbb{Z}) \oplus H^2(S_1, \mathbb{Z})^+ \bigoplus_{(r,s)} \langle D_{r,s} \rangle .$$

2.20. L'application miroir a été construite explicitement pour les courbes elliptiques (cf. [10], [2]). Cette application que nous noterons $\left(m_1, m_2^{-1}\right)$ pour la distinguer de la précédente, est l'involution sur l'espace paramétrant la donnée d'une courbe elliptique $E \simeq \mathbb{C}/\Gamma$, où $\Gamma = \langle e_1, e_2 \rangle$ est un réseau d'orientation positive dans $\mathbb{C}$, et d'une classe de type (1,1) α_E telle que $\mathrm{Re}\left(\int_E \alpha_E\right) > 0$, simplement définie de la façon suivante: soit $e_2 = \tau e_1, \tau \in \mathbb{C}$; alors $\mathrm{Im}\,\tau > 0$ et j détermine E. D'autre part soit $\lambda = \int_E \alpha_E$. Alors $\mathrm{Re}\,\lambda > 0$ et λ détermine α_E. On associe alors à (E_1, α_{E_1}) le couple (α_{E_2}, E_2) , où E_2 est définie par: $\tau_2 = i\lambda_1$ et α_{E_2} est définie par: $\lambda_2 = \int_E \alpha_{E_2} = -i\tau_1$. On a utilisé la notation $\left(m_1, m_2^{-1}\right)$ par analogie avec l'application M, et pour souligner le fait que l'application miroir préserve la structure de produit de l'espace considéré ci-dessus en échangeant les facteurs. Ici $m_1(\tau_1) = \lambda_2 = -i\tau_1$, et $m_2 = m_1$.

2.21. On effectue maintenant la synthèse du théorème 2.17, et de la construction 2.20.

Donnons nous une deux forme sur X_1 de la forme $\alpha_{E_1} + \alpha_{S_1}$ où $\mathrm{Re}\int_E \alpha_{E_1} > 0$ et $(\mathrm{Re}\,\alpha_{S_1})^2 > 0$ (plus précisément $\alpha_{S_1} \in M_2(U_2')$ où U_2' est définie dans la remarque 2.1.8.c). Alors via m_2^{-1} α_{E_1} détermine une courbe elliptique marquée E_2 , et via M_2^{-1} α_{S_1} détermine une surface $K3$ marquée S_2 munie d'une involution i_2 agissant comme $H(i_2)$ sur $H^2(S_2, \mathbb{Z})$ qui est marqué. De même pour une structure complexe sur X_1, on a la droite projective $H^{3,0}(X_1) \subset H^3(X_1, \mathbb{C})$ qui détermine, grâce au marquage, les droites projectives $H^{1,0}(E_1) \subset H^1(E_1, \mathbb{C})$, et $H^{2,0}(S_1) \subset H^2(S_1, \mathbb{C})^-$. Via m_1 et M_1 on alors des formes $\alpha_{E_2} \in H^1(\Omega_{E_2})$, et $\alpha_{S_2} \in H^1(\Omega_{S_2})^+$, avec $\mathrm{Re}\int_{E_2} \alpha_{E_2} > 0, (\mathrm{Re}\,\alpha_{S_2})^2 > 0$ d'où une deux forme $\alpha_{E_2} + \alpha_{S_2}$ sur X_2. On définit le miroir de $(X_1, \alpha_{E_1}, \alpha_{S_1})$ comme étant $(X_2, \alpha_{E_2}, \alpha_{S_2})$ avec $X_2 = E_2 \widetilde{\times} S_2/(j_2, i_2)$ où j_2 est l'involution $(-1)_{E_2} \simeq \mathbb{C}/\Gamma_2$.

2.22. Comme on a $N_2 = N_1'$ et $N_2' = N_1$, le lemme 1.8 montre que X_1 et X_2 satisfont $b_2(X_1) = h^{2,1}(X_2)$ et $b_2(X_2) = h^{2,1}(X_1)$. L'application miroir que l'on a construite ici est donc parfaitement conforme à la "mirror symmetry" prédite par les physiciens (0.3). On peut noter cependant que l'on n'a construit ici l'application miroir que sur un sous-espace $H_1 \times K_1$ du produit { structure complexe marque sur X_1} × { formes de type (1, 1) sur X_1 satisfaisant certaines conditions de positivité }, H_1 étant l'ensemble des structures complexes pour lesquelles X_1 est du type $E_1 \widetilde{\times} S_1/(j_1, i_1)$ et K_1 correspondant à l'ensemble des formes sur X_1 de type $\alpha_{E_1} + \alpha_{S_1}$. Cete application est à valeurs dans les sous-espace $H_2 \times K_2$ correspondant pour X_2.

2.23. Si l'on tient pour vraie la conjecture sur l'existence de la "mirror symmetry", on peut imaginer que l'on a construit ici l'application miroir des physiciens, restreinte à un sous-espace du type $H_1 \times K_1'$, mais il est possible que K_1' ne soit pas l'espace K_1 considéré ci-dessus mais par exemple un translaté de K_2 par une constante $c(X_1) \in H^1(\Omega_{X_1})$ supportée par les diviseurs exceptionnels de X_1. On propose même dans la section suivante une valeur de $c(X_1)$ qui rendrait valide les prédictions des physiciens concernant la comparaison des accouplements de Yukawa et la forme d'intersection, en prenant la limite de 0.4.1 de façon adéquate. Mais ceci est très spéculatif et il est peut-être préférable de considérer le problème suivant comme ouvert:

2.24. Problème : A quel sous-espace de $H^1(\Omega_{X_1})$, naturellement isomorphe à K_1, correspond (via le miroir des physiciens supposé égal au notre) le sous-espace de l'espace des structure complexes sur X_2 constitué des variété du type $\widetilde{E_2 \times S_2}/(j_2, i_2)$?

§3 Accouplements de Yukawa et formes d'intersection.

3.1. On se propose dans cette section de "tester" la formule 0.4.1 pour l'application miroir construite en 2.17, 2.21, au sens suivant: supposons que la formule 0.4.1 soit bien donnée par une série convergente; supposons aussi que α' satisfasse: $\operatorname{Re}\alpha'$ est une forme de Kähler; alors on doit pouvoir prouver que l'accouplement de Yukawa sur $X_t :=$ miroir de $(X', t\alpha')$, où t est un réel positif tendant vers $+\infty$, correctement normalisé, converge vers une forme cubique isomorphe à la forme d'intersection $v \to \int_{X'} v^3$ sur $H^1(\Omega_{X'})$. (Ce type de calcul a été aussi effectué dans [3] pour certaines variétés du type 0.5).

3.2. On travaillera avec l'hypothèse naïve 2.23. Cela signifie qu'on supposera que le miroir $(X_2, \alpha_{E_2} + \alpha_{S_2})$ de $(X_1, \alpha_{E_1} + \alpha_{S_1})$ construit en 2.21 s'identifie au miroir des physiciens $(X_2, \alpha_2) \to (X_1, \alpha_1)$, à condition de poser $\alpha_2 = \alpha_{E_2} + \alpha_{S_2} + c(X_2)$, $\alpha_1 = \alpha_{E_1} + \alpha_{S_1} + c(X_1)$ où $c(X_2) \in H^1(\Omega_{X_2})$ (resp. $c(X_1) \in H^1(\Omega_{X_1})$) est une constante supportée sur les diviseurs exceptionnels de X_2 (resp. de X_1). A ce moment là on doit avoir la variante suivante de 3.1: Supposons $\alpha_{E_2} > 0, \alpha_{S_2} > 0$; alors pour $t \in \mathbb{R}^+, t \to +\infty, e^{-\int_{\mathbb{P}^1} tf^*(\alpha_{E_2}+\alpha_{S_2})+f^*c(X_2)} \to 0$ dès que $f : \mathbb{P}^1 \to X_2$ n'est pas un revêtement d'une courbe rationnelle $\mathbb{P}^1_{r,s} \subset X_2$, fibre de l'une des applications $\tau_{r,s} : D_{r,s} \to C_{r,s}$ de 1.7. On doit donc pouvoir prouver que l'accouplement de Yukawa sur $X_{1,t} :=$ miroir de $(X_2, t(\alpha_{E_2} + \alpha_{S_2}) + c(X_2))$ converge vers une forme cubique isomorphe à la forme sur $H^1(\Omega_{X_2})$:

$$(3.2.1)\quad v \mapsto \int_{X_2} v^3 + \sum_{f_{k,r,s}:\mathbb{P}^1 \to \mathbb{P}^1_{r,s}} e^{-\int_{\mathbb{P}^1} f^*_{k,r,s}(c(X_2))} \left(\int_{\mathbb{P}^1} f^*_{k,r,s} v\right)^3 \times n(f_{k,r,s})$$

où $f_{k,r,s}$ est la composante de l'ensemble des applications holomorphes de $\mathbb{P}^1$ dans X_2 donnée par les revêtements ramifiés de degrés k de $\mathbb{P}^1_{r,s}$. Evidemment il faut pouvoir donner un sens à la somme de cette série. On peut utiliser pour cela le résultat de Aspinwall et Morrison [4], et le lemme suivant :

