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1 Introduction

The notion of *quantum cohomology ring* of a Kähler manifold V naturally appears in theoretical physics in the consideration of the so called *topological sigma model* associated with V ([16], 3a-b). If the canonical line bundle $\mathcal{K}_V$ of V is negative, then one recovers the multiplicative structure of the quantum cohomology ring of V from the intersection theory on the moduli space $\mathcal{I}_\lambda$ of holomorphic mappings f of the Riemann sphere $f : S^2 \cong \mathbf{CP}^1 \to V$ where λ is the homology class in $H_2(V, \mathbf{Z})$ of $f(\mathbf{CP}^1)$.

If the canonical bundle $\mathcal{K}_V$ is trivial, the quantum cohomology ring was considered by C. Vafa as an important tool for explaining the mirror symmetry for Calabi-Yau manifolds [15].

The quantum cohomology ring $QH_\varphi(V, \mathbf{C})$ of a Kähler manifold V, unlike the ordinary cohomology ring, have the multiplicative structure which depends on the class φ of the Kähler $(1,1)$-form corresponding to a Kähler metric g on V. When we rescale the metric $g \to tg$ and put $t \to \infty$, the quantum ring becomes the classical cohomology ring. For example, for the topological sigma model on the complex projective line $\mathbf{CP}^1$ itself, the classical cohomology ring is generated by the class x of a Kahler $(1,1)$-form, where x satisfies the quadratic equation

$$x^2 = 0, \tag{1}$$

while the quantum cohomology ring is also generated by x, but the equation satisfied by x is different:

$$x^2 = \exp(-\int_\lambda \varphi), \tag{2}$$

λ is a non-zero effective 2-cycle. Similarly, the quantum cohomology ring of d-dimensional complex projective space is generated by the element x satisfying the equation

$$x^{d+1} = \exp(-\int_{\lambda} \varphi). \tag{3}$$

The main purpose of this paper is to construct and investigate the quantum cohomology ring $QH^*_{\varphi}(\mathbf{P}_{\Sigma}, \mathbf{C})$ of an arbitrary smooth compact d-dimensional toric manifold $\mathbf{P}_{\Sigma}$ where φ is an element of the ordinary second cohomology group $H^2(\mathbf{P}_{\Sigma}, \mathbf{C})$. Since all projective spaces are toric manifolds, we obtain a generalization of above examples of quantum cohomology rings.

According to the physical interpretation, a quantum cohomology ring is a closed operator algebra acting on the fermionic Hilbert space. For example, the equation (3) one should better write as an equations for a linear operator $\mathcal{X}$ corresponding to the cohomology class x:

$$\mathcal{X}^{d+1} = \exp(-\int_{\lambda} \varphi) id. \tag{4}$$

It is convenient to define quantum rings by polynomial equations among generators.

Definition 1.1 Let

$$h(t, x) = \sum_{n \in \mathcal{N}} c_n(t) x^n$$

be a one-parameter family of polynomials in the polynomial ring $\mathbf{C}[x]$, where $x = \{x_i\}_{i \in I}$ is a set of variables indexed by I, t is a positive real number, $\mathcal{N}$ is a fixed finite set of exponents. We say that the polynomial

$$h^{\infty}(x) = \sum_{n \in \mathcal{N}} c_n^{\infty} x^n$$

is the limit of the family $h(t, x)$ as $t \to \infty$, if the point $\{c_n^{\infty}\}_{n \in \mathcal{N}}$ of the $(\mid \mathcal{N} \mid -1)$-dimensional complex projective space is the limit of the one-parameter family of points with homogeneous coordinates $\{c_n(t)\}_{n \in \mathcal{N}}$.

Definition 1.2 Let R_t be a one-parameter family of commutative algebras over $\mathbf{C}$ with a fixed set of generators $\{r_i\}$, $t \in \mathbf{R}_{>0}$. We denote by J_t the ideal in $\mathbf{C}[x]$ consisting of all polynomial relations among $\{r_i\}$, i.e., the kernel of the surjective homomorphism $\mathbf{C}[x] \to R_t$. We say that the ideal J^{∞} is the limit of J_t as $t \to \infty$, if any one-parameter family of polynomials $h(t, x) \in J_t$ (as in 1.1) has a limit, and J^{∞} is generated as $\mathbf{C}$-vector space by all these limits. The $\mathbf{C}$-algebra

$$R^{\infty} = \mathbf{C}[x]/J^{\infty}$$

will be called the *limit* of R_t.

Remark 1.3 In general, it is not true that if $J^\infty = \lim_{t\to\infty} J_t$, and J_t is generated by a finite set of polynomials $\{h_1(t,x)\ldots, h_k(t,x)\}$, then J^∞ is generated by the limits $\{h_1^\infty(x),\ldots,h_k^\infty(x)\}$. The limit ideal J^∞ is generated by the limits $h_i^\infty(x)$ *only if* the set of polynomials $\{h_i(t,x)\}$ form a *Gröbner-type basis* for J_t.

In this paper, we establish the following basic properties of quantum cohomology rings of toric manifolds:

I : If φ is an element in the interior of the Kähler cone $K(\mathbf{P}_\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{C})$, then there exists a limit of $QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C})$ as $t \to \infty$, and this limit is isomorphic to the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma, \mathbf{C})$ (Corollary 5.5).

II : Assume that two smooth projective toric manifolds $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are isomorphic in codimension 1, for instance, that $\mathbf{P}_{\Sigma_1}$ is obtained from $\mathbf{P}_{\Sigma_2}$ by a flop-type birational transformation. Then the natural isomorphism $H^2(\mathbf{P}_{\Sigma_1}, \mathbf{C}) \cong H^2(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ induces the isomorphism between the quantum cohomology rings

$$QH^*_\varphi(\mathbf{P}_{\Sigma_1}, \mathbf{C}) \cong QH^*_\varphi(\mathbf{P}_{\Sigma_2}, \mathbf{C})$$

(Theorem 6.1). We notice that ordinary cohomology rings of $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are not isomorphic in general.

III : Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ of $\mathbf{P}_\Sigma$ belongs to the closed Kähler cone $K(\mathbf{P}_\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{C})$. Then the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ is isomorphic to the Jacobian ring of a Laurent polynomial $f_\varphi(X)$ such that the equation $f_\varphi(X) = 0$ defines an affine Calabi-Yau hypersurface Z_f in the d-dimensional algebraic torus $(\mathbf{C}^*)^d$ where Z_f is *mirror symmetric* with respect to Calabi-Yau hypersurfaces in $\mathbf{P}_\Sigma$ (Theorem 8.4). Here by the *mirror symmetry* we mean the correspondence, based on the polar duality [6], between families of Calabi-Yau hypersurfaces in toric varieties.

The properties II and III give a general view on the recent result of P. Aspinwall, B. Greene, and D. Morrison [3] who have shown, for a family of Calabi-Yau 3-folds W that their quantum cohomology ring $QH^*_\varphi(W, \mathbf{C})$ does not change under a flop-type birational transformation (see also [1, 2]).

IV: Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ of $\mathbf{P}_\Sigma$ is divisible by r, i.e., there exists an element $h \in H^2(\mathbf{P}_\Sigma, \mathbf{Z})$ such that $c_1(\mathbf{P}_\Sigma) = rh$. Then $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has a natural $\mathbf{Z}/r\mathbf{Z}$-grading (Theorem 5.7). We remark that the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no $\mathbf{Z}$-grading.

The paper is organized as follows. In Sections 2-4, we recall a definition of toric manifolds and standard facts about them. In Section 5, we define the quantum cohomology ring of toric manifolds and prove their properties.

In Section 6, we consider examples of the behavior of quantum cohomology rings under elementary birational transformations such as blow-ups and flops, we also consider the case of singular toric varieties. In Section 7, we give an combinatorial interpretation of the relation between the quantum cohomology rings and the ordinary cohomology rings. In Section 8, we show that the quantum cohomology ring can be interpreted as a Jacobian ring of some Laurent polynomial. Finally, in Section 9, we prove that our quantum cohomology rings coincide with the quantum cohomology rings defined by sigma models on toric manifolds.

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2 A definition of compact toric manifolds

Toric varieties were considered in full generality in [9, 11]. For the general definition of toric variety which includes affine and quasi-projective toric varieties with singularities, it is more convenient to use the language of schemes. However, for our purposes, it will be sufficient to have a simplified more classical version of the definition for *smooth and compact toric varieties* over $\mathbf{C}$. This approach to compact toric manifolds was first proposed by M. Audin [4], and developed by D. Cox [8].