3.3 Lemme: Pour une déformation générique X_2' de X_2, et pour $g(C_{2,s}) \geq 1$, les courbes rationnelles obtenues par déformation de $\mathbb{P}^1_{r,s}$ consistent en exactement $2g(C_{2,s}) - 2$ courbes rationnelles $\mathbb{P}^1 \subset X_2'$, de fibré normal $N_{\mathbb{P}^1}X_2' \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Démonstration: On reprend les notations de 1.16. X_2 est la désingularisation du revêtement double de $\mathbb{P}^1 \times T_2$, ramifié le long de la surface d'équation $A_2F_2 = 0$ où $A_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ et $F_2 \in H^0(-2K_{T_2})$. Soit $C_2 = \bigcup_{s'} C_{2,s'}$ la courbe d'équation $F_2 = 0$ dans T_2. $C_{2,s}$ est une composante de C_2. L'application composée: $H^0(-K_{T_2}) \to H^0(\Omega_{C_2}) \to H^0(\Omega_{C_{2,s}})$ est surjective, et il existe donc $G_2 \in H^0(-K_{T_2})$, telle que pour chaque s tel que $g(C_{2,s}) \geq 1$ la restriction de G_2 à $C_{2,s}$ ait exactement $2g(C_{2,s}) - 2$ zéros. Soit $B_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ générique. On vérifie facilement que pour $t \in \mathbb{C}$ assez petit, la surface d'équation $A_2F_2 + tB_2G_2^2$ est singulière seulement le long de $A_2 = F_2 = G_2 = 0$ avec des singularités quadratiques ordinaires de rang 3 (nœuds) aux points de $p_{2,r} \times C_{2,s} \cap \{G_2 = 0\}$ pour $g(C_{2,s}) \geq 1$, et avec des singularités quadratiques ordinaires de rang 2 le long des courbes $p_r \times C_{2,s}$, avec $g(C_{2,s}) = 0$. On pose $t = v^2$ et on considère la variété de dimension quatre W obtenue comme le revêtement double de $\mathbb{P}^1 \times T \times \Delta$ (Δ un petit disque, muni de la coordonnée v) ramifié le long de l'hypersurface d'équation $A_2F_2 + v^2B_2G_2^2$. Il n'est pas trop difficile de montrer que W admet une désingularisation $\widetilde{W}$, satisfaisant:

i) les fibres de l'application naturelle $\widetilde{W} \xrightarrow{\pi} \Delta$ donnée par la coordonnée v sont lisses;

ii) pour $v \neq 0$ $\pi^{-1}(v)$ est une désingularisation du revêtement double de $\mathbb{P}^1 \times T$ ramifié le long de la surface d'equation $A_2F_2 + v^2B_2G_2^2$, obtenue par éclatement des courbes singulières $p_r \times C_{2,s}$ avec $C_{2,s}$ rationnelle et par petite résolution des nœuds au dessus des points $p_r \times C_{2,s} \cap \{G_2 = 0\}$ pour $g(C_{2,s}) \geq 1$.

iii) $\pi^{-1}(0)$ est isomorphe à X_2.

On a donc montré que dans $\pi^{-1}(v)$ les courbes $\mathbb{P}^1_{r,s}$ sont remplacées par les $2g(C_{2,s})-2$ courbes exceptionnelles de la petite résolution de W_v, pour $g(C_{2,s})\geq 1$.

3.4. Par le Lemme 3.3, et par le résultat de Morrison et Aspinwall, on peut faire alors $n(f_{k,r,s}) = 2g(C_{2,s})-2$ pour tout $k\geq 1$ et $2g(C_{2,s})\geq 1$, dans la formule 3.2.1. On admettra que ceci reste vrai pour $g(C_{2,s})=0$.

On peut alors pour les valeurs adéquates de $c(X_2)$ assurant la convergence de la série, remplacer 3.2.1 par :

$$\psi'_2 : v \mapsto \int_{X_2} v^3 \tag{3.4.2}$$
$$+\sum_{r,s}(2g(C_{2,s})-2)\, e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)} \Big/ \left(1-e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)}\right)\left(\int_{\mathbb{P}^1_{r,s}} v\right)^3$$

3.5. La forme ψ'_2 est facile à décrire dans la base $\bigoplus_{(r,s)}\langle D_{2,r,s}\rangle\,\mathbb{Q}\oplus H^2(T_2,\mathbb{Q})\oplus H^2(E_2,\mathbb{Q})$ de $H^2(X_2,\mathbb{Q})$ compte-tenu de $0=\int_{\mathbb{P}^1_{r,s}}\alpha=\int_{\mathbb{P}^1_{r,s}}\beta$, pour $\alpha\in H^2(T_2,\mathbb{Q})$, $\beta\in H^2(E_2,\mathbb{Q})$, et $\int_{\mathbb{P}^1_{r,s}}[D_{2,r',s'}]=0$, pour $(r',s')\neq(r,s)$, $\int_{\mathbb{P}^1_{r,s}}[D_{2,r,s}]=-2$. Du Lemme 1.11, on tire immédiatement :

(3.5.1)
$$\psi'_2(d+\alpha+\beta)=$$
$$\sum\nolimits_{r,s} d^3_{r,s}\left[(D_{2,r,s})^3_X - 8\,(2g(C_{2,s})-2)\,e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)} \Big/ \left(1-e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)}\right)\right]$$
$$+3(d^2\alpha)_X+3(\alpha^2\beta)_X$$

où $d=\sum_{r,s} d_{r,s}D_{2,r,s}$.

3.6. D'après 1.11, on a $D^3_{2,r,s}=8-8g(C_{2,s})$, et l'on voit donc que le premier terme s'annule pour $e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)} / \left(1 - e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)}\right) = -\frac{1}{2}$, c'est-à-dire $e^{-\int_{\mathbb{P}^1_{r,s}} c(X_2)} = -1$, soit $\int_{\mathbb{P}^1_{r,s}} c(X_2) = i\pi$, ou encore: $c(X_2) = -i\pi/2 \sum_{r,s} [D_{2,r,s}]$.

Pour cette valeur de la constante $c(X_2)$ l'accouplement ψ_2' prend la forme simplifiée :

3.6.1 $\psi_2'(d + \alpha + \beta) = 3\left(d^2\alpha\right)_{X_2} + 3\left(d^2\beta\right)_{X_2}$.

Or d'après la proposition 1.17 c'est précisément la forme générale de l'accouplement de Yukawa ψ pour X_1, puisque l'on a dans la base $\bigoplus_{(r,s')} H^0\left(\Omega_{C_1,s'}\right) \oplus H^1\left(\Omega_{S_1}\right)^- \oplus H^1\left(\Omega_{E_1}\right)$ de $H^1\left(T_{X_1}\right) : \psi\left(w + u_{S_1} + u_{E_1}\right) = 3\left(\nu'(w^2)u_{S_1}\right) + 3\int_{E_1} u_{E_1} \cdot \int_{S_1} u_{S_1}^2$, et que :

$$\operatorname{rang} \bigoplus_{(r,s)} \langle D_{2,r,s} \rangle \cdot \mathbb{C} = 4N_2 = 4N_1' = \operatorname{rang} \bigoplus_{(r,s')} \left(\Omega_{C_1,s'}\right)$$
$$\operatorname{rang} H^2\left(T_2, \mathbb{C}\right) = 10 + N_2 - N_2' = \operatorname{rang} H^1\left(\Omega_{S_1}\right)^-.$$

On continue désormais en supposant dans l'hypothèse 2.23 que $c(X_2) = -i\pi/2 \sum_{r,s} D_{2,r,s}$.

3.7. Il reste maintenant à raffiner 3.6, en montrant que les accouplements ψ_2' et ψ deviennent bien isomorphes "à la limite", c'est-à-dire lorsque la structure complexe sur X_1 dégénère, ou lorsque la forme a_{S_2} de X_2 tend vers l'infini.