In order to obtain a d-dimensional compact toric manifold V, we need a combinatorial object Σ, a *complete fan of regular cones*, in a d-dimensional vector space over $\mathbf{R}$.

Let N, $M = \mathrm{Hom}(N, \mathbf{Z})$ be dual lattices of rank d, and $N_{\mathbf{R}}$, $M_{\mathbf{R}}$ their $\mathbf{R}$-scalar extensions to d-dimensional real vector spaces.

Definition 2.1 A convex subset $\sigma \subset N_{\mathbf{R}}$ is called a *regular k-dimensional cone* ($k \geq 1$) if there exist k linearly independent elements $v_1, \ldots, v_k \in N$ such that

$$\sigma = \{\mu_1 v_1 + \cdots + \mu_k v_k \mid \mu_i \in \mathbf{R}, \mu_i \geq 0\},$$

and $\{v_1, \ldots, v_k\}$ is a subset of some $\mathbf{Z}$-basis of N. In this case, we call $v_1, \ldots, v_k \in N$ the *integral generators of* σ.

The origin $0 \in N_{\mathbf{R}}$ we call the *regular* 0*-dimensional cone.* By definition, the set of integral generators of this cone is empty.

Definition 2.2 A regular cone σ' is called *a face* of a regular cone σ (we write $\sigma' \prec \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.

Definition 2.3 A finite system $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of regular cones in $N_{\mathbf{R}}$ is called *a complete d-dimensional fan* of regular cones, if the following conditions are satisfied:

(i) if $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma' \in \Sigma$;

(ii) if σ, σ' are in Σ, then $\sigma \cap \sigma' \prec \sigma$ and $\sigma \cap \sigma' \prec \sigma'$;

(iii) $N_{\mathbf{R}} = \sigma_1 \cup \cdots \cup \sigma_s$.

The set of all k-dimensional cones in Σ will be denoted by $\Sigma^{(k)}$.

Example 2.4 Choose $d+1$ vectors $v_1, \ldots, v_{d+1}$ in a d-dimensional real space E such that E is spanned by $v_1, \ldots, v_{d+1}$ and there exists the linear relation

$$v_1 + \cdots + v_{d+1} = 0.$$

Define N to be the lattice in E consisting of all integral linear combinations of $v_1, \ldots, v_{d+1}$. Obviously, $N_{\mathbf{R}} = E$. Then any k-element subset $I \subset \{v_1, \ldots, v_{d+1}\}$ $(k \leq d)$ generates a k-dimensional regular cone $\sigma(I)$. The set $\Sigma(d)$ consisting of $2^{d+1} - 1$ cones $\sigma(I)$ generated by I is a complete d-dimensional fan of regular cones.

Definition 2.5 (cf.[5]) Let Σ be a complete d-dimensional fan of regular cones. Denote by $G(\Sigma) = \{v_1, \ldots, v_n\}$ the set of all generators of 1-dimensional cones in Σ ($n = \mathrm{Card}\, \Sigma^{(1)}$). We call a subset $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\} \subset G(\Sigma)$ a *primitive collection* if $\{v_{i_1}, \ldots, v_{i_p}\}$ is not the set of generators of a p-dimensional simplicial cone in Σ, while for all k $(0 \leq k < p)$ each k-element subset of $\mathcal{P}$ generates a k-dimensional cone in Σ.

Example 2.6 Let Σ be a fan $\Sigma(d)$ from Example 2.4. Then there exists the unique primitive collection $\mathcal{P} = G(\Sigma(d))$.

Definition 2.7 Let $\mathbf{C}^n$ be n-dimensional affine space over $\mathbf{C}$ with the set of coordinates $z_1, \ldots, z_n$ which are in the one-to-one correspondence $z_i \leftrightarrow v_i$ with elements of $G(\Sigma)$. Let $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\}$ be a primitive collection in $G(\Sigma)$. Denote by $\mathbf{A}(\mathcal{P})$ the $(n-p)$-dimensional affine subspace in $\mathbf{C}^n$ defined by the equations

$$z_{i_1} = \cdots = z_{i_p} = 0.$$

Remark 2.8 Since every primitive collection $\mathcal{P}$ has at least two elements, the codimension of $\mathbf{A}(\mathcal{P})$ is at least 2.

Definition 2.9 Define the closed algebraic subset $Z(\Sigma)$ in $\mathbf{C}^n$ as follows

$$Z(\Sigma) = \bigcup_{\mathcal{P}} \mathbf{A}(\mathcal{P}),$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$. Put

$$U(\Sigma) = \mathbf{C}^n \setminus Z(\Sigma).$$

Definition 2.10 Two complete d-dimensional fans of regular cones Σ and Σ' are called *combinatorially equivalent* if there exists a bijective mapping $\Sigma \to \Sigma'$ respecting the face-relation "$\prec$" (see 2.2).

Remark 2.11 It is easy to see that the open subset $U(\Sigma) \subset \mathbf{C}^n$ depends only on the combinatorial structure of Σ, i.e., for any two combinatorially equivalent fans Σ and Σ', one has $U(\Sigma) \cong U(\Sigma')$.

Definition 2.12 Let $R(\Sigma)$ be the subgroup in $\mathbf{Z}^n$ consisting of all lattice vectors $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$.

Obvioulsy, $R(\Sigma)$ is isomorphic to $\mathbf{Z}^{n-d}$.

Definition 2.13 Let Σ be a complete d-dimensional fan of regular cones. Define $\mathbf{D}(\Sigma)$ to be the connected commutative subgroup in $(\mathbf{C}^*)^n$ generated by all one-parameter subgroups

$$\begin{aligned} a_\lambda \ &: \ \mathbf{C}^* \to (\mathbf{C}^*)^n, \\ t &\to (t^{\lambda_1}, \ldots, t^{\lambda_n}) \end{aligned}$$

where $\lambda \in R(\Sigma)$.

Remark 2.14 Choosing a $\mathbf{Z}$-basis in $R(\Sigma)$, one easily obtains an isomorphism between $\mathbf{D}(\Sigma)$ and $(\mathbf{C}^*)^{n-d}$.

Now we are ready to give the definition of the compact toric manifold $\mathbf{P}_\Sigma$ associated with a complete d-dimensional fan of regular cones Σ.

Definition 2.15 Let Σ be a complete d-dimensional fan of regular cones. Then quotient

$$\mathbf{P}_\Sigma = U(\Sigma)/\mathbf{D}(\Sigma)$$

is called the *compact toric manifold associated with* Σ.

Example 2.16 Let Σ be a fan $\Sigma(d)$ from Example 2.4. By 2.6, $U(\Sigma(d)) = \mathbf{C}^{d+1} \setminus \{0\}$. By the definition of $\Sigma(d)$, the subgroup $R(\Sigma) \subset \mathbf{Z}^n$ is generated by $(1, \ldots, 1) \in \mathbf{Z}^{d+1}$. Thus, $\mathbf{D}(\Sigma) \subset (\mathbf{C}^*)^n$ consists of the elements $(t, \ldots, t)$, where $t \in \mathbf{C}^*$. So the toric manifold associated with $\Sigma(d)$ is the ordinary d-dimensional projective space.

A priori, it is not obvious that the quotient space $\mathbf{P}_\Sigma = U(\Sigma)/\mathbf{D}(\Sigma)$ always exists as the space of orbits of the group $\mathbf{D}(\Sigma)$ acting free on $U(\Sigma)$, and that $\mathbf{P}_\Sigma$ is smooth and compact. However, these facts are easy to check if we take the d-dimensional projective space $\mathbf{P}_{\Sigma(d)}$ as a model example.