On reprend d'abord les notations de §2, et on étudie la différentielle de $M_1 : D_1 \to D_2'$ (cf. 2.14), et sa compatibilité avec les formes d'intersection sur $TD_{1(\omega)} = \omega^\perp / \langle \omega \rangle$ et $TD'_{2(\eta)} = H^1\left(\Omega_{S_2}\right)^+$, où $\eta = M_1(\omega)$. On a:

3.8 Proposition: Soit $\eta = M_1(\omega)$, et pour $t \in \mathbb{R}^+$ soit $\omega_t = M_1^{-1}(t\eta)$. Alors lorsque t tend vers $+\infty$, ω_t converge vers $\alpha' - \beta'$, et convenablement multipliée, la différentielle $\left(dM_1\right)_{\omega_t}$ tend vers un isomorphisme $(a' - \beta')^\perp / \langle \alpha' - \beta' \rangle \simeq$

$H^1(\Omega_{S_2})^+$, compatible avec les formes d'intersection.

Démonstration: On reprend les notations de 2.10. 2.13. On a donc $\omega_t = \lambda'_t\alpha' + \mu'_t\beta' + \chi \in \mathbb{P}(Q_\omega) = \mathbb{P}(Q_\eta)$, et $t\eta = i(\lambda'_t - \mu'_t)\chi$, avec: $\lambda'^2_t - \mu'^2_t + 1 = 0$.

Soit $\eta = \varepsilon\chi$; on a donc: $\lambda'_t - \mu'_t = -it\varepsilon$ et $\lambda'_t + \mu'_t = \frac{1}{it\varepsilon}$, soit $\lambda'_t = \frac{1}{2}\left(\frac{1}{it\varepsilon} - it\varepsilon\right)$ et $\mu'_t = \frac{1}{2}\left(\frac{1}{it\varepsilon} + it\varepsilon\right)$. Lorsque t tend vers $+\infty$, $-\left(\frac{2}{it\varepsilon}\right)\omega_t$ converge donc vers $\alpha' - \beta'$.

On calcule maintenant la différentielle $dM_{1(\omega)} : d\omega \to d\eta$, avec $d\omega \in \omega^\perp/\langle\omega\rangle$, où $\omega^\perp$ est l'orthogonal de ω dans $H^2(S_1, \mathbb{C})^-$. On a $\lambda' = \omega \cdot \alpha', \mu' = -\omega \cdot \beta'$, et $\eta = i(\lambda' - \mu')\chi$. En différentiant ces relations, on obtient:

3.8.1 $d\omega = d\lambda'\alpha' + d\mu'\beta' + d\chi, d\lambda' = d\omega \cdot \alpha', d\mu' = -d\omega \cdot \beta'$, $d\eta = i(d\lambda' - d\mu')\chi + i(\lambda' - \mu')d\chi$; d'où:

3.8.2 $d\eta = i(d\lambda' - d\mu')\chi + i(\lambda' - \mu')(d\omega - d\lambda'\alpha' - d\mu'\beta')$. Soit $u = (\mu' + \lambda')(\alpha' + \beta') + \chi \in \omega^\perp$; alors l'espace engendré par u et $Q_\omega^\perp$, l'orthogonal de Q_ω dans $H^2(S_1, \mathbb{C})^-$, est naturellement isomorphe à $\omega^\perp/\omega$, et l'on a en appliquant 3.8.1 et 3.8.2.

3.8.3 $dM_{1(\omega)}(v) = i(\lambda' - \mu')v$ pour $v \in Q_\omega^\perp$, et

$$dM_{1(w)}(u) = i(\lambda' - \mu')u + i(u \cdot \alpha' + u \cdot \beta')\chi - i(\lambda' - \mu')$$
$$((u \cdot \alpha') \cdot \alpha' - (u \cdot \beta') \cdot \beta') = i(\lambda' - \mu')(u - ((\mu' + \lambda')\alpha' + (\mu' + \lambda')\beta').$$

Ceci fournit encore:

3.8.4 $\frac{1}{i(\lambda' - \mu')}dM_{1(w)}(v) = v$, pour $v \in Q_w^\perp$, et $\frac{1}{i(\lambda' - \mu')}dM_{1(\omega)}(u) = u - (\mu' + \lambda')(\alpha' + \beta')$.

Faisons maintenant $\eta_t = t\eta$ avec $t \to +\infty$, et soit $\omega_t = M_1^{-1}(\eta_t)$.

On a alors $\lambda'_t + \mu'_t = \frac{1}{it\varepsilon} \to 0$, $u_t \to \chi$, et par le calcul précédent $-\left(\frac{2}{it\varepsilon}\right)\omega \to \alpha' - \beta'$.

$\langle u_t, Q_\omega^\perp \rangle$ converge donc vers $\langle \chi, W_\omega^\perp \rangle$, qui est naturellement isomorphe a $\langle \alpha' - \beta' \rangle^\perp /$
$\langle \alpha' - \beta' \rangle$, par l'inclusion $\langle \chi, W_\omega^\perp \rangle \subset \langle \alpha' - \beta' \rangle^\perp$ et par 3.8.4 $\frac{1}{i(\lambda' - \mu')} dM_{1(\omega_t)}$ converge vers l'isomorphisme composé: $\langle \alpha' - \beta' \rangle^\perp / \langle \alpha' - \beta' \rangle \simeq \langle \chi, Q_\omega^\perp \rangle \simeq H^2(S_1, \mathbb{C})^- \cap P^\perp = H^2(S_2, \mathbb{C}^+) \simeq TD_2'$. Il est clair que cet isomorphisme préserve les formes d'intersection sur ces espaces, induites par celle de $H^2(S_1, \mathbb{C})^-$. La proposition 3.8 est donc montrée.

3.9. La proposition 3.8 montre que la différentielle de l'application miroir, restreinte au sous-espace $H^1(T_{E_1}) \oplus H^1(T_{S_1})^+$ de $H^1(T_{X_1})$ et à valeur dans $H^1(\Omega_{E_2}) \oplus H^1(\Omega_{S_2})^+ \subset H^1(\Omega_{X_2})$ converge, à condition de prendre la limite dans le précisé en 3.8, vers une application transformant la limite de l'accouplement de Yukawa ψ de X_1, restreinte à $H^1(T_{E_1}) \oplus H^1(T_{S_1})^+$, en l'accouplement ψ_2' de 3.6.1 sur $H^1(\Omega_{E_2}) \oplus H^1(\Omega_{S_2})^+$, lorsque α_{S_2} "tend vers l'infini".

Il reste maintenant à étudier le comportement asymptotique des accouplements de Yukawa du type $\psi(w, w, u_{S_1})$ (ici ψ est la forme trilinéaire associée à ψ, $w \in \bigoplus_{(r,s)} H^1(\Omega_{C_{1,s}})$, $u_{S_1} \in H^1(\Omega_{S_1})^-$). On ne fera pas cette étude en considérant des limites du type $\alpha_{S_2} \to t\alpha_{S_2}$, $t \in \mathbb{R}, t \to +\infty$, mais en construisant des dégénérations adéquates de S_1; pour compléter ce travail il resterait encore à montrer qu'on peut réaliser la limite 3.8 et les limites 3.10, 3.14 en même temps. De plus les propositions 3.13, 3.17, 3.19 ne donneront une confirmation du fait que ψ devient isomorphe à ψ_2' "à la limite" que dans les cas où $C_1 = \bigcup_s C_{1,s}$ n'a pas de composantes rationnelles et sous des hypothèses géométriques supplémentaires sur T_1. (cf. 3.20.1). On va d'abord étudier le comportement de l'application ν' (1.14.1) lors d'une dégénération de Lefschetz de la courbe de ramification $C_1 \subset T_1$, telle que la fibre centrale ait p nœuds imposant les conditions indépendantes à $H^0(-K_{T_1})$. On utilisera pour cela le lemme 1.15.

3.10. Soit $F_1 \in H^0(-2K_{T_1})$ définissant une courbe ayant p nœuds $q_1, \cdots, q_p$ imposant des conditions indépendantes à $H^0(-K_{T_1})$.

Soit $F_1' \in H^0(-2K_{T_1})$ telle que $F_1'(q_i) \neq 0$.