There exists a simple open covering of $U(\Sigma)$ by affine algebraic varieties:

Proposition 2.17 *Let σ be a k-dimensional cone in Σ, $\{v_{i_1}, \ldots, v_{i_k}\}$ the set of generators of σ. Define the open subset $U(\sigma) \subset \mathbf{C}^n$ by the conditions $z_j \neq 0$ for all $j \notin \{i_1, \ldots, i_k\}$. Then the open sets $U(\sigma)$ ($\sigma \in \Sigma$) have the properties:*

(i)
$$U(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma);$$

(ii) *if $\sigma \prec \sigma'$, then $U(\sigma) \subset U(\sigma')$;*

(iii) *for any two cone $\sigma_1, \sigma_2 \in \Sigma$, one has $U(\sigma_1) \cap U(\sigma_2) = U(\sigma_1 \cap \sigma_2)$; in particular,*
$$U(\Sigma) = \bigcup_{\sigma \in \Sigma^{(d)}} U(\sigma).$$

Proposition 2.18 *Let σ be a d-dimensional cone in $\Sigma^{(d)}$, $\{v_{i_1}, \ldots, v_{i_d}\}$ the set of generators of σ. Denote by $u_{i_1}, \ldots, u_{i_d}$ the dual to $v_{i_1}, \ldots, v_{i_d}$ $\mathbf{Z}$-basis of the lattice M, i.e, $\langle v_{i_k}, u_{i_l} \rangle = \delta_{k,l}$, where $\langle *, * \rangle : N \times M \to \mathbf{Z}$ is the canonical pairing between lattices N and M.*

Then the affine open subset $U(\sigma)$ is isomorphic to $\mathbf{C}^d \times (\mathbf{C}^*)^{n-d}$, the action of $\mathbf{D}(\Sigma)$ on $U(\sigma)$ is free, and the space of $\mathbf{D}(\Sigma)$-orbits is isomorphic to the affine space $U_\sigma = \mathbf{C}^d$ whose coordinate functions $x_1^\sigma, \ldots, x_d^\sigma$ are the following Laurent monomials in $z_1, \ldots, z_n$:

$$x_1^\sigma = z_1^{\langle v_1, u_{i_1} \rangle} \cdots z_n^{\langle v_n, u_{i_1} \rangle}, \ \ldots \ , x_d^\sigma = z_1^{\langle v_1, u_{i_d} \rangle} \cdots z_n^{\langle v_n, u_{i_d} \rangle}.$$

The last statement yields a general formula for the local affine coordinates $x_1^\sigma, \ldots, x_d^\sigma$ of a point $p \in U_\sigma$ as functions of its *homogeneous coordinates* $z_1, \ldots, z_n$ (see also [8]).

Compactness of $\mathbf{P}_\Sigma$ follows from the fact that the local polydiscs

$$D_\sigma = \{x \in U_\sigma \ : \mid x_1^\sigma \mid \leq 1, \ldots, \mid x_d^\sigma \mid \leq 1\}, \ \ \sigma \in \Sigma^{(d)}$$

form a finite compact covering of $\mathbf{P}_\Sigma$.

3 Cohomology of toric manifolds

Let Σ be a complete d-dimensional fan of regular cones.

Definition 3.1 A continuous function $\varphi \ : \ N_\mathbf{R} \to \mathbf{R}$ is called Σ-*piecewise linear*, if φ is a linear function on every cone $\sigma \in \Sigma$.

Remark 3.2 It is clear that any Σ-piecewise linear function φ is uniquely defined by its values on elements v_i of $G(\Sigma)$. So the space of all Σ-piecewise linear functions $PL(\Sigma)$ is canonically isomorphic to $\mathbf{R}^n$: $\varphi \to (\varphi(v_1), \ldots, \varphi(v_n))$.

Theorem 3.3 *The space $PL(\Sigma)/M_\mathbf{R}$ of all Σ-piecewise linear functions modulo the d-dimensional subspace of globally linear functions on $N_\mathbf{R}$ is canonically isomorphic to the cohomology space $H^2(\mathbf{P}_\Sigma, \mathbf{R})$. Moreover, the first Chern class $c_1(\mathbf{P}_\Sigma)$, as an element of $H^2(\mathbf{P}_\Sigma, \mathbf{Z})$, is represented by the class of the Σ-piecewise linear function $\alpha_\Sigma \in PL(\Sigma)$ such that $\alpha_\Sigma(v_1) = \cdots = \alpha_\Sigma(v_n) = 1$.*

Theorem 3.4 *Let $R(\Sigma)_\mathbf{R}$ be the $\mathbf{R}$-scalar extension of the abelian group $R(\Sigma)$. Then the space $R(\Sigma)_\mathbf{R}$ is canonically isomorphic to $H_2(\mathbf{P}_\Sigma, \mathbf{R})$.*

Definition 3.5 Let φ be an element of $PL(\Sigma)$, λ an element of $R(\Sigma)_\mathbf{R}$. Define the *degree of λ relative to φ* as

$$\deg_\varphi(\lambda) = \sum_{i=1}^{n} \lambda_i \varphi(v_i).$$

It is easy to see that for any $\varphi \in M_\mathbf{R}$ and for any $\lambda \in R(\Sigma)_\mathbf{R}$, one has $\deg_\varphi(\lambda) = 0$. Moreover, the degree-mapping induces the nondegenerate pairing

$$\deg \ : \ PL(\Sigma)/M_\mathbf{R} \times R(\Sigma)_\mathbf{R} \to \mathbf{R}$$

which coincides with the canonical intersection pairing

$$H^2(\mathbf{P}_\Sigma, \mathbf{R}) \times H_2(\mathbf{P}_\Sigma, \mathbf{R}) \to \mathbf{R}.$$

Definition 3.6 Let $\mathbf{C}[z]$ be the polynomial ring in n variables $z_1, \ldots, z_n$. Denote by $SR(\Sigma)$ the ideal in $\mathbf{C}[z]$ generated by all monomials

$$\prod_{v_j \in \mathcal{P}} z_j,$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$. The ideal $SR(\Sigma)$ is usually called the *Stenley-Reisner ideal* of Σ.

Definition 3.7 Let $u_1, \ldots, u_d$ be any $\mathbf{Z}$-basis of the lattice M. Denote by $P(\Sigma)$ the ideal in $\mathbf{C}[z]$ generated by d elements

$$\sum_{i=1}^{n} \langle v_i, u_1 \rangle z_i, \ldots, \sum_{i=1}^{n} \langle v_i, u_d \rangle z_i.$$

Obviously, the ideal $P(\Sigma)$ does not depend on the choice of basis of M.

Theorem 3.8 *The cohomology ring of the compact toric manifold* $\mathbf{P}_\Sigma$ *is canonically isomorphic to the quotient of* $\mathbf{C}[z]$ *by the sum of two ideals* $P(\Sigma)$ *and* $SR(\Sigma)$:

$$H^*(\mathbf{P}_\Sigma, \mathbf{C}) \cong \mathbf{C}[z]/(P(\Sigma) + SR(\Sigma)).$$

Moreover, the canonical embedding $H^2(\mathbf{P}_\Sigma, \mathbf{C}) \hookrightarrow H^*(\mathbf{P}_\Sigma, \mathbf{C})$ *is induced by the linear mapping*

$$PL(\Sigma) \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C}[z], \quad \varphi \mapsto \sum_{i=1}^{n} \varphi_i(v_i) z_i.$$

In particular, the first Chern class of $\mathbf{P}_\Sigma$ *is represented by the sum* $z_1 + \cdots + z_n$.

Example 3.9 Let $\mathbf{P}_\Sigma$ be d-dimensional projective space defined by the fan $\Sigma(d)$ (see 2.4). Then

$$P(\Sigma(d)) = < (z_1 - z_{d+1}), \ldots, (z_d - z_{d+1}) >, \; SR(\Sigma(d)) = < \prod_{i=1}^{d+1} z_i > .$$

So we obtain

$$\mathbf{C}[z_1, \ldots, z_{d+1}]/(P(\Sigma(d)) + SR(\Sigma(d)) \cong \mathbf{C}[x]/x^{d+1}.$$

4 Line bundles and Kähler classes

Let $\pi : U(\Sigma) \to \mathbf{P}_\Sigma$ be the canonical projection whose fibers are principal homogeneous spaces of $\mathbf{D}(\Sigma)$. For any line bundle $\mathcal{L}$ over $\mathbf{P}_\Sigma$, the pullback $\pi^*\mathcal{L}$ is a line bundle over $U(\Sigma)$. By 2.8, $\pi^*\mathcal{L}$ is isomorphic to $\mathcal{O}_{U(\Sigma)}$. Therefore, the Picard group of $\mathbf{P}_\Sigma$ is isomorphic to the group of all $\mathbf{D}$-linearization of $\mathcal{O}_{U(\Sigma)}$, or to the group of all characters $\chi : \mathbf{D}(\Sigma) \to \mathbf{C}^*$. The latter is isomorphic to the group $\mathbf{Z}^n/M$ where $\mathbf{Z}^n$ is the group of all Σ-piecewise linear functions φ such that $\varphi(N) \subset \mathbf{Z}$.