Pour t petit, la courbe C_1^t d'équation $F_1 + tF_1'$ est lisse; soit $t = v^2$, et soit $X \xrightarrow{\pi} T_1 \times \Delta$ le revêtement double de $T_1 \times \Delta$ ramifié le long de la surface d'équation $F_1 + v^2 F_1'$. X a des nœuds au dessus des points $(q_i, 0)$, et un choix de petite résolution en chacun des nœuds fournit une application lisse $\widehat{X} \xrightarrow{\widehat{\pi}} T_1 \times \Delta$. La fibre centrale $\widehat{X}_0$ est la désingularisation minimale de X_0 et admet une involution i_1^0 dont le quotient est l'éclaté de T_1 aux points q_i. On a $H^2\left(\widehat{X}_0, \mathbb{Z}\right) \simeq H^2\left(\widehat{X}_v, \mathbb{Z}\right)$ mais sous cet isomorphisme on a : $H^2\left(\widehat{X}_v, \mathbb{Q}\right)^-_{i_1} = H^2\left(\widehat{X}_0, \mathbb{Q}\right)^-_{i_1^0} \bigoplus_i [E_{q_i}]\, \mathbb{Q}$ où E_{q_i} est la courbe exceptionnelle de $\widehat{X}_0$ au dessus de q_i.

Ici i_1 est l'involution générique de $X_{1,v}$ au dessus de T_1. Sur Δ, on a le fibré vectoriel $\mathcal{H}^{1,1}$ de fibre $\mathcal{H}^{1,1}_{(v)} \simeq H^1\left(\Omega_{\widehat{X}_v}\right)$ et le sous-fibré $\mathcal{H}^{1,1^-}$ défini sur Δ^* par $\mathcal{H}^{1,1^-}_{(v)} = \mathcal{H}^1\left(\Omega_{\widehat{X}_v}\right)^-_{i_1}$ se prolonge naturellement en $\mathcal{H}^{1,1^-} \subset \mathcal{H}^{1,1}$ tel que $\mathcal{H}^{1,1^-}_{(0)} \simeq H^1\left(\Omega_{\widehat{X}_0}\right)^-_{i_0} \oplus \bigoplus_i [E_{q_i}]\, \mathbb{C}$, où $[E_{q_i}]$ est la classe de E_{q_i} vue comme un élément de $H^1\left(\Omega_{\widehat{X}_0}\right)$.

Soit $(\omega_v)_{v\in\Delta}$ une section partout non nulle du fibré $\mathcal{H}^{2,0}$ sur Δ de fibre $\mathcal{H}^{2,0}_{(v)} = H^0\left(\Omega^2_{\widehat{X}_v}\right)$. Pour $P \in (-2K_T)$, on définit une section φ_P de $\mathcal{H}^{1,1^-}$ sur Δ^* par $\varphi_P(v) = \nu_v'(P)$, où ν_v' est l'application ν' de 1.14.1, pour la courbe d'équation $F_1 + v^2 F_1'$, et pour $\omega = \omega_v$. Par le lemme 1.15, $\varphi_P(v)$ est la projection dans $H^1\left(\Omega_{\widehat{X}_v}\right)^-_X$ de la classe de la forme méromorphe $P\omega_v / \left(F_1 + v^2 F_1'\right)$ sur $\widehat{X}_v$. On a le lemme suivant:

3.11 Lemme: i) Si P s'annule en $q_i, \forall i \in \{1, \cdots, p\}, \varphi_P$ se prolonge holomorphiquement en 0.

ii) De plus si P s'annule doublement en q_i, $\varphi_P(0) \in H^1\left(\Omega_{\widehat{X}_0}\right)^-_{i_0}$, et $\varphi_P(0) = \nu_0'(P)$, où ν_0' est l'application ν' de 1.14.1, pour la courbe normalisée $\mathcal{C}_1^0 \subset \widehat{X}_0$, qui est la courbe de ramification du revêtement double $\widehat{X_0} \to \widehat{T_1}$, et pour la deux forme w_0. ($P_{|C_1^0}$ est alors considéré comme une section de $\Omega^{\otimes 2}_{C_1^0}$).

iii) en général $v\varphi_P$ se prolonge holomorphiquement en 0, et il existe un isomorphisme naturel $\oplus\,\mathrm{res}_{q_i} : \oplus H^0\left(\mathcal{O}_{q_i}(-2K_T)\right) \simeq \oplus H^0\left(\mathcal{O}_{q_i}\right)$ tel que $v\varphi_P(0) = \sum_i \mathrm{res}_{q_i}\left(P_{|q_i}\right)[E_{q_i}]$.

Démonstration: Au voisinage de q_i, on pose $f_1 = F_1/F_1'$. On peut trouver des coordonnées locales x, y pour T_1, telles que $f_1 = -(x^2+y^2)$. Alors, au voisinage de $(q_i, 0), X$ est décrit par l'équation $u^2 = -(x^2+y^2)+v^2$, dans des coordonnées (u,x,y,v) paramétrant un ouvert de $\mathbb{C}^4$ centré en 0. La famille de formes méromorphes $p\omega_v/\left(F_1+v^2F_1'\right)$ fournit une famille continue de deux-formes fermées sur les ouverts $X\backslash B\cap X_v$, où B est un voisinage dans X de la surface de ramification $\bigcup_1 C_1^v$. Donc si $(\gamma_v)_{v\in\Delta}$ est une famille continue de classes d'homologie supportées en dehors de $B\cap X_v$, $\int_{\gamma_v} P\omega_v/\left(F_1+v^2F_1'\right)$ se prolonge continûment en 0, et $\int_{\gamma_v} vP\omega_v/\left(F_1+v^2F_1'\right)$ tend vers 0 avec v.

D'autre part, $H_2\left(X_v, \mathbb{Z}\right)^-$ est engendré par les classes supportées sur $X_v\backslash B\cap X_v$ (pour B petit) et les classes des sphères évanescentes $S^2_{q_i}(v)$, décrites dans les coordonnées (u,x,y,v) par: $u = vu_0, x = vx_0, y = vy_0$, avec u_0, x_0, y_0 réels et $u_0^2+x_0^2+y_0^2 = 1$. La limite de $S^2_{q_i}(v)$ est égale à la classe de E_{q_i} dans $H^2(\widehat{X}_0, \mathbb{Z})$. $S^2_{q_i}(v)$ est aussi homologue au contour suivant: on remplace $S^2_{q_i}(v)$ par la réunion $T_{q_i}(v)$ de $S^2_{q_i}(v)\cap|u_0|\geq\frac{1}{2}$ et de l'ensemble $\{(u,x,y,v)/u=(v/2)e^{i\theta'}, x = ve^{i\theta}x_1, x_1$ réel, $y = ve^{i\theta}y_1, y_1$ réel avec $\theta, \theta'\in[0,\pi]$ et $e^{2i\theta}\left(x_1^2+y_1^2\right) = 1-\frac{1}{4}e^{2i\theta'}\}$, qui est une partie d'un fibré en cercle au dessus du cercle évanescent de C_1^v, décrit par: $u=0,\ x = vx_1,\ y = vy_1, x_1,\ y_1$ réels, et $x_1^2+y_1^2=1$. $T_{q_i}(v)$ est construit pour éviter l'ensemble $\{u=0\}$. On vérifie facilement que, au voisinage de q_i, on a $\omega_v = \varphi\frac{dx\wedge dy}{v}$ où φ est une fonction non nulle. Il reste à montrer:

3.11.1 i) si $\psi(x,y)$ s'annule en 0, $\displaystyle\int_{T_{q_i(v)}} \frac{\psi dx\wedge dy}{u^3}$ a une limite finie quand v tend vers 0, nulle si ψ s'annule à l'ordre 2.

ii) si $\psi(0)\neq 0$, $\displaystyle\int_{T_{q_i(v)}} \frac{v\psi dx\wedge dy}{u^3}$ a une limite finie non nulle lorsque v

tend vers 0.

3.11.2 i) peut se montrer en supposant que ψ est homogène de degré 1 ou 2 en x et y. Comme les $T_{q_i(v)}$ sont isomorphes par la multiplication par v, on voit immédiatement que dans le premier cas $\int_{T_{q_i(v)}} \frac{\psi dx\wedge dy}{u^3}$ est constant, tandis que dans le second cas $\frac{1}{v}\int_{T_{q_i(v)}} \frac{\psi dx\wedge dy}{u^3}$ est constant.

Pour 3.11.1 ii), on suppose $\psi = 1$, alors $v\displaystyle\int_{T_{q_i(v)}} \frac{\psi dx\wedge dy}{u^3}$ est constant.

Pour voir que $v\displaystyle\int_{T_{q_i(v)}} \frac{dx\wedge dy}{u^3}$ est non nul, on se place dans le quadrique projective de dimension deux Q d'équation $U^2+X^2+Y^2 = V^2$ dans $\mathbf{P}^3$ de coordonnées homogènes (U,X,Y,V): elle contient la quadrique affine $u^2+x^2+y^2 = 1$ et on vérifie que $\frac{dx\wedge dy}{u^3}$ engendre $H^1(\Omega_Q)^{\text{prim}}$, ce qui se montre par la construction 1.15 appliqué au revêtement double $Q \to \mathbf{P}^2, (U,X,Y,V)\to(X,Y,V)$, tandis qu'il est bien connu que $S^2(1)$, donc $T(1)$ engendre $H_2(Q,\mathbf{Z})^{\text{prim}}$.

Le Lemme 3.11 résulte de 3.11.1 de la façon suivante:

i) si P s'annule en zéro $\displaystyle\int_{\gamma_v} P\omega_v/\left(F_1+v^2F_1'\right)$ a une limite finie pour γ_v supportée dans $X_v\backslash B\cap X_v$, et aussi pour $\gamma_v = T_{q_i}(v)$; donc φ_P se prolonge holomorphiquement en zéro.

ii) si P s'annule doublement en zéro, on a $\displaystyle\lim_{v\to 0}\int_{T_{q_i(v)}} P\omega_v/\left(F_1+v^2F_1'\right) = 0 = \int_{E_{q_i}}\varphi_P(0)$, tandis que par la définition de ν_0', (cf 1.15) on a $\displaystyle\lim_{v\to 0}\int_{\gamma_v} P\omega_v/\left(F_1+v^2F_1'\right) = \int_{\gamma_0}\varphi_P(0) = \int_{\gamma_0}\nu_0'(P)$ pour $(\gamma_v)_{v\in\Delta}$ une famille continue de classes d'homologie supportées dans $X_v\backslash B\cap X_v$. Comme $\int_{E_{q_i}}\nu_0'(P) = 0$, on a donc $\varphi_P(0) = \nu_0'(P)$.

iii) l'isomorphisme $\oplus\,\mathrm{Res}_{q_i}$ est fourni par 3.11.1 ii) $P_{|q_i} \mapsto -\frac{1}{2}\left(P/F_1'(q_i)\times v\int_{T_{q_i(v)}}\frac{dx\wedge dy}{u^3}\right)$. On a alors $\int_{E_{q_i}}\varphi_P(0) = -2\,\mathrm{Res}_{q_i}(P), \int_\gamma v\varphi_P = 0$ pour γ supportée dans $X_0\backslash B\cap X_0$. Ce qui montre 3.11. iii).

3.12. On travaille encore avec les hypothèses et les notations de 3.10. On note $\Delta_w(resp\Delta_v)$ le disque de coordonnées w(resp.v); soit $\rho = \Delta_w \to \Delta_v$ le revêtement double donné par $v = w^2$. Sur Δ_v on a la famille de courbes $\mathcal{C}_1 \xrightarrow{\pi} \Delta_v$ de fibre C_1^v. On a aussi les fibrés $\mathcal{F} = R^0\pi_*(k^*(-2K_{T_1}))$, où $k : \mathcal{C}_1 \to T_1$ est l'application naturelle, et $\mathcal{H}^{1,1^-}$. Notons $\mathcal{F}'$ le fibré vectoriel sur Δ_v, défini comme le noyau de l'application d'évaluation composée de $\mathcal{F} \to H^0\left(-2K_{T_1|C_1^0}\right)$ et de $H^0\left(-2K_{T_1|C_1^0}\right) \to \bigoplus_i (\mathcal{O}_{q_i}(-2K_{T_1}))$. D'après 3.11, l'application $\nu' = (\nu'_v)_{v\in\Delta^*}$ se prolonge en une application que l'on notera encore $\nu' : \mathcal{F}' \to \mathcal{H}^{1,1^-}$. On en déduit donc aussi une application $\nu' : \rho^*\mathcal{F}' \to (\mathcal{H}^{1,1^-})$. Considérons maintenant sur Δ_w la famille $\rho^*\mathcal{C}^1 \xrightarrow{\pi^w} \Delta_w$, et soit $\mathcal{G}$ le fibré vectoriel $R^0\pi_*^w(k^{w*}(-K_T))$, où $k^w = k \circ \rho : \rho^*\mathcal{C}^1 \to T_1$. Notons $\mathcal{G}'$ le sous-fibré de $\mathcal{G}$ défini comme le noyau de l'application d'évaluation, composée de $\mathcal{G} \to H^0\left(-K_{T_1|C_1^0}\right)$ et de $H^0\left(-K_{T_1|C_1^0}\right) \to \bigoplus_i H^0(\mathcal{O}_{q_i}(-K_{T_1}))$. L'application naturelle donnée par le produit: $\mu : S^2\mathcal{G} \to \rho^*\mathcal{F}$ induit alors une application: $\mu' : S^2\mathcal{G}' \to \rho^*\mathcal{F}'$, et l'on a une application composée $\mu'' = \nu' \circ \mu' : S^2\mathcal{G}' \to \rho^*\mathcal{H}^{1,1^-}$. Pour $w \neq 0$, d'après 1.14, on a pour $\eta, \gamma \in \mathcal{G}'_w \simeq H^0(\Omega_{p_r\times C_1^w})$ et $\alpha \in \mathcal{H}^{1,1^-}_{(w)} \simeq H^1(\Omega_{X_w})^-$ l'égalité: $\psi_w(\eta,\gamma,\alpha) = \mu''(\eta\otimes\gamma)\cdot_{X_w}\alpha$, où ψ_w est la forme trilinéaire associée à l'accouplement de Yukawa sur $X_{1,w} := \widetilde{E_1 \times X_w}/\left(\widetilde{j_1, i_1}\right)$.

Les accouplements $\mu_0''(\eta_0\otimes\gamma_0)\cdot_{\widetilde{X}_w}\alpha_0$, pour $\eta_0,\gamma_0 \in \mathcal{G}'_0$ et $\alpha_0 \in H^1\left(\Omega_{\widetilde{X}_0}\right) \oplus \Sigma\langle E_{q_i}\rangle\cdot\mathbb{C}$ décrivent donc une limite des accouplements ψ_w.

Cette limite est complètement décrite dans la proposition suivante:

3.13 Proposition. $\mathcal{G}'_0$ est isomorphe à la somme directe $\mathcal{G}''_0 \oplus \sum_i \mathbb{C}_{q_i}$, où $\mathcal{G}''_0 \simeq \operatorname{Ker} H^0\left(-K_{T|C_1^0}\right) \to \bigoplus_i (-K_{T|q_i})$ et pour $\eta_0 = \eta''_0 + \sum_i \alpha_{q_i} \in \mathcal{G}'_0$, où $\eta''_0 \in \mathcal{G}''_0$ et $\alpha_{q_i} \in \mathbb{C}$, on a

$$\mu_0''\left(\eta_0^{\otimes 2}\right) = \nu'_0\left(\eta_0^{''\otimes 2}\right) + \sum_i \alpha_{q_i}^2 \cdot E_{q_i}.$$

Démonstration. Choississons des éléments $f_i \in H^0(-K_{T_1})$ tels que $f_i(q_i) \neq 0, f_i(q_j) = 0$, pour $i \neq j$. Alors $\mathcal{G}'$ admet une base de sections holomorphes donnée par $w \cdot f_i$ pour $i = 1, \cdots, p$ et $1 \times g$ pour $g \in \mathcal{G}_0''$ (on considère $\mathcal{G}'$ comme un sous-fibré du fibré trivial $H^0(-K_{T_1}) \otimes \mathcal{O}_{\Delta_w}$). $\mathcal{G}_0''$ et $(w \cdot f_i)_0$ donnent une base de $\mathcal{G}_0'$. Pour $g \in \mathcal{G}_0'', g^2$ s'annule doublement en q_i, $\forall i$, et donc $\mu_0''(g^2) = \nu_0'(g^2)$ par 3.11. ii). Pour $g \in \mathcal{G}_0'', i = 1 \cdots p, gf_i$ s'annule en $q_j, \forall j$, et donc par 3.11.i, $\lim_{w \to 0} \nu_w'(wg \cdot f_i) = 0$, soit $\mu_0''(g \cdot (wf_i)_0) = 0$.

De même, si $i \neq j$, $f_i f_j$ s'annule en $q_k, \forall k$, et donc $\lim_{w \to 0} \nu_w'(vf_i \cdot f_j) = 0$, soit $\mu_0''((wf_i)_0 (wf_j)_0) = 0$. Enfin on a pour $i = 1, \cdots, p$, $\mu_0''\left((wf_i)_0^{\otimes 2}\right) = \lim_{w \to 0} v\nu'(f_i^2) = \mathrm{Res}_{q_i}(f_i^2) \cdot E_i$ par 3.11.iii, ce qui montre 3.13, à condition de choisir de façon adéquate l'isomorphisme $\bigoplus_i (wf_i)_0 \cdot \mathbb{C} \simeq \bigoplus_i (wf_i) \mathbb{C}_{q_i}$.