Proposition 4.1 *Assume that a character χ is represented by the class of an integral Σ-piecewise linear function φ. Then the space $H^0(\mathbf{P}_\Sigma, \mathcal{L}_\chi)$ of global sections of the corresponding line bundle $\mathcal{L}_\chi$, is canonically isomorphic to the space of all polynomials $F(z_1, \ldots, z_n) \in \mathbf{C}[z]$ satisfying the condition*

$$F(t^{\lambda_1} z_1, \ldots, t^{\lambda_n} z_n) = t^{\deg_\varphi \lambda} F(z_1, \ldots, z_n)$$

$$for \ \ all \ \ \lambda \in R(\Sigma), \ t \in \mathbf{C}^*.$$

The exponents $(m_1, \ldots, m_n)$ of the monomials satisfying the above condition can be identified with integral points in the convex polyhedron:

$$\Delta_\varphi = \{(x_1, \ldots, x_n) \in \mathbf{R}^n_{\geq 0} : \deg_\varphi \lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n, \ \ \lambda \in R(\Sigma)\}$$

Definition 4.2 A Σ-piecewise linear function $\varphi \in PL(\Sigma)$ is called a *strictly convex support* function for the fan Σ, if φ satisfies the properties

(i) φ is an *upper convex function*, i.e.,

$$\varphi(x) + \varphi(y) \geq \varphi(x + y);$$

(ii) for any two different d-dimensional cones σ_1, $\sigma_2 \in \Sigma$, the restrictions $\varphi|_\sigma$ and $\varphi|_{\sigma'}$ are different linear functions.

Proposition 4.3 *If φ is a strictly convex support function, then the polyhedron Δ_φ is simple (i.e., any vertex of Δ_φ is contained in d-faces of codimension 1), and the fan Σ can be uniquely recovered from Δ_φ using the property:*

$$\Delta_\varphi \cong \{x \in M_\mathbf{R} \ : \ \langle v_i, x \rangle \geq -\varphi(v_i)\}.$$

Definition 4.4 Denote by $K(\Sigma)$ the cone in $H^2(\mathbf{P}_\Sigma, \mathbf{R}) = PL(\Sigma)/M_\mathbf{R}$ consisting of the classes of all upper convex Σ-piecewise linear functions $\varphi \in PL(\Sigma)$. We denote by $K^0(\Sigma)$ the interior of $K(\Sigma)$, i.e., the cone consisting of the classes of all strictly convex support functions in $PL(\Sigma)$.

Theorem 4.5 *The open cone $K^0(\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{R})$ consists of classes of Kähler $(1,1)$-forms on $\mathbf{P}_\Sigma$, i.e., $K(\Sigma)$ is isomorphic to the closed Kähler cone of $\mathbf{P}_\Sigma$.*

Next theorem will play the central role in the sequel. Its statement is contained implicitly in [12, 13]:

Theorem 4.6 *A Σ-piecewise linear function φ is a strictly convex support function, i.e., $\varphi \in K^0(\Sigma)$, if and only if*

$$\varphi(v_{i_1}) + \cdots + \varphi(v_{i_k}) > \varphi(v_{i_1} + \cdots + v_{i_k})$$

for all primitive collections $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_k}\}$ in $G(\Sigma)$.

5 Quantum cohomology rings

Definition 5.1 Let φ be a Σ-piecewise linear function with complex values, or an element of the complexified space $PL(\Sigma)_{\mathbf{C}} = PL(\Sigma) \otimes_{\mathbf{R}} \mathbf{C}$. Define the quantum cohomology ring as the quotient of the polynomial ring $\mathbf{C}[z]$ by the sum of ideals $P(\Sigma)$ and $Q_\varphi(\Sigma)$:

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}) := \mathbf{C}[z]/(P(\Sigma) + Q_\varphi(\Sigma))$$

where $Q_\varphi(\Sigma)$ is generated by binomials

$$\exp\left(\sum_{i=1}^n a_i \varphi(v_i)\right) \prod_{i=1}^n z_i^{a_i} - \exp\left(\sum_{j=1}^n b_j \varphi(v_j)\right) \prod_{j=1}^n z_j^{b_j}$$

running over all possible linear relations

$$\sum_{i=1}^n a_i v_i = \sum_{j=1}^n b_j v_j,$$

where all coefficients a_i and b_j are non-negative and integral.

Definition 5.2 Let $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_k}\} \subset G(\Sigma)$ be a primitive collection, $\sigma_\mathcal{P}$ the minimal cone in Σ containing the sum

$$v_\mathcal{P} = v_{i_1} + \ldots + v_{i_k},$$

$v_{j_1}, \ldots, v_{j_l}$ generators of $\sigma_{\mathcal{P}}$. Let l be the dimension of $\sigma_{\mathcal{P}}$. By 2.3(iii), there exists the unique representation of $v_{\mathcal{P}}$ as an integral linear combination of generators $v_{j_1}, \ldots, v_{j_l}$ with positive integral coefficients $c_1, \ldots, c_l$:

$$v_{\mathcal{P}} = c_1 v_{j_1} + \cdots + c_l v_{j_l},$$

We put

$$E_{\varphi}(\mathcal{P}) = \exp(\varphi(v_{i_1} + \ldots + v_{i_k}) - \varphi(v_{i_1}) - \ldots - \varphi(v_{i_k}))$$

$$= \exp(c_1\varphi(v_{j_1}) + \cdots + c_l\varphi(v_{j_l}) - \varphi(v_{i_1}) - \ldots - \varphi(v_{i_k})).$$

Theorem 5.3 *Assume that the Kähler cone $K(\Sigma)$ has non-empty interior, i.e., $\mathbf{P}_{\Sigma}$ is projective. Then the ideal $Q_{\varphi}(\Sigma)$ is generated by the binomials*

$$B_{\varphi}(\mathcal{P}) = z_{i_1} \cdots z_{i_k} - E_{\varphi}(\mathcal{P}) z_{j_1}^{c_1} \cdots z_{j_l}^{c_l},$$

where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$.

Proof. We use some ideas from [14]. Let ϕ be an element in $PL(\Sigma)$ representing an interior point of $K(\Sigma)$. Define the weights $\omega_1, \ldots, \omega_n$ of $z_1, \ldots, z_n$ as

$$\omega_i = \phi(v_i) \ \ (1 \leq i \leq n).$$

We claim that binomials $B_{\varphi}(\mathcal{P})$ form a reduced Gröbner basis for $Q_{\varphi}(\Sigma)$ relative to the weight vector

$$\omega = (\omega_1, \ldots, \omega_n).$$

Notice that the weight of the monomial $z_{i_1} \cdots z_{i_k}$ is greater than the weight of the monomial $z_{j_1}^{c_1} \cdots z_{j_l}^{c_l}$, because

$$\phi(v_{i_1}) + \cdots + \phi(v_{i_k}) > \phi(v_{i_1} + \cdots + v_{i_k}) = c_1\phi(v_{j_1}) + \cdots c_l\phi(v_{j_l})$$

(Theorem 4.6). So the initial ideal $init_{\omega}\langle B_{\varphi}(\mathcal{P})\rangle$ of the ideal $\langle B_{\varphi}(\mathcal{P})\rangle$ generated by $B_{\varphi}(\mathcal{P})$ coincides with the ideal $SR(\Sigma)$. It suffices to show that the initial ideal $init_{\omega}Q_{\varphi}(\Sigma)$ also equals $SR(\Sigma)$. The latter again follows from Theorem 4.6. □

Definition 5.4 The tube domain in the cohomology space $H^2(\mathbf{P}_{\Sigma}, \mathbf{C})$:

$$K(\Sigma)_{\mathbf{C}} = K(\Sigma) + iH^2(\mathbf{P}_{\Sigma}, \mathbf{R})$$

we call the *complexified Kähler cone of* $\mathbf{P}_{\Sigma}$.

Corollary 5.5 *Let φ be an element of $H^2(\mathbf{P}_\Sigma, \mathbf{C})$, t a positive real number. Then all generators $B_{t\varphi}(\mathcal{P})$ of the ideal $Q_{t\varphi}(\Sigma)$ have finite limits as $t \to \infty$ if and only if $\varphi \in K(\Sigma)_{\mathbf{C}}$. Moreover, if $\varphi \in K(\Sigma)_{\mathbf{C}}$, then the limit of $QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C})$ is the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma, \mathbf{C})$.*

Proof. Applying Theorem 4.6, we obtain:

$$\lim_{t\to\infty} B_{t\varphi}(\mathcal{P}) = z_{i_1} \cdots z_{i_k}.$$

Thus,

$$\lim_{t\to\infty} Q_{t\varphi}(\Sigma) = SR(\Sigma).$$

By Theorem 3.8,

$$\lim_{t\to\infty} QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C}) = H^*(\mathbf{P}_\Sigma, \mathbf{C}).$$

□

Example 5.6 Consider the fan $\Sigma(d)$ defining d-dimensional projective space (see 2.4). Then we obtain

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}) \cong \mathbf{C}[x]/(x^{d+1} - \exp(-\deg_\varphi \lambda)),$$

where $\lambda = (1, \ldots, 1)$ is the generator of $R(\Sigma(d))$. This shows the quantum cohomology ring $QH^*_\varphi(\mathbf{CP}^d, \mathbf{C})$ coincides with the quantum cohomology ring for $\mathbf{CP}^d$ proposed by physicists.