3.14. On va maintenant considérer un type complètement différent de dégénérations de S_1. On suppose dans ce qui suit que la courbe C_1 de ramification de $\varphi : S_1 \to T_1$ n'a que des composantes elliptiques. C_1 peut avoir une ou deux composantes dans ce cas, d'après 1.1.1, et dans la suite "cas i" et cas ii, signifient que la première, ou la seconde de ces conditions respectivement, est satisfaite.

Dans le cas i), $h^0(-K_{T_1}) = 1$ et le diviseur de l'unique section σ de $-K_{T_1}$ est une courbe elliptique connexe, qu'on supposera lisse. Dans le cas ii), $h^0(-K_{T_1}) = 2$, et le diviseur d'une section générique σ de $-K_{T_1}$ est une courbe elliptique connexe et lisse, puisqu'en particulier chaque composante de C_1 appartient à $|-K_{T_1}|$. Dans les deux cas on choisit σ telle que $V(\sigma) =: E_\sigma$ est lisse elliptique connexe et ne rencontre pas C_1. Soit $F_1 \in H^0(-2K_{T_1})$ une équation pour C_1. Soit Δ un disque de coordonnée t. La surface $\mathcal{C}$ d'équation $\sigma^2 + tF_1$ dans $T_1 \times \Delta$ est lisse. Soit $\varphi : X \to T_1 \times \Delta$ le revêtement double ramifié le long de $\mathcal{C}$. X est lisse, $\pi : X \to \Delta$ est lisse au dessus de Δ^*, pour Δ petit et ses fibres sont des surfaces $S_1^t \to T_1$ comme ci-dessus. Pour $t = 0$, X_0 est la surface à croisements normaux obtenue en récoltant deux copies de T_1 le long de E_σ. Sur X_0 l'involution de X agit en inversant les deux composantes de X_0.

Pour $t \neq 0$, on reprend la notation $\nu'_t : H^0\left(-2K_{T_1|C_1^t}\right) \to H^1\left(\Omega_{S_1^t}\right)$ de 1.14; la proposition suivante décrit le comportement asymptotique de $(\nu'_t)_{t\in\Delta^*}$.

3.15 Proposition:

i) Le fibré $\mathcal{H}_*^{1,1}$ sur Δ^*, de fibre $\mathcal{H}_{*t}^{1,1} \simeq H^1\left(\Omega_{S_1^t}\right)$ s'étend en un fibré $\mathcal{H}^{1,1}$ sur Δ, tel que l'involution $(i_1^t)_{t\in\Delta^*}$ sur $\mathcal{H}^{1,1}$ s'étende en une involution i_1 sur $\mathcal{H}^{1,1}$, et que la forme d'intersection $\langle i_1^t\rangle_t$ non dégénérée, donnée par la dualité de Serre sur $\mathcal{H}_t^{1,1} \simeq H^1\left(\Omega_{S_1^t}\right)$, s'étende en une forme d'intersection non dégénérée sur $\mathcal{H}_0^{1,1}$, et $\mathcal{H}_0^{1,1^-}$, qu'on notera $\langle\ \rangle_0$.

ii) Pour $P(t)$ une section holomorphe de $H^0(-2K_{T_1})\otimes\mathcal{O}_\Delta$, telle que $P(0)\neq 0$ soit un multiple non nul de $\sigma^2, \nu'_t(P(t)) \in \mathcal{H}_t^{1,1^-}$ s'étend en une section holomorphe de $\mathcal{H}^{1,1^-}$, et $\nu'_0(P(0))$ satisfait: $\langle\nu'_0(P(0)),\nu'_0(P(0))\rangle_0 = 0, \nu'_0(P(0))\neq 0$.

Démonstration: (cf. [28]) On considère sur X le fibré vectoriel de rang 2 $\Omega_{X/\Delta}(\operatorname{Log}X_0) := \Omega_X(\operatorname{Log}X_0)/\pi^*(\Omega_\Delta(\operatorname{Log}0))$; En coordonnées locales X est décrit par $u^2=\sigma^2+t$, où σ est une coordonnée sur T_1, u une coordonnée sur X: posant $x=u-\sigma, y=u+\sigma$ on a donc $xy=t$. $\Omega_X(\operatorname{Log}X_0)$ est engendré localement par $\frac{dx}{x}$, $\frac{dy}{y}$ et dz, où z est une coordonnée supplémentaire sur T_1, tandis que $\Omega_\Delta(\operatorname{Log}0)$ est engendré par $\frac{dt}{t}$. Donc $\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0)$ est engendré par $\frac{dx}{x}, \frac{dy}{y}, dz$, avec la relation $\frac{dx}{x}+\frac{dy}{y}=0$. Pour $t\neq 0$, on a $\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0)_{|X_t} \simeq \Omega_{S_1^t}$, et sur Δ^* on a $\mathcal{H}^{1,1} \simeq R^1\pi_*(\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0))$. On vérifie facilement que sur $X_0 \simeq T_1^1 U_{E_\sigma} T_1^2$ on a: $\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0)_{|T_1^1} \simeq \Omega_{T_1^1}(\operatorname{Log}E_\sigma)$ et $\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0)_{|T_1^2} \simeq \Omega_{T_1^2}(\operatorname{Log}E_\sigma)$. On a donc une suite exacte:

3.15.1
$$\begin{aligned}&0 \to (\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0))_{|X_0}\\ &\to \Omega_{T_1^1}(\operatorname{Log}E_\sigma)\oplus\Omega_{T_1^2}(\operatorname{Log}E_\sigma) \to \Omega_{T_1}(\operatorname{Log}E_\sigma)_{|E_\sigma} \to 0.\end{aligned}$$

Comme $H^0\left(\Omega_{T_1^1}(\operatorname{Log}E_\sigma)\right)=0$, on en déduit

$$H^0(\Omega_X(\operatorname{Log}X_0)/\pi^*\Omega_\Delta(\operatorname{Log}0))_{|X_0} = 0.$$

Comme le faisceau dualisant de X_0 est trivial et que le déterminant de $\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)$ est également trivial on a aussi

$$H^2\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)\right)_{|X_0} = 0$$

donc $H^1\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)\right)_{|X_t}$ est constant et

$$\mathcal{H}^{1,1} := R^1\pi_*\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)\right)$$

est localement libre et fournit l'extension cherchée de $\mathcal{H}^{1,1}_*$. Clairement l'involution i agissant sur X au-dessus $T_1 \times \Delta$ agit sur

$$\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)$$

et donc aussi sur $\mathcal{H}^{1,1}$.

Finalement, comme le fibré dualisant relatif de X/Δ est trivial, et que $R^1\pi_*\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)\right)$ est libre de fibre $\mathcal{H}^{1,1}_{t*} = H^1\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)_{|X_t}\right), \forall t \in \Delta$, la dualité de Serre relative fournit $\mathcal{H}^{1,1} \simeq \left(\mathcal{H}^{1,1}\right)^\vee$. Ceci montre i), avec l'information supplémentaire suivante:

3.15.2: $\mathcal{H}^{1,1}_0 = H^1\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)_{|X_0}\right)$

Pour montrer ii) on utilise la description suivante de ν'_t (cf. preuve de 1.15). Soit $C_1^t \subset S_1^t$ la courbe de ramification du revêtement double $S_1^t \to T_1$. En utilisant l'application i_1^t qui agit sur $\Omega^t_{S_1|C_1^t}$, on obtient une décomposition canonique:

3.15.3: $\Omega^t_{S_1|C_1^t} \simeq \Omega_{C_1^t} \oplus \left(N^t_{C_1}\right)^\vee \simeq -K_{T_1|C_1^t} \oplus K_{T_1|C_1^t}$. Ceci fournit une inclusion naturelle: $H^0\left(-2K_{T_1|C_1^t}\right) \to H^0\left(\Omega_{S_1^t}(-K_{T_1})_{|C_1^t}\right)$ qui composée avec le cobord
$\partial : H^0\left(\Omega_{S_1^t}(-K_{T_1})_{|C_1^t}\right) \to H^1\left(\Omega_{S_1^t}\right)$ associée à la suite exacte: $0 \to \Omega_{S_1^t} \to \Omega_{S_1^t}(-K_{T_1}) \to \Omega_{S_1^t}(-K_{T_1})_{|C_1^t} \to 0$ fournit ν'_t.