It is important to remark that the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no any $\mathbf{Z}$-grading, but it is possible to define a $\mathbf{Z}_N$-grading on it.

Theorem 5.7 Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ is divisible by r. Then the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has a natural $\mathbf{Z}/r\mathbf{Z}$-grading.

Proof. A linear relation

$$\sum_{i=1}^n a_i v_i = \sum_{j=1}^n b_j v_j$$

gives rise to an element

$$\lambda = (a_1 - b_1, \ldots, a_n - b_n) \in R(\Sigma).$$

By our assumption,

$$\deg_{\alpha_\Sigma} \lambda = \sum_{i=1}^n a_i - \sum_{j=1}^n b_j$$

is the intersection number of $c_1(\mathbf{P}_\Sigma)$ and $\lambda \in H_2(\mathbf{P}_\Sigma, \mathbf{C})$, i.e., it is divisible by r. This means that the binomials

$$\exp\left(\sum_{i=1}^{n} a_i\varphi(v_i)\right)\prod_{i=1}^{n} z_i^{a_i} - \exp\left(\sum_{j=1}^{n} b_j\varphi(v_j)\right)\prod_{j=1}^{n} z_j^{b_j}$$

are $\mathbf{Z}/r\mathbf{Z}$-homogeneous. □

Although, the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no any $\mathbf{Z}$-grading, it is possible to define a graded version of this quantum cohomology ring over the Laurent polynomial ring $\mathbf{C}[z_0, z_0^{-1}]$.

Definition 5.8 Let φ be a Σ-piecewise linear function with complex values from the complexified space $PL(\Sigma)_{\mathbf{C}} = PL(\Sigma) \otimes_{\mathbf{R}} \mathbf{C}$. Define the quantum cohomology ring

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}])$$

as the quotient of the Laurent polynomial extension $\mathbf{C}[z][z_0, z_0^{-1}]$ by the sum of ideals $Q_{\varphi,z_0}(\Sigma)$ and $P(\Sigma)$: where $Q_{\varphi,z_0}(\Sigma)$ is generated by binomials

$$\exp\left(\sum_{i=1}^{n} a_i\varphi(v_i)\right) z_0^{(-\sum_{i=1}^{n} a_i)}\prod_{i=1}^{n} z_i^{a_i} - \exp\left(\sum_{j=1}^{n} b_j\varphi(v_j)\right) z_0^{(-\sum_{j=1}^{n} b_j)}\prod_{l=1}^{n} z_j^{b_j}$$

running over all possible linear relations

$$\sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{n} b_j v_j$$

with non-negative integer coefficients a_i and b_j.

The properties of the $\mathbf{Z}$-graded quantum cohomology ring

$$QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}])$$

are analogous to the properties of $QH^*(\mathbf{P}_\Sigma, \mathbf{C})$:

Theorem 5.9 *For every binomial $B_\varphi(\mathcal{P})$, take the corresponding homogeneous binomial in variables $z_0, z_1, \ldots, z_n$*

$$B_{\varphi,z_0}(\mathcal{P}) = z_{i_1}\cdots z_{i_k} - E_\varphi(\mathcal{P}) z_{j_1}^{c_1}\cdots z_{j_l}^{c_l} z_0^{(k-\sum_{s=1}^{l} c_s)}.$$

Then the elements $B_{\varphi,z_0}(\mathcal{P})$ generate the ideal $Q_{\varphi,z_0}(\Sigma)$, and Kähler limits of

$$QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}]), \quad t \to \infty$$

are isomorphic to the Laurent polynomial extension $H^*(\mathbf{P}_\Sigma, \mathbf{C})[z_0, z_0^{-1}]$ of the odinary cohomology ring.

Finally, if the first Chern class of $\mathbf{P}_\Sigma$ belongs to the Kähler cone, i.e., $\alpha_\Sigma \in PL(\Sigma)$ is upper convex, then it is possible to define the quantum deformations of the cohomology ring of $\mathbf{P}_\Sigma$ over the polynomial ring $\mathbf{C}[z_0]$.

Definition 5.10 Assume that $\alpha_\Sigma \in PL(\Sigma)$ is upper convex. We define the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$ over $\mathbf{C}[z_0]$ as the quotient of the polynomial ring $\mathbf{C}[z_0, z_1, \dots, z_n]$ by the sum of the ideal $P(\Sigma)[z_0]$ and the ideal

$$\mathbf{C}[z_0, z_1, \dots, z_n] \cap Q_{\varphi, z_0}(\Sigma)$$

Theorem 5.11 *The ideal*

$$\mathbf{C}[z_0, z_1, \dots, z_n] \cap Q_{\varphi, z_0}(\Sigma)$$

is generated by homogeneous binomials

$$B_{\varphi, z_0}(\mathcal{P}) = z_{i_1} \cdots z_{i_k} - E_\varphi(\mathcal{P}) z_{j_1}^{c_1} \cdots z_{j_l}^{c_l} z_0^{(k - \sum_{s=1}^{l} c_s)}$$

where $\mathcal{P}$ runs over all primitive collections $\mathcal{P} \subset G(\Sigma)$. (Notice that convexity of α_Σ implies $k - \sum_{s=1}^{l} c_s \geq 0$.)

Kähler limits of the quantum cohomology ring

$$QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C}[z_0]), \quad t \to \infty$$

are isomorphic to the polynomial extension

$$H^*(\mathbf{P}_\Sigma, \mathbf{C})[z_0]$$

of the odinary cohomology ring.

6 Birational transformations

It may look strange that we defined the quantum cohomology rings using infinitely many generators for the ideals $Q_\varphi(\Sigma)$ and $Q_{\varphi, z_0}(\Sigma)$, while these ideals have only finite number of generators indexed by primitive collections in $G(\Sigma)$. The reason for that is the following important theorem:

Theorem 6.1 *Let Σ_1 and Σ_2 be two complete fans of regular cones such that $G(\Sigma_1) = G(\Sigma_2)$, then the quantum cohomology rings $QH^*_\varphi(\mathbf{P}_{\Sigma_1}, \mathbf{C})$ and $QH^*_\varphi(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ are isomorphic.*

Proof. Our definitions of quantum cohomology rings does not depend on the combinatiorial structure of the fan Σ, one needs to know only all lattice vectors $v_1, \ldots, v_n \in G(\Sigma)$, but not the combinatorial structure of the fan Σ. □

Since the equality $G(\Sigma_1) = G(\Sigma_2)$ means that two toric varieties $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are isomorphic in codimension 1, we obtain

Corollary 6.2 *Let $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ be two smooth compact toric manifolds which are isomorphic in codimension 1, then the rings $QH^*_\varphi(\mathbf{P}_1, \mathbf{C})$ and $QH^*_\varphi(\mathbf{P}_2, \mathbf{C})$ are isomorphic.*

Example 6.3 Consider two 3-dimensional fans Σ_1 and Σ_2 in $\mathbf{R}^3$ such that $G(\Sigma_1) = G(\Sigma_2) = \{v_1, \ldots, v_6\}$ where

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1),$$

$$v_4 = (-1,0,0),\ v_5 = (0,-1,0),\ v_6 = (1,1,-1).$$

We define the combinatorial structure of Σ_1 by the primitive collections

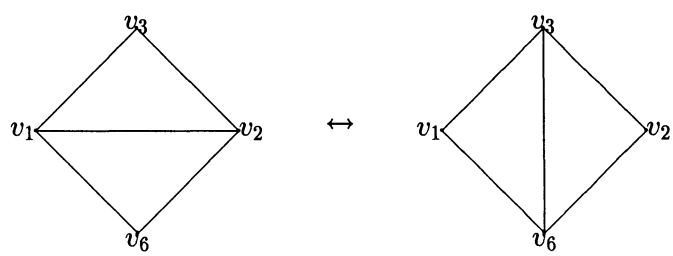
$$\mathcal{P}_1 = \{v_1, v_4\},\ \mathcal{P}_2 = \{v_2, v_5\},\ \mathcal{P}_3 = \{v_3, v_6\},$$

and the combinatorial structure of Σ_2 by the primitive collections

$$\mathcal{P}'_1 = \{v_1, v_4\},\ \mathcal{P}'_2 = \{v_2, v_5\},\ \mathcal{P}'_3 = \{v_1, v_2\},$$

$$\mathcal{P}'_4 = \{v_3, v_5, v_6\},\ \mathcal{P}'_5 = \{v_3, v_4, v_6\}.$$

The flop between two toric manifolds is described by the diagrams:



The ordinary cohomology rings $H^*(\mathbf{P}_{\Sigma_1}, \mathbf{C})$ and $H^*(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ are not isomorphic, because their homogeneous ideals of polynomial relations among $z_1, \ldots, z_6$ have different numbers of minimal generators. There exists the polynomial relation in the quantum cohomology ring:

$$\exp(\varphi(v_1) + \varphi(v_2))z_1z_2 = \exp(\varphi(v_3) + \varphi(v_6))z_3z_6.$$

If $\varphi(v_1) + \varphi(v_2) < \varphi(v_3) + \varphi(v_6)$, then we obtain the element $z_3z_6 \in SR(\Sigma_1)$ as the limit for $t\varphi$, when $t \to \infty$. On the other hand, if $\varphi(v_1) + \varphi(v_2) > \varphi(v_3) + \varphi(v_6)$, taking the same limit, we obtain $z_1z_2 \in SR(\Sigma_2)$.

Let us consider another simple example of birational tranformation.

Example 6.4 The quantum cohomology ring of the 2-dimensional toric variety F_1 which is the blow-up of a point p on $\mathbf{P}^2$ is isomorphic to the quotient of the polynomial ring $\mathbf{C}[x_1, x_2]$ by the ideal generated by two binomials

$$x_1(x_1 + x_2) = \exp(-\phi_2);\ x_2^2 = \exp(-\phi_1)x_1,$$

where x_1 is the class of the (-1)-curve C_1 on F_1, x_2 is the class of the fiber C_2 of the projection of F_1 on $\mathbf{P}^1$. The numbers ϕ_1 and ϕ_2 are respectively degrees of the restriction of the Kähler class φ on C_1 and C_2.

Remark 6.5 The definition of the quantum cohomology ring for smooth toric manifolds immediatelly extends to the case of singular toric varieties. However, the ordinary cohomology ring of singular toric varieties is not anymore the Kähler limit of the quantum cohomology ring. In some cases, the quantum cohomology ring of singular toric varieties V contains information about the ordinary cohomology ring of special desingularizations V' of V. For instance, if we assume that there exists a projective desingularization $\psi : V' \to V$ such that $\psi^*\mathcal{K}_V = \mathcal{K}_{V'}$. Then for every Kähler class $\varphi \in H^2(V, \mathbf{C})$, one has

$$\dim_{\mathbf{C}} QH^*_\varphi(V, \mathbf{C}) = \dim_{\mathbf{C}} H^*(V', \mathbf{C}).$$

7 Geometric interpretation of quantum rings

The spectra of the quantum cohomology ring $\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$, and its two polynomial versions

$$\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}]),\ \operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$$

have simple geometric interpretations.

Definition 7.1 Denote by $\Pi(\Sigma)$ the $(n-d)$-dimensional affine subspace in $\mathbf{C}^n$ defined by the ideal $P(\Sigma)$.

Definition 7.2 Choose any isomorphism $N \cong \mathbf{Z}^d$, so that any element $v \in N$ defines a Laurent monomial X^v in d variables $X_1, \ldots, X_d$. Consider the embedding of the d-dimensional torus $T(\Sigma) \cong (\mathbf{C}^*)^d$ in $(\mathbf{C}^*)^n$:

$$(X_1, \ldots, X_d) \to (X^{v_1}, \ldots, X^{v_n}).$$

Denote by $\Theta(\Sigma)$ the $(n-d)$-dimensional algebraic torus $(\mathbf{C}^*)^n/T(\Sigma)$.

Definition 7.3 Denote by Exp the analytical exponential mapping

$$Exp \ : \ \mathcal{G} \to \mathbf{G}$$

where $\mathbf{G}$ is a complex analytic Lie group, and $\mathcal{G}$ is its Lie algebra.

For example, one has the exponential mapping

$$Exp \ : \ PL(\Sigma)_{\mathbf{C}} \to (\mathbf{C}^*)^n$$

$$\varphi \mapsto (e^{\varphi(v_1)}, \ldots, e^{\varphi(v_n)})$$

which descends to the exponential mapping

$$Exp \ : \ H^2(\mathbf{P}_\Sigma, \mathbf{C}) \to \Theta(\Sigma).$$

Proposition 7.4 *The $\mathbf{T}(\Sigma)$-orbit $T_\varphi(\Sigma)$ of the point $Exp(\varphi) \in (\mathbf{C}^*)^n$ is closed, and its ideal is canonically isomorphic to $Q_\varphi(\Sigma)$.*

Corollary 7.5 *The scheme* $\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ *is the scheme-theoretic intersection of the d-dimensional subvariety $\overline{T}_\varphi(\Sigma) \subset \mathbf{C}^n$ and the $(n-d)$-dimensional subspace $\Pi(\Sigma)$.*

Definition 7.6 Let $\tilde{N} = \mathbf{Z} \oplus N$. For any $v \in N$, define $\tilde{v} \in \tilde{N}$ as $\tilde{v} = (1, v)$. Define the embedding of the $(d+1)$-dimensional torus $T^\circ(\Sigma) \cong (\mathbf{C}^*)^{d+1}$ in $(\mathbf{C}^*)^{n+1}$:

$$(X_0, X_1, \ldots, X_d) \to (X_0, X^{\tilde{v_1}}, \ldots, X^{\tilde{v_n}}).$$

The quotient $(\mathbf{C}^*)^{n+1}/T^\circ(\Sigma)$ is again isomorphic to $\Theta(\Sigma)$.

Proposition 7.7 *The ideal of the $T^\circ(\Sigma)$-orbit*

$$T^\circ_\varphi(\Sigma) \subset \mathbf{C}^* \times \mathbf{C}^n$$

of the of the point $(1, Exp\,(\varphi)) \in (\mathbf{C}^*)^{n+1}$ is canonically isomorphic to $Q_{\varphi, z_0}(\Sigma)$.

Corollary 7.8 *The scheme* $\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0, z_0^{-1}])$ *is the scheme-theoretic intersection of the* $(d+1)$*-dimensional subvariety* $T^\circ_\varphi(\Sigma) \subset \mathbf{C}^* \times \mathbf{C}^n$ *and the* $(n-d+1)$*-dimensional subvariety* $\mathbf{C}^* \times \Pi(\Sigma) \subset \mathbf{C}^* \times \mathbf{C}^n$.

Similarly, one obtain the geometric interpretation of $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$, when the first Chern class of $\mathbf{P}_\Sigma$ belongs to the Kähler cone $K(\Sigma)$.

Proposition 7.9 *The scheme* $\operatorname{Spec} QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C}[z_0])$ *is the scheme-theoretic intersection in* $\mathbf{C}^{n+1}$ *of the* $(d+1)$*-dimensional* $T^\circ(\Sigma)$*-orbit of* $(1, Exp(\varphi))$ *and the* $(n-d+1)$*-dimensional affine subspace* $\mathbf{C} \times \Pi(\Sigma) \subset \mathbf{C}^{n+1}$.

The limits of quantum cohomology rings have also geometric interpretations. One obtains, for instance, the spectrum of the ordinary cohomology ring of $\mathbf{P}_\Sigma$ as the scheme-theoretic intersection of the affine subspace $\Pi(\Sigma)$ with a *toric degeneration* of closures of $\mathbf{T}(\Sigma)$-orbits $\overline{T}_\varphi(\Sigma) \subset \mathbf{C}^n$. Such an interpretation allows to apply methods of M. Kapranov, B. Sturmfels, and A. Zelevinsky (see [10], Theorem 5.3) to establish connection between vertices of Chow polytope (secondary polyhedron) and Kähler limits of quantum cohomology rings.

8 Calabi-Yau hypersurfaces and Jacobian rings

Throughout in this section we fix a complete d-dimensional fan of regular cones, and we assume that $\mathbf{P} = \mathbf{P}_\Sigma$ is a toric manifold whose first Chern class belongs to the closed Kähler cone $K(\Sigma)$, i.e., $\alpha := \alpha_\Sigma$ is a convex Σ-piecewise linear function.

Let $\Delta = \Delta_\alpha$, the convex polyhedron in $M_{\mathbf{R}}$ (see 4.1). For any sufficiently general section S of the anticanonical sheaf $\mathcal{K}_V^{-1}$ on $\mathbf{P}$ represented by homogeneous polynomial $F(z)$, the set $Z = \{\pi(z) \in \mathbf{P} \,:\, F(z) = 0\}$ in $\mathbf{P}$ is a Calabi-Yau manifold $(c_1(\mathcal{K}_V^{-1}) = c_1(\mathbf{P}))$.