Pour mettre cette construction en famille, on considère dans X la surface d'équation $u = 0$, lieu de ramification de $\varphi : X \to T_1 \times \Delta$. Cette surface est clairement isomorphe à un voisinage de E_σ dans T_1. L'involution i de X agit sur $\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)_{|\{u=0\}}$ ce qui fournit une décomposition:

3.15.4 $\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)_{|\{u=0\}} \simeq L_1 \oplus L_2$ où L_1 et L_2 sont des fibrés en droites sur $\{u = 0\}$. Par restriction à la courbe C_1^t d'équations $u = 0$, $\sigma^2 + tF_1 = 0$, 3.15.4 induit 3.15.3. Cependant on vérifie facilement que $L_1 \simeq \mathcal{O}_{T_1}, L_2 \simeq \mathcal{O}_{T_2}$, (et non pas $L_1 \simeq -K_{T_1}, L_2 \simeq K_{T_1}$), de sorte que la restriction de 3.15.4 à C_1^t ne fournit 3.15.3 que modulo l'isomorphisme $\mathcal{O}_{C_1^t} \simeq -K_{T_1|C_1^t}$ fourni par $\sigma \in H^0(-K_{T_1})$.

3.15.5 Considérons la courbe $E_\sigma^{(2)} \subset \{u = 0\}$ définie par $\sigma^2 = 0$. On peut aussi l'identifier à un diviseur de X_0, faisant partie du système linéaire $|-K_{T_1}|$ sur X_0, puisqu'elle est décrite par l'équation $u = 0$ dans X_0. Alors 3.15.4, restreint à $E_\sigma^{(2)}$, fournit:

$$\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)_{|E_\sigma^{(2)}} \simeq \mathcal{O} \oplus \mathcal{O}, \text{ d'où une application:}$$

$$j_0 : H^0\left((-K_{T_1})_{|E_\sigma^{(2)}}\right) \to H^0\left((\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)(-K_{T_1}))_{|E_\sigma^{(2)}}\right)$$

donnée par l'inclusion du premier facteur; finalement la suite exacte:

$$\begin{gathered} 0 \to (\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0))_{|X_0} \\ \xrightarrow{u} (\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)(-K_{T_1}))_{|X_0} \to \\ (\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)(-K_{T_1}))_{|E_\sigma^{(2)}} \to 0 \end{gathered}$$

fournit :

$$\begin{gathered} \partial_0 : H^0\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)(-K_{T_1})_{|E_\sigma^{(2)}}\right) \to \\ \to H^1\left(\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0)(-K_{T_1})_{|X_0}\right) \end{gathered}$$

d'où finalement une flèche

$$\nu_0' = \partial_0 \circ j_0 : H^0\left(-K_{T_1|E_\sigma^{(2)}}\right) \to H^1\left((\Omega_X(\operatorname{Log} X_0)/\pi^*\Omega_\Delta(\operatorname{Log} 0))_{|X_0}\right)$$

qui par ce qui précède est la limite des flèches :

$$\nu_t' \circ \sigma : H^0\left(-K_{T_1|C_1^t}\right) \xrightarrow{\sigma} H^0\left(-2K_{T_1|C_1^t}\right) \xrightarrow{\nu_t'} H^1\left(\Omega_{S_1^t}\right).$$

3.15.6. On termine la preuve de 3.15. ii)

Soit $P(t) = \alpha\sigma^2 + tQ(t)$ une section de $H^0(-2K_{T_1}) \otimes \mathcal{O}_\Delta$. Alors $\frac{1}{\sigma}P(t)_{|C_1^t}$ converge vers $\alpha\sigma \in H^0\left(-K_{T_1|E_\sigma^{(2)}}\right)$. On en déduit que $\lim\limits_{t\to 0} \nu_t'\left(P(t)_{|C_1^t}\right) = \nu_0'(\alpha\sigma)$.

Notons que $\alpha\sigma \in \mathrm{Ker}\left(-K_{T_1|E_\sigma^{(2)}}\right) \to H^0\left(-K_{T_1|E_\sigma}\right)$ donc

$$j_0(\alpha\sigma) \in \Big[\mathrm{Ker}\, H^0\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)(-K_{T_1})\right)_{|E_\sigma^{(2)}}\right) \to$$

$$H^0\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)(-K_{T_1})\right)_{|E_\sigma}\right)\Big] \simeq$$

$$H^0\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)(-K_{T_1})\right)_{|E_\sigma}\right).$$

Notons $j_0'(\alpha\sigma) \in H^0\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)\right)_{|E_\sigma}\right)$ l'image de $j_0(\alpha\sigma)$ par cet isomorphisme. Considérons la suite exacte: (cf 3.15.1)

3.15.7
$$0 \to \left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)_{|X_0}\right) \to$$
$$\Omega_{T_1^1}(\mathrm{Log}\, E_\sigma) \oplus \Omega_{T_1^2}(\mathrm{Log}\, E_\sigma) \to \left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)\right)_{|E_\sigma} \to 0.$$

Elle donne

$$\partial_0' : H^0\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)\right)_{|E_\sigma}\right) \to H^1\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)\right)_{|X_0}\right)$$

et il immédiat de montrer que :

$$\partial_0 \circ j_0(\alpha\sigma) = \partial_0'\left(j_0'(\alpha\sigma)\right)$$

dans

$$H^1\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)\right)_{|X_0}\right).$$

On a donc: $\lim\limits_{t\to 0} \nu_t'\left(P(t)_{|C_1^t}\right) = \partial_0'\left(j_0'(\alpha\sigma)\right)$. Comme il résulte de 3.15.7 que ∂_0' est injective, on a $\delta_0'\left(j_0'(\alpha\sigma)\right) \neq 0$, pour $\alpha \neq 0$. Enfin il est facile de vérifier que $\delta_0'\left(j_0'(\alpha\sigma)\right)$ est de self intersection 0 dans $H^1\left(\left(\Omega_X(\mathrm{Log}\, X_0)/\pi^*\Omega_\Delta(\mathrm{Log}\, 0)\right)_{|X_0}\right)$ pour $\langle\ \rangle_0$ donnée par la dualité de Serre. Donc 3.15 (ii) est prouvé.

3.16. On utilise maintenant la proposition 1.17 et la proposition 3.15 pour étudier la limite des accouplements de Yukawa sur $X_1^t := E_1 \times S_1^t/(j_1, i_1^t)$, du type $\psi(w,\eta,\gamma)$ pour $w \in H^1\left(\Omega_{S_1}^t\right), \eta, \gamma \in \bigoplus_r H^0\left(\Omega_{p_r \times C_1^t}\right)$, lorsque S_1^t dégénère comme en 3.14. On considère d'abord le cas i). On a d'abord en reprenant les notations précédentes, la conséquence immédiate suivante de 1.17 et 3.15.

3.17. Proposition:

(Cas i) Soit $(w_t)_{t\in\Delta}$ une section holomorphe de $\mathcal{H}^{1,1^-}$ sur Δ. Soit σ le générateur de $H^0(-K_{T_1})$: alors $\sigma_{|C_1^t}$ est un générateur de $H^0\left(\Omega_{C_1^t}\right) \simeq \left(-K_{T_1|C_1^t}\right)$, pour $t \neq 0$, et donc $\alpha = (\alpha_{p_1}\sigma, \cdots, \alpha_{p_4}\sigma)$, pour $\alpha_{p_r} \in \mathbb{C}$ peut être considéré comme un élément de $\bigoplus_r H^0\left(\Omega_{p_r \times C_1^t}\right)$ pour $t \neq 0$. On a alors $\lim\limits_{t\to 0} \psi(w_t,\alpha,\alpha) = \sum_r \alpha_r^2 \left\langle w_0, \nu_0'(\sigma^2)\right\rangle_0$ où $w_0 \in H_0^{1,1}$, $\nu_0'(\sigma^2) \in \mathcal{H}_0^{1,1}$ est défini en 3.15. ii, et est non nul de self intersection 0.