Since the first Chern class of $\mathbf{P}$ in the ordinary cohomology ring $H^*(\mathbf{P}, \mathbf{C})$ is the class of the sum $(z_1 + \cdots + z_n)$, we obtain:

Proposition 8.1 *The image of* $H^*(\mathbf{P}, \mathbf{C})$ *under the restriction mapping to* $H^*(Z, \mathbf{C})$ *is isomorphic to the quotient*

$$H^*(\mathbf{P}, \mathbf{C}) / \operatorname{Ann}(z_1 + \cdots + z_n),$$

where $\operatorname{Ann}(z_1 + \cdots + z_n)$ *denotes the annulet of the class of* $(z_1 + \cdots + z_n)$ *in* $H^*(\mathbf{P}, \mathbf{C})$.

In general, Proposition 8.1 allows us to calculate only a part of the ordinary cohomology ring of a Calabi-Yau hypersurface Z in toric variety $\mathbf{P}$. If the first Chern class of $\mathbf{P}$ is in the interior of the Kähler cone $K(\Sigma)$, then Z is an ample divisor. For $d \geq 4$, by Lefschetz theorem, the restriction mapping $H^2(\mathbf{P}, \mathbf{C}) \to H^2(Z, \mathbf{C})$ is isomorphism. Thus, using Proposition 8.1, we can calculate cup-products of any $(1,1)$-forms on Z.

Definition 8.2 Denote by Δ^* the convex hull of the set $G(\Sigma)$ of all generators, or equivalently,

$$\Delta^* = \{v \in N_{\mathbf{R}} \mid \alpha(v) \leq 1\}.$$

Remark 8.3 The polyhedron Δ^* is dual to Δ reflexive polyhedron (see [6]).

Theorem 8.4 *There exists the canonical isomorphism between the quantum cohomology ring*

$$QH^*_{\varphi}(\mathbf{P}, \mathbf{C})$$

and the Jacobian ring

$$\mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]/(X_1 \partial f/\partial X_1, \ldots, X_d \partial f/\partial X_d)$$

of the Laurent polynomial

$$f_{\varphi}(X) = -1 + \sum_{i=1}^{n} \exp(\varphi(v_i))^{-1} X^{v_i}.$$

This isomorphism is induced by the correspondence

$$z_i \to X^{v_i}/\exp(\varphi(v_i)) \quad (1 \leq i \leq n).$$

In particular, it maps the first Chern class $(z_1 + \ldots + z_n)$ *of* $\mathbf{P}$ *to* $f_{\varphi}(X) + 1$.

Proof. Let

$$\mathcal{H} \; : \; \mathbf{C}[z_1, \ldots, z_n] \to \mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$$

be the homomorphism defined by the correspondence

$$z_i \to X^{v_i}/\exp(\varphi(v_i)).$$

By 2.3(iii), $\mathcal{H}$ is surjective. It is clear that $Q_{\varphi}(\Sigma)$ is the kernel of $\mathcal{H}$. On the other hand, if we a $\mathbf{Z}$-basis $\{u_1, \ldots, u_d\} \subset M$ which establishes isomorphisms $M \cong \mathbf{Z}^d$ and $N \cong \mathbf{Z}^d$, we obtain:

$$\mathcal{H}(P(\Sigma)) =< X_1 \partial f/\partial X_1, \ldots, X_d \partial f/\partial X_d > .$$

□

Definition 8.5 Let S_{Δ^*} be the affine coordinate ring of the $T^{\circ}(\Sigma)$-orbit of the point $(1,\dots,1) \in \mathbf{C}^{n+1}$ (see 7.6).

Definition 8.6 For any Laurent polynomial

$$f(X) = a_0 + \sum_{i=1}^{n} a_i X^{v_i},$$

we define elements

$$F_0, F_1, \dots, F_d \in S_{\Delta^*}$$

as $F_i = \partial X_0 f(X) \partial X_0$, $(0 \leq i \leq d)$.

Remark 8.7 The ring S_{Δ^*} is a subring of $\mathbf{C}[X_0, X_1^{\pm 1}, \dots, X_d^{\pm 1}]$. There exists the canonical grading of S_{Δ^*} by degree of X_0.

It is easy to see that the correspondence

$$z_0 \to -X_0,$$

$$z_i \to X_0 X^{v_i}/(\exp(\varphi(v_i)))$$

defines the isomorphism

$$\mathbf{C}[z]/Q_{\varphi}(\Sigma) \cong S_{\Delta^*}.$$

This isomorphism maps $(-z_0 + z_1 + \cdots z_n)$ to F_0.

Theorem 8.8 ([7]) *Let*

$$R_f = S_{\Delta^*} / < F_0, F_1, \dots, F_d > .$$

Then the quotient

$$R_f / \mathrm{Ann}\,(X_0)$$

is isomorphic to the $(d-1)$-weight subspace $W_{d-1}H^{d-1}(Z_f, \mathbf{C})$ in the cohomology space $H^{d-1}(Z_f, \mathbf{C})$ of the affine Calabi-Yau hypersurface in $T(\Sigma)$ defined by the Laurent polynomial $f(X)$.

For any Laurent polynomial $f(X) = a_0 + \sum_{i=1}^{n} a_i X^{v_i}$, we can find an element $\varphi \in PL(\Sigma)_{\mathbf{C}}$ such that

$$\frac{-a_i}{a_0} = \exp(-\varphi(v_i)).$$

A one-parameter family $t\varphi$ in $PL(\Sigma)$ induces the one-parameter family of Laurent polynomials

$$f_t(X) = -1 + \sum_{i=1}^{n} \exp(-t\varphi(v_i)) X^{v_i}.$$

Applying the isomorphism in 8.7 and the statement in Theorem 5.3, we obtain the following:

Theorem 8.9 *Assume that φ is in the interior of the Kähler cone $K(\Sigma)$. Then the limit*

$$R_{f_t}/\mathrm{Ann}(X_0)$$

is isomorphic to

$$H^*(\mathbf{P},\mathbf{C})/\mathrm{Ann}(z_1+\cdots+z_n).$$

The last statement shows the relation, established in [3], between the *toric* part of the topological cohomology rings of Calabi-Yau 3-folds in toric varieties and limits of the multiplicative structure on $(d-1)$-weight part of the Jacobian rings of their *mirrors*.

9 Topological sigma models on toric manifolds

So far we have not explained why the ring $QH^*_\varphi(\mathbf{P}_\Sigma,\mathbf{C})$ coincides with the quantum cohomology ring corresponding to the topological sigma model on V. In this section we want to establish the relations between the ring $QH^*_\varphi(\mathbf{P}_\Sigma,\mathbf{C})$ and the quantum cohomology rings considered by physicists.

In order to apply the general construction of the correlation functions in sigma models ([16], 3a), we need the following information on the structure of the space of holomorphic mappings of $\mathbf{CP}^1$ to a d-dimensional toric manifold $\mathbf{P}_\Sigma$.

Theorem 9.1 *Let $\mathcal{I}$ be the moduli space of holomorphic mappings $f : \mathbf{CP}^1 \to \mathbf{P}_\Sigma$. The space $\mathcal{I}$ consists of infinitely many algebraic varieties $\mathcal{I}_\lambda$ indexed by elements*

$$\lambda = (\lambda_1,\ldots,\lambda_n) \in R(\Sigma),$$

where the numbers λ_i are equal to the intersection numbers $\deg_{\mathbf{CP}^1} f^*\mathcal{O}(Z_i)$ with divisors $Z_i \subset \mathbf{P}_\Sigma$ such that $\pi^{-1}(Z_i)$ is defined by the equation $z_i = 0$ in $U(\Sigma)$. Moreover, if all $\lambda_i \geq 0$, then $\mathcal{I}_\lambda$ is irreducible and the virtual dimension of $\mathcal{I}_\lambda$ equals

$$d_\lambda = \dim_{\mathbf{C}}\mathcal{I}_\lambda = d + \sum_{i=1}^{n}\lambda_i.$$

Proof. The first statement follows immediatelly from the description of the intersection product on $\mathbf{P}_\Sigma$ (3.5).

Assume now that all λ_i are non-negative. This means that the preimage $f^{-1}(Z_i)$ consists of λ_i points including their multiplicities. Let $\mathcal{F}_\Sigma$ be the tangent bundle over $\mathbf{P}_\Sigma$. There exists the generalized Euler exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}}^{n-d} \to \mathcal{O}_{\mathbf{P}}(Z_1)\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}}(Z_n) \to \mathcal{F}_\Sigma \to 0.$$

Applying f^*, we obtain the short exact sequence of vector bundles on $\mathbf{CP}^1$.

$$0 \to \mathcal{O}_{\mathbf{CP}^1}^{n-d} \to \mathcal{O}_{\mathbf{CP}^1}(\lambda_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{CP}^1}(\lambda_n) \to f^*\mathcal{F}_\Sigma \to 0.$$

This implies that $h^1(\mathbf{CP}^1, f^*\mathcal{F}_\Sigma) = 0$, and $h^0(\mathbf{CP}^1, f^*\mathcal{F}_\Sigma) = d + \lambda_1 + \cdots + \lambda_n$.

The irreducibility of $\mathcal{I}_\lambda$ for $\lambda \geq 0$ follows from the explicit geometrical construction of mappings $f \in \mathcal{I}_\lambda$:

Choose n polynomials $f_1(t), \ldots, f_n(t)$ such that $\deg f_i(t) = \lambda_i$ $(i = 1, \ldots, n)$. If all $\mid \lambda \mid = \lambda_1 + \cdots + \lambda_n$ roots of $\{f_i\}$ are distinct, then these polynomials define the mapping

$$g : \mathbf{C} \to U(\Sigma) \subset \mathbf{C}^n.$$

The composition $\pi \circ g$ extends to the mapping f of $\mathbf{CP}^1$ to $\mathbf{P}_\Sigma$ whose homology class is λ. □

Definition 9.2 Let

$$\Phi : \mathcal{I} \times \mathbf{CP}^1 \to \mathbf{P}_\Sigma$$

be the universal mapping. For every point $x \in \mathbf{CP}^1$ we denote by Φ_x the restriction of Φ to $\mathcal{I} \times x$. Choose the cohomology classes $z_1 = [Z_1], \ldots, z_n = [Z_n]$ of divisors $Z_1, \ldots, Z_n$ on $\mathbf{P}_\Sigma$ in the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma)$. We determine the divisors $W_{z_1}, \ldots, W_{z_n}$ on $\mathcal{I}$ whose cohomology classes are independent of choice of $x \in \mathbf{CP}^1$ as follows

$$W_{z_i} = \Phi_x^{-1}(Z_i) = \{f \in \mathcal{I} \mid f(x) \in Z_i\}.$$

The quantum cohomology ring of the sigma model with the target space $\mathbf{P}_\Sigma$ is defined by the relations

$$W_{\alpha_1} \cdot W_{\alpha_2} \cdot \cdots \cdot W_{\alpha_k} = \sum_{\lambda \in W_{\alpha_1} \cap \cdots \cap W_{\alpha_k}} \exp(-\deg_\varphi \lambda),$$

where α_i are cycles on $\mathbf{P}_\Sigma$ and $W_{\alpha_i} = \Phi_x^{-1}(\alpha_i)$, and the intersection $W_{\alpha_1} \cap \cdots \cap W_{\alpha_k}$ on the moduli space $\mathcal{I}$ is assumed to be of virtual dimension zero.

Theorem 9.3 *Let $\mathbf{P}_\Sigma$ be a d-dimensional toric manifold, $\varphi \in H^2(\mathbf{P}_\Sigma, \mathbf{C})$ a Kähler class. Let $\lambda^0 = (\lambda_1^0, \ldots, \lambda_n^0)$ be a non-negative element in $R(\Sigma)$, $\Omega \in H^{2d}(\mathbf{P}_\Sigma, \mathbf{C})$ the fundamental class of the toric manifold $\mathbf{P}_\Sigma$. Then the intersection number on the moduli space $\mathcal{I}$*

$$(W_\Omega) \cdot (W_{z_1})^{\lambda_1^0} \cdot (W_{z_2})^{\lambda_2} \cdot \cdots \cdot (W_{z_n})^{\lambda_n^0}$$

vanishes for all components $\mathcal{I}_\lambda$ except from $\lambda = \lambda_0$. In the latter case, this number equals

$$\exp(-\deg_\varphi \lambda).$$

Proof. Since the fundamental class Ω is involved in the considered intersection number, this number is zero for all $\mathcal{I}_\lambda$ such that the rational curves in the class λ do not cover a dense Zariski open subset in $\mathbf{P}_\Sigma$. Thus, we must consider only non-negative classes λ. Moreover, the factors $(W_{z_i})^{\lambda_i^0}$ show that we must consider only those $\lambda = (\lambda_1, \ldots, \lambda_n) \in R(\Sigma)$ such that $\lambda_i \geq \lambda_i^0$, i.e., a mapping $f \in \mathcal{I}_\lambda$ is defined by polynomials $f_1, \ldots, f_n$ such that $\deg f_i \geq \lambda_i$.

There is a general principle that non-zero contributions to the intersection product

$$(W_{\alpha_1} \cdot W_{\alpha_2} \cdot \cdots \cdot W_{\alpha_k})_{\mathcal{I}}$$

appear only from the components whose virtual $\mathbf{R}$-dimension is equal to

$$\sum_{i=1} \deg \alpha_i.$$

In our case, the last number is $d + \lambda_1^0 + \ldots + \lambda_n^0$. Therefore, a non-zero contribution appears only if $\lambda = \lambda^0$.

It remains to notice that this contribution equals $\exp(-\deg_\varphi \lambda_0)$. The last statement follows from the observation that the points $f^{-1}(Z_i) \subset \mathbf{CP}^1$ $(i = 1, \ldots, n)$ define the mapping $f : \mathbf{CP}^1 \to \mathbf{P}_\Sigma$ uniquely up to the action of the d-dimensional torus $\mathbf{T} = \mathbf{P}_\Sigma \setminus (Z_1 \cup \cdots \cup Z_n)$, and the weight of the mapping f in the intersection product is $\int_{\mathbf{CP}^1} f^*(\varphi)$. □

Corollary 9.4 *Let $\mathcal{Z}_i$ be the quantum operator corresponding to the class $[Z_i] \in H^2(\mathbf{P}_\Sigma, \mathbf{C})$ $(i = 1, \ldots, n)$ considered as an element of the quantum cohomology ring. Then for every non-negative element $\lambda \in R(\Sigma)$, one has the algebraic relation*

$$\mathcal{Z}_1^{\lambda_1} \circ \cdots \circ \mathcal{Z}_n^{\lambda_n} = \exp(-\deg_\varphi \lambda)\, id.$$

It turns out that the polynomial relations of above type are sufficient to recover the quantum cohomology ring $QH_\varphi^*(\mathbf{P}_\Sigma, \mathbf{C})$:

Theorem 9.5 *Let $A_\varphi(\Sigma)$ be the quotient of the polynomial ring $\mathbf{C}[z]$ by the sum of two ideals: $P(\Sigma)$ and the ideal generated by all polynomials*

$$B_\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n} - \exp(-\deg_\varphi \lambda)$$

where λ runs over all non-negative elements of $R(\Sigma)$. Then $A_\varphi(\Sigma)$ is isomorphic to $QH_\varphi^*(\mathbf{P}_\Sigma, \mathbf{C})$.

Proof. Let $B_\varphi(\Sigma)$ be the ideal generated by all binomials B_λ. By definition, $B_\varphi(\Sigma) \subset Q_\varphi(\Sigma)$. So it is sufficient to prove that $Q_\varphi(\Sigma) \subset B_\varphi(\Sigma)$.

Let

$$\sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{n} b_j v_j$$

be a linear relation among $v_1, \ldots, v_n$ such that a_i, $b_j \geq 0$. Since the set of all nonnegative elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in R(\Sigma)$ $(\lambda_i \geq 0)$ generates a convex cone of maximal dimension in $H^2(\mathbf{P}_\Sigma, \mathbf{C})$, there exist two nonnegative vectors λ, $\lambda' \in R(\Sigma)$ such that

$$\lambda - \lambda' = (\lambda_1 - \lambda'_1, \ldots, \lambda_n - \lambda'_n) = (a_1 - b_1, \ldots, a_n - b_n).$$

By definition, two binomials P_λ and $P_{\lambda'}$ are contained in $Q_\varphi(\Sigma)$. Hence, the classes of $z_1, \ldots, z_n$ in $\mathbf{C}[z]/B_\varphi(\Sigma)$ are invertible elements. Thus, the class of the binomial

$$\exp\left(\sum_{i=1}^{n} a_i \varphi(v_i)\right) \prod_{i=1}^{n} z_i^{a_i} - \exp\left(\sum_{j=1}^{n} b_j \varphi(v_j)\right) \prod_{j=1}^{n} z_j^{b_j}$$

is zero in $\mathbf{C}[z]/B_\varphi(\Sigma)$. Thus, $B_\varphi(\Sigma) = Q_\varphi(\Sigma)$. □.

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