3.18. Le cas ii) est un peu plus compliqué: les courbes C_1^t ont alors deux composantes, et il est naturel si l'on veut obtenir une limite correcte de l'accouplement de Yukawa de x_1^t, de les ordonner en passant à un revêtement double de Δ, c'est-à-dire à un disque Δ_w, muni de $\rho : \Delta_w \to \Delta$ donnée par $t = w^2$. Alors l'équation $\sigma^2 + tF_1$ peut s'écrire sous la forme $(\sigma + w\varphi_1(w))(\sigma + w\varphi_2(w))$ où $\varphi_2(w), \varphi_1(w) \in H^0(-K_{T_1})$ sont holomorphes en w. $C_1^{w^2}$ est la réunion de $C_{1,1}^w$ (décrite par l'équation $\sigma + w\varphi_1(w) \in H^0(-K_{T_1})$ et $C_{1,2}^w$ (décrite par l'équation $\sigma + w\varphi_2(w)$).

Alors $\sigma + w\varphi_2(w)_{|C_{1,1}^w}$ est un générateur de $H^0\left(-K_{T_1|C_{1,1}^w}\right)$, et s'annule sur $C_{1,2}^w$ de sorte qu'il est naturel de l'identifier à un générateur de $H^0\left(\Omega_{C_{1,1}^w}\right) \subset H^0\left(\Omega_{C_{1,1}^w}\right) \oplus H^0\left(\Omega_{C_{1,2}^w}\right) \simeq H^0\left(\Omega_{C_1^w}\right)$. On a donc une trivialisation naturelle de $\bigoplus\limits_{(r,s)} H^0\left(\Omega_{p_r} \times C_{1,s}^w\right)$ (où $r \in \{1,\cdots,4\}, s \in \{1,2\}$, donnée par:

$$\alpha = (\alpha_{p_1,1}, \alpha_{p_1,2}, \cdots, \alpha_{p_4,1}, \alpha_{p_4,2}) \to \sum_{(r,s)} \alpha_{p_r,s}\left(\sigma + w\varphi_{s'|C_{1,s}^w \times p_r}\right)$$

où $s' = 2$ pour $s = 1, s' = 1$ pour $s = 2$.

La proposition 3.15 et la proposition 1.17 donnent alors immédiatement (avec les notations de 3.16):

3.19. Proposition: (Cas ii) Soit $(\chi_t)_{t\in\Delta}$ une section holomorphe de $\mathcal{H}^{1,1^-}$. Alors on a $\lim_{w\to 0} \psi(\chi_t,\alpha,\alpha) = \sum_{(r,s)} \alpha^2_{p_i,s} \langle \chi_0, \nu_0'(\sigma^2)\rangle_0$ où $\nu_0'(\sigma^2) \in \mathcal{H}_0^{1,1^-}$ satisfait: $\nu_0'(\sigma^2) \neq 0$ et $\langle \nu_0'(\sigma^2), \nu_0'(\sigma^2)\rangle_0 = 0$.

3.20. On va maintenant rassembler les résultats 3.13 et 3.17, 3.19. On fera l'hypothèse suivante sur T_1:

3.20.1

i) C_1 n'a pas de composantes rationnelles.

ii) Si C_1 a une composante de genre > 1 (donc C_1 est connexe par (1.1), C_1 admet une dégénération de Lefschetz sur une courbe $C_{1,0}$ dont la normalisée $\widehat{C_{1,0}}$ est elliptique et connexe.

iii) Soit $\widehat{T}_1$ la surface obtenue en éclatant les nœuds q_i de $C_{1,0}$; alors l'unique section de $-K_{\widehat{T}_1}$ définit une courbe lisse E_σ. On a alors sous les hypothèses 3.20.1:

3.21. Théorème: Si $g(C_1) > 1$, considérons une dégénération de Lefschetz de C_1 comme en 3.20 ii; puis faisons dégénérer la courbe $\widehat{C_{1,0}}$ sur $2E_\sigma$, comme en 3.14. Si C_1 a deux composantes elliptiques, faisons dégénérer C_1 sur $2E_\sigma$. Alors quitte à passer à un revêtement ramifié de la base de la dégénération, l'accouplement de Yukawa ψ de $X_1 = \widetilde{E_1 \times S_1}/(j_1, i_1)$ admet une limite naturelle, qui est isomorphe (comme cubique) à la forme d'intersection corrigée ψ_2' sur $H^1(\Omega_{X_2})$, et l'isomorphisme $M : \lim H^1(\Omega^2_{X_1}) \to H^1(\Omega_{X_2})$ qui transforme $\lim\psi$ en ψ_2' peut-être choisi de la façon suivante:

- M induit: $\bigoplus_s H^0(\Omega_{p_r\times C_s}) \xrightarrow{\sim} \bigoplus_{s'} \langle D_{r,s'}\rangle\, \mathbb{C}, \forall r\in\{1,\cdots,4\}$

- M induit $H^0(\Omega_{E_1}) \simeq H^1(\Omega_{E_1})$ et $H^1(\Omega_{S_1})^- \simeq H^1(\Omega_{S_2})^+$, le dernier isomorphisme étant compatible avec les formes d'intersection.

Démonstration: Dans le premier cas en faisant la synthèse de 3.13 et 3.17, on voit qu'il existe (après passage à un revêtement ramifié de la base), une limite naturelle de $\bigoplus_s H^0(\Omega_{p_r} \times C_s)$ isomorphe à $\bigoplus_{i=1,C=1}^{N_1'-1} \mathbb{C}_{q_i,r} \oplus \mathbb{C} \cdot \sigma$, où $\langle\sigma\rangle = H^0\left(-K_{\widehat{T_1}}\right)$, une limite $\lim H^1(\Omega_{S_1})^-$ munie d'une forme d'intersection limite non dégénérée, et des classes $E_{q_i}, \nu_0'(\sigma^2) \in H^1(\Omega_{S_1})^-$ satisfaisant $E_{q_i}^2 = -2, \nu_0'(\sigma^2)^2 = 0, \nu_0'(\sigma^2) \neq 0, E_{q_i} \cdot E_{q_j} = 0,\ E_{q_i} \cdot \nu_0'(\sigma^2) = 0$, telles que: pour $w \in \lim H^1(\Omega_{S_1})^-, \eta \in \bigoplus_{r,s} H^0(\Omega_{p_r} \times C_s), \eta = (\alpha_{q_i,r}, \alpha_r)$, on ait $(\lim\psi)(w,\eta,\eta) = \left\langle w\left(\sum_{i,r} \alpha_{q_i,r}^2 E_{q_i} + \sum_r \alpha^2 \nu_0'(\sigma^2)\right)\right\rangle$. D'autre part $(\lim\psi)(w + \eta + u_E) = 3(\lim\psi)(w,\eta,\eta) + 3(w \cdot w) \times \gamma(u_E)$ où γ est une forme linéaire non nulle sur $H^0(\Omega_E)$ (cf. 1.17).

Rappelons d'autre part la formule 3.6.1 qui décrit ψ_2' : $\psi_2'(d + \chi + \beta) = 3(d^2\chi)_{X_2} + 3(\chi,\chi)_{S_2} \cdot \int_{E_2} \beta$, pour $\beta \in H^1(\Omega_E), \chi \in H^1(\Omega_{S_2})^+, d = \sum_{r=1,s'=1}^{4,N_2} d_{r,s'} D_{2,r,s'}$, avec la relation (Lemme 1.11) $(d^2\chi)_{X_2} = -2\sum_{r,s'} d_{r,s'}^2 (C_{s'} \cdot \chi)_{S_2}$.

Il reste simplement à noter que X_1 a par hypothèse les invariants $N_1 = 1, N_1'$, et X_2 a donc les invariants $N_2 = N_1', N_2' = 1$. Donc nécessairement la courbe C_2 a une composante elliptique et $N_2 - 1 = N_1' - 1$ composantes rationnelles.

L'existence d'un isomorphisme M transformant $\lim\psi$ en ψ_2' et jouissant des propriétés énoncées dans le théorème 3.21 est alors claire:

Supposons que la composante elliptique de C_2 est la composante $C_{2,N_1'}$. Il suffit alors de faire:

a) $M\left(1_{q_i,r}\right) = \frac{i}{\sqrt{2}} D_{2,r,i}$ pour $i = 1, N_1' - 1$, $M(\sigma_{p_r}) = D_{2,r,N_1'}$.

b) $M(E_{q_i}) = C_{2,i}$ pour $i = 1, \cdots, N_1' - 1$, $M\left(\nu_0'(\sigma^2)\right) = C_{2,N_1'}$, puis d'étendre l'application M construite en b) $\langle E_{q_i}, \nu'_0(\sigma^2)\rangle \simeq \langle C_{2,i}\rangle$ en un isomorphisme $M : \lim \left(H^1\left(\Omega_{S_1}\right)^-\right) \to H^1\left(\Omega_{S_2}\right)^+$ préservant les formes d'intersection.

Le second cas se montre de façon similaire.

